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Existence of solutions to Sturm–Liouville boundary value problems

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Abstract. We study the solvability of Sturm–Liouville boundary value problems for $x'' = f(t,x,x'), t \in (0,1)$. The nonlinearity can be defined on a bounded set and is required to be continuous on its subset. The results obtained are based on combinations of well-known conditions with barrier strip type conditions.

Keywords: nonlinear boundary value problem, Sturm–Liouville boundary conditions, existence, barrier strips conditions.

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1 Introduction

This paper is devoted to the solvability of boundary value problems (BVPs) for the equation

$$x'' = f(t,x,x'), \qquad t \in (0,1), \tag{1.1}$$

with Sturm–Liouville boundary conditions (BCs) either

$$-\alpha x(0) + \beta x'(0) = A, \qquad a x(1) + b x'(1) = B, \tag{1.2}$$

$$x'(0) = A, \qquad a x(1) + b x'(1) = B, \tag{1.3}$$

or

$$-\alpha x(0) + \beta x'(0) = A, \qquad x'(1) = B, \tag{1.4}$$

where $f : [0,1] \times D_x \times D_p \to \mathbb{R}$, $D_x, D_p \subseteq \mathbb{R}$, $\alpha, \beta, a, b > 0$, and $A, B \in \mathbb{R}$.

This paper is motivated by A. Granas et al. [6]. The authors prove that BVP (1.1), (1.2) has a solution in $C^2[0,1]$ assuming that the function $f(t,x,p)$ is continuous on $[0,1] \times \mathbb{R}^2$ and there is a constant $M \geq 0$ such that

$$x f(t,x,0) \geq M \quad \text{for } t \in [0,1] \text{ and } |x| > M \tag{1.5}$$

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and the well-known Bernstein's growth condition holds, that is, there exist positive constants $G_i, i = 1,2$, for which

$$|f(t,x,p)| \le G_1 p^2 + G_2 \quad \text{for } (t,x) \in [0,1] \times [-M_0, M_0], \tag{1.6}$$

where $M_0 = \max\{|A/\alpha|, |B/a|, M\}$. A similar result guarantees $C^2[0,1]$-solutions to BVPs (1.1), (1.3) and (1.1), (1.4). In it, (1.5) is replaced by the assumption that $f(t,x,p)$ is differentiable with respect to x and there is a constant $K > 0$ such that

$$f_x(t,x,p) \ge K \quad \text{for } (t,x,p) \in [0,1] \times \mathbb{R} \times \{C\}, \tag{1.7}$$

where $C = A$ or $C = B$. The conditions imposed guarantee a priori bounds. Moreover, each of the conditions (1.5) and (1.7) provide a priori bound for the solutions to the considered BVP, that is, for $|x(t)|$, and (1.6) provides the bound for $|x'(t)|$. The established a priori bounds are needed for applying the Topological transversality theorem.

One of the BVPs studied in C. Tisdell [10] is (1.1), (1.2). Here, the Leray–Schauder degree theory and a priori bounds are used. The well known condition

$$f(t,-R_1,0) < 0 \quad \text{and} \quad f(t,R_2,0) > 0, \qquad t \in [0,1],$$

where R_1 and R_2 are some positive constants with $\min\{R_1,R_2\} > \max\{|A/\alpha|, |B/a|\}$, gives the bound for $|x(t)|$, and this for $|x'(t)|$ follows from the assumption that there exist non-negative constants α and K such that

$$|f(t,x,p)| \le \alpha f(t,x,p) + K \quad \text{for all } t \in [0,1], |x| \le R, p \in \mathbb{R},$$

where $R = \max\{R_1, R_2\}$.

Other results on the solvability of BVPs for various equations with Sturm–Liouville boundary conditions can be found, for example, in M. Dobkevich and F. Sadyrbaev [3], A. M. A. El-Sayed et al. [4,5], T. Xue et al. [12], Y. Liu et al. [9], F. H. Wong et al. [11] and L. Zhang et al. [13].

The purpose of this paper is to give sufficient conditions for the existence of solutions in which growth restrictions on $f(t,x,p)$ are not imposed, that is, we do not use condition (1.6). It is replaced by sign conditions of barrier strips type.

The existence discussion is based on the basic existence theorem proved in R. P. Agarwal et al. [2], which is a variant of [6, Chapter V, Theorem 1.1]. Let us prepare its wording.

Consider the BVP

$$\begin{cases} x^{(n)} + \sum_{k=0}^{n-1} s_k(t) x^{(k)} = f(t,x,x',\dots,x^{(n-1)}), \ t \in [0,1], \\ V_i(x) = A_i, \ i = \overline{1,n}, \end{cases} \tag{1.8}$$

of which the considered boundary value problems (1.1), (1.2)–(1.4) are special cases. Here $s_k(t), k = \overline{0,n-1}$, are continuous on $[0,1]$, $f : [0,1] \times D_0 \times D_1 \times \cdots \times D_{n-1} \to \mathbb{R}$,

$$V_i(x) \equiv \sum_{j=0}^{n-1} [a_{ij} x^{(j)}(0) + b_{ij} x^{(j)}(1)], \qquad i = \overline{1,n},$$

with constants a_{ij} and b_{ij} for which $\sum_{j=0}^{n-1}(a_{ij}^2 + b_{ij}^2) > 0, i = \overline{1,n}$, and $A_i \in \mathbb{R}$.

For $\lambda \in [0,1]$, consider also the family of BVPs

$$\begin{cases} x^{(n)} + \sum_{k=0}^{n-1} s_k(t) x^{(k)} = g(t,x,x',\dots,x^{(n-1)},\lambda), \ t \in [0,1], \\ V_i(x) = A_i, \ i = \overline{1,n}, \end{cases} \tag{1.9}$$

where $g : [0,1] \times D_0 \times D_1 \times \cdots \times D_{n-1} \times [0,1] \to \mathbb{R}$, and $s_k(t), k = \overline{0,n-1}, V_i, A_i, i = \overline{1,n}$, are as above.

Let, as usual, $C[0,1]$ be the Banach space of continuous functions on $[0,1]$ with the norm $\|x\|_0 = \sup_{t\in[0,1]} |x(t)|$, and $C^n[0,1]$ be the Banach space of n-times continuously differentiable functions with $\|x\|_n = \max\{\|x\|_0, \ldots, \|x^{(n)}\|_0\}$.

Let $\mathcal{B}$ denote the set of functions that satisfy the BCs $V_i(x) = A_i$, $i = \overline{1,n}$, and $\mathcal{B}_0$ be the set of functions satisfying $V_i(x) = 0, i = \overline{1,n}$. Finally, let $C^n_{\mathcal{B}}[0,1] = C^n[0,1] \cap \mathcal{B}$ and $C^n_{\mathcal{B}_0}[0,1] = C^n[0,1] \cap \mathcal{B}_0$.

We are now ready to formulate the basic existence theorem.

Theorem 1.1 ([2, Theorem 4]). *Assume that:*

(i) *For $\lambda = 0$ problem (1.9) has a unique solution in $C^n[0,1]$.*

(ii) *Problems (1.8) and (1.9) are equivalent when $\lambda = 1$.*

(iii) *The map $\Lambda_h : C^n_{\mathcal{B}_0}[0,1] \to C[0,1]$, defined by*

$$\Lambda_h x = x^{(n)} + \sum_{k=0}^{n-1} s_k(t) x^{(k)},$$

is one-to-one.

(iv) *Each solution $x \in C^n[0,1]$ to family (1.9) satisfies the bounds*

$$m_i \le x^{(i)}(t) \le M_i \quad \text{for } t \in [0,1],\ i = \overline{0,n},$$

where the constants $-\infty < m_i, M_i < \infty$, $i = \overline{0,n}$, are independent of λ and x.

(v) *There is a sufficiently small $\delta > 0$ such that*

$$[m_i - \delta, M_i + \delta] \subseteq D_i, \qquad i = \overline{0,n-1},$$

and the function $g(t, p_0, \ldots, p_{n-1}, \lambda)$ is continuous on $[0,1] \times D \times [0,1]$, where

$$D = [m_0 - \delta, M_0 + \delta] \times [m_1 - \delta, M_1 + \delta] \times \cdots \times [m_{n-1} - \delta, M_{n-1} + \delta];$$

$m_i, M_i, i = \overline{0,n-1}$, are as in (iv).

Then, BVP (1.8) has at least one solution in $C^n[0,1]$.

To apply Theorem 1.1 for studying the considered BVPs, we use families of BVPs for

$$x'' = \lambda f(t,x,x') + (1-\lambda)(x - x'), \qquad t \in (0,1), \tag{1.1}$$

and

$$x'' = \lambda f(t,x,x') + (1-\lambda)(x + x'), \qquad t \in (0,1), \tag{1.1}$$

where $\lambda \in [0,1]$. They are adapted to the application of the barrier strips type conditions used here, namely conditions **(B_1)** and **(B_2)** below.

In order to achieve the a priori bounds of condition *(iv)* of Theorem 1.1, we impose three sets of conditions. The conditions of set **A**, these are (1.5) and (1.7), guarantee the a priori bounds for each eventual $C^2[0,1]$-solution $x(t)$ to the used families, those of the set **B** give the bounds for $x'(t)$, and **C** ensures the bounds for $x''(t)$.

Following are the hypotheses used in this article.

(A$_1$) There is a constant $M \geq 0$ such that $[-M, M] \subseteq D_x$ and

$$xf(t, x, 0) \geq 0 \quad \text{for } t \in [0, 1] \text{ and } x \in D_x \setminus [-M, M].$$

In the formulation of the next hypothesis, we use the functions

$$v(t) = At + \frac{B - A(a + b)}{a} \quad \text{and} \quad w(t) = Bt + \frac{B\beta - A}{\alpha}, \qquad t \in [0, 1],$$

and more precisely the constants $m_v = \min_{[0,1]} v(t)$, $M_v = \max_{[0,1]} v(t)$, $m_w = \min_{[0,1]} w(t)$ and $M_w = \max_{[0,1]} w(t)$.

(A$_2$) $J \subseteq D_x$, here $J = [m_v, M_v]$ for (1.3) and $J = [m_w, M_w]$ for (1.4), and there is a constant $K > 0$ such that

$$f_x(t, x, C) \geq K \quad \text{for } (t, x) \in [0, 1] \times D_x,$$

where $C = A$ for (1.3) and $C = B$ for (1.4).

For some constants M_0, G_m and G_M, they will be specified later for each problem, suppose that:

(B$_1$) There are constants $F_i, L_i, i = 1, 2$, such that $[F_2, L_2] \subseteq D_p$,

$$F_2 < F_1 \leq \min\{-M_0, G_m\}, \qquad \max\{M_0, G_M\} \leq L_1 < L_2,$$

$$f(t, x, p) \leq 0 \quad \text{for } (t, x, p) \in [0, 1] \times [-M_0, M_0] \times [L_1, L_2], \tag{1.10}$$

$$f(t, x, p) \geq 0 \quad \text{for } (t, x, p) \in [0, 1] \times [-M_0, M_0] \times [F_2, F_1]. \tag{1.11}$$

(B$_2$) There are constants $F_i', L_i', i = 1, 2$, such that $[F_2', L_2'] \subseteq D_p$,

$$F_2' < F_1' \leq \min\{-M_0, G_m\}, \qquad \max\{M_0, G_M\} \leq L_1' < L_2',$$

$$f(t, x, p) \geq 0 \quad \text{for } (t, x, p) \in [0, 1] \times [-M_0, M_0] \times [L_1', L_2'], \tag{1.12}$$

$$f(t, x, p) \leq 0 \quad \text{for } (t, x, p) \in [0, 1] \times [-M_0, M_0] \times [F_2', F_1'].$$

(C) There are constants $m_i \leq M_i, i = 0, 1$, and a sufficiently small $\delta > 0$ such that

$$[m_0 - \delta, M_0 + \delta] \subseteq D_x, \qquad [m_1 - \delta, M_1 + \delta] \subseteq D_p,$$

and $f(t, x, p)$ is continuous on $[0, 1] \times [m_0 - \delta, M_0 + \delta] \times [m_1 - \delta, M_1 + \delta]$.

The paper is organized as follows. In Section 2 we establish a priori bounds for $x(t)$ and $x'(t)$ for each solution $x \in C^2[0, 1]$ to the families of BVPs for $(1.1)_\lambda^-$ with BCs (1.2) or (1.3) and for $(1.1)_\lambda^+$ with BCs (1.2) or (1.4). In Section 3 we use the obtained bounds to prove existence results for the considered BVPs. Examples illustrate the application of the obtained results in Section 4.

2 Auxiliary results

We need a well known maximum principle, see for example [6, Chapter II, Lemma 1.1], concerning equations of the form

$$x'' = h(t, x, x'), \qquad t \in [0, 1]. \tag{2.1}$$

It is based on the following assumption.

(A) There is a constant $M \geq 0$ such that $[-M, M] \subseteq D_x$ and

$$xh(t, x, 0) > 0 \quad \text{for } t \in [0, 1] \text{ and } x \in D_x \setminus [-M, M].$$

Lemma 2.1. *Let $x \in C^2[0, 1]$ be a solution to equation (2.1) such that $|x(t)|$ does not achieve its maximum at $t = 0$ or $t = 1$. Assume further that* **(A)** *holds. Then $x(t)$ satisfies the bound*

$$|x(t)| \leq M \text{ for } t \in [0, 1].$$

Proof. By the assumption of the lemma, $|x(t)|$ must achieve a positive maximum at a point $t_0 \in (0, 1)$. Clearly, the function $y(t) = (x(t))^2$ also has a maximum at t_0. Thus,

$$y''(t_0) = 2x(t_0)x''(t_0) = 2x(t_0)h(t_0, x(t_0), 0) \leq 0.$$

Next, reasoning by contradiction, assume $|x(t_0)| > M$. Then from **(A)** it follows

$$x(t_0)h(t_0, x(t_0), 0) > 0$$

and the derived contradiction proves the lemma. □

The proofs of the following two lemmas follow the idea of proof of [6, Chapter II, Lemma 1.2].

Lemma 2.2. *Let* **(A)** *hold. Then each solution $x \in C^2[0, 1]$ to (2.1), (1.3) with $A = B = 0$ satisfies the bound*

$$|x(t)| \leq M, \qquad t \in [0, 1].$$

Proof. Suppose that $|x(0)|$ is the maximum value of $|x(t)|$. We claim that $|x(0)| > M$ is impossible. To verify this, by contradiction, assume it is true. Then,

$$x(0)x''(0) = x(0)h(0, x(0), 0) > 0.$$

Now, if $x(0) < 0$, then $x''(0) < 0$. Because of the continuity of $x''(t)$ on $[0, 1]$, there is a neighborhood $N_0 \subseteq [0, 1]$ of $t = 0$ where $x''(t) < 0$. This means that $x'(t)$ is strictly decreasing on N_0 and so $x'(t) < x'(0) = 0$ for $t \in N_0$. Consequently $x(t)$ is also strictly decreasing on N_0 and so $|x(0)|$ can not be the maximum of $|x(t)|$ on $[0, 1]$, a contradiction. If $x(0) > 0$, then $x''(0) > 0$, from where conclude $x'(t) > x'(0) = 0$ for $t \in U_0$, where $U_0 \subseteq [0, 1]$ is any neighborhood of $t = 0$. Thus, $x(t) > x(0) > 0$ for $t \in U_0$, which means that $|x(t)| > |x(0)|, t \in U_0$, again a contradiction. So, $|x(0)| \leq M$.

Let $|x(1)|$ be the maximum value of $|x(t)|, t \in [0, 1]$. Then, $x(1)x'(1) \geq 0$ and

$$0 \leq x(1)bx'(1) = x(1)(-ax(1)) = -a(x(1))^2 \leq 0,$$

which is possible if $x(1) = 0$. So, we have $\max_{[0,1]} |x(t)| = 0$, which means $x(t) = 0$ for $t \in [0,1]$ and the lemma is true.

Finally, if $|x(t)|$ achieve its maximum in $(0,1)$, then

$$|x(t)| \leq M, \qquad t \in [0,1],$$

by Lemma 2.1. □

Lemma 2.3. *Let* **(A)** *hold. Then each solution* $x \in C^2[0,1]$ *to* (2.1), (1.4) *with* $A = B = 0$ *satisfies the bound*

$$|x(t)| \leq M, \qquad t \in [0,1].$$

Proof. If $|x(0)|$ is the maximum value of $|x(t)|$, we have $x(0)x'(0) \leq 0$ from where obtain

$$0 \geq x(0)\beta x'(0) = x(0)(\alpha x(0)) = \alpha(x(0))^2 \geq 0.$$

This implies $\max_{[0,1]} |x(t)| = |x(0)| = 0$ which means $x(t) = 0$ on $[0,1]$ and so the lemma is true.

If $|x(t)|$ achieves its maximum at $t = 1$, the assumption that $|x(1)| > M$ yields

$$x(1)x''(1) = x(1)h(1, x(1), 0) > 0.$$

Next, following the proof of Lemma 2.2, we derive contradictions in the cases $x(1) < 0$ and $x(1) > 0$ and conclude that $|x(1)| \leq M$.

Finally, if $|x(t)|$ achieves its maximum somewhere at the interval $(0,1)$, then the bound

$$|x(t)| \leq M, \qquad t \in [0,1],$$

follows from Lemma 2.1. □

Following the proof of [6, Chapter II, Theorem 3.3], establish the assertion.

Lemma 2.4. *Let* **(A)** *hold for* (1.1). *Then each solution* $x \in C^2[0,1]$ *to* $(1.1)^-_\lambda$, (1.2) *or to* $(1.1)^+_\lambda$, (1.2) *satisfies the bound*

$$|x(t)| \leq \max\{|A/\alpha|, |B/a|, M\}, \qquad t \in [0,1].$$

Proof. We will prove the assertion about family $(1.1)^-_\lambda$, (1.2), the proof for family $(1.1)^+_\lambda$, (1.2) is practically the same.

If $|x(0)|$ is the maximum value of $|x(t)|$ on $[0,1]$ we have $x(0)x'(0) \leq 0$, from where obtain

$$0 \geq x(0)\beta x'(0) = x(0)(A + \alpha x(0)) = \alpha(x(0))^2\Big[\frac{A}{\alpha x(0)} + 1\Big].$$

This yields consecutively

$$\frac{A}{\alpha x(0)} + 1 \leq 0, \qquad \left|\frac{A}{\alpha x(0)}\right| \geq 1 \quad \text{and} \quad |x(0)| \leq |A/\alpha|.$$

Likewise, if $|x(t)|$ achieves its maximum at $t = 1$, we obtain

$$|x(1)| \leq |B/a|.$$

Now, let $|x(t)|$ achieve its maximum in an interior point of $(0,1)$. For the right-hand side of equation $(1.1)_\lambda^-$ we have

$$\lambda x f(t,x,0) + (1-\lambda)x^2 > 0 \quad \text{for each} \quad \lambda \in [0,1], t \in [0,1] \quad \text{and} \quad x \in D_x \setminus [-M,M],$$

that is, **(A)** is satisfied for $(1.1)_\lambda^-$. Thus,

$$|x(t)| \le M, \qquad t \in [0,1],$$

by Lemma 2.1. □

Now, let

$$L_- = \max\{\max|f(t,x,A)|, \max|x-A|\},$$

where the maximums are computed for $(t,x) \in [0,1] \times [m_v, M_v]$.

Lemma 2.5. *Let* **(A_2)** *hold for* $J = [m_v, M_v]$ *and* $C = A$. *Then each solution* $x \in C^2[0,1]$ *to* $(1.1)_\lambda^-$, (1.3) *satisfies the bound*

$$|x(t)| \le \frac{L_-}{\min\{1,K\}} + \max\{|m_v|, |M_v|\}, \qquad t \in [0,1].$$

Proof. Define the function $y(t) = x(t) - v(t), t \in [0,1]$. Since $v(t)$ satisfies BC (1.3), $y(t)$ is a $C^2[0,1]$-solution to the homogeneous BVP

$$\begin{aligned} y'' &= h(t,y,y',\lambda), \qquad t \in (0,1), \\ y'(0) &= 0, \qquad ay(1) + by'(1) = 0, \end{aligned}$$

where $h(t,y,y',\lambda) = \lambda f(t,y+v,y'+v') + (1-\lambda)(y+v-y'-v'),\ \lambda[0,1]$. Besides,

$$\begin{aligned} yh(t,y,0,\lambda) &= y\Big(\lambda f(t,y+v,v') + (1-\lambda)(y+v-v')\Big) \\ &= y\Big(\lambda f(t,y+v,A) + (1-\lambda)(y+v-A) - \lambda f(t,v,A) \\ &\quad - (1-\lambda)(v-A) + \lambda f(t,v,A) + (1-\lambda)(v-A)\Big) \\ &= y\Big(\lambda f_x(t,\theta y+v,A)y + (1-\lambda)y\Big) + y\Big(\lambda f(t,v,A) + (1-\lambda)(v-A)\Big) \end{aligned}$$

for any $\theta \in (0,1)$. Let us note, $f_x(t,\theta y+v,A), t \in [0,1]$, is well defined since for $t \in [0,1]$ $\theta y(t) + v(t) \in \big[\min\{m_v, \min_{[0,1]} x(t)\}, \max\{M_v, \max_{[0,1]} x(t)\}\big] \subseteq D_x$. Then

$$yh(t,y,0,\lambda) \ge y^2(\lambda K + (1-\lambda)) - |y|L_- \ge |y|(|y|\min\{1,K\} - L_-),$$

from where it follows

$$yh(t,y,0,\lambda) > 0 \quad \text{for } t \in [0,1] \text{ and } |y| > \frac{L_-}{\min\{1,k\}}$$

and for each $\lambda \in [0,1]$. Thus,

$$|y(t)| \le \frac{L_-}{\min\{1,k\}} \quad \text{for } t \in [0,1],$$

by Lemma 2.2. Keeping in mind that $x(t) = y(t) + v(t)$, we obtain the lemma. □

Now, let

$$L_+ = \max\{\max|f(t,x,B)|, \max|x+B|\},$$

where the maximums are computed over $(t,x) \in [0,1] \times [m_w, M_w]$.

Lemma 2.6. *Let* **(A_2)** *hold for* $J = [m_w, M_w]$ *and* $C = B$. *Then each solution* $x \in C^2[0,1]$ *to* $(1.1)^+_\lambda$, (1.4) *satisfies the bound*

$$|x(t)| \le \frac{L_+}{\min\{1,K\}} + \max\{|m_w|, |M_w|\}, \qquad t \in [0,1].$$

Proof. Following the proof of Lemma 2.5, introduce the function $y(t) = x(t) - w(t), t \in [0,1]$. Since $w(t)$ satisfies BC (1.4), $y(t)$ is a $C^2[0,1]$-solution to the homogeneous BVP

$$\begin{aligned} y'' &= h(t,y,y',\lambda), \quad t \in (0,1), \\ -\alpha y(0) + \beta y'(0) &= 0, \quad y'(0) = 0, \end{aligned}$$

where $h(t,y,y',\lambda) = \lambda f(t,y+w,y'+w') + (1-\lambda)(y+w+y'+w')$. Besides,

$$\begin{aligned} yh(t,y,0,\lambda) &= y\Big(\lambda f(t,y+w,w') + (1-\lambda)(y+w+w')\Big) \\ &= y\Big(\lambda f(t,y+w,B) + (1-\lambda)(y+w+B) - \lambda f(t,w,B) \\ &\qquad - (1-\lambda)(w+B) + \lambda f(t,w,B) + (1-\lambda)(w+B)\Big) \\ &= y\Big(\lambda f_x(t,\theta y+w,B)y + (1-\lambda)y\Big) + y\Big(\lambda f(t,w,B) + (1-\lambda)(w+B)\Big) \end{aligned}$$

for any $\theta \in (0,1)$. Then

$$yh(t,y,0,\lambda) \ge y^2(\lambda K + (1-\lambda)) - |y|L_+ \ge |y|(|y|\min\{1,K\} - L_+),$$

from where it follows

$$yh(t,y,0,\lambda) > 0 \text{ for } t \in [0,1] \text{ and } |y| > \frac{L_+}{\min\{1,k\}}$$

and for each $\lambda \in [0,1]$. Thus,

$$|y(t)| \le \frac{L_+}{\min\{1,k\}} \quad \text{for } t \in [0,1],$$

by Lemma 2.3, which yields the bound for $|x(t)|$. □

The considerations used in the proofs of the following statements are standard for proofs of lemmas based on barrier strips conditions. These results provide a priori bounds for the $(n-1)$th derivative of the nth-order differential equations, see for example Lemma 2 of R. P. Agarwal and P. Kelevedjiev [1], Lemma 3.1 of P. Kelevedjiev and T. Todorov [7] and Theorem 3.1 of P. Kelevedjiev [8].

Lemma 2.7. *Let* **(B_1)** *hold for constants* M_0, $G_m = (A - \alpha M_0)\beta^{-1}$ *and* $G_M = (A + \alpha M_0)\beta^{-1}$. *Then each solution* $x \in C^2[0,1]$ *to* $(1.1)^-_\lambda$, (1.2) *with the property*

$$-M_0 \le x(t) \le M_0, \qquad t \in [0,1],$$

satisfies the bound

$$F_1 \le x'(t) \le L_1, \qquad t \in [0,1].$$

Proof. Reasoning by contradiction, assume that $x'(t) > L_1$ for some $t \in (0,1]$. Then, the continuity of $x'(t)$ on $[0,1]$ and

$$x'(0) = \frac{A + \alpha x(0)}{\beta} \leq \frac{A + \alpha M_0}{\beta} \leq L_1$$

imply that the set

$$S_- = \{t \in [0,1] : L_1 < x'(t) \leq L_2\}$$

is not empty. Also, that there is a $\gamma \in S_-$ such that

$$x''(\gamma) > 0.$$

Because $x(t)$ is a $C^2[0,1]$-solution to $(1.1)^-_\lambda$, we have in particular

$$x''(\gamma) = \lambda f(\gamma, x(\gamma), x'(\gamma)) + (1-\lambda)(x(\gamma) - x'(\gamma)).$$

Now, from $(\gamma, x(\gamma), x'(\gamma)) \in S_- \times [-M_0, M_0] \times (L_1, L_2]$ and (1.10) we get

$$\lambda f(\gamma, x(\gamma), x'(\gamma)) \leq 0,$$

and $x'(\gamma) > L_1 \geq M_0 \geq x(\gamma)$ yields $x(\gamma) - x'(\gamma) < 0$. As a result

$$x''(\gamma) \leq 0,$$

a contradiction. Thus,

$$x'(t) \leq L_1 \text{ for } t \in [0,1].$$

Similarly, assuming that $x'(t) < F_1$ for some $t \in (0,1]$ and using that

$$x'(0) = \frac{A + \alpha x(0)}{\beta} \geq \frac{A - \alpha M_0}{\beta} \geq F_1,$$

we establish that

$$S_+ = \{t \in [0,1] : F_2 \leq x'(t) < F_1\}$$

is the empty set and so

$$x'(t) \geq F_1 \quad \text{for } t \in [0,1]. \qquad \square$$

Lemma 2.8. *Let* **(B_2)** *hold for constants* M_0, $G_m = (B - aM_0)b^{-1}$ *and* $G_M = (B + aM_0)b^{-1}$. *Then each solution* $x \in C^2[0,1]$ *to* $(1.1)^+_\lambda$, (1.2) *with the property*

$$-M_0 \leq x(t) \leq M_0, \qquad t \in [0,1],$$

satisfies the bound

$$F_1' \leq x'(t) \leq L_1', \qquad t \in [0,1].$$

Proof. By contradiction, assume that $x'(t) > L_1'$ for some $t \in [0,1)$. This means that the set

$$S_+ = \{t \in [0,1] : L_1' < x'(t) \leq L_2'\}$$

is not empty because $x'(t)$ is continuous on $[0,1]$ and

$$x'(1) = \frac{B - ax(1)}{b} \leq \frac{B + aM_0}{b} \leq L_1'.$$

Also, there is a $\gamma \in S_+$ such that

$$x''(\gamma) < 0.$$

As $x(t)$ is a $C^2[0,1]$-solution to $(1.1)^+_\lambda$,

$$x''(\gamma) = \lambda f(\gamma, x(\gamma), x'(\gamma)) + (1-\lambda)(x(\gamma) + x'(\gamma)).$$

Now, $(\gamma, x(\gamma), x'(\gamma)) \in S_+ \times [-M_0, M_0] \times (L_1', L_2']$ and (1.12) imply

$$\lambda f(\gamma, x(\gamma), x'(\gamma)) \geq 0.$$

Besides, $x(\gamma) + x'(\gamma) > 0$ because $x'(\gamma) > L_1' \geq M_0 \geq x(\gamma) \geq -M_0$. Thus,

$$x''(\gamma) \geq 0,$$

a contradiction. Consequently,

$$x'(t) \leq L_1' \quad \text{for } t \in [0,1].$$

Along similar lines, assuming on the contrary that the set

$$S_- = \{t \in [0,1] : F_2' \leq x'(t) < F_1'\}$$

is not empty and using that

$$x'(1) = \frac{B - ax(0)}{b} \geq \frac{B - aM_0}{b} \geq F_1',$$

we achieve a contradiction which implies that

$$F_1' \leq x'(t) \quad \text{for } t \in [0,1]. \qquad \square$$

Lemma 2.9. *Let* **(B_1)** *hold for constants* M_0, $G_m = \min\{-M_0, A\}$ *and* $G_M = \max\{M_0, A\}$. *Then each solution* $x \in C^2[0,1]$ *to* $(1.1)^-_\lambda$, (1.3) *with the property*

$$-M_0 \leq x(t) \leq M_0, \qquad t \in [0,1],$$

satisfies the bound

$$F_1 \leq x'(t) \leq L_1, \qquad t \in [0,1].$$

Proof. Let, on the contrary, $x'(t) < F_1$ for some $t \in (0,1]$. Then, the continuity of $x'(t)$ on $[0,1]$ and $x'(0) = A \geq F_1$ imply that the set

$$S_+ = \{t \in [0,1] : F_2 \leq x'(t) < F_1\}$$

is not empty and there is a $\gamma \in S_+$ such that

$$x''(\gamma) < 0.$$

On the other hand, since $x(t)$ is a $C^2[0,1]$-solution to $(1.1)^-_\lambda$, we have in particular

$$x''(\gamma) = \lambda f(\gamma, x(\gamma), x'(\gamma)) + (1-\lambda)(x(\gamma) - x'(\gamma)).$$

But, $(\gamma, x(\gamma), x'(\gamma)) \in S_+ \times [-M_0, M_0] \times [F_2, F_1)$ and (1.11) imply

$$\lambda f(\gamma, x(\gamma), x'(\gamma)) \geq 0,$$

and $x(\gamma) - x'(\gamma) > 0$, because $x'(\gamma) < F_1 \le -M_0 \le x(\gamma)$. As a result

$$x''(\gamma) \ge 0$$

and a contradiction is achieved. Thus,

$$x'(t) \ge F_1 \quad \text{for } t \in [0,1].$$

Similar reasoning based on the use of (1.10) shows that

$$S_- = \{t \in [0,1] : L_1 < x'(t) \le L_2\}$$

is empty and so

$$x'(t) \le L_1 \quad \text{for } t \in [0,1]. \qquad \square$$

The proof of the next assertion is omitted, it follows the lines of proof of Lemmas 2.7, 2.8 and 2.9.

Lemma 2.10. *Let* **(B_2)** *hold for constants* M_0, $G_m = \min\{-M_0, B\}$ *and* $G_M = \max\{M_0, B\}$. *Then each solution* $x \in C^2[0,1]$ *to* $(1.1)^+_\lambda$, (1.4) *with the property*

$$-M_0 \le x(t) \le M_0, \qquad t \in [0,1],$$

satisfies the bound

$$F_1' \le x'(t) \le L_1', \qquad t \in [0,1].$$

3 Existence results

BVP $(1.1)^-_0$, (1.2), arising from $(1.1)^-_\lambda$, (1.2) for $\lambda = 0$, has the form

$$\begin{aligned} x'' + x' - x &= 0, \qquad t \in (0,1), \\ -\alpha x(0) + \beta x'(0) &= A, \qquad a x(1) + b x'(1) = B. \end{aligned}$$

By standard reasoning, we obtain that the solutions of the differential equation are the functions $x(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$, and the system for C_1 and C_2 gives a unique solution to the BVP if

$$\begin{vmatrix} \beta r_1 - \alpha & \beta r_2 - \alpha \\ (a + b r_1) e^{r_1} & (a + b r_2) e^{r_2} \end{vmatrix} \ne 0, \tag{3.1}$$

where $r_1 = -\frac{1+\sqrt{5}}{2}$ and $r_2 = \frac{-1+\sqrt{5}}{2}$ are the roots of the characteristic equation.

Theorem 3.1. *Let* **(A_1)** *hold,* **(B_1)** *hold for*

$$M_0 = \max\{|A/\alpha|, |B/a|, M\}, \qquad G_m = (A - \alpha M_0)\beta^{-1} \quad \textit{and} \quad G_M = (A + \alpha M_0)\beta^{-1},$$

(C) *hold for the same constant* M_0 *and for* $m_0 = -M_0, m_1 = F_1, M_1 = L_1$ *and* (3.1) *be satisfied. Then BVP* (1.1), (1.2) *has at least one solution in* $C^2[0,1]$.

Proof. We divide the proof into two steps.

First step. We will check the hypotheses of Theorem 1.1 for the family of BVPs $(1.1)^-_\lambda$, (1.2) and BVP (1.1), (1.2) assuming firstly that the inequality in **(A$_1$)** is strong, that is,

$$xf(t,x,0) > 0 \quad \text{for } t \in [0,1] \text{ and } x \in D_x \setminus [-M,M]. \tag{3.2}$$

Condition *(i)* is fulfilled because we know from above that (3.1) guarantees a unique $C^2[0,1]$-solution to BVP $(1.1)^-_0$, (1.2). Apparently *(ii)* also holds. To check *(iii)* we establish, by standard reasoning, that for an arbitrary $y(t) \in C[0,1]$ the BVP

$$\begin{gathered} x'' = y(t) \\ -\alpha x(0) + \beta x'(0) = 0, \qquad ax(1) + bx'(1) = 0, \end{gathered}$$

has a unique solution in $C^2[0,1]$, which means that the map $\Lambda_h : C^2_{\mathcal{B}_0}[0,1] \to C[0,1]$, defined by $\Lambda_h x = x''$, is one-to-one. Besides, for each solution $x(t) \in C^2[0,1]$ to $(1.1)^-_\lambda$, (1.2) we have

$$m_0 \le x(t) \le M_0, \qquad t \in [0,1], \quad \text{by Lemma 2.4,}$$

and

$$m_1 \le x'(t) \le M_1, \qquad t \in [0,1], \quad \text{by Lemma 2.7.}$$

In view of **(C)**, the function $f(t,x,p)$ is continuous on $[0,1] \times [m_0,M_0] \times [m_1,M_1]$. Thus, there are constants m_2 and M_2 such that

$$\begin{gathered} m_2 \le \lambda f(t,x,p) + (1-\lambda)(x-p) \le M_2 \\ \text{for} \quad \lambda \in [0,1] \quad \text{and} \quad (t,x,p) \in [0,1] \times [m_0,M_0] \times [m_1,M_1]. \end{gathered}$$

Since for $t \in [0,1]$ we have $(x(t),x'(t)) \in [m_0,M_0] \times [m_1,M_1]$, the equation $(1.1)^-_\lambda$ implies

$$m_2 \le x''(t) \le M_2 \quad \text{for } t \in [0,1].$$

Hence, *(iv)* also holds. Finally, *(v)* follows again from **(C)**. Therefore, we can apply Theorem 1.1 to conclude that assertion is true when we have (3.2).

Second step. Now, assuming that **(A$_1$)** is in the form given, consider the family of BVPs

$$\begin{cases} x'' = f_n(t,x,x'),\ t \in (0,1), \\ -\alpha x(0) + \beta x'(0) = A,\ ax(1) + bx'(1) = B, \end{cases} \tag{3.3}$$

where $f_n(t,x,x') = f(t,x,x') + (x-x') : n,\ n = 1,2,3,\ldots$. Clearly,

$$xf_n(t,x,0) = xf(t,x,0) + \frac{x^2}{n} > 0 \text{ for } t \in [0,1], x \in D_x \setminus [-M,M],$$

that is, (3.2) is satisfied for each $n = 1,2,3,\ldots$. Besides,

$$\begin{aligned} &\frac{x-p}{n} \le 0 \quad \text{for } x \in [-M_0,M_0] \text{ and } p \ge L_1 \ge M_0, \\ &\frac{x-p}{n} \ge 0 \quad \text{for } x \in [-M_0,M_0] \text{ and } p \le F_1 \le -M_0, \end{aligned}$$

and **(B$_1$)** imply

$$f_n(t,x,p) \le 0 \quad \text{for } (t,x,p) \in [0,1] \times [-M_0,M_0] \times [L_1,L_2],$$

and

$$f_n(t,x,p) \geq 0 \quad \text{for } (t,x,p) \in [0,1] \times [-M_0, M_0] \times [F_2, F_1].$$

Thus, family (3.3) satisfies **(B_1)** for all $n = 1,2,3,\ldots$. Obviously, (3.3) satisfies **(C)** as well. Finally, to verify (3.1), we just have to consider that for each fixed $n = 1,2,3,\ldots$, the λ-family corresponding to (3.3) is

$$x'' = \lambda\left(f(t,x,x') + \frac{x-x'}{n}\right) + (1-\lambda)(x-x'), \qquad \lambda \in [0,1],$$
$$-\alpha x(0) + \beta x'(0) = A, \qquad ax(1) + bx'(1) = B,$$

and to see that for $\lambda = 0$ we have again BVP $(1.1)_0^-$, (1.2). So, taking into consideration the proved in *First step*, we conclude that family (3.3) has a solution $x_n \in C^2[0,1]$ for each $n = 1,2,3,\ldots$.

Further, just as in the proof of Lemma 2.4, we find that

$$|x_n(t)| \leq M_0 \quad \text{for } t \in [0,1],$$

and just as in the proof of Lemma 2.7 establish that

$$F_1 \leq x_n'(t) \leq L_1 \quad \text{for } t \in [0,1]$$

for each $n = 1,2,3,\ldots$. Finally,

$$|f_n(t,x,p)| \leq |f(t,x,p)| + \frac{|x-p|}{n} \leq |f(t,x,p)| + |x-p|$$

together with the continuity of $f(t,x,p)$ on $[0,1] \times [-M_0, M_0] \times [F_1, L_1]$ implies that there is a constant M_2 independent of n such that

$$|x_n''(t)| \leq M_2 \quad \text{for } t \in [0,1].$$

The estimates for $|x_n(t)|$, $|x_n'(t)|$ and $|x_n''(t)|$ allow us to use the Arzelà–Ascoli theorem to extract a subsequence $\{x_{n_k}\}$, $k = 1,2,3,\ldots$, of $\{x_n\}$ converging uniformly on $[0,1]$ to a function $x \in C^1[0,1]$. We will show that $x(t)$ is a $C^2[0,1]$-solution to BVP (1.1), (1.2). For this purpose introduce the sequence $\{y_{n_k}\}$, $k = 1,2,3,\ldots$, where $y_{n_k}(t) = x_{n_k}(t) - r(t)$, $t \in [0,1]$, with

$$r(t) = \frac{\alpha B + aA}{\alpha a + \alpha b + a\beta} t - \frac{Aa + Ab - \beta B}{\alpha a + \alpha b + a\beta}.$$

Clearly, $\{y_{n_k}\}$ converges uniformly on $[0,1]$ to the function $y(t) = x(t) - r(t)$ and $y \in C^1[0,1]$. Besides, since $r(t)$ satisfies BCs (1.2), $y_{n_k}(t)$ is a solution to the BVP

$$y_{n_k}'' = f_{n_k}(t, y_{n_k} + r, y_{n_k}' + r'), \qquad t \in (0,1),$$
$$-\alpha y_{n_k}(0) + \beta y_{n_k}'(0) = 0, \qquad a y_{n_k}(1) + b y_{n_k}'(1) = 0,$$

and its integral form

$$y_{n_k}(t) = \int_0^1 G(t,s) f_{n_k}(s, y_{n_k}(s) + r(s), y_{n_k}'(s) + r'(s))\,ds, \tag{3.4}$$

for $k = 1,2,3,\ldots$, where $G(t,s)$ is the Green function for the BVP

$$x'' = 0, \qquad t \in (0,1),$$

$$-\alpha x(0)+\beta x'(0)=0, \qquad ax(1)+bx'(1)=0,$$

and

$$f_{n_k}(t,y_{n_k}+r,y'_{n_k}+r')=f(t,y_{n_k}+r,y'_{n_k}+r')+\frac{y_{n_k}+r-y'_{n_k}-r'}{n_k}.$$

Since $|y_{n_k}+r-y'_{n_k}-r'|=|x_{n_k}-x'_{n_k}|\le M_0+\max\{|F_1|,L_1\}$, letting $k\to\infty$ in (3.4), we obtain

$$y(t)=\int_0^1 G(t,s)f(s,y(s)+r(s),y'(s)+r'(s))ds.$$

Thus,

$$x(t)=\int_0^1 G(t,s)f(s,x(s),x'(s))ds+r(t),$$

which means that $x(t)$ is a $C^2[0,1]$-solution to BVP (1.1), (1.2). □

Now let us look at BVP $(1.1)_0^+$, (1.2)

$$\begin{aligned} x''-x'-x&=0, \qquad t\in(0,1),\\ -\alpha x(0)+\beta x'(0)&=A, \qquad ax(1)+bx'(1)=B.\end{aligned}$$

It has a unique solution if

$$\begin{vmatrix} \beta s_1-\alpha & \beta s_2-\alpha \\ (a+bs_1)e^{s_1} & (a+bs_2)e^{s_2}\end{vmatrix}\neq 0, \tag{3.5}$$

where $s_1=\frac{1-\sqrt{5}}{2}$ and $s_2=\frac{1+\sqrt{5}}{2}$ are the roots of the characteristic equation.

Theorem 3.2. *Let* **(A_1)** *hold,* **(B_2)** *hold for*

$$M_0=\max\{|A/\alpha|,|B/a|,M\}, \qquad G_m=(B-aM_0)b^{-1} \quad \text{and} \quad G_M=(B+aM_0)b^{-1},$$

(C) *hold for the same constant M_0 and for $m_0=-M_0, m_1=F_1', M_1=L_1'$ and (3.5) be satisfied. Then BVP (1.1), (1.2) has at least one solution in $C^2[0,1]$.*

Proof. It is similar to the proof of the previous theorem. Now we will apply Theorem 1.1 on the family of BVPs $(1.1)_\lambda^+$, (1.2).

First step. Again assume firstly that the inequality in **(A_1)** is strong, that is, **(A)** holds.

The condition *(i)* follows from (3.5). The condition *(ii)* is again obvious, and the verification of *(iii)* is as in Theorem 3.1. Now, the bounds

$$m_0\le x(t)\le M_0 \quad \text{and} \quad m_1\le x'(t)\le M_1, \qquad t\in[0,1],$$

for each solution $x(t)\in C^2[0,1]$ to $(1.1)_\lambda^+$, (1.2), follow by Lemmas 2.4 and 2.8, respectively, and the bound

$$m_2\le x''(t)\le M_2 \quad \text{for } t\in[0,1]$$

follows by arguments similar to those in the proof of Theorem 3.1. Thus, *(iv)* holds. Finally, *(v)* follows again from **(C)**. So, the assertion is true by Theorem 1.1.

Second step. Now, **(A_1)** is in the form given. Consider the family of BVPs

$$\begin{cases} x''=f_n(t,x,x'),\ t\in(0,1),\\ -\alpha x(0)+\beta x'(0)=A,\ ax(1)+bx'(1)=B,\end{cases} \tag{3.6}$$

where $f_n(t,x,x') = f(t,x,x') + (x+x') : n,\ n = 1,2,3,\dots$. Clearly,

$$xf_n(t,x,0) = xf(t,x,0) + \frac{x^2}{n} > 0 \quad \text{for } t \in [0,1], x \in D_x \setminus [-M,M],$$

that is, **(A)** is satisfied for each $n = 1,2,3,\dots$. Besides,

$$\frac{x+p}{n} \geq 0 \quad \text{for } x \in [-M_0,M_0] \text{ and } p \geq L_1' \geq M_0,$$
$$\frac{x+p}{n} \leq 0 \quad \text{for } x \in [-M_0,M_0] \text{ and } p \leq F_1' \leq -M_0,$$

and **(B_2)** imply

$$f_n(t,x,p) \geq 0 \quad \text{for } (t,x,p) \in [0,1] \times [-M_0,M_0] \times [L_1', L_2'],$$

and

$$f_n(t,x,p) \leq 0 \quad \text{for } (t,x,p) \in [0,1] \times [-M_0,M_0] \times [F_2', F_1'].$$

Thus, family (3.6) satisfies **(B_2)** for all $n = 1,2,3,\dots$. Obviously, (3.6) satisfies and **(C)**. As in the proof of Theorem 3.1, we notice that (3.5) is also fulfilled. Consequently, according to what we proved in the *First step*, each boundary value problem arising from the family (3.6) for $n = 1,2,3,\dots$ has a solution $x_n \in C^2[0,1]$.

Further, just as in the proof of Lemma 2.4, we find that

$$|x_n(t)| \leq M_0 \quad \text{for } t \in [0,1],$$

and just as in the proof of Lemma 2.8 establish that

$$F_1' \leq x_n'(t) \leq L_1' \quad \text{for } t \in [0,1]$$

for each $n = 1,2,3,\dots$ Besides, the continuity of $|f(t,x,p)| + |x+p|$ on the compact set $[0,1] \times [-M_0,M_0] \times [F_1', L_1']$ implies that there is a constant M_2 independent of n such that

$$|x_n''(t)| \leq M_2 \quad \text{for } t \in [0,1].$$

By the Arzelà–Ascoli theorem the sequence $\{x_n\}$ has a subsequence $\{x_{n_k}\}$, $k = 1,2,3,\dots$, converging uniformly on $[0,1]$ to a function $x \in C^1[0,1]$. Next, arguing as in the proof of Theorem 3.1, establish that $x(t)$ is a $C^2[0,1]$-solution to BVP (1.1), (1.2). □

It is standardly found that the BVP

$$x'' + x' - x = 0, \quad t \in (0,1),$$
$$x'(0) = A, \quad ax(1) + bx'(1) = B,$$

in fact this is $(1.1)_0^-$, (1.3), has a unique solution in $C^2[0,1]$ if

$$\begin{vmatrix} r_1 & r_2 \\ (a+br_1)e^{r_1} & (a+br_2)e^{r_2} \end{vmatrix} \neq 0, \tag{3.7}$$

where r_1 and r_2 are as in (3.1).

Theorem 3.3. *Let* **(A_2)** *hold for* $J = [m_v, M_v]$ *and* $C = A$, **(B_1)** *hold for constants*

$$M_0 = \frac{L_-}{\min\{1, K\}} + \max\{|m_v|, |M_v|\}, \qquad G_m = \min\{-M_0, A\} \quad \text{and} \quad G_M = \max\{M_0, A\},$$

(C) *hold for the same constant* M_0 *and for* $m_0 = -M_0, m_1 = F_1$ *and* $M_1 = L_1$ *and* (3.7) *be satisfied. Then BVP* (1.1), (1.3) *has at least one solution in* $C^2[0,1]$.

Proof. We will apply Theorem 1.1 on family $(1.1)^-_\lambda$, (1.3). The hypothesis *(i)* follows immediately from (3.7). Besides, *(ii)* is obvious. The map $\Lambda_h : C^2_{\mathcal{B}_0}[0,1] \to C[0,1]$, defined by $\Lambda_h x = x''$ is one-to-one, because for an arbitrary $y(t) \in C[0,1]$ the BVP

$$\begin{aligned} x'' &= y(t) \\ x'(0) = 0, &\qquad ax(1) + bx'(1) = 0, \end{aligned}$$

has a unique solution in $C^2[0,1]$, *(iii)* is also fulfilled. For each solution $x(t) \in C^2[0,1]$ to $(1.1)^-_\lambda$, (1.3) we have

$$\begin{aligned} m_0 \le x(t) \le M_0, &\qquad t \in [0,1], \quad \text{by Lemma 2.5,} \\ m_1 \le x'(t) \le M_1, &\qquad t \in [0,1], \quad \text{by Lemma 2.9,} \end{aligned}$$

and the bound

$$m_2 \le x''(t) \le M_2 \quad \text{for } t \in [0,1]$$

follows as in the proof of Theorem 3.1. So, *(iv)* also holds. Finally, **(C)** ensures the validity of *(v)*. Therefore, we can apply Theorem 1.1 to conclude that the assertion is true. □

It is easily established that BVP $(1.1)^+_0$, (1.4), namely,

$$\begin{aligned} x'' - x' - x = 0, &\qquad t \in (0,1), \\ x'(0) = A, &\qquad ax(1) + bx'(1) = B, \end{aligned}$$

has a unique solution if

$$\begin{vmatrix} \beta s_1 - \alpha & \beta s_2 - \alpha \\ s_1 e^{s_1} & s_2 e^{s_2} \end{vmatrix} \neq 0, \tag{3.8}$$

where s_1 and s_2 are as in (3.5).

Theorem 3.4. *Let* **(A_2)** *hold for* $J = [m_w, M_w]$ *and* $C = B$, **(B_2)** *hold for constants*

$$M_0 = \frac{L_+}{\min\{1, K\}} + \max\{|m_w|, |M_w|\}, \qquad G_m = \min\{-M_0, B\} \quad \text{and} \quad G_M = \max\{M_0, B\},$$

(C) *hold for the same constant* M_0 *and for* $m_0 = -M_0, m_1 = F_1', M_1 = L_1'$ *and* (3.8) *be satisfied. Then BVP* (1.1), (1.4) *has at least one solution in* $C^2[0,1]$.

Proof. The proof is virtually the same as that of Theorem 3.3. Now, apply Theorem 1.1 on family $(1.1)^+_\lambda$, (1.4). *(i)* follows from (3.8), to check *(iii)* show that for an arbitrary $y(t) \in C[0,1]$ the BVP

$$\begin{aligned} x'' &= y(t) \\ -\alpha x(0) + \beta x'(0) = 0, &\qquad x'(1) = 0, \end{aligned}$$

has a unique solution in $C^2[0,1]$, and the bounds from *(iv)*,

$$\begin{aligned} m_0 \le x(t) \le M_0, &\qquad t \in [0,1], \\ m_1 \le x'(t) \le M_1, &\qquad t \in [0,1], \end{aligned}$$

for each solution $x(t) \in C^2[0,1]$ to $(1.1)^+_\lambda$, (1.4) follow from Lemmas 2.6 and 2.10, respectively. □

4 Examples

The following examples illustrate the application of the obtained results.

Example 4.1. Consider the BVPs for the equation

$$x'' = x'(x'-10) + (x-2)^2(x+2), \qquad t \in (0,1),$$
$$-3x(0) + 2x'(0) = 6, \qquad 2x(1) + x'(1) = -3.$$

We easily check that **(A_1)** holds for $M = 2$. Calculate $M_0 = 2, G_m = 0$ and $G_M = 6$. Next, keeping in mind that

$$0 \le (x-2)^2(x+2) < 1 \quad \text{for } x \in [-2,2],$$

we can choose, for example, $F_2 = -4, F_1 = -3, L_1 = 7$ and $L_2 = 8$ to see that **(B_1)** also hold. **(C)** is obvious, and (3.1) is also easily verifiable. So, we can apply Theorem 3.1 to conclude that the considered BVP has at least one solution in $C^2[0,1]$.

Example 4.2. Consider the BVP

$$x'' = x'^3 + (x-1)\sqrt{x+10}, \qquad t \in (0,1),$$
$$-x(0) + 4x'(0) = 5, \qquad 2x(1) + x'(1) = 4.$$

First observe that

$$xf(t,x,0) = x(x-1)\sqrt{x+10} \ge 0 \text{ for } x \in [-10,\infty) \setminus [-1,1],$$

which means that **(A_1)** is satisfied for $M = 1$. Then $M_0 = 5, G_m = -6, G_M = 14$. Besides,

$$-6\sqrt{5} \le (x-1)\sqrt{x+10} \le 4\sqrt{15} \quad \text{for } x \in [-5,5].$$

So, we can choose $F_2' = -8, F_1' = -7, L_1' = 15$ and $L_2' = 16$ to check **(B_2)**. In addition $f(t,x,p) = p^3 + (x-1)\sqrt{x+10}$ is continuous on the set $[0,1] \times [-5-\delta, 5+\delta] \times [-7-\delta, 15+\delta]$, where $\delta > 0$ is sufficiently small, to say $\delta = 0.1$, that is, **(C)** is satisfied. Checking (3.5) also presents no difficulty. So, we can apply Theorem 3.2 to conclude that the considered BVP has a solution in $C^2[0,1]$.

Example 4.3. Consider the BVP

$$x'' = \phi(t,x) + P_n(x'), \qquad t \in (0,1),$$
$$x'(0) = A, \qquad ax(1) + bx'(1) = B,$$

where the function $\phi\colon [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous, differentiable with respect to x and there is a constant $K > 0$ for which

$$\phi_x(t,x) \ge K > 0 \quad \text{for } (t,x) \in [0,1] \times \mathbb{R},$$

the polynomial $P_n(p) = \sum_{k=0}^{n} a_k p^k, n = 2s+1, s \in \mathbf{N}$, is such that $a_n < 0$, and a and b are such that (3.7) holds.

Clearly, **(A_2)** is satisfied. Further, if

$$f_{\max} := \max |\phi(t,x) + P_n(A)| \quad \text{for } (t,x) \in [0,1] \times [m_v, M_v],$$

where m_v and M_v are the constants used in Theorem 3.3, and

$$g_{\max} := \max|x - A| \quad \text{for } (t,x) \in [0,1] \times [m_v, M_v],$$

we determine first

$$L_- = \max\{f_{\max}, g_{\max}\}$$

and then M_0. Because the continuity of ϕ, there are constants $\phi_m, \phi_M \in \mathbb{R}$ such that

$$\phi_m \le \phi(t,x) \le \phi_M \quad \text{for } (t,x) \in [0,1] \times [-M_0, M_0].$$

Then, for sufficiently large $p > \max\{M_0, A\}$ we have

$$P_n(p) < -\max\{|\phi_m|, |\phi_M|\},$$

which implies that the constants L_1 and L_2 of **($\mathbf{B_1}$)** exist. Besides, for sufficiently small $p < \min\{-M_0, A\}$ we have

$$P_n(p) > \max\{|\phi_m|, |\phi_M|\},$$

which means that the constants F_1 and F_2 of **($\mathbf{B_1}$)** also exist. Finally, **(C)** is obvious and so the considered BVP has a solution in $C^2[0,1]$ by Theorem 3.3.

Example 4.4. Consider the BVP

$$\begin{aligned} x'' &= \phi(t,x) + P_n(x'), \qquad t \in (0,1), \\ -\alpha x'(0) + \beta x'(0) &= A, \qquad x'(1) = B, \end{aligned}$$

where the function $\phi[0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous, differentiable with respect to x and there is a constant $K > 0$ for which

$$\phi_x(t,x) \ge K > 0 \quad \text{for } (t,x) \in [0,1] \times \mathbb{R},$$

the polynomial $P_n(p) = \sum_{k=0}^{n} a_k p^k, n = 2s+1, s \in \mathbf{N}$, is such that $a_n > 0$, and α and β are such that (3.8) holds.

An analysis similar to that of Example 4.3 shows that we can apply Theorem 3.4 to conclude that this BVP has a solution in $C^2[0,1]$.

Example 4.5. Consider the BVP

$$\begin{aligned} x'' &= x + 10^{-1} x' \sqrt{(x'+14)(11-x')}, \qquad t \in (0,1), \\ -x(0) + x'(0) &= 3, \qquad x'(1) = 2. \end{aligned}$$

We will check the conditions of Theorem 3.4. Here $w(t) = 2t - 1$ with $m_w = -1$ and $M_w = 1$. Then **($\mathbf{A_2}$)** is satisfied for $K = 1$, $J = [-1,1]$ and $C = 2$, because $f_x(t,x,2) = 1$ for $(t,x) \in [0,1] \times \mathbb{R}$. Form $f(t,x,2) = x + 2.4$. Then, for $(t,x) \in [0,1] \times [-1,1]$ we have $\max|f(t,x,2)| = 3.4$ and $\max|x+2| = 3$. We calculate $L_+ = 3.4, M_0 = 4.4, G_m = -4.4$ and $G_M = 4.4$. We are now ready to check that **($\mathbf{B_2}$)** is satisfied for $F_2' = -6, F_1' = -5, L_1' = 5$ and $L_2' = 6$. Keeping in mind that $f(t,x,p)$ is defined and continuous for $(t,x,p) \in [0,1] \times \mathbb{R} \times [-14,11]$, we easily conclude that **(C)** also holds for $m_0 = -4.4, M_0 = 4.4, m_1 = -5, M_1 = 5$ and, to say, $\delta = 0.01$. Finally, to check that (3.8) is satisfied, we establish that the determinant

$$\begin{vmatrix} \frac{1-\sqrt{5}}{2} - 1 & \frac{1+\sqrt{5}}{2} - 1 \\ \frac{1-\sqrt{5}}{2} e^{\frac{1-\sqrt{5}}{2}} & \frac{1+\sqrt{5}}{2} e^{\frac{1+\sqrt{5}}{2}} \end{vmatrix}$$

is different from zero. So, we can apply Theorem 3.4 to conclude that the considered BVP has a solution in $C^2[0,1]$.

5 Conclusions

Here we will comment on conditions (3.1), (3.5), (3.7) and (3.8). In fact, they are not essential. If, for example, (3.1) is not fulfilled, we can replace $(1.1)_\lambda^-$ with

$$x'' = \lambda f(t, x, x') + (1 - \lambda)(x - kx'), \qquad t \in (0, 1),$$

where $\lambda \in [0, 1]$ and $k > 0$. Now $(1.1)_0^-$ has the form

$$x'' + kx' - x = 0$$

and we can choose k such that its characteristic equation has roots r_1 and r_2 for which (3.1) is satisfied. This necessitates a slight change in **(B_1)**, namely

$$F_2 < F_1 \leq \min\{-M_0/k, G_m\}, \qquad \max\{M_0/k, G_M\} \leq L_1 < L_2,$$

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Existence and multiplicity of solutions for a p-Laplacian fractional system with logarithmic nonlinearity

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Abstract. This paper is concerned with the existence and multiplicity of a ground state solution for the following class of elliptic fractional type problems given by

$$\begin{cases} (-\Delta)_p^s u + |u|^{p-2}u = \lambda h_1(x)|u|^{\theta-2}u \ln|u| + \dfrac{q}{q+r} b_1(x)|v|^r|u|^{q-2}u & \text{in } \Omega, \\ (-\Delta)_p^t v + |v|^{p-2}v = \mu h_2(x)|v|^{\theta-2}v \ln|v| + \dfrac{q}{q+r} b_2(x)|u|^r|v|^{q-2}v & \text{in } \Omega, \\ u = v = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz boundary, $s,t \in (0,1)$, $N > \max\{ps, pt\}$, $\lambda, \mu > 0$, $p \le \theta$, $2 < q + r < \min\left\{\frac{pN}{N-ps}, \frac{pN}{N-pt}\right\}$, and the additional weights $h_1, h_2, b_1, b_2 \in C(\overline{\Omega})$ are such that: $b_1(x), b_2(x)$ are positive functions and $h_1(x), h_2(x)$ are sign-changing functions. The operators $(-\Delta)_p^s$ and $(-\Delta)_p^t$ represents, both, fractional p-Laplacian operator, a generalization for the fractional Laplacian $(-\Delta)^s, 0 < s < 1$ $(p = 2)$, defined in a integral way as

$$(-\Delta)^s u(x) := \frac{c(n,s)}{2} \int_{\mathbb{R}^N} \frac{2u(x) - (x+y) - u(x-y)}{|y|^{n+2s}} dy, \qquad x \in \mathbb{R}^N,$$

where $c(n,s)$ is a positive normalizing constant, and another fractional operator.

Specifically, the operators $(-\Delta)_p^s$ and $(-\Delta)_p^t$ are defined, up to a normalization constant, by the formula

$$(-\Delta)_p^\ell u(x) := \lim_{\varepsilon \to 0^+} 2 \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x-y|^{N+sp}} dy,$$

for all $u \in C_0^\infty(\mathbb{R}^N), x \in \mathbb{R}^N$, and $\ell \in \{s, t\}$.

Keywords: fractional p-Laplacian operator, logarithmic nonlinearity, Nehari manifold, mountain pass theorem.

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1 Introduction and main results

Since logarithmic nonlinearities play a significant role in depicting the mathematical and physical phenomena, they have received much attention in PDEs (see [15, 16, 19] and references therein).

In [15], the authors studied the existence and multiplicity of a class of fractional Laplacian systems with logarithmic nonlinearity in which three types of weights with certain regularity are involved.

$$\begin{cases} (-\Delta)^s u = \lambda h_1(x) u \ln(|u|) + \dfrac{p}{p+q} b(x)|v|^q |u|^{p-2}u & \text{in } \Omega, \\ (-\Delta)^t v = \mu h_2(x) v \ln(|v|) + \dfrac{r}{q+r} b(x)|u|^r |v|^{q-2}v & \text{in } \Omega, \\ u = v = 0, & \text{in } \mathbb{R}^N \setminus \Omega \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz boundary, $s, t \in (0,1)$, $N > \max\{2s, 2t\}$, $\lambda, \mu > 0$, $2 < q + p < \min\left\{\frac{2N}{N-2s}, \frac{2N}{N-2t}\right\}$, and the additional weights $h_1, h_2, b \in C(\overline{\Omega})$ are such that: $b(x)$ are positive functions and $h_1(x), h_2(x)$ are sign-changing functions.

In [16], the authors studied a class of systems of equations where they showed the existence and multiplicity of solutions for a mixed local-nonlocal system with logarithmic nonlinearities

$$\begin{cases} (-\Delta)^s u_j + u_j = \lambda_j a_j(x) u_j \ln|u_j| + \displaystyle\sum_{i \neq j} \beta_{ij} |u_j|^{q-2} u_j & \text{in } \Omega, \\ u = v = 0, & \text{in } \mathbb{R}^N \setminus \Omega \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz boundary, $s \in (0,1)$, $N > 4$, λ_j are parameters, $\beta_{ij} > 0$ for all $1 \leq i < j \leq k$, $\beta_{ij} = \beta_{ji}$ for all $i, j = 1, \ldots, k$, $a_j \in C(\overline{\Omega})$. When $1 < 2q < 2 < 2^* = \frac{2N}{N-2}$ and a_j they are functions that change sign, they obtained two different solutions using Nehari method. When $1 < q < 2 < 2^* = \frac{2N}{N-2}$ and a_j are positive constant functions, the existence of the ground state solution is obtained using the minimization method.

In [9], the authors studied a class of fractional Laplacian systems where they showed the existence of solutions using the Nehari method and multiplicity of solutions using the Lusternik–Schnirelmann category, with polynomial nonlinearity.

$$\begin{cases} (-\Delta)^s u = \lambda |u|^{p-2}u + \dfrac{2\alpha}{\alpha+\beta} |v|^\beta |u|^{\alpha-2}u & \text{in } \Omega, \\ (-\Delta)^s v = \mu |u|^{p-2}u + \dfrac{2\beta}{\alpha+\beta} |u|^\alpha |v|^{\beta-2}v & \text{in } \Omega, \\ u = v = 0, & \text{in } \mathbb{R}^N \setminus \Omega \end{cases} \tag{1.3}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, $s \in (0,1)$, $N > 4s$, $\lambda, \mu > 0$ are parameters, $\alpha + \beta = \frac{2N}{N-2s}$ is the critical Sobolev exponent.

In [6], the authors showed the existence of solutions for the fractional critical system (p, q)

Laplacian using variational method

$$\begin{cases} (-\Delta)_p^{s_1} u + (-\Delta)_q^{s_2} u = \lambda Q(x)|u|^{r-2}u + \dfrac{2\alpha}{\alpha+\beta}|v|^{\beta}|u|^{\alpha-2}u & \text{in } \Omega, \\ (-\Delta)_p^{s_1} v + (-\Delta)_q^{s_2} v = \mu Q(x)|u|^{r-2}u + \dfrac{2\beta}{\alpha+\beta}|u|^{\alpha}|v|^{\beta-2}v & \text{in } \Omega, \\ u = v = 0, & \text{in } \mathbb{R}^N \setminus \Omega \end{cases} \tag{1.4}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, $0 < s_2 < s_1 < 1$, $1 < q < p < r < p_{s_1}^*$, $N > ps_1$, $\lambda, \mu > 0$ are parameters and $\alpha > 1$ $\beta > 1$ satisfy $\alpha + \beta = p_{s_1}^*$ with $\frac{Np}{N-ps_1}$ is the critical Sobolev exponent, and $(-\Delta)_t^s$ is the fractional t-Laplacian operator.

In [19] was concerned with the existence and asymptotic behavior of normalized ground states solutions for the following coupled Schrödinger system with logarithmic terms:

$$\begin{cases} -\Delta u_1 + \omega_1 u_1 = \mu_1 u_1 \log u_1^2 + \dfrac{p}{p+q}|u_2|^q |u_1|^{p-2}u_1, \\ -\Delta u_2 + \omega_2 u_2 = \mu_2 u_2 \log u_2^2 + \dfrac{q}{p+q}|u_1|^p |u_2|^{q-2}u_2, \\ \displaystyle\int_\Omega |u_i|^2 dx = \rho_i, \ i = 1, 2, \end{cases} \tag{1.5}$$

where $\Omega = \mathbb{R}^N$ or $\Omega \subset \mathbb{R}^N (N \geq 3)$ is a bounded smooth domain, $\omega_i \in \mathbb{R}, \mu_i, \rho_i > 0, i = 1, 2$. Moreover, $p, q \geq 1, 2 \leq p + q \leq 2^*$ where $2^* := \frac{2N}{N-2}$.

In [17] the authors studied the existence of least energy solutions to the following fractional Kirchhoff problem with logarithmic nonlinearity

$$\begin{cases} M([u]_{s,t}^p)(-\Delta)_p^s u = h(x)|u|^{\theta p-2}u \ln|u| + \lambda|u|^{q-2}u, & x \in \Omega \\ u = 0, & x \in \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.6}$$

where $s \in (0,1)$, $1 < p < Np$, $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz boundary, $M([u]_{s,p}^p) = [u]_{s,p}^{(\theta-1)p}$, with $\theta \geq 1$ and $u, h \in C(\overline{\Omega})$ may change sign, $\lambda > 0$ and $q \in (1, p_s^*)$.

On the other hand, parabolic and hyperbolic type equations with logarithmic nonlinearity have been studied extensively in recent years. Here we only refer some results involving fractional Laplacian. [2] considered the Cauchy problem of the following Schrödinger equation

$$iu_t - (-\Delta)^s u + u \log|u|^2 = 0, \quad x \in \Omega, \ t > 0. \tag{1.7}$$

The existence of global solutions was obtained by using a compactness method. Moreover, the author obtained the existence of ground states by the Nehari manifold approach. Xiang et al. [18] studied the initial boundary value problem of the following parabolic equation involving logarithmic nonlinearity

$$u_t + [u]_s^{2(\kappa-1)}(-\Delta)^s u = |u|^{q-2}u \ln|u|^2, \quad x \in \Omega, \ t > 0. \tag{1.8}$$

where $\kappa \in (1, 2_s^*/2)$ and $2\kappa < q < 2^*$. By the potential well theory, the existence of global solutions and blow-up properties of local solutions were discussed. Particularly, using the Nehari manifold approach, the existence of ground state solutions for above stationary problem was investigated.

The main tool used in this paper is the so-called fibering method introduced by Pohozaev [11], [12] and [13], and applied to a single equation of p-Laplacian type by Drábek and Pohozaev in [8]. Bozhkov and Mitidieri [4] used this method to study the existence for multiple solution for a quasilinear system. Fibering method is very used to proof existence of multiple solution for a large class of equations. For example, Brown and Wu [5] used this method to show the existence of at least two positive solutions for the semilinear elliptic boundary-value problem. For more recent applying of fibering method we indicate [1], used to show existence of multiple solution for a class of Schrödinger equations involving indefinite weight functions, and other interesting work involving the fractional operator is [7]

In this paper motivated by [9, 15, 16, 19] we deal with the existence and multiplicity of ground state solutions for the subcritical for the following class of problems

$$\begin{cases} (-\Delta)_p^s u + |u|^{p-2}u = \lambda h_1(x)|u|^{\theta-2}u\ln(|u|) + \dfrac{q}{q+r}b_1(x)|v|^r|u|^{q-2}u & \text{in } \Omega, \\ (-\Delta)_p^t v + |v|^{p-2}v = \mu h_2(x)|v|^{\theta-2}v\ln(|v|) + \dfrac{r}{q+r}b_2(x)|u|^r|v|^{q-2}v & \text{in } \Omega, \\ u = v = 0, & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{P}$$

Taking $W_0^{s,j} = \{u \in W^{s,j}(\mathbb{R}^N) : u = 0 \text{ in } \mathbb{R}^N \setminus \Omega\}$, we define the normed space W, denoted by $W := W_0^{s,p}(\Omega) \times W_0^{s,t}(\Omega)$ (see Section 3), the main results of this paper is writing as follows:

Theorem 1.1. *Problem (P) has a nontrivial ground state solution in W for $s,t \in (0,1)$, $2 < p+q < \min\{p_s^*, p_t^*\}$ and $h_1(x), h_2(x)$ positive functions in $C(\overline{\Omega})$.*

Theorem 1.2. *Problem (P) has a nontrivial ground state solution in W for $s,t \in (0,1)$, $2 < p+q < \min\{p_s^*, p_t^*\}$ and $h_1(x), h_2(x)$ two sign-changing functions in $C(\overline{\Omega})$.*

The paper is organized as follows. In Section 2 we study the variational framework. In Section 3 study Nehari manifold and fibering map analysis. In Section 4 we prove Theorem 1.1. In Section 5 we prove Theorem 1.2.

2 The variational framework

First of all, in this section we introduce the fractional Sobolev space where lies the solution for Problem (P). After defining this space, we introduce some technical results that will be used to proof the main theorems.

Let $\Omega \subset \mathbb{R}^N$ be a domain. For for $p \in [1,\infty)$ and $s \in (0,1)$, we define the fractional Sobolev space

$$W^{s,p}(\Omega) := \left\{ u \in L^p(\Omega) : \frac{|u(x)-u(y)|}{|x-y|^{\frac{N}{p}+s}} \in L^p(\Omega\times\Omega) \right\}; \tag{2.1}$$

i.e., an intermediary Banach space between $L^p(\Omega)$ and $W^{1,p}(\Omega)$, endowed with the natural norm

$$\|u\|_s = \|u\|_{W^{s,p}(\Omega)} = \left(\int_\Omega |u|^p dx + \int_\Omega\int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy \right)^{1/p}$$

where

$$[u]_s = [u]_{W^{s,p}(\Omega)} := \left(\int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x-y|^{N+ps}} dxdy \right)^{1/p}$$

is the so-called Gagliardo seminorm of u.

Moreover, we define the corresponding local fractional Sobolev spaces by

$$W^{s,p}_{\text{loc}}(\Omega) := \{u \in L^p_{\text{loc}}(\Omega) : \ u \in W^{s,p}(\Omega') \text{ for any } \Omega' \subset\subset \Omega\}.$$

Also, we define the space

$$W^{s,p}_0(\Omega) := \{u \in W^{s,p}(\mathbb{R}^N) : \ u = 0 \text{ in } \mathbb{R}^N \setminus \Omega\}.$$

Now we are able to define the space $W := W^{s,p}_0(\Omega) \times W^{s,t}_0(\Omega)$ where lies the solutions of the Problem (P) in the Theorems 1.1 and 1.2. One can check that $(W, |\cdot|)$ is a normed space, with norm given by

$$\|(u,v)\| := \|(u,v)\|_W = \left([u]^p_s + |u|^p_p + [v]^p_t + |v|^p_p\right)^{1/p}.$$

Motivated by [10] we show the following lemma for system.

Lemma 2.1. *W is continuously embedding on $L^{s,p}(\Omega) \times L^{s,t}(\Omega)$.*

Proof. Let $u \in W^{s,p}(\Omega)$. Since $\Omega \subset \mathbb{R}^N$ is an extension domain for $W^{s,p}$, then there exists a constant $C_1 = C_1(N,p,s,\Omega) > 0$ such that

$$\|\tilde{u}\|_{W^{s,p}(\mathbb{R}^N)} \leq C_1 \|u\|_{W^{s,p}(\Omega)} \tag{2.2}$$

with $\tilde{u}$ such that $\tilde{u}(x) = u(x)$ for x a.e. in Ω. On the other hand, by [10, Theorem 6.7], the space $W^{s,p}(\mathbb{R}^N)$ is continuously embedded in $L^r(\mathbb{R}^N)$, for any $r \in [p, p^*_s]$; i.e., there exists a constant $C_2 = C_2(N,p,s) > 0$ such that

$$\|\tilde{u}\|_{L^r(\mathbb{R}^N)} \leq C_2 \|\tilde{u}\|_{W^{s,p}(\mathbb{R}^N)}. \tag{2.3}$$

Using (2.2) and (2.3), we get

$$\|u\|_{L^r(\Omega)} = \|\tilde{u}\|_{L^r(\Omega)} \leq \|\tilde{u}\|_{L^r(\mathbb{R}^N)} \leq C_2 \|\tilde{u}\|_{W^{s,p}(\mathbb{R}^N)} \leq C_2 C_1 \|u\|_{W^{s,p}(\Omega)}. \tag{2.4}$$

Similarly we have the following inequality

$$\|v\|_{L^\eta(\Omega)} = \|\tilde{v}\|_{L^\eta(\Omega)} \leq \|\tilde{v}\|_{L^\eta(\mathbb{R}^N)} \leq C_3 \|\tilde{v}\|_{W^{t,p}(\mathbb{R}^N)} \leq C_3 C_4 \|v\|_{W^{t,p}(\Omega)}. \tag{2.5}$$

Finally using (2.4) and (2.5) we have

$$\|(u,v)\|_{L^r(\Omega)\times L^\eta(\Omega)} \leq \tilde{C} \|(u,v)\|_W \tag{2.6}$$

for some $C \in (0,\infty)$, from which follows the proposition. □

Lemma 2.2. *If $q + r < \min\{p^*_s, p^*_t\}$ then there exists a positive constant c such that*

$$\left(\int_\Omega |u|^r |v|^q dx \right)^{\frac{1}{q+r}} \leq c \|(u,v)\|.$$

Proof. The proof follows from the definition

$$S_{r,q} := \inf_{(u,v)\in W\setminus\{(0,0)\}} \left\{ \frac{\|(u,v)\|^p}{\left(\int_\Omega |u|^r|v|^q dx\right)^{\frac{p}{q+r}}} \right\}. \tag{2.7}$$

and the inequality $|u|^r|v|^q \leq \frac{r}{r+q}|u|^{r+q} + \frac{q}{r+q}|v|^{r+q} \leq |u|^{r+q} + |v|^{r+q}$. □

Lemma 2.3. *Let $\xi, \eta \in \mathbb{R}^N$. Then,*

$$|\xi-\eta|^p \leq \begin{cases} 2^p\left(|\xi|^{p-2}\xi + |\eta|^{p-2}\eta\right)(\xi-\eta), & \text{for } p \geq 2, \\ \dfrac{1}{(p-1)} \dfrac{\left[\left(|\xi|^{p-2}\xi + |\eta|^{p-2}\eta\right)(\xi-\eta)\right]^{p/2}}{\left(|\xi|^p + |\eta|^p\right)^{(p-2)/p}}, & \text{for } 1 < p < 2. \end{cases} \tag{2.8}$$

The proof of this lemma can be found in [14, Lemma 6].

Definition 2.4. We say that $(u,v) \in W$ is a (weak) solution of Problem (P) if

$$\begin{aligned} &\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+ps}}dxdy + \int_\Omega |u|^{p-2}u\varphi dx \\ &\quad + \int_\Omega\int_\Omega \frac{|v(x)-v(y)|^{p-2}(v(x)-v(y))(\phi(x)-\phi(y))}{|x-y|^{N+pt}}dxdy + \int_\Omega |v|^{p-2}v\phi dx \\ &= \lambda\int_\Omega h_1(x)|u|^{\theta-2}u\ln|u|\varphi dx + \mu\int_\Omega h_2(x)|v|^{\theta-2}v\ln|v|\phi dx \\ &\quad + \frac{q}{q+r}\int_\Omega b_1(x)|v|^r|u|^{q-2}u\varphi dx + \frac{r}{q+r}\int_\Omega b_2(x)|u|^q|v|^{r-2}v\phi dx, \end{aligned}$$

for any $(\varphi,\phi) \in W$.

Now we consider the energy functional for Problem (P), $E : W \to \mathbb{R}$ defined by

$$\begin{aligned} E(u,v) = &\frac{1}{p}[u]^p_{s,p} + \frac{1}{p}\int_\Omega |u|^p dx + \frac{1}{p}[v]^p_{s,t} + \frac{1}{p}\int_\Omega |v|^p dx - \frac{\lambda}{\theta}\int_\Omega h_1(x)|u|^\theta \ln|u|dx \\ &+ \frac{\lambda}{\theta^2}\int_\Omega h_1(x)|u|^\theta dx - \frac{\mu}{\theta}\int_\Omega h_2(x)|v|^\theta\ln|v|dx + \frac{\mu}{\theta^2}\int_\Omega h_2(x)|v|^\theta dx \\ &- \frac{1}{q+r}\int_\Omega b_1(x)|u|^q|v|^r dx - \frac{1}{q+r}\int_\Omega b_2(x)|u|^q|v|^r dx. \end{aligned}$$

We also consider the functional

$$\begin{aligned} I(u,v) = &[u]^p_{s,p} + \int_\Omega |u|^p dx + [v]^p_{s,t} + \int_\Omega |v|^p dx - \lambda\int_\Omega h_1(x)|u|^\theta\ln|u|dx \\ &- \mu\int_\Omega h_2(x)|v|^\theta\ln|v|dx - \int_\Omega b_1(x)|u|^q|v|^r dx - \int_\Omega b_2(x)|u|^q|v|^r dx. \end{aligned}$$

As consequence of embedding on Remark 2.1 and Lemma 2.2, we obtain that the functional I

is well-defined and $E, I \in C^1(W, \mathbb{R})$. Moreover, note that, for all $(\varphi, \phi) \in W$,

$$\begin{aligned}
E'(u,v)(\varphi,\phi) = &\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+ps}}dxdy + \int_\Omega |u|^{p-2}u\varphi dx \\
&+ \int_\Omega\int_\Omega \frac{|v(x)-v(y)|^{p-2}(v(x)-v(y))(\phi(x)-\phi(y))}{|x-y|^{N+pt}}dxdy + \int_\Omega |v|^{p-2}v\phi dx \\
&- \lambda\int_\Omega h_1(x)|u|^{\theta-2}u\ln|u|\varphi dx - \mu\int_\Omega h_2(x)|v|^{\theta-2}v\ln|v|\phi dx \\
&- \frac{q}{q+r}\int_\Omega b_1(x)|v|^r|u|^{q-2}u\varphi dx - \frac{r}{q+r}\int_\Omega b_2(x)|u|^q|v|^{r-2}v\phi dx
\end{aligned}$$

and

$$\begin{aligned}
I'(u,v)(\varphi,\phi) = &\, p\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+ps}}dxdy + p\int_\Omega |u|^{p-2}u\varphi dx \\
&+ p\int_\Omega\int_\Omega \frac{|v(x)-v(y)|^{p-2}(v(x)-v(y))(\phi(x)-\phi(y))}{|x-y|^{N+pt}}dxdy + p\int_\Omega |v|^{p-2}v\phi dx \\
&- \lambda\theta\int_\Omega h_1(x)|u|^{\theta-2}u\varphi\ln|u|dx - \lambda\int_\Omega h_1(x)|u|^{\theta-2}u\varphi dx \\
&- \mu\theta\int_\Omega h_2(x)|v|^{\theta-2}v\phi\ln|v|dx - \mu\int_\Omega h_2(x)|v|^{\theta-2}v\phi dx \\
&- (q+r)\int_\Omega b_1(x)|v|^r|u|^{q-2}u\varphi dx - (q+r)\int_\Omega b_2(x)|u|^q|v|^{r-2}v\phi dx.
\end{aligned}$$

Some important results will be used in this paper, including those concerning Sobolev's Logarithmic Inequalities to obtain some estimates of the problem and to resolve some arisen difficult in the logarithmic term:

Lemma 2.5. *Let ρ be a positive real number. Then we have the following inequalities*

$$\ln t \le \frac{t^\rho}{e\rho} \quad \text{for all } 1 \le t \tag{2.9}$$

and

$$|t^\rho \ln t| \le \frac{1}{e\rho} \quad \text{for all } 0 < t < 1. \tag{2.10}$$

where e is Euler logarithm basis.

Proof. We consider the following function $h : [1, \infty) \to \mathbb{R}$ defined by

$$h(t) = \ln t - \frac{1}{e\rho}t^\rho \quad \text{for all } t \ge 1.$$

With respect to t, just by taking a simple derivative, we deduce

$$h'(t) = \frac{1}{t} - \frac{1}{e}t^{\rho-1} \quad \text{for all } t \ge 1.$$

Then, $t_* = e^{1/\rho}$ is the unique maximum point of function h. Thus, $h(t) \le h(t_*) = 0$ for all $t \ge 1$ and the inequality (2.9) is valid.

To prove the inequality (2.10), we consider $g : (0, 1] \to \mathbb{R}$ defined by

$$g(t) = t^\rho|\ln t| \quad \text{for all } t \in (0, 1].$$

We can verify that $t_* = e^{-1/\rho}$ is the unique minimum point of the function g. We can also check that $\lim_{t\to 0^+} g(t) = 0$. Thus,

$$g(t) \geq g(t_*) = -\frac{1}{e\rho}$$

and the proof is complete. □

Lemma 2.6. *Assume that (u_n, v_n) is bounded in W such that u_n converges to u a.e. and v_n converges to v in Ω. Then*

$$\lim_{n\to\infty} \int_\Omega h_1(x)|u_n|^\theta \ln|u_n| dx = \int_\Omega h_1(x)|u|^\theta \ln|u| dx.$$

and

$$\lim_{n\to\infty} \int_\Omega h_2(x)|v_n|^\theta \ln|v_n| dx = \int_\Omega h_2(x)|v|^\theta \ln|v| dx,$$

for $2 < \theta < q + r$.

Proof. Since that (u_n, v_n) is bounded in W, we get that $\|(u_n, v_n)\| \leq C$, for all $n \in \mathbb{N}$, we first need to discuss the two cases of $|u_n|$ as follows.

Case I. If $\frac{|u_n(x)|}{\|(u_n,v_n)\|} < 1$. By Lemma 2.5 with (2.9) with $\rho = \theta < p_s^*$, from the Sobolev Inequalities, we get

$$\begin{aligned}\left|h_1(x)|u_n|^\theta \ln\left(\frac{|u(x)|}{\|(u_n,v_n)\|}\right)\right| &\leq C_{h_1}\left||u|^\theta \ln\left(\frac{|u(x)|}{\|(u_n,v_n)\|}\right)\right| \\ &= C_{h_1}\|(u_n,v_n)\|^\theta \left|\frac{|u_n(x)|}{\|(u_n,v_n)\|}\right|^\theta \left|\ln\left(\frac{|u_n(x)|}{\|(u_n,v_n)\|}\right)\right| \\ &\leq \frac{C_{h_1}}{e\theta}\|(u_n,v_n)\|^\theta =: g_n^1. \end{aligned} \tag{2.11}$$

Since $(u_n, v_n) \rightharpoonup (0,0)$ in W, then $(u_n, v_n) \to (0,0)$ in $L^\theta(\Omega) \times L^\theta(\Omega)$ for all $\theta \in (1, \min\{p_s^*, p_t^*\})$ using the Sobolev embeddings, we have

$$\int_\Omega |g_n^1(x)| dx \leq \frac{C_{H_1}}{e\theta}\|(u_n,v_n)\|^\theta \to 0, \quad \text{as } n \to \infty. \tag{2.12}$$

Thus, $|g_n^1| \to 0$ in $L^1(\Omega)$, which means that there exists some $g_1 \in L^1(\Omega)$ such that $|g_n^1(x)| \leq g^1(x)$ a.e. in Ω for all $n \in \mathbb{N}$.

Case II. If $\frac{|u_n(x)|}{\|(u_n,v_n)\|} > 1$. Using Lemma 2.5 with (2.9) and Hölder's inequality, we get

$$\begin{aligned}\left|h_1(x)|u_n|^\theta \ln\left(\frac{|u(x)|}{\|(u_n,v_n)\|}\right)\right| &\leq C_{h_1}\left||u_n|^\theta \ln\left(\frac{|u(x)|}{\|(u_n,v_n)\|}\right)\right| \\ &= C_{h_1}\|(u_n,v_n)\|^{\theta-\rho}|u_n|^\rho \left|\frac{|u_n(x)|}{\|(u_n,v_n)\|}\right|^{\theta-\rho} \left|\ln\left(\frac{|u_n(x)|}{\|(u,v)\|}\right)\right| \\ &\leq C_{h_1}\frac{\|(u_n,v_n)\|^{\theta-\rho}}{e(\theta-\rho)}|u_n|^\rho := g_2^n. \end{aligned} \tag{2.13}$$

Since $(u_n, v_n) \rightharpoonup (0,0)$ in W, then $(u_n, v_n) \to (0,0)$ in $L^\theta(\Omega) \times L^\theta(\Omega)$ for all $\theta \in (1, \min\{p_s^*, p_t^*\})$ using the Sobolev embeddings, we have

$$\int_\Omega |g_n^2(x)| dx \leq \frac{C_{h_1}}{e(\theta-\rho)}\|(u_n,v_n)\|^\theta \to 0, \quad \text{as } n \to \infty. \tag{2.14}$$

Thus, $|g_n^2| \to 0$ in $L^1(\Omega)$, which means that there exists some $g_1 \in L^1(\Omega)$ such that $|g_n^2(x)| \leq g^1(x)$ a.e. in Ω for all $n \in \mathbb{N}$. Now, defining $f_n := |u_n(x)|^\theta|\varphi|$, we also have $f_n \to 0$ in $L^1(\Omega)$, which provides $f \in L^1(\Omega)$ such that $|f_n(x)| \leq f(x)$ a.e. in $\mathbb{N}$ for all $n \in \mathbb{N}$.
Combining (2.11) and (2.13) , we have the following estimate

$$\begin{aligned}\left|h_1(x)|u_n(x)|^\theta \ln\left(\frac{|u_n(x)|\cdot\|(u_n,v_n)\|}{\|(u_n,v_n)\|}\right)\right| &\leq C_{h_1}\left[\left||u_n(x)|^\theta \ln\left(\frac{|u(x)|}{\|(u_n,v_n)\|}\right)\right|\right.\\ &\quad\left.+\ln\|(u_n,v_n)\||u_n(x)|^\theta\right]\\ &\leq g_n^1+g_n^2+f_n, \end{aligned} \tag{2.15}$$

where $g_n^1+g_n^2+f_n \in L^1(\Omega)$, for all $n \in \mathbb{N}$. For other hand, since $\{(u_n,v_n)\}$ is bounded in W, up to a subsequence, we may assume that $(u_n(x),v_n(x)) \to (u(x),v(x))$ a.e in Ω. This implies that

$$h_1(x)|u_n(x)|^\theta \ln|u_n(x)| \to h_1(x)|u(x)|^\theta \ln|u(x)|, \quad \text{a.e. in } \Omega. \tag{2.16}$$

Therefore, using (2.15) and (2.16) the Lebesgue Dominated Convergence Theorem yields that

$$\lim_{n\to+\infty}\int_\Omega h_1(x)|u_n|^\theta \ln|u_n|dx = \int_\Omega h_1(x)|u|^\theta \ln|u|dx. \tag{2.17}$$

Using the same idea of equation (2.17), we have the following convergence

$$\lim_{n\to+\infty}\int_\Omega h_2(x)|v_n|^\theta \ln|v_n|dx = \int_\Omega h_2(x)|v|^\theta \ln|v|dx. \tag{2.18}$$

The proof is concluded. □

Lemma 2.7. *Let $\Omega \subset \mathbb{R}^N$ be a domain, $q \in [0,\infty)$ and (u_n) be a bounded sequence in $L^r(\Omega)$. If $u_n \to u$ almost everywhere on Ω as $n \to +\infty$, then for every $q \in [1,r]$.*

$$\lim_{n\to+\infty}\int_\Omega \left||u_n|^q - |u_n-u|^q - |u|^q\right|^{\frac{r}{q}}dx = 0. \tag{2.19}$$

Also recall that pointwise convergence of a bounded sequence implies weak converge (see for example [3, Proposition 4.7.12])

Lemma 2.8. *Let the sequences (u_n) and (v_n) be in $\in W_0^{s,p}(\Omega)$ such that $u_n \rightharpoonup u$, $v_n \rightharpoonup v$ in $W_0^{s,p}(\Omega)$ and $u_n(x) \to u(x)$, $v_n(x) \to v(x)$ a.e. in Ω. Then,*

$$\lim_{n\to\infty}\int_\Omega \left[|u_n|^q|v_n|^r - |u_n-u|^q|v_n-v|^r\right]dx = \int_\Omega |u|^q|v|^r dx.$$

Proof. By direct calculation we have the following equality of integrals

$$\begin{aligned}&\int_\Omega \Big(|u_n|^q|v_n|^r - |u_n-u|^q|v_n-v|^r\Big)dx\\ &\quad= \int_\Omega \Big(|u_n|^q - |u_n-u|^q)|v_n|^r + |u_n-u|^q(|v_n|^r - |v_n-v|^r\Big)dx.\end{aligned}$$

Since that $u_n \rightharpoonup u$, $v_n \rightharpoonup v$ in $W_0^{s,p}(\Omega)$ and using Lemma 2.7, we get

$$\lim_{n\to+\infty}\int_\Omega \Big(|u_n|^q - |u_n-u|^q - |u|^q\Big)^{\frac{q+r}{q}}dx = 0. \tag{2.20}$$

which means that $|u_n|^q - |u_n - u|^q \to |u|^q$ in $L^{\frac{q+r}{q}}(\Omega)$, for other hand $|v_n|^r \rightharpoonup |v|^r$ in $L^{\frac{q+r}{r}}(\Omega)$, this implies that

$$\lim_{n\to+\infty}\int_\Omega \Big(|u_n|^q - |u_n-u|^q - |u|^q\Big)|v_n|^r dx = 0.$$

Also, note that $|v_n|^r - |v_n - v|^r \to |v|^q$ in $L^{\frac{q+r}{r}}(\Omega)$. As $|u_n - u|^q \rightharpoonup 0$ in $L^{\frac{q+r}{q}}(\Omega)$, this implies that

$$\lim_{n\to+\infty}\int_\Omega |u_n-u|^q\Big(|v_n|^r - |v_n-v|^r\Big)dx = 0. \tag{2.21}$$

This concludes the proof. □

Lemma 2.9. *Let $(u,v) \in W \setminus \{(0,0)\}$. Then*

$$\begin{aligned}&\lambda\int_\Omega h_1(x)|u|^\theta \ln|u|dx + \mu\int_\Omega h_2(x)|v|^\theta\ln|v|dx\\ &\quad\le (\lambda C_{h_1} + \mu C_{h_2})L\|(u,v)\|^p + \ln\|(u,v)\|\left[\lambda\int_\Omega h_1(x)|u|^\theta dx + \mu\int_\Omega h_2(x)|v|^\theta dx\right]\end{aligned}$$

where

$$L := \frac{|\Omega|}{e\theta} + \frac{1}{e(p_s^*-\theta)}p_{p_s^*}^{p_{p_s^*}} + \frac{1}{e(p_t^*-\theta)}p_{p_t^*}^{p_{p_t^*}},\quad C_{h_1} := \max_{x\in\overline{\Omega}}|h_1(x)|,\quad C_{h_2} := \max_{x\in\overline{\Omega}}|h_2(x)|,$$

and $S_{p_s^*}^{p_{p_s^*}}, S_{p_t^*}^{p_{p_t^*}} > 0$ denote the best constants of embeddings from $W_0^{s,p}(\Omega)$, and $W_0^{t,p}(\Omega)$, respectively.

Proof. Let us consider $\Omega = \Omega_1 \cup \Omega_2$, using integration properties over Ω, where $\Omega_1 = \{x \in \Omega : |u(x)| \le \|(u,v)\|\}$ and $\Omega_2 = \{x\in\Omega : |u(x)| > \|(u,v)\|\}$. Then

$$\int_\Omega h_1(x)|u|^\theta \ln\frac{|u(x)|}{\|(u,v)\|}dx = \int_{\Omega_1} h_1(x)|u|^\theta\ln\frac{|u(x)|}{\|(u,v)\|}dx + \int_{\Omega_2} h_1(x)|u|^\theta\ln\frac{|u(x)|}{\|(u,v)\|}dx.$$

Using (2.10) by direct calculation gives that

$$\begin{aligned}\int_{\Omega_1} h_1(x)|u|^\theta\ln\frac{|u(x)|}{\|(u,v)\|}dx &\le C_{h_1}\|(u,v)\|^\theta\int_\Omega\left|\frac{u(x)}{\|(u,v)\|}\right|^\theta\left|\ln\frac{|u(x)|}{\|(u,v)\|}\right|dx\\ &\le C_{h_1}\|(u,v)\|^\theta\int_\Omega\frac{1}{e\theta}dx = C_{h_1}\|(u,v)\|^\theta|\Omega|\frac{1}{e\theta}.\end{aligned}$$

By Lemma 2.5 for $\rho = p_s^* - \theta > 0$ and direct calculation

$$\begin{aligned}\int_{\Omega_2} h_1(x)|u|^\theta\ln\frac{|u(x)|}{\|(u,v)\|}dx &\le C_{h_1}\int_\Omega |u|^\theta\ln\frac{|u(x)|}{\|(u,v)\|}dx\\ &\le \frac{C_{h_1}}{e(p_s^*-\theta)}\int_\Omega|u|^\theta\left[\frac{|u(x)|}{\|(u,v)\|}\right]^{p_s^*-\theta}dx\\ &\le \frac{C_{h_1}}{e(p_s^*-\theta)}\frac{1}{\|(u,v)\|^{p_s^*-\theta}}\int_\Omega|u|^{p_s^*}dx\\ &\le \frac{C_{h_1}}{e(p_s^*-\theta)}\frac{1}{\|(u,v)\|^{p_s^*-\theta}}S_{p_s^*}^{p_s^*}\|u\|_{p_s^*}^{p_s^*}\\ &\le \frac{C_{h_1}}{e(p_s^*-\theta)}\frac{1}{\|(u,v)\|^{p_s^*-\theta}}S_{p_s^*}^{p_s^*}\|(u,v)\|^{p_s^*}\\ &= \frac{C_{h_1}}{e(p_s^*-\theta)}S_{p_s^*}^{p_s^*}\|(u,v)\|^\theta.\end{aligned}$$

Consequently we get

$$\int_\Omega |u|^\theta \ln\left[\frac{|u(x)|}{\|(u,v)\|}\right]dx \le C_{h_1}\left[\frac{|\Omega|}{e\theta}+\frac{S_{p_s^*}^{p_s^*}}{e(p_s^*-\theta)}\right]\|(u,v)\|^\theta.$$

For other hand

$$\begin{aligned}\lambda\int_\Omega h_1(x)|u|^\theta\ln|u|dx &= \lambda\int_\Omega h_1(x)|u|^\theta\ln\left[\frac{|u(x)|}{\|(u,v)\|}\right]dx+\ln(\|(u,v)\|)\int_\Omega \lambda h_1(x)|u|^\theta dx\\ &\le \lambda C_{h_1}\left[\frac{|\Omega|}{e\theta}+\frac{S^{p_s^*}}{e(p_s^*-\theta)}\right]\|(u,v)\|^\theta + C_{h_1}\ln(\|(u,v)\|)\int_\Omega \lambda|u|^\theta dx.\end{aligned}$$

Similarly,

$$\mu\int_\Omega h_2(x)|v|^\theta\ln|v|dx \le \mu C_{h_2}\left[\frac{|\Omega|}{e\theta}+\frac{S^{p_s^*}}{e(p_t^*-\theta)}\right]\|(u,v)\|^\theta + C_{h_2}\ln(\|(u,v)\|)\int_\Omega \mu|v|^\theta dx.$$

Adding the integrals for Ω_1 and Ω_2 we have

$$\begin{aligned}&\lambda\int_\Omega h_1(x)|u|^\theta\ln|u|dx+\mu\int_\Omega h_2(x)|v|^\theta\ln|v|dx\\ &\quad\le (\lambda C_{h_1}+\mu C_{h_2})\left[\frac{|\Omega|}{e\theta}+\frac{S^{p_s^*}}{e(p_s^*-\theta)}+\frac{S^{p_s^*}}{e(p_t^*-\theta)}\right]\|(u,v)\|^\theta\\ &\qquad+\ln\|(u,v)\|\int_\Omega\left[\lambda h_1(x)|u|^\theta+\mu h_2(x)|v|^\theta\right]dx.\end{aligned}$$

□

3 Nehari manifold and fibering map analysis

The main tool used in this paper is the so-called fibering method introduced by Pohozaev [11], [12] and [13]. In this section, we assume $\lambda,\mu>0$, and the functions $h_1,h_2,b_1,b_2\in C(\overline{\Omega})$.

We define the Nehari manifolds as

$$\mathcal{N}:=\{(u,v)\in W\setminus\{(0,0)\}\mid I(u,v)=0\}.$$

For all $(u,v)\in\mathcal{N}$, we have $(u,v)\neq 0$ and

$$\|(u,v)\|^p=\lambda\int_\Omega h_1(x)|u|^\theta\ln|u|dx+\mu\int_\Omega h_2(x)|v|^\theta\ln|v|dx+\int_\Omega(b_1(x)+b_2(x))|u|^q|v|^r dx.$$

Now we define the $C^2(0,\infty)$ fibering map $\Phi_{(u,v)}:(0,\infty)\to\mathbb{R}$ as $\Phi_{(u,v)}(k):=E(k(u,v))$, for $k>0$, it is

$$\begin{aligned}\Phi_{(u,v)}(k)=&\frac{k^p}{p}\|(u,v)\|^p-\frac{\lambda k^\theta}{\theta}\ln|k|\int_\Omega h_1(x)|u|^\theta dx-\frac{\lambda k^\theta}{\theta}\int_\Omega h_1(x)|u|^\theta\ln|u|dx\\ &+\frac{\lambda k^\theta}{\theta^2}\int_\Omega h_1(x)|u|^\theta dx-\frac{\mu k^\theta}{\theta}\ln|k|\int_\Omega h_2(x)|v|^\theta dx-\frac{\mu k^\theta}{\theta}\int_\Omega h_2(x)|v|^\theta\ln|v|dx\\ &+\frac{\mu k^\theta}{\theta^2}\int_\Omega h_2(x)|v|^\theta dx-\frac{k^{q+r}}{q+r}\int_\Omega(b_1(x)+b_2(x))|u|^q|v|^r dx.\end{aligned}$$

Consequentially:

$$\begin{aligned}\Phi'_{(u,v)}(k)=&k^{p-1}\|(u,v)\|^p-\lambda k^{\theta-1}\int_\Omega h_1(x)|u|^\theta\ln|ku|dx-\mu k^{\theta-1}\int_\Omega h_2(x)|v|^\theta\ln|kv|dx\\ &-k^{q+r-1}\int_\Omega(b_1(x)+b_2(x))|u|^q|v|^r dx\end{aligned}$$

and also,

$$\begin{aligned}\Phi''_{(u,v)}(k) &= (p-1)k^{p-2}\|(u,v)\|^p - (\theta-1)k^{\theta-2}\lambda\int_\Omega h_1(x)|u|^\theta \ln|ku|dx \\ &\quad - (\theta-1)k^{\theta-2}\mu\int_\Omega h_2(x)|v|^\theta \ln|kv|dx - \lambda k^{\theta-2}\int_\Omega h_1(x)|u|^\theta dx \\ &\quad - \mu k^{\theta-2}\int_\Omega h_2(x)|v|^\theta dx - (q+r-1)k^{q+r-2}\int_\Omega (b_1(x)+b_2(x))|u|^q|v|^r dx.\end{aligned}$$

Thus, one can easily verify that for all $(u,v)\in\mathcal{N}$,

$$\Phi'_{(u,v)}(1) = 0,$$

and

$$\begin{aligned}\Phi''_{(u,v)}(1) &= (p-1)\|(u,v)\|^p - \lambda(\theta-1)\int_\Omega h_1(x)|u|^\theta \ln|u|dx - \mu(\theta-1)\int_\Omega h_2(x)|v|^\theta \ln|v|dx \\ &\quad - \lambda\int_\Omega h_1(x)|u|^\theta dx - \mu\int_\Omega h_2(x)|v|^\theta dx - (q+r-1)\int_\Omega (b_1(x)+b_2(x))|u|^q|v|^r dx,\end{aligned}$$

consequently

$$\begin{aligned}\Phi''_{(u,v)}(1) &= (p-1)\lambda\int_\Omega h_1(x)|u|^\theta \ln|u|dx + (p-1)\mu\int_\Omega h_2(x)|v|^\theta \ln|v|dx \\ &\quad + (p-1)\int_\Omega (b_1(x)+b_2(x))|u|^q|v|^r dx - \lambda(\theta-1)\int_\Omega h_1(x)|u|^\theta \ln|u|dx \\ &\quad - \mu(\theta-1)\int_\Omega h_2(x)|v|^\theta \ln|v|dx - \lambda\int_\Omega h_1(x)|u|^\theta dx - \mu\int_\Omega h_2(x)|v|^\theta dx \\ &\quad - (q+r-1)\int_\Omega (b_1(x)+b_2(x))|u|^q|v|^r dx,\end{aligned}$$

which implies that

$$\begin{aligned}\Phi''_{(u,v)}(1) &= \lambda(p-\theta)\int_\Omega h_1(x)|u|^\theta \ln|u|dx + \mu(p-\theta)\int_\Omega h_2(x)|v|^\theta \ln|v|dx \\ &\quad - \lambda\int_\Omega h_1(x)|u|^\theta dx - \mu\int_\Omega h_2(x)|v|^\theta dx - (q+r-p)\int_\Omega (b_1+b_2)(x)|u|^q|v|^r dx.\end{aligned}$$

As a consequence of the previously calculus, it's make possible to rewrite the Nehari manifold $\mathcal{N}$ as

$$\mathcal{N} = \mathcal{N}^+ \cup \mathcal{N}^0 \cup \mathcal{N}^-,$$

where

$$\begin{aligned}\mathcal{N}^+ &= \left\{(u,v)\in\mathcal{N} \mid \Phi''_{(u,v)}(1) > 0\right\}; \\ \mathcal{N}^0 &= \left\{(u,v)\in\mathcal{N} \mid \Phi''_{(u,v)}(1) = 0\right\};\end{aligned}$$

and

$$\mathcal{N}^- = \left\{(u,v)\in\mathcal{N} \mid \Phi''_{(u,v)}(1) < 0\right\}.$$

Lemma 3.1. *Assume $(u,v)\in W\setminus\{(0,0)\}$ and $k>0$. Then $(ku,kv)\in\mathcal{N}$ if, and only if, $\Phi'_{(u,v)}(k)=0$.*

Proof. If $k(u,v) \in \mathcal{N}$, for $k > 0$ we have $I(ku, kv) = 0$. So

$$\begin{aligned} 0 = k^p \|(u,v)\|^p - \lambda k^\theta \int_\Omega h_1(x)|u|^\theta \ln|ku| dx - \mu k^\theta \int_\Omega h_2(x)|u|^\theta \ln|kv| dx \\ - k^{q+r} \int_\Omega b_1(x)|u|^q |v|^r dx - k^{q+r} \int_\Omega b_2(x)|u|^q |v|^r dx. \end{aligned}$$

Dividing the above equation for $k > 0$, we have

$$\begin{aligned} 0 &= k^{p-1} \|(u,v)\|^p - \lambda k^{\theta-1} \int_\Omega h_1(x)|u|^\theta \ln|ku| dx - \mu k^{\theta-1} \int_\Omega h_2(x)|u|^\theta \ln|kv| dx \\ &\quad - k^{q+r-1} \int_\Omega (b_1 + b_2)(x)|u|^q |v|^r dx \\ &= \Phi'(k). \end{aligned}$$

This completes the proof. □

Lemma 3.2. *If (u,v) is a local minimizer for the functional E on $\mathcal{N}$, with $(u,v) \neq \mathcal{N}^0$, then $E'(u,v) = 0$.*

Proof. By the assumption for $u \in \mathcal{N}$, applying Lagrange's multipliers, there exists $\gamma \in \mathbb{R}$ such that

$$E'(u,v)(u,v) = \gamma I'(u,v)(u,v). \tag{3.1}$$

But because of $(u,v) \in \mathcal{N}$, we get

$$\begin{aligned} E'(u,v)(u,v) &= \|(u,v)\|^p - \lambda \int_\Omega h_1(x)|u|^\theta \ln|u| dx - \mu \int_\Omega h_2(x)|v|^\theta \ln|v| dx \\ &\quad - \int_\Omega (b_1 + b_2)(x)|u|^q |v|^r dx \\ &= 0. \end{aligned}$$

Thus, $\gamma I'(u,v)(u,v) = 0$. Now since $(u,v) \in \mathcal{N}^0$, we get

$$\begin{aligned} I'(u,v)(u,v) &= \lambda(p-\theta)\int_\Omega h_1(x)|u|^\theta \ln|u| dx + \mu(p-\theta)\int_\Omega h_2(x)|v|^\theta \ln|v| dx - \lambda \int_\Omega h_1(x)|u|^\theta dx \\ &\quad - \mu \int_\Omega h_2(x)|v|^\theta dx + (p-q-r)\int_\Omega (b_1+b_2)(x)|u|^q|v|^r dx \\ &= \Phi''_{(u,v)}(1) \neq 0. \end{aligned}$$

Thus, $\gamma \neq 0$, and $E'(u,v) = 0$. □

Lemma 3.3. *Let $0 < \lambda C_{h_1} + \mu C_{h_2} < \frac{1}{L}$ and $2 < q + r < \min\{p_s^*, p_t^*\}$. If $b_1, b_2 \in L^\infty(\Omega)$ are non-negative functions satisfying $C_B < K$, where $C_B := \max_{x \in \overline{\Omega}} |b_1(x) + b_2(x)|$ then, for any $(u,v) \in W \setminus \{(0,0)\}$, we get*

1) *If $\lambda \int_\Omega h_1(x)|u|^\theta + \mu \int_\Omega h_2(x)|v|^\theta dx \geq 0$, then there exists a unique $k_{(u,v)} > 0$ such that $\Phi'_{(u,v)}(k_{(u,v)}) = 0$ and $k_{(u,v)}(u,v) \in \mathcal{N}^-$. Moreover,*

$$E(k_{(u,v)}u, k_{(u,v)}v) = \sup_{k>0} E(ku, kv).$$

2) *If* $\lambda \int_\Omega h_1(x)|u|^\theta + \mu \int_\Omega h_2(x)|v|^\theta dx < 0$, *then there exists* $k_1, k_2 > 0$ *such that* $0 < k_1 < k_{\max} < K_2 < \infty$ *where* $\Phi'_{(u,v)}(k_1) = 0 = \Phi'_{(u,v)}(k_2)$ *witch* $(k_1u, k_1v) \in \mathcal{N}^+$ *and* $(k_2u, k_2v) \in \mathcal{N}^-$. *Moreover,*

$$E(k_1u, k_1v) = \inf_{0<k<k_{\max}} E(ku, kv) \quad \text{and} \quad E(k_2u, k_2v) = \sup_{k>0} E(ku, kv).$$

Proof. Define, for $k > 0$, a function $f : \mathbb{R} \to W$ given by

$$f(k) = k^{p-q-r}\|(u,v)\|^p - k^{p-q-r}\Big(\lambda \int_\Omega h_1(x)|u|^\theta \ln|ku| dx + \mu \int_\Omega h_2(x)|v|^\theta \ln|kv| dx\Big).$$

We can rewrite $f(k)$, for $k > 0$, as

$$\begin{aligned} f(k) = k^{p-q-r}\|(u,v)\|^p &- k^{p-q-r}\ln|k|\Big(\lambda \int_\Omega h_1(x)|u|^\theta dx\Big) - k^{p-q-r}\Big(\lambda \int_\Omega h_1(x)|u|^\theta \ln|u| dx\Big) \\ &- k^{p-q-r}\ln|k|\Big(\lambda \int_\Omega h_2(x)|v|^\theta dx\Big) - k^{p-q-r}\Big(\lambda \int_\Omega h_2(x)|v|^\theta \ln|v| dx\Big). \end{aligned}$$

A direct computation shows that

$$\begin{aligned} f'(k) = k^{p-q-r-1}\Bigg[&(p-q-r)\|(u,v)\|^p - \lambda \int_\Omega h_1(x)|u|^\theta dx - \lambda(p-q-r)\int_\Omega h_1(x)|u|^\theta \ln|u| dx \\ &- \mu \int_\Omega h_2(x)|v|^\theta dx - \mu(p-q-r)\int_\Omega h_2(x)|v|^\theta \ln|v| dx \\ &- (p-q-r)\ln|k|\Big(\lambda \int_\Omega h_1(x)|u|^\theta dx + \mu \int_\Omega h_2(x)|v|^\theta dx\Big)\Bigg]. \end{aligned}$$

Now we analyze all the possibilities:

i) If $\lambda \int_\Omega h_1(x)|u|^\theta dx + \mu \int_\Omega h_2(x)|v|^\theta dx > 0$, then $f \in C(0,\infty)$ and because $q + r > p$, we have

$$\lim_{k\to 0^+} f(k) = +\infty \quad \text{and} \quad \lim_{k\to\infty} f(k) = 0.$$

then there exists a unique minimum point $k_{\min} > 0$ such that $f'(k_{\min}) = 0$. Because $k_{\min} > 0$,

$$\begin{aligned} &(p-q-r)\ln|k_{\min}|\Big(\lambda \int_\Omega h_1(x)|u|^\theta dx + \mu \int_\Omega h_2(x)|v|^\theta dx\Big) \\ &\quad = (p-q-r)\|(u,v)\|^p - \lambda \int_\Omega h_1(x)|u|^\theta dx - \lambda(p-q-r)\int_\Omega h_1(x)|u|^\theta \ln|u| dx \\ &\qquad - \mu \int_\Omega h_2(x)|u|^\theta dx - \mu(p-q-r)\int_\Omega h_2(x)|v|^\theta \ln|v| dx. \end{aligned}$$

Thus

$$k_{\min} = \exp\left(\frac{\|(u,v)\|^p - \lambda \int_\Omega h_1(x)|u|^\theta \ln|u| dx - \mu \int_\Omega h_2(x)|v|^\theta \ln|v| dx}{\lambda \int_\Omega h_1(x)|u|^\theta dx + \mu \int_\Omega h_2(x)|v|^\theta dx} - \frac{1}{p-q-r}\right) \tag{3.2}$$

Obviously, f is decreasing on $(0, k_{\min})$ and increasing on $(k_{\min}, \infty)$. Then, because equation (3.2), the fact that $k_{\min} > 0$ and that $\lambda \int_\Omega h_1(x)|u|^\theta dx + \mu \int_\Omega h_2(x)|v|^\theta dx > 0$, we get

$$\begin{aligned} f(k_{\min}) &= k_{\min}^{p-q-r}\Bigg[\|(u,v)\|^p - \ln|k_{\min}|\Big(\lambda \int_\Omega h_1(x)|u|^\theta dx + \mu \int_\Omega h_2(x)|v|^\theta dx\Big) \\ &\qquad - \Big(\lambda \int_\Omega h_1(x)|u|^\theta \ln|u| dx + \mu \int_\Omega h_2(x)|v|^\theta \ln|v| dx\Big)\Bigg] \\ &= k_{\min}^{p-q-r}\Bigg[\frac{1}{p-q-r}\Big(\lambda \int_\Omega h_1(x)|u|^\theta dx + \mu \int_\Omega h_2(x)|v|^\theta dx\Big)\Bigg] \\ &< 0. \end{aligned}$$

Since we have

$$\int_\Omega (b_1+b_2)(x)|u|^q|v|^r dx > 0, \quad \text{and} \quad \lambda \int_\Omega h_1(x)|u|^\theta \ln|u| dx + \mu \int_\Omega h_2(x)|v|^\theta \ln|v| dx > 0,$$

and because

$$\begin{aligned}&\Phi'_{(u,v)}(k)\\ &\quad = k^{q+r+\theta-p-1}\left[f(k) - (k^{2p-q-r-\theta} + k^{p-q-r})\|(u,v)\|^p - k^{p-\theta}\int_\Omega (b_1+b_2)(x)|u|^q|v|^r dx\right],\end{aligned}$$

there exists a unique $k_{(u,v)}$ such that $0 < k_{(u,v)} < k_{\min}$ such that

$$f(k_{(u,v)}) = \left(k_{(u,v)}^{2p-q-r-\theta} + k_{(u,v)}^{p-q-r}\right)\|(u,v)\|^p + k_{(u,v)}^{p-\theta}\int_\Omega (b_1+b_2)(x)|u|^q|v|^r dx,$$

and $f'(k_{(u,v)}) < 0$, we get $k_{(u,v)}(u,v) \in \mathcal{N}^-$. Moreover, it follows from $f(k) < f(k_{(u,v)})$, for all $k > k_{(u,v)}$ and $f(k) > f(k_{(u,v)})$, for all $k < k_{(u,v)}$, that

$$E(k_{(u,v)}(u,v)) = \sup_{k>0} E(ku,kv).$$

ii) If $\lambda \int_\Omega h_1(x)|u|^\theta dx + \mu \int_\Omega h_2(x)|v|^\theta dx = 0$, it follows from Lemma 2.9 there exists a unique $k_{(u,v)} > 0$ such that $k_{(u,v)}(u,v) \in \mathcal{N}^-$, and

$$E(k_{(u,v)}) = \sup_{k>0} E(ku,kv).$$

iii) If $\lambda \int_\Omega h_1(x)|u|^\theta dx + \mu \int_\Omega h_2(x)|v|^\theta dx < 0$, then $f \in C(0,\infty)$, $\lim_{k\to 0^+} f(k) = -\infty$ and $\lim_{k\to\infty} f(k) = 0$. Then f has a unique maximum point $k_{\max} > 0$ which is given by

$$k_{\max} = \exp\left(\frac{\|(u,v)\|^p - \lambda\int_\Omega h_1(x)|u|^\theta \ln|u| dx + \mu\int_\Omega h_2(x)|v|^\theta \ln|v| dx}{\lambda\int_\Omega h_1(x)|u|^\theta dx + \mu\int_\Omega h_2(x)|v|^\theta dx} - \frac{1}{p-q-r}\right).$$

Moreover, f is increasing on $(0,k_{\max})$ and decreasing on $(k_{\max},\infty)$. By Lemma (2.9), we get

$$k_{\max} \geq \exp\left(\frac{(1-(\lambda C_{h_1}+\mu C_{h_2})L)\|(u,v)\|^p}{\lambda\int_\Omega h_1(x)|u|^\theta dx + \mu\int_\Omega h_2(x)|v|^\theta dx} - \ln\|(u,v)\| - \frac{1}{p-q-r}\right).$$

So,

$$k_{\max}^{p-q-r} \geq \exp\left((p-q-r)\frac{(1-(\lambda C_{h_1}+\mu C_{h_2})L)\|(u,v)\|^p}{\lambda\int_\Omega h_1(x)|u|^\theta dx + \mu\int_\Omega h_2(x)|v|^\theta dx} - 1\right)\|(u,v)\|^{q+r}.$$

It is known that the following inequality holds:

$$\exp(k-1) \geq k, \qquad \forall\, k \geq 0.$$

Then

$$k_{\max}^{p-q-r} \geq (p-q-r)\frac{1-(\lambda C_{h_1}+\mu C_{h_2})L}{\lambda\int_\Omega h_1(x)|u|^\theta dx + \mu\int_\Omega h_2(x)|v|^\theta dx}\|(u,v)\|^{q+r}.$$

Therefore

$$f(k_{\max}) = k_{\max}^{p-q-r}\frac{\lambda\int_\Omega h_1(x)|u|^\theta dx + \mu\int_\Omega h_2(x)|v|^\theta dx}{p-q-r}$$
$$\geq (1-(\lambda C_{h_1}+\mu C_{h_2})L)\|(u,v)\|^{q+r}.$$

Because $\left(k_{(u,v)}^{2p-q-r-\theta}+k_{(u,v)}^{p-q-r}\right)\|(u,v)\|^p + k_{(u,v)}^{p-\theta}\int_\Omega (b_1+b_2)(x)|u|^q|v|^r dx, > 0$. By Hölder's inequality,

$$\int_\Omega (b_1+b_2)(x)|u|^p|v|^r dx \leq C_B|\Omega|^{1-\frac{q}{p_s^*}-\frac{r}{p_t^*}}\left(\int_\Omega |u|^{2_s^*}\right)^{\frac{q}{p_s^*}}\left(\int_\Omega |v|^{2_t^*}\right)^{\frac{r}{p_t^*}}$$
$$\leq C_B|\Omega|^{1-\frac{q}{p_s^*}-\frac{r}{p_t^*}} S_{p_s^*}^q S_{p_t^*}^r \|(u,v)\|^{q+r},$$

for all $(u,v)\in W$, where $C_B = \max_{x\in\overline{\Omega}}|b_1(x)+b_2(x)|$ and $S_{p_s^*}^q, S_{p_t^*}^r > 0$ denote the best constants of embeddings $W_0^{s,p}\hookrightarrow L^{p_s^*}$ and $W_0^{t,p}\hookrightarrow L^{p_t^*}$, respectively.

Then

$$(1-(\lambda C_{h_1}+\mu C_{h_2})L)S_{p_s^*}^q S_{p_t^*}^r |\Omega|^{\frac{q}{p_s^*}+\frac{r}{p_t^*}-1} > C_B,$$

it implies that

$$f(k_{\max}) > \left(k_{(u,v)}^{2p-q-r-\theta}+k_{(u,v)}^{p-q-r}\right)\|(u,v)\|^p + k_{(u,v)}^{p-\theta}\int_\Omega (b_1+b_2)(x)|u|^q|v|^r dx > 0 = \lim_{k\to\infty} f(k),$$

and it show us there exists k_1, k_2 in which,

$$0 < k_1 < k_{\max} < k_2 < \infty,$$

such that

$$f(k_1) = \left(k_{(u,v)}^{2p-q-r-\theta}+k_{(u,v)}^{p-q-r}\right)\|(u,v)\|^p + k_{(u,v)}^{p-\theta}\int_\Omega (b_1+b_2)(x)|u|^q|v|^r dx = f(k_2).$$

It show us that $\Phi'_{(u,v)}(k_1) = 0 = \Phi'_{(u,v)}(k_2)$. Moreover, f is increasing on $(0,k_{\max})$ and decreasing on $(k_{\max},\infty)$. So

$$k_1(u,v)\in\mathcal{N}^+ \quad \text{and} \quad k_2(u,v)\in\mathcal{N}^-.$$

Moreover, we have

$$f(k) \geq \left(k^{2p-q-r-\theta}+k^{p-q-r}\right)\|(u,v)\|^p + k^{p-\theta}\int_\Omega (b_1+b_2)(x)|u|^q|v|^r dx, \qquad \forall k\in[k_1,k_2],$$

and

$$f(k) < \left(k^{2p-q-r-\theta}+k^{p-q-r}\right)\|(u,v)\|^p + k^{p-\theta}\int_\Omega (b_1+b_2)(x)|u|^q|v|^r dx, \qquad \forall k\in\mathbb{R}_+^*\setminus[k_1,k_2].$$

Thus

$$E(k_1(u,v)) = \inf_{0<k\leq k_{\max}} E(k(u,v)) \quad \text{and} \quad E(k_2(u,v)) = \sup_{k>0} E(k(u,v)).$$

The proof is complete. □

Remark 3.4. In what follows, we define

$$M := \frac{|\Omega|^{1-\frac{q}{p_s^*}}S_{p_s^*}^q + |\Omega|^{1-\frac{q}{p_s^*}}S_{p_t^*}^q}{q+r-p}, \quad \text{and} \quad K := \frac{1-(\lambda C_{h_1}+\mu C_{h_2})(L+M)}{|\Omega|^{1-\frac{q}{p_s^*}-\frac{r}{p_t^*}}S_{p_s^*}^q S_{p_t^*}^r}.$$

Lemma 3.5. *If* $0 < \lambda C_{h_1} + \mu C_{h_2} < \frac{1}{L+M}$ *and* $C_B < K$, *then* $\mathcal{N}^0 = \emptyset$.

Proof. Arguing by contradiction, let $(u,v) \in \mathcal{N}^0$. Then $I(u,v) = 0$, and

$$\|(u,v)\|^p - \lambda \int_\Omega h_1(x)|u|^\theta dx - \mu \int_\Omega h_2(x)|v|^\theta dx = \int_\Omega (b_1+b_2)(x)|u|^q|v|^r dx \tag{3.3}$$

and

$$\begin{aligned} 0 = {} & \lambda(p-\theta)\int_\Omega h_1(x)|u|^\theta \ln|u| dx + \mu(p-\theta)\int_\Omega h_2(x)|v|^\theta \ln|v| dx \\ & - \lambda \int_\Omega h_1(x)|u|^\theta dx - \mu \int_\Omega h_2(x)|v|^\theta dx \\ & + (p-q-r)\int_\Omega (b_1+b_2)(x)|u|^q|v|^r dx. \end{aligned} \tag{3.4}$$

From equations (3.3) and (3.4), and Lemma 2.9, we get

$$\begin{aligned} & \|(u,v)\|^p - \int_\Omega (b_1+b_2)(x)|u|^\theta dx \\ & \quad \le (\lambda C_{h_1} + \mu C_{h_2}) L \|(u,v)\|^p + \ln\|(u,v)\| \Big(\lambda \int_\Omega h_1(x)|u|^\theta dx + \mu \int_\Omega h_2(x)|v|^\theta dx \Big). \end{aligned}$$

Then, again by (3.4)

$$\begin{aligned} (1-(\lambda C_{h_1} + \mu C_{h_2})L)\|(u,v)\|^p \le {} & \ln\|(u,v)\| \Big(\lambda \int_\Omega h_1(x)|u|^\theta dx + \mu \int_\Omega h_2(x)|v|^\theta dx \Big) \\ & + \int_\Omega (b_1+b_2)(x)|v|^q|v|^r dx \\ \le {} & \ln\|(u,v)\| \Big(\lambda \int_\Omega h_1(x)|u|^\theta dx + \mu \int_\Omega h_2(x)|v|^\theta dx \Big) \\ & + \frac{1}{p-q-r}\Big[-\lambda(p-\theta)\int_\Omega h_1(x)|u|^\theta \ln|u| dx \\ & - \mu(p-\theta)\int_\Omega h_2(x)|v|^\theta \ln|v| dx \\ & + \lambda \int_\Omega h_1(x)|u|^\theta dx + \mu \int_\Omega h_2(x)|v|^\theta dx \Big]. \end{aligned}$$

That is,

$$\begin{aligned} (1-(\lambda C_{h_1} + \mu C_{h_2})L)\|(u,v)\|^p \le {} & \ln\|(u,v)\| \Big(\lambda \int_\Omega h_1(x)|u|^\theta dx + \mu \int_\Omega h_2(x)|v|^\theta dx \Big) \\ & + \frac{1}{p-q-r}\Big[-\lambda(p-\theta)\int_\Omega h_1(x)|u|^\theta \ln|u| dx \\ & - \mu(p-\theta)\int_\Omega h_2(x)|v|^\theta \ln|v| dx \Big] \\ & + \frac{\lambda C_{h_1} + \mu C_{h_2}}{q+r-p}\Big(|\Omega|^{1-\frac{q}{p_s^*}} S_{p_s^*}^q + |\Omega|^{1-\frac{q}{p_s^*}} S_{p_t^*}^q \Big)\|(u,v)\|^p, \end{aligned} \tag{3.5}$$

which means that

$$\begin{aligned} & \Big[1 - (\lambda C_{h_1} + \mu C_{h_2})L - \frac{\lambda C_{h_1} + \mu C_{h_2}}{q+r-p}\Big(|\Omega|^{1-\frac{q}{p_s^*}} S_{p_s^*}^q + |\Omega|^{1-\frac{q}{p_s^*}} S_{p_t^*}^q \Big) \Big] \|(u,v)\|^p \\ & \quad \le \ln\|(u,v)\| \Big(\lambda \int_\Omega h_1(x)|u|^\theta dx + \mu \int_\Omega h_2(x)|v|^\theta dx \Big) \\ & \qquad + \frac{1}{p-q-r}\Big(-\lambda(p-\theta)\int_\Omega h_1(x)|u|^\theta \ln|u| dx - \mu(p-\theta)\int_\Omega h_2(x)|v|^\theta \ln|v| dx \Big). \end{aligned}$$

Because $p < q + r$, equation (3.4) show us that

$$0 > -\lambda(p-\theta)\int_\Omega h_1(x)|u|^\theta \ln|u|dx - \mu(p-\theta)\int_\Omega h_2(x)|v|^\theta \ln|v|dx + \lambda\int_\Omega h_1(x)|u|^\theta dx + \mu\int_\Omega h_2(x)|v|^\theta dx.$$

Then, we have

$$0 > \ln\|(u,v)\|\Big(\lambda\int_\Omega h_1(x)|u|^\theta dx + \mu\int_\Omega h_2(x)|v|^\theta dx\Big) + \frac{1}{p-q-r}\Big(-\lambda(p-\theta)\int_\Omega h_1(x)|u|^\theta \ln|u|dx - \mu(p-\theta)\int_\Omega h_2(x)|v|^\theta \ln|v|dx\Big).$$

Thus, it follows from $\lambda C_{h_1} + \mu C_{h_2} < \frac{1}{L+M}$ that $\|(u,v)\| \leq 1$, where, here we consider M as on Remark 3.4.

Otherwise, using (3.5), and one more time (3.3) and (3.4),

$$\begin{aligned}[1-(\lambda C_{h_1}+\mu C_{h_2})L]\|(u,v)\|^p &\leq \ln\|(u,v)\|\Big(\lambda\int_\Omega h_1(x)|u|^\theta dx + \mu\int_\Omega h_2(x)|v|^\theta dx\Big) \\ &\quad + \int_\Omega (b_1+b_2)(x)|u|^q|v|^r dx \\ &\leq (p-q-r)\ln\|(u,v)\|\int_\Omega (b_1+b_2)(x)|u|^q|v|^r dx \\ &\quad + (\lambda C_{h_1}+\mu C_{h_2})M\|(u,v)\|^p. \end{aligned} \tag{3.6}$$

Now, we can note that

$$\int_\Omega (b_1+b_2)(x)|u|^q|v|^r dx \leq C_B|\Omega|^{1-\frac{q}{p_s^*}-\frac{r}{p_t^*}} S_{p_s^*}^q S_{p_t^*}^r \|(u,v)\|^{q+r}, \tag{3.7}$$

Thus

$$[1-(\lambda C_{h_1}+\mu C_{h_2})(L+M) - C_B|\Omega|^{1-\frac{q}{p_s^*}-\frac{r}{p_t^*}} S_{p_s^*}^q S_{p_t^*}^r]\|(u,v)\|^p \leq 0.$$

Since $(\lambda C_{h_1}+\mu C_{h_2}) < \frac{1}{L+M}$, and

$$C_B < K := \frac{1-(\lambda C_{h_1}+\mu C_{h_2})(L+M)}{|\Omega|^{1-\frac{q}{p_s^*}-\frac{r}{p_t^*}} S_{p_s^*}^q S_{p_t^*}^r},$$

it follows that $\|(u,v)\| = 0$. Then $(u,v) = (0,0)$, a contradiction. Therefore $\mathcal{N}^0 = \emptyset$. □

Remark 3.6. For next result, we consider

$$\Lambda_{\lambda,\mu} := \min\left\{1, \left(\frac{1-(\lambda C_{h_1}+\mu C_{h_2})L - C_B|\Omega|^{1-\frac{q}{p_s^*}-\frac{r}{p_t^*}} S_{p_s^*}^q S_{p_t^*}^r}{C_B|\Omega|^{1-\frac{q}{p_s^*}-\frac{r}{p_t^*}} S_{p_s^*}^q S_{p_t^*}^r}\right)^{\frac{1}{q+r-p}}\right\}.$$

Lemma 3.7. *If $0 < \lambda C_{h_1} + \mu C_{h_2} < \frac{1}{L+M}$ and $C_B < K$, then $\|(U,V)\| \geq \Lambda_{\lambda,\mu}$, for all $(U,V) \in \mathcal{N}^-$ and $\|(u,v)\| \leq 1$, for all $(u,v) \in \mathcal{N}^+$.*

Proof. Let $(U,V) \in \mathcal{N}^-$. Then

$$\|(U,V)\|^p - v\lambda \int_\Omega h_1(x)|U|^\theta \ln|U|dx - \mu \int_\Omega h_2(x)|V|^\theta \ln|V|dx = \int_\Omega (b_1+b_2)(x)|U|^q|V|^r dx, \quad (3.8)$$

and

$$\begin{aligned}(p-q-r)&\int_\Omega (b_1+b_2)(x)|U|^q|V|^r dx \\ &< \lambda \int_\Omega h_1(x)|U|^\theta dx + \mu \int_\Omega h_2(x)|V|^\theta dx \\ &\quad - (p-\theta)\Big(\lambda \int_\Omega h_1(x)|U|^\theta \ln|U|dx + \mu \int_\Omega h_2(x)|V|^\theta \ln|V|dx\Big). \end{aligned} \quad (3.9)$$

Similar to equation (3.6) in Lemma 3.5, we have

$$\begin{aligned}[1-(\lambda C_{h_1}+\mu C_{h_2})L]\|(U,V)\|^p \le \ln\|(U,V)\|&\Big(\lambda \int_\Omega h_1(x)|U|^\theta dx + \mu \int_\Omega h_2(x)|V|^\theta dx\Big) \\ &+ C_B|\Omega|^{1-\frac{q}{p_s^*}-\frac{r}{p_t^*}} S_{p_s^*}^q S_{p_t^*}^r \|(U,V)\|^{q+r}. \end{aligned} \quad (3.10)$$

We consider two cases:

i) If $\lambda \int_\Omega h_1(x)|U|^\theta dx + \mu \int_\Omega h_2(x)|V|^\theta dx > 0$, then $\|(U,V)\| \ge 1$, because otherwise, being $\|(U,V)\| < 1$, we have

$$[1-(\lambda C_{h_1}+\mu C_{h_2})L]\|(U,V)\|^p < C_B|\Omega|^{1-\frac{q}{p_s^*}-\frac{r}{p_t^*}} S_{p_s^*}^q S_{p_t^*}^r \|(U,V)\|^{q+r},$$

and together with $C_B < K$, we get

$$\|(U,V)\| > \left(\frac{1-(\lambda C_{h_1}+\mu C_{h_2})L}{C_B|\Omega|^{1-\frac{q}{p_s^*}-\frac{r}{p_t^*}} S_{p_s^*}^q S_{p_t^*}^r}\right)^{\frac{1}{q+r-p}} > 1, \quad (3.11)$$

a contradiction. Thus $\|(U,V)\| \ge 1$.

ii) If $\lambda \int_\Omega h_1(x)|U|^\theta dx + \mu \int_\Omega h_2(x)|V|^\theta dx < 0$, we have two more cases to analyze:

2.1) If $\|(U,V)\| > 1$, similar to case i) above, we get the same equation (3.11).

2.2) If $\|(U,V)\| < 1$, we have by equations (3.8) and (3.9),

$$\begin{aligned}&[1-(\lambda C_{h_1}+\mu C_{h_2})L]\|(U,V)\|^p \\ &< C_B|\Omega|^{1-\frac{q}{p_s^*}-\frac{r}{p_t^*}} S_{p_s^*}^q S_{p_t^*}^r \|(U,V)\|^p \\ &\quad + (p-\theta)\Big(\lambda \int_\Omega h_1(x)|U|^\theta \ln|U|dx + \mu \int_\Omega h_2(x)|V|^\theta \ln|V|dx\Big) \\ &\quad + C_B|\Omega|^{1-\frac{q}{p_s^*}-\frac{r}{p_t^*}} S_{p_s^*}^q S_{p_t^*}^r \|(U,V)\|^{q+r}, \end{aligned}$$

So, because $p \le \theta$,

$$\|(U,V)\| > \left(\frac{1-(\lambda C_{h_1}+\mu C_{h_2})L - C_B|\Omega|^{1-\frac{q}{p_s^*}-\frac{r}{p_t^*}} S_{p_s^*}^q S_{p_t^*}^r}{C_B|\Omega|^{1-\frac{q}{p_s^*}-\frac{r}{p_t^*}} S_{p_s^*}^q S_{p_t^*}^r}\right)^{\frac{1}{q+r-p}} > 1.$$

Thus, $\|(U,V)\| \ge \Lambda_{\lambda,\mu}$, for all $(U,V) \in \mathcal{N}^-$.

Now, if $(u,v) \in \mathcal{N}^+$, a similar discussion show us that $\|(u,v)\| \leq 1$. □

Lemma 3.8. *If $0 < \lambda C_{h_1} + \mu C_{h_2} < \frac{1}{L+M}$ and $C_B < K$, then $\mathcal{N}^-$ is a closed subset o W.*

Proof. The proof follows directly from Lemmas 3.5 and 3.7. □

Lemma 3.9. *If $0 < \lambda C_{h_1} + \mu C_{h_2} < \frac{1}{L+M}$. Then the functional E is bounded from below on $\mathcal{N}$.*

Proof. Because of the relationship between the functionals E and I, for $(u,v) \in \mathcal{N}$, we have

$$\begin{aligned} E(u,v) = & \left(\frac{1}{p} - \frac{1}{\theta}\right)\|(u,v)\|^p + \frac{1}{\theta^2}\Big(\lambda\int_\Omega h_1(x)|u|^\theta dx + \mu\int_\Omega h_2(x)|v|^\theta dx\Big) \\ & - \left(\frac{1}{q+r} - \frac{1}{\theta}\right)\int_\Omega (b_1+b_2)(x)|u|^q|v|^r dx. \end{aligned}$$

Because $(u,v) \in \mathcal{N}$, we have also

$$\int_\Omega (b_1+b_2)(x)|u|^q|v|^r dx = \|(u,v)\|^p - \lambda\int_\Omega h_1(x)|u|^\theta \ln|u| dx - \mu\int_\Omega h_2(x)|v|^\theta \ln|v| dx.$$

So

$$\begin{aligned} E(u,v) = & \left(\frac{1}{p} - \frac{1}{q+r}\right)\|(u,v)\|^p + \frac{1}{\theta^2}\Big(\lambda\int_\Omega h_1(x)|u|^\theta dx + \mu\int_\Omega h_2(x)|v|^\theta dx\Big) \\ & + \left(\frac{1}{\theta} - \frac{1}{q+r}\right)\Big(\lambda\int_\Omega h_1(x)|u|^\theta \ln|u| dx + \mu\int_\Omega h_2(x)|v|^\theta \ln|v| dx\Big). \end{aligned}$$

Now, concerning to exposed above, we consider two cases:

i) If $\lambda\int_\Omega h_1(x)|u|^\theta dx + \mu\int_\Omega h_2(x)|v|^\theta dx > 0$, because $0 < p < \theta < q+r$, it follows that

$$E(u,v) > \left(\frac{1}{\theta} - \frac{1}{q+r}\right)\int_\Omega (b_1+b_2)(x)|u|^q|v|^r dx \geq 0.$$

ii) If $\lambda\int_\Omega h_1(x)|u|^\theta dx + \mu\int_\Omega h_2(x)|v|^\theta dx \leq 0$, it follows from Lemma (2.9) that

$$\begin{aligned} E(u,v) = & \left(\frac{1}{p} - \frac{1}{q+r}\right)[1 - (\lambda C_{h_1} + \mu C_{h_2})L]\|(u,v)\|^p \\ & + \left[\frac{1}{\theta^2} - \left(\frac{1}{\theta} - \frac{1}{q+r}\right)\ln\|(u,v)\|\right]\Big(\lambda\int_\Omega h_1(x)|u|^\theta dx + \mu\int_\Omega h_2(x)|v|^\theta dx\Big). \end{aligned}$$

Now, if $\frac{1}{\theta^2} - (\frac{1}{\theta} - \frac{1}{q+r})\ln\|(u,v)\| \geq 0$, then $E(u,v) \geq 0$.
Otherwise, if $\frac{1}{\theta^2} - (\frac{1}{\theta} - \frac{1}{q+r})\ln\|(u,v)\| < 0$, there exists a constant $C > 0$ such that

$$\begin{aligned} E(u,v) \geq & \left(\frac{1}{p} - \frac{1}{q+r}\right)[1 - (\lambda C_{h_1} + \mu C_{h_2})L]\|(u,v)\|^p \\ & + \left[\frac{1}{\theta^2} - \left(\frac{1}{\theta} - \frac{1}{q+r}\right)\ln\|(u,v)\|\right]\Big(\lambda\int_\Omega h_1(x)|u|^\theta dx + \mu\int_\Omega h_2(x)|v|^\theta dx\Big) \\ \geq & \left(\frac{1}{p} - \frac{1}{q+r}\right)[1 - (\lambda C_{h_1} + \mu C_{h_2})L]\|(u,v)\|^p \\ & - C\left[\frac{1}{\theta^2} - \left(\frac{1}{\theta} - \frac{1}{q+r}\right)\ln\|(u,v)\|\right] \\ \geq & -C. \end{aligned}$$

Thus, E is bounded from below on $\mathcal{N}$. □

4 Nontrivial ground state solution for positive weight function

In this section, we establish the existence of a ground state solution for problem (P) when the weights functions $h_1, h_2 > 0$ by Mountain Pass theorem and the existence of a level $c_* \in W$, where the functional E satisfy the $(PS)_{c_*}$ condition.

Lemma 4.1. *Let $h_1, h_2 \in C(\overline{\Omega})$ and $h_1(x), h_2(x) > 0$ for all $x \in \Omega$. Then there are exist $\eta, \zeta > 0$, such that*

(i) $E(u,v) \geq \eta > 0$ for all $\|(u,v)\| = \zeta$,

(ii) There exists $(u,v) \in W$ such that $E(u,v) < 0$ if $\|(u,v)\| > \zeta$.

Proof. Because the definition of E and inequality (3.7), we have

$$\begin{aligned} E(u,v) \geq & \left[\frac{1}{p} - \frac{1}{\theta}(\lambda C_{h_1} + \mu C_{h_2})L\right] \|(u,v)\|^p \\ & - \frac{1}{\theta}\ln\|(u,v)\|\left(\lambda \int_\Omega h_1(x)|u|^\theta dx + \mu \int_\Omega h_2(x)|v|^\theta dx\right) \\ & - \frac{C_B|\Omega|^{1-\frac{q}{p_s^*}-\frac{r}{p_t^*}} S_{p_s^*}^q S_{p_t^*}^r}{q+r}\|(u,v)\|^{q+r}, \end{aligned}$$

what implies that, for all $(u,v) \in W$ with $0 < \|(u,v)\| \leq 1$,we have

$$E(u,v) \geq \left[\frac{1}{p} - \frac{1}{\theta}(\lambda C_{h_1} + \mu C_{h_2})L\right] \|(u,v)\|^p - \frac{C_B|\Omega|^{1-\frac{q}{p_s^*}-\frac{r}{p_t^*}} S_{p_s^*}^q S_{p_t^*}^r}{q+r}\|(u,v)\|^{q+r}.$$

Choosing $\zeta \in (0,1]$ small enough, such that

$$\left(\frac{1}{p} - \frac{1}{\theta}(\lambda C_{h_1} + \mu C_{h_2})L\right) - \frac{C_B|\Omega|^{1-\frac{q}{p_s^*}-\frac{r}{p_t^*}} S_{p_s^*}^q S_{p_t^*}^r}{q+r}\zeta^{q+r} > 0,$$

we have

$$E(u,v) \geq \left[\left(\frac{1}{p} - \frac{1}{\theta}(\lambda C_{h_1} + \mu C_{h_2})L\right) - \frac{C_B|\Omega|^{1-\frac{q}{p_s^*}-\frac{r}{p_t^*}} S_{p_s^*}^q S_{p_t^*}^r}{q+r}\zeta^{q+r}\right]\zeta^p > 0,$$

for all $(u,v) \in W$, with $\|(u,v)\| = \zeta$. Thus (i) holds.

Otherwise, for all $(u,v) \in W \setminus \{0,0\}$ and $k > 0$, we have

$$\begin{aligned} & E(ku,kv) \\ & = k^{q+r}\left[\frac{k^{p-q-r}}{p}\|(u,v)\|^p\right. \\ & \quad - \frac{k^{\theta-q-r}\ln|k|}{\theta}\int_\Omega h_1(x)|u|^\theta dx - \frac{k^{\theta-q-r}}{\theta}\lambda\int_\Omega h_1(x)|u|^\theta \ln|u|dx + \frac{k^{\theta-q-r}}{\theta^2}\lambda\int_\Omega h_1(x)|u|^\theta dx \\ & \quad \left. - \frac{k^{\theta-q-r}\ln|k|}{\theta}\int_\Omega h_2(x)|v|^\theta dx - \frac{k^{\theta-q-r}}{\theta}\mu\int_\Omega h_2(x)|v|^\theta \ln|v|dx + \frac{k^{\theta-q-r}}{\theta^2}\mu\int_\Omega h_2(x)|v|^\theta dx\right], \end{aligned}$$

and because $2 < p < \theta < q+r$, it implies that there exists $k_0 > 0$ large enough such that $\|(k_0u, k_0v)\| > \zeta$ and $E(k_0u, k_0v) < 0$. So, taking $(u,v) = (k_0u, k_0v)$, item (ii) holds □

Lemma 4.2. *Let (u_n, v_n) be a $(PS)_{c_*}$ sequence of the functional E. Then, a sequence (u_n, v_n) is bounded in W.*

Proof. For $c_* \in \mathbb{R}$, we assume that $\{(u_n, v_n)\}_n \subset W$ with $\|(u_n, v_n)\| > 1$ is a $(PS)_{c_*}$ sequence with, it is

$$E(u_n, v_n) \to c_* \quad \text{and} \quad E'(u_n, v_n)(u_n, v_n) \to 0,$$

as $n \to \infty$. Since (u_n, v_n) is a $(PS)_{c_*}$ sequence for functional E, we have

$$\begin{aligned} c_* + o_n(1) + C\|(u_n, v_n)\| &\geq E(u_n, v_n) - \frac{1}{p}E'(u_n, v_n)(u_n, v_n) \\ &= \frac{1}{p^2}\Big(\lambda \int_\Omega h_1(x)|u_n|^\theta dx + \mu \int_\Omega h_2(x)|v_n|^\theta dx\Big) \\ &\quad + \left(\frac{1}{p} - \frac{1}{q+r}\right)\int_\Omega (b_1 + b_2)(x)|u_n|^q|v_n|^r dx \\ &= \frac{1}{p^2}\Big(\lambda \int_\Omega h_1(x)|u_n|^\theta dx + \mu \int_\Omega h_2(x)|v_n|^\theta dx\Big). \end{aligned} \tag{4.1}$$

We have also

$$\begin{aligned} &c_* + o_n(1) + C\|(u_n, v_n)\| \\ &\quad \geq E(u_n, v_n) - \frac{1}{q+r}E'(u_n, v_n)(u_n, v_n) \\ &\quad = \left(\frac{1}{p} - \frac{1}{q+r}\right)\|(u_n, v_n)\|^p \\ &\qquad - \left(\frac{1}{p} - \frac{1}{q+r}\right)\left(\lambda \int_\Omega h_1(x)|u_n|^\theta \ln u_n dx + \mu \int_\Omega h_2(x)|v_n|^\theta \ln v_n dx\right) \\ &\qquad + \frac{1}{p^2}\left(\lambda \int_\Omega h_1(x)|u_n|^\theta dx + \mu \int_\Omega h_2(x)|v_n|^\theta dx\right) \\ &\quad \geq \left(\frac{1}{p} - \frac{1}{q+r}\right)\|(u_n, v_n)\|^p \\ &\qquad - \left(\frac{1}{p} - \frac{1}{q+r}\right)\left(\lambda \int_\Omega h_1(x)|u_n|^\theta \ln u_n dx + \mu \int_\Omega h_2(x)|v_n|^\theta \ln v_n dx\right). \end{aligned} \tag{4.2}$$

Combining equations (4.1) and (4.2), Lemma (2.5) and Lemma (2.9), we have

$$\begin{aligned} &\left[\left(\frac{1}{p} - \frac{1}{q+r}\right)[1 - (\lambda C_{h_1} + \mu C_{h_2})L]\right]\|(u, v)\| \\ &\quad \leq \left(\frac{1}{p} - \frac{1}{q+r}\right)\frac{p^2[(u_n, v_n)]^\sigma}{e\sigma}\left[c_* + C\|(u_n, v_n)\| + o(1)\right] \\ &\qquad + c_* + C\|(u_n, v_n)\| + o(1) \\ &\quad \leq C\left(1 + \|(u_n, v_n)\|^{1+\sigma}\right) + o(1), \end{aligned}$$

where $\sigma \in (0, p-1)$. Thus $\{(u_n, v_n)\}_n$ is bounded in W. □

Lemma 4.3. *Let (u_n, v_n) be a $(PS)_{c_*}$ sequence of the functional E. Then, functional E satisfies the $(PS)_{c_*}$ condition at any level c_*.*

Proof. By Lemmas 2.6 and 2.8, we get

$$\lim_{n\to\infty}\int_\Omega h_1(x)u|u_n|^{\theta-1}\ln|u|dx=\int_\Omega h_1(x)|u|^\theta\ln|u|dx, \tag{4.3}$$

$$\lim_{n\to\infty}\int_\Omega h_2(x)v|v_n|^{\theta-1}\ln|v|dx=\int_\Omega h_2(x)|v|^\theta\ln|v|dx, \tag{4.4}$$

$$\lim_{n\to\infty}\int_\Omega (b_1+b_2)(x)|u_n|^{q-2}u_n|v_n|^r(u_n-u)dx=0, \tag{4.5}$$

$$\lim_{n\to\infty}\int_\Omega (b_1+b_2)(x)|v_n|^{r-2}v_n|u_n|^q(v_n-v)dx=0. \tag{4.6}$$

Note that

$$\begin{aligned}\|(u_n-u,v_n-v)\|^p=&(E'(u_n,v_n)-E'(u,v))(u_n-u,v_n-v)\\&+\lambda\int_\Omega h_1(x)\left(|u_n|^{\theta-1}\ln|u_n|-|u|^{\theta-1}\ln|u|\right)(u_n-u)dx\\&+\mu\int_\Omega h_2(x)\left(|v_n|^{\theta-1}\ln|v_n|-|v|^{\theta-1}\ln|v|\right)(v_n-v)dx\\&+\frac{q}{q+r}\int_\Omega (b_1+b_2)(x)\left(|v_n|^r u_n^{q-2}u_n-|v|^r|u|^{q-2}u\right)(u_n-u)dx\\&+\frac{r}{q+r}\int_\Omega (b_1+b_2)(x)\left(|u_n|^q v_n^{r-2}v_n-|u|^q|v|^{r-2}v^{r-2}v\right)(v_n-v)dx.\end{aligned}$$

Since $(E'(u_n,v_n)-E'(u,v))(u_n-u,v_n-v)\to 0$ as $n\to\infty$, it follow from equations (4.3), (4.4), (4.5), (4.6) and Lemma 2.3, that $(u_n,v_n)\to(u,v)$ in W. It yields the proof. □

Now we can prove one of our main results, Theorem 1.1.

4.1 Proof of Theorem 1.1

Define $\mathfrak{N}:=\{(u,v)\in W\setminus\{0,0\}\mid E'(u,v)(u,v)=0\}$. By the previous lemmas, $\mathfrak{N}$ is nonempty. Let $(u,v)\in\mathfrak{N}$, from Lemma (2.9), we have

$$\begin{aligned}0=&\|(u,v)\|^p-\left(\lambda\int_\Omega h_1(x)|u|^\theta\ln|u|dx+\mu\int_\Omega h_2(x)|v|^\theta\ln|v|dx\right)-\int_\Omega (b_1+b_2)(x)|u|^q|v|^r dx\\\geq&(1-(\lambda C_{h_1}+\mu C_{h_2})L)\|(u,v)\|^p-\ln\|(u,v)\|\left(\lambda\int_\Omega h_1(x)|u|^\theta\ln|u|dx+\mu\int_\Omega h_2(x)|v|^\theta\ln|v|dx\right)\\&-\int_\Omega (b_1+b_2)(x)|u|^q|v|^r dx\end{aligned} \tag{4.7}$$

If $\|(u,v)\|\leq 1$, it follows from the above equation (4.7) that

$$[1-(\lambda C_{h_1}+\mu C_{h_2})]\|(u,v)\|^p\leq\int_\Omega (b_1+b_2)(x)|u|^q|v|^r dx\leq C_B|\Omega|^{1-\frac{q}{p_s^*}-\frac{r}{p_t^*}}\|(u,v)\|^{q+r}.$$

Then,

$$\|(u,v)\|\geq\left[\frac{1-(\lambda C_{h_1}+\mu C_{h_2})L}{C_B|\Omega|^{1-\frac{q}{p_s^*}-\frac{r}{p_t^*}}S_{p_s^*}^q S_{p_t^*}^r}\right]^{\frac{1}{q+r-p}},$$

which implies that

$$\int_\Omega (b_1+b_2)(x)|u|^q|v|^r dx \geq [1-(\lambda C_{h_1}+\mu C_{h_2})L]\|(u,v)\|^p \geq \left[\frac{[1-(\lambda C_{h_1}+\mu C_{h_2})L]^{q+r}}{\left(C_B|\Omega|^{1-\frac{q}{p_s^*}-\frac{r}{p_t^*}}S_{p_s^*}^q S_{p_t^*}^r\right)^p}\right]^{\frac{1}{q+r-p}}. \tag{4.8}$$

Define

$$\tilde{C} := \left[\frac{[1-(\lambda C_{h_1}+\mu C_{h_2})L]^{q+r}}{\left(C_B|\Omega|^{1-\frac{q}{p_s^*}-\frac{r}{p_t^*}}S_{p_s^*}^q S_{p_t^*}^r\right)^p}\right]^{\frac{1}{q+r-p}} \quad \text{and} \quad \mathfrak{C} := \inf\{E(u,v) \mid (\tilde{U}_n,\tilde{V}_n)\in\mathfrak{N}\},$$

then $\mathfrak{C}>0$, because otherwise, there exists $\{(\tilde{U}_n,\tilde{V}_n)\}_n \subset \mathfrak{N}$ such that $E'(\tilde{U}_n,\tilde{V}_n)\to 0$.

It follows from

$$\begin{aligned} E(u,v) = &\left(\frac{1}{p}-\frac{1}{\theta}\right)\|(u,v)\|^p + \frac{1}{\theta}I(u,v) \\ &+\frac{1}{\theta^2}\left(\lambda\int_\Omega h_1(x)|u|^\theta dx + \mu\int_\Omega h_2(x)|v|^\theta dx\right) \\ &-\left(\frac{1}{q+r}-\frac{1}{p}\right)\int_\Omega (b_1+b_2)(x)|u|^q|v|^r dx, \end{aligned}$$

that

$$\frac{1}{\theta^2}\left(\lambda\int_\Omega h_1(x)|\tilde{U}_n|^\theta dx + \mu\int_\Omega h_2(x)|\tilde{V}_n|^\theta dx\right) - \left(\frac{1}{q+r}-\frac{1}{p}\right)\int_\Omega (b_1+b_2)(x)|\tilde{U}_n|^q|\tilde{V}_n|^r dx \to 0,$$

when $n\to\infty$, and by Lemma 4.2. But, from equation (4.8) follows that

$$\int_\Omega (b_1+b_2)(x)|\tilde{U}_n|^q|\tilde{V}_n|^r dx \geq \tilde{C} > 0,$$

and it implies that $0\geq\tilde{C}>0$, a contradiction. Thus, $\mathfrak{C}>0$.

Finally, let $\{(u_n,v_n)\}_n \subset \mathfrak{N}$ be a minimizing sequence. Then $E'(u_n,v_n)(u_n,v_n)=0$ and $\lim_{n\to\infty} E(u_n,v_n)=\mathfrak{C}>0$. Again by Lemma 4.2, there exists $(u_0,v_0)\in W\setminus\{0,0\}$ such that

$$(u_n,v_n)\to(u_0,v_0)$$

in W. Hence $E(u_0,v_0)=\mathfrak{C}$ and $E'(u_0,v_0)=0$, and it means that (u_0,v_0) is a nontrivial ground state solution of problem (P).

5 Nontrivial solutions for sing-changing weight functions

Lemma 5.1. *E has a nontrivial and nonnegative minimizer on $\mathcal{N}^+$.*

Proof. The proof of this lemma will be shown with two steps.

Step 1. The strong convergence of minimizing sequence. By Lemma 3.9, we get

$$c^+ = \inf_{(u,v)\in\mathcal{N}^+} E(u,v).$$

We claim that $c^+ < 0$. Indeed, for each $(u,v) \in \mathcal{N}^+$, we get

$$\begin{aligned}\|(u,v)\|^p = \lambda \int_\Omega h_1(x)|u|^\theta \ln|u|dx + \mu \int_\Omega h_2(x)|v|^\theta \ln|v|dx \\ + \int_\Omega (b_1(x)+b_2(x))|u|^q|v|^r dx. \end{aligned} \tag{5.1}$$

Substituting (5.1), in the energy functional E, we have the following expression

$$\begin{aligned}E(u,v) &= \frac{1}{p}\|(u,v)\|^p - \frac{\lambda}{\theta}\int_\Omega h_1(x)|u|^\theta \ln|u|dx + \frac{\lambda}{\theta^2}\int_\Omega h_1(x)|u|^\theta dx - \frac{\mu}{\theta}\int_\Omega h_2(x)|v|^\theta \ln|v|dx \\ &\quad + \frac{\mu}{\theta^2}\int_\Omega h_2(x)|v|^\theta dx - \frac{1}{q+r}\int_\Omega (b_1(x)+b_2(x))|u|^q|v|^r dx \\ &= \frac{1}{p}\left[\lambda\int_\Omega h_1(x)|u|^\theta \ln|u|dx + \mu\int_\Omega h_2(x)|v|^\theta \ln|v|dx + \int_\Omega (b_1(x)+b_2(x))|u|^q|v|^r dx\right] \\ &\quad - \frac{\lambda}{\theta}\int_\Omega h_1(x)|u|^\theta \ln|u|dx + \frac{\lambda}{\theta^2}\int_\Omega h_1(x)|u|^\theta dx - \frac{\mu}{\theta}\int_\Omega h_2(x)|v|^\theta \ln|v|dx \\ &\quad + \frac{\mu}{\theta^2}\int_\Omega h_2(x)|v|^\theta dx - \frac{1}{q+r}\int_\Omega (b_1(x)+b_2(x))|u|^q|v|^r dx \\ &= \left(\frac{1}{p}-\frac{1}{\theta}\right)\left[\lambda\int_\Omega h_1(x)|u|^\theta \ln|u|dx + \mu\int_\Omega h_2(x)|v|^\theta \ln|v|dx\right] \\ &\quad + \frac{1}{\theta^2}\left[\lambda\int_\Omega h_1(x)|u|^\theta dx + \mu\int_\Omega h_2(x)|v|^\theta dx\right] \\ &\quad + \left(\frac{1}{p}-\frac{1}{q+r}\right)\int_\Omega (b_1(x)+b_2(x))|u|^q|v|^r dx.\end{aligned}$$

Using the logarithmic estimates for positive weight functions in addition with $p < \theta$ and $p < q+r$, we have the following inequality

$$\begin{aligned}&(p-\theta)\left[\lambda\int_\Omega h_1(x)|u_n|^\theta \ln|u|dx + \mu\int_\Omega h_2(x)|v_n|^\theta \ln|v|dx\right] \\ &\quad > (q+r-p)\int_\Omega (b_1(x)+b_2(x))|u|^q|v|^r dx + \lambda\int_\Omega h_1(x)|u|^\theta dx + \mu\int_\Omega h_2(x)|v|^\theta dx.\end{aligned}$$

From the last inequality, we have that

$$\begin{aligned}\lambda\int_\Omega h_1(x)|u_n|^\theta \ln|u|dx + \mu\int_\Omega h_2(x)|v_n|^\theta \ln|v|dx \\ < -\frac{q+r-p}{\theta-p}\int_\Omega (b_1(x)+b_2(x))|u|^q|v|^r dx \\ - \frac{1}{\theta-p}\left[\lambda\int_\Omega h_1(x)|u|^\theta dx + \mu\int_\Omega h_2(x)|v|^\theta dx\right].\end{aligned} \tag{5.2}$$

Finally, using (5.2) in the functional E, it follows that

$$\begin{aligned}E(u,v) = \left(\frac{1}{p}-\frac{1}{\theta}\right)\left[\lambda\int_\Omega h_1(x)|u|^\theta \ln|u|dx + \mu\int_\Omega h_2(x)|v|^\theta \ln|v|dx\right] \\ + \frac{1}{\theta^2}\left[\lambda\int_\Omega h_1(x)|u|^\theta dx + \mu\int_\Omega h_2(x)|v|^\theta dx\right] \\ + \left(\frac{1}{p}-\frac{1}{q+r}\right)\int_\Omega (b_1(x)+b_2(x))|u|^q|v|^r dx\end{aligned}$$

$$< (p-q-r)\left(\frac{1}{\theta-p}-\frac{1}{p(q+r)}\right)\int_\Omega (b_1(x)+b_2(x))|u|^q|v|^r dx$$
$$+\left(\frac{1}{\theta^2}-\frac{1}{\theta-p}\right)\left[\lambda\int_\Omega h_1(x)|u|^\theta dx+\mu\int_\Omega h_2(x)|v|^\theta dx\right]<0.$$

Let $\{(u_n,v_n)\}\subset\mathcal{N}^+$ be a minimizing sequence. Then, we get

$$\|(u_n,v_n)\|^p=\lambda\int_\Omega h_1(x)|u_n|^\theta\ln|u|dx+\mu\int_\Omega h_2(x)|v_n|^\theta\ln|v|dx$$
$$+\int_\Omega (b_1(x)+b_2(x))|u|^q|v|^r dx \tag{5.3}$$

and

$$\Phi''_{(u_n,v_n)}(1)=\|(u_n,v_n)\|^p-\lambda\int_\Omega h_1(x)|u_n|^\theta\ln|u_n|dx-\mu\int_\Omega h_2(x)|v|^\theta\ln|v|dx$$
$$-\int_\Omega (b_1+b_2)(x)|u_n|^q|v_n|^r dx-\lambda\int_\Omega h_1(x)|u_n|^\theta dx$$
$$-\mu\int_\Omega h_2(x)|v_n|^\theta dx+(p-q-r)\int_\Omega (b_1(x)+b_2(x))|u|^q|v|^r dx. \tag{5.4}$$

Using $\{(u_n,v_n)\}\subset\mathcal{N}^+$ and $\Phi''_{(u_n,v_n)}(1)>0$, we conclude that

$$-\lambda\int_\Omega h_1(x)|u_n|^\theta dx-\mu\int_\Omega h_2(x)|v_n|^\theta dx+(p-q-r)\int_\Omega (b_1(x)+b_2(x))|u|^q|v|^r dx>0. \tag{5.5}$$

Since $\{(u_n,v_n)\}$ is bounded by Lemma 3.8, up to subsequence we assume that

$$\begin{cases}(u_n,v_n)\rightharpoonup(u^+,v^+) & \text{in } W;\\ u_n\to u^+, v_n\to v^+ & \text{strongly in } L^t(\Omega),\ \text{for } 1\le t<\min\{p_s^*,p_t^*\};\\ u_n(x)\to u^+(x), v_n(x)\to v^+(x) & \text{a.e. in } \Omega.\end{cases} \tag{5.6}$$

Similar to Lemma 4.3, we obtain the following convergences:

$$\lim_{n\to\infty}\int_\Omega h_1(x)u_n|u_n|^\theta\ln|u_n|dx=\int_\Omega h_1(x)|u^+|^\theta\ln|u^+|dx, \tag{5.7}$$

and

$$\lim_{n\to\infty}\int_\Omega h_2(x)v_n|v_n|^\theta\ln|v_n|dx=\int_\Omega h_2(x)|v^+|^\theta\ln|v^+|dx, \tag{5.8}$$

Furthermore, the dominated convergence theorem of Lebesgue is valid for the product of functions $|u|^q$ and $|v|^r$, hence we have that

$$\lim_{n\to\infty}\int_\Omega b_1(x)|u_n|^q|v_n|^r dx=\int_\Omega b_2(x)|u^+|^q|v^+|^r dx, \tag{5.9}$$

and

$$\lim_{n\to\infty}\int_\Omega b_2(x)|u_n|^q|v_n|^r dx=\int_\Omega b_2(x)|u^+|^q|v^+|^r dx. \tag{5.10}$$

If $(u_n,v_n)\rightharpoonup(u^+,v^+)$ in W, then

$$\|(u^+,v^+)\|^2\le\liminf_{n\to+\infty}\|(u_n,v_n)\|^2.$$

This implies that

$$\begin{aligned}
&\|(u^+,v^+)\|^p - \lambda\int_\Omega h_1(x)|u^+|^\theta \ln|u^+|dx - \mu\int_\Omega h_1(x)|v^+|^\theta \ln|v^+|dx\\
&\quad - \int_\Omega b_1(x)|u^+|^q|v^+|^r dx - \int_\Omega b_2(x)|u^+|^q|v^+|^r dx\\
&< \liminf_{n\to+\infty}\Big[\|(u_n,v_n)\|^2 - \lambda\int_\Omega h_1(x)|u_n|^\theta \ln|u_n|dx - \mu\int_\Omega h_1(x)|v_n|^\theta \ln|v_n|dx\\
&\quad - \int_\Omega b_1(x)|u_n|^q|v_n|^r dx - \int_\Omega b_2(x)|u_n|^q|v_n|^r dx\Big] = 0. \qquad (5.11)
\end{aligned}$$

Now we prove that for (u^+,v^+) there exists $0 < t_{(u^+,v^+)} \neq 1$ such that

$$t_{(u^+,v^+)}(u^+,v^+) \in \mathcal{N}^+.$$

Since $c^+ < 0$, one can show that $(u^+,v^+) \neq (0,0)$. By (5.5), we deduce that

$$\lambda\int_\Omega b_1(x)|u^+|^q dx + \mu\int_\Omega b_2(x)|v^+|^r dx < 0,$$

then by Lemma 3.3 there exists $t_{(u^+,v^+)} > 0$ such that

$$t_{(u^+,v^+)}(u^+,v^+) \in \mathcal{N}^+ \quad \text{and} \quad \Psi'_{(u^+,v^+)}(t_{(u^+,v^+)}) = 0.$$

By (5.11), $\Psi'_{(u^+,v^+)}(1)) < 0$. Thus, $t_{(u^+,v^+)} \neq 1$. Note that $t_{(u^+,v^+)}(u^+,v^+)$ is minimizer of $g(t) = E(\tau(u^+,\tau v^+)$. Thus,

$$E(t_{(u^+,v^+)}u^+, t_{(u^+,v^+)}v^+) < E(u^+,v^+)) \leq \lim_{n\to+\infty} E(u_n,v_n) = \inf_{(u,v)\in\mathcal{N}^+} E(u,v),$$

and this is absurd. Therefore, we obtain $(u_n,v_n) \to (u^+,v^+)$ in W.

Step 2. Existence of nonnegative minimizers. If $(u_n,v_n) \to (u^+,v^+)$ in W and Lemma 3.9, we get

$$-\lambda\int_\Omega h_1(x)|u^+|^\theta dx - \mu\int_\Omega h_2(x)|v^+|^\theta dx + (p-q-r)\int_\Omega (b_1+b_2)(x)|u^+|^q|v^+|^r dx > 0. \quad (5.12)$$

Thus, we obtain that $(u^+,v^+) \in \mathcal{N}^+$. This gives that (u^+,v^+) is a minimizer of E on $\mathcal{N}^+$. Proof, we can prove $(|u^+|,|v^+|)$ is also a minimizer $\mathcal{N}^+$. Since

$$E(|u^+|,|v^+|) \leq E(u^+,v^+)$$

and using (5.12) hold for $(|u^+|,|v^+|)$, it suffices to show that

$$\begin{aligned}
&\|(u^+,v^+)\|^p - \lambda\int_\Omega h_1(x)|u^+|^\theta \ln|u^+|dx - \mu\int_\Omega h_1(x)|v^+|^\theta \ln|v^+|dx\\
&\quad = \int_\Omega b_1(x)|u^+|^q|v^+|^r dx + \int_\Omega b_2(x)|u^+|^q|v^+|^r dx. \qquad (5.13)
\end{aligned}$$

Using $[|u^+|]_s^p \leq [u^+]_s^p$, we obtain

$$\begin{aligned}
&\|(u^+,v^+)\|^p - \lambda\int_\Omega h_1(x)|u^+|^\theta \ln|u^+|dx - \mu\int_\Omega h_1(x)|v^+|^\theta \ln|v^+|dx\\
&\quad \leq \int_\Omega b_1(x)|u^+|^q|v^+|^r dx + \int_\Omega b_2(x)|u^+|^q|v^+|^r dx. \qquad (5.14)
\end{aligned}$$

If

$$\|(u^+,v^+)\|^p - \lambda\int_\Omega h_1(x)|u^+|^\theta \ln|u^+|dx - \mu\int_\Omega h_1(x)|v^+|^\theta\ln|v^+|dx$$
$$< \int_\Omega b_1(x)|u^+|^q|v^+|^r dx + \int_\Omega b_2(x)|u^+|^q|v^+|^r dx, \tag{5.15}$$

then $\Psi'_{(|u^+|,|v^+|)}(1)) < 0$. For $(|u^+|,|v^+|)$, by Lemma 3.3 there exist $t_{(|u^+|,|v^+|)} \in \mathcal{N}^+$ and $\Psi'_{(|u^+|,|v^+|)}t_{(|u^+|,|v^+|)} = 0$. Thus, $t_{(|u^+|,|v^+|)} \neq 1$. For other hand $t_{(|u^+|,|v^+|)}$ is a minimizer of

$$\xi(t) := E(u^+,v^+).$$

Thus,

$$E(t_{(|u^+|,|v^+|)}|u^+|, t_{(|u^+|,|v^+|)}|v^+| \leq E(|u^+|,|v^+|) \leq E(u^+,v^+) = \inf_{(u,v)\in\mathcal{N}^+} E(u,v).$$

This is absurd. Thus, $(|u^+|,|v^+|) \in \mathcal{N}^+$ and

$$E(|u^+|,|v^+|) = \inf_{(u,v)\in\mathcal{N}^+} E(u,v)$$

In conclusion, we get nonnegative minimizer E on $\mathcal{N}^+$. □

Lemma 5.2. *E has a nontrivial and nonnegative minimizer on $\mathcal{N}^-$.*

Proof. By Lemma 3.9, we know $c^- := \inf_{(u,v)\in\mathcal{N}^-} E(u,v)$ is attained. Let $\{(u_n,v_n)\}_n \subset \mathcal{N}^-$ be a minimizing sequence such that $E(u_n,v_n) \to c^-$. Then

$$\|(u_n,v_n)\|^p - \lambda\int_\Omega h_1(x)|u_n|^\theta dx - \mu\int_\Omega h_2(x)|v_n|^\theta dx = \int_\Omega (b_1+b_2)(x)|u_n|^q|v_n|^r dx, \tag{5.16}$$

and,

$$0 > \lambda(p-\theta)\int_\Omega h_1(x)|u_n|^\theta\ln|u_n|dx + \mu(p-\theta)\int_\Omega h_2(x)|v_n|^\theta\ln|v_n|dx$$
$$-\lambda(p-\theta)\int_\Omega h_1(x)|u_n|^\theta dx - \mu(p-\theta)\int_\Omega h_2(x)|v_n|^\theta dx$$
$$-(q+r-p)\int_\Omega (b_1+b_2)(x)|u_n|^q|v_n|^r dx. \tag{5.17}$$

We claim that $\{(u_n,v_n)\}_n$ is bounded in $\mathcal{N}^-$.

Without loss of generality, we assume that $\|(u_n,v_n)\| \geq 1$. Then

$$c^- + o_n(1) = E(u_n,v_n) - \frac{1}{p}E'(u_n,v_n)(u_n,v_n)$$
$$\left(\frac{1}{p}-\frac{1}{\theta}\right)\left(\lambda\int_\Omega h_1(x)|u_n|^\theta\ln|u_n|dx + \mu\int_\Omega h_2(x)|v_n|^\theta\ln|v_n|dx\right)$$
$$+\frac{1}{\theta^2}\left(\lambda\int_\Omega h_1(x)|u_n|^\theta dx + \mu\int_\Omega h_2(x)|v_n|^\theta dx\right)$$
$$+\left(\frac{1}{p}-\frac{1}{q+r}\right)\int_\Omega (b_1+b_2)(x)|u_n|^q|v_n|^r dx. \tag{5.18}$$

So

$$c^- + o_n(1) \geq \left(\frac{1}{p} - \frac{1}{\theta}\right)\left(\lambda \int_\Omega h_1(x)|u_n|^\theta \ln|u_n|dx + \mu \int_\Omega h_2(x)|v_n|^\theta \ln|v_n|dx\right) + \frac{1}{\theta^2}\left(\lambda \int_\Omega h_1(x)|u_n|^\theta dx + \mu \int_\Omega h_2(x)|v_n|^\theta dx\right). \tag{5.19}$$

If $\lambda \int_\Omega h_1(x)|u_n|^\theta dx + \mu \int_\Omega h_2(x)|v_n|^\theta dx \geq 0$, then from equations (5.18), (5.19), we have

$$\|(u,v)\| \leq \left(\frac{1}{\frac{1}{p} - \frac{1}{\theta}(\lambda C_{h_1} + \mu C_{h_2})L}\right)^{\frac{1}{p-2}}.$$

If, $\lambda \int_\Omega h_1(x)|u_n|^\theta dx + \mu \int_\Omega h_2(x)|v_n|^\theta dx < 0$, from equation (5.19), we have

$$\begin{aligned} c^- + o_n(1) &= E(u_n, v_n) \\ &\geq \left[\frac{1}{p} - \frac{1}{\theta}(\lambda C_{h_1} + \mu C_{h_2})L\right]\|(u_n,v_n)\|^p \\ &\quad + \left[\frac{1}{\theta^2} - \left(\frac{1}{\theta} - \frac{1}{q+r}\right)\ln\|(u_n,v_n)\|\right]\left(\lambda \int_\Omega h_1(x)|u_n|^\theta dx + \mu \int_\Omega h_2(x)|v_n|^\theta dx\right). \end{aligned}$$

We have two cases to analyze:

i) If $\frac{1}{\theta^2} - \left(\frac{1}{\theta} - \frac{1}{q+r}\right)\ln\|(u_n,v_n)\| \leq 0$, we have

$$c^- + o_n(1) = \left(\frac{1}{p} - \frac{1}{q+r}\right)[1 - (\lambda C_{h_1} + \mu C_{h_2})L]\|(u_n,v_n)\|^p,$$

which implies that

$$\|(u_n,v_n)\| \leq \left[\frac{p(q+r)(c^- + o_n(1))}{(q+r-p)[1-(\lambda C_{h_1} + \mu C_{h_2})L]}\right]^{\frac{1}{p}}.$$

ii) If $\frac{1}{\theta^2} - \left(\frac{1}{\theta} - \frac{1}{q+r}\right)\ln\|(u_n,v_n)\| > 0$, then $\|(u_n,v_n)\| < \exp\left(\frac{q+r-\theta}{\theta^3(q+r)}\right)$. So there exists $C > 0$ such that $\|(u_n,v_n)\| < C$.

Thus, every minimizer sequence of E on $\mathcal{N}^-$ is bounded.
Now, since $\{(u_n,v_n)\}$ is bounded in $\mathcal{N}^-$, up to a subsequence, we may assume that

$$\begin{aligned} &(u_n,v_n) \rightharpoonup (u_-,v_-) && \text{weakly in } W; \\ &(u_n,v_n) \to (u_-,v_-) && \text{strongly in } L^\nu(\Omega)\times L^\tau(\Omega); \\ &(u_n,v_n) \to (u_+,v_+) && \text{a.e in } \Omega, \end{aligned}$$

where $1 < \nu, \tau < \min\{p_s^*, p_t^*\}$. It implies that

$$\begin{aligned} \lim_{n\to\infty}\int_\Omega h_1(x)|u_n|^\theta \ln|u_n|dx &= \int_\Omega h_1(x)|u_-|^\theta \ln|u_-|dx; \\ \lim_{n\to\infty}\int_\Omega h_2(x)|v_n|^\theta \ln|v_n|dx &= \int_\Omega h_2(x)|v_-|^\theta \ln|v_-|dx; \\ \lim_{n\to\infty}\int_\Omega h_1(x)|u_n|^\theta dx &= \int_\Omega h_1(x)|u_-|^\theta dx; \\ \lim_{n\to\infty}\int_\Omega h_1(x)|v_n|^\theta dx &= \int_\Omega h_2(x)|v_-|^\theta dx. \end{aligned}$$

An easy computation, combined with equation (5.17), shows that

$$I(u_-, v_-) \leq \liminf_{n\to\infty} I(u_n, v_n) = 0.$$

It means that $(u_-, v_-) \in \mathcal{N}^-$. Now, by Lemma 3.3 (1), there exists $k_{(u,v)} > 0$ such that $I'_{(u_-,v_-)}k_{(u_-,v_-)} = 0$ and $k_{(u_-,v_-)} \neq 1$. Since $(u_n, v_n) \not\to (u_-, v_-)$, we get

$$k_{(u_-,v_-)}(u_n, v_n) \not\to k_{(u_-,v_-)}(u_-, v_-)$$

in W. Another easy computation shows that

$$E(k_{(u_-,v_-)}u_-, k_{(u_-,v_-)}v_-) \leq E(k_{(u_-,v_-)}u_n, k_{(u_-,v_-)}v_n).$$

Now, observe that the function $z(k) := E(ku_n, kv_n)$ attains its maximum at $k = k_{(u_-,v_-)}$. So

$$\begin{aligned} E(k_{(u_-,v_-)}u_-, k_{(u_-,v_-)}v_-) &< \liminf_{n\to\infty} E(k_{(u_-,v_-)}u_n, k_{(u_-,v_-)}v_n) \\ &\leq \lim_{n\to\infty} E(u_n, v_n) \\ &= \inf_{(u,v)\in\mathcal{N}^-} E(u, v), \end{aligned}$$

a contradiction. Thus $(u_n, v_n) \to (u_-, v_-)$ in W.

Because $(u_-, v_-) \in \mathcal{N}^-$, then (u_-, v_-) is a minimizer of E on $\mathcal{N}^-$. Moreover, a similar discussion as Theorem 5.1 - *step* 2- one can show that $(|u_-|, |v_-|)$ is a minimizer of E on $\mathcal{N}^-$. This yields the proof. □

Now it is possible to prove Theorem 1.2.

5.1 Proof of Theorem 1.2

From Theorems 5.1 and 5.2, E has two non-negative minimizers $(u_+, v_+) \in \mathcal{N}^+$ and $(u_-, v_-) \in \mathcal{N}^-$. Then, from Theorem 3.2, E has two non-negative critical points on W, which is non-trivial and non-negative local least energy solution of Problem (P). This two solution are distinct, because is obviously that $\mathcal{N}^- \cap \mathcal{N}^+ = \emptyset$.

We claim that (u_+, v_+) and (u_-, v_-) are not semi-trivial solution.

Supposing, otherwise, $v_t = 0$ in (u_+, v_+), we get that u_+ is a non-trivial solution of the problem

$$\begin{cases} (-\Delta)^s_p u + |u|^{p-2}u = \lambda h_1(x)|u|^{\theta-2}u \ln|u| & \text{in } \Omega, \\ (u, v) \in W_0^{s,p}(\Omega) \times W_0^{t,p}(\Omega) \end{cases} \tag{P'}$$

Then $\|(u_+, 0)\|^p = [u_+]^p_s = \lambda \int_\Omega h_1(x)|u_+|^\theta \ln|u_+|dx$, and because $(u_+, 0) \in \mathcal{N}^+$ and $\Phi''_{(u,v)}(1) > 0$, we get

$$\lambda(p-\theta)\int_\Omega h_1(x)|u_+|^\theta dx < \lambda \int_\Omega h_1(x)|u_+|^\theta dx.$$

Because $p < \theta$, we have $\int_\Omega h_1(x)|u_+|^\theta dx < 0$.

Now we choose $w \in W_0^{t,p}(\Omega) \setminus \{0\}$ such that $\int_\Omega h_1(x)|w|^\theta dx < 0$.

For (u_+, w), by Lemma 3.3, there exists a unique $k_1 > 0$ such that $k_1(u_+, w) \in \mathcal{N}^+$. Moreover, we have

$$k_{\max} = \exp\left(\frac{\|(u_+, w)\|^p - \lambda\int_\Omega h_1(x)|u_+|^\theta \ln|u_+|^\theta dx + \mu\int_\Omega h_2(x)|w|^\theta \ln|w|^\theta dx}{\lambda\int_\Omega h_1(x)^\theta dx + \mu\int_\Omega h_2(x)|w|^\theta dx} - \frac{1}{p-q-r}\right),$$

and

$$E(k_1u_+, k_1w) = \inf_{0<k<k_{\max}} E(ku_+, kw).$$

It follows that

$$c^+ \leq E(k_1u_+, k_1w) < E(u_+, w) < E(u_+, 0) = c^+,$$

a contradiction. Thus (u_+, v_+) is not a semi-trivial solution for problem (P).

Otherwise, (u_-, v_-) is not a semi-trivial solution for problem (P), by using the same above argument, but this time assuming $v_- = 0$. In this case $(u_-, 0)$ is a nontrivial solution for problem (P') and $\int_\Omega h_1(x)|u_-|^\theta dx > 0$ and $w \in W_0^{t,p}(\Omega) \setminus \{0\}$ is taking such that

$$\lambda \int_\Omega h_1(x)|u_-|^\theta dx + \mu \int_\Omega h_2(x)|w|^\theta dx < 0.$$

In this way we concluded what we wanted.

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Existence of solutions for discontinuous systems of Stieltjes differential equations

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Abstract. We introduce new transversality conditions in terms of derivatives with respect to a nondecreasing function $g : \mathbb{R} \longrightarrow \mathbb{R}$, and we use them to prove the existence of solutions for discontinuous systems of Stieltjes differential equations, i.e. differential equations where usual derivatives are replaced with derivatives with respect to the derivator g. The proof of our main existence result leans on some interesting new results on g-derivatives also proven in this paper.

Keywords: discontinuous differential equations, Carathéodory solutions, Stieltjes differential equations.

2020 Mathematics Subject Classification: 34A06, 34A36.

1 Introduction

We are concerned with initial value problems for first order systems of Stieltjes differential equations of the form

$$x'_g = f(t,x), \quad t \in I = [t_0, t_0 + L), \ x(t_0) = x_0, \tag{1.1}$$

where $g : \mathbb{R} \longrightarrow \mathbb{R}$ is a nondecreasing function, $t_0, L \in \mathbb{R}$, $L > 0$, $x_0 \in \mathbb{R}^n$ ($n \in \mathbb{N}$) and $f = (f_1, f_2, \ldots, f_n) : I \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ need not be continuous.

As usual, x'_g stands for the derivative with respect to g which, roughly speaking, means

$$x'_g(t) = \lim_{s \to t} \frac{x(s) - x(t)}{g(s) - g(t)},$$

and it allows us to study difference and differential equations, with or without impulses, under the unified formulation of (1.1).

Existence of Carathéodory type solutions for (1.1) has recently been well studied for the case when $f(t,x)$ is continuous with respect to the x variable. See [5, 6, 11–14]. In this paper,

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we intend to improve on the previous references by allowing the nonlinear part $f(t,x)$ to present many discontinuities. To do so, we show that discontinuities do not matter as long as they can be covered by some subsets of $I \times \mathbb{R}^n$ satisfying a novel transversality condition with respect to the g-differential equation in (1.1).

Our new transversality condition, inspired by an idea by Bressan and Shen [2], later generalized in [10], leans on some new results on g-differentiation which are interesting in its own right. We manage to translate the ideas the authors brought in [10] from usual differential equations to Stieltjes differental equations, generalising their results even further. We achieve this by introducing some new interesting properties on the g-derivatives.

This paper is organized as follows. In Section 2 we include the basic preliminaries on g-differential equations so that this paper be self-contained; in Section 3 we prove some new results on g-differentiation, specially, on a chain rule for functions of several variables which is essential for our definition of local transversality conditions; in Section 4 we prove our main existence result for (1.1).

2 Preliminaries on Stieltjes derivatives and differential equations

In what follows, $g : \mathbb{R} \longrightarrow \mathbb{R}$ is monotone, nondecreasing and continuous from the left everywhere. Let us consider the set of points around which g is constant, namely

$$C_g = \{t \in \mathbb{R} \,:\, g \text{ is constant on } (t-\varepsilon, t+\varepsilon) \text{ for some } \varepsilon > 0\}, \tag{2.1}$$

and the set of discontinuity points of g, which is at most countable and shall be denoted by

$$D_g = \{t \in \mathbb{R} \,:\, g(t^+) - g(t) > 0\}, \tag{2.2}$$

where, as usual, $g(t^+) = \lim_{s \to t^+} g(s) \in \mathbb{R}$.

Following [11], we define the *derivative with respect to* g (or *g-derivative*) of a real-valued real function f at a point $t_0 \in \mathbb{R} \setminus C_g$ as follows, provided that the corresponding limit exists:

$$f'_g(t_0) = \lim_{t \to t_0} \frac{f(t) - f(t_0)}{g(t) - g(t_0)} \quad \text{if } t_0 \notin D_g, \text{ or} \tag{2.3}$$

$$f'_g(t_0) = \lim_{t \to t_0^+} \frac{f(t) - f(t_0)}{g(t) - g(t_0)} \quad \text{if } t_0 \in D_g, \tag{2.4}$$

where C_g and D_g are defined in (2.1) and (2.2) respectively. Notice that the g-derivative at a point $t_0 \in D_g$ exists if, and only if, $f(t_0^+)$ exists and (2.4) can be rewritten as

$$f'_g(t_0) = \frac{f(t_0^+) - f(t_0)}{g(t_0^+) - g(t_0)}. \tag{2.5}$$

We say that f is *g-differentiable at t_0* if $f'_g(t_0)$ exists, and we say that f is *g-differentiable in a set $A \subset \mathbb{R}$* when f is g-differentiable at every $t_0 \in A \setminus C_g$. There will be no need to define g-derivatives for the points in C_g because, according to [10, Proposition 2.5], we have $\mu_g(C_g) = 0$, where μ_g stands for the Lebesgue–Stieltjes measure induced by g. However, it is possible to give a reasonable definition of a g-derivative everywhere, see [13], but we shall not need it in this paper.

Roughly speaking, g-differentiable functions can be recovered by integrating their g-derivatives in the Stieltjes sense with respect to g. More precisely, we have the following Fundamental Theorem of Calculus for the Lebesgue–Stieltjes integral [10, Theorem 5.4].

Theorem 2.1 (Fundamental Theorem of Calculus for the Lebesgue–Stieltjes integral)**.** *Let $a, b \in \mathbb{R}$, $a < b$, and $F : [a, b] \longrightarrow \mathbb{R}$. The following conditions are equivalent.*

(1) The function F is absolutely continuous with respect to g *(or g-*absolutely continuous*) according to the following definition: to each $\varepsilon > 0$, there is some $\delta > 0$ such that, for any family $\{(a_n, b_n)\}_{n=1}^m$ of pairwise disjoint open subintervals of $[a, b]$, the inequality*

$$\sum_{n=1}^{m} (g(b_n) - g(a_n)) < \delta$$

implies

$$\sum_{n=1}^{m} |F(b_n) - F(a_n)| < \varepsilon.$$

(2) The function F fulfills the following properties:

(a) there exists $F'_g(t)$ for g-almost all $t \in [a, b)$ (i.e., for all t except on a set of μ_g measure zero);

(b) $F'_g \in \mathcal{L}^1_g([a, b))$, the set of Lebesgue–Stieltjes integrable functions with respect to μ_g;

(c) for each $t \in [a, b]$, we have

$$F(t) = F(a) + \int_{[a,t)} F'_g(s)\, d\mu_g. \tag{2.6}$$

We remark that $\mu_g(\{t\}) = g(t^+) - g(t) > 0$ when $t \in D_g$ (see equation (6) in [10]). In particular, if $t \in [a, b) \cap D_g$, then, (2.6) implies that

$$F(t^+) - F(t) = \int_{\{t\}} F'_g(s)\, d\mu_g = F'_g(t)(g(t^+) - g(t)),$$

which may be nonzero. Therefore, g-absolutely continuous functions need not be continuous at discontinuity points of g.

However, g-absolutely continuous functions have some nice properties.

Proposition 2.2 ([10, Proposition 5.3])**.** *If F is g-absolutely continuous on $[a, b]$, then it has bounded variation and it is continuous from the left at every $x \in [a, b)$.*

Moreover, F is continuous in $[a, b] \setminus D_g$, where D_g is the set of discontinuity points of g, and if g is constant on some $(\alpha, \beta) \subset [a, b]$, then F is constant on (α, β) as well.

We say that a vector-valued function $F : [a, b] \longrightarrow \mathbb{R}^n$ is g-absolutely continuous on $[a, b]$ if each of its components is g-absolutely continuous. We denote by $\mathcal{AC}_g([a, b])$ the set of g-absolutely continuous functions on $[a, b]$ with values in $\mathbb{R}^n$.

Definition 2.3. A (Carathéodory) *solution* of (1.1) is a g-absolutely continuous function $x : [t_0, t_0 + L] \longrightarrow \mathbb{R}^n$ such that $x(t_0) = x_0$, and

$$x'_g(t) = f(t, x(t)) \quad \text{for } g\text{-almost all } (g\text{-a.a.}) \ t \in [t_0, t_0 + L).$$

Next we introduce a useful splitting of g into its continuous and discrete parts. This is well known for bounded variation functions on bounded intervals, see, for instance, [1, Theorem 2.6.1], but we have to adjust it for functions defined on the whole real line.

Denote

$$\Delta g(t) = g(t^+) - g(t),$$

for all $t \in \mathbb{R}$ and define the function $g^B : \mathbb{R} \to \mathbb{R}$ as follows:

$$g^B(t) = \begin{cases} \sum_{s\in[0,t)} \Delta g(s), & t > 0, \\ -\sum_{s\in[t,0)} \Delta g(s), & t \leq 0. \end{cases} \tag{2.7}$$

We have that g^B is nondecreasing and left-continuous. We will say that g^B is the *discontinuous* or *jump part* of g. We say g is *totally discontinuous* when $g = g^B \neq 0$. We define the *continuous part* of g as follows:

$$g^C(t) := g(t) - g^B(t), \quad \forall t \in \mathbb{R}.$$

Thus, g^C is nondecreasing and continuous (in the usual sense).

By definition we have that $g = g^C + g^B$. In particular we have, over Borel sets, that $\mu_g = \mu_{g^C} + \mu_{g^B}$. It holds that

$$g^C(s) - g^C(t) \leq g(s) - g(t)$$
$$g^B(s) - g^B(t) \leq g(s) - g(t)$$

for all $t, s \in \mathbb{R}$, $t < s$.

Now we recall the notion of g-topology in $\mathbb{R}$, introduced for the first time in [6].

Definition 2.4. Given any $t \in \mathbb{R}$ and $\varepsilon > 0$, we define the g-ball of center t and radius ε as the set

$$B_g(t, \varepsilon) := \{s \in \mathbb{R} \mid |g(s) - g(t)| < \varepsilon\}.$$

Define the g-topology as

$$\tau_g := \{\ U \subset \mathbb{R} \ \mid\ \forall t \in U\ \ \exists \varepsilon > 0 :\ B_g(t, \varepsilon) \subset U\ \}.$$

For any $\lambda > 0$ and $B_g(t, \varepsilon)$ denote $\lambda B_g(t, \varepsilon) = B_g(t, \lambda\varepsilon)$. Denote also $B_g[t, \varepsilon]$ as the closed ball

$$B_g[t, \varepsilon] = \{s \in \mathbb{R} \mid |g(s) - g(t)| \leq \varepsilon\},$$

which need not be a closed set in the usual topology (for instance, $B_g(1, 1/2) = (0, \infty)$ if $g(t) = 0$ for $t \leq 0$ and $g(t) = 1$ for $t > 0$).

Finally, since C_g is open, we can express it as a countable union of disjoint open intervals

$$C_g = \bigcup_{n=1}^{\infty} (a_n, b_n),$$

and we denote by N_g the set of the real endpoints of all the intervals (a_n, b_n). The following technical information will be used in Section 3.

Proposition 2.5. *Let $t \in \mathbb{R}$ be such that $t \notin D_g \cup N_g \cup C_g$. Then any set N is a usual neighborhood of t iff N is a neighborhood of t in the topology τ_g.*

Proof. First, note that $\mathbb{R} - (C_g \cup N_g) = \{t \in \mathbb{R} : g(s) \neq g(t)\ \forall s \in \mathbb{R},\ s \neq t\}$. Since $t \notin D_g$, g is continuous at t. Hence,

$$\forall \epsilon > 0\ \exists \delta > 0 : (t - \delta, t + \delta) \subset B_g(t, \epsilon).$$

Fix $\epsilon > 0$ now, take $\delta = \min\{|g(t) - g(t - \epsilon)|, |g(t + \epsilon) - g(t)|\} > 0$, thus

$$B_g(t, \delta) \subset (t - \epsilon, t + \epsilon).$$

Equivalently, the function Id is g-continuous at t. In fact, g-continuity at t is equivalent to usual continuity at t. Every local basis of t (usual or on τ_g) is a local basis on both topologies. The claim is then proved. □

Remark 2.6. Let f be a function defined on a neighborhood $(t-\epsilon, t+\epsilon)$ of $t \notin D_g \cup N_g \cup C_g$. If $f'_g(t)$ exists, then f is continuous at t.

To prove it, simply observe that there exists $\delta > 0$ such that

$$|f(t) - f(s)| \leq (|f'_g(t)| + 1)|g(t) - g(s)| \quad \forall s \in (t-\delta, t+\delta),$$

which implies that f is continuous at t because so g is.

3 Covering lemmas and fine results on g-derivatives

This section is devoted to proving some technical results on g-null sets and g-derivatives which will play a fundamental role in the proof of our main existence result for (1.1).

We start with the following lemma on existence of special subcollections of arbitrary collections of sets.

Lemma 3.1. *Let X be a nonempty set and let $\mathcal{B}$ be a collection of subsets of X containing at least one nonempty set. Then, there exists a subcollection of subsets $\mathcal{C} \subset \mathcal{B}$ satisfying the following conditions:*

(1) $A \cap B = \emptyset$, for all $A, B \in \mathcal{C}$, $A \neq B$.

(2) For any $B \in \mathcal{B}$, there exists some $C \in \mathcal{C}$ such that $B \cap C \neq \emptyset$.

Proof. Let Ω be the set of all disjoint subcollections of $\mathcal{B}$ equipped with the partial order $\subset$. Clearly $\Omega \neq \emptyset$ as $\{B\} \in \Omega$ for any $B \in \mathcal{B}$. Let Λ be a nonempty chain in Ω. If it were to happen that

$$\mathcal{D} = \bigcup_{\mathcal{A} \in \Lambda} \mathcal{A} \in \Omega,$$

then $\mathcal{D}$ would be an upper bound of Λ. Let us prove the above. Clearly, $\mathcal{D} \subset \mathcal{B}$ since $\mathcal{A} \subset \mathcal{B}$ for all $\mathcal{A} \in \Lambda$. Besides, for any $A, B \in \mathcal{D}$, $A \neq B$, there exists some $\mathcal{A} \in \Lambda$ such that $A, B \in \mathcal{A}$, therefore, $A \cap B = \emptyset$. Hence, $\mathcal{D} \in \Omega$. By Zorn's lemma there exists some maximal element $\mathcal{C} \in \Omega$. $\mathcal{C}$ satisfies (1) and (2). □

As we will prove now, we can recover the Vitali Covering Theorem using the balls defined with the derivator. Our proof is basically identical to [4, Theorem 1.24], since the classical proof holds for the Stieltjes case as well.

Theorem 3.2 (The Stieltjes–Vitali Covering Theorem)**.** *Let Ω be an arbitrary collection of balls such that*

$$R = \sup\{\, \mathrm{rad}(B_g) \ : \ B_g \in \Omega \,\} < \infty$$

where $\mathrm{rad}(B_g)$ denotes the radius of B_g. Then, there exists a countable disjoint subcollection $\Lambda \subset \Omega$ of balls such that

$$\bigcup_{B_g \in \Omega} B_g \subset \bigcup_{A_g \in \Lambda} 5A_g.$$

Besides, each $B_g \in \Omega$ intersects some $A_g \in \Lambda$ such that $B_g \subset 5A_g$.

Proof. Denote

$$F_j = \left\{ B_g \in \Omega \ : \ \mathrm{rad}(B_g) \in \left(\frac{R}{2^{j+1}}, \frac{R}{2^j} \right] \right\} \quad \text{for} \quad j = 0, 1, 2, \ldots.$$

We define Λ_n as follows:

(1) Let Λ_0 be any maximal disjoint subcollection of F_0, which exists by Lemma 3.1.

(2) Suppose $\Lambda_0, \dots, \Lambda_{n-1}$ are already defined, define Λ_n to be the maximal disjoint subcollection of balls in

$$\left\{ B_g \in F_n \ : \ B_g \cap A_g = \emptyset, \ \forall A_g \in \bigcup_{k=0}^{n-1} \Lambda_k \right\}$$

given by Lemma 3.1.

Define now

$$\Lambda = \bigcup_{n=0}^{\infty} \Lambda_n.$$

By construction, Λ is a disjoint subcollection of balls in Ω, and hence, countable, since τ_g is second countable [3, Proposition 2.5]. Take any $B_g \in \Omega$, there exist some index $n \geq 0$ such that $B_g \in F_n$. There must exist then some $A_g \in \cup_{k=0}^{n} \Lambda_k$ such that $B_g \cap A_g \neq \emptyset$. Hence, $\operatorname{rad}(B_g) \leq 2 \operatorname{rad}(A_g)$. Let $x \in B_g \cap A_g$ and x_1, x_2 be the centers of B_g and A_g respectively. Then, for every, $y \in B_g$,

$$d_g(y, x_2) \leq d_g(y, x_1) + d_g(x_1, x) + d_g(x, x_2) \leq 2 \operatorname{rad}(B_g) + \operatorname{rad}(A_g) \leq 5 \operatorname{rad}(A_g).$$

Thus, $B_g \subset 5A_g$. □

Since g is nondecreasing, every ball B_g is connected and thus an interval on $\mathbb{R}$. Fix any $t \in \mathbb{R}$ and $\varepsilon > 0$ and consider the closed ball $B_g[t, \varepsilon]$. Assume $B_g[t, \varepsilon]$ to be bounded, take $a = \inf B_g[t, \varepsilon]$ and $b = \sup B_g[t, \varepsilon]$. Hence

$$(a, b) \subset B_g[t, \varepsilon]. \tag{3.1}$$

Recall that B_g must be non degenerate since g is left-continuous, so $a \neq b$. We have that,

$$g(s) \leq g(t) + \varepsilon$$

for all $s < b$, hence $b \in B_g$ by left-continuity. Now, a may or may not be on $B_g[t, \varepsilon]$ depending whether $\Delta g(a)$ equals zero or not. In any case

$$\mu_g(B_g[t, \varepsilon]) = \mu_g((a, b]) = g(b^+) - g(a^+) = \Delta g(b) + g(b) - g(a^+) \leq \Delta g(b) + 2\varepsilon.$$

Note that, since g^C is continuous, singletons have zero measure. So,

$$\mu_{g^C}(B_g[t, \varepsilon]) = \mu_{g^C}((a, b]) = \mu_{g^C}((a, b)) = g^C(b) - g^C(a^+) \leq g(b) - g(a^+) \leq 2\varepsilon. \tag{3.2}$$

Lemma 3.3. *Let $\Lambda = \{ B_g(t_k, \varepsilon_k) \ : \ k = 1, \dots, n \}$ be a finite disjoint collection of open balls. Assume there exists some $t \in \mathbb{R}$ and $\varepsilon > 0$ such that*

$$\bigcup_{B_g \in \Lambda} B_g \subset B_g(t, \varepsilon).$$

Then, $\varepsilon > \frac{1}{4} \sum_{k=1}^{n} \varepsilon_k$.

Proof. Let us look first at the case $n = 2$. Suppose then we have two balls of center t_1 and t_2 and radius ε_1 and ε_2 respectively. Since the balls are disjoint, we have that $t_2 \notin B_g(t_1, \varepsilon_1)$ and $t_1 \notin B_g(t_2, \varepsilon_2)$. Hence

$$\max\{\varepsilon_1, \varepsilon_2\} \leq |g(t_2) - g(t_1)|.$$

Go back to the general case and assume without loss of generality that the center of the balls $t_1, \ldots, t_n$ are ordered from left to right. We have that

$$\begin{aligned} g(t_n) - g(t_1) &= \sum_{k=1}^{n-1} g(t_{k+1}) - g(t_k) \geq \sum_{k=1}^{n-1} \max\{\varepsilon_{k+1}, \varepsilon_k\} \geq \sum_{k=1}^{n-1} \frac{\varepsilon_{k+1} + \varepsilon_k}{2} \\ &\geq \sum_{k=2}^{n-1} \varepsilon_k + \frac{1}{2}\varepsilon_1 + \frac{1}{2}\varepsilon_n \geq \frac{1}{2} \sum_{k=1}^{n} \varepsilon_k. \end{aligned}$$

Now $t_1, t_n \in B_g(t, \varepsilon)$, hence

$$\frac{1}{2} \sum_{k=1}^{n} \varepsilon_k \leq g(t_n) - g(t_1) < 2\varepsilon. \qquad \square$$

Remark 3.4. The bounds

$$\frac{\varepsilon_1 + \varepsilon_2}{2} \leq \max\{\varepsilon_1, \varepsilon_2\}, \quad \frac{1}{2} \sum_{k=1}^{n} \varepsilon_k \leq \sum_{k=2}^{n-1} \varepsilon_k + \frac{1}{2}\varepsilon_1 + \frac{1}{2}\varepsilon_n$$

made on Lemma 3.3 are just arbitrary and of course suboptimal. We wish for our bounds just to be comfortable to manage with series as this next result is the whole purpose of the above.

Corollary 3.5. *Let $\Lambda = \{ B_g(t_k, \varepsilon_k) \mid k \in \mathbb{N}\}$ be a countable disjoint collection of open balls. If*

$$\bigcup_{B_g \in \Lambda} B_g \subset B_g(t, \varepsilon)$$

for some $t \in \mathbb{R}$ and $\varepsilon > 0$, the series $\sum_{k=1}^{\infty} \varepsilon_k$ converges.

Theorem 3.6. *Let X be a nonempty subset of $\mathbb{R}$. Let Ω be a collection of closed balls whose union covers X and satisfies the following property:*

For all $t \in X$ and $\delta > 0$, there exists some $A_g \in \Omega$ such that $A_g \subset B_g(t, \delta)$.

Then there exists a countable disjoint subcollection $\Lambda \subset \Omega$ which covers X up to a g^C-null measure set. In other words, such that

$$\mu_{g^C}\left(X - \bigcup_{B_g \in \Lambda} B_g \right) = 0.$$

Proof. Without loss of generality assume all balls in Ω have radius less than 1. Denote as Λ the countable disjoint subcollection given by Theorem 3.2. We show now that

$$E = X - \bigcup_{B_g \in \Lambda} B_g$$

has null g^C-measure. Take $E_n = E \cap B_g(0, n)$ for $n \in \mathbb{N}$, is enough to show that $\mu_{g^C}(E_n) = 0$ for all $n \in \mathbb{N}$. Denote $\Lambda_n = \{ B_g^k \equiv B_g[t_k, \varepsilon_k] \mid k \in \mathbb{N}\}$ the subcollection of balls of Λ that meet $B_g(0, n)$. Let $t \in E_n$ and $m \in \mathbb{N}$, since $K = \bigcup_{k=0}^{m} B_g^k$ is closed in τ_g, there exists some $A_g \in \Omega$

containing t, contained in $B_g(0,n)$ and disjoint from K. By the property of Λ, A_g intersects some $B_g^i \in \Lambda_n$ for which $A_g \subset 5B_g^i$. This means, $i > m$ and therefore,

$$E_n \subset \bigcup_{k>m} 5B_g^k.$$

This gives the inequality

$$\mu_{g^C}^*(E_n) \leq \mu_{g^C}\left(\bigcup_{k>m} 5B_g^k\right) \leq \sum_{k>m} \mu_{g^C}(5B_g^k) \tag{3.3}$$

for all $m \in \mathbb{N}$. Now, the balls Λ_n are all contained in $B_g(0,n+2)$. From Corollary 3.5, $\sum_{k=1}^{\infty} \varepsilon_k < \infty$, where $\varepsilon_k = \operatorname{rad}(B_g^k)$. Hence, by equation (3.2)

$$\sum_{k=1}^{\infty} \mu_{g^C}(5B_g[t_k,\varepsilon_k]) \leq 10 \sum_{k=1}^{\infty} \varepsilon_k < \infty.$$

From (3.3), $\mu_{g^C}^*(E_n) = 0$. □

Theorem 3.7. *Let $f : [a,b] \longrightarrow \mathbb{R}$ be an arbitrary function and E the set of points of $[a,b]$ at which f has nonzero g-derivative (ie. the g-derivative exists and takes nonzero values). Then, for every null Lebesgue measure set Z, the set $f^{-1}(Z) \cap E$ has null g^C-measure.*

Proof. It suffices to prove the result for the set of points where the g-derivative is positive, which we shall denote again by E. The result for the set of points where the g-derivative is negative follows from the other one applied to the function $-f$ and the null set $-Z$.

Moreover, since $\mu_{g^C}(N_g \cup C_g \cup D_g \cup \{a,b\}) = 0$, we can redefine E to be

$$E = \{t \in (a,b) : f_g'(t) \text{ exists and is positive, } g \text{ is continuous at } t \text{ and } g(s) \neq g(t)\ \forall s \in \mathbb{R},\ s \neq t\}.$$

We can express

$$E = \bigcup_{n=1}^{\infty} \left\{ t \in E \,:\, f_g'(t) > \frac{1}{n} \right\},$$

so it suffices to prove that, for any $\delta > 0$, the set

$$E_\delta = \left\{ t \in E \,:\, f_g'(t) > \delta \right\}$$

has a g^C-null intersection with $f^{-1}(Z)$, for any Z as in the statement.

For a fixed $\delta > 0$ and $q, p \in \mathbb{Q} \cap (a,b)$, $q < p$, we define

$$A_{q,p} = \left\{ t \in E \cap (q,p) : \frac{f(t) - f(s)}{g(t) - g(s)} > \delta,\ \forall s \in (q,p),\ s \neq t \right\},$$

and we have

$$E_\delta \subset \bigcup_{\substack{q,p \in \mathbb{Q} \cap (a,b) \\ q<p}} A_{q,p}.$$

This reduces the proof to showing that for each fixed null Lebesgue measure set Z, for each fixed $\delta > 0$ and for each fixed pair $q, p \in \mathbb{Q} \cap (a,b)$, $q < p$, we have $\mu_{g^C}(f^{-1}(Z) \cap A_{q,p}) = 0$.

First, observe that for $t_1, t_2 \in A_{q,p}$, $t_1 < t_2$, we have

$$\frac{f(t_1)-f(t_2)}{g(t_1)-g(t_2)} > \delta \Rightarrow f(t_1)-f(t_2) < \delta(g(t_1)-g(t_2)) < 0 \Rightarrow f(t_1) < f(t_2),$$

so f is increasing on $A_{q,p}$.

Let $\epsilon > 0$ be fixed and let U be a usual open set of measure less than ϵ that contains Z.

The definition of $A_{q,p}$ guarantees that for any $t \in A_{q,p}$ we can use Proposition 2.5 to ensure that the g-balls $B_g[t,r]$, $r > 0$, are usual neighborhoods of t and, by Remark 2.6, that f is continuous at t. Moreover, $B_g[t,r]$ shrinks to $\{t\}$ as r tends to zero. Therefore, for each $t \in f^{-1}(Z) \cap A_{q,p}$ there exists $r(t) > 0$ such that the usual closure $\overline{B_g[t,r(t)]} \subset (q,p)$ and $f\left(\overline{B_g[t,r(t)]}\right) \subset U$.

Define Ω as the collection of all closed g-balls $B_{t,k} = B_g[t, r(t)/k]$, $t \in f^{-1}(Z) \cap A_{q,p}$, $k \in \mathbb{N}$. Plainly, $f^{-1}(Z) \cap A_{q,p} \subset \bigcup_{t,k} B_{t,k}$ and, moreover, for each $t \in f^{-1}(Z) \cap A_{q,p}$ and each $\rho > 0$ there exists $k \in \mathbb{N}$ such that $B_{t,k} \subset B_g(t,\rho)$. By Vitali's Theorem 3.6 there exists a countable and disjoint subfamily of Ω that covers $f^{-1}(Z) \cap A_{q,p}$ up to a null g^C-measure set. Denote this subcollection as $\{B_g[t_n, r_n] : n \in \mathbb{N}\}$ and put

$$[a_n, b_n] = \overline{B_g[t_n, r_n]}, \quad n \in \mathbb{N}.$$

Since $[a_n, b_n] \subset (q,p)$ and $t_n \in A_{q,p}$,

$$\begin{aligned} f(b_n)-f(a_n) &= (f(b_n)-f(t_n)) + (f(t_n)-f(a_n)) \\ &> \delta((g(b_n)-g(t_n)) + (g(t_n)-g(a_n))) = \delta(g(b_n)-g(a_n)), \end{aligned} \tag{3.4}$$

and, moreover, $[f(a_n), f(b_n)] \subset U$.

Now, take $m, k \in \mathbb{N}$ such that $m \neq k$. Since the intervals $[t_m, b_m]$ and $[t_k, b_k]$ are disjoint we can suppose that $b_m < t_k$. Therefore,

$$f(t_m) < f(b_m) < f(t_k) < f(b_k)$$

since t_m and t_k belong in $A_{q,p}$. Hence, $\{(f(t_n), f(b_n)), n \in \mathbb{N}\}$ forms a disjoint family of nondegenerate intervals such that the sum of their lengths is less than ϵ since they are all contained in U. A similar argument proves the same for the intervals $(f(a_n), f(t_n))$, $n \in \mathbb{N}$. We deduce from (3.4) that

$$\begin{aligned} \mu_{g^C}\left(\bigcup_{n=1}^{\infty} B_g[t_n, r_n]\right) &\leq \sum_{n=1}^{\infty} \mu_{g^C}([a_n, b_n]) \leq \sum_{n=1}^{\infty} g(b_n) - g(a_n) \\ &< \frac{1}{\delta}\sum_{n=1}^{\infty} f(b_n) - f(a_n) = \frac{1}{\delta}\sum_{n=1}^{\infty} f(b_n) - f(t_n) + f(t_n) - f(a_n) \\ &= \frac{1}{\delta}\sum_{n=1}^{\infty} f(b_n) - f(t_n) + \frac{1}{\delta}\sum_{n=1}^{\infty} f(t_n) - f(a_n) \leq 2\frac{\epsilon}{\delta}, \end{aligned}$$

which implies that $\mu_{g^C}\left(f^{-1}(Z) \cap A_{a,p}\right) < 2\varepsilon/\delta$ for any $\epsilon > 0$. □

4 Chain rule

Here we shall prove a new version of the chain rule in higher dimensions which is crucial in the transversality conditions to be introduced in our main existence result for (1.1).

Proposition 4.1. *Let $t, \varepsilon \in \mathbb{R}$, $\varepsilon > 0$, be fixed. Let $h : (t-\epsilon, t+\epsilon) \longrightarrow \mathbb{R}^n$, $f : U \longrightarrow \mathbb{R}$ where U is a usual open set of $\mathbb{R}^n$ such that $h(t-\varepsilon, t+\varepsilon) \subset U$.*

If $t \notin D_g \cup C_g$, h is g-differentiable at t and f is differentiable at $h(t)$, then $f \circ h$ is g-differentiable at t and

$$(f \circ h)'_g(t) = \nabla f(h(t)) \cdot h'_g(t),$$

where $\cdot$ denotes the usual inner product.

Proof. Since $t \notin C_g$ we have $g(s) < g(t)$ for all $s < t$, or $g(s) > g(t)$ for all $s > t$, or both. Since $t \notin D_g$, the existence of $h'_g(t)$ means, in any of the previous three cases, that there exists

$$h'_g(t) = \lim_{\substack{s \to t \\ g(s) \neq g(t)}} \frac{h(s) - h(t)}{g(s) - g(t)}$$

and we want to prove that

$$\lim_{\substack{s \to t \\ g(s) \neq g(t)}} \frac{f(h(s)) - f(h(t))}{g(s) - g(t)} = \nabla f(h(t)) \cdot h'_g(t).$$

Since g is continuous at t, there exists

$$\lim_{\substack{s \to t \\ g(s) \neq g(t)}} h(s) - h(t) = \lim_{\substack{s \to t \\ g(s) \neq g(t)}} \frac{h(s) - h(t)}{g(s) - g(t)} (g(s) - g(t)) = 0. \tag{4.1}$$

Observe that, as long as $g(s) \neq g(t)$, we have

$$\left| \frac{f(h(s)) - f(h(t))}{g(s) - g(t)} - \nabla f(h(t)) \cdot h'_g(t) \right| \leq \underbrace{\left| \frac{f(h(s)) - f(h(t)) - \nabla f(h(t)) \cdot (h(s) - h(t))}{g(s) - g(t)} \right|}_{=A(s)}$$
$$+ \underbrace{\|\nabla f(h(t))\| \left\| \frac{h(s) - h(t)}{g(s) - g(t)} - h'_g(t) \right\|}_{=B(s)},$$

and $B(s) \to 0$ as $s \to t$ ($g(s) \neq g(t)$) by definition of $h'_g(t)$. On the other hand, if $h(s) = h(t)$, then $A(s) = 0$, and otherwise we have

$$A(s) = \frac{|f(h(s)) - f(h(t)) - \nabla f(h(t)) \cdot (h(s) - h(t))|}{\|h(s) - h(t)\|} \left\| \frac{h(s) - h(t)}{g(s) - g(t)} \right\|,$$

which tends to zero as $s \to t$ ($g(s) \neq g(t)$) by definition of $\nabla f(h(t))$ and (4.1). □

Corollary 4.2. *Let $t, \varepsilon \in \mathbb{R}$, $\varepsilon > 0$ be fixed. Let $x : (t-\epsilon, t+\epsilon) \longrightarrow \mathbb{R}^n$, $\tau : (t-\epsilon, t+\epsilon) \times U \longrightarrow \mathbb{R}$ where U is a usual open set of $\mathbb{R}^n$ such that $x(t-\varepsilon, t+\varepsilon) \subset U$.*

If $t \notin D_g \cup C_g$, $g'(t) > 0$, x is g-differentiable at t and τ is differentiable at $(t, x(t))$, then $\tau(\cdot, x(\cdot))$ is g-differentiable at t and

$$\frac{d}{dg(t)} \tau(t, x(t)) = \nabla \tau(t, x(t)) \cdot (1/g'(t), x'_g(t)).$$

Proof. Apply Proposition 4.1 with $h(t) = (t, x(t))$ and $f(y) = \tau(y)$ for $y = (t, x) \in (t - \varepsilon, t + \varepsilon) \times \mathbb{R}^n$. The condition $g'(t) > 0$ implies that h is g-differentiable at t and $h'_g(t) = (1/g'(t), x'_g(t))$. □

5 Main result

We are already in a position to prove our new existence result of Carathéodory solutions of (1.1). Our arguments lean on the chain rule, established in Section 4, along with a transversality condition of the g-differential equation with respect to the sets where discontinuities are allowed for $f(t,x)$.

A Krasovskij solution of (1.1) is defined as a g-absolutely continuous function $x : I \longrightarrow \mathbb{R}^n$ such that $x(t_0) = x_0$ and

$$x'_g(t) \in \bigcap_{\varepsilon>0} \overline{\text{co}}\, f\,(t, B_\varepsilon(x(t))) \quad \text{for } g\text{-a.a. } t \in [t_0, t_0+L), \tag{5.1}$$

where $\overline{\text{co}}$ means closed convex hull and $B_\varepsilon(x) = \{y \in \mathbb{R}^n \,:\, \|y-x\| < \varepsilon\}$. Observe that, in the scalar case ($n=1$), we have $B_\varepsilon(x) = (x-\varepsilon, x+\varepsilon)$.

We are now in a position to prove a result on the existence of Carathéodory solutions for (1.1).

Theorem 5.1. *Assume that for all $x \in \mathbb{R}^n$ the mapping $f(\cdot,x)$ is g-measurable, for all $t \in [t_0,t_0+L) \cap D_g$ the mapping $f(t,\cdot)$ is continuous, and there exists $M \in L^1_g([t_0,t_0+L))$ such that for g-almost all $t \in [t_0,t_0+L)$ and all $x \in \mathbb{R}^n$ we have $\|f(t,x)\| \le M(t)$.*

Moreover, assume that there exist null Lebesgue measure sets $A_k \subset \mathbb{R}$, $k \in \mathcal{C} \subset \mathbb{N}$, and differentiable mappings $\tau_k : [a_k,b_k] \times \mathbb{R}^n \longrightarrow \mathbb{R}$, $[a_k,b_k] \subset I$, such that for g-a.a. $t \in [t_0,t_0+L) \setminus D_g$ the following conditions hold:

(a) $g'(t) > 0$;

(b) There exists a set $N(t) \subset \mathbb{R}^n$ such that $f(t,\cdot)$ is continuous in $\mathbb{R}^n \setminus N(t)$;

(c) For each $x \in N(t)$ there exists $k \in \mathcal{C}$ such that $t \in [a_k,b_k]$, $\tau_k(t,x) \in A_k$, and

$$\nabla \tau_k(t,x) \cdot (1/g'(t), z) \neq 0 \quad \textit{for all } z \in \bigcap_{\varepsilon>0} \overline{\text{co}}\, f\,(t, B_\varepsilon(x)). \tag{5.2}$$

Then, problem (1.1) has at least one Carathéodory solution.

Proof. The assumptions imply that the multivalued mapping

$$(t,x) \longmapsto \mathcal{F}(t,x) = \bigcap_{\varepsilon>0} \overline{\text{co}}\, f\,(t, B_\varepsilon(x))$$

is compact and convex-valued, and it is upper semicontinuous. Hence, Theorem 3.9 in [8] ensures that problem (1.1) has at least one Krasovskij solution $x : I \longrightarrow \mathbb{R}^n$. Let us prove that x is a Carathéodory solution of (1.1).

Let $E \subset I$ be a g-null measure set such that, first, conditions (a), (b) and (c) hold for all $t \in I \setminus E$, and, second,

$$x'_g(t) \in \mathcal{F}(t,x(t)) \quad \text{for all } t \in I \setminus E.$$

Observe that for each $t \in I \setminus E$ such that $t \in D_g$ or $x(t) \notin N(t)$ the assumptions ensure that $f(t,\cdot)$ is continuous at $x(t)$, and therefore

$$x'_g(t) \in \mathcal{F}(t,x(t)) = \{f(t,x(t))\}.$$

Hence, it suffices to prove that the set $J = \{t \in I \setminus E,\, t \notin D_g \,:\, x(t) \in N(t)\}$ is g-null.

We deduce from condition (c) that

$$J \subset \bigcup_{k\in\mathcal{C}} \{t \in [a_k, b_k] \setminus E,\ t \notin D_g\ :\ \tau_k(t, x(t)) \in A_k\},$$

so the proof is reduced to showing that each $J_k = \{t \in [a_k, b_k] \setminus E,\ t \notin D_g\ :\ \tau_k(t, x(t)) \in A_k\}$ is a null g-measure set. For an arbitrarily fixed $k \in \mathcal{C}$, we define $\varphi(t) = \tau_k(t, x(t))$ for all $t \in [a_k, b_k]$, so that $J_k \subset \varphi^{-1}(A_k) \setminus D_g$ and it suffices to prove that $\varphi^{-1}(A_k) \setminus D_g$ is g-null. Since $m(A_k) = 0$, Theorem 3.7 guarantees the existence of a set $B \subset \varphi^{-1}(A_k)$, with null g^c-measure, such that for every $t \in \varphi^{-1}(A_k) \setminus B$ we have $\varphi'_g(t) = 0$, i.e. $\varphi'_g(t) = 0$ for g^c-a.a. $t \in \varphi^{-1}(A_k)$ or, equivalently, for g-a.a. $t \in \varphi^{-1}(A_k) \setminus D_g$ we have

$$\frac{d}{dg(t)} \tau_k(t, x(t)) = 0. \tag{5.3}$$

Let us prove that $\varphi^{-1}(A_k) \setminus D_g \subset E$, thus showing that $\varphi^{-1}(A_k) \setminus D_g$ is g-null. Reasoning by contradiction, we assume that there is some $t \in \varphi^{-1}(A_k) \setminus D_g$ such that $t \notin E$, and then we can use the chain rule in (5.3) to deduce that

$$\nabla \tau_k(t, x(t)) \cdot (1/g'(t), x'_g(t)) = 0,$$

a contradiction with condition (5.2). □

Checking the transversality condition (5.2) might be a difficult task in practical situations. Our next corollary gives a more friendly sufficient condition for it. Basically, it tells us that (5.2) is satisfied provided that $\partial\tau_k/\partial t$ is far from zero and $g'(t)$ is small enough. This is an interesting condition from the viewpoint of Stieltjes differential equations because it leans precisely on the slopes of the derivator.

Corollary 5.2. *Assume that for all $x \in \mathbb{R}^n$ the mapping $f(\cdot, x)$ is g-measurable, for all $t \in [t_0, t_0 + L) \cap D_g$ the mapping $f(t, \cdot)$ is continuous, and there exists $M \in L^1_g([t_0, t_0 + L))$ such that for g-almost all $t \in [t_0, t_0 + L)$ and all $x \in \mathbb{R}^n$ we have $\|f(t, x)\| \leq M(t)$.*

Moreover, assume that there exist null Lebesgue measure sets $A_k \subset \mathbb{R}$, $k \in \mathcal{C} \subset \mathbb{N}$, and differentiable mappings $\tau_k : [a_k, b_k] \times \mathbb{R}^n \longrightarrow \mathbb{R}$, $[a_k, b_k] \subset I$, such that for g-a.a. $t \in [t_0, t_0 + L) \setminus D_g$ the following conditions hold:

(a) $g'(t) > 0$;

(b) There exists a set $N(t) \subset \mathbb{R}^n$ such that $f(t, \cdot)$ is continuous in $\mathbb{R}^n \setminus N(t)$;

(c) For each $x \in N(t)$ there exists $k \in \mathcal{C}$ such that $t \in [a_k, b_k]$, $\tau_k(t, x) \in A_k$, and*

$$\left|\frac{\partial \tau_k}{\partial t}(t, x)\right| > g'(t)\, M(t) \sum_{j=1}^{n} \left|\frac{\partial \tau_k}{\partial x_j}(t, x)\right|. \tag{5.4}$$

Then, problem (1.1) has at least one Carathéodory solution.

Proof. We only have to check that condition $(c*)$ implies condition (c) in Theorem 5.1. To do it, we take x as in (c^*) and we observe that $z = (z_1, z_2, \ldots, z_n) \in \bigcap_{\varepsilon>0} \overline{\text{co}}\, f(t, B_\varepsilon(x))$ implies

that $|z_j| \le M(t)$ for $j = 1, 2, \dots, n$, so we have

$$\begin{aligned}
\left|\nabla\tau_k(t,x)\cdot(1/g'(t),z)\right| &= \left|\frac{1}{g'(t)}\frac{\partial\tau_k}{\partial t}(t,x) + \sum_{j=1}^{n} z_j \frac{\partial\tau_k}{\partial x_j}(t,x)\right| \\
&= \left|\frac{1}{g'(t)}\frac{\partial\tau_k}{\partial t}(t,x) - \sum_{j=1}^{n}(-z_j)\frac{\partial\tau_k}{\partial x_j}(t,x)\right| \\
&\ge \frac{1}{g'(t)}\left|\frac{\partial\tau_k}{\partial t}(t,x)\right| - \left|\sum_{j=1}^{n}(-z_j)\frac{\partial\tau_k}{\partial x_j}(t,x)\right| \\
&\ge \frac{1}{g'(t)}\left|\frac{\partial\tau_k}{\partial t}(t,x)\right| - M(t)\sum_{j=1}^{n}\left|\frac{\partial\tau_k}{\partial x_j}(t,x)\right|
\end{aligned}$$

and the last term is positive thanks to condition (5.4). □

Observe that condition (5.4) is fulfilled when $\partial\tau_k/\partial t$ is positive and either g' is small enough or $\partial\tau_k/\partial t$ is large enough. Corollary 5.2 is new even in case $g(t) = t$, i.e. for ordinary differential equations, where condition (5.4) reduces to

$$\left|\frac{\partial\tau_k}{\partial t}(t,x)\right| > M(t)\sum_{j=1}^{n}\left|\frac{\partial\tau_k}{\partial x_j}(t,x)\right|.$$

Finally, we get an interesting consequence for the scalar case in connection with bounded variation (not necessarily continuous) nonlinear parts of the g-differential equations.

Corollary 5.3. *The scalar problem*

$$x'_g = f(t,x), \quad t \in I = [t_0, t_0 + L),\ x(t_0) = x_0, \tag{5.5}$$

has at least one Carathéodory solution provided that there exists $M > 0$ such that $|f(t,x)| \le M$ for all $(t,x) \in I \times \mathbb{R}$, $f(t,\cdot)$ is continuous whenever $t \in [t_0, t_0 + L) \cap D_g$, and for g-a.a. $t \in [t_0, t_0 + L) \setminus D_g$ we have $g'(t) > 0$ and we can express $f(t,x) = F(\tau(t,x))$ for some functions F and τ satisfying the following assumptions:

(i) $F : \mathbb{R} \longrightarrow \mathbb{R}$ *has bounded variation on any compact interval;*

(ii) $\tau : I \times \mathbb{R} \longrightarrow \mathbb{R}$ *is differentiable and*

$$\left|\frac{\partial\tau}{\partial t}(t,x)\right| > M\,g'(t)\left|\frac{\partial\tau}{\partial x}(t,x)\right|, \quad (t,x) \in ([t_0, t_0 + L) \setminus D_g) \times \mathbb{R}. \tag{5.6}$$

Proof. The result follows from Corollary 5.2. First, observe that for any $x \in \mathbb{R}$ the composition $f(\cdot,x) = F(\tau(\cdot,x))$ is Borel measurable, hence g-measurable in the Lebesgue–Stieltjes sense. On the other hand, for any $t \in I \setminus D_g$ the function $f(t,\cdot) = F(\tau(t,\cdot))$ is continuous everywhere except at those $x \in \mathbb{R}$ such that $\tau(t,x)$ is a discontinuity point of F. Let A be a countable set such that F is continuous in $\mathbb{R} \setminus A$; then $f(t,\cdot)$ is continuous in $\mathbb{R} \setminus N(t)$, where

$$N(t) = \{x \in \mathbb{R} : \tau(t,x) \in A\} = \tau(t,\cdot)^{-1}(A).$$

Obviously, for each $x \in N(t)$ we have $\tau(t,x) \in A$, and condition (5.6) implies condition (5.4). □

Remark 5.4. In the conditions of Corollary 5.3, solutions $x(t)$ of (5.5) satisfy $|x(t) - x_0| \le M\,(g(t) - g(t_0)) \le M\,(g(t_0 + L) - g(t_0))$ for all $t \in [t_0, t_0 + L)$. Hence, condition (5.6) is really needed only for all $(t,x) \in ([t_0, t_0 + L) \setminus D_g) \times \mathbb{R}$ such that $|x - x_0| \le M\,(g(t) - g(t_0))$.

5.1 An example

Let $g : \mathbb{R} \longrightarrow \mathbb{R}$ be nondecreasing, left-continuous everywhere and assume that $g'(t) > 0$ for g-a.a. $t \in [t_0, t_0 + L) \setminus D_g$.

We consider the following particular case of problem (5.5):

$$x_g' = F(\alpha\, t + \beta\, x)\chi_{I\setminus D_g}(t) + F(t)\chi_{I\cap D_g}(t), \quad t \in I = [t_0, t_0 + L),\ x(t_0) = x_0, \tag{5.7}$$

where $\alpha, \beta \in \mathbb{R}$, χ_A denotes the characteristic function of a set $A \subset \mathbb{R}$, and $F : \mathbb{R} \longrightarrow \mathbb{R}$ is given by

$$F(y) = \sum_{r_n < y} 2^{-n}, \quad y \in \mathbb{R},$$

where $r_n = r(n)$, $n \in \mathbb{N}$, and $r : \mathbb{N} \longrightarrow \mathbb{Q}$ is a bijection.

Observe that F is increasing, $0 < F(y) < 1$ for all $y \in \mathbb{R}$, and F is discontinuous at every rational number.

It is easy to check that Corollary 5.3 applies if

$$|\alpha| > g'(t)|\beta| \quad \text{for } g\text{-a.a. } t \in I \setminus D_g,$$

so in that case (5.7) has at least one solution.

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Multiple solutions for a parametric Steklov problem involving the $p(x)$-Laplacian operator

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Abstract. In this paper, we study the existence and multiplicity of weak solutions for a Steklov problem involving $p(x)$-Laplacian operator in a bounded domain $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ with smooth boundary $\partial\Omega$. The boundary equation is perturbed with some weight functions belonging to approriate generalized Lebesgue spaces and two real parameters. Our arguments are based on variational method, using "Mountain Pass Theorem", "Fountain Theorem" and "Dual Fountain Theorem" combined with the critical points theory, we prove several existence results.

Keywords: Steklov problem, $p(x)$-Laplacian operator, generalized Lebesgue–Sobolev spaces, variational method, Mountain Pass Theorem, Fountain Theorem, Dual Fountain Theorem.

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1 Introduction

We are interested in the existence and multiplicity of weak solutions for the following quasi-linear elliptic problem with Steklov boundary condition involving variable exponents

$$\begin{aligned} \Delta_{p(x)}u &= a(x)|u|^{p(x)-2}u \quad \text{in } \Omega, \\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial \nu} &= \lambda V(x)|u|^{q(x)-2}u + \mu W(x)|u|^{r(x)-2}u \quad \text{on } \partial\Omega, \end{aligned} \tag{$S_{\lambda,\mu}$}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded domain with smooth boundary $\partial\Omega$, ν is the outward unit normal to $\partial\Omega$, λ, μ are real parameters, $p, q, r : \overline{\Omega} \to (1, +\infty)$ are continuous functions, $a \in L^\infty(\Omega)$ is a positive function, V, W are positive measurable weight functions belonging to some generalized Lebesgue space and $\Delta_{p(x)}u$ is the $p(x)$-Laplacian operator defined by $\Delta_{p(x)}u = \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ which is a natural extension of the p-Laplacian operator obtained in the constant case $p(x) \equiv p$, if $p \equiv 2$ we have the classical Laplacian operator $\Delta u = \operatorname{div}(\nabla u)$.

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In recent years, nonlinear boundary value problems with variable exponents have received an increasing attention. This is partly due to their frequent appearance in a lot of models in mathematical physics such as the modeling of electrorheological fluids, elasticity problems, flow in porous medium and image processing, see [17, 42, 44, 51]. The study of variable exponent problems is more difficult than the constant one, since the $p(x)$-Laplacian operator possesses more complicated structures than the p-Laplacian operator, for example it is inhomogeneous. Recently, boundary value problems involving a such operator have been a subject of many researchers, we refer to [1, 15, 24, 29, 33, 34, 36, 39], in which the authors have extended some well known results from constant to variable exponents case.

Problems of Steklov type appear in the modeling of an elastic membrane problem whose mass is concentrated on the boundary [45]. In the past decades, many works have extended the results of [45] from Laplacian operator to p-Laplacian operator [7–10, 22, 38, 40, 47] and more general $p(x)$-Laplacian operator [2–6, 11, 12, 14, 19, 43].

In [11], problem $(S_{\lambda,\mu})$ is considered with $a(x) \equiv 1$, $V(x) \equiv 1$ and $\mu = 0$. For any $\lambda > 0$, if $q^+ < p^-$, the authors proved the existence of a sequence of nontrivial weak solutions, and if $p^+ < q^- \leq q^+ < p^\partial(x)$, the existence of nontrivial weak solution. In [14], the authors considered problem $(S_{\lambda,\mu})$ with $\mu = 0$ and $V(x)$ belonging to some generalized Lebesgue space. Under condition $1 \leq q^+ < p^-$, the authors proved the existence of at least one non trivial weak solution with negative energy for all $\lambda > 0$, and non trivial weak solution for all $\lambda \in (0, \lambda^*)$, where λ^* is a positive constant. It should be noticed that in [24, 36], problem like $(S_{\lambda,\mu})$ was considered but with Dirichlet boundary condition. Under appropriate conditions on the variable exponents, the authors proved several existence and multiplicity results by changing the role of parameters λ, μ with respect to their signs.

Inspired by the above mentioned works [11, 14, 24, 36], we study the Steklov problem $(S_{\lambda,\mu})$ with some weight functions and two parameters. The appropriate functional spaces and their basic properties used in this work are guaranteed by the paper of S. G. Deng [19], namely, the compact embedding from the generalized Sobolev space to the generalized Lebesgue space with weight. Under relaxed conditions on the variable exponents and real parameters λ, μ that the ones use in [11, 14], we obtain new criteria on existence and multiplicity results of $(S_{\lambda,\mu})$.

We make the following basic hypotheses on the data of $(S_{\lambda,\mu})$:

$$\begin{gathered} s, t : \overline{\Omega} \to (1, +\infty) \quad \text{are variable exponents such that :} \\ s(x), t(x) > \frac{N-1}{p(x)-1}, \quad \text{for a.e. } x \in \partial\Omega, \\ 1 \leq q(x) < \frac{s(x)-1}{s(x)} p^\partial(x), \qquad 1 \leq r(x) < \frac{t(x)-1}{t(x)} p^\partial(x), \quad \text{for a.e. } x \in \partial\Omega, \end{gathered} \tag{$\mathbf{H_1}$}$$

where $p^\partial(x)$ is the critical Sobolev exponent on $\partial\Omega$;

$$a \in L^\infty(\Omega), \quad \text{with } \operatorname*{ess\,inf}_{x\in\Omega} a(x) = a^- > 0; \tag{$\mathbf{H_2}$}$$

$$V \in L^{s(x)}(\partial\Omega), \qquad W \in L^{t(x)}(\partial\Omega), \quad \text{with } V(x), W(x) > 0 \quad \text{for a.e. } x \in \partial\Omega. \tag{$\mathbf{H_3}$}$$

Our main results in this paper are given by the following theorems.

Theorem 1.1. *Assume that hypotheses* ($\mathbf{H_1}$)–($\mathbf{H_2}$)–($\mathbf{H_3}$) *are fulfilled. Denote by* p^+, q^+, r^+ *(resp.* p^-, q^-, r^-*) the maximum (resp. the minimum) of functions* $p(x)$, $q(x)$, $r(x)$ *on* $\overline{\Omega}$*, respectively.*

(i) *If* $\max\{q^+, r^+\} < p^-$, *then for any* $\lambda, \mu \in \mathbb{R}$, *problem* $(S_{\lambda,\mu})$ *has a nonnegative solution.*

(ii) *If* $\min\{q^-, r^-\} > p^+$, *then for any* $\lambda, \mu \geq 0$, *problem* $(S_{\lambda,\mu})$ *has a nontrivial nonnegative solution.*

Theorem 1.2. *Assume that hypotheses* $(\mathbf{H_1})$–$(\mathbf{H_2})$–$(\mathbf{H_3})$ *are fulfilled. Denote by* Ψ *the energy fuctional associated to* $(S_{\lambda,\mu})$ *and* p^+, q^+, r^+ *(resp.* p^-, q^-, r^-*) the maximum (resp. the minimum) of functions* $p(x)$, $q(x)$, $r(x)$ *on* $\overline{\Omega}$, *respectively.*

(i) *If* $1 < p^- \leq p^+ < r^- \leq q^-$, *then for any* $\lambda, \mu \geq 0$, *problem* $(S_{\lambda,\mu})$ *has a sequence of solutions* $(\pm u_n)$ *such that* $\Psi(\pm u_n) \to \infty$ *as* $n \to \infty$.

(ii) *If* $1 < q^- \leq q^+ < p^- \leq p^+ < r^-$, *then for any* $\lambda \in \mathbb{R}$ *and* $\mu > 0$, *problem* $(S_{\lambda,\mu})$ *has a sequence of solutions* $(\pm u_n)$ *such that* $\Psi(\pm u_n) \to \infty$ *as* $n \to \infty$.

(iii) *If* $1 < q^- \leq q^+ = p^- \leq p^+ < r^-$, *then for any* $\lambda \in \mathbb{R}$ *with* $|\lambda| < \lambda_0 := \frac{\frac{1}{p^+} - \frac{1}{r^-}}{(\frac{1}{q^-} - \frac{1}{h^-})C^{q^+}}$ *and* $\mu > 0$, *problem* $(S_{\lambda,\mu})$ *has a sequence of solutions* $(\pm u_n)$ *such that* $\Psi(\pm u_n) \to \infty$ *as* $n \to \infty$.

(iv) *If* $1 < q^- \leq q^+ < p^- \leq p^+ < r^-$, *then for any* $\lambda > 0$ *and* $\mu \in \mathbb{R}$, *problem* $(S_{\lambda,\mu})$ *has a sequence of solutions* $(\pm v_n)$ *such that* $\Psi(\pm v_n) < 0$, *for all* n, *and* $\Psi(\pm v_n) \to 0$ *as* $n \to \infty$.

(v) *If* $1 < q^- \leq q^+ < \min\{p^-, r^-\}$ *and* $p^+ < r^+$, *then for any* $\lambda > 0$ *and* $\mu \leq 0$, *problem* $(S_{\lambda,\mu})$ *has a sequence of solutions* $(\pm v_n)$ *such that* $\Psi(\pm v_n) < 0$, *for all* n, *and* $\Psi(\pm v_n) \to 0$ *as* $n \to \infty$.

To prove these results, our approach is variational combined with the critical point theory. Theorem 1.1 is established by using the global minimizer method for the case (i) and Mountain Pass Theorem of Ambrosetti–Rabinowitz for the case (ii). For the proof of Theorem 1.2 we need to use the fountain theorem established by T. Bartsch in 1993 and the dual of fountain theorem established by T. Bartsch and M. Willem in 1995, which are both simple and powerful tools in searching solutions for large energy and small energy of several boundary value problems.

The remainder of this paper is organized in the following way. In Section 2, we give some preliminaries on variable exponent Lebesgue and Sobolev spaces. In particular, we give a weighted variable exponent Sobolev trace compact embedding theorem which will be useful in this work. In Section 3, we begin by the preliminary results of $(S_{\lambda,\mu})$, state some abstract abstract critical point theorems and establish the technical lemmas which allow us to use the abstract theorems in order to guarantee the existence of solutions. Section 4 is devoted to the proof of our main results : Theorem 1.1 and Theorem 1.2. Finally, in the Appendix we give some examples of variable exponents illustrating Theorem 1.1 and Theorem 1.2.

2 Preliminaries on generalized Lebesgue–Sobolev spaces

We need to recall here some basic properties of variable exponent Lebesgue and Sobolev spaces which will be used later. We refer to [19, 21, 24, 29, 30, 37] for more details on the properties of these space.

Let Ω be a bounded domain of $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$. Let us denote by

$$C_+(\overline{\Omega}) = \{p \in C(\overline{\Omega}) : p(x) > 1 \text{ for any } x \in \overline{\Omega}\};$$

for $p \in C_+(\overline{\Omega})$,

$$p^- = \min_{x\in\overline{\Omega}} p(x) \quad \text{and} \quad p^+ = \max_{x\in\overline{\Omega}} p(x).$$

Let $p \in C_+(\overline{\Omega})$, the variable exponent Lebesgue spaces $L^{p(x)}(\Omega)$ is defined by

$$L^{p(x)}(\Omega) = \left\{ u \colon \Omega \to \mathbb{R} \text{ measurable function } : \int_\Omega |u(x)|^{p(x)} dx < \infty \right\},$$

endowed with the so-called Luxemburg norm given by

$$\|u\|_{L^{p(x)}(\Omega)} = \|u\|_{p(x)} = \inf \left\{ \lambda > 0 : \int_\Omega \left| \frac{u(x)}{\lambda} \right|^{p(x)} dx \le 1 \right\}.$$

Remark 2.1. The spaces $\left(L^{p(x)}(\Omega), \|\cdot\|_{p(x)}\right)$ resembles to the classical Lebesgue spaces in many case, namely, they are separable Banach spaces, the Hölder inequality holds, the inclusions between Lebesgue spaces are also valid, that is, if $0 < \text{meas}(\Omega) < \infty$ and $p(x) < q(x)$ a.e. in Ω, then there exists a continuous embedding $L^{q(x)}(\Omega) \hookrightarrow L^{p(x)}(\Omega)$.

The variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ is defined by

$$W^{1,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega) \right\},$$

endowed with the norm

$$\|u\|_{1,p(x)} = \inf \left\{ \lambda > 0 : \int_\Omega \left(\left| \frac{\nabla u(x)}{\lambda} \right|^{p(x)} + \left| \frac{u(x)}{\lambda} \right|^{p(x)} \right) dx \le 1 \right\},$$

that is $\|u\|_{1,p(x)} = \|\nabla u\|_{p(x)} + \|u\|_{p(x)}$.

Proposition 2.2 ([30]). *Both $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ are separable, reflexive and uniformly convex Banach spaces.*

Let $L^{p'(x)}(\Omega)$ be the conjugate space of $L^{p(x)}(\Omega)$, with $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$ for all $x \in \overline{\Omega}$.

Proposition 2.3 ([30]). *The Hölder inequality holds, namely*

$$\int_\Omega |u(x)||v(x)|dx \le \left(\frac{1}{p^-} + \frac{1}{p'^-} \right) \|u\|_{p(x)} \|v\|_{p'(x)} \le 2\|u\|_{p(x)} \|v\|_{p'(x)},$$

for all $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$.

In what follows, for brevity we write $X = W^{1,p(x)}$ and denote X^* its dual space.
Let $a \in L^\infty(\Omega)$ with $\operatorname{ess\,inf}_{x\in\Omega} a(x) > 0$. For any $u \in X$, let us define

$$\|u\|_a = \inf \left\{ \lambda > 0 : \int_\Omega \left(\left| \frac{\nabla u(x)}{\lambda} \right|^{p(x)} + a(x) \left| \frac{u(x)}{\lambda} \right|^{p(x)} \right) dx \le 1 \right\}.$$

Then, $\|\cdot\|_a$ is a norm on X equivalent to the norm $\|\cdot\|_{1,p(x)}$ (see [2, 19]). In what follows, we will use $\|\cdot\|_a$ instead of $\|\cdot\|_{1,p(x)}$ on X.

The modular is an important tool in studying problems involving generalized Lebesgue–Sobolev spaces, it is a mapping ρ_a defined on X by

$$\rho_a(u) = \int_\Omega \left(|\nabla u(x)|^{p(x)} + a(x)|u(x)|^{p(x)} \right) dx.$$

The following proposition connect $\|u\|_a$ with $\rho_a(u)$, it will be very useful in this work.

Proposition 2.4 ([20,30]). *Then, for all $u, u_n \in X$ $(n = 1, 2, \dots)$, we have*

(i) $\|u\|_a < 1 (= 1; > 1) \Leftrightarrow \rho_a(u) < 1 (= 1; > 1)$;

(ii) $\|u\|_a \leq 1 \Rightarrow \|u\|_a^{p^+} \leq \rho_a(u) \leq \|u\|_a^{p^-}$;

(iii) $\|u\|_a \geq 1 \Rightarrow \|u\|_a^{p^-} \leq \rho_a(u) \leq \|u\|_a^{p^+}$;

(iv) $\|u_n\|_a \to 0 \Leftrightarrow \rho_a(u_n) \to 0$ *(as $n \to \infty$).*

Consider the real valued fuctional defined on X by

$$\phi(u) = \int_\Omega \frac{1}{p(x)} \left(|\nabla u|^{p(x)} + a(x)|u|^{p(x)} \right) dx.$$

The following properties of ϕ are very interesting in the study of variational problems.

Proposition 2.5 ([25,26]). *Then, ϕ is convex, sequentially weakly lower semi-continuous and continuously Fréchet differentiable such that*

$$\langle \phi'(u), v \rangle = \int_\Omega \left(|\nabla u|^{p(x)-2} \nabla u \nabla v + a(x)|u|^{p(x)-2} uv \right) dx, \quad \text{for all } u, v \in X.$$

Moreover, The derivative function ϕ' satisfies

(i) $\phi' : X \to X^*$ *is continuous, bounded and strictly monotone operator;*

(ii) $\phi' : X \to X^*$ *is a mapping of type (S^+), that is, if $u_n \xrightarrow{weakly} u$ in X and $\limsup_{n\to\infty} \langle \phi'(u_n) - \phi'(u), u_n - u \rangle \leq 0$, then $u_n \xrightarrow{strongly} u$ in X;*

(iii) $\phi' : X \to X^*$ *is a homeomorphism.*

Let $b : \partial\Omega \to \mathbb{R}$ be a measurable function. The weighted variable exponent Lebesgue space is defined by

$$L_{b(x)}^{p(x)}(\partial\Omega) = \left\{ u \colon \partial\Omega \to \mathbb{R} \text{ measurable function} : \int_{\partial\Omega} |b(x)||u(x)|^{p(x)} d\sigma < \infty \right\},$$

endowed with the norm

$$\|u\|_{(p(x),b(x))} = \|u\|_{L_{b(x)}^{p(x)}(\partial\Omega)} = \inf \left\{ \tau > 0 : \int_{\partial\Omega} |b(x)| \left| \frac{u(x)}{\tau} \right|^{p(x)} d\sigma \leq 1 \right\},$$

where $d\sigma$ is the surface measure on the boundary.

Remark 2.6. The space $L_{b(x)}^{p(x)}(\partial\Omega)$ is a Banach space. In particular, when $b(x) \equiv 1$ on $\partial\Omega$ or more general $b(x) \in L^\infty(\partial\Omega)$, $L_{b(x)}^{p(x)}(\partial\Omega) = L^{p(x)}(\partial\Omega)$.

For $u \in L_{b(x)}^{p(x)}(\partial\Omega)$, consider the real valued functional

$$\rho(u) = \int_{\partial\Omega} |b(x)||u(x)|^{p(x)} d\sigma.$$

The following proposition connect $\|u\|_{(p(x),b(x))}$ with $\rho(u)$.

Proposition 2.7 ([19]). *Then, for all $u, u_n \in L^{p(x)}_{b(x)}(\partial\Omega)$ $(n = 1, 2, \dots)$, we have*

(i) $\|u\|_{(p(x),b(x))} \leq 1 \Rightarrow \|u\|^{p^+}_{(p(x),b(x))} \leq \rho(u) \leq \|u\|^{p^-}_{(p(x),b(x))}$;

(ii) $\|u\|_{(p(x),b(x))} \geq 1 \Rightarrow \|u\|^{p^-}_{(p(x),b(x))} \leq \rho(u) \leq \|u\|^{p^+}_{(p(x),b(x))}$;

(iii) $\|u_n\|_{(p(x),b(x))} \to 0 \Leftrightarrow \rho(u_n) \to 0 (as\ n \to \infty)$.

Consequently,

$$\|u\|^{p^-}_{(p(x),b(x))} - 1 \leq \rho(u) \leq \|u\|^{p^+}_{(p(x),b(x))} + 1, \quad \text{for all } u \in L^{p(x)}_{b(x)}(\partial\Omega).$$

Proposition 2.8 ([13]). *The map $u \in L^{p(x)}(\partial\Omega) \longmapsto |u|^{q(x)-2}u \in L^{p'(x)}(\partial\Omega)$ is continuous.*

For $p \in C_+(\overline{\Omega})$ and $r \in C_+(\partial\Omega)$, the Sobolev critical exponents are defined by

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N, \\ +\infty & \text{if } p(x) \geq N, \end{cases} \quad \text{and} \quad p^\partial(x) = \begin{cases} \frac{(N-1)p(x)}{N-p(x)} & \text{if } p(x) < N, \\ +\infty & \text{if } p(x) \geq N. \end{cases}$$

Let $p^\partial_{r(x)}(x)$ defined by

$$p^\partial_{r(x)}(x) = \frac{r(x)-1}{r(x)} p^\partial(x).$$

Proposition 2.9 ([19, 28]). *Assume that the boundary of Ω possesses the cone property and $p \in C_+(\overline{\Omega})$. Suppose that $b \in L^{r(x)}(\partial\Omega)$, $r \in C_+(\partial\Omega)$ with $r(x) > \frac{p^\partial(x)}{p^\partial(x)-1}$ for all $x \in \partial\Omega$. Then, we have the following properties.*

(i) If $q \in C_+(\overline{\Omega})$ and $1 \leq q(x) < p^*(x)$ for any $x \in \overline{\Omega}$. Then, the embedding from $W^{1,p(x)}(\Omega)$ into $L^{q(x)}(\Omega)$ is compact and continuous.

(ii) If $q \in C_+(\partial\Omega)$ and $1 \leq q(x) < p^\partial_{r(x)}(x)$ for any $x \in \partial\Omega$. Then, the embedding from $W^{1,p(x)}(\Omega)$ into $L^{q(x)}_{b(x)}(\partial\Omega)$ is compact and continuous. In particular, there is a compact and continuous embedding from $W^{1,p(x)}(\Omega)$ into $L^{q(x)}(\partial\Omega)$, where $1 \leq q(x) < p^\partial(x)$, for any $x \in \partial\Omega$.

3 Preliminary results, abstract theorems and technical lemmas

We investigate here the preliminary results, state some well known abstract critical point theorems and establish the technical lemmas which allow us to prove our main results. Here and henceforth, we denote by X the generalized Sobolev space $W^{1,p(x)}(\Omega)$ equipped with the norm $\|\cdot\|_a$ and X^* its dual space.

3.1 Preliminary results

Definition 3.1. We say that $u \in X$ is a weak solution of the Steklov problem $(S_{\lambda,\mu})$ if and only if for all $v \in X$

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v dx + \int_\Omega a(x)|u|^{p(x)-2}uvdx = \lambda \int_{\partial\Omega} V(x)|u|^{q(x)-2}uvd\sigma + \mu \int_{\partial\Omega} W(x)|u|^{r(x)-2}uvd\sigma,$$

where $d\sigma$ is the surface measure on the boundary of Ω.

Denote by $\Psi : X \to \mathbb{R}$ the Euler–Lagrange functional (energy functional) associated with the problem ($S_{\lambda,\mu}$), it is defined by

$$\Psi(u) = \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx + \int_\Omega \frac{a(x)}{p(x)}|u|^{p(x)}dx - \lambda \int_{\partial\Omega} \frac{V(x)}{q(x)}|u|^{q(x)}d\sigma - \mu \int_{\partial\Omega} \frac{W(x)}{r(x)}|u|^{r(x)}d\sigma.$$

For $u \in X$, we denote by $u^+ = \max\{u, 0\}$, $u^- = \max\{-u, 0\}$ and we define the functional

$$\Psi_+(u) = \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx + \int_\Omega \frac{a(x)}{p(x)}|u|^{p(x)}dx - \lambda \int_{\partial\Omega} \frac{V(x)}{q(x)}(u^+)^{q(x)}d\sigma - \mu \int_{\partial\Omega} \frac{W(x)}{r(x)}(u^+)^{r(x)}d\sigma.$$

Introduce the functionals $\phi, \psi : X \to \mathbb{R}$ defined by

$$\phi(u) = \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx + \int_\Omega \frac{a(x)}{p(x)}|u|^{p(x)}dx$$
$$\psi(u) = \lambda \int_{\partial\Omega} \frac{V(x)}{q(x)}|u|^{q(x)}d\sigma + \mu \int_{\partial\Omega} \frac{W(x)}{r(x)}|u|^{r(x)}d\sigma.$$

From assumptions ($\mathbf{H_1}$)–($\mathbf{H_2}$)–($\mathbf{H_3}$) and Proposition 2.9, we deduce the following compact and continuous embeddings

$$X \hookrightarrow L^{q(x)}_{V(x)}(\partial\Omega) \quad \text{and} \quad X \hookrightarrow L^{r(x)}_{W(x)}(\partial\Omega),$$

that is, there exists a positive constant C such that

$$|u|_{(q(x),V(x))} \le C\|u\|_a \quad \text{and} \quad |u|_{(r(x),W(x))} \le C\|u\|_a, \quad \text{for all } u \in X. \tag{3.1}$$

Without loss of generality, we can suppose that $C > 1$.

Remark 3.2. If $u \in X$, then we have

$$V(x)^{\frac{1}{q(x)}}u \in L^{q(x)}(\partial\Omega) \quad \text{and} \quad V(x)^{\frac{1}{q'(x)}}u^{q(x)-1} \in L^{q'(x)}(\partial\Omega).$$

Moreover, $\|V(x)^{\frac{1}{q(x)}}u\|_{q(x),\partial} = \|u\|_{(q(x),V(x))}$.

Proposition 3.3. *The functionals $\Psi, \Psi_+ \in C^1(X, \mathbb{R})$ (i.e., Ψ and Ψ_+ are continuously Fréchet differentiable) and $u \in X$ is a critical point of Ψ (respectively Ψ_+) if and only if u is a solution (respectively a nonnegative solution) of the Steklov problem ($S_{\lambda,\mu}$).*

Proof. From Proposition 2.5, it is clear that the functional $\phi \in C^1(X, \mathbb{R})$. Then, it is enough to show that $\psi \in C^1(X, \mathbb{R})$. Consider the functional $J : X \to \mathbb{R}$ defined by

$$J(u) = \int_{\partial\Omega} \frac{V(x)}{q(x)}|u|^{q(x)}d\sigma.$$

We prove that J is Fréchet differentiable, i.e., at any $u \in X$ there exists $J'(u) \in X^*$ such that

$$\lim_{t\to 0} \frac{J(u+tv) - J(u)}{t} = \langle J'(u), v\rangle \quad \text{for all } v \in X \quad \text{and} \quad J' : X \to X^* \quad \text{is continuous.}$$

For $\epsilon > 0$ small enough, let $\gamma : [-\epsilon, \epsilon] \to \mathbb{R}$ defined by $\gamma(t) = J(u + tv)$, then we have

$$\begin{aligned}\frac{d\gamma(t)}{dt} &= \frac{dJ(u+tv)}{dt} = \frac{d}{dt}\int_{\partial\Omega}\frac{V(x)}{q(x)}|u+tv|^{q(x)}d\sigma \\ &= \int_{\partial\Omega}\frac{\partial}{\partial t}\left(\frac{V(x)}{q(x)}|u+tv|^{q(x)}\right)d\sigma \\ &= \int_{\partial\Omega} V(x)|u+tv|^{q(x)-2}(u+tv)vd\sigma.\end{aligned}$$

Indeed, the differentiation under the integral is allowed for t close to zero, since for $|t| < 1$ we have

$$|V(x)|u+tv|^{q(x)-2}(u+tv)v| \leq V(x)(|u|+|v|)^{q(x)-1}|v| \in L^1(\partial\Omega),$$

because of Remark 3.2. It follows that

$$\lim_{t\to 0}\frac{J(u+tv)-J(u)}{t} = \lim_{t\to 0}\frac{\gamma(t)-\gamma(0)}{t} = \frac{d\gamma(t)}{dt}|t=0 = \int_{\partial\Omega} V(x)|u|^{q(x)-2}uvd\sigma.$$

For $u \in X$, let $J'(u) : X \to \mathbb{R}$ be an operator defined by

$$\langle J'(u), v\rangle = \int_{\partial\Omega} V(x)|u|^{q(x)-2}uvd\sigma, \quad \text{for all } v \in X.$$

Obviously, $J'(u)$ is a linear operator for each given $u \in X$. Using Remark 3.2, Hölder's inequality, Proposition 2.7, and inequality (3.1), the following inequalities hold true

$$\begin{aligned}|\langle J'(u), v\rangle| &\leq \int_{\partial\Omega} V(x)|u|^{q(x)-1}|v|d\sigma \\ &= \int_{\partial\Omega} V(x)^{\frac{1}{q'(x)}}|u|^{q(x)-1}V(x)^{\frac{1}{q(x)}}|v|d\sigma \\ &\leq 2\left|V(x)^{\frac{1}{q'(x)}}|u|^{q(x)-1}\right|_{q'(x),\partial}\left|V(x)^{\frac{1}{q(x)}}|v|\right|_{q(x),\partial} \\ &\leq 2\left(1+\int_{\partial\Omega} V(x)|u|^{q(x)}d\sigma\right)^{\frac{1}{(q')^-}}|v|_{(q(x),V(x))} \\ &\leq 2\left(2+|u|^{q^+}_{(q(x),V(x))}\right)^{\frac{1}{(q')^-}}|v|_{(q(x),V(x))} \\ &\leq 2C\left(2+|u|^{q^+}_{(q(x),V(x))}\right)^{\frac{1}{(q')^-}}\|v\|_a \\ &\leq M\|v\|_a,\end{aligned}$$

where $M = 2C\left(2+|u|^{q^+}_{(q(x),V(x))}\right)^{\frac{1}{(q')^-}}$ is a positive constant. So, $J'(u)$ is a linear and bounded operator, hence continuous, i.e., $J'(u) \in X^*$. The functional J is then Gâteaux differentiable at any $u \in X$ and its Gâteaux derivative is $J'(u)$. Next, we will prove that for $(u_n)_{n\in\mathbb{N}} \subset X$ and $u \in X$, we have

$$u_n \longrightarrow u \quad \text{as } n \longrightarrow \infty \text{ in } X \quad \Longrightarrow \quad J'(u_n) \longrightarrow J'(u) \quad \text{as } n \longrightarrow \infty \text{ in } X^*.$$

Using Remark 3.2, Hölder's inequality, Proposition 2.7, and inequality (3.1), the following

inequalities hold true for all $v \in X$

$$\begin{aligned}\left|\langle J'(u_n) - J'(u), v\rangle\right| &\leq \int_{\partial\Omega} V(x)\left||u_n|^{q(x)-2}u_n - |u|^{q(x)-2}u\right||v|d\sigma \\ &\leq 2\left|V(x)^{\frac{1}{q'(x)}}|u_n|^{q(x)-2}u_n - V(x)^{\frac{1}{q'(x)}}|u|^{q(x)-2}u\right|_{q'(x),\partial}\left|V(x)^{\frac{1}{q(x)}}|v|\right|_{q(x),\partial} \\ &\leq 2C\left|V(x)^{\frac{1}{q'(x)}}|u_n|^{q(x)-2}u_n - V(x)^{\frac{1}{q'(x)}}|u|^{q(x)-2}u\right|_{q'(x),\partial}\|v\|_a.\end{aligned}$$

So, we have

$$\|J'(u_n) - J'(u)\|_{X^*} \leq 2C\left|V(x)^{\frac{1}{q'(x)}}|u_n|^{q(x)-2}u_n - V(x)^{\frac{1}{q'(x)}}|u|^{q(x)-2}u\right|_{q'(x),\partial}. \tag{3.2}$$

Since the mapping $u \mapsto |u|^{q(x)-2}u$ from $L^{q(x)}(\partial\Omega)$ into $L^{q'(x)}(\partial\Omega)$ is continuous, using the compact and continuous embedding $X \hookrightarrow L^{q(x)}_{V(x)}(\partial\Omega)$, we deduce that

$$V(x)^{\frac{1}{q(x)}}u_n \longrightarrow V(x)^{\frac{1}{q(x)}}u \quad \text{as } n \longrightarrow \infty \text{ in } L^{q(x)}(\partial\Omega),$$

and then

$$V(x)^{\frac{1}{q'(x)}}|u_n|^{q(x)-2}u_n \longrightarrow V(x)^{\frac{1}{q'(x)}}|u|^{q(x)-2}u \quad \text{as } n \longrightarrow \infty \text{ in } L^{q'(x)}(\partial\Omega). \tag{3.3}$$

From (3.2) and (3.3), we obtain

$$\|J'(u_n) - J'(u)\|_{X^*} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty,$$

that is, the mapping $J' : X \to X^*$ is continuous. Then $J \in C^1(X, \mathbb{R})$, i.e., J is Fréchet differentiable. By similar computation the functional $\int_{\partial\Omega}\frac{W(x)}{r(x)}|u|^{r(x)}d\sigma$ is of class $C^1(X, \mathbb{R})$. Then $\Psi \in C^1(X, \mathbb{R})$, and for $u, v \in X$, we have

$$\begin{aligned}\langle \Psi'(u), v\rangle = &\int_\Omega \left(|\nabla u|^{p(x)-2}\nabla u\nabla v + a(x)|u|^{p(x)-2}uv\right)dx \\ &- \lambda\int_{\partial\Omega} V(x)|u|^{q(x)-2}uvd\sigma - \mu\int_{\partial\Omega} W(x)|u|^{r(x)-2}uvd\sigma.\end{aligned}$$

Similarly, one can show that $\Psi_+ \in C^1(X, \mathbb{R})$, and for $u, v \in X$, we have

$$\begin{aligned}\langle \Psi'_+(u), v\rangle = &\int_\Omega \left(|\nabla u|^{p(x)-2}\nabla u\nabla v + a(x)|u|^{p(x)-2}uv\right)dx \\ &- \lambda\int_{\partial\Omega} V(x)(u^+)^{q(x)-1}vd\sigma - \mu\int_{\partial\Omega} W(x)(u^+)^{r(x)-1}vd\sigma.\end{aligned}$$

It is easy to see that $u \in X$ is a weak solution (rep. nonnegative weak solution) of $(S_{\lambda,\mu})$ if and only if u is a critical point of Ψ (resp. Ψ_+). Indeed, let $u \in X$ be a critical point of Ψ_+, then

$$\begin{aligned}\int_\Omega \left(|\nabla u|^{p(x)-2}\nabla u\nabla u^- + a(x)|u|^{p(x)-2}uu^-\right)dx - \lambda\int_{\partial\Omega} V(x)(u^+)^{q(x)-1}u^-d\sigma \\ - \mu\int_{\partial\Omega} W(x)(u^+)^{r(x)-1}u^-d\sigma = 0,\end{aligned}$$

which gives $\int_\Omega \left(|\nabla u^-|^{p(x)} + a(x)|u^-|^{p(x)}\right)dx = 0$. From Proposition 2.4 (iv), it follows that $\|u^-\|_a = 0$ and then $u^- = 0$ in X, i.e., u is nonnegative function. Since u is nonnegative, we have $u^+ = u$, then u is also a critical point of Ψ. □

Proposition 3.4. *Ψ and Ψ_+ are sequentially weakly lower semicontinuous functionals.*

Proof. By Proposition 2.5, the functional ϕ is sequentially weakly lower semicontinuous. We have only to show that ψ is also sequentially weakly lower semicontinuous. Let $(u_n)_{n\in\mathbb{N}} \subset X$ be a sequence weakly convergent to some u in X. Let us prove that, up to a subsequence, $(\psi(u_n))_n$ converge strongly to $\psi(u)$ in $\mathbb{R}$. By virtue of the compact embedding $X \hookrightarrow L^{q(x)}_{V(x)}(\partial\Omega)$, we have

$$|u_n - u|_{(q(x),V(x))} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty \quad \Longleftrightarrow \quad \int_{\partial\Omega} V(x)|u_n - u|^{q(x)} d\sigma \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.$$

According to the reciprocal of Lebesque's dominated convergence theorem, there is a subsequence of $(u_n)_n$ still denoted by $(u_n)_n$ such that

$$u_n(x) \longrightarrow u(x) \quad \text{as } n \longrightarrow \infty \text{ for a.e. } x \in \partial\Omega, \qquad V(x)|u_n - u|^{q(x)} \leq g,$$

with $g \in L^1(\partial\Omega)$. Then, we deduce that

$$\frac{V(x)}{q(x)}|u_n|^{q(x)} \longrightarrow \frac{V(x)}{q(x)}|u|^{q(x)} \quad \text{as } n \longrightarrow \infty \text{ for a.e. } x \in \partial\Omega.$$

Using the inequality $(a+b)^q \leq 2^{q-1}(a^q + b^q)$, for $a, b \geq 0$ and $q \geq 1$, we have

$$\begin{aligned}
\frac{V(x)}{q(x)}|u_n|^{q(x)} &\leq V(x)|u_n|^{q(x)} \leq V(x)\left(|u| + |u_n - u|\right)^{q(x)} \\
&\leq 2^{q(x)-1}(V(x)|u|^{q(x)} + V(x)|u_n - u|^{q(x)}) \\
&\leq 2^{q^+-1}(V(x)|u|^{q(x)} + g(x)).
\end{aligned}$$

By Lebesgue's dominated convergence theorem, we obtain

$$\lim_{n\to\infty} \int_{\partial\Omega} \frac{V(x)}{q(x)}|u_n|^{q(x)} d\sigma = \int_{\partial\Omega} \frac{V(x)}{q(x)}|u|^{q(x)} d\sigma. \tag{3.4}$$

Similarly, we also have

$$\lim_{n\to\infty} \int_{\partial\Omega} \frac{W(x)}{r(x)}|u_n|^{r(x)} d\sigma = \int_{\partial\Omega} \frac{W(x)}{r(x)}|u|^{r(x)} d\sigma. \tag{3.5}$$

Then, $\lim_{n\to\infty} \psi(u_n) = \psi(u)$. Since $\liminf_{n\to\infty} \phi(u_n) \geq \phi(u)$, we have

$$\liminf_{n\to\infty} \Psi(u_n) \geq \Psi(u).$$

By similar steps one can show that the functional Ψ_+ is also sequentially weakly lower semicontinuous. □

Proposition 3.5. *The derivative functions Ψ' and Ψ'_+ are of (S^+) type operators.*

Proof. Let $(u_n)_{n\in\mathbb{N}} \subset X$ be a sequence weakly convergent to some u in X such that

$$\limsup_{n\to\infty} \langle \Psi'(u_n) - \Psi'(u), u_n - u \rangle \leq 0.$$

As in the proof of Lemma 3.3, we obtain the following inequalities

$$\begin{aligned}\left|\langle J'(u_n)-J'(u),u_n-u\rangle\right| &\leq \int_{\partial\Omega} V(x)\left||u_n(x)|^{q(x)-2}u_n(x)-|u(x)|^{q(x)-2}u(x)\right||u_n-u|d\sigma\\ &\leq 2\left|V(x)^{\frac{1}{q'(x)}}|u_n|^{q(x)-2}u_n - V^{\frac{1}{q'(x)}}|u|^{q(x)-2}u\right|_{q'(x),\partial}|u_n-u|_{(q(x),V(x))}.\end{aligned}$$

From the compact embedding $X \hookrightarrow L^{q(x)}_{V(x)}(\partial\Omega)$ and the continuity of the mapping $u \mapsto |u|^{q(x)-2}u$ from $L^{q(x)}(\partial\Omega)$ into $L^{q'(x)}(\partial\Omega)$, we obtain

$$\langle J'(u_n)-J'(u),u_n-u\rangle \longrightarrow 0 \quad \text{as } n\longrightarrow\infty.$$

Similarly for the functional $u\mapsto\int_{\partial\Omega}\frac{W(x)}{r(x)}|u|^{r(x)}d\sigma$. Then, $\lim_{n\to\infty}\langle\psi'(u_n)-\psi'(u),u_n-u\rangle=0$. We have

$$\limsup_{n\to\infty}\langle\phi'(u_n)-\phi'(u),u_n-u\rangle=\limsup_{n\to\infty}\langle\Psi'(u_n)-\Psi'(u),u_n-u\rangle\leq 0.$$

Combine the last inequality and the (S^+) property of the operator ϕ' (see Proposition 2.5, it follows that $(u_n)_n$ converge strongly to u in X. The operator Ψ' is then of (S^+) type. Similarly, one can show that the operator Ψ'_+ is also of (S^+) type. □

We need to recall some classical well known results of the theory of Banach spaces which will be useful in the sequel. Let X be a reflexive and separable Banach space, there exist $(e_n)_{n\in\mathbb{N}^*}\subseteq X$ and $(f_n)_{n\in\mathbb{N}^*}\subseteq X^*$ such that $\langle f_n,e_m\rangle=\delta_{nm}$ for all $n,m\in\mathbb{N}^*$, $X=\overline{\text{Span}\{e_n\mid n\in\mathbb{N}^*\}}$, $X^*=\overline{\text{Span}\{f_n\mid n\in\mathbb{N}^*\}}^{w^*}$ (see [49], Section 17), where X^* denote the dual space of X. For $k\in\mathbb{N}^*$, consider the subsets of X :

$$X_k=\text{Span}\{e_k\},\qquad Y_k=\bigoplus_{j=1}^{k}X_j,\qquad Z_k=\overline{\bigoplus_{k}^{\infty}X_j}.$$

Proposition 3.6 ([27]). *Assume that $\psi: X\to\mathbb{R}$ is weakly-strongly continuous and $\psi(0)=0$. Let $r>0$ be given and set*

$$\beta_k=\beta_k(r)=\sup\{|\psi(u)|\mid u\in Z_k,\|u\|_a\leq r\}.$$

Then $\beta_k\to 0$ as $k\to\infty$.

Definition 3.7. Let X be a reflexive Banach space and $\Phi: X\to\mathbb{R}$ be a functional such that $\Phi\in C^1(X,\mathbb{R})$.

(i) The C^1-functional Φ is said to satisfy the Palais–Smale condition (in short (PS) condition) if any sequence $(u_n)_{n\in\mathbb{N}}\subset X$ for which $(\Phi(u_n))_{n\in\mathbb{N}}\subset\mathbb{R}$ is bounded and $\Phi'(u_n)\to 0$ as $n\to\infty$, has a convergent subsequence.

(ii) The C^1-functional Φ is said satisfy the $(PS)_c$ condition, where $c\in\mathbb{R}$, if any sequence $(u_n)_{n\in\mathbb{N}}\subset X$ for which $\Phi(u_n)\to c$ and $\Phi'(u_n)\to 0$ as $n\to\infty$, has a convergent subsequence.

(iii) Let Y_n, $n\in\mathbb{N}^*$, as defined above. The C^1-functional Φ is said to satisfy the $(PS)^*_c$ condition, where $c\in\mathbb{R}$, if any sequence $(u_n)_{n\in\mathbb{N}^*}\subset X$ for which $u_n\in Y_n$ for any $n\in\mathbb{N}^*$, $\Phi(u_n)\to c$ and $(\Phi|_{Y_n})'(u_n)\to 0$ as $n\to\infty$, has a subsequence convergent to a critical point of Φ.

Remark 3.8. It is easy to remark that if Φ satisfies the (PS) condition, then Φ satisfies the $(PS)_c$ condition for every $c\in\mathbb{R}$.

3.2 Abstract theorems

Theorem 3.9 ([46]). *Let X be a reflexive Banach space and $\Phi : X \to \mathbb{R}$ be a functional such that*

(i) Φ is sequentially weakly lower semicontinuous, i.e., for any sequence $(u_n)_n$ and u in X

$$u_n \xrightarrow{\text{weakly}} u \quad \text{as } n \longrightarrow \infty \quad \Longrightarrow \quad \Phi(u) \leq \liminf_{n\to\infty} \Phi(u_n);$$

(ii) Φ is coercive, i.e.,

$$\Phi(u) \longrightarrow \infty \quad \text{as } \|u\|_X \longrightarrow \infty.$$

Then Φ is bounded below and its minimum is achieved at a point $u_0 \in X$.

Theorem 3.10 (Mountain Pass Theorem, [48]). *Let X be a real Banach space and $\Phi \in C^1(X, \mathbb{R})$ be a functional satisfying the Palais-Smale (PS) condition. Suppose $\Phi(0) = 0$ and*

(i) there are constants $\alpha, \rho > 0$ such that $\Phi|_{\|x\|=\alpha} \geq \rho$;

(ii) there is an element $e \in X$ such that $\Phi(e) \leq 0$.

Then Φ possesses a critical value $c \geq \rho$. Moreover, c can be characterized as

$$c = \inf_{g\in\Gamma} \max_{t\in[0,1]} \Phi(g(t)) \text{ where } \Gamma = \{g \in C([0,1], X) : g(0) = 0,\ g(1) = e\}.$$

Theorem 3.11 (Fountain Theorem, [48]). *Let X be a reflexive and separable Banach space, $\Phi \in C^1(X, \mathbb{R})$ an even functional and the subspaces X_k, Y_k, Z_k as defined above. If for each $k \in \mathbb{N}^*$ there exist $\rho_k > r_k > 0$ such that*

(F1) $\inf_{x\in Z_k,\ \|x\|=r_k} \Phi(x) \to \infty$ *as* $k \to \infty$;

(F2) $\max_{x\in Y_k,\ \|x\|=\rho_k} \Phi(x) \leq 0$;

(F3) Φ satisfies the $(PS)_c$ condition for every $c > 0$.

Then, Φ has a sequence of critical values tending to $+\infty$.

Theorem 3.12 (Dual Fountain Theorem, [48]). *Let X be a reflexive and separable Banach space, $\Phi \in C^1(X, \mathbb{R})$ an even functional and the subspaces X_k, Y_k, Z_k as defined above. Assume there is a $k_0 \in \mathbb{N}^*$ such that, for each $k \in \mathbb{N}^*$, $k \geq k_0$, there exist $\rho_k > r_k > 0$ such that*

(D1) $\inf_{x\in Z_k,\ \|x\|=\rho_k} \Phi(x) \geq 0$;

(D2) $b_k := \max_{x\in Y_k,\ \|x\|=r_k} \Phi(x) < 0$;

(D3) $d_k := \inf_{x\in Z_k,\ \|x\|\leq\rho_k} \Phi(x) \to 0$ *as* $k \to \infty$;

(D4) Φ satisfies the $(PS)^*_c$ condition for every $c \in [d_{k_0}, 0)$.

Then, Φ has a sequence of negative critical values converging to 0.

3.3 Technical lemmas

Lemma 3.13. *Assume that hypotheses* ($\mathbf{H_1}$)–($\mathbf{H_2}$)–($\mathbf{H_3}$) *are fulfilled. If* $\min\{q^-, r^-\} > p^+$ *and* $\lambda, \mu \geq 0$, *then there exists* $\varphi \in X \setminus \{0\}$ *such that* $\varphi \geq 0$ *and* $\Psi_+(t\varphi) \to -\infty$ *as* $t \to +\infty$.

Proof. Suppose that $\theta = \min\{q^-, r^-\} > p^+$ and $\lambda, \mu \geq 0$. Choose $\varphi \in C_0^\infty(\Omega)$, $\varphi \geq 0$ and $\varphi \not\equiv 0$. For any $t > 1$, we have

$$\begin{aligned}
\Psi_+(t\varphi) &= \int_\Omega \frac{t^{p(x)}}{p(x)} \left(|\nabla \varphi(x)|^{p(x)} + a(x)|\varphi(x)|^{p(x)} \right) dx \\
&\quad - \lambda \int_{\partial\Omega} \frac{t^{q(x)}}{q(x)} V(x)\varphi(x)^{q(x)} d\sigma - \mu \int_{\partial\Omega} \frac{t^{r(x)}}{r(x)} W(x)\varphi(x)^{r(x)} d\sigma \\
&\leq \frac{t^{p^+}}{p^-} \int_\Omega \left(|\nabla \varphi(x)|^{p(x)} + a(x)|\varphi(x)|^{p(x)} \right) dx \\
&\quad - \frac{\lambda t^{q^-}}{q^+} \int_{\partial\Omega} W(x)\varphi(x)^{q(x)} d\sigma - \frac{\mu t^{r^-}}{r^+} \int_{\partial\Omega} W(x)\varphi(x)^{r(x)} d\sigma \\
&\leq \frac{t^{p^+}}{p^-} \int_\Omega \left(|\nabla \varphi(x)|^{p(x)} + a(x)|\varphi(x)|^{p(x)} \right) dx \\
&\quad - t^\theta \left(\frac{\lambda}{q^+} \int_{\partial\Omega} V(x)\varphi(x)^{q(x)} d\sigma + \frac{\mu}{r^+} \int_{\partial\Omega} W(x)\varphi(x)^{r(x)} d\sigma \right).
\end{aligned}$$

Since $\theta > p^+$, it follows that $\Psi_+(t\varphi) \to -\infty$ as $t \to +\infty$. □

Lemma 3.14. *Assume that hypotheses* ($\mathbf{H_1}$)–($\mathbf{H_2}$)–($\mathbf{H_3}$) *are fulfilled.*

(i) If $\min\{q^-, r^-\} > p^+$ *and* $\lambda, \mu \geq 0$, *then there exist* $\tau, \rho > 0$ *such that* $\Psi_+(u) \geq \tau$ *if* $\|u\|_a = \rho$.

(ii) If $p^+ < r^-$, *then for each given* $\mu \in \mathbb{R}$, *there exists* $\lambda^0 = \lambda^0(\mu) > 0$ *such that, for any* $\lambda < \lambda^0$, *there exist* $\tau, \rho > 0$ *such that* $\Psi_+(u) \geq \tau$ *if* $\|u\|_a = \rho$.

(iii) If $q^+ < p^-$, *then for each given* $\lambda \in \mathbb{R}$, *there exists* $\mu^0 = \mu^0(\lambda) > 0$ *such that, for any* $\mu < \mu^0$, *there exist* $\overline{\tau}, \overline{\rho} > 0$ *such that* $\Psi_+(u) \geq \overline{\tau}$ *if* $\|u\|_a = \overline{\rho}$.

Proof. We have seen, from (3.1), that there exists a positive constant $C > 1$ such that

$$|u|_{(q(x),V(x))} \leq C\|u\|_a \text{ and } |u|_{(r(x),W(x))} \leq C\|u\|_a, \text{ for all } u \in X.$$

(i) Suppose that $\min\{q^-, r^-\} > p^+$ and $\lambda, \mu \geq 0$. Fix a positive constant ρ such that

$$\rho < \frac{1}{C} < 1 \text{ and } \rho \leq \min\left\{ \left(\frac{q^-}{4\lambda p^+ C^{q^-}} \right)^{\frac{1}{q^- - p^+}}, \left(\frac{r^-}{4\mu p^+ C^{r^-}} \right)^{\frac{1}{r^- - p^+}} \right\}.$$

Then, using Proposition 2.4, Proposition 2.7 and inequalities (3.1), for all $u \in X$ with $\|u\|_a =$

$\rho < 1$, we have

$$\begin{aligned}
\Psi_+(u) \geq & \frac{1}{p^+}\int_\Omega \left(|\nabla u(x)|^{p(x)} + a(x)|u(x)|^{p(x)}\right) dx \\
& - \frac{\lambda}{q^-}\int_{\partial\Omega} V(x)|u(x)|^{q(x)}d\sigma - \frac{\mu}{r^-}\int_{\partial\Omega} W(x)|u(x)|^{r(x)}d\sigma \\
\geq & \frac{1}{p^+}\rho_a(u) - \frac{\lambda}{q^-}|u|^{q^-}_{(q(x),V(x))} - \frac{\mu}{r^-}|u|^{r^-}_{(r(x),W(x))}\} \\
\geq & \frac{1}{p^+}\|u\|_a^{p^+} - \frac{\lambda C^{q^-}}{q^-}\|u\|_a^{q^-} - \frac{\mu C^{r^-}}{r^-}\|u\|_a^{r^-} \\
= & \frac{1}{p^+}\rho^{p^+} - \frac{\lambda C^{q^-}}{q^-}\rho^{q^-} - \frac{\mu C^{r^-}}{r^-}\rho^{r^-} \\
\geq & \frac{1}{p^+}\rho^{p^+} - \frac{1}{4p^+}\rho^{p^+} - \frac{1}{4p^+}\rho^{p^+} \\
= & \frac{1}{2p^+}\rho^{p^+}.
\end{aligned}$$

Then, there exist $\rho > 0$ and $\tau = \frac{1}{2p^+}\rho^{p^+} > 0$ such that $\Psi_+(u) \geq \tau$ for all $u \in X$ with $\|u\|_a = \rho$.

(ii) Suppose that $p^+ < h^-$. For each given $\mu \in \mathbb{R}$, fix ρ such that

$$0 < \rho < \frac{1}{C} \quad \text{and} \quad \frac{|\mu|C^{r^-}}{r^-}\rho^{r^-} \leq \frac{1}{2p^+}\rho^{p^+}.$$

Then, using Proposition 2.4, Proposition 2.7 and inequalities (3.1), for all $u \in X$ with $\|u\|_a = \rho < 1$, we have

$$\begin{aligned}
\Psi_+(u) \geq & \frac{1}{p^+}\int_\Omega \left(|\nabla u(x)|^{p(x)} + a(x)|u(x)|^{p(x)}\right) dx \\
& - \frac{\max\{\lambda,0\}}{q^-}\int_{\partial\Omega} V(x)|u(x)|^{q(x)}d\sigma - \frac{|\mu|}{r^-}\int_{\partial\Omega} W(x)|u(x)|^{r(x)}d\sigma \\
\geq & \frac{1}{p^+}\rho_a(u) - \frac{\max\{\lambda,0\}}{q^-}|u|^{q^-}_{(q(x),V(x))} - \frac{|\mu|}{r^-}|u|^{r^-}_{(r(x),W(x))} \\
\geq & \frac{1}{p^+}\|u\|_a^{p^+} - \frac{\max\{\lambda,0\}C^{q^-}}{q^-}\|u\|_a^{q^-} - \frac{|\mu|C^{r^-}}{r^-}\|u\|_a^{r^-} \\
\geq & \frac{1}{2p^+}\rho^{p^+} - \frac{\max\{\lambda,0\}C^{q^-}}{q^-}\rho^{q^-}.
\end{aligned}$$

We can choose

$$\lambda^0 = \frac{q^-\rho^{p^+-q^-}}{4p^+C^{q^-}} \quad \text{and} \quad \tau = \frac{1}{4p^+}\rho^{p^+}.$$

Then, for any $\lambda < \lambda^0$, $\Psi_+(u) \geq \tau > 0$ for all $u \in X$ with $\|u\|_a = \rho$.

(iii) From Proposition 2.7 and inequalities (3.1), for all $u \in X$, we have

$$\begin{aligned}
&\int_{\partial\Omega} V(x)|u(x)|^{q(x)}d\sigma \leq \max\{C^{q^-}\|u\|_a^{q^-}, C^{q^+}\|u\|_a^{q^+}\}, \\
&\int_{\partial\Omega} W(x)|u(x)|^{r(x)}d\sigma \leq \max\{C^{r^-}\|u\|_a^{r^-}, C^{r^+}\|u\|_a^{r^+}\}.
\end{aligned} \tag{3.6}$$

Suppose that $q^+ < p^-$. For each given $\lambda \in \mathbb{R}$, setting $\overline{\rho} \geq 1$ and $\frac{|\lambda| C^{q^+}}{q^-}\overline{\rho}^{q^+} \leq \frac{1}{2p^+}\overline{\rho}^{p^-}$. Then, using Proposition 2.4 and inequalities (3.6), for all $u \in X$ with $\|u\|_a = \overline{\rho} \geq 1$, we have

$$\begin{aligned}\Psi_+(u) &\geq \frac{1}{p^+}\int_\Omega \left(|\nabla u(x)|^{p(x)} + a(x)|u(x)|^{p(x)}\right) dx \\ &\quad - \frac{|\lambda|}{q^-}\int_{\partial\Omega} V(x)|u(x)|^{q(x)}d\sigma - \frac{\max\{\mu,0\}}{h^-}\int_{\partial\Omega} W(x)|u(x)|^{r(x)}d\sigma \\ &\geq \frac{1}{p^+}\|u\|_a^{p^-} - \frac{|\lambda|C^{q^+}}{q^-}\|u\|_a^{q^+} - \frac{\max\{\mu,0\}C^{r^+}}{r^-}\|u\|_a^{r^+} \\ &\geq \frac{1}{2p^+}\overline{\rho}^{p^-} - \frac{\max\{\mu,0\}C^{r^+}}{r^-}\overline{\rho}^{r^+}.\end{aligned} \tag{3.7}$$

We can choose

$$\mu^0 = \frac{r^-\overline{\rho}^{p^- - r^+}}{4p^+C^{r^+}} \quad \text{and} \quad \overline{\tau} = \frac{1}{4p^+}\overline{\rho}^{p^-}.$$

Then, for any $\mu < \mu^0$, $\Psi_+(u) \geq \overline{\tau} > 0$ for all $u \in X$ with $\|u\|_a = \overline{\rho}$. □

Lemma 3.15. *Assume that hypotheses* ($\mathbf{H_1}$)–($\mathbf{H_2}$)–($\mathbf{H_3}$) *are fulfilled. Then, the functional* Ψ_+ *satisfies* (PS) *condition if one of the following assertions is satisfied :*

(i) $\min\{q^-, r^-\} > p^+$ *and* $\lambda, \mu \geq 0$;

(ii) $q^+ < p^- \leq p^+ < r^-$ *and* $\lambda \in \mathbb{R}$, $\mu \geq 0$;

(iii) $q^+ = p^- \leq p^+ < r^-$ *and* $\lambda \in \mathbb{R}$ *with* $|\lambda| < \lambda_0 := \frac{\frac{1}{p^+} - \frac{1}{r^-}}{(\frac{1}{q^-} - \frac{1}{r^-})C^{q^+}}$, $\mu \geq 0$;

(iv) $p^+ < r^-$, $q^+ \leq r^-$ *and* $\lambda \leq 0$, $\mu \geq 0$.

Proof. In the first time, we show that every (PS) sequence of Ψ_+ is bounded in X under condition (i)–(iv). Let $(u_n)_{n\in\mathbb{N}}$ be a (PS) sequence of Ψ_+, that is

$$|\Psi_+(u_n)| < M, \quad \text{for all } n \in \mathbb{N}, \text{ with } M > 0 \quad \text{and} \quad \Psi'_+(u_n) \to 0 \quad \text{in } X^* \text{ as } n \to \infty.$$

Arguing by contradiction, let us suppose that

$$\|u_n\|_a \to +\infty \quad \text{as} \quad n \to \infty.$$

Case (i), assume that $\theta = \min\{q^-, r^-\} > p^+$ and $\lambda, \mu \geq 0$. If $\|u_n\|_a > 1$ for n large enough, using Proposition 2.4, we have

$$\begin{aligned}M + \|u_n\|_a &\geq \Psi_+(u_n) - \frac{1}{\theta}\langle \Psi'_+(u_n), u_n\rangle \\ &\geq \frac{1}{p^+}\int_\Omega \left(|\nabla u_n(x)|^{p(x)} + a(x)|u_n(x)|^{p(x)}\right) dx \\ &\quad - \frac{\lambda}{q^-}\int_{\partial\Omega} V(x)(u_n^+)^{q(x)}d\sigma - \frac{\mu}{r^-}\int_{\partial\Omega} W(x)(u_n^+)^{r(x)}d\sigma \\ &\quad - \frac{1}{\theta}\int_\Omega \left(|\nabla u_n(x)|^{p(x)} + a(x)|u_n(x)|^{p(x)}\right) dx \\ &\quad + \frac{\lambda}{\theta}\int_{\partial\Omega} V(x)(u_n^+)^{q(x)}d\sigma + \frac{\mu}{\theta}\int_{\partial\Omega} W(x)(u_n^+)^{r(x)}d\sigma\end{aligned}$$

$$\geq \left(\frac{1}{p^+} - \frac{1}{\theta}\right) \int_\Omega \left(|\nabla u_n(x)|^{p(x)} + a(x)|u_n(x)|^{p(x)}\right) dx$$
$$\geq \left(\frac{1}{p^+} - \frac{1}{\theta}\right) \|u_n\|_a^{p^-}.$$

From the last inequality, for $n \in \mathbb{N}$ large, we obtain a contradiction since $p^- > 1$. So, the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in X.
Cases (ii) and (iii), assume that $q^+, p^+ \leq r^-$ and $\lambda \in \mathbb{R}, \mu \geq 0$. If $\|u_n\|_a > 1$ for any $n \in \mathbb{N}$, then

$$\begin{aligned} M + \|u_n\|_a &\geq \Psi_+(u_n) - \frac{1}{r^-}\langle \Psi'_+(u_n), u_n\rangle \\ &\geq \left(\frac{1}{p^+} - \frac{1}{r^-}\right) \int_\Omega \left(|\nabla u_n(x)|^{p(x)} + a(x)|u_n(x)|^{p(x)}\right) dx \\ &\quad - \lambda \int_{\partial\Omega} \frac{1}{q(x)} V(x)(u_n^+)^{q(x)} d\sigma + \frac{\lambda}{r^-} \int_{\partial\Omega} V(x)(u_n^+)^{q(x)} d\sigma \\ &\geq \left(\frac{1}{p^+} - \frac{1}{r^-}\right) \int_\Omega \left(|\nabla u_n(x)|^{p(x)} + a(x)|u_n(x)|^{p(x)}\right) dx \\ &\quad - |\lambda| \left(\frac{1}{q^-} - \frac{1}{r^-}\right) \int_{\partial\Omega} V(x)(u_n^+)^{q(x)} d\sigma. \end{aligned} \tag{3.8}$$

Taking into account inequalities (3.6), inequality (3.8) and Proposition 2.4 imply that

$$M + \|u_n\|_a \geq \left(\frac{1}{p^+} - \frac{1}{r^-}\right) \|u_n\|_a^{p^-} - \left(\frac{1}{q^-} - \frac{1}{r^-}\right) |\lambda| C^{q^+} \|u_n\|_a^{q^+}.$$

This last inequality yields a contradiction if $n \in \mathbb{N}$ is large, since $p^- > 1$, $q^+ \leq p^-$ and $|\lambda| < \lambda_0$. So, the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in X in both cases (ii) and (iii).
Case (iv), by a similar computation as above, we have

$$M + \|u_n\|_a \geq \left(\frac{1}{p^+} - \frac{1}{r^-}\right) \|u_n\|_a^{p^-},$$

which gives a contradiction if $n \in \mathbb{N}$ is large, since $p^- > 1$.
Thus, the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in X.

Next, we show that there is a subsequence of $(u_n)_{n\in\mathbb{N}}$ strongly convergent in X. Because of the reflexiveness of X, since $(u_n)_{n\in\mathbb{N}}$ is bounded in X, we can extract a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ weakly convergent to some $u \in X$. We have

$$\begin{aligned} |\langle \Psi'_+(u_{n_k}) - \Psi'_+(u), u_{n_k} - u\rangle| &\leq |\langle \Psi'_+(u_{n_k}), u_{n_k} - u\rangle| + |\langle \Psi'_+(u), u_{n_k} - u\rangle| \\ &\leq |\langle \Psi'_+(u_{n_k}), u_{n_k}\rangle| + |\langle \Psi'_+(u_{n_k}), u\rangle| \\ &\quad + |\langle \Psi'_+(u), u_{n_k}\rangle - \langle \Psi'_+(u), u\rangle| \\ &\leq \|\Psi'_+(u_{n_k})\|_{X^*} \|u_{n_k}\|_a + \|\Psi'_+(u_{n_k})\|_{X^*} \|u\|_a \\ &\quad + |\langle \Psi'_+(u), u_{n_k}\rangle - \langle \Psi'_+(u), u\rangle|. \end{aligned} \tag{3.9}$$

Since $\Psi'_+(u_{n_k}) \longrightarrow 0$ in X^* as $k \longrightarrow \infty$ and $(u_{n_k})_{k\in\mathbb{N}}$ is bounded in X, inequality (3.9) and the weak convergence $u_{n_k} \xrightarrow{\text{weakly}} u$ as $k \to \infty$ yield

$$\lim_{k\to\infty} \langle \Psi'_+(u_{n_k}) - \Psi'_+(u), u_{n_k} - u\rangle = 0. \tag{3.10}$$

Combine (3.10) and the (S^+) property of Ψ'_+, we obtain that the subsequence $(u_{n_k})_{k\in\mathbb{N}}$ converges strongly to u in X. □

Remark 3.16. All statements of Lemmas 3.13–3.15 remain valid if we replace Ψ_+ by Ψ.

4 Proof of main theorems

4.1 Proof of Theorem 1.1

(i) Assume that $\max\{q^+, r^+\} < p^-$ and $\lambda, \mu \in \mathbb{R}$. From Lemma 3.4 the C^1-functional Ψ_+ is sequentially weakly lower semicontinuous, we only have to show that Ψ_+ is coercive to apply Theorem 3.9. Let $u \in X$ be such that $\|u\|_a > 1$, from Proposition 2.4 and inequalities (3.6), we have

$$\begin{aligned}\Psi_+(u) &\geq \frac{1}{p^+}\int_\Omega \left(|\nabla u(x)|^{p(x)} + a(x)|u(x)|^{p(x)}\right) dx \\ &\quad - \frac{|\lambda|}{q^-}\int_{\partial\Omega} V(x)|u(x)|^{q(x)}d\sigma - \frac{|\mu|}{r^-}\int_{\partial\Omega} W(x)|u(x)|^{r(x)}d\sigma \\ &\geq \frac{1}{p^+}\|u\|_a^{p^-} - \frac{|\lambda|C^{q^+}}{q^-}\|u\|_a^{q^+} - \frac{|\mu|C^{r^+}}{r^-}\|u\|_a^{r^+},\end{aligned}$$

Since $\max\{q^+, r^+\} < p^-$, we have $\Psi_+(u) \to \infty$ as $\|u\|_a \to \infty$, then Ψ_+ is coercive. Applying Theorem 3.9, the functional Ψ_+ has a minimum point u_0 in X which is a nonnegative solution of $(S_{\lambda,\mu})$, as a critical point of Ψ_+.

(ii) Assume that $\min\{q^-, r^-\} > p^+$ and $\lambda, \mu \geq 0$. From Lemma 3.14 (i), there exist $\tau, \rho > 0$ such that $\Psi_+(u) \geq \tau$ if $\|u\|_a = \rho$. From Lemma 3.13, there exists $\varphi \in X \setminus \{0\}$ such that $\varphi \geq 0$ and $\Psi_+(t\varphi) \to -\infty$ as $t \to +\infty$. Therefore, for $t > 1$ large enough, there is $e = t\varphi \in X$ such that $\|e\|_a > \rho$ and $\Psi_+(e) < 0$. Moreover, from Lemma 3.15 (i), Ψ_+ satisfies the (PS) condition. Then, the functional Ψ_+ satisfies all requirements of the mountain geometry. Consequently, according to the Mountain Pass Theorem (see Theorem 3.10), Ψ_+ admits a critical value $c \geq \tau$ which is characterized by

$$c = \inf_{g\in\Gamma}\max_{t\in[0,1]} \Psi_+(g(t)), \quad \text{where } \Gamma = \{g \in C([0,1], X) : g(0) = 0,\ g(1) = e\}.$$

We deduce the existence of a critical point u of Ψ_+ associated with the critical value c satisfying $\Psi_+(u) \geq \tau$. Hence, the Steklov problem $(S_{\lambda,\mu})$ has a nontrivial nonnegative solution.

4.2 Proof of Theorem 1.2

(i) Assume $1 < p^- \leq p^+ < r^- \leq q^-$ and $\lambda, \mu \geq 0$. We will prove that the functional Ψ satisfies the conditions of Theorem 3.11. Obviously, from Lemma 3.3 and Lemma 3.15, $\Psi \in C^1(X, \mathbb{R})$ is an even functional and satisfies (PS) condition. Then, Ψ satisfies also $(PS)_c$ condition for every $c > 0$. Next, we will prove that if k is large enough, then there exist $\rho_k > r_k > 0$ such that conditions (F1) and (F2) of Theorem 3.11 hold. For $r > 0$ and $k \in \mathbb{N}^*$, let

$$\beta_k(r) = \sup\{|\psi(u)| \mid u \in Z_k, \|u\|_a \leq r\},$$

with $\psi(u) = \lambda\int_{\partial\Omega}\frac{V(x)}{q(x)}|u|^{q(x)}d\sigma + \mu\int_{\partial\Omega}\frac{W(x)}{r(x)}|u|^{r(x)}d\sigma$. The functional ψ is weakly-strongly continuous (see Lemma 3.4). Since $\beta_k(r) \to 0$ as $k \to \infty$ (see Proposition 3.6), it follows that, for each $n \in \mathbb{N}^*$, there exists $k_0(n) \in \mathbb{N}^*$ such that $\beta_k(n) \leq 1$ if $k \geq k_0(n)$. Without loss of generality, we may assume that $k_0(n) < k_0(n+1)$ for each $n \in \mathbb{N}^*$. Let us define the sequence r_k, $k \in \mathbb{N}^*$, by

$$r_k = \begin{cases} 1 & \text{if } 1 \leq k < k_0(1), \\ n & \text{if } k_0(n) \leq k < k_0(n+1). \end{cases}$$

It is easy to see that $r_k \to \infty$ as $k \to \infty$. For $k \in \mathbb{N}^*$ large enough, let $u \in Z_k$ with $\|u\|_a = r_k$, we have

$$\Psi(u) = \int_\Omega \frac{1}{p(x)} \left(|\nabla u|^{p(x)} dx + a(x)|u|^{p(x)} \right) dx - \psi(u) \geq \frac{1}{p^+} (r_k)^{p^-} - 1,$$

because $\psi(u) \leq \beta_k(n) \leq 1$. Therefore,

$$\inf_{u \in Z_k,\ \|u\|_a = r_k} \Psi(u) \to \infty \quad \text{as } k \to \infty.$$

So, the condition (F1) of Theorem 3.11 is satisfied.

Since Y_k is a finite dimensional space, the norms $\|\cdot\|_a$, $|\cdot|_{(q(x),V(x))}$ and $|\cdot|_{(r(x),W(x))}$ are equivalent. Then, there exist positive constants δ_1, δ_2 such that $|u|_{(q(x),V(x))} \geq \delta_1 \|u\|_a$ and $|u|_{(r(x),W(x))} \geq \delta_2 \|u\|_a$, for all $u \in Y_k$. For $k \in \mathbb{N}^*$, let $u \in Y_k$ with $\|u\|_a > 1$ large enough such that $|u|_{(q(x),V(x))} > 1$ and $|u|_{(r(x),W(x))} > 1$, from Proposition 2.7, we deduce that

$$\begin{aligned}
\Psi(u) &\leq \frac{1}{p^-} \|u\|_a^{p^+} - \frac{\lambda}{q^+} \int_{\partial\Omega} V(x)|u|^{q(x)} d\sigma - \frac{\mu}{r^+} \int_{\partial\Omega} W(x)|u|^{r(x)} d\sigma \\
&\leq \frac{1}{p^-} \|u\|_a^{p^+} - \frac{\lambda}{q^+} |u|^{q^-}_{(q(x),V(x))} - \frac{\mu}{r^+} |u|^{r^-}_{(r(x),W(x))} \\
&\leq \frac{1}{p^-} \|u\|_a^{p^+} - \frac{\lambda}{q^+} (|u|^{q^-}_{(q(x),V(x))} - 1) - \frac{\mu}{r^+} (|u|^{r^-}_{(r(x),W(x))} - 1 \\
&\leq \frac{1}{p^-} \|u\|_a^{p^+} - \frac{\lambda}{q^+} (\delta_1^{q^-} \|u\|_a^{q^-} - 1) - \frac{\mu}{r^+} (\delta_2^{r^-} \|u\|_a^{r^-} - 1) \\
&\leq \frac{1}{p^-} \|u\|_a^{p^+} - (\frac{\lambda \delta_1^{q^-}}{q^+} + \frac{\mu \delta_2^{r^-}}{r^+}) \|u\|_a^\theta + C(\lambda, \mu, q^+, r^+),
\end{aligned} \tag{4.1}$$

where $\theta = \min\{q^-, r^-\}$ and $C(\lambda, \mu, q^+, r^+) > 0$ a constant. From the last inequality, since $\theta > p^+$, we obtain that $\Psi(u) \to -\infty$ as $\|u\|_a \to \infty$. Thus for each $k \in \mathbb{N}^*$, there exists a sufficiently large $\rho_k > r_k$ such that $\Psi(u) < 0$ for all $u \in Y_k$ with $\|u\|_a = \rho_k$. Therefore,

$$\max_{u \in Y_k,\ \|u\|_a = \rho_k} \Psi(u) \leq 0.$$

So, the condition (F2) of Theorem 3.11 is satisfied.

According to the Fountain Theorem 3.11, the functional Ψ has a sequence of critical values $(c_n)_n$ tending to $+\infty$, which correspond to a sequence $(u_n)_n \subset X$ of critical points of Ψ such that $\Psi(u_n) \longrightarrow +\infty$ as $n \longrightarrow +\infty$.

(ii)–(iii) Assume $q^+ \leq p^- \leq p^+ < h^-$, $\lambda \in \mathbb{R}$ and $\mu > 0$. We will prove that the functional Ψ satisfies the conditions of Theorem 3.11. As above, there exists $r_k \to \infty$ as $k \to \infty$ such that for $k \in \mathbb{N}^*$ large enough, $u \in Z_k$ with $\|u\|_a = r_k$, we have

$$\Psi(u) = \int_\Omega \frac{1}{p(x)} \left(|\nabla u|^{p(x)} dx + a(x)|u|^{p(x)} \right) dx - \psi(u) \geq \frac{1}{p^+} (r_k)^{p^-} - 1.$$

Therefore,

$$\inf_{u \in Z_k,\ \|u\|_a = r_k} \Psi(u) \to \infty \quad \text{as } k \to \infty.$$

So, the condition (F1) of Theorem 3.11 is satisfied.

For $k \in \mathbb{N}^*$, let $u \in Y_k$ with $\|u\|_a > 1$, from Proposition 2.7, we deduce that

$$\begin{aligned}\Psi(u) &\leq \frac{1}{p^-}\|u\|_a^{p^+} + \frac{|\lambda|}{q^-}\left(|u|^{q^+}_{(q(x),V(x))} + 1\right) - \frac{\mu}{r^+}\left(|u|^{r^-}_{(r(x),W(x))} - 1\right) \\ &\leq \frac{1}{p^-}\|u\|_a^{p^+} + \frac{|\lambda|\delta_1^{q^+}}{q^-}\|u\|_a^{q^+} - \frac{\mu\delta_2^{r^-}}{r^+}\|u\|_a^{r^-} + C(\lambda,\mu,q^-,r^+),\end{aligned} \tag{4.2}$$

where $C(\lambda,\mu,q^-,r^+) > 0$ is a constant and δ_1, δ_2 as in the proof of (i). From the last inequality, since $q^+ \leq p^+ < r^-$, we obtain that $\Psi(u) \to -\infty$ as $\|u\|_a \to \infty$. Thus for each $k \in \mathbb{N}^*$, there exists a sufficiently large $\rho_k > r_k$ such that $\Psi(u) < 0$ for all $u \in Y_k$ with $\|u\|_a = \rho_k$. Therefore,

$$\max_{u\in Y_k,\ \|u\|_a=\rho_k} \Psi(u) \leq 0.$$

So, the condition (F2) of Theorem 3.11 is satisfied.

By Theorem 3.11, there is a sequence $(u_n)_n \subset X$ of critical points of Ψ such that $\Psi(u_n) \longrightarrow +\infty$ as $n \longrightarrow +\infty$.

(iv)–(v) Assume $q^+ < \min\{p^-, r^-\}$, $\lambda > 0$ and $\mu \in \mathbb{R}$. We will prove that the functional Ψ satisfies the conditions of Theorem 3.12. We have already seen that $\Psi \in C^1(X,\mathbb{R})$ is an even functional. For $r > 0$ and $k \in \mathbb{N}^*$, let

$$\beta_k(r) = \sup\{|\psi(u)| \mid u \in Z_k, \|u\|_a \leq r\},$$

with $\psi(u) = \lambda\int_{\partial\Omega}\frac{V(x)}{q(x)}|u|^{q(x)}d\sigma + \mu\int_{\partial\Omega}\frac{W(x)}{r(x)}|u|^{r(x)}d\sigma$. The functional ψ is weakly-strongly continuous (see Lemma 3.4). Since $\beta_k(r) \to 0$ as $k \to \infty$ (see Proposition 3.6), then there exists $k_0 \in \mathbb{N}^*$ such that $\beta_k(1) \leq \frac{1}{2p^+}$ if $k \geq k_0$. Setting $\rho_k = 1$, then for $k \geq k_0$ and $u \in Z_k$ with $\|u\|_a = \rho_k = 1$, we have

$$\Psi(u) \geq \frac{1}{p^+}\rho_a(u) - \psi(u) \geq \frac{1}{p^+} - \frac{1}{2p^+} = \frac{1}{2p^+} > 0.$$

Therefore,

$$\inf_{u\in Z_k,\ \|u\|_a=\rho_k} \Psi(u) \geq 0.$$

So, the condition (D1) of Theorem 3.12 is satisfied.

Let r be a positive small real number such that $0 < r < \frac{1}{C}$, where $C > 1$ appears in (3.1). As above, since Y_k is a finite dimensional space, there are positive constants δ_1, δ_2 such that for all $u \in Y_k$ with $\|u\|_a = r < 1$, we have

$$\Psi(u) \leq \frac{1}{p^-}\|u\|_a^{p^-} - \frac{\lambda\delta_1^{q^+}}{q^+}\|u\|_a^{q^+} + \frac{|\mu|\delta_2^{r^-}}{r^-}\|u\|_a^{r^-}.$$

Since $q^+ < \min\{p^-, r^-\}$, we can choose $r_k \in (0, \rho_k)$ small enough such that $\Psi(u) < 0$ for all $u \in Y_k$ with $\|u\|_a = r_k$. Therefore,

$$\max_{u\in Y_k,\ \|u\|_a=r_k} \Psi(u) < 0.$$

So, the condition (D2) of Theorem 3.12 is satisfied.

From the definition of Y_k and Z_k, it follows that $Y_k \cap Z_k \neq \emptyset$ for each $k \in \mathbb{N}^*$. Let $u_0 \in Y_k \cap Z_k$ with $u_0 \not\equiv 0$, and set $u_k = \frac{r_k}{\|u_0\|_a}u_0$ for each $k \geq k_0$, then $\|u_k\|_a = r_k$. For $k \geq k_0$, since $0 < r_k < \rho_k$, we have

$$d_k := \inf_{u\in Z_k,\ \|u\|_a\leq\rho_k} \Psi(u) \leq \Psi(u_k) \leq b_k := \max_{u\in Y_k,\ \|u\|_a=r_k} \Psi(u) < 0.$$

For $u \in Z_k$ with $\|u\|_a \leq \rho_k = 1$, we have

$$\Psi(u) = \phi(u) - \psi(u) \geq -\psi(u) \geq -\beta_k(1),$$

and then

$$-\beta_k(1) \leq d_k := \inf_{u \in Z_k,\ \|u\|_a \leq \rho_k} \Psi(u) < 0.$$

Since $\beta_k(1) \to 0$ as $k \to \infty$, we deduce from the last inequality that

$$d_k := \inf_{u \in Z_k,\ \|u\|_a \leq \rho_k} \Psi(u) \to 0 \text{ as } k \to \infty.$$

So, the condition (D3) of Theorem 3.12 is satisfied.

Next, we will prove that Ψ satisfy condition $(PS)_c^*$ for every $c \in \mathbb{R}$. Let $(u_n)_{n \in \mathbb{N}^*} \subset X$ and $c \in \mathbb{R}$ such that

$$u_n \in Y_n \quad \text{for each } n \in \mathbb{N}^*, \qquad \Psi(u_n) \longrightarrow c \quad \text{and} \quad (\Psi|_{Y_n})'(u_n) \longrightarrow 0 \quad \text{as } n \to \infty.$$

Let us show that the sequence (u_n) is bounded in X. Arguing by contradiction, assume $\|u_n\|_a \to \infty$ as $n \to \infty$. For $\lambda > 0, \mu \geq 0$ and $n \in \mathbb{N}^*$ large enough such that $\|u_n\|_a > 1$, we have

$$\begin{aligned} c + 1 + \|u_n\|_a \geq & \Psi(u_n) - \frac{1}{r^-} \langle (\Psi|_{Y_n})'(u_n), u_n \rangle \\ = & \Psi(u_n) - \frac{1}{r^-} \langle \Psi'(u_n), u_n \rangle \\ \geq & \left(\frac{1}{p^+} - \frac{1}{r^-} \right) \|u_n\|_a^{p^-} - \lambda \left(\frac{1}{q^-} - \frac{1}{r^-} \right) C^{q^+} \|u_n\|_a^{q^+}. \end{aligned}$$

Since $p^- > 1$, we obtain a contradiction for a large $n \in \mathbb{N}^*$. For $\lambda > 0, \mu \leq 0$ and $n \in \mathbb{N}^*$ large enough such that $\|u_n\|_a > 1$, we have

$$\begin{aligned} c + 1 + \|u_n\|_a \geq & \Psi(u_n) - \frac{1}{r^+} \langle (\Psi|_{Y_n})'(u_n), u_n \rangle \\ = & \Psi(u_n) - \frac{1}{r^+} \langle \Psi'(u_n), u_n \rangle \\ \geq & \left(\frac{1}{p^+} - \frac{1}{r^+} \right) \|u_n\|_a^{p^-} - \lambda \left(\frac{1}{q^-} - \frac{1}{r^+} \right) C^{q^+} \|u_n\|_a^{q^+}. \end{aligned}$$

Since $p^- > 1$, we obtain a contradiction for a large $n \in \mathbb{N}^*$. Then, (u_n) is bounded in X. As X is a reflexive Banach space, we can extract a subsequence $(u_{n_k})_{k \in \mathbb{N}^*}$ of $(u_n)_{n \in \mathbb{N}^*}$ weakly convergent to some $u \in X$. We can write X as

$$X = \overline{\bigcup_{n \in \mathbb{N}^*} Y_n}.$$

By choosing $(v_n)_{n \in \mathbb{N}^*}$ such that

$$v_n \in Y_n \quad \text{for each } n \in \mathbb{N}^* \quad \text{and} \quad \lim_{n \to \infty} v_n = u \quad \text{in } X,$$

we obtain

$$\begin{aligned} \lim_{k \to \infty} \langle \Psi'(u_{n_k}), u_{n_k} - u \rangle &= \lim_{k \to \infty} \langle \Psi'(u_{n_k}), u_{n_k} - v_{n_k} \rangle + \lim_{k \to \infty} \langle \Psi'(u_{n_k}), v_{n_k} - u \rangle \\ &= \lim_{k \to \infty} \langle (\Psi|_{Y_{n_k}})'(u_{n_k}), u_{n_k} - v_{n_k} \rangle = 0, \end{aligned}$$

because of $(\Psi|_{Y_{n_k}})'(u_{n_k}) \to 0$ as $k \to \infty$, $u_{n_k} - v_{n_k} \rightharpoonup 0$ in Y_{n_k} and $v_{n_k} \to u \in X$. Then, we deduce that

$$\lim_{k\to\infty} \langle \Psi'(u_{n_k}) - \Psi'(u), u_{n_k} - u \rangle = 0.$$

By the (S^+) property of the functional Ψ, we get the strong convergence of $(u_{n_k})_{k\in\mathbb{N}^*}$ to u in X. It remains to prove that u is a critical point of Ψ. Choosing an arbitrary $w_n \in Y_n$, for any $n_k \geq n$, we have

$$\begin{aligned} \langle \Psi'(u), w_n \rangle =& \langle \Psi'(u) - \Psi'(u_{n_k}), w_n \rangle + \langle \Psi'(u_{n_k}), w_n \rangle \\ =& \langle \Psi'(u) - \Psi'(u_{n_k}), w_n \rangle + \langle (\Psi|_{Y_{n_k}})'(u_{n_k}), w_n \rangle. \end{aligned} \tag{4.3}$$

Since $\Psi \in C^1(X, \mathbb{R})$ and $\lim_{k\to\infty} u_{n_k} = u$ in X, then $\lim_{k\to\infty} \Psi'(u_{n_k}) = \Psi'(u)$. In expression (4.3), letting $k \to \infty$ we deduce that $\langle \Psi'(u), w_n \rangle = 0$, for all $w_n \in Y_n$, it follows that $\Psi'(u) = 0$. Consequently, Ψ satisfies the $(PS)^*_c$ condition for every $c \in \mathbb{R}$.

According to the Dual Fountain Theorem 3.12, the functional Ψ has a sequence of negative critical values $(c_n)_n$ converging to 0, which correspond to a sequence $(u_n)_n \subset X$ of critical points of Ψ such that $\Psi(u_n) \leq 0$, for each $n \in \mathbb{N}$ and $\Psi(u_n) \longrightarrow 0$ as $n \longrightarrow +\infty$.

Appendix

Some examples of variable exponents illustrating (i)–(ii) of Theorem 1.1

Example 1 : Suppose that $N = 1$ and $\Omega = [a, b]$ the closed set of $\mathbb{R}$.

(i) Consider the continuous functions $p, q, r : \Omega = [a, b] \longrightarrow (1, +\infty)$ defined by :

$$p(x) = 4 + \frac{1}{1 + (x - c)^2}, \text{ with } c \in [a, b];\ q(x) = 1 + \sin(x);\ r(x) = 2 + \cos(x).$$

We have

$$\max\{q^+, r^+\} = \max\{2, 3\} = 3 < 4 < p^-.$$

(ii) Consider the continuous functions $p, q, r : \Omega = [a, b] \longrightarrow (1, +\infty)$ defined by :

$$p(x) = 1 + \frac{1}{1 + (x - c)^2}, \text{ with } c \in [a, b];\ q(x) = 4 + \sin(x);\ r(x) = 5 + \cos(x).$$

We have

$$\min\{q^-, r^-\} = \min\{3, 4\} = 3 > p^+ = 2.$$

Example 2 : Suppose that $N = 2$ and $\Omega = [0, 1] \times [0, 1]$ the closed set of $\mathbb{R}^2$.

(i) Consider the continuous functions $p, q, r : \Omega \longrightarrow (1, +\infty)$ defined by :

$$\begin{aligned} p(x, y) &= 5 + \frac{1}{1 + x^2 + y^2}; \\ q(x, y) &= 2 + \sin(\pi x)\cos(\pi x); \\ r(x, y) &= 3 + \cos(\pi x)\sin(\pi x). \end{aligned}$$

We have

$$\max\{q^+, r^+\} = \max\{3, 4\} = 4 < p^- = \frac{16}{3}.$$

(ii) Consider the continuous functions $p, q, r : \Omega = [a, b] \longrightarrow (1, +\infty)$ defined by :

$$\begin{aligned} p(x,y) &= 2 + \frac{1}{1+x^2+y^2}; \\ q(x,y) &= 5 + \sin(\pi x)\cos(\pi x); \\ r(x,y) &= 6 + \cos(\pi x)\sin(\pi x). \end{aligned}$$

We have

$$\min\{q^-, r^-\} = \max\{4, 5\} = 4 > p^+ = 3.$$

Some examples of variable exponents illustrating (i)–(v) of Theorem 1.2

Example 1 : Suppose that $N = 1$ and $\Omega = [a, b]$ the closed set of $\mathbb{R}$.

(i) Consider the continuous functions $p, q, r : \Omega = [a, b] \longrightarrow (1, +\infty)$ defined by :

$$p(x) = 1 + \frac{1}{1+(x-c)^2}, \quad \text{with } c \in [a, b]; \qquad q(x) = 5 + \sin(x); \qquad r(x) = 4 + \cos(x).$$

We have

$$p^+ = 2 < r^- = 3 < q^- = 4.$$

(ii) Consider the continuous functions $p, q, r : \Omega = [a, b] \longrightarrow (1, +\infty)$ defined by :

$$p(x) = 4 + \sin(x); \qquad q(x) = 1 + \frac{1}{1+(x-c)^2}, \quad \text{with } c \in [a, b]; \qquad r(x) = 7 + \cos(x).$$

We have

$$q^+ = 2 < p^- = 3 < p^+ = 5 < r^- = 6.$$

(iii) Consider the continuous functions $p, q, r : \Omega = [a, b] \longrightarrow (1, +\infty)$ defined by :

$$p(x) = 5 + \sin(x); \qquad q(x) = 3 + \frac{1}{1+(x-c)^2}, \quad \text{with } c \in [a, b]; \qquad r(x) = 6 + \cos(x).$$

We have

$$q^+ = p^- = 4 < r^- = 5.$$

(iv) The same example of (ii).

(v) Consider the continuous functions $p, q, r : \Omega = [a, b] \longrightarrow (1, +\infty)$ defined by :

$$p(x) = 4 + \sin(x); \qquad q(x) = 1 + \frac{1}{1+(x-c)^2}, \quad \text{with } c \in [a, b]; \qquad r(x) = 5 + \cos(x).$$

We have

$$q^+ = 2 < \min\{p^-, r^-\} = 3 \quad \text{and} \quad p^+ = 5 < r^+ = 6.$$

Example 2 : Suppose that $N = 2$ and $\Omega = [0, 1] \times [0, 1]$ the closed set of $\mathbb{R}^2$.

(i) Consider the continuous functions $p, q, r : \Omega = [a, b] \longrightarrow (1, +\infty)$ defined by :

$$\begin{aligned} p(x,y) &= 2 + \frac{1}{1+x^2+y^2}; \\ (x,y) &= 6 + \sin(\pi x)\cos(\pi y); \\ r(x,y) &= 5 + \cos(\pi x)\sin(\pi y). \end{aligned}$$

We have

$$p^+ = 3 < r^- = 4 < q^- = 5.$$

(ii) Consider the continuous functions $p, q, r : \Omega \longrightarrow (1, +\infty)$ defined by :

$$\begin{aligned} p(x,y) &= 5 + \sin(\pi x)\cos(\pi y); \\ q(x,y) &= 2 + \frac{1}{1+x^2+y^2}; \\ r(x,y) &= 8 + \cos(\pi x)\sin(\pi x). \end{aligned}$$

We have

$$q^+ = 3 < p^- = 4 < p^+ = 6 < r^- = 7.$$

(iii) Consider the continuous functions $p, q, r : \Omega \longrightarrow (1, +\infty)$ defined by :

$$\begin{aligned} p(x,y) &= 5 + \sin(\pi x)\cos(\pi y); \\ q(x,y) &= 3 + \frac{1}{1+x^2+y^2}; \\ r(x,y) &= 8 + \cos(\pi x)\sin(\pi x). \end{aligned}$$

We have

$$q^+ = 4 = p^- < p^+ = 6 < r^- = 7.$$

(iv) The same example of (ii).

(v) Consider the continuous functions $p, q, r : \Omega \longrightarrow (1, +\infty)$ defined by :

$$\begin{aligned} p(x,y) &= 5 + \sin(\pi x)\cos(\pi y); \\ q(x,y) &= 2 + \frac{1}{1+x^2+y^2}; \\ r(x,y) &= 6 + \cos(\pi x)\sin(\pi x). \end{aligned}$$

We have

$$q^+ = 3 < \min\{p^-, r^-\} = 4 \quad \text{and} \quad p^+ = 6 < r^+ = 7.$$

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A critical elliptic equation with a logarithmic type perturbation

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Abstract. In this note, we consider a critical elliptic equation perturbed by a logarithmic type subcritical term in $\mathbb{R}^4$, and investigate how the logarithmic term affects the existence of weak solutions to such a problem. Since the logarithmic term does not satisfy the standard monotonicity condition, essential difficulty arises when one looks for weak solutions to this problem in the variational framework. After some delicate estimates on the logarithmic term we can control the mountain pass level of the corresponding functional so that it satisfies the local compactness condition. Then a positive weak solution follows with the application of the Mountain Pass Lemma and the Brézis–Lieb Lemma. Our result implies that the logarithmic term plays a positive role for the problem to admit positive solutions.

Keywords: critical, logarithmic type perturbation, Brézis–Lieb Lemma.

2020 Mathematics Subject Classification: 35J61, 35J20.

1 Introduction and the main results

In this note, we confine ourselves to the following critical elliptic problem with a logarithmic type perturbation

$$\begin{cases} -\Delta u = \lambda |u|^{q-2} u \ln u^2 + u^3, & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^4$ is a bounded smooth domain with boundary $\partial\Omega$, $q \in (2,4)$ and $\lambda > 0$ is a parameter. Without loss of generality, we may assume that $0 \in \Omega$.

It is clear that problem (1.1) is a special form of the following elliptic boundary value problem

$$\begin{cases} (-\Delta)^m u = \mu u + f(x,u) + |u|^{p-2}u, & x \in \Omega, \\ D^\alpha u = 0, \quad for\ |\alpha| \le m-1, & x \in \partial\Omega, \end{cases} \tag{1.2}$$

where $m \in \mathbb{N}$, $\Omega \subset \mathbb{R}^N (N > 2m)$ is a bounded smooth domain, $\mu \in \mathbb{R}$, $p = \frac{2N}{N-2m}$ and $f(x,u)$ is a nonlinear term. One of the main features of the equation in (1.2) is that it involves a critical

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term in the sense that the embedding $H_0^m(\Omega) \hookrightarrow L^p(\Omega)$ is not compact, which brings essential difficulty in proving the existence of weak solutions to (1.2). Therefore, the study of elliptic problems with critical exponents is not only interesting, but also challenging. However, it was mainly after the pioneering work of Brézis and Nirenberg [2] that such problems were extensively investigated, and remarkable progress has been made on the existence, nonexistence and multiplicity of weak solutions to such problems.

On the other hand, partial differential equations with logarithmic nonlinearities have also been attracting more and more attention in recent years due to their wide applications in many applied sciences. Interested readers may refer to [3,5,8–11] and the references therein for the study of different kinds of equations with logarithmic nonlinearity. For example, when $m = 1$, $f(x,u) = \mu u \ln u^2$ ($\mu \neq 0$), Deng et al. [3] considered the existence and nonexistence of positive solutions to problem (1.2). With the help of some delicate estimates on the logarithmic term, they showed, among many other interesting results, that problem (1.2) admits a positive mountain pass type solution when $N \geq 4$, $\lambda \in \mathbb{R}$ and $\mu > 0$, which is also a ground state solution. Later, the main results in [3] were extended to second order elliptic system by Hajaiej et al. [4] and to critical bi-harmonic equation by Li et al. [5].

It is worth pointing out that the exponent of the logarithmic term in all the above mentioned literature is $q = 2$, and it seems that there is no result for $q > 2$. Motivated mainly by [3,5,11] we will consider problem (1.1) and investigate what role the logarithmic term plays for the problem to admit weak solutions. Since the logarithmic term does not satisfy the standard monotonicity condition, additional difficulty arises when we look for weak solutions to problem (1.1) by using variational methods. A key step in this process is to control the mountain pass level of the corresponding energy functional from above by a small constant such that it satisfies the (PS) condition locally. This is done with the help of some very delicate estimates on the logarithmic function of the truncated Talenti's functions. Then by combining the Mountain Pass Lemma with the Brézis–Lieb Lemma, we can show for $q \in (2,4)$ that problem (1.1) admits a positive mountain pass type solution for all $\lambda > 0$, which, compared with the case $\lambda = 0$ (see Remark 1.2 below), means that the logarithmic type subcritical perturbation plays a positive role for problem (1.1) to admit positive solutions.

The notations used in this paper are almost standard. We use $\|\cdot\|_p$ to denote the $L^p(\Omega)$ norm for $1 \leq p \leq \infty$, and equip the Sobolev space $H_0^1(\Omega)$ with the norm $\|u\| := \|u\|_{H_0^1(\Omega)} = \|\nabla u\|_2$. The dual space of $H_0^1(\Omega)$ is denoted by $H^{-1}(\Omega)$ and the dual pair between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$ is written as $\langle\cdot,\cdot\rangle$. For each Banach space B, we use $\to$ and $\rightharpoonup$ to denote the strong and weak convergence in it, respectively. We also use C, C_1, C_2, ... to denote (possibly different) positive constants. The positive constant S denotes the best embedding constant from $H_0^1(\Omega)$ to $L^4(\Omega)$, i.e.,

$$S = \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\|u\|^2}{\|u\|_4^2}. \tag{1.3}$$

Finally, $B_r(x_0)$ is a ball of radius r centered at x_0 and $B_r(0)$ is simply written as B_r when no confusion arises. The symbol $O(t)$ means $|\frac{O(t)}{t}| \leq C$ as $t \to 0$ and $o_n(1)$ is an infinitesimal as $n \to \infty$.

The energy functional associated with problem (1.1) and its Fréchet derivative are denoted respectively by $I(u)$ and $I'(u)$, i.e.,

$$I(u) = \frac{1}{2}\|u\|^2 + \frac{2\lambda}{q^2}\|u\|_q^q - \frac{\lambda}{q}\int_\Omega |u|^q \ln u^2 \mathrm{d}x - \frac{1}{4}\|u\|_4^4, \qquad u \in H_0^1(\Omega), \tag{1.4}$$

and

$$\langle I'(u),\phi\rangle = \int_\Omega \nabla u\nabla\phi \mathrm{d}x - \lambda\int_\Omega |u|^{q-2}u\phi \ln u^2 \mathrm{d}x - \int_\Omega u^3\phi \mathrm{d}x, \qquad u,\phi\in H_0^1(\Omega). \tag{1.5}$$

It is obviously that $I(u)$ is a C^1 functional on $H_0^1(\Omega)$, and each critical point of $I(u)$ is also a weak solution to problem (1.1). Since $I(u) = I(|u|)$, we may assume that $u \geq 0$ in the sequel. The main result of this paper is the following theorem.

Theorem 1.1. *Assume that $q \in (2,4)$. Then problem (1.1) admits a positive mountain pass type solution u for all $\lambda > 0$.*

Remark 1.2. In [6], Pohožaev showed that problem (1.1) with $\lambda = 0$ admits no positive solution if Ω is star-shaped. However, Theorem 1.1 shows that this situation can be reversed when the logarithmic term is introduced. This means that the logarithmic term plays a positive role for problem (1.1) to admit positive solutions.

Remark 1.3. The solution obtained in Theorem 1.1 is actually a ground state solution to problem (1.1). To see this, set

$$\mathcal{N} = \left\{u\in H_0^1(\Omega)\setminus\{0\} : \langle I'(u),u\rangle = 0\right\},$$

$c_1 = \inf_{u\in\mathcal{N}} I(u)$ and $c_2 = \inf_{u\in H_0^1(\Omega)\setminus\{0\}}\max_{t\geq 0} I(tu)$. Then it is easily to verify that $c_1 = c_2$. Let c_0 be the mountain pass level defined in Lemma 2.7. Since $q\in(2,4)$, one sees that $I(tu) < 0$ for $u\in H_0^1(\Omega)\setminus\{0\}$ and t large enough, which implies that $c_0 \leq c_2$. Noticing that the Nehari's manifold $\mathcal{N}$ separates $H_0^1(\Omega)$ into two components. The component containing the origin also contains a small ball around the origin. Moreover $I(u)\geq 0$ for all u in this component, by the assumption $q\in(2,4)$. Thus every $\gamma\in\Gamma$ has to cross $\mathcal{N}$ and $c_1 \leq c_0$. Therefore, $c_0 = c_1$ and the solution u obtained in Theorem 1.1 is a ground state solution.

Remark 1.4. When $q = 2$ and $N = 3,4$, Deng et al. [3] also considered the case that $\lambda < 0$, and obtained a positive solution by using the Mountain Pass Lemma without compactness condition. However, it seems quite difficult to derive the same conclusion for $q\in(2,4)$. This can be seen from (2.23) and (2.24). Indeed, from (2.23) and (2.24) we know that compared with $\int_\Omega |u_\varepsilon|^q \ln u_\varepsilon^2 \mathrm{d}x$, $\|u_\varepsilon\|_q^q$ is an infinitesimal of higher order as $\varepsilon\to 0$ and consequently we can not control the mountain pass level as was done in [3]. So the treatment for the case $q\in(2,4)$ must be different from those in [3] and new ideas and techniques may be needed.

2 Proof of the main result

We begin this section with some definitions, lemmas and basic inequalities, which will be used in the proof of the main result.

Definition 2.1. ($(PS)_c$ condition) Assume that B is a real Banach space, $I : B\to\mathbb{R}$ is a C^1 functional and $c\in\mathbb{R}$. We say that I satisfies the $(PS)_c$ condition if any sequence $\{u_n\}\subset B$ such that

$$I(u_n)\to c \quad\text{and}\quad I'(u_n)\to 0 \quad \text{in } B' \text{ (the dual space of } B) \quad \text{as } n\to\infty$$

has a convergent subsequence.

Lemma 2.2 (Mountain Pass Lemma [5,7]). *Assume that $(B, \|\cdot\|_B)$ is a real Banach space, $I : B \to \mathbb{R}$ is a C^1 functional and there exist $\alpha > 0$ and $\rho > 0$ such that I satisfies the following mountain pass geometry*

(i) $I(u) \geq \alpha > 0$ *if* $\|u\|_B = \rho$;

(ii) There exists a $\overline{u} \in B$ *such that* $\|\overline{u}\|_B > \rho$ *and* $I(\overline{u}) < 0$.

Then there exist a sequence $\{u_n\} \subset B$ *such that* $I(u_n) \to c_0$ *and* $I'(u_n) \to 0$ *in* B' *as* $n \to \infty$, *where*

$$c_0 := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I(\gamma(t)) \geq \alpha, \qquad \Gamma = \{\gamma \in C([0,1], B) : \gamma(0) = 0, \gamma(1) = \overline{u}\}.$$

Furthermore, c_0 *(which is usually called the mountain level) is a critical value of* I *if* I *satisfies the* $(PS)_{c_0}$ *condition.*

Lemma 2.3 (Brézis–Lieb Lemma [1]). *Let* $p \in (1, \infty)$. *Suppose that* $\{u_n\}$ *is a bounded sequence in* $L^p(\Omega)$ *and* $u_n \to u$ *a.e. in* Ω. *Then*

$$\lim_{n\to\infty} (\|u_n\|_p^p - \|u_n - u\|_p^p) = \|u\|_p^p.$$

Lemma 2.4. *For any* $\delta > 0$, *there exists a positive constant* C_δ *such that*

$$|\ln t| \leq C_\delta(t^\delta + t^{-\delta}), \qquad \forall\, t > 0, \tag{2.1}$$

$$\ln t \leq \frac{t^\delta}{e\delta}, \qquad \forall\, t \geq 1. \tag{2.2}$$

In order to apply the Mountain Pass Lemma, we then verify the mountain pass geometry for I around 0 when $q \in (2, 4)$ and $\lambda > 0$.

Lemma 2.5. *Assume that* $q \in (2, 4)$ *and* $\lambda > 0$. *Then* $I(u)$ *satisfies the mountain pass geometry around* 0.

Proof. By recalling (2.2) with $\delta \in (0, 4 - q)$ and making use of the Sobolev embedding inequality, one has

$$\begin{aligned} I(u) &= \frac{1}{2}\|u\|^2 + \frac{2\lambda}{q^2}\|u\|_q^q - \frac{\lambda}{q}\int_\Omega |u|^q \ln u^2 \mathrm{d}x - \frac{1}{4}\|u\|_4^4 \\ &\geq \frac{1}{2}\|u\|^2 - \frac{\lambda}{q}\int_{\Omega_1} |u|^q \ln u^2 \mathrm{d}x - \frac{\lambda}{q}\int_{\Omega_2} |u|^q \ln u^2 \mathrm{d}x - \frac{1}{4}\|u\|_4^4 \\ &\geq \frac{1}{2}\|u\|^2 - C\|u\|^{q+\delta} - \frac{1}{4}S^{-2}\|u\|^4 \\ &= \|u\|^2\left(\frac{1}{2} - C\|u\|^{q+\delta-2} - \frac{1}{4}S^{-2}\|u\|^2\right), \end{aligned}$$

where $\Omega_1 = \{x \in \Omega : |u(x)| < 1\}$, $\Omega_2 = \{x \in \Omega : |u(x)| \geq 1\}$. Hence, there exist positive constants α and ρ such that

$$I(u) \geq \alpha \quad \text{for all } \|u\| = \rho.$$

On the other hand, for any $v \in H_0^1(\Omega) \setminus \{0\}$, one has $\lim_{t\to\infty} I(tv) = -\infty$ since $q < 4$, which ensures that there exists a $t(v) > 0$ such that $\|t(v)v\| > \rho$ and $I(t(v)v) < 0$. Thus, I satisfies the mountain pass geometry around 0. The proof is complete. □

Next, we shall verify that I satisfies the $(PS)_c$ condition when $c < \frac{1}{4}S^2$. This condition will play a key role in showing the existence of weak solutions to problem (1.1).

Lemma 2.6. *Assume that $q \in (2,4)$ and $\lambda > 0$. Then $I(u)$ satisfies the $(PS)_c$ condition when $c < \frac{1}{4}S^2$.*

Proof. Let $\{u_n\} \subset H_0^1(\Omega)$ be a (PS) sequence for $I(u)$ at the level c with $c < \frac{1}{4}S^2$, i.e., $I(u_n) \to c$ and $I'(u_n) \to 0$ in $H^{-1}(\Omega)$ as $n \to \infty$. First, we claim that $\{u_n\}$ is bounded. Indeed, from the definition of the (PS) sequence, one obtains, for n suitably large, that

$$\begin{aligned} c + 1 + o(1)\|u_n\| &\geq I(u_n) - \frac{1}{q}\langle I'(u_n), u_n\rangle \\ &= \left(\frac{1}{2} - \frac{1}{q}\right)\|u_n\|^2 + \frac{2\lambda}{q^2}\|u_n\|_q^q + \left(\frac{1}{q} - \frac{1}{4}\right)\|u_n\|_4^4 \\ &\geq \left(\frac{1}{2} - \frac{1}{q}\right)\|u_n\|^2, \end{aligned}$$

which implies that $\{u_n\}$ is bounded in $H_0^1(\Omega)$. Consequently, there is a subsequence of $\{u_n\}$ (still denoted by $\{u_n\}$) such that, as $n \to \infty$,

$$\begin{cases} u_n \rightharpoonup u & \text{in } H_0^1(\Omega), \\ u_n \to u & \text{in } L^r(\Omega)\ (1 \leq r < 4), \\ u_n \rightharpoonup u & \text{in } L^4(\Omega), \\ |u_n|^2 u_n \rightharpoonup |u|^2 u & \text{in } L^{\frac{4}{3}}(\Omega), \\ u_n \to u & \text{a.e. in } \Omega. \end{cases} \tag{2.3}$$

Next, we show that

$$\lim_{n\to\infty} \int_\Omega |u_n|^q \ln u_n^2 \mathrm{d}x = \int_\Omega |u|^q \ln u^2 \mathrm{d}x. \tag{2.4}$$

Indeed, since $u_n \to u$ a.e. in Ω as $n \to \infty$, we get

$$|u_n|^q \ln u_n^2 \to |u|^q \ln u^2 \quad \text{a.e. in } \Omega. \tag{2.5}$$

Moreover, by recalling (2.2) with $\delta < 4 - q$ and (2.3), one has

$$\big||u_n|^q \ln u_n^2\big| \leq C_\delta(|u_n|^{q-\delta} + |u_n|^{q+\delta}) \to C_\delta(|u|^{q-\delta} + |u|^{q+\delta}) \quad \text{in } L^1(\Omega). \tag{2.6}$$

Then (2.4) follows from (2.5), (2.6) and Lebesgue's dominated convergence theorem. Similarly,

$$\lim_{n\to\infty} \int_\Omega |u_n|^{q-2} u_n \phi \ln u_n^2 \mathrm{d}x = \int_\Omega |u|^{q-2} u\phi \ln u^2 \mathrm{d}x, \qquad \forall\ \phi \in H_0^1(\Omega). \tag{2.7}$$

To prove that $u_n \to u$ in $H_0^1(\Omega)$ as $n \to \infty$, set $w_n = u_n - u$. Then $\{w_n\}$ is also a bounded sequence in $H_0^1(\Omega)$, and therefore there exists a subsequence of $\{w_n\}$ (which we still denote by $\{w_n\}$) such that

$$\lim_{n\to\infty} \|w_n\|^2 = l \geq 0. \tag{2.8}$$

We claim that $l = 0$. Indeed, in view of the weak convergence $u_n \rightharpoonup u$ in $H_0^1(\Omega)$, we have

$$\|u_n\|^2 = \|w_n\|^2 + \|u\|^2 + o_n(1), \qquad n \to \infty. \tag{2.9}$$

Moreover, (2.3) implies that we can apply the Brézis–Lieb Lemma to obtain

$$\|u_n\|_4^4 = \|w_n\|_4^4 + \|u\|_4^4 + o_n(1), \qquad n \to \infty. \tag{2.10}$$

With (2.3), (2.4), (2.7), (2.9), (2.10) at hand and recalling the boundedness of $\{u_n\}$ and the assumption that $I'(u_n) \to 0$ in $H^{-1}(\Omega)$, we have

$$o_n(1) = \langle I'(u_n), u_n\rangle = \langle I'(u), u\rangle + \|w_n\|^2 - \|w_n\|_4^4 + o_n(1), \qquad n \to \infty, \tag{2.11}$$

and

$$o_n(1) = \langle I'(u_n), \phi\rangle = \langle I'(u), \phi\rangle + o_n(1), \qquad n \to \infty,\ \forall\ \phi \in H_0^1(\Omega). \tag{2.12}$$

It follows from (2.12) that u is a weak solution to problem (1.1).

Choosing $\phi = u$ in (2.12), one obtains

$$\langle I'(u), u\rangle = 0, \tag{2.13}$$

which, together with (2.11), implies that

$$\|w_n\|^2 - \|w_n\|_4^4 = o_n(1), \qquad n \to \infty. \tag{2.14}$$

In addition, by the Sobolev embedding one has

$$\|w_n\|_4^4 \le S^{-2}\|w_n\|^4, \qquad \forall\ n \in \mathbb{N}. \tag{2.15}$$

Letting $n \to \infty$ on both sides of (2.15) and recalling (2.8) and (2.14), one arrives at

$$l \le S^{-2} l^2. \tag{2.16}$$

If $l > 0$, then

$$l \ge S^2. \tag{2.17}$$

On one hand, by virtue of (2.13), we have

$$I(u) = I(u) - \frac{1}{q}\langle I'(u), u\rangle = \left(\frac{1}{2} - \frac{1}{q}\right)\|u\|^2 + \frac{2\lambda}{q^2}\|u\|_q^q + \left(\frac{1}{q} - \frac{1}{4}\right)\|u\|_4^4 \ge 0,$$

since $q \in (2,4)$ and $\lambda > 0$.

On the other hand, in view of (2.3), (2.4), (2.9), (2.10) and the fact that $I(u_n) = c + o_n(1)$ as $n \to \infty$, we deduce that

$$o_n(1) + c = I(u_n) = I(u) + \frac{1}{2}\|w_n\|^2 - \frac{1}{4}\|w_n\|_4^4 + o_n(1), \qquad n \to \infty,$$

which yields

$$I(u) = c - \frac{1}{2}\|w_n\|^2 + \frac{1}{4}\|w_n\|_4^4 + o_n(1), \qquad n \to \infty.$$

From this and recalling (2.8), (2.14) and (2.17) one has

$$I(u) = c - \left(\frac{1}{2} - \frac{1}{4}\right) l \le c - \frac{1}{4}S^2 < 0,$$

which is a contradiction. Therefore, we have $\lim_{n\to\infty}\|w_n\|^2 = l = 0$, i.e., $u_n \to u$ in $H_0^1(\Omega)$ as $n \to \infty$. The proof is complete. □

Lemma 2.7. *Assume that $q \in (2,4)$ and $\lambda > 0$. If there exists a $v \in H_0^1(\Omega) \setminus \{0\}$ such that*

$$\sup_{t\geq 0} I(tv) < \frac{1}{4}S^2, \tag{2.18}$$

then problem (1.1) possesses a nontrivial weak solution.

Proof. From Lemma 2.5, we know that I satisfies the mountain pass geometry around 0. Consequently, in view of Lemma 2.2, there exists a sequence $\{u_n\} \subset H_0^1(\Omega)$ such that $I(u_n) \to c_0$ and $I'(u_n) \to 0$ in $H^{-1}(\Omega)$ as $n \to \infty$, where

$$0 < \alpha \leq c_0 = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I(\gamma(t)), \quad \Gamma = \{\gamma \in C([0,1], H_0^1(\Omega)) : \gamma(0) = 0, \gamma(1) = t(v)v\}, \tag{2.19}$$

and $t(v) > 0$ is the constant determined in the proof of Lemma 2.5. According to (2.18), we have

$$c_0 \leq \max_{t\in[0,1]} I(tt(v)v) \leq \sup_{t\geq 0} I(tv) < \frac{1}{4}S^2. \tag{2.20}$$

Then by (2.20) and Lemma 2.6, we know that there exists a convergent subsequence of $\{u_n\}$ (still denoted by $\{u_n\}$), such that $u_n \to u$ in $H_0^1(\Omega)$ as $n \to \infty$, which implies that $I(u) = c_0$ and $I'(u) = 0$, i.e., u is a nontrivial mountain pass type solution to problem (1.1). The proof is complete. □

From Lemma 2.7 it is easily seen that a nontrivial nonnegative mountain pass type solution to problem (1.1) follows once we can find a $v \in H_0^1(\Omega) \setminus \{0\}$ such that (2.18) holds. This is done with the help of some delicate estimates on the truncated Talenti's functions, inspired mainly by Brézis and Nirenberg [2], Deng et al. [3] and Li et al. [5].

For any $\varepsilon > 0$, define

$$U_\varepsilon(x) = \frac{\sqrt{8}\varepsilon}{\varepsilon^2 + |x|^2}, \qquad x \in \mathbb{R}^4. \tag{2.21}$$

It is well known that $U_\varepsilon(x)$ is a solution to the critical problem

$$-\Delta u = u^3, \qquad x \in \mathbb{R}^4,$$

which is also a minimizer for S.

Lemma 2.8. *Let $\varphi \in C_0^\infty(\Omega)$ satisfy $0 \leq \varphi(x) \leq 1$ in Ω and*

$$\varphi(x) = \begin{cases} 1, & |x| < R, \\ 0, & |x| > 2R, \end{cases}$$

where $R > 0$ is a constant such that $B_{2R}(0) \subset \Omega$. Set $u_\varepsilon(x) = \varphi(x)U_\varepsilon(x)$. Then, as $\varepsilon \to 0$,

$$\begin{aligned} \|u_\varepsilon\|^2 &= S^2 + O(\varepsilon^2), \\ \|u_\varepsilon\|_4^4 &= S^2 + O(\varepsilon^4), \end{aligned} \tag{2.22}$$

$$\|u_\varepsilon\|_q^q = O(\varepsilon^{4-q}), \qquad 2 < q < 4, \tag{2.23}$$

and

$$\int_\Omega |u_\varepsilon|^q \ln u_\varepsilon^2 dx = C\varepsilon^{4-q}\ln\left(\frac{1}{\varepsilon}\right) + O(\varepsilon^{4-q}). \tag{2.24}$$

Proof. (2.22) is well known and can be referred, for example, to [2] and [3]. We only prove (2.23) and (2.24).

By the definition of u_ε, one has

$$\begin{aligned}\|u_\varepsilon\|_q^q &= C\int_{B_{2R}} \frac{\varphi^q(x)\varepsilon^q}{(\varepsilon^2+|x|^2)^q}\mathrm{d}x\\ &= C\int_{B_R} \frac{\varepsilon^q}{(\varepsilon^2+|x|^2)^q}\mathrm{d}x + C\int_{B_{2R}\setminus B_R} \frac{\varphi^q(x)\varepsilon^q}{(\varepsilon^2+|x|^2)^q}\mathrm{d}x\\ &= I_1 + O(\varepsilon^q).\end{aligned}$$

By direct computation, we obtain

$$\begin{aligned}I_1 &= C\int_{B_{R/\varepsilon}} \frac{\varepsilon^q\cdot\varepsilon^4}{\varepsilon^{2q}(1+|y|^2)^q}\mathrm{d}y = C\varepsilon^{4-q}\int_0^{R/\varepsilon}\frac{r^3}{(1+r^2)^q}\mathrm{d}r\\ &= C\varepsilon^{4-q}\left(\int_0^\infty \frac{r^3}{(1+r^2)^q}\mathrm{d}r - \int_{R/\varepsilon}^\infty \frac{r^3}{(1+r^2)^q}\mathrm{d}r\right)\\ &= C\varepsilon^{4-q} - O(\varepsilon^q),\end{aligned}$$

where we have used the fact that

$$\left|\int_{R/\varepsilon}^\infty \frac{r^3}{(1+r^2)^q}\mathrm{d}r\right| \le \int_{R/\varepsilon}^\infty r^{3-2q}\mathrm{d}r = O(\varepsilon^{2q-4}).$$

Therefore,

$$\|u_\varepsilon\|_q^q = C\varepsilon^{4-q} + O(\varepsilon^q) = O(\varepsilon^{4-q}).$$

Next we prove (2.24). Set

$$\begin{aligned}J &= \int_\Omega |u_\varepsilon|^q \ln u_\varepsilon^2 \mathrm{d}x\\ &= \int_\Omega \varphi^q(x)U_\varepsilon^q(x)\ln\varphi^2(x)\mathrm{d}x + \int_\Omega \varphi^q(x)U_\varepsilon^q(x)\ln U_\varepsilon^2(x)\mathrm{d}x\\ &= \int_\Omega \varphi^q(x)U_\varepsilon^q(x)\ln\varphi^2(x)\mathrm{d}x + \int_{B_R} U_\varepsilon^q(x)\ln U_\varepsilon^2(x)\mathrm{d}x + \int_{B_{2R}\setminus B_R}\varphi^q(x)U_\varepsilon^q(x)\ln U_\varepsilon^2(x)\mathrm{d}x\\ &= J_1 + J_2 + J_3,\end{aligned} \tag{2.25}$$

where

$$|J_1| = \left|\int_{B_{2R}\setminus B_R}\varphi^q(x)U_\varepsilon^q(x)\ln\varphi^2(x)\mathrm{d}x\right| \le C\int_{B_{2R}\setminus B_R} U_\varepsilon^q(x)\mathrm{d}x = O(\varepsilon^q). \tag{2.26}$$

By applying (2.1) with $\delta_1 \in (0, 2q-4)$ to J_3, one has

$$|J_3| \le C_{\delta_1}\int_{B_{2R}\setminus B_R}\left(U_\varepsilon^{q-\delta_1}(x) + U_\varepsilon^{q+\delta_1}(x)\right)\mathrm{d}x = O(\varepsilon^{q-\delta_1}). \tag{2.27}$$

To estimate J_2, we rewrite it as follows:

$$\begin{aligned}
J_2 =& \int_{B_R} U_\varepsilon^q(x)\ln U_\varepsilon^2(x)\mathrm{d}x \\
=& C\int_{B_R}\frac{\varepsilon^q}{(\varepsilon^2+|x|^2)^q}\ln\left(\frac{C\varepsilon^2}{(\varepsilon^2+|x|^2)^2}\right)\mathrm{d}x \\
=& C\varepsilon^{4-q}\int_{B_{R/\varepsilon}}\frac{1}{(1+|y|^2)^q}\ln\left(\frac{C}{\varepsilon^2(1+|y|^2)^2}\right)\mathrm{d}y \\
=& C\varepsilon^{4-q}\ln\left(\frac{1}{\varepsilon}\right)\int_{B_{R/\varepsilon}}\frac{1}{(1+|y|^2)^q}\mathrm{d}y + C\varepsilon^{4-q}\int_{B_{R/\varepsilon}}\frac{1}{(1+|y|^2)^q}\ln\left(\frac{C}{1+|y|^2}\right)\mathrm{d}y \\
=& C\varepsilon^{4-q}\ln\left(\frac{1}{\varepsilon}\right)\left[\int_{\mathbb{R}^4}\frac{1}{(1+|y|^2)^q}\mathrm{d}y - \int_{B^c_{R/\varepsilon}}\frac{1}{(1+|y|^2)^q}\mathrm{d}y\right] \\
&+ C\varepsilon^{4-q}\int_{B_{R/\varepsilon}}\frac{1}{(1+|y|^2)^q}\ln\left(\frac{C}{1+|y|^2}\right)\mathrm{d}y \\
=& C\varepsilon^{4-q}\ln\left(\frac{1}{\varepsilon}\right) - C\varepsilon^{4-q}\ln\left(\frac{1}{\varepsilon}\right)\int_{B^c_{R/\varepsilon}}\frac{1}{(1+|y|^2)^q}\mathrm{d}y \\
&+ C\varepsilon^{4-q}\int_{B_{R/\varepsilon}}\frac{1}{(1+|y|^2)^q}\ln\left(\frac{C}{1+|y|^2}\right)\mathrm{d}y,
\end{aligned} \tag{2.28}$$

where

$$\left|\int_{B^c_{R/\varepsilon}}\frac{1}{(1+|y|^2)^q}\mathrm{d}y\right| \le C\int_{R/\varepsilon}^{\infty}\frac{r^3}{(1+r^2)^q}\mathrm{d}r = O(\varepsilon^{2q-4}). \tag{2.29}$$

The last term in (2.28) can be estimated as follows:

$$\begin{aligned}
&\left|\int_{B_{R/\varepsilon}}\frac{1}{(1+|y|^2)^q}\ln\left(\frac{C}{1+|y|^2}\right)\mathrm{d}y\right| \\
&= \left|\int_{\mathbb{R}^4}\frac{1}{(1+|y|^2)^q}\ln\left(\frac{C}{1+|y|^2}\right)\mathrm{d}y - \int_{B^c_{R/\varepsilon}}\frac{1}{(1+|y|^2)^q}\ln\left(\frac{C}{1+|y|^2}\right)\mathrm{d}y\right| \\
&= \left|C - \int_{B^c_{R/\varepsilon}}\frac{1}{(1+|y|^2)^q}\ln\left(\frac{C}{1+|y|^2}\right)\mathrm{d}y\right| \\
&\le C + C_{\delta_2}\int_{B^c_{R/\varepsilon}}\left(\frac{1}{(1+|y|^2)^{q-\delta_2}} + \frac{1}{(1+|y|^2)^{q+\delta_2}}\right)\mathrm{d}y \\
&= C + O(\varepsilon^{2q-4-2\delta_2}),
\end{aligned} \tag{2.30}$$

where we have used (2.1) with $\delta_2 \in (0, q-2)$. Substituting (2.26)–(2.30) into (2.25) and noticing the choice of δ_1 and δ_2 one sees that

$$\begin{aligned}
J &= O(\varepsilon^q) + C\varepsilon^{4-q}\ln\left(\frac{1}{\varepsilon}\right) + O\left(\varepsilon^q\ln\left(\frac{1}{\varepsilon}\right)\right) + O(\varepsilon^{4-q}) + O(\varepsilon^{q-2\delta_2}) + O(\varepsilon^{q-\delta_1}) \\
&= C\varepsilon^{4-q}\ln\left(\frac{1}{\varepsilon}\right) + O(\varepsilon^{4-q}).
\end{aligned}$$

Therefore, (2.24) is valid. The proof is complete. □

On the basis of the estimates in Lemma 2.8, we can find a $v \in H_0^1(\Omega)\setminus\{0\}$ such that (2.18) holds.

Lemma 2.9. *Assume that $q \in (2,4)$ and $\lambda > 0$. Then for $\varepsilon > 0$ suitably small, we have*

$$\sup_{t\geq 0} I(tu_\varepsilon) < \frac{1}{4}S^2. \tag{2.31}$$

Proof. Let u_ε be given in Lemma 2.8 and set $\psi_{u_\varepsilon}(t) = I(tu_\varepsilon)$. Then $\lim_{t\to 0}\psi_{u_\varepsilon}(t) = 0$, $\psi_{u_\varepsilon}(t) > 0$ for suitably small $t > 0$ and $\lim_{t\to+\infty}\psi_{u_\varepsilon}(t) = -\infty$ uniformly for $\varepsilon \in (0,\varepsilon_1)$, where $\varepsilon_1 > 0$ is a suitably small but fixed number. Consequently, for each $\varepsilon \in (0,\varepsilon_1)$, there exists a $t_\varepsilon \in (0,+\infty)$ such that

$$\sup_{t\geq 0} I(tu_\varepsilon) = \sup_{t\geq 0}\psi_{u_\varepsilon}(t) = \psi_{u_\varepsilon}(t_\varepsilon),$$

and

$$\psi'_{u_\varepsilon}(t_\varepsilon) = t_\varepsilon\left(\|u_\varepsilon\|^2 - \lambda t_\varepsilon^{q-2}\int_\Omega |u_\varepsilon|^q \ln(t_\varepsilon u_\varepsilon)^2 \mathrm{d}x - t_\varepsilon^2\|u_\varepsilon\|_4^4\right) = 0,$$

which implies that

$$\|u_\varepsilon\|^2 = t_\varepsilon^2\|u_\varepsilon\|_4^4 + \lambda t_\varepsilon^{q-2}\int_\Omega |u_\varepsilon|^q \ln u_\varepsilon^2 \mathrm{d}x + \lambda t_\varepsilon^{q-2}\ln t_\varepsilon^2\|u_\varepsilon\|_q^q. \tag{2.32}$$

Combining (2.22)–(2.24) with (2.32) we see that there exist $0 < T_0 < T^0$ such that, for $\varepsilon > 0$ uniformly small,

$$T_0 \leq t_\varepsilon \leq T^0. \tag{2.33}$$

By the boundedness of t_ε and (2.22)–(2.24) one gets

$$\begin{aligned}
\sup_{t\geq 0} I(tu_\varepsilon) = I(t_\varepsilon u_\varepsilon) &= \frac{1}{2}t_\varepsilon^2\|u_\varepsilon\|^2 - \frac{1}{4}t_\varepsilon^4\|u_\varepsilon\|_2^4 + \frac{2\lambda}{q^2}t_\varepsilon^q\|u_\varepsilon\|_q^q - \frac{\lambda}{q}\int_\Omega |t_\varepsilon u_\varepsilon|^q \ln(t_\varepsilon u_\varepsilon)^2 \mathrm{d}x \\
&= \frac{1}{2}t_\varepsilon^2\|u_\varepsilon\|^2 - \frac{1}{4}t_\varepsilon^4\|u_\varepsilon\|_2^4 + \frac{\lambda}{q}t_\varepsilon^q\left(\frac{2}{q} - \ln t_\varepsilon^2\right)\|u_\varepsilon\|_q^q - \frac{\lambda}{q}t_\varepsilon^q\int_\Omega |u_\varepsilon|^q \ln u_\varepsilon^2 \mathrm{d}x \\
&= \left(\frac{1}{2}t_\varepsilon^2 - \frac{1}{4}t_\varepsilon^4\right)\left(S^2 + O(\varepsilon^2)\right) + O(\varepsilon^{4-q}) - \frac{\lambda}{q}t_\varepsilon^q\left(C\varepsilon^{4-q}\ln\left(\frac{1}{\varepsilon}\right) + O(\varepsilon^{4-q})\right) \\
&\leq \frac{1}{4}S^2 + O(\varepsilon^2) + O(\varepsilon^{4-q}) - C\varepsilon^{4-q}\ln\left(\frac{1}{\varepsilon}\right).
\end{aligned} \tag{2.34}$$

Since $2 < q < 4$, we see that

$$\lim_{\varepsilon\to 0}\frac{\varepsilon^2}{\varepsilon^{4-q}\ln\left(\frac{1}{\varepsilon}\right)} = 0, \qquad \lim_{\varepsilon\to 0}\frac{\varepsilon^{4-q}}{\varepsilon^{4-q}\ln\left(\frac{1}{\varepsilon}\right)} = 0. \tag{2.35}$$

Consequently, it follows from (2.34) and (2.35) that $\sup_{t\geq 0} I(tu_\varepsilon) < \frac{1}{4}S^2$ for sufficiently small $\varepsilon > 0$. The proof is complete. □

With the above lemmas at hand, we are now in the position to prove Theorem 1.1.

Proof of Theorem 1.1. Assume that $q \in (2,4)$ and $\lambda > 0$. By virtue of Lemmas 2.5, 2.7 and 2.9 we know that problem (1.1) admits a nontrivial mountain pass type solution u. The nonnegativity of u follows from the nonnegativity of the (PS) sequence $\{u_n\}$ which we have assumed without loss of generality to be valid since the energy functional $I(u)$ satisfies $I(u) = I(|u|)$ for any $u \in H_0^1(\Omega)$. Finally, by applying the standard arguments used in the proof of Theorem 1.2 in [3] we can show that the solution is positive in Ω. The proof is complete. □

Remark 2.10. Although we only consider problem (1.1) when the space dimension $N = 4$, the same result can be obtained for general $N \geq 5$ with little modification when $q \in (2, \frac{2N}{N-2})$ and $\lambda > 0$. Interested readers may check it themselves.

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Critical points of locally Lipschitz functionals inside and outside the ordered interval

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Abstract. In this paper, under the condition that there exists an ordered interval composed of two internal ordered intervals which have the location similar to that of Amann's three-solution theorem, we add some simple conditions and then we obtain some results about the existence of multiple critical points inside and outside the ordered interval. The main results of this paper can be regarded as an extension of the classical Amann three-solution theorem and the mountain pass lemma on the ordered interval of Shujie Li and Zhiqiang Wang. To show our main results, we extend the method of invariant sets of descending flow that proposed by Jingxian Sun for smooth functionals to the locally Lipschitz functionals. Our main results can be applied to the study of differential inclusion problems with concave-convex nonlinearity. In this way, we partially extend some relevant results concerning the differential equation boundary value problems with a concave-convex nonlinearity that was first studied by A. Ambrosetti, H. Brezis and G. Cerami.

Keywords: locally Lipschitz functionals, critical points, differential inclusion problems.

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1 Introduction

In the study of nonlinear functional, ordered intervals are often used to study the existence of critical points or fixed points. It is well known that under simpler conditions, one can get the result that there is at least one critical point or fixed point in the ordered interval. If the ordered interval has a finer internal structure, one can often get multiple fixed points or critical points. A famous result of this is Amann's three-solution theorem [1]. In the case of the ordered interval composed of two pairs of upper and lower solutions, Amann used the fixed point index method to prove that there are at least three fixed points in the ordered interval. Later, Li and Wang [20] generalize this result to smooth functionals using the method of invariant set of descending flow, and establish the mountain pass theorem in the ordered interval.

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In [20], as an application of their mountain pass theorem in the ordered interval, Li and Wang studied the multiple solutions of the boundary value problem

$$\begin{cases} -\Delta u = \lambda |u|^{q-2}u + g(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1$_\lambda$}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, $\lambda > 0$ is a parameter, $1 < q < 2$ and $g(u)$ is a superlinear term. The nonlinearity of (1.1$_\lambda$) may involve a combination of concave and convex terms. This type of nonlinearity has been studied for a long times and has been extensively studied by many authors; See [2, 10, 16, 20, 28] and the references therein. A. Ambrosetti, H. Brezis, G. Cerami in their well known paper [2] firstly considered the boundary value problem with a concave-convex terms nonlinearity

$$\begin{cases} -\Delta u = \lambda u^q + u^p & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2$_\lambda$}$$

where $0 < q < 1 < p$ and $\lambda > 0$ be a parameter. They fund a $\Lambda > 0$ such that (1.2$_\lambda$) has two positive solutions for all $\lambda \in (0, \Lambda)$, at least one positive solution for $\lambda = \Lambda$ and has no positive for all $\lambda > \Lambda$. Li and Wang [20] fund a $\lambda^* > 0$ such that for $\lambda \in (0, \lambda^*)$, (1.1$_\lambda$) has at least two positive solutions, at least two negative solutions and at least two sign-changing solutions. Their proof method is to combine the mountain pass lemma in the ordered interval with the invariant set method of descending flow. They first established two pairs of upper and lower solutions which have the location similar to those of H. Amann's three solution theorem. Then, by using the mountain pass theorem in the ordered interval they obtained the result of at least one positive critical point, one negative critical point and one sign-changing critical point inside the ordered interval, and by using the invariant set method of descending flow obtained the result of at least one positive critical point, one negative critical point and one sign-changing critical point outside the ordered interval.

Inspired by Li and Wang's idea of proof, in the present paper we will establish some results about multiple critical points for locally Lipschitz functionals inside and outside the ordered interval composed of two internal ordered intervals, which have the location similar to that of Amann's three-solution theorem. In the last 40 years, critical point theories and their applications for locally Lipschitz functionals have been extensively studied by many peoples; see [5–8, 12, 13, 17, 18, 23–25, 27] . Our main results in this paper can be thought as an extension of Amann's three solutions theorem and Li and Wang's mountain pass theorem in ordered intervals. By adding some simple conditions, we obtain the existence of critical points both inside and outside the ordered interval. Also, in this paper we introduce an ordering in the Banach space and give the locations of theses critical points.

To show our main results, we will employ the invariant set method of descending flow proposed by Sun [32]. For theories of invariant set of descending flow of C^1 functionals, one can also refer to [22]. According to this method, finding different critical points may be attributed to finding different invariant sets of descending flow. So far, it has been widely used to study the existence of solutions of various of elliptic equation boundary value problems. However, to our best knowledge, there are no one study the existence of critical points for locally Lipschitz functionals by using this method. There are two main difficulties to use the method of invariant sets of descending flow finding critical points of locally Lipschitz functionals. As is well known, to use the method of invariant sets of descending flow for C^1 functionals one need first construct a pseudogradient vector field over the Banach space. However, for the

Lipschitz functional case, one usually only can construct locally a pseudogradient vector field over a subset of the Banach space rather than the whole Banach space. This is the first difficulty need to overcome. In the course of the study, one need to determine whether a closed convex set is an invariant set of the descending flow generated by the pseudogradient vector field. In the cases of the functionals being of C^1, one can use the Schauder invariance condition presented in the literatures [22, 32, 33]. However, in the cases of the functionals being of locally Lipschitz, no one has yet used the Schauder invariance condition to establish invariant flows on convex closed sets. This is the second difficulty we face.

In this paper, we will analyze the energy of the possible critical points of the locally Lipschitz functionals in advance, construct locally a pseudogradient vector field on a neighborhood of the energy, and then extend this pseudogradient vector field to the entire Banach space. We can then use the method of invariant sets of descending flow that Sun have proposed to obtain the existence results for the critical points of the locally Lipschitz functionals. By this way we overcome the first difficulty. For the second difficulty, based on the conclusion in [25] about the relationship between the critical points on the whole space and the critical points on the closed convex sets under the Schauder invariance condition, and using the Von Neumann theorem to establish the descending flow on closed convex sets, we get the result that some closed convex sets are descending flow invariant sets. In this paper we get the theoretical results for the existence of at least one sign-changing, at least one nontrivial, at least two positive and at least two negative critical points of the locally Lipschitz functionals considered. The theoretical results can be applied to the study of the existence of sign-changing solutions, positive solutions and negative solutions for differential inclusion problems with a concave-convex nonlinearity terms. In this way, we extend the relevant results on concave-convex nonlinearity terms in literatures.

The critical points of this paper are in the sense of Chang [6]. The study of critical points of non-smooth functional has been greatly developed in the last thirty years. Some peoples have studied the critical point of the Motreanu–Panagiotopopoulos functionals [26, 29, 34]. This kind of functional has the form: $f := h + \psi$, with h a locally Lipschitz functional and ψ a covex,proper and l.s.c. functional. The Motreanu–Panagiotopopoulos critical point theory contain as particular cases both the the critical point theory in the sense of Chang as well as in the sense of Szulkin [34]. Obviously, how to study the critical point theory of the Motreanu–Panagiotopopoulos functionals in Banach space by using the invariant set method of descending flow is an interesting problem worthy of further study.

2 Preliminaries

In what follows we will let X and E be two real Banach spaces with the norms $\|\cdot\|$ and $\|\cdot\|_1$, respectively. Assume that X is reflexive, E is densely and continuously embedded in X. Let X^* be the topological dual of X and $\langle\cdot,\cdot\rangle$ denote the duality pairing between X^* and X. Let P be a cone of X, that is, P is closed convex set in X, $\lambda x \in P$ for all $x \in P$ and $\lambda \geqslant 0$, and $P \cap (-P) = \{0\}$. The P is said to be generating if $X = P - P$. Let $P_1 = P \cap E$. Then P_1 is a cone in E. We assume that P_1 has a nonempty interior in the E topology, and denote its interior in the E topology by $\operatorname{int} P_1$. For each $x, y \in X$, let us define the ordering $\leqslant$ in X by

$$x \leqslant y \quad \text{if and only if} \quad y - x \in P.$$

For each $x, y \in E$, if $y - x \in \operatorname{int} P_1$, we write $x \ll y$. For $x \in X$ and $A \subset X$, let $\operatorname{dist}(x, A) = \inf_{y\in A} \|x - y\|$. For any $R > 0$, let $B(0, R) = \{x \in X : \|x\| < R\}$ and $S_R = \{x \in X : \|x\| = R\}$.

Given $B \subset A \subset E$, we write $\partial_A B$ for the boundary of B in A and $\operatorname{int}_A B$ for the interior of B in A.

Let us recall some theories concerning the sub-differential theory of locally Lipschitz functionals due to Clarke [7]. A functional $\varphi : X \to \mathbb{R}$ is said to be locally Lipschitz, if for every $x \in X$, there exists a neighborhood U of x and a constant $k > 0$ depending on U such that $|\varphi(z) - \varphi(y)| \leqslant k\|z - y\|$ for all $z, y \in U$. For such a functional we define generalized directional derivative $\varphi^0(x; h)$ at $x \in X$ in the direction $h \in X$ by

$$\varphi^0(x; h) = \limsup_{x' \to x \ \lambda \downarrow 0^+} \frac{\varphi(x' + \lambda h) - \varphi(x')}{\lambda}.$$

The function $h \mapsto \varphi^0(x; h)$ is sublinear, continuous. So by the Hahn–Banach theorem we know that $\varphi^0(x; \cdot)$ is the support function of a nonempty, convex and w^*-compact set

$$\partial\varphi(x) = \{x^* \in X^* : \langle x^*, h\rangle \leqslant \varphi^0(x; h) \text{ for all } h \in X\}.$$

The set $\partial\varphi(x)$ is called the generalized or Clarke sub-differential of φ at x. A point $x \in X$ is a critical point of φ if $0 \in \partial\varphi(x)$. Let $\mathbb{K} = \{x \in X : 0 \in \partial\varphi(x)\}$.

Proposition 2.1 ([5, 12]). *1) If $\varphi, \psi : X \to \mathbb{R}$ are locally Lipschitz functionals, then $\partial(\varphi + \psi)(x) \subset \partial\varphi(x) + \partial\psi(x)$, while for any $\lambda \in \mathbb{R}$ we have $\partial(\lambda\varphi)(x) = \lambda\partial\varphi(x)$; 2) If $\varphi : X \to \mathbb{R}$ is also convex, then this sub-differential coincides with the sub-differential in the sense of convex analysis. If φ is strictly differentiable, then $\partial\varphi(x) = \{\varphi'(x)\}$; 3) If $\varphi : X \to \mathbb{R}$ is locally Lipschitz functional, $\partial\varphi(u)$ is a weakly*-compact subset of X^* which is bounded by the Lipschitz constant $k > 0$ of φ near u.*

S. T. Kyritsi and N. S. Papageorgiou [18] developed a critical point theory for non-smooth locally Lipschitz functionals defined on a closed, convex set extending this way the work of Struwe. Let $C \subset X$ be a nonempty, non-singleton, closed and convex set. For $x \in C$ we define

$$m_C(x) = \inf_{x^*} \sup_{y} \{\langle x^*, x - y\rangle : y \in C, \|x - y\| < 1, x^* \in \partial\varphi(x)\}.$$

Evidently, $m_C(x) \geqslant 0$ for all $x \in C$. This quantity can be viewed as a measure of the generalized slope of φ at $x \in C$. If φ admits an extension $\hat{\varphi} \in C^1(X)$, then $\partial\varphi(x) = \{\varphi'(x)\}$ and so we have

$$m_C(x) = \sup\{\langle \varphi'(x), x - y\rangle : y \in C, \|x - y\| < 1\},$$

which is the quantity used by Struwe [31, p. 147]. Also if $C = X$, then we have

$$m_C(x) = m(x) = \inf\{\|x^*\|_* : x^* \in \partial\varphi(x)\},$$

which is the quantity used by Chang [6].

Now let us introduce the outwardly directed condition and the Schauder invariance condition for set value mappings in a manner as in [25]. Let X be reflexive. As usual, we will identify X^{**} with X while $\mathsf{F} : X^* \mapsto 2^X$ will denote the duality map, given by

$$\mathsf{F}(x^*) := \{x \in X : \langle x^*, x\rangle = \|x^*\|_*^2 = \|x\|^2\}, \qquad \forall x^* \in X^*.$$

The set $\mathsf{F}(x^*)$ turns out to be nonempty, convex, and closed; see, e.g. [13, pp. 311–319]. Define

$$\nabla\varphi(x) := \mathsf{F}(\partial\varphi(x)), \qquad x \in X. \tag{2.1}$$

Clearly, $\nabla\varphi(x)$ depends on the choice of the duality pairing between X and X^* whenever it is compatible with the topology of X. If X is a Hilbert space, the duality paring becomes the scalar product and (2.1) gives the usual gradient. Write I for the identity operator on X.

Suppose C is a convex and closed set of X. Let $\delta_C : X \to \mathbb{R} \cup \{+\infty\}$ be the indicator function of C, namely

$$\delta_C(x) := \begin{cases} 0, & \text{if } x \in C, \\ +\infty, & \text{otherwise.} \end{cases}$$

Then we have

$$\partial\delta_C(x) = \left\{x^* \in X^* : \langle x^*, z - x\rangle \leqslant 0, \forall z \in C\right\}$$

The set $\partial\delta_C(x)$ is usually called normal cone to C at x.

Definition 2.2. Suppose X is reflexive, C is a convex and closed set of X, $\varphi : X \mapsto \mathbb{R}$ is locally Lipschitz continuous. If $(-\partial\delta_C(x)) \cap \partial\varphi(x) \subset \{0\}$ for any $x \in \partial C$ when $\operatorname{int} C \neq \emptyset$, or for any $x \in C$ when $\operatorname{int} C = \emptyset$, then we say that $\partial\varphi$ turns out to be outwardly directed on C. This clearly rewrites as

$$\forall z^* \in \partial\varphi(x)\backslash\{0\} \quad \text{there exists} \quad z \in C \quad \text{fulfilling} \quad \langle z^*, z - x\rangle < 0.$$

Definition 2.3 (Schauder invariance condition)**.** Suppose X is reflexive, C is a convex and closed set of X, $\varphi : X \mapsto \mathbb{R}$ is locally Lipschitz continuous. Then we say φ satisfies the Schauder invariance condition on C if $\big(I - \nabla\varphi\big)(\partial C) \subset C$ when $\operatorname{int} C \neq \emptyset$, if $\big(I - \nabla\varphi\big)(C) \subset C$ when $\operatorname{int} C = \emptyset$

Remark 2.4. Definition 2.2 and 2.3 essentially come from [25]. However, there are some subtle differences here and [25]. The well known Schauder invariance condition for a C^1-functional φ on a Hilbert space X reads as $(I - \varphi')(C) \subset C$; see [15, 22, 32, 33]. It has been extend to Banach spaces in [19]. The notation of Schauder invariance condition was firstly put forward by J. Sun in [33].

It follows from [25, Theorem 4.4 and 4.5] we have the following Lemma 2.5.

Lemma 2.5. *Suppose X is reflexive, C is a convex and closed set of X, $\varphi : X \mapsto \mathbb{R}$ is locally Lipschitz continuous, φ is outwardly directed on C or φ satisfies the Schauder invariance condition on C. Then $m_C(x) = 0$ if and only if $m(x) = 0$.*

Lemma 2.6 ([30], von Neumann)**.** *Let X, Y be two Hausdorff topological linear spaces, $C \subset X$, $D \subset Y$ be two convex and compact sets. Let $\psi : X \times Y \mapsto \mathbb{R}$ satisfy:*

1) $x \mapsto \psi(x, y)$ is upper semi-continuous (usc.) and concave;

2) $y \mapsto \psi(x, y)$ is lower semi-continuous (lsc.) and convex.

Then ψ has at least one saddle point $(\bar{x}, \bar{y}) \in C \times D$, that is

$$\psi(x, \bar{y}) \leqslant \psi(\bar{x}, \bar{y}) \leqslant \psi(\bar{x}, y) \quad \text{for } (x, y) \in C \times D.$$

Definition 2.7. Let D be a nonempty closed subset of X. We say that φ satisfies the non-smooth CPS-condition on D, denoted by $(CPS)_D$, if every sequence $\{x_n\} \subset D$ such that $\{\varphi(x_n)\}$ is bounded and $(1 + \|x_n\|)m_D(x_n) \to 0$ as $n \to \infty$, has a convergent subsequence. If $D = X$, denoted simply by (CPS).

The following Lemma 2.8 comes from [4, p. 63], in which X is required to be a Hilbert space. However, we can easily see that the conclusion also holds if we assume that X is a real Banach space.

Lemma 2.8 ([4]). *Let C be a convex set of X. Then for all $x \in \operatorname{int} C$ and $y \in \bar{C}$, $\alpha x + (1-\alpha)y \subset \operatorname{int} C$ for all $\alpha \in (0,1]$.*

Proposition 2.9. *Assume that X is reflexive, C is a convex and closed set of X, $x_0 \in C$, $\varphi : X \mapsto \mathbb{R}$ is locally Lipschitz. Then there exists $(x_0^*, u(x_0)) \in \partial\varphi(x_0) \times \big((x_0 - C) \cap \bar{B}(0,1)\big)$ such that $m_C(x_0) = \langle x_0^*, u(x_0)\rangle$. In particular, $m(x_0) = \langle x_0^*, u(x_0)\rangle$ for some $(x_0^*, u(x_0)) \in \partial\varphi(x_0) \times \bar{B}(x_0, 1)$ if $X = C$.*

Proof. Let X_w and X_w^* be the spaces X and X^* equipped with their weak topology respectively. Since X is reflexive, it follows from Proposition 2.1 that $\partial\varphi(x_0)$ is a compact set in X_w^*. Obviously, $\big(x_0 - C\big) \cap \bar{B}(0,1)$ is a compact set in X_w. Let $\psi : \big((x_0 - C) \cap \bar{B}(0,1)\big) \times \partial\varphi(x_0) \mapsto \mathbb{R}$ be defined by $\psi(x, y^*) = \langle y^*, x\rangle$ for any $(x, y^*) \in \big((x_0 - C) \cap \bar{B}(0,1)\big) \times \partial\varphi(x_0)$. It follows from Lemma 2.6 that ψ has at least one saddle point $(u(x_0), x_0^*) \in \big((x_0 - C) \cap \bar{B}(0,1)\big) \times \partial\varphi(x_0)$, that is

$$\langle x_0^*, x\rangle \leqslant \langle x_0^*, u(x_0)\rangle \leqslant \langle x^*, u(x_0)\rangle, \qquad \forall x^* \in \partial\varphi(x_0), x \in (x_0 - C) \cap \bar{B}(0,1).$$

Hence, $m_C(x_0) = \langle x_0^*, u(x_0)\rangle$. The proof is complete. □

Remark 2.10. By using Proposition 2.9 we can find $(u(x_0), x_0^*) \in \big((x_0 - C) \cap \bar{B}(0,1)\big) \times \partial\varphi(x_0)$ such that $m_C(x_0) = \langle x_0^*, u(x_0)\rangle$. By this way we give an unified treatment for the case $C = X$ in [6] and the case C being a proper closed and convex set in [18]. It should be pointed out that in order to prove Proposition 2.9, we only need C to be weakly sequence compact not weakly compact. A direct proof of Proposition 2.9 can be found in the Appendix.

Proposition 2.11. *Assume that X is reflexive, C is a convex and closed set of X, $\varphi : X \mapsto \mathbb{R}$ is locally Lipschitz, $(I - \nabla\varphi)(C) \subset C$. Then*

$$m_C(x) \geqslant \min\left\{\frac{1}{2}, m(x)\right\} m(x) \quad \text{for all } x \in C. \tag{2.2}$$

Proof. It follows from of Proposition 2.9 that for each $x_0 \in C$, there exists $(u(x_0), x_0^*) \in \big(\{x_0\} - C\big) \cap \bar{B}(0,1)\big) \times \partial\varphi(x_0)$ satisfying $m_C(x_0) = \langle x_0^*, u(x_0))$. Take $y_0 \in \{x_0\} - \mathsf{F}(x_0^*)$. It follows from the condition $(I - \nabla\varphi)(C) \subset C$ that $y_0 \in C$. So, if $\|x_0 - y_0\| < 1$,

$$m_C(x_0) = \langle x_0^*, u(x_0))\rangle \geqslant \langle x_0^*, x_0 - y_0\rangle = \langle x_0^*, \mathsf{F}(x_0^*)\rangle = \|x_0^*\|_*^2 \geqslant m^2(x_0);$$

if $\|x_0 - y_0\| \geqslant 1$, let $z_0 = x_0 + \frac{y_0 - x_0}{2\|x_0 - y_0\|}$, we have $z_0 \in C$ and $\|x_0 - z_0\| = \frac{1}{2}$, and so

$$\begin{aligned} m_C(x_0) = \langle x_0^*, u(x_0))\rangle &\geqslant \langle x_0^*, x_0 - z_0\rangle \\ &= \frac{1}{2\|x_0 - y_0\|}\langle x_0^*, x_0 - y_0\rangle \\ &= \frac{1}{2\|x_0\|_*}\|x_0^*\|_*^2 = \frac{1}{2}\|x_0^*\|_* \geqslant \frac{1}{2}m(x_0) \end{aligned}$$

So, (2.2) holds. The proof is complete. □

Definition 2.12. Let D be a nonempty closed subset of X. We say that φ satisfies the nonsmooth PS-condition on D, denoted by $(PS)_D$, if every sequence $\{x_n\} \subset D$ such that $\{\varphi(x_n)\}$ is bounded and $m_D(x_n) \to 0$ as $n \to \infty$, has a convergent subsequence. If $D = X$, denoted simply by (PS).

Remark 2.13. Assume that all conditions of Proposition 2.11 hold. Then, by (2.2), we see that φ satisfies the condition $(PS)_C$ if and only if φ satisfies the condition (PS) for each convex and closed set C. Moreover, $m_C(x_0) = 0$ if and only if $m(x_0) = 0$ for $x_0 \in C$. So, we can deduce Lemma 2.5 by (2.2) if φ satisfies the Schauder invariance condition.

Lemma 2.14 ([10]). *Assume U is bounded connected open set of $\mathbb{R}^2$ and $(0,0) \in U$, then there exists a connected component Γ' of the boundary of U, such that each one side ray l emitting from the origin satisfies $l \cap \Gamma' \neq \emptyset$.*

3 Main results

3.1 Critical points outside the ordered interval

Let $u_0, v_1 \in E$ be such that $u_0 \ll 0 \ll v_1$. Set $D_1 = [u_0, v_1]$ and $G_1 = D_1 \cap E$. Now let us introduce the following conditions:

(H_1) $\varphi^{-1}([a,b]) \cap \mathbb{K}$ is compact in E for each $a < b$;

(H_2) φ satisfies the conditions (CPS), $(CPS)_{\pm P}$ and $(CPS)_D$ for any ordered interval D in X;

(H_3) there exist subspace E_1 of E and $R_0 > 0$ such that $\dim E_1 = 2$,

$$E_1 \cap D_1 \subset B(0, R_0), \ E_1 \cap \operatorname{int} P_1 \neq \emptyset \tag{3.1}$$

and

$$\alpha_0 := \max_{u \in S_{R_0} \cap E_1} \varphi(u) < \beta_0 := \inf_{u \in D_1} \varphi(u). \tag{3.2}$$

We have the following result concerning the existence of multiple critical points outside the ordered interval D_1.

Theorem 3.1. *Suppose that (H_1)$\sim$(H_3) hold, φ has no nontrivial critical point on $\partial_E(\pm P_1)$ and $\partial_E G_1$. Moreover, either*

(H_4') φ is outwardly directed on $\pm P$, D_1 and $\pm P \cap D_1$, and P is generating; or

(H_5') $(I - \nabla\varphi)(\pm P) \subset \pm P$ and $(I - \nabla\varphi)(D_1) \subset D_1$.

Then φ has at least one positive critical point $\bar{u}_1$, one negative critical point $\bar{u}_2$ and one sign-changing critical point $\bar{u}_3$ outside D_1.

Remark 3.2. The condition (H_5') is stronger than (H_4'). In fact, it follows from [25, Theorem 4.5] that φ is outwardly directed on $\pm P$ if $(I - \nabla\varphi)(\pm P) \subset \pm P$.

Let $\alpha_1 = \sup_{u \in B(0,R_0) \cap E_1} \varphi(u)$, $D_0 = \varphi^{-1}([\alpha_0 - 1, \alpha_1 + 1])$, and $K = D_0 \cap \mathbb{K}$. Assume without loss of generality that $D_0 \backslash K \neq \emptyset$. Since φ has no nontrivial critical point on $\partial_E(\pm P_1)$ and $\partial_E G_1$, by using the conditions (H_1) we may take $\delta > 0$ small enough such that $D_0^\delta \neq \emptyset$, and

$$K_{3\delta} \cap \left(\partial_E P_1 \cup \partial_E(-P_1) \cup \partial_E G_1\right) = \emptyset \tag{3.3}$$

where $K_{3\delta} = \{x \in D_0 : \operatorname{dist}(x, K) < 3\delta\}$ and $D_0^\delta = D_0 \backslash K_\delta$.

In what follows of this Subsection 3.1, we assume that all conditions of Theorem 3.1 hold.

Lemma 3.3. *There exists a locally Lipschitz mapping $v : D_0^\delta \mapsto X$ such that $\|v(x)\| \leqslant 2(1+\|x\|)$ for any $x \in D_0^\delta$, $\langle x^*, v(x)\rangle \geqslant \frac{\gamma}{16}$ for some $\gamma > 0$ and all $x^* \in \partial\varphi(x)$. Moreover, $v : D_0^\delta \cap E \mapsto E$ is locally Lipschitz, and*

$$x - \frac{1}{1+\|x\|}v(x) \in \pm P_1 \quad \textit{for any} \quad x \in \pm P \cap D_0^\delta, \tag{3.4}$$

$$x - \frac{1}{1+\|x\|}v(x) \in G_1 \quad \textit{for any} \quad x \in D_1 \cap D_0^\delta, \tag{3.5}$$

$$x - \frac{1}{1+\|x\|}v(x) \in \pm P_1 \cap G_1 \quad \textit{for any} \quad x \in \pm P \cap D_1 \cap D_0^\delta. \tag{3.6}$$

Proof. Let $S = D_0^\delta \backslash \big(P \cup (-P) \cup D_1\big)$ and $\widetilde{S} = D_0^\delta \cap (D_1 \backslash (P \cup (-P)))$. First we claim that for some $\gamma > 0$,

$$(1+\|x\|)m(x) \geqslant \gamma, \qquad \forall x \in S, \tag{3.7}$$

$$(1+\|x\|)m_P(x) \geqslant \gamma, \qquad \forall x \in (D_0^\delta \cap P)\backslash D_1, \tag{3.8}$$

$$(1+\|x\|)m_{-P}(x) \geqslant \gamma, \qquad \forall x \in (D_0^\delta \cap (-P))\backslash D_1, \tag{3.9}$$

$$(1+\|x\|)m_{D_1}(x) \geqslant \gamma, \qquad \forall x \in \widetilde{S}, \tag{3.10}$$

$$(1+\|x\|)m_{P\cap D_1}(x) \geqslant \gamma, \qquad \forall x \in D_0^\delta \cap P \cap D_1, \tag{3.11}$$

$$(1+\|x\|)m_{-P\cap D_1}(x) \geqslant \gamma, \qquad \forall x \in D_0^\delta \cap (-P) \cap D_1. \tag{3.12}$$

We only show that (3.11) holds. In a similar way we can show that (3.7)$\sim$(3.10) and (3.12) hold. Arguing by contradiction, assume that (3.11) does not hold. Then there exists an sequence $\{x_n\} \subset D_0^\delta \cap D_1 \cap P$ such that $(1+\|x_n\|)m_{P\cap D_1}(x_n) \to 0$ as $n \to \infty$. Obviously, $\{\varphi(x_n) : n = 1, 2, \ldots\} \subset [\alpha_0 - 1, \alpha_1 + 1]$. It follows from (H_2) that φ satisfies the condition $(CPS)_{D_1\cap P}$. Thus, up to a subsequence if necessary, $x_n \to x_0$ as $n \to \infty$ for some $x_0 \in D_0^\delta \cap D_1 \cap P$. Since $m_{D_1\cap P} : D_1 \cap P \mapsto \mathbb{R}$ is lsc., we have $m_{D_1\cap P}(x_0) = 0$. It follows from (H_4') or (H_5') and Lemma 2.5 that $m(x_0) = 0$, which is a contradiction. Thus, (3.11) holds.

Pick $x_0 \in S$. It follows from Proposition 2.9 and (3.7) that there exist $(u_1(x_0), x_0^*) \in \bar{B}(x_0, 1) \times \partial\varphi(x_0)$ such that for any $y^* \in \partial\varphi(x_0)$,

$$\langle y^*, u_1(x_0)\rangle \geqslant \langle x_0^*, u_1(x_0)\rangle = m(x_0) > \frac{\gamma}{2(1+\|x_0\|)}.$$

Since the map $x \mapsto \partial\varphi(x)$ is usc. from X into X_w^*, we may take an open neighborhood $B_1(x_0, r_1(x_0))$ of x_0, such that

$$\langle y^*, u_1(x_0)\rangle > \frac{\gamma}{4(1+\|y\|)}, \qquad \forall y^* \in \partial\varphi(y),\ y \in U_1(x_0), \tag{3.13}$$

where $U_1(x_0) = B_1(x_0, r_1(x_0)) \cap S$. Since S is an open subset of D_0^δ, we may take $r_1(x_0) > 0$ small enough such that $U_1(x_0) \subset S$.

Pick $x_0 \in (D_0^\delta \cap P)\backslash D_1$. It follows from Proposition 2.9 and inequality (3.8) that there exist $(u_2(x_0), x_0^*) \in \big((x_0 - P) \cap \bar{B}(0,1)\big) \times \partial\varphi(x_0)$,such that for any $x^* \in \partial\varphi(x_0)$,

$$\langle x^*, u_2(x_0)\rangle \geqslant \langle x_0^*, u_2(x_0)\rangle = m_P(x_0) > \frac{\gamma}{2(1+\|x_0\|)}.$$

Again by using the fact that $x \mapsto \partial\varphi(x)$ is usc., we know that there exists an open neighborhood $B_2(x_0, r_2(x_0))$ of x_0 such that for any $y \in U_2(x_0)$, $y^* \in \partial\varphi(y)$,

$$\langle y^*, u_2(x_0)\rangle > \frac{\gamma}{4(1+\|y\|)}. \tag{3.14}$$

where $U_2(x_0) = B_2(x_0, r_2(x_0)) \cap D_0^\delta$. Since $x_0 \in (D_0^\delta \cap P)\backslash D_1$ and $D_0^\delta\backslash(D_1 \cup (-P))$ is an open set of D_0^δ, we may take $r_2(x_0) > 0$ small such that $U_2(x_0) \cap \big(D_1 \cup (-P)\big) = \emptyset$.

Similarly, by (3.9) and Proposition 2.9 we can show that for each $x_0 \in (D_0^\delta \cap (-P))\backslash D_1$, there exist $U_3(x_0) := B_3(x_0, r_3(x_0)) \cap D_0^\delta$, $u_3(x_0) \in X$ with $\|u_3(x_0)\| \leqslant 1$, such that

$$x_0 - u_3(x_0) \in -P, \qquad U_3(x_0) \cap (D_1 \cup P) = \emptyset$$

and for any $y \in U_3(x_0)$, $y^* \in \partial\varphi(y)$,

$$\langle y^*, u_3(x_0)\rangle > \frac{\gamma}{4(1+\|y\|)}. \tag{3.15}$$

By (3.10) we can show that for each $x_0 \in \widetilde{S}$, there exist $U_4(x_0) := B_4(x_0, r_4(x_0)) \cap D_0^\delta$, $u_4(x_0) \in X$ with $\|u_4(x_0)\| \leqslant 1$, such that

$$x_0 - u_4(x_0) \in D_1, \qquad U_4(x_0) \cap (P \cup (-P)) = \emptyset$$

and for any $y \in U_4(x_0)$, $y^* \in \partial\varphi(y)$,

$$\langle y^*, u_4(x_0)\rangle > \frac{\gamma}{4(1+\|y\|)}. \tag{3.16}$$

It follows from (H_4') or (H_5') that $\partial\varphi(0) = \{0\}$. Then 0 is a critical point. Pick $x_0 \in D_0^\delta \cap D_1 \cap P$. Then $x_0 \neq 0$. By using Proposition 2.9 and (3.11) we obtain that there exist $U_5(x_0) := B_5(x_0, r_5(x_0)) \cap D_0^\delta$, $u_5(x_0) \in X$ with $\|u_5(x_0)\| \leqslant 1$, such that

$$x_0 - u_5(x_0) \in D_1 \cap P, \qquad U_5(x_0) \cap (-P) = \emptyset$$

and for any $y \in U_5(x_0)$, $y^* \in \partial\varphi(y)$,

$$\langle y^*, u_5(x_0)\rangle > \frac{\gamma}{4(1+\|y\|)}. \tag{3.17}$$

Similarly, by (3.12) we can show that for each $x_0 \in D_0^\delta \cap D_1 \cap (-P)$, there exist $U_6(x_0) := B_6(x_0, r_6(x_0)) \cap D_0^\delta$, $u_6(x_0) \in X$ with $\|u_6(x_0)\| \leqslant 1$, such that

$$x_0 - u_6(x_0) \in D_1 \cap (-P), \qquad U_6(x_0) \cap P = \emptyset$$

and for any $y \in U_6(x_0)$, $y^* \in \partial\varphi(y)$,

$$\langle y^*, u_6(x_0)\rangle > \frac{\gamma}{4(1+\|y\|)}. \tag{3.18}$$

By 3) in Proposition 2.1, we may assume that $\|x^*\|_* \leqslant L_{\alpha,i}$ for some $L_{\alpha,i} > 0$ and any $x \in U_i(x_\alpha)$ with $i = 1, 2, \ldots, 6$ and $x^* \in \partial\varphi(x)$. Also, we assume that for $i = 1, 2, \ldots, 6$, $B_i(x_\alpha, r_i(x_\alpha))$ has a small radius $r_i(x_\alpha) > 0$ such that $(1 + \|x\|)(1 + \|x_\alpha\|)^{-1} \leqslant 2$ for each $x \in U_i(x_\alpha)$, and

$$0 < r_i(x_\alpha) \leqslant \min\left\{\frac{1}{2}, \frac{\gamma}{64(1+\|x_\alpha\|)L_{\alpha,i}}\right\}.$$

Let

$$\begin{aligned}
\mathscr{A}_1 &= \big\{U_1(x_0) : x_0 \in S\big\},\\
\mathscr{A}_2 &= \big\{U_2(x_0) : x_0 \in (D_0^\delta \cap P)\backslash D_1\big\},\\
\mathscr{A}_3 &= \big\{U_3(x_0) : x_0 \in (D_0^\delta \cap (-P))\backslash D_1\big\},\\
\mathscr{A}_4 &= \big\{U_4(x_0) : x_0 \in \widetilde{S}\big\},\\
\mathscr{A}_5 &= \big\{U_5(x_0) : x_0 \in D_0^\delta \cap P \cap D_1\big\},\\
\mathscr{A}_6 &= \big\{U_6(x_0) : x_0 \in D_0^\delta \cap (-P) \cap D_1\big\},
\end{aligned}$$

and $\mathscr{A} = \bigcup_{i=1,2,\ldots,6} \mathscr{A}_i$. Then $\mathscr{A}$ is an open cover of D_0^δ.

By paracompactness we can find a locally finite refinement $\mathscr{B} = \{V_\alpha : \alpha \in \Lambda\}$ and a locally Lipschitz partition of unit $\{\gamma_\alpha : \alpha \in \Lambda\}$ sub-ordinate to it, with $\operatorname{supp}\gamma_\alpha \subset V_\alpha$. For each $\alpha \in \Lambda$ we can find $x_\alpha \in D_0^\delta$ such that $V_\alpha \subset U_{i(\alpha)}(x_\alpha)$ for some $i(\alpha) \in \{1,2,\ldots,6\}$, and $U_{i(\alpha)}(x_\alpha) \in \mathscr{A}$. To this $x_\alpha \in D_0^\delta$ corresponds the element $w_\alpha^{i(\alpha)}$ such that $\|w_\alpha^{i(\alpha)}\| \leqslant 1$, and $w_\alpha^{i(\alpha)} = u_{i(\alpha)}(x_\alpha)$ if $V_\alpha \subset U_{i(\alpha)}(x_\alpha)$ for some $i(\alpha) \in \{1,2,\ldots,6\}$. Since E is densely embedded in X, and so $P_1, -P_1, G_1$ are densely embedded in $P, -P, D_1$, respectively. Thus, we may take $\bar{x}_\alpha, \bar{w}_\alpha^{i(\alpha)} \in E$ with $\|\bar{w}_\alpha^{i(\alpha)}\| \leqslant 1$ such that

$$\max\left\{\|w_\alpha^{i(\alpha)} - \bar{w}_\alpha^{i(\alpha)}\|, \|x_\alpha - \bar{x}_\alpha\|\right\} < \min\left\{\frac{1}{2}, \frac{\gamma}{64(1+\|x_\alpha\|)L_{\alpha,i(\alpha)}}\right\}. \tag{3.19}$$

and

$$\bar{x}_\alpha - \bar{w}_\alpha^{i(\alpha)} \in \begin{cases} P_1 & \text{if } x_\alpha \in P \cap D_0^\delta; \\ -P_1 & \text{if } x_\alpha \in -P \cap D_0^\delta; \\ G_1 & \text{if } x_\alpha \in \widetilde{S}; \\ G_1 \cap P_1 & \text{if } x_\alpha \in P \cap D_1 \cap D_0^\delta; \\ G_1 \cap (-P_1) & \text{if } x_\alpha \in -P \cap D_1 \cap D_0^\delta. \end{cases} \tag{3.20}$$

Now, let $v : D_0^\delta \mapsto X$ be defined by

$$v(x) = (1+\|x\|)\sum_{\alpha\in\Lambda}\gamma_\alpha(x)(\bar{w}_\alpha^{i(\alpha)} - \bar{x}_\alpha + x). \tag{3.21}$$

Then, $v : D_0^\delta \mapsto X$ is locally Lipschitz. Since $E \hookrightarrow X$, $\gamma_\alpha : D_0^\delta \cap E \mapsto \mathbb{R}$ is also locally Lipschitz. Thus, $v : D_0^\delta \cap E \mapsto E$ is locally Lipschitz.

By (3.19) and (3.21) , we have for any $x \in D_0^\delta$,

$$\|v(x)\| \leqslant (1+\|x\|)\sum_{\alpha\in\Lambda}\gamma_\alpha(x)\big(\|\bar{w}_\alpha^{i(\alpha)}\| + \|\bar{x}_\alpha - x_\alpha\| + \|x_\alpha - x\|\big) \leqslant 2(1+\|x\|).$$

Moreover, we have for any $x \in D_0^\delta$ and $x^* \in \partial\varphi(x)$,

$$\begin{aligned}
&(1+\|x\|)\Big|\sum_{\alpha\in\Lambda}\gamma_\alpha(x)\langle x^*, x - \bar{x}_\alpha\rangle\Big| \\
&\quad\leqslant \sum_{\alpha\in\Lambda}\frac{1+\|x\|}{1+\|x_\alpha\|}(1+\|x_\alpha\|)\gamma_\alpha(x)\|x^*\|_*\big(\|x - x_\alpha\| + \|x_\alpha - \bar{x}_\alpha\|\big) \\
&\quad\leqslant 2\sum_{\alpha\in\Lambda}L_{\alpha,i(\alpha)}\big(\|x - x_\alpha\| + \|x_\alpha - \bar{x}_\alpha\|\big)(1+\|x_\alpha\|) \\
&\quad< \frac{\gamma}{16}
\end{aligned}$$

and

$$\begin{aligned}
\left|\sum_{\alpha\in\Lambda}\gamma_\alpha(x)(1+\|x\|)\langle x^*, \bar{w}_\alpha^{i(\alpha)} - w_\alpha^{i(\alpha)}\rangle\right| &\leqslant \sum_{\alpha\in\Lambda}\frac{1+\|x\|}{1+\|x_\alpha\|}(1+\|x_\alpha\|)\gamma_\alpha(x)\|x^*\|_*\|\bar{w}_\alpha^{i(\alpha)} - w_\alpha^{i(\alpha)}\| \\
&\leqslant 2\sum_{\alpha\in\Lambda}(1+\|x_\alpha\|)L_{\alpha,i(\alpha)}\|\bar{w}_\alpha^{i(\alpha)} - w_\alpha^{i(\alpha)}\| < \frac{\gamma}{16}.
\end{aligned}$$

So, by (3.13)∼(3.18) we have for any $x \in D_0^\delta$ and $x^* \in \partial\varphi(x)$,

$$\begin{aligned}\sum_{\alpha\in\Lambda}\gamma_\alpha(x)(1+\|x\|)\langle x^*, \bar{w}_\alpha^{i(\alpha)}\rangle &= \sum_{\alpha\in\Lambda}\gamma_\alpha(x)(1+\|x\|)\langle x^*, \bar{w}_\alpha^{i(\alpha)} - w_\alpha^{i(\alpha)} + w_\alpha^{i(\alpha)}\rangle\\ &= \sum_{\alpha\in\Lambda}\gamma_\alpha(x)(1+\|x\|)\langle x^*, w_\alpha^{i(\alpha)}\rangle\\ &\quad + \sum_{\alpha\in\Lambda}\gamma_\alpha(x)(1+\|x\|)\langle x^*, \bar{w}_\alpha^{i(\alpha)} - w_\alpha^{i(\alpha)}\rangle\\ &\geqslant \frac{\gamma}{4} - \frac{\gamma}{16} \geqslant \frac{\gamma}{8}.\end{aligned}$$

Hence, we have for any $x \in D_0^\delta$ and $x^* \in \partial\varphi(x)$,

$$\begin{aligned}\langle x^*, v(x)\rangle = &\sum_{\alpha\in\Lambda}\gamma_\alpha(x)(1+\|x\|)\langle x^*, \bar{w}_\alpha^{i(\alpha)}\rangle\\ &+ \sum_{\alpha\in\Lambda}\gamma_\alpha(x)(1+\|x\|)\langle x^*, x-\bar{x}_\alpha\rangle \geqslant \frac{\gamma}{16}.\end{aligned}$$

For a given $x \in D_0^\delta$, we assume that $x \in U_{i(\alpha)}(x_\alpha)$ for some $U_{i(\alpha)}(x_\alpha) \in \mathscr{A}$. Recalling the construction of $\mathscr{A}$, we have

$$U_{i(\alpha)}(x_\alpha) \in \begin{cases}\mathscr{A}_2 \cup \mathscr{A}_5 & \text{if } x \in (D_0^\delta \cap P)\backslash D_1;\\ \mathscr{A}_3 \cup \mathscr{A}_6 & \text{if } x \in (D_0^\delta \cap (-P))\backslash D_1;\\ \mathscr{A}_4 \cup \mathscr{A}_5 \cup \mathscr{A}_6 & \text{if } x \in \widetilde{S};\\ \mathscr{A}_5 & \text{if } x \in D_0^\delta \cap P \cap D_1;\\ \mathscr{A}_6 & \text{if } x \in D_0^\delta \cap (-P) \cap D_1.\end{cases} \tag{3.22}$$

It follows from (3.21) and (3.22) that (3.4)∼(3.6) hold. The proof is complete. □

Let $l_1, l_2 : X \mapsto \mathbb{R}$ be defined by

$$\begin{aligned}l_1(x) &= \frac{\text{dist }(x, K_\delta)}{\text{dist }(x, K_\delta) + \text{dist }(x, X\backslash K_{2\delta})},\\ l_2(x) &= \frac{\text{dist }(x, \Gamma_1)}{\text{dist }(x, \Gamma_2) + \text{dist }(x, \Gamma_1)},\end{aligned} \tag{3.23}$$

$$v_1(x) = \begin{cases} v(x), & x \in D_0^\delta;\\ 0, & \text{otherwise,}\end{cases}$$

and $V(x) = l_1(x)l_2(x)v_1(x)$ for any $x \in X$, where

$$\Gamma_1 = \varphi^{-1}\left(\left[\alpha_1 + \frac{1}{2}, +\infty\right)\right) \cup \varphi^{-1}\left(\left(-\infty, \alpha_0 - \frac{1}{2}\right]\right), \qquad \Gamma_2 = \varphi^{-1}\left(\left[\alpha_0 - \frac{1}{4}, \alpha_1 + \frac{1}{4}\right]\right).$$

Then $l_1, l_2 : X \mapsto \mathbb{R}$ are locally Lipschitz, and so $V : X \mapsto X$ is locally Lipschitz. Since $E \hookrightarrow X$, $l_1, l_2 : E \mapsto \mathbb{R}$ are also locally Lipschitz, and so $V : E \mapsto E$ is locally Lipschitz.

Consider the following initial value problem

$$\begin{cases}\frac{du}{dt} = -V(u),\\ u(0) = v_0 \in X.\end{cases} \tag{3.24}$$

By the theories for initial value problems of ordinary equations in Banach space, we see that (3.24) has a unique solution $\sigma(t, v_0)$ in X, with its right maximal existence interval $[0, T(v_0))$, and its right maximal existence interval $[0, T_1(v_0))$ in E. Obviously, we have $T_1(v_0) \leqslant T(v_0)$. Concerning the solution $\sigma(t, v_0)$ of (3.24), we have the following Lemmas 3.4∼3.6.

Lemma 3.4. *For each $v_0 \in X$, $T(v_0) = +\infty$*

Proof. The proof is standard. For the reader's convenience we give detailed process. Arguing by contradiction, let us assume that $T(v_0) < +\infty$. By (3.24) we have

$$\|\sigma(t, v_0) - v_0\| \leqslant \int_0^t \|V(\sigma(s, v_0))\| ds \leqslant 2 \int_0^t \big(1 + \|\sigma(s, v_0)\|\big) ds.$$

So, we have

$$\begin{aligned} \frac{1}{2}\|\sigma(t, v_0) - v_0\| &\leqslant \int_0^t (1 + \|\sigma(s, v_0)\|) ds \\ &\leqslant \int_0^t \|\sigma(s, v_0) - v_0\| ds + (1 + \|v_0\|)t, \end{aligned}$$

By the well known Gronwall's inequality, we have

$$\begin{aligned} \frac{1}{2}\|\sigma(t, v_0) - v_0\| &\leqslant \int_0^t (1 + \|v_0\|)e^{t-s} ds + (1 + \|v_0\|)t \\ &\leqslant (1 + \|v_0\|)(e^t - 1) + (1 + \|v_0\|)t \\ &\leqslant (1 + \|v_0\|)(t + e^t - 1) \\ &\leqslant (1 + \|v_0\|)(T(v_0) + e^{T(v_0)}). \end{aligned}$$

So, we have

$$\|\sigma(t, v_0)\| \leqslant 2(1 + \|v_0\|)(T(v_0) + e^{T(v_0)}) + 2\|v_0\| =: M_1.$$

Take $\{t_n\} \subset [0, T(v_0))$ such that $t_n \to T^-(v_0)$ and for $n = 1, 2, \ldots$,

$$|t_n - t_{n-1}| < \frac{1}{2 \cdot 2^n (1 + M_1)}.$$

Then we have

$$\begin{aligned} \|\sigma(t_n, v_0) - \sigma(t_{n-1}, v_0)\| &\leqslant \int_{t_{n-1}}^{t_n} \|V(\sigma(s, v_0)\| ds \\ &\leqslant 2 \int_{t_{n-1}}^{t_n} \big(1 + \|\sigma(s, v_0)\|\big) ds \\ &\leqslant 2(1 + M_1)(t_n - t_{n-1}) < \frac{1}{2^n}. \end{aligned}$$

This implies that $\{\sigma(t_n, v_0)\}$ is a Cauchy sequence in X. Thus, there exists $\bar{u} \in X$ such that $\sigma(t_n, v_0) \to \bar{u}$ as $t_n \to T^-(v_0)$. Then we can show that $\sigma(t, v_0) \to \bar{u}$ as $t \to T^-(v_0)$.

Now we consider the initial value problem

$$\begin{cases} \frac{du}{dt} = -V(u), \\ u(0) = \bar{u}. \end{cases} \tag{3.25}$$

Then (3.25) has a unique solution on $[0, \bar{\delta})$ for some $\bar{\delta} > 0$, and so (3.24) has a unique solution on $[0, T(v_0) + \bar{\delta})$, which is a contradiction. Thus, we have $T(v_0) = +\infty$. The proof is complete. □

Lemma 3.5. *If* $o(v_0) := \{\sigma(t, v_0) \subset X : t \in [0, +\infty)\} \subset \Gamma_2 \backslash K_{2\delta}$, *then* $T_1(v_0) = +\infty$.

Proof. Let the operator $A : E \to X$ be defined by

$$Ax = x - \frac{1}{1 + \|x\|} V(x), \qquad \forall x \in E.$$

Then we have

$$Ax = \sum_{\alpha \in \Lambda} \gamma_\alpha(x)(\bar{x}_\alpha - \bar{w}_\alpha^{i(\alpha)}), \qquad \forall x \in \Gamma_2 \backslash K_{2\delta}. \tag{3.26}$$

Let

$$\mu(t) = \int_0^t \big(1 + \|\sigma(s, v_0)\|\big) ds \quad \text{for } t \in [0, +\infty).$$

Obviously, $\mu : [0, +\infty) \to [0, +\infty)$ is strictly increasing. And so, μ^{-1}, the inverse function of μ, exists. Then we have

$$\begin{cases} \frac{d}{dt}\big(e^{\mu(t)}\sigma(t, v_0)\big) = e^{\mu(t)}\big(1 + \|\sigma(t, v_0)\|\big) A\sigma(t, v_0), \\ \sigma(0, v_0) = v_0. \end{cases}$$

By direct computation, we have

$$\sigma(t, v_0) = e^{-\mu(t)} v_0 + e^{-\mu(t)} \int_0^t e^{\mu(s)} \big(1 + \|\sigma(s, v_0)\|\big) A\sigma(s, v_0) ds. \tag{3.27}$$

where the integral is in the sense of X topology. Now we show that $T_1(v_0) = +\infty$ when $o(v_0) \subset \Gamma_2 \backslash K_{2\delta}$. Arguing by contradiction, let us assume that $T_1(v_0) < +\infty$. Take $T > T_1(v_0)$. It follows from (3.26) that $\{A(\sigma(t, v_0)) : t \in [0, T]\}$ is contained in a finite-dimensional subspace of X. Then, there exists $M_1(T) > 0$ such that for $s \in [0, T]$,

$$\|e^{\mu(s)}\big(1 + \|\sigma(s, v_0)\|\big) A\sigma(s, v_0)\|_1 \leqslant M_1(T).$$

Let $\{t_n\} \subset [0, T_1(v_0))$ such that $t_n \to T_1^-(v_0)$ as $n \to \infty$. Note (3.27) also holds in which the integral is in the sense of E topology for any $t \in [0, T_1(v_0))$. Assume without loss of generality that $\{t_n\}$ is increasing. Then we have

$$\begin{aligned} \|\sigma(t_n, v_0) - \sigma(t_{n-1}, v_0)\|_1 &\leqslant |e^{-\mu(t_n)} - e^{-\mu(t_{n-1})}| \Big(\|v_0\|_1 \\ &\quad + \int_0^{t_n} e^{\mu(s)} \big(1 + \|\sigma(s, v_0)\|\big) \|A\sigma(s, v_0)\|_1 ds \Big) \\ &\quad + e^{-\mu(t_{n-1})} \int_{t_{n-1}}^{t_n} e^{\mu(s)} \big(1 + \|\sigma(s, v_0)\|\big) \|A\sigma(s, v_0)\|_1 ds \\ &\leqslant |e^{-\mu(t_n)} - e^{-\mu(t_{n-1})}| \big(\|v_0\|_1 + TM_1(T) \big) + M_1(T)(t_n - t_{n-1}). \end{aligned}$$

So, $\{\sigma(t_n, v_0)\}$ is a Cauchy sequence in E. Assume that that $\sigma(t_n, v_0) \to \bar{u}$ in E as $t \to T_1^-(v_0)$. Since $V : E \mapsto E$ is locally Lipschitz. Then we can easily obtain a contradiction as in the proof of $T(v_0) = +\infty$. Thus, we have $T_1(v_0) = +\infty$. The proof is complete. □

Lemma 3.6. $\varphi(\sigma(t, v_0))$ *is non-increasing in* $t \in [0, T_1(v_0))$.

Proof. Let $h(t,v_0)=\varphi(\sigma(t,v_0))$ for all $t\in[0,T_1(v_0))$. It is easy to see that $h(t,v_0)$ is locally Lipschitz in $t\in[0,T_1(v_0))$, hence differentiable almost everywhere. According to Lebourg's Mean Theorem we have

$$\begin{aligned}\frac{\partial}{\partial s}h(s,v_0)&\leqslant\max\left\{\left\langle w^*,\frac{\partial}{\partial s}\sigma(s,v_0)\right\rangle : w^*\in\partial\varphi(\sigma(s,v_0))\right\}\quad\text{a.e.}\\&=-\min\left\{\langle w^*,V(\sigma(s,v_0))\rangle : w^*\in\partial\varphi(\sigma(s,v_0))\right\}\quad\text{a.e.}\\&\leqslant\begin{cases}-\frac{\gamma}{16}, & \text{if } \sigma(s,v_0)\in\varphi^{-1}([\alpha_0-\frac14,\alpha_1+\frac14])\backslash K_{2\delta};\\ 0, & \text{otherwise.}\end{cases}\end{aligned}\tag{3.28}$$

Consequently, $\varphi(\sigma(t,v_0))$ is non-increasing in $t\in[0,T_1(v_0))$. The proof is complete. □

As in [22], we give the following Definition 3.7 and 3.8

Definition 3.7. A nonempty subset D of E is called an invariant set of descending flow of (3.24) if $o(v_0)\subset D$ for all $v_0\in D$, where $o(v_0)=\{\sigma(t,v_0):t\in[0,T_1(v_0))\}$.

Definition 3.8. Let $M\subset E$ be a connected invariant set of the descending flow of (3.24), D be an open subset of M and be an invariant set of descending flow of (3.24). Denote

$$C_M(D)=\{v_0:v_0\in D \text{ or } v_0\in M\backslash D \text{ and there exists } t'\in(0,T_1(v_0)) \text{ such that } \sigma(t',v_0)\in D\}.$$

If $D=C_M(D)$, then D is called a complete invariant set of descending flow of (3.24) in M.

Lemma 3.9. *$\pm P_1$, $\operatorname{int}(\pm P_1)$, G_1 and $\operatorname{int}G_1$ are all invariant sets of descending flow of (3.24).*

Proof. 1) P_1, $-P_1$ and G_1 are all invariant sets of descending flow of (3.24).

For $u\in\partial_E G_1$, it follows from Lemma 3.3 that for $\lambda>0$ small enough,

$$u+\lambda(-V(u))=\lambda g(u)\left(u-\frac{v_1(u)}{1+\|u\|}\right)+(1-\lambda g(u))\,u\in G_1,$$

where $g(u)=l_1(u)l_2(u)(1+\|u\|)$. It follows from the theorem due to Brezis–Martin (see [11]) that G_1 is an invariant sets of descending flow of (3.24).

In a similar way we can show that $P_1,-P_1$ are also invariant sets of descending flow of (3.24).

2) $\operatorname{int}P_1$, $\operatorname{int}(-P_1)$ and $\operatorname{int}G_1$ are all invariant sets of descending flow of (3.24)

Take $v_0\in\operatorname{int}G_1$. Note (3.27) also holds where the integral is in the sense of E topology for $t\in[0,T_1(v_0))$. Make a variable change $\tau=e^{\mu(s)}-1$ in (3.27). Then we have

$$s=\mu^{-1}(\ln(1+\tau)),\qquad ds=\frac{e^{-\mu(s)}}{\mu'(s)}d\tau$$

and

$$\begin{aligned}&e^{-\mu(t)}\int_0^{e^{\mu(t)}-1}A\sigma\left(\mu^{-1}(\ln(1+\tau)),v_0\right)d\tau\quad\text{(the integral is in the sense of } E\text{ topology)}\\&=\lim_{n\to\infty}\frac1n(1-e^{-\mu(t)})\sum_{k=0}^{n-1}A\sigma\left(\mu^{-1}\left(\ln(1+\frac{k(e^{\mu(t)}-1)}{n}\right),v_0\right).\end{aligned}$$

For any $x \in G_1$, we have

$$Ax = \begin{cases} x, & \text{if } x \in \big(E\backslash\varphi^{-1}(\alpha_0 - \frac{1}{2}, \alpha_1 + \frac{1}{2})\big) \cup K_\delta; \\ x - \frac{1}{1+\|x\|}v(x), & \text{if } x \in \varphi^{-1}([\alpha_0 - \frac{1}{4}, \alpha_1 + \frac{1}{4}])\backslash K_{2\delta}; \\ l(x)\Big(x - \frac{1}{1+\|x\|}v_1(x)\Big) + (1 - l(x))x, & \text{otherwise.} \end{cases} \in G_1, \tag{3.29}$$

where $l(x) := l_1(x)l_2(x) \in (0,1)$, that is $A(G_1) \subset G_1$. Since G_1 is an invariant set of descending flow of (3.24), by (3.29) we have

$$\bar{v} := \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} A\sigma\left(\mu^{-1}\left(\ln(1 + \frac{k(e^{\mu(t)} - 1)}{n}\right), v_0\right) \in G_1.$$

It follows from (3.27) that

$$\sigma(t, v_0) = e^{-\mu(t)}v_0 + (1 - e^{-\mu(t)})\bar{v}. \tag{3.30}$$

Since $v_0 \in \operatorname{int} G_1$ and $e^{-\mu(t)} \in (0,1)$, it follows from Lemma 2.8 and (3.30) that $\sigma(t, v_0) \in \operatorname{int} G_1$ for $t \in [0, T_1(v_0))$. Thus, $\operatorname{int} G_1$ is an invariant set of descending flow of (3.24).

Similarly, we can show that $\operatorname{int}(\pm P_1)$ are invariant sets for the descending flow of (3.24). The proof is complete. □

Lemma 3.10. *For each $v_0 \in \varphi^{-1}\big((-\infty, \alpha_1 + \frac{1}{4}]\big) \cap E$ with $\inf_{u\in o(v_0)} \varphi(u) \geqslant \alpha_0 - \frac{1}{4}$, there exists $\tau(v_0) \geqslant 0$ such that $\sigma(\tau(v_0), v_0) \in K_{2\delta}$.*

Proof. Assume that $o(v_0) \cap K_{2\delta} = \emptyset$. It follows from (3.28) that

$$\varphi(v_0) - \varphi(\sigma(t, v_0)) = -\int_0^t \frac{\partial}{\partial s} h(s, v_0) ds \geqslant \frac{\gamma}{16} t \quad \text{for } t \in [0, T_1(v_0)).$$

It follows from Lemma 3.5 that the maximal existence interval of $\sigma(t, v_0)$ in E is $[0, +\infty)$. So, we may take $t_0 = 16\gamma^{-1}(\alpha_1 - \alpha_0 + 2)$, and have

$$\varphi(\sigma(t_0, v_0)) \leqslant \varphi(v_0) - \frac{\gamma}{16} t_0 \leqslant \alpha_1 + \frac{1}{4} - \frac{\gamma}{16} t_0 < \alpha_0 - 1,$$

which contradicts to $\inf_{u\in o(v_0)} \varphi(u) \geqslant \alpha_0 - \frac{1}{4}$. The proof is complete. □

Similar to the proof of Lemma 3.1 in [22] we have the following Lemma 3.11.

Lemma 3.11. *Let $G \subset E$ be a connected and invariant set of (3.24), and D be an open invariant subset of G. Then the following assertions hold:*

1) *$C_G(D)$ is an open subset of G;*

2) *$\partial_G C_G(D)$ is an invariant set of descending flow of (3.24);*

3) *$\inf_{u\in\partial_G C_G(D)} \varphi(u) \geqslant \inf_{u\in\partial_G D} \varphi(u)$.*

Proof of Theorem 3.1. It follows from Lemma 3.9 that $\operatorname{int}(\pm P_1)$ and $\operatorname{int} G_1$ are all invariant sets of descending flow of (3.24). So, $\operatorname{int}(P_1 \cap G_1)$ and $\operatorname{int}(-P_1 \cap G_1)$ are invariant sets of descending flow of (3.24). It follows from Lemma 3.11 that $C_E(\operatorname{int} G_1)$ is an open invariant set of

descending flow of (3.24). It follows from (3.2) that $C_E(\operatorname{int} G_1) \neq E$, and so $\partial_E C_E(\operatorname{int} G_1) \neq \emptyset$. By Lemma 3.11 we have

$$\inf_{u\in\partial_E C_E(\operatorname{int} G_1)} \varphi(u) \geqslant \inf_{u\in\partial_E G_1} \varphi(u) \geqslant \inf_{u\in D_1} \varphi(u) = \beta_0 > \alpha_0 - \frac{1}{4}. \tag{3.31}$$

Since $C_E(\operatorname{int} G_1)$ is open in E, $C_E(\operatorname{int} G_1) \cap E_1 \subset B(0, R_0)$ is an open and bounded subset of E_1 containing 0. It follows from Lemma 2.14 that there exists a connected component Γ' of the boundary of $C_E(\operatorname{int} G_1) \cap E_1$, such that each one side ray l emitting from the origin satisfies $l \cap \Gamma' \neq \emptyset$. Let Γ be the connected component of $\partial_E C_E(\operatorname{int} G_1)$ containing Γ'. It follows from Lemma 3.11 that Γ is an invariant set of descending flow of (3.24).

It follows from (3.1) that $S_{R_0} \cap E_1 \cap \operatorname{int} P_1 \neq \emptyset$. Thus, $\Gamma \cap \operatorname{int} P_1$ is an invariant set of descending flow of (3.24). Take $\widetilde{v}_1 \in S_{R_0} \cap E_1 \cap \operatorname{int} P_1$ and let $\widetilde{l}$ be the ray emitting from the origin and passing through $\widetilde{v}_1$. Then, we have $\widetilde{l} \cap (\Gamma \cap P_1) \neq \emptyset$, and so $\Gamma \cap P_1 \cap \varphi^{-1}\big((-\infty, \alpha_1 + \frac{1}{4}]\big) \neq \emptyset$. Take $v_0 \in \Gamma \cap P_1 \cap \varphi^{-1}\big((-\infty, \alpha_1 + \frac{1}{4}]\big)$. It follows from (3.31) that $\inf_{u\in o(v_0)} \varphi(u) \geqslant \alpha_0 - \frac{1}{4}$, that is $o(v_0) \subset \varphi^{-1}\big([\alpha_0 - \frac{1}{4}, \alpha_1 + \frac{1}{4}]\big)$.

Now we have the following two cases:

1) If $v_0 \in K_{2\delta}$, it follows from (3.3) that there must exists a u_1 with

$$\bar{u}_1 \in P_1 \cap \varphi^{-1}\left(\left[\alpha_0 - \frac{1}{4}, \alpha_1\right]\right) \cap (K\backslash\{0\})$$

and $\bar{u}_1 \in E\backslash(D_1)_\delta$.

2) If $v_0 \in \big(\Gamma \cap P_1 \cap \varphi^{-1}([\alpha_0 - \frac{1}{4}, \alpha_1 + \frac{1}{4}])\big)\backslash K_{2\delta}$, by Lemma 3.10 we see that $\sigma(\tau(v_0), v_0) \in K_{2\delta} \cap P_1$ for some $\tau(v_0) > 0$, also by (3.3) we see that there must exists a $\bar{u}_1$ with

$$u_1 \in P_1 \cap \varphi^{-1}\left(\left[\alpha_0 - \frac{1}{4}, \alpha_1 + \frac{1}{4}\right]\right) \cap (K\backslash\{0\}),$$

and $\bar{u}_1 \in E\backslash(D_1)_\delta$. Hence, φ has at least one positive critical point $\bar{u}_1$ outside D_1.

Similarly, we can show that φ has at least one negative critical point $\bar{u}_2$ outside D_1.

Now we show that φ has at least one sign-changing critical point $\bar{u}_3$. Obviously, $\Gamma \cap \operatorname{int} P_1$ and $\Gamma \cap \operatorname{int}(-P_1)$ are two open invariant sets of descending flow of (3.24) in Γ. It follows from Lemma 3.11 that $C_\Gamma(\Gamma \cap \operatorname{int} P_1)$ and $C_\Gamma(\Gamma \cap \operatorname{int}(-P_1))$ are two open invariant sets of descending flow of (3.24) in Γ. By the connectedness of Γ, we see that

$$O_1 := \Gamma\backslash\big(C_\Gamma(\Gamma \cap \operatorname{int} P_1) \cup C_\Gamma(\Gamma \cap \operatorname{int}(-P_1)\big) \neq \emptyset.$$

Let

$$O_2 := \Gamma'\backslash\big(C_\Gamma(\Gamma \cap \operatorname{int} P_1) \cup C_\Gamma(\Gamma \cap \operatorname{int}(-P_1)\big).$$

Obviously, $O_2 \subset O_1$. Also by the connectedness of Γ', we have $O_2 \neq \emptyset$. Take $v_0 \in O_2$. It follows from (3.31) that $\inf_{u\in o(v_0)} \varphi(u) \geqslant \alpha_0 - \frac{1}{4}$. Then we can show that φ has a sign-changing critical point $\bar{u}_3$. Indeed, if $v_0 \in K_{2\delta}$, by (3.31) there must exist a $\bar{u}_3 \in S \cap \varphi^{-1}([\alpha_0 - \frac{1}{4}, \alpha_1 + \frac{1}{4}]) \cap K$, where S is defined as in Lemma 3.3. If $v_0 \in \varphi^{-1}([\alpha_0 - \frac{1}{4}, \alpha_1 + \frac{1}{4}])\backslash K_{2\delta}$, it follows from Lemma 3.10 that $\sigma(\tau(v_0), v_0) \in K_{2\delta}$ for some $\tau(v_0) > 0$. By (3.3), we see that there must exists a $\bar{u}_3$ with

$$\bar{u}_3 \in (E\backslash(P \cup (-P))) \cap \varphi^{-1}\left(\left[\alpha_0 - \frac{1}{4}, \alpha_1 + \frac{1}{4}\right]\right) \cap (K\backslash\{0\})$$

and $u_3 \in E\backslash(D_1)_\delta$. Hence, φ has at least one sign-changing critical point $\bar{u}_3$ outside D_1. The proof is complete. □

Remark 3.12. According to Remark 2.13, if (H_5') holds, then (H_2) can be substituted with the condition that φ satisfies the condition (PS). So, we have the following Corollary 3.13.

Corollary 3.13. *it Suppose that (H_1), (H_3) and (H_5') hold. Moreover, φ satisfies the condition (PS) and has no nontrivial critical point on $\partial_E(\pm P_1)$ and $\partial_E G_1$. Then the conclusion in Theorem 3.1 holds.*

3.2 Critical points inside the ordered interval

Next in this section we will give the existence results for critical points inside the ordered interval D_1. Assume that $u_0, v_0, u_1, v_1 \in E$ such that $u_0 \ll v_0 \ll 0 \ll u_1 \ll v_1$. Set $D_2 = [u_0, v_0]$, $D_3 = [u_1, v_1]$ and $G_i = D_i \cap E$ for $i = 2, 3$.

Theorem 3.14. *Suppose that (H_1) holds, φ has no nontrivial critical point on $\partial_E G_i$ for $i = 1, 2, 3$, and φ satisfies $(CPS)_D$ for any ordered interval D in X. Moreover, either*

(H_4'') φ is outwardly directed on D_i for $i = 1, 2, 3$; or

(H_5'') $(I - \nabla\varphi)(D_i) \subset D_i$ for $i = 1, 2, 3$.

Then φ has at least one positive critical point $\bar{u}_4$, one negative critical point $\bar{u}_5$ and one critical point $\bar{u}_6$ inside D_1. Moreover, $\bar{u}_6$ is nontrivial if there exists a curve l in E_1 such that $l \cap D_2 \neq \emptyset$, $l \cap D_3 \neq \emptyset$ and $\max_{u\in l} \varphi(u) < \varphi(0)$.

Corollary 3.15. *Suppose that (H_1) holds, φ has no nontrivial critical point on $\partial_E G_i$ for $i = 1, 2, 3$, φ satisfies (PS) and $(I - \nabla\varphi)(D_i) \subset D_i$ for $i = 1, 2, 3$. Then the conclusion in Theorem 3.14 holds.*

Set

$$\begin{aligned}
\widetilde{K}_r &= \{x \in D_1 : \varphi(x) = r, m(x) = 0\} \text{ for any } r \in \mathbb{R},\\
(\widetilde{K}_r)_\delta &= \{x \in D_1 : d(x, \widetilde{K}_r) < \delta\}, (\widetilde{K}_r)^c_\delta = D_1 \backslash (\widetilde{K}_r)_\delta \text{ for any } r \in \mathbb{R}, \delta > 0,\\
D_1(r, \varepsilon, \delta) &= \{x \in D_1 : r - \varepsilon \leqslant \varphi(x) \leqslant r + \varepsilon, x \in (\widetilde{K}_r)^c_\delta\} \text{ for any } r \in \mathbb{R}, \delta > 0, \varepsilon > 0.
\end{aligned}$$

Lemma 3.16. *Assume that all conditions in Theorem 3.14 hold. Let $r \in \mathbb{R}, \delta > 0$ be given. Then there exist $\bar{\varepsilon} > 0$, $\gamma > 0$ and a locally Lipschitz mapping $\widetilde{v} : D_1(r, \bar{\varepsilon}, \delta) \mapsto D_1$ such that $\|\widetilde{v}(x)\| \leqslant 2(1 + \|x\|)$ for any $x \in D_1(r, \bar{\varepsilon}, \delta)$, $\langle x^*, \widetilde{v}(x)\rangle \geqslant \frac{\gamma}{16}$ for any $x \in D_1(r, \bar{\varepsilon}, \delta)$ and $x^* \in \partial\varphi(x)$. Moreover, $\widetilde{v} : D_1(r, \bar{\varepsilon}, \delta) \cap E \mapsto E$ is locally Lipschitz, and for $i = 1, 2, 3$,*

$$x - \frac{1}{1 + \|x\|}\widetilde{v}(x) \in G_i \quad \text{for any } x \in D_i \cap D_1(r, \bar{\varepsilon}, \delta). \tag{3.32}$$

Proof. The proof is similar to Lemma 3.3. Now we only sketch it. First we can show that for $i = 1, 2, 3$ and some $\bar{\varepsilon}, \gamma > 0$,

$$(1 + \|x\|)m_{D_i}(x) \geqslant \gamma, \qquad \forall x \in D_1(r, \bar{\varepsilon}, \delta) \cap D_i. \tag{3.33}$$

Then, by using (3.33), as the proof in Lemma 3.3, we can find a $\widetilde{u}_i(x) \in X$ for $i = 1, 2, 3$, an open neighborhood $\widetilde{U}_1(x)$ for $x \in D_1(r, \bar{\varepsilon}, \delta)\backslash(D_2 \cup D_3)$, an open neighborhood $\widetilde{U}_2(x)$ for $x \in D_2 \cap D_1(r, \bar{\varepsilon}, \delta)$, and an open neighborhood $\widetilde{U}_3(x)$ for $x \in D_3 \cap D_1(r, \bar{\varepsilon}, \delta)$ such that, $\|\widetilde{u}_i(x)\| \leqslant 1, x - \widetilde{u}_i(x) \in D_i$,

$$\widetilde{U}_1(x) \cap (D_2 \cup D_3) = \emptyset, \qquad \widetilde{U}_2(x) \cap D_3 = \emptyset, \qquad \widetilde{U}_3(x) \cap D_2 = \emptyset,$$

and

$$\langle y^*, \widetilde{u}_i(x_0)\rangle > \frac{\gamma}{4(1+\|y\|)}, \qquad \forall y^* \in \partial\varphi(y),\ y \in \widetilde{U}_i(x), \tag{3.34}$$

where $\widetilde{U}_i(x) = B(x, r_i(x)) \cap D_1(r, \bar{\varepsilon}, \delta)$ for $i = 1, 2, 3$.

Let $\mathscr{A}_1 = \{\widetilde{U}_1(x) : x \in D_1(r, \bar{\varepsilon}, \delta)\backslash(D_2 \cup D_3)\}$, $\mathscr{A}_i = \{\widetilde{U}_i(x) : x \in D_i \cap D_1(r, \bar{\varepsilon}, \delta)\}$ for $i = 2, 3$ and $\mathscr{A} = \mathscr{A}_1 \cup \mathscr{A}_2 \cup \mathscr{A}_3$. The collection $\mathscr{A}$ is an open cover of the set $D_1(r, \bar{\varepsilon}, \delta)$. So, by the paracompactness we can find a locally finite refinement $\{V_\alpha : \alpha \in \Lambda\}$ and a locally Lipschitz partition of unit $\{\gamma_\alpha : \alpha \in \Lambda\}$ sub-ordinate to it with $\operatorname{supp}\gamma_\alpha \subset V_\alpha$ for each $\alpha \in \Lambda$. For each $\alpha \in \Lambda$, we can find $x_\alpha \in D_1(r, \bar{\varepsilon}, \delta)$ such that $V_\alpha \subset \widetilde{U}_{i(\alpha)}(x_\alpha)$ for some $i(\alpha) \in \{1, 2, 3\}$, and $\widetilde{U}_{i(\alpha)}(x_\alpha) \in \mathscr{A}$. To this $x_\alpha \in D_1(r, \bar{\varepsilon}, \delta)$ corresponds the element $w_\alpha^{i(\alpha)}$ such that $\|w_\alpha^{i(\alpha)}\| \leqslant 1$, and $w_\alpha^{i(\alpha)} = \bar{u}_{i(\alpha)}(x_\alpha)$ if $V_\alpha \subset U_{i(\alpha)}(x_\alpha)$ for some $i(\alpha) \in \{1, 2, 3\}$. Since E is densely embedded in X, and so G_1, G_2, G_3 are densely embedded in D_1, D_2, D_3, respectively. Thus, we may take $\bar{x}_\alpha, \bar{w}_\alpha^{i(\alpha)} \in E$ with $\|\bar{w}_\alpha^{i(\alpha)}\| \leqslant 1$ such that

$$\max\left\{\|w_\alpha^{i(\alpha)} - \bar{w}_\alpha^{i(\alpha)}\|, \|x_\alpha - \bar{x}_\alpha\|\right\} < \min\left\{\frac{1}{2}, \frac{\gamma}{64(1+\|x_\alpha\|)L_{\alpha,i(\alpha)}}\right\}. \tag{3.35}$$

and

$$\bar{x}_\alpha - \bar{w}_\alpha^{i(\alpha)} \in G_i \quad \text{if } x_\alpha \in D_i \cap D_1(r, \bar{\varepsilon}, \delta) \text{ for } i = 1, 2, 3. \tag{3.36}$$

Now, let $\widetilde{v} : D_1(r, \bar{\varepsilon}, \delta) \mapsto X$ be defined by

$$\widetilde{v}(x) = (1 + \|x\|) \sum_{\alpha\in\Lambda} \gamma_\alpha(x)(\bar{w}_\alpha^{i(\alpha)} - \bar{x}_\alpha + x). \tag{3.37}$$

Then, by (3.34), (3.35) and (3.37) we have $\|\widetilde{v}(x)\| \leqslant 2(1 + \|x\|)$, and $\langle x^*, \widetilde{v}(x)\rangle \geqslant \frac{\gamma}{16}$ for any $x \in D_1(r, \bar{\varepsilon}, \delta)$ and $x^* \in \partial\varphi(x)$.

For a given $x \in D_1(r, \bar{\varepsilon}, \delta)$, we assume that $x \in U_{i(\alpha)}(x_\alpha)$ for some $U_{i(\alpha)}(x_\alpha) \in \mathscr{A}$. Recalling the construction of $\mathscr{A}$, we have

$$U_{i(\alpha)}(x_\alpha) \in \begin{cases} \mathscr{A}_1 \cup \mathscr{A}_2 \cup \mathscr{A}_3 & \text{if } x \in D_1(r, \bar{\varepsilon}, \delta)\backslash(D_2 \cup D_3); \\ \mathscr{A}_2 & \text{if } x \in D_2 \cap D_1(r, \bar{\varepsilon}, \delta); \\ \mathscr{A}_3 & \text{if } x \in D_3 \cap D_1(r, \bar{\varepsilon}, \delta). \end{cases} \tag{3.38}$$

It follows from (3.36), (3.37), (3.38) that (3.32) holds. The proof of is complete. □

Since φ has no nontrivial critical point on $\partial_E G_i$ for $i = 1, 2, 3$, by using the conditions (H_1) we may take $\delta > 0$ small enough such that

$$(\widetilde{K}_r)_{3\delta} \cap (\partial_{G_1} G_2 \cup \partial_{G_1} G_3) = \emptyset. \tag{3.39}$$

Let $O = G_1\backslash(G_2 \cup G_3)$ and $\widetilde{K}_r^0 = \widetilde{K}_r \cap O$.

Lemma 3.17. *Assume that all the conditions in Theorem 3.14 hold. Let $r \in \mathbb{R}, \delta > 0$ be such that (3.39) holds. Then there exists $\bar{\varepsilon}_0 > 0$ such that for any $0 < \varepsilon < \frac{1}{3}\bar{\varepsilon}_0$ and any compact set $B \subset \varphi^{r+\varepsilon} \cap G_1$, there exists $\eta \in C([0, 1] \times G_1, G_1)$ such that*

1) *$\eta(t, x) = x$ for $t = 0$; or $x \notin G_1 \cap D_1(r, \bar{\varepsilon}_0, \delta)$;*

2) *$\eta(1, B\backslash(\widetilde{K}_r^0)_{3\delta}) \subset (O \cap \varphi^{r-\varepsilon}) \cup \operatorname{int}_{G_1} G_2 \cup \operatorname{int}_{G_1} G_3$;*

3) $\eta(t,\cdot)$ *is a homeomorphism of* G_1 *for* $t \in [0,1]$;

4) $\varphi(\eta(\cdot,x))$ *is non-increasing for any* $x \in G_1$;

5) $\eta(1,G_2) \subset G_2$, $\eta(1,G_3) \subset G_3$;

6) $\eta(1,\mathrm{int}_{G_1} G_2) \subset \mathrm{int}_{G_1} G_2$, $\eta(1,\mathrm{int}_{G_1} G_3) \subset \mathrm{int}_{G_1} G_3$.

Proof. It follows from Lemma 3.16 that there exists $\widetilde{v} : D_1(r,\bar{\varepsilon},\delta) \to X$ such that such that $\|\widetilde{v}(x)\| \leqslant 2(1+\|x\|)$ for any $x \in D_1(r,\bar{\varepsilon},\delta)$, $\langle x^*, \widetilde{v}(x)\rangle \geqslant \frac{\gamma}{16}$ for some $\gamma > 0$ and all $x^* \in \partial\varphi(x)$, and (3.32) holds. Since $\widetilde{K}_r$ is compact in X, we may choose $R_0 > 0$ such that $\|x\| \leqslant R_0$ for all $x \in (\widetilde{K}_r)_{3\delta}$. Take

$$0 < \bar{\varepsilon}_0 < \min\left\{\bar{\varepsilon}, \frac{\gamma\delta}{32(1+R_0)}\right\}. \tag{3.40}$$

Take $0 < \varepsilon < \frac{1}{3}\bar{\varepsilon}_0$. Let $l_i : D_1 \to \mathbb{R}$ for $i = 1,2$ be defined by

$$l_1(x) = \frac{d\big(x, D_1\backslash\varphi_{r-\bar{\varepsilon}_0}^{r+\bar{\varepsilon}_0}\big)}{d\big(x, D_1\backslash\varphi_{r-\bar{\varepsilon}_0}^{r+\bar{\varepsilon}_0}\big) + d\big(x, D_1 \cap \varphi_{r-2\varepsilon}^{r+2\varepsilon}\big)},$$

and

$$l_2(x) = \frac{d\big(x, D_1 \cap (\widetilde{K}_r)_\delta\big)}{d\big(x, D_1 \cap (\widetilde{K}_r)_\delta\big) + d\big(x, D_1\backslash(\widetilde{K}_r)_{2\delta}\big)}.$$

Then $l_i : D_1 \to \mathbb{R}$ for $i = 1,2$ is locally Lipschitz continuous in X. It is easy to see that $l_i : G_1 \to \mathbb{R}$ for $i = 1,2$ is locally Lipschitz continuous in E. Let $V(x) = l_1(x)l_2(x)\widetilde{v}_1(x)$ for all $x \in D_1$, where

$$\widetilde{v}_1(x) = \begin{cases} \widetilde{v}(x), & x \in D_1(r,\bar{\varepsilon},\delta); \\ 0, & \text{otherwise.} \end{cases}$$

Then $V : D_1 \to X$ is locally Lipschitz, $V : G_1 \to E$ is also locally Lipschitz.

Consider the initial value problem in X

$$\begin{cases} \frac{d\sigma}{dt} = -V(\sigma(t,u)), \\ \sigma(0) = u \in G_1. \end{cases} \tag{3.41}$$

Obviously, (3.41) has a unique solution $\sigma(t,u)$ in X, with its right maximal existence interval $[0,T(u))$, where $0 < T(u) \leqslant +\infty$. Since $V : G_1 \to E$ is also locally Lipschitz, (3.41) has a unique solution $\sigma(t,u)$ in E, with its right maximal existence interval $[0,T_1(u))$, where $0 < T_1(u) \leqslant T(u) = +\infty$.

In a similar way as in the proof of Lemmas 3.4, 3.6, 3.9,we can show the following conclusions hold: 1) $T(u) = +\infty$; 2). G_1, G_2, G_3, $\mathrm{int}_{G_1} G_2$ and $\mathrm{int}_{G_1} G_3$ are invariant sets for the flow of (3.41). Moreover, the following inequality hold: for any $[t_1,t_2] \subset [0,T_1(u))$,

$$\begin{aligned} \varphi\big(\sigma(t_1,u)\big) - \varphi\big(\sigma(t_2,u)\big) &= -\int_{t_1}^{t_2} \frac{d\varphi(\sigma(s,u))}{ds} ds \\ &\geqslant \frac{\gamma}{16}(t_2 - t_1) \ \text{ if } \ \sigma(s,u) \in G_1 \cap D_1(r,2\varepsilon,2\delta) \text{ for } s \in [t_1,t_2]. \end{aligned} \tag{3.42}$$

Let $B \subset \varphi^{r+\varepsilon} \cap G_1$ be a compact set, $u_0 \in B\backslash(K_r^0)_{3\delta}$ and $o(u_0) = \{\sigma(t,u_0) : t \in [0,T_1(u_0))\}$. Then we have the following cases:

1) $o(u_0) \cap \mathrm{int}_{G_1} G_2 \neq \emptyset$. Assume that $u_1 := \sigma(t(u_0), u_0) \in \mathrm{int}_{G_1} G_2$ for some $t(u_0) \in [0, T_1(u_0))$. Take a neighborhood $U(u_1)$ of u_1 in G_1 such that $U(u_1) \subset \mathrm{int}_{G_1} G_2$. By the continuous dependence of ordinary differential equations on initial data, there exists an open neighborhood $U(u_0)$ of u_0, such that $\sigma(t(u_0), u) \in U(u_1) \subset \mathrm{int}_{G_1} G_2$ for any $u \in U(u_0)$. Since $\mathrm{int}_{G_1} G_2$ is an invariant set, $\sigma(t, u) \in \mathrm{int}_{G_1} G_2$ for all $u \in U(u_0)$ and $t \geqslant t(u_0)$.

2) $o(u_0) \cap \mathrm{int}_{G_1} G_3 \neq \emptyset$. A similar argument as above yields that, there is a neighborhood $U(u_0)$ of u_0 and $t(u_0) > 0$, such that $\sigma(t, u) \in \mathrm{int}_{G_1} G_3$ for any $u \in U(u_0)$ and $t \geqslant t(u_0)$.

3) $o(u_0) \cap \big(\mathrm{int}_{G_1} G_2 \cup \mathrm{int}_{G_1} G_3\big) = \emptyset$. In this case we have the following two subcases:

3a) $o(u_0) \cap (\widetilde{K}_r^0)_{2\delta} = \emptyset$. First we show that $o(u_0) \cap \{x \in X : \varphi(x) < r - \varepsilon\} \neq \emptyset$. Assume indirectly that this is not the case, then $o(u_0) \cap \{x \in X : \varphi(x) < r - \varepsilon\} = \emptyset$. Assume without loss of generality that $o(u_0) \subset \varphi^{-1}([r - \varepsilon, r + \varepsilon])$. Let $Ax = x - \frac{1}{1+\|x\|} V(x)$ for all $x \in D_1$. Note that

$$Ax = \sum_{\alpha \in \Lambda} \gamma_\alpha(x)(\bar{x}_\alpha - \bar{w}_\alpha), \qquad \forall x \in D_1 \cap \varphi^{-1}([r - \varepsilon, r + \varepsilon]) \backslash (\widetilde{K}_r)_{2\delta}.$$

So, in a similar way as in the proof of Lemma 3.5 we can easily show that $T_1(u_0) = +\infty$. Then, we can take $t_1 = \frac{32\varepsilon}{\gamma}$. By (3.42) we have

$$vr(\sigma(t_1, u_0)) \leqslant \varphi(u_0) - \frac{\gamma}{16} t_1 \leqslant \varphi(u_0) - \frac{\gamma}{16} t_1 < r - \varepsilon. \tag{3.43}$$

which is a contradiction. Thus, $o(u_0) \cap \{x \in X : \varphi(x) < r - \varepsilon\} \neq \emptyset$. Assume that $u_1 := \sigma(t(u_0), u_0) \in \{x \in X : \varphi(x) < r - \varepsilon\}$ for some $t(u_0) \geqslant 0$. Take a neighborhood $U(u_1)$ of u_1 such that $U(u_1) \subset \{x \in X : \varphi(x) < r - \varepsilon\}$. Then, there exists an open neighborhood $U(u_0)$ of u_0, such that $\sigma(t(u_0), u) \in U(u_1) \subset \{x \in X : \varphi(x) < r - \varepsilon\}$ for any $u \in U(u_0)$. Since $\sigma(t, u)$ is non-increasing in $t \in [0, +\infty)$, we have $\sigma(t, u) \subset \{x \in X : \varphi(x) < r - \varepsilon\}$ for any $u \in U(u_0)$ and $t \geqslant t(u_0)$.

3b) $o(u_0) \cap (\widetilde{K}_r^0)_{2\delta} \neq \emptyset$. In this case, we may take $[t_1, t_2] \subset [0, +\infty)$ such that $\sigma(t_1, u_0) \in \partial(\widetilde{K}_r)_{3\delta}$, $\sigma(t_2, u_0) \in \partial(\widetilde{K}_r)_{2\delta}$, and $\sigma(t, u_0) \in (\widetilde{K}_r^0)_{3\delta} \backslash (\widetilde{K}_r^0)_{2\delta}$ for $t \in [t_1, t_2]$. By (3.41) we have

$$\begin{aligned} \|\sigma(t_2, u_0) - \sigma(t_1, u_0)\| &\leqslant \int_{t_1}^{t_2} \|V(\sigma(s, u_0)\| ds \\ &\leqslant 2 \int_{t_1}^{t_2} \big(1 + \|\sigma(s, u_0)\|\big) ds \\ &\leqslant 2\big(1 + R_0\big)(t_2 - t_1). \end{aligned}$$

So, we have

$$t_2 - t_1 \geqslant \frac{\|\sigma(t_2, u_0) - \sigma(t_1, u_0)\|}{2\big(1 + R_0\big)} \geqslant \frac{\delta}{2\big(1 + R_0\big)}. \tag{3.44}$$

Now we show that there must exist $t(u_0) \in [t_1, t_2]$ such that $\sigma(t(u_0), u_0) \in \{x \in X : \varphi(x) < r - \varepsilon\}$. Assume indirectly that this is not the case, then

$$\{\sigma(t, u_0) : t \in [t_1, t_2]\} \cap \{x \in X : \varphi(x) < r - \varepsilon\} = \emptyset.$$

Then, by (3.42) and (3.44) we have

$$\begin{aligned} \varphi(\sigma(t_2, u_0)) &\leqslant \varphi(\sigma(t_1, u_0)) - \frac{\gamma}{16}(t_2 - t_1) \\ &\leqslant \varphi(u_0) - \frac{\gamma}{16}(t_2 - t_1) \\ &\leqslant r + \varepsilon - \frac{\gamma\delta}{32\big(1 + R_0\big)} \\ &< r + \varepsilon - \bar{\varepsilon}_0 < r - \varepsilon, \end{aligned}$$

which is a contradiction. Thus, there must exist $t(u_0) \in [t_1, t_2]$ such that $\sigma(t(u_0), u_0) \in \{x \in X : \varphi(x) < r - \varepsilon\}$. Then, as the arguments in the case 3a), we can show that there exists a open neighborhood of $U(u_0)$ of u_0 such that $\sigma(t, u) \subset \{x \in X : \varphi(x) < r - \varepsilon\}$ for any $u \in U(u_0)$ and $t \geqslant t(u_0)$.

Let $\mathscr{B} = \{U(u_0) : u_0 \in B\backslash(\widetilde{K}_r^0)_{3\delta}\}$. Then $\mathscr{B}$ is an open cover of $B\backslash(\widetilde{K}_r^0)_{3\delta}$. Since $B\backslash(\widetilde{K}_r^0)_{3\delta}$ is a compact set, then there exist finite sets of $\mathscr{B}$, say $U(u_1), U(u_2), \ldots, U(u_m)$, such that $B\backslash(\widetilde{K}_r^0)_{3\delta} \subset \bigcup_{i=1}^m U(u_i)$. Let $T = \max\{t(u_1), t(u_2), \ldots, t(u_m)\}$ and $\eta(Tt, u)$ for all $t \in [0,1]$ and $u \in B\backslash(\widetilde{K}_r^0)_{3\delta}$. Then we can easily check that all conclusions hold. The proof is complete. □

Proof of Theorem 3.14. Let

$$r = \inf_{\gamma\in\Gamma} \sup_{u\in\gamma([0,1])\cap O} \varphi(u),$$

where

$$\Gamma = \{\gamma \in C([0,1], G_1) : \gamma(0) \in \mathrm{int}_{G_1} G_2, \gamma(1) \in \mathrm{int}_{G_1} G_3\}.$$

Now we show that $\widetilde{K}_r \cap O \neq \emptyset$. Arguing by contradiction that $O \cap \widetilde{K}_r = \emptyset$. Take $\delta > 0$ small enough such that (3.39) holds. By using Lemma 3.17, there exists $\bar{\varepsilon}_0 > 0$ such that for any $0 < \varepsilon < \frac{1}{3}\bar{\varepsilon}_0$ and any compact set $B \subset \varphi^{r+\varepsilon} \cap G_1$, there exists $\eta \in C([0,1] \times G_1, G_1)$ such that the conclusions 1)∼6) in Lemma 3.17 hold.

Take $\gamma \in \Gamma$ such that $\sup_{u\in\gamma([0,1])\cap O} \varphi(u) < r + \varepsilon$. Let $\gamma_1 = \eta(1, \gamma(\cdot))$. Since

$$\eta(1, \mathrm{int}_{G_1} G_2) \subset \mathrm{int}_{G_1} G_2, \qquad \eta(1, \mathrm{int}_{G_1} G_3) \subset \mathrm{int}_{G_1} G_3,$$

$$\gamma(0) \in \mathrm{int}_{G_1} G_2, \qquad \gamma(1) \in \mathrm{int}_{G_1} G_3,$$

then we have $\gamma_1(0) \in \mathrm{int}_{G_1} G_2$ and $\gamma_1(1) \in \mathrm{int}_{G_1} G_3$. So, $\gamma_1 \in \Gamma$.

On the other hand, we have by 2) in Lemma 3.17,

$$r \leqslant \sup_{u\in\gamma_1([0,1])\cap O} \varphi(u) \leqslant \sup_{u\in(\varphi^{r-\varepsilon}\cup\mathrm{int}_{G_1} G_2\cup\mathrm{int}_{G_1} G_3)\cap O} \varphi(u) \leqslant \sup_{u\in\varphi^{r-\varepsilon}} \varphi(u) \leqslant r - \varepsilon,$$

which is a contradiction. So, $\widetilde{K}_r \cap O \neq \emptyset$.

If there exists a curve l in E_1 such that $l \cap D_2 \neq \emptyset$, $l \cap D_3 \neq \emptyset$ and $\max_{u\in l} \varphi(u) < \varphi(0)$, then we have $r < \varphi(0)$. So, $\widetilde{K}_r \cap (O\backslash\{0\}) \neq \emptyset$. The proof is complete. □

Remark 3.18. Here in Theorem 3.14 we have proved a three critical points theorem in ordered interval, which can be thought as a mountain pass theorem in ordered interval. It should be pointed out that Theorem 3.14 can also be obtained by the method showing Theorem 3.1.

3.3 Critical points inside and outside the ordered interval

By Theorem 3.1 and 3.14 we can get the following Theorem 3.19.

Theorem 3.19. *Suppose that (H_1)∼(H_3) hold, φ has no nontrivial critical point on $\partial_E(\pm P_1)$ and $\partial_E(D_i \cap E)$ for $i = 1, 2, 3$. Moreover, either*

(H_4) φ is outwardly directed on $\pm P$, D_i for $i = 1, 2, 3$ and $\pm P \cap D_1$; or

(H_5) $(I - \nabla\varphi)(\pm P) \subset \pm P$ and $(I - \nabla\varphi)(D_i) \subset D_i$ for $i = 1, 2, 3$.

Then φ has at least one positive critical point $\bar{u}_1$, one negative critical point $\bar{u}_2$ and one sign-changing critical point $\bar{u}_3$ outside D_1; 2) φ has at least one positive critical point $\bar{u}_4$, one negative critical point $\bar{u}_5$ and one critical point $\bar{u}_6$ inside D_1; moreover, $\bar{u}_6$ is nontrivial if there exists a curve l in E_1 such that $l \cap D_2 \neq \emptyset$, $l \cap D_3 \neq \emptyset$ and $\max_{u\in l} \varphi(u) < \varphi(0)$.

Remark 3.20. Obviously, by the method above we can study the existence of multiple critical points inside and outside the ordered interval which not contain the origin. Moreover, we can give results for at least one or two critical points exist outside the ordered interval, and we can also use the concept of parallel pair of upper and lower solutions to establish the results for multiple critical points. For the sake of brevity, we will only list two of such kind results and not give their proofs.

Theorem 3.21. *Let $u_0 \ll v_1$, $D_1 = [u_0, v_1]$ and $G_1 = D_1 \cap E$. Suppose that (H_1) and (H_3) hold, φ satisfies the condition (PS) and has no critical points on $\partial_E G_1$, $(I - \nabla\varphi)(D_1) \subset D_1$. Then φ has at least one critical point outside D_1, and at least one critical point inside D_1.*

Theorem 3.22. *Let $u_0 \ll u_1 \ll v_1$, $D_1 = [u_0, v_1]$, $D_2 = [u_1, v_1]$, $G_1 = D_1 \cap E$, $G_2 = D_2 \cap E$ and $O_1 = \{x \in X : x \geqslant u_1\}$. Suppose that (H_1) and (H_3) hold, φ satisfies the condition (PS) and has no critical points on $\partial_E G_1 \cup \partial_E(O_1 \cap E) \cup \partial_E G_2$, $(I - \nabla\varphi)(D_1) \subset D_1$, $(I - \nabla\varphi)(D_2) \subset D_2$, and $(I - \nabla\varphi)(O_1) \subset O_1$. Then φ has at least two critical points $\bar{u}_1, \bar{u}_2$ outside D_1 and at least one critical point $\bar{u}_3$ inside D_1 such that $\bar{u}_1 \gg u_1$, $\bar{u}_2 \not\gg u_1$, and $u_1 \ll \bar{u}_3 \ll v_1$.*

Remark 3.23. Obviously, we can also study the existence of fixed points for set value operators using the methods described above. Moreover, our main results also be applicable for φ being of C^1 class, and some of our main results are new even for the case of φ being of C^1 class.

4 Application to the differential inclusion problem with a convex-concave nonlinearity

As the application of Theorem 3.19, in this section we will show multiple solutions of a differential inclusion problem with a convex-concave nonlinearity. The main result of this section extend some relevant results concerning the differential equation boundary value problems with a concave-convex nonlinearity that was first studied by A. Ambrosetti, H. Brezis and G. Cerami [2].

Let

$$\|u\| = \left(\int_\Omega \|Du\|^p dx\right)^{\frac{1}{p}}, \qquad |u|_k = \left(\int_\Omega |u|^k dx\right)^{\frac{1}{k}}$$

be the standard norms of $W_0^{1,p}(\Omega)$, respectively $L^k(\Omega)$ for $1 < k < p^*$. Let $X = W_0^{1,p}(\Omega)$ and $E = C_0^1(\Omega)$.

Consider the following Dirichlet problem for differential inclusion problem

$$\begin{cases} -\mathrm{div}\big(\|Du(x)\|^{p-2}Du(x)\big) - \lambda|u(x)|^{q-2}u(x) \in \partial j(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where Ω is a bounded open domain in $\mathbb{R}^N$ with a smooth boundary, $1 < q < p < +\infty$, the reaction term $\partial j(x,s)$ is the generalized gradient of a non-smooth potential $s \mapsto j(x,s)$, which is subject to the following conditions.

(H_j) $j : \Omega \times \mathbb{R} \mapsto \mathbb{R}$ is a Carathéodory function and there exist constants $a_1 > 0$, $q < p < r < p^*$ such that

(i) $j(x,\cdot)$ is locally Lipschitz for almost every $x \in \Omega$, $j(x,z) \geqslant 0$ for all $z \in \mathbb{R}$ and $\partial j(x,0) = \{0\}$ a.e. on Ω;

(ii) $|\xi| \leqslant a_1(1+|s|^{r-1})$ a.e. in Ω and for all $s \in \mathbb{R}$, $\xi \in \partial j(x,s)$;

(iii) there exist constants $\mu > p$ and $M > 0$ such that

$$\inf_{x\in\Omega} j(x,M) > 0 \quad \text{and} \quad \mu j(x,z) \leqslant -j^o(x,z;-z) \quad \text{a.e. on } \Omega \text{ all } z \geqslant M;$$

(iv) there exist $M_+ > 0$, $M_- < 0$ such that

$$\max \partial j(x, M_+^{\frac{1}{p-1}} e) < \frac{M_+}{2} \quad \text{and} \quad \min \partial j(x, M_-^{\frac{1}{p-1}} e) > \frac{M_-}{2} \quad \text{a.e. in } \Omega,$$

where $e \in W_0^{1,p}(\Omega)$ such that $-\Delta_p e = 1$.

(v) both $\min \partial j(x,z) + m|z|^{p-2}z$ and $\max \partial j(x,z) + m|z|^{p-2}z$ are nondecreasing in $z \in \mathbb{R}$ for a.e. $x \in \Omega$ and some $m \geqslant 0$.

Remark 4.1. (H_j) (iii) was firstly put forward by [27]. It is a super-linear condition; see (4.14) below. So, (4.1) has a concave-convex nonlinearity. The condition (H_j) (iv) assures the existence of a pair of strict upper and lower solution of (4.1).

For $\lambda > 0$, we introduce the energy functional $\varphi_\lambda : X \mapsto \mathbb{R}$ by

$$\varphi_\lambda(u) = \frac{1}{p}\|u\|_p^p - \frac{\lambda}{q}|u|_q^q - \int_\Omega j(x,u(x))dx,$$

Let $P = \{u \in X : u(x) \geqslant 0 \text{ a.e. } x \in \Omega\}$ and $P_1 = P \cap E$. Given $\lambda > 0$, we say that $u \in X$ is a (weak) solution of (4.1_λ) if $\Delta_p u \in L^{r'}(\Omega)$, where $\frac{1}{r} + \frac{1}{r'} = 1$, and

$$-\Delta_p u(x) \in \lambda|u(x)|^{q-2}u(x) + \partial j(x,u(x)) \quad \text{for almost every } x \in \Omega.$$

Let $K_\lambda = \{x \in X : 0 \in \partial\varphi_\lambda(x)\}$ for $\lambda > 0$.

Theorem 4.2. *Assume (H_j) holds. Then there exists $\lambda^* > 0$, such that for $\lambda \in (0,\lambda^*)$, (4.1) has at least two positive solutions $\bar{u}_1, \bar{u}_2 \in \operatorname{int} P_1$, two negative solutions $\bar{u}_3, \bar{u}_4 \in \operatorname{int}(-P_1)$, one sign-changing solution $\bar{u}_5 \in C_0^1(\bar{\Omega})$ and one nontrivial solution $\bar{u}_6 \in C_0^1(\bar{\Omega})$.*

Example 4.3. Assume $\partial j(x,u) := g(u)$ satisfies:

(a) $g : \mathbb{R} \to \mathbb{R}$ is continuous;

(b) $|g(u)| \leqslant a_1(1+|u|^{r-1})$ for some $a_1 > 0$ and $q < p < r < p^*$;

(c) there exists $\mu > p$ and $M \geqslant 0$ such that $0 < \mu G(u) \leqslant ug(u)$ for all $u \geqslant M$, where $G(u) = \int_0^u g(s)ds$;

(d) $\lim_{u\to 0} \frac{g(u)}{|u|^{p-2}u} = 0$;

(e) there exists $m_0 \geqslant 0$ such that $g(u) + m_0|u|^{p-2}u$ is nondecreasing in u.

Note $e \in L^\infty(\Omega)$ if $e \in W_0^{1,p}(\Omega)$ and $-\Delta_p e = 1$. By the condition (d) we can see the condition (H_j) (iv) holds. Thus, all of the conditions in (H_j) hold if all conditions (a)$\sim$(e) hold. According to Theorem 4.2, we see that, for small enough $\lambda > 0$, the following boundary value problem

$$\begin{cases} -\operatorname{div}\big(\|Du(x)\|^{p-2}Du(x)\big) - \lambda|u(x)|^{q-2}u(x) = g(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

has at least two positive solutions $\bar{u}_1, \bar{u}_2 \in \operatorname{int} P_1$, two negative solutions $\bar{u}_3, \bar{u}_4 \in \operatorname{int}(-P_1)$, one sign-changing solution $\bar{u}_5 \in C_0^1(\bar{\Omega})$ and one nontrivial solution $\bar{u}_6 \in C_0^1(\bar{\Omega})$.

We should point out that in this case $\bar{u}_6$ may be taken as a sign-changing solution. The result of this kind has been obtained in [20]. Hence, Theorem 4.2 can be thought as an extension of some main results in [20].

Example 4.4. Let

$$j(s) = \begin{cases} \frac{a}{r_1}|s|^{r_1}, & \text{if } s < 1, \\ \frac{b}{r_2}|s|^{r_2} + \left(\frac{a}{r_1} - \frac{b}{r_2}\right), & \text{if } s \geqslant 1, \end{cases}$$

where $p < r_1 < r_2 < p^*$, $0 < a < b$. Let $Q = [a, b]$, then we have

$$\partial j(s) = \begin{cases} a|s|^{r_1-2}s, & \text{if } s < 1, \\ Q, & \text{if } s = 1, \\ b|s|^{r_2-2}s, & \text{if } s > 1. \end{cases}$$

It is easy to check that (H_j) holds for the above j.

To show Theorem 4.2 we will apply Theorem 3.19. To this end, we need to check that all conditions in Theorem 3.19 hold. By now, many of these proofs are standard and well known. Nevertheless, for the convenience of the reader, we will give detailed proofs of some lemmas. Some of our proofs refer to [17, 27]. In what follows we will assume that $m = 0$. It is not difficult to show the result holds in the case of $m > 0$.

Let $A : W_0^{1,p}(\Omega) \mapsto W^{-1,p'}(\Omega)$ be defined by

$$\langle A(u), v\rangle = \int_\Omega \|Du(z)\|^{p-2}(Du(z), Dv(z))_{\mathbb{R}^N} dz \quad \text{for } u, v \in W_0^{1,p}(\Omega).$$

The following Lemma 4.5 can be found in [17, p. 435].

Lemma 4.5. *The mapping $A : W^{1,p} \mapsto W^{-1,p'}(\Omega)$ is continuous and has the $(S)_+$ property, i.e., if $\{u_n\}$ is a sequence in $W^{1,p}(\Omega)$ such that $u_n \rightharpoonup u$ in $W^{1,p}(\Omega)$ and*

$$\limsup_{n\to\infty} \langle A(u_n), u_n - u\rangle \leqslant 0,$$

then $u_n \to u$ in $W^{1,p}(\Omega)$.

Recall some facts about the spectrum of the p-Laplacian with Dirichlet boundary condition. Consider the nonlinear eigenvalue problem

$$\begin{cases} -\operatorname{div}\big(\|Du(x)\|^{p-2}Du(x)\big) = \lambda|u(x)|^{p-2}u(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.2}$$

Let λ_1 be the principal eigenvalue of $(-\Delta_p, W_0^{1,p}(\Omega))$. Then λ_1 is positive, isolated and simple. There is the following variational characterization of λ_1 using Rayleigh quotient:

$$\lambda_1 = \inf\left\{\frac{\|Du\|_p^p}{|u|_p^p} : u \in W_0^{1,p}(\Omega), u \neq 0\right\}.$$

This minimum is actually realized at normalized eigenfunction $\widetilde{u}_1$. The Ljusternik–Schnirelmann theory gives, in addition to λ_1, a whole strictly increasing sequence of positive numbers $\lambda_1 < \lambda_2 \leqslant \lambda_3 \leqslant \cdots \leqslant \lambda_k \leqslant \cdots$ for which there exist nontrivial solutions for problem (4.2). In what follows we let $\widetilde{u}_2 \in W_0^{1,p}(\Omega)$ be a nontrivial solutions for problem (4.2) corresponding to λ_2, and $E_1 = \operatorname{span}\{\widetilde{u}_1, \widetilde{u}_2\}$.

Lemma 4.6. *If $u \in K_\lambda$, then $u \in C_0^1(\bar{\Omega})$ and u solves (4.1). Moreover, if $u \in \pm P \cap K_\lambda$ and $u \neq 0$, then $u \in \operatorname{int}(\pm P_1) \cap K_\lambda$.*

Proof. The proof is similar to Proposition 3.1 and 3.2 in [17]. Obviously, $u \mapsto \frac{1}{p}\|u\|^p$ is a C^1-functional whose derivative is the operator A. Aubin–Clarke's Theorem ensures that the functional

$$u \mapsto \int_\Omega j(x,u)dx$$

is Lipschitz continuous on any bounded subset of $L^r(\Omega)$ and its gradient is included in the set

$$N(u) = \{w \in L^{r'}(\Omega) : w(x) \in \partial j(x, u(x)) \text{ for almost every } x \in \Omega\}.$$

Since X continuously embedded in $L^r(\Omega)$, the function φ_λ turns out to be locally Lipschitz on X. So, we have

$$\partial\varphi_\lambda(u) \subset A(u) - \lambda|u|^{q-2}u - N(u). \tag{4.3}$$

Now, if $u \in X$ complies with $0 \in \partial\varphi_\lambda(u)$ then

$$A(u) = \lambda|u|^{q-2}u + w \quad \text{in } X^*$$

for some $w \in N(u)$. Hence, $\Delta_p u \in L^{r'}(\Omega)$ and u solves (4.1). By the condition (H_j) (ii) and (4.3) we get the estimate

$$-u\Delta_p u \leqslant a_1(|u| + |u|^r) \quad \text{a.e. in } \Omega.$$

Hence, by [12, Theorem 1.5.5], we have $u \in L^\infty(\Omega)$. From (H_j) (ii) it follows $\Delta_p u \in L^\infty(\Omega)$. So, by [12, Theorem 1.5.6], we have $u \in C_0^1(\bar{\Omega})$.

Let $u \in P \cap K_\lambda$ and $u \neq 0$. By (H_j) (v), we can find a constant $c_0 > 0$

$$\Delta_p u = -\lambda u^{q-1} - w \leqslant c_0 u^{p-1}$$

for some $w \in \partial j(x,u)$. The Vázquez maximum principle yields $u \in \operatorname{int} P_1$.

Similarly, if $u \in -P \cap K_\lambda$ and $u \neq 0$, then $u \in \operatorname{int}(-P_1) \cap K_\lambda$. The proof is complete. □

Lemma 4.7. *If $\{x_n\} \subset W_0^{1,p}(\Omega)$ is bounded, and either $(1 + \|x_n\|)m(x_n) \to 0$, or $(1 + \|x_n\|)m_{\pm P}(x_n) \to 0$, or $(1 + \|x_n\|)m_D(x_n) \to 0$ as $n \to +\infty$, where D is an ordered interval in X, then $\{x_n\}$ has a convergent subsequence.*

Proof. We only consider the case of $(1+\|x_n\|)m(x_n)\to 0$ as $n\to+\infty$. In a similar way we can prove other cases.

Since $\{x_n\}$ is bounded, by passing to a subsequence if necessary, we may assume

$$x_n \rightharpoonup x \quad \text{in } W_0^{1,p}(\Omega), \quad x_n \to x \quad \text{in } L^k(\Omega) \quad \text{for } 1<k<p^*, \quad x_n(z)\to x(z) \quad \text{a.e. on } \Omega$$

and $|x_n(z)|\leqslant g(z)$ a.e. on Ω, for all $n\geqslant 1$, with $g\in L^r(\Omega)$. Take $x_n^*\in\partial\varphi_\lambda(x_n)$ such that $m(x_n)=\|x_n^*\|_*$ for $n\geqslant 1$. Then we have

$$x_n^* = A(x_n)-\lambda|x_n|^{q-2}x_n-u_n \tag{4.4}$$

with $u_n\in L^{r'}(\Omega)$, where $\frac{1}{r}+\frac{1}{r'}=1$, satisfying $u_n(x)\in\partial\varphi(x,x_n(x))$ a.e. on Ω.

Now, we can deduce from $(1+\|x_n\|)m(x_n)\to 0$ that $|\langle x_n^*, x_n-x\rangle|\leqslant\frac{1}{n}\|x_n-x\|$. This reads

$$\left|\langle A(x_n),x_n-x\rangle-\lambda\int_\Omega|x_n|^{q-2}x_n(x_n-x)dz-\int_\Omega u_n(x_n-x)dz\right|\leqslant\frac{1}{n}\|x_n-x\|.$$

Then, we have

$$\lambda\int_\Omega|x_n|^{q-2}x_n(x_n-x)dz\to 0 \quad\text{and}\quad \int_\Omega u_n(x_n-x)dz\to 0 \quad \text{as } n\to\infty,$$

and so,

$$\lim_{n\to\infty}\langle A(x_n),x_n-x\rangle=0.$$

It follows from Lemma 4.5 that $x_n\to x$ in $W_0^{1,p}(\Omega)$. The proof is complete. □

Lemma 4.8. *For $\lambda>0$ small enough, the functional $\varphi_\lambda: X\mapsto\mathbb{R}$ satisfies the conditions (CPS), $(CPS)_{\pm P}$ and $(CPS)_D$ for any ordered interval $D\subset X$.*

Proof. The proof is similar to claim 1 of Theorem 3.1 in [27]. We only prove that $\varphi_\lambda: X\mapsto\mathbb{R}$ satisfies the conditions $(CPS)_P$ and $(CPS)_D$ for any interval $D\subset X$.

Let $c_1>0$ is the best embedding constant of $L^p(\Omega)\hookrightarrow L^q(\Omega)$. Assume that $\widetilde{\lambda}>0$ be such that

$$\left(\frac{\mu}{p}-1\right)-\frac{\widetilde{\lambda}c_1^p}{\lambda_1^{\frac{q}{p}}}\left(\frac{\mu}{q}-1\right)>0.$$

Let $\lambda\in(0,\widetilde{\lambda})$. In what follows $c_2,\dots,c_8$ denote some positive constants. Let $\{x_n\}\subset P$ be such that $|\varphi_\lambda(x_n)|\leqslant M_1$ for some $M_1>0$, and $(1+\|x_n\|)m_P(x_n)\to 0$ as $n\to\infty$. By Proposition 2.9 that $m_P(x_n)=\langle x_n^*,u(x_n)\rangle$ for some $x_n^*\in\partial\varphi_\lambda(x_n)$ and $u(x_n)\in(x_n-P)\cap\bar{B}(0,1)$. So, we have

$$\langle x_n^*,x_n-y\rangle\leqslant\langle x_n^*,u(x_n)\rangle \quad \text{for all } y\in P, \|x_n-y\|<1. \tag{4.5}$$

Let $y_n=x_n+\frac{1}{2\|x_n\|}x_n$ for $n\geqslant 1$. Then, $y_n\in P$, $\|x_n-y_n\|<1$. And so, by (4.5) we have

$$\begin{aligned}(1+\|x_n\|)\langle x_n^*,x_n-y_n\rangle &= -\frac{(1+\|x_n\|)}{2\|x_n\|}\langle x_n^*,x_n\rangle\\ &= -\frac{(1+\|x_n\|)}{2\|x_n\|}\left(\langle A(x_n),x_n\rangle-\lambda|x_n|_q^q+\int_\Omega j^o(z,x_n(z);-x_n(z))dz\right)\\ &\leqslant (1+\|x_n\|)\langle x_n^*,u(x_n)\rangle=:\varepsilon_n \quad \text{with } \varepsilon_n\downarrow 0.\end{aligned}$$

So we obtain

$$-\left(\langle A(x_n), x_n\rangle - \lambda|x_n|_q^q + \int_\Omega j^o(z, x_n(z); -x_n(z))dz\right) \leqslant \frac{2\|x_n\|}{(1+\|x_n\|)}\varepsilon_n.$$

which leads

$$-\|Dx_n\|_p^p + \lambda|x_n|_q^q - \int_\Omega j^o(z, x_n(z); -x_n(z))dz \leqslant \frac{2\|x_n\|}{(1+\|x_n\|)}\varepsilon_n = \varepsilon_n', \varepsilon_n' \downarrow 0. \tag{4.6}$$

Assume without loss of generality that $\varepsilon_n' \leqslant 1$ for all n. Since $|\varphi_\lambda(x_n)| \leqslant M_1$ for all $n \geqslant 1$, we have

$$\frac{1}{p}\|Dx_n\|_p^p - \frac{\lambda}{q}|x_n|_q^q - \int_\Omega j(z, x_n(z))dz \leqslant M_1. \tag{4.7}$$

By (4.6) and (4.7) we obtain that if $\|x_n\| \geqslant 1$,

$$\begin{aligned}\left(\frac{\mu}{p}-1\right)\|Dx_n\|_p^p - \lambda\left(\frac{\mu}{q}-1\right)|x_n|_q^q - \int_\Omega \left(\mu j(z, x_n(z)) + j^o(z, x_n(z); -x_n(z))\right)dz \\ \leqslant \mu M_1 + \varepsilon_n'.\end{aligned} \tag{4.8}$$

By (H_j) (ii)(iii) we have for some $\beta_1 > 0$ (see [27, 2531]),

$$\int_\Omega \left(\mu j(z, x_n(z)) + j^o(z, x_n(z); -x_n(z))\right)dz \geqslant -\beta_1. \tag{4.9}$$

It follows from (4.8) and (4.9) that for $n \geqslant 1$,

$$\left[\left(\frac{\mu}{p}-1\right) - \frac{\lambda c_1^q}{\lambda_1^{\frac{q}{p}}}\left(\frac{\mu}{q}-1\right)\right]\|Dx_n\|_p^p \leqslant \mu M_1 + 1 + \beta_1.$$

Then, we infer that $\{x_n\} \subset W_0^{1,p}(\Omega)$ is bounded. So, by Lemma 4.7 we see that $\{x_n\}$ has a convergent subsequence. Thus, φ_λ satisfies the condition $(CPS)_P$.

Now we show that φ_λ satisfies the condition $(CPS)_D$ for any ordered interval $D \subset X$. Let $\{x_n\} \subset D$ be such that $|\varphi_\lambda(x_n)| \leqslant M_2$ for some $M_2 > 0$ and $(1+\|x_n\|)m_D(x_n) \to 0$ as $n \to \infty$. Obviously, we see that $\{|x_n|_\mu\}$ is bounded. Take $s > 1$ such that $r < s < \min\left\{p^*, \frac{p\max\{N,p\}+\mu)}{\min\{N,p\}}\right\}$. By (H_j) (ii) we have for a.e. $x \in \Omega$, and $z \in \mathbb{R}$,

$$j(x,z) \leqslant c_2 + c_3|z|^s, c_2, c_3 > 0.$$

Take

$$\theta = \begin{cases}\frac{p^*(s-\mu)}{s(p^*-\mu)}, & N > p,\\ 1-\frac{\mu}{s}, & N \leqslant p.\end{cases}$$

Then we have $0 < \theta < 1$, and

$$\frac{1}{s} = \frac{1-\theta}{\mu} + \frac{\theta}{p^*}.$$

Thus, by using the interpolation inequality and the Sobolev embedding theorem, we obtain

$$|x_n|_s \leqslant |x_n|_\mu^{1-\theta}|x_n|_{p^*}^\theta \leqslant c_4\|x_n\|^\theta \quad \text{for some} \quad c_4 > 0.$$

Since $|\varphi_\lambda(x_n)| \leqslant M_1$, $L^s(\Omega) \hookrightarrow L^p(\Omega)$ and $L^s(\Omega) \hookrightarrow L^q(\Omega)$, by using the well known Young's inequality we have

$$\begin{aligned}\frac{1}{p}\|Dx_n\|_p^p &\leqslant \frac{1}{q}|x_n|_q^q + c_2|\Omega| + c_3|x_n|_s^s + M_1\\ &\leqslant c_5 + c_6|x_n|_s^s \leqslant c_7 + c_8\|Dx_n\|_p^{\theta s}.\end{aligned}$$

If $N > p$, then $Ns < p(N+\mu)$, and so we have

$$\theta s = \frac{p^*(s-\mu)}{p^*-\mu} = \frac{pN}{N-p}\cdot\frac{(s-\mu)(N-p)}{pN-N\mu+p\mu} < \frac{pN}{N-p}\cdot\frac{(s-\mu)(N-p)}{Ns-N\mu} = p.$$

If $N \leqslant p$, then $s < p+\mu$. So,

$$\theta s = \Big(1-\frac{\mu}{s}\Big)s = s-\mu < p.$$

Thus, in both cases we have $\theta s < p$. Consequently, $\{\|x_n\|\}$ is bounded. So, by Lemma 4.6 we see that $\{x_n\}$ has a convergent subsequence. Thus, φ_λ satisfies the condition $(CPS)_D$. The proof is complete. □

By the condition (H_j)(iv), we may take $\rho_+ > 1$ and $\lambda_+ > 0$ such that

$$\max\partial j(x, M_+^{\frac{1}{p-1}}e) < \frac{M_+}{2\rho_+} \text{ and } \lambda_+(M_+)^{\frac{q-p}{p-1}}\|e\|_\infty^{q-1} < \frac{1}{2\rho_+}.$$

Then we have for $\lambda \in (0, \lambda_+)$,

$$\rho_+^{-1}\big(-\Delta_p(M_+^{\frac{1}{p-1}}e)\big) > \lambda|M_+^{\frac{1}{p-1}}e|^{q-2}(M_+^{\frac{1}{p-1}}e) + \max\partial j(x, M_+^{\frac{1}{p-1}}e) \quad \text{a.e. in } \Omega. \tag{4.10}$$

Similarly, there exist $\rho_- > 1, \lambda_- > 0$ such that for $\lambda \in (0, \lambda_-)$,

$$\rho_-\big(-\Delta_p(M_-^{\frac{1}{p-1}}e)\big) < \lambda|M_-^{\frac{1}{p-1}}e|^{q-2}(M_-^{\frac{1}{p-1}}e) + \min\partial j(x, M_-^{\frac{1}{p-1}}e) \quad \text{a.e. in } \Omega.$$

Let $\rho = \min\{\rho_+, \rho_-\}$ and $\lambda^* = \min\{\lambda_-, \lambda_+, \widetilde{\lambda}\}$. For each $\lambda \in (0, \lambda^*)$, let

$$\varepsilon_0(\lambda) = \left(\frac{\lambda^{\frac{1}{p-q}}}{(\rho\lambda_1)^{\frac{1}{p-q}}\|\widetilde{u}_1\|_\infty}\right)^{p-1}.$$

Then for each $\varepsilon \in (0, \varepsilon_0(\lambda))$, we have

$$\begin{aligned}\rho\big(-\Delta_p(\varepsilon^{\frac{1}{p-1}}\widetilde{u}_1)\big) &= \rho\varepsilon\lambda_1\widetilde{u}_1^{p-1} < \lambda(\varepsilon^{\frac{1}{p-1}}\widetilde{u}_1)^{q-1}\\ &\leqslant \lambda(\varepsilon^{\frac{1}{p-1}}\widetilde{u}_1)^{q-1} + \min\partial j(x, \varepsilon^{\frac{1}{p-1}}\widetilde{u}_1) \quad \text{a.e. in } \Omega,\end{aligned} \tag{4.11}$$

and

$$\begin{aligned}\rho^{-1}\big(-\Delta_p(-\varepsilon^{\frac{1}{p-1}}\widetilde{u}_1)\big) &= -\varepsilon\lambda_1|\widetilde{u}_1|^{p-2}\widetilde{u}_1\\ &> -\lambda|\varepsilon^{\frac{1}{p-1}}\widetilde{u}_1|^{q-2}\varepsilon^{\frac{1}{p-1}}\widetilde{u}_1 + \max\partial j(x, -\varepsilon^{\frac{1}{p-1}}\widetilde{u}_1) \quad \text{a.e. in } \Omega.\end{aligned}$$

Since $\widetilde{u}_1, e \in P_1$, there exists $\varepsilon_1(\lambda) \in (0, \varepsilon_0(\lambda))$ such that for each $\lambda \in (0, \lambda^*)$ and $\varepsilon \in (0, \varepsilon_1(\lambda))$,

$$M_-^{\frac{1}{p-1}}e \ll -\varepsilon^{\frac{1}{p-1}}\widetilde{u}_1 \ll 0 \ll \varepsilon^{\frac{1}{p-1}}\widetilde{u}_1 \ll M_+^{\frac{1}{p-1}}e.$$

Let

$$u_0 = M_-^{\frac{1}{p-1}}e, v_0 = -\varepsilon^{\frac{1}{p-1}}\widetilde{u}_1, u_1 = \varepsilon^{\frac{1}{p-1}}\widetilde{u}_1, v_1 = M_+^{\frac{1}{p-1}}e \tag{4.12}$$

and $D_1 = [u_0, v_1], D_2 = [u_0, v_0], D_3 = [u_1, v_1]$.

Lemma 4.9.

1) There exists $R_0 > 0$ such that for $\lambda \in (0, \lambda^*)$,

$$\sup_{u \in S_{R_0} \cap E_1} \varphi_\lambda(u) < \varphi(0). \tag{4.13}$$

2) Let $l_\varepsilon = S_\varepsilon \cap E_1$ such that $l_\varepsilon \cap G_2 \neq \emptyset$ and $l_\varepsilon \cap G_3 \neq \emptyset$. Then there exits $\bar{\varepsilon}_0 > 0$ such that $\alpha(\varepsilon) := \max_{u \in l_\varepsilon} \varphi_\lambda(u) < 0$ for $\varepsilon \in (0, \bar{\varepsilon}_0)$.

Proof. 1) The proof is similar to claim 3 of Theorem 3.1 in [27]. For almost all $x \in \Omega$ and all $z \in \mathbb{R}$, the function $s \mapsto \frac{1}{s^\mu} j(x, sz)$ is locally Lipschitz on $(0, +\infty)$. Using the mean value theorem for locally Lipschitz functions, for $s > 1$ we can find $\theta \in (1, s)$ such that

$$\begin{aligned} \frac{1}{s^\mu} j(x, sz) - j(x, z) &\in \Big(- \frac{\mu}{\theta^{\mu+1}} j(x, \theta z) + \frac{1}{\theta^\mu} \partial_z j(x, \theta z) z \Big)(s - 1) \\ &= \frac{s-1}{\theta^{\mu+1}} \big(- \mu j(x, \theta z) + \partial_z j(x, \theta z) \theta z \big). \end{aligned}$$

By (H_j) (iii), for almost all $x \in \Omega$ and all $z \geqslant M$, we have

$$\frac{1}{s^\mu} j(x, sz) - j(x, z) \geqslant \frac{s-1}{\theta^{\mu+1}} \big(- \mu j(x, \theta z) - j^o(x, \theta z; -\theta z) \big) \geqslant 0.$$

Then for almost all $x \in \Omega$ and all $z \geqslant M$, we have

$$j(x, z) = j\Big(x, \frac{z}{M} M\Big) \geqslant \Big(\frac{z}{M}\Big)^\mu j(x, M) \geqslant \Big(\frac{z}{M}\Big)^\mu \inf_{x \in \Omega} j(x, M).$$

Take $p_1 \in (p, \mu)$. So, it is seen that for a given $\eta > 0$ we can find a constant $c_\eta > 0$ such that

$$j(x, z) \geqslant \frac{\eta}{p} z^{p_1} - c_\eta \quad \text{for a.e. } x \in \Omega. \tag{4.14}$$

Let $\bar{u} \in S_1 := \{u \in W_0^{1,p}(\Omega) : \|Du\|_p = 1\}$. It follows from (4.14) that for $t \in [0, +\infty)$,

$$\begin{aligned} \varphi_\lambda(t\bar{u}) &= \frac{1}{p} \|D(t\bar{u})\|_p^p - \frac{\lambda t^q}{q} |\bar{u}|_q^q - \int_\Omega j\big(x, t\bar{u}(x)\big) dx \\ &\leqslant \frac{t^p}{p} - \frac{\lambda t^q}{q} |\bar{u}|_q^q - \frac{\eta t^{p_1}}{p} |\bar{u}|_{p_1}^{p_1} + c_\eta |\Omega|. \end{aligned} \tag{4.15}$$

This implies that

$$\lim_{t \to +\infty} \varphi_\lambda(t\bar{u}) = -\infty.$$

Since $E_1 \cap S_1$ is compact, there exists $R_0 > 0$ such that (4.13) holds.

2) For each $u \in l_\varepsilon$, let $\bar{u} = \frac{u}{\|u\|} \in S_1$. Since $j(x, z) \geqslant 0$ for a.e. $x \in \Omega$ and $z \in \mathbb{R}$, as in the proof of (4.15) we have

$$\begin{aligned} \varphi_\lambda(\varepsilon\bar{u}) &= \frac{1}{p} \|D(\varepsilon\bar{u})\|_p^p - \frac{\lambda \varepsilon^q}{q} |\bar{u}|_q^q - \int_\Omega j\big(x, \bar{u}(x)\big) dx \\ &\leqslant \frac{\varepsilon^p}{p} - \frac{\lambda \varepsilon^q}{q} |\bar{u}|_q^q \end{aligned}$$

Thus, there exists $\bar{\varepsilon}_0 > 0$ such that $\alpha(\varepsilon) < 0$ for $\varepsilon \in (0, \bar{\varepsilon}_0)$. The proof is complete. □

Lemma 4.10. *φ_λ is outwardly directed on $\pm P$, D_i and $\pm P \cap D_i$ for $i = 1,2,3$ and $\lambda \in (0,\lambda^*)$. Moreover, φ_λ has no critical point at $\partial_E(\pm P_1)$, and $\partial_E(D_i \cap E)$ for $i = 1,2,3$ and $\lambda \in (0,\lambda^*)$, where D_i be defined by (4.12).*

Proof. We only prove that φ_λ is outwardly directed on $[u_1, v_1]$ and has no critical point on $\partial_E([u_1, v_1] \cap E)$ for $\lambda \in (0,\lambda^*)$. In the same way we can show other cases.

Assume to the contrary that $\partial_E([u_1, v_1] \cap E) \cap K_\lambda \neq \emptyset$. Take $u \in \partial_E([u_1, v_1] \cap E) \cap K_\lambda$. Then we have $u = (-\Delta_p)^{-1}\big(\lambda|u|^{q-2}u + w\big)$ for some $w \in N(u)$. It follows from $u_1, v_1 \in E$ and $u_1 \leqslant u \leqslant v_1$ that $u \in L^\infty(\Omega)$. Then we have by (4.10) and (4.11),

$$\begin{aligned} -\Delta_p u &= \lambda|u|^{q-2}u + w \geqslant \lambda|u|^{q-2}u + \min \partial j(x,u) \\ &\geqslant \lambda|u_1|^{q-2}u_1 + \min \partial j(x,u_1) \geqslant \rho\big(-\Delta_p u_1\big), \end{aligned}$$

and

$$\begin{aligned} -\Delta_p u &= \lambda|u|^{q-2}u + w \leqslant \lambda|u|^{q-2}u + \max \partial j(x,u) \\ &\leqslant \lambda|v_1|^{q-2}v_1 + \max \partial j(x,v_1) \leqslant \rho^{-1}\big(-\Delta_p v_1\big). \end{aligned}$$

As a consequence of the weak comparison theorem [35] we obtain $u \geqslant \rho^{\frac{1}{p-1}} u_1 \gg u_1$ and $v_1 \geqslant \rho^{\frac{1}{p-1}} u \gg u$. This implies that $u \in \mathrm{int}([u_1, v_1] \cap E)$, which is a contradiction. Hence, φ_λ has no critical point on $\partial_E([u_1, v_1] \cap E)$.

The following elementary inequality is well known:

$$\big(|y|^{p-2}y - |h|^{p-2}h, y-h\big)_{\mathbb{R}^N} \geqslant \begin{cases} c_1(p)(|y|+|h|)^{p-2}|y-h|^2 & \text{if } 1 < p < 2, \\ c_2(p)|y-h|^p & \text{if } p \geqslant 2 \end{cases}$$

for all $y, h \in \mathbb{R}^N$, where $c_1(p), c_2(p) > 0$ are constants.

For $u^* = Au - \lambda|u|^{q-2}u - w \in \partial\varphi_\lambda(u)$, we let $v = (-\Delta_p)^{-1}\big(\lambda|u|^{q-2}u + w\big)$, where $w \in L^{r'}(\Omega)$ and $w \in \partial j(x,u)$. If $u^* \neq 0$, then $u \neq v$, and so

$$\begin{aligned} \langle u^*, u-v\rangle &= \langle Au - \lambda|u|^{q-2}u - w, u-v\rangle \\ &= \langle Au + \Delta_p v, u-v\rangle = \langle Au, u-v\rangle + \langle \Delta_p v, u-v\rangle \\ &= \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla(u-v) + \int_\Omega (u-v)\Delta_p v \\ &= \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla(u-v) - \int_\Omega |\nabla v|^{p-2}\nabla v \cdot \nabla(u-v) \\ &\geqslant \begin{cases} c_1(p)\int_\Omega (|\nabla u| + |\nabla v|)^{p-2}|\nabla(u-v)|^2 & \text{if } 1 < p < 2, \\ c_2(p)\int_\Omega |\nabla(u-v)|^p & \text{if } p \geqslant 2 \end{cases} \\ &> 0. \end{aligned}$$

This implies that φ_λ is outwardly directed on $[u_1, v_1]$. The proof is complete. □

Lemma 4.11. *For any $a, b \in \mathbb{R}$ with $a < b$, $K_\lambda \cap \varphi_\lambda^{-1}([a,b])$ is compact in E.*

Proof. We follow some ideas in Lemma 3.2 and 3.3 in [3]. For each $u \in K_\lambda \cap \varphi_\lambda^{-1}([a,b])$, it follows from the proof of Lemma 4.6 that $u = (-\Delta_p)^{-1}(\lambda|u|^{q-2}u + w)$ for some $w \in N(u)$. By a similar way as the proof of Lemma 4.8 we can prove that $K_\lambda \cap \varphi_\lambda^{-1}([a,b])$ is bounded in X. Let

$$B(\lambda) := \big\{\lambda|u|^{q-2}u + w : u \in K_\lambda \cap \varphi_\lambda^{-1}([a,b]), w \in N(u)\big\}.$$

If $p > N$ then $X \hookrightarrow L^\infty(\Omega)$. So, $K_\lambda \cap \varphi_\lambda^{-1}([a,b])$ is bounded in $L^\infty(\Omega)$ if $p > N$. It follows the condition (H_j) (ii) that the set $B(\lambda)$ is bounded in $L^\infty(\Omega)$. According to [21], there exists $0 < \alpha < 1$ and $c_1 > 0$ such that

$$\|(-\Delta_p)^{-1}u\|_{C^{1,\alpha}} \leqslant c_1\|u\|_\infty^{\frac{1}{p-1}} \quad \text{for all } u \in L^\infty(\Omega). \tag{4.16}$$

Hence, $K_\lambda \cap \varphi_\lambda^{-1}([a,b])$ is bounded in $C^{1,\alpha}(\bar{\Omega})$, and is compact in E.

If $1 < p \leqslant N$, take $p^* > \widetilde{r} > \frac{(r-1)N}{p}$. According to [14] we have

$$\|(-\Delta_p)^{-1}u\|_\infty \leqslant c_2\|u\|_{\widetilde{r}}^{\frac{1}{p-1}} \quad \text{for all } u \in L^{\widetilde{r}}(\Omega).$$

So, for each $u \in K_\lambda \cap \varphi_\lambda^{-1}([a,b])$, $u = (-\Delta_p)^{-1}\big(\lambda|u|^{q-2}u + w\big)$ for some $w \in N(u)$, we have

$$\|u\|_\infty = \|(-\Delta_p)^{-1}(\lambda|u|^{q-2}u + w)\|_\infty \leqslant c_2\|(\lambda|u|^{q-2}u + w)\|_{\widetilde{r}}^{\frac{1}{p-1}}. \tag{4.17}$$

It follows from the Sobolev embedding theorem and the condition (H_j) (ii) that the set $B(\lambda)$ is bounded in $L^{\widetilde{r}}(\Omega)$. Thus, it follows from (4.17) that $K_\lambda \cap \varphi_\lambda^{-1}([a,b])$ is bounded in $L^\infty(\Omega)$. Then, by (4.16) , we see that $K_\lambda \cap \varphi_\lambda^{-1}([a,b])$ is compact in E. The proof is complete. □

Proof of Theorem 4.2. It follows from Lemmas 4.5∼4.11 that all conditions in Theorem 3.19 hold. According to Theorem 3.19, (4.1) has at least two positive solutions $\bar{u}_1, \bar{u}_2$, two negative solutions $\bar{u}_3, \bar{u}_4$, one sign-changing solution $\bar{u}_5$ and one nontrivial solutions $\bar{u}_6$. The proof is complete. □

Remark 4.12. There have been some papers studied the existence for sign-changing solutions of differential inclusion problems; see [3,8,17,36] and the references therein. For example, by combing variational methods with truncation techniques the paper [17] obtained the existence of positive, negative and nodal solutions to differential inclusion problems with a parameter. Here, our method is different to that in [8,17].

Remark 4.13. Here, we cannot be sure that the nontrivial solution $\bar{u}_6$ is a sign-changing solution. How to get the nontrivial solution to be a sign-changing solution under the condition of $q < p$ is a problem that needs to be further discussed.

5 Appendix: Another proof of Proposition 2.9

Proof. We follow some ideas to show von Neumann–Sion Saddle-point Theorem. Set $D = \partial\varphi(x_0)$. Let X_w^* and X_w denote the spaces X^* and X furnished their weak topology, respectively. Then $\big((x_0 - C) \cap \bar{B}(0,1)\big)$ is a sequentially compact set in the space X_w. Define $h : D \to \mathbb{R}$ by $h(y^*) = \sup_{x \in \big((x_0-C)\cap\bar{B}(0,1)\big)} \langle y^*, x\rangle$ for $y^* \in D$. Then $h : D \to \mathbb{R}$ is lower weakly semi-continuous. Note that D is a compact set of X_w^*. Hence, $\alpha = \min_{y^* \in D} \max_{x \in ((x_0-C)\cap\bar{B}(0,1))} \langle y^*, x\rangle$ exists.

For each $\delta > 0$ and $x \in \big((x_0 - C) \cap \bar{B}(0,1)\big)$, let $g_x(y^*) = \langle y^*, x\rangle - \alpha + \delta$ for all $y^* \in D$. For each $x \in \big((x_0 - C) \cap \bar{B}(0,1)\big)$ and $\varepsilon > 0$, let

$$G_{x,\varepsilon} = \{y^* \in D : g_x(y^*) > \varepsilon\}.$$

Since g_x is continuous in X_w^*, $G_{x,\varepsilon}$ is an open subset in X_w^* for each $x \in \big((x_0 - C) \cap \bar{B}(0,1)\big)$ and $\varepsilon > 0$. Note that the inequalities

$$g_x(y^*) \leqslant 0, \qquad \forall x \in \big((x_0 - C) \cap \bar{B}(0,1)\big)$$

has no solution in D. Then, for each $y^* \in D$, there at least exist $x \in \big((x_0 - C) \cap \bar{B}(0,1)\big)$ such that $g_x(y^*) > 0$. Hence, $\mathscr{A} = \{G_{x,\varepsilon} : x \in \big((x_0 - C) \cap \bar{B}(0,1)\big), \varepsilon > 0\}$ is an open cover of D in X_w^*. Since D is compact in X_w^*, then there exist finite sets in $\mathscr{A}$, say $G_{x_1,\varepsilon_1}, G_{x_2,\varepsilon_2}, \ldots, G_{x_m,\varepsilon_m}$, such that $D \subset \bigcup_{i=1}^m G_{x_i,\varepsilon_i}$. Let $\varepsilon_0 = \min\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m\}$. Then we have $D \subset \bigcup_{i=1}^m G_{x_i,\varepsilon_0}$.

Let $T : D \to \mathbb{R}^m$ be defined by

$$T(y^*) = \big(g_{x_1}(y^*), g_{x_2}(y^*), \ldots, g_{x_m}(y^*)\big).$$

Set $A = \{(y_1, y_2, \ldots, y_m) : y_i < \varepsilon_0, i = 1, 2, \ldots, m\}$. Then, we have $\operatorname{co}\big(T(D)\big) \cap A = \emptyset$. In fact, for any $y = (y_1, y_2, \ldots, y_m) \in \operatorname{co}\big(T(D)\big)$, there exist $z_1, z_2, \ldots, z_\gamma \in T(D)$ and $t_i \geqslant 0$ for $1 \leqslant i \leqslant \gamma$, $\sum_{i=1}^m t_i = 1$ such that $y = \sum_{i=1}^\gamma t_i z_i$. Moreover, for each $z_i \in T(D)$, $1 \leqslant i \leqslant \gamma$, there exist $y_i^* (1 \leqslant i \leqslant \gamma)$ such that

$$z_i = \big(g_{x_1}(y_i^*), g_{x_2}(y_i^*), \ldots, g_{x_m}(y_i^*)\big), \qquad i = 1, 2, \ldots, \gamma.$$

Thus, we have

$$y_j = \sum_{i=1}^{\gamma} t_i g_{x_j}(y_i^*) = g_{x_j}\left(\sum_{i=1}^{\gamma} t_i y_i^*\right), \qquad j = 1, 2, \ldots, m.$$

Since $\sum_{i=1}^\gamma t_i y_i^* \in D$, then there exists $j_0 \in \{1, 2, \ldots, m\}$ such that $y_{j_0} = g_{x_{j_0}}\big(\sum_{i=1}^\gamma t_i y_i^*\big) > \varepsilon_0$. This implies that $y \notin A$, and so $\operatorname{co}\,(T(D)) \cap A = \emptyset$.

By using the Eidelheit convex separation theorem, there exists $\lambda_1', \lambda_2', \ldots, \lambda_n'$, such that $\sum_{i+1}^m \lambda_i' y_i \leqslant 1$ for all $(y_1, y_2, \ldots, y_m) \in A$, and $\sum_{i=1}^m \lambda_i' g_{x_i}(y^*) \geqslant 1$ for each $y^* \in D$ and $T(y^*) = \big(g_{x_1}(y^*), g_{x_2}(y^*), \ldots, g_{x_m}(y^*)\big) \in T(D)$. It is easy to see that $\lambda_i' \geqslant 0$ and $(\lambda_1', \lambda_2', \ldots, \lambda_m') \neq (0, 0, \ldots, 0)$. Let $\lambda = \sum_{i=1}^m \lambda_i'$ and $\lambda_i = \frac{\lambda_i'}{\lambda}$ for $i = 1, 2, \ldots, m$. Then we have for all $y^* \in D$,

$$g(y^*) := \sum_{i=1}^{m} \lambda_i g_{x_i}(y^*) \geqslant \frac{1}{\lambda} > 0.$$

Note that

$$g(y^*) = \sum_{i=1}^{m} \lambda_i \big(\langle y^*, x_i\rangle - \alpha + \delta\big) = \left\langle y^*, \sum_{i=1}^{m} \lambda_i x_i \right\rangle - \alpha + \delta.$$

Thus, we have for all $y^* \in D$,

$$\left\langle y^*, \sum_{i=1}^{m} \lambda_i x_i \right\rangle > \alpha - \delta.$$

So, we have

$$\min_{y^* \in D} \left\langle y^*, \sum_{i=1}^{m} \lambda_i x_i \right\rangle \geqslant \alpha - \delta.$$

Since $\delta > 0$ is arbitrarily given, we have

$$\max_{x \in \big((x_0 - C) \cap \bar{B}(0,1)\big)} \min_{y^* \in D} \langle y^*, x\rangle \geqslant \alpha.$$

On the other hand, we have

$$\min_{z^*\in D}\langle z^*, x\rangle \leqslant \langle y^*, x\rangle, \qquad \forall y^* \in D, x \in \big((x_0 - C)\cap \bar{B}(0,1)\big),$$

and so

$$\max_{x\in\left((x_0-C)\cap\bar{B}(0,1)\right)} \min_{z^*\in D}\langle z^*, x\rangle \leqslant \max_{x\in\left((x_0-C)\cap\bar{B}(0,1)\right)} \langle y^*, x\rangle.$$

Hence,

$$\max_{x\in\left((x_0-C)\cap\bar{B}(0,1)\right)} \min_{z^*\in D}\langle z^*, x\rangle \leqslant \min_{y^*\in D}\max_{x\in\left((x_0-C)\cap\bar{B}(0,1)\right)} \langle y^*, x\rangle = \alpha.$$

Then we have

$$\max_{x\in\left((x_0-C)\cap\bar{B}(0,1)\right)} \min_{y^*\in D}\langle y^*, x\rangle = \min_{y^*\in D}\max_{x\in\left((x_0-C)\cap\bar{B}(0,1)\right)} \langle y^*, x\rangle = \alpha.$$

Moreover, there exists $(x_0^*, x_0) \in D \times \big((x_0 - C)\cap \bar{B}(0,1)\big)$ such that

$$\max_{x\in\left((x_0-C)\cap\bar{B}(0,1)\right)} \min_{y^*\in D}\langle y^*, x\rangle = \langle x_0^*, x_0\rangle = \min_{y^*\in D}\max_{x\in\left((x_0-C)\cap\bar{B}(0,1)\right)} \langle y^*, x\rangle.$$

This implies that for all $x \in \big((x_0 - C)\cap \bar{B}(0,1)\big)$ and $y^* \in D$,

$$\langle x_0^*, x\rangle \leqslant \langle x_0^*, x_0\rangle \leqslant \langle y^*, x_0\rangle.$$

The proof is complete. □

Declarations

- **Data availability** Data sharing not applicable to this article as no data sets were generated or analyzed during the current study.
- **Conflict of interests** On behalf of all authors, the corresponding author states that there is no conflict of interest.

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Multiple solutions for a class of Kirchhoff-type equation with critical growth

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Abstract. In this paper, we study the multiplicity of solutions to a class of Kirchhoff-type equation with critical growth

$$-\left(a+b\int_{\mathbb{R}^3}|\nabla u|^2dx\right)\Delta u+V(x)u=\lambda h(x)f(u)+g(x)u^5 \quad \text{in } \mathbb{R}^3,$$

where a, $b>0$, λ is a positive parameter and f is a continuous nonlinearity with subcritical growth. Under suitable conditions on the potentials $V(x)$, $h(x)$ and $g(x)$, we prove the multiplicity results and investigate the relation between the number of solutions with the topology of the set where g attains its maximum value for small values of the parameter λ. The proofs are based on Nehari manifold and Lusternik–Schnirelmann theory.

Keywords: Kirchhoff-type problem, critical growth, variational method.

2020 Mathematics Subject Classification: 35B33, 35J20, 35J60.

1 Introduction

Consider the following Kirchhoff-type problem

$$-\left(a+b\int_{\mathbb{R}^3}|\nabla u|^2dx\right)\Delta u+V(x)u=\lambda h(x)f(u)+g(x)u^5 \quad \text{in } \mathbb{R}^3, \tag{1.1}$$

where a, $b>0$ are constants and $\lambda>0$ is a parameter. The Kirchhoff-type problem is primarily introduced in [10] to generalize the classical D'Alembert wave equation for free vibrations of elastic strings. More precisely, the original equation is

$$h\rho\frac{\partial^2u}{\partial t^2}-\left(P_0+\frac{Eh}{2L}\int_0^L\left|\frac{\partial u}{\partial x}\right|^2dx\right)\frac{\partial^2u}{\partial x^2}+\delta\frac{\partial u}{\partial t}+f(x,u)=0 \tag{1.2}$$

for $t\geq 0$ and $0<x<L$, where $u=u(t,x)$ is the lateral displacement at the time t and at the space coordinate x, L the length of the string, h the cross-section area, E the Young modulus

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of the material, ρ the mass density, P_0 the initial axial tension, δ the resistance modulus and f the external force. When $\delta = f = 0$, equation (1.2) is introduced by Kirchhoff [10]. For more physical and mathematical background on Kirchhoff-type problems, we refer the readers to [2,7] and the references therein.

If we set $V = 0$ and replace $\mathbb{R}^3$ by a smooth bounded domain $\Omega \subset \mathbb{R}^N (N \geq 3)$, then problem (1.1) becomes a special case of the following Kirchhoff Dirichlet problem

$$\begin{cases} -\left(a + b\int_\Omega |\nabla u|^2 dx\right)\Delta u = \hat{f}(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

Problem (1.3) is often referred to be nonlocal because of the presence of the term $\left(\int_\Omega |\nabla u|^2 dx\right)\Delta u$ which implies that (1.3) is no longer a pointwise identity. This phenomenon causes some mathematical difficulties, which make the study of problem (1.3) particularly interesting. After Lions [12] proposed an abstract functional analysis framework, problem (1.3) had attracted much attention, see, for example, [6,17–19] and the references therein. In [19], Qin et al. considered (1.3) in the case where $\hat{f}(x,u) := Q(x)u^5 + \lambda|u|^{p-1}u$ $(3 < p < 5)$, and proved the existence of one ground state solution by using variational methods that are constrained to the Nehari manifold. The relation between the number of maxima of Q and the number of positive solutions for the problem was also investigated. In [17], Naimen generalized the result of Brézis and Nirenberg ([5]) to problem (1.3) for the case when $\hat{f}(x,u) := \lambda f(x,u) + |u|^{2^*-2}u$, a, $b \geq 0$ and $a + b > 0$. Some existence results as well as nonexistence results were obtained. In [18], the authors further studied the high dimensional case $(N \geq 5)$, and proved the multiplicity of positive solutions of problem (1.3) when $\hat{f}(x,u) := \lambda u^p + u^{2^*-1}$ with $q \in [1, 2^* - 1)$. By combining the variational method and Lusternik–Schnirelmann theory, Cai et al. [6] discussed problem (1.3), where $N = 3$ and $\hat{f}(x,u) := |u|^{4-\varepsilon}u - \lambda u$ with $\varepsilon \in (0,2)$ and $\lambda \geq 0$, and obtained the existence of multiple positive solutions.

Recently, many researchers focused on the existence, multiplicity and asymptotic behavior of solutions of the following problem

$$\begin{cases} -\left(a + b\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)\Delta u + V(x)u = \hat{f}(x,u) & \text{in } \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3), \end{cases} \tag{1.4}$$

where $V : \mathbb{R}^3 \to \mathbb{R}$ is a potential function and $\hat{f} \in C(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$, see [8,11,23–27] and the references therein. In [25], Zhang studied problem (1.4) in the case where $V = 1$ and $\hat{f}(x,u) = a(x)|u|^{p-2}u + \lambda b(x)|u|^{q-2}u + u^5$ with $p, q \in (4,6)$. Besides some other conditions, he assumed that a, $b \in C(\mathbb{R}^3, \mathbb{R})$, $\lim_{|x|\to\infty} a(x) = a_\infty$, $\lim_{|x|\to\infty} b(x) = 0$ and $a(x) \geq a_\infty - Ce^{-a_0|x|}$ for some $a_0 > 0$ and $x \in \mathbb{R}^3$, and proved the existence of one ground state solution for each $\lambda > 0$. It was also proven the existence of two nontrivial solutions for $\lambda > 0$ small. Fan [8] discussed problem (1.4) when $V = 1$ and $\hat{f}(x,u) = \lambda f(x)u^{p-2} + g(x)u^5$ with $(4 < p < 6)$. With the help of Nehari manifold and Lusternik–Schnirelmann theory, he obtained a relationship between the number of positive solutions and the topology of the global maximum set of g. Later, by using a technique introduced by Adachi and Tanaka [1], Zhang et al. [27] obtained the existence of two nontrivial solutions for problem (1.4) when $\hat{f}(x,u) := \lambda f(u) + g(x)u^5$ with f belongs to $C^1(\mathbb{R}, \mathbb{R})$, V has a positive lower bound and satisfies the condition

$$\exists r > 0 \text{ such that } \lim_{|y|\to\infty} \operatorname{meas}\left\{x \in \mathbb{R}^3 : |x - y| < r, V(x) \leq M\right\} = 0, \qquad \forall M > 0.$$

Following [27], Zhang et al. [28] studied the multiplicity of solutions for the ciritical fractional Schrödinger equation with a small superlinear term of the form $(-\Delta)^s u + V(x)u = \lambda f(x,u) + g(x)|u|^{2_s^*-2}u$ in $\mathbb{R}^N$, where $N \geq 3$, $s \in (0,1)$ and $2_s^* = \frac{2N}{N-2s}$ is the critical exponent. Li et al. [11] studied the existence and concentration of positive solutions for the following nonlinear Kirchhoff-type problem

$$-\left(\varepsilon^2 a + \varepsilon b \int_{\mathbb{R}^3} |\nabla u|^2 dx\right)\Delta u + V(x)u = P(x)f(u) + Q(x)u^5 \quad \text{in } \mathbb{R}^3,$$

where $a, b > 0$, $\varepsilon > 0$ is a parameter and f is a continuous subcritical nonlinearity. As $\varepsilon \to 0$, they explored the asymptotic behavior of the semiclassical solutions. See also [26, 30] for related results.

Motivated by the works mentioned above, in this paper, we consider the multiplicity of solutions for the critical Kirchhoff-type problem (1.1) under more general conditions. Precisely, we make the following hypotheses:

(V) $V \in C(\mathbb{R}^3, \mathbb{R})$, $\inf_{x\in\mathbb{R}^3} V(x) := V_0 > 0$ and $\lim_{|x|\to\infty} V(x) = V_\infty > 0$.

(h) $h \in C(\mathbb{R}^3, \mathbb{R})$, $\inf_{x\in\mathbb{R}^3} h(x) := h_0 > 0$ and $\lim_{|x|\to\infty} h(x) = h_\infty > 0$.

(f_1) $f \in C(\mathbb{R}, \mathbb{R})$ and $\lim_{s\to 0} \frac{f(s)}{s} = \lim_{|s|\to\infty} \frac{f(s)}{s^5} = 0$.

(f_2) $\frac{f(s)}{s^3}$ is positive for $s \neq 0$, nonincreasing on $(-\infty, 0)$ and nondecreasing on $(0, +\infty)$.

(f_3) $\lim_{s\to+\infty} \frac{F(s)}{|s|^4} = +\infty$, where $F(s) = \int_0^s f(t)dt$.

(g_1) $g \in C(\mathbb{R}^3, \mathbb{R})$, $g_0 := \inf_{x\in\mathbb{R}^3} g(x) > 0$, $g_M := \sup_{x\in\mathbb{R}^3} g(x) < +\infty$ and $g_\infty := \liminf_{|x|\to\infty} g(x) < g_M$.

(g_2) There exists $\rho_0 > 0$ such that $g(x) = g_M$ for $\rho_0 < |x| < 2\rho_0$. Moreover, $g(0) < g_M$.

For dealing with the multiplicity of solutions to problem (1.1), we recall the Lusternik–Schnirelmann category theory. Suppose that Y is a closed subset of a topological space X, we denoted by $\mathrm{cat}_X(Y)$ the Lusternik–Schnirelmann category of Y in X, that is the least number of closed and contractible sets in X which cover Y; see [4] for more details. Denote

$$\Lambda := \{y \in \mathbb{R}^3 : g(y) = g_M\} \quad \text{and} \quad \Lambda_d := \{x \in \mathbb{R}^3 : \mathrm{dist}(x, \Lambda) < d\} \quad \text{for } d > 0.$$

We assume that

(g_3) The set Λ is nonempty and bounded, there exists $\rho \geq 1$ such that $g(x) - g(y) = O(|x-y|^\rho)$ as $x \to y$ uniformly for $y \in \Lambda$.

The main results of this paper are the following.

Theorem 1.1. *Assume that (V), (h), (f_1)–(f_3) and (g_1)–(g_2) are satisfied. Then there exists $\lambda_0 > 0$ such that problem (1.1) has at least two nontrivial solutions for $\lambda \in (0, \lambda_0)$.*

Theorem 1.2. *Assume that (V), (h), (f_1)–(f_3), (g_1) and (g_3) are satisfied. Then for any $d > 0$, there exists $\lambda_d > 0$ such that, for any $\lambda \in (0, \lambda_d)$, problem (1.1) has at least $\mathrm{cat}_{\Lambda_d}(\Lambda)$ nontrivial solutions.*

Remark 1.3. By assumption (g_3), there exist $C, r > 0$ such that for any $y \in \Lambda$,

$$|g(x) - g(y)| \le C|x - y|^\rho, \qquad \forall x \in B_r(y),$$

where $B_r(y)$ denotes the ball in $\mathbb{R}^3$ with radius r and center y.

Remark 1.4. We point out that, in some special cases, Theorem 1.2 permits to find an arbitrarily large number of solutions of problem (1.1). For example, suppose that (V), (h), (f_1)–(f_3) hold, $g \in C(\mathbb{R}^3, (0, +\infty))$ satisfying $0 < g_0 \le g_\infty < g_M$, and there exist k points $x_1, x_2, \ldots, x_k$ in $\mathbb{R}^3$ such that $g(x_i)$ are strict local maxima satisfying $g(x_i) = g_M = \max_{x\in\mathbb{R}^3} g(x)$, and

$$|g(x) - g(x_i)| = O(|x - x_i|^\rho) \quad \text{as } x \to x_i$$

for each $i = 1, 2, \ldots, k$ and some $\rho \ge 1$. Then it is easy to check that there exists $d = d(k) > 0$ such that $\mathrm{cat}_{\Lambda_d}(\Lambda) \ge k$. By Theorem 1.2, problem (1.1) has at least k solutions for any $\lambda \in (0, \lambda_d)$.

The proofs of Theorem 1.1 and Theorem 1.2 are based on variational methods. Since f is only continuous, we can not use the Nehari manifold arguments developed in [9, 14, 16] in which the condition $f \in C^1$ is required and to overcome this difficulty, we apply some variants of critical point theorems due to Szulkin and Weth [20]. Moreover, there are two main difficulties to prove our result. First, the lack of compactness which caused by the unbounded domain and the critical growth terms makes the bounded (PS) sequences could not converge. Second, the appearance of the nonlocal term, it would be natural to consider how the interaction between the nonlocal term and the critical nonlinear term will effect the existence and multiplicity of solutions of problem (1.1). To overcome these difficulties, we adapt a technique introduced by Benci and Cerami [4] and use the Lusternik–Schnirelmann category.

The paper is organized as follows. In Section 2, we present some technique lemmas and make the estimations for the functionals associated to problem (1.1). In Sections 3 and 4, we show the multiplicity results and complete the proofs of Theorems 1.1 and 1.2, respectively.

Throughout the paper, we make use of the following **notations**. $H^1(\mathbb{R}^3)$ is the Hilbert space endowed with the norm $\|u\|^2 = \int_{\mathbb{R}^3}(|\nabla u|^2 + u^2)dx$. $L^s(\mathbb{R}^3)$, $1 \le s \le +\infty$, denotes the usual Lebesgue space with the norm $\|\cdot\|_s$. $\mathcal{D}^{1,2}(\mathbb{R}^3)$ is the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm $\|u\|^2_{\mathcal{D}^{1,2}} := \int_{\mathbb{R}^3} |\nabla u|^2 dx$. S denotes the best Sobolev constant $S := \inf_{u\in\mathcal{D}^{1,2}(\mathbb{R}^3)\setminus\{0\}} \|u\|^2_{\mathcal{D}^{1,2}} \setminus \|u\|^2_6$. Finally, $C, C_1, C_2, \ldots$ denote different positive constants whose exact value is inessential.

2 Preliminaries

Let $E = H^1(\mathbb{R}^3)$ and $\|u\| = \left(\int_{\mathbb{R}^3}(a|\nabla u|^2 + V(x)u^2)dx\right)^{1/2}$. Then, by (V), $\|\cdot\|$ is an equivalent norm on E. We defined the functional on E by

$$I_\lambda(u) = \frac{1}{2}\|u\|^2 + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)^2 - \int_{\mathbb{R}^3}\left(\lambda h(x)F(u) + \frac{1}{6}g(x)|u|^6\right)dx.$$

It follows from (f_1) that for any $\varepsilon > 0$, $p \in (2, 6)$, there exists $C_\varepsilon > 0$ such that

$$\max\{|F(u)|, |f(u)u|\} \le \varepsilon|u|^2 + C_\varepsilon|u|^6, \qquad \forall u \in \mathbb{R}, \tag{2.1}$$

$$\max\{|F(u)|, |f(u)u|\} \leq \varepsilon(|u|^2 + |u|^6) + C_\varepsilon |u|^p, \qquad \forall u \in \mathbb{R}. \tag{2.2}$$

By (f_2), we derive that

$$\frac{1}{4} f(u)u \geq F(u) \geq 0, \qquad \forall u \in \mathbb{R} \tag{2.3}$$

and

$$\frac{1}{4} f(t)t - F(t) \text{ is nondecreasing in } t > 0 \text{ and nonincreasing in } t < 0. \tag{2.4}$$

Indeed, for $0 \leq s \leq t$, we have

$$\begin{aligned}
\left(\frac{1}{4} f(t)t - F(t)\right) - \left(\frac{1}{4} f(s)s - F(s)\right) &= \frac{1}{4}\left(f(t)t - f(s)s\right) - \left(F(t) - F(s)\right) \\
&= \int_0^t \frac{f(t)}{t^3}\tau^3 d\tau - \int_0^s \frac{f(s)}{s^3}\tau^3 d\tau - \int_s^t f(\tau) d\tau \\
&= \int_0^s \left(\frac{f(t)}{t^3} - \frac{f(s)}{s^3}\right)\tau^3 d\tau + \int_s^t \left(\frac{f(t)}{t^3} - \frac{f(\tau)}{\tau^3}\right)\tau^3 d\tau \\
&\geq 0.
\end{aligned}$$

Arguing similarly for the case $t \leq s \leq 0$.

In order to find the critical points of I_λ, we consider the Nehari manifold

$$\mathcal{M}_\lambda = \left\{u \in E \setminus \{0\} : \langle I'_\lambda(u), u\rangle = 0\right\}.$$

Obviously, $\mathcal{M}_\lambda$ contains all nontrivial critical points of I_λ. Since it is not assumed that f is differentiable, $\mathcal{M}_\lambda$ may not be a C^1-manifold. To overcome the non-differentiability of $\mathcal{M}_\lambda$, we adapt a technique developed in Szulkin and Weth [20].

Lemma 2.1. *Under conditions (V), (h), (f_1)–(f_2) and (g_1), for $\lambda \in (0,1)$, we have*

(i) for each $u \in E \setminus \{0\}$, there exists a unique $t_u > 0$ such that $t_u u \in \mathcal{M}_\lambda$. Moreover, the point t_u is a maximum for $t \to I_\lambda(tu)$;

(ii) the set $\mathcal{M}_\lambda$ is bounded away from 0;

(iii) let $S_1 = \{u \in E : \|u\| = 1\}$, then there exists $\alpha > 0$ such that $t_u \geq \alpha$ for each $u \in S_1$ and, for each compact subset $K \subset S_1$, there exists a constant $C_K > 0$ such that $t_u \leq C_K$ for all $u \in K$;

(iv) the mapping m_λ is a homeomorphism between S_1 and $\mathcal{M}_\lambda$, and for every $u \in \mathcal{M}_\lambda$, $m_\lambda^{-1}(u) = \frac{u}{\|u\|} \in S_1$.

Proof. (i) For each $u \in E \setminus \{0\}$ and $t > 0$, set $g(t) = I_\lambda(tu)$. It is easy to see that $g(0) = 0$, $g(t) > 0$ for $t > 0$ small and $g(t) < 0$ for $t > 0$ large. Thus g has a positive maximum at $t = t_u > 0$ such that $g'(t_u) = 0$ and $t_u u \in \mathcal{M}_\lambda$. Noticing

$$g'(t) = t\left[\|u\|^2 + t^2\left(\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)^2\right] - t^3\left[\lambda \int_{\mathbb{R}^3} h(x)\frac{f(tu)}{(tu)^3}u^4 dx + t^2 \int_{\mathbb{R}^3} g(x)|u|^6 dx\right],$$

we have that t_u is unique. Indeed, suppose $t'_u > t_u > 0$ such that $t'_u u$, $t_u u \in \mathcal{M}_\lambda$. Then we deduce

$$\frac{\|u\|^2}{t_u^2} + b\|\nabla u\|_2^4 = \lambda \int_{\mathbb{R}^3} h(x)\frac{f(t_u u)}{(t_u u)^3}u^4 dx + t_u^2 \int_{\mathbb{R}^3} g(x)|u|^6 dx,$$

$$\frac{\|u\|^2}{t_u'^2}+b\|\nabla u\|_2^4=\lambda\int_{\mathbb{R}^3}h(x)\frac{f(t_u'u)}{(t_u'u)^3}u^4dx+t_u'^2\int_{\mathbb{R}^3}g(x)|u|^6dx,$$

and hence,

$$\left(\frac{1}{t_u^2}-\frac{1}{t_u'^2}\right)\|u\|^2=\lambda\int_{\mathbb{R}^3}h(x)\left(\frac{f(t_uu)}{(t_uu)^3}-\frac{f(t_u'u)}{(t_u'u)^3}\right)u^4dx+\left(t_u^2-t_u'^2\right)\int_{\mathbb{R}^3}g(x)|u|^6dx,$$

which is impossible in view of (f_2) and $t_u'>t_u>0$.

(ii) By using (2.1), (h) and (g_1), we deduce that for any $u\in\mathcal{M}_\lambda$,

$$\begin{aligned}\|u\|^2&\le\int_{\mathbb{R}^3}\left(\lambda h(x)f(u)u+g(x)|u|^6\right)dx\\&\le C\varepsilon\int_{\mathbb{R}^3}|u|^2dx+(C_1C_\varepsilon+g_M)\int_{\mathbb{R}^3}|u|^6dx\\&\le\frac{C\varepsilon}{V_0}\|u\|^2+\frac{C_1C_\varepsilon+g_M}{(aS)^3}\|u\|^6,\end{aligned}$$

which implies that $\|u\|^2\ge C_2$ for some $C_2>0$.

(iii) For each $u\in S_1$, there exists $t_u>0$ such that $t_uu\in\mathcal{M}_\lambda$. By (ii), we have

$$t_u=\|t_uu\|\ge\alpha.$$

Now we prove that $t_u\le C_K$ for all $u\in K\subset S_1$. Arguing indirectly, assume that there exists $\{u_n\}\subset K\subset S_1$ such that $t_{u_n}\to\infty$. Since K is compact, we have $u_n\to u\in K$ and $\int_{\mathbb{R}^3}|u_n|^6dx\to\int_{\mathbb{R}^3}|u|^6dx>0$. Then,

$$I_\lambda(t_{u_n}u_n)\le\frac{t_{u_n}^2}{2}\|u_n\|^2+\frac{bt_{u_n}^4}{4}\left(\int_{\mathbb{R}^3}|\nabla u_n|^2dx\right)^2-\frac{t_{u_n}^6}{6}\int_{\mathbb{R}^3}g_0|u_n|^6dx\to-\infty$$

as $n\to\infty$, which leads to a contradiction because (2.3) implies that, for all $u\in\mathcal{M}_\lambda$,

$$\begin{aligned}I_\lambda(u)&=I_\lambda(u)-\frac{1}{4}\langle I_\lambda'(u),u\rangle,\\&=\frac{1}{4}\|u\|^2+\lambda\int_{\mathbb{R}^3}h(x)\left(\frac{1}{4}f(u)u-F(u)\right)dx+\frac{1}{12}\int_{\mathbb{R}^3}g(x)|u|^6dx\\&\ge0.\end{aligned}$$

(iv) Let us define the maps $\hat{m}_\lambda:E\setminus\{0\}\to\mathcal{M}_\lambda$ and $m_\lambda:S_1\to\mathcal{M}_\lambda$ by setting

$$\hat{m}_\lambda(u)=t_uu\quad\text{and}\quad m_\lambda=\hat{m}_\lambda|_{S_1}.\tag{2.5}$$

By virtue of (i)–(iii) and [20, Proposition 3.1], we deduce that m_λ is a homemorphism between S_1 and $\mathcal{M}_\lambda$, and the inverse of m_λ is given by $m_\lambda^{-1}(u)=\frac{u}{\|u\|}$. □

Now we define the functionals $\hat{J}_\lambda:E\setminus\{0\}\to\mathbb{R}$ and $J_\lambda:S\to\mathbb{R}$, as

$$\hat{J}_\lambda(u)=I_\lambda(\hat{m}_\lambda(u))\quad\text{and}\quad J_\lambda(u)=\hat{J}_\lambda|_{S_1},$$

where $\hat{m}_\lambda(u)=t_uu$ is given in (2.5). As in [20], we have the following conclusion.

Lemma 2.2. *Under the conditions of Lemma 2.1, for $\lambda\in(0,1)$, we have*

(i) $J_\lambda \in C^1(S_1, \mathbb{R})$ *and for each* $v \in T_u(S_1) := \{v \in E : \langle u, v\rangle = 0\}$,

$$\langle J'_\lambda(u), v\rangle = \|m_\lambda(u)\|\langle I'_\lambda(m_\lambda(u)), v\rangle;$$

(ii) $\{u_n\}$ *is a Palais–Smale sequence for* J_λ *if and only if* $\{m_\lambda(u_n)\}$ *is a Palais–Smale sequence for* I_λ. *If* $\{u_n\} \subset \mathcal{M}_\lambda$ *is a bounded Palais–Smale sequence for* I_λ, *then* $\{m_\lambda^{-1}(u_n)\}$ *is a Palais–Smale sequence for* J_λ;

(iii) $u \in S_1$ *is a critical point of* J_λ *if and only if* $m_\lambda(u)$ *is a nontrivial critical point of* I_λ. *Moreover, the corresponding values coincide and* $\inf_{S_1} J_\lambda = \inf_{\mathcal{M}_\lambda} I_\lambda$.

Taking

$$c^* := \frac{a}{3}\left(\frac{bS^3 + \sqrt{(bS^3)^2 + 4aS^3 g_M}}{2g_M}\right) + \frac{b}{12}\left(\frac{bS^3 + \sqrt{(bS^3)^2 + 4aS^3 g_M}}{2g_M}\right)^2$$

and $m_\lambda^\infty := \inf_{u\in\mathcal{M}_\lambda^\infty} I_\lambda^\infty(u)$, where

$$I_\lambda^\infty(u) = \frac{1}{2}\|u\|_{V_\infty}^2 + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)^2 - \int_{\mathbb{R}^3}\left(\lambda h_\infty F(u) + \frac{1}{6}g_\infty |u|^6\right) dx,$$

$\mathcal{M}_\lambda^\infty = \{u \in E \setminus \{0\} : \langle I_\lambda^{\infty\prime}(u), u\rangle = 0\}$ and $\|u\|_{V_\infty} = \left(\int_{\mathbb{R}^3}(a\nabla u \cdot \nabla v + V_\infty uv)dx\right)^{\frac{1}{2}}$. We have the following local compactness result for I_λ.

Lemma 2.3. *Assume that conditions* (V), (h), (f_1)–(f_2) *and* (g_1) *are satisfied. Let* $\lambda > 0$ *and* $\{u_n\} \subset E$ *be a sequence such that* $I_\lambda(u_n) \to c_\lambda \in (-\infty, \min\{c^*, m_\lambda^\infty\})$ *and* $I'_\lambda(u_n) \to 0$ *as* $n \to \infty$. *Then* (u_n) *has a strongly convergent subsequence.*

Proof. By (V) and (2.3), we have

$$c_\lambda + o(1) + o(1)\|u_n\| = I_\lambda(u_n) - \frac{1}{4}\langle I'_\lambda(u_n), u_n\rangle \geq \frac{1}{4}\|u_n\|^2,$$

which implies that $\{u_n\}_{n\in\mathbb{N}} \subset E$ is bounded. Going if necessary to a subsequence, we may assume that there is $u \in E$ such that for each bounded domain $\Omega \subset \mathbb{R}^3$,

$$\begin{aligned} u_n &\rightharpoonup u \quad \text{in } E, \qquad u_n(x) \to u(x) \ \text{ a.e. } x \in \mathbb{R}^3, \\ u_n &\to u \quad \text{in } L^s(\Omega) \ \ (2 < s < 6), \\ |u_n(x)| &\leq w(x) \quad \text{for some } w \in L^s(\Omega). \end{aligned} \tag{2.6}$$

Take $A = \lim_{n\to\infty}\int_{\mathbb{R}^3}|\nabla u_n|^2 dx$. We define the functionals G, H, Φ, Ψ on E by

$$\begin{aligned} G(u) &= \frac{1}{2}\|u\|^2 + \frac{bA}{2}\int_{\mathbb{R}^3}|\nabla u|^2 dx - \int_{\mathbb{R}^3}\left(\lambda h(x)F(u) + \frac{1}{6}g(x)u^6\right)dx, \\ H(u) &= \frac{1}{2}\|u\|_{V_\infty}^2 + \frac{bA}{2}\int_{\mathbb{R}^3}|\nabla u|^2 dx - \int_{\mathbb{R}^3}\left(\lambda h_\infty F(u) + \frac{1}{6}g(x)u^6\right)dx, \\ \Phi(u) &= \frac{1}{2}\|u\|^2 + \frac{bA}{4}\int_{\mathbb{R}^3}|\nabla u|^2 dx - \int_{\mathbb{R}^3}\left(\lambda h(x)F(u) + \frac{1}{6}g(x)u^6\right)dx, \\ \Psi(u) &= \frac{1}{2}\|u\|_{V_\infty}^2 + \frac{bA}{4}\int_{\mathbb{R}^3}|\nabla u|^2 dx - \int_{\mathbb{R}^3}\left(\lambda h_\infty F(u) + \frac{1}{6}g(x)u^6\right)dx. \end{aligned}$$

We claim that $G'(u) = 0$, i.e., $\langle G'(u), \varphi\rangle = 0$ for any $\varphi \in C_0^\infty(\mathbb{R}^3)$. Assume that $1 \le p, q, r, s < +\infty$, Ω is a bounded domain and $h \in C(\Omega \times \mathbb{R})$ satisfying $|h(x,u)| \le C(|u|^{p/r} + |u|^{q/s})$, then, according to [22, Theorem A.4], the operator

$$A : L^p(\Omega)\bigcap L^q(\Omega) \longrightarrow L^r(\Omega) + L^s(\Omega) : u \to h(x,u)$$

is continuous, where $L^p(\Omega)\bigcap L^q(\Omega)$ is the space endowed with the norm $|u|_{p\wedge q} = \|u\|_{L^p(\Omega)} + \|u\|_{L^q(\Omega)}$ and $L^r(\Omega) + L^s(\Omega)$ endowed with the norm

$$|u|_{r\vee s} = \inf\left\{\|v\|_{L^r(\Omega)} + \|w\|_{L^s(\Omega)} : u = v + w, v \in L^r(\Omega), w \in L^s(\Omega)\right\}.$$

Now set $p = r = 2$, $q \in (5,6)$, $s = q/5$ and $h(x,u) = \lambda h(x) f(u)u + g(x)u^5$. By (h), (g) and (f_1), we have

$$|h(x,u)| \le C(|u|^{\frac{2}{2}} + |u|^{\frac{q}{s}}), \qquad \forall (x,u) \in \mathbb{R}^3 \times \mathbb{R}.$$

Since $\varphi \in C_0^\infty(\mathbb{R}^3)$ has a compact support Ω_0, $u_n \rightharpoonup u$ in E implies that $u_n \to u$ in $L^2(\Omega_0)\bigcap L^q(\Omega_0)$. So, by virtue of [22, Theorem A.4],

$$h(x,u_n) \to h(x,u) \quad \text{in } L^2(\Omega_0) + L^s(\Omega_0).$$

Hence

$$\begin{aligned}\int_{\mathbb{R}^3} |(h(x,u_n) - h(x,u))\varphi| dx &= \int_{\Omega_0} |(h(x,u_n) - h(x,u))\varphi| dx \\ &\le |h(x,u_n) - h(x,u)|_{2\vee s} |\varphi|_{2\wedge s'} \xrightarrow{n} 0,\end{aligned}$$

where $1/s + 1/s' = 1$. Combining this and (2.6), we get that $o(1) = \langle I'_\lambda(u_n), \varphi\rangle = \langle G'(u), \varphi\rangle + o(1)$ for any $\varphi \in C_0^\infty(\mathbb{R}^3)$. Thus $G'(u) = 0$.

Let $v_n := u_n - u$. It follows from the Brézis–Lieb lemma, [31, Lemma 2.2] and (f_1) that

$$\begin{aligned}\|u_n\|^2 - \|v_n\|^2 - \|u\|^2 &= o(1),\\ \int_{\mathbb{R}^3} g(x)(u_n^6 - u^6 - v_n^6)dx &= o(1),\\ \int_{\mathbb{R}^3} h(x)\left(F(u_n) - F(u) - F(v_n)\right)dx &= o(1).\end{aligned} \tag{2.7}$$

Noting u is a critical point of G, arguing as in [29, Lemma 2.3], we can conclude that u is locally bounded. Hence, for each $\xi \in E$, by [22, Lemma 8.9], we get

$$\left|\int_{\mathbb{R}^3} g(x)(u_n^5 - u^5 - v_n^5)\xi dx\right| = o(1)\|\xi\|, \tag{2.8}$$

and, similar to [22, Lemma 8.1],

$$\left|\int_{\mathbb{R}^3} h(x)(f(u_n) - f(u) - f(v_n))\xi dx\right| = o(1)\|\xi\|. \tag{2.9}$$

Since $v_n \rightharpoonup 0$ in E, by (V), (h) and (2.2), we deduce that

$$\int_{\mathbb{R}^3} (V(x) - V_\infty)v_n^2 dx \to 0, \qquad \int_{\mathbb{R}^3} (h(x) - h_\infty)F(v_n)dx \to 0, \tag{2.10}$$

and

$$\int_{\mathbb{R}^3} (V(x) - V_\infty)v_n\xi dx \to 0, \qquad \int_{\mathbb{R}^3} (h(x) - h_\infty)f(v_n)\xi dx, \qquad \forall \xi \in E \tag{2.11}$$

as $n \to \infty$. Hence we have

$$\begin{aligned} c_\lambda + o(1) &= I_\lambda(u_n) \\ &= \frac{1}{2}(\|u\|^2 + \|v_n\|^2) + \frac{bA}{4}\int_{\mathbb{R}^3}(|\nabla u|^2 + |\nabla v_n|^2)dx \\ &\quad - \int_{\mathbb{R}^3}\lambda h(x)(F(u) + F(v_n))dx - \int_{\mathbb{R}^3}\frac{1}{6}g(x)(u^6 + v_n^6)dx + o(1) \\ &= \frac{1}{2}(\|u\|^2 + \|v_n\|_{V_\infty}^2) + \frac{bA}{4}\int_{\mathbb{R}^3}(|\nabla u|^2 + |\nabla v_n|^2)dx \\ &\quad - \int_{\mathbb{R}^3}\lambda(h(x)F(u) + h_\infty F(v_n))dx - \int_{\mathbb{R}^3}\frac{1}{6}g(x)(u^6 + v_n^6)dx + o(1) \\ &= \Phi(u) + \Psi(v_n) + o(1), \end{aligned}$$

by (2.10) and (2.7). Moreover, noting $G'(u) = 0$, by (h), (g_1) and (2.3) we have

$$\begin{aligned} \Phi(u) &= \Phi(u) - \frac{1}{4}\langle G'(u), u\rangle \\ &= \frac{1}{4}\|u\|^2 + \int_{\mathbb{R}^3}\left[\lambda h(x)\left(\frac{1}{4}f(u)u - F(u)\right) + \frac{1}{12}g(x)u^6\right]dx \\ &\geq 0, \end{aligned}$$

and hence

$$c_\lambda + o(1) \geq \Psi(v_n). \tag{2.12}$$

Combining (2.8), (2.9) and (2.11), we obtain that

$$\begin{aligned} o(1) &= \langle I'_\lambda(u_n), \xi\rangle - \langle G'(u), \xi\rangle \\ &= (v_n, \xi) + bA\int_{\mathbb{R}^3}\nabla v_n \nabla\xi dx - \int_{\mathbb{R}^3}(\lambda h(x)f(v_n)\xi + g(x)v_n^5\xi)dx + o(1) \\ &= \int_{\mathbb{R}^3}(a\nabla v_n\nabla\xi + V_\infty v_n\xi)dx + bA\int_{\mathbb{R}^3}\nabla v_n\nabla\xi dx - \int_{\mathbb{R}^3}(\lambda h_\infty f(v_n)\xi + g(x)v_n^5\xi)dx + o(1) \\ &= \langle H'(v_n), \xi\rangle + o(1), \qquad \forall \xi \in E, \end{aligned}$$

which implies that

$$H'(v_n) = o(1). \tag{2.13}$$

Next we prove that $v_n \to 0$ in E. According to [31, Lemma 2.1], for some subsequence of $\{v_n\}$, either "vanishing" or "nonvanishing" holds. If "nonvanishing" occurs, we can find $(y_n) \subset \mathbb{R}^3$ with $y_n \overset{n}{\to} \infty$ such that, for $w_n(x) := v_n(x + y_n)$, there is $w \in E \setminus \{0\}$ satisfying

$$\begin{aligned} w_n &\rightharpoonup w \qquad \text{in } E, \\ w_n &\to w \qquad \text{in } L^s_{loc}(\mathbb{R}^3) \ \ (2 \leq s < 6), \\ w_n(x) &\to w(x) \quad \text{a.e. } x \in \mathbb{R}^3. \end{aligned} \tag{2.14}$$

We claim that

$$L'(w) = 0, \tag{2.15}$$

where

$$L(u) = \frac{1}{2}\|u\|_{V_\infty}^2 + \frac{bA}{2}\int_{\mathbb{R}^3}|\nabla u|^2 dx - \int_{\mathbb{R}^3}\left(\lambda h_\infty F(u) + \frac{1}{6}g_\infty u^6\right)dx.$$

Indeed, for every $\xi \in E$, set $\xi_n(x) = \xi(x - y_n)$. We have $\|\xi_n\|_{H^1} = \|\xi\|_{H^1}$, and hence, by (2.13) and (2.14),

$$\begin{aligned}
&|\langle H'(v_n), \xi_n\rangle| \\
&\quad = \left| \int_{\mathbb{R}^3} \left(a\nabla v_n \cdot \nabla \xi_n + V_\infty v_n \xi_n + bA\nabla v_n \cdot \nabla \xi_n - \lambda h_\infty f(v_n)\xi_n - g(x)v_n^5 \xi_n\right) dx \right| \\
&\quad = \left| \int_{\mathbb{R}^3} \left(a\nabla w_n \cdot \nabla \xi + V_\infty w_n \xi + bA\nabla w_n \cdot \nabla \xi - \lambda h_\infty f(w_n)\xi - g(x+y_n)w_n^5 \xi\right) dx \right| \\
&\quad = \left| \int_{\mathbb{R}^3} \left(a\nabla w \cdot \nabla \xi + V_\infty w \xi + bA\nabla w \cdot \nabla \xi\right) dx - \int_{supp\xi} \left(\lambda h_\infty f(w)\xi + g_\infty w^5 \xi\right) dx \right| + o(1) \\
&\quad = |\langle L'(w), \xi\rangle| + o(1),
\end{aligned}$$

and

$$|\langle H'(v_n), \xi_n\rangle| \le \|H'(v_n)\|\|\xi_n\| \le C\|H'(v_n)\|\|\xi_n\|_{H^1} \le C\|H'(v_n)\|\|\xi\|_{H^1} \xrightarrow{n} 0.$$

So (2.15) holds. From (2.15), we see that

$$\|w\|_{V_\infty}^2 + bA\int_{\mathbb{R}^3} |\nabla w|^2 dx = \int_{\mathbb{R}^3} (\lambda h_\infty f(w)w + g_\infty |w|^6) dx. \tag{2.16}$$

Since $w \neq 0$, there exists a unique $t > 0$ such that $tw \in \mathcal{M}_\lambda^\infty$, i.e.,

$$t^2\|w\|_{V_\infty}^2 + bt^4\left(\int_{\mathbb{R}^3} |\nabla w|^2 dx\right)^2 = \int_{\mathbb{R}^3} (\lambda h_\infty f(tw)tw + t^6 g_\infty |w|^6) dx. \tag{2.17}$$

We claim that $t \le 1$. For otherwise $t > 1$, then it follows from (2.17), (2.16), (f_2) and the fact $A \ge \int_{\mathbb{R}^3} |\nabla w|^2 dx$ that

$$\begin{aligned}
t^2\|w\|_{V_\infty}^2 + bt^4\left(\int_{\mathbb{R}^3} |\nabla w|^2 dx\right)^2 &< t^4\left[\|w\|_{V_\infty}^2 + b\left(\int_{\mathbb{R}^3} |\nabla w|^2 dx\right)^2\right] \\
&\le t^4\left(\|w\|_{V_\infty}^2 + bA\int_{\mathbb{R}^3} |\nabla w|^2 dx\right) \\
&= t^4 \int_{\mathbb{R}^3} (\lambda h_\infty f(w)w + g_\infty |w|^6) dx \\
&\le \int_{\mathbb{R}^3} (\lambda h_\infty f(w)t^4 w + t^6 g_\infty |w|^6) dx \\
&\le \int_{\mathbb{R}^3} \left(\lambda h_\infty \frac{f(tw)}{(tw)^3} t^4 w^4 + t^6 g_\infty |w|^6\right) dx \\
&= \int_{\mathbb{R}^3} (\lambda h_\infty f(tw)tw + t^6 g_\infty |w|^6) dx \\
&= t^2\|w\|_{V_\infty}^2 + bt^4\left(\int_{\mathbb{R}^3} |\nabla w|^2 dx\right)^2,
\end{aligned}$$

a contradiction. Thus $t \le 1$. Combining this with (2.4), (2.12), (2.13) and Fatou's lemma, we deduce that

$$\begin{aligned}
c_\lambda + o(1) &\ge \Psi(v_n) - \frac{1}{4}\langle H'(v_n), v_n\rangle \\
&= \frac{1}{4}\|v_n\|_{V_\infty}^2 + \int_{\mathbb{R}^3} \lambda h_\infty \left(\frac{1}{4}f(v_n)v_n - F(v_n)\right) dx + \frac{1}{12}\int_{\mathbb{R}^3} g(x)|v_n|^6 dx
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{4}\|w_n\|^2_{V_\infty} + \int_{\mathbb{R}^3} \lambda h_\infty \left(\frac{1}{4}f(w_n)w_n - F(w_n)\right) dx + \frac{1}{12}\int_{\mathbb{R}^3} g(x+y_n)|w_n|^6 dx \\
&\geq \frac{1}{4}\|w\|^2_{V_\infty} + \int_{\mathbb{R}^3} \lambda h_\infty \left(\frac{1}{4}f(w)w - F(w)\right) dx + \frac{1}{12}\int_{\mathbb{R}^3} g_\infty |w|^6 dx + o(1) \\
&\geq \frac{1}{4}\|tw\|^2_{V_\infty} + \int_{\mathbb{R}^3} \lambda h_\infty \left(\frac{1}{4}f(tw)tw - F(tw)\right) dx + \frac{1}{12}\int_{\mathbb{R}^3} g_\infty |tw|^6 dx + o(1) \\
&= I^\infty_\lambda(tw) - \frac{1}{4}\langle I^{\infty\prime}_\lambda(tw), tw\rangle + o(1) \\
&= I^\infty_\lambda(tw) + o(1) \\
&\geq m^\infty_\lambda + o(1),
\end{aligned}
$$

which contradicts $c_\lambda < m^\infty_\lambda$.

Thus, "nonvanishing" cannot occur, and then we have only the "vanishing" case. In this case, $v_n \to 0$ in $L^s(\mathbb{R}^3)$ $(2 < s < 6)$, and hence, by (2.2), we see that

$$
\int_{\mathbb{R}^3} h(x)F(v_n)dx \to 0 \quad \text{and} \quad \int_{\mathbb{R}^3} h(x)f(v_n)v_n dx \to 0
$$

as $n \to \infty$. Combining this and (2.12)–(2.13), we obtain

$$
c_\lambda + o(1) \geq \Psi(v_n) = \frac{1}{2}\|v_n\|^2_{V_\infty} + \frac{bA}{4}\int_{\mathbb{R}^3} |v_n|^2 dx - \frac{1}{6}\int_{\mathbb{R}^3} g(x)|v_n|^6 dx + o(1), \tag{2.18}
$$

$$
o(1) = \langle H'(v_n), v_n\rangle = \|v_n\|^2_{V_\infty} + bA\int_{\mathbb{R}^3} |\nabla v_n|^2 dx - \int_{\mathbb{R}^3} g(x)|v_n|^6 dx + o(1). \tag{2.19}
$$

Set $l = \lim_{n\to\infty}\left(\int_{\mathbb{R}^3} g(x)|v_n|^6 dx\right)^{\frac{1}{3}}$. If $l > 0$, then using (2.19) and the fact $g(x) \leq g_M$, we have

$$
\begin{aligned}
\int_{\mathbb{R}^3} g(x)|v_n|^6 dx &\geq \int_{\mathbb{R}^3} a|\nabla v_n|^2 dx + b\left(\int_{\mathbb{R}^3} |\nabla v_n|^2 dx\right)^2 + o(1) \\
&\geq aS\left(\int_{\mathbb{R}^3} |v_n|^6 dx\right)^{\frac{1}{3}} + bS^2\left(\int_{\mathbb{R}^3} |v_n|^6 dx\right)^{\frac{2}{3}} + o(1) \\
&\geq \frac{aS}{g_M^{\frac{1}{3}}}\left(\int_{\mathbb{R}^3} g(x)|v_n|^6 dx\right)^{\frac{1}{3}} + \frac{bS^2}{g_M^{\frac{2}{3}}}\left(\int_{\mathbb{R}^3} g(x)|v_n|^6 dx\right)^{\frac{2}{3}} + o(1),
\end{aligned}
$$

which implies that $l \geq \frac{bS^2 + \sqrt{(bS^2)^2 + 4aSg_M}}{2g_M^{\frac{2}{3}}}$. Combining this and (2.18), (2.19), we deduce that

$$
\begin{aligned}
c_\lambda + o(1) &\geq \Psi(v_n) - \frac{1}{6}\langle H'(v_n), v_n\rangle \\
&\geq \frac{a}{3}\int_{\mathbb{R}^3} |\nabla v_n|^2 dx + \frac{b}{12}\left(\int_{\mathbb{R}^3} |\nabla v_n|^2 dx\right)^2 \\
&\geq \frac{aS}{3g_M^{\frac{1}{3}}}\left(\int_{\mathbb{R}^3} g(x)|v_n|^6 dx\right)^{\frac{1}{3}} + \frac{bS^2}{12g_M^{\frac{2}{3}}}\left(\int_{\mathbb{R}^3} g(x)|v_n|^6 dx\right)^{\frac{2}{3}} \\
&\geq \frac{a}{3}\left(\frac{bS^3 + \sqrt{(bS^3)^2 + 4aS^3g_M}}{2g_M}\right) + \frac{b}{12}\left(\frac{bS^3 + \sqrt{(bS^3)^2 + 4aS^3g_M}}{2g_M}\right)^2 + o(1) \\
&= c^* + o(1),
\end{aligned}
$$

which is a contradiction. Thus $l = 0$, which, together with (2.19), yields that $v_n \to 0$ in E. Therefore $u_n \to u$ in E and the proof is complete. □

Lemma 2.4. *Under the conditions of Lemma 2.3, then there exists $\lambda_1 > 0$ such that $c^* < m_\lambda^\infty$ for $\lambda \in (0, \lambda_1)$.*

Proof. Suppose by contradiction that there is $\lambda_n \to 0$ such that $m_{\lambda_n}^\infty \le c^*$ for all n. In view of [13], $m_{\lambda_n}^\infty$ is attained by a positive solution $u_n \in M_{\lambda_n}^\infty$ such that $I_{\lambda_n}^\infty(u_n) = m_{\lambda_n}^\infty$. We claim that there exist C_3, $C_4 > 0$ (independent of λ) such that $C_3 \le \|u_n\|_{V_\infty} \le C_4$ for all n. Indeed, by (2.3), we have

$$c^* \ge m_{\lambda_n}^\infty = I_{\lambda_n}^\infty(u_n) - \frac{1}{4}\langle (I_{\lambda_n}^\infty)'(u_n), u_n\rangle \ge \frac{1}{4}\|u_n\|_{V_\infty}^2$$

for all n, that is, $\|u_n\|_{V_\infty}^2 \le 4c^*$ for all n. On the other hand, since $u_n \in M_{\lambda_n}^\infty$, by condition (g_1) and (2.1), we obtain for $\varepsilon \in (0, \frac{V_\infty}{2h_\infty})$,

$$\|u_n\|_{V_\infty}^2 \le \lambda_n h_\infty \varepsilon \int_{\mathbb{R}^3} |u_n|^2 dx + (\lambda_n h_\infty C_\varepsilon + g_\infty) \int_{\mathbb{R}^3} |u_n|^6 dx$$

and then,

$$\frac{1}{2}\|u_n\|_{V_\infty}^2 \le (h_\infty C_\varepsilon + g_\infty)(aS)^{-3}\|u_n\|_{V_\infty}^6$$

for large n, which implies that

$$\|u_n\|_{V_\infty}^2 \ge \frac{(aS)^{\frac{3}{2}}}{\sqrt{2(h_\infty C_\varepsilon + g_\infty)}} \tag{2.20}$$

for n large. Then, noting $\lambda_n \to 0$, we deduce that $\lambda_n \int_{\mathbb{R}^3} h(x)F(u_n)dx = o(1)$ and $\lambda_n \int_{\mathbb{R}^3} h(x)f(u_n)u_n dx = o(1)$. Hence

$$\begin{aligned} m_{\lambda_n}^\infty &= \frac{1}{2}\|u_n\|_{V_\infty}^2 + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla u_n|^2 dx\right)^2 - \frac{1}{6}\int_{\mathbb{R}^3} g_\infty |u_n|^6 dx + o(1), \\ 0 &= \|u_n\|_{V_\infty}^2 + b\left(\int_{\mathbb{R}^3} |\nabla u_n|^2 dx\right)^2 - \int_{\mathbb{R}^3} g_\infty |u_n|^6 dx + o(1). \end{aligned} \tag{2.21}$$

Set $\lim_{n\to\infty} \int_{\mathbb{R}^3} |\nabla u_n|^2 dx = D$. One has $D > 0$. Indeed, if $D = 0$, then $\int_{\mathbb{R}^3} |u_n|^6 dx \to 0$ as $n \to \infty$, and thus, by (2.21), $\|u_n\|_{V_\infty}^2 \to 0$. This gives a contradiction to (2.20). It follows from (2.21) and the definition of S that $D \ge \frac{bS^3 + \sqrt{(bS^3)^2 + 4aS^3 g_\infty}}{2g_\infty}$. Hence

$$\begin{aligned} m_{\lambda_n}^\infty &= I_{\lambda_n}^\infty(u_n) - \frac{1}{6}\langle (I_{\lambda_n}^\infty)'(u_n), u_n\rangle \\ &\ge \frac{a}{3}\int_{\mathbb{R}^3} |\nabla u_n|^2 dx + \frac{b}{12}\left(\int_{\mathbb{R}^3} |\nabla u_n|^2 dx\right)^2 + o(1) \\ &\ge \frac{a}{3}\left(\frac{bS^3 + \sqrt{(bS^3)^2 + 4aS^3 g_\infty}}{2g_\infty}\right) + \frac{b}{12}\left(\frac{bS^3 + \sqrt{(bS^3)^2 + 4aS^3 g_\infty}}{2g_\infty}\right)^2 + o(1), \end{aligned}$$

a contradiction with $m_{\lambda_n}^\infty \le c^*$ and $g_\infty < g_M$. □

Corollary 2.5. *Under the conditions of Lemma 2.3, for each $\lambda \in (0, \lambda_1)$, we have J_λ satisfies the Palais–Smale condition for $c_\lambda < c^*$.*

Proof. Let $\{u_n\} \subset S_1$ be a Palais–Smale sequence for J_λ. By Lemma 2.2, $\{m_\lambda(u_n)\} \subset \mathcal{M}_\lambda$ is a Palais–Smale sequence for I_λ, and then using Lemma 2.3, we deduce that $w_n := m_\lambda(u_n) \to w$ in E after passing to a subsequence. Since the mapping m_λ is a homeomorphism between S_1 and $\mathcal{M}_\lambda$, we see that m_λ^{-1} is continuous. Hence $u_n = m_\lambda^{-1}(w_n) \to m_\lambda^{-1}(w)$ in E. The proof is complete. □

For $\varepsilon > 0$ and $y \in \Lambda$, let

$$u_{\varepsilon,y}(x) = \frac{\psi(x)\varepsilon^{\frac{1}{4}}}{(\varepsilon + |x-y|^2)^{\frac{1}{2}}},$$

where $\psi \in C_0^\infty(B_{2r_0}(y))$ such that $\psi(x) = 1$ for $|x-y| \le r_0$, $0 \le \psi(x) \le 1$ and $|\nabla\psi| \le 2$. It is well known that S is attained by the function $\frac{\varepsilon^{1/4}}{(\varepsilon+|x|^2)^{1/2}}$. For $\varepsilon > 0$ small, we have (see [22]):

$$\int_{\mathbb{R}^3} |\nabla u_{\varepsilon,y}|^2 dx = K_1 + O(\varepsilon^{\frac{1}{2}}), \qquad \int_{\mathbb{R}^3} |u_{\varepsilon,y}|^6 dx = K_2 + O(\varepsilon^{\frac{3}{2}}) \tag{2.22}$$

and

$$\int_{\mathbb{R}^3} |u_{\varepsilon,y}|^s dx = \begin{cases} O(\varepsilon^{\frac{6-s}{4}}), & s \in (3,6), \\ O(\varepsilon^{\frac{3}{4}}|\ln\varepsilon|), & s = 3, \\ O(\varepsilon^{\frac{s}{4}}), & s \in [2,3), \end{cases} \tag{2.23}$$

where K_1, K_2 are positive constants and $S = K_1/K_2^{1/3}$.

Lemma 2.6. *Assume that conditions (V), (h), (f_1), (f_3) and (g_3) are satisfied. Then there exist C_0, $\varepsilon_0 > 0$ independent of $y \in \Lambda$ such that for $\varepsilon \in (0, \varepsilon_0)$, $\sup_{t\ge 0} I_\lambda(tu_{\varepsilon,y}) \le c^* - C_0\varepsilon^{\frac{1}{2}}$.*

Proof. For $y \in \Lambda$, we get

$$\int_{\mathbb{R}^3} g(x)|u_{\varepsilon,y}|^6 dx = \int_{\mathbb{R}^3} (g(x) - g(y))|u_{\varepsilon,y}|^6 dx + \int_{\mathbb{R}^3} g_M |u_{\varepsilon,y}|^6 dx. \tag{2.24}$$

By (g_3), there exist $r_1 \in (0, 2r_0)$ and $C > 0$ such that $|g(x) - g(y)| \le C|x-y|^\rho$ for $|x-y| < r_1$ and for $y \in \Lambda$. Then we have

$$\begin{aligned}
\int_{\mathbb{R}^3} |g(x) - g(y)||u_{\varepsilon,y}|^6 dx &= \int_{|x-y|\le 2r_0} |g(x) - g(y)||u_{\varepsilon,y}|^6 dx \\
&\le \int_{|x-y|<r_1} C|x-y|^\rho \frac{\varepsilon^{\frac{3}{2}}}{(\varepsilon + |x-y|^2)^3} dx \\
&\quad + \int_{r_1 \le |x-y| \le 2r_0} \frac{2g_M\varepsilon^{\frac{3}{2}}}{(\varepsilon + |x-y|^2)^3} dx \\
&\le C\int_0^{r_1} \frac{\varepsilon^{\frac{3}{2}} r^{2+\rho}}{(\varepsilon + r^2)^3} dr + C\int_{r_1}^{2r_0} \frac{\varepsilon^{\frac{3}{2}} r^2}{(\varepsilon + r^2)^3} dr \\
&\le C\varepsilon^{\frac{\rho}{2}} \int_0^{\frac{r_1}{\sqrt{\varepsilon}}} \frac{r^{2+\rho}}{(1+r^2)^3} dr + C\int_{\frac{r_1}{\sqrt{\varepsilon}}}^{\frac{2r_0}{\sqrt{\varepsilon}}} \frac{r^2}{(1+r^2)^3} dr \\
&\le C_5 h(\varepsilon),
\end{aligned} \tag{2.25}$$

where

$$h(\varepsilon) = \begin{cases} \varepsilon^{\frac{\rho}{2}}, & 1 \le \rho < 3, \\ \varepsilon^{\frac{3}{2}}|\ln\varepsilon|, & \rho = 3, \\ \varepsilon^{\frac{3}{2}}, & \rho > 3. \end{cases}$$

From (2.24) and (2.25), we obtain that

$$\int_{\mathbb{R}^3} g(x)|u_{\varepsilon,y}|^6 dx = g_M \int_{\mathbb{R}^3} |u_{\varepsilon,y}|^6 dx + O(h(\varepsilon)). \tag{2.26}$$

It follows from (2.22) and (2.23) that there exists $\varepsilon_1 > 0$ (independent of $y \in \Lambda$) such that for $\varepsilon \in (0, \varepsilon_1)$,

$$\int_{\mathbb{R}^3} |\nabla u_{\varepsilon,y}|^2 dx \le \frac{3K_1}{2}, \qquad \|u_{\varepsilon,y}\|^2 \le \frac{3aK_1}{2}, \qquad \int_{\mathbb{R}^3} g(x)|u_{\varepsilon,y}|^6 dx \ge \frac{g_0 K_2}{2}.$$

Then, using (2.3),

$$\begin{aligned} I_\lambda(tu_{\varepsilon,y}) &\le \frac{t^2}{2}\|u_{\varepsilon,y}\|^2 + \frac{bt^4}{4}\left(\int_{\mathbb{R}^3} |\nabla u_{\varepsilon,y}|^2 dx\right)^2 - \frac{t^6}{6}\int_{\mathbb{R}^3} g(x)|u_{\varepsilon,y}|^6 dx \\ &\le \frac{3aK_1}{4}t^2 + \frac{9bK_1^2}{16}t^4 - \frac{g_0 K_2}{12}t^6, \end{aligned}$$

which implies that there are $t_1 > 0$ small and $t_2 > 0$ large (independent of ε) such that

$$\sup_{t\in[0,t_1]\cup[t_2,+\infty)} I_\lambda(tu_{\varepsilon,y}) \le \frac{c^*}{2}. \tag{2.27}$$

Set $B_\varepsilon = \frac{\int_{\mathbb{R}^3} |\nabla u_{\varepsilon,y}|^2 dx}{\left(\int_{\mathbb{R}^3} g(x)|u_{\varepsilon,y}|^6 dx\right)^{1/3}}$. By (2.26) and (2.22), we have

$$B_\varepsilon = \frac{K_1 + O(\varepsilon^{\frac{1}{2}})}{(g_M K_2 + O(h(\varepsilon)))^{\frac{1}{3}}} \le \frac{S}{g_M^{\frac{1}{3}}} + O(\varepsilon^{\frac{1}{2}}). \tag{2.28}$$

Take $k(t) = \frac{a}{2}t^2\|\nabla u_{\varepsilon,y}\|_2^2 + \frac{b}{4}t^4\|\nabla u_{\varepsilon,y}\|_2^4 - \frac{t^6}{6}\int_{\mathbb{R}^3} g(x)|u_{\varepsilon,y}|^6 dx$. Then

$$k'(t) = t\left(a\|\nabla u_{\varepsilon,y}\|_2^2 + bt^2\|\nabla u_{\varepsilon,y}\|_2^4 - t^4\int_{\mathbb{R}^3} g(x)|u_{\varepsilon,y}|^6 dx\right),$$

and k attains its maximum at

$$t_0 = \left(\frac{b\|\nabla u_{\varepsilon,y}\|_2^4 + \sqrt{b^2\|\nabla u_{\varepsilon,y}\|_2^8 + 4a\|\nabla u_{\varepsilon,y}\|_2^2\int_{\mathbb{R}^3} g(x)|u_{\varepsilon,y}|^6 dx}}{2\int_{\mathbb{R}^3} g(x)|u_{\varepsilon,y}|^6 dx}\right)^{\frac{1}{2}}.$$

A direct calculation shows that

$$\begin{aligned} \max_{t\ge 0} k(t) &= k(t_0) \\ &= \frac{a\|\nabla u_{\varepsilon,y}\|_2^2\left(b\|\nabla u_{\varepsilon,y}\|_2^4 + \sqrt{b^2\|\nabla u_{\varepsilon,y}\|_2^8 + 4a\|\nabla u_{\varepsilon,y}\|_2^2\int_{\mathbb{R}^3} g(x)|u_{\varepsilon,y}|^6 dx}\right)}{6\int_{\mathbb{R}^3} g(x)|u_{\varepsilon,y}|^6 dx} \\ &\quad + \frac{b\|\nabla u_{\varepsilon,y}\|_2^4\left(b\|\nabla u_{\varepsilon,y}\|_2^4 + \sqrt{b^2\|\nabla u_{\varepsilon,y}\|_2^8 + 4a\|\nabla u_{\varepsilon,y}\|_2^2\int_{\mathbb{R}^3} g(x)|u_{\varepsilon,y}|^6 dx}\right)^2}{12\left(2\int_{\mathbb{R}^3} g(x)|u_{\varepsilon,y}|^6 dx\right)^2} \\ &= \frac{a}{3}\left(\frac{bB_\varepsilon^3 + \sqrt{b^2B_\varepsilon^6 + 4aB_\varepsilon^3}}{2}\right) + \frac{b}{12}\left(\frac{bB_\varepsilon^3 + \sqrt{b^2B_\varepsilon^6 + 4aB_\varepsilon^3}}{2}\right)^2 \\ &\le \frac{a}{3}\left(\frac{bS^3 + \sqrt{(bS^3)^2 + 4ag_M S^3}}{2g_M}\right) + \frac{b}{12}\left(\frac{bS^3 + \sqrt{(bS^3)^2 + 4ag_M S^3}}{2g_M}\right)^2 + C\varepsilon^{\frac{1}{2}} \\ &= c^* + C\varepsilon^{\frac{1}{2}} \end{aligned}$$

by (2.28). Hence, there exist $C_0 > 0$ and $\varepsilon_2 \in (0, \varepsilon_1)$ (independent of $y \in \Lambda$) such that for $\varepsilon \in (0, \varepsilon_2)$,

$$\begin{aligned}\sup_{t\in[t_1,t_2]} I_\lambda(tu_{\varepsilon,y}) &\le \sup_{t\ge 0} k(t) + \frac{t_2^2}{2}\int_{\mathbb{R}^3} V(x)|u_{\varepsilon,y}|^2dx - \inf_{t\in[t_1,t_2]} \lambda\int_{\mathbb{R}^3} h(x)F(tu_{\varepsilon,y})dx \\ &\le c^* + C_0\varepsilon^{\frac{1}{2}} - \inf_{t\in[t_1,t_2]} \lambda\int_{\mathbb{R}^3} h_0F(tu_{\varepsilon,y})dx. \end{aligned} \tag{2.29}$$

From (f_3), for any $L > 0$, there is $R_L > 0$ such that $F(u) \ge L|u|^4$ for all $u \ge R_L$. Now choosing $\varepsilon_0 \in \big(0, \min\big\{\varepsilon_2, r_0^2, \big(\frac{t_1}{\sqrt{2}R_L}\big)^4\big\}\big)$, we have for $\varepsilon \in (0, \varepsilon_0)$,

$$u_{\varepsilon,y}(x) = \frac{\varepsilon^{\frac{1}{4}}}{(\varepsilon + |x-y|^2)^{\frac{1}{2}}} \ge \frac{1}{\sqrt{2}\varepsilon^{\frac{1}{4}}}, \qquad \forall |x-y| \le \sqrt{\varepsilon},$$

and then,

$$\begin{aligned}\inf_{t\in[t_1,t_2]}\int_{\mathbb{R}^3} F(tu_{\varepsilon,y})dx &\ge \inf_{t\in[t_1,t_2]}\int_{|x-y|\le\sqrt{\varepsilon}} F(tu_{\varepsilon,y})dx \\ &\ge \frac{Lt_1^4}{4\varepsilon}\int_{|x-y|\le\sqrt{\varepsilon}} dx \\ &= \frac{1}{4}Lt_1^4\varepsilon^{\frac{1}{2}}\int_{|x|\le 1} dx,\end{aligned}$$

which, together with (2.29), shows that

$$\sup_{t\in[t_1,t_2]} I_\lambda(tu_{\varepsilon,y}) \le c^* + C_0\varepsilon^{\frac{1}{2}} - \frac{\lambda h_0}{4}Lt_1^4\varepsilon^{\frac{1}{2}}\int_{|x|\le 1} dx.$$

Choosing $L > 0$ large enough, we derive that there exists $\varepsilon_0 \in (0, \varepsilon_2)$ uniformly in y such that for $\varepsilon \in (0, \varepsilon_0)$, $\sup_{t\in[t_1,t_2]} I_\lambda(tu_{\varepsilon,y}) \le c^* - C_0\varepsilon^{\frac{1}{2}}$. Combining this and (2.27), we get the conclusion. □

3 Proof of Theorem 1.1

In this section, we suppose all the conditions of Theorem 1.1 are satisfied. Define

$$\hat{I}(u) = \frac{1}{2}\|u\|^2 + \frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla u|^2dx\right)^2 - \frac{1}{6}\int_{\mathbb{R}^3} g(x)|u|^6dx$$

for $u \in E$. The following lemma plays a key role in proving Theorem 1.1.

Lemma 3.1. *There exists $\lambda_0 \in (0, \lambda_1)$ such that if $\lambda \in (0, \lambda_0)$, then $\int_{\mathbb{R}^3} \frac{x}{|x|}|u|^6dx \ne 0$ for all $u \in S_1$ with $J_\lambda(u) < c^*$.*

Proof. We adapt an argument in [27]. Assume by contradiction that there exist $\lambda_n \downarrow 0$ and $\{u_n\} \subset S_1$ such that $J_{\lambda_n}(u_n) < c^*$ and $\int_{\mathbb{R}^3} \frac{x}{|x|}|u_n|^6dx = 0$. By Lemma 2.1, there exists $t_n > 0$ such that $v_n := t_nu_n \in \mathcal{M}_{\lambda_n}$. Then one has $I_{\lambda_n}(v_n) = J_{\lambda_n}(u_n) < c^*$ and $\int_{\mathbb{R}^3} \frac{x}{|x|}|v_n|^6dx = 0$. Since $\{v_n\} \subset \mathcal{M}_\lambda$, it follows from Lemma 2.1 (ii) that $\{v_n\}$ is bounded, and then, by $\lambda_n \to 0$,

$$\lambda_n\int_{\mathbb{R}^3} h(x)F(v_n)dx = o(1) \quad \text{and} \quad \lambda_n\int_{\mathbb{R}^3} h(x)f(v_n)v_ndx = o(1).$$

Therefore,

$$c^* \geq \hat{I}(v_n) + o(1) \quad \text{and} \quad \langle \hat{I}'(v_n), v_n \rangle = o(1). \tag{3.1}$$

Take $l = \lim_{n\to\infty} \left(\int_{\mathbb{R}^3} g(x)|v_n|^6 dx\right)^{\frac{1}{3}}$. By the proof of (ii) of Lemma 2.1, one infers that $\|v_n\|_6 \geq C$ for each $n \in \mathbb{N}$ and some constant $C > 0$. Therefore, $l > 0$, which, jointly with (3.1) and the fact $g(x) \leq g_M$ for all $x \in \mathbb{R}^3$, we deduce that $l \geq \left(bS^2 + \sqrt{(bS^2)^2 + 4aSg_M}\right)/2g_M^{\frac{2}{3}}$, and hence

$$\begin{aligned}
c^* + o(1) \geq & \hat{I}(v_n) - \frac{1}{6}\langle \hat{I}'(v_n), v_n \rangle \\
= & \frac{a}{3}\int_{\mathbb{R}^3} |\nabla v_n|^2 dx + \frac{1}{3}\int_{\mathbb{R}^3} V(x)|v_n|^2 dx + \frac{b}{12}\left(\int_{\mathbb{R}^3} |\nabla v_n|^2 dx\right)^2 \\
\geq & \frac{aS}{3}\left(\int_{\mathbb{R}^3} |v_n|^6 dx\right)^{\frac{1}{3}} + \frac{bS^2}{12}\left(\int_{\mathbb{R}^3} |v_n|^6 dx\right)^{\frac{2}{3}} \\
\geq & \frac{aS}{3g_M^{\frac{1}{3}}}\left(\int_{\mathbb{R}^3} g(x)|v_n|^6 dx\right)^{\frac{1}{3}} + \frac{bS^2}{12g_M^{\frac{2}{3}}}\left(\int_{\mathbb{R}^3} g(x)|v_n|^6 dx\right)^{\frac{2}{3}} \\
\geq & \frac{aS}{3g_M^{\frac{1}{3}}} l + \frac{bS^2}{12g_M^{\frac{2}{3}}} l^2 + o(1) \\
\geq & c^* + o(1),
\end{aligned}$$

which implies that $\int_{\mathbb{R}^3} V(x)|v_n|^2 dx \to 0$, $\int_{\mathbb{R}^3} |\nabla v_n|^2 dx \to \frac{bS^3 + \sqrt{(bS^3)^2 + 4aS^3 g_M}}{2g_M}$ and

$$\lim_{n\to\infty}\left(\int_{\mathbb{R}^3} g(x)|v_n|^6 dx\right)^{\frac{1}{3}} = \lim_{n\to\infty}\left(\int_{\mathbb{R}^3} g_M|v_n|^6\right)^{\frac{1}{3}} = \frac{bS^2 + \sqrt{(bS^2)^2 + 4aSg_M}}{2g_M^{\frac{2}{3}}}. \tag{3.2}$$

Set $w_n = v_n/|v_n|_6$. Then $\int_{\mathbb{R}^3} |\nabla w_n|^2 dx \to S$ and $\int_{\mathbb{R}^3} |w_n|^6 dx = 1$. From [22, Theorem 1.41], there exist $w \in E$, $\{z_n\} \subset \mathbb{R}^3$ and $\mu_n \in (0, +\infty)$ such that $\|\mu_n^{\frac{1}{2}} w_n(\mu_n x + z_n) - w\|_{D^{1,2}} \to 0$ up to a subsequence, i.e.,

$$\left\| w_n - \frac{1}{\mu_n^{\frac{1}{2}}} w\left(\frac{x - z_n}{\mu_n}\right)\right\|_{D^{1,2}} \xrightarrow{n} 0. \tag{3.3}$$

Hence $\int_{\mathbb{R}^3} |\nabla w|^2 dx = S$ and $\int_{\mathbb{R}^3} |w|^6 dx = 1$, i.e., S is achieved by w. From [21], the minimizers of S are of the form $\frac{c_0}{(1+h_0(x-x_0)^2)^{\frac{1}{2}}}$, where $c_0 \neq 0$, $h_0 > 0$ and $x_0 \in \mathbb{R}^3$. Thus

$$\left\| w_n - \frac{c_0 \mu_n^{\frac{1}{2}}}{\left(\mu_n^2 + h_0^2|\cdot - z_n - x_0\mu_n|^2\right)^{\frac{1}{2}}}\right\|_{D^{1,2}} \xrightarrow{n} 0. \tag{3.4}$$

Observing $g(0) < g_M$, we see that there is $\delta > 0$ such that $g(x) \leq \frac{g(0)+g_M}{2}$ for $|x| \leq \delta$. We distinguish two cases.

Case 1. $\mu_n \to \mu_0 \in (0, +\infty]$ as $n \to \infty$.

Since $\int_{\mathbb{R}^3} V(x)|v_n|^2 dx \to 0$ and $V(x) \geq V_0 (> 0)$, one has $\int_{\mathbb{R}^3} |v_n(x)|^2 dx \to 0$, and hence $\int_{\mathbb{R}^3} |w_n(x)|^2 dx \to 0$ $(n \to \infty)$. Setting $x = \mu_n y + z_n$, it follows that

$$\mu_n^2 \int_{\mathbb{R}^3} |\mu_n^{\frac{1}{2}} w_n(\mu_n y + z_n)|^2 dy \to 0.$$

This, together with the fact $\mu_n \to \mu_0 \in (0, +\infty]$ and (3.3), gives $\int_{\mathbb{R}^3} |w|^2 dx = 0$, which is a contradiction.

Case 2. $\mu_n \to 0$ as $n \to \infty$. We will further consider two cases.

(i) $|z_n| \le \delta$ for large n.

Letting $z_n \to z_0$ $(n \to \infty)$, then $|z_0| \le \delta$ and $g(z_0) \le \frac{g(0)+g_M}{2}$. It follows from (3.2) and (3.3) that

$$\begin{aligned} o(1) &= \int_{\mathbb{R}^3} (g_M - g(x))|w_n|^6 dx \\ &= \int_{\mathbb{R}^3} (g_M - g(x)) \left| \frac{1}{\mu_n^2} w\left(\frac{x - z_n}{\mu_n}\right)\right|^6 dx + o(1) \\ &= \int_{\mathbb{R}^3} (g_M - g(\mu_n x + z_n))|w|^6 dx + o(1). \end{aligned}$$

Using the Lebesgue dominated convergence theorem, we have

$$0 = \int_{\mathbb{R}^3} (g_M - g(z_0))|w|^6 dx \ge \frac{g_M - g(0)}{2} \int_{\mathbb{R}^3} |w|^6 dx = \frac{g_M - g(0)}{2} > 0,$$

a contradiction.

(ii) There is a subsequence $\{z_{n_k}\} \subset \{z_n\}$ such that $|z_{n_k}| \ge \delta$ for all k. Without loss of generality, we assume that $|z_n| \ge \delta$ for all n. Since $\int_{\mathbb{R}^3} \frac{x}{|x|} |v_n|^6 dx = 0$, one has $\int_{\mathbb{R}^3} \frac{x}{|x|} |w_n|^6 dx = 0$. Hence, by (3.4),

$$\begin{aligned} o(1) &= \int_{\mathbb{R}^3} \frac{x}{|x|} \frac{c_0^6 \mu_n^3}{\left(\mu_n^2 + h_0^2 |x - z_n - x_0\mu_n|^2\right)^3} dx \\ &= \int_{\mathbb{R}^3} \left(\frac{x}{|x|} - \frac{z_n + x_0\mu_n}{|z_n + x_0\mu_n|}\right) \frac{c_0^6 \mu_n^3}{\left(\mu_n^2 + h_0^2 |x - z_n - x_0\mu_n|^2\right)^3} dx \\ &\quad + \frac{z_n + x_0\mu_n}{|z_n + x_0\mu_n|} \int_{\mathbb{R}^3} \frac{c_0^6 \mu_n^3}{\left(\mu_n^2 + h_0^2 |x - z_n - x_0\mu_n|^2\right)^3} dx. \end{aligned} \tag{3.5}$$

Since $\mu_n \to 0$ and $|z_n| \ge \delta$ for all n, we have

$$|z_n + x_0\mu_n| \ge |z_n| - \mu_n |x_0| \ge \frac{\delta}{2}$$

for large n. Combining this and the fact

$$\left| \frac{x}{|x|} - \frac{z}{|z|} \right| \le \frac{|x(|z| - |x|) + |x|(x - z)|}{|x||z|} \le \frac{2|x - z|}{|z|}$$

for all $x, z \in \mathbb{R}^3 \setminus \{0\}$, we deduce that

$$\begin{aligned} &\int_{|x-(z_n+x_0\mu_n)| \le \mu_n} \left| \frac{x}{|x|} - \frac{z_n + x_0\mu_n}{|z_n + x_0\mu_n|} \right| \frac{c_0^6 \mu_n^3}{\left(\mu_n^2 + h_0^2 |x - z_n - x_0\mu_n|^2\right)^3} dx \\ &\le \int_{|x-(z_n+x_0\mu_n)| \le \mu_n} \frac{2|x - (z_n + x_0\mu_n)|}{|z_n + x_0\mu_n|} \frac{c_0^6 \mu_n^3}{\left(\mu_n^2 + h_0^2 |x - z_n - x_0\mu_n|^2\right)^3} dx \\ &\le \frac{4\mu_n}{\delta} \int_{\mathbb{R}^3} \frac{c_0^6}{\left(1 + h_0^2 |x|^2\right)^3} dx \\ &\le C_6 \mu_n \end{aligned} \tag{3.6}$$

and

$$\begin{aligned}
&\int_{|x-(z_n+x_0\mu_n)|\geq\mu_n}\left|\frac{x}{|x|}-\frac{z_n+x_0\mu_n}{|z_n+x_0\mu_n|}\right|\frac{c_0^6\mu_n^3}{\left(\mu_n^2+h_0^2|x-z_n-x_0\mu_n|^2\right)^3}dx\\
&\quad\leq\frac{2}{|z_n+x_0\mu_n|}\int_{|x-(z_n+x_0\mu_n)|\geq\mu_n}\frac{c_0^6\mu_n^3|x-(z_n+x_0\mu_n)|}{\left(\mu_n^2+h_0^2|x-z_n-x_0\mu_n|^2\right)^3}dx\\
&\quad\leq\frac{4\mu_n}{\delta}\int_{|x|\geq1}\frac{c_0^6|x|}{\left(1+h_0^2|x|^2\right)^3}dx\\
&\quad\leq C_7\mu_n. \qquad (3.7)
\end{aligned}$$

Hence we obtain, by (3.5)–(3.7),

$$\begin{aligned}
0&=\lim_{n\to\infty}\left|\frac{z_n+x_0\mu_n}{|z_n+x_0\mu_n|}\int_{\mathbb{R}^3}\frac{c_0^6\mu_n^3}{\left(\mu_n^2+h_0^2|x-z_n-x_0\mu_n|^2\right)^3}dx\right|\\
&=\int_{\mathbb{R}^3}\frac{c_0^6}{\left(1+h_0^2|x|^2\right)^3}dx>0,
\end{aligned}$$

which is a contradiction. □

To prove Theorem 1.1, we recall a multiplicity result for critical points involving Ljusternik–Schnirelman category, which has been widely used in dealing with semilinear elliptic equations.

Lemma 3.2 (see Proposition 2.4 in [1]). *Let M be a Hilbert manifold and $I\in C^1(M,\mathbb{R})$. If there exist $c_0\in\mathbb{R}$ and $k\in\mathbb{N}$ such that $I(u)$ satisfies the (PS) condition for $c\leq c_0$ and $\mathrm{cat}(\{u\in M: I(u)\leq c_0\})\geq k$, then $I(u)$ admits at least k critical points in $\{u\in M: I(u)\leq c_0\}$.*

Lemma 3.3 (see Theorem 2.5 in [1]). *Let X be a topological space. Assume that there exist two continuous mappings*

$$F:\mathbb{S}^2=\{y\in\mathbb{R}^3:|y|=1\}\to X,\qquad G:X\to\mathbb{S}^2$$

such that $G\circ F$ is homotopic to identity, that is, there is a continuous mapping $\zeta:[0,1]\times\mathbb{S}^2\to\mathbb{S}^2$ such that $\zeta(0,x)=(G\circ F)(x)$ for $x\in\mathbb{S}^2$ and $\zeta(1,x)=x$ for $x\in\mathbb{S}^2$. Then $cat(X)\geq2$.

Proof of Theorem 1.1. Let $\lambda\in(0,\lambda_0)$ with λ_0 given in Lemma 3.1. Take $y=\frac{3}{2}\rho_0 z$, where ρ_0 is the constant given in (g_2) and $z\in\mathbb{S}^2$. Let $r_1<\frac{1}{4}\rho_0$. By (g_2), one has $g(x)=g_M$ for $|x-\frac{3}{2}\rho_0 z|\leq 2r_1$. Noting

$$u_{\varepsilon,y}(x)=\frac{\psi(x)\varepsilon^{\frac{1}{4}}}{(\varepsilon+|x-\frac{3}{2}\rho_0 z|^2)^{\frac{1}{2}}},$$

where $\psi\in C_0^\infty(B_{2r_1}(\frac{3}{2}\rho_0 z))$ such that $\psi(x)=1$ for $|x-\frac{3}{2}\rho_0 z|\leq r_1$ and $0\leq\psi(x)\leq1$, we deduce that

$$\begin{aligned}
\int_{\mathbb{R}^3}g(x)|u_{\varepsilon,y}|^6dx&=\int_{|x-\frac{3}{2}\rho_0 z|<2r_1}g(x)|u_{\varepsilon,y}|^6dx\\
&=g_M\int_{|x-\frac{3}{2}\rho_0 z|<2r_1}|u_{\varepsilon,y}|^6dx\\
&=g_M\int_{\mathbb{R}^3}|u_{\varepsilon,y}|^6dx.
\end{aligned}$$

Arguing as in the proof of Lemma 2.6, we still conclude that $\sup_{t\geq 0} I_\lambda(tu_{\varepsilon,y}) \leq c^* - C_0\varepsilon^{\frac{1}{2}}$. Set $h(t) = I_\lambda\big(\frac{tu_{\varepsilon,y}}{\|u_{\varepsilon,y}\|}\big)$, where $t > 0$ and $\varepsilon \in (0,\varepsilon_0)$ with ε_0 given in Lemma 2.6. It follows from Lemma 2.1 that $h(t)$ attains its maximum at a unique point t_y and $t_y u_{\varepsilon,y} \in \mathcal{M}_\lambda$. By Lemma 2.6,

$$J_\lambda\left(\frac{u_{\varepsilon,y}}{\|u_{\varepsilon,y}\|}\right) = I_\lambda(t_y u_{\varepsilon,y}) = \sup_{t\geq 0} I_\lambda(tu_{\varepsilon,y}) \leq c^* - C_0\varepsilon^{\frac{1}{2}} < c^*.$$

Define $F : \mathbb{S}^2 \to S_1$ by $F(z) = \frac{u_{\varepsilon,y}}{\|u_{\varepsilon,y}\|}$. Then

$$F : \mathbb{S}^2 \to \{u \in S_1 : J_\lambda(u) < c^*\}.$$

Let $G : \{u \in S_1 : J_\lambda(u) < c^*\} \to \mathbb{S}^2$ by $G(u) = \frac{\int_{\mathbb{R}^3} \frac{x}{|x|}|u|^6 dx}{\left|\int_{\mathbb{R}^3} \frac{x}{|x|}|u|^6 dx\right|}$. Then G is well defined and continuous by virtue of Lemma 3.1. Define $\zeta(\theta,z) : [0,1]\times\mathbb{S}^2 \to \mathbb{S}^2$ such that $\zeta(\theta,z) = G\big(\frac{u_{(1-\theta)\varepsilon,y}}{\|u_{(1-\theta)\varepsilon,y}\|}\big)$ for $\theta \in [0,1)$ and $\zeta(1,z) = z$. It follows from (2.22) that

$$\begin{aligned}
&\lim_{\theta\to 1^-}\int_{\mathbb{R}^3}\frac{x}{|x|}|u_{(1-\theta)\varepsilon,y}|^6 dx \\
&= \lim_{\theta\to 1^-}\int_{\mathbb{R}^3}\left(\frac{x}{|x|} - \frac{\frac{3}{2}\rho_0 z}{|\frac{3}{2}\rho_0 z|}\right)|u_{(1-\theta)\varepsilon,y}|^6 dx + \lim_{\theta\to 1^-}\frac{\frac{3}{2}\rho_0 z}{|\frac{3}{2}\rho_0 z|}\int_{\mathbb{R}^3}|u_{(1-\theta)\varepsilon,y}|^6 dx \\
&= \lim_{\theta\to 1^-}\int_{\mathbb{R}^3}\left(\frac{x}{|x|} - \frac{\frac{3}{2}\rho_0 z}{|\frac{3}{2}\rho_0 z|}\right)|u_{(1-\theta)\varepsilon,y}|^6 dx + K_2 z. \qquad (3.8)
\end{aligned}$$

Since $|u_{(1-\theta)\varepsilon,y}|^6 \leq \frac{((1-\theta)\varepsilon)^{\frac{3}{2}}}{\left((1-\theta)\varepsilon+|x-\frac{3}{2}\rho_0 z|^2\right)^3}$ and since $\left|\frac{x}{|x|} - \frac{\frac{3}{2}\rho_0 z}{|\frac{3}{2}\rho_0 z|}\right| \leq \frac{2|x-\frac{3}{2}\rho_0 z|}{|\frac{3}{2}\rho_0 z|}$, we deduce that

$$\begin{aligned}
&\int_{|x-\frac{2}{3}\rho_0 z|\leq\sqrt{(1-\theta)\varepsilon}}\left|\frac{x}{|x|} - \frac{\frac{3}{2}\rho_0 z}{|\frac{3}{2}\rho_0 z|}\right||u_{(1-\theta)\varepsilon,y}|^6 dx \\
&\leq \int_{|x-\frac{2}{3}\rho_0 z|\leq\sqrt{(1-\theta)\varepsilon}}\frac{4\left|x-\frac{2}{3}\rho_0 z\right|}{3\rho_0}\frac{1}{((1-\theta)\varepsilon)^{\frac{3}{2}}\left(1+\left|\frac{x-\frac{2}{3}\rho_0 z}{\sqrt{(1-\theta)\varepsilon}}\right|^2\right)^3}dx \\
&\leq \frac{4\sqrt{(1-\theta)\varepsilon}}{3\rho_0}\int_{|x|\leq 1}\frac{1}{(1+|x|^2)^3}dx \\
&\leq C_8\sqrt{(1-\theta)\varepsilon} \qquad (3.9)
\end{aligned}$$

and

$$\begin{aligned}
&\int_{|x-\frac{2}{3}\rho_0 z|\geq\sqrt{(1-\theta)\varepsilon}}\left|\frac{x}{|x|} - \frac{\frac{3}{2}\rho_0 z}{|\frac{3}{2}\rho_0 z|}\right||u_{(1-\theta)\varepsilon,y}|^6 dx \\
&\leq \frac{4}{3\rho_0}\int_{|x-\frac{2}{3}\rho_0 z|\geq\sqrt{(1-\theta)\varepsilon}}\frac{\left|x-\frac{3}{2}\rho_0 z\right|}{((1-\theta)\varepsilon)^{\frac{3}{2}}\left(1+\left|\frac{x-\frac{2}{3}\rho_0 z}{\sqrt{(1-\theta)\varepsilon}}\right|^2\right)^3}dx \\
&\leq \frac{4\sqrt{(1-\theta)\varepsilon}}{3\rho_0}\int_{|x|\geq 1}\frac{|x|}{(1+|x|^2)^3}dx \\
&\leq C_9\sqrt{(1-\theta)\varepsilon}. \qquad (3.10)
\end{aligned}$$

Combining (3.8)–(3.10), we have

$$\lim_{\theta\to 1^-}\int_{\mathbb{R}^3}\frac{x}{|x|}|u_{(1-\theta)\varepsilon,y}|^6dx = K_2 z,$$

which, together with the continuous of G, gives that $\zeta \in C([0,1]\times\mathbb{S}^2,\mathbb{S}^2)$. Noting $\zeta(0,z) = G\big(\frac{u_{\varepsilon,y}}{\|u_{\varepsilon,y}\|}\big) = G\circ F(z)$ and $\zeta(1,z) = z$ for $z\in\mathbb{S}^2$, one has $G\circ F : \mathbb{S}^2\to\mathbb{S}^2$, $z\to G\circ F(z)$ is homotopic to the identity. Thus, by Lemma 3.3,

$$\operatorname{cat}\left\{u\in S_1 : J_\lambda(u)\le c^* - C_0\varepsilon^{\frac{1}{2}}\right\}\ge 2.$$

Therefore, using Corollary 2.5 and Lemma 3.2, we deduce that J_λ has at least two nontrivial critical points, and thus I_λ has at least two nontrivial critical points. This completes the proof. □

4 Proof of Theorem 1.2

In this section, we suppose that all the conditions of Theorem 1.2 are satisfied. By (g_1), there is $R_0>0$ such that $g(x)\le\frac{1}{2}(g_M+g_\infty)$ for all $|x|\ge R_0$. For any $d>0$, let $\rho=\rho(d)>R_0$ be such that $\Lambda_d\subset B_\rho(0)$. Define $\chi:\mathbb{R}^3\to\mathbb{R}^3$ as $\chi(x)=x$ for $|x|\le\rho$ and $\chi(x)=\rho x/|x|$ for $|x|>\rho$. We consider the barycenter map $\beta: E\setminus\{0\}\to\mathbb{R}^3$ given by

$$\beta(u)=\frac{\int_{\mathbb{R}^3}\chi(x)|u(x)|^6dx}{\int_{\mathbb{R}^3}|u(x)|^6dx}.$$

Since $\Lambda_d\subset B_\rho(0)$, by the definition of χ and Lebesgue's theorem, we have the following conclusion.

Lemma 4.1. *For any $d>0$, there exists $\lambda_d>0$ such that, if $\lambda\in(0,\lambda_d)$ and $u\in S_1$ with $J_\lambda(u)<c^*$, then $\beta(u)\in\Lambda_d$.*

Proof. Arguing indirectly, we assume that there exist $d_0>0$, $\lambda_n\downarrow 0$ and $(u_n)\subset S_1$ with $J_{\lambda_n}(u_n)<c^*$, but $\beta(u_n)\notin\Lambda_{d_0}$. From Lemma 2.1, there exists a unique $t_n>0$ such that $t_nu_n\in M_{\lambda_n}$. Take $v_n=t_nu_n$ and $w_n=v_n/|v_n|_6$. Following the steps contained in the proof of Lemma 3.1, we deduce

$$\int_{\mathbb{R}^3}|\nabla w_n|^2dx\to S,\qquad \int_{\mathbb{R}^3}|w_n|^6dx=1\quad\text{and}\quad\int_{\mathbb{R}^3}g(x)|w_n|^6dx\to g_M. \tag{4.1}$$

So, there exist $\{z_n\}\subset\mathbb{R}^3$, $\mu_n\in(0,+\infty)$ and $w\in D^{1,2}(\mathbb{R}^3)$ such that

$$\left\|w_n-\frac{1}{\mu_n^{\frac{1}{2}}}w\left(\frac{x-z_n}{\mu_n}\right)\right\|_{D^{1,2}}\to 0\qquad(n\to\infty). \tag{4.2}$$

Thus $\int_{\mathbb{R}^3}|\nabla w|^2dx=S$ and $\int_{\mathbb{R}^3}|w|^6dx=1$. Arguing as in Lemma 3.1 (Case 1), if $\mu_n\to\mu_0\in(0,+\infty]$ $(n\to\infty)$, one obtains a contradiction. Hence $\mu_n\to 0$ as $n\to\infty$, and we distinguish into two cases.

Case 1. $\mu_n\to 0$ as $n\to\infty$ and $|z_n|\le R_0$ for large n. Suppose that $z_n\to z_0$ as $n\to\infty$. Then $|z_0|\le R_0$ and $\chi(z_0)=z_0$. Applying (4.1), (4.2) and the Lebesgue dominated convergence

theorem, we obtain

$$\begin{aligned} g_M &= \lim_{n\to\infty} \int_{\mathbb{R}^3} g(x) \left| \frac{1}{\mu_n^{\frac{1}{2}}} w\left(\frac{x-z_n}{\mu_n}\right)\right|^6 dx \\ &= \lim_{n\to\infty} \int_{\mathbb{R}^3} g(\mu_n x + z_n)|w|^6 dx \\ &= g(z_0) \int_{\mathbb{R}^3} |w|^6 dx \\ &= g(z_0), \end{aligned}$$

which implies that $z_0 \in \Lambda$. Moreover, by (4.2) and using the fact $\chi(z_0) = z_0$, we conclude that

$$\begin{aligned} \beta(w_n) &= \int_{\mathbb{R}^3} \chi(x)|w_n|^6 dx \\ &= \int_{\mathbb{R}^3} \chi(x) \left| \frac{1}{\mu_n^{\frac{1}{2}}} w\left(\frac{x-z_n}{\mu_n}\right)\right|^6 dx + o(1) \\ &= \int_{\mathbb{R}^3} \chi(\mu_n x + z_n)|w|^6 dx + o(1) \\ &= z_0 \int_{\mathbb{R}^3} |w|^6 dx + o(1) \\ &= z_0 + o(1), \end{aligned}$$

which, together with $z_0 \in \Lambda$, yields that $\beta(w_n) \in \Lambda_{d_0}$ for large n. This contradicts the assumption that $\beta(w_n) = \beta(u_n) \notin \Lambda_{d_0}$ for all n.

Case 2. $\mu_n \to 0$ as $n \to \infty$ and there exits a subsequence of $\{z_n\}$ (still denoted by $\{z_n\}$) such that $|z_n| \geq R_0$ for all n. Applying (4.1) and (4.2), we deduce that

$$\begin{aligned} o(1) &= \int_{\mathbb{R}^3} (g_M - g(x))|w_n|^6 dx \\ &= \int_{\mathbb{R}^3} (g_M - g(x)) \left| \frac{1}{\mu_n^{\frac{1}{2}}} w\left(\frac{x-z_n}{\mu_n}\right)\right|^6 dx + o(1) \\ &= \int_{\mathbb{R}^3} (g_M - g(\mu_n x + z_n))|w|^6 dx + o(1). \end{aligned}$$

Recall that $g(x) \leq \frac{1}{2}(g_\infty + g_M)$ for $|x| \geq R_0$. It follows from the Lebesgue dominated convergence theorem that

$$0 = \lim_{n\to\infty} \int_{\mathbb{R}^3} (g_M - g(\mu_n x + z_n))|w|^6 dx \geq \frac{1}{2}(g_M - g_\infty) \int_{\mathbb{R}^3} |w|^6 dx = \frac{1}{2}(g_M - g_\infty) > 0,$$

which is a contradiction. □

Now we are in a position to show that problem (1.1) admits at least $\mathrm{cat}_{\Lambda_d}(\Lambda)$ solutions. For this aim, we compare the topology of Λ and the topology of a suitable energy sublevel, and use the maps J_λ and β as they are introduced before. Moreover, we shall utilize a multiplicity result for critical points involving Lusternik–Schnirelmann category, e.g. see [15].

Lemma 4.2 (see [15]). *Let M be a $C^{1,1}$ complete Riemannian manifold (modelled on a Hilbert space) and assume that $\Phi \in C^1(M,\mathbb{R})$ bounded from below and satisfies $-\infty < \inf_M \Phi < a < b < +\infty$. Suppose that Φ satisfies the Palais–Smale condition on the sublevel $\{u \in M : \Phi(u) \le b\}$ and that a is not a critical level for Φ. Then Φ has at least $cat_{\Phi^a}(\Phi^a)$ critical points in Φ^a, where $\Phi^a := \{u \in M : \Phi(u) \le a\}$.*

Lemma 4.3 (see [3,4]). *Let A, B, M be closed sets with $A \subset B$. Let $F : A \to M$ and $G : M \to B$ be two continuous maps such that $G \circ F$ is homotopically equivalent to the embedding $J : A \to B$. Then $cat_M(M) \ge cat_B(A)$.*

Remark 4.4. Since $\mathcal{M}_\lambda$ is not a C^1 submanifold of E, we can not apply Lemma 4.2 directly. Fortunately, from Lemma 2.1, we know that the mapping m_λ is a homeomorphism between $\mathcal{M}_\lambda$ and S_1, but S_1 is a C^1 submanifold of E. Thus we can apply Lemma 4.2 to $J_\lambda(u) = I_\lambda(\hat{m}_\lambda(u))\,|_{S_1} = I_\lambda(m_\lambda(u))$, where J_λ is given in Lemma 2.2.

Proof of Theorem 1.2. For any $d > 0$, let $\lambda \in (0, \min\{\lambda_1, \lambda_d\})$ with λ_1 is given in Lemma 2.4 and λ_d is given in Lemma 4.1. It is easy to see that S_1 is a $C^{1,1}$ complete Riemann manifold and $J_\lambda \in C^1(S_1,\mathbb{R})$ is bounded from below. Set $l(t) = I_\lambda(tu_{\varepsilon,y})$, where $t > 0$, $y \in \Lambda$ and $\varepsilon \in (0,\varepsilon_0)$ with ε_0 is given in Lemma 2.6. In view of Lemma 2.1, $l(t)$ admits its maximum at a unique point t_y and $t_y u_{\varepsilon,y} \in \mathcal{M}_\lambda$. Hence, by Lemma 2.6,

$$J_\lambda\left(\frac{u_{\varepsilon,y}}{\|u_{\varepsilon,y}\|}\right) = I_\lambda(t_y u_{\varepsilon,y}) = \sup_{t\ge 0} I_\lambda(tu_{\varepsilon,y}) \le c^* - \eta_0, \tag{4.3}$$

where $\eta_0 > 0$ is a constant independent of $y \in \Lambda$. From Corollary 2.5, we see that J_λ satisfies the (PS) condition on $\{u \in S_1 : J_\lambda(u) < c^*\}$. Therefore, by Lemma 4.2, J_λ has at least $\mathrm{cat}_{S_1(\eta_0)}(S_1(\eta_0))$ critical points, where $S_1(\eta_0) = \{u \in S_1 : J_\lambda(u) \le c^* - \eta_0\}$.

Define the mappings $F : \Lambda \to S_1$ and $G : S_1(\eta_0) \to \mathbb{R}^3$ by

$$F(y) = \frac{u_{\varepsilon,y}}{\|u_{\varepsilon,y}\|}, \qquad G(u) = \beta(u).$$

Then F and G are continuous. It follows from Lemma 4.1 and (4.3) that $F(\Lambda) \subset S_1(\eta_0)$ and $G(S_1(\eta_0)) \subset \Lambda_d$. Define $\xi : [0,1] \times \Lambda \to \Lambda_d$ by

$$\xi(\theta,y) = \begin{cases} G\left(\frac{u_{(1-\theta)\varepsilon,y}}{\|u_{(1-\theta)\varepsilon,y}\|}\right), & \theta \in [0,1), \\ y, & \theta = 1. \end{cases}$$

Noting $y \in \Lambda \subset B_\rho(0)$, we obtain $\chi(y) = y$ and

$$\begin{aligned} \lim_{\theta\to 1^-} G\left(\frac{u_{(1-\theta)\varepsilon,y}}{\|u_{(1-\theta)\varepsilon,y}\|}\right) &= \lim_{\theta\to 1^-} \frac{\int_{\mathbb{R}^3} \chi(x)|u_{(1-\theta)\varepsilon,y}|^6 dx}{\int_{\mathbb{R}^3} |u_{(1-\theta)\varepsilon,y}|^6 dx} \\ &= \lim_{\theta\to 1^-} \frac{\int_{\mathbb{R}^3} \frac{\chi\left(\sqrt{(1-\theta)\varepsilon}z+y\right)\left|\psi\left(\sqrt{(1-\theta)\varepsilon}z+y\right)\right|^6}{(1+|z|^2)^3} dz}{\int_{\mathbb{R}^3} \frac{\left|\psi\left(\sqrt{(1-\theta)\varepsilon}z+y\right)\right|^6}{(1+|z|^2)^3} dz} = y. \end{aligned}$$

Thus $\xi \in C([0,1]\times\Lambda, \Lambda_d)$. Then we see that $\xi(\theta,y)$ with $(\theta,y) \in [0,1]\times\Lambda$ is a homotopy between $G \circ F$ and the inclusion map $j : \Lambda \to \Lambda_d$. This fact and Lemma 4.3 yield $\mathrm{cat}_{S_1(\eta_0)}(S_1(\eta_0)) \ge \mathrm{cat}_{\Lambda_d}(\Lambda)$. Hence, J_λ has at least $\mathrm{cat}_{\Lambda_d}(\Lambda)$ critical points. Then, in view of Lemma 2.2 (iii), we conclude that I_λ has at least $\mathrm{cat}_{\Lambda_d}(\Lambda)$ nontrivial critical points. Thus, problem (1.1) has at least $\mathrm{cat}_{\Lambda_d}(\Lambda)$ nontrivial solutions. This completes the proof. □

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Qualitative properties of solutions to a generalized Fisher–KPP equation

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Abstract. The following Fisher–KPP type equation

$$u_t = Ku_{xx} - Bu^q + Au^p, \quad (x,t) \in \mathbb{R} \times (0,\infty),$$

with $p > q > 0$ and A, B, K positive coefficients, is considered. For both $p > q > 1$ and $p > 1$, $q = 1$, we construct stationary solutions, establish their behavior as $|x| \to \infty$ and prove that they are separatrices between solutions decreasing to zero in infinite time and solutions presenting blow-up in finite time. We also establish decay rates for the solutions that decay to zero as $t \to \infty$.

Keywords: Fisher–KPP equation, stationary solutions, separatrix, decay rates, finite time blow-up.

2020 Mathematics Subject Classification: 35A17, 35B30, 35B44, 35K57, 35Q92

1 Introduction

The aim of this paper is to bring into light some qualitative properties related to the dynamics of solutions to the following generalized Fisher–KPP model:

$$u_t = Ku_{xx} - Bu^q + Au^p, \tag{1.1}$$

posed for $(x,t) \in \mathbb{R} \times (0,\infty)$, in the generic range of exponents $p > q > 0$, $p > 1$ and with A, B, K positive coefficients. The main feature of Eq. (1.1) is the competition between three terms having different effects for large times: a source term with exponent $p > 1$ which, when alone, usually leads to finite time blow-up (see for example [25] for a thorough study of this phenomenon), an absorption term with exponent $q > 0$ which, when alone, implies a dissipation, that is, a loss of the L^1 norm (leading also to a finite time extinction if $q < 1$), and a diffusion term which is conservative with respect to the L^1 norm. Thus, the balance between

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reaction and absorption will determine the main features related to the large time behavior of the solutions. In order to simplify the model, we can perform the following scaling

$$x = a\overline{x}, \quad t = b\overline{t}, \quad u = c\overline{u}, \tag{1.2}$$

with coefficients

$$c = \left(\frac{B}{A}\right)^{1/(p-q)}, \quad a = \left(\frac{Kc^{1-p}}{A}\right)^{1/2}, \quad b = \frac{c^{1-p}}{A}, \tag{1.3}$$

in order to obtain that, in the new variables $(\overline{x}, \overline{t})$, the rescaled function $\overline{u}$ is a solution to

$$\overline{u}_{\overline{t}} = (\overline{u})_{\overline{x}\overline{x}} - \overline{u}^q + \overline{u}^p. \tag{1.4}$$

We shall thus work, without any loss of generality, with Eq. (1.4), and for simplicity we drop from the notation the overlines, that is, we relabel the solution and the variables in Eq. (1.4) as (u, x, t). Eq. (1.4) is a very well-established model when the reaction and absorption exponents satisfy the opposite order, that is, $0 < p < q$, stemming from the seminal works by Fisher [10] and Kolmogorov, Petrovsky and Piscounoff [21], dealing with the specific case $p = 1$ and $q = 2$ as a model in mathematical biology. Nowadays, the Fisher–KPP equation with $0 < p < q$ is rather well understood, new applications have been proposed and, from the mathematical point of view, it has been noticed that the dynamics of the solutions is well represented by solutions in form of traveling waves with a speed $c > 0$, that is

$$u(x,t) = f(x - ct), \quad c > 0, \tag{1.5}$$

where $f(\cdot)$ is the profile of the wave, see for example [1,7,9,14,22,26] and references therein. In particular, a very famous result (see for example the classical papers [1,27] for a deduction of c^* in different cases) is the existence of a critical speed $c^* > 0$ such that traveling waves only exist if the speed is $c \geq c^*$ and do not exist when $c < c^*$.

The opposite case of ordering between the reaction and absorption exponents, that is, $p > q > 0$, has been also proposed, more recently, in a number of models arising in applied sciences. As a general model, Eq. (1.4) appears in growth and diffusion models as established, for example, in the book [3]. More specific applications in the mathematical modeling of tumor growth have been proposed by authors such as Marusic and Bajzer and their collaborators, see for example [2,23,24] and references therein, and a similar simplified model in [31].

Despite these applications, we discovered that there are still some gaps in the mathematical study of Eq. (1.4). As it is already well-established, in the more "standard" Fisher–KPP equation, that is, with exponents ordered as $q > p$, traveling waves, although they are still not stable in the L^∞ norm, are asymptotically stable with respect to suitable norms involving exponential weights, as shown for example in [28,30]. In stark contrast to the Fisher–KPP case, in our case it appears that traveling waves (and other explicit or semi-explicit solutions, as we shall see in the present work) are unstable with respect to the dynamics of general solutions to Eq. (1.4); that is, even if an initial condition $u_0 \in C(\mathbb{R})$ is sufficiently close (with respect to the L^∞ norm) to one of several explicit solutions, it will evolve either by growing up (and then blowing up in finite time) or by decaying as $t \to \infty$. We mention here [25, Chapter 17] where solutions to Eq. (1.4) with $q = 1$ are studied and the short note [18] devoted to the range $p > 1 > q > 0$ and $m > 1$, where the transition from finite time extinction to blow-up is established. The existence of a separatrix in the form of a stationary solution has been thoroughly investigated for the supercritical semilinear heat equation in [4,15,16,32] (see also

references therein), while some more general equations related to Eq. (1.4), or particular cases of it, have been considered in [17,33].

Putting into light this instability of several stationary solutions is actually the core of this work; indeed, we construct several solutions to Eq. (1.4) (in either explicit or implicit form) and we then prove that they serve as separatrices for the dynamics of general solutions to the Cauchy problem associated to Eq. (1.4) with suitable initial conditions. More precisely, solutions with data lying *above* the specific solution (even very close to it) *blow up in finite time*, while solutions with data lying *below* the specific solution *decay as* $t \to \infty$ (and even *vanish in finite time* if $0 < q < 1$, as shown in [18]) and we give some decay estimates. To fix the notation, we consider throughout this work the following family of initial conditions:

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}, \quad u_0 \in L^\infty(\mathbb{R}) \cap C(\mathbb{R}), \quad u_0 \geq 0, \quad u_0 \not\equiv 0. \tag{1.6}$$

Since we are dealing with a semilinear problem, solutions will be taken in classical sense. We say that u is a *subsolution* (respectively *supersolution*) to Eq. (1.4) if the sign of equality is replaced with $\leq$ (respectively $\geq$) in Eq. (1.4) and we say that u is a subsolution (respectively supersolution) to the Cauchy problem (1.4)–(1.6) if u is a subsolution (respectively supersolution) to Eq. (1.4) and, furthermore, $u(x,0) \leq u_0(x)$ for any $x \in \mathbb{R}$ (respectively $u(x,0) \geq u_0(x)$ for any $x \in \mathbb{R}$).

Thus, our main tool in the forthcoming proofs is the *comparison principle*, which is a well established property of Eq. (1.4) and even of much more general models of analogous type, see for example [6,8].

Another motivation for writing this paper is that, in a forthcoming work, we are able to map by some transformations a rather general family of a priori more complex partial differential equations into various cases of Eq. (1.4), and we did not find a proper reference in literature giving the precise information we need on Eq. (1.4). We thus decided to fill in this gap, at the same time the current work serving for completing the study of the ranges of q not considered in the short note [18] devoted to absorption exponents $q \in (0,1)$.

Concerning the organization of the material, the paper is divided into two sections, related to, by order of appearance, the ranges $q = 1$ and $q \in (1,p)$, followed by a short discussion at the end. The two main sections are further divided into subsections with the following scheme: a first subsection establishing some exact solutions (either explicit or implicit), and then one or two subsections establishing, on the one hand, the decay rate of solutions lying below the constant solution and, on the other hand, the character of separatrix of the stationary solution established in the first subsection in any of these cases. The *main difficulty* stems from the fact that, for data u_0 which are very close to the stationary solution, in any of the cases, a rather fine construction of subsolutions and supersolutions is required. We are now in a position to begin our analysis.

2 The exponent $q = 1$

As indicated in the title, throughout this section, we fix $q = 1$ in Eq. (1.4), that is,

$$u_t = u_{xx} - u + u^p. \tag{2.1}$$

We follow the plan mentioned at the end of the Introduction.

2.1 Some explicit solutions

We derive below some exact solutions to Eq. (2.1), all of them in explicit form.

• **constant solution:** it is obvious that $u(x,t) = 1$ is the unique non-trivial constant solution.

• **solutions depending only on time:** we look for solutions in the form $u(x,t) = h(t)$. With this ansatz, Eq. (2.1) becomes the differential equation

$$h'(t) = -h(t) + h(t)^p, \quad t > 0,$$

which can be integrated explicitly, leading to the family of explicit solutions

$$U(x,t;C) = h(t;C) = \left[1 + Ce^{(p-1)t}\right]^{-1/(p-1)}, \quad C \in (-1,\infty). \tag{2.2}$$

Let us notice here that the behavior of $U(\cdot,\cdot;C)$ as $t \to \infty$ depends on the sign of C. Indeed, it is obvious that U decreases as $t \to \infty$ if $C > 0$, while it blows up in finite time if $C \in (-1,0)$, noticing that we recover the constant solution $U \equiv 1$ for $C = 0$.

• **stationary solutions:** we look for solutions in the form $u(x,t) = g(x)$. With this ansatz, Eq. (2.1) becomes the differential equation

$$g''(x) - g(x) + g(x)^p = 0, \quad x \in \mathbb{R}. \tag{2.3}$$

By multiplying (2.3) by $g'(x)$ and integrating, we find that

$$(g')^2(x) - g^2(x) + \frac{2}{p+1} g^{p+1}(x) = K, \quad K \in \mathbb{R}. \tag{2.4}$$

Since we consider solutions such that $g \in L^\infty(\mathbb{R})$ and $g(x) \to 0$ as $x \to \pm\infty$, we also infer from (2.3) that $g'' \in L^\infty(\mathbb{R})$ and an application of Barbalat's Lemma [29] readily leads to $K = 0$ in (2.4). After some easy manipulations and an implicit integration, we obtain

$$C \pm x = -\frac{2}{p-1} \operatorname{arctanh} \sqrt{1 - \frac{2g(x)^{p-1}}{p+1}},$$

which can be written in an explicit form as follows:

$$g(x;C) = \left\{ \frac{p+1}{2} \left[1 - \tanh^2 \left(C + \frac{p-1}{2} x \right) \right] \right\}^{1/(p-1)}, \quad C \in \mathbb{R}, \tag{2.5}$$

an expression similar to the ones identified at the end of [18, Section 2]. Recalling the identity

$$1 - \tanh^2 \theta = \frac{4}{2 + e^{2\theta} + e^{-2\theta}}, \quad \theta \in \mathbb{R},$$

we deduce from (2.5) that the stationary solutions $g(\cdot;C)$ have an exponential decay as $|x| \to \infty$, more precisely

$$g(x;C) \sim K(C,p) e^{-|x|}, \quad K(C,p) := \left(\frac{p+1}{2} \right)^{1/(p-1)} e^{2C/(p-1)}, \quad C \in \mathbb{R}.$$

In particular, the stationary solution belongs to $L^1(\mathbb{R})$. In the forthcoming analysis, we will let for simplicity $C = 0$ and we will work with the even stationary solution $g(\cdot;0)$, but, apart from longer and more tedious calculations, the separatrix property of every stationary solution $g(\cdot;C)$ will follow in a completely analogous way.

2.2 Decay rate and large time behavior below the constant solution

The aim of this section is to show that solutions starting from data which are smaller than one decay as $t \to \infty$ and stabilize towards a profile linked with the heat equation.

Theorem 2.1. *Let $u_0 \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$ be such that $0 < \|u_0\|_\infty < 1$. Then there exists $C > 0$ such that the solution u to the Cauchy problem (2.1)–(1.6) satisfies*

$$\|u(t)\|_\infty \le Ce^{-t}, \quad (x,t) \in \mathbb{R} \times (0,\infty). \tag{2.6}$$

If furthermore $u_0 \in L^1(\mathbb{R})$, we have the following large time behavior for the solution u to the Cauchy problem (2.1)–(1.6)

$$\lim_{t\to\infty} t^{1/2}\|e^t u(t) - G(t)\|_\infty = 0, \tag{2.7}$$

where $G(t)$ is the heat kernel

$$G(x,t) = \frac{M}{\sqrt{4\pi t}} e^{-|x|^2/4t}, \quad M = \|u_0\|_1.$$

Proof. Assume first that $u_0 \in L^\infty(\mathbb{R}) \cap C(\mathbb{R})$ and $\|u_0\|_\infty \in (0,1)$. Pick $C_0 > 0$ such that

$$\|u_0\|_\infty < (1+C_0)^{-1/(p-1)}.$$

Recalling the solution $u(\cdot,\cdot;C_0)$ defined in (2.2), the comparison principle then entails that

$$u(x,t) \le U(x,t;C_0) = \left[1 + C_0 e^{(p-1)t}\right]^{-1/(p-1)} \le C_0^{-1/(p-1)} e^{-t}, \tag{2.8}$$

for any $(x,t) \in \mathbb{R} \times (0,\infty)$, and thus we have proved the estimate (2.6).

Set next

$$w(x,t) := e^t u(x,t), \quad (x,t) \in \mathbb{R} \times (0,\infty).$$

Notice that $w(x,0) = u(x,0) = u_0(x)$, for any $x \in \mathbb{R}$, and a straightforward calculation shows that w is a solution to the following equation

$$w_t = w_{xx} + e^{-(p-1)t} w^p, \quad (x,t) \in \mathbb{R} \times (0,\infty). \tag{2.9}$$

Assume now that, moreover, $u_0 \in L^1(\mathbb{R})$. The equation (2.9) strongly suggests that an asymptotic simplification to the heat equation as $t \to \infty$ is expected to take place. In order to prove it in a rigorous way, the simplest path is to apply the stability theorem by Galaktionov and Vázquez [12,13]. Indeed, if we let v to be the solution to the standard heat equation $v_t = v_{xx}$ with the same initial condition $v(x,0) = u_0(x)$, $x \in \mathbb{R}$, we easily observe that w is a supersolution to this problem, so that the comparison principle applied to the heat equation, together with (2.8), ensure that

$$v(x,t) \le w(x,t) \le C_0^{-1/(p-1)}, \quad (x,t) \in \mathbb{R} \times (0,\infty). \tag{2.10}$$

The estimates (2.10), together with well known properties of the heat equation, readily imply that the hypothesis required for the application of the above mentioned stability theorem are in force. The stability theorem thus gives that the ω-limit set of the orbits $w(\cdot;t)$ as $t \to \infty$ are contained in the solutions to the standard heat equation. However, since the initial condition is u_0, the uniqueness of the solution to the Cauchy problem for the heat equation implies that $w(t)$ approaches $v(t)$ as $t \to \infty$. Since $u_0 \in L^1(\mathbb{R})$, the well known asymptotic behavior as $t \to \infty$ for the integrable solutions to the heat equation leads to the convergence (2.7). □

Remark. We can observe by comparison with a suitable Gaussian function, which is a subsolution to (2.9), that, even if $u_0 \in L^\infty(\mathbb{R})$ but $u_0 \notin L^1(\mathbb{R})$, we have

$$Ct^{-1/2}e^{-t} \le \|u(t)\|_\infty, \quad t > 0.$$

2.3 The stationary solution as a separatrix

As commented in Section 2.1, we fix for simplicity $C = 0$ and denote by $g_0(x) = g(x;0)$, $x \in \mathbb{R}$, the stationary solution defined in (2.5) with $C = 0$. In this section, we prove that this solution (and, analogously or by a simple translation, any other stationary solution $g(\cdot;C)$) plays the role of a separatrix for the large time behavior of the solutions to the Cauchy problem (2.1)–(1.6): that is, an initial condition strictly above it produces a solution whose L^∞ norm increases with time (and in the end blows up in finite time), while an initial condition strictly below it produces a solution decaying in time, for which the outcome of Theorem 2.1 applies. This is made precise in the next statement.

Theorem 2.2.

(a) Let $u_0 \in L^\infty(\mathbb{R}) \cap C(\mathbb{R})$ such that

$$\inf_{x\in\mathbb{R}} \frac{u_0(x)}{g_0(x)} = \kappa_0 > 1. \tag{2.11}$$

Then the solution u to the Cauchy problem (2.1)–(1.6) with initial condition u_0 blows up in finite time.

(b) Let $u_0 \in L^\infty(\mathbb{R}) \cap C(\mathbb{R})$ such that

$$\sup_{x\in\mathbb{R}} \frac{u_0(x)}{g_0(x)} = \kappa^0 < 1. \tag{2.12}$$

Then the solution u to the Cauchy problem (2.1)–(1.6) with initial condition u_0 decays to zero as $t \to \infty$ and behaves as in Theorem 2.1.

Remark. Before going to the proof, let us observe that conditions (2.11) and (2.12) are in fact related to a separation of the tails as $|x| \to \infty$. Indeed, due to the strong maximum principle, another solution cannot touch from above or from below the stationary solution g_0 at a time $t > 0$, and for a fixed compact subset $K \subset \mathbb{R}$ one could really find $\kappa_0(K)$, respectively $\kappa^0(K)$ (depending in this case on the compact K) so that the previous conditions hold true in K. It is thus as $|x| \to \infty$ where a separation has to be required, as it does not follow from a maximum principle, and this is the sense of the two conditions (2.11) and (2.12).

Proof of Theorem 2.2. (a) Let us consider the function

$$G(x,t) = (t+T)^\delta g_0((t+T)^\gamma x),$$

where $T > 1$, $\delta > 0$ and $\gamma > 0$ are parameters to be determined later, in order for G to be a subsolution to the Cauchy problem (2.1)–(1.6). A direct calculations gives

$$G_t(x,t) = \delta(t+T)^{\delta-1} g_0(\zeta) + \gamma(t+T)^{\delta-1}\zeta g_0'(\zeta), \quad \zeta := (t+T)^\gamma x.$$

Since g_0 is an even function with a decreasing profile with respect to $x > 0$, we deduce that

$$G_t(x,t) \le \delta(t+T)^{\delta-1} g_0(\zeta). \tag{2.13}$$

Moreover, employing the equation (2.3) satisfied by g_0, we find that

$$G_{xx}(x,t) = (t+T)^{\delta+2\gamma} g_0''(\zeta) = (t+T)^{\delta+2\gamma}(g_0(\zeta) - g_0^p(\zeta)). \tag{2.14}$$

Gathering the outcome of (2.13) and (2.14), we obtain

$$G_t(x,t) - G_{xx}(x,t) + G(x,t) - G^p(x,t) \le (t+T)^{\delta-1}\left[\delta + T + t - (t+T)^{2\gamma+1}\right] g_0(\zeta) + (t+T)^{\delta+2\gamma}\left[1 - (t+T)^{\delta(p-1)-2\gamma}\right] g_0^p(\zeta).$$

We next choose the (up to now) free parameters as follows: fix first

$$0 < \delta < \kappa_0^{(p-1)/2} - 1, \quad \gamma = \frac{\delta(p-1)}{4}. \tag{2.15}$$

With the previous choices, we are in a position to also choose

$$T := \kappa_0^{1/\delta} > 1.$$

Let us first observe that the terms in brackets in the previous calculation are negative with these choices, for any $t \ge 0$. On the one hand, since $\delta(p-1) - 2\gamma > 0$, we immediately get that

$$1 - (t+T)^{\delta(p-1)-2\gamma} \le 1 - T^{\delta(p-1)-2\gamma} < 0, \quad t \ge 0.$$

On the other hand, since $\gamma > 0$, we have

$$(T+t)^{2\gamma+1} - (t+T) = (t+T)((T+t)^{2\gamma} - 1) \ge T(T^{2\gamma} - 1) > \kappa_0^{(p-1)/2} - 1 > \delta,$$

whence

$$\delta + T + t - (t+T)^{2\gamma+1} < 0, \quad t \ge 0.$$

We have thus proved that $G(\cdot,t)$ is a subsolution to Eq. (2.1) for any $t \ge 0$. Moreover, taking into account that g_0 is decreasing with respect to $|x|$ and that $T > 1$, $\gamma > 0$, we have

$$G(x,0) = T^\delta g_0(T^\gamma x) = \kappa_0 g_0(T^\gamma x) \le \kappa_0 g_0(x) \le u_0(x), \quad x \in \mathbb{R},$$

hence G is a subsolution to the Cauchy problem (2.1)–(1.6) with initial condition $u_0(x)$. The comparison principle then gives

$$u(x,t) \ge (t+T)^\delta g_0((t+T)^\gamma x), \quad (x,t) \in \mathbb{R} \times (0,\infty).$$

Defining then the following energy

$$E(u(t)) = \frac{1}{2}\int_{\mathbb{R}} (|u_x|^2(x,t) + u^2(x,t))\,dx - \frac{1}{p+1}\int_{\mathbb{R}} u^{p+1}(x,t)\,dx,$$

we observe that

$$\begin{aligned} E(G(t)) &= \frac{1}{2}\int_{\mathbb{R}} \left[(T+t)^{2(\delta+\gamma)}|g_0'(x(T+t)^\gamma)|^2\,dx + (T+t)^{2\delta} g_0(x(T+t)^\gamma)^2\right] dx \\ &\quad - \frac{(T+t)^{\delta(p+1)}}{p+1}\int_{\mathbb{R}} g_0^{p+1}(x(T+t)^\gamma)\,dx \\ &= \frac{1}{2}(T+t)^{2\delta+\gamma}\int_{\mathbb{R}} |g_0'(y)|^2\,dy + \frac{1}{2}(T+t)^{2\delta-\gamma}\int_{\mathbb{R}} g_0^2(y)\,dy \\ &\quad - \frac{1}{p+1}(T+t)^{(p+1)\delta-\gamma}\int_{\mathbb{R}} g_0^{p+1}\,dy < 0, \end{aligned}$$

provided $t > 0$ is taken sufficiently large, since the choice of δ and γ in (2.15) implies

$$(p+1)\delta - \gamma > 2\delta + \gamma > 2\delta - \gamma.$$

We then infer from [25, Theorem 17.6] that the solution to Eq. (2.1) with initial condition $G(t_0)$ for t_0 sufficiently large such that $E(G(t_0)) < 0$ blows up in finite time. It follows by comparison that also u blows up in finite time, as claimed.

(b) Working "in the mirror" with respect to part (a), let us consider the function

$$H(x,t) = (t+T)^{-\delta} g_0((t+T)^{-\gamma} x),$$

where $T > 1$, $\delta > 0$ and $\gamma > 0$ are parameters to be determined later, in order for H to be a supersolution to the Cauchy problem (2.1)–(1.6). A direct calculations gives

$$H_t(x,t) = -\delta(t+T)^{-\delta-1} g_0(\zeta) - \gamma(t+T)^{-\delta-1} \zeta g_0'(\zeta), \quad \zeta := (t+T)^{-\gamma} x.$$

Since g_0 is an even function with a decreasing profile with respect to $x > 0$, we deduce that

$$H_t(x,t) \geq -\delta(t+T)^{-\delta-1} g_0(\zeta). \tag{2.16}$$

Moreover, employing the equation (2.3) satisfied by g_0, we find that

$$H_{xx}(x,t) = (t+T)^{-\delta-2\gamma} g_0''(\zeta) = (t+T)^{-\delta-2\gamma}(g_0(\zeta) - g_0^p(\zeta)). \tag{2.17}$$

Gathering the outcome of (2.16) and (2.17), we obtain

$$\begin{aligned} H_t(x,t) - H_{xx}(x,t) + H(x,t) - H^p(x,t) \geq (t+T)^{-\delta-1} \left[-\delta + T + t - (t+T)^{1-2\gamma} \right] g_0(\zeta) \\ + (t+T)^{-\delta-2\gamma} \left[1 - (t+T)^{-\delta(p-1)+2\gamma} \right] g_0^p(\zeta). \end{aligned}$$

Proceeding as in part (a), we next choose the parameters as follows: fix first

$$0 < \delta < 1 - (\kappa^0)^{p-1}, \quad \gamma = \frac{\delta(p-1)}{2}. \tag{2.18}$$

With the previous choices, we are in a position to also choose

$$T := (\kappa^0)^{-1/\delta} > 1.$$

Notice that $T^{-\delta} = \kappa^0$, whence

$$H(x,0) = T^{-\delta} g_0(\zeta) = \kappa^0 g_0(T^{-\gamma} x) \geq \kappa^0 g_0(x) \geq u_0(x), \quad x \in \mathbb{R},$$

where we have employed once more the fact that g_0 is decreasing with respect to $|x|$. Moreover, since $2\gamma = \delta(p-1)$ by (2.18), we have

$$1 - (t+T)^{-\delta(p-1)+2\gamma} = 0,$$

for any $t \geq 0$, while the first term in brackets is also non-negative, since

$$T + t - (T+t)^{1-2\gamma} - \delta = (T+t)\left[1 - (T+t)^{-2\gamma}\right] - \delta \geq 1 - T^{-2\gamma} - \delta = 1 - (\kappa^0)^{(p-1)} - \delta > 0,$$

according to (2.18). We infer that H is a supersolution to the Cauchy problem (2.1)–(1.6) and thus, by comparison,

$$u(x,t) \leq (t+T)^{-\delta} g_0((t+T)^{-\gamma} x), \quad (x,t) \in \mathbb{R} \times (0,\infty). \tag{2.19}$$

We then find from (2.19) that there is $t_0 > 0$ sufficiently large such that

$$\|u(t_0)\|_\infty \leq (t_0+T)^{-\delta} \|g_0\|_\infty < 1,$$

and an application of Theorem 2.1 starting with $t = t_0$ as initial time completes the proof. □

3 The range $1 < q < p$

Throughout this section, we work with absorption exponents $q \in (1, p)$. This choice does no longer allow for explicit integrations of the differential equations giving rise to solutions depending only on time or on the space variable, in contrast to the calculations in Section 2.1. Because of this technical problem, the forthcoming analysis is more involved than the previous one. We follow the same program as in Section 2.

3.1 Some special solutions in implicit form

We examine below the properties of solutions either depending only on time or only on space, to Eq. (1.4). We have:

- **constant solution:** once more, $u(x,t) \equiv 1$ is a constant solution to Eq. (1.4).
- **solutions depending only on time:** we look for solutions of the form $u(x,t) = h(t)$, $t > 0$. With this ansatz, Eq. (1.4) becomes

$$h'(t) = h^p(t) - h^q(t). \tag{3.1}$$

It is easy to observe that $h(0) > 1$ implies $h'(t) > 0$ for any $t > 0$, while $0 < h(0) < 1$ implies $h'(t) < 0$ for any $t > 0$. We next give a more precise description of the properties of solutions to (3.1). Assume first that $h(0) > 1$, hence $h(t) > h(0)$ for any $t > 0$ and thus

$$h^p(t) > h'(t) = h^p(t)(1 - h^{q-p}(t)) > h^p(t)(1 - h(0)^{q-p}).$$

The second inequality already implies finite time blow-up of h, and let us denote by $T \in (0, \infty)$ its blow-up time. We have

$$1 - h(0)^{q-p} < h^{-p}(t)h'(t) < 1$$

and a straightforward argument of integration on (t, T) leads to the blow-up rate

$$(p-1)^{-1/(p-1)}(T-t)^{-1/(p-1)} < h(t) < [(p-1)(1-h(0)^{q-p})]^{-1/(p-1)}(T-t)^{-1/(p-1)}. \tag{3.2}$$

Assume now that $h(0) < 1$. Then we get from (3.1) that

$$-h^q(t) < h'(t) < (h(0)^{p-q} - 1)h^q(t) < 0,$$

or equivalently

$$(1-q)(h(0)^{p-q} - 1) < (h^{1-q})'(t) < -(1-q)$$

and by integration on $(0, t)$ and straightforward manipulations, we deduce the decay rate of the solution h as $t \to \infty$ as follows:

$$[(q-1)t + h(0)^{1-q}]^{-1/(q-1)} < h(t) < [(q-1)(1-h(0)^{p-q})t + h(0)^{1-q}]^{-1/(q-1)}, \tag{3.3}$$

which means in particular that $h(t)$ decays like $t^{-1/(q-1)}$ as $t \to \infty$.

- **stationary solutions:** we look for solutions of the form $u(x,t) = \psi(x)$, $x \in \mathbb{R}$. With this ansatz, Eq. (1.4) becomes

$$\psi''(x) - \psi^q(x) + \psi^p(x) = 0,$$

which, after a multiplication by $\psi'(x)$ and an obvious integration term by term (setting the integration constant to zero), reads

$$(\psi')^2(x) = \frac{2}{q+1}\psi^{q+1}(x) - \frac{2}{p+1}\psi^{p+1}(x). \tag{3.4}$$

Notice first that, at a maximum point (which we assume to be at $x = 0$, as we want an even solution for simplicity), we obtain from (3.4)

$$\psi(0) = \|\psi\|_\infty = \left(\frac{p+1}{q+1}\right)^{1/(p-q)} > 1. \tag{3.5}$$

The standard existence and uniqueness theorem applied to the Cauchy problem (3.4) (taking the square root with a minus sign in front) with initial condition (3.5) gives the existence of a stationary solution for $x > 0$, which can be then extended by symmetry to $x < 0$, having its maximum at $x = 0$ and a decreasing profile, as desired. From now on, we denote this stationary solution by ψ_0. Its asymptotic properties are stated in the following result.

Proposition 3.1. *The following behavior of ψ_0 as $|x| \to \infty$ holds true:*

$$\lim_{|x|\to\infty} |x|^{2/(q-1)}\psi_0(x) = \left[\frac{2}{q-1}\sqrt{\frac{q+1}{2}}\right]^{2/(q-1)}, \quad \lim_{|x|\to\infty} \frac{|x|\psi_0'(x)}{\psi_0(x)} = -\frac{2}{q-1}. \tag{3.6}$$

Proof. Taking into account the radial symmetry, we can work on the half-plane $x > 0$. We deduce on the one hand that

$$\psi_0'(x) = -\sqrt{\frac{2}{q+1}\psi_0^{q+1}(x) - \frac{2}{p+1}\psi_0^{p+1}(x)} > -\sqrt{\frac{2}{q+1}}\psi_0^{(q+1)/2}(x), \tag{3.7}$$

and on the other hand, for $x > 1$,

$$\begin{aligned}\psi_0'(x) &= -\sqrt{\frac{2}{q+1}\psi_0^{q+1}(x)\left[1 - \frac{q+1}{p+1}\psi_0^{p-q}(x)\right]} \\ &< -\sqrt{\frac{2}{q+1}\left[1 - \frac{q+1}{p+1}\psi_0(1)^{p-q}\right]}\psi_0^{(q+1)/2}(x).\end{aligned} \tag{3.8}$$

Gathering the estimates (3.7) and (3.8), we deduce that there exist $C_1 > C_2 > 0$ such that

$$-C_1\psi_0^{(q+1)/2}(x) < \psi_0'(x) < -C_2\psi_0^{(q+1)/2}(x), \quad x \in (1,\infty).$$

Multiplying the previous inequalities by $\psi_0^{-(q+1)/2}(x)$ and integrating on $(1,x)$ leads to the bounds

$$[\overline{C}_1(x-1) + \psi_0(1)^{(1-q)/2}]^{-2/(q-1)} < \psi_0(x) < [\overline{C}_2(x-1) + \psi_0(1)^{(1-q)/2}]^{-2/(q-1)}, \quad x > 1,$$

with $\overline{C}_i = (q-1)C_i/2$, $i = 1,2$, which in particular give that

$$0 < \overline{C}_1^{-2/(q-1)} < \liminf_{x\to\infty} x^{2/(q-1)}\psi_0(x) \le \limsup_{x\to\infty} x^{2/(q-1)}\psi_0(x) < \overline{C}_2^{-2/(q-1)}. \tag{3.9}$$

Set now $\phi_0(x) := x^{2/(q-1)}\psi_0(x)$. Straightforward calculations lead to the differential equation solved by ϕ_0, that is,

$$x\phi_0'(x) - \frac{2}{q-1}\phi_0(x) = -\phi_0^{(q+1)/2}(x)\sqrt{\frac{2}{q+1} - \frac{2}{p+1}x^{-2(p-q)/(q-1)}\phi_0^{p-q}(x)}. \tag{3.10}$$

We next have two possibilities:

• either there exists $\lim_{x\to\infty}\phi_0(x) = L \in (0,\infty)$, and in this case, an application of a standard calculus fact (see [20, Lemma 2.9] for a precise statement and proof) gives that there is a sequence $(x_j)_{j\geq 1}$ such that $x_j \to \infty$ and $x_j\phi_0'(x_j) \to 0$ as $j\to\infty$. Since $\phi_0(x_j)\to L$ as $j\to\infty$, we obtain by evaluating (3.10) at $x = x_j$ and letting $j\to\infty$ that

$$-\frac{2}{q-1}L = -L^{(q+1)/2}\sqrt{\frac{2}{q+1}}, \quad \text{that is,} \quad L = \left[\frac{2}{q-1}\sqrt{\frac{q+1}{2}}\right]^{2/(q-1)},$$

as claimed.

• or the limit of $\phi_0(x)$ as $x\to\infty$ does not exist. Since ϕ_0 is bounded according to (3.9), it follows that it oscillates infinitely many times between two extremal values and thus there are sequences $(x_j^m)_{j\geq 1}$ and $(x_j^M)_{j\geq 1}$ of local minima, respectively local maxima for ϕ_0, such that $x_j^m\to\infty$, $x_j^M\to\infty$ as $j\to\infty$ and that

$$\lim_{j\to\infty}\phi_0(x_j^m) = \liminf_{x\to\infty}\phi_0(x), \quad \lim_{j\to\infty}\phi_0(x_j^M) = \limsup_{x\to\infty}\phi_0(x).$$

By evaluating (3.10) at $x = x_j^m$, respectively $x = x_j^M$, and letting $j\to\infty$, we readily obtain that

$$\liminf_{x\to\infty}\phi_0(x) = \limsup_{x\to\infty}\phi_0(x) = \left[\frac{2}{q-1}\sqrt{\frac{q+1}{2}}\right]^{2/(q-1)},$$

which is a contradiction with the non-existence of the limit, showing that this case is not possible. Thus, the proof of the first limit in (3.6) is complete. For the second limit in (3.6), it is enough to replace $\psi_0'(x)$ by the right hand side in (3.7), obtaining thus an expression depending only on $\psi_0(x)$, and then employ the first limit in (3.6) to get the result. We leave the easy details to the reader. □

3.2 The stationary solution as a separatrix

Similarly as we did in Section 2.3, but technically more involved as we work with non-explicit solutions, we show next that the stationary solution constructed in the previous section plays the role of a separatrix for the qualitative properties of solutions to the Cauchy problem (1.4)–(1.6). This is made precise in the following statement.

Theorem 3.2.

(a) Let $u_0 \in L^\infty(\mathbb{R})\cap C(\mathbb{R})$ be an initial condition such that $\|u_0\|_\infty < 1$. Then the solution u to the Cauchy problem (1.4)–(1.6) decays as $t\to\infty$ and, more precisely, there exists $C>0$ such that

$$\|u(t)\|_\infty \leq Ct^{-1/(q-1)}, \quad t>0. \tag{3.11}$$

(b) Let $u_0 \in L^\infty(\mathbb{R})\cap C(\mathbb{R})$ such that

$$\inf_{x\in\mathbb{R}}\frac{u_0(x)}{\psi_0(x)} = \kappa_0 > 1. \tag{3.12}$$

Then the solution u to the Cauchy problem (1.4)–(1.6) with initial condition u_0 blows up in finite time.

(c) Let $u_0 \in L^\infty(\mathbb{R}) \cap C(\mathbb{R})$ such that

$$\sup_{x\in\mathbb{R}} \frac{u_0(x)}{\psi_0(x)} = \kappa^0 < 1. \tag{3.13}$$

Then the solution u to the Cauchy problem (1.4)–(1.6) with initial condition u_0 decays to zero as $t \to \infty$ as in (3.11).

Proof. (a) This follows directly by comparison with a solution $h(t)$ to (3.1) with $\|u_0\|_\infty < h(0) < 1$. The comparison principle and the estimate (3.3) lead to the conclusion.

(b) Following similar ideas as in the proof of Theorem 2.2, we construct a subsolution to the Cauchy problem (1.4)–(1.6) in the form

$$\Psi(x,t) = (T+t)^\delta \psi_0((T+t)^\gamma x), \quad (x,t) \in [0,\infty)\times[0,\infty),$$

with $T > 1$, $\delta > 0$ and $\gamma > 0$ to be determined. Setting $\zeta = (T+t)^\gamma x$, we obtain by direct calculations

$$\begin{aligned}
\Psi_t(x,t) - \Psi_{xx}(x,t) + \Psi^q(x,t) - \Psi^p(x,t) &= \delta(T+t)^{\delta-1}\psi_0(\zeta) \\
&+ \gamma(T+t)^{\delta-1}\zeta\psi_0'(\zeta) - (t+T)^{\delta+2\gamma}\psi_0^q(\zeta) + (t+T)^{\delta+2\gamma}\psi_0^p(\zeta) \\
&+ (t+T)^{q\delta}\Psi_0^q(\zeta) - (T+t)^{p\delta}\psi_0^p(\zeta) \\
= (T+t)^{\delta-1}&\left[\delta - \gamma\zeta\frac{\psi_0'(\zeta)}{\psi_0(\zeta)}\right]\psi_0(\zeta) \\
&+ (T+t)^{\delta+2\gamma}\left[1-(t+T)^{\delta(p-1)-2\gamma}\right]\psi_0^p(\zeta) \\
&+ (T+t)^{q\delta}\left[1-(t+T)^{2\gamma-\delta(q-1)}\right]\psi_0^q(\zeta) = T_1+T_2+T_3,
\end{aligned} \tag{3.14}$$

where in the formula for the term T_1 we have employed the equation (3.4) satisfied by $\psi_0'(\zeta)$ for $\zeta \in (0,\infty)$. Since we want that our function Ψ remains a subsolution for any $t > 0$, the expressions of the terms T_2 and T_3 in (3.14) imply that a necessary condition is to fix

$$\frac{\delta(q-1)}{2} < \gamma < \frac{\delta(p-1)}{2}. \tag{3.15}$$

We next fix T as follows:

$$T = \kappa_0^{1/\delta} > 1, \tag{3.16}$$

remaining thus to choose δ as the only (still) free parameter. We infer from (3.15) and (3.16) that, on the one hand,

$$(t+T)^{2\gamma-\delta(q-1)} \geq T^{2\gamma-\delta(q-1)} > 1, \quad t \geq 0$$

and, on the other hand,

$$(t+T)^{\delta(p-1)-2\gamma} \geq T^{\delta(p-1)-2\gamma} > 1, \quad t \geq 0,$$

whence $T_2 < 0$ and $T_3 < 0$ in (3.14), for any $t \geq 0$. We are left with estimating the term T_1 in (3.14). We derive from (3.6) that

$$\zeta\sqrt{\frac{2}{q+1}\psi_0^{q-1}(\zeta) - \frac{2}{p+1}\psi_0^{p-1}(\zeta)} = -\frac{\zeta\psi_0'(\zeta)}{\psi_0(\zeta)} \to \frac{2}{q-1},$$

as $\zeta \to \infty$. Since $\delta/\gamma < 2/(q-1)$ by (3.15), we deduce that there exists $R_0 > 0$ such that

$$\frac{\delta}{\gamma} < \zeta \sqrt{\frac{2}{q+1}\psi_0^{q-1}(\zeta) - \frac{2}{p+1}\psi_0^{p-1}(\zeta)}, \quad |\zeta| > R_0, \tag{3.17}$$

which in particular implies that $T_1 < 0$ if $|\zeta| > R_0$. Set next

$$L_0 := \inf_{\zeta \in [-R_0, R_0]} \psi_0(\zeta) > 0, \tag{3.18}$$

and notice that R_0 and L_0 depend only on p and q, if we fix, for example,

$$\frac{\delta}{\gamma} = \frac{1}{p-1} + \frac{1}{q-1}, \tag{3.19}$$

a choice that satisfies (3.15). We are left with estimating T_1 in (3.14) in the closed interval $[-R_0, R_0]$, and to this end we change a bit the strategy: instead of estimating T_1 alone, we recall that $\zeta\psi_0'(\zeta) \le 0$ for any $\zeta \in \mathbb{R}$ and we compensate T_1 with the negativity of T_2 in this interval. More precisely, discarding the already negative contribution of $\gamma(T+t)^{\delta-1}\zeta\psi_0'(\zeta)$, we want to choose $\delta > 0$ such that, for any $\zeta \in [-R_0, R_0]$ and $t \ge 0$, we have

$$\delta(T+t)^{\delta-1} - (T+t)^{\delta+2\gamma}\left[(T+t)^{\delta(p-1)-2\gamma} - 1\right]\psi_0^{p-1}(\zeta) < 0,$$

for which a sufficient condition is to pick δ such that

$$0 < \delta < \left(\kappa_0^{p-1-2\gamma/\delta} - 1\right) L_0^{p-1} = \left(\kappa_0^{(p-1)(p-q)/(p+q-2)} - 1\right) L_0^{p-1}, \tag{3.20}$$

taking into account (3.19) and that $T > 1$ and $\kappa_0 > 1$. It thus follows that Ψ is a subsolution to Eq. (1.4) and also satisfies (taking into account that ψ_0 has a decreasing profile with respect to $|x|$ and that $T > 1$, $\gamma > 0$)

$$\Psi(x,0) = \kappa_0\psi_0(T^\gamma x) \le \kappa_0\psi_0(x) \le u_0(x), \quad x \in \mathbb{R},$$

hence the comparison principle entails that

$$\Psi(x,t) = (T+t)^\delta \psi_0(\zeta) \le u(x,t), \quad (x,t) \in \mathbb{R} \times (0,\infty).$$

In order to prove the finite time blow-up, we adapt the argument at the end of the proof of Theorem 2.2 with a slightly changed energy functional in order to cope with the term u^q instead of u, that is,

$$E(u(t)) = \frac{1}{2}\int_{\mathbb{R}} |u_x|^2(x,t)\,dx + \frac{1}{q+1}\int_{\mathbb{R}} u^{q+1}(x,t)\,dx - \frac{1}{p+1}\int_{\mathbb{R}} u^{p+1}(x,t)\,dx.$$

The proof of [25, Theorem 17.6, (ii)] straightforwardly adapts to this energy to show that, if for a general solution u to Eq. (1.4) in $\mathbb{R} \times (0,\infty)$ we have $E(u(t)) < 0$ for some $t \ge 0$, then the solution u blows up in finite time. In our case, similarly as at the end of the proof of Theorem 2.2, we compute

$$\begin{aligned} E(\Psi(t)) = {} & \frac{1}{2}(T+t)^{2\delta+\gamma}\int_{\mathbb{R}} |\psi_0'(y)|^2\,dy + \frac{1}{q+1}(T+t)^{\delta(q+1)-\gamma}\int_{\mathbb{R}} \psi_0^{q+1}(y)\,dy \\ & - \frac{1}{p+1}(T+t)^{\delta(p+1)-\gamma}\int_{\mathbb{R}} \psi_0^{p+1}(y)\,dy < 0, \end{aligned}$$

provided $t > t_0 > 0$ is sufficiently large, since the fact that $p > q$ and (3.15) ensure that

$$(p+1)\delta - \gamma > (q+1)\delta - \gamma, \quad (p+1)\delta - \gamma > 2\delta + \gamma.$$

Thus, the solution to Eq. (1.4) with initial condition $\Psi(t_0)$ such that $E(\Psi(t_0)) < 0$ blows up in finite time. We conclude then from the comparison principle that u also blows up in finite time.

(c) We want to construct a supersolution to the Cauchy problem (1.4)–(1.6) decaying in time, and we work again "in the mirror" with respect to the construction performed in part (b), so that we will only give a sketch of it below. We consider

$$\Phi(x,t) = (T+t)^{-\delta}\psi_0(x(T+t)^{-\gamma}), \quad \zeta = (T+t)^{-\gamma}x,$$

and we want to choose the parameters and exponents T, δ, γ such that Φ is a supersolution. Direct calculations similar to the ones performed in part (b) lead to the following expression

$$\begin{aligned}
&\Phi_t(x,t) - \Phi_{xx}(x,t) + \Phi^q(x,t) - \Phi^p(x,t) \\
&\quad = -\gamma(T+t)^{-\delta-1}\left[\frac{\delta}{\gamma} + \frac{\zeta\psi_0'(\zeta)}{\psi_0(\zeta)}\right]\psi_0(\zeta) \\
&\qquad + (T+t)^{-\delta-2\gamma}\left[1 - (t+T)^{-\delta(p-1)+2\gamma}\right]\psi_0^p(\zeta) \\
&\qquad + (T+t)^{-q\delta}\left[1 - (t+T)^{-2\gamma+\delta(q-1)}\right]\psi_0^q(\zeta) \\
&\quad = T_1 + T_2 + T_3.
\end{aligned} \tag{3.21}$$

Since we want a supersolution for any $t \geq 0$, we are forced to choose δ, γ such that the powers of $T+t$ in the second and third line of (3.21) are negative, thus (3.19) remains in force. Moreover, with the experience gained in part (b), we fix

$$T = (\kappa^0)^{-1/\delta} > 1. \tag{3.22}$$

With the choices in (3.19) and (3.22), we observe that, on the one hand,

$$(t+T)^{-\delta(p-1)+2\gamma} \leq T^{-\delta(p-1)+2\gamma} < 1, \quad t \geq 0,$$

and on the other hand

$$(t+T)^{-2\gamma+\delta(q-1)} \leq T^{-2\gamma+\delta(q-1)} < 1, \quad t \geq 0.$$

This ensures that $T_2 > 0$ and $T_3 > 0$ for any $t \geq 0$. Moreover, since ψ_0 has a decreasing profile with respect to $|x|$ and $T > 1$, $\gamma > 0$, at $t = 0$ we have the comparison between initial conditions:

$$\Phi(x,0) = T^{-\delta}\psi_0(T^{-\gamma}x) = \kappa^0\psi_0(T^{-\gamma}x) \geq \kappa^0\psi_0(x) \geq u_0(x), \quad x \in \mathbb{R}. \tag{3.23}$$

It only remains to choose $\delta > 0$ such that $T_1 > 0$ too. Noticing that (3.19) and (3.6) imply

$$\frac{\delta}{\gamma} < \frac{2}{q-1} = -\lim_{\zeta\to\infty}\frac{\zeta\psi_0'(\zeta)}{\psi_0(\zeta)},$$

we infer that there is $R_0 > 0$ such that $T_1 > 0$ for $\zeta \in \mathbb{R}$ such that $|\zeta| > R_0$. For the compact interval $[-R_0, R_0]$, the proof follows by compensating T_1 by T_2 in order to choose δ, following

exactly the same lines as in the final part of the proof of part (b), this compensation, together with the choices in (3.19) and (3.22) leading to the condition

$$\delta < \min\left\{\frac{1}{2}\left(\frac{1}{p-1}+\frac{1}{q-1}\right), \left[1-(\kappa^0)^{(p-1)(p-q)/(p+q-2)}\right] L_0^{p-1}\right\},$$

with L_0 defined as in (3.18), noticing that the previous election also entails that $\gamma \in (0, 1/2)$. The comparison principle and (3.23) then entail that

$$\Phi(x,t) = (T+t)^{-\delta}\psi_0(x(T+t)^{-\gamma}) \geq u(x,t), \quad (x,t) \in \mathbb{R}\times(0,\infty),$$

and in particular

$$\|u(t)\|_\infty \leq (T+t)^{-\delta}\|\psi_0\|_\infty < 1,$$

provided $t \geq t_0$ sufficiently large. This fact together with part (a) lead to the conclusion. □

Discussion and extensions

The previous analysis raises a few questions related to a finer description of the dynamics of solutions to Eq. (1.4), that we comment in the next lines.

• **Blow-up rates and patterns.** Due to the fact that, for large values of u (close to a blow-up point), u^q is negligible compared to u^p, it is strongly expected that the blow-up behavior of solutions to Eq. (1.4) presenting finite time blow-up is totally similar to the one of the usual reaction-diffusion equation $u_t = u_{xx} + u^p$. In fact, if $T \in (0,\infty)$ is the blow-up time of the solution to the Cauchy problem (1.4)–(1.6) with an initial condition u_0 satisfying either Theorem 2.2(a) or Theorem 3.2(b), then it cannot lie completely below the solution given by either (2.2) (if $q = 1$) or to (3.1) (if $q \in (1, p)$) having the same blow-up time T. We then deduce the lower blow-up rate from (3.2), that is,

$$\|u(t)\|_\infty \geq C(T-t)^{-1/(p-1)}, \quad t \in (0,T),$$

for some $C > 0$. As for the upper rate and other fine properties of blow-up, the theory in [25] should apply without many changes. In particular, for $q = 1$ the previous rate follows as a particular case of the results in the recent paper [33].

• **Large time behavior for solutions with decay as $t \to \infty$.** As we have seen, for $q = 1$ we have obtained the large time behavior by an asymptotic simplification. The same must hold true for $q \in (1, p)$ since for u very small, the reaction u^p is negligible with respect to the absorption u^q, so that the asymptotic simplification is practically obvious at a formal level. We do not enter this discussion in detail, since the large time behavior of solutions to the pure absorption-diffusion equation (neglecting u^p) is not straightforward, but given by either a Gaussian, as a further asymptotic simplification, if $q > 3$, or some very singular solutions constructed in [5] if $1 < q < 3$.

• **Extinction behavior for $q \in (0,1)$.** As established in [18], for $q \in (0,1)$, initial conditions u_0 lying below the stationary solution lead to finite time extinction, that is, there exists $T_e \in (0,\infty)$ such that $u(t) \not\equiv 0$ for $t \in (0, T_e)$, but $u(T_e) \equiv 0$. Establishing the extinction rate (that is not studied in [18]) follows rather straightforwardly, since one can construct solutions depending only on time $h(t)$, solving the differential equation

$$h'(t) = h^p(t) - h^q(t), \quad h(0) < 1,$$

and, by estimates in the same style as the ones leading to (3.2), find that

$$[(1-q)(1-h(0)^{p-q})]^{1/(1-q)}(T_e-t)^{1/(1-q)} < h(t) < (1-q)^{1/(1-q)}(T_e-t)^{1/(1-q)}.$$

Moreover, a solution to Eq. (1.4) is a supersolution to the pure absorption-diffusion equation obtained by neglecting u^p, and about which it is known that the extinction rate of general solutions is

$$\|u(t)\|_\infty \sim C(T_e-t)^{1/(1-q)}, \quad \text{as } t \to T_e,$$

so that the same rate will be in force for Eq. (1.4) with $p > 1$ and $q \in (0,1)$. However, a finer analysis of the extinction phenomenon is rather complex, as it should be at least similar to (if not technically more involved than) the one for the absorption-diffusion equation

$$u_t = u_{xx} - u^q, \quad 0 < q < 1,$$

studied in detail in classical but quite complicated works such as [11, 19]. We leave this discussion here.

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Multiplicity of weak solutions to degenerate weighted quasilinear elliptic equations with nonlocal terms and variable exponents

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Abstract. This paper studies the existence and multiplicity of weak solutions to degenerate weighted quasilinear elliptic equations with nonlocal nonlinearities and variable exponents. The equation involves a degenerate nonlinear operator with variable exponents, a nonlocal term, and growth conditions on the nonlinearity. Using critical point theory, we prove the existence of at least three weak solutions under general assumptions, extending the applicability of the results to a broad class of nonlinear problems in mathematical physics.

Keywords: variational methods, Hardy inequality, degenerate $p(x)$-Laplacian operators.

2020 Mathematics Subject Classification: 35J15, 35J20, 35J25.

1 Introduction

Significant challenges in the analysis and behavior of solutions arise from the presence of singularities and degeneracies in elliptic equations. Singularities, especially those arising at the origin or the boundary, can dramatically alter the properties of the operator, causing the solutions to become more sensitive to changes in the domain. For example, when $1 < p < N$, it is known that $u/|x| \in L^p(\mathbb{R}^N)$ if $u \in W^{1,p}(\mathbb{R}^N)$, or $u/|x| \in L^p(\Omega)$ when $u \in W^{1,p}(\Omega)$, where Ω is a bounded domain. This leads to the emergence of Hardy-type inequalities that control the singular behavior of solutions near critical points, particularly when the equation includes singular potential terms (see, e.g., [11, 13, 15, 16]).

In addition to the challenges posed by singularities, the inclusion of nonlocal terms further complicates the situation. Nonlocal interactions, which typically arise from integral expressions or global coupling terms, create dependencies that affect the solution at each point in the domain, as well as its values across the entire domain. The nonlocal nature of these terms requires the use of advanced techniques that go beyond classical local methods to analyze the solution structure and its multiplicity.

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Furthermore, the operator's degeneracy, especially when coupled with a weighted function $\omega(x)$ in the p-Laplacian or $p(x)$-Laplacian operator, adds another layer of complexity. The degeneracy introduced by $\omega(x)$, whether singular or merely bounded, necessitates a shift in the choice of functional spaces. In such cases, traditional Sobolev spaces such as $W^{1,p}(\Omega)$ or $W^{1,p(x)}(\Omega)$ may no longer be suitable. Instead, we must consider alternative Sobolev spaces that accommodate the weight function, such as $W^{1,p}(\omega,\Omega)$, to handle the singularities or degeneracies (see [6] for further details).

This paper addresses these challenges by studying a class of weighted quasilinear elliptic equations with nonlocal nonlinearities and variable exponents. The goal is to establish the existence and multiplicity of weak solutions while taking into account the degeneracy of the operator, the Hardy-type singularities, and the nonlocal interactions that frequently arise in applied mathematical models.

In this paper, we investigate the existence of generalized solutions to a class of weighted quasilinear elliptic equations of the form:

$$\begin{cases} -\Delta_{p(x),a(x,u)}u + \frac{b(x)|u|^{q-2}u}{|x|^q} = \lambda f(x,u)\left(\int_\Omega F(x,u)\,dx\right)^r & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where, $1 < q < N$, $\Omega \subset \mathbb{R}^N$ (with $N \geq 3$) is a bounded open subset, $\partial\Omega$ denotes its smooth boundary and $b(x)$ is a non-negative measurable function. The unknown function u satisfies a weighted quasilinear elliptic equation involving a variable exponent $p(x)$, a nonlinear term $f(x,u)$, and a non-local term involving an integral expression with a positive exponent r.

The operator $\Delta_{p(x),a(x,u)}u$ is a nonlinear generalization of the standard Laplacian, defined as:

$$\Delta_{p(x),a(x,u)}u = \operatorname{div}\left(a(x,u)|\nabla u|^{p(x)-2}\nabla u\right),$$

where $a(x,u)$ is a Carathéodory function satisfying the inequality

$$a_1\omega(x) \leq a(x,u) \leq a_2\omega(x) \quad \text{for some constants } a_1, a_2 > 0,$$

and a positive measurable function $\omega(x)$. The function $\omega(x)$ is assumed to belong to the local Lebesgue space $L^1_{\text{loc}}(\Omega)$ and satisfy additional growth conditions, such as $\omega^{-h(x)} \in L^1(\Omega)$, where $h(x)$ belongs to a range related to the variable exponent $p(x)$, more precisely, we assume that

(ω) $\omega^{-h(x)} \in L^1(\Omega)$, for $h(x) \in C(\overline{\Omega})$ and $h(x) \in (N/p(x), +\infty) \cap [1/(p(x)-1), +\infty)$,

The nonlinearity in the equation is characterized by the function $f(x,u)$, which satisfies growth conditions of the form:

$$(f_1) \qquad M_1|u|^{\alpha(x)-1} \leq |f(x,u)| \leq M_2|u|^{\beta(x)-1},$$

with $1 < \alpha(x) \leq \beta(x) < p_h^*(x)$, where $p_h^*(x) = \frac{Np_h(x)}{N-p_h(x)}$ and $p_h(x) = \frac{h(x)p(x)}{h(x)+1}$. These conditions allow for a broad class of nonlinearities, including those that are locally integrable and exhibit power-like behavior at infinity.

The parameter $\lambda > 0$ scales the non-local term, which depends on the integral

$$\int_\Omega F(x,u)\,dx$$

of another function $F(x,u)$, which itself is assumed to satisfy certain conditions. These non-local terms introduce further complexity into the problem, as they may give rise to both local and global interactions within the domain Ω.

The main objective of this paper is to establish the existence of at least three weak solutions to the problem under very general assumptions on the weighted function $\omega(x)$ and the non-linear non-local term. To achieve this, we apply critical point theory to the associated energy functional, which is constructed by integrating the relevant terms of the equation over Ω. The application of critical point theorems allows us to prove the existence of solutions without the need for strict assumptions on the regularity or structure of the nonlinearity, making the result highly general and applicable to a wide variety of problems in mathematical physics and differential equations.

2 Preliminaries and variational structure

Set

$$C_+(\overline{\Omega}) = \{p(x) | p(x) \in C(\overline{\Omega}),\ p(x) > 1,\ \forall x \in \overline{\Omega}\},$$
$$p^+ = \max_{x\in\overline{\Omega}} p(x), \quad p^- = \min_{x\in\overline{\Omega}} p(x).$$

For $k > 0$, and $p(x) \in C_+(\overline{\Omega})$, we use the following notations

$$k^{\hat{p}} = \max\{k^{p^-},\ k^{p^+}\}, \quad k^{\check{p}} = \min\{k^{p^-},\ k^{p^+}\}.$$

It is easy to verify that (one can see [3], for further details)

(i) $k^{\frac{1}{\hat{p}}} = \max\left\{k^{\frac{1}{p^-}}, k^{\frac{1}{p^+}}\right\}$,

(ii) $k^{\frac{1}{\check{p}}} = \min\left\{k^{\frac{1}{p^-}}, k^{\frac{1}{p^+}}\right\}$,

(iii) $k^{\frac{1}{\hat{p}}} = a \Longleftrightarrow k = a^{\hat{p}}, k^{\frac{1}{\check{p}}} = a \Longleftrightarrow k = a^{\check{p}}$,

(iv) $(k\beta)^{\check{p}} \leq k^{\check{p}}\beta^{\check{p}} \leq (k\beta)^{\hat{p}} \leq k^{\hat{p}}\beta^{\hat{p}}$.

Denote

$$L^{p(x)}(\omega,\Omega) = \left\{u :\ \text{measurable in } \Omega \text{ such that } \int_\Omega \omega(x)|u(x)|^{p(x)}\,dx < \infty\right\}$$

with a Luxemburg-type norm defined by

$$\|u\|_{L^{p(x)}(\omega,\Omega)} = \inf\left\{\eta > 0 : \int_\Omega \omega(x)\left|\frac{u(x)}{\eta}\right|^{p(x)} dx \leq 1\right\}.$$

Now, we define the variable exponent Sobolev space

$$W^{1,p(x)}(\Omega) = \left\{u \in L^{p(x)}(\Omega) :\ |\nabla u| \in L^{p(x)}(\Omega)\right\},$$

with the norm

$$\|u\|_{W^{1,p(x)}(\Omega)} = \|\nabla u\|_{p(x)} + \|u\|_{p(x)},$$

where $\|\nabla u\|_{p(x)} = \||\nabla u|\|_{p(x)}, |\nabla u| = \left(\sum_{i=1}^N \left|\frac{\partial u}{\partial x_i}\right|^2\right)^{\frac{1}{2}}, \nabla u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial u}{\partial x_N}\right)$ is the gradient of u at $(x_1, x_2, \ldots, x_N)$.

Let

$$W^{1,p(x)}(\omega,\Omega)=\left\{u\in L^{p(x)}(\Omega):\omega^{\frac{1}{p(x)}}|\nabla u|\in L^{p(x)}(\Omega)\right\}$$

be the weighted Sobolev space, and denote by $W_0^{1,p(x)}(\omega,\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\omega,\Omega)$ with the norm

$$\|u\|=\inf\left\{\eta>0:\int_\Omega \omega(x)\left|\frac{\nabla u(x)}{\eta}\right|^{p(x)}dx\le 1\right\}.$$

Lemma 2.1 ([9]). *If $p_1(x),p_2(x)\in C_+(\overline{\Omega})$ such that $p_1(x)\le p_2(x)$ a.e. $x\in\Omega$, then there exists the continuous embedding $W^{1,p_2(x)}(\Omega)\hookrightarrow W^{1,p_1(x)}(\Omega)$.*

Proposition 2.2 ([10]). *For $p(x)\in C_+(\overline{\Omega}),u,u_n\in L^{p(x)}(\Omega)$, we have*

$$\min\left\{\|u\|_{p(x)}^{p^-},\|u\|_{p(x)}^{p^+}\right\}\le\int_\Omega|u(x)|^{p(x)}dx\le\max\left\{\|u\|_{p(x)}^{p^-},\|u\|_{p(x)}^{p^+}\right\}.$$

Let $0<d(x)\in S(\Omega)$ and $S(\Omega)$ be the set of all measurable real functions defined on Ω. Define

$$L_{d(x)}^{p(x)}(\Omega)=L^{p(x)}(d,\Omega)=\left\{u\in S(\Omega):\int_\Omega d(x)|u(x)|^{p(x)}\,dx<\infty\right\}$$

with a Luxemburg-type norm defined by

$$\|u\|_{L_{d(x)}^{p(x)}(\Omega)}=\|u\|_{(p(x),d(x))}:=\inf\left\{\eta>0:\int_\Omega d(x)\left|\frac{u(x)}{\eta}\right|^{p(x)}\,dx\le 1\right\}.$$

Proposition 2.3 ([7]). *If $p\in C_+(\overline{\Omega})$. Then*

$$\min\left\{\|u\|_{(p(x),d(x))}^{p^-},\|u\|_{(p(x),d(x))}^{p^+}\right\}\le\int_\Omega d(x)|u(x)|^{p(x)}dx\le\max\left\{\|u\|_{(p(x),d(x))}^{p^-},\|u\|_{(p(x),d(x))}^{p^+}\right\}$$

for any $u\in L_{d(x)}^{p(x)}(\Omega)$ and for a.e. $x\in\Omega$.

Combining Proposition 2.2 with Proposition 2.3, one has

Lemma 2.4. *Let*

$$\rho_\omega(u)=\int_\Omega\omega(x)\,|\nabla u(x)|^{p(x)}\,dx.$$

For $p\in C_+(\overline{\Omega}),u\in W^{1,p(x)}(\omega,\Omega)$, we have

$$\min\left\{\|u\|^{p^-},\|u\|^{p^+}\right\}\le\rho_\omega(u)\le\max\left\{\|u\|^{p^-},\|u\|^{p^+}\right\}.$$

From Proposition 2.4 of [13], if (ω) holds, $W^{1,p(x)}(\omega,\Omega)$ is a separable and reflexive Banach space.

From Theorem 2.11 of [14], if (ω) holds, the following embedding

$$W^{1,p(x)}(\omega,\Omega)\hookrightarrow W^{1,p_h(x)}(\Omega)\tag{2.1}$$

is continuous, where

$$p_h(x)=\frac{p(x)h(x)}{h(x)+1}<p(x).$$

Combining (2.1) with Proposition 2.7 and Proposition 2.8 in [8], we get the following embedding

$$W^{1,p(x)}(\omega,\Omega) \hookrightarrow L^{r(x)}(\Omega)$$

is continuous, where

$$1 \le r(x) \le p_h^*(x) = \frac{Np_h(x)}{N-p_h(x)} = \frac{Np(x)h(x)}{Nh(x)+N-p(x)h(x)}.$$

Furthermore, the following embedding

$$W^{1,p(x)}(\omega,\Omega) \hookrightarrow\hookrightarrow L^{t(x)}(\Omega)$$

is compact, when $1 \le t(x) < p_h^*(x)$.

Define the functional $\mathcal{I}_\lambda \colon W_0^{1,p(x)}(\omega,\Omega) \to \mathbb{R}$ by

$$\mathcal{I}_\lambda(u) := \Phi(u) - \lambda\Psi(u),$$

where

$$\Phi(u) := \int_\Omega \frac{a(x,u)}{p(x)}|\nabla u|^{p(x)}dx + \frac{1}{q}\int_\Omega \frac{b(x)|u|^q}{|x|^q}dx,$$
$$\Psi(u) := \frac{1}{r+1}\Big(\int_\Omega F(x,u(x))dx\Big)^{r+1},$$

and $F(x,u) = \int_0^u f(x,\tau)d\tau, \forall (x,u) \in \Omega\times\mathbb{R}$.

It is clear that functionals Φ and Ψ are continuously Gâteaux differentiable with

$$\Phi'(u)(v) = \int_\Omega a(x,u)|\nabla u|^{p(x)-2}\nabla u\nabla v dx + \int_\Omega \frac{b(x)|u|^{q-2}uv}{|x|^q}dx,$$

and

$$\Psi'(u)(v) = \left(\int_\Omega F(x,u)\,dx\right)^r \int_\Omega f(x,u)v dx, \quad \forall u,v \in W_0^{1,p(x)}(\omega,\Omega).$$

We say that $u \in W_0^{1,p(x)}(\omega,\Omega)$ is a generalized solution of the problem (1.1) if

$$\mathcal{I}'_\lambda(u)(v) = \Phi'(u)(v) - \lambda\Psi'(u)(v) = 0, \quad \forall v \in W_0^{1,p(x)}(\omega,\Omega).$$

Lemma 2.5. *The functional Φ' is coercive and strictly monotone in $W_0^{1,p(x)}(\omega,\Omega)$.*

Proof. For any $u \in W_0^{1,p(x)}(\omega,\Omega)\setminus\{0\}$, by Lemma 2.4 one has

$$\begin{aligned}\Phi'(u)(u) &= \int_\Omega a(x,u)|\nabla u|^{p(x)-2}\nabla u\nabla u dx + \int_\Omega \frac{b(x)|u|^{q-2}u^2}{|x|^q}dx\\ &\ge a_1\rho_\omega(u)\\ &\ge a_1\cdot\min\{\|u\|^{p^+},\|u\|^{p^-}\},\end{aligned}$$

thus

$$\lim_{\|u\|\to\infty}\frac{\Phi'(u)(u)}{\|u\|} \ge a_1\cdot\lim_{\|u\|\to\infty}\frac{\min\{\|u\|^{p^+},\|u\|^{p^-}\}}{\|u\|} = +\infty,$$

then Φ' is coercive in view of $p(x) \in C_+(\overline{\Omega})$.

According to (2.2) of [17], for all $x, y \in \mathbb{R}^N$, there is a positive constant C_p such that

$$\langle |x|^{p-2}x - |y|^{p-2}y, x-y\rangle \geq C_p|x-y|^p, \quad \text{if } p \geq 2,$$

and

$$\langle |x|^{p-2}x - |y|^{p-2}y, x-y\rangle \geq \frac{C_p|x-y|^2}{(|x|+|y|)^{2-p}}, \quad \text{if } 1 < p < 2, \quad \text{and} \quad (x,y) \neq (0,0),$$

where $\langle .,. \rangle$ is the usual inner product in $\mathbb{R}^N$. Thus, for any $u, v \in X$ satisfying $u \neq v$, by standard arguments we can obtain

$$\begin{aligned}\langle \Phi'(u) - \Phi'(v), u-v\rangle = &\int_\Omega a(x,u)(|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v)(\nabla u - \nabla v)dx \\ &+ \int_\Omega \frac{b(x)}{|x|^q}(|u|^{q-2}u - |v|^{q-2}v)(u-v))dx \\ &> 0,\end{aligned}$$

hence we have Φ' is strictly monotone in $W_0^{1,p(x)}(\omega, \Omega)$. □

Lemma 2.6. *The functional Φ' is a mapping of (S_+)-type, i.e. if $u_n \rightharpoonup u$ in $W_0^{1,p(x)}(\omega,\Omega)$, and $\overline{\lim}_{n\to\infty}\langle \Phi'(u_n) - \Phi'(u), u_n - u)\rangle \leq 0$, then $u_n \to u$ in $W_0^{1,p(x)}(\omega,\Omega)$.*

Proof. Let $u_n \rightharpoonup u$ in $W_0^{1,p(x)}(\omega,\Omega)$, and $\overline{\lim}_{n\to\infty}\langle \Phi'(u_n) - \Phi'(u), u_n - u\rangle \leq 0$.

Noting that Φ' is strictly monotone in $W_0^{1,p(x)}(\omega,\Omega)$, one has

$$\lim_{n\to\infty}\langle \Phi'(u_n) - \Phi'(u), u_n - u\rangle = 0,$$

while

$$\begin{aligned}\langle \Phi'(u_n) - \Phi'(u), u_n - u\rangle = &\int_\Omega a(x,u)(|\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla u|^{p(x)-2}\nabla u)(\nabla u_n - \nabla u)dx \\ &+ \int_\Omega \Big(\frac{b(x)|u_n|^{q-2}}{|x|^q}u_n(u_n-u) - \frac{b(x)|u|^{q-2}}{|x|^q}u(u_n-u)\Big)dx,\end{aligned}$$

thus we get

$$\overline{\lim}_{n\to\infty}\int_\Omega a(x,u)(|\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla u|^{p(x)-2}\nabla u)(\nabla u_n - \nabla u)dx \leq 0.$$

Further, by (1.2) one has

$$\overline{\lim}_{n\to\infty}\int_\Omega \omega(x)(|\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla u|^{p(x)-2}\nabla u)(\nabla u_n - \nabla u)dx \leq 0,$$

then $u_n \to u$ in $W_0^{1,p(x)}(\omega,\Omega)$ via Lemma 3.2 in [12]. □

Lemma 2.7. *The functional Φ' is a homeomorphism.*

Proof. The strict monotonicity of Φ' implies that it is injective. Since Φ' is coercive, it is also surjective, and hence Φ' has an inverse mapping.

Next, we show that the inverse mapping $(\Phi')^{-1}$ is continuous.

Let $\tilde{f}_n, \tilde{f} \in (W_0^{1,p(x)}(\omega,\Omega))^*$ such that $\tilde{f}_n \to \tilde{f}$. We aim to prove that $(\Phi')^{-1}(\tilde{f}_n) \to (\Phi')^{-1}(\tilde{f})$.

Indeed, let $(\Phi')^{-1}(\tilde{f}_n) = u_n$ and $(\Phi')^{-1}(\tilde{f}) = u$, so that $\Phi'(u_n) = \tilde{f}_n$ and $\Phi'(u) = \tilde{f}$. By the coercivity of Φ', the sequence u_n is bounded. Without loss of generality, assume $u_n \rightharpoonup u_0$, which implies

$$\lim_{n\to\infty} \left(\Phi'(u_n) - \Phi'(u), u_n - u_0\right) = \lim_{n\to\infty} \left(\tilde{f}_n - \tilde{f}, u_n - u_0\right) = 0.$$

Thus, $u_n \to u_0$ because Φ' is of (S_+)-type, which ensures that $\Phi'(u_n) \to \Phi'(u_0)$. Combining this with $\Phi'(u_n) \to \Phi'(u)$, we deduce that $\Phi'(u) = \Phi'(u_0)$. Since Φ' is injective, it follows that $u = u_0$, and hence $u_n \to u$. Therefore, we have $(\Phi')^{-1}(\tilde{f}_n) \to (\Phi')^{-1}(\tilde{f})$, proving that $(\Phi')^{-1}$ is continuous. □

Lemma 2.8 (Hölder type inequality [1,8]). *Let $p, q, s \geq 1$ be measurable functions defined on Ω and*

$$\frac{1}{s(x)} = \frac{1}{p(x)} + \frac{1}{q(x)}, \quad \text{for a.e. } x \in \Omega.$$

If $f \in L^{p(x)}(\Omega)$ and $g \in L^{q(x)}(\Omega)$, then $fg \in L^{s(x)}(\Omega)$ and

$$\|fg\|_{s(x)} \leq 2\|f\|_{p(x)}\|g\|_{q(x)}. \tag{2.2}$$

Lemma 2.9. *The functional $\Psi' : X := W_0^{1,p(x)}(\omega,\Omega) \to (W_0^{1,p(x)}(\omega,\Omega))^*$ is compact.*

Proof. The condition (f_1) and the compact embedding $W_0^{1,p(x)}(\omega,\Omega) \hookrightarrow\hookrightarrow L^{\beta(x)}(\Omega)$, where $1 \leq \beta(x) < p_h^*(x)$, imply the compactness of $\Psi'(u)$. Specifically, let $(u_k)_k \subset X$ be a sequence such that $u_k \rightharpoonup u$. Since the embedding $W_0^{1,p(x)}(\omega,\Omega) \hookrightarrow\hookrightarrow L^{\beta(x)}(\Omega)$ is compact, there exists a subsequence, still denoted by $(u_k)_k$, such that $u_k \to u$ strongly in $L^{\beta(x)}(\Omega)$ and $u_k(x) \to u(x)$ almost everywhere. The continuity of $F(x,u)$ with respect to u ensures that

$$F(x,u_k) \to F(x,u) \quad \text{for almost every } x.$$

Moreover, there exists $C > 0$ such that

$$|F(x,u_k)| \leq C|u_k|^{\beta(x)}.$$

Applying the dominated Convergence theorem we can conclude

$$\int_\Omega F(x,u_k)\,dx \to \int_\Omega F(x,u)\,dx \quad \text{as } k \to +\infty. \tag{2.3}$$

From condition (f_1), it follows that the Nemytskii operator $N_f(u)(x) = f(x,u(x))$ is continuous, as $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function that satisfies (f_1). Consequently, we conclude that $N_f(u_k) \to N_f(u)$ in $L^{\frac{\beta(x)}{\beta(x)-1}}(\Omega)$.

Using Hölder's inequality, for any $v \in W_0^{1,p(x)}(\omega,\Omega)$, we obtain

$$\begin{aligned}\left|\int_\Omega f(x,u_k)v dx - \int_\Omega f(x,u)v dx\right| &\leq \int_\Omega |(f(x,u_k) - f(x,u))v| dx \\ &\leq 2\|N_f(u_k) - N_f(u)\|_{\frac{\beta(x)}{\beta(x)-1}}\|v\|_{\beta(x)} \\ &\leq 2c_\beta\|N_f(u_k) - N_f(u)\|_{\frac{\beta(x)}{\beta(x)-1}}\|v\|,\end{aligned}$$

where c_β is the embedding constant of the embedding $W_0^{1,p(x)}(\omega,\Omega) \hookrightarrow L^{\beta(x)}(\Omega), 1 \leq \beta(x) < p_h^*(x)$. Thus, from (2.3) and the above inequality, $\Psi'(u_k) \to \Psi'(u)$ in X^*, i.e. Ψ' is completely continuous, thus Ψ' is compact. □

The following critical point theorems constitute the principal tools used to obtain our result.

Theorem 2.10 ([5, Theorem 3.6]). *Let X be a reflexive real Banach space, and let $\Phi : X \to \mathbb{R}$ be a coercive functional that is continuously Gâteaux differentiable and weakly lower semicontinuous in the sequential sense. Assume that the Gâteaux derivative of Φ has a continuous inverse on the dual space X^*. Additionally, let $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Assume the following conditions hold:*

(a_0) $$\inf_X \Phi = \Phi(0) = \Psi(0) = 0.$$

There exist constants $d > 0$ and a point $\overline{x} \in X$ such that $d < \Phi(\overline{x})$, and the following conditions are satisfied:

(a_1) $$\frac{\sup_{\Phi(x)<d} \Psi(x)}{d} < \frac{\Psi(\overline{x})}{\Phi(\overline{x})},$$

(a_2) *For each $\lambda \in \Lambda_d := \left(\frac{\Phi(\overline{x})}{\Psi(\overline{x})}, \frac{d}{\sup_{\Phi(x)\leq d} \Psi(x)}\right)$, the functional $I_\lambda := \Phi - \lambda\Psi$ is coercive.*

Then, for every $\lambda \in \Lambda_d$, the functional $\Phi - \lambda\Psi$ has at least three distinct critical points in X.

3 Main results

In this section, a theorem about the existence of at least three weak solutions to the problem (1.1) are obtained.

Recall the Hardy inequality (see Lemma 2.1 in [11] for more details), which states that for $1 < p < N$, the following inequality holds:

$$\int_\Omega \frac{|u(x)|^p}{|x|^p}\,dx \leq \frac{1}{H}\int_\Omega |\nabla u|^p\,dx, \quad \forall u \in W_0^{1,p}(\Omega),$$

where $H = \left(\frac{N-p}{p}\right)^p$ is the optimal constant.

By combining this with Lemma 2.1 and using the fact that $1 < q < p_h(x) < N$, we get the continuous embeddings

$$W_0^{1,p(x)}(\omega,\Omega) \hookrightarrow W_0^{1,p_h(x)}(\Omega) \hookrightarrow W_0^{1,q}(\Omega),$$

which leads to the inequality

$$\int_\Omega \frac{|u(x)|^q}{|x|^q}\,dx \leq \frac{1}{H}\int_\Omega |\nabla u|^q\,dx, \quad \forall u \in W_0^{1,p(x)}(\omega,\Omega),$$

where $H = \left(\frac{N-q}{q}\right)^q$.

We are now ready to present our primary result. To this end, we define

$$D(x) := \sup\{D > 0 \mid B(x,D) \subseteq \Omega\}$$

for each $x \in \Omega$, where $B(x, D)$ denotes a ball centered at x with radius D. It is clear that there exists a point $x^0 \in \Omega$ such that $B(x^0, R) \subseteq \Omega$, where

$$R = \sup_{x \in \Omega} D(x).$$

In the remainder of the paper, the symbol m will represent the constant

$$m = \frac{\pi^{\frac{N}{2}}}{\frac{N}{2}\Gamma\left(\frac{N}{2}\right)},$$

with Γ denoting the Gamma function.

Theorem 3.1. *Assume that $p^- > \beta^+(r+1)$ and that there exist $d, \delta > 0$, such that*

$$\frac{a_1}{p^+}\left(\frac{2\delta}{R}\right)^{\check{p}} \|\omega\|_{L^1(\mathfrak{B})} \geq d,$$

and

$$A_\delta := \frac{\frac{a_2}{p^-}\left(\frac{2\delta}{R}\right)^{\hat{p}} \|\omega\|_{L^1(\mathfrak{B})} + \left(\frac{2\delta}{R}\right)^{q} \frac{\|b\|_\infty}{qH} m\left(R^N - \left(\frac{R}{2}\right)^N\right)}{\frac{M_1^{r+1}}{(r+1)(\alpha^+)^{r+1}}\left((\delta)^{\check{\alpha}} m(\frac{R}{2})^N\right)^{r+1}} < B_d := \frac{d}{\frac{(M_2)^{r+1}\left(c_\beta^{\hat{\beta}}\right)^{(r+1)}}{(r+1)(\beta^-)^{r+1}}\left(\left[\left(\frac{p^+}{a_1}d\right)^{\frac{1}{\hat{p}}}\right]^{\hat{\beta}}\right)^{r+1}},$$

then for any $\lambda \in]A_\delta, B_d[$, problem (1.1) admits at least three weak solutions.

Proof. It is worth noting that Φ and Ψ satisfy the regularity assumptions outlined in Theorem 2.10. We will now establish the fulfillment of conditions (a_1) and (a_2). To this end, let's consider

$$\frac{a_1}{p^+}\left(\frac{2\delta}{R}\right)^{\check{p}} \|\omega\|_{L^1(\mathfrak{B})} \geq d$$

and consider $v_\delta \in X$ such that

$$v_\delta(x) := \begin{cases} 0 & x \in \Omega \backslash B\left(x^0, R\right) \\ \frac{2\delta}{R}\left(R - \left|x - x^0\right|\right) & x \in \mathfrak{B} := \overline{B}\left(x^0, R\right) \backslash B\left(x^0, \frac{R}{2}\right), \\ \delta & x \in \overline{B}\left(x^0, \frac{R}{2}\right). \end{cases}$$

Then, by the definition of Φ , we have

$$\begin{aligned} \frac{a_1}{p^+}\left(\frac{2\delta}{R}\right)^{\check{p}} \|\omega\|_{L^1(\mathfrak{B})} &< \Phi(v_\delta) \\ &\leq \frac{a_2}{p^-}\left(\frac{2\delta}{R}\right)^{\hat{p}} \|\omega\|_{L^1(\mathfrak{B})} + \left(\frac{2\delta}{R}\right)^{q} \frac{\|b\|_\infty}{qH} m\left(R^N - \left(\frac{R}{2}\right)^N\right) \end{aligned}$$

Therefore, $\Phi(v_\delta) > d$. However, it is important to consider the following

$$\begin{aligned} \Psi(v_\delta) \geq \frac{1}{r+1}\left(\int_\Omega F(x, v_\delta)dx\right)^{r+1} &\geq \frac{M_1^{r+1}}{(r+1)(\alpha^+)^{r+1}}\left(\int_{B(x^0, \frac{R}{2})} |\delta|^{\alpha(x)}dx\right)^{r+1}, \\ &\geq \frac{M_1^{r+1}}{(r+1)(\alpha^+)^{r+1}}\left((\delta)^{\check{\alpha}} m\left(\frac{R}{2}\right)^N\right)^{r+1}, \end{aligned} \tag{3.1}$$

which yields to $\frac{\Phi(v_\delta)}{\Psi(v_\delta)} \leq A_\delta < \lambda$. In addition, for every $u \in \Phi^{-1}(]-\infty, d])$, one has the following

$$\frac{a_1}{p^+}\|u\|^{\check{p}} \leq d. \tag{3.2}$$

therefore,

$$\|u\| \leq \left(\frac{p^+}{a_1}\Phi(u)\right)^{\frac{1}{\check{p}}} < \left(\frac{p^+}{a_1}d\right)^{\frac{1}{\check{p}}}.$$

Furthermore, under the assumption (f_1), we can conclude the following

$$\begin{aligned}
\Psi(u) &\leq \frac{1}{r+1}\Big(\int_\Omega F(x,u(x))dx\Big)^{r+1}, \\
&\leq \frac{(M_2)^{r+1}}{(r+1)(\beta^-)^{r+1}}\Big(\int_\Omega |u|^{\beta(x)}dx\Big)^{r+1}, \\
&\leq \frac{(M_2)^{r+1}}{(r+1)(\beta^-)^{r+1}}\big(\|u\|_\beta^{\hat{\beta}}\big)^{r+1}, \\
&\leq \frac{(M_2)^{r+1}\big(c_\beta^{\hat{\beta}}\big)^{(r+1)}}{(r+1)(\beta^-)^{r+1}}\big(\|u\|^{\hat{\beta}}\big)^{r+1}.
\end{aligned} \tag{3.3}$$

This leads to the following result

$$\sup_{\Phi(u)<d}\Psi(u) \leq \frac{(M_2)^{r+1}\big(c_\beta^{\hat{\beta}}\big)^{(r+1)}}{(r+1)(\beta^-)^{r+1}}\left(\left[\left(\frac{p^+}{a_1}d\right)^{\frac{1}{\check{p}}}\right]^{\hat{\beta}}\right)^{r+1},$$

and

$$\frac{1}{d}\sup_{\Phi(u)<d}\Psi(u) < \frac{1}{\lambda}.$$

Furthermore, we can establish the coerciveness of $\mathcal{I}_\lambda$ for any positive value of λ by employing inequality (3.3) once more. This yields the following result

$$\Psi(u) \leq \frac{(M_2)^{r+1}\big(c_\beta^{\hat{\beta}}\big)^{(r+1)}}{(r+1)(\beta^-)^{r+1}}\big(\|u\|^{\hat{\beta}}\big)^{r+1}.$$

When $\|u\|$ is great enough, the following can be inferred

$$\Phi(u) - \lambda\Psi(u) \geq \frac{a_1}{p^+}\|u\|^{p^-} - \lambda\frac{(M_2)^{r+1}\big(c_\beta^{\hat{\beta}}\big)^{(r+1)}}{(r+1)(\beta^-)^{r+1}}\big(\|u\|^{\hat{\beta}}\big)^{r+1}.$$

By considering the fact that $p^- > \beta^+(r+1)$, we can reach the desired conclusion. In conclusion, considering the aforementioned fact that

$$\bar{\Lambda}_{d,\delta} := (A_\delta, B_d) \subseteq \left(\frac{\Phi(v_\delta)}{\Psi(v_\delta)}, \frac{d}{\sup_{\Phi(u)<d}\Psi(u)}\right),$$

Based on Theorem 2.10, it can be deduced that for any $\lambda \in \bar{\Lambda}_{d,\delta}$, the function $\Phi - \lambda\Psi$ possesses at least three critical points in $X := W_0^{1,p}(\omega,\Omega)$. These critical points correspond to weak solutions of problem (1.1). □

Remark 3.2. Setting $d = \frac{a_1}{p^+}$, then B_d becomes

$$B_d = \frac{\frac{a_1}{p^+}}{\frac{(M_2)^{r+1}\left(c_\beta^{\hat{\beta}}\right)^{(r+1)}}{(r+1)(\beta^-)^{r+1}}},$$

moreover, one has

$$p^+ \geq p^- > \beta^+(r+1) \geq \alpha^+(r+1) \quad \text{and} \quad q < p^+,$$

thus $\lim_{\delta\to 0^+} A_\delta = 0$, consequently, $]A_\delta, B_d[\neq \emptyset$ and Theorem 3.1, can be rewritten as follows: Assume that $p^- > \beta^+(r+1)$ and that there exist $\delta > 0$, such that

$$\delta \geq \frac{R}{2}\left(\frac{1}{\|\omega\|_{L^1(\mathfrak{B})}}\right)^{\frac{1}{\hat{p}}}$$

and

$$A_\delta := \frac{\frac{a_2}{p^-}\left(\frac{2\delta}{R}\right)^{\hat{p}}\|\omega\|_{L^1(\mathfrak{B})} + \left(\frac{2\delta}{R}\right)^{q}\frac{\|b\|_\infty}{qH}m\left(R^N - \left(\frac{R}{2}\right)^N\right)}{\frac{M_1^{r+1}}{(r+1)(\alpha^+)^{r+1}}\left((\delta)^{\check{\alpha}}m(\frac{R}{2})^N\right)^{r+1}} < B_d := \frac{\frac{a_1}{p^+}}{\frac{(M_2)^{r+1}\left(c_\beta^{\hat{\beta}}\right)^{(r+1)}}{(r+1)(\beta^-)^{r+1}}},$$

then for any $\lambda \in]A_\delta, B_d[$, problem (1.1) admits at least three weak solutions.

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Periodic solutions in a linear delay difference system

Dedicated to the memory of Professor István Győri (1943–2022)

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Abstract. The paper investigates periodicity properties of a linear autonomous difference system with two delayed terms. Assuming that the system matrices are simultaneously triangularizable, we formulate necessary and sufficient conditions guaranteeing the existence of a nonzero periodic solution (with an *a priori* given period) of the studied system. The analytical form of such conditions is shown to generalize the existing results on this topic. Moreover, it is supported by a geometric reformulation, offering a better understanding of the derived periodicity conditions. Information on the form of the searched periodic solution (including its prime period) is also provided.

Keywords: difference equation, delay, periodic solution.

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1 Introduction and preliminaries

The study of periodic solutions in differential and difference systems belongs among frequent research topics. It is well known that the appearance of periodic solutions forms an important part of the bifurcation analysis of dynamical systems that is often accompanied by other interesting phenomena (such as the loss of asymptotic stability of the corresponding equilibria). There are various methods for detecting a periodic behavior in the studied systems, often based on fixed-point theorems. Since the procedures utilizing this approach do not usually give analytical expressions of the searched periodic solutions, other methods were found and applied to obtain the desired expressions (see, e.g., the Carvalho method, initially developed for delay differential equations but applicable also to discrete equations [6,7,14]). We refer to [2,8–12,17–19,21,22] for a survey of some results and techniques used in the existence theory of periodic solutions.

In the autonomous linear case, the problem of the existence of periodic solutions is related to the existence of a specific characteristic root. In particular, the delay differential system

$$y'(t) + \sum_{s=1}^{k} A_s y(t-s) = 0, \qquad t \in \mathbb{R}^+,$$

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where $A_s \in \mathbb{R}^{d\times d}$, has a q-periodic solution $y : \mathbb{R}^+ \to \mathbb{R}^d$ (i.e., a solution satisfying $y(t+q) = y(t)$ for all $t \in \mathbb{R}^+$ and some $q \in \mathbb{R}^+$) if and only if the corresponding characteristic quasi-polynomial

$$F(\lambda) = \det\left(\lambda I + \sum_{s=1}^{k} \exp(-\lambda s) A_s\right)$$

(I being the identity matrix) has a pure imaginary root. Similarly, the difference system

$$x(n) + \sum_{s=1}^{k} A_s x(n-s) = 0, \qquad n \in \mathbb{Z}^+, \tag{1.1}$$

$A_s \in \mathbb{R}^{d\times d}$, admits a q-periodic solution $x : \mathbb{Z}^+ \to \mathbb{R}^d$ (i.e., a solution satisfying $x(n+q) = x(n)$ for all $n \in \mathbb{Z}^+$ and some $q \in \mathbb{Z}^+$) if and only if the family of roots of

$$P(\lambda) = \det\left(\lambda^k I + \sum_{s=1}^{k} \lambda^{k-s} A_s\right)$$

involves a q-th root of unity. That is, there exists a root $\lambda_{j,q}$ of P in the form

$$\lambda_{j,q} = \exp\left(\mathrm{i}\frac{2\pi j}{q}\right) \quad \text{for some } j = \left\lfloor -\frac{q}{2} \right\rfloor + 1, \left\lfloor -\frac{q}{2} \right\rfloor + 2, \ldots, \left\lfloor \frac{q}{2} \right\rfloor. \tag{1.2}$$

However, converting these theoretical conditions into efficient forms is usually regarded as questionable, therefore other approaches are preferred.

The aim of this paper is to pose arguments supporting a deeper analysis of the above stated polynomial properties when detecting a periodic behavior of the studied systems. To do this, we consider a particular case of the difference system (1.1) and apply a simple method converting the above polynomial condition into a more efficient form. In the sequel, we precise what is meant by this particular case.

If $k = 1$ in (1.1), then the roots of P are eigenvalues of the only system matrix, namely $-A_1$. Thus, (1.1) with $k = 1$ admits a periodic solution just when $-A_1$ (hence also A_1) has an eigenvalue with a unitary modulus and a rational argument. We can take this classical result as a pattern for our next investigations, and search for periodicity conditions in terms of eigenvalues of the system matrices.

Further, we consider (1.1) with two nonzero system matrices, i.e., the system

$$x(n) + Ax(n-m) + Bx(n-k) = 0 \tag{1.3}$$

with $A, B \in \mathbb{R}^{d\times d}$ and coprime $k, m \in \mathbb{Z}^+$, $k > m$ (note that if $\gcd(k,m) > 1$, then (1.3) can be easily converted to the case with coprime delays). Now, the characteristic polynomial P becomes

$$P^*(\lambda) = \det(\lambda^k I + \lambda^{k-m} A + B).$$

Contrary to the case $k = 1$, the formulation of efficient conditions on eigenvalues of A, B, ensuring that P^* has a root (1.2), seems to be extremely complicated (and perhaps lying beyond theoretical possibilities). Therefore, other methods have been developed to detect a periodic behavior in (1.3). In particular, the paper [10] successfully employed the method of circulant matrices and, for specific choices $A = -I$ and $m = 1$, extended some earlier results (see, e.g., [15,16]) to obtain complete periodicity conditions for the studied system. We recall Theorem 4.1 of [10] as the most relevant result to the topic of our paper.

Theorem 1.1. *The system (1.3) with $A = -I$ and $m = 1$ has a nontrivial periodic solution x if and only if at least one of the following conditions holds:*

(a) *B has a zero eigenvalue or has a real eigenvalue $\beta = b$ such that*

$$b \in \left\{ 2(-1)^{\ell} \sin\left(\frac{(2\ell+1)\pi}{2(2k-1)} \right),\ \ell = 0, 1, \dots, k-1 \right\}; \tag{1.4}$$

(b) *B has a pure imaginary eigenvalue $\beta = \mathrm{i}b$ ($b \in \mathbb{R}$) such that*

$$b \in \left\{ 2(-1)^{\ell} \sin\left(\frac{\ell\pi}{2k-1} \right),\ \ell = 1, \dots, 2k-2 \right\}; \tag{1.5}$$

(c) *B has a complex eigenvalue $\beta = b\exp(\mathrm{i}\phi)$ ($b, \phi \in \mathbb{R}$, $|\phi| < \pi/2$) such that, for some positive integers $\xi < \eta$,*

$$|\phi| = \frac{\xi\pi}{2\eta} \quad \text{and} \quad b \in \left\{ 2(-1)^{\ell} \sin\left(\frac{(\eta-\xi)\pi}{2\eta(2k-1)} + \frac{\ell\pi}{2k-1} \right),\ \ell = 0, 1, \dots, 2k-2 \right\}. \tag{1.6}$$

Following the conclusions of Theorem 1.1, we return to the characteristic polynomial approach. In particular, we show that Theorem 1.1 can be obtained from the root analysis of P^* as well, and moreover, this analysis appears to be applicable also in some more general situations. In fact, it utilizes a straightforward procedure how to simplify the form of P^* under a specific restriction imposed on A, B, namely their simultaneous triangularizability. This property assumes the existence of a regular matrix T such that both $T^{-1}AT$ and $T^{-1}BT$ are upper triangular. The existence of such a matrix T is guaranteed whenever A, B are commuting (for some less restrictive conditions on A, B, see, e.g., [3]). Notice that the matrices $A = -I$ and B considered in Theorem 1.1 obviously share this property. Further, if A, B are simultaneously triangularizable, then, after a few straightforward computational steps, one can decompose P^* into a significantly simpler form, namely

$$P^{**}(\lambda) = \prod_{i=1}^{d} (\lambda^k + \alpha_i \lambda^{k-m} + \beta_i), \tag{1.7}$$

where $(\alpha_i, \beta_i) \in \mathbb{C}^2$, $i = 1, \dots, d$, are simultaneously ordered couples of eigenvalues of A, B (i.e., couples involving mutually corresponding diagonal elements of the matrices $T^{-1}AT$ and $T^{-1}BT$, respectively). Thus, (1.3) with simultaneously triangularizable matrices A, B has a q-periodic solution x if and only if at least one of the trinomials involved in the product (1.7) has a root of the form (1.2).

To summarize the previous considerations, the mathematical core of discussions on the existence of periodic solutions of (1.3) with simultaneously triangularizable A, B consists in analyzing the following problem: *Find conditions under which a complex trinomial*

$$Q(\lambda) = \lambda^k + \alpha\lambda^{k-m} + \beta \tag{1.8}$$

admits a root (1.2).

Based on the previous notes, the paper is organized as follows. In Section 2, we consider (1.3) with simultaneously triangularizable A, B, and formulate a general result on the existence of a q-periodic solution of (1.3), including its form and prime period. Section 3 reveals a geometric background of this result, namely a relationship between the derived periodicity conditions and analytical description of some roulette curves. Two particular cases of (1.3)

are investigated in Section 4. Here, we provide a generalization of Theorem 1.1 to (1.3) with $A = -I$ and a general m, and derive fully explicit conditions for detecting its periodic solution. An application of the presented results in control theory is stated as well. The final section, involving survey remarks and possible perspectives of the research, concludes the paper.

2 Basic analytical conditions for the existence of a periodic solution of (1.3)

First, we consider the characteristic polynomial P that is associated with the general linear system (1.1). A simple but efficient method for analyzing the distribution of characteristic roots with respect to the unit circle is the D-partition method (in the discrete case, it is also referred to as the boundary locus technique). This method is standardly employed in stability analysis of (1.1) but can also be applied to reveal periodic behavior.

Indeed, assume that P has a unimodular root $\lambda = \exp(\mathrm{i}\phi)$, $-\pi < \phi \leq \pi$. A direct substitution into $P(\lambda) = 0$ yields

$$\det\left(\exp(\mathrm{i}k\phi)I + \sum_{s=1}^{k} \exp(\mathrm{i}(k-s)\phi)A_s\right) = 0. \tag{2.1}$$

In particular, if $\lambda = \lambda_{j,q}$ is given by (1.2), i.e., if we put $\phi = 2\pi j/q$, then (2.1) becomes

$$\det\left(\exp\left(\mathrm{i}\frac{2\pi jk}{q}\right)I + \sum_{s=1}^{k} \exp\left(\mathrm{i}\frac{2\pi j(k-s)}{q}\right)A_s\right) = 0,$$

hence,

$$\det\left(I + \sum_{s=1}^{k} \exp\left(-\mathrm{i}\frac{2\pi js}{q}\right)A_s\right) = 0. \tag{2.2}$$

Thus, (1.1) admits a q-periodic solution if and only if there exists an integer j (specified in (1.2)) such that (2.2) holds. Notice that the same conclusion was formulated in Theorem 3.1 of [10] by the use of tools of circulant matrices (contrary to Theorem 3.1 of [10], now we do not need to assume pairwise commutativity of the system matrices).

Of course, (2.2) is still rather a theoretical condition. The following assertion converts (2.2) into an efficient form if we consider (1.3) instead of (1.1), and assume that A, B are simultaneously triangularizable. In what follows, Arg denotes the principal argument of a complex number (i.e., $-\pi < \mathrm{Arg}(z) \leq \pi$ for $z \in \mathbb{C}$), and we introduce a function $(\cdot)_{2\pi} : \mathbb{R} \to (-\pi, \pi]$ as the 2π-periodic extension of the identity function defined on $(-\pi, \pi]$ (i.e., $(\cdot)_{2\pi}$ provides modulo 2π operation).

Lemma 2.1. *Let $A, B \in \mathbb{R}^{d\times d}$ be simultaneously triangularizable matrices, $k, m \in \mathbb{Z}^+$, $k > m$, be coprime and let $q \in \mathbb{Z}^+$. Then (1.3) has a nontrivial q-periodic solution if and only if there exists a couple of simultaneously ordered eigenvalues (α, β) of A, B satisfying, for some integer j such that $\lfloor -q/2 \rfloor + 1 \leq j \leq \lfloor q/2 \rfloor$, any of the following conditions:*

(i) $\alpha = 0, \quad |\beta| = 1, \quad \mathrm{Arg}(\beta) = \left(\pi + \dfrac{2\pi jk}{q}\right)_{2\pi}$;

(ii) $\beta = 0, \quad |\alpha| = 1, \quad \mathrm{Arg}(\alpha) = \left(\pi + \dfrac{2\pi jm}{q}\right)_{2\pi}$;

(iii) $\alpha\beta \neq 0$, $\big||\alpha| - |\beta|\big| \leq 1 \leq |\alpha| + |\beta|$, *and*

$$\begin{aligned} \cos\left(\operatorname{Arg}(\alpha) - \frac{2\pi jm}{q}\right) &= \frac{|\beta|^2 - |\alpha|^2 - 1}{2|\alpha|}, \\ \cos\left(\operatorname{Arg}(\beta) - \frac{2\pi jk}{q}\right) &= \frac{|\alpha|^2 - |\beta|^2 - 1}{2|\beta|}, \end{aligned} \tag{2.3}$$

where $(\operatorname{Arg}(\alpha) - 2\pi jm/q)_{2\pi}$, $(\operatorname{Arg}(\beta) - 2\pi jk/q)_{2\pi}$ *have opposite signs if both are nonzero.*

Proof. Because of the reduction of P into P^{**}, we need to describe the situation when the complex trinomial Q (with appropriate eigenvalues α and β of A and B, respectively) has a unimodular root (1.2). First note that Q has a unimodular root $\lambda = \exp(\mathrm{i}\phi)$, $-\pi < \phi \leq \pi$, just when

$$\exp(\mathrm{i}k\phi) + \alpha \exp(\mathrm{i}(k-m)\phi) + \beta = 0 \tag{2.4}$$

(see also (2.1)). Since λ should be a q-th root of unity, we put again $\phi = 2\pi j/q$, and (2.4) becomes

$$\exp\left(\mathrm{i}\frac{2\pi jk}{q}\right) + \alpha \exp\left(\mathrm{i}\frac{2\pi j(k-m)}{q}\right) + \beta = 0. \tag{2.5}$$

This immediately implies the lemma's conditions (i) and (ii).

Now assume $\alpha\beta \neq 0$. Then (2.5) can be rewritten as

$$1 + |\alpha| \exp\left(\mathrm{i}\left(\operatorname{Arg}(\alpha) - \frac{2\pi jm}{q}\right)\right) = -|\beta| \exp\left(\mathrm{i}\left(\operatorname{Arg}(\beta) - \frac{2\pi jk}{q}\right)\right)$$

which is (by comparing the real and imaginary parts) equivalent to the system

$$\begin{aligned} 1 + |\alpha| \cos\left(\operatorname{Arg}(\alpha) - \frac{2\pi jm}{q}\right) &= -|\beta| \cos\left(\operatorname{Arg}(\beta) - \frac{2\pi jk}{q}\right), \\ |\alpha| \sin\left(\operatorname{Arg}(\alpha) - \frac{2\pi jm}{q}\right) &= -|\beta| \sin\left(\operatorname{Arg}(\beta) - \frac{2\pi jk}{q}\right). \end{aligned} \tag{2.6}$$

Summing the squares of both the lines of (2.6) yields

$$1 + 2|\alpha| \cos\left(\operatorname{Arg}(\alpha) - \frac{2\pi jm}{q}\right) + |\alpha|^2 = |\beta|^2,$$

which implies $(2.3)_1$.

Now, substituting $(2.3)_1$ to the left-hand side of $(2.6)_1$, we obtain

$$1 + |\alpha| \frac{|\beta|^2 - |\alpha|^2 - 1}{2|\alpha|} = -|\beta| \cos\left(\operatorname{Arg}(\beta) - \frac{2\pi jk}{q}\right),$$

which, after a straightforward rearrangement, becomes $(2.3)_2$. Moreover, the couple (α, β) has to also satisfy the stated sign correspondence of the cosine arguments because $(2.6)_2$ implies that $(\operatorname{Arg}(\alpha) - 2\pi jm/q)_{2\pi}$ is in $(0, \pi)$ (that is positive) just when $(\operatorname{Arg}(\beta) - 2\pi jk/q)_{2\pi}$ is in $(-\pi, 0)$ (that is negative). □

Remark 2.2. If we fix $\alpha \neq 0$, then (2.3) allows to express β explicitly (in terms of its modulus and argument). Indeed, taking into account the sign relation between $(\mathrm{Arg}(\alpha) - 2\pi jm/q)_{2\pi}$ and $(\mathrm{Arg}(\beta) - 2\pi jk/q)_{2\pi}$, we get

$$\begin{aligned} |\beta| &= \sqrt{1 + |\alpha|^2 + 2|\alpha| \cos\left(\mathrm{Arg}(\alpha) - \frac{2\pi jm}{q}\right)}, \\ \mathrm{Arg}(\beta) &= \left(\kappa \arccos\left(\frac{|\alpha|^2 - |\beta|^2 - 1}{2|\beta|}\right) + \frac{2\pi jk}{q}\right)_{2\pi}, \end{aligned} \tag{2.7}$$

where $\kappa = -1$ if $(\mathrm{Arg}(\alpha) - 2\pi jm/q)_{2\pi}$ is in $(0, \pi]$, and $\kappa = 1$ if $(\mathrm{Arg}(\alpha) - 2\pi jm/q)_{2\pi}$ is in $(-\pi, 0]$. It means that, depending on the integer j, we have (for a given α) at most q possible positions of β in the complex plane such that a q-th root of unity belongs among the roots of Q.

Analogous explicit expressions can be obtained for α when $\beta \neq 0$ is fixed. More precisely, $(2.3)_2$, $(2.3)_1$ imply

$$\begin{aligned} |\alpha| &= \sqrt{1 + |\beta|^2 + 2|\beta| \cos\left(\mathrm{Arg}(\beta) - \frac{2\pi jk}{q}\right)}, \\ \mathrm{Arg}(\alpha) &= \left(\kappa \arccos\left(\frac{|\beta|^2 - |\alpha|^2 - 1}{2|\alpha|}\right) + \frac{2\pi jm}{q}\right)_{2\pi}, \end{aligned} \tag{2.8}$$

where $\kappa = -1$ if $(\mathrm{Arg}(\beta) - 2\pi jk/q)_{2\pi}$ belongs to $(0, \pi]$, and $\kappa = 1$ if $(\mathrm{Arg}(\beta) - 2\pi jk/q)_{2\pi}$ belongs to $(-\pi, 0]$. We add that a nice geometric interpretation of the conditions (2.7) and (2.8) will be discussed in the next section.

To complete the above basic periodic investigations of (1.3), we reveal the structure of the discussed q-periodic solution. As a first step, we state the following root property of P^{**}.

Proposition 2.3. *If $\lambda_{j,q} = \exp(\mathrm{i}2\pi j/q)$ (see (1.2)) is a root of P^{**}, then the complex conjugate number $\overline{\lambda}_{j,q} = \exp(-\mathrm{i}2\pi j/q)$ is also a root of P^{**}. Moreover, if v is a characteristic vector corresponding to $\lambda_{j,q}$, i.e., v solves the system $\big((\lambda_{j,q})^k I + (\lambda_{j,q})^{k-m} A + B\big)v = 0$, then its complex conjugate $\overline{v}$ is a characteristic vector corresponding to $\overline{\lambda}_{j,q}$, i.e., $\overline{v}$ satisfies $\big((\overline{\lambda}_{j,q})^k I + (\overline{\lambda}_{j,q})^{k-m} A + B\big)\overline{v} = 0$.*

Proof. Let $\lambda_{j,q}$ be a root of P^{**}. Since P^{**} and P^* are identically equal, and P^* is a real polynomial, we immediately get that $\overline{\lambda}_{j,q}$ is a root of P^{**} as well.

The complex conjugation of the corresponding characteristic vectors can be shown analogously as in the case of standard eigenvectors. Indeed, let v be a characteristic vector corresponding to $\lambda_{j,q}$, i.e., v satisfies

$$\big((\lambda_{j,q})^k I + (\lambda_{j,q})^{k-m} A + B\big)v = 0.$$

Then, taking the conjugates of both sides, we have

$$\overline{\big((\lambda_{j,q})^k I + (\lambda_{j,q})^{k-m} A + B\big)v} = \overline{0} \quad \Longrightarrow \quad \big((\overline{\lambda}_{j,q})^k I + (\overline{\lambda}_{j,q})^{k-m} A + B\big)\overline{v} = 0.$$

Therefore, $\overline{v}$ is a characteristic vector corresponding to $\overline{\lambda}_{j,q}$. □

Now, using Proposition 2.3, we can easily describe the family of all q-periodic solutions of (1.3) generated by $\lambda_{j,q}$, v, and their conjugates. Obviously, (1.3) has two such (linearly independent) solutions in the form

$$x_1^*(n) = v \exp\left(\mathrm{i}\frac{2\pi jn}{q}\right) \quad \text{and} \quad x_2^*(n) = \overline{v} \exp\left(-\mathrm{i}\frac{2\pi jn}{q}\right).$$

To avoid solutions with complex values, we put

$$x_1(n) = \frac{1}{2}\big(x_1^*(n) + x_2^*(n)\big) \quad \text{and} \quad x_2(n) = \frac{1}{\mathrm{i}2}\big(x_1^*(n) - x_2^*(n)\big)$$

to get a new pair of (linearly independent) real solutions

$$x_1(n) = v_1 \cos\left(\frac{2\pi jn}{q}\right) - v_2 \sin\left(\frac{2\pi jn}{q}\right), \qquad x_2(n) = v_2 \cos\left(\frac{2\pi jn}{q}\right) + v_1 \sin\left(\frac{2\pi jn}{q}\right), \tag{2.9}$$

where $v_1 = \Re(v)$, $v_2 = \Im(v)$. Then, any q-periodic solution x of (1.3) generated by $\lambda_{j,q}$, v, and their conjugates is given by a linear combination $x(n) = C_1x_1(n) + C_2x_2(n)$, $C_1, C_2 \in \mathbb{R}$. It is clear that the corresponding initial vectors

$$x^0(-k+1), x^0(-k+2), \dots, x^0(0)$$

have to match the form (2.9), i.e., we put

$$x_1^0(n) = v_1 \cos\left(\frac{2\pi jn}{q}\right) - v_2 \sin\left(\frac{2\pi jn}{q}\right), \qquad x_2^0(n) = v_2 \cos\left(\frac{2\pi jn}{q}\right) + v_1 \sin\left(\frac{2\pi jn}{q}\right) \tag{2.10}$$

for $n = -k+1, -k+2, \dots, 0$.

Moreover, we can easily decide whether the prescribed period q is also a prime period of the solutions (2.9). Obviously, x_1, x_2 are p-periodic ($p \in \mathbb{Z}^+$) whenever pj/q is an integer. Therefore, such a number p is a prime period if $p = q/\gcd(j,q)$. It means that if j and q are coprime, then q is a prime period.

3 Some geometric observations

In this section, we clarify a geometric background of the conditions (2.7) and (2.8). Doing this, we come back to the condition (2.4) characterizing the set of all complex α, β such that Q has a unimodular root $\lambda = \exp(\mathrm{i}\phi)$, $-\pi < \phi \le \pi$. As observed in [20], the geometric nature of such characterizations can be described in terms of some roulette curves. In the sequel, we elaborate on these issues in detail.

If we fix $\alpha \neq 0$ (as in Remark 2.2), the set of all β satisfying (2.4) is alternatively given by

$$\beta = |\alpha| \exp\big(\mathrm{i}(\pi + \operatorname{Arg}(\alpha) + (k-m)\phi)\big) - \exp(\mathrm{i}k\phi). \tag{3.1}$$

Set $s = \pi + \operatorname{Arg}(\alpha) + (k-m)\phi$ to obtain

$$\beta = |\alpha| \exp(\mathrm{i}s) - \exp\left(\mathrm{i}\frac{k}{k-m}\Big(s - (\pi + \operatorname{Arg}(\alpha))\Big)\right). \tag{3.2}$$

Further, the substitution $t = s - (\pi + \operatorname{Arg}(\alpha))k/m$ converts (3.2) to the form

$$\begin{aligned} \beta &= |\alpha| \exp\left(\mathrm{i}t + \mathrm{i}\frac{k}{m}\Big(\pi + \operatorname{Arg}(\alpha)\Big)\right) - \exp\left(\mathrm{i}\frac{k}{k-m}t + \mathrm{i}\frac{k}{m}\Big(\pi + \operatorname{Arg}(\alpha)\Big)\right) \\ &= \exp\left(\mathrm{i}\frac{k}{m}\Big(\pi + \operatorname{Arg}(\alpha)\Big)\right)\left(|\alpha| \exp(\mathrm{i}t) - \exp\left(\mathrm{i}\frac{k}{k-m}t\right)\right). \end{aligned} \tag{3.3}$$

Finally, we introduce the parameters $R = |\alpha|m/k$, $r = |\alpha|(k-m)/k$, $\delta = 1$ to rewrite (3.3) as

$$\beta = \exp\left(\mathrm{i}\frac{k}{m}\Big(\pi + \operatorname{Arg}(\alpha)\Big)\right)\left((R+r)\exp(\mathrm{i}t) - \delta \exp\left(\mathrm{i}\frac{R+r}{r}t\right)\right). \tag{3.4}$$

The second term in (3.4) represents the complex canonical parametrization of an epitrochoid, which is a roulette traced by a point attached to a circle of radius r, having distance δ from its center and rolling around the outside of a fixed circle of radius R. In addition, the appearance of the first term in (3.4) implies that the epitrochoid is rotated by the angle of rotation $\omega = \left(\frac{k}{m}(\pi + \operatorname{Arg}(\alpha))\right)_{2\pi}$. Thus, (3.1) is converted to the parametric equation of a rotated epitrochoid

$$\beta = \exp(\mathrm{i}\omega)\left((R+r)\exp(\mathrm{i}t) - \delta\exp\left(\mathrm{i}\frac{R+r}{r}t\right)\right), \tag{3.5}$$

where $t = s - (\pi + \operatorname{Arg}(\alpha))k/m = (k-m)\phi - \big(\pi + \operatorname{Arg}(\alpha)\big)(k-m)/m$. Note also that an epitrochoid is closed whenever r/R is a rational number. In our case, $r/R = (k-m)/m \in \mathbb{Q}$, and it is not difficult to check that the parametrization (3.5) has the prime period $2\pi(k-m)$.

To summarize, the complex number β satisfies (3.1) just when β is located on the (rotated) epitrochoid (3.5) with

$$R = |\alpha|\frac{m}{k}, \quad r = |\alpha|\frac{k-m}{k}, \quad \delta = 1, \quad \omega = \left(\frac{k}{m}\big(\pi + \operatorname{Arg}(\alpha)\big)\right)_{2\pi}. \tag{3.6}$$

In particular, Q has a unimodular root (1.2) just when the position of β on this epitrochoid is specified via

$$t = (k-m)\frac{2\pi j}{q} - \big(\pi + \operatorname{Arg}(\alpha)\big)\frac{k-m}{m} \quad \text{for some } j = \left\lfloor -\frac{q}{2}\right\rfloor + 1, \left\lfloor -\frac{q}{2}\right\rfloor + 2, \ldots, \left\lfloor \frac{q}{2}\right\rfloor. \tag{3.7}$$

Similarly, if we fix $\beta \neq 0$, then the set of all α satisfying (2.4) is given by

$$\alpha = |\beta|\exp\big(\mathrm{i}(\pi + \operatorname{Arg}(\beta) + (m-k)\phi)\big) - \exp(\mathrm{i}m\phi), \qquad -\pi < \phi \leq \pi.$$

Putting $s = \pi + \operatorname{Arg}(\beta) + (m-k)\phi$, we obtain

$$\alpha = |\beta|\exp(\mathrm{i}s) - \exp\left(\mathrm{i}\frac{m}{m-k}\Big(s - \big(\pi + \operatorname{Arg}(\beta)\big)\Big)\right). \tag{3.8}$$

Now, let

$$t = \frac{m}{m-k}\left(s - \Big(\pi + \operatorname{Arg}(\beta)\Big)\right) - \frac{m}{k}\operatorname{Arg}(\beta).$$

Then, (3.8) becomes

$$\begin{aligned}\alpha &= |\beta|\exp\left(\mathrm{i}\frac{m-k}{k}t + \mathrm{i}\Big(\pi + \frac{m}{k}\operatorname{Arg}(\beta)\Big)\right) - \exp\left(\mathrm{i}t + \mathrm{i}\frac{m}{k}\operatorname{Arg}(\beta)\right)\\ &= \exp\left(\mathrm{i}\Big(\pi + \frac{m}{k}\operatorname{Arg}(\beta)\Big)\right)\left(\exp(\mathrm{i}t) + |\beta|\exp\left(\mathrm{i}\frac{m-k}{m}t\right)\right).\end{aligned}$$

If we introduce $R = k/(k-m)$, $r = m/(k-m)$, $\delta = |\beta|$, and $\omega = (\pi + \operatorname{Arg}(\beta)m/k)_{2\pi}$, then

$$\alpha = \exp(\mathrm{i}\omega)\left((R-r)\exp(\mathrm{i}t) + \delta\exp\left(\mathrm{i}\frac{r-R}{r}t\right)\right), \tag{3.9}$$

where $t = \big(s - (\pi + \operatorname{Arg}(\beta))\big)m/(m-k) - \operatorname{Arg}(\beta)m/k = m\phi - \operatorname{Arg}(\beta)m/k$. Notice that the second term in (3.9) represents another roulette curve – hypotrochoid, which is a roulette traced by a point attached to a circle of radius r, having distance δ from its center, and rolling around the inside of a fixed circle of radius R ($R > r$). The first term in (3.9) again provides

a rotation of the curve by the angle ω. Since the ratio $r/R = m/k$ is a rational number, the parametrization (3.9) is periodic and its prime period is $2\pi m$ (i.e., (3.9) represents a closed curve).

Overall, analogously as above, one can conclude that Q has a unimodular root (1.2) just when α is located on the hypotrochoid (3.9) with

$$R = \frac{k}{k-m}, \quad r = \frac{m}{k-m}, \quad \delta = |\beta|, \quad \omega = \left(\pi + \frac{m}{k}\operatorname{Arg}(\beta)\right)_{2\pi}, \tag{3.10}$$

where

$$t = m\frac{2\pi j}{q} - \frac{m}{k}\operatorname{Arg}(\beta) \quad \text{for some } j = \left\lfloor -\frac{q}{2}\right\rfloor + 1, \left\lfloor -\frac{q}{2}\right\rfloor + 2, \ldots, \left\lfloor \frac{q}{2}\right\rfloor. \tag{3.11}$$

Thus, Lemma 2.1 and Remark 2.2, supported by the previous considerations, yield the following analytical and geometric characterization of the existence of a periodic solution of (1.3).

Theorem 3.1. *Let $A, B \in \mathbb{R}^{d\times d}$ be simultaneously triangularizable matrices, $k, m \in \mathbb{Z}^+$, $k > m$, be coprime and let $q \in \mathbb{Z}^+$. Then the following statements are equivalent.*

(i) *The system* (1.3) *has a nontrivial q-periodic solution.*

(ii) *Either* (iia) *or* (iib) *holds, where:*

- (iia) *There exists a zero eigenvalue α of A such that the corresponding eigenvalue β of B (in the sense of simultaneous ordering) is lying on a unit circle centered at the origin and the admissible values of arguments of β are specified via the conditions* (i) *of Lemma* 2.1.
- (iib) *There exists a nonzero eigenvalue α of A such that the corresponding eigenvalue β of B (in the sense of simultaneous ordering) is lying on the rotated epitrochoid* (3.5) *with R, r, δ, ω given by* (3.6)*; moreover, all the admissible values of the parameter t are specified via* (3.7)*. Alternatively, all such positions of β on this epitrochoid are described by* (2.7).

(iii) *Either* (iiia) *or* (iiib) *holds, where:*

- (iiia) *There exists a zero eigenvalue β of B such that the corresponding eigenvalue α of A (in the sense of simultaneous ordering) is lying on a unit circle centered at the origin and the admissible values of arguments of α are specified via the conditions* (ii) *of Lemma* 2.1.
- (iiib) *There exists a nonzero eigenvalue β of B such that the corresponding eigenvalue α of A (in the sense of simultaneous ordering) is lying on the rotated hypotrochoid* (3.9) *with R, r, δ, ω given by* (3.10)*; moreover, all the admissible values of the parameter t are specified via* (3.11)*. Alternatively, all such positions of α on this hypotrochoid are described by* (2.8).

Example 3.2. Consider the (planar) system (1.3) with $k = 5, m = 3$ and A, B given by

$$A = \frac{\sqrt{3}}{3}\begin{pmatrix} 8 & 9 \\ -6 & -7 \end{pmatrix}, \qquad B = \frac{\sqrt{3}}{3}\begin{pmatrix} -1 & -3 \\ 2 & 4 \end{pmatrix}.$$

It is easy to check that A and B are commuting (therefore simultaneously triangularizable), and the pairs of their simultaneously ordered eigenvalues are

$$(\alpha_1, \beta_1) = \left(-\frac{\sqrt{3}}{3}, \frac{2\sqrt{3}}{3}\right), \qquad (\alpha_2, \beta_2) = \left(\frac{2\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right).$$

Considering the first pair (α_1, β_1), we can check that the corresponding trinomial

$$\lambda^5 - \frac{\sqrt{3}}{3}\lambda^2 + \frac{2\sqrt{3}}{3},$$

appearing in the decomposition (1.7), has a unique unimodular root. More precisely, $\beta_1 = 2\sqrt{3}/3$ is located on the epitrochoid (3.5) with $R = \sqrt{3}/5$, $r = 2\sqrt{3}/15$, $\delta = 1$ and $\omega = -2\pi/3$. Moreover, its position corresponds to $t = 4\pi j/q - 4\pi/3$, where $j = 1$ and $q = 12$, see Figure 3.1 (left). Equivalently, $\alpha_1 = -\sqrt{3}/3$ is located on the hypotrochoid (3.9) with $R = 5/2$, $r = 3/2$, $\delta = 2\sqrt{3}/3$, $\omega = \pi$, and its position corresponds to $t = 6\pi j/q$, where $j = 1$ and $q = 12$, see Figure 3.1 (right). Thus, the eigenvalues β_1, α_1 obey (2.7), (2.8) with $j = 1$ and $q = 12$.

Considering the second pair (α_2, β_2), one can see that the corresponding trinomial

$$\lambda^5 + \frac{2\sqrt{3}}{3}\lambda^2 + \frac{\sqrt{3}}{3},$$

appearing in (1.7), has no unimodular root, hence β_2, α_2 do not meet (2.7), (2.8) for any integers j and q.

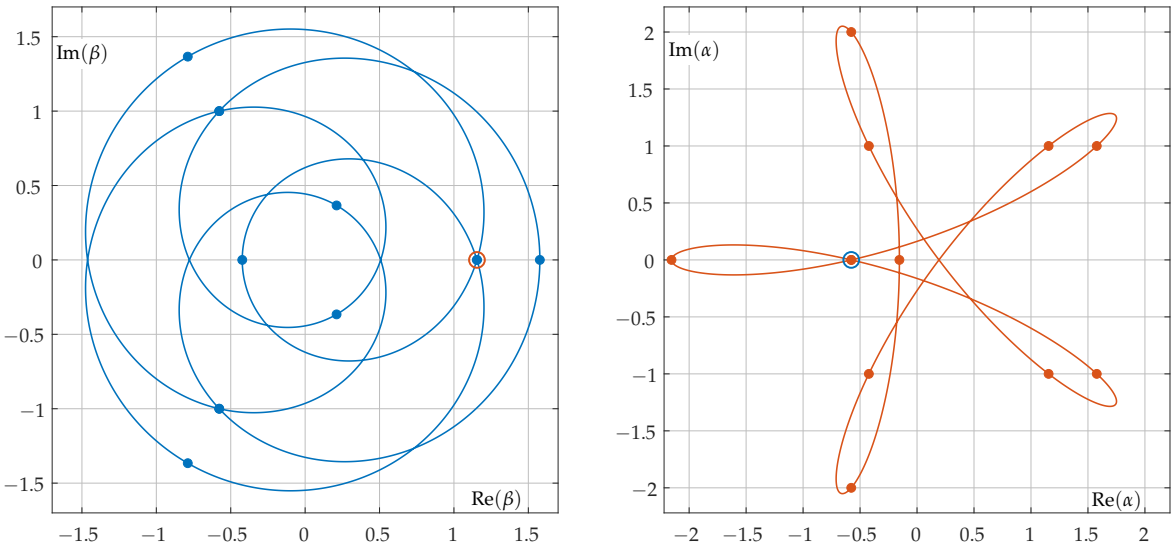


Figure 3.1: An epitrochoid (left) and a hypotrochoid (right) corresponding to the data of Example 3.2. The blue points in the left figure depict all the 12 possible positions of numbers β which can be an eigenvalue of B in a simultaneously ordered pair with α_1 of A (in fact, we have 9 positions only as one triple coincides with another one). Emphasized is the eigenvalue β_1 of B. The orange points in the right figure depict all the 12 positions of numbers α which can be an eigenvalue of A in a simultaneously ordered pair with β_1 of B (in fact, we have 11 positions only as one couple coincides). Emphasized is the eigenvalue α_1 of A.

By Theorem 3.1, the system has just a 12-periodic solution (and 12 is a prime period as $j = 1$ and $q = 12$ are coprime). Moreover, since it holds

$$(\lambda_{1,12})^k I + (\lambda_{1,12})^{k-m} A + B = \begin{pmatrix} \frac{\sqrt{3}}{2} + \mathrm{i}\frac{9}{2} & \frac{\sqrt{3}}{2} + \mathrm{i}\frac{9}{2} \\ -\frac{\sqrt{3}}{3} - \mathrm{i}3 & -\frac{\sqrt{3}}{3} - \mathrm{i}3 \end{pmatrix},$$

where $\lambda_{1,12} = \exp(\mathrm{i}2\pi/12)$, the characteristic vector corresponding to $\lambda_{1,12}$ can be taken simply as $v = (1,-1)^T$ (i.e., $v_1 = \Re(v) = (1,-1)^T$, $v_2 = \Im(v) = (0,0)^T$). Thus, using the

considerations performed at the end of Section 2, all (12-periodic) solutions of the studied system are of the form

$$x(n) = C_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos\left(\frac{\pi n}{6}\right) + C_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \sin\left(\frac{\pi n}{6}\right), \qquad C_1, C_2 \in \mathbb{R}$$

(five initial vectors $x^0(-4), x^0(-3), x^0(-2), x^0(-1), x^0(0)$ have to be prescribed accordingly). No other periodic solution can be detected in this system. Figure 3.2 depicts both components of the periodic solution x with $C_1 = 1$ and $C_2 = 2$.

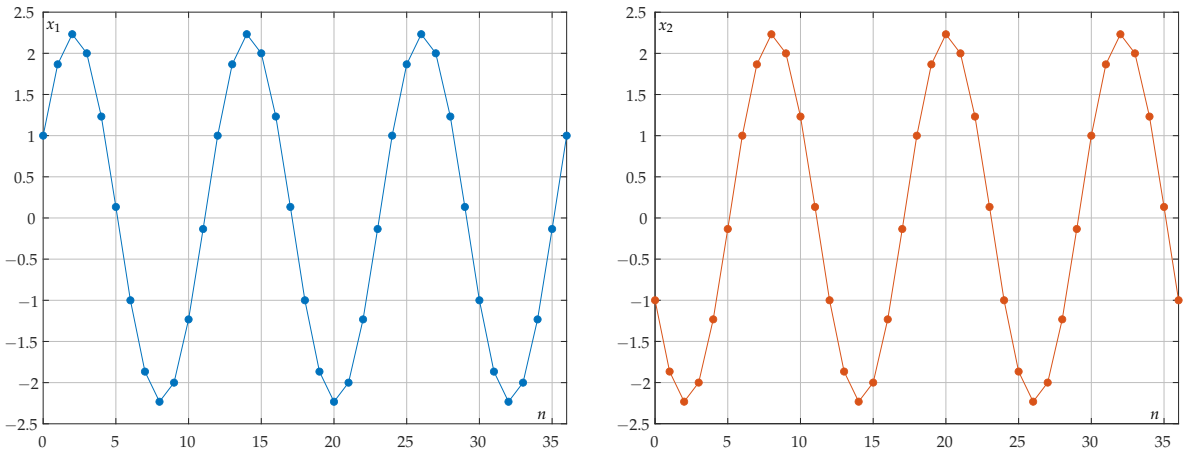


Figure 3.2: The first (left) and the second (right) component of the 12-periodic solution with respect to the data of Example 3.2.

4 Periodicity problem in two particular cases of (1.3)

In this section, we elaborate on the conclusions of Theorem 3.1 under specific choices of A (while B remains free). First, we consider the difference system

$$x(n) - x(n-m) + Bx(n-k) = 0, \tag{4.1}$$

and generalize the conditions (1.4)–(1.6) of Theorem 1.1 for arbitrary coprime integers k, m, $k > m$. Moreover, we give a new insight into a geometric structure of these conditions.

Theorem 4.1. *Let $B \in \mathbb{R}^{d\times d}$, and let $k, m \in \mathbb{Z}^+$, $k > m$, be coprime. Then the following statements are equivalent.*

(i) *The system (4.1) has a nontrivial periodic solution.*

(ii) *There exists an eigenvalue β of B lying on the epitrochoid (3.5), where $R = m/k$, $r = (k-m)/k$, $\delta = 1$, $\omega = (2\pi k/m)_{2\pi}$, and β corresponds to $t = 2\pi(k-m)(jm-q)/(qm)$ for some $q \in \mathbb{Z}^+$ and some $j = \lfloor -q/2 \rfloor + 1, \lfloor -q/2 \rfloor + 2, \ldots, \lfloor q/2 \rfloor$.*

(iii) *B has a zero eigenvalue or has an eigenvalue $\beta = b\exp(\mathrm{i}\phi)$ such that, for some $\xi \in \mathbb{Z}_0^+, \eta \in \mathbb{Z}^+$, $\xi \le \eta$, we have*

$$|\phi| = \frac{\pi\xi}{2\eta} \qquad \text{and} \qquad b = 2(-1)^{\ell} \sin\left(\frac{\pi(\eta-\xi)m}{2\eta(2k-m)} + \frac{\pi\ell m}{2k-m}\right),$$

where ℓ is any integer satisfying

$$-k+\frac{m}{2}-\frac{1}{2}\left(1-\frac{\xi}{\eta}\right)\le \ell \le k-\frac{m}{2}-\frac{1}{2}\left(1-\frac{\xi}{\eta}\right),$$
$$\ell \neq \left(\frac{2k}{m}-1\right)s-\frac{1}{2}\left(1-\frac{\xi}{\eta}\right) \qquad \text{for all integers } s \text{ such that } \quad -\left\lfloor\frac{m}{2}\right\rfloor \le s \le \left\lfloor\frac{m}{2}\right\rfloor. \tag{4.2}$$

Proof. If we put $A=-I$, then all the eigenvalues α_i of A are equal to -1 (i.e., $|\alpha_i|=1$, $\mathrm{Arg}(\alpha_i)=\pi$), and the equivalence between (i) and (ii) follows immediately from Theorem 3.1, the branches (i) and (iib). We show the equivalence between (i) and (iii).

If $\beta=0$ is an eigenvalue of B, then the corresponding trinomial (1.8) reduces to $Q(\lambda)=\lambda^k-\lambda^{k-m}$. Such a trinomial has a root $\lambda=1$ which is a q-th root of unity (for any $q\in\mathbb{Z}^+$), and the existence of a nontrivial periodic (constant) solution is straightforward.

Now, let $\beta=b\exp(\mathrm{i}\phi)$, $b\neq 0$, $|\phi|\le\pi/2$, be an eigenvalue of B. The existence of a periodic solution is guaranteed by Theorem 3.1 (branches (i) and (iib) – its alternative supplement). Hence, let us find all the possible values of b and ϕ such that the conditions (2.7) are satisfied for some $q\in\mathbb{Z}^+$ and some $j\in\mathbb{Z}$ such that $\lfloor -q/2\rfloor+1\le j\le\lfloor q/2\rfloor$. Since $\alpha_i=-1$, then, taking into account the identity $2\cos^2(t)=1+\cos(2t)$, (2.7) simplifies to

$$\begin{aligned} |\beta|=|b|&=2\left|\cos\left(\frac{\pi}{2}-\frac{\pi jm}{q}\right)\right|,\\ \mathrm{Arg}(\beta)&=\left(\kappa\arccos\left(-\frac{|b|}{2}\right)+\frac{2\pi jk}{q}\right)_{2\pi}. \end{aligned} \tag{4.3}$$

If j is such that $\cos(\pi/2-\pi jm/q)>0$, then we can write $(4.3)_1$ as

$$\frac{|b|}{2}=\cos\left(\frac{\pi}{2}-\frac{\pi jm}{q}\right) \tag{4.4}$$

and $(4.3)_2$, with help of the identity $\arccos(-t)=\pi-\arccos(t)$, becomes

$$\mathrm{Arg}(\beta)=-\pi+\arccos\left(\frac{|b|}{2}\right)+\frac{2\pi jk}{q}+2\pi p_j=-\frac{\pi}{2}+(2k-m)\frac{\pi j}{q}+2\pi p_j \tag{4.5}$$

for a suitable $p_j\in\mathbb{Z}$. On the other hand, if j is such that $\cos(\pi/2-\pi jm/q)<0$, then

$$-\frac{|b|}{2}=\cos\left(\frac{\pi}{2}-\frac{\pi jm}{q}\right) \tag{4.6}$$

and $(4.3)_2$ becomes

$$\mathrm{Arg}(\beta)=\frac{\pi}{2}+(2k-m)\frac{\pi j}{q}+2\pi p_j \tag{4.7}$$

for a suitable $p_j\in\mathbb{Z}$. Clearly, in both the cases, $\mathrm{Arg}(\beta)$ has to be a rational multiple of π, therefore, let us write $|\phi|=\frac{\pi\xi}{2\eta}$ where ξ, η are integers such that $\eta>0$ and $0\le\xi\le\eta$. We distinguish four cases according to the sign of b and ϕ.

Let $b>0$, $\phi>0$ and $\cos(\pi/2-\pi jm/q)>0$. Then $\mathrm{Arg}(\beta)=\phi=\frac{\pi\xi}{2\eta}$ and (4.5), (4.4) yield

$$\frac{j}{q}=\frac{\xi+\eta-4\eta p_j}{2\eta(2k-m)}, \tag{4.8}$$

$$b=2\cos\left(\frac{\pi}{2}-\frac{\pi jm}{q}\right)=-2\sin\left(-\frac{\pi jm}{q}\right). \tag{4.9}$$

Substituting (4.8) into (4.9), we have

$$b = 2(-1)^{\ell_j} \sin\left(\frac{\pi(\eta-\xi)m}{2\eta(2k-m)} + \frac{\pi\ell_j m}{2k-m}\right), \tag{4.10}$$

where $\ell_j = 2p_j - 1$ is an odd integer.

Similarly, if $\cos(\pi/2 - \pi jm/q) < 0$, with help of (4.7) and (4.6) we get (4.10), where $\ell_j = 2p_j$ is now an even integer.

A discussion of the remaining cases ($b > 0$ and $\phi \le 0$, $b < 0$ and $\phi > 0$, $b < 0$ and $\phi \le 0$) is quite analogous, leading to the identical description (4.10) of the values b independently of the $\cos(\pi/2 - \pi jm/q)$ sign (both odd and even integers ℓ_j are always involved).

It remains to determine the range of the integers ℓ_j, which follows easily from the fact that $-\lfloor q/2\rfloor < j \le \lfloor q/2\rfloor$. Since q can be chosen as $q = 2\eta(2k-m)$, see (4.8), we actually have $\lfloor q/2\rfloor = q/2$. Thus,

$$\left|\frac{\pi jm}{q}\right| \le \frac{\pi m}{2},$$

which implies

$$-k + \frac{m}{2} - \frac{1}{2}\left(1 - \frac{\xi}{\eta}\right) \le \ell_j \le k - \frac{m}{2} - \frac{1}{2}\left(1 - \frac{\xi}{\eta}\right).$$

Finally, the condition (4.2) excludes those integers ℓ for which the argument of sine function in (4.10) is an integer multiple of π. Therefore, the case of $b = 0$ does not occur. □

Now, we put $m = 1$ and formulate a consequence of Theorem 4.1 for the case studied in [10].

Corollary 4.2. *Let $B \in \mathbb{R}^{d\times d}$, and let $k \in \mathbb{Z}^+$. Then the following statements are equivalent.*

(i) *The system (4.1) with $m = 1$ has a nontrivial periodic solution.*

(ii) *There exists an eigenvalue β of B lying on the epitrochoid (3.5), where $R = 1/k$, $r = (k-1)/k$, $\delta = 1$, $\omega = 0$, and $t = 2\pi(k-1)(j-q)/q$ for some $q \in \mathbb{Z}^+$ and some $j = \lfloor -q/2\rfloor + 1, \lfloor -q/2\rfloor + 2, \dots, \lfloor q/2\rfloor$.*

(iii) *B has a zero eigenvalue or has an eigenvalue $\beta = b\exp(\mathrm{i}\phi)$ such that, for some $\xi \in \mathbb{Z}_0^+, \eta \in \mathbb{Z}^+$, $\xi \le \eta$, and $\ell \in \mathbb{Z}_0^+$, we have*

$$|\phi| = \frac{\pi\xi}{2\eta}, \quad b = 2(-1)^{\ell} \sin\left(\frac{\pi(\eta-\xi)}{2\eta(2k-1)} + \frac{\pi\ell}{2k-1}\right), \quad -\frac{1}{2}\left(1 - \frac{\xi}{\eta}\right) < \ell \le 2k-2. \tag{4.11}$$

Proof. The stated equivalencies follow directly from Theorem 4.1 considering $m = 1$. Indeed, while the first two statements are straightforward, the conditions (iii) of Theorem 4.1 simplify to

$$|\phi| = \frac{\pi\xi}{2\eta} \quad \text{and} \quad b = 2(-1)^{\ell} \sin\left(\frac{\pi(\eta-\xi)}{2\eta(2k-1)} + \frac{\pi\ell}{2k-1}\right),$$

where ℓ is an integer such that

$$-k + \frac{\xi}{2\eta} \le \ell < -\frac{1}{2}\left(1 - \frac{\xi}{\eta}\right) \quad \text{or} \quad -\frac{1}{2}\left(1 - \frac{\xi}{\eta}\right) < \ell \le k - 1 + \frac{\xi}{2\eta}. \tag{4.12}$$

Now, to adjust the bounds from $(4.12)_1$, we can write

$$b = 2(-1)^{\ell}\sin\left(\frac{\pi(\eta-\xi)}{2\eta(2k-1)}+\frac{\pi\ell}{2k-1}\right) = 2(-1)^{\ell+1}\sin\left(\pi+\frac{\pi(\eta-\xi)}{2\eta(2k-1)}+\frac{\pi\ell}{2k-1}\right)$$
$$= 2(-1)^{\ell'}\sin\left(\frac{\pi(\eta-\xi)}{2\eta(2k-1)}+\frac{\pi\ell'}{2k-1}\right), \quad \text{where} \quad \ell' = \ell+2k-1.$$

Thus,

$$-k+\frac{\xi}{2\eta} \le \ell < -\frac{1}{2}\left(1-\frac{\xi}{\eta}\right) \quad \text{if and only if} \quad k-1+\frac{\xi}{2\eta} \le \ell' < 2k-\frac{1}{2}\left(3-\frac{\xi}{\eta}\right).$$

Since $2 \le 3-\frac{\xi}{\eta} \le 3$, the last inequality can be written as $\ell' \le 2k-2$. □

Remark 4.3. To compare the conclusions of Corollary 4.2 and Theorem 1.1 (Theorem 4.1 of [10]), one can observe that the equivalency between (i) and (iii) of Corollary 4.2 actually agrees with the statement of Theorem 1.1. More precisely, when $\xi = 0$, we obtain $|\phi| = 0$ (i.e., $\beta = b$ is a real eigenvalue) and (4.11) reduces to

$$b = 2(-1)^{\ell}\sin\left(\frac{\pi(2\ell+1)}{2(2k-1)}\right), \qquad \ell = 0,1,\dots,2k-2.$$

It is not difficult to check that the values of b corresponding to $\ell = k, k+1, \dots, 2k-2$ are already contained in the set of all b corresponding to $\ell = 0,1,\dots,k-1$, hence we get (1.4). Further, if $\xi = \eta$, then the conditions (4.11) directly imply (1.5), and if $0 < \xi < \eta$, then the conditions (4.11) agree with (1.6).

In addition to the assertion of Theorem 1.1, the part (ii) of Corollary 4.2 reveals a geometric nature of the condition (iii).

As the second illustration of Theorem 3.1, we consider a specific control problem for the linear autonomous difference system

$$x(n) + Ax(n-1) = 0. \tag{4.13}$$

We again restrict to the planar case and assume that A has the Jordan form

$$A = \begin{pmatrix} a_1 & a_2 \\ -a_2 & a_1 \end{pmatrix}, \qquad a_1, a_2 \in \mathbb{R} \tag{4.14}$$

with the spectral norm $\|A\| = \sqrt{a_1^2+a_2^2} \ne 1$. Thus, the trivial equilibrium of this system is a focus (attractive if $\|A\| < 1$ and repelling if $\|A\| > 1$). Obviously, such a system does not admit any (nontrivial) periodic solution.

Now we extend this system by a control term $u(n) = Bx(n)$, acting with an integer delay k, to obtain a delay feedback controlled system

$$x(n) + Ax(n-1) + Bx(n-k) = 0.$$

We put $B = \rho\Phi$ with a real scalar $\rho > 0$ and a rotation matrix

$$\Phi = \begin{pmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{pmatrix}, \qquad -\pi < \vartheta \le \pi.$$

Our task is to describe the set of all control parameters ρ, ϑ and k such that the controlled system

$$x(n) + \begin{pmatrix} a_1 & a_2 \\ -a_2 & a_1 \end{pmatrix} x(n-1) + \rho \begin{pmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{pmatrix} x(n-k) = 0 \tag{4.15}$$

admits a (nontrivial) q-periodic solution, a positive integer q being prescribed.

From the mathematical side, we analyze the existence of a q-periodic solution of (1.3) with the above specified A, B and $m = 1$. Obviously, the matrices A, B are commuting, and the simultaneous ordering of their eigenvalues is $(a_1 + \mathrm{i}a_2, \rho\exp(\mathrm{i}\vartheta)), (a_1 - \mathrm{i}a_2, \rho\exp(-\mathrm{i}\vartheta))$. Thus, if (ρ, ϑ) is a control couple, then $(\rho, -\vartheta)$ is also a control couple, and we can restrict to one sign variant only, say $(a_1 + \mathrm{i}a_2, \rho\exp(\mathrm{i}\vartheta))$.

Now, Theorem 3.1 (see also Remark 2.2) yields that the controlled system (4.15) has a q-periodic solution if and only if the control parameters ρ, ϑ, k meet, for an integer j such that $\lfloor -q/2 \rfloor + 1 \le j \le \lfloor q/2 \rfloor$, the conditions

$$\begin{aligned} \rho^2 &= 1 + \|A\|^2 + 2\|A\| \cos\left(\mathrm{Arg}(a_1 + \mathrm{i}a_2) - \frac{2\pi j}{q} \right), \\ \vartheta &= \left(\kappa \arccos\left(\frac{\|A\|^2 - \rho^2 - 1}{2\rho} \right) + \frac{2\pi jk}{q} \right)_{2\pi} \end{aligned} \tag{4.16}$$

where

$$\kappa = -1 \quad \text{if} \quad 0 < \left(\mathrm{Arg}(a_1 + \mathrm{i}a_2) - \frac{2\pi j}{q} \right)_{2\pi} \le \pi$$

and

$$\kappa = 1 \quad \text{if} \quad -\pi < \left(\mathrm{Arg}(a_1 + \mathrm{i}a_2) - \frac{2\pi j}{q} \right)_{2\pi} \le 0.$$

In particular, if $k = q$, then we directly have

$$\vartheta = \kappa \arccos\left(\frac{\|A\|^2 - \rho^2 - 1}{2\rho} \right).$$

Alternatively, Theorem 3.1 implies that such a control couple is lying on the epitrochoid (3.5) with $R = \|A\|/k$, $r = \|A\|(k-1)/k$, $\delta = 1$, $\omega = \left(k\pi + k\,\mathrm{Arg}(a_1 + \mathrm{i}a_2)\right)_{2\pi}$ and $t = (k-1)(2\pi j/q - \pi - \mathrm{Arg}(a_1 + \mathrm{i}a_2))$ for some $j = \lfloor -q/2 \rfloor + 1, \lfloor -q/2 \rfloor + 2, \ldots, \lfloor q/2 \rfloor$.

Example 4.4. Consider the system (4.13) with A given by (4.14) and put $a_1 = -0.4$, $a_2 = -1$. The zero equilibrium of such a system is an unstable focus, therefore $\limsup_{n\to\infty} \|x(n)\| = \infty$ for any nontrivial solution x of (4.13), see Figure 4.1 (left). We wish to find control parameters ρ, ϑ, k generating a solution x of (4.15) with the period $q = 22$.

If we put $k = q = 22$ and $j = 3$, then (4.16) yields the control parameters $\rho \approx 0.353005$ and $\vartheta \approx 1.520652$. Under this choice of ρ, ϑ, k and j, the 22-nd root of unity $\lambda_{3,22} = \exp(\mathrm{i}6\pi/22)$ is also a root of the characteristic polynomial associated with the controlled system (4.15), hence, this system admits a 22-periodic solution ($q = 22$ is even a prime period since $\gcd(3, 22) = 1$). In view of the notes at the end of Section 2, any such periodic solution is a linear combination

$$x(n) = C_1 x_1(n) + C_2 x_2(n), \qquad C_1, C_2 \in \mathbb{R},$$

where x_1, x_2 are given by (2.9) with $q = 22$, $j = 3$, $v_1 = (1, 0)^T$, $v_2 = (0, 1)^T$, and the initial vectors are prescribed according to (2.10). Figure 4.1 (right) depicts a particular trajectory for $C_1 = -1$, $C_2 = 1$ (such a choice of C_1, C_2 implies $x_1(0) = -1$, $x_2(0) = 1$).

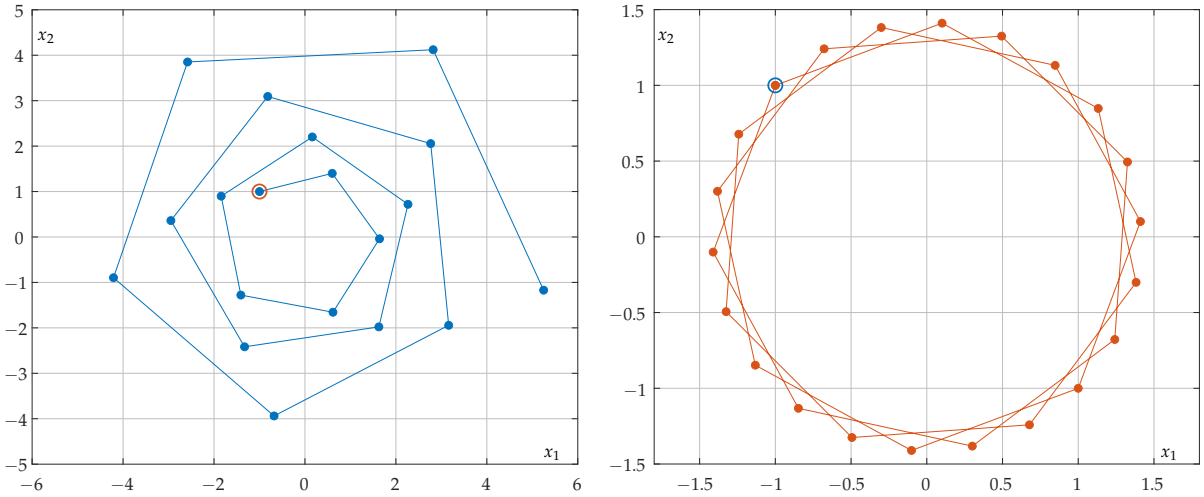


Figure 4.1: The left figure depicts a trajectory of the (uncontrolled) system (4.13) in the (x_1, x_2)-plane for the data of Example 4.4 and the initial vector $(-1, 1)^T$. The right figure then depicts a periodic trajectory (oriented clockwise) of the controlled system (4.15) (note that the prime period is $22/\gcd(3, 22) = 22$).

5 Final remarks

In this paper, we focused on detecting a periodic behavior in the linear difference system (1.3) with simultaneously triangularizable matrices A, B. Based on the analysis of the characteristic polynomial, we formulated efficient conditions for the existence of a periodic solution of (1.3). These conditions are given in terms of eigenvalues of A, B, and supported by its geometric characterization, calculating the revealed periodic solutions and their prime period.

Extending the periodicity problem from this paper to the general case of (1.3) (i.e., without the assumption on simultaneous triangularizability of A, B), or even to the system (1.1), is probably very complicated. Nice periodicity results were obtained for a scalar version of (1.3) in [11]. Similarly as in [10], the authors utilized the theory of circulant matrices in their investigations (note that those results seem to be achievable also by the D-partition technique applied in this paper). However, in a vector case, the formulation of such periodicity results in terms of eigenvalues of the system matrices is a considerably more challenging task.

The difficulty of the problem is underlined by the fact that similar problems for (1.3) and (1.1) remain open also in the related stability area. In this connection, we refer to the interesting paper [13] suggesting a method for detecting periodic solutions in some linear difference equations without knowledge of their characteristic roots. Of course, efficiency of the method for more general cases is questionable.

Taking into account recent developments on locating the trinomial roots (see, e.g., [1, 4, 5, 20]), we believe that the approach based on a suitable decomposition of the characteristic polynomial and the following analysis of the corresponding sparse polynomials has a promising potential in periodicity (as well as stability) theory of autonomous difference equations.

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On the surjectivity conditions of a linear functional differential operator of the second order

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Abstract. Linear second-order functional differential equations without the Volterra condition are studied. Sufficient conditions for the everywhere solvability of the equation (surjectivity of the corresponding functional differential operator) are obtained in terms of norms of the positive and negative parts of the functional operators. These conditions are shown to be sharp in the sense that if they are not satisfied, then there exists an equation with no solution. The obtained solvability conditions are formulated directly for the equation itself, without considering specific boundary value problems.

Keywords: functional differential equations, solvability, boundary value problems, surjectivity.

2020 Mathematics Subject Classification: 34K05, 34K06, 34K10.

1 Introduction.

Functional differential equations are very important because they successfully model many phenomena in physics, engineering, biology, and economics [3, 7, 13, 20] (of course, they are also important as an independent mathematical object [3, 4, 6, 32]). Often, the most important aspect of the study is the study of solutions of boundary value problems for functional differential equations. Many works are devoted to the theory of boundary value problems for such equations [6, 16, 31]. However, given the widespread use of modeling in all fields of knowledge, a functional differential equation can be interesting even without being tied to a specific boundary value problem. In this case, first of all, the question of the existence of a solution to the equation for any right-hand side (the surjectivity of the operator) is of interest; only then can the equation be used for appropriate modeling.

So, we consider linear functional differential equations

$$(\mathcal{L}x)(t) = f(t), \quad t \in [0,1], \tag{1.1}$$

where the operator $\mathcal{L}$ acts from the space $\mathbf{AC}^1$ (real functions that are absolutely continuous together with their first derivative on the interval $[0,1]$) to the space $\mathbf{L}$ of integrable real functions and is defined by the equality

$$(\mathcal{L}x)(t) \equiv \ddot{x}(t) - (T^+x)(t) + (T^-x)(t), \quad t \in [0,1],$$

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where T^+, T^- are linear bounded positive operators acting from the space of continuous real functions $\mathbf{C}$ into the space $\mathbf{L}$. A linear operator $T : \mathbf{C} \to \mathbf{L}$ is called positive if it maps non-negative functions into almost everywhere non-negative ones. The norm of such a positive operator is defined by the equality

$$\| T\|_{\mathbf{C}\to\mathbf{L}} = \int_0^1 (T\mathbf{1})(s)\, ds,$$

where $\mathbf{1} : \mathbb{R} \to \mathbb{R}$ is the unit function. Here in the spaces $\mathbf{AC}^1$, $\mathbf{C}$, $\mathbf{L}$ the standard norms are used:

$$\| x\|_{\mathbf{AC}^1} = |x(0)| + |\dot{x}(0)| + \int_0^1 |\ddot{x}(t)|\, dt,\ \| x\|_{\mathbf{C}} = \max_{t\in[0,1]} |x(t)|,\ \| z\|_{\mathbf{L}} = \int_0^1 |z(t)|\, dt.$$

Next, we will also need the space $\mathbf{L}_\infty[0,1]$ of real functions essentially bounded on $[0,1]$ and the space $\mathbf{AC}_\infty$ of absolutely continuous real functions on $[0,1]$ with derivative $\dot{x} \in \mathbf{L}_\infty[0,1]$,

$$\| z\|_{\mathbf{L}_\infty} = \operatorname*{ess\,sup}_{s\in[0,1]} |z(t)|, \quad \| x\|_{\mathbf{AC}_\infty} = |x(0)| + \| \dot{x}\|_{\mathbf{L}_\infty}.$$

Definition 1.1. A solution of equation (1.1) is any function $x \in \mathbf{AC}^1$ that satisfies equation (1.1) for almost all $t \in [0,1]$.

Definition 1.2. We call equation (1.1) everywhere solvable if it has a solution for all $f \in \mathbf{L}$. In this case, the operator $\mathcal{L} : \mathbf{AC}^1 \to \mathbf{L}$ is surjective.

Definition 1.3. A linear boundary value problem for equation (1.1) is a system of equations consisting of equation (1.1) and two equations of the form

$$\ell_1 x = \alpha_1, \quad \ell_2 x = \alpha_2, \tag{1.2}$$

where $\ell_1, \ell_2 : \mathbf{AC}^1 \to \mathbb{R}$ are linearly independent linear bounded functionals, α_1, $\alpha_2 \in \mathbb{R}$ are given numbers.

Definition 1.4. Boundary value problem (1.1)–(1.2) is called Fredholm if the operator of this problem $[\mathcal{L}, \ell_1, \ell_2] : \mathbf{AC}^1 \to \mathbf{L} \times \mathbb{R}^2$ is a Noetherian operator of zero index [5,21].

The space $\mathbf{AC}^1$ is compactly embedded in the space $\mathbf{C}$ (see, for example, [25]). Hence, any boundary value problem (1.1)–(1.2) is Fredholm. So, a necessary and sufficient condition for the unique solvability of this problem is that the corresponding homogeneous boundary value problem

$$\mathcal{L}x = 0, \quad \ell_1 x = 0, \quad \ell_2 = 0,$$

has only the trivial solution. in the space $\mathbf{AC}^1$. Note that in [5] it was shown that even under weaker natural assumptions on the operator $\mathcal{L}$ boundary value problem (1.1)–(1.2) is Fredholm.

We are interested not in the solvability condition of boundary value problems for equation (1.1), but rather in the solvability conditions of equation (1.1) itself. Since we do not require the Volterra property [5, p. 200] of the operators T^+, T^-, the properties of our equation may not directly correspond to those of linear ordinary differential equations [5, p. 100], for which the unique solvability of the Cauchy problem guarantees solvability everywhere. It is easy to give an example of an unsolvable equation (1.1). Consider the equation

$$\ddot{x}(t) - 4\,x(0) + 8\,x(1/2) - 4\,x(1) = 1, \quad t \in [0,1]. \tag{1.3}$$

This equation has form (1.1) for

$$(T^+x)(t) = 4\,x(0) + 4\,x(1), \quad (T^-x)(t) = 8x(1/2), \quad t \in [0,1],$$

thus, $\|\,T^+\|_{\mathbf{C}\to\mathbf{L}} = 8$, $\|\,T^-\|_{\mathbf{C}\to\mathbf{L}} = 8$. Since for the solution x of equation (1.3) the function $\ddot{x}$ must be constant, then $x(t) = a_0 + a_1\,t + a_2\,t^2$, $t \in [0,1]$, for some constants a_i. It is easy to check that $\mathcal{L}x = 0$ for this x. Then $\mathcal{L}x(t) \equiv 0 \neq 1$. The contradiction shows that equation (1.3) has no solution.

Definition 1.5. Let $\mathcal{P} \geqslant 0$, $\mathcal{Q} \geqslant 0$. Denote by $\mathcal{M}_{\mathcal{P},\mathcal{Q}}$ the set of operators $\mathcal{L} : \mathbf{AC}^1 \to \mathbf{L}$ defined by the equality

$$(\mathcal{L}x)(t) \equiv \ddot{x}(t) - (T^+x)(t) + (T^-x)(t), \quad t \in [0,1], \tag{1.4}$$

such that

$$\|\,T^+\|_{\mathbf{C}\to\mathbf{L}} = \mathcal{P}, \quad \|\,T^-\|_{\mathbf{C}\to\mathbf{L}} = \mathcal{Q}. \tag{1.5}$$

Denote by $\mathcal{S}$ the set of surjective operators (1.4).

Definition 1.6. The set $\Omega \equiv \{(\mathcal{P},\mathcal{Q}) \in [0,\infty)^2 \mid \mathcal{M}_{\mathcal{P},\mathcal{Q}} \subset \mathcal{S}\}$ is called the set of everywhere solvability (or the set of surjectivity).

Remark 1.7. From the proof of Lemmas 2.3, 2.4, 2.5, and Theorem 3.1 it follows that if $(\mathcal{P},\mathcal{Q}) \in \Omega$, then $(\mathcal{P}_1,\mathcal{Q}_1) \in \Omega$ for all $\mathcal{P}_1 \in [0,\mathcal{P}]$, $\mathcal{Q}_1 \in [0,\mathcal{Q}]$.

Therefore, the equalities (1.5) in Definition 1.5 can be replaced by the inequalities

$$\|\,T^+\|_{\mathbf{C}\to\mathbf{L}} \leqslant \mathcal{P}, \quad \|\,T^-\|_{\mathbf{C}\to\mathbf{L}} \leqslant \mathcal{Q}.$$

Our goal is to find the set Ω. The following Theorem 3.1 gives a complete description of Ω.

The problem of constructing a set of everywhere solvability, apparently, has not attracted attention due to the fact that there is a simple necessary and sufficient condition for the operator $\mathcal{L}$ to be everywhere solvable.

Lemma 1.8 ([5])**.** *Equation* (1.1) *is everywhere solvable if and only if there exist linearly independent linear bounded functionals* $\ell_1, \ell_2 : \mathbf{AC}^1 \to \mathbb{R}$ *such that boundary value problem* (1.1)–(1.2) *is uniquely solvable.*

By choosing different functionals ℓ_1, ℓ_2 and finding solvability conditions for boundary value problem (1.1)–(1.2) in terms of the norms of the operators T^+ and T^-, using Lemma 1.8 we can construct only subsets of the set of everywhere solvability Ω. Finding the set of everywhere solvability is, apparently, a new problem that has not yet been solved in this formulation. Its solution can be useful in modeling various processes, including in economics, when it is required that a solution exists for each external influence from the space under consideration.

In the work [18], surjectivity was studied for a partial differential equation with constant and distributed delay. In [1], the surjectivity of functional differential operators was studied using the surjectivity module [28, p. 26] $q(\mathcal{L}) = \inf_{y\neq 0} \|\,\mathcal{L}^*y\|\,/\,\|\,y\|$, which is related to the minimal norm of the Green operator of boundary value problems for the equation $\mathcal{L}x = f$. Here it is possible to estimate the norm of the adjoint operator from below only in very special cases. For example, one effective condition for the surjectivity of a neutral functional differential operator was obtained in [2]. In the papers [29,30], conditions for surjectivity of operators in a Banach space are obtained, and the results are applied to boundary value problems. In

[14,15], perturbations of linear surjective operators are studied. Thus, surjectivity conditions for linear operators can also be useful in studying boundary value problems. Coefficient surjectivity criteria (including those for families of equations) without special requirements for functional operators have not been obtained, as far as the author knows.

A convenient criterion for surjectivity is formulated using the adjoint operator.

Lemma 1.9 ([5]). *The following statements are equivalent:*

1) *the operator $\mathcal{L}$ is surjective;*

2) $\dim\ker\mathcal{L}=2$;

3) $\ker\mathcal{L}^*=\{0\}$.

The construction of the solvability set Ω is apparently a rather difficult problem. To the author's knowledge, it has only been solved for the first-order equation [9]. The method used here is completely different from the method of [9]. It is based on the fact that if the family $\mathcal{M}_{\mathcal{P},\mathcal{Q}}$ contains a non-surjective operator, then this family contains finite-dimensional operators T^+ and T^- for which the adjoint boundary value problem has a non-trivial solution.

There are relatively many works on the conditions for the solvability of boundary value problems for functional differential equations (see, for example, [8,10–12,26,27]).

If in Lemma 1.8 we take the periodic boundary value problem $x(0)=x(1)$, $\dot{x}(0)=\dot{x}(1)$, then the solvability condition [22] of this problem gives the following conditions for the surjectivity of $\mathcal{L}$:

$$\frac{4\mathcal{P}}{4-\mathcal{P}}\leqslant\mathcal{Q}\leqslant 8+4\sqrt{4-\mathcal{P}},\ \mathcal{P}<4,\quad\text{or}\quad\frac{4\mathcal{Q}}{4-\mathcal{Q}}\leqslant\mathcal{P}\leqslant 8+4\sqrt{4-\mathcal{Q}},\ \mathcal{Q}<4. \tag{1.6}$$

Our main result, formulated in Theorem 3.1 (section 3), will significantly improve inequalities (1.6) (see Fig. 3.1, Corollaries (3.2), (3.3), (3.4)).

The general scheme of this work is as following. First we show that if $(\mathcal{P},\mathcal{Q})\notin\Omega$, there exists an operator $\mathcal{L}\in\mathcal{M}_{\mathcal{P},\mathcal{Q}}$ such that the equation $\mathcal{L}^*g=0$ has a non-trivial solution (Lemmas 2.1, 2.2). Then we prove that in this case for some $\mathcal{P}_1\leqslant\mathcal{P}$, $\mathcal{Q}_1\leqslant\mathcal{Q}$ there exists an operator $\mathcal{L}\in\mathcal{M}_{\mathcal{P}_1,\mathcal{Q}_1}$ such that the equation $\mathcal{L}^*g=0$ has a non-trivial piecewise linear solution (Lemma 2.3). In Lemmas 2.4, 2.5, is proved that it is sufficient to consider the case of a piecewise linear solution g whose graph consists of three line segments. This last case admits a simple explicit solution of the adjoint equation, which allows us to explicitly describe the set Ω (Theorem 3.1).

2 Auxiliary statements

If $\mathcal{L}\in\mathcal{M}_{\mathcal{P},\mathcal{Q}}$, then

$$\mathcal{L}x=\ddot{x}-Tx, \tag{2.1}$$

where the operator $T:T^+-T^-$ has the representation [19, p. 303–304] in the form of the Riemann–Stieltjes integral

$$(Tx)(t)=\int_0^1 x(s)\,d_sr(t,s),\quad t\in[0,1],$$

$$r(t,s)=p^+(t)k^+(t,s)-p^-(t)k^-(t,s),\quad t\in[0,1],$$

– the functions p^+, $p^- \in \mathbf{L}$ are nonnegative,

– $\| p^+ \|_{\mathbf{L}} = \mathcal{P}$, $\| p^- \|_{\mathbf{L}} = \mathcal{Q}$,

– the functions $k^+(t,\cdot)$, $k^-(t,\cdot)$ are nondecreasing for all $t \in [0,1]$,

– $k^+(t,0) = 0$, $k^-(t,0) = 0$, $k^+(t,1) = 1$, $k^-(t,1) = 1$.

The operators T^+, T^- have the representations

$$(T^{+/-}x)(t) = \int_0^1 x(s) d_s r^{+/-}(t,s), \quad t \in [0,1],$$

where $r^+(t,s) = p^+(t)k^+(t,s)$, $\quad r^-(t,s) = p^-(t)k^-(t,s)$, $\quad s,t \in [0,1]$, $\quad \int_0^1 r^+(t,1)\,dt = \mathcal{P}$, $\int_0^1 r^-(t,1)\,dt = \mathcal{Q}$.

2.1 Adjoint equation

Let $\mathcal{L}^* : (\mathbf{L})^* \to (\mathbf{AC}^1)^*$ be the adjoint operator. The $\mathbf{AC}^1$ is isomorphic to the space $\mathbf{L} \times \mathbb{R}^2$. Indeed, let $(\delta x)(t) = \ddot{x}(t)$, $t \in [0,1]$; $r_0 x = x(0)$, $r_1 x = \dot{x}(0)$. Then the operator $[\delta, r_0, r_1] : \mathbf{AC}^1 \to \mathbf{L}_1[0,1] \times \mathbb{R}^2$ defined by the equality

$$[\delta, r_0, r_1]x = \{\ddot{x}, x(0), \dot{x}(0)\},$$

is continuously invertible. Let $(\Lambda z)(t) = \int_0^t (t-s)z(s)\,ds$, $t \in [0,1]$, for every $z \in \mathbf{L}$. Then $\Lambda : \mathbf{L} \to \mathbf{AC}^1$ is a bounded operator and the Cauchy problem

$$\begin{cases} \delta x = z \\ r_0 x = c_0, \quad r_1 x = c_1, \end{cases}$$

has a unique solution

$$x(t) = c_0 + c_1 t + (\Lambda z)(t), \quad t \in [0,1]. \tag{2.2}$$

So, we have

$$\left([\delta, r_0, r_1]^{-1}\{z, c_0, c_1\}\right)(t) = c_0 + c_1\, t + (\Lambda z)(t), \quad t \in [0,1].$$

Thus, $(\mathbf{AC}^1)^* \simeq (\mathbf{L} \times \mathbb{R}^2)^* = (\mathbf{L})^* \times \mathbb{R}^2 = \mathbf{L}_\infty[0,1] \times \mathbb{R}^2$. Therefore, $\mathcal{L}^*$ acts from $(\mathbf{L})^* \simeq \mathbf{L}_\infty[0,1]$ into the space $\mathbf{L}_\infty[0,1] \times \mathbb{R}^2$.

Lemma 2.1. *Let an operator $\mathcal{L} : \mathbf{AC}^1 \to \mathbf{L}$ be defined by (2.1). Then $\mathcal{L}$ is not surjective if and only if there exists a non-trivial $g \in \mathbf{L}_\infty[0,1]$ satisfying the homogeneous adjoint equation $\mathcal{L}^* g = 0$:*

$$g(t) - \int_0^1 \left(\int_t^1 (\tau - t) d_\tau r(s,t) \right) g(s)\,ds = 0, \quad t \in [0,1], \tag{2.3}$$

$$\int_0^1 \int_0^1 d_s r(t,s) g(t)\,dt = \int_0^1 r(s,1) g(s)\,ds = 0, \tag{2.4}$$

$$\int_0^1 \int_0^1 s\, d_s r(t,s) g(t)\,dt = 0. \tag{2.5}$$

Proof. Let $\mathcal{L} \in \mathcal{M}_{\mathcal{P},\mathcal{Q}}$ from (2.1), $x \in \mathbf{AC}^1$, $g \in \mathbf{L}_\infty[0,1]$. Using the representation

$$x(t) = x(0) + t\dot{x}(0) + \int_0^1 (t-s)\ddot{x}(s)\,ds, \quad t \in [0,1],$$

we obtain

$$\begin{aligned}
&\int_0^1 g(t)(\mathcal{L}x)(t)\,dt \\
&= \int_0^1 g(t)\left[\ddot{x}(t) - \int_0^1 d_s r(t,s)\,x(0) - \int_0^1 s\,d_s r(t,s)\,\dot{x}(0) - \int_0^1 \left(\int_0^s (s-\tau)\ddot{x}(\tau)\,d\tau\right) d_s r(t,s)\right] dt \\
&= \int_0^1 \ddot{x}(t)\left[g(t) - \int_0^1 \left(\int_t^1 (s-t) d_s r(t,s)\right) g(\tau)\,d\tau\right] dt \\
&\quad - \int_0^1 \int_0^1 d_s r(t,s) g(t)\,dt\,x(0) - \int_0^1 \int_0^1 s\,d_s r(t,s) g(t)\,dt\,\dot{x}(0) = (\mathcal{L}^* g)(x), \qquad (2.6)
\end{aligned}$$

where $\mathcal{L}^* : \mathbf{L}_\infty[0,1] \to \mathbf{L}_\infty[0,1] \times \mathbb{R}^2$ is the adjoint operator of the operator $\mathcal{L}$.

If the operator $\mathcal{L}$ is not surjective, then there exists a non-trivial $g \in \mathbf{L}_\infty[0,1]$ such that $\int_0^1 g(t)(\mathcal{L}x)(t)\,dt = 0$ for all $x \in \mathbf{AC}^1$. From (2.6), it follows that $\int_0^1 g(t)(\mathcal{L}x)(t)\,dt = 0$ for all $\ddot{x} \in \mathbf{L}$, all $x(0) \in \mathbb{R}$, for all $\dot{x}(0) \in \mathbb{R}$. Therefore, in this case, there exists a non-zero solution $g \in \mathbf{L}_\infty[0,1]$ to the adjoint equation $\mathcal{L}^* g = 0$, that is g satisfies equalities (2.3), (2.4), (2.5).

If the system (2.3), (2.4), (2.5) has only the trivial solution, then there is no nontrivial g such that $\int_0^1 g(t)(\mathcal{L}x)(t)\,dt = 0$ for all $x \in \mathbf{AC}^1$. Therefore, the operator $\mathcal{L}$ is surjective. □

From equation (2.3) it follows that its solution g is differentiable and $\dot{g} \in \mathbf{L}_\infty$. Therefore, $g \in \mathbf{AC}_\infty$. Substituting $t = 1$ into equation (2.3), we obtain $g(1) = 0$. Substituting $t = 0$ into equation (2.3) and taking into account equality (2.5), we obtain $g(0) = 0$. Differentiating equality (2.3) and taking into account equality (2.4), we obtain

$$\dot{g}(t) = -\int_0^1 \left(\int_t^1 d_\tau r(s,t)\right) g(s)\,ds = -\int_0^1 (r(s,1) - r(s,t)) g(s)\,ds = \int_0^1 r(s,t) g(s)\,ds.$$

Therefore, system (2.3), (2.4), (2.5) is equivalent to the system

$$\dot{g}(t) = \int_0^1 r(s,t) g(s)\,ds, \quad t \in [0,1], \qquad (2.7)$$

$$g(0) = 0, \quad g(1) = 0 \qquad (2.8)$$

$$\int_0^1 r(s,1) g(s)\,ds = 0 \qquad (2.9)$$

where $g \in \mathbf{AC}_\infty$.

From here we obtain the following lemma.

Lemma 2.2. *Let an operator $\mathcal{L} : \mathbf{AC}^1 \to \mathbf{L}$ be defined by (2.1). Then $\mathcal{L}$ is not surjective if and only if there exists a non-trivial $g \in \mathbf{AC}_\infty$ satisfying to equalities (2.7), (2.8), (2.9) that is satisfying the homogeneous adjoint equation $\mathcal{L}^* g = 0$.*

In simplest case, when

$$(\mathcal{L}x)(t) = \ddot{x}(t) - \sum_{j=1}^n p_j(t) x(t_j), \quad t \in [0,1].$$

constructing the adjoint operator is not difficult. Let assume that $g \in (\mathbf{L})^* = \mathbf{L}_\infty[0,1]$. We use

the representation (2.2) for $x \in \mathbf{AC}^1$. We have

$$\begin{aligned} g(\mathcal{L}x) &= \int_0^1 \ddot{x}(t)g(t)\,dt - \sum_{j=1}^n p_j(t)\left(x(0) + t_j\dot{x}(0) + \int_0^{t_j}(t_j - s)\ddot{x}(s)\,ds\right) \\ &= \int_0^1 \ddot{x}(s)\left(g(s) - \sum_{j=1}^n (t_j - s)^+ \int_0^1 g(\tau)p_j(\tau)\,d\tau\right)ds \\ &\quad - x(0)\int_0^1 g(t)\sum_{j=1}^n p_j(t)\,dt - \dot{x}(0)\int_0^1 g(t)\sum_{j=1}^n t_j p_j(t)\,dt = (\mathcal{L}^*g)(x). \end{aligned}$$

Thus, $g \in \ker \mathcal{L}^*$ if and only if $g(\mathcal{L}x) = 0$ for all $x(0) \in \mathbb{R}$, all $\dot{x}(0) \in \mathbb{R}$, and for all $\ddot{x} \in \mathbf{L}$, that is

$$g(s) - \sum_{j=1}^n (t_j - s)^+ \int_0^1 g(\tau)p_j(\tau)\,d\tau = 0, \quad s \in [0,1], \tag{2.10}$$

$$\int_0^1 g(t)\sum_{j=1}^n p_j(t)\,dt = 0, \tag{2.11}$$

$$\int_0^1 g(t)\sum_{j=1}^n t_j p_j(t)\,dt = 0, \tag{2.12}$$

where

$$\alpha^+ = \begin{cases} \alpha & \text{if } \alpha > 0; \\ 0 & \text{if } \alpha \leqslant 0. \end{cases}$$

From (2.10), it follows that any solution to $\mathcal{L}^*g = 0$ is piecewise linear. Moreover, $g(1) = 0$ from (2.10) and $g(0) = 0$ from (2.10) and (2.12).

2.2 Lemmas on the existence of piecewise linear solutions of a homogeneous adjoint equation

Lemma 2.3. *Let $(\mathcal{P}, \mathcal{Q}) \notin \Omega$. Then there exists an operator $\mathcal{L} \in \mathcal{M}_{\mathcal{P},\mathcal{Q}}$ such that the equation $\mathcal{L}^*g = 0$ has a non-trivial solution g in the form*

$$g(t) = \sum_{j\in J} \varphi_j(t), \quad t \in [0,1], \tag{2.13}$$

where $J \subset \{1, 2, \ldots\}$,

$$\varphi_j(t) = \begin{cases} \frac{t-a_j}{c_j-a_j}F_j, & t \in (a_j, c_j), \\ \frac{b_j-t}{b_j-a_j}F_j, & t \in (c_j, b_j), \\ 0, & t \in [0,1]\setminus(a_j, b_j), \end{cases} \tag{2.14}$$

$c_j \in (a_j, b_j)$, $F_j \in \mathbb{R}$, $j \in J$, the intervals (a_j, b_j) are disjoint.

Proof. Since $(P, Q) \notin \Omega$, there exists an operator $\mathcal{L} \in \mathcal{M}_{\mathcal{P},\mathcal{Q}}$ such that the problem (2.7)–(2.9) has a nontrivial solution g. Denote by $\{e_j\}_{j\in J}$ the set of intervals within which g preserves its strict sign: $e_j = [a_j, b_j]$, $g(a_i) = g(b_j) = 0$, $g(t) \neq 0$ for $t \in (a_j, b_j)$, where a_j, b_j are pairs of adjacent zeros of g. There exist at most countably many such pairs. We will demonstrate how to modify the operator $\mathcal{L}$ such that the adjoint equation $\tilde{\mathcal{L}}^*\tilde{g} = 0$ of the modified operator $\tilde{\mathcal{L}}$

has a solution g satisfying the conditions of the lemma. It turns out that $|\tilde{g}(t)| \geqslant |g(t)|$ for each $t \in e_j$.

Let a, b be adjacent zeros of g, and $g(t) > 0$ for $t \in (a,b)$ (the case $g(t) < 0$, $t \in (a,b)$ is similar).

Denote $E^+ = \{\tau \in [0,1] : g(\tau) \geqslant 0\}$, $E^- = \{\tau \in [0,1] : g(\tau) < 0\}$.

We define a function $L : [a,b] \to \mathbb{R}$ by the following equalities: $L(a) = 0$; and for all $t \in (a,b]$

$$L(t) = \int_0^1 \max_{\substack{k^+(s,a) \leqslant \tilde{k}^+(s,t) \leqslant k^+(s,t) \\ k^-(s,a) \leqslant \tilde{k}^-(s,t) \leqslant k^-(s,t)}} (p^+(s)\tilde{k}^+(s,t) - p^-(s)\tilde{k}^-(s,t))g(s)\,ds$$
$$= \int_{E^+} (p^+(s)k^+(s,t) - p^-(s)k^-(s,a))g(s)ds + \int_{E^-} (p^+(s)k^+(s,a) - p^-(s)k^-(s,t))g(s)ds.$$

Since the functions $k^+(s,\cdot)$, $k^-(s,\cdot)$ are non-decreasing, the functions L and $\tilde{L}(t) \equiv \int_0^t L(s)\,ds$ are also non-decreasing, and the function $\tilde{L}$ is convex downwards. Therefore

$$\frac{\tilde{L}(c)(t-a)}{c-a} \leqslant \tilde{L}(t), \quad c \in (a,b], \quad t \in [a,c].$$

From the definition of the function L, it follows that

$$L(t) \geqslant \dot{g}(t), \quad \tilde{L}(t) \geqslant g(t), \quad t \in [a,b].$$

Similarly, we define the function R: $R(b) = 0$, when $t \in [a,b)$

$$R(t) = \int_0^1 \max_{\substack{k^+(s,t) \leqslant \tilde{k}^+(s,t) \leqslant k^+(s,b) \\ k^-(s,t) \leqslant \tilde{k}^-(s,t) \leqslant k^-(s,b)}} (p^+(s)\tilde{k}^+(s,t) - p^-(s)\tilde{k}^-(s,t))g(s)\,ds$$
$$= \int_{E^+} (p^+(s)k^+(s,t) - p^-(s)k^-(s,b))g(s)ds + \int_{E^-} (p^+(s)k^+(s,b) - p^-(s)k^-(s,t))g(s)ds.$$

It is obvious that $R(t) \leqslant \dot{g}(t)$, $t \in [a,b)$, R is non-decreasing, the function $\tilde{R}(t) \equiv -\int_t^b R(s)\,ds$ is non-increasing and convex downwards. Therefore

$$\frac{\tilde{R}(c)(b-t)}{b-c} \geqslant R(t) \geqslant g(t), \quad c \in [a,b), \quad t \in [c,b].$$

It follows from the definition of the function R that

$$R(t) \leqslant \dot{g}(t), \quad \tilde{R}(t) \geqslant g(t), \quad t \in [a,b].$$

Let $c \in (a,b)$ be a unique solution to the equation $\tilde{L}(c) = \tilde{R}(c)$.

Define a new function

$$\tilde{g}(t) \equiv \begin{cases} g(t), & t \in [0,1] \setminus [a,b], \\ \frac{\tilde{L}(c)(t-a)}{c-a}, & t \in [a,c], \\ \frac{\tilde{R}(c)(b-t)}{b-c}, & t \in [c,b]. \end{cases}$$

Define a new operator $\tilde{\mathcal{L}}$ by the equality $\tilde{\mathcal{L}}x \equiv \ddot{x} - \int_0^1 x(s)\,d_s\tilde{r}(\cdot,s)$, where

$$\tilde{r}(t,s) \equiv \tilde{p}^+(s)\tilde{k}^+(s,t) - \tilde{p}^-(s)\tilde{k}^-(s,t), \quad s,t \in [0,1],$$

and

$$\tilde p^+(t) \equiv \begin{cases} p^+(t), & t \in [0,1] \setminus (a,b), \\ \frac{p^+(t)g(t)}{\tilde g(t)}, & t \in (a,b), \end{cases} \qquad \tilde p^-(t) \equiv \begin{cases} p^-(t), & t \in [0,1] \setminus (a,b), \\ \frac{p^-(t)g(t)}{\tilde g(t)}, & t \in (a,b), \end{cases}$$

$$\tilde k^+(s,t) \equiv \begin{cases} k^+(s,a), \ s \in E^-, \ t \in [a,c], \\ k^+(s,b), \ s \in E^-, \ t \in [c,b], \\ k^+(s,t), \ \text{otherwise}, \end{cases} \qquad \tilde k^-(s,t) \equiv \begin{cases} k^-(s,a), \ s \in E^+, \ t \in [a,c], \\ k^-(s,b), \ s \in E^+, \ t \in [c,b], \\ k^-(s,t), \ \text{otherwise}. \end{cases}$$

Then $\tilde{\mathcal{L}} \in \mathcal{M}_{\mathcal{P},\mathcal{Q}}$, since $\tilde p^+(s) \leqslant p^+(s)$, $\tilde p^-(s) \leqslant p^-(s)$, $s \in [0,1]$. Moreover, the function $\tilde g$ is a solution to the equation $\tilde{\mathcal{L}}^* \tilde g = 0$.

The procedure described above can be applied to all adjacent zeros (of which there are at most countably many). After this, we obtain an operator, which we denote by $\mathcal{L} \in \mathcal{M}_{\mathcal{P},\mathcal{Q}}$, such that the equation $\mathcal{L}^* g = 0$ has a solution of the form (2.13), where ϕ_j are defined by (2.14). □

Lemma 2.4. *Let $(\mathcal{P},\mathcal{Q}) \notin \Omega$. Then there exist points $0 \leqslant a_1 < c_1 < b_1 \leqslant a_2 < c_2 < b_2 \leqslant 1$ and an operator $\mathcal{L} \in \mathcal{M}_{\mathcal{P},\mathcal{Q}}$ such that the adjoint equation $\mathcal{L}^* g = 0$ has a piecewise linear solution satisfying conditions $g(t) = 0$ for $t \in [0,1] \cup [b_1,a_2] \cup [b_2,1]$, $g(t) < 0$ for $t \in (a_1,b_1)$, $g(t) > 0$ for $t \in (a_2,b_2)$.*

Proof. By Lemma 2.3 there exists an operator $\mathcal{L} \in \mathcal{M}_{\mathcal{P},\mathcal{Q}}$ such that the adjoint equation $\mathcal{L}^* g = 0$ has a piecewise linear solution g of the form (2.13). Let $\{e_j = [a_j,b_j]\}_{j\in J}$ be the set of all intervals between adjacent zeros of the function g, $\{e_j\}_{j\in J^+}$ be the set of all intervals within which $g(t) > 0$, $J^- = J \setminus J^+$.

Assume that the solution g takes its minimum and maximum at the points $c_1 < c_2$ respectively:

$$g_{\min} \equiv \min_{t\in[0,1]} g(t) = g(c_1), \quad g_{\max} \equiv \max_{t\in[0,1]} g(t) = g(c_2).$$

Let

$$\widetilde e_1 = [a_1,c_1], \ \widetilde e_2 = [c_1,b_1], \ \widetilde e_3 = [a_2,c_2], \ \widetilde e_4 = [c_2,b_2].$$

For $t \in \widetilde e_i$, $i = 1,2,3,4$, we have

$$r(s,t) = p^+(s)k_i^+(s) - p^-(s)k_i^-(s), \quad s \in [0,1],$$

where $k_i^+(s) \leqslant k_{i+1}^+(s)$, $k_i^-(s) \leqslant k_{i+1}^-(s)$, $i = 1,2,3$.

We will write the equation for g at $t \in \widetilde e_i$:

$$\begin{aligned} \dot x(t) &= \int_0^1 r(s,t)g(s)\,ds = \int_0^1 (p^+(s)k_i^+(s) - p^-(s)k_i^-(s))\,ds \\ &= \sum_{j\in J^+} \int_{e_j} (p^+(s)k_i^+(s) - p^-(s)k_i^-(s))g(s)\,ds + \sum_{j\in J^-} \int_{e_j} (p^+(s)k_i^+(s) - p^-(s)k_i^-(s))g(s)\,ds. \end{aligned}$$

Let us stretch or compress (that is, make a suitable change of variable) all segments $[a_j,c_j]$, $[c_j,b_j]$ so that they coincide with the segments $[a_1,c_1]$, $[c_1,b_1]$, respectively, if $j \in J^-$, and coincide with the segments $[a_2,c_2]$, $[c_2,b_2]$ if $j \in J^+$. On each of these segments we multiply the functions p^+, p^- by the ratio of the corresponding segments so that the values

$$\int_{e_j} p^+(s)\,ds, \quad \int_{e_j} p^-(s)\,ds, \quad \int_{e_j} (p^+(s) - p^-(s))\,ds, \ \int_{e_j} (p^+(s)k_i^+(s) - p^-(s)k_i^-(s))\,ds,$$

do not change after the change of variable.

Therefore, for $t \in \widetilde{e}_i$ we have

$$\begin{aligned}
\dot{g}(t) = & \sum_{j\in J^-} \int\limits_{e_1} (p_j^+(s)k_{i,j}^+(s) - p_j^-(s)k_{i,j}^-(s))\, g(s) \frac{g_j}{g_{\min}} ds \\
& + \sum_{j\in J^+} \int\limits_{e_2} (p_j^+(s)k_{i,j}^+(s) - p_j^-(s)k_{i,j}^-(s))\, g(s) \frac{g_j}{g_{\max}} ds \\
= & \int_{e_1} \left(\sum_{j\in J^-} \frac{p_j^+(s)g_j}{g_{\min}} k_{i,j}^+(s) - \sum_{j\in J^-} \frac{p_j^-(s)g_j}{g_{\min}} k_{i,j}^-(s) \right) g(s) ds \\
& + \int_{e_2} \left(\sum_{j\in J^+} \frac{p_j^+(s)g_j}{g_{\max}} k_{i,j}^+(s) - \sum_{j\in J^+} \frac{p_j^-(s)g_j}{g_{\max}} k_{i,j}^-(s) \right) g(s) ds \\
= & \int_{e_1} \left(\left[\sum_{j\in J^-} \frac{p_j^+(s)g_j}{g_{\min}} \right] \widetilde{k}_i^+(s) - \left[\sum_{j\in J^-} \frac{p_j^-(s)g_j}{g_{\min}} \right] \widetilde{k}_i^-(s) \right) g(s) ds \\
& + \int_{e_2} \left(\left[\sum_{j\in J^+} \frac{p_j^+(s)g_j}{g_{\max}} \right] \widetilde{k}_i^+(s) - \left[\sum_{j\in J^+} \frac{p_j^-(s)g_j}{g_{\max}} \right] \widetilde{k}_i^-(s) \right) g(s) ds.
\end{aligned}$$

Here $p_j^{+/-}$, $k_{i,j}^{+/-}$ denote the functions $p^{+/-}$, $k_i^{+/-}$ restricted on the set e_j in the coordinates of one of the corresponding segments $\widetilde{e}_i$, $i = 1,2,3,4$; $g_j = g(c_j)$; the functions $\widetilde{k}_i^{+/-}$ are determined by the equalities

$$\sum_{j\in J^{+/-}} \frac{p_j^{+/-}(s)g_j}{g_{\max/\min}} k_{i,j}^{+/-}(s) = \left(\sum_{j\in J^{+/-}} \frac{p_j^{+/-}(s)g_j}{g_{\max/\min}} \right) \widetilde{k}_i^{+/-}(s).$$

Moreover,

$$\widetilde{k}_i^{+/-}(s) \leqslant \widetilde{k}_{i+1}^{+/-}(s), \quad s \in [0,1], \quad i = 1,2,3,$$

since

$$k_{i,j}^{+/-}(s) \leqslant k_{i+1,j}^{+/-}(s), \quad s \in [0,1], \quad i = 1,2,3, \quad j \in J.$$

We can now define

$$\widetilde{p}^+(s) \equiv \begin{cases} \sum\limits_{j\in J^-} \frac{p_j^+(s)g_j}{g_{\min}}, & s \in e_1, \\ \sum\limits_{j\in J^+} \frac{p_j^+(s)g_j}{g_{\max}}, & s \in e_2, \\ 0, & s \in [0,1] \setminus (e_1 \cup e_2), \end{cases} \qquad \widetilde{p}^-(s) \equiv \begin{cases} \sum\limits_{j\in J^-} \frac{p_j^-(s)g_j}{g_{\min}}, & s \in e_1, \\ \sum\limits_{j\in J^+} \frac{p_j^-(s)g_j}{g_{\max}}, & s \in e_2, \\ 0, & s \in [0,1] \setminus (e_1 \cup e_2). \end{cases}$$

Let $\widetilde{k}^{+/-}(s,t) = k_i(s)$ for $s \in [0,1]$, $t \in \widetilde{e}_j$. On the remaining part of the interval with respect to t, the function $\widetilde{k}^{+/-}(s,t)$ is extended in such a way that the functions $\widetilde{k}^{+/-}(s,\cdot)$ are non-decreasing for each $s \in [0,1]$.

In this case $\| \widetilde{p}^+ \|_{\mathbf{L}} \leqslant \| p^+ \|_{\mathbf{L}}$, $\| \widetilde{p}^- \|_{\mathbf{L}} \leqslant \| p^- \|_{\mathbf{L}}$, and

$$\widetilde{g}(t) = \begin{cases} g(t), & t \in e_1 \cup e_2, \\ 0, & t \in [0,1] \setminus (e_1 \cup e_2), \end{cases}$$

is a solution to the boundary value problem

$$\dot{\widetilde{g}}(t) = \int_0^1 \widetilde{r}(s,t)\widetilde{g}(s)\,ds, \quad t \in [0,1], \quad \widetilde{g}(0) = 0, \quad \widetilde{g}(1) = 0,$$
$$\int_0^1 (\widetilde{p}^+(s) - \widetilde{p}^-(s))g(s)\,ds = 0.$$

Thus, $(\mathcal{P},\mathcal{Q}) \notin \Omega$, and there exists an operator $\widetilde{\mathcal{L}} \in \mathcal{M}_{\mathcal{P},\mathcal{Q}}$ such that the equation $\widetilde{\mathcal{L}}^*\widetilde{g} = 0$ has a piecewise linear solution satisfying the conditions: $\widetilde{g}(t) = 0$ if $t \in [0,a_1] \cup [b_1,a_2] \cup [b_2,1]$, $\widetilde{g}(t) < 0$ if $t \in (a_1,b_1)$, $\widetilde{g}(t) > 0$ if $t \in (a_2,b_2)$, where $0 \leqslant a_1 < c_1 < b_1 \leqslant a_2 < c_2 < b_2 \leqslant 1$. □

Lemma 2.5. *Let $(\mathcal{P},\mathcal{Q}) \notin \Omega$. There exist points $0 = t_1 < t_2 < t_3 < t_4 = 1$, and functions $p_j \in \mathbf{L}$ such that the operator $\mathcal{L} : \mathbf{AC}^1 \to \mathbf{L}$ defined by the equality*

$$\mathcal{L}x \equiv \ddot{x}(t) - \sum_{j=1}^{4} p_j(t)x(t_j), \quad t \in [0,1],$$

belongs to the set $\mathcal{M}_{\mathcal{P},\mathcal{Q}}$ and is not surjective.

Proof. The operator $\mathcal{L}$, whose existence is asserted in the Lemma 2.4, has the form

$$(\mathcal{L}x)(t) = \ddot{x}(t) - \sum_{j\in J_1} p_j(t)x(t_j), \quad t \in [0,1],$$

where $p_j = p_j^+ - p_j^-$, p_j^+, $p_j^- \in \mathbf{L}$, $p_j^+(t) \geqslant 0$, $p_j^-(t) \geqslant 0$, $t \in [0,1]$, $\sum_{j\in J_1} \| p_j^+ \|_{\mathbf{L}} \leqslant \mathcal{P}$, $\sum_{j\in J_1} \| p_j^- \|_{\mathbf{L}} \leqslant \mathcal{Q}$. The set $\{t_j | j \in J_1\} = \{0, a_1, c_1, b_1, a_2, c_2, b_2, 1\}$ is finite.

The adjoint boundary value problem has the form (2.10), (2.11), (2.12).

By Lemma 2.4 the adjoint boundary value problem

$$g(s) - \sum_{j\in J_1} (t_j - s)^+ \int_0^1 g(\tau)p_j(\tau)\,d\tau = 0, \quad s \in [0,1], \tag{2.15}$$

$$\int_0^1 g(t) \sum_{j\in J_1} p_j(t)\,dt = 0, \tag{2.16}$$

$$\int_0^1 g(t) \sum_{j\in J_1} t_j p_j(t)\,dt = 0. \tag{2.17}$$

has a nontrivial piecewise linear solution g.

We see that problem (2.15)–(2.17) depends only on the values $A_j = \int_0^1 g(t)p_j(t)\,dt$. We know that $\min_{t\in[0,1]} g(t) = g(c_1)$, $\max_{t\in[0,1]} g(t) = g(c_2)$.

It is obvious that we can change the functions p_j so that the values A_j, $\| p_j^+ \|_L$, $\| p_j^- \|_L$ are preserved, and the support of each of the functions p_j is in any sufficiently small neighborhoods of four points, $t = 0$, $t = 1$, c_1, c_2. Let this condition be satisfied.

We will show that from the set $\{t_j | j \in J_1\}$ we can successively exclude the points a_1, b_1, a_2, b_2 by setting $p_j = 0$ for the corresponding j. Let $t_k \in (0,1)$ be one of such points, and let the supports of all functions p_k not intersect the interval (t_{k-1}, t_{k+1}). Then for some $\xi \in (0,1)$ the equality $t_k = \xi t_{k-1} + (1-\xi)t_{k+1}$ holds. Let $\widetilde{p}_k = 0$, $\widetilde{p}_{k-1} = p_{k-1} + \xi p_k$, $\widetilde{p}_{k+1} = p_{k+1} + (1-\xi)p_k$, $\widetilde{p}_i = p_i$ for $i \in J_1 \setminus \{t_{k-1}, t_k, t_{k+1}\}$. Let

$$\widetilde{g}(s) - \sum_{j\in J_1} (t_j - s)^+ \int_0^1 g(\tau)\widetilde{p}_j(\tau)\,d\tau = 0, \quad s \in [0,1].$$

It is easy to check that $\widetilde{g}$ satisfies the equation $\widetilde{\mathcal{L}}^*\widetilde{g} = 0$ for

$$(\widetilde{\mathcal{L}}x)(t) = \ddot{x}(t) - \sum_{j\in J_1} \widetilde{p}_j(t)x(t_j), \quad t \in [0,1].$$

Obviously, $\widetilde{g}$ satisfies the boundary conditions (2.16), (2.17). □

3 Main result

Let us formulate the main assertion.

Theorem 3.1. *Let non-negative $\mathcal{P}$ and $\mathcal{Q}$ be given. $(\mathcal{P},\mathcal{Q}) \in \Omega$ if and only if*

$$\mathcal{Q} \in [0,4], \quad \mathcal{P} \leqslant \min_{k\in(0,1]} \frac{(1+\sqrt{k}+\sqrt{k+1})^2(k+1) - \mathcal{Q}}{k},$$

or

$$\mathcal{Q} \in (4, 12+8\sqrt{2}], \quad \mathcal{P} \leqslant \min_{k\in[0,1]} \left((1+\sqrt{k}+\sqrt{k+1})^2(k+1) - \mathcal{Q}\,k\right).$$

Proof. By Lemma 2.5, if $(\mathcal{P},\mathcal{Q}) \notin \Omega$, then there exists a non-surjective operator $\mathcal{L} \in \mathcal{M}_{\mathcal{P},\mathcal{Q}}$ defined by

$$(\mathcal{L}x)(t) \equiv \ddot{x}(t) - \sum_{j=1}^{4} p_j(t)x(t_j), \quad t \in [0,1], \tag{3.1}$$

where $0 = t_1 < t_2 < t_3 < t_4 = 1$,

$$p_j = p_j^+ - p_j^-, \quad p_j^+, p_j^- \in \mathbf{L}, \quad p_j^+(t) \geqslant 0, \quad p_j^-(t) \geqslant 0, \quad t \in [0,1].$$

We will assume that

$$\int_0^1 p_j^+(s)\,ds = \mathcal{P}_j, \quad j = 1,\dots,4, \quad \sum_{j=1}^{4} \mathcal{P}_j = \mathcal{P},$$

$$\int_0^1 p_j^-(s)\,ds = \mathcal{Q}_j, \quad j = 1,\dots,4, \quad \sum_{j=1}^{4} \mathcal{Q}_j = \mathcal{Q}.$$

For the equation (3.1) the adjoint equation is constructed in part 2.1. Its solution satisfies the equalities (2.10), (2.11), (2.12). For convenience, we present here the adjoint boundary value problem

$$g(t) = \sum_{j=2,3,4} (t_j - t)^+ \int_0^1 g(s)p_j(s)\,ds, \quad t \in [0,1], \tag{3.2}$$

$$g(0) = 0, \quad g(1) = 0, \tag{3.3}$$

$$\sum_{j=1}^{4} \int_0^1 g(s)p_j(s)\,ds = 0, \tag{3.4}$$

$$\sum_{j=2}^{4} t_j \int_0^1 g(s)p_j(s)\,ds = 0, \tag{3.5}$$

which has a non-trivial solution g. Let us denote the maximum of this solution by $M \geqslant 0$ (taken at the point $t = t_3$) and the minimum by $-m \leqslant 0$ (taken at the point $t = t_2$). The

function g is linear on each of the intervals $[0, t_2]$, $[t_2, t_3]$, $[t_3, 1]$. At least one of the numbers M and m is positive. Let $m > 0$.

Let us find for what minimal $\mathcal{P}$, $\mathcal{Q}$ a non-zero solution to the problem (3.2)–(3.5) can exist.

Let $A_j = \int_0^1 g(s) p_j(s)\, ds$, $j = 1, 2, 3, 4$.

From (3.2) for $t \in [t_3, 1]$ it follows that

$$\dot{g}(t) = -A_4 = \frac{-M}{1 - t_3}, \tag{3.6}$$

$t \in (t_3, 1)$, therefore $A_4 \geqslant 0$.

From (3.2) for $t \in (t_2, t_3)$ it follows that $\dot{g}(t) = -A_4 - A_3 = \frac{M+m}{t_3 - t_2} > 0$. Since $A_4 \geqslant 0$, then $A_3 < 0$.

From (3.2) for $t \in (0, t_2)$ it follows that

$$\dot{g}(t) = -A_4 - A_3 - A_2 = \frac{-m}{t_2} < 0. \tag{3.7}$$

Therefore, from (3.4) it follows that $\dot{g}(t) = A_1 = \frac{-m}{t_2}$, therefore $A_1 < 0$. In addition, we have $A_3 + A_4 < 0$, $A_4 + A_3 + A_2 > 0$. Therefore, $A_2 > 0$.

So, $A_1 < 0$, $A_2 > 0$, $A_3 < 0$, $A_4 \geqslant 0$.

To satisfy these conditions with minimal $\mathcal{P}$ and $\mathcal{Q}$, the supports of the functions p_1^+, p_2^-, p_3^+, p_4^- should be concentrated in a sufficiently small neighborhood of the point $t = t_2$, which is the minimum point of the function g. Similarly, the supports of the functions p_1^-, p_2^+, p_3^-, p_4^+ should be concentrated in a sufficiently small neighborhood of the point $t = t_3$, which is the maximum point of the function g. We have

$$\inf_{\substack{\| p_i^+ \|_{\mathrm{L}} = P_i, \\ \| p_i^- \|_{\mathrm{L}} = Q_i}} A_i = -M Q_i - m P_i, \quad i = 1, 3; \qquad \sup_{\substack{\| p_i^+ \|_{\mathrm{L}} = P_i, \\ \| p_i^- \|_{\mathrm{L}} = Q_i}} A_i = M P_i + m Q_i, \quad i = 2, 4, \tag{3.8}$$

while inf and sup in (3.8) cannot be achieved for the integrable functions p_j.

Consider the limiting case

$$A_i = -M Q_i - m P_i, \quad i = 1, 3; \quad A_i = M P_i + m Q_i, \quad i = 2, 4. \tag{3.9}$$

From (3.4) and (3.9), it follows that

$$A_1 + A_2 + A_3 + A_4 = M (P_2 + P_4 - Q_1 - Q_3) - m (P_1 + P_3 - Q_2 - Q_4) = 0. \tag{3.10}$$

From (3.5) and (3.9), we obtain

$$t_2 (M P_2 + m Q_2) - t_3 (M Q_3 + m P_3) + M P_4 + m Q_4 = 0. \tag{3.11}$$

Now, for $k \equiv \frac{M}{-m} \geqslant 0$ from (3.6), (3.7), (3.10), and (3.11), it follows that

$$k = \frac{Q_4(1 - t_3)}{1 - P_4(1 - t_3)} = \frac{1 - P_1 t_2}{Q_1 t_2} = \frac{P_1 + P_3 - Q_2 - Q_4}{P_2 + P_4 - Q_1 - Q_3} = \frac{P_3 t_2 - Q_2 t_2 - Q_4}{P_2 t_2 + P_4 - Q_3 t_3}.$$

These inequalities are equivalent to the system

$$\begin{cases} P_1 + k Q_1 = \gamma_1, \\ k P_2 + Q_2 = \gamma_1 + (1 + k)\gamma_2, \\ P_3 + k Q_3 = k \gamma_3 + (1 + k)\gamma_2, \\ k P_4 + Q_4 = k \gamma_3, \end{cases} \tag{3.12}$$

where $\gamma_i \equiv \frac{1}{\Delta_i}$, $\Delta_i \equiv t_{i+1} - t_i$, $i = 1, 2, 3$.

From here and from the non-negativity of P_i, we obtain the conditions for Q_i:

$$\begin{cases} Q_1 \leqslant \frac{\gamma_1}{k}, \\ Q_2 \leqslant \gamma_1 + (1+k)\gamma_2, \\ Q_3 \leqslant \gamma_3 + \frac{(1+k)}{k}\gamma_2, \\ Q_4 \leqslant k\,\gamma_3. \end{cases} \tag{3.13}$$

From (3.12) we get

$$\mathcal{P} = \mathcal{P}_1 + \mathcal{P}_2 + \mathcal{P}_3 + \mathcal{P}_4 = \mathcal{R} - \frac{1}{k}(Q_2 + Q_4) - k(Q_1 + Q_3), \tag{3.14}$$

where

$$\mathcal{R} \equiv \gamma_1 \frac{1+k}{k} + \gamma_2 \frac{(1+k)^2}{k} + \gamma_3(1+k).$$

We need to find the minimum of $\mathcal{P}$ over $k \in (0, 1]$, over all γ_i, $i = 1, 2, 3$, $\frac{1}{\gamma_1} + \frac{1}{\gamma_2} + \frac{1}{\gamma_3} \leqslant 1$, and over all $Q_i \geqslant 0$ such that $Q_1 + Q_2 + Q_3 + Q_4 = \mathcal{Q}$ and conditions (3.13) are satisfied. Let us find the maximum value of Q for which conditions (3.13) are satisfied. We have

$$Q = Q_1 + Q_2 + Q_3 + Q_4 \leqslant \mathcal{R}.$$

Set $\Delta_2 = 1 - \Delta_1 - \Delta_3$ and find the minimum

$$\mathcal{R} = (1+k)\left(\frac{1}{\Delta_1 k} + \frac{1+k}{(1-\Delta_1-\Delta_3)k} + \frac{1}{\Delta_3}\right)$$

relative to the variables Δ_1, Δ_3, with the other arguments fixed. From the conditions $\frac{\partial \mathcal{P}}{\partial \Delta_1} = \frac{\partial \mathcal{P}}{\partial \Delta_3} = 0$, we have

$$\Delta_1 = \frac{1}{1 + \sqrt{k} + \sqrt{k+1}}, \quad \Delta_3 = \Delta_1\sqrt{k}, \quad \Delta_2 = \Delta_1\sqrt{k+1}.$$

Introduce the notations

$$G(k) \equiv (1 + \sqrt{k} + \sqrt{k+1})^2(k+1), \tag{3.15}$$

and for every Q

$$H_1(k) \equiv \frac{G(k) - Q}{k}, \tag{3.16}$$

$$H_2(k) \equiv G(k) - k\,Q. \tag{3.17}$$

Then

$$\mathcal{R} = (1+k)\frac{(1 + \sqrt{k} - \sqrt{k+1})^2}{(\sqrt{k+1} - \sqrt{k})^2(\sqrt{k+1} - 1)^2} = \frac{G(k)}{k}.$$

We can compute the derivative:

$$\frac{dR}{dk} = k^{-5/2}\frac{1 + \sqrt{k} + \sqrt{k+1}}{\sqrt{k+1}}(k^2(\sqrt{k} + \sqrt{k+1}) - \sqrt{k}(\sqrt{k+1} + 1) < 0.$$

It is easy to check that $\frac{dR}{dk} < 0$ for all $k \in (0, 1)$. Therefore, $\min_{k\in(0,1]} \mathcal{R}(k) = \mathcal{R}(1) = 2(2 + \sqrt{2})^2 = 12 + 8\sqrt{2} \equiv \mathcal{Q}^{\max}$.

Thus, if $\mathcal{Q} = \mathcal{Q}^{\max}$, then equalities (3.12) are possible only if $\mathcal{P} = 0$ and not for integrable functions p_j. If $\mathcal{Q} > \mathcal{Q}^{\max}$, then equalities (3.12) are impossible.

Let $\mathcal{Q} \in [0, \mathcal{Q}^{\max})$. Find the minimum of $\mathcal{P}$ in (3.14) for $\mathcal{Q}_i \geqslant 0$ such that $\sum_{i=1}^{4} \mathcal{Q}_i = \mathcal{Q}$ and with the other variables fixed. Increasing Q_2 and Q_4 until the constraints (3.13) are satisfied and decreasing Q_1 and Q_3 to zero, while keeping $\mathcal{Q}$ the same, we obtain one of the following two cases:

i) if $\mathcal{Q} \leqslant \gamma_1 + \gamma_2(1+k) + k\gamma_3$, then $\mathcal{Q}_1 = \mathcal{Q}_3 = 0$ and

$$\mathcal{P} = \mathcal{R} - \frac{\mathcal{Q}}{k},$$

ii) if $\mathcal{Q} \geqslant \gamma_1 + \gamma_2(1+k) + k\gamma_3$, then $\mathcal{Q}_2 = \gamma_1 + \gamma_2(1+k)$, $\mathcal{Q}_4 = k\gamma_3$ and

$$\mathcal{P} = \mathcal{R} - k(\mathcal{Q} - (\gamma_1 + \gamma_2(1+k) + k\gamma_3)) - \frac{\gamma_1 + \gamma_2(1+k) + k\gamma_3}{k} = k\mathcal{R} - k\mathcal{Q}.$$

For a fixed k, the minimum value of $\mathcal{R}$ is $\frac{G(k)}{k}$, and the following equality holds for the constraint on $\mathcal{Q}$

$$\gamma_1 + \gamma_2(1+k) + k\gamma_3 = (1 + \sqrt{k} + \sqrt{k+1})^2.$$

Thus, in case i) one should minimize $\mathcal{P} = H_1(k)$ subject to

$$\mathcal{Q} \leqslant (1 + \sqrt{k} + \sqrt{k+1})^2, \tag{3.18}$$

in the second case, one should minimize $\mathcal{P} = H_2(k)$ subject to

$$\mathcal{Q} \geqslant (1 + \sqrt{k} + \sqrt{k+1})^2. \tag{3.19}$$

It is obvious that for $\mathcal{Q} \leqslant 4$ at the minimum point of $H_1(k)$, $k \in (0,1]$, the inequality (3.18) is satisfied.

It is easy to check that at the minimum point of $H_2(k)$, $k \in (0,1]$, the inequality (3.19) is satisfied if $\mathcal{Q} \geqslant 4$.

Since the limiting case (3.9) is not achieved on integrable coefficients p_j, the obtained boundaries belong to the set of everywhere solvability Ω. □

Now we obtain a representation for the boundaries of the set Ω, which in the notation (3.15), (3.16), (3.17) in accordance with Theorem 3.1 look like this: $(\mathcal{P}, \mathcal{Q}) \in \Omega$ if and only if

$$\mathcal{Q} \in [0,4], \quad \mathcal{P} \leqslant \widetilde{\mathcal{P}_1}(\mathcal{Q}) \equiv \min_{k \in (0,1]} H_1(k).$$

or

$$\mathcal{Q} \in (4, 12 + 8\sqrt{2}], \quad \mathcal{P} \leqslant \widetilde{\mathcal{P}_2}(\mathcal{Q}) \equiv \min_{k \in [0,1]} H_2(k).$$

For all $\mathcal{Q} \in [0,4]$, the function H_1 takes its minimum at the point $k \in [0,1]$, which satisfies the equation $H_1'(k) = 0$, that is if

$$G'(k)k - G(k) + \mathcal{Q} = 0 \tag{3.20}$$

(here and below $(\cdot)' \equiv \frac{d(\cdot)}{dk}$). Therefore, the dependence $\widetilde{\mathcal{P}_1}(\mathcal{Q})$ for $\mathcal{Q} \in [0,4]$ can be defined parametrically using the equalities (3.16), (3.20) .

First, we define the number $k_0 = k_1^2 \approx 0.43$, where $k_1 \in [0,1]$ is the only root of the equation $k^4 + 6k^3 + 5k^2 - k = 0$ on the interval $[0,1]$. The number k_0 satisfies the equation (3.20) for $\mathcal{Q} = 4$. The function $\mathcal{Q}(k) = G(k) - G'(k)k$ maps $[k_0, 1]$ onto $[0,4]$.

Let us define the function $\widetilde{\mathcal{P}_1}(\mathcal{Q})$ parametrically:

$$\mathcal{Q} = G(k) - G'(k)k, \quad \widetilde{\mathcal{P}_1} \equiv G'(k), \quad k \in [k_0, 1]. \tag{3.21}$$

Denote $\mathcal{P}_4 \equiv \widetilde{\mathcal{P}_1}(4) = G'(k_0) \approx 17.7$.
We have, in particular,

$$\widetilde{\mathcal{P}_1}(0) = 12 + 8\sqrt{2} \quad (k = 1), \quad \widetilde{\mathcal{P}_1}(27/8) = 19 \quad (k = 9/16), \quad \widetilde{\mathcal{P}_1}(4) = \mathcal{P}_4.$$

For $\mathcal{Q} \in (4, \mathcal{P}_4]$, the function $H_2(k)$ takes a minimum value for $k \in [0, 1]$ at the point $k = 0$, therefore $\widetilde{\mathcal{P}_2}(\mathcal{Q}) = H_2(0) = G(0) = 4$. If $\mathcal{Q} \in (\mathcal{P}_4, 12 + 8\sqrt{2}]$, then the function $H_2(k)$ takes a minimum value at the maximum point satisfying the equality

$$H_2'(k) = G'(k) - \mathcal{Q} = 0. \tag{3.22}$$

Therefore, the dependence $\widetilde{\mathcal{P}_2}(\mathcal{Q})$ for $\mathcal{Q} \in (\mathcal{P}_4, 12 + 8\sqrt{2}]$ can also be specified parametrically using the equalities (3.17),(3.22):

$$\mathcal{Q} = G'(k), \quad \widetilde{\mathcal{P}_2} = G(k) - G'(k)k, \quad k \in [k_0, 1]. \tag{3.23}$$

From the parameterizations (3.21) and (3.23) it is clear that the dependencies $\widetilde{\mathcal{P}_1}(Q)$ and $\widetilde{\mathcal{P}_2}(\mathcal{Q})$ are mutually inverse.

If $\mathcal{Q} > 12 + 8\sqrt{2}$, the minimum value of $H_2(k)$ is negative (and is accepted for $k = 1$). Thus, operator (1.4) cannot be surjective for all linear positive operators satisfying equalities (1.5) for any P.

Fig. 3.1 shows the boundaries of the set Ω.

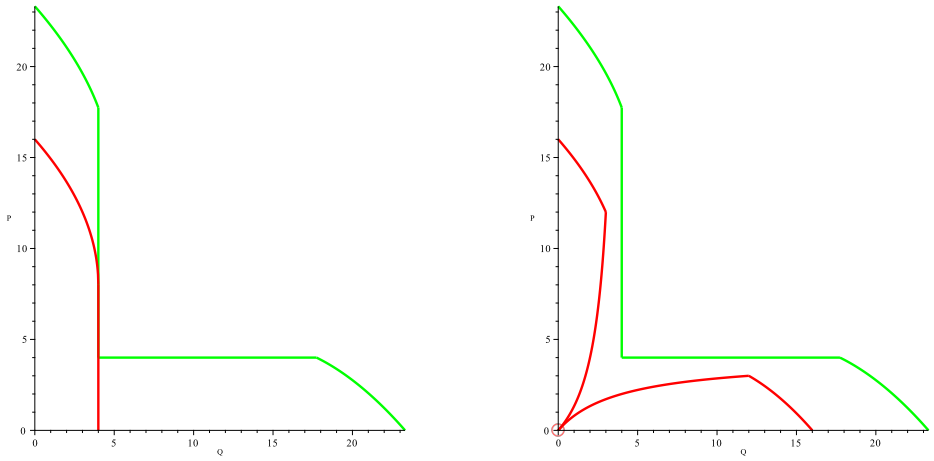


Figure 3.1: The boundaries of the set Ω are marked in green. On the left, the boundaries of the set of solvability of the two-point problem $x(0) = 0$, $x(1) = 0$ [23, 24] are marked in red. On the right, the boundaries of the set of solvability of the periodic problem (1.6) [17] are marked in red.

Let us formulate the statements just obtained as consequences of Theorem 3.1.

Corollary 3.2. *If* $(\mathcal{P},\mathcal{Q})\in\Omega$, *then* $(\mathcal{Q},\mathcal{P})\in\Omega$.

Corollary 3.3.

1. *If* $\mathcal{Q}=0$, *then* $(\mathcal{P},\mathcal{Q})\in\Omega$ *if and only if* $0\leqslant\mathcal{P}\leqslant 12+8\sqrt{2}$.
2. *If* $\mathcal{Q}\in[0,4]$, *then* $(\mathcal{P},\mathcal{Q})\in\Omega$ *if and only if* $\mathcal{P}\in[0,G'(k)]$, *where* $k\in(0,1]$ *is a unique solution to the equation* $G(k)-G'(k)k=\mathcal{Q}$.
3. *If* $\mathcal{Q}\in[4,\mathcal{P}_4]$, *then* $(\mathcal{P},\mathcal{Q})\in\Omega$ *if and only if* $\mathcal{P}\in[0,4]$.
4. *If* $\mathcal{Q}\in[\mathcal{P}_4,12+8\sqrt{2}]$, *then* $(\mathcal{P},\mathcal{Q})\in\Omega$ *if and only if* $\mathcal{P}\in[0,G(k)-G'(k)k]$, *where* $k\in(0,1]$ *is the maximum root of the equation* $G'(k)=\mathcal{Q}$.

There are also rational points on the boundary of Ω.

Corollary 3.4. *If* $\mathcal{Q}=27/8$, *then* $(\mathcal{P},\mathcal{Q})\in\Omega$ *if and only if* $\mathcal{P}\in[0,19]$.
If $\mathcal{Q}=19$, *then* $(\mathcal{P},\mathcal{Q})\in\Omega$ *if and only if* $\mathcal{P}\in[0,27/8]$.

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Attractivity with asymptotic phase of local center manifolds and an application to one-parameter bifurcation for integral equations with infinite delay

Dedicated to Professor Satoru Murakami on the occasion of his 70th birthday

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Abstract. For autonomous C^1-smooth integral equations with infinite delay, exponential attractivity with asymptotic phase of the local center manifolds of the equilibrium 0, together with a reduction principle, is proved by means of a dynamical systems approach based on the variation-of-constants formula in the phase space established in [*Funkcial. Ekvac.* **55**(2012), 479–520]. As its application to one-parameter family of integral equations, it is also shown that saddle-node and pitchfork bifurcations occur when the equilibrium 0 (the zero solution) of the linearized equation changes its stability properties.

Keywords: integral equations with infinite delay, attractivity with asymptotic phase, center manifolds, reduction principle, bifurcation, a variation-of-constants formula.

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1 Introduction

We consider in this paper the integral equation with infinite delay

$$x(t) = \int_{-\infty}^{t} K(t-s)x(s)ds + f(x_t), \tag{E}$$

where K is a measurable $m \times m$ matrix valued function with complex components that satisfies the conditions

$$\int_0^\infty \|K(t)\|e^{\rho t}dt < \infty \quad \text{and} \quad \operatorname{ess\,sup}\{\|K(t)\|e^{\rho t} : t \geq 0\} < \infty,$$

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and f belongs to the space $C^1(X;\mathbb{C}^m)$, the set of all continuously (Fréchet) differentiable functions mapping X into $\mathbb{C}^m$, with the property that $f(0) = 0$ and $Df(0) = 0$; here, ρ is a positive constant which is fixed throughout the paper. Let $X := L^1_\rho(\mathbb{R}^-;\mathbb{C}^m)$, $\mathbb{R}^- := (-\infty, 0]$, be a Banach space which will be introduced in the next section as the phase space for Eq. (E), and x_t denotes the element in X defined as $x_t(\theta) = x(t+\theta)$ for $\theta \in \mathbb{R}^-$. In [21] the first author et al. established a "variation-of-constants formula" for integral equations (VCF, for short) in the phase space, which allows us to study behaviors of solutions to Eq. (E) in the dynamical systems framework. Indeed, by means of VCF, the present authors have proved center manifold theorem; that is, the existence, the (local) exponential attractivity and so on of the (local) center manifolds of the equilibrium point 0 of Eq. (E). For VCF in the phase space for abstract functional differential equations with infinite delay and its applications to study of such as almost periodic solutions and invariant manifold theory, see [12–14, 25–27]. We should also refer the reader for treatments of Eq. (E) by adjoint semigroup theory to the pioneering works due to Diekmann and Gyllenberg [5], in which several important results, together with the principle of linearized stability for integral equations, are established.

One of the purposes of the paper is to improve a part of our preceding results on center manifolds for Eq. (E) [23, Theorems 4 and 5]; more precisely, to show that the local center manifolds are exponentially attractive with asymptotic phase under the assumption that the nonlinear term f is of class C^1. Thus, it turns out that the behavior of any solution of Eq. (E) in a neighborhood of the local center manifold of the (nonhyperbolic) equilibrium 0 is determined by the dynamics on this manifold which is described by a finite dimensional ordinary differential equation, which we call the *central equation* of Eq. (E). As its application we also intend to discuss one-parameter bifurcation structures for integral equations. These subjects have been extensively studied and now are popular for ordinary differential equations, functional differential equations, parabolic partial differential equations and so on (see [1, 2, 4, 6–11, 15–17, 19, 20, 24, 28, 29] and the literature cited therein). However, to the best of our knowledge, it seems that they have remained to be open problems to integral equations. Another purpose of the paper is to show that bifurcations of equilibria can occur for one-parameter family of integral equations with infinite delay. Our analysis will be thoroughly done in a dynamical systems viewpoint, based on VCF in the phase space.

The paper is organized as follows. In Section 2, we present preliminary results, e.g., VCF, the center manifold theorem for Eq. (E), and formal adjoint theory for the linear part of Eq. (E) ([21–23]), which are necessary for our later arguments. Section 3 proves the exponential attractivity with asymptotic phase of the local center manifolds of the equilibrium 0 for Eq. (E) under the assumption that the unstable subspace (of the phase space) is trivial (Theorem 3.3 (a)). In addition a reduction principle will be obtained, which is also a refinement of [23, Theorem 6]; so that, besides the asymptotic stability of the equilibrium 0 of Eq. (E), its stability also follows from that of the central equation of Eq. (E) (Theorem 3.3 (b)). We discuss in Section 4 one-parameter bifurcation problems for integral equations. For this we introduce an extended system of integral equations including the parameter as a state variable. By an application of Theorem 3.3 (a), combined with the study of the central equation of the extended system, we show that under some additional assumptions on the integral kernel, saddle-node and pitchfork bifurcations can be observed as the parameter takes the critical value (Theorem 4.4). In Section 5, we will give an example of a scalar integral equation to which our results in the previous section are applicable.

2 Notations and preliminary results

Let $\mathbb{N}$, $\mathbb{R}^-$, $\mathbb{R}^+$, $\mathbb{R}$ and $\mathbb{C}$ be the set of natural numbers, nonpositive real numbers, nonnegative real numbers, real numbers and complex numbers, respectively. For an $m \in \mathbb{N}$, we denote by $\mathbb{C}^m$ the space of all m-column vectors whose components are complex numbers, with the Euclidean norm $|\cdot|$.

Given Banach spaces $(U, \|\cdot\|_U)$ and $(V, \|\cdot\|_V)$ and positive integer n, we denote by $\mathcal{L}^n(U;V)$ the space of bounded n-linear mappings from U to W with norm

$$\|Q\|_{\mathcal{L}^n(U;V)} := \sup\{\|Q(u_1,\dots,u_n)\|_V : \|u_j\|_U \le 1,\ u_j \in U,\ j = 1,\dots,n\}$$

for $Q \in \mathcal{L}^n(U;V)$. We use the symbol $\mathcal{L}(U;V)$, the space of bounded linear operators, rather than $\mathcal{L}^1(U;V)$; and simply write $\mathcal{L}(U)$ in place of $\mathcal{L}(U;U)$. In particular, for $m \times m$-matrix M with complex components, $\|M\|$ means its operator norm $\|M\|_{\mathcal{L}(\mathbb{C}^m)}$.

For arbitrary linear operator A, the symbols $\mathcal{N}(A)$ and $\mathcal{R}(A)$ stand for the null space and the range of A, respectively.

Given an interval $J \subset \mathbb{R}$ and a Banach space U, we denote by $C(J;U)$ the space of U-valued continuous functions on J, and by $BC(J;U)$ its subspace of bounded continuous functions on J. We also use the notation $B_U(u_0;r)$ which stands for the open ball in U at the center u_0 with radius $r > 0$, that is, $B_U(u_0;r) = \{u \in U : \|u - u_0\|_U < r\}$. If $u_0 = 0$, we simply write $B_U(r)$ rather than $B_U(0;r)$. Also, $\overline{B}_U(u_0;r)$ denotes the closure of $B_U(u_0;r)$.

2.1 Phase space and initial value problems

Let ρ be a fixed positive constant, and let us introduce the function space

$$X := L^1_\rho(\mathbb{R}^-;\mathbb{C}^m) = \{\phi : \mathbb{R}^- \to \mathbb{C}^m : \phi(\theta)e^{\rho\theta} \text{ is integrable on } \mathbb{R}^-\}.$$

X is a Banach space endowed with norm

$$\|\phi\|_X := \int_{-\infty}^0 |\phi(\theta)|e^{\rho\theta}d\theta, \quad \phi \in X.$$

For any function $x : (-\infty, a) \to \mathbb{C}^m$ and $t < a$, we define a function $x_t : \mathbb{R}^- \to \mathbb{C}^m$ by $x_t(\theta) := x(t+\theta)$ for $\theta \in \mathbb{R}^-$; the function x_t is called the t-segment of $x(t)$.

Consider the integral equations

$$x(t) = \int_{-\infty}^t K(t-s)x(s)ds + p(t) \tag{2.1}$$

and

$$x(t) = \int_{-\infty}^t K(t-s)x(s)ds + f(x_t), \tag{E}$$

where we assume, throughout the paper, that the kernel K is a measurable $m \times m$-matrix valued function with complex components satisfying the condition

$$\|K\|_{1,\rho} := \int_0^\infty \|K(t)\|e^{\rho t}dt < \infty, \quad \|K\|_{\infty,\rho} := \operatorname{ess\,sup}\{\|K(t)\|e^{\rho t} : t \ge 0\} < \infty,$$

$p \in C(\mathbb{R};\mathbb{C}^m)$ and $f : X \to \mathbb{C}^m$ is of class C^1. Then Eq. (2.1) (resp. (E)) can be formulated as an abstract equation on the space X of the form

$$x(t) = F(t, x_t), \tag{2.2}$$

with $F(t,\phi) = L(\phi) + p(t)$ (resp. $L(\phi) + f(\phi)$) for $(t,\phi) \in \mathbb{R} \times X$, where

$$L(\phi) := \int_{-\infty}^{0} K(-\theta)\phi(\theta)d\theta, \quad \phi \in X.$$

Note that, in each case, $F(t,\phi)$ is well-defined because of

$$|L(\phi)| \le \int_{-\infty}^{0} \|K(-\theta)\|e^{-\rho\theta}|\phi(\theta)|e^{\rho\theta}d\theta \le \|K\|_{\infty,\rho}\|\phi\|_X.$$

Thus, X may be viewed as the phase space for Eq.'s (2.1) and (E); in what follows we will call X the phase space.

Now let $F : [b,\infty) \times X \to \mathbb{C}^m$ be any continuous function, and consider the equation (2.2) with the initial condition

$$x_\sigma = \phi, \ \text{ that is, } \ x(\sigma+\theta) = \phi(\theta) \ \text{ for } \ \theta \in \mathbb{R}^-, \tag{2.3}$$

where $(\sigma,\phi) \in [b,\infty) \times X$ is given arbitrarily. A function $x : (-\infty, a) \to \mathbb{C}^m$ is said to be a solution of the initial value problem (2.2)–(2.3) on the interval (σ, a) if x satisfies the following conditions:

(i) $x_\sigma = \phi$, that is, $x(\sigma+\theta) = \phi(\theta)$ for $\theta \in \mathbb{R}^-$;

(ii) $x \in L^1_{\mathrm{loc}}[\sigma, a)$, x is locally integrable on $[\sigma, a)$;

(iii) $x(t) = F(t, x_t)$ for $t \in (\sigma, a)$.

If $F(t,\phi)$ is locally Lipschitz continuous in ϕ, by [21, Proposition 1] the initial value problem (2.2)–(2.3) has a unique (local) solution, which is defined globally if $F(t,\phi)$ is globally Lipschitz continuous in ϕ ([21, Proposition 3]). So for any $(\sigma,\phi) \in \mathbb{R} \times X$ (2.1)–(2.3) has a unique global solution, denoted $x(t;\sigma,\phi,p)$, which is called the solution of Eq. (2.1) through (σ,ϕ). Similarly, (E)–(2.3) has a unique (local) solution, which is denoted by $x(t;\sigma,\phi,f)$. Note also that if $x(t)$ is a solution of Eq. (2.2) on (σ, a), then x_t is an X-valued continuous function on $[\sigma, a)$ (cf. [21, Lemma 1]).

When J is an interval in $\mathbb{R}$, a $\mathbb{C}^m$-valued function $\xi(t)$ is called a solution of Eq. (2.1) on J, if $\xi_t \in X$ is defined for all $t \in J$ and if it satisfies $x(t;\sigma,\xi_\sigma,p) = \xi(t)$ for all t and σ in J with $t \ge \sigma$; and, similarly, a $\mathbb{C}^m$-valued function $\xi(t)$ is called a solution of Eq. (E) on J whenever $\xi_t \in X$ for $t \in J$, and $x(t;\sigma,\xi_\sigma,f) = \xi(t)$ holds for all t and σ in J with $t \ge \sigma$.

2.2 A variation-of-constant formula and decomposition of the phase space

Now, for any $t \ge 0$ and $\phi \in X$, we define $T(t)\phi \in X$ by

$$\left[T(t)\phi\right](\theta) := x_t(\theta;,0,\phi,0) = \begin{cases} x(t+\theta;0,\phi,0), & -t < \theta \le 0, \\ \phi(t+\theta), & \theta \le -t. \end{cases}$$

Then $T(t)$ defines a bounded linear operator on X. We call $T(t)$ the solution operator of the homogeneous integral equation

$$x(t) = \int_{-\infty}^{t} K(t-s)x(s)ds. \tag{2.4}$$

$\{T(t)\}_{t\geq 0}$ is a strongly continuous semigroup of bounded linear operators on X, called a solution semigroup for Eq. (2.4).

Given a positive integer n, we introduce a continuous function $\Gamma^n : \mathbb{R}^- \to \mathbb{R}^+$ which is of compact support with $\operatorname{supp}\Gamma^n \subset [-1/n, 0]$ and satisfies $\int_{-\infty}^{0} \Gamma^n(\theta)d\theta = 1$. Obviously, $\Gamma^n x \in X$ for $x \in \mathbb{C}^m$ and the inequality $\|\Gamma^n x\|_X \leq |x|$ holds.

The following theorem plays a crucial role throughout the paper, which gives a representation formula for solutions of Eq. (2.1) in the phase space X, and is called the variation-of-constants formula (VCF, for short) in the phase space.

Theorem 2.1 ([21, Theorem 3]). *The segment $x_t(\sigma, \phi, p)$ of solution $x(\cdot; \sigma, \phi, p)$ of Eq. (2.1) satisfies the following relation in X:*

$$x_t(\sigma, \phi, p) = T(t-\sigma)\phi + \lim_{n\to\infty} \int_\sigma^t T(t-s)(\Gamma^n p(s))ds, \quad t \geq \sigma.$$

Let $\overline{X}$ be a subset of X of elements $\phi \in X$ which are continuous on $[-\varepsilon_\phi, 0]$ for some $\varepsilon_\phi > 0$, and set

$$X_0 := \{\psi \in X : \psi = \phi \text{ a.e. on } \mathbb{R}^- \text{ for some } \phi \in \overline{X}\}.$$

Then for any $\psi \in X_0$ we can define the value of ψ at $\theta = 0$ by $\psi[0] := \phi(0)$, where ϕ is an element in $\overline{X}$ satisfying $\psi = \phi$ a.e. on $\mathbb{R}^-$. $\psi[0]$ is well-defined, and X_0 is a normed linear space with norm

$$\|\psi\|_{X_0} := \|\psi\|_X + |\psi[0]|, \quad \psi \in X_0.$$

By [21, Lemma 1], we note that the solution $x(\cdot; \sigma, \phi, p)$ of Eq. (2.1) through $(\sigma, \phi) \in \mathbb{R} \times X$ satisfies $x_t(\sigma, \phi, p) \in X_0$ with $(x_t(\sigma, \phi, p))[0] = x(t; \sigma, \psi, p)$ for $t > \sigma$.

The next theorem describes an intimate relation between solutions of Eq. (2.1) and X-valued functions satisfying an integral equation which arises from the variation-of-constants formula in the phase space.

Theorem 2.2 ([21, Theorem 4]). *Let $p \in C(\mathbb{R}; \mathbb{C}^m)$.*

(i) *If $x(t)$ is a solution of Eq. (2.1) on the entire $\mathbb{R}$, then the X-valued function $\xi(t) := x_t$ satisfies the relations*

  (a) $\xi(t) = T(t-\sigma)\xi(\sigma) + \lim_{n\to\infty} \int_\sigma^t T(t-s)(\Gamma^n p(s))ds$, $\forall\, (t, \sigma) \in \mathbb{R}^2$ *with* $t \geq \sigma$, *in* X;

  (b) $\xi \in C(\mathbb{R}; X_0)$.

(ii) *Conversely, if a function $\xi : \mathbb{R} \to X$ satisfies the relation*

$$\xi(t) = T(t-\sigma)\xi(\sigma) + \lim_{n\to\infty} \int_\sigma^t T(t-s)(\Gamma^n p(s))ds, \quad \forall\, (t, \sigma) \in \mathbb{R}^2 \text{ with } t \geq \sigma,$$

then

  (c) $\xi \in C(\mathbb{R}; X_0)$;

  (d) *if we set $u(t) = (\xi(t))[0]$ for $t \in \mathbb{R}$, then $u \in C(\mathbb{R}; \mathbb{C}^m)$, $u_t = \xi(t)$ (in X) for any $t \in \mathbb{R}$ and u is a solution of Eq. (2.1) on $\mathbb{R}$.*

Based on spectral analysis of the generator A of the solution semigroup $\{T(t)\}_{t\geq 0}$, we also have established the decomposition theorem of the phase space X ([21]); let $\sigma(A)$ and $P_\sigma(A)$

be the spectrum and the point spectrum of the generator A, respectively. Then the following relation holds between the spectrum of A and the characteristic roots of Eq. (2.4)

$$\sigma(A) \cap \mathbb{C}_{-\rho} = P_\sigma(A) \cap \mathbb{C}_{-\rho} = \{\lambda \in \mathbb{C}_{-\rho} : \det \Delta(\lambda) = 0\},$$

where $\mathbb{C}_{-\rho} := \{z \in \mathbb{C} : \mathrm{Re}\, z > -\rho\}$, and $\Delta(\lambda)$ is the characteristic operator of Eq. (2.4), i.e., $\Delta(\lambda) := E_m - \int_0^\infty K(t)e^{-\lambda t}dt$, E_m being the $m \times m$-unit matrix ([21, Proposition 4]). Moreover, for ess (A), the essential spectrum of A, we have $\sup_{\lambda \in \mathrm{ess}\,(A)} \mathrm{Re}\,\lambda \leq -\rho$ ([21, Corollarly 2]). Now set $\Sigma^u := \{\lambda \in \sigma(A) : \mathrm{Re}\,\lambda > 0\}$, $\Sigma^c := \{\lambda \in \sigma(A) : \mathrm{Re}\,\lambda = 0\}$, and $\Sigma^s := \sigma(A)\backslash(\Sigma^c \cup \Sigma^u)$. The decomposition theorem of X is the following.

Theorem 2.3 ([21, Theorem 2]). *Let $\{T(t)\}_{t\geq 0}$ be the solution semigroup of Eq. (2.4). Then X is decomposed as a direct sum of closed subspaces E^u, E^c, and E^s*

$$X = E^u \oplus E^c \oplus E^s$$

with the following properties:

(i) $\dim(E^u \oplus E^c) < \infty$,

(ii) $T(t)E^u \subset E^u$, $T(t)E^c \subset E^c$, *and* $T(t)E^s \subset E^s$ *for* $t \in \mathbb{R}^+$,

(iii) $\sigma(A|_{E^u}) = \Sigma^u$, $\sigma(A|_{E^c}) = \Sigma^c$ *and* $\sigma(A|_{E^s \cap \mathcal{D}(A)}) = \Sigma^s$,

(iv) $T^u(t) := T(t)|_{E^u}$ *and* $T^c(t) := T(t)|_{E^c}$ *are extendable for* $t \in \mathbb{R}$ *as groups of bounded linear operators on* E^u *and* E^c, *respectively,*

(v) $T^s(t) := T(t)|_{E^s}$ *is a strongly continuous semigroup of bounded linear operators on* E^s, *and its generator is identical with* $A|_{E^s \cap \mathcal{D}(A)}$,

(vi) *there exist positive constants* α, ε *with* $\alpha > \varepsilon$ *and a constant* $C \geq 1$ *such that*

$$\begin{aligned} \|T^s(t)\|_{\mathcal{L}(X)} &\leq Ce^{-\alpha t}, \quad t \in \mathbb{R}^+, \\ \|T^u(t)\|_{\mathcal{L}(X)} &\leq Ce^{\alpha t}, \quad t \in \mathbb{R}^-, \\ \|T^c(t)\|_{\mathcal{L}(X)} &\leq Ce^{\varepsilon|t|}, \quad t \in \mathbb{R}. \end{aligned}$$

Note that in (vi) of the theorem above C is a constant depending only on the positive numbers α and ε, and that ε can be taken arbitrarily small. We will use the notations $E^{cu} = E^c \oplus E^u$, $E^{su} = E^s \oplus E^u$ etc, and denote by Π^s the projection from X onto E^s along E^{cu}, and likewise for Π^u, Π^{cu} etc. Also, we set

$$C_1 := \|\Pi^s\|_{\mathcal{L}(X)} + \|\Pi^c\|_{\mathcal{L}(X)} + \|\Pi^u\|_{\mathcal{L}(X)}.$$

By an *equilibrium (or equilibrium point) of the integral equation* (E) we mean that of the semi-dynamical system on the phase space X induced by Eq. (E); namely, let $U : \mathbb{R}^+ \times X \to X$ be the map defined by

$$U(t, \phi) := x_t(0, \phi, f), \quad (t, \phi) \in \mathbb{R}^+ \times X.$$

We then call $\phi \in X$ an equilibrium (or equilibrium point) of Eq. (E) if $U(t, \phi) = \phi$ holds for all $t \in \mathbb{R}^+$. If ϕ is an equilibrium, then $\phi(\theta) =$ const. a.e. $\theta \in \mathbb{R}^-$. Indeed, put $u(t) := x(t; 0, \phi, f)$, then $U(t, \phi) = \phi$ implies that $u(t + \theta) = \phi(\theta) = u(\theta)$ a.e. on $\mathbb{R}^-$ for all $t \in \mathbb{R}^+$; hence any weak derivative of $u(\theta)$ is 0 a.e., so that $\phi(\theta) = u(\theta) =$ const. a.e. on $\mathbb{R}^-$. When 0 is an equilibrium, we often call it the zero solution of the integral equation.

If $f \in C^1(X; \mathbb{C}^m)$ satisfies $f(0) = 0$ and $Df(0) = 0$, Eq. (2.4) is the linearized equation of Eq.(E) around the equilibrium 0. The equilibrium 0 (or the zero solution) of Eq. (E) is said to be *hyperbolic* provided that $\Delta(\lambda)$ is invertible on the imaginary axis; that is, $\Sigma^c = \emptyset$.

2.3 Center manifold for an integral equation with modified nonlinear term

Suppose that $f \in C^1(X;\mathbb{C}^m)$ satisfies $f(0) = 0$ and $Df(0) = 0$. Under the assumption we have established the existence of local center manifolds of the equilibrium 0 of Eq. (*E*) and proved its exponential attractivity ([23, Theorem 5]). To this end, we discussed the corresponding problems for a modified equation of (*E*)

$$x(t) = \int_{-\infty}^{t} K(t-s)x(s)ds + f_\delta(x_t), \tag{E_δ}$$

where $f_\delta : X \to \mathbb{C}^m$ ($\delta > 0$) is defined by

$$f_\delta(\phi) := \chi\big(\|\Pi^{su}\phi\|_X/\delta\big)\chi\big(\|\Pi^c\phi\|_X/\delta\big)f(\phi), \quad \phi \in X,$$

χ being a C^∞-function on $\mathbb{R}$ satisfying $\chi(t) = 1$ ($|t| \le 2$) and $\chi(t) = 0$ ($|t| \ge 3$). Note that f_δ is continuous on X, and is of class C^1 when restricted to the open set $S_\delta := \{\phi \in X : \|\Pi^{su}\phi\|_X < \delta\}$ since we may assume that $\|\Pi^c\phi\|_X$ is of class C^1 for $\phi \neq 0$ because of $\dim E^c < \infty$. Also, by the assumption $f(0) = Df(0) = 0$, there exist a $\delta_1 > 0$ and a nondecreasing continuous function $\zeta_* : (0,\delta_1] \to \mathbb{R}^+$ such that $\zeta_*(+0) = 0$,

$$\|f_\delta(\phi)\|_X \le \delta\zeta_*(\delta), \quad \text{and} \quad \|f_\delta(\phi) - f_\delta(\psi)\|_X \le \zeta_*(\delta)\|\phi - \psi\|_X$$

for $\phi, \psi \in X$ and $\delta \in (0,\delta_1]$. Indeed, we may put

$$\zeta_*(\delta) = \left(\sup_{\|\phi\|_X \le 3\delta} \|Df(\phi)\|_{\mathcal{L}(X;\mathbb{C}^m)}\right)\left(1 + 3\sup_{0\le t\le 3}|\chi'(t)|\right)$$

(cf. [4, Lemma 4.1]).

Fix a positive number η such that $\varepsilon < \eta < \alpha$, where ε and α are the constants in Theorem 2.3. For the existence of center manifold for Eq. (E_δ) and its exponential attractivity property, we have established the following:

Theorem 2.4 ([23, Theorem 4]). *There exist a positive number δ and a C^1-map $F_{*,\delta} : E^c \to E^{su}$ with $F_{*,\delta}(0) = 0$ such that the following properties hold:*

(i) $W^c_\delta := \operatorname{graph} F_{*,\delta}$ *is tangent to E^c at zero,*

(ii) W^c_δ *is invariant for Eq. (E_δ), that is, if $\xi \in W^c_\delta$, then $x_t(0,\xi,f) \in W^c_\delta$ for $t \in \mathbb{R}$.*

(iii) *Assume moreover that $\Sigma^u = \emptyset$. Then there exists a positive constant β_0 with the property that if x is a solution of Eq. (E_δ) on an interval $J = [t_0, t_1]$, then the inequality*

$$\|\Pi^s x_t - F_{*,\delta}(\Pi^c x_t)\|_X \le C\|\Pi^s x_{t_0} - F_{*,\delta}(\Pi^c x_{t_0})\|_X e^{-\beta_0(t-t_0)}, \quad t \in J$$

holds true. In particular, if x is a solution on an interval $[t_0,\infty)$, x_t tends to W^c_δ exponentially as $t \to \infty$.

It has also been proved that W^c_δ has the same smoothness as the nonlinear term $f(\phi)$ does ([23, Appendix]; see also [27] for details). Let us briefly recall the outline of the proof of the existence part. Take a $\delta_1 > 0$ sufficiently small in such a way that

$$\zeta_*(\delta_1)CC_1\left(\frac{1}{\eta-\varepsilon} + \frac{2}{\alpha+\eta} + \frac{2}{\alpha-\eta}\right) < \frac{1}{2}$$

holds. Let Y_η be the Banach space

$$Y_\eta := \{y \in C(\mathbb{R}; X) : \sup_{t\in\mathbb{R}} \|y(t)\|_X e^{-\eta|t|} < \infty\}$$

with norm

$$\|y\|_{Y_\eta} := \sup_{t\in\mathbb{R}} \|y(t)\|_X e^{-\eta|t|}, \quad y \in Y_\eta,$$

and consider the map $\mathcal{F}_\delta : E^c \times Y_\eta \to Y_\eta$ defined by

$$\begin{aligned}\mathcal{F}_\delta(\psi, y)(t) := T^c(t)\psi + \lim_{n\to\infty} \int_0^t T^c(t-s)\Pi^c\Gamma^n f_\delta(y(s))ds \\ - \lim_{n\to\infty} \int_t^\infty T^u(t-s)\Pi^u\Gamma^n f_\delta(y(s))ds + \lim_{n\to\infty} \int_{-\infty}^t T^s(t-s)\Pi^s\Gamma^n f_\delta(y(s))ds\end{aligned} \tag{2.5}$$

for $(\psi, y) \in E^c \times Y_\eta$ and $t \in \mathbb{R}$. Then, for each $\psi \in E^c$, $\mathcal{F}_\delta(\psi, \cdot)$ is a contraction map from Y_η into itself provided that $0 < \delta \le \delta_1$, and therefore has a unique fixed point, say $\Lambda_{*,\delta}(\psi) \in Y_\eta$.

Now define $F_{*,\delta} : E^c \to E^{su}$ by $F_{*,\delta}(\psi) := \Pi^{su}\big(\Lambda_{*,\delta}(\psi)(0)\big)$ for $\psi \in E^c$. One can see from (2.5) and the relation $\Lambda_{*,\delta}(\psi)(0) = \mathcal{F}_\delta(\psi, \Lambda_{*,\delta}(\psi))(0)$ that

$$\begin{aligned}F_{*,\delta}(\psi) = & - \lim_{n\to\infty} \int_0^\infty T^u(-s)\Pi^u\Gamma^n f_\delta(\Lambda_{*,\delta}(\psi)(s))ds \\ & + \lim_{n\to\infty} \int_{-\infty}^0 T^s(-s)\Pi^s\Gamma^n f_\delta(\Lambda_{*,\delta}(\psi)(s))ds.\end{aligned} \tag{2.6}$$

The map $F_{*,\delta}$ is the required one and the center manifold is given by

$$W^c_\delta := \operatorname{graph} F_{*,\delta} = \{\psi + F_{*,\delta}(\psi) : \psi \in E^c\}.$$

The definition (2.6) of $F_{*,\delta}$ shall be used in the arguments in Subsection 4.4; W^c_δ will be also denoted by $W^c_\delta(0)$ in later sections.

Let r be a positive number with $r \le \delta$, set $F_* := F_{*,\delta}|_{B_{E^c}(r)}$ and $\Omega_0 := \{\phi \in X : \|\Pi^{su}\phi\|_X < \delta,\ \|\Pi^c\phi\|_X < r\}$. Then, $f \equiv f_\delta$ on Ω_0, and therefore, Theorem 2.4 assures that $W^c_{\mathrm{loc}}(0) := \operatorname{graph} F_*$ is a local center manifold for Eq. (E) (see [23, Theorem 5]).

The following proposition is often used in the subsequent sections.

Proposition 2.5 ([23, Propositions 2 and 3]). *The maps $\Lambda_{*,\delta}$ and $F_{*,\delta}$ are (globally) Lipschitz continuous and have the following properties:*

(i) $\|F_{*,\delta}(\psi_1) - F_{*,\delta}(\psi_2)\|_X \le L(\delta)\|\psi_1 - \psi_2\|_X$ *for* $\psi_1, \psi_2 \in E^c$, *where* $L(\delta)$ *is the constant given by* $L(\delta) := 4C^2C_1\zeta_*(\delta)/(\alpha - \eta)$.

(ii) *For* $\hat\phi \in W^c_\delta$ *and* $\tau \in \mathbb{R}$,

$$\Pi^{su}x_t(\tau, \hat\phi, f_\delta) = F_{*,\delta}\big(\Pi^e x_t(\tau, \hat\phi, f_\delta)\big), \quad t \in \mathbb{R}.$$

In particular W^c_δ is invariant for (E_δ), that is, $x_t(\tau, \hat\phi, f_\delta) \in W^c_\delta$ for $t \in \mathbb{R}$, provided that $\hat\phi \in W^c_\delta$.

2.4 The projection onto the center-unstable subspace via formal adjoint theory

Let $\Sigma^{cu} = \{\lambda_1, \dots, \lambda_r\}$; then each λ_i is a normal eigenvalue of the generator A, and hence, its generalized eigenspace $\mathcal{M}_{\lambda_i}(A)$ is of the form $\mathcal{N}((A-\lambda_i I)^{p_i})$, p_i being the ascent of λ_i. The center-unstable subspace E^{cu} $(= E^c \oplus E^u)$ is then expressed as

$$E^{cu} = \bigoplus_{i=1}^{r} \mathcal{M}_{\lambda_i}(A) = \bigoplus_{i=1}^{r} \mathcal{N}((A-\lambda_i I)^{p_i}) \tag{2.7}$$

([22, Subsection 2.2]); and, in addition, each direct summand is characterized by the following proposition. Given $\lambda \in \mathbb{C}_{-\rho}$ and $k \in \mathbb{N}$, consider a function $w_k(\lambda) : \mathbb{R}^- \to \mathbb{C}$ and a $(km) \times (km)$ matrix $D_k(\lambda)$ defined by

$$w_k(\lambda)(\theta) := \frac{\theta^{k-1}}{(k-1)!} e^{\lambda\theta}, \quad \theta \le 0, \tag{2.8}$$

$$D_k(\lambda) := \begin{pmatrix} \Delta(\lambda) & \Delta'(\lambda) & \cdots & \Delta^{(k-1)}(\lambda)/(k-1)! \\ 0 & \Delta(\lambda) & \cdots & \Delta^{(k-2)}(\lambda)/(k-2)! \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \Delta(\lambda) \end{pmatrix}, \tag{2.9}$$

where $\Delta^{(n)}(z) := (d^n/dz^n)\Delta(z)$ for $n \in \mathbb{N}$.

Proposition 2.6 ([22, Proposition 3.1])**.** *Let $\lambda \in \mathbb{C}_{-\rho}$ and $k \in \mathbb{N}$. Then the following statements are equivalent:*

(i) $\phi \in \mathcal{N}((A-\lambda I)^k)$,

(ii) $\phi = \sum_{j=1}^{k} w_j(\lambda)\eta_j$ *in* X, *where* $\eta_1, \dots, \eta_k$ *belong to* $\mathbb{C}^m$ *satisfying the relation*

$$D_k(\lambda)\,\mathrm{col}\,(\eta_1, \dots, \eta_k) = \mathrm{col}\,(0, \dots, 0).$$

Moreover, in [22] the formal adjoint operator $A^\sharp$ was introduced and some duality properties between A and $A^\sharp$ were observed. $A^\sharp$ is indeed defined as follows. Let $\mathbb{C}^{m*}$ be the space of all m-dimensional row vectors with complex components with the usual operator norm $|\cdot|$ as the dual space of $\mathbb{C}^m$. Let $X^\sharp$ be the Banach space defined by

$$X^\sharp := L^1_\rho(\mathbb{R}^+; \mathbb{C}^{m*}) = \left\{\alpha : \mathbb{R}^+ \to \mathbb{C}^{m*} : \alpha(s)e^{-\rho s} \text{ is integrable on } \mathbb{R}^+\right\}$$

with norm

$$\|\alpha\|_{X^\sharp} = \int_0^\infty |\alpha(s)| e^{-\rho s} ds, \quad \alpha \in X^\sharp,$$

and $\tilde{X}^\sharp$ be the subspace

$$\tilde{X}^\sharp = \Big\{\tilde{\alpha} \in X^\sharp : \tilde{\alpha} \text{ is locally absolutely continuous on } \mathbb{R}^+, \\ \frac{d}{ds}\tilde{\alpha} \in X^\sharp \text{ and } \tilde{\alpha}(0) = \int_0^\infty \tilde{\alpha}(s)K(s)ds\Big\}.$$

We define $A^\sharp : X^\sharp \supset \mathcal{D}(A^\sharp) \to X^\sharp$ by

$$A^\sharp\alpha := -\frac{d}{ds}\tilde{\alpha}, \quad \alpha \in \mathcal{D}(A^\sharp),$$

where $\mathcal{D}(A^\sharp) := \{\alpha \in X^\sharp : \alpha(s) = \tilde{\alpha}(s) \text{ a.e. } s \in \mathbb{R}^+ \text{ for some } \tilde{\alpha} \in \tilde{X}^\sharp\}$. In fact, the operator $A^\sharp$ is identical with the infinitesimal generator of the solution semigroup on $X^\sharp$ induced by an adjoint integral equation of (2.4) in some sense (see [22] for details).

A characterization of the space $\mathcal{N}((A^\sharp - \lambda I)^k)$ is obtained in a similar fashion to Proposition 2.6. For any $\lambda \in \mathbb{C}_{-\rho}$ and $k \in \mathbb{N}$, define $w_k^\sharp(\lambda) : \mathbb{R}^+ \to \mathbb{C}$ by

$$w_k^\sharp(\lambda)(s) := w_k(\lambda)(-s) = \frac{(-s)^{k-1}}{(k-1)!}e^{-\lambda s}, \quad s \geq 0. \tag{2.10}$$

Proposition 2.7 ([22, Proposition 3.4]). *Let $\lambda \in \mathbb{C}_{-\rho}$ and $k \in \mathbb{N}$. Then the following statements are equivalent:*

(i) $\alpha \in \mathcal{N}((A^\sharp - \lambda I)^k)$,

(ii) $\alpha = \sum_{j=1}^k w_j^\sharp(\lambda)\zeta_{k+1-j}$ *in* $X^\sharp$, *where* $\zeta_1, \dots, \zeta_k$ *belong to* $\mathbb{C}^{m*}$ *satisfying the relation*

$$(\zeta_1, \dots, \zeta_k)D_k(\lambda) = (0, \dots, 0).$$

Now, let us consider the bilinear form $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ on $X^\sharp \times X$ given by

$$\langle\!\langle \alpha, \phi \rangle\!\rangle := \int_{-\infty}^0 \int_\theta^0 \alpha(\xi - \theta)K(-\theta)\phi(\xi)d\xi d\theta, \quad \phi \in X, \ \alpha \in X^\sharp,$$

which is well-defined and bounded. Then $\mathcal{R}((A - \lambda_i I)^{p_i})$ is characterized as the annihilator of $\mathcal{N}((A^\sharp - \lambda_i I)^{p_i})$ with respect to this pairing ([22, Proposition 3.5]). So, by virtue of the fact $X = \mathcal{N}((A - \lambda_i I)^{p_i}) \oplus \mathcal{R}((A - \lambda_i I)^{p_i})$ and the fact that $\dim \mathcal{N}((A - \lambda_i I)^{p_i}) = \dim \mathcal{N}((A^\sharp - \lambda_i I)^{p_i})$, which follows from Propositions 2.6 and 2.7, one may think of $\mathcal{N}((A^\sharp - \lambda_i I)^{p_i})$ as the dual space of $\mathcal{N}((A - \lambda_i I)^{p_i})$. Hence, $\mathcal{N}^\sharp := \bigoplus_{i=1}^r \mathcal{N}((A^\sharp - \lambda_i I)^{p_i})$ plays a role of the dual space of E^{cu} due to (2.7).

So, let $\{\phi_1, \dots, \phi_d\}$ and $\{\psi_1, \dots, \psi_d\}$ be bases for E^{cu} and $\mathcal{N}^\sharp$, respectively; set $\Phi = (\phi_1, \dots, \phi_d)$ and $\Psi = \operatorname{col}(\psi_1, \dots, \psi_d)$. We denote by $\langle\!\langle \Psi, \Phi \rangle\!\rangle$ the $d \times d$ matrix whose (i,j)-component is $\langle\!\langle \psi_i, \phi_j \rangle\!\rangle$, and by $\langle\!\langle \Psi, \phi \rangle\!\rangle$ the column vector $\operatorname{col}(\langle\!\langle \psi_1, \phi \rangle\!\rangle, \dots, \langle\!\langle \psi_d, \phi \rangle\!\rangle)$ for any $\phi \in X$. Then, we have:

Theorem 2.8 ([22, Theorem 3.1]). *Let Φ, Ψ be the ones cited above. Then the matrix $\langle\!\langle \Psi, \Phi \rangle\!\rangle$ is nonsingular, and the projection $\Pi^{cu} : X \to E^{cu}$ is given by*

$$\Pi^{cu}\phi = \Phi\langle\!\langle \Psi, \Phi \rangle\!\rangle^{-1}\langle\!\langle \Psi, \phi \rangle\!\rangle, \quad \phi \in X.$$

3 Exponential attractivity with asymptotic phase of the local center manifold and a reduction principle

In this section we prove the global exponential attractivity with asymptotic phase of the center manifold $W_\delta^c(0)$ and a reduction principle for Eq. (*E*), which are refinements of Theorem 4 (iii) and [23, Theorem 6], respectively. Thus, behaviors of the solutions of Eq. (E_δ), including stability properties, are completely described by the dynamics on $W_\delta^c(0)$. As a corollary the corresponding results are obtained for Eq. (*E*).

3.1 Main theorems and preparatory propositions

Consider the integral equation

$$x(t) = \int_{-\infty}^{t} K(t-s)x(s)ds + f_\delta(x_t). \tag{E_δ}$$

Now, assume that $\Sigma^u = \emptyset$. Let d be the dimension of E^c, and Φ, Ψ the ones in the previous subsection. Since $\{T^c(t)\}_{t\geq 0}$ is a strongly continuous semigroup on the d-dimensional space E^c, there exists a $d \times d$ matrix G_c such that $T^c(t)\Phi = \Phi e^{tG_c}$ for $t \geq 0$ and $\sigma(G_c)$, the spectrum of G_c, is identical with Σ^c. Let us consider the ordinary differential equation on $\mathbb{C}^d$

$$z'(t) = G_c z(t) + H_c f_\delta(\Phi z(t) + F_{*,\delta}(\Phi z(t))), \tag{CE_δ}$$

where H_c is the $d \times m$ matrix given by

$$H_c x := \lim_{n\to\infty} \langle\!\langle \Psi, \Gamma^n x \rangle\!\rangle, \quad x \in \mathbb{C}^m.$$

We call Eq. (CE_δ) the *central equation* of Eq. (E_δ) (cf. [23, Subsection 3.2]).

Proposition 3.1. *Let $z(t)$ be a solution of Eq. (CE_δ) with $z(t_0) = z_0$ defined on an interval J, and $x(t)$ the solution of Eq. (E_δ) with $x_{t_0} = \Phi z_0 + F_{*,\delta}(\Phi z_0)$. Then $\Phi z(t) + F_{*,\delta}(\Phi z(t))$ is the segment of $x(t)$, that is,*

$$x_t(t_0, \Phi z_0 + F_{*,\delta}(\Phi z_0), f_\delta) = \Phi z(t) + F_{*,\delta}(\Phi z(t)), \quad t \in J.$$

Proof. Put $\hat{\phi} := \Phi z_0 + F_{*,\delta}(\Phi z_0))$, and let $\hat{z}(t)$ be the function determined by $\Phi\hat{z}(t) = \Pi^c x_t(t_0, \hat{\phi}, f_\delta)$. Then, by virtue of [21, Proposition 7], $\hat{z}(t)$ is a solution of

$$z'(t) = G_c z(t) + H_c f_\delta(\Phi z(t) + \Pi^{su} x_t(t_0, \hat{\phi}, f_\delta)).$$

Moreover, it follows from Proposition 2.5 (ii) that $\Pi^{su} x_t(t_0, \hat{\phi}, f_\delta) = F_{*,\delta}(\Pi^c x_t(t_0, \hat{\phi}, f_\delta)) = F_{*,\delta}(\Phi\hat{z}(t))$. Hence, $\hat{z}(t)$ is the solution of (CE_δ) with $\Phi\hat{z}(t_0) = \Pi^c\hat{\phi} = \Phi z_0$, i.e., $\hat{z}(t_0) = z_0$, so that $z(t) = \hat{z}(t)$ $(t \in J)$ due to the uniqueness of solutions of (CE_δ). Consequently, $x_t(t_0, \hat{\phi}, f_\delta) = \Pi^c x_t(t_0, \hat{\phi}, f_\delta) + \Pi^{su} x_t(t_0, \hat{\phi}, f_\delta) = \Phi z(t) + F_{*,\delta}(\Phi z(t))$ $(t \in J)$. □

One of our main results is the following reduction principle for the modified integral equation (E_δ).

Theorem 3.2. *Assume that $f \in C^1(X; \mathbb{C}^m)$ with $f(0) = Df(0) = 0$, and furthermore that $\Sigma^u = \emptyset$. Then for small $\delta > 0$ the following statements hold.*

(a) *Let β be a positive number satisfying $\varepsilon < \beta < \alpha$. If $x(t)$ is a solution of Eq. (E_δ) defined on $J := [t_0, \infty)$, then there exists a unique solution $z(t)$ of Eq. (CE_δ) on J satisfying*

$$\begin{aligned} \|\Pi^c x_t - \Phi z(t)\|_X &\leq C_0 \|\Pi^s x_{t_0} - F_{*,\delta}(\Phi z(t_0))\|_X e^{-\beta(t-t_0)}, \\ \|\Pi^s x_t - F_{*,\delta}(\Phi z(t))\|_X &\leq C_0 \|\Pi^s x_{t_0} - F_{*,\delta}(\Phi z(t_0))\|_X e^{-\beta(t-t_0)} \end{aligned} \tag{3.1}$$

for $t \in J$, where $C_0 > C$ is a constant which can be chosen as close to C as one expects by taking δ small. In particular, we have the estimate

$$\|x_t - x_t(t_0, \hat{\phi}, f_\delta)\|_X \leq 2C_0 \|\Pi^s x_{t_0} - F_{*,\delta}(\Phi z(t_0))\|_X e^{-\beta(t-t_0)}, \quad t \in J, \tag{3.2}$$

where $\hat{\phi} = \Phi z(t_0) + F_{*,\delta}(\Phi z(t_0)) \in W^c_\delta(0)$.

(b) *If the zero solution of Eq.* (CE_δ) *is stable (resp. asymptotically stable, unstable), the zero solution of Eq.* (E_δ) *is stable (resp. asymptotically stable, unstable).*

As a corollary, we obtain the following theorem for Eq. (*E*).

Theorem 3.3. *Assume that* $f \in C^1(X;\mathbb{C}^m)$ *with* $f(0) = Df(0) = 0$, *and furthermore that* $\Sigma^u = \emptyset$. *Then the following statements hold.*

(a) *Let* β *be a positive number satisfying* $\varepsilon < \beta < \alpha$. *Then there exists an open neighborhood* Ω_0 *of* 0 *in* X *such that if* $x(t)$ *is a solution of Eq.* (*E*) *defined on* $J := [t_0, t_1]$ *satisfying* $x_t \in \Omega_0$ *for* $t \in J$, *then there exists a solution* $z(t)$ *of Eq.* (CE) *on* J *with the property*

$$\begin{aligned} \|\Pi^c x_t - \Phi z(t)\|_X &\le C_0 \|\Pi^s x_{t_0} - F_*(\Phi z(t_0))\|_X e^{-\beta(t-t_0)}, \\ \|\Pi^s x_t - F_*(\Phi z(t))\|_X &\le C_0 \|\Pi^s x_{t_0} - F_*(\Phi z(t_0))\|_X e^{-\beta(t-t_0)} \end{aligned} \tag{3.3}$$

for $t \in J$, *where* $C_0 > C$ *is a constant which can be chosen as close to* C *as one expects by taking* Ω_0 *small. In particular, we have the estimate*

$$\|x_t - x_t(t_0, \phi^0, f)\|_X \le 2C_0 \|\Pi^s x_{t_0} - F_*(\Phi z(t_0))\|_X e^{-\beta(t-t_0)}, \quad t \in J, \tag{3.4}$$

where $\phi^0 = \Phi z(t_0) + F_*(\Phi z(t_0)) \in W^c_{\mathrm{loc}}(0)$.

(b) *If the zero solution of Eq.* (CE) *is stable (resp. asymptotically stable, unstable), the zero solution of Eq.* (*E*) *is stable (resp. asymptotically stable, unstable).*

Remark 3.4. Since Eq. (E_δ) is autonomous, it follows that $x_t(t_0, \phi, f_\delta) = x_{t-t_0}(0, \phi, f_\delta)$ for $t \ge t_0$. So it is sufficient to prove part (a) of Theorem 3.2 in case that $t_0 = 0$; and likewise for that of Theorem 3.3. Also, by the same reasoning stability (resp. asymptotic stability) in statement (b) of each theorem means actually uniform stability (resp. uniform asymptotic stability).

For the proof of part (a) of Theorem 3.2 we need the following lemma, which is a modification of [23, Lemma 1].

Lemma 3.5. *The solutions of Eq.* (E_δ) *satisfying* (3.1) *with* $t_0 = 0$ *are characterized by a system of integral equations in* X; *more precisely,*

(i) *Suppose that* $x(t)$ *is a solution of Eq.* (E_δ) *defined on* $\mathbb{R}^+$ *such that there exists a solution* $z(t)$ *of Eq.* (CE_δ) *with the properties*

$$\sup_{t\in\mathbb{R}^+} \|\Pi^c x_t - \Phi z(t)\|_X e^{-\beta t} < \infty, \text{ and } \sup_{t\in\mathbb{R}^+} \|\Pi^s x_t - F_{*,\delta}(\Phi z(t))\|_X e^{-\beta t} < \infty. \tag{3.5}$$

Then the X*-valued functions* $\xi(t)$ *and* $y(t)$ *defined by*

$$\xi(t) := \Pi^c x_t - \Phi z(t) \quad \text{and} \quad y(t) := \Pi^s x_t,$$

respectively satisfy

$$y(t) = T^s(t)\phi^s + \lim_{n\to\infty} \int_0^t T^s(t-s)\Pi^s \Gamma^n f_\delta\big(\Phi z(s) + \xi(t) + y(s)\big) ds, \tag{3.6}$$

$$\xi(t) = -\lim_{n\to\infty} \int_t^\infty T^c(t-s)\Gamma^n \Pi^c g_\delta(s) ds \tag{3.7}$$

for $t \in \mathbb{R}^+$, *where* $\phi := \Phi z(0) + \Pi^s x_0$ *and* $g_\delta(t)$ *is the function defined by*

$$g_\delta(t) := f_\delta\big(\Phi z(t) + \xi(t) + y(t)\big) - f_\delta\big(\Phi z(t) + F_{*,\delta}(\Phi z(t))\big)$$

for $t \in \mathbb{R}^+$. *Moreover* ξ *and* y *belong to* $C((0,\infty); X_0)$.

(ii) *Conversely, suppose that* $z(t)$ *is a solution of Eq.* (CE_δ), *and that* ξ *and* y *are elements of* $C(\mathbb{R}^+;X)$ *with the properties*

$$\sup_{t\in\mathbb{R}^+}\|\xi(t)\|_X e^{\beta t}<\infty,\quad \textit{and}\quad \sup_{t\in\mathbb{R}^+}\|y(t)-F_{*,\delta}(\Phi z(t))\|_X e^{\beta t}<\infty \tag{3.8}$$

which satisfy (3.6) *and* (3.7), *where* $\phi:=\Phi z(0)+y(0)$. *Then* ξ *and* y *belong to* $C((0,\infty);X_0)$ *and the function* $x(t)$ *defined by*

$$x(t):=\begin{cases}\big(\Phi z(t)+\xi(t)+y(t)\big)[0], & t>0;\\ \big(\Phi z(0)+\xi(0)+y(0)\big)(t), & t\le 0\end{cases}$$

is a solution of Eq. (E_δ) *on* $\mathbb{R}^+$ *that satisfies* (3.5) *and* $x_t=\Phi z(t)+\xi(t)+y(t)$ *for* $t\in\mathbb{R}^+$.

Proof. (i) We know by VCF (Theorem 2.1)

$$x_t=T(t-\tau)x_\tau+\lim_{n\to\infty}\int_\tau^t T(t-s)\Gamma^n f_\delta(x_s)ds,\quad t\ge\tau\ge 0, \tag{3.9}$$

and hence it follows that

$$y(t)=T^s(t)y(0)+\lim_{n\to\infty}\int_0^t T^s(t-s)\Pi^s\Gamma^n f_\delta\big(\Phi z(s)+\xi(s)+y(s)\big)ds, \tag{3.10}$$

where we used the relation $x_t=\Phi z(t)+\xi(t)+y(t)$ for $t\in\mathbb{R}^+$. By the definition of ϕ one can readily see $\phi^s=y(0)$ to get (3.6). Also, (3.9) yields

$$\Pi^c x_t=T^c(t-\tau)\Pi^c x_\tau+\lim_{n\to\infty}\int_\tau^t T^c(t-s)\Pi^c\Gamma^n f_\delta(x_s)ds.$$

Since, by Proposition 3.1, $\Phi z(t)+F_{*,\delta}(\Phi z(t))$ is the segment of the solution of Eq. (E_δ) through $(0,\phi^c+F_{*,\delta}(\phi^c))$, we deduce from VCF that

$$\Phi z(t)=T^c(t-\tau)\Phi z(\tau)+\lim_{n\to\infty}\int_\tau^t T^c(t-s)\Pi^c\Gamma^n f_\delta\big(\Phi z(s)+F_{*,\delta}(\Phi z(s))\big)ds. \tag{3.11}$$

So, $\xi(t)$ satisfies

$$\xi(t)=T^c(t-\tau)\xi(\tau)+\lim_{n\to\infty}\int_\tau^t T^c(t-s)\Pi^c\Gamma^n g_\delta(s)ds. \tag{3.12}$$

The group property of $\{T^c(t)\}_{t\in\mathbb{R}}$, then implies

$$\xi(\tau)=T^c(\tau-t)\xi(t)-\lim_{n\to\infty}\int_\tau^t T^c(\tau-s)\Pi^c\Gamma^n g_\delta(s)ds,\quad t\ge\tau\ge 0. \tag{3.13}$$

On the other hand, by the assumption (3.5) there exists a constant $C_*\ge 0$ such that $\|\xi(t)\|_X\le C_*e^{-\beta t}$ and $\|y(t)-F_{*,\delta}(\Phi z(t))\|_X\le C_*e^{-\beta t}$ for $t\in\mathbb{R}^+$; it follows from Theorem 2.3 that

$$\|T^c(\tau-t)\xi(t)\|_X\le Ce^{\varepsilon|\tau-t|}C_*e^{-\beta t}=CC_*e^{-\varepsilon\tau}e^{-(\beta-\varepsilon)t},\quad t\ge\tau.$$

Letting $t\to\infty$ in (3.13), we see

$$\xi(\tau)=-\lim_{t\to\infty}\lim_{n\to\infty}\int_\tau^t T^c(\tau-s)\Pi^c\Gamma^n g_\delta(s)ds.$$

Moreover, by Proposition 2.5 we have

$$\|g_\delta(s)\|_X \le \zeta_*(\delta)\|\xi(s) + y(s) - F_{*,\delta}(\Phi z(s))\|_X \le 2\zeta_*(\delta)C_* e^{-\beta s},$$

and hence

$$\|T^c(\tau - s)\Pi^c\Gamma^n g_\delta(s)\|_X \le 2CC_1C_*\zeta_*(\delta)e^{-\varepsilon\tau-(\beta-\varepsilon)s}, \quad s \ge \tau.$$

This means that $\int_\tau^\infty T^c(\tau - s)\Pi^c\Gamma^n g_\delta(s)ds$ is convergent in X uniformly in n; and in particular that

$$\left\| \int_t^\infty T^c(\tau - s)\Pi^c\Gamma^n g_\delta(s)ds \right\|_X \le C(\delta)e^{-(\beta-\varepsilon)(t-\tau)}, \quad t \ge \tau,$$

where $C(\delta) := 2CC_1C_*\zeta_*(\delta)/(\beta - \varepsilon)$. Thus, for $n, m \in \mathbb{N}$

$$\begin{aligned}
&\left\| \int_\tau^\infty T^c(\tau - s)\Pi^c\Gamma^n g_\delta(s)ds - \int_\tau^\infty T^c(\tau - s)\Pi^c\Gamma^m g_\delta(s)ds \right\|_X \\
&\quad \le \left\| \int_t^\infty T^c(\tau - s)\Pi^c\Gamma^n g_\delta(s)ds \right\|_X + \left\| \int_t^\infty T^c(\tau - s)\Pi^c\Gamma^m g_\delta(s)ds \right\|_X \\
&\qquad + \left\| \int_\tau^t T^c(\tau - s)\Pi^c\Gamma^n g_\delta(s)ds - \int_\tau^t T^c(\tau - s)\Pi^c\Gamma^m g_\delta(s)ds \right\|_X \\
&\quad \le 2C(\delta)e^{-(\beta-\varepsilon)(t-\tau)} + \left\| T^c(\tau - t)\left(\int_\tau^t T^c(t - s)\Pi^c\Gamma^n g_\delta(s)ds - \int_\tau^t T^c(t - s)\Pi^c\Gamma^m g_\delta(s)ds \right) \right\|_X.
\end{aligned}$$

Notice that $\lim_{n\to\infty} \int_\tau^t T^c(t - s)\Pi^c\Gamma^n g_\delta(s)ds = \Pi^c x_t(\tau, 0, g_\delta)$ (Theorem 2.1), which yields

$$\limsup_{n,m\to\infty} \left\| \int_\tau^\infty T^c(\tau - s)\Pi^c\Gamma^n g_\delta(s)ds - \int_\tau^\infty T^c(\tau - s)\Pi^c\Gamma^m g_\delta(s)ds \right\|_X \le 2C(\delta)e^{-(\beta-\varepsilon)(t-\tau)}.$$

Since $t \ge \tau$ is arbitrary, $\int_\tau^\infty T^c(\tau - s)\Gamma^n g_\delta(s)ds$ turns out to converge in X as $n \to \infty$.

Consequently, the argument in the last paragraph gives

$$\begin{aligned}
\lim_{n\to\infty} \int_\tau^\infty T^c(\tau - s)\Gamma^n g_\delta(s)ds &= \lim_{n\to\infty} \lim_{t\to\infty} \int_\tau^t T^c(\tau - s)\Gamma^n g_\delta(s)ds \\
&= \lim_{t\to\infty} \lim_{n\to\infty} \int_\tau^t T^c(\tau - s)\Gamma^n g_\delta(s)ds = -\xi(\tau), \quad \tau \in \mathbb{R}^+,
\end{aligned}$$

i.e., (3.7) holds. In view of (3.10) and (3.12), combined with the fact that $\xi \in C(\mathbb{R}^+; E^c)$ and $y \in C(\mathbb{R}^+; E^s)$, the latter part of (i) immediately follows from Theorem 2.2 or [21, Theorem 4].

(ii) We observe from the assumption (3.7) that (3.12) holds for $t \ge \tau \ge 0$. So, by the same reasoning as the proof of the latter part of (i), we see that ξ and y belong to $C((0,\infty); X_0)$. Now set $u(t) := \Phi z(t) + \xi(t) + y(t)$. Adding (3.6), (3.11) and (3.12) with $\tau = 0$, and using $y(0) = \phi^s$, we have

$$\begin{aligned}
u(t) &= T(t)(\Phi z(0) + \xi(0) + y(0)) + \lim_{n\to\infty} \int_0^t T(t - s)\Gamma^n f_\delta\big(\Phi z(s) + \xi(s) + y(s)\big)ds \\
&= T(t)u(0) + \lim_{n\to\infty} \int_0^t T(t - s)\Gamma^n f_\delta(u(s))ds, \quad t \in \mathbb{R}^+.
\end{aligned}$$

Then, Theorem 2.2 implies $x(t) = \big(u(t)\big)[0] \equiv x(t; 0, u(0), f_\delta)$ and $x_t = u(t) = \Phi z(t) + \xi(t) + y(t)$ for $t \in \mathbb{R}^+$. Thus, $x(t)$ is a solution of Eq. (E_δ) defined on $\mathbb{R}^+$ with $\Pi^c x_t - \Phi z(t) = \xi(t)$ and $\Pi^s x_t = y(t)$ for $t \in \mathbb{R}^+$. Therefore, the assertion of (ii) directly follows from (3.8). □

Assume that $\delta > 0$ is small enough such that

$$K_\delta := CC_1\zeta_*(\delta) \le \min\left((\alpha-\beta)/2,\ (\beta-\varepsilon)/4\right) \tag{3.14}$$

holds as well as

$$K_\delta(1+L(\delta)) < \varepsilon. \tag{3.15}$$

We will give preparatory propositions below so as to prove Theorem 3.2. Given $\phi \in X$, denote by $z(t,\phi)$, $t \in \mathbb{R}^+$, the solution of Eq.(CE_δ) with $\Phi z(0) = \phi^c$. One can see $z(t,\phi)$ satisfies

$$\Phi z(t,\phi) = T^c(t)\phi^c + \lim_{n\to\infty}\int_0^t T^c(t-s)\Pi^c\Gamma^n f_\delta\big(\Phi z(s,\phi) + F_{*,\delta}(\Phi z(s,\phi)\big)ds \tag{3.16}$$

for $t \in \mathbb{R}^+$ (see (3.11)). Then we have:

Proposition 3.6. *Let $\phi \in X$ and $\xi \in C(\mathbb{R}^+;E^c)$ with $\sup_{t\in\mathbb{R}^+}\|\xi(t)\|_X e^{\beta t} < \infty$. Then there exists one and only one $y \in C(\mathbb{R}^+;E^s)$ that satisfies*

$$y(t) = T^s(t)\phi^s + \lim_{n\to\infty}\int_0^t T^s(t-s)\Pi^s\Gamma^n f_\delta\big(\Phi z(s,\phi) + \xi(s) + y(s)\big)ds \tag{3.17}$$

for $t \in \mathbb{R}^+$.

Proof. Let η_0 be a positive number with $\eta_0 > 2\varepsilon$, and consider the Banach space

$$Z^s_{\eta_0} := BC^{-\eta_0}(\mathbb{R}^+;E^s) = \left\{y \in C(\mathbb{R}^+;E^s) : \sup_{t\in\mathbb{R}^+}\|y(s)\|_X e^{-\eta_0 t} < \infty\right\}$$

with norm $\|y\|_{Z^s_{\eta_0}} := \sup_{t\in\mathbb{R}^+}\|y(s)\|_X e^{-\eta_0 t}$ for $y \in Z^s_{\eta_0}$. Set

$$r_0 := \max\left(\|\phi\|_X,\ \sup_{t\in\mathbb{R}^+}\|\xi(t)\|_X e^{\beta t}\right),$$

and take an $r > 0$ such that

$$r > 2r_0\left\{CC_1 + K_\delta\left(\frac{1}{\alpha-\beta} + \frac{CC_1}{\alpha+2\varepsilon}\right)\right\}.$$

Now let $A_r := \overline{B}_{Z^s_{\eta_0}}(r)$, and for each $y \in A_r$ define $\mathcal{G}_{\phi,\xi}(y) \in C(\mathbb{R}^+;E^s)$ by

$$\big(\mathcal{G}_{\phi,\xi}(y)\big)(t) := T^s(t)\phi^s + \lim_{n\to\infty}\int_0^t T^s(t-s)\Pi^s\Gamma^n f_\delta\big(\Phi z(s,\phi) + \xi(s) + y(s)\big)ds \tag{3.18}$$

for $t \in \mathbb{R}^+$. Then, $\mathcal{G}_{\phi,\xi} : y \longmapsto \mathcal{G}_{\phi,\xi}(y)$ is a contraction map from A_r to itself. Indeed, by (3.16) and Proposition 2.5 we have

$$\begin{aligned}\|\Phi z(t,\phi)\|_X &\le Ce^{\varepsilon t}\|\phi^c\|_X + \int_0^t CC_1\zeta_*(\delta)e^{\varepsilon(t-s)}\|\Phi z(s,\phi) + F_{*,\delta}(\Phi z(s,\phi)\|_X ds\\ &\le Ce^{\varepsilon t}\|\phi^c\|_X + \int_0^t K_\delta(1+L(\delta))e^{\varepsilon(t-s)}\|\Phi z(s,\phi)\|_X ds,\end{aligned}$$

or equivalently, by (3.15)

$$e^{-\varepsilon t}\|\Phi z(t,\phi)\|_X \le C\|\phi^c\|_X + \int_0^t \varepsilon e^{-\varepsilon s}\|\Phi z(s,\phi)\|_X ds,$$

so that Gronwall's inequality yields

$$\|\Phi z(t,\phi)\|_X \le C\|\phi^c\|_X e^{2\varepsilon t}, \quad t \in \mathbb{R}^+.$$

Thus, for $y \in A_r$,

$$\|(\mathcal{G}_{\phi,\xi}(y))(t)\|_X \le C\|\phi^s\|_X e^{-\alpha t} + \int_0^t K_\delta e^{-\alpha(t-s)}(C\|\phi^c\|_X e^{2\varepsilon s} + r_0 e^{-\beta s} + re^{\eta_0 s})ds,$$

and hence

$$\begin{aligned}\|(\mathcal{G}_{\phi,\xi}(y))(t)\|_X e^{-\eta_0 t} &\le CC_1 r_0 + K_\delta\Big(\frac{CC_1 r_0}{\alpha+2\varepsilon} + \frac{r_0}{\alpha-\beta} + \frac{r}{\alpha+\eta_0}\Big)\\ &\le \frac{r}{2} + \frac{K_\delta r}{\alpha+\eta_0} \le r, \quad t \in \mathbb{R}^+.\end{aligned}$$

Therefore $\mathcal{G}_{\phi,\xi}(y) \in A_r$; so, we have $\mathcal{G}_{\phi,\xi}(A_r) \subset A_r$. Next, let y_1, $y_2 \in A_r$ be given. Then

$$\begin{aligned}\|(\mathcal{G}_{\phi,\xi}(y_1))(t) - (\mathcal{G}_{\phi,\xi}(y_2))(t)\|_X &\le \int_0^t K_\delta e^{-\alpha(t-s)}\|y_1(s) - y_2(s)\|_X ds\\ &\le \int_0^t K_\delta e^{-\alpha(t-s)}\|y_1 - y_2\|_{Z^s_{\eta_0}} e^{\eta_0 s} ds \le \frac{K_\delta}{\alpha+\eta_0}\|y_1 - y_2\|_{Z^s_{\eta_0}} e^{\eta_0 t}, \quad t \in \mathbb{R}^+,\end{aligned}$$

which, combined with (3.14), implies

$$\|\mathcal{G}_{\phi,\xi}(y_1) - \mathcal{G}_{\phi,\xi}(y_2)\|_{Z^s_{\eta_0}} \le \frac{1}{2}\|y_1 - y_2\|_{Z^s_{\eta_0}}.$$

Consequently, $\mathcal{G}_{\phi,\xi}$ is a contraction map from A_r to itself; hence, it has a unique fixed point in A_r, which is a solution of Eq. (3.17) on $\mathbb{R}^+$. This proves the proposition since r can be chosen arbitrarily large. □

Let us denote by $y(\phi,\xi)(t)$ the solution of Eq. (3.17).

Proposition 3.7. *If $\phi \in X$ and $\xi \in C(\mathbb{R}^+; E^c)$ with $\sup_{t\in\mathbb{R}^+}\|\xi(t)\|_X e^{\beta t} < \infty$, then we have for $t \in \mathbb{R}^+$*

$$\|y(\phi,\xi)(t) - F_{*,\delta}(\Phi z(t,\phi))\|_X \le \Big(C\|\phi^s - F_{*,\delta}(\phi^c)\|_X + \sup_{\tau\in\mathbb{R}^+}\|\xi(\tau)\|_X e^{\beta\tau}\Big)e^{-\beta t}.$$

Proof. By the invariance of the center manifold $W^c_\delta(0)$ (Theorem 2.4) and VCF,

$$\begin{aligned}F_{*,\delta}(\Phi z(t,\phi)) &= \Pi^s x_t(0, \phi^c + F_{*,\delta}(\phi^c), f_\delta)\\ &= T^s(t)F_{*,\delta}(\phi^c) + \lim_{n\to\infty}\int_0^t T^s(t-s)\Pi^s\Gamma^n f_\delta\big(\Phi z(s,\phi) + F_{*,\delta}(\Phi z(s,\phi))\big)ds.\end{aligned}$$

Since $y(\phi,\xi) = \mathcal{G}_{\phi,\xi}(y(\phi,\xi))$, we have

$$\begin{aligned}y(\phi,\xi)(t) - F_{*,\delta}(\Phi z(t,\phi)) = T^s(t)(\phi^s - F_{*,\delta}(\phi^c)) + \lim_{n\to\infty}\int_0^t T^s(t-s)\Pi^s\Gamma^n\big\{f_\delta\big(\Phi z(s,\phi)\\ + \xi(s) + y(\phi,\xi)(s)\big) - f_\delta\big(\Phi z(s,\phi) + F_{*,\delta}(\Phi z(s,\phi))\big)\big\}ds,\end{aligned}$$

and therefore

$$\begin{aligned}
&\|y(\phi,\xi)(t) - F_{*,\delta}(\Phi z(t,\phi))\|_X \\
&\quad \le Ce^{-\alpha t}\|\phi^s - F_{*,\delta}(\phi^c)\|_X + \int_0^t K_\delta e^{-\alpha(t-s)}(\|\xi(s)\|_X + \|y(\phi,\xi)(s) - F_{*,\delta}(\Phi z(s,\phi))\|_X)ds \\
&\quad \le Ce^{-\alpha t}\|\phi^s - F_{*,\delta}(\phi^c)\|_X + \int_0^t K_\delta e^{-\alpha(t-s)}\big(\sup_{\tau\in\mathbb{R}^+}\|\xi(\tau)\|_X e^{\beta\tau}\big)e^{-\beta s}ds \\
&\qquad + \int_0^t K_\delta e^{-\alpha(t-s)}\|y(\phi,\xi)(s) - F_{*,\delta}(\Phi z(s,\phi))\|_X ds.
\end{aligned}$$

By an application of Lemma 3.8 below, combined with (3.14), we obtain

$$\begin{aligned}
&\|y(\phi,\xi)(t) - F_{*,\delta}(\Phi z(t,\phi))\|_X \\
&\quad \le C\|\phi^s - F_{*,\delta}(\phi^c)\|_X e^{-(\alpha-K_\delta)t} + \frac{K_\delta}{\alpha-\beta-K_\delta}\big(\sup_{\tau\in\mathbb{R}^+}\|\xi(\tau)\|_X e^{\beta\tau}\big)e^{-\beta t} \\
&\quad \le \big(C\|\phi^s - F_{*,\delta}(\phi^c)\|_X + \sup_{\tau\in\mathbb{R}^+}\|\xi(\tau)\|_X e^{\beta\tau}\big)e^{-\beta t}, \quad t\in\mathbb{R}^+,
\end{aligned}$$

which proves the assertion. □

The following lemma is an analogue of [23, Lemma 2] and we omit the proof.

Lemma 3.8. *Let $g, r \in C(\mathbb{R}^+;\mathbb{R})$ and $h \in C^1(\mathbb{R}^+;\mathbb{R})$ satisfy $r(t) \ge 0$ $(t\in\mathbb{R}^+)$ and*

$$g(t) \le h(t) + \int_0^t r(s)g(s)ds, \quad t\in\mathbb{R}^+.$$

Then we have

$$g(t) \le h(0)\exp\left(\int_0^t r(s)ds\right) + \int_0^t h'(s)\exp\left(\int_s^t r(u)du\right)ds, \quad t\in\mathbb{R}^+.$$

Let Z^c_β be the Banach space defined by

$$Z^c_\beta := BC^\beta(\mathbb{R}^+;E^c) = \{\xi\in C(\mathbb{R}^+;E^c) : \sup_{t\in\mathbb{R}^+}\|\xi(t)\|_X e^{\beta t} < \infty\},$$

with norm $\|\xi\|_{Z^c_\beta} := \sup_{t\in\mathbb{R}^+}\|\xi(t)\|_X e^{\beta t}$. Also, set

$$h_\delta(t,\phi,\xi) := f_\delta\big(\Phi z(t,\phi) + \xi(t) + y(\phi,\xi)(t)\big) - f_\delta\big(\Phi z(t,\phi) + F_{*,\delta}(\Phi z(t,\phi))\big)$$

for $t\in\mathbb{R}^+$, $\phi\in X$ and $\xi\in Z^c_\beta$.

Proposition 3.9. *For each $\phi\in X$ the equation*

$$\xi(t) = -\lim_{n\to\infty}\int_t^\infty T^c(t-s)\Pi^c\Gamma^n h_\delta(s,\phi,\xi)ds, \quad t\in\mathbb{R}^+ \tag{3.19}$$

has a unique solution in Z^c_β.

Proof. Let $r_0 > \|\phi\|_X$, and take a positive number r_1 satisfying

$$r_1 > \frac{1}{2}CC_1(1+L(\delta))r_0. \tag{3.20}$$

We define a map $\mathcal{H} : \overline{B}_X(r_0) \times \overline{B}_{Z^c_\beta}(r_1) \to C(\mathbb{R}^+; E^c)$ by

$$(\mathcal{H}(\phi,\xi))(t) := -\lim_{n\to\infty}\int_t^\infty T^c(t-s)\Pi^c\Gamma^n h_\delta(s,\phi,\xi)ds$$

for $t \in \mathbb{R}^+$, $\phi \in \overline{B}_X(r_0)$, and $\xi \in \overline{B}_{Z^c_\beta}(r_1)$. In view of Proposition 3.7,

$$\begin{aligned}\|h_\delta(s,\phi,\xi)\|_X &\le \zeta_*(\delta)\big(\|\xi\|_X + \|y(\phi,\xi)(s) - F_{*,\delta}(\Phi z(s,\phi))\|_X\big)\\ &\le \zeta_*(\delta)\big\{\|\xi\|_{Z^c_\beta}e^{-\beta s} + \big(C\|\phi^s - F_{*,\delta}(\phi^c)\|_X + \|\xi\|_{Z^c_\beta}\big)e^{-\beta s}\big\}\\ &= \zeta_*(\delta)\big(CC_1(1+L(\delta))\|\phi\|_X + 2\|\xi\|_{Z^c_\beta}\big)e^{-\beta s},\end{aligned} \tag{3.21}$$

and therefore

$$\begin{aligned}\|(\mathcal{H}(\phi,\xi))(t)\|_X &\le \int_t^\infty K_\delta e^{-\varepsilon(t-s)}\big(CC_1(1+L(\delta))\|\phi\|_X + 2\|\xi\|_{Z^c_\beta}\big)e^{-\beta s}ds\\ &\le \frac{K_\delta}{\beta-\varepsilon}(CC_1(1+L(\delta))r_0 + 2r_1)e^{-\beta t} \le r_1 e^{-\beta t}\end{aligned} \tag{3.22}$$

for $t \in \mathbb{R}^+$, where we used (3.14) and (3.20). So, $\mathcal{H}(\phi,\xi)$ belongs to Z^c_β with $\|\mathcal{H}(\phi,\xi)\|_{Z^c_\beta} \le r_1$; hence for each $\phi \in \overline{B}_X(r_0)$, $\mathcal{H}(\phi,\cdot)$ defines a map from $\overline{B}_{Z^c_\beta}(r_1)$ to itself. We will prove the proposition by showing that $\mathcal{H}(\phi,\cdot)$ is a contraction map. Indeed, let $\xi_1, \xi_2 \in \overline{B}_{Z^c_\beta}(r_1)$. Then

$$\begin{aligned}&\|(\mathcal{H}(\phi,\xi_1))(t) - (\mathcal{H}(\phi,\xi_2))(t)\|_X\\ &\quad\le \int_t^\infty K_\delta e^{-\varepsilon(t-s)}(\|\xi_1(s) - \xi_2(s)\|_X + \|y(\phi,\xi_1)(s) - y(\phi,\xi_2)(s)\|_X)ds.\end{aligned} \tag{3.23}$$

We know $y(\phi,\xi_i) = \mathcal{G}_{\phi,\xi_i}(y(\phi,\xi_i))$ $(i = 1,2)$, so that

$$\begin{aligned}y(\phi,\xi_1)(t) - y(\phi,\xi_2)(t) = \lim_{n\to\infty}\int_0^t T^s(t-s)\Pi^s\Gamma^n\big\{&f_\delta\big(\Phi z(s,\phi) + \xi_1(s) + y(\phi,\xi_1)(s)\big)\\ &- f_\delta\big(\Phi z(s,\phi) + \xi_2(s) + y(\phi,\xi_2(s))\big)\big\}ds.\end{aligned}$$

Hence

$$\begin{aligned}&\|y(\phi,\xi_1)(t) - y(\phi,\xi_2)(t)\|_X\\ &\quad\le \int_0^t K_\delta e^{-\alpha(t-s)}(\|\xi_1(s) - \xi_2(s)\|_X + \|y(\phi,\xi_1)(s) - y(\phi,\xi_2)(s)\|_X)ds\\ &\quad\le \int_0^t K_\delta e^{-\alpha(t-s)}\|\xi_1 - \xi_2\|_{Z^c_\beta}e^{-\beta s}ds + \int_0^t K_\delta e^{-\alpha(t-s)}\|y(\phi,\xi_1)(s) - y(\phi,\xi_2)(s)\|_X ds,\end{aligned}$$

that is,

$$\begin{aligned}&e^{\alpha t}\|y(\phi,\xi_1)(t) - y(\phi,\xi_2)(t)\|_X\\ &\quad\le \int_0^t K_\delta\|\xi_1 - \xi_2\|_{Z^c_\beta}e^{(\alpha-\beta)s}ds + \int_0^t K_\delta e^{\alpha s}\|y(\phi,\xi_1)(s) - y(\phi,\xi_2)(s)\|_X ds\end{aligned}$$

holds for $t \in \mathbb{R}^+$. It follows from Lemma 3.8 and (3.14) that

$$\|y(\phi,\xi_1)(t) - y(\phi,\xi_2)(t)\|_X \le \frac{K_\delta}{\alpha-\beta-K_\delta}\|\xi_1 - \xi_2\|_{Z^c_\beta}e^{-\beta t} \le \|\xi_1 - \xi_2\|_{Z^c_\beta}e^{-\beta t}, \quad t \in \mathbb{R}^+.$$

Consequently, we see from (3.23) that

$$\begin{aligned}\|(\mathcal{H}(\phi,\xi_1))(t)-(\mathcal{H}(\phi,\xi_2))(t)\|_X &\le \int_t^\infty K_\delta e^{-\varepsilon(t-s)}\,2\|\xi_1-\xi_2\|_{Z^c_\beta}e^{-\beta s}ds\\ &= \frac{2K_\delta}{\beta-\varepsilon}\|\xi_1-\xi_2\|_{Z^c_\beta}e^{-\beta t},\quad t\in\mathbb{R}^+,\end{aligned}$$

which, together with (3.14), implies

$$\|\mathcal{H}(\phi,\xi_1)-\mathcal{H}(\phi,\xi_2)\|_{Z^c_\beta}\le\frac{1}{2}\|\xi_1-\xi_2\|_{Z^c_\beta}.$$

Thus, $\mathcal{H}(\phi,\cdot)$ is a contraction map from $\overline{B}_{Z^c_\beta}(r_1)$ to itself; and therefore has a unique fixed point in $\overline{B}_{Z^c_\beta}(r_1)$ for $\phi\in\overline{B}_X(r_0)$. Since one can choose r_1 arbitrarily large, the proposition immediately follows. □

For $\phi\in X$ denote $\xi(\phi)(t)$ the solution of Eq. (3.19).

Proposition 3.10. *Define a map $\mathcal{K}: X\to X$ by $\mathcal{K}(\phi):=\xi(\phi)(0)$, $\phi\in X$. Then $\mathcal{K}$ is continuous.*

Proof. Since Z^c_β is a subspace of the Banach space $BC(\mathbb{R}^+;E^c)$, it is sufficient to show that $\xi:\phi\longmapsto\xi(\phi)$ is continuous as a map from X to $BC(\mathbb{R}^+;E^c)$. So letting ϕ_0 be an element of X and $\{\phi_l\}\subset X$ an arbitrary sequence converging to ϕ_0, we show $\xi(\phi_l)\to\xi(\phi_0)$ in $BC(\mathbb{R}^+;E^c)$ as $l\to\infty$. The proof will be divided into several steps.

Step 1. $\{\xi(\phi_l)\}$ is relatively compact in $BC(\mathbb{R}^+;E^c)$. To confirm the assertion, take positive numbers r_0 and r_1 in such a way that $\phi_l\in B_X(r_0)$ $(l\in\mathbb{N})$ and (3.20) are satisfied. Since $\|\xi(\phi_l)(t)\|_X\le\|\xi(\phi_l)\|_{Z^c_\beta}e^{-\beta t}\le r_1e^{-\beta t}$ $(t\in\mathbb{R}^+)$ follows from the proof of Proposition 3.9, the sequence $\{\xi(\phi_l)(t)\}$ is uniformly bounded in $\mathbb{R}^+$, and is equi-convergent (to 0) as $t\to\infty$. We next verify that $\{\xi(\phi_l)(t)\}$ is equi-continuous in $\mathbb{R}^+$. Let $t_2\ge t_1\ge 0$. In view of the relation $\xi(\phi_l)=\mathcal{H}(\phi_l,\xi(\phi_l))$ (cf. Proposition 3.9),

$$\begin{aligned}\xi(\phi_l)(t_2)-\xi(\phi_l)(t_1) &= \lim_{n\to\infty}\int_{t_1}^{t_2}T^c(t_1-s)\Pi^c\Gamma^n h_\delta(s,\phi_l,\xi(\phi_l))ds\\ &\quad+\lim_{n\to\infty}\int_{t_2}^{\infty}(T^c(t_1-s)-T^c(t_2-s))\Pi^c\Gamma^n h_\delta(s,\phi_l,\xi(\phi_l))ds\\ &= \lim_{n\to\infty}\int_{t_1}^{t_2}T^c(t_1-s)\Pi^c\Gamma^n h_\delta(s,\phi_l,\xi(\phi_l))ds\\ &\quad+\lim_{n\to\infty}\int_{t_2}^{\infty}T^c(t_1-s)(I_{E^c}-T^c(t_2-t_1))\Pi^c\Gamma^n h_\delta(s,\phi_l,\xi(\phi_l))ds,\end{aligned}$$

where I_{E^c} is the identity map of E^c. Hence it follows from (3.21) and (3.20) that

$$\begin{aligned}\|\xi(\phi_l)(t_2)-\xi(\phi_l)(t_1)\|_X &\le \int_{t_1}^{t_2}CC_1e^{\varepsilon(s-t_1)}\|h_\delta(s,\phi_l,\xi(\phi_l))\|_X ds\\ &\quad+\int_{t_2}^{\infty}CC_1e^{\varepsilon(s-t_1)}\|I_{E^c}-T^c(t_2-t_1)\|\,\|h_\delta(s,\phi_l,\xi(\phi_l))\|_X ds\\ &\le \big(C\|\phi_l^s-F_{*,\delta}(\phi_l^c)\|_X+2\|\xi(\phi_l)\|_{Z^c_\beta}\big)\\ &\quad\times\left(\int_{t_1}^{t_2}K_\delta e^{\varepsilon(s-t_1)}e^{-\beta s}ds+\int_{t_2}^{\infty}K_\delta e^{\varepsilon(s-t_1)}\|T^c(t_2-t_1)-I_{E^c}\|e^{-\beta s}ds\right)\\ &\le \frac{K_\delta}{\beta-\varepsilon}\big(CC_1(1+L(\delta))r_0+2r_1\big)\big(|t_2-t_1|+\|T^c(t_2-t_1)-I_{E^c}\|_{\mathcal{L}(E^c)}\big)\\ &\le r_1\big(|t_2-t_1|+\|T^c(t_2-t_1)-I_{E^c}\|_{\mathcal{L}(E^c)}\big),\end{aligned}$$

which shows the equi-continuity in $\mathbb{R}^+$ of $\{\xi(\phi_l)(t)\}$ since E^c is a finite-dimensional subspace.

By the properties of $\{\xi(\phi_l)(t)\}$ above and the fact that $\dim E^c < \infty$, a variant of the Arzelà–Ascoli theorem yields that $\{\xi(\phi_l)\}$ is relatively compact in $BC(\mathbb{R}^+; E^c) = Z_0^c$. So there is a subsequence of $\{\phi_l\}$, denoted $\{\phi_l\}$ again, such that $\|\xi(\phi_l) - \xi_*\|_{Z_0^c} \to 0$ $(l \to \infty)$ for some $\xi_* \in BC(\mathbb{R}^+; E^c)$.

Step 2. $\{y(\phi_l, \xi(\phi_l))(t)\}$ converges to $y(\phi_0, \xi_*)(t)$ uniformly in any compact set of $\mathbb{R}^+$ as $l \to \infty$. Indeed, since $y(\phi_l, \xi(\phi_l))$ and $y(\phi_0, \xi_*)$ are the fixed points of $\mathcal{G}_{\phi_l, \xi(\phi_l)}$ and $\mathcal{G}_{\phi_0, \xi_*}$ respectively (cf. (3.18)),

$$\begin{aligned} &y(\phi_l, \xi(\phi_l))(t) - y(\phi_0, \xi_*)(t) \\ &\quad = T(t)\phi_l^s - T(t)\phi_0^s + \lim_{n\to\infty} \int_0^t T^s(t-s)\Pi^s\Gamma^n \big\{ f_\delta\big(\Phi z(s, \phi_l) + \xi(\phi_l)(s) \\ &\qquad + y(\phi_l, \xi(\phi_l)(s)\big) - f_\delta\big(\Phi z(s, \phi_0) + \xi_*(s) + y(\phi_0, \xi_*)(s)\big)\big\} ds. \end{aligned}$$

So, letting $u_l(t) := y(\phi_l, \xi(\phi_l))(t) - y(\phi_0, \xi_*)(t)$, we have

$$\begin{aligned} \|u_l(t)\|_X \le CC_1\|\phi_l - \phi_0\|_X e^{-\alpha} + \int_0^t K_\delta e^{-\alpha(t-s)} \big\{ &\|\Phi z(s, \phi_l) - \Phi z(s, \phi_0)\|_X \\ &+ \|\xi(\phi_l)(s) - \xi_*(s)\|_X + \|u_l(s)\|_X \big\} ds. \end{aligned} \tag{3.24}$$

Notice from (3.16) that

$$\|\Phi z(t, \phi_l) - \Phi z(t, \phi_0)\|_X \le Ce^{\varepsilon t}\|\phi_l^c - \phi_0^c\|_X + \int_0^t K_\delta e^{\varepsilon(t-s)}(1 + L(\delta))\|\Phi z(s, \phi_l) - \Phi z(s, \phi_0)\|_X ds;$$

then one can see from Gronwall's inequality, combined with (3.15), that

$$\|\Phi z(t, \phi_l) - \Phi z(t, \phi_0)\|_X \le C\|\phi_l^c - \phi_0^c\|_X e^{(\varepsilon + K_\delta(1+L(\delta))t} \le C\|\phi_l^c - \phi_0^c\|_X e^{2\varepsilon t}, \quad t \in \mathbb{R}^+. \tag{3.25}$$

Thus (3.24) implies

$$\begin{aligned} \|u_l(t)\|_X e^{\alpha t} \le CC_1\|\phi_l - \phi_0\|_X + \int_0^t K_\delta e^{\alpha s}(C\|\phi_l^c - \phi_0^c\|_X e^{2\varepsilon s} \\ + \|\xi(\phi_l)(s) - \xi_*(s)\|_X) ds + \int_0^t K_\delta e^{\alpha s}\|u_l(s)\|_X ds. \end{aligned}$$

By Lemma 3.8,

$$\begin{aligned} \|u_l(t)\|_X e^{\alpha t} &\le CC_1\|\phi_l - \phi_0\|_X e^{K_\delta t} + \int_0^t K_\delta e^{\alpha s}\big(C\|\phi_l^c - \phi_0^c\|_X e^{2\varepsilon s} \\ &\quad + \|\xi(\phi_l)(s) - \xi_*(s)\|_X\big) e^{K_\delta(t-s)} ds \\ &\le CC_1\|\phi_l - \phi_0\|_X e^{K_\delta t} + \frac{K_\delta C}{\alpha + 2\varepsilon - K_\delta}\|\phi_l^c - \phi_0^c\|_X e^{(\alpha+2\varepsilon)t} \\ &\quad + \frac{K_\delta}{\alpha - K_\delta}\|\xi(\phi_l) - \xi_*\|_{Z_0^c} e^{\alpha t}, \end{aligned}$$

and therefore from (3.14)

$$\|u_l(t)\|_X \le CC_1\|\phi_l - \phi_0\|_X e^{-(\alpha - K_\delta)t} + C\|\phi_l^c - \phi_0^c\|_X e^{2\varepsilon t} + \|\xi(\phi_l) - \xi_*\|_{Z_0^c}$$

holds for $t \in \mathbb{R}^+$. It follows from Step 1 that for any $\tau_* > 0$, the sequence $\{y(\phi_l, \xi(\phi_l))(t)\}$ is uniformly convergent to $y(\phi_0, \xi_*)(t)$ in $[0, \tau_*]$ as $l \to \infty$, which proves Step 2.

Step 3. ξ_* coincides with $\xi(\phi_0)$. Indeed, given positive numbers ε_* and τ, choose a τ_* in such a way that $\tau_* > \tau$ and

$$e^{-(\beta-\varepsilon)\tau_*} < \frac{\varepsilon_*}{2Cr_1} \tag{3.26}$$

hold. Observe that

$$\begin{aligned}\xi(\phi_l)(t) - \mathcal{H}(\phi_0,\xi_*)(t) = &-\lim_{n\to\infty}\int_t^{\tau_*} T^c(t-s)\Pi^c\Gamma^n(h_\delta(s,\phi_l,\xi(\phi_l)) - h_\delta(s,\phi_0,\xi_*))ds\\ &+ T^c(t-\tau_*)(\xi(\phi_l)(\tau_*) - \mathcal{H}(\phi_0,\xi_*)(\tau_*)).\end{aligned}$$

Since neither $\|\xi(\phi_l)(\tau_*)\|_X$ nor $\|\mathcal{H}(\phi_0,\xi_*)(\tau_*)\|_X$ is greater than $r_1e^{-\beta\tau_*}$ (see (3.22)), it follows from (3.26) that

$$\begin{aligned}\left\|T^c(t-\tau_*)\left(\xi(\phi_l)(\tau_*) - \mathcal{H}(\phi_0,\xi_*)(\tau_*)\right)\right\|_X &\le Ce^{\varepsilon(\tau_*-t)}(r_1e^{-\beta\tau_*} + r_1e^{-\beta\tau_*})\\ &< \varepsilon_*, \quad t\in[0,\tau_*].\end{aligned}$$

Consequently,

$$\begin{aligned}&\|\xi(\phi_l)(t) - \mathcal{H}(\phi_0,\xi_*)(t)\|_X\\ &\quad< \int_t^{\tau_*} K_\delta Ce^{\varepsilon(s-t)}\big\{(2+L(\delta))\|\Phi z(s,\phi_l) - \Phi z(s,\phi_0)\|_X\\ &\qquad+ \|\xi(\phi_l)(s) - \xi_*(s)\|_X + \|y(\phi_l,\xi(\phi_l))(s) - y(\phi_0,\xi_*)(s)\|_X\big\}ds + \varepsilon_*\\ &\quad\le \int_t^{\tau_*} K_\delta Ce^{\varepsilon(s-t)}\big\{C_1(2+L(\delta))\|\phi_l - \phi_0\|_Xe^{2\varepsilon s} + \|\xi(\phi_l) - \xi_*\|_{Z_0^c}\\ &\qquad+ \|y(\phi_l,\xi(\phi_l))(s) - y(\phi_0,\xi_*)(s)\|_X\big\}ds + \varepsilon_*\end{aligned}$$

for $0 \le t \le \tau_*$ because of (3.25). Passing to the limit as $l\to\infty$, we deduce from Steps 1 and 2 that

$$\|\xi_*(t) - \mathcal{H}(\phi_0,\xi_*)(t)\|_X = \lim_{m\to\infty}\|\xi(\phi_l)(t) - \mathcal{H}(\phi_0,\xi_*)(t)\|_X \le \varepsilon_*, \quad t\in[0,\tau].$$

Since ε_* is arbitrary, we have $\|\xi_*(t) - \mathcal{H}(\phi_0,\xi_*)(t)\|_X = 0$ for $t\in[0,\tau]$; hence ξ_* must coincide with $\mathcal{H}(\phi_0,\xi_*)$, for τ is also arbitrary. The uniqueness of fixed points of $\mathcal{H}(\phi_0,\cdot)$ then implies $\xi_* = \xi(\phi_0)$.

Step 4. The argument above shows that given a subsequence of $\{\phi_l\}$, one can choose its subsequence, say $\{\phi_{l_k}\}$, such that $\xi(\phi_{l_k})$ is convergent to $\xi(\phi_0)$ with norm $\|\cdot\|_{Z_0^c}$ as $k\to\infty$. It therefore turns out that $\{\xi(\phi_l)\}$ is itself convergent to $\xi(\phi_0)$ in $BC(\mathbb{R}^+;E^c)$. Thus, $\xi:\phi\longmapsto\xi(\phi)$ is continuous as a map from X to $BC(\mathbb{R}^+;E^c)$, which completes the proof. □

3.2 Proofs of Theorems 3.2 and 3.3

We are now able to give a proof of Theorem 3.2.

Proof of Theorem 3.2. (a) Without loss of generality, we may prove part (a) in case $t_0 = 0$ (see Remark 3.4). For each $\phi\in X$, we set $y(\phi)(t) := y(\phi,\xi(\phi))(t)$ for $t\in\mathbb{R}^+$. Then it follows from Propositions 3.6, 3.9 and 3.7 that the X-valued functions $\xi(\phi)$ and $y(\phi)$ satisfy (3.6) and (3.7) together with (3.8). Hence, by virtue of Lemma 3.5, the $\mathbb{C}^m$-valued function $x(\phi)(t)$ on $\mathbb{R}$, defined by

$$x(\phi)(t) := \begin{cases}\big(\Phi z(t,\phi) + \xi(\phi)(t) + y(\phi)(t)\big)[0], & t>0,\\ \big(\Phi z(0,\phi) + \xi(\phi)(0) + y(\phi)(0)\big)(t), & t\le 0,\end{cases}$$

is a solution of Eq. (E_δ) on $\mathbb{R}^+$, which satisfies both of

$$\|\Pi^c(x(\phi))_t - \Phi z(t,\phi)\|_X \le C_* e^{-\beta t},$$
$$\|\Pi^s(x(\phi))_t - F_{*,\delta}(\Phi z(t,\phi))\|_X \le C_* e^{-\beta t}$$

and $(x(\phi))_t = \Phi z(t,\phi) + \xi(\phi)(t) + y(\phi)(t)$ for $t \in \mathbb{R}^+$, where C_* is a nonnegative constant. So, if one can find a $\phi \in X$ with $x_0 = (x(\phi))_0$, the uniqueness of solutions for Eq. (E_δ) ensures that $x(t) \equiv x(\phi)(t)$ for $t \in \mathbb{R}^+$; so that $z(t) = z(t,\phi)$ is a solution of Eq. (CE_δ) satisfying

$$\|\Pi^c x_t - \Phi z(t)\|_X \le C_* e^{-\beta t} \quad \text{and} \quad \|\Pi^s x_t - F_{*,\delta}(\Phi z(t))\|_X \le C_* e^{-\beta t} \tag{3.27}$$

for $t \in \mathbb{R}^+$. Now consider the map $\widehat{g} : X \to X$ defined by

$$\widehat{g}(\phi) := (x(\phi))_0, \quad \phi \in X. \tag{3.28}$$

Claim 1. $\widehat{g}$ is a bijection from X to itself if $\delta > 0$ is sufficiently small.

Proof of Claim 1. We first verify the surjectivity. Note, by definition, that

$$(x(\phi))_0 = \Phi z(0,\phi) + \xi(\phi)(0) + y(\phi)(0) = \phi^c + \mathcal{K}(\phi) + \phi^s \tag{3.29}$$

for $\phi \in X$ (see also (3.16) and (3.17)), and therefore that $\widehat{g}(\phi) = \phi + \mathcal{K}(\phi)$.

Let $M(\delta) := 2CK_\delta/(\beta - \varepsilon)$ and take a δ so small that

$$C_1 M(\delta)(1 + L(\delta)) < \frac{1}{2}. \tag{3.30}$$

In the same way as (3.22), we get

$$\|\xi(\phi)\|_{Z^c_\beta} \le \frac{K_\delta}{\beta - \varepsilon}(C\|\phi^s - F_{*,\delta}(\phi^c)\|_X + 2\|\xi(\phi)\|_{Z^c_\beta})$$

and in particular by (3.14)

$$\|\mathcal{K}(\phi)\|_X \le \|\xi(\phi)\|_{Z^c_\beta} \le M(\delta)\|\phi^s - F_{*,\delta}(\phi^c)\|_X \le C_1 M(\delta)(1 + L(\delta))\|\phi\|_X. \tag{3.31}$$

Now given $\bar\phi \in \operatorname{graph} F_{*,\delta}$ and $r_0 > 0$, define $H : \overline{B}_X(\bar\phi; r_0) \times [0,1] \to X$ by

$$H(\phi,\lambda) := \phi + \lambda\mathcal{K}(\phi), \quad \phi \in \overline{B}_X(\bar\phi; r_0), \ \ \lambda \in [0,1].$$

Since $\mathcal{K}$ is continuous (Proposition 3.10), bounded on each bounded set of X, and the range of $\mathcal{K}$ is contained in E^c, which is finite-dimensional, we deduce that the map $\mathcal{K}$ is compact. In addition, observe from (3.31) and the fact $\bar\phi^s = F_{*,\delta}(\bar\phi^c)$ that

$$\begin{aligned}\|\mathcal{K}(\phi)\|_X &\le M(\delta)(\|\phi^s - \bar\phi^s\|_X + \|F_{*,\delta}(\bar\phi^c) - F_{*,\delta}(\phi^c)\|_X)\\ &\le M(\delta)(\|\phi^s - \bar\phi^s\|_X + L(\delta)\|\phi^c - \bar\phi^c\|_X)\\ &\le C_1 M(\delta)(1 + L(\delta))\|\phi - \bar\phi\|_X.\end{aligned} \tag{3.32}$$

Let $r_0 > 0$ and $\psi \in \overline{B}_X(\bar\phi; r_0/2)$, then it follows from (3.30) and (3.32) that for $\phi \in \partial B_X(\bar\phi; r_0)$ and $\lambda \in [0,1]$,

$$\begin{aligned}\|H(\phi,\lambda) - \psi\|_X &\ge \|\phi - \bar\phi\|_X - \|\psi - \bar\phi\|_X - \|\mathcal{K}(\phi)\|_X\\ &\ge \frac{r_0}{2} - C_1 M(\delta)(1 + L(\delta))r_0 > 0.\end{aligned}$$

Hence, $\deg(H(\cdot,\lambda), B_X(\bar{\phi};r_0), \psi)$, the Leray–Schauder degree ([3]) of $H(\cdot,\lambda)$, is well-defined and is independent of λ. Since H is a homotopy between $\widehat{g}$ and the identity I_X as maps from $\overline{B}_X(\bar{\phi};r_0)$ to X, it follows that

$$\deg(\widehat{g}, B_X(\bar{\phi};r_0), \psi) = \deg(I_X, B_X(\bar{\phi};r_0), \psi) = 1,$$

which shows that $\widehat{g}^{-1}(\psi) \cap B_X(\bar{\phi};r_0) \neq \emptyset$ for each $\psi \in B_X(\bar{\phi};r_0/2)$; in other words,

$$B_X(\bar{\phi};r_0/2) \subset \widehat{g}(B_X(\bar{\phi};r_0)). \tag{3.33}$$

Thus, $\widehat{g}$ is a surjection from X onto itself since $r_0 > 0$ is arbitrary.

We next prove the injectivity. Let $\delta > 0$ be so small that

$$\mu(\delta) := \frac{K_\delta(1+L(\delta))}{\beta-\varepsilon} < 1, \tag{3.34}$$

and assume that $\widehat{g}(\phi) = \widehat{g}(\tilde{\phi})$ for some ϕ and $\tilde{\phi}$ in X. Then by the definition of $\widehat{g}$ we obtain

$$\|\Pi^c x_t(0, \widehat{g}(\phi), f_\delta) - \Phi_c z(t,\phi)\|_X \le C_* e^{-\beta t}$$

and

$$\|\Pi^c x_t(0, \widehat{g}(\tilde{\phi}), f_\delta) - \Phi z(t,\tilde{\phi})\|_X \le \tilde{C}_* e^{-\beta t}$$

for $t \in \mathbb{R}^+$, where C_* and $\tilde{C}_*$ are some nonnegative constant. In particular,

$$\|\Phi z(t,\phi) - \Phi z(t,\tilde{\phi})\|_X \le C^* e^{-\beta t}, \quad t \in \mathbb{R}^+ \tag{3.35}$$

with $C^* := C_* + \tilde{C}_*$. We know by (3.16) that $\Phi z(t,\phi)$ satisfies

$$\Phi z(t,\phi) = T^c(t-\tau)\Phi z(\tau,\phi) + \lim_{n\to\infty} \int_\tau^t T^c(t-s)\Pi^c f_\delta(\Phi z(s,\phi) + F_{*,\delta}(\Phi z(s,\phi)))ds$$

for $t \ge \tau \ge 0$. So, by the group property of $\{T^c(t)\}_{t\in\mathbb{R}}$, we see

$$\Phi z(\tau,\phi) = T^c(\tau-t)\Phi z(t,\phi) - \lim_{n\to\infty} \int_\tau^t T^c(\tau-s)\Pi^c f_\delta(\Phi z(s,\phi) + F_{*,\delta}(\Phi z(s,\phi)))ds$$

and likewise for $\Phi_c z(\tau,\tilde{\phi})$. Hence it follows from (3.35) that

$$\begin{aligned}
&\|\Phi z(\tau,\phi) - \Phi z(\tau,\tilde{\phi}) - T^c(\tau-t)\big(\Phi z(t,\phi) - \Phi z(t,\tilde{\phi})\big)\|_X \\
&\quad\le \int_\tau^t K_\delta(1+L(\delta))e^{\varepsilon|\tau-s|}\|\Phi z(s,\phi) - \Phi z(s,\tilde{\phi})\|_X ds \\
&\quad\le \int_\tau^t K_\delta(1+L(\delta))e^{\varepsilon(s-\tau)}\cdot C^* e^{-\beta s} ds \le \frac{C^* K_\delta(1+L(\delta))}{\beta-\varepsilon} e^{-\beta\tau},
\end{aligned}$$

so that

$$\|\Phi z(\tau,\phi) - \Phi z(\tau,\tilde{\phi})\|_X \le CC^* e^{-\varepsilon\tau-(\beta-\varepsilon)t} + C^*\mu(\delta)e^{-\beta\tau}, \quad t \ge \tau \ge 0.$$

Passing to the limit as $t \to \infty$, we get

$$\|\Phi z(\tau,\phi) - \Phi z(\tau,\tilde{\phi})\|_X \le C^*\mu(\delta)e^{-\beta\tau}, \quad \tau \in \mathbb{R}^+. \tag{3.36}$$

Thus, (3.35) implies (3.36). Applying the same argument to (3.36) (in place of (3.35)), we have $\|\Phi z(t,\phi) - \Phi z(t,\tilde{\phi})\|_X \le C^*\mu(\delta)^2 e^{-\beta t}$ $(t \in \mathbb{R}^+)$. By the repetition of this procedure, one

reaches $\|\Phi z(t,\phi) - \Phi z(t,\tilde{\phi})\|_X \le C^* \mu(\delta)^n e^{-\beta t}$ $(t \in \mathbb{R}^+, n = 1,2,\dots)$, which yields $\|\Phi z(t,\phi) - \Phi z(t,\tilde{\phi})\|_X = 0$ $(t \in \mathbb{R}^+)$ because of (3.34). In particular, $\phi^c = \Phi z(0,\phi) = \Phi(0,\tilde{\phi}) = \tilde{\phi}^c$. Moreover, since $\Pi^s \mathcal{K}(\phi) = \Pi^s \mathcal{K}(\tilde{\phi}) = 0$, it also follows that $\phi^s = \Pi^s \widehat{g}(\phi) = \Pi^s \widehat{g}(\tilde{\phi}) = \tilde{\phi}^s$, and therefore that $\phi = \tilde{\phi}$. Consequently the claim is proved. □

In view of Claim 1, given any solution $x(t)$ of Eq. (E_δ) defined on $\mathbb{R}^+$, there exists a unique solution $z(t)$ of Eq. (CE_δ) satisfying (3.27), that is, $z(t) = z(t,\phi)$ with $\widehat{g}(\phi) = x_0$. We will estimate the constant C_* in (3.27). Since $x_0 = \widehat{g}(\phi)$ for some $\phi \in X$, $x_t = (x(\phi))_t = \Phi z(t,\phi) + \xi(\phi)(t) + y(\phi)(t)$ for $t \in \mathbb{R}^+$. Hence (3.29) yields $\phi^s = \Pi^s x_0$. It then follows from Proposition 3.7 and (3.31) that

$$\begin{aligned} \|y(\phi)(t) - F_{*,\delta}(\Phi z(t,\phi))\|_X &\le (C\|\phi^s - F_{*,\delta}(\phi^c)\|_X + \|\xi(\phi)\|_{Z^c_\beta})e^{-\beta t} \\ &\le C_0 \|\Pi^s x_0 - F_{*,\delta}(\Phi z(0,\phi))\|_X e^{-\beta t} \end{aligned} \tag{3.37}$$

for $t \in \mathbb{R}^+$, where $C_0 := C + M(\delta)$. Thus we obtain

$$\begin{aligned} \|\Pi^c x_t - \Phi z(t,\phi)\|_X = \|\xi(\phi)(t)\|_X &\le M(\delta)\|\phi^s - F_{*,\delta}(\phi^c)\|_X e^{-\beta t} \\ &\le C_0 \|\Pi^s x_0 - F_{*,\delta}(\Phi z(0,\phi))\|_X e^{-\beta t}, \end{aligned}$$

and by (3.37)

$$\|\Pi^s x_t - F_{*,\delta}(\Phi z(t,\phi))\|_X \le C_0 \|\Pi^s x_0 - F_{*,\delta}(\Phi z(0,\phi))\|_X e^{-\beta t}$$

for $t \in \mathbb{R}^+$. Obviously, C_0 can be chosen as close to C as one expects by taking $\delta > 0$ small. Moreover, we know from Proposition 3.1 that $\Phi z(t,\phi) + F_{*,\delta}(\Phi z(t,\phi)) = x_t(0,\hat{\phi},f_\delta)$ with $\hat{\phi} = \Phi z(0,\phi) + F_{*,\delta}(\Phi z(0,\phi))$, so that (3.2) readily follows. This proves part (a).

(b) Given $t_0 \in \mathbb{R}$ and $z^0 \in \mathbb{C}^d$, write the solution of Eq. (CE_δ) through (t_0,z^0) as $z(t;t_0,z^0)$; in other words, $z(t;t_0,z^0) := z(t-t_0,\Phi z^0)$, $t \in [t_0,\infty)$. Moreover we set

$$\|\Phi\|^* := \left(\sum_{j=1}^d \|\phi_j\|_X^2\right)^{1/2}, \quad \text{and} \quad \|\Phi\|_* := \inf\{\|\Phi z\|_X : |z| = 1,\ z \in \mathbb{C}^d\}.$$

Suppose that the zero solution of Eq. (CE_δ) is stable. Then for arbitrary $\hat{\varepsilon} > 0$ there exists a $\delta_0 > 0$ such that $|z(t;t_0,z^0)| < \hat{\varepsilon}/(2(1+L(\delta))\|\Phi\|^*)$ for every $t \in [t_0,\infty)$ and $z^0 \in \mathbb{C}^d$ with $|z^0| < \delta_0$. Take a $\delta_* > 0$ such that

$$\delta_* < \min\left(\frac{\|\Phi\|_*\delta_0}{2C_1}, \frac{\hat{\varepsilon}}{4C_0C_1(1+2L(\delta))}\right).$$

Now let $\psi \in B_X(\delta_*)$. Then by Claim 1 and (3.33) there exists a (unique) $\phi \in B_X(2\delta_*)$ such that $\psi = \widehat{g}(\phi)$. Let z^0 be the element of $\mathbb{C}^d$ determined by $\phi^c = \Phi z^0$. Since $\|\Phi\|_*|z^0| \le \|\phi^c\|_X \le C_1\|\phi\|_X < 2C_1\delta_*$, i.e., $|z^0| < \delta_0$, we see

$$\|\Phi z(t-t_0,\phi)\|_X \le \|\Phi\|^*|z(t;t_0,z^0)| < \frac{\hat{\varepsilon}}{2(1+L(\delta))}, \quad t \in [t_0,\infty). \tag{3.38}$$

Consider the solution $x(t) = x(t+t_0;t_0,\psi,f_\delta)$ of Eq. (E_δ). Since $x_0 = x_{t_0}(t_0,\psi,f_\delta) = \psi$ and $\Phi z(0,\phi) = \phi^c$, we have

$$\begin{aligned} \|\Pi^s x_0 - F_{*,\delta}(\Phi z(0,\phi))\|_X = \|\psi^s - F_{*,\delta}(\phi^c)\|_X &\le \|\psi^s\|_X + L(\delta)\|\phi^c\|_X \\ &\le C_1(1+2L(\delta))\delta_* < \frac{\hat{\varepsilon}}{4C_0}. \end{aligned} \tag{3.39}$$

So, by letting $\hat{\phi} := \Phi z^0 + F_{*,\delta}(\Phi z^0)$, it follows from Proposition 3.1 and (3.38) that

$$\begin{aligned}\|x_t(t_0,\hat{\phi},f_\delta)\|_X &= \|\Phi z(t-t_0,\phi) + F_{*,\delta}(\Phi z(t-t_0,\phi)\|_X \\ &\le (1+L(\delta))\|\Phi z(t-t_0,\phi)\|_X < \frac{\hat{\varepsilon}}{2},\end{aligned}$$

and hence from part (a) and (3.39) that

$$\begin{aligned}\|x_t(t_0,\psi,f_\delta)\|_X &\le \|x_t(t_0,\psi,f_\delta) - x_t(t_0,\hat{\phi},f_\delta)\|_X + \|x_t(t_0,\hat{\phi},f_\delta)\|_X \\ &< \frac{\hat{\varepsilon}}{2}e^{-\beta(t-t_0)} + \frac{\hat{\varepsilon}}{2} < \hat{\varepsilon}\end{aligned} \tag{3.40}$$

for $t \in [t_0,\infty)$. Hence the zero solution of Eq. (E_δ) is stable.

Although we had proved the asymptotic stability part in [23, Theorem 6] via construction of a Liapunov function, we will give another proof below. Let us next assume that the zero solution of Eq. (CE_δ) is asymptotically stable. Then the stability of the zero solution of Eq. (E_δ) follows from the argument above. By the attractivity of the zero solution of Eq. (CE_δ) there exists an $R_0 > 0$ with the property that given $\hat{\varepsilon} > 0$, there exists a $\tau > 0$ such that $|z(t;t_0,z^0)| < \hat{\varepsilon}/(2(1+L(\delta))\|\Phi\|^*)$ for every $t \ge t_0+\tau$ and $z^0 \in \mathbb{C}^d$ with $|z^0| \le R_0$. Now choose an $R_* > 0$ and a $\tau_* > 0$ in such a way that $R_* < \|\Phi\|_* R_0/2$ and

$$\tau_* > \max\left(\tau,\, \beta^{-1}\log\frac{4C_0C_1(1+L(\delta))R_*}{\hat{\varepsilon}}\right).$$

If $\psi \in B_X(R_*)$, then, in a similar fashion to the last paragraph, ψ can be written as $\psi = \widehat{g}(\phi)$ with some $\phi \in B_X(2R_*)$. Corresponding to (3.39), we get this time

$$\|\Pi^s x_0 - F_{*,\delta}(\Phi z(0,\phi))\|_X \le C_1(1+2L(\delta))R_* < \frac{\hat{\varepsilon}}{4C_0}e^{\beta\tau_*}.$$

and hence, in a similar way to (3.40),

$$\begin{aligned}\|x_t(t_0,\psi,f_\delta)\|_X &\le \frac{\hat{\varepsilon}}{2}e^{-\beta(t-t_0-\tau_*)} + (1+L(\delta))\|\Phi\|^*|z(t-t_0,\phi)| \\ &< \frac{\hat{\varepsilon}}{2} + \frac{\hat{\varepsilon}}{2} = \hat{\varepsilon}\end{aligned}$$

for $t \ge t_0+\tau_*$. Thus, the zero solution of Eq. (E_δ) is also asymptotically stable.

The instability part immediately follows from the invariance of $W^c_\delta(0)$. □

Proof of Theorem 3.3. (a) Let us take a $\delta > 0$ sufficiently small such that $\|F_{*,\delta}(\psi)\|_X < \delta$ for any $\psi \in B_{E^c}(\delta)$. This is possible by Proposition 2.5 and $\zeta_*(+0) = 0$ (in fact, $F_{*,\delta}(0) = 0$ and $DF_{*,\delta}(0) = 0$). Set $F_* := F_{*,\delta}|_{B_{E^c}(\delta)}$, and let Ω and Ω_0 be open neighborhoods of 0 in X defined by $\Omega := \{\phi \in X : \|\Pi^s\phi\|_X < \delta,\ \|\Pi^c\phi\|_X < \delta\}$ and $\Omega_0 := \{\phi \in X : \|\Pi^s\phi\|_X < \delta/3,\ \|\Pi^c\phi\|_X < \delta/3\}$, respectively. Observe that

$$f(\phi) = f_\delta(\phi) \quad \text{for } \phi \in \Omega. \tag{3.41}$$

Since $x(t)$ is a solution of (E) satisfying $x_t \in \Omega_0$ $(t \in J)$, by virtue of VCF, one can readily see that $x(t)$ is also a solution of (E_δ) on J. Then by Theorem 3.2 there exists a solution of (CE_δ), say $z(t)$, satisfying (3.1). So, by Claim 1 in the proof of Theorem 3.2, there exists a (unique) $\phi \in X$ such that $x_0 = \widehat{g}(\phi)$; hence $x(t) = x(\phi)(t)$, $z(t) = z(t,\phi)$, and $x_t = \Phi z(t) + \xi(\phi)(t) + y(\phi)(t)$ hold for $t \in J$. In view of (3.30) and (3.31), note that $\|\xi(\phi)\|_{Z^c_\beta} \le (1/2)\|\phi\|_X$. Also, by the

relation $B_X(r_0/2) \subset \widehat{g}(B_X(r_0))$ for $r_0 > 0$ (cf. (3.33)), one can see $\|\phi\|_X \le 2\|x_0\|_X$, which is valid even if $x_0 = 0$. Since $\Pi^c x_t = \Phi z(t) + \xi(\phi)(t)$, we have

$$\|\Phi z(t)\|_X \le \|\Pi^c x_t\|_X + \|\xi(\phi)\|_{Z^c_\beta} < \frac{\delta}{3} + \frac{1}{2}\|\phi\|_X \le \frac{\delta}{3} + \|x_0\|_X.$$

Besides, noting that

$$\|x_0\|_X \le \|\Pi^c x_0\|_X + \|\Pi^s x_0\|_X < \frac{\delta}{3} + \frac{\delta}{3} = \frac{2\delta}{3},$$

we obtain $\|\Phi z(t)\|_X < \delta$ $(t \in J)$; and therefore $\|F_{*,\delta}(\Phi z(t))\|_X < \delta$ for $t \in J$ by the choice of δ. In particular, $\Phi z(t) + F_{*,\delta}(\Phi z(t)) \in \Omega$ for $t \in J$. So, $F_{*,\delta}(\Phi z(t)) = F_*(\Phi z(t))$, and

$$f_\delta(\Phi z(t) + F_{*,\delta}(\Phi z(t))) = f(\Phi z(t) + F_*(\Phi z(t)))$$

for $t \in J$, which implies that $z(t)$ is also a solution of (CE) on J. Hence, (3.3) and (3.4) directly follow from the estimates (3.1) and (3.2). This proves part (a).

(b) Suppose that the zero solution of (CE) is stable (asymptotically stable, unstable). Then one can see from (3.41) that the zero solution of (CE_δ) is stable (asymptotically stable, unstable); so is the zero solution of (E_δ) by Theorem 3.2 (b). By (3.41) again, so is that of (E). This completes the proof. □

4 One-parameter bifurcation structures

4.1 Statement of the results

Employing Theorem 3.3, we shall in this section discuss bifurcation structures of equilibria for the parametrized integral equation of the form

$$x(t) = \lambda \int_{-\infty}^{t} P(t-s)x(s)ds + f(x_t), \quad \lambda \in \mathbb{R}, \tag{PE}$$

where P is a measurable $m \times m$-matrix valued function on $\mathbb{R}^+$ with real components that satisfies

$$\int_0^\infty \|P(t)\|e^{\rho t}dt < \infty \quad \text{and} \quad \operatorname{ess\,sup}\{\|P(t)\|e^{\rho t} : t \in \mathbb{R}^+\} < \infty, \tag{4.1}$$

and $f \in C^1(X;\mathbb{R}^m)$ satisfies $f(0) = 0$ and $Df(0) = 0$. Also, in this section we set $X := L^1_\rho(\mathbb{R}^-;\mathbb{R}^m)$, and put

$$P_0 := \int_0^\infty P(t)dt, \quad P_1 := \int_0^\infty tP(t)dt.$$

More specifically, we assume the following conditions (A_1) through (A_4) throughout this section, and show that Eq. (PE) possesses a saddle-node bifurcation structure of equilibria as well as pitchfork one when the parameter λ varies in a neighborhood of 1.

(A_1) 1 is a simple eigenvalue of P_0;

(A_2) $\mathcal{R}(E_m - P_0) \oplus P_1\big(\mathcal{N}(E_m - P_0)\big) = \mathbb{R}^m$;

(A_3) $\det \Delta(\sigma + i\omega) \neq 0$ for $(\sigma,\omega) \in \mathbb{R}^+ \times \mathbb{R}^+ \setminus \{(0,0)\}$;

(A_4) $f : X \to \mathbb{R}^m$ is given by

$$f(\phi) = Q(\phi) + g(\phi), \quad \phi \in X, \tag{4.2}$$

where $Q : X \to \mathbb{R}^m$ is defined by

$$Q(\phi) = Q_*(\phi, \phi, \dots, \phi), \quad \phi \in X \tag{4.3}$$

with Q_* a bounded n-linear map from X^n to $\mathbb{R}^m$, and $g \in C^1(X; \mathbb{R}^m)$ satisfies

$$g(\phi) = o(\|\phi\|_X^n) \quad \text{and} \quad Dg(\phi) = o(\|\phi\|_X^{n-1}) \quad \text{as } \phi \to 0 \text{ in } X \tag{4.4}$$

for some integer n greater than 1.

Remark 4.1. The assumption (A_2) is equivalent to $\mathcal{R}(E_m - P_0) + P_1\big(\mathcal{N}(E_m - P_0)\big) = \mathbb{R}^m$ because of $\dim \mathcal{N}(E_m - P_0) + \dim \mathcal{R}(E_m - P_0) = m$.

Remark 4.2. (A_3) holds for instance under the following conditions:

(a) $\int_0^\infty \|P(t)\| dt \le 1$;

(b) $\det \Delta(i\omega) \neq 0$ for $\omega > 0$.

Indeed, (a) implies

$$\|E_m - \Delta(\sigma + i\omega)\| = \Big\| \int_0^\infty P(t) e^{-(\sigma+i\omega)t} dt \Big\| \le \int_0^\infty \|P(t)\| e^{-\sigma t} dt < 1$$

for $\sigma > 0$; hence $\Delta(\sigma + i\omega)$ is invertible for $\sigma > 0$. Thus, (A_3) follows from (a) and (b).

Remark 4.3. A typical example of Q_* in (A_4) is of the form

$$Q_*(\phi_1, \phi_2, \dots, \phi_n) = \sum_{j=1}^{l} \Big(\prod_{k=1}^{n} \int_{-\infty}^{0} q_k^{(j)}(-\theta) \phi_k(\theta) d\theta \Big) v_j,$$

with $\phi_k \in X$ ($k = 1, 2, \dots, n$), where each $q_k^{(j)}$ is a measurable $\mathbb{R}^{m*}$-valued function on $\mathbb{R}^+$ that satisfies

$$\int_0^\infty |q_k^{(j)}(t)| e^{\rho t} dt < \infty \quad \text{and} \quad \operatorname{ess\,sup}\{|q_k^{(j)}(t)| e^{\rho t} : t \ge 0\} < \infty$$

($k = 1, 2, \dots, n$) and $v_j \in \mathbb{R}^m$ ($j = 1, 2, \dots, l$).

Now let η_* be an eigenvector of P_0 associated with eigenvalue 1 and ζ^* an eigenvector of P_0^*, the adjoint of P_0; and put

$$q_* := -\frac{\langle \zeta^*, P_1 \eta_* \rangle}{\langle \zeta^*, \eta_* \rangle}, \quad c_0 := \langle \zeta^*, Q(w_1(0)\eta_*) \rangle. \tag{4.5}$$

Note that $\langle \zeta^*, \eta_* \rangle \neq 0$, and q_* does not depend on the choice of η_* and ζ^* since both $\mathcal{N}(E_m - P_0)$ and $\mathcal{N}(E_m - P_0^*)$ are one-dimensional spaces due to (A_1).

Our main theorem on bifurcation structure is as follows:

Theorem 4.4. *Suppose that the assumptions (A_1) through (A_4) are satisfied. Suppose furthermore that $c_0 \neq 0$. Then, the following statements hold:*

(i) *Let n be even.*

(ia) *If $\lambda = 1$, then there exists an open neighborhood $\mathcal{W}_0$ of 0 in X such that Eq. (PE) has no equilibria in $\mathcal{W}_0$ other than 0. The equilibrium 0 is also unstable. Moreover, there exist an $\varepsilon^* > 0$ and a continuous map $\phi_* : (1-\varepsilon^*, 1+\varepsilon^*)\backslash\{1\} \to X\backslash\{0\}$ such that for each λ, ϕ_*^λ is an equilibrium of Eq. (PE), and $\phi_*^\lambda \to 0$ as $\lambda \to 1$. (Here we used the notation ϕ_*^λ rather than $\phi_*(\lambda)$ for $\lambda \in (1-\varepsilon^*, 1+\varepsilon^*)$.)*

(ib) *If $q_* < 0$, then the equilibria ϕ_*^λ and 0 are asymptotically stable (resp. unstable) in case $\lambda < 1$ (resp. $\lambda > 1$); and if $q_* > 0$, then ϕ_*^λ and 0 are unstable (resp. asymptotically stable) in case $\lambda < 1$ (resp. $\lambda > 1$).*

(ic) *There exists an open neighborhood $\mathcal{W}$ of 0 in X such that if $\phi_* \in \mathcal{W}$ is an equilibrium of Eq. (PE) with $|\lambda - 1|$ small, then $\phi_* = 0$ or $\phi_* = \phi_*^\lambda$.*

(ii) *Let n be odd and c_0 positive.*

(iia) *If $\lambda = 1$, then there exists an open neighborhood $\mathcal{W}_0$ of 0 in X such that Eq. (PE) has no equilibria in $\mathcal{W}_0$ other than 0. The equilibrium 0 is also unstable. Moreover, there exist an $\varepsilon^* > 0$ and continuous maps $\phi_\pm : (1, 1+\varepsilon^*) \to \mathcal{W}_0\backslash\{0\}$ such that for each λ, ϕ_+^λ and ϕ_-^λ are two distinct equilibria of Eq. (PE), and $\phi_\pm^\lambda \to 0$ as $\lambda \to 1+0$.*

(iib) *If $q_* < 0$, then the equilibria ϕ_+^λ and ϕ_-^λ are unstable for $\lambda > 1$, whereas 0 is asymptotically stable for $\lambda > 1$; and if $q_* > 0$, then ϕ_+^λ and ϕ_-^λ are asymptotically stable and 0 is unstable for $\lambda > 1$.*

(iic) *There exists an open neighborhood $\mathcal{W}$ of 0 in X such that if $\phi_* \in \mathcal{W}$ is an equilibrium of Eq. (PE) with $\lambda - 1 > 0$ small, then ϕ^* coincides with one of 0, ϕ_+^λ and ϕ_-^λ.*

(iii) *Let n be odd and c_0 negative.*

(iiia) *If $\lambda = 1$, then there exists an open neighborhood $\mathcal{W}_0$ of 0 in X such that Eq. (PE) has no equilibria in $\mathcal{W}_0$ other than 0. The equilibrium 0 is also asymptotically stable. Moreover, there exist an $\varepsilon^* > 0$ and continuous maps $\phi_\pm : (1-\varepsilon^*, 1) \to \mathcal{W}_0\backslash\{0\}$ such that for each λ, ϕ_+^λ and ϕ_-^λ are two distinct equilibria of Eq. (PE), and $\phi_\pm^\lambda \to 0$ as $\lambda \to 1-0$.*

(iiib) *If $q_* < 0$, then the equilibria ϕ_+^λ and ϕ_-^λ are asymptotically stable for $\lambda < 1$, whereas 0 is unstable for $\lambda < 1$; and if $q_* > 0$, then ϕ_+^λ and ϕ_-^λ are unstable and 0 is asymptotically stable for $\lambda < 1$.*

(iiic) *There exists an open neighborhood $\mathcal{W}$ of 0 in X such that if $\phi_* \in \mathcal{W}$ is an equilibrium of Eq. (PE) with $1 - \lambda > 0$ small, then ϕ^* coincides with one of 0, ϕ_+^λ and ϕ_-^λ.*

4.2 Extended system and the projection onto its center subspace

For the convenience, put $\lambda := 1 + \varepsilon$. Then, Eq. (PE) becomes

$$x(t) = \int_{-\infty}^{t} P(t-s)x(s)ds + \varepsilon \int_{-\infty}^{t} P(t-s)x(s)ds + f(x_t). \tag{4.6}$$

For the proof of Theorem 4.4, we will treat an extended system of (4.6), and apply Theorem 3.2. To do so, consider the system of integral equations

$$\begin{aligned} \varepsilon(t) &= \int_{-\infty}^{t} p_0(t-s)\varepsilon(s)ds, \\ x(t) &= \int_{-\infty}^{t} P(t-s)x(s)ds + \left(\int_{-\infty}^{t} p_0(t-s)\varepsilon(s)ds \right) \left(\int_{-\infty}^{t} P(t-s)x(s)ds \right) + f(x_t), \end{aligned} \tag{4.7}$$

where $p_0(t)$ is the function on $\mathbb{R}^+$ defined by

$$p_0(t) := \rho_0 e^{-\rho_0 t}, \quad t \in \mathbb{R}^+,$$

and ρ_0 is a positive number satisfying $\rho_0 > \rho$. Note that

$$\int_0^\infty p_0(t)dt = 1 \tag{4.8}$$

and that

$$\int_0^\infty |p_0(t)|e^{-\rho t}dt < \infty \quad \text{and} \quad \operatorname{ess\,sup}\{|p_0(t)|e^{\rho t} : t \geq 0\} < \infty.$$

Let us denote

$$\widetilde{P}(t) := \operatorname{diag}\big(p_0(t),\, P(t)\big) = \begin{pmatrix} p_0(t) & 0 \\ 0 & P(t) \end{pmatrix}, \quad t \in \mathbb{R}^+.$$

Set $\widetilde{X} := L^1_\rho(\mathbb{R}^-;\mathbb{R}^{m+1})$ and $X_1 := L^1_\rho(\mathbb{R}^-;\mathbb{R})$ (see Subsection 2.1); then $\widetilde{X}$ can be naturally identified with $X_1 \times X$. We denote by Π_1 and Π_2 the projections from $\widetilde{X}$ to X_1 and X, respectively. Moreover, define $\widetilde{G} : \widetilde{X} \to \mathbb{R}^{m+1}$ by

$$\widetilde{G}(\tilde{\phi}) := \operatorname{col}\big(0,\, G^{(2)}(\tilde{\phi})\big), \quad \tilde{\phi} \in \widetilde{X},$$

where $G^{(2)}$ is an $\mathbb{R}^m$-valued function given by

$$G^{(2)}(\tilde{\phi}) := \left(\int_{-\infty}^0 p_0(-\theta)\phi^{(1)}(\theta)d\theta\right)\left(\int_{-\infty}^0 P(-\theta)\phi^{(2)}(\theta)d\theta\right) + f(\phi^{(2)}), \tag{4.9}$$

that is, $G^{(2)}(\tilde{\phi}) = \Pi_2\widetilde{G}(\tilde{\phi})$. Here we used the notation

$$\tilde{\phi} = \operatorname{col}\,(\phi^{(1)},\, \phi^{(2)}) \quad \text{with } \phi^{(1)} \in X_1 \text{ and } \phi^{(2)} \in X. \tag{4.10}$$

By letting $\tilde{x}(t) := \operatorname{col}\,(\varepsilon(t), x(t))$, Eq. (4.7) can then be rewritten as

$$\tilde{x}(t) = \int_{-\infty}^t \widetilde{P}(t-s)\tilde{x}(s)ds + \widetilde{G}(\tilde{x}_t). \tag{$\widetilde{E}$}$$

In connection with Eq. $(\widetilde{E})$ we will also consider the integral equation

$$\tilde{x}(t) = \int_{-\infty}^t \widetilde{P}(t-s)\tilde{x}(s)ds + \widetilde{G}_\delta(\tilde{x}_t), \tag{$\widetilde{E}_\delta$}$$

where $\widetilde{G}_\delta : \widetilde{X} \to \mathbb{R}^{m+1}$ is defined by

$$\widetilde{G}_\delta(\tilde{\phi}) := \chi\big(\|\Pi^s\tilde{\phi}\|_{\widetilde{X}}/\delta\big)\chi\big(\|\Pi^c\tilde{\phi}\|_{\widetilde{X}}/\delta\big)G(\tilde{\phi}), \quad \tilde{\phi} \in \widetilde{X}$$

(see Subsection 2.3). By (4.2), (4.3) and (4.4), it is clear that $\widetilde{G}$ belongs to $C^1(\widetilde{X};\mathbb{R}^{m+1})$ with $\widetilde{G}(0) = 0$ and $D\widetilde{G}(0) = 0$. Hence Theorem 2.4 implies that there exists the (global) center manifold of the equilibrium 0 of Eq. $(\widetilde{E}_\delta)$, denoted $\widetilde{W}^c_\delta(0)$, which is given as the graph of some C^1 map $\widetilde{F}_{*,\delta} : \widetilde{E}^c \to \widetilde{E}^s$:

$$\widetilde{W}^c_\delta(0) := \big\{\psi + \widetilde{F}_{*,\delta}(\psi) : \psi \in \widetilde{E}^c\big\}. \tag{4.11}$$

Recall that $\widetilde{F}_{*,\delta}$ is defined by

$$\widetilde{F}_{*,\delta}(\psi) = \lim_{n\to\infty}\int_{-\infty}^0 \widetilde{T}^s(-s)\Pi^s\Gamma^n\widetilde{G}_\delta(\Lambda_{*,\delta}(\psi)(s))ds, \quad \psi \in \widetilde{E}^c, \tag{4.12}$$

where $\{\widetilde{T}(t)\}_{t\geq 0}$ is the solution semigroup of the integral equation

$$\widetilde{x}(t) = \int_{-\infty}^{t} \widetilde{P}(t-s)\widetilde{x}(s)ds \tag{4.13}$$

(see (2.6)). Let $\widetilde{F}_* : B_{\widetilde{E}^c}(\delta) \to \widetilde{X}$ be the restriction of $\widetilde{F}_{*,\delta}$ to $B_{\widetilde{E}^c}(\delta)$. The local center manifold $\widetilde{W}^c_{\mathrm{loc}}(0)$ of the equilibrium 0 of Eq. $(\widetilde{E})$ is then given by $\widetilde{W}^c_{\mathrm{loc}}(0) := \operatorname{graph} \widetilde{F}_* \subset \widetilde{W}^c_\delta(0)$ (see [23, Theorem 5]). Henceforth, we set

$$\widetilde{\Omega} := \{\tilde{\phi} \in \widetilde{X} : \|\Pi^c\tilde{\phi}\|_{\widetilde{X}} < \delta,\ \|\Pi^s\tilde{\phi}\|_{\widetilde{X}} < \delta\},$$

and

$$\widetilde{\Omega}_0 := \{\tilde{\phi} \in \widetilde{X} : \|\Pi^c\tilde{\phi}\|_{\widetilde{X}} < \delta/3,\ \|\Pi^s\tilde{\phi}\|_{\widetilde{X}} < \delta/3\}.$$

Proposition 4.5. *Given any $t_0 \in \mathbb{R}$ and $\tilde{\phi} \in \widetilde{X}$, let $\widetilde{x}(t; t_0, \tilde{\phi}, \widetilde{G}_\delta)$ be the solution of Eq. $(\widetilde{E}_\delta)$ with $\widetilde{x}_{t_0} = \tilde{\phi}$; and $\widetilde{x}(t; t_0, \tilde{\phi}, \widetilde{G})$ the one of Eq. $(\widetilde{E})$ with $\widetilde{x}_{t_0} = \tilde{\phi}$. Let $\varepsilon \in \mathbb{R}$ and $j_\varepsilon : X \to \widetilde{X}$ be the map defined by*

$$j_\varepsilon(\phi) := \operatorname{col}\big(w_1(0)\varepsilon, \phi\big), \quad \phi \in X \tag{4.14}$$

where $w_1(0)$ is the function defined in (2.8) (see Subsection 2.4). Then we have

(i) *Given any $\varepsilon \in \mathbb{R}$ and $\phi \in X$, $\widetilde{x}_t(t_0, j_\varepsilon(\phi), \widetilde{G}_\delta) \in j_\varepsilon(X)$ holds for $t \geq t_0$.*

(ii) *Let $x(t; t_0, \phi, f)$ be the solution on $J = [t_0, t_1]$ of Eq. (PE) satisfying $x_{t_0} = \phi \in X$. Then*

$$\widetilde{x}(t; t_0, j_\varepsilon(\phi), \widetilde{G}) = \operatorname{col}\big(\varepsilon, x(t; t_0, \phi, f)\big), \quad t \in J;$$

so that $\widetilde{x}_t(t_0, j_\varepsilon(\phi), \widetilde{G}) = j_\varepsilon(x_t(t_0, \phi, f))$ holds for every $t \in J$ and $\varepsilon \in \mathbb{R}$.

In particular, $j_\varepsilon(X)$ is positively invariant for both Eq. $(\widetilde{E}_\delta)$ and Eq. $(\widetilde{E})$.

Proof. (i) Set $\chi_0(\tilde{\phi}) := \chi\big(\|\Pi^s\tilde{\phi}\|_{\widetilde{X}}/\delta\big)\chi\big(\|\Pi^c\tilde{\phi}\|_{\widetilde{X}}/\delta\big)$ for $\tilde{\phi} \in \widetilde{X}$ and let $u(t)$ be the solution of

$$u(t) = \big(1 + \varepsilon\chi_0(j_\varepsilon(u_t))\big)\int_{-\infty}^{t} P(t-s)u(s)ds + \chi_0(j_\varepsilon(u_t))f(u_t)$$

with $u_{t_0} = \phi$. The existence and uniqueness of $u(t)$ is due to [21, Proposition 3]. Define $\widetilde{x} : \mathbb{R} \to \widetilde{X}$ by

$$\widetilde{x}(t) = \operatorname{col}\big(x^{(1)}(t), x^{(2)}(t)\big) := \operatorname{col}\big(\varepsilon, u(t)\big) \in X_1 \times X, \quad t \in \mathbb{R}.$$

Then, it readily follows from (4.8) that

$$x^{(1)}(t) = \int_{-\infty}^{t} p_0(t-s)x^{(1)}(s)ds.$$

Besides, since $\widetilde{x}_t = j_\varepsilon(u_t)$ $(t \geq t_0)$, $x^{(2)}(t)$ satisfies

$$x^{(2)}(t) = \int_{-\infty}^{t} P(t-s)x^{(2)}(s)ds + \chi_0(\widetilde{x}_t)\left(\int_{-\infty}^{t} p_0(t-s)\varepsilon ds\right)\left(\int_{-\infty}^{t} P(t-s)x^{(2)}(s)ds\right) + \chi_0(\widetilde{x}_t)f\big(x^{(2)}_t\big)$$

for $t \geq t_0$. Therefore $\widetilde{x}(t)$ is the solution of Eq. $(\widetilde{E}_\delta)$ with $\widetilde{x}_{t_0} = j_\varepsilon(u_{t_0}) = j_\varepsilon(\phi)$. Thus, we obtain $\widetilde{x}(t) = \widetilde{x}(t; t_0, j_\varepsilon(\phi), \widetilde{G}_\delta)$; hence

$$\widetilde{x}_t(t_0, j_\varepsilon(\phi), \widetilde{G}_\delta) = \widetilde{x}_t = j_\varepsilon(u_t) \in j_\varepsilon(X) \quad \text{for } t \geq t_0.$$

(ii) One can easily see that the argument above is valid with $\chi_0 = 1$, $\widetilde{G}_\delta = \widetilde{G}$ and $u(t) = x(t;t_0,\phi,f)$. Hence, it follows that $\tilde{x}(t;t_0,j_\varepsilon(\phi),\widetilde{G}) = \mathrm{col}\,(\varepsilon, x(t;t_0,\phi,f))$. So,

$$\widetilde{x}_t(t_0,j_\varepsilon(\phi),\widetilde{G}) = \mathrm{col}\,(w_1(0)\varepsilon, x_t(t_0,\phi,f)) = j_\varepsilon(x_t(t_0,\phi,f)) \in j_\varepsilon(X)$$

for $t \in J$. Thus, $j_\varepsilon(X)$ is positively invariant for Eq. ($\widetilde{E}_\delta$) and Eq. ($\widetilde{E}$). □

The bilinear form, induced in the formal adjoint theory (Subsection 2.4; see also [22]) associated with Eq. (4.13), the linear part of Eq. ($\widetilde{E}$), is given by

$$\langle\!\langle \widetilde{\psi}, \widetilde{\phi} \rangle\!\rangle := \int_{-\infty}^{0} \Big(\int_{\theta}^{0} \widetilde{\psi}(\xi-\theta)\widetilde{P}(-\theta)\widetilde{\phi}(\xi)d\xi \Big) d\theta, \quad \widetilde{\phi} \in \widetilde{X},\ \ \widetilde{\psi} \in \widetilde{X}^\sharp, \tag{4.15}$$

where $\widetilde{X}^\sharp := L^1_\rho(\mathbb{R}^+;\mathbb{R}^{m+1})$. Using the notations (4.10) and $\widetilde{\psi} = (\psi^{(1)},\ \psi^{(2)})$ with $\psi^{(1)} \in X_1^\sharp := L^1_\rho(\mathbb{R}^+;\mathbb{R})$ and $\psi^{(2)} \in X^\sharp := L^1_\rho(\mathbb{R}^+;\mathbb{R}^m)$, we have

$$\widetilde{\psi}(\xi-\theta)\widetilde{P}(-\theta)\widetilde{\phi}(\eta) = (\psi^{(1)}(\xi-\theta),\ \psi^{(2)}(\xi-\theta)) \begin{pmatrix} p_0(-\theta) & 0 \\ 0 & P(-\theta) \end{pmatrix} \begin{pmatrix} \phi^{(1)}(\theta) \\ \phi^{(2)}(\theta) \end{pmatrix};$$

hence (4.15) becomes

$$\langle\!\langle \widetilde{\psi}, \widetilde{\phi} \rangle\!\rangle = \int_{-\infty}^{0} \Big(\int_{\theta}^{0} (\psi^{(1)}(\xi-\theta)p_0(-\theta)\phi^{(1)}(\xi) + \psi^{(2)}(\xi-\theta)P(-\theta)\phi^{(2)}(\xi))d\xi \Big) d\theta. \tag{4.16}$$

The characteristic operator of Eq. (4.13) is

$$\begin{aligned} \widetilde{\Delta}(z) &= E_{m+1} - \int_0^\infty \widetilde{P}(t)e^{-zt}dt \\ &= \begin{pmatrix} 1 - \int_0^\infty p_0(t)e^{-zt}dt & 0 \\ 0 & E_m - \int_0^\infty P(t)e^{-zt}dt \end{pmatrix} = \begin{pmatrix} z/(z+\rho_0) & 0 \\ 0 & \Delta(z) \end{pmatrix}. \end{aligned}$$

So, it follows that

$$\det \widetilde{\Delta}(z) = \frac{z}{z+\rho_0} \det \Delta(z),$$

and in view of $\Delta(0) = E_m - P_0$ and (A_1), 0 is a characteristic root of Eq. (4.13) whose order as a zero of $\det \widetilde{\Delta}(z)$ is at least greater than 1 (Recall that $\det \Delta(z)$ is analytic in the domain $\mathbb{C}_{-\rho}$).

Proposition 4.6. *$\det \Delta(z)$ has 0 as a zero of order 1 (i.e., $\det \widetilde{\Delta}(z)$ has 0 as the one of order 2) if and only if (A_1) and (A_2) hold.*

Proof. Let us denote $\Delta(0) = E_m - P_0 = (a_1\ a_2\ \cdots\ a_m)$ and $P_1 = (p_1\ p_2\ \cdots\ p_m)$ with $a_j, p_j \in \mathbb{R}^m$ $(j = 1,2,\ldots,m)$. By virtue of the first inequality of (4.1), we have

$$\Delta^{(k)}(z) = -\frac{d^k}{dz^k}\int_0^\infty P(t)e^{-zt}dt = (-1)^{k-1}\int_0^\infty t^k P(t)e^{-zt}dt, \quad z \in \mathbb{C}_{-\rho},$$

$(k = 1,2,\ldots)$ and hence $\Delta(z)$ is expressed as

$$\Delta(z) = \Delta(0) + z\Delta'(0) + o(z) = E_m - P_0 + zP_1 + o(z).$$

Put $d(z) := \det \Delta(z)$. Then, $d(0) = 0$ and

$$\begin{aligned} d'(0) &= \frac{d}{dz}\Big|_{z=0} \det \Big(a_1 + zp_1 + o(z) \quad a_2 + zp_2 + o(z) \quad \cdots \quad a_m + zp_m + o(z) \Big) \\ &= \sum_{j=1}^m \det (a_1\ \cdots\ \underset{\widehat{j}}{p_j}\ \cdots\ a_m). \end{aligned}$$

Suppose that $\det \Delta(z)$ has 0 as a zero of order 1, that is, $d'(0) \neq 0$. Let $k := \dim \mathcal{R}(E_m - P_0)$ and assume that $k \leq m-2$. Without loss of generality, we may assume that $\mathcal{R}(E_m - P_0) = \operatorname{span}\{a_1, a_2, \ldots, a_k\}$. Since $\operatorname{span}\{a_1, \ldots, a_{j-1}, p_j, a_{j+1}, \ldots, a_m\} = \operatorname{span}\{a_1, \ldots, a_k, p_j\}$, we have $\dim \operatorname{span}\{a_1, \ldots, a_{j-1}, p_j, a_{j+1}, \ldots, a_m\} \leq k+1 \leq m-1$ so that

$$\det \begin{pmatrix} a_1 & \cdots & \underset{\widehat{j}}{p_j} & \cdots & a_m \end{pmatrix} = 0, \quad j = 1, 2, \ldots, m.$$

Hence, we get $d'(0) = 0$, contradicting to our assumption. Thus, we obtain $k = m-1$, and therefore (A_1) holds. Moreover, in this case, we may assume that a_m can be written as $a_m = \sum_{k=1}^{m-1} c_k a_k$ with some $c_k \in \mathbb{R}$ $(k = 1, 2, \ldots, m-1)$. Then,

$$\begin{aligned} d'(0) &= \sum_{j=1}^{m} \sum_{k=1}^{m-1} \det \begin{pmatrix} a_1 & \cdots & \underset{\widehat{j}}{p_j} & \cdots & c_k a_k \end{pmatrix} \\ &= \det \begin{pmatrix} a_1 & \cdots & a_{m-1} & p_m \end{pmatrix} + \sum_{j=1}^{m-1} \det \begin{pmatrix} a_1 & \cdots & \underset{\widehat{j}}{p_j} & \cdots & a_{m-1} & c_j a_j \end{pmatrix} \\ &= \det \begin{pmatrix} a_1 & \cdots & a_{m-1} & p_m \end{pmatrix} + \sum_{j=1}^{m-1} \det \begin{pmatrix} a_1 & \cdots & \underset{\widehat{j}}{a_j} & \cdots & a_{m-1} & -c_j p_j \end{pmatrix} \\ &= \det \begin{pmatrix} a_1 & a_2 & \cdots & a_{m-1} & p_m - \sum_{j=1}^{m-1} c_j p_j \end{pmatrix}. \end{aligned}$$

Since $\operatorname{col}(c_1, \ldots, c_{m-1}, -1) \in \mathcal{N}(E_m - P_0)$, we see $p_m - \sum_{j=1}^{m-1} c_j p_j$ belongs to $P_1\big(\mathcal{N}(E_m - P_0)\big)$. Consequently, it follows from $d'(0) \neq 0$ that (A_2) is also valid.

Conversely, suppose that (A_1) and (A_2) hold. Then $d'(0) \neq 0$ is clear from the argument above. □

Note that Eq. (4.13) has no characteristic roots with positive real parts. Indeed, (A_3) implies $\det \Delta(z) \neq 0$ for $\operatorname{Re} z > 0$ since $\Delta(z) = \overline{\Delta(\bar{z})}$. Thus, $\Sigma^u = \emptyset$, and therefore we have

$$\widetilde{X} = \widetilde{E}^c \oplus \widetilde{E}^s, \tag{4.17}$$

where $\widetilde{E}^c$ and $\widetilde{E}^s$ are the center subspace and the stable subspace of the equilibrium 0 for Eq. (4.13), respectively. Also, notice from (A_3) that $\Sigma^c = \{0\}$.

By (2.7) of Subsection 2.4, $\widetilde{E}^c$ is given by the generalized eigenspace of $\widetilde{A}$ associated with eigenvalue 0:

$$\widetilde{E}^c = \mathcal{M}_0(\widetilde{A}) = \bigcup_{k \geq 1} \mathcal{N}(\widetilde{A}^k),$$

where $\widetilde{A}$ is the generator of the solution semigroup $\{\widetilde{T}(t)\}_{t \geq 0}$ of Eq. (4.13).

Proposition 4.7. *We have* $\mathcal{M}_0(\widetilde{A}) = \mathcal{N}(\widetilde{A}) = \operatorname{span}\{w_1(0)e_1, w_1(0)\tilde{\eta}_*\}$, *where* e_1 *and* $\tilde{\eta}_*$ *are the elements of* $\mathbb{R}^{m+1}$ *given by* $e_1 := \operatorname{col}(1, 0, \ldots, 0)$, $\tilde{\eta}_* = \operatorname{col}(0, \eta_*)$, η_* *being an eigenvector of* P_0 *corresponding to eigenvalue* 1, *and* $w_1(0)$ *is the one given by* (2.8).

Proof. By virtue of Proposition 2.6 we know that $\widetilde{\phi} \in \mathcal{N}(\widetilde{A})$ if and only if $\widetilde{\phi} = w_1(0)\eta_1$ for some $\tilde{\eta}_1 \in \mathbb{R}^{m+1}$ with $\widetilde{\Delta}(0)\tilde{\eta}_1 = 0$. Since

$$D_1(0) = \widetilde{\Delta}(0) = \begin{pmatrix} 0 & 0 \\ 0 & E_m - P_0 \end{pmatrix}$$

(see ((2.9)), one can see from (A_1) that $\mathcal{N}(\widetilde{A}) = \operatorname{span}\{w_1(0)e_1,\, w_1(0)\tilde{\eta}_*\}$.

Recall the fact that

$$\mathcal{N}(\widetilde{A}^2) = \left\{ w_1(0)\tilde{\eta}_1 + w_2(0)\tilde{\eta}_2 : D_2(0)\begin{pmatrix}\tilde{\eta}_1\\ \tilde{\eta}_2\end{pmatrix} = 0,\ \tilde{\eta}_1, \tilde{\eta}_2 \in \mathbb{R}^{m+1}\right\},$$

where $D_2(0)$ is the matrix defined by

$$D_2(0) = \begin{pmatrix}\widetilde{\Delta}(0) & \widetilde{\Delta}'(0)\\ 0 & \widetilde{\Delta}(0)\end{pmatrix}$$

(see (2.9)). So, $w_1(0)\tilde{\eta}_1 + w_2(0)\tilde{\eta}_2 \in \mathcal{N}(\widetilde{A}^2)$ is equivalent to

$$\widetilde{\Delta}(0)\tilde{\eta}_1 + \widetilde{\Delta}'(0)\tilde{\eta}_2 = 0 \quad \text{and} \quad \widetilde{\Delta}(0)\tilde{\eta}_2 = 0. \tag{4.18}$$

It is also easily seen that

$$\widetilde{\Delta}'(z) = \begin{pmatrix}\rho_0/(z+\rho_0)^2 & 0\\ 0 & \int_0^\infty tP(t)e^{-zt}dt\end{pmatrix}; \quad \text{hence} \quad \widetilde{\Delta}'(0) = \begin{pmatrix}1/\rho_0 & 0\\ 0 & P_1\end{pmatrix}.$$

(4.18) implies $\widetilde{\Delta}'(0)\tilde{\eta}_2 \in \widetilde{\Delta}'(0)\big(\mathcal{N}(\widetilde{\Delta}(0))\big)$, so that $\widetilde{\Delta}(0)\tilde{\eta}_1 = 0$ and $\widetilde{\Delta}'(0)\tilde{\eta}_2 = 0$ because

$$\mathcal{R}(\widetilde{\Delta}(0)) \oplus \widetilde{\Delta}'(0)\big(\mathcal{N}(\widetilde{\Delta}(0))\big) = \mathbb{R}^{m+1} \tag{4.19}$$

follows from (A_3) immediately. In particular, $\tilde{\eta}_2 = 0$. Indeed, assume that $\tilde{\eta}_2 \neq 0$. Since $\tilde{\eta}_2$ belongs to $\mathcal{N}(\widetilde{\Delta}(0))$, which is a two-dimensional subspace spanned by e_1 and $\tilde{\eta}_*$, the fact that $\widetilde{\Delta}'(0)\tilde{\eta}_2 = 0$ means $\dim \widetilde{\Delta}'(0)\big(\mathcal{N}(\widetilde{\Delta}(0))\big) \leq 1$. On the other hand, in view of (A_1), $\dim \mathcal{R}(\widetilde{\Delta}(0)) = \dim \mathcal{R}(\Delta(0)) = m-1$, which contradicts to (4.19). Hence, we obtain $\tilde{\eta}_2 = 0$, and therefore (4.18) is equivalent to $\widetilde{\Delta}(0)\tilde{\eta}_1 = 0$ and $\tilde{\eta}_2 = 0$. Consequently, $\mathcal{N}(\widetilde{A}^2) = \{w_1(0)\tilde{\eta}_1 : \widetilde{\Delta}(0)\tilde{\eta}_1 = 0,\ \tilde{\eta}_1 \in \mathbb{R}^{m+1}\} = \mathcal{N}(\widetilde{A})$ and the proof is completed. □

Remark 4.8. By virtue of Proposition 4.6, together with [22, Cororally 3.1], one may also conclude that $\dim \mathcal{M}_0(\widetilde{A}) = 2$ under the assumptions (A_1) and (A_3).

Similarly, the generalized eigenspace of the formal adjoint operator $\widetilde{A}^\sharp$ is identical with its eigenspace.

Proposition 4.9. *We have* $\mathcal{N}((\widetilde{A}^\sharp)^k) = \mathcal{N}(\widetilde{A}^\sharp)$ $(k = 2, 3, \ldots)$ *and*

$$\mathcal{N}(\widetilde{A}^\sharp) = \operatorname{span}\{w_1^\sharp(0)e_1^*,\, w_1^\sharp(0)\tilde{\zeta}^*\},$$

where e_1^* *and* $\tilde{\zeta}^*$ *are the elements of* $\mathbb{R}^{m+1\,*}$ *given by* $e_1^* := (1, 0, \ldots, 0)$, $\tilde{\zeta}^* = (0, \zeta^*)$ *being an eigenvector of* P_0^* *associated with eigenvalue* 1, *that is,* $\zeta^* \in \mathcal{N}(\Delta(0)^*)\backslash\{0\}$, *and* $w_1^\sharp(0)$ *is the one given in* (2.10). *Here* $\Delta(0)^*$ *and* P_0^* *denote the adjoint operators of* $\Delta(0)$ *and* P_0, *respectively.*

Proof. Recall from Proposition 2.7 that $\mathcal{N}(\widetilde{A}^\sharp) = \operatorname{span}\{w_1^\sharp(0)\tilde{\zeta} : \tilde{\zeta}D_1(0) = 0,\ \tilde{\zeta} \in \mathbb{R}^{m+1\,*}\}$ and

$$\mathcal{N}((\widetilde{A}^\sharp)^2) = \{w_1^\sharp(0)\tilde{\zeta}_2 + w_2^\sharp(0)\tilde{\zeta}_1 : (\tilde{\zeta}_1, \tilde{\zeta}_2)D_2(0) = 0,\ \tilde{\zeta}_1, \tilde{\zeta}_2 \in \mathbb{R}^{m+1\,*}\}.$$

One can readily verify that $\mathcal{N}(\widetilde{A}^\sharp) = \operatorname{span}\{w_1^\sharp(0)e_1^*,\, w_1^\sharp(0)\tilde{\zeta}_*\}$. Also, $(\tilde{\zeta}_1, \tilde{\zeta}_2)D_2(0) = 0$ implies

$$\tilde{\zeta}_1\widetilde{\Delta}(0) = 0 \quad \text{and} \quad \tilde{\zeta}_1\widetilde{\Delta}'(0) + \tilde{\zeta}_2\widetilde{\Delta}(0) = 0. \tag{4.20}$$

Hence, $\tilde{\zeta}_1$ vanishes on the subspace $\mathcal{R}(\widetilde{\Delta}(0))$ of $\mathbb{R}^{m+1}$ and on $\widetilde{\Delta}'(0)(\mathcal{N}(\widetilde{\Delta}(0)))$, as well. Therefore, by (4.19), we must have $\tilde{\zeta}_1 = 0$. Thus, (4.20) is equivalent to $\tilde{\zeta}_1 = 0$ and $\tilde{\zeta}_2\widetilde{\Delta}(0) = 0$, so that $\mathcal{N}((\widetilde{A}^\sharp)^2) = \{w_1^\sharp(0)\tilde{\zeta}_2 : \tilde{\zeta}_2\widetilde{\Delta}(0) = 0,\ \tilde{\zeta}_2 \in \mathbb{R}^{m+1\,*}\} = \mathcal{N}(\widetilde{A}^\sharp)$. This completes the proof. □

By Theorem 2.8 the projection $\Pi^c : \widetilde{X} \to \widetilde{E}^c$ along the decomposition (4.17) is given as follows:

Proposition 4.10.

$$\Pi^c\widetilde{\phi} = w_1(0)\begin{pmatrix} \rho_0 \int_{-\infty}^0 p_0(-\theta)\Big(\int_\theta^0 \phi^{(1)}(\xi)d\xi\Big)d\theta \\ (1/r)\int_{-\infty}^0 \zeta^* P(-\theta)\Big(\int_\theta^0 \phi^{(2)}(\xi)d\xi\Big)d\theta\cdot\eta_* \end{pmatrix}$$

for $\widetilde{\phi} = \mathrm{col}\,(\phi^{(1)}, \phi^{(2)}) \in \widetilde{X}$, *where* r *denotes the constant* $\langle\zeta^*, P_1\eta_*\rangle$.

Proof. We first verify $r \neq 0$. Suppose by contradiction that $r = 0$; then $\zeta^*|_{\Delta'(0)(\mathcal{N}(\Delta(0)))} = 0$ because of $\mathcal{N}(\Delta(0)) = \mathrm{span}\,\{\eta_*\}$. Note that $\zeta^* \in \mathcal{N}(\Delta(0)^*) = \mathcal{R}(\Delta(0))^\perp$, where for any subspace W of $\mathbb{R}^{m+1}$, $W^\perp$ stands for the annihilator of W. By the same reasoning as the proof of Proposition 4.9, it follows from (A_3) that $\zeta^* = 0$, contradicting to the fact that ζ^* is an eigenvector of P_0^* corresponding to eigenvalue 1.

We next consider the representation of Π^c. In virtue of (4.16), it is easy to see

$$\begin{aligned}
\langle\!\langle w_1^\sharp(0)e_1^*, w_1(0)e_1\rangle\!\rangle &= \int_{-\infty}^0\Big(\int_\theta^0 p_0(-\theta)d\xi\Big)d\theta = \int_{-\infty}^0(-\theta)p_0(-\theta)d\theta = \int_0^\infty t p_0(t)dt = \frac{1}{\rho_0},\\
\langle\!\langle w_1^\sharp(0)e_1^*, w_1(0)\tilde{\eta}_*\rangle\!\rangle &= \langle\!\langle w_1^\sharp(0)\tilde{\zeta}^*, w_1(0)e_1\rangle\!\rangle = 0,\\
\langle\!\langle w_1^\sharp(0)\tilde{\zeta}^*, w_1(0)\tilde{\eta}_*\rangle\!\rangle &= \int_{-\infty}^0\Big(\int_\theta^0 \zeta^* P(-\theta)\eta_* d\xi\Big)d\theta = \zeta^*\Big(\int_{-\infty}^0(-\theta)P(-\theta)d\theta\Big)\eta_*\\
&= \zeta^*\Big(\int_0^\infty tP(t)dt\Big)\eta_* = \langle\zeta^*, P_1\eta_*\rangle = r.
\end{aligned}$$

Therefore, by letting

$$\Phi_c := (w_1(0)e_1, w_1(0)\eta_*), \quad \Psi_c := \begin{pmatrix} w_1^\sharp(0)e_1^* \\ w_1^\sharp(0)\zeta^*\end{pmatrix},$$

we have

$$\langle\!\langle\Phi_c, \Psi_c\rangle\!\rangle = \begin{pmatrix} 1/\rho_0 & 0 \\ 0 & r\end{pmatrix},$$

and so,

$$\langle\!\langle\Phi_c, \Psi_c\rangle\!\rangle^{-1}\langle\!\langle\Psi_c, \widetilde{\phi}\rangle\!\rangle = \begin{pmatrix}\rho_0 & 0\\ 0 & 1/r\end{pmatrix}\begin{pmatrix}\langle\!\langle w_1^\sharp(0)e_1^*, \widetilde{\phi}\rangle\!\rangle \\ \langle\!\langle w_1^\sharp(0)\zeta^*, \widetilde{\phi}\rangle\!\rangle\end{pmatrix} = \begin{pmatrix}\langle\!\langle w_1^\sharp(0)(\rho_0 e_1^*), \widetilde{\phi}\rangle\!\rangle \\ \langle\!\langle w_1^\sharp(0)((1/r)\zeta^*), \widetilde{\phi}\rangle\!\rangle\end{pmatrix}.$$

Hence, Theorem 2.8 yields

$$\Pi^c\widetilde{\phi} = \Phi_c\langle\!\langle\Phi_c, \Psi_c\rangle\!\rangle^{-1}\langle\!\langle\Psi_c, \widetilde{\phi}\rangle\!\rangle = w_1(0)\begin{pmatrix}\langle\!\langle w_1^\sharp(0)(\rho_0 e_1^*), \widetilde{\phi}\rangle\!\rangle \\ \langle\!\langle w_1^\sharp(0)((1/r)\zeta^*), \widetilde{\phi}\rangle\!\rangle\eta_*\end{pmatrix}.$$

Since

$$\langle\!\langle w_1^\sharp(0)(\rho_0 e_1^*), \widetilde{\phi}\rangle\!\rangle = \int_{-\infty}^0\Big(\int_\theta^0 \rho_0 p_0(-\theta)\phi^{(1)}(\xi)d\xi\Big)d\theta$$

and

$$\langle\!\langle w_1^\sharp(0)((1/r)\zeta^*), \widetilde{\phi}\rangle\!\rangle = \int_{-\infty}^0\Big(\int_\theta^0 (1/r)\zeta^* P(-\theta)\phi^{(2)}(\xi)d\xi\Big)d\theta,$$

we have

$$\Pi^c\widetilde{\phi} = w_1(0)\begin{pmatrix} \rho_0\int_{-\infty}^0 p_0(-\theta)\Big(\int_\theta^0 \phi^{(1)}(\xi)d\xi\Big)d\theta \\ (1/r)\int_{-\infty}^0 \zeta^* P(-\theta)\Big(\int_\theta^0 \phi^{(2)}(\xi)d\xi\Big)d\theta\cdot\eta_*\end{pmatrix}.$$

This completes the proof. □

4.3 Central equation for the extended system

Now we shall derive the central equation for Eq. $(\widetilde{E})$. Let $\widetilde{x}(t) = \mathrm{col}\,(\varepsilon(t), x(t))$ be a solution of Eq. $(\widetilde{E})$. Then, by Proposition 4.10,

$$\Pi^c\widetilde{x}_t = \Pi^c\begin{pmatrix}\varepsilon_t\\ x_t\end{pmatrix} = w_1(0)\begin{pmatrix} \rho_0\int_{-\infty}^0 p_0(-\theta)\Big(\int_\theta^0 \varepsilon(t+\xi)d\xi\Big)d\theta \\ (1/r)\int_{-\infty}^0 \zeta_* P(-\theta)\Big(\int_\theta^0 x(t+\xi)d\xi\Big)d\theta\cdot\eta_*\end{pmatrix}.$$

Since

$$\begin{aligned}&\int_{-\infty}^0 p_0(-\theta)\Big(\int_\theta^0 \varepsilon(t+\xi)d\xi\Big)d\theta\\ &\quad= \int_{-\infty}^0 p_0(-\theta)\Big(\int_{t+\theta}^t \varepsilon(s)ds\Big)d\theta = \int_{-\infty}^t p_0(t-\tau)\Big(\int_\tau^t \varepsilon(s)ds\Big)d\tau\\ &\quad= \int_{-\infty}^t\Big(\int_{-\infty}^s p_0(t-\tau)d\tau\Big)\varepsilon(s)ds = \int_{-\infty}^t\Big(\int_{t-s}^\infty p_0(w)dw\Big)\varepsilon(s)ds\end{aligned}$$

and similarly

$$\int_{-\infty}^0 \zeta^* P(-\theta)\Big(\int_\theta^0 x(t+\xi)d\xi\Big)d\theta = \int_{-\infty}^t\Big(\int_{t-s}^\infty \zeta^* P(w)dw\Big)x(s)ds,$$

it follows that

$$\begin{aligned}\Pi^c\widetilde{x}_t &= w_1(0)\begin{pmatrix} \rho_0\int_{-\infty}^t\Big(\int_{t-s}^\infty p_0(w)dw\Big)\varepsilon(s)ds \\ (1/r)\int_{-\infty}^t\Big(\int_{t-s}^\infty \zeta^* P(w)dw\Big)x(s)ds\cdot\eta_*\end{pmatrix}\\ &= \Phi_c\begin{pmatrix} \rho_0\int_{-\infty}^t\Big(\int_{t-s}^\infty p_0(w)dw\Big)\varepsilon(s)ds \\ (1/r)\int_{-\infty}^t\Big(\int_{t-s}^\infty \zeta^* P(w)dw\Big)x(s)ds\end{pmatrix}.\end{aligned}$$

Hence, in view of $\Pi^c\widetilde{x}_t = \Phi_c z_c(t)$, we obtain

$$z_c(t) = \begin{pmatrix} \rho_0\int_{-\infty}^t\Big(\int_{t-s}^\infty p_0(w)dw\Big)\varepsilon(s)ds \\ (1/r)\int_{-\infty}^t\Big(\int_{t-s}^\infty \zeta^* P(w)dw\Big)x(s)ds\end{pmatrix},$$

which, together with Eq. $(\widetilde{E})$ (or (4.7)), leads to

$$\begin{aligned}\frac{d}{dt}z_c(t) &= \begin{pmatrix} \rho_0\Big(\varepsilon(t) - \int_{-\infty}^t p_0(t-s)\varepsilon(s)ds\Big) \\ (1/r)\zeta^*\Big(x(t) - \int_{-\infty}^t P(t-s)x(s)ds\Big)\end{pmatrix}\\ &= \frac{1}{r}\begin{pmatrix}0\\ \zeta^* G^{(2)}(\widetilde{x}_t)\end{pmatrix} = \frac{1}{r}\begin{pmatrix}0\\ \zeta^* G^{(2)}\big(\Phi_c z_c(t) + \widetilde{F}_*(\Phi_c z_c(t))\big)\end{pmatrix}.\end{aligned}$$

Summarizing, we have:

Proposition 4.11. *The central equation of the equilibrium* 0 *of Eq.* ($\widetilde{E}$) *is described as the* 2*-dimensional ODE*

$$z' = H(z), \tag{4.21}$$

where H *is an* $\mathbb{R}^2$*-valued function defined on a neighborhood of* 0 *of* $\mathbb{R}^2$ *by*

$$H(w) := H_c\widetilde{G}\big(\Phi_c w + \widetilde{F}_*(\Phi_c w)\big) \quad \textit{for small } |w|,$$

and H_c *is the* $2 \times (m+1)$ *matrix given by* $H_c := r^{-1}\mathrm{col}\,(0, \tilde{\zeta}^*)$.

Let us denote $w = \mathrm{col}\,(\varepsilon, s) \in \mathbb{R}^2$. Then $\Phi_c w = w_1(0)(\varepsilon e_1 + s\eta_*) = \mathrm{col}\,\big(w_1(0)\varepsilon,\, w_1(0)(s\eta_*)\big)$ and hence by (4.9),

$$\begin{aligned} G^{(2)}(\Phi_c w) &= \left(\int_{-\infty}^0 p_0(-\theta)\big(w_1(0)\varepsilon\big)(\theta)d\theta\right)\left(\int_{-\infty}^0 P(-\theta)\big(w_1(0)(s\eta_*)\big)(\theta)d\theta\right) \\ &\quad + Q(w_1(0)(s\eta_*)) + g(w_1(0)(s\eta_*)) \\ &= \left(\int_{-\infty}^0 p_0(-\theta)\varepsilon d\theta\right)\left(\int_{-\infty}^0 P(-\theta)(s\eta_*)d\theta\right) + Q(w_1(0)\eta_*)s^n \\ &\quad + o(\|w_1(0)(s\eta_*)\|_X^n) \\ &= \varepsilon s\eta_* + s^n v_0 + o(|s|^n) \quad \text{as } (\varepsilon, s) \to (0,0), \end{aligned} \tag{4.22}$$

where we used the fact $P_0\eta_* = \eta_*$ and put $v_0 := Q(w_1(0)\eta_*)$.

Proposition 4.12. $\widetilde{F}_*\big(w_1(0)(\varepsilon e_1)\big) = 0$ *for sufficiently small* $|\varepsilon|$.

Proof. $\tilde{x}^\varepsilon(t) := \varepsilon e_1$ $(t \in \mathbb{R})$ is obviously a solution on $\mathbb{R}^+$ of ($\widetilde{E}$) with $\tilde{x}_t^\varepsilon = w_1(0)(\varepsilon e_1)$ $(= \Phi_c(\varepsilon e_1))$. So, if $|\varepsilon| < \rho\delta/3$, we have $\tilde{x}_t^\varepsilon \in \widetilde{\Omega}_0$ for $t \in \mathbb{R}^+$, and hence Theorem 3.3 means $\mathrm{dist}\,\big(\tilde{x}_t^\varepsilon, W_{\mathrm{loc}}^c(0)\big) \to 0$ as $t \to \infty$. Thus, $w_1(0)(\varepsilon e_1) \in W_{\mathrm{loc}}^c(0) = \mathrm{graph}\,\widetilde{F}_*$, that is,

$$w_1(0)(\varepsilon e_1) = \Pi^c(w_1(0)(\varepsilon e_1)) + \widetilde{F}_*\big(\Pi^c(w_1(0)(\varepsilon e_1))\big),$$

which proves the proposition because $\Pi^c(w_1(0)(\varepsilon e_1)) = w_1(0)(\varepsilon e_1)$. □

Proposition 4.13. $G^{(2)}(\Phi_c w + \widetilde{F}_*(\Phi_c w))$ *satisfies*

(i) $G^{(2)}(\Phi_c w + \widetilde{F}_*(\Phi_c w)) = \varepsilon s\eta_* + s^n v_0 + o(|s|^n) + o(|\varepsilon s|)$,

(ii) $(\partial/\partial s)G^{(2)}(\Phi_c w + \widetilde{F}_*(\Phi_c w)) = \varepsilon + o(|\varepsilon|) + ns^{n-1}v_0 + o(|s|^{n-1})$

as $w = \mathrm{col}\,(\varepsilon, s) \to (0,0)$.

Proof. We assume that $\|\Phi_c w\|_{\widetilde{X}} < \delta$ so that $\widetilde{F}_*(\Phi_c w)$ is well-defined. Let us denote $\widetilde{F}_*(\Phi_c w) = \mathrm{col}\,(\phi^w, \psi^w) \in \widetilde{X}$ with $\phi^w \in X_1$ and $\psi^w \in X$. By Proposition 4.12,

$$\begin{aligned} \widetilde{F}_*(\Phi_c w) &= \widetilde{F}_*(w_1(0)(\varepsilon e_1 + s\eta_*)) - \widetilde{F}_*(w_1(0)(\varepsilon e_1)) \\ &= \int_0^1 \frac{d}{d\tau}\widetilde{F}_*(w_1(0)(\varepsilon e_1 + \tau s\eta_*))d\tau \\ &= \int_0^1 D\widetilde{F}_*\big(w_1(0)(\varepsilon e_1 + \tau s\eta_*)\big)(w_1(0)(s\eta_*))d\tau, \end{aligned}$$

and hence

$$\|\widetilde{F}_*(\Phi_c w)\|_{\widetilde{X}} \le \rho^{-1}|s||\eta_*| \sup_{0\le\tau\le 1} \big\|D\widetilde{F}_*\big(w_1(0)(\varepsilon e_1 + \tau s\eta_*)\big)\big\|_{\mathcal{L}(\widetilde{E}^c;\widetilde{E}^s)}.$$

This yields $\widetilde{F}_*(\Phi_c w) = o(|s|)$ as $(\varepsilon, s) \to (0,0)$ since $\widetilde{F}_*$ is of class C^1 with $D\widetilde{F}_*(0) = 0$; so, in view of $\|\psi^w\|_X \le \|\widetilde{F}_*(\Phi_c w)\|_{\widetilde{X}}$,

$$\psi^w = o(|s|), \quad (\varepsilon, s) \to (0,0). \tag{4.23}$$

In addition, since

$$\frac{\partial}{\partial s}\widetilde{F}_*(\Phi_c w) = D\widetilde{F}_*(\Phi_c w)(w_1(0)\eta_*) = o(1) \quad \text{as } (\varepsilon, s) \to (0,0)$$

and $\|\partial\psi^w/\partial s\|_X \le \|(\partial/\partial s)\widetilde{F}_*(\Phi_c w)\|_{\widetilde{X}}$, it follows that

$$\frac{\partial}{\partial s}\psi^w = o(1), \quad (\varepsilon, s) \to (0,0). \tag{4.24}$$

(i) We first observe that $\Pi^c \widetilde{F}_*(\Phi_c w) = 0$ because of the fact $\widetilde{F}_*(\Phi_c w) \in \widetilde{E}^s$; so Proposition 4.10 yields

$$\begin{aligned} 0 &= \rho_0 \int_{-\infty}^0 p_0(-\theta)\left(\int_\theta^0 \phi^w(\xi)d\xi\right)d\theta \\ &= \int_{-\infty}^0 \left(\int_{-\infty}^\xi \rho_0 p_0(-\theta)d\theta\right)\phi^w(\xi)d\xi = \int_{-\infty}^0 p_0(-\xi)\phi^w(\xi)d\xi. \end{aligned}$$

Hence, one can see from (4.9) that

$$\begin{aligned} &G^{(2)}(\Phi_c w + \widetilde{F}_*(\Phi_c w)) - \widetilde{G}^{(2)}(\Phi_c w) \\ &= \left(\int_{-\infty}^0 p_0(-\theta)(\phi^w(\theta) + \varepsilon)d\theta\right)\left(\int_{-\infty}^0 P(-\theta)(\psi^w(\theta) + s\eta_*)d\theta\right) + f(\psi^w + w_1(0)(s\eta_*)) \\ &\quad -\left(\int_{-\infty}^0 p_0(-\theta)\varepsilon d\theta\right)\left(\int_{-\infty}^0 P(-\theta)(s\eta_*)d\theta\right) - f(w_1(0)(s\eta_*)) \\ &= \varepsilon\int_{-\infty}^0 P(-\theta)\psi^w(\theta)d\theta + f(\psi^w + w_1(0)(s\eta_*)) - f(w_1(0)(s\eta_*)). \end{aligned} \tag{4.25}$$

Notice from (4.23) that the first term is estimated by

$$\left|\varepsilon\int_{-\infty}^0 P(-\theta)\psi^w(\theta)d\theta\right| \le |\varepsilon|\|P\|_{\infty,\rho}\|\psi^w\|_X = o(|\varepsilon s|) \tag{4.26}$$

as $(\varepsilon, s) \to (0,0)$. On the other hand, we have

$$\begin{aligned} |f(\psi^w + w_1(0)(s\eta_*)) - f(w_1(0)(s\eta_*))| &\le |Q(\psi^w + w_1(0)(s\eta_*)) - Q(w_1(0)(s\eta_*))| \\ &\quad + |g(\psi^w + w_1(0)(s\eta_*)) - g(w_1(0)(s\eta_*))| =: d_1 + d_2. \end{aligned}$$

Since

$$d_2 \le \int_0^1 \|Dg(\psi^w + \tau w_1(0)(s\eta_*))\|_{\mathcal{L}(X;\mathbb{C}^m)}\|\psi^w\|_X d\tau$$

and, by (4.23) again,

$$\begin{aligned} \|\psi^w + \tau w_1(0)(s\eta_*)\|_X &\le \|\psi^w\|_X + \|w_1(0)(s\eta_*)\|_X \\ &= o(|s|) + \rho^{-1}|s||\eta_*| \le C_3|s|, \quad \tau \in [0,1] \end{aligned} \tag{4.27}$$

hold with some positive constant C_3, it follows from the assumption (4.4) that

$$d_2 = o(|s|^{n-1})\, o(|s|) = o(|s|^n) \quad \text{as} \quad (\varepsilon, s) \to (0,0).$$

Also, by the multi-linearity of Q_* and (4.27),

$$\begin{aligned}
d_1 &= |Q_*(\psi^w + w_1(0)(s\eta_*), \dots, \psi^w + w_1(0)(s\eta_*)) - Q_*(w_1(0)(s\eta_*), \dots, w_1(0)(s\eta_*))| \\
&\le \sum_{k=1}^{n} \big| Q_*(\psi^w, \dots, \psi^w, \underset{\widehat{k}}{\psi^w + w_1(0)(s\eta_*)}, \dots, \psi^w + w_1(0)s) \\
&\quad - Q_*(\psi^w, \dots, \psi^w, \underset{\widehat{k}}{w_1(0)(s\eta_*)}, \psi^w + w_1(0)(s\eta_*), \dots, \psi^w + w_1(0)(s\eta_*)) \big| \\
&= \sum_{k=1}^{n} \big| Q_*(\psi^w, \dots, \psi^w, \underset{\widehat{k}}{\psi^w}, \psi^w + w_1(0)(s\eta_*), \dots, \psi^w + w_1(0)(s\eta_*)) \big| \\
&\le \sum_{k=1}^{n} \|Q_*\|_{\mathcal{L}^n(X;\mathbb{R}^m)} \|\psi^w\|_X^k \, \|\psi^w + w_1(0)(s\eta_*)\|_X^{n-k} \\
&\le \|Q_*\|_{\mathcal{L}^n(X;\mathbb{R}^m)} \sum_{k=1}^{n} \|\psi^w\|_X^k \, (C_3|s|)^{n-k},
\end{aligned}$$

and hence (4.23) implies $d_1 = o(|s|^n)$ as $(\varepsilon, s) \to (0,0)$. By (4.25), (4.26) and the estimates of d_i $(i = 1, 2)$, we obtain

$$G^{(2)}(\Phi_c w + \widetilde{F}_*(\Phi_c w)) - G^{(2)}(\Phi_c w) = o(|\varepsilon s|) + o(|s|^m),$$

and so, by (4.22),

$$G^{(2)}(\Phi_c w + \widetilde{F}_*(\Phi_c w)) = \varepsilon s + \mu s^n v_0 + o(|s|^n) + o(\varepsilon s|) \quad \text{as} \quad (\varepsilon, s) \to (0,0).$$

(ii) We know that

$$\begin{aligned}
&G^{(2)}(\Phi_c w + \widetilde{F}_*(\Phi_c w)) \\
&\quad = \left(\int_{-\infty}^{0} p_0(\theta)(\phi^w(\theta) + \varepsilon) d\theta \right) \left(\int_{-\infty}^{0} P(\theta)(\psi^w(\theta) + s\eta_*) d\theta \right) + f(\psi^w + w_1(0)(s\eta_*)) \\
&\quad = \varepsilon s\eta_* + \varepsilon \int_{-\infty}^{0} P(\theta)\psi^w(\theta) d\theta + \mu Q(\psi^w + w_1(0)(s\eta_*)) + g(\psi^w + w_1(0)(s\eta_*)),
\end{aligned}$$

and hence

$$\begin{aligned}
\frac{\partial}{\partial s} G^{(2)}(\Phi_c w + \widetilde{F}_*(\Phi_c w)) &= \varepsilon\eta_* + \varepsilon \int_{-\infty}^{0} P(\theta) \frac{\partial}{\partial s}\psi^w(\theta) d\theta + \mu \frac{\partial}{\partial s} Q(\psi^w + w_1(0)(s\eta_*)) \\
&\quad + Dg(\psi^w + w_1(0)(s\eta_*)) \left(\frac{\partial}{\partial s}\psi^w + w_1(0)\eta_* \right).
\end{aligned} \tag{4.28}$$

In view of (4.24),

$$\left| \varepsilon \int_{-\infty}^{0} P(\theta) \frac{\partial}{\partial s}\psi^w(\theta) d\theta \right| \le |\varepsilon| \|P\|_{\infty,\rho} \left\| \frac{\partial \psi^w}{\partial s} \right\|_X = o(|\varepsilon|), \quad (\varepsilon, s) \to (0,0).$$

Moreover, observe that

$$Q(\psi^w + w_1(0)(s\eta_*)) = s^n v_0 + \sum_{(\phi_1, \dots, \phi_m) \in B} Q_*(\phi_1, \dots, \phi_m),$$

where we used the fact that $Q(w_1(0)(s\eta_*)) = s^n v_0$, and denoted by B the subset of X^n, $\{\psi^w, w_1(0)(s\eta_*)\}^n \backslash \{(w_1(0)(s\eta_*), \dots, w_1(0)(s\eta_*))\}$. Hence,

$$\frac{\partial}{\partial s} Q(\psi^w + w_1(0)(s\eta_*)) = ns^{n-1}v_0 + \sum_{(\phi_1,\dots,\phi_n)\in B} \sum_{k=1}^{n} Q_*(\phi_1, \dots, \partial\phi_k/\partial s, \dots, \phi_n).$$

One can readily deduce from (4.23) and (4.24) that for any $(\phi_1, \dots, \phi_n) \in B$,

$$Q_*(\phi_1, \dots, \partial\phi_k/\partial s, \dots, \phi_n) = o(|s|^{n-1}) \quad \text{as } (\varepsilon, s) \to (0,0),$$

so that

$$\frac{\partial}{\partial s} Q(\psi^w + w_1(0)(s\eta_*)) = ns^{n-1}v_0 + o(|s|^{n-1}).$$

Also, by (4.24) again, there exists a positive constant C_4 such that

$$\left\| \frac{\partial}{\partial s}\psi^w + w_1(0)\eta_* \right\|_X \le C_4 \quad \text{for small } |\varepsilon| \text{ and } |s|,$$

which, together with (4.27) and the assumption (4.4), means that

$$\left\| Dg(\psi^w + w_1(0)(s\eta_*)) \left(\frac{\partial}{\partial s}\psi^w + w_1(0)\eta_* \right) \right\|_X \le C_4 \|Dg(\psi^w + w_1(0)(s\eta_*)\|_X = o(|s|^{n-1}).$$

Consequently (4.28), together with the estimates above, leads to

$$\frac{\partial}{\partial s} G^{(2)}(\Phi_c w + \widetilde{F}_*(\Phi_c w)) = \varepsilon\eta_* + o(|\varepsilon|) + ns^{n-1}v_0 + o(|s|^{n-1})$$

as $(\varepsilon, s) \to (0,0)$, and this completes the proof. □

In virtue of Proposition 4.13, the central equation (4.21) turns out to be of the form

$$\begin{aligned} \varepsilon' &= 0, \\ s' &= h(\varepsilon, s), \end{aligned} \tag{$C\widetilde{E}$}$$

where $h(\varepsilon, s) = (1/r)\langle \zeta^*, G^{(2)}(\Phi_c w + \widetilde{F}_*(\Phi_c w))\rangle$ is of class C^1 and satisfies

$$\begin{aligned} h(\varepsilon, s) &= \frac{1}{r}\langle \zeta^*, \varepsilon s\eta_* + s^n v_0 + o(|s|^n) + o(\varepsilon s|)\rangle \\ &= -\frac{1}{q_*}\varepsilon s + c_0 s^n + o(|s|^n) + o(|\varepsilon s|) \end{aligned} \tag{4.29}$$

as $(\varepsilon, s) \to (0,0)$. Furthermore, its derivative with respect to s has the property

$$\frac{\partial h}{\partial s}(\varepsilon, s) = -\frac{1}{q_*}\varepsilon + nc_0 s^{n-1} + o(|s|^{n-1} + |\varepsilon|), \quad (\varepsilon, s) \to (0,0), \tag{4.30}$$

where q_* and c_0 are constants introduced in (4.5), that is,

$$q_* = -\langle \zeta^*, P_1\eta_*\rangle / \langle \zeta^*, \eta_*\rangle, \text{ and } c_0 = \langle \zeta^*, Q(w_1(0)\eta_*)\rangle.$$

4.4 Proof of Theorem 4.4

In this subsection we will prove Theorem 4.4. Under the conditions (4.29) and (4.30), the behaviors of solutions of ($C\widetilde{E}$) are determined by the following lemma, which is a slight refinement of [11, Lemma 6.3.1].

Lemma 4.14. *Let $h(\varepsilon, s)$ be a real valued function of class C^1 defined in a neighborhood U_0 of $(0,0)$ of $\mathbb{R}^2$ that satisfies*

$$h(\varepsilon, s) = -\alpha s\varepsilon + \beta s^m + o(|s|^m + |\varepsilon s|), \tag{4.31}$$

and

$$\frac{\partial h}{\partial s}(\varepsilon, s) = -\alpha\varepsilon + m\beta s^{m-1} + o(|s|^{m-1} + |\varepsilon|) \tag{4.32}$$

as $(s,\varepsilon) \to (0,0)$, where $\alpha > 0$, $\beta \neq 0$ and $m = 2, 3, \ldots$ Then we have:

(a) *If m is even, then*

(a_1) *there exists an $s^* > 0$ such that $h(0,s)$ is positive (resp. negative) definite for $0 < |s| < s^*$ in the case of $\beta > 0$ (resp. $\beta < 0$). Moreover, there exist an $\varepsilon^* > 0$ and a continuous function $s : (-\varepsilon^*, \varepsilon^*)\backslash\{0\} \to \mathbb{R}\backslash\{0\}$ such that $h(\varepsilon, s_\varepsilon) = 0$ for $0 < |\varepsilon| < \varepsilon^*$, $s_\varepsilon \to 0$ as $\varepsilon \to 0$, and*

$$\frac{\partial h}{\partial s}(\varepsilon, s_\varepsilon) \begin{cases} < 0 & (\varepsilon < 0), \\ > 0 & (\varepsilon > 0) \end{cases} \tag{4.33}$$

holds for small $|\varepsilon|$.

(a_2) *There exists an open subset U_1 of U_0 containing $(0,0)$ such that $h(\hat{\varepsilon}, \hat{s}) = 0$ with some $(\hat{\varepsilon}, \hat{s}) \in U_1$ implies $\hat{s} = 0$ or $\hat{s} = s_{\hat{\varepsilon}}$.*

(b) *If m is odd and $\beta > 0$, then*

(b_1) *there exists an $s^* > 0$ such that $sh(0,s) > 0$ for $0 < |s| < s^*$. Moreover, if $\varepsilon \leq 0$ with $|\varepsilon|$ small, $h(\varepsilon, \cdot)$ has no small zeros other than 0; on the other hand, there exist an $\varepsilon^* > 0$ and continuous functions $s^\pm : (0, \varepsilon^*) \to \mathbb{R}\backslash\{0\}$ such that $h(\varepsilon, s^+_\varepsilon) = h(\varepsilon, s^-_\varepsilon) = 0$ for $0 < \varepsilon < \varepsilon^*$, $s^\pm_\varepsilon \to 0$ as $\varepsilon \to +0$, and*

$$\frac{\partial h}{\partial s}(\varepsilon, s^\pm_\varepsilon) > 0 \quad \text{for small } \varepsilon > 0. \tag{4.34}$$

(b_2) *There exists an open subset U_1 of U_0 containing $(0,0)$ such that $h(\hat{\varepsilon}, \hat{s}) = 0$ with $(\hat{\varepsilon}, \hat{s}) \in U_1$ implies that $\hat{s}$ coincides with 0, $s^+_{\hat{\varepsilon}}$ or $s^-_{\hat{\varepsilon}}$.*

(c) *If m is odd and $\beta < 0$, then*

(c_1) *there exists an $s^* > 0$ such that $sh(0,s) < 0$ for $0 < |s| < s^*$. Moreover, if $\varepsilon \geq 0$ with $|\varepsilon|$ small, $h(\varepsilon, \cdot)$ has no small zeros other than 0; on the other hand, there exist an $\varepsilon^* > 0$ and continuous functions $s^\pm : (-\varepsilon^*, 0) \to \mathbb{R}\backslash\{0\}$ such that $h(\varepsilon, s^+_\varepsilon) = h(\varepsilon, s^-_\varepsilon) = 0$ for $-\varepsilon^* < \varepsilon < 0$, $s^\pm_\varepsilon \to 0$ as $\varepsilon \to -0$, and*

$$\frac{\partial h}{\partial s}(\varepsilon, s^\pm_\varepsilon) < 0 \quad \text{for } \varepsilon < 0 \text{ with } |\varepsilon| \text{ small.}$$

(c_2) *There exists an open subset U_1 of U_0 containing $(0,0)$ s such that $h(\hat{\varepsilon}, \hat{s}) = 0$ with $(\hat{\varepsilon}, \hat{s}) \in U_1$ implies that $\hat{s}$ coincides with 0, $s^+_{\hat{\varepsilon}}$, or $s^-_{\hat{\varepsilon}}$.*

We will first prove Theorem 4.4 and, for completeness, give a proof of Lemma 4.14 at the end of this subsection. Note that any equilibrium in $\widetilde{\Omega}_0$ of Eq. $(\widetilde{E})$ must be contained in $\widetilde{W}^c_{\mathrm{loc}}(0)$ because of the attractivity of $\widetilde{W}^c_{\mathrm{loc}}(0)$ (see [23, Theorem 5], or Theorem 3.3); so in view of Proposition 3.1, $\tilde{\phi}_* \in \widetilde{\Omega}$ is an equilibrium of Eq. $(\widetilde{E})$ if and only if there exists an equilibrium z_* of the central equation $(C\widetilde{E})$ (or (4.21)) such that

$$\tilde{\phi}_* = \Phi_c z_* + \widetilde{F}_*(\Phi_c z_*). \tag{4.35}$$

Remark 4.15. If $\tilde{\phi}_*$ is an equilibrium of Eq. $(\widetilde{E})$ (or (4.7)), then $\phi_* := \Pi_2\tilde{\phi}_*$ is an equilibrium of Eq. (PE) with $\lambda = 1+\varepsilon$, where ε is given by $w_1(0)\varepsilon = \Pi_1\tilde{\phi}_*$. Conversely, if ϕ_* is an equilibrium of Eq. (PE), then $\tilde{\phi}_* := j_{\lambda-1}(\phi_*)$ is an equilibrium of Eq. $(\widetilde{E})$ (or (4.7)), where $j_\varepsilon : X \to \widetilde{X}$ is the map defined in (4.14) (see Proposition 4.5).

Moreover, throughout this subsection, we assume that

$$L(\delta) < 1. \tag{4.36}$$

by taking $\delta > 0$ sufficiently small.

Proof of Theorem 4.4. We know by (4.29) and (4.30) that $h(\varepsilon, s)$ in Eq. $(C\widetilde{E})$ satisfies the assumptions (4.31) and (4.32).

(i) Let $\lambda = 1$, that is, $\varepsilon := \lambda - 1 = 0$, and $s^* > 0$ be the one in Lemma 4.14 (a_1). Then, $s' = h(0, s)$, the second equation of $(C\widetilde{E})$ with $\varepsilon = 0$, has no equilibria in $(-s^*, s^*)\backslash\{0\}$. By taking s^* small if necessary, we may assume that

$$s^* < \frac{\rho\delta}{3|\eta_*|}. \tag{4.37}$$

(ia) Note that $\|\Phi_c z\|_{\widetilde{X}} = \|sw_1(0)\tilde{\eta}_*\|_{\widetilde{X}} = \rho^{-1}|s||\eta_*|$ for $z = \mathrm{col}\,(0, s)$. So such a z with $s \in (-s^*, s^*)$ satisfies $\|\Phi_c z\|_{\widetilde{X}} < \delta/3$, and hence

$$\|\Phi_c z + \widetilde{F}_*(\Phi_c z)\|_{\widetilde{X}} \geq (1 - L(\delta))\|\Phi_c z\|_{\widetilde{X}} = (1 - L(\delta))\rho^{-1}|s||\eta_*.$$

Let $\rho_1 := (1 - L(\delta))\rho^{-1}s^*|\eta_*|$; then $0 < \rho_1 < \delta/3$ due to (4.36) and (4.37). Therefore, one can see from (4.35) that Eq. $(\widetilde{E})$ has no equilibria in $B_{\widetilde{X}}(\rho_1)\backslash\{0\}$. So, by Remark 4.15, Eq. (PE) has no equilibria in the set $j_0^{-1}(B_{\widetilde{X}}(\rho_1)\backslash\{0\}) = B_X(\rho_1)\backslash\{0\}$, where we used the fact that j_ε is an isometry for $\varepsilon \in \mathbb{R}$. Thus, the former part of (ia) is valid with $\mathcal{W}_0 = B_X(\rho_1)$.

Let $s : (-\varepsilon^*, \varepsilon^*)\backslash\{0\} \to \mathbb{R}\backslash\{0\}$ be the function in Lemma 4.14 (a_1) and set

$$\tilde{\phi}^\lambda_* := \Phi_c z^{\lambda-1} + \widetilde{F}_*(\Phi_c z^{\lambda-1}), \quad \lambda \in (1 - \varepsilon^*, 1 + \varepsilon^*)\backslash\{1\}, \tag{4.38}$$

where we used the notation $z^\varepsilon := \mathrm{col}\,(\varepsilon, s_\varepsilon)$ for $\varepsilon \in (-\varepsilon^*, \varepsilon^*)\backslash\{0\}$. We may assume

$$\|\Phi_c z^\varepsilon\|_{\widetilde{X}} < \delta/3 \quad \text{for } \varepsilon \in (-\varepsilon^*, \varepsilon^*)\backslash\{0\}. \tag{4.39}$$

Since z^ε is an equilibrium of Eq. $(C\widetilde{E})$, $\tilde{\phi}^\lambda_*$ is also an equilibrium of Eq. $(\widetilde{E})$ by (4.35).

Note that $\widetilde{T}(t) = \mathrm{diag}\,\big(T_0(t), T(t)\big)$, where $\{T_0(t)\}_{t\geq 0}$ is the solution semigroup of the first equation of (4.7). So, $\widetilde{T}^s(t) = \mathrm{diag}\,\big(T^s_0(t), T^s(t)\big)$ because $X_1 \times \{0\}$ and $\{0\} \times X$ are $\widetilde{T}(t)$-invariant subspaces of $\widetilde{X}$ for $t \geq 0$. Since $\Gamma^n\widetilde{G}_\delta(\Lambda_{*,\delta}(\psi)(s)) = \mathrm{col}\,\big(0, \Gamma^n G^{(2)}_\delta(\Lambda_{*,\delta}(\psi)(s))\big)$ with $G^{(2)}_\delta(\tilde{\phi}) := \Pi_2\widetilde{G}_\delta(\tilde{\phi})$, we see $\Pi_1\big(\Pi^s\Gamma^n\widetilde{G}_\delta(\Lambda_{*,\delta}(\psi)(s))\big) = 0$. Hence, (4.12) means that for $\psi \in \widetilde{E}^c$,

$$\Pi_1(\widetilde{F}_{*,\delta}(\psi)) = \lim_{n\to\infty}\int_{-\infty}^0 T^s_0(-s)\Pi_1\big(\Pi^s\Gamma^n\widetilde{G}_\delta(\Lambda_{*,\delta}(\psi)(s))\big)ds = 0. \tag{4.40}$$

It also follows from Proposition 4.10 that $\Phi_c z^{\lambda-1} = w_1(0)\big((\lambda-1)e_1 + s_{\lambda-1}\tilde{\eta}_*\big)$; so that

$$\Pi_1(\tilde{\phi}_*^\lambda) = \Pi_1(\Phi_c z^{\lambda-1}) + \Pi_1(\widetilde{F}_*(\Phi_c z^{\lambda-1})) = w_1(0)(\lambda-1)$$

because of (4.38) and the fact that $\widetilde{F}_* \equiv \widetilde{F}_{*,\delta}$ on $B_{\widetilde{E}^c}(\delta)$; in other words, $\Pi_1(\tilde{\phi}_*^\lambda)(\theta) = \lambda - 1$ for $\theta \in \mathbb{R}^-$.

Define $\phi_* : (1-\varepsilon^*, 1+\varepsilon^*)\backslash\{1\} \to X\backslash\{0\}$ by

$$\phi_*^\lambda := \Pi_2(\tilde{\phi}_*^\lambda), \quad \lambda \in (1-\varepsilon^*, 1+\varepsilon^*)\backslash\{1\}.$$

Then, ϕ_*^λ is seen to be an equilibrium of Eq. (*PE*) by Remark 4.15. In addition, ϕ_*^λ is continuous in λ and satisfies $\phi_*^\lambda \to 0$ $(\lambda \to 1)$ since $z^{\lambda-1} \to 0$ as $\lambda \to 1$.

We next show the instability of the equilibrium 0. When $\varepsilon = 0$, the second equation of $(C\widetilde{E})$ is $s' = c_0 s^n + o(|s|^n)$; so its zero solution, and hence the equilibrium 0 of $(C\widetilde{E})$, is unstable because n is even. Given $t_0 \in \mathbb{R}$ and $s_0 \in \mathbb{R}\backslash\{0\}$, let $z(t)$ be the solution of $(C\widetilde{E})$ with $z(t_0) = \mathrm{col}\,(0, s_0)$ and set $\widetilde{\psi} := \Phi_c z(t_0) + \widetilde{F}_*(\Phi_c z(t_0))$, where we assume $|s_0|$ is sufficiently small in such a way that $\|\Phi_c z(t_0)\|_X < \delta$ holds. Then it follows from (4.40) that $\Pi_1\widetilde{\psi} = 0$, so that $\widetilde{\psi} = j_0(\psi)$ with $\psi = \Pi_2\widetilde{\psi}$. So we deduce from Proposition 4.5 that

$$j_0(x_t(t_0, \psi, f)) = \widetilde{x}_t(t_0, \widetilde{\psi}, \widetilde{G}) = \Phi_c z(t) + \widetilde{F}_*(\Phi_c z(t))$$

for $t \geq t_0$ with $t - t_0$ small. Hence the instability of the equilibrium 0 of (*PE*) with $\lambda = 1$ follows from that of the equilibrium 0 of $(C\widetilde{E})$. Thus, (ia) holds.

(ib) The central equation of the equilibrium 0 of Eq. $(\widetilde{E}_\delta)$ is of the form

$$z' = H_c\widetilde{G}_\delta(\Phi_c z + \widetilde{F}_{*,\delta}(\Phi_c z)) \tag{$C\widetilde{E}_\delta$}$$

(cf. Proposition 4.11 and (4.11)). Let $h_\delta(z)$ be the second component of $H_c\widetilde{G}_\delta(\Phi_c z + \widetilde{F}_{*,\delta}(\Phi_c z))$. In view of (4.36), $\|\Phi_c z\|_{\widetilde{X}} < \delta$ implies

$$\|\widetilde{F}_{*,\delta}(\Phi_c z)\|_{\widetilde{X}} \leq L(\delta)\|\Phi_c z\|_{\widetilde{X}} < \delta,$$

and hence $\Phi_c z + \widetilde{F}_{*,\delta}(\Phi_c z) \in \widetilde{\Omega}$. So, $h(z) = h_\delta(z)$ for $\|\Phi_c z\|_{\widetilde{X}} < \delta$; in particular, by (4.39), $z^{\lambda-1}$ is also an equilibrium of Eq. $(C\widetilde{E}_\delta)$ for $\lambda \in (1-\varepsilon^*, 1+\varepsilon^*)$.

Suppose that $q_* < 0$. Notice from Lemma 4.14 (a_1) and the principle of linearized stability that $s_{\lambda-1}$ $(\lambda \neq 1)$ is the nonzero equilibrium of $s' = h(\lambda-1, s)$, hence of $s' = h_\delta(\lambda-1, s)$, which is asymptotically stable (resp. unstable) if $\lambda < 1$ (resp. $\lambda > 1$).

Now let $\lambda < 1$. Since $\{\lambda-1\} \times \mathbb{R}$ is an invariant set for Eq. $(C\widetilde{E}_\delta)$, the stability of $s_{\lambda-1}$ implies that for arbitrary $t_0 \in \mathbb{R}^+$ and $\hat{\varepsilon} > 0$, there exists a $\delta_0(\hat{\varepsilon}) > 0$ such that if $z^* \in \{\lambda-1\} \times \mathbb{R}$ satisfies $|z^* - z^{\lambda-1}| < \delta_0(\hat{\varepsilon})$, then $z(t; t_0, z^*) \in \{\lambda-1\} \times \mathbb{R}$ and $|z(t; t_0, z^*) - z^{\lambda-1}| < \hat{\varepsilon}$ hold for $t \geq t_0$, where $z(t; t_0, z^*)$ denotes the solution of Eq. $(C\widetilde{E}_\delta)$ with $z(t_0) = z^*$.

Let us denote by Π_X^c and Π_X^s the projections defined by $\Pi_X^c := \Pi_2 \circ \Pi^c \circ j_0$ and $\Pi_X^s := \Pi_2 \circ \Pi^s \circ j_0$, respectively. It is easy to see that $\Pi^c \circ j_\varepsilon = j_\varepsilon \circ \Pi_X^c$ and $\Pi^s \circ j_\varepsilon = j_0 \circ \Pi_X^s$ for $\varepsilon \in \mathbb{R}$. Set

$$\phi_*^{\lambda,c} := \Pi_X^c\phi_*^\lambda, \quad \text{and} \quad \phi_*^{\lambda,s} := \Pi_X^s\phi_*^\lambda.$$

Similarly, given $\phi \in X$ and $\tilde{\phi} \in \widetilde{X}$, we put $\phi^c := \Pi_X^c\phi$ and $\tilde{\phi}^c := \Pi^c\tilde{\phi}$; and likewise for ϕ^s and $\tilde{\phi}^s$. Given $\phi \in X$, let z^* be the point of $\mathbb{R}^2$ satisfying $\Phi_c z^* = (j_{\lambda-1}(\phi))^c = j_{\lambda-1}(\phi^c)$. Then $z(t; t_0, z^*) = z(t - t_0, j_{\lambda-1}(\phi))$ for $t \geq t_0$ (cf. (3.16)). Observe that $\Phi_c z^{\lambda-1} = \Pi^c\tilde{\phi}_*^\lambda = \Pi^c(j_{\lambda-1}(\phi_*^\lambda)) = j_{\lambda-1}(\phi_*^{\lambda,c})$ and

$$\begin{aligned}\|\Phi_c\|_*|z^* - z^{\lambda-1}| &\leq \|\Phi_c z^* - \Phi_c z^{\lambda-1}\|_{\widetilde{X}} \\ &= \|j_{\lambda-1}(\phi^c) - j_{\lambda-1}(\phi_*^{\lambda,c})\|_{\widetilde{X}} = \|\phi^c - \phi_*^{\lambda,c}\|_X,\end{aligned}$$

where the last equality is due to the fact that j_ε is an isometry. So, if $\phi \in X$ satisfies $\|\phi^c - \phi_*^{\lambda,c}\|_X < \|\Phi_c\|_* \, \delta_0(\hat{\varepsilon}/\|\Phi_c\|^*)$, then

$$\begin{aligned}\|\Phi_c z(t-t_0, j_{\lambda-1}(\phi)) - j_{\lambda-1}(\phi_*^{\lambda,c})\|_{\widetilde{X}} &= \|\Phi_c z(t;t_0,z^*) - \Phi_c z^{\lambda-1}\|_{\widetilde{X}} \\ &\le \|\Phi_c\|^* |z(t;t_0,z^*) - z^{\lambda-1}| < \hat{\varepsilon}\end{aligned} \tag{4.41}$$

for $t \ge t_0$. For $\hat{\varepsilon} > 0$, set

$$\hat{\delta}(\hat{\varepsilon}) := \min\left(\frac{\hat{\varepsilon}}{3C_0C_1(1+2L(\delta))}, \frac{\|\Phi_c\|_*}{2C_1}\delta_0\left(\frac{\hat{\varepsilon}}{3(1+L(\delta))\|\Phi_c\|^*}\right)\right). \tag{4.42}$$

Given any $\hat{\varepsilon}$ with $0 < \hat{\varepsilon} < \delta$, let $\psi \in B_X(\phi_*^\lambda; \hat{\delta}(\hat{\varepsilon}))$. Then $j_{\lambda-1}(\psi) \in B_{\widetilde{X}}(j_{\lambda-1}(\phi_*^\lambda); \hat{\delta}(\hat{\varepsilon}))$, and hence, it follows from Claim 1 (Subsection 3.2) and (3.33) that there exists a $\tilde{\phi} \in B_{\widetilde{X}}(j_{\lambda-1}(\phi_*^\lambda); 2\hat{\delta}(\hat{\varepsilon}))$ such that $\widehat{g}(\tilde{\phi}) = j_{\lambda-1}(\psi)$, where $\widehat{g}$ is the one in (3.28) applied for Eq. ($\widetilde{E}_\delta$). In particular,

$$\begin{aligned}&\|\Pi^c \widetilde{x}_t(t_0, j_{\lambda-1}(\psi), \widetilde{G}_\delta) - \Phi_c z(t-t_0, \tilde{\phi})\|_{\widetilde{X}} \\ &\quad \le C_0 \|\Pi^s \widetilde{x}_{t_0}(t_0, j_{\lambda-1}(\psi), \widetilde{G}_\delta) - \widetilde{F}_{*,\delta}(\Phi_c z(0,\tilde{\phi}))\|_{\widetilde{X}} e^{-\beta(t-t_0)}\end{aligned} \tag{4.43}$$

holds for $t \ge t_0$ because of Theorem 3.2 (a) and its proof. By Proposition 4.5 we have

$$\Pi^c \widetilde{x}_t(t_0, j_{\lambda-1}(\psi), \widetilde{G}_\delta) = \Pi^c j_{\lambda-1}(x_t(t_0,\psi,f_\delta)) = j_{\lambda-1}\big(\Pi_X^c x_t(t_0,\psi,f_\delta)\big) \in j_{\lambda-1}(X). \tag{4.44}$$

Moreover, let $\tilde{z} = \operatorname{col}(\tilde{\varepsilon}, \tilde{s})$ be the point of $\mathbb{R}^2$ determined by $\Pi^c\tilde{\phi} = \Phi_c\tilde{z}$. Then it readily follows from Eq. ($C\widetilde{E}$) that $\Phi_c z(t-t_0,\tilde{\phi}) \in j_{\tilde{\varepsilon}}(X)$. Consequently, we must have $\lambda - 1 = \tilde{\varepsilon}$ due to (4.43) and (4.44), that is, $\tilde{\phi} = j_{\lambda-1}(\phi)$ for some $\phi \in X$ with $\|\phi - \phi_*^\lambda\|_X < 2\hat{\delta}(\hat{\varepsilon})$.

Noting that $\tilde{\phi}_*^{\lambda,s} = \widetilde{F}_{*,\delta}(\tilde{\phi}_*^{\lambda,c})$ follows from $\tilde{\phi}_*^\lambda \in \widetilde{W}_\delta^c(0)$, and the fact that $\tilde{\phi}_*^\lambda = j_{\lambda-1}(\phi_*^\lambda)$, one can see $(j_{\lambda-1}(\phi_*^\lambda))^s = \widetilde{F}_{*,\delta}(j_{\lambda-1}(\phi_*^{\lambda,c}))$, so that

$$\begin{aligned}&\|\Pi^s \widetilde{x}_{t_0}(t_0, j_{\lambda-1}(\psi), \widetilde{G}_\delta) - \widetilde{F}_{*,\delta}(\Phi_c z(0, j_{\lambda-1}(\phi)))\|_{\widetilde{X}} \\ &\quad = \big\|(j_{\lambda-1}(\psi))^s - \widetilde{F}_{*,\delta}\big((j_{\lambda-1}(\phi))^c\big)\big\|_{\widetilde{X}} \\ &\quad \le \|(j_{\lambda-1}(\psi))^s - (j_{\lambda-1}(\phi_*^\lambda))^s\|_{\widetilde{X}} + \|\widetilde{F}_{*,\delta}(j_{\lambda-1}(\phi_*^{\lambda,c})) - \widetilde{F}_{*,\delta}(j_{\lambda-1}(\phi^c))\|_{\widetilde{X}} \\ &\quad \le C_1\|j_{\lambda-1}(\psi) - j_{\lambda-1}(\phi_*^\lambda)\|_{\widetilde{X}} + L(\delta)\|(j_{\lambda-1}(\phi_*^\lambda))^c - (j_{\lambda-1}(\phi))^c\|_{\widetilde{X}} \\ &\quad \le C_1(\|\psi - \phi_*^\lambda\|_X + L(\delta)\|\phi_*^\lambda - \phi\|_X) < C_1(1+2L(\delta))\hat{\delta}(\hat{\varepsilon}).\end{aligned} \tag{4.45}$$

Since

$$\|\phi^c - \phi_*^{\lambda,c}\|_X \le C_1\|\phi - \phi_*^\lambda\|_X < 2C_1\hat{\delta}(\hat{\varepsilon}) \le \|\Phi_c\|_*\delta_0\left(\frac{\hat{\varepsilon}}{3(1+L(\delta))\|\Phi_c\|^*}\right), \tag{4.46}$$

it follows from (4.41), combined with $j_{\lambda-1}(\phi_*^{\lambda,c}) = \tilde{\phi}_*^{\lambda,c}$, (4.38) and (4.39), that

$$\|\Phi_c z(t-t_0, j_{\lambda-1}(\phi))\|_{\widetilde{X}} < \|\tilde{\phi}_*^{\lambda,c}\|_{\widetilde{X}} + \frac{\hat{\varepsilon}}{3} < \frac{\delta}{3} + \frac{\delta}{3} = \frac{2\delta}{3}. \tag{4.47}$$

Hence, by (4.43) and (4.45), together with $\tilde{\phi} = j_{\lambda-1}(\phi)$,

$$\begin{aligned}\|\Pi^c \widetilde{x}_t(t_0, j_{\lambda-1}(\psi), \widetilde{G}_\delta)\|_{\widetilde{X}} &\le \|\Phi_c z(t-t_0, j_{\lambda-1}(\phi))\|_{\widetilde{X}} + C_0C_1(1+2L(\delta))\hat{\delta}(\hat{\varepsilon})e^{-\beta(t-t_0)} \\ &< \frac{2\delta}{3} + \frac{\hat{\varepsilon}}{3} < \delta, \quad t \ge t_0.\end{aligned}$$

Moreover, (3.1), (4.36), (4.45) and (4.47) yield

$$\begin{aligned}\|\Pi^s \widetilde{x}_t(t_0, j_{\lambda-1}(\psi), \widetilde{G}_\delta)\|_{\widetilde{X}} &\le \|\widetilde{F}_{*,\delta}(\Phi_c z(t-t_0, j_{\lambda-1}(\phi)))\|_{\widetilde{X}} + C_0C_1(1+2L(\delta))\hat{\delta}(\hat{\varepsilon})e^{-\beta(t-t_0)} \\ &\le \|\Phi_c z(t-t_0, j_{\lambda-1}(\phi))\|_{\widetilde{X}} + \frac{\hat{\varepsilon}}{3} < \delta, \quad t \ge t_0.\end{aligned}$$

Thus, we deduce $\widetilde{x}_t(t_0, j_{\lambda-1}(\psi), \widetilde{G}_\delta) \in \widetilde{\Omega}$ $(t \ge t_0)$. Noting that $\widetilde{G}_\delta \equiv \widetilde{G}$ on $\widetilde{\Omega}$, we have $\widetilde{x}_t(t_0, j_{\lambda-1}(\psi), \widetilde{G}_\delta) = \widetilde{x}_t(t_0, j_{\lambda-1}(\psi), \widetilde{G})$. Therefore, Proposition 4.5 ensures that the solution $x(t; t_0, \psi, f)$ is defined on $[t_0, \infty)$ and satisfies $j_{\lambda-1}\big(x_t(t_0, \psi, f)\big) = \widetilde{x}_t(t_0, j_{\lambda-1}(\psi), \widetilde{G})$ there.

We also know from (3.2) and (4.45) that

$$\begin{aligned}\|\widetilde{x}_t(t_0, j_{\lambda-1}(\psi), \widetilde{G}_\delta) - \widetilde{x}_t(t_0, j_{\lambda-1}(\phi), \widetilde{G}_\delta)\|_{\widetilde{X}} &< 2C_0C_1(1+2L(\delta))\hat{\delta}(\hat{\varepsilon})e^{-\beta(t-t_0)} \\ &\le \frac{2\hat{\varepsilon}}{3}e^{-\beta(t-t_0)}.\end{aligned} \tag{4.48}$$

So, in view of (4.48), (4.41), (4.46) and Proposition 3.1, combined with the fact that

$$j_{\lambda-1}(\phi_*^\lambda) = \tilde{\phi}_*^\lambda = \tilde{\phi}_*^{\lambda,c} + \tilde{\phi}_*^{\lambda,s} = j_{\lambda-1}(\phi_*^{\lambda,c}) + \widetilde{F}_{*,\delta}(j_{\lambda-1}(\phi_*^{\lambda,c})),$$

we conclude that if $\psi \in B_X(\phi_*^\lambda; \hat{\delta}(\hat{\varepsilon}))$, then

$$\begin{aligned}\|x_t(t_0, \psi, f) - \phi_*^\lambda\|_X &= \|j_{\lambda-1}\big(x_t(t_0, \psi, f)\big) - j_{\lambda-1}(\phi_*^\lambda)\|_{\widetilde{X}} = \|\widetilde{x}_t(t_0, j_{\lambda-1}(\psi), \widetilde{G}) - \tilde{\phi}_*^\lambda\|_{\widetilde{X}} \\ &\le \|\widetilde{x}_t(t_0, j_{\lambda-1}(\psi), \widetilde{G}_\delta) - \widetilde{x}_t(t_0, j_{\lambda-1}(\phi), \widetilde{G}_\delta)\|_{\widetilde{X}} \\ &\quad + \|\Phi_c z(t-t_0, j_{\lambda-1}(\phi)) - j_{\lambda-1}(\phi_*^{\lambda,c})\|_{\widetilde{X}} \\ &\quad + \|\widetilde{F}_{*,\delta}(\Phi_c z(t-t_0, j_{\lambda-1}(\phi))) - \widetilde{F}_{*,\delta}(j_{\lambda-1}(\phi_*^{\lambda,c}))\|_{\widetilde{X}} \\ &< \frac{2\hat{\varepsilon}}{3}e^{-\beta(t-t_0)} + (1+L(\delta))\|\Phi_c z(t-t_0, j_{\lambda-1}(\phi)) - j_{\lambda-1}(\phi_*^{\lambda,c})\|_{\widetilde{X}} < \hat{\varepsilon}\end{aligned}$$

for $t \ge t_0$. The equilibrium ϕ_*^λ is therefore stable.

We next verify the attractivity of ϕ_*^λ. By the asymptotic stability of $s_{\lambda-1}$ as an equilibrium of $s' = h_\delta(\lambda-1, s)$, together with the invariance of $\{\lambda-1\} \times \mathbb{R}$ for Eq. $(C\widetilde{E}_\delta)$, there exist an $R_0 > 0$ with the property that to any $\hat{\varepsilon} > 0$, there corresponds a $\tau(\hat{\varepsilon}) > 0$ such that $|z(t; t_0, z^*) - z^{\lambda-1}| < \hat{\varepsilon}$ for every $t \ge t_0 + \tau(\hat{\varepsilon})$ and $z^* \in \{\lambda-1\} \times \mathbb{R}$ with $|z^* - z^{\lambda-1}| \le R_0$. Let $\hat{R} := \min\big(\hat{\delta}(\delta), \|\Phi_c\|_* R_0/(2C_1)\big)$, where $\hat{\delta}(\cdot)$ is the one in (4.42). Given any $\hat{\varepsilon} \in (0, \delta)$, set

$$\hat{\tau}(\hat{\varepsilon}) := \max\left(\left(\frac{1}{\beta}\log\frac{4C_0C_1(1+2L(\delta))\hat{R}}{\hat{\varepsilon}}, \tau\left(\frac{\hat{\varepsilon}}{2(1+L(\delta))\|\Phi_c\|^*}\right)\right)\right)$$

and assume that $\psi \in B_X(\phi_*^\lambda, \hat{R})$. Then $j_{\lambda-1}(\psi) \in B_{\widetilde{X}}(j_{\lambda-1}(\phi_*^\lambda); \hat{R})$, and there exists a $\tilde{\phi} \in B_{\widetilde{X}}(j_{\lambda-1}(\phi_*^\lambda); 2\hat{R})$ with $\widehat{g}(\tilde{\phi}) = j_{\lambda-1}(\psi)$. Since $\hat{R} \le \hat{\delta}(\delta)$, we know from the former part that $\widetilde{x}_t(t_0, j_{\lambda-1}(\psi), \widetilde{G}_\delta) \in \widetilde{\Omega}$ $(t \ge t_0)$. So, in the same way as above, $\hat{\phi}$ can be written as $\hat{\phi} = j_{\lambda-1}(\phi)$ for some $\phi \in X$ with $\|\phi - \phi_*^\lambda\|_X < 2\hat{\delta}(\delta)$ and the following estimate is valid:

$$\begin{aligned}\|x_t(t_0, \psi, f) - \phi_*^\lambda\|_X &\le \|\widetilde{x}_t(t_0, j_{\lambda-1}(\psi), \widetilde{G}_\delta) - \widetilde{x}_t(t_0, j_{\lambda-1}(\phi), \widetilde{G}_\delta)\|_{\widetilde{X}} \\ &\quad + (1+L(\delta))\|\Phi_c z(t-t_0, j_{\lambda-1}(\phi)) - j_{\lambda-1}(\phi_*^{\lambda,c})\|_{\widetilde{X}} \\ &\le 2C_0C_1(1+2L(\delta))\hat{R}e^{-\beta(t-t_0)} + (1+L(\delta))\|\Phi_c\|^*|z(t; t_0, z^*) - z^{\lambda-1}| \\ &< \frac{\hat{\varepsilon}}{2} + \frac{\hat{\varepsilon}}{2} = \hat{\varepsilon} \quad \text{for } t \ge t_0 + \hat{\tau}(\hat{\varepsilon}),\end{aligned}$$

where z^* is again the point of $\mathbb{R}^2$ determined by $\Phi_c z^* = j_{\lambda-1}(\phi^c)$. Thus, ϕ^λ_* is an attractive equilibrium of Eq. (*PE*), and hence, is asymptotically stable.

Similarly, the equilibrium 0 of Eq. $s' = h(\lambda - 1, s)$ is asymptotically stable because

$$\frac{\partial h}{\partial s}(\varepsilon, 0) = -\frac{1}{q_*}\varepsilon + o(|\varepsilon|) < 0$$

for $\varepsilon < 0$ with $|\varepsilon|$ small. By the same reasoning as above, one can see the asymptotic stability of the equilibrium 0 of Eq. (*PE*).

Suppose now that $\lambda > 1$. Then $z^{\lambda-1} = \mathrm{col}\,(\lambda - 1, s_{\lambda-1})$ is a unstable equilibrium of Eq. ($C\widetilde{E}$). So, by Theorem 3.3, $\tilde{\phi}^\lambda_*$ is a unstable equilibrium of Eq. ($\widetilde{E}$). The invariance of $j_{\lambda-1}(X)$ then implies the instability of ϕ^λ_*, and likewise for the equilibrium 0.

If $q_* > 0$, then consider the function $\check{h}(\varepsilon, s) := h(\varepsilon, -s)$ $((\varepsilon, s) \in \mathbb{R}^2)$ in place of $h(\varepsilon, s)$ in Eq. ($C\widetilde{E}$). One can readily verify that $\check{h}(\varepsilon, s)$ satisfies the conditions (4.31) and (4.32) of Lemma 4.14. Applying the lemma to $\check{h}(\varepsilon, s)$ and noting that $(\partial h/\partial s)(\varepsilon, s) = -(\partial \check{h}/\partial s)(\varepsilon, -s)$, we see that the equilibrium $s_{\lambda-1}$ of the scalar equation $s' = h(\lambda - 1, s)$ is unstable (resp. asymptotically stable) if $\lambda < 1$ (resp. $\lambda > 1$). So, the argument above yields the desired stability properties of ϕ^λ_*; and similarly for the equilibrium 0.

(ic) Let U_1 be the open set of $\mathbb{R}^2$ in Lemma 4.14 (a_2). We may assume that U_1 is a rectangle $U(r) = (-r, r) \times (-r, r)$, where $r > 0$ satisfies $r < \rho\delta/3 - s^*|\eta_*|$ (see the proof of the lemma and notice (4.37)). Set

$$\mathcal{W} := \Pi_2\big(\widetilde{\Omega}_0 \cap (\Pi^c)^{-1}(\Phi_c U_1)\big);$$

then $\mathcal{W}$ is an open set of X since Φ_c is a homeomorphism as a map from $\mathbb{R}^2$ to $\widetilde{E}^c$, and Π_2 is an open map. Now assume that $\phi_* \in \mathcal{W}\backslash\{0\}$ is an equilibrium of Eq. (*PE*) with $|\lambda - 1| < r$. Since $\phi_* \in \mathcal{W}$, ϕ_* can be written as $\phi_* = \Pi_2\tilde{\phi}^\circ$, where $\tilde{\phi}^\circ \in \widetilde{\Omega}_0$ satisfies $\Pi^c\tilde{\phi}^\circ = \Phi_c z^\circ$ for some $z^\circ = \mathrm{col}\,(\lambda^\circ - 1, s^\circ) \in U_1$. Hence, $\Pi^c_X\phi_* = \Pi^c_X\Pi_2\tilde{\phi}^\circ = \Pi_2\Pi^c\tilde{\phi}^\circ = \Pi_2\Phi_c z^\circ = w_1(0)(s^\circ\eta_*)$.

On the other hand, in view of Remark 4.15, $\tilde{\phi}_* := j_{\lambda-1}(\phi_*)$ is an equilibrium of Eq. ($\widetilde{E}$). Since $\Pi^c\tilde{\phi}_* = j_{\lambda-1}(\Pi^c_X\phi_*) = w_1(0)\big((\lambda - 1)e_1 + s^\circ\eta_*\big)$ and $\Pi^s\tilde{\phi}_* = j_0(\Pi^s_X\phi_*) = j_0(\Pi_2\Pi^s\tilde{\phi}_\circ)$, we have

$$\|\Pi^c\tilde{\phi}_*\|_{\widetilde{X}} \le \rho^{-1}(|\lambda - 1| + |s^\circ||\eta_*|) < \rho^{-1}(r + s^*|\eta_*|) < \frac{\delta}{3},$$

and

$$\|\Pi^s\tilde{\phi}_*\|_{\widetilde{X}} = \|\Pi_2\Pi^s\tilde{\phi}^\circ\|_X \le \|\Pi^s\tilde{\phi}^\circ\|_{\widetilde{X}} < \frac{\delta}{3}$$

(because of $\phi^\circ \in \widetilde{\Omega}_0$). Hence $\tilde{\phi}_*$ is an equilibrium of Eq. ($\widetilde{E}$) in $\widetilde{\Omega}_0$, and therefore, by (4.35), $\tilde{\phi}_* = \Phi_c z_* + \widetilde{F}_*(\Phi_c z_*)$ for some equilibrium z_* of Eq. ($C\widetilde{E}$). Thus, we have $\Pi_1\tilde{\phi}_* = \Pi_1 j_{\lambda-1}(\phi_*) = w_1(0)(\lambda - 1)$, and $\Pi_1\tilde{\phi}_* = \Pi_1\Phi_c z_*$ as well (see (4.40)); in particular, z_* is of the form $z_* = \mathrm{col}\,(\lambda - 1, s_*) \in \mathbb{R}^2$. One can also obtain that $\Pi^c_X\phi_* = \Pi^c_X\Pi_2\tilde{\phi}_* = \Pi_2\Pi^c\tilde{\phi}_* = \Pi_2\Phi_c z_* = w_1(0)(s_*\eta_*)$, so that $s^\circ = s_*$. Consequently, $z_* = \mathrm{col}\,(\lambda - 1, s^\circ) \in U_1$. It therefore follows from Lemma 4.14 (a_2) that $z_* = z^{\lambda-1}$, and hence that $\tilde{\phi}_* = \Phi_c z^{\lambda-1} + \widetilde{F}_*(\Phi_c z^{\lambda-1}) = \tilde{\phi}^\lambda_*$, which yields $\phi_* = \phi^\lambda_*$.

(ii) (iia) Let $\lambda = 1$ and $s^* > 0$ be the one in Lemma 4.14 (b_1). In a similar manner to the proof of (ia), one can see that $\mathcal{W}_0 := B_X(\rho_1)$ is a required open set. Now let s^ε_+ and s^ε_- the functions from $(0, \varepsilon^*)$ to $\mathbb{R}\backslash\{0\}$ given in Lemma 4.14 (b_1), and set $z^\varepsilon_\pm := \mathrm{col}\,(\varepsilon, s^\varepsilon_\pm)$. Define the functions $\phi_\pm : (1, 1 + \varepsilon^*) \to X$ by

$$\phi^\lambda_\pm := \Pi_2\big(\Phi_c z^{\lambda-1}_\pm + \widetilde{F}_*(\Phi_c z^{\lambda-1}_\pm)\big), \quad \lambda \in (1, 1 + \varepsilon^*),$$

where we choose ε^* small if necessary. Then ϕ_+^λ and ϕ_-^λ are distinct equilibria of Eq. (*PE*) which tends to 0 as $\lambda \to 1+0$.

The proof of the instability of the equilibrium 0 is quite similar to that of (1a).

(iib) By the same reasoning as (iia), the stability properties of $\phi_\pm^\lambda$ and 0 directly follow from Lemma 4.14 (b_1).

(iic) Let U_1 be the open set of $\mathbb{R}^2$ in Lemma 4.14 (b_2), which may also be thought of a rectangle $U(r) = (-r, r) \times (-r, r)$ with $r > 0$ small. Then, $\mathcal{W} := \Pi_2\big(\widetilde{\Omega}_0 \cap (\Pi^c)^{-1}(\Phi_c U_1)\big)$ is a desired open set of X; the proof being similar to that of (ic).

(iii) The proof is quite the same as that of (ii), and hence we will omit it. □

Now we prove Lemma 4.14. For this, we will employ the following lemma, which is the implicit function theorem for Lipschitz continuous maps.

Lemma 4.16. *Let U and V be open sets of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, and (u_0, v_0) a point of $U \times V$. Suppose that $G : U \times V \to \mathbb{R}^n$ is a continuous map satisfying the followings:*

(i) *For each $v \in V$, $G(\cdot, v)$ is a Lipschitz continuous map from U to $\mathbb{R}^n$; and in addition there exists an $l_* \in [0, 1)$ such that the Lipschitz constant of $E_n - G(\cdot, v)$ does not exceed l_* for $v \in V$, E_n being the $n \times n$ unit matrix.*

(ii) $G(u_0, v_0) = 0$.

Then there exist an $r > 0$ and a continuous map $\varphi : B_{\mathbb{R}^n}(u_0; r) \to V$ with the properties that $\varphi(u_0) = v_0$ and

$$G(u, \varphi(u)) = 0, \quad u \in B_{\mathbb{R}^n}(u_0; r).$$

Moreover, if $r > 0$ and $\tilde{r} > 0$ are sufficiently small, then $G(u, v) = 0$ with $(u, v) \in B_{\mathbb{R}^n}(u_0; r) \times B_{\mathbb{R}^m}(v_0; \tilde{r})$ implies $v = \varphi(u)$.

Proof of Lemma 4.14. Set $\Delta(\varepsilon, s) := h(\varepsilon, s) + \alpha s\varepsilon - \beta s^m$. By letting $s := \sigma\varepsilon^{1/(m-1)}$, $h(\varepsilon, s)$ can be written as

$$\widetilde{h}(\varepsilon, \sigma) := h(\varepsilon, \sigma\varepsilon^{1/(m-1)}) = \varepsilon^{m/(m-1)}(-\alpha\sigma + \beta\sigma^m) + \Delta(\varepsilon, \sigma\varepsilon^{1/(m-1)}).$$

So,

$$\widetilde{h}(\varepsilon, \sigma) = \varepsilon^{m/(m-1)}\big(-\alpha\sigma + \beta\sigma^m + R(\varepsilon, \sigma)\big), \quad (\varepsilon, \sigma) \in \widetilde{U}_0 := \widetilde{\tau}^{-1}(U_0), \tag{4.49}$$

where $R(\varepsilon, \sigma)$ is the function defined by

$$R(\varepsilon, \sigma) := \begin{cases} \varepsilon^{-m/(m-1)}\Delta(\varepsilon, \sigma\varepsilon^{1/(m-1)}), & \varepsilon \neq 0, \\ 0, & \varepsilon = 0, \end{cases}$$

and $\widetilde{\tau}$ is the one defined by $\widetilde{\tau}(\varepsilon, \sigma) := (\varepsilon, \sigma\varepsilon^{m/(m-1)})$. Then we have:

Claim 2. $R(\varepsilon, \sigma)$ has the following properties.

(R_1) $R(\varepsilon, \sigma)$ is of class C^1 in $\widetilde{U}_0 \backslash \{\varepsilon = 0\}$.

(R_2) $R(\varepsilon, \sigma)$ and the partial derivative $R_\sigma(\varepsilon, \sigma)$ are continuous in $\widetilde{U}_0$.

Proof of Claim 2. (R_1) is evident.

(R_2) Since m is even,

$$R(\varepsilon,\sigma)=\frac{\Delta(\varepsilon,\sigma\varepsilon^{1/(m-1)})}{|\sigma\varepsilon^{1/(m-1)}|^m+|\varepsilon\sigma\varepsilon^{1/(m-1)}|}\,(|\sigma|^m+|\sigma|),\quad \varepsilon\neq 0. \tag{4.50}$$

(4.31) therefore implies that $R(\varepsilon,\sigma)$ converges to 0 uniformly for $|\sigma|\le\sigma^*$ as $\varepsilon\to 0$, where $\sigma^*>0$ is arbitrary; and in particular $R(\varepsilon,\sigma)$ is continuous at the points in $\widetilde{U}_0\cap\{\varepsilon=0\}$, which, together with ($R_1$), yields the continuity in $\widetilde{U}_0$ of $R(\varepsilon,\sigma)$. In virtue of (4.32) and the fact that

$$R_\sigma(\varepsilon,\sigma)=\varepsilon^{-1}\frac{\partial\Delta}{\partial s}(\varepsilon,\sigma\varepsilon^{1/(m-1)})=\frac{\Delta_s(\varepsilon,\sigma\varepsilon^{1/(m-1)})}{|\sigma\varepsilon^{1/(m-1)}|^{m-1}+|\varepsilon|}\,(|\sigma|^{m-1}+1)$$

for $\varepsilon\neq 0$, $R_\sigma(\sigma,\varepsilon)$ converges to $0=R_\sigma(0,\sigma)$ uniformly for $|\sigma|\le\sigma^*$ as $\varepsilon\to 0$; hence, by the same reasoning as the former part, $R_\sigma(\varepsilon,\sigma)$ is continuous in $\widetilde{U}_0$. This proves Claim 2.

Now let

$$F(\varepsilon,\sigma):=-\alpha\sigma+\beta\sigma^m+R(\varepsilon,\sigma),\quad (\varepsilon,\sigma)\in\widetilde{U}_0.$$

(a) (a_1) The first part is obvious. Since m is even, $F(0,\sigma)$ has always two zeros, i.e., $\sigma=0$ and

$$\sigma_0:=\left(\frac{\alpha}{\beta}\right)^{1/(m-1)}. \tag{4.51}$$

Notice that $F_\sigma(0,\sigma_0)=-\alpha+m\beta\sigma_0^{m-1}=(m-1)\alpha\neq 0$, and define $G(\varepsilon,\sigma)$ by

$$G(\varepsilon,\sigma):=\frac{1}{F_\sigma(0,\sigma_0)}F(\varepsilon,\sigma),\quad (\varepsilon,\sigma)\in\widetilde{U}_0.$$

G is a continuous function with $G(0,\sigma_0)=0$ and $G(\varepsilon,\cdot)$ is locally Lipschitz continuous for each ε. In view of (R_2) and the fact that $G_\sigma(0,\sigma_0)=1$, there exist open intervals U and V containing 0 and σ_0, respectively, and an $l_0\in[0,1)$ such that $\mathrm{Lip}\,(1-G(\varepsilon,\cdot))$, the Lipschitz constant of the restriction $(1-G(\varepsilon,\cdot))\big|_V$, satisfies

$$\mathrm{Lip}\,(1-G(\varepsilon,\cdot))=\sup_{(\varepsilon,\sigma)\in U\times V}|1-G_\sigma(\varepsilon,\sigma)|\le l_0,\quad \varepsilon\in U.$$

Therefore it follows from Lemma 4.16 that there exist an $\varepsilon^*>0$ and a continuous function $\sigma:(-\varepsilon^*,\varepsilon^*)\to V$ with the properties that $\sigma(0)=\sigma_0$ and $G(\varepsilon,\sigma(\varepsilon))=0$ for $\varepsilon\in(-\varepsilon^*,\varepsilon^*)$. Hence, $\widetilde{h}(\varepsilon.\sigma(\varepsilon))=\varepsilon^{m/(m-1)}F(\varepsilon,\sigma(\varepsilon))=0$ for $\varepsilon\in(-\varepsilon^*,\varepsilon^*)\backslash\{0\}$. We may also assume that $\sigma(\varepsilon)\neq 0$ for $\varepsilon\in(-\varepsilon^*,\varepsilon^*)$ since $\sigma_0\neq 0$ and $\sigma(\varepsilon)$ is continuous. Set $s_\varepsilon:=\sigma(\varepsilon)\varepsilon^{1/(m-1)}$. Then $s:(-\varepsilon^*,\varepsilon^*)\to\mathbb{R}$ is a continuous function which maps $(-\varepsilon^*,\varepsilon^*)\backslash\{0\}$ into $\mathbb{R}\backslash\{0\}$, and satisfies

$$h(\varepsilon,s_\varepsilon)=\widetilde{h}(\varepsilon,\sigma(\varepsilon))=0,\quad \varepsilon\in(-\varepsilon^*,\varepsilon^*)\backslash\{0\}$$

together with $s_\varepsilon\to 0$ as $\varepsilon\to 0$. Moreover, it follows from (4.32) that

$$\begin{aligned}\frac{\partial h}{\partial s}(\varepsilon,s_\varepsilon)&=-\alpha\varepsilon+m\beta\sigma(\varepsilon)^{m-1}\varepsilon+o(\varepsilon)\\&=-\alpha\varepsilon+m\beta\sigma_0^{m-1}\varepsilon+m\beta(\sigma(\varepsilon)^{m-1}-\sigma(0)^{m-1})\varepsilon+o(\varepsilon)\\&=(m-1)\alpha\varepsilon+o(\varepsilon)\quad\text{as }\varepsilon\to 0,\end{aligned} \tag{4.52}$$

where we used (4.51) and the continuity of $\sigma(\varepsilon)$; so that (4.33) holds.

Also, by Lemma 4.16 $G(\varepsilon,\sigma)=0$ with $(\varepsilon,\sigma)\in(-\varepsilon^*,\varepsilon^*)\times(\sigma_0-\sigma^*,\sigma_0+\sigma^*)$ yields $\sigma=\sigma(\varepsilon)$, provided that $\varepsilon^*>0$ and $\sigma^*>0$ are small enough.

We next consider $\sigma=0$ as the other zero of $F(0,\sigma)$. Since $F_\sigma(0,0)=-\alpha\neq 0$, by applying the same argument as above to the function

$$\widetilde{G}(\varepsilon,\sigma):=\frac{1}{F_\sigma(0,0)}F(\varepsilon,\sigma),\quad (\varepsilon,\sigma)\in\widetilde{U}_0,$$

one can see that there exist an $\tilde{\varepsilon}^*>0$ and a continuous function $\tilde{\sigma}:(-\tilde{\varepsilon}^*,\tilde{\varepsilon}^*)\to V$ with the properties that $\tilde{\sigma}(0)=0$ and $\widetilde{G}(\varepsilon,\tilde{\sigma}(\varepsilon))=0$ for $\varepsilon\in(-\tilde{\varepsilon}^*,\tilde{\varepsilon}^*)$. On the other hand, it follows from (4.31) that $\Delta(\varepsilon,0)=0$, and hence $\widetilde{G}(\varepsilon,0)=0$. By the same reasoning as the last paragraph we obtain $\tilde{\sigma}(\varepsilon)=0$ for $\varepsilon\in(-\tilde{\varepsilon}^*,\tilde{\varepsilon}^*)$ with $\tilde{\varepsilon}^*>0$ small.

(a_2) Let $U(r):=(-r,r)\times(-r,r)$ and

$$d_r:=\sup\left\{|\Delta(\varepsilon,s)|/(|s|^m+|s\varepsilon|):(\varepsilon,s)\in U(r)\backslash\{s=0\}\right\}\tag{4.53}$$

for $r>0$. Then $d_r\to 0$ as $r\to 0$ by (4.31). Put

$$c_*:=\inf_{|\sigma-\sigma_0|\geq\sigma^*}|-\alpha+\beta\sigma^{m-1}|\tag{4.54}$$

and take an $r>0$ small enough so that $r<\varepsilon^*$ and

$$d_r<\min\left(\frac{|\beta|}{2},\frac{c_*}{2(|\sigma_0|^{m-1}+1)}\right).\tag{4.55}$$

Now suppose that $h(\hat{\varepsilon},\hat{s})=0$ for some $(\hat{\varepsilon},\hat{s})\in U(r)$ with $\hat{s}\neq 0$, and set $\hat{\sigma}:=\hat{s}\hat{\varepsilon}^{-1/(m-1)}$. (Notice that $\hat{\varepsilon}\neq 0$ due to (4.55).) Since $F(\hat{\varepsilon},\hat{\sigma})=0$, it follows from (4.50) that

$$|-\alpha\hat{\sigma}+\beta\hat{\sigma}^m|=|R(\hat{\varepsilon},\hat{\sigma})|\leq d_r(|\hat{\sigma}|^m+|\hat{\sigma}|).\tag{4.56}$$

Hence, in view of $\hat{\sigma}\neq 0$ and (4.55),

$$\frac{|\beta|}{2}|\hat{\sigma}|^{m-1}\leq(|\beta|-d_r)|\hat{\sigma}|^{m-1}\leq\alpha+d_r\leq\alpha+\frac{|\beta|}{2},$$

so that $|\hat{\sigma}|^{m-1}\leq 2|\sigma_0|^{m-1}+1$, and therefore (4.56) and (4.55) imply

$$|-\alpha+\beta\hat{\sigma}^{m-1}|\leq d_r(|\hat{\sigma}|^{m-1}+1)\leq 2d_r(|\sigma_0|^{m-1}+1)<c_*,$$

from which and (4.54) we deduce that $\hat{\sigma}\in(\sigma_0-\sigma^*,\sigma_0+\sigma^*)$. Thus, we arrive at $G(\hat{\varepsilon},\hat{\sigma})=0$ with $(\hat{\varepsilon},\hat{\sigma})\in(-\varepsilon^*,\varepsilon^*)\times(\sigma_0-\sigma^*,\sigma_0+\sigma^*)$. The argument in the proof of (a_1) then yields $\hat{\sigma}=\sigma(\hat{\varepsilon})$ and therefore $\hat{s}=s_{\hat{\varepsilon}}$. Consequently, ($a_2$) holds true with $U_1=U(r)$.

(b) (b_1) The first part is obvious. Let us take $r>0$ so small that $d_r<\min(\alpha,\beta)$, d_r being the one in (4.53). Then, it follows that for $(\varepsilon,s)\in U(r)$ with $s\neq 0$ and $\varepsilon\leq 0$,

$$\begin{aligned}sh(\varepsilon,s)&=s(-\alpha s\varepsilon+\beta s^m+\Delta(\varepsilon,s))\\&\geq\alpha|\varepsilon|s^2+\beta|s|^{m+1}-|s||\Delta(\varepsilon,s)|\\&\geq(\alpha-d_r)|\varepsilon|s^2+(\beta-d_r)|s|^{m+1}>0.\end{aligned}$$

Thus, $h(\varepsilon,\cdot)$ has no zeros in $(-r,r)$ other than $s=0$, provided that $-r<\varepsilon\leq 0$.

Assume now that $\varepsilon > 0$. Since m is odd, $F(0,\sigma) = 0$ has three zeros, i.e., $\sigma = 0$, $\sigma_0^+ := (\alpha/\beta)^{1/(m-1)}$ and $\sigma_0^- := -(\alpha/\beta)^{1/(m-1)}$. Define $h^+ : U_0 \to \mathbb{R}$ by

$$h^+(\varepsilon, s) := -\alpha s\varepsilon + \beta|s|^m + \Delta(\varepsilon, s), \quad (\varepsilon, s) \in U_0.$$

Then, h^+ is of class C^1 and by (4.32),

$$\frac{\partial h^+}{\partial s}(\varepsilon, s) = -\alpha\varepsilon + m\beta|s|^{m-1}\operatorname{sgn} s + o(|s|^{m-1} + |\varepsilon|),$$

where $\operatorname{sgn} s$ denotes the sign of $s \in \mathbb{R}$ with $\operatorname{sgn} 0 := 0$. Moreover, by letting $s := \sigma|\varepsilon|^{1/(m-1)}\operatorname{sgn}\varepsilon$, we have

$$\widetilde{h}^+(\varepsilon, \sigma) := h^+(\varepsilon, \sigma|\varepsilon|^{1/(m-1)}\operatorname{sgn}\varepsilon) = |\varepsilon|^{m/(m-1)}\big(-\alpha\sigma + \beta|\sigma|^m + R^+(\varepsilon, \sigma)\big),$$

according to (4.49), where

$$R^+(\varepsilon, \sigma) := \begin{cases} |\varepsilon|^{-m/(m-1)}\Delta\big(\varepsilon, \sigma|\varepsilon|^{1/(m-1)}\operatorname{sgn}\varepsilon\big), & \varepsilon \neq 0, \\ 0, & \varepsilon = 0. \end{cases}$$

Consequently, the arguments in the proof of Claim 2 (resp. (a)) are valid for R^+ (resp. h^+ with σ_0^+ in place of σ_0), and hence there exists a function $s : (-\varepsilon^*, \varepsilon^*) \to \mathbb{R}$ satisfying the properties (a_1) and (a_2). In particular, it follows from $\sigma_0^+ > 0$ that $s_\varepsilon > 0$ for $\varepsilon \in (0, \varepsilon^*)$, which, combined with the fact that $h^+(\varepsilon, s) = h(\varepsilon, s)$ for $s \geq 0$, implies that $s^+ := s|_{(0,\varepsilon^*)}$ is one of the desired functions.

For the existence of $s^- : (0, \varepsilon^*) \to \mathbb{R}$, we have only to consider the function

$$h^-(\varepsilon, s) := -\alpha s\varepsilon - \beta|s|^m + \Delta(\varepsilon, s), \quad (\varepsilon, s) \in U_0.$$

In the same way as (4.52), we also obtain

$$\begin{aligned} \frac{\partial h^\pm}{\partial s}(\varepsilon, s_\varepsilon^\pm) &= -\alpha\varepsilon + m\beta|s_\varepsilon^\pm|^{m-1} + o(\varepsilon) = -\alpha\varepsilon + m\beta|\sigma_0^\pm|^{m-1}\varepsilon + o(\varepsilon) \\ &= (m-1)\alpha\varepsilon + o(\varepsilon) \quad \text{as } \varepsilon \to +0, \end{aligned}$$

and (4.34) follows from

$$\frac{\partial h^+}{\partial s}(\varepsilon, s) = \frac{\partial h}{\partial s}(\varepsilon, s) \quad (s \geq 0), \quad \text{and} \quad \frac{\partial h^-}{\partial s}(\varepsilon, s) = \frac{\partial h}{\partial s}(\varepsilon, s) \quad (s \leq 0).$$

(b_2) Since the statement of (a_2) holds for both $h^+(\varepsilon, s)$ and $h^-(\varepsilon, s)$, the definition of s^+ and s^-, together with the argument of the first paragraph, imply the existence of a desired open set of U_0.

(c) The proof is quite similar to that of (b), and is omitted. □

5 Examples

We will show an example on one-parameter bifurcation to illustrate our Theorem 4.4.

Example 5.1. Let us consider a nonlinear scalar integral equation

$$x(t) = \lambda \int_{-\infty}^{t} p(t-s)x(s)ds + f(x_t), \tag{5.1}$$

where λ is a nonnegative parameter and p is a nonnegative continuous function on $\mathbb{R}^+$ that satisfies

$$\int_0^\infty p(t)dt = 1 \tag{5.2}$$

together with

$$\|p\|_{1,\rho} = \int_0^\infty p(t)e^{\rho t}dt < \infty \quad \text{and} \quad \|p\|_{\infty,\rho} = \operatorname{ess\,sup}\{p(t)e^{\rho t} : t \geq 0\} < \infty$$

for some positive constant ρ. Let $X := L^1_\rho(\mathbb{R}^-;\mathbb{R})$ and $f : X \to \mathbb{R}$ be of the form

$$f(\phi) = \left(\int_{-\infty}^{0} q(-\theta)\phi(\theta)d\theta \right)^3 + g(\phi),$$

where $q : \mathbb{R}^+ \to \mathbb{R}$ satisfies

$$\int_0^\infty q(t)dt > 0, \quad \int_0^\infty |q(t)|e^{\rho t}dt < \infty \quad \text{and} \quad \operatorname{ess\,sup}\{|q(t)|e^{\rho t} : t \geq 0\} < \infty$$

and $g \in C^1(X;\mathbb{R})$ with $g(\phi) = o(\|\phi\|_X^3)$ as $\|\phi\|_X \to 0$.

Since $f \in C^1(X;\mathbb{R})$ with $f(0) = 0$ and $Df(0) = 0$, so [23, Proposition 8] implies the following fact:

- If $0 \leq \lambda < 1$, then the zero solution of Eq. (5.1) is exponentially stable (in L^1_ρ);
- If $\lambda > 1$, then the zero solution of Eq. (5.1) is unstable (in L^1_ρ).

We will verify that $\lambda = 1$ is a bifurcation point and a pitchfork bifurcation occurs when λ exceeds 1. Notice that $P_0 = \int_0^\infty p(t)dt = 1$ and $P_1 = \int_0^\infty tp(t)dt > 0$. So 0 is a simple eigenvalue of P_0 and $P_1\big(\mathcal{N}(1-P_0)\big) = \mathbb{R}$; hence (A_1) and (A_2) hold together with (A_4). In addition, since the characteristic operator is $\Delta(z) = 1 - \lambda \int_0^\infty p(t)e^{-zt}dt$, we see from (5.2) that

$$\Delta(i\omega) = \int_0^\infty p(t)(1-\cos\omega t)dt + i\int_0^\infty p(t)\sin\omega t dt, \quad \omega \in \mathbb{R}.$$

So, if $\det\Delta(i\omega) = \Delta(i\omega) = 0$ for some $\omega > 0$, then

$$\int_0^\infty p(t)(1-\cos\omega t)dt = \int_0^\infty p(t)\sin\omega t dt = 0.$$

Hence, $p(t) = 0$ on $\mathbb{R}^+$ follows, for $p(t)$ is continuous and nonnegative on $\mathbb{R}^+$ and $1-\cos\omega t > 0$ ($t \in \mathbb{R}\backslash(2\pi\omega^{-1}\mathbb{Z})$); see also [22, Example 4.1]. This contradicts to (5.2), so that $\det\Delta(i\omega) \neq 0$ for $\omega > 0$; and therefore (A_3) is also valid due to Remark 4.2 and (5.2).

Since one can choose $\eta_* = 1$ and $\zeta^* = 1$, so $q_* = -P_1 < 0$ and

$$c_0 = \left(\int_{-\infty}^{0} q(-\theta)(w_1(0))(\theta)d\theta \right)^3 = \left(\int_0^\infty q(t)dt \right)^3 > 0$$

(see (4.5)). Consequently by Theorem 4.4 (iib) a pitchfork bifurcation is observed when the parameter λ exceeds 1; the equilibrium 0 becomes unstable and two more equilibria appear in a neighborhood of 0, each of which is asymptotically stable for $\lambda > 1$.

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Existence and regularity of solutions for a singular anisotropic (p,q)-Laplacian with variable exponent

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Abstract. In this paper, we investigate the existence and regularity of positive solutions for certain singular problems that involve an anisotropic (p,q)-Laplacian-type operator and a singular term with a variable exponent, under zero Dirichlet boundary conditions on $\partial\Omega$. The main equation we analyze is

$$-\sum_{i=1}^{N}\partial_i\left(|\partial_i u(x)|^{p_i-2}\partial_i u(x)\right)-\sum_{i=1}^{N}\partial_i\left(|\partial_i u(x)|^{q_i-2}\partial_i u(x)\right)=\frac{f(x)}{u(x)^{\gamma(x)}}\quad\text{in }\Omega,$$

where Ω is a bounded, regular domain in $\mathbb{R}^N$, f is a positive function belonging to a specific Lebesgue space, and $\gamma(x)$ is a positive continuous function on $\overline{\Omega}$. In our study, we do not make comparisons between p_i and q_i, and as a result, we show that the solution belongs to either $W_0^{1,\vec{p}}(\Omega)\cap W_0^{1,\vec{q}}(\Omega)$ or $W_{loc}^{1,\vec{p}}(\Omega)\cap W_{loc}^{1,\vec{q}}(\Omega)$ depending on the summability of $f(x)$ and the values of $\gamma(x)$. The results are achieved using approximation techniques that include truncation, comparison, and variational methods.

Keywords: anisotropic singular (p,q)-equations, approximation, variational methods, existence and regularity results.

2020 Mathematics Subject Classification: 35J60, 35J75, 35J30, 35B51.

1 Introduction and main results

This study investigates elliptic partial differential equations (PDEs) characterized by anisotropic (p,q)-Laplacian-type operators, focusing on their mathematical properties and physical relevance. Anisotropic (p,q)-Laplacians extend the classical Laplacian by incorporating directional dependence, making them instrumental in modeling complex phenomena in heterogeneous materials, such as heat conduction, fluid dynamics, and nonlinear elasticity. The research also highlights the importance of reaction-diffusion systems in plasma physics and

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chemical reaction design, which are driven by parabolic reaction-diffusion equations.

$$u_t - \operatorname{div}\left[|\nabla u|^{p-2}\nabla u\right] - \operatorname{div}\left[|\nabla u|^{q-2}\nabla u\right] = \mathfrak{G}(x,u).$$

These systems are fundamental for understanding pattern formation, wave propagation, and turbulence control in various scientific and engineering applications. For more technical details, we point out the following refs. [4,7,21,27,47]. By examining the theoretical framework surrounding anisotropic elliptic (PDEs) and referencing foundational works, this study underscores the critical role of these mathematical constructs in solving real-world problems in fields such as material science, image processing, and biological modeling. Can also refer to Refs. [32,34,46] and the references therein.

In this paper, we focus on the study of the following anisotropic singular problem:

$$\begin{cases} -\sum_{i=1}^{N}\partial_i\left(|\partial_i u|^{p_i-2}\partial_i u\right) - \sum_{i=1}^{N}\partial_i\left(|\partial_i u|^{q_i-2}\partial_i u\right) = \dfrac{f(x)}{u^{\gamma(x)}} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with a C^2 boundary $\partial\Omega$ and $N \geq 3$. The exponent $\gamma(x)$ is a positive continuous function on $\overline{\Omega}$, and $f(x) \in L^r(\Omega)$ for some $r \geq 1$.

This study is inspired by the significant work of Giacomoni, Kumar, and Sreenadh [23], who investigate a singular quasilinear elliptic equation involving the (p,q)-Laplacian operator. In their work, the equation under consideration is

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) - \operatorname{div}(|\nabla u|^{q-2}\nabla u) = \frac{f(x)}{u^{\delta}} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $1 < q < p < \infty$, $\delta > 0$, and $f(x)$ is a nonnegative function that exhibits singular behavior near the boundary of Ω. Their paper proves the existence of weak solutions in appropriate Sobolev spaces and establishes the solutions' boundary behavior for different values of δ. Specifically, they derive optimal Sobolev regularity, prove uniqueness under certain conditions, and provide non-existence results in other regimes. Moreover, they obtain Hölder regularity estimates for the gradient of weak solutions, offering significant insights into equations with singular nonlinearities. For further details, the reader is referred to [12,14] and related references. Arruda and Nascimento [1] generalize the results of Giacomoni, Kumar, and Sreenadh by considering a broader class of quasilinear operators beyond the standard p-Laplacian. They study a class of nonlinear, nonhomogeneous singular elliptic equations involving the (p,q)-Laplacian operator given by

$$\begin{cases} -\operatorname{div}(a(|\nabla u|^p)|\nabla u|^{p-2}\nabla u) = \frac{h(x)}{u^{\delta}} + f(x,u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where Ω is a bounded domain in $\mathbb{R}^N$ (with $N \geq 3$), $2 \leq p < N$, and $\delta > 0$. By using the sub–supersolution method combined with variational techniques, they establish the existence of at least one weak solution and, under additional assumptions on f, prove the existence of two distinct solutions. Key contributions include addressing nonhomogeneous

operators without truncation techniques and accommodating general singular terms for all $\gamma > 0$. These results apply to a broad class of quasilinear operators, including the p-Laplacian (when $a(t) \equiv 1$) and the (p,q)-Laplacian (when $a(t) = 1 + t^{q-p}/p$). Interested readers are referred to [37] and [45] for additional details.

Our work extends the (p,q)-Laplacian operators studied in (1.2) and (1.3) by introducing two anisotropic operators with distinct vector exponents $\{p_i\}_{i=1}^N$ and $\{q_i\}_{i=1}^N$. This generalization allows for direction-dependent nonlinearities, unlike the isotropic (p,q)-Laplacian. Additionally, the right-hand side of our problem (1.1) features a variable exponent $\gamma(x)$ in the singular term $\frac{f(x)}{u^{\gamma(x)}}$, which introduces spatial heterogeneity in the singularity strength. Moreover, we establish L^∞-regularity for solutions to (1.1), proving that $u \in L^\infty(\Omega)$ despite the anisotropic operators and the singular term.

The singular (q,p)-Laplacian denotes a class of singular differential equations that involve both the p-Laplacian and the q-Laplacian operators. These problems frequently arise in the study of various physical and mathematical phenomena, serving as models for steady-state solutions of reaction-diffusion problems encountered in biophysics, plasma physics, and chemical reaction studies (see, for example, [5, 7]). In the context of (p,q)-Laplacian-driven reaction-diffusion equations, the (p,q)-Laplacian term represents the diffusion component of the equation, which is given by:

$$|\partial_i u|^{p-2}|\partial_i u| + |\partial_i u|^{q-2}|\partial_i u| \quad \text{in } \Omega,$$

where $\partial_i u$ denotes the partial derivative of u with respect to x_i, $2 \le q < p$. The terms $|\partial_i u|^{p-2}|\partial_i u|$ and $|\partial_i u|^{q-2}|\partial_i u|$ represent the diffusion terms associated with the (p,q)-Laplacian operator.

Researchers have explored various aspects of the (p,q)-Laplacian, including the existence and uniqueness of solutions, regularity properties, and the behavior of solutions under different conditions. Recent results in this area can be found in [2, 12, 31, 35, 36, 38]. Our research builds on these findings by focusing on the properties of solutions in anisotropic singular problems.

The study of elliptic equations with singular nonlinearities has garnered significant attention from researchers. A pioneering contribution was made in 1976 by Crandall, Rabinowitz, and Tartar [11], where the authors explored nonlinear elliptic boundary value problems characterized by singular terms. Their work established the existence of both classical and generalized solutions under various conditions for problems associated with $-\Delta$ and Dirichlet boundary conditions. Specifically, when $p_i = q_i = 2$, the problem (1.1) reduces to the well-known semilinear elliptic equation that has been extensively studied:

$$\begin{cases} -\operatorname{div}(\nabla u) = \frac{f(x)}{u^{\gamma(x)}} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

with f is a non-negative function in a suitable Lebesgue space. In 1991, Lazer and McKenna published an important paper [28], focusing on the case where f is continuous and $\gamma(x) = \gamma$. They proved existence and regularity results concerning the behavior of solutions near the boundary. Boccardo and Orsina later extended this work in [3], where they used approximation methods by truncating the singular term to prove existence, regularity, and nonexistence results for problems modeled by (1.4) with $\gamma(x) = \gamma$. These results were further generalized by Chu, Y. Gao, and W. Gao [8] as well as by Carmona and Martínez-Aparicio [6], who studied

the case where $\gamma(x)$ varies as a function of x. Their research examined how the summability of f and the values of $\gamma(x)$ affect the existence and regularity of solutions to problem (1.4).

For general p, the following p-Laplacian problem has been widely studied by numerous researchers:

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \frac{f(x)}{u^{\gamma(x)}} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

where f is a positive function and $p > 1$. For example, in the paper [9], Chu and Gao applied the approximation method to analyze this problem involving the p-Laplacian operator with a constant exponent $\gamma(x) = \gamma$. Subsequently, Chu, Gao, and Sun extended these results in [10], where considering cases $\gamma(x)$ varies as a function of x. Their research explored various scenarios based on the behavior of $\gamma(x)$ and f, demonstrating that solutions to problem (1.5) exist in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

There has been increasing interest in singular problems involving anisotropic operators, which correspond to a specific case of our problem (1.1) when $p_i = q_i$. For instance, in [30], the following problem was studied:

$$\begin{cases} -\sum_{i=1}^{N} \partial_i \left(|\partial_i u|^{p_i-2}\partial_i u \right) = \frac{f(x)}{u^{\gamma(x)}} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.6}$$

where $\gamma(x) = \gamma$ is a positive constant and f belongs to an appropriate Lebesgue space. Using perturbation techniques, Leggat and Miri demonstrated the existence of positive solutions for problem (1.6). Additionally, Miri extended this analysis in [33], considering $\gamma(x) > 0$ as a variable function and f as a positive function in $L^m(\Omega)$. They established the existence and regularity of solutions in $W_0^{1,\vec{p}}(\Omega)$ when $\gamma(x) \leq 1$ and $m = \bar{p}^{*\prime}$, and in $W_{loc}^{1,\vec{p}}(\Omega)$ when $\|\gamma\|_{L^\infty(\Omega)} \leq \gamma^*$, with $\gamma^* > 1$.

It is important to note that the results in [30] and [33] are largely based on the strong maximum principle. Additional results on singular anisotropic problems, particularly using the sub-supersolution method, can be found in works like [15] and [16].

In our previous work [25], we extended existing results on similar problems (1.6) by incorporating variable exponent singularities and critical growth within an anisotropic framework. The problem is formulated as follows:

$$\begin{cases} -\sum_{i=1}^{N} M\left(\int_\Omega |\partial_i u|^{p_i} \, dx \right) \partial_i \left(|\partial_i u|^{p_i-2}\partial_i u \right) = \frac{f_1(x)}{u^{\beta(x)}} + \lambda \, f_2(x) \, u^{\bar{p}^*-1} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Here, Ω is a bounded regular domain in $\mathbb{R}^N$ with $N > \bar{p}$. The critical Sobolev exponent $\bar{p}^*$ is given by $\bar{p}^* = \frac{N\bar{p}}{N-\bar{p}}$, where $\bar{p} = \frac{N}{\sum_{i=1}^{N} \frac{1}{p_i}}$. The parameter $\lambda > 0$ is a positive constant, while the exponent function satisfies $0 < \beta(x) < 1$. The functions f_1 and f_2 have specific properties, and M represents the Kirchhoff coefficient.

Our main result establishes the existence of at least two weak solutions with opposite energy signs. The methodology is based on the fibering method using the Nehari manifold.

While our variational framework shares similarities with previous works, this study extends the analysis by considering a double anisotropic operator, singular reaction terms and regularity analysis. To handle these challenges, we employ techniques such as truncation methods and weighted Sobolev embeddings. In addition to establishing existence, we discuss various cases of the function $\gamma(x)$ to determine the corresponding solution in each scenario. Under further assumptions on the function f, it is proven that the obtained solutions belong to $L^\infty(\Omega)$, ensuring their boundedness and physical relevance. The incorporation of a double anisotropic operator enriches the analysis, providing deeper insights into the behavior of solutions in complex settings.

The techniques developed in this work not only generalize classical results but also pave the way for future research in problems involving variable exponent spaces, nonlocal operators, and anisotropic phenomena. The rigorous approach presented here contributes to the broader understanding of non-standard growth conditions in partial differential equations.

There is a substantial amount of literature and increasing interest in anisotropic (p,q)-Laplacian problems. Noteworthy recent contributions can be found in works such as [29,39, 40,43]. For example, in [39], Razani and Figueiredo studied the following problem:

$$\begin{cases} -\sum_{i=1}^{N} \partial_i \left(|\partial_i u|^{p_i-2} \partial_i u \right) - \sum_{i=1}^{N} \partial_i \left(|\partial_i u|^{q_i-2} \partial_i u \right) = \mathfrak{H}(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.7}$$

where $\mathfrak{H}(x,u) = \lambda u^{\gamma-1}$ with $\gamma > 1$. By applying a sub-supersolution approach along with minimization techniques in convex sets, the authors proved the existence of positive solutions for problem (1.7). A similar approach was used in [43], where Tavares considered problem (1.7) with $\mathfrak{H}(x,u) = k(x)u^{\alpha-1} + f(x,u)$, where $\alpha > 1$. Under a broad set of conditions, the existence and multiplicity of solutions were demonstrated. Additionally, Leggat and Miri [29], used the variational method to prove multiplicity results for the case where $\mathfrak{H}(x,u) = \lambda f(u)$.

We now introduce the key notations and assumptions relevant to problem (1.1). Specifically, we define the following parameters

$$\vec{p} = (p_1, \dots, p_N), \quad \bar{p} := \frac{N}{\sum_{i=1}^{N} 1/p_i}, \quad \bar{p}^* := \frac{N\bar{p}}{N-\bar{p}}, \quad p_\infty := \max\{\bar{p}^*, p_N\},$$
$$\vec{q} = (q_1, \dots, q_N), \quad \bar{q} := \frac{N}{\sum_{i=1}^{N} 1/q_i}, \quad \bar{q}^* := \frac{N\bar{q}}{N-\bar{q}}, \quad q_\infty := \max\{\bar{q}^*, q_N\}.$$

Throughout this paper, we impose the following assumptions

$$2 \le p_1 \le p_2 \le \cdots \le p_N < \bar{p}^*, \quad \text{and} \quad \sum_{i=1}^{N} \frac{1}{p_i} > 1,$$
$$2 \le q_1 \le q_2 \le \cdots \le q_N < \bar{q}^*, \quad \text{and} \quad \sum_{i=1}^{N} \frac{1}{q_i} > 1,$$
$$\gamma^+ = \sup_{x\in\overline{\Omega}} \gamma(x), \quad \text{and} \quad \gamma^- = \inf_{x\in\overline{\Omega}} \gamma(x).$$

We denote by X the anisotropic Sobolev space associated with problem (1.1), defined as

$$X := W_0^{1,\vec{p}}(\Omega) \cap W_0^{1,\vec{q}}(\Omega),$$

where $\Omega \subseteq \mathbb{R}^N$ is a bounded and regular domain with $N \geq 3$. Furthermore, the space X is given by the closure

$$X = \overline{C_0^\infty(\Omega)}^{\|\cdot\|_X},$$

where the norm $\|\cdot\|_X$ is defined as

$$\|u\|_X := \|u\|_{W_0^{1,\vec{p}}(\Omega)} + \|u\|_{W_0^{1,\vec{q}}(\Omega)} = \sum_{i=1}^N \|\partial_i u\|_{L^{p_i}(\Omega)} + \sum_{i=1}^N \|\partial_i u\|_{L^{q_i}(\Omega)}.$$

Moreover, $(X, \|\cdot\|_X)$ is a uniformly convex Banach space, and consequently, it is reflexive. For further details, see [19].

Additionally, we assume that f is a nontrivial measurable function satisfying the condition

$$(H_f) \qquad \operatorname*{ess\,inf}_{\Omega} f(x) > 0.$$

The primary objective of this article is to investigate the existence and regularity of positive weak solutions for the singular problem (1.1) in the space X by considering various cases of the function $\gamma(x)$. Our strategy is to establish the existence of critical points for the energy functional

$$\Psi(u) = \sum_{i=1}^N \frac{1}{p_i} \int_\Omega |\partial_i u|^{p_i}\, dx + \sum_{i=1}^N \frac{1}{q_i} \int_\Omega |\partial_i u|^{q_i}\, dx - \int_\Omega f|u|^{1-\gamma(x)}\, dx.$$

The singular term $u^{-\gamma(x)}$ introduces significant challenges by causing a loss of Gateaux differentiability, despite Ψ being weakly lower semi-continuous (in fact, continuous). With this framework in place, we are now ready to rigorously define the notion of a solution for the singular problem.

Definition 1.1. A function $u \in X$ is called a weak solution of (1.1) if it satisfies the following conditions:

- **(i)** $u > 0$ a.e. in Ω, $fu^{-\gamma(x)} \in L^1_{loc}(\Omega)$, and
- **(ii)** $\sum_{i=1}^N \int_\Omega |\partial_i u|^{p_i-2} \partial_i u \partial_i \phi dx + \sum_{i=1}^N \int_\Omega |\partial_i u|^{q_i-2} \partial_i u \partial_i \phi dx = \int_\Omega \frac{f\phi}{u^{\gamma(x)}} dx,$

  for all $\phi \in C_0^1(\Omega)$.

The principal theorems established in this work are as follows:

Theorem 1.2. *Suppose that f is a positive function in $L^1(\Omega)$, and (H_f) holds, with $0 < \gamma^- \leq \gamma(x) \leq \gamma^+ < 1$. Then, the problem (1.1) has a solution in X.*

Theorem 1.3. *Let f be a positive function in $L^1(\Omega)$ and assume that condition (H_f) holds, with $0 < \gamma^- < 1 < \gamma^+$. Then problem (1.1) admits a solution in the space $X_{loc} := W^{1,\vec{p}}_{loc}(\Omega) \cap W^{1,\vec{q}}_{loc}(\Omega)$. Moreover, the solution u belongs to $L^r(\Omega)$, where $r = \frac{N(\gamma^+ - 1 + \overline{m})}{(N - \overline{m})}$, with $\overline{m} = \overline{p}, \overline{q}$.*

Theorem 1.4. *Assume that f is a positive function in $L^1(\Omega)$ and condition (H_f) is satisfied, with $1 < \gamma^- \leq \gamma(x) \leq \gamma^+$. Then, problem (1.1) admits a solution in the space X_{loc}.*

Theorem 1.5. *Assume that (H_f) holds and $f \in L^r(\Omega)$ with $r \geq \frac{\min\{\bar{p}^*, \bar{q}^*\}}{\min\{\bar{p}^*, \bar{q}^*\} - \max\{\bar{p}, \bar{q}\}}$, and let $\gamma(x) > 0$. Then, the solution u of problem (1.1), as established by Theorems 1.2, 1.3, and 1.4, belongs to $L^\infty(\Omega)$.*

This paper is organized as follows. In Section 2 establishes several technical results that are essential for proving our main theorem. We begin by presenting a regularity result for the solution of the anisotropic (p,q)-Laplacian problem. Next, we introduce a comparison principle, and finally, we demonstrate that the solution remains strictly greater than a positive term. Section 3 applies the variational method and leverages the auxiliary results from Section 2 to prove that the approximate problem admits a solution in $X \cap L^{\infty}(\Omega)$. Section 4 is dedicated to the proof of the main results, while Section 5 explores possible generalizations of our existence result and discusses future research perspectives.

2 Preliminaries and technical results

This section is dedicated to establishing fundamental results essential for proving our main theorems. We begin by addressing the uniqueness and L^{∞}-regularity of solutions to the anisotropic (p,q)-Laplacian problem under consideration. Next, we introduce a comparison principle and conclude with a boundary behavior lemma, which ensures that a solution u remains bounded below by a positive quantity proportional to the distance to the boundary. Together, these results provide crucial insights into how the anisotropic structure influences the solution's regularity and its interaction with the domain's geometry. We start by examining the following anisotropic (p,q)-Laplacian problem

$$\begin{cases} -\sum_{i=1}^{N} \partial_i \left(|\partial_i u|^{p_i-2} \partial_i u \right) - \sum_{i=1}^{N} \partial_i \left(|\partial_i u|^{q_i-2} \partial_i u \right) = h(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

Lemma 2.1 ([43]). *Let $h \in X'$. Then, the problem (2.1) has a unique solution in X.*

The proof follows from Minty–Browder's Theorem, as stated in Lemma 3.3 of [43].

Lemma 2.2. *Let $h \in L^r(\Omega)$ with $r > \frac{\max\{\bar{p}^*,\bar{q}^*\}}{\max\{\bar{p}^*,\bar{q}^*\}-\max\{p_N,q_N\}}$. The solution u to the problem (2.1) belongs to $L^{\infty}(\Omega)$.*

Proof. Let $\Omega_k := \{x \in \Omega : |u(x)| > k\}$ for $k > 0$. Define the truncated test function

$$\xi_k := \operatorname{sign}(u) \cdot (|u| - k)^+,$$

where $(\cdot)^+$ denotes the positive part. Let $\beta := \max\{\overline{p}^*, \overline{q}^*\}$, where $\overline{p}^*$ and $\overline{q}^*$ are the critical Sobolev exponents associated with the anisotropic exponents $\{p_i\}$ and $\{q_i\}$, respectively. Observe that $\xi_k \in X$ and $\partial_i \xi_k = \partial_i u$ in Ω_k, while $\partial_i \xi_k = 0$ outside Ω_k. Let $|\Omega_k|$ denote the Lebesgue measure of Ω_k.

Using ξ_k as a test function in the weak formulation of problem (2.1) and applying Hölder's inequality, we can derive the following results

$$\begin{aligned} \sum_{i=1}^{N} \int_{\Omega_k} |\partial_i \xi_k|^{p_i} \, dx + \sum_{i=1}^{N} \int_{\Omega_k} |\partial_i \xi_k|^{q_i} \, dx &= \int_{\Omega} h \xi_k dx \\ &\leq \left(\int_{\Omega} |\xi_k|^{\beta} \, dx \right)^{\frac{1}{\beta}} \left(\int_{\Omega} |h|^r dx \right)^{\frac{1}{r}} |\Omega_k|^{1-\frac{1}{\beta}-\frac{1}{r}}. \end{aligned} \tag{2.2}$$

Define the energy minimization problem

$$\mathfrak{B} := \inf_{\substack{u\in D^{1,\vec{p}}(\mathbb{R}^N)\cap D^{1,\vec{q}}(\mathbb{R}^N)\\ \|u\|_{L^\beta(\mathbb{R}^N)}=1}} \sum_{i=1}^{N}\frac{1}{p_i}\|\partial_i u\|^{p_i}_{L^{p_i}(\mathbb{R}^N)} + \sum_{i=1}^{N}\frac{1}{q_i}\|\partial_i u\|^{q_i}_{L^{q_i}(\mathbb{R}^N)}, \tag{2.3}$$

where

$$\mathcal{D}^{1,\vec{p}}\left(\mathbb{R}^N\right) := \left\{u \in L^{\bar{p}^*}\left(\mathbb{R}^N\right) : |\partial_i u| \in L^{p_i}\left(\mathbb{R}^N\right)\right\}.$$

Form Lemma 3 in [18], if $\beta = \bar{p}^*$, we have

$$\mathfrak{B} \geq \inf_{\substack{u\in D^{1,\vec{p}}(\mathbb{R}^N)\\ \|u\|_{L^{\bar{p}^*}(\mathbb{R}^N)}=1}} \sum_{i=1}^{N}\frac{1}{p_i}\|\partial_i u\|^{p_i}_{L^{p_i}(\mathbb{R}^N)} > 0.$$

Similarly, if $\beta = \bar{q}^*$, the same argument yields $\mathfrak{B} > 0$.

By direct computation, we deduce that for every $u \in X$

$$\begin{aligned}
&\sum_{i=1}^{N}\int_\Omega |\partial_i u|^{p_i}\,dx + \sum_{i=1}^{N}\int_\Omega |\partial_i u|^{q_i}\,dx \\
&\quad\geq \|u\|^\theta_{L^\beta(\Omega)}\sum_{i=1}^{N}\int_\Omega \left|\partial_i\left(\frac{u}{\|u\|_{L^\beta(\Omega)}}\right)\right|^{p_i}dx + \|u\|^\theta_{L^\beta(\Omega)}\sum_{i=1}^{N}\int_\Omega \left|\partial_i\left(\frac{u}{\|u\|_{L^\beta(\Omega)}}\right)\right|^{q_i}dx \\
&\quad\geq \|u\|^\theta_{L^\beta(\Omega)}\left[\sum_{i=1}^{N}\frac{1}{p_i}\int_\Omega \left|\partial_i\left(\frac{u}{\|u\|_{L^\beta(\Omega)}}\right)\right|^{p_i}dx + \sum_{i=1}^{N}\frac{1}{q_i}\int_\Omega \left|\partial_i\left(\frac{u}{\|u\|_{L^\beta(\Omega)}}\right)\right|^{q_i}dx\right],
\end{aligned}$$

where

$$\theta = \begin{cases} \min\{p_1, q_1\} & \text{if } \|u\|_{L^\beta(\Omega)} > 1, \\ \max\{p_N, q_N\} & \text{if } \|u\|_{L^\beta(\Omega)} \leq 1. \end{cases}$$

Furthermore, from (2.3), we derive

$$\sum_{i=1}^{N}\int_\Omega |\partial_i u|^{p_i}\,dx + \sum_{i=1}^{N}\int_\Omega |\partial_i u|^{q_i}\,dx \geq \mathfrak{B}\times\|u\|^\theta_{L^\beta(\Omega)}. \tag{2.4}$$

Combining (2.2) and (2.4), we obtain

$$\mathfrak{B}\left(\int_{\Omega_k}|\xi_k|^\beta\,dx\right)^{\frac{\theta-1}{\beta}} \leq \|h\|_{L^r(\Omega)}|\Omega_k|^{1-\frac{1}{\beta}-\frac{1}{r}}. \tag{2.5}$$

Additionally, if $0 < k < \ell$, we have $\Omega_\ell \subset \Omega_k$, and

$$|\Omega_\ell|^{\frac{1}{\beta}}(\ell-k) = \left(\int_{\Omega_\ell}(\ell-k)^\beta dx\right)^{\frac{1}{\beta}} \leq \left(\int_{\Omega_k}|\xi_k|^\beta\,dx\right)^{\frac{1}{\beta}}.$$

Given that $\theta \geq 2$, we conclude

$$\mathfrak{B}|\Omega_\ell|^{\frac{\theta-1}{\beta}}(\ell-k)^{\theta-1} \leq \mathfrak{B}\left(\int_{\Omega_k}|\xi_k|^\beta\,dx\right)^{\frac{\theta-1}{\beta}}, \tag{2.6}$$

and by combining (2.5) and (2.6), we infer

$$|\Omega_\ell| \leq \frac{1}{(\ell-k)^\beta \mathfrak{B}^{\frac{\beta}{\theta-1}}} \|h\|_{L^r(\Omega)}^{\frac{\beta}{\theta-1}} |\Omega_k|^{\frac{\beta}{\theta-1}\left(1-\frac{1}{\beta}-\frac{1}{r}\right)}.$$

Since $r > \frac{\max\{\bar{p}^*,\bar{q}^*\}}{\max\{\bar{p}^*,\bar{q}^*\}-\max\{p_N,q_N\}} \geq \frac{\beta}{\beta-\theta}$, we deduce that $\frac{\beta}{\theta-1}\left(1-\frac{1}{\beta}-\frac{1}{r}\right) > 1$. Thus, defining

$$\begin{cases} \psi(h) = |\Omega_\ell|, \\ \sigma = \frac{\beta}{\theta-1}\left(1-\frac{1}{\beta}-\frac{1}{r}\right), \\ k_0 = 0, \end{cases}$$

we see that ψ is a positive, non-increasing function. Moreover,

$$\psi(\ell) \leq \frac{C}{(\ell-k)^\beta}\psi(k)^\beta, \qquad \text{for all } \ell > k > 0.$$

Using ((i), Lemma 4.1, in [42]), we conclude that $\psi(d) = 0$ with $d^\beta = \frac{C\|h\|_{L^r(\Omega)}^{\frac{\beta}{\theta-1}} |\Omega|^{\sigma-1}}{\mathfrak{B}^{\frac{\beta}{\theta-1}}}$. Thus, we obtain

$$\|u\|_{L^\infty(\Omega)} \leq \frac{C\|h\|_{L^r(\Omega)}^{\frac{1}{\theta-1}} |\Omega|^{\frac{\sigma-1}{\beta}}}{\mathfrak{B}^{\frac{1}{\theta-1}}}.$$

Therefore, we have established that $u \in L^\infty(\Omega)$. □

Lemma 2.3 ([43]). *Consider $u, v \in X$ satisfying the following in the weak sense*

$$\begin{cases} -\sum_{i=1}^N \partial_i\left(|\partial_i u|^{p_i-2}\partial_i u\right) - \sum_{i=1}^N \partial_i\left(|\partial_i u|^{q_i-2}\partial_i u\right) \\ \qquad \leq -\sum_{i=1}^N \partial_i\left(|\partial_i v|^{p_i-2}\partial_i v\right) - \sum_{i=1}^N \partial_i\left(|\partial_i v|^{q_i-2}\partial_i v\right) \quad \text{in } \Omega, \\ (u-v)^+ \in X. \end{cases}$$

Then, it follows that $u \leq v$ almost everywhere in Ω.

Regarding the proof, we refer the reader to Lemma 3.5 in [43].

Lemma 2.4. *Let μ be a positive constant and $u \in X$ be the unique solution to the following problem*

$$\begin{cases} -\sum_{i=1}^N \partial_i\left(|\partial_i u|^{p_i-2}\partial_i u\right) - \sum_{i=1}^N \partial_i\left(|\partial_i u|^{q_i-2}\partial_i u\right) = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.7}$$

Then, there exist constants $\eta \in (0,1]$ and M, which are independent of μ and u, such that

$$u(x) \geq M \min\left\{\mu^{\max\left\{\frac{1}{p_1-1},\frac{1}{q_1-1}\right\}}, \mu^{\min\left\{\frac{1}{p_N-1},\frac{1}{q_N-1}\right\}}\right\} \min\{\eta, d(x)\},$$

with $d(x) := \operatorname{dist}(x,\partial\Omega), x \in \Omega$.

Proof. Let $\rho > 0$ define the set

$$A_\rho := \{x \in \Omega \mid d(x) < \rho\},$$

where $d(x) = \operatorname{dist}(x, \partial\Omega)$ denotes the distance from x to the boundary $\partial\Omega$. Since the boundary $\partial\Omega$ is of class C^2, there exists a sufficiently small constant $\eta \in (0, 1]$ such that

$$d \in C^2\left(\overline{A_{3\eta}}\right) \quad \text{and} \quad |\nabla d(x)| \equiv 1,$$

as stated in Lemma 14.16 of Gilbarg and Trudinger [24].

For

$$\beta > \frac{1}{\min\{p_1, q_1\} - 1},$$

we define the function

$$\varsigma(x) = \begin{cases} \kappa d(x), & \text{if } d(x) < \eta, \\ \kappa\eta + \displaystyle\int_\eta^{d(x)} \kappa \left(\frac{2\eta - t}{\eta}\right)^\beta dt, & \text{if } \eta \le d(x) < 2\eta, \\ \kappa\eta + \displaystyle\int_\eta^{2\eta} \kappa \left(\frac{2\eta - t}{\eta}\right)^\beta dt, & \text{if } d(x) \ge 2\eta, \end{cases}$$

where $\kappa > 0$ is a constant to be determined later. It is evident that $\varsigma \in C_0^1(\overline{\Omega})$.

According to direct calculations, if $x \in \Omega$ satisfies $d(x) < \eta$ and $\partial_i d(x) \neq 0$, then we have

$$\begin{aligned} &-\sum_{i=1}^N \left[\partial_i \left(|\partial_i(\kappa d)|^{p_i - 2}\, \partial_i(\kappa d)\right) + \partial_i \left(|\partial_i(\kappa d)|^{q_i - 2}\, \partial_i(\kappa d)\right)\right] \\ &\quad = -\sum_{i=1}^N \left[\left(\operatorname{sgn}\left(\partial_i d\right)\right) \kappa^{p_i - 1} \partial_i \left(\operatorname{sgn}\left(\partial_i d\right)^{p_i - 1} \left(\partial_i d\right)^{p_i - 1}\right)\right] \\ &\qquad -\sum_{i=1}^N \left[\left(\operatorname{sgn}\left(\partial_i d\right)\right) \kappa^{q_i - 1} \partial_i \left(\operatorname{sgn}\left(\partial_i d\right)^{q_i - 1} \left(\partial_i d\right)^{q_i - 1}\right)\right] \\ &\quad = -\sum_{i=1}^N \left[\kappa^{p_i - 1} \left(p_i - 1\right) \left(\left(\operatorname{sgn}\left(\partial_i d\right)\right) \partial_i d\right)^{p_i - 2} \partial_i^2 d\right] \\ &\qquad -\sum_{i=1}^N \left[\kappa^{q_i - 1} \left(q_i - 1\right) \left(\left(\operatorname{sgn}\left(\partial_i d\right)\right) \partial_i d\right)^{q_i - 2} \partial_i^2 d\right] \\ &\quad =: \Upsilon_1(x), \end{aligned}$$

where $\operatorname{sgn}(x) = 1$ if $x \ge 0$ and $\operatorname{sgn}(x) = -1$ if $x < 0$. In the case where $\eta < d(x) < 2\eta$ and

$\partial_i d(x) \neq 0$, we find that

$$
\begin{aligned}
&-\sum_{i=1}^{N}\partial_i\left(\left|\partial_i\left(\kappa\eta+\int_{\eta}^{d(x)}\kappa\left(\frac{2\eta-t}{\eta}\right)^{\beta}dt\right)\right|^{p_i-2}\partial_i\left(\kappa\eta+\int_{\eta}^{d(x)}\kappa\left(\frac{2\eta-t}{\eta}\right)^{\beta}dt\right)\right)\\
&\quad-\sum_{i=1}^{N}\partial_i\left(\left|\partial_i\left(\kappa\eta+\int_{\eta}^{d(x)}\kappa\left(\frac{2\eta-t}{\eta}\right)^{\beta}dt\right)\right|^{q_i-2}\partial_i\left(\kappa\eta+\int_{\eta}^{d(x)}\kappa\left(\frac{2\eta-t}{\eta}\right)^{\beta}dt\right)\right)\\
&=-\sum_{i=1}^{N}\partial_i\left(\left|\partial_i(d(x))\kappa\left(\frac{2\eta-d(x)}{\eta}\right)^{\beta}\right|^{p_i-2}\partial_i(d(x))\kappa\left(\frac{2\eta-d(x)}{\eta}\right)^{\beta}\right)\\
&\quad-\sum_{i=1}^{N}\partial_i\left(\left|\partial_i(d(x))\kappa\left(\frac{2\eta-d(x)}{\eta}\right)^{\beta}\right|^{q_i-2}\partial_i(d(x))\kappa\left(\frac{2\eta-d(x)}{\eta}\right)^{\beta}\right)\\
&=-\sum_{i=1}^{N}\kappa^{p_i-1}\beta\left(p_i-1\right)\left(\frac{2\eta-d(x)}{\eta}\right)^{\beta(p_i-1)-1}\left(\frac{-1}{\eta}\right)\left(\left(\operatorname{sgn}\left(\partial_i d\right)\right)\partial_i d\right)^{p_i-2}\left(\partial_i d\right)^2\\
&\quad-\sum_{i=1}^{N}\kappa^{p_i-1}\left(\frac{2\eta-d(x)}{\eta}\right)^{\beta(p_i-1)}\left(p_i-1\right)\left|\partial_i d\right|^{p_i-2}\partial_i^2(d(x))\\
&\quad-\sum_{i=1}^{N}\kappa^{q_i-1}\beta\left(q_i-1\right)\left(\frac{2\eta-d(x)}{\eta}\right)^{\beta(q_i-1)-1}\left(\frac{-1}{\eta}\right)\left(\left(\operatorname{sgn}\left(\partial_i d\right)\right)\partial_i d\right)^{q_i-2}\left(\partial_i d\right)^2\\
&\quad-\sum_{i=1}^{N}\kappa^{q_i-1}\left(\frac{2\eta-d(x)}{\eta}\right)^{\beta(q_i-1)}\left(q_i-1\right)\left|\partial_i d\right|^{q_i-2}\partial_i^2(d(x))\\
&=:\mathrm{Y}_2(x).
\end{aligned}
$$

According to the previous cases, we have

$$
\sum_{i=1}^{N}\int_{\Omega}\left|\partial_i(\varsigma)\right|^{p_i-2}\partial_i(\varsigma)\partial_i\phi dx+\sum_{i=1}^{N}\int_{\Omega}\left|\partial_i(\varsigma)\right|^{q_i-2}\partial_i(\varsigma)\partial_i\phi dx=\int_{\Omega}\mathrm{Y}\phi dx\quad\forall\phi\in X,
$$

where

$$
\mathrm{Y}(x)=\begin{cases}\mathrm{Y}_1(x) & \text{if } d(x)<\eta,\ \partial_i d(x)\neq 0,\\ \mathrm{Y}_2(x) & \text{if } \eta<d(x)<2\eta,\ \partial_i d(x)\neq 0,\\ 0 & \text{if } d(x)>2\eta \text{ or } \partial_i d(x)=0.\end{cases}
$$

Given that $\beta>1/(\min\{p_1,q_1\}-1)$ and $p_i,q_i\geq 2$ for all $i=1,\ldots,N$, we conclude that

$$
\begin{aligned}
&\sum_{i=1}^{N}\int_{\Omega}\left|\partial_i(\varsigma)\right|^{p_i-2}\partial_i(\varsigma)\partial_i\phi dx+\sum_{i=1}^{N}\int_{\Omega}\left|\partial_i(\varsigma)\right|^{q_i-2}\partial_i(\varsigma)\partial_i\phi dx\\
&\qquad\leq M_0\max\left\{\kappa^{p_1-1},\ldots,\kappa^{p_N-1}\right\}+M_1\max\left\{\kappa^{q_1-1},\ldots,\kappa^{q_N-1}\right\}\\
&\qquad\leq M\max\left\{\kappa^{p_1-1},\ldots,\kappa^{p_N-1},\kappa^{q_1-1},\ldots,\kappa^{q_N-1}\right\},
\end{aligned}
$$

in the weak sense, where M_0 and M_1 are the constants from Lemma 2.4 and $M=2\max\{M_0,M_1\}$.

Let u be the solution to problem (2.7) and $\mu>0$. Now we distinguish two cases:
In the first case, if $\frac{\mu}{M}<1$, we choose $\kappa\in(0,1)$ such that

$$
\max\left\{\kappa^{p_1-1},\ldots,\kappa^{p_N-1},\kappa^{q_1-1},\ldots,\kappa^{q_N-1}\right\}=\kappa^{\min\{p_1-1,q_1-1\}}=\frac{\mu}{M}.
$$

Applying Lemma 2.3 yields

$$u(x) \geq \varsigma(x) \geq \kappa \min\{\eta, d(x)\} = \left(\frac{\mu}{M}\right)^{\max\{\frac{1}{p_1^{-1}}, \frac{1}{q_1^{-1}}\}} \min\{\eta, d(x)\}.$$

In the second case, when $\frac{\mu}{M} \geq 1$ we can employ the same reasoning as in the first case to obtain

$$u(x) \geq \varsigma(x) \geq \kappa \min\{\eta, d(x)\} = \left(\frac{\mu}{M}\right)^{\min\{\frac{1}{p_N^{-1}}, \frac{1}{q_N^{-1}}\}} \min\{\eta, d(x)\}.$$

Consequently, we arrive at

$$u(x) \geq \min\left\{\left(\frac{\mu}{M}\right)^{\max\{\frac{1}{p_1^{-1}}, \frac{1}{q_1^{-1}}\}}, \left(\frac{\mu}{M}\right)^{\min\{\frac{1}{p_N^{-1}}, \frac{1}{q_N^{-1}}\}}\right\} \min\{\eta, d(x)\}.$$

Thus, the statement is proven. □

3 Approximation problems

To formulate a minimization problem, the energy functional Ψ must be bounded from below in X, which, unfortunately, is not the case here. Additionally, the presence of a singular term introduces non-differentiability in the energy functional, creating an additional challenge in defining the natural constrained set. As a result, we cannot directly apply the standard critical point theory to Ψ to solve problem (1.1). To overcome this difficulty and obtain our result, we study an associated approximating problem. Specifically, for any $n \in \mathbb{N}^*$, we consider the following perturbed problem

$$\begin{cases} -\sum_{i=1}^{N} \partial_i \left(|\partial_i u_n|^{p_i - 2} \partial_i u_n\right) - \sum_{i=1}^{N} \partial_i \left(|\partial_i u_n|^{q_i - 2} \partial_i u_n\right) = \dfrac{f_n}{\left(u_n + \frac{1}{n}\right)^{\gamma(x)}} & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{$\mathcal{P}_n$}$$

where $f_n(x) = \min\{f(x)\ ,\ n\}$.

Lemma 3.1. *Suppose that $\gamma \in C^1(\overline{\Omega})$. Then the problem ($\mathcal{P}_n$) has a non-negative solution $u_n \in X \cap L^\infty(\Omega)$.*

Proof. While many studies rely on Schauder's fixed point theorem to establish the existence of solutions, this paper adopts a variational approach. We define the functions
$f : \Omega \times (0, \infty) \longrightarrow \mathbb{R}$ and $f_n : \Omega \times [0, \infty) \longrightarrow \mathbb{R}$ by

$$f(x,t) = \frac{f(x)}{t^{\gamma(x)}}, \quad \text{and} \quad f_n(x,t) = \frac{f_n(x)}{(t + 1/n)^{\gamma(x)}}, \qquad \text{for all } n > 0.$$

It is important to note that f_n is a Carathéodory function, and its primitive is given by the function F_n, where

$$F_n(x,t) = \int_0^t f_n(x,s)ds.$$

Remark 3.2. For $t \geq 0$, the following inequalities hold

$$0 < f_n(x,t) \leq f_n(x) n^{\|\gamma\|_{L^\infty(\Omega)}}, \quad \text{and} \quad F_n(x,t) \leq f_n(x) n^{\|\gamma\|_{L^\infty(\Omega)}} t.$$

We now define the functional $\Psi_n : X \longrightarrow \mathbb{R}$ associated with the problem $(\mathcal{P}_n)$ as follows

$$\Psi_n(u) = \sum_{i=1}^{N} \frac{1}{p_i} \int_\Omega |\partial_i u|^{p_i} dx + \sum_{i=1}^{N} \frac{1}{q_i} \int_\Omega |\partial_i u|^{q_i} dx - \int_\Omega F_n(x,u) dx.$$

Since it holds that

$$W_0^{1,\vec{p}}(\Omega) \overset{\text{compact}}{\hookrightarrow} L^r(\Omega) \quad \text{for all} \quad r \in [1, p_\infty),$$

and

$$W_0^{1,\vec{q}}(\Omega) \overset{\text{compact}}{\hookrightarrow} L^r(\Omega) \quad \text{for all} \quad r \in [1, q_\infty).$$

as established in [20]. This implies that the functional Ψ_n is sequentially weakly lower semi-continuous and coercive. Indeed, we have

$$\begin{aligned}
\Psi_n(u) &= \sum_{i=1}^{N} \frac{1}{p_i} \int_\Omega |\partial_i u|^{p_i} dx + \sum_{i=1}^{N} \frac{1}{q_i} \int_\Omega |\partial_i u|^{q_i} dx - \int_\Omega F_n(x,u) dx \\
&\geq \frac{1}{p_N N^{p_N - 1}} \left(\sum_{i=1}^{N} |\partial_i u|_{L^{p_i}(\Omega)} \right)^{p_0} + \frac{1}{q_N N^{q_N - 1}} \left(\sum_{i=1}^{N} |\partial_i u|_{L^{q_i}(\Omega)} \right)^{q_0} \\
&\quad - n^{\|\gamma\|_{L^\infty(\Omega)}} \int_\Omega f_n(x) u dx - 2N \\
&\geq \frac{1}{q_N p_N N^{p_N - 1} N^{q_N - 1}} \left(\|u\|^{p_0}_{W_0^{1,\vec{p}}(\Omega)} + \|u\|^{q_0}_{W_0^{1,\vec{q}}(\Omega)} \right) - n^{\|\gamma\|_{L^\infty(\Omega)}} \int_\Omega f_n(x) u dx - 2N.
\end{aligned}$$

Here,

$$\begin{aligned}
&p_0 = p_1 \quad \text{if } \|u\|_{W_0^{1,\vec{p}}(\Omega)} > 1 \qquad \text{and} \qquad p_0 = p_N \quad \text{if } \|u\|_{W_0^{1,\vec{p}}(\Omega)} \leq 1, \\
&q_0 = q_1 \quad \text{if } \|u\|_{W_0^{1,\vec{q}}(\Omega)} > 1 \qquad \text{and} \qquad q_0 = q_N \quad \text{if } \|u\|_{W_0^{1,\vec{q}}(\Omega)} \leq 1.
\end{aligned}$$

Observe that $W_0^{1,\vec{p}}(\Omega) \overset{\text{continuous}}{\hookrightarrow} L^{p_1}(\Omega)$, implying that there exists a constant C such that, $\|u\|_{L^{p_1}(\Omega)} \leq C \|u\|_{W_0^{1,\vec{p}}(\Omega)}$. This leads to the following inequality

$$\Psi_n(u) \geq \frac{1}{q_N p_N N^{p_N - 1} N^{q_N - 1}} \left(\|u\|^{p_0}_{W_0^{1,\vec{p}}(\Omega)} + \|u\|^{q_0}_{W_0^{1,\vec{q}}(\Omega)} \right) - C n^{\|\gamma\|_{L^\infty(\Omega)}} \|f\|_{L^{p_1'}(\Omega)} \|u\|_{W_0^{1,\vec{p}}(\Omega)} - 2N.$$

Since $p_0, q_0 \geq 2$, we have

$$\Psi_n(u) \longrightarrow +\infty \quad \text{as } \|u\|_X \longrightarrow +\infty.$$

Consequently, Ψ_n has a global minimum, denoted u_n. Moreover, $\Psi_n \in C^1(X)$ with derivative at u given by

$$\langle \Psi_n'(u), \phi \rangle \sum_{i=1}^{N} \int_\Omega |\partial_i(u)|^{p_i - 2} \partial_i(u) \partial_i \phi dx + \sum_{i=1}^{N} \int_\Omega |\partial_i(u)|^{q_i - 2} \partial_i(u) \partial_i \phi dx - \int_\Omega f_n(x,u) \phi dx.$$

Hence, this global minimum is a critical point and thus constitutes a weak solution to the problem $(\mathcal{P}_n)$. Next, we consider the following problem

$$\begin{cases}
-\sum_{i=1}^{N} \partial_i \left(|\partial_i w_n|^{p_i - 2} \partial_i w_n \right) - \sum_{i=1}^{N} \partial_i \left(|\partial_i w_n|^{q_i - 2} \partial_i w_n \right) = n^{\|\gamma\|_{L^\infty(\Omega)}} f_n & \text{in } \Omega, \\
w_n = 0 & \text{on } \partial\Omega.
\end{cases}$$

According to Lemmas 2.1 and 2.2 the aforementioned problem has a solution $w_n \in L^\infty(\Omega)$, whenever the right-hand side belongs to $L^s(\Omega)$ with $s > \max\{\bar{p}^*, \bar{q}^*\}'$. Here, $\max\{\bar{p}^*, \bar{q}^*\}'$ denotes the Hölder conjugate exponent, defined by the relation $\frac{1}{\max\{\bar{p}^*, \bar{q}^*\}} + \frac{1}{\max\{\bar{p}^*, \bar{q}^*\}'} = 1$.

Since $\frac{f_n}{(u_n+\frac{1}{n})^{\gamma(x)}} \geq 0$ a.e. in Ω, by applying Lemma 2.3, we show that $u_n \geq 0$ a.e. in Ω.

Furthermore, using Lemma 2.3 again, along with the inequality $f_n(x,u) \leq n^{\|\gamma\|_{L^\infty(\Omega)}} f_n(x)$ a.e. in Ω for all $n \in \mathbb{N}^*$, we conclude that $u_n \leq w_n$ a.e. in Ω, for all $n \in \mathbb{N}^*$. This leads us to deduce that $u_n \in L^\infty(\Omega)$. □

Remark 3.3. If u_n and v_n are two solutions of problem ($\mathcal{P}_n$), then by a straightforward calculation using Lemma 2.3, we can demonstrate that $u_n \leq v_n$. Due to symmetry, this implies that the solution to problem ($\mathcal{P}_n$) is unique.

Lemma 3.4. *The sequence $\{u_n\}$ is increasing with respect to $n, u_n \geq 0$ a.e in Ω, and there exists $K > 0$ (independent of n) such that*

$$u_n(x) \geq Kd(x) > 0 \quad a.e.\ x \in \Omega, \tag{3.1}$$

for every $n \in \mathbb{N}$, where $d(x) := \operatorname{dist}(x, \partial\Omega)$.

Proof. The function $f_n(x) = \min\{f(x)\ ,\ n\}$ provides a pointwise truncation of $f(x)$. Additionally, $\gamma(x) > 0$ in Ω, observing that

$$-\sum_{i=1}^{N} \partial_i \left(|\partial_i u_n|^{p_i-2}\partial_i u_n\right) - \sum_{i=1}^{N} \partial_i \left(|\partial_i u_n|^{q_i-2}\partial_i u_n\right) = \frac{f_n}{\left(u_n + \frac{1}{n}\right)^{\gamma(x)}} \leq \frac{f_{n+1}}{\left(u_n + \frac{1}{n+1}\right)^{\gamma(x)}}$$

and

$$-\sum_{i=1}^{N} \partial_i \left(|\partial_i u_{n+1}|^{p_i-2}\partial_i u_{n+1}\right) - \sum_{i=1}^{N} \partial_i \left(|\partial_i u_{n+1}|^{q_i-2}\partial_i u_{n+1}\right) = \frac{f_{n+1}}{\left(u_{n+1} + \frac{1}{n+1}\right)^{\gamma(x)}},$$

which implies

$$\begin{aligned}
&-\sum_{i=1}^{N} \partial_i \left(|\partial_i u_n|^{p_i-2}\partial_i u_n\right) - \sum_{i=1}^{N} \partial_i \left(|\partial_i u_n|^{q_i-2}\partial_i u_n\right) \\
&\qquad + \sum_{i=1}^{N} \partial_i \left(|\partial_i u_{n+1}|^{p_i-2}\partial_i u_{n+1}\right) + \sum_{i=1}^{N} \partial_i \left(|\partial_i u_{n+1}|^{q_i-2}\partial_i u_{n+1}\right) \\
&\leq \frac{f_{n+1}}{\left(u_n + \frac{1}{n+1}\right)^{\gamma(x)}} - \frac{f_{n+1}}{\left(u_{n+1} + \frac{1}{n+1}\right)^{\gamma(x)}} \\
&\leq \left(\frac{\left(u_{n+1} + \frac{1}{n+1}\right)^{\gamma(x)} - \left(u_n + \frac{1}{n+1}\right)^{\gamma(x)}}{\left(u_n + \frac{1}{n+1}\right)^{\gamma(x)} \left(u_{n+1} + \frac{1}{n+1}\right)^{\gamma(x)}} \right) f_{n+1}.
\end{aligned} \tag{3.2}$$

In the last inequality, applying $(u_n - u_{n+1})^+$ as a test function, the right-hand side shows

$$\left(\frac{\left(u_{n+1} + \frac{1}{n+1}\right)^{\gamma(x)} - \left(u_n + \frac{1}{n+1}\right)^{\gamma(x)}}{\left(u_n + \frac{1}{n+1}\right)^{\gamma(x)} \left(u_{n+1} + \frac{1}{n+1}\right)^{\gamma(x)}} \right) (u_n - u_{n+1})^+ f_{n+1} \leq 0 \quad \text{a.e. in } \Omega. \tag{3.3}$$

Combining (3.3) and (3.2) yields

$$\sum_{i=1}^{N}\int_{\Omega}\left(|\partial_i u_n|^{p_i-2}\partial_i u_n-|\partial_i u_{n+1}|^{p_i-2}\partial_i u_{n+1}\right)\partial_i\left(u_n-u_{n+1}\right)^{+}$$
$$+\sum_{i=1}^{N}\int_{\Omega}\left(|\partial_i u_n|^{q_i-2}\partial_i u_n-|\partial_i u_{n+1}|^{q_i-2}\partial_i u_{n+1}\right)\partial_i\left(u_n-u_{n+1}\right)^{+}\leq 0.$$

We recall the following algebraic inequality (see [41], page 210)

$$\left[|\xi_1|^{m-2}\xi_1-|\xi_2|^{m-2}\xi_2\right]\cdot(\xi_1-\xi_2)\geq C\,|\xi_1-\xi_2|^{m},$$

where $\xi_1,\xi_2\in\mathbb{R}^N$, $m\geq 2$, and C is a positive constant.

Since $p_i\geq 2$ for all $i=1,\dots,N$, we apply the above inequality with $\xi_1=\partial_i u_n$ and $\xi_2=\partial_i u_{n+1}$, which yields

$$0\leq\sum_{i=1}^{N}\int_{\Omega}|\partial_i\left(u_n-u_{n+1}\right)^{+}|^{p_i}dx+\sum_{i=1}^{N}\int_{\Omega}|\partial_i\left(u_n-u_{n+1}\right)^{+}|^{q_i}dx\leq 0.$$

Consequently, we find that $\left(u_n-u_{n+1}\right)^{+}=0$ a.e. in Ω, which implies $u_n\leq u_{n+1}$ for all $n\in\mathbb{N}$.

Given that the sequence u_n is increasing with respect to n, it suffices to prove that inequality (3.1) holds for u_1. We know that u_1 is a solution of the problem $(\mathcal{P}_n)$ when $n=1$, which implies, in the weak sense

$$-\sum_{i=1}^{N}\partial_i\left(|\partial_i u_1|^{p_i-2}\partial_i u_1\right)-\sum_{i=1}^{N}\partial_i\left(|\partial_i u_1|^{q_i-2}\partial_i u_1\right)=\frac{f_1}{\left(u_1+1\right)^{\gamma(x)}}.$$

According to Lemma 3.1 and hypothesis (H_f), we have

$$\frac{f_1}{\left(u_1+1\right)^{\gamma(x)}}\geq\frac{\operatorname{ess\,inf}_{\Omega}f_1}{\left(\|u_1\|_{L^{\infty}(\Omega)}+1\right)^{\gamma^{-}}}>0.$$

Let $\lambda>0$ be such that

$$\frac{\operatorname{ess\,inf}_{\Omega}f_1}{\left(\|u_1\|_{L^{\infty}(\Omega)}+1\right)^{\gamma^{-}}}\geq\lambda. \tag{3.4}$$

By applying Lemma 2.4, there exists a unique solution $\underline{u}\in X$, to the following problem

$$\begin{cases}-\sum_{i=1}^{N}\partial_i\left(|\partial_i\underline{u}|^{p_i-2}\partial_i\underline{u}\right)-\sum_{i=1}^{N}\partial_i\left(|\partial_i\underline{u}|^{q_i-2}\partial_i\underline{u}\right)=\lambda & \text{in }\Omega,\\ \underline{u}=0 & \text{on }\partial\Omega,\end{cases}$$

such that

$$\underline{u}(x)\geq Kd(x), \tag{3.5}$$

where $d(x):=\operatorname{dist}(x,\partial\Omega)$ for $x\in\Omega$, and K is a positive constant independent of $\underline{u}$.

From (3.4), we obtain

$$-\sum_{i=1}^{N}\partial_i\left(|\partial_i u_1|^{p_i-2}\partial_i u_1\right)-\sum_{i=1}^{N}\partial_i\left(|\partial_i u_1|^{q_i-2}\partial_i u_1\right)\geq-\sum_{i=1}^{N}\partial_i\left(|\partial_i\underline{u}|^{p_i-2}\partial_i\underline{u}\right)-\sum_{i=1}^{N}\partial_i\left(|\partial_i\underline{u}|^{q_i-2}\partial_i\underline{u}\right).$$

Using Lemma 2.3 together with inequality (3.5), we conclude

$$u_1 \geq \underline{u} \geq Kd(x), \quad \text{a.e. in } \Omega. \tag{3.6}$$

Due to the monotonicity of the sequence $\{u_n\}_n$ and the result in (3.6), it follows that for all $n \in \mathbb{N}^*$

$$u_n \geq Kd(x), \quad \text{a.e. in } \Omega. \qquad \square$$

4 Proof of the main results

In this section, contingent on the behavior of the function $\gamma(x)$, we establish the main results of our work by analyzing the convergence of solutions to the regularized problems $(\mathcal{P}_n)$ and investigating the regularity properties of these solutions.

Proof of Theorem 1.2.

When $0 < \gamma(x) < 1$, the existence of solutions in X for problem (1.1) is established by showing that the sequence of solutions to the approximate problem $(\mathcal{P}_n)$ remains bounded in X. Leveraging the reflexivity of X and applying Lemma 3.4, we then conclude the existence of a solution.

Lemma 4.1. *Suppose that (H_f) holds, $f \in L^1(\Omega)$, and let u_n be the solution of $(\mathcal{P}_n)$ with $0 < \gamma^- \leq \gamma(x) \leq \gamma^+ < 1$. Then the sequence $\{u_n\}_n$ is bounded in X.*

Proof. Begin by using u_n as a test function in $(\mathcal{P}_n)$. We have

$$\begin{aligned}
\sum_{i=1}^{N} \int_\Omega |\partial_i u_n|^{p_i} dx + \sum_{i=1}^{N} \int_\Omega |\partial_i u_n|^{q_i} dx &= \int_\Omega \frac{f_n u_n}{\left(u_n + \frac{1}{n}\right)^{\gamma(x)}} dx \\
&\leq \int_\Omega f u_n^{1-\gamma(x)} dx \\
&\leq \int_{\Omega \cap \{u_n \leq 1\}} f dx + \int_{\Omega \cap \{u_n \geq 1\}} f u_n dx \\
&\leq \|f\|_{L^1(\Omega)} + \|f\|_{L^1(\Omega)} \|u_n\|_{L^\infty(\Omega)}.
\end{aligned} \tag{4.1}$$

Now, referencing the proof of Lemma 3.1, we can derive the following inequality:

$$\sum_{i=1}^{N} \int_\Omega |\partial_i u_n|^{p_i} dx + \sum_{i=1}^{N} \int_\Omega |\partial_i u_n|^{q_i} dx \geq \frac{\|u_n\|^{p_0}_{W_0^{1,\vec{p}}(\Omega)}}{N^{p_N - 1}} + \frac{\|u_n\|^{q_0}_{W_0^{1,\vec{q}}(\Omega)}}{N^{q_N - 1}} - 2N, \tag{4.2}$$

where p_0 and q_0 are identical in the proof of Lemma 3.1. Based on the previous inequalities and (4.1), we obtain

$$\begin{aligned}
\|u_n\|^{p_0}_{W_0^{1,\vec{p}}(\Omega)} + \|u_n\|^{q_0}_{W_0^{1,\vec{q}}(\Omega)} &\leq \max\left\{N^{p_N - 1}, N^{q_N - 1}\right\} \left(\|f\|_{L^1(\Omega)} + \|f\|_{L^1(\Omega)} \|u_n\|_{L^\infty(\Omega)} + 2N\right) \\
&\leq \mathfrak{C}_1(f, \varphi).
\end{aligned}$$

The fact that $u_n \in L^\infty(\Omega)$ implies that $\{u_n\}$ is bounded in X. $\square$

Since the embidding $X \hookrightarrow L^s(\Omega)$ is compact for all $s \in [1, \min\{\bar{p}^*, \bar{q}^*\})$, and given that $\{u_n\}$ is bounded in X by Lemma 4.1, we can conclude that, up to a subsequence, $u_n \rightharpoonup u$

weakly in X, where $u \in X$. Moreover, $u_n \to u$ strongly in $L^s(\Omega)$ for $1 \leq s < \min\{\bar{p}^*, \bar{q}^*\}$, and $u_n(x) \to u(x)$ almost everywhere in Ω. Thus, for every $\phi \in C_0^1(\Omega)$, we have

$$\lim_{n\longrightarrow\infty} \sum_{i=1}^{N} \int_\Omega |\partial_i u_n|^{p_i-2}\partial_i u_n \partial_i \phi dx + \sum_{i=1}^{N} \int_\Omega |\partial_i u_n|^{q_i-2}\partial_i u_n \partial_i \phi dx$$
$$= \sum_{i=1}^{N} \int_\Omega |\partial_i u|^{p_i-2}\partial_i u \partial_i \phi dx + \sum_{i=1}^{N} \int_\Omega |\partial_i u|^{q_i-2}\partial_i u \partial_i \phi dx. \quad (4.3)$$

According to Lemma 3.4, we have $u_n \geq kd(x)$ a.e. in Ω, which implies that

$$0 \leq \left| \frac{f_n \phi}{\left(u_n + \frac{1}{n}\right)^{\gamma(x)}} \right| \leq \left(\| \phi \left(kd(\cdot)\right)^{-\gamma^+} \|_{L^\infty(\Omega)} + \| \phi \left(kd(\cdot)\right)^{-\gamma^-} \|_{L^\infty(\Omega)} \right) f,$$

for all $\phi \in C_0^1(\Omega)$. By using the Dominated Lebesgue's theorem, we have

$$\lim_{n\longrightarrow\infty} \int_\Omega \frac{f_n \phi}{\left(u_n + \frac{1}{n}\right)^{\gamma(x)}} dx = \int_\Omega \frac{f\phi}{(u)^{\gamma(x)}} dx. \quad (4.4)$$

The fact that u_n is the solution of $(\mathcal{P}_n)$ allows us to conclude, using (4.3) and (4.4), that

$$\sum_{i=1}^{N} \int_\Omega |\partial_i u|^{p_i-2}\partial_i u \partial_i \phi dx + \sum_{i=1}^{N} \int_\Omega |\partial_i u|^{q_i-2}\partial_i u \partial_i \phi dx = \int_\Omega \frac{f\phi}{u^{\gamma(x)}} dx, \quad (4.5)$$

for all $\phi \in C_0^1(\Omega)$. This proves that (1.1) has a solution u in X. □

Proof of Theorem 1.3.
In the case $\gamma^- < 1 < \gamma^+$, we choose a test function $u_n^{\gamma^+}$ to demonstrate that u_n is bounded in X_{loc}, and also bounded in $L^r(\Omega)$ for some appropriate value of r. By utilizing the same arguments as in the case $0 < \gamma(x) < 1$ (Theorem 1.2), we can establish the existence and regularity of the solutions.

Lemma 4.2. *Assume that (H_f) is satisfied and that f is a nonnegative function in $L^1(\Omega)$. Let u_n denote the solution to $(\mathcal{P}_n)$ for the parameters $0 < \gamma^- < 1 < \gamma^+$. Then, the sequence $\{u_n\}_n$ is bounded in both X_{loc} and $L^r(\Omega)$, where*

$$r = \frac{N\left(\gamma^+ - 1 + \overline{m}\right)}{(N - \overline{m})},$$

with $\overline{m}$ being either $\overline{p}$ or $\overline{q}$.

Proof. Taking $u_n^{\gamma^+}$ as a test function, we obtain the following by using $(\mathcal{P}_n)$ and Lemma 3.4. Given that $u_n \in L^\infty(\Omega)$ and $\gamma^+ - \gamma^- > 0$, we obtain

$$\sum_{i=1}^{N} \gamma^+ \int_\Omega |\partial_i u_n|^{p_i} u_n^{\gamma^+-1} dx + \sum_{i=1}^{N} \gamma^+ \int_\Omega |\partial_i u_n|^{q_i} u_n^{\gamma^+-1} dx = \int_\Omega \frac{f_n u_n^{\gamma^+}}{\left(u_n + \frac{1}{n}\right)^{\gamma(x)}} dx$$
$$\leq \int_\Omega f u_n^{\gamma^+-\gamma^-} dx + \int_\Omega f dx$$
$$\leq \|f\|_{L^1(\Omega)} \left(1 + \|u_n^{\gamma^+-\gamma^-}\|_{L^\infty(\Omega)}\right)$$
$$\leq \mathfrak{C}_2(f, \gamma, \Omega). \quad (4.6)$$

Furthermore, applying Lemma 3.4 for any compact set $K \subset \Omega$ and since $\gamma^+ - 1 > 0$, we conclude

$$\sum_{i=1}^{N} \int_{\Omega} |\partial_i u_n|^{p_i} u_n^{\gamma^+ - 1} dx + \sum_{i=1}^{N} \int_{\Omega} |\partial_i u_n|^{q_i} u_n^{\gamma^+ - 1} dx$$
$$\geq \min_K (kd(x))^{\gamma^+ - 1} \left[\sum_{i=1}^{N} \int_K |\partial_i u_n|^{p_i} dx + \sum_{i=1}^{N} \int_K |\partial_i u_n|^{q_i} dx \right].$$

By the last inequalities and (4.6), we obtain

$$\sum_{i=1}^{N} \int_K |\partial_i u_n|^{p_i} dx + \sum_{i=1}^{N} \int_K |\partial_i u_n|^{q_i} dx \leq \frac{1}{\min_K (kd(x))^{\gamma^+ - 1}} \mathfrak{C}_2(f, \gamma, \Omega), \tag{4.7}$$

for any compact set $K \subset \Omega$. We conclude that u_n is bounded in X_{loc}.

Now from (4.6), we get for every $i = 1, \ldots, N$

$$\int_{\Omega} |\partial_i u_n|^{m_i} u_n^{\gamma^+ - 1} dx \leq \mathfrak{C}_2(f, \gamma, \Omega),$$

where m_i being either p_i or q_i, which implies

$$\left[\int_{\Omega} |\partial_i u_n|^{m_i} u_n^{\gamma^+ - 1} dx \right]^{\frac{1}{m_i}} \leq \mathfrak{C}_2(f, \gamma, \Omega)^{\frac{1}{m_i}}.$$

Hence,

$$\prod_{i=1}^{N} \left[\int_{\Omega} |\partial_i u_n|^{m_i} u_n^{\gamma^+ - 1} dx \right]^{\frac{1}{m_i}} \leq \mathfrak{C}_2(f, \gamma, \Omega)^{\frac{N}{\overline{m}}},$$

where $\frac{N}{\overline{m}} = \sum_{i=1}^{N} \frac{1}{m_i}$.

Now, we need to use the Sobolev inequality provided in [33, Theorem 1.2], with the following choice of exponents:

$$t_i m_i = \gamma^+ - 1, \quad r = \frac{N(\gamma^+ - 1 + \overline{m})}{(N - \overline{m})} \quad \text{and} \quad \frac{1}{r} = \frac{\gamma_i (N-1) - 1 + \frac{1}{m_i}}{t_i + 1},$$

where γ_i and t_i are defined as in [33, Theorem 1.2] . We deduce that

$$\left[\int_{\Omega} u_n^r dx \right]^{\frac{N}{\overline{m}} - 1} \leq \mathfrak{C}_2(f, \gamma, \Omega)^{\frac{N}{\overline{m}}}.$$

The fact that $N > \overline{m}$ allows us to conclude that u_n is bounded in $L^r(\Omega)$ with $r = \frac{N(\gamma^+ - 1 + \overline{m})}{(N - \overline{m})}$. □

Given that u_n is bounded in X_{loc}, we can prove the existence of a solution u in X_{loc} by following the proof of Theorem 1.2. Moreover, since u_n is bounded in $L^r(\Omega)$ with $r = \frac{N(\gamma^+ - 1 + \overline{m})}{(N - \overline{m})}$, it follows that u is also in $L^r(\Omega)$.

Proof of Theorem 1.4.

Finally, when $\gamma(x) > 1$, we select a test function $u_n(\varphi^2 + \epsilon^{\gamma^+})$ to show that u_n is bounded in X_{loc}. By adapting the arguments from the case $0 < \gamma(x) < 1$, we can conclude the existence of solutions in X_{loc}.

Lemma 4.3. *Assume that (H_f) is satisfied and that $f \in L^1(\Omega)$ is a nonnegative function. Let u_n represent the solution to $(\mathcal{P}_n)$ under the condition $1 < \gamma^- \leq \gamma(x) \leq \gamma^+$. Then, the sequence $\{u_n\}_n$ is bounded in X_{loc}.*

Proof. By using the test function $u_n(\varphi^2 + \epsilon^{\gamma^+})$ with $\varphi \in C^1(\overline{\Omega})$ and $\operatorname{supp}\varphi \subset \Omega$, and under the condition $0 < \epsilon < \frac{1}{n}$ for a fixed n, we can apply the properties from $(\mathcal{P}_n)$ and Lemma 3.4. We obtain

$$\begin{aligned}
&\sum_{i=1}^{N}\left[\int_\Omega \left(|\partial_i u_n|^{p_i} + |\partial_i u_n|^{q_i}\right)(\varphi^2+\epsilon^{\gamma^+})dx + 2\int_\Omega u_n\varphi\left(|\partial_i u_n|^{p_i-2} + |\partial_i u_n|^{q_i-2}\right)\partial_i u_n\partial_i\varphi dx\right] \\
&\quad = \int_\Omega \frac{f_n u_n(\varphi^2+\epsilon^{\gamma^+})}{\left(u_n+\frac{1}{n}\right)^{\gamma(x)}}dx \\
&\quad \leq \int_\Omega \frac{f_n u_n\varphi^2}{(kd(x))^{\gamma(x)}}dx + \int_\Omega \frac{f_n u_n\epsilon^{\gamma^+}}{\left(\frac{1}{n}\right)^{\gamma^+}}dx \\
&\quad \leq \int_{\operatorname{supp}\varphi} \frac{f u_n\varphi^2}{\left(k\min\limits_{\operatorname{supp}\varphi} d(x)\right)^{\gamma(x)}}dx + \int_\Omega f_n u_n dx,
\end{aligned}$$

where $k > 0$ is a constant. Using Hölder's inequality,

$$\begin{aligned}
&\sum_{i=1}^{N}\left[\int_\Omega \left(|\partial_i u_n|^{p_i} + |\partial_i u_n|^{q_i}\right)(\varphi^2+\epsilon^{\gamma^+})dx + 2\int_\Omega u_n\varphi\left(|\partial_i u_n|^{p_i-2} + |\partial_i u_n|^{q_i-2}\right)\partial_i u_n\partial_i\varphi dx\right] \\
&\quad \leq \left[\|\varphi^2\|_{L^\infty(\Omega)}\min\left\{\left(k\min_{\operatorname{supp}\varphi} d(x)\right)^{-\gamma^-}, \left(k\min_{\operatorname{supp}\varphi} d(x)\right)^{-\gamma^+}\right\} + 1\right]\|f\|_{L^1(\Omega)}\|u_n\|_{L^\infty(\Omega)}.
\end{aligned} \tag{4.8}$$

Let $0 < \lambda < \frac{\epsilon^{\gamma^+}}{\left(\|\varphi\|_{L^\infty(\Omega)}+1\right)^{\frac{\max\{q_N,p_N\}}{\min\{q_1,p_1\}}-1}}$. By applying Young's inequality, we get

$$\begin{aligned}
&\sum_{i=1}^{N} 2\int_{\operatorname{supp}\varphi} |\partial_i u_n|^{m_i-1}|\partial_i\varphi||u_n\varphi|dx \\
&\quad \leq \sum_{i=1}^{N}\left\{\frac{m_i-1}{m_i}\int_{\operatorname{supp}\varphi}\lambda|\partial_i u_n|^{m_i}|\varphi|^{\frac{m_i}{m_i-1}}dx + \frac{2^{m_i}}{m_i}\int_{\operatorname{supp}\varphi}\frac{1}{\lambda^{m_i-1}}|\partial_i\varphi|^{m_i}|u_n|^{m_i}dx\right\},
\end{aligned}$$

which implies

$$\begin{aligned}
&\sum_{i=1}^{N} 2\int_\Omega u_n\varphi|\partial_i u_n|^{m_i-2}\partial_i u_n\partial_i\varphi dx \\
&\quad \geq -\sum_{i=1}^{N} 2\int_{\operatorname{supp}\varphi}|\partial_i u_n|^{m_i-1}|\partial_i\varphi||u_n\varphi|dx \\
&\quad \geq -\sum_{i=1}^{N}\left\{\int_{\operatorname{supp}\varphi}\lambda|\partial_i u_n|^{m_i}|\varphi|^{\frac{m_i}{m_i-1}}dx + \frac{2^{m_i}}{m_i\lambda^{m_i-1}}\int_{\operatorname{supp}\varphi}|\partial_i\varphi|^{m_i}|u_n|^{m_i}dx\right\},
\end{aligned}$$

where $m_i = p_i$ or $m_i = q_i$. By combining the last inequalities and (4.8) yields

$$\begin{aligned}
&\sum_{i=1}^{N}\int_{\operatorname{supp}\varphi}|\partial_i u_n|^{p_i}\left(\varphi^2+\epsilon^{\gamma^+}-\lambda|\varphi|^{\frac{p_i}{p_i-1}}\right)dx+\sum_{i=1}^{N}\int_{\operatorname{supp}\varphi}|\partial_i u_n|^{q_i}\left(\varphi^2+\epsilon^{\gamma^+}-\lambda|\varphi|^{\frac{q_i}{q_i-1}}\right)dx\\
&\quad\leq\left[\|\varphi^2\|_{L^\infty(\Omega)}\min\left\{\left(k\min_{\operatorname{supp}\varphi}d(x)\right)^{-\gamma^-},\left(k\min_{\operatorname{supp}\varphi}d(x)\right)^{-\gamma^+}\right\}+1\right]\|f\|_{L^1(\Omega)}\|u_n\|_{L^\infty(\Omega)}\\
&\qquad+\sum_{i=1}^{N}\frac{2^{p_i}}{p_i\lambda^{p_i-1}}\|\partial_i\varphi\|^{p_i}_{L^\infty(\Omega)}\|u_n\|^{p_i}_{L^{p_i}(\Omega)}+\sum_{i=1}^{N}\frac{2^{q_i}}{q_i\lambda^{q_i-1}}\|\partial_i\varphi\|^{q_i}_{L^\infty(\Omega)}\|u_n\|^{q_i}_{L^{q_i}(\Omega)}\\
&\quad\leq\mathfrak{C}_3(f,\varphi),
\end{aligned}\tag{4.9}$$

where the boundedness of the sequences $\|u_n\|_{L^\infty(\Omega)}$, $\|u_n\|_{L^{p_i}(\Omega)}$, and $\|u_n\|_{L^{q_i}(\Omega)}$ for each $i=1,\dots,N$ ensures the validity of the inequality.

At this point, noting that

$$0<\lambda<\frac{\epsilon^{\gamma^+}}{\left(\|\varphi\|_{L^\infty(\Omega)}+1\right)^{\frac{\max\{q_N,p_N\}}{\min\{q_1,p_1\}-1}}},$$

we can deduce that

$$\begin{aligned}
&\sum_{i=1}^{N}\int_{\operatorname{supp}\varphi}|\partial_i u_n|^{p_i}\left(\varphi^2+\epsilon^{\gamma^+}-\lambda|\varphi|^{\frac{p_i}{p_i-1}}\right)dx+\sum_{i=1}^{N}\int_{\operatorname{supp}\varphi}|\partial_i u_n|^{q_i}\left(\varphi^2+\epsilon^{\gamma^+}-\lambda|\varphi|^{\frac{q_i}{q_i-1}}\right)dx\\
&\quad\geq\sum_{i=1}^{N}\int_{\operatorname{supp}\varphi}|\partial_i u_n|^{p_i}\left(\varphi^2+\epsilon^{\gamma^+}-\epsilon^{\gamma^+}\frac{|\varphi|^{\frac{p_i}{p_i-1}}}{\left(\|\varphi\|_{L^\infty(\Omega)}+1\right)^{\frac{\max\{q_N,p_N\}}{\min\{q_1,p_1\}-1}}}\right)dx\\
&\qquad+\sum_{i=1}^{N}\int_{\operatorname{supp}\varphi}|\partial_i u_n|^{q_i}\left(\varphi^2+\epsilon^{\gamma^+}-\epsilon^{\gamma^+}\frac{|\varphi|^{\frac{q_i}{q_i-1}}}{\left(\|\varphi\|_{L^\infty(\Omega)}+1\right)^{\frac{\max\{q_N,p_N\}}{\min\{q_1,p_1\}-1}}}\right)dx\\
&\quad\geq\sum_{i=1}^{N}\int_{\operatorname{supp}\varphi}|\partial_i u_n|^{p_i}\varphi^2dx+\sum_{i=1}^{N}\int_{\operatorname{supp}\varphi}|\partial_i u_n|^{q_i}\varphi^2dx,
\end{aligned}\tag{4.10}$$

since the inequality

$$\frac{|\varphi|^{\frac{m_i}{m_i-1}}}{\left(\|\varphi\|_{L^\infty(\Omega)}+1\right)^{\frac{\max\{q_N,p_N\}}{\min\{q_1,p_1\}-1}}}<1$$

holds for $m_i=q_i$ or $m_i=p_i$ for every $i=1,\dots,N$, it follows that, in view of (4.9) and (4.10), we obtain

$$\sum_{i=1}^{N}\int_{\Omega}|\partial_i u_n|^{p_i}\varphi^2dx+\sum_{i=1}^{N}\int_{\Omega}|\partial_i u_n|^{q_i}\varphi^2dx\leq\mathfrak{C}_3(f,\varphi).$$

Thus, the sequence $\{u_n\}_n$ is bounded in X_{loc} as desired. □

Minor adjustments in the proof of Theorem 1.2 allow us to prove the existence of u in X_{loc}.

Proof of Theorem 1.5.

By using $(u-\delta)^+$ as a test function in problem (1.1), we demonstrate that for all $\gamma(x) > 0$ and $f \in L^r(\Omega)$ for some suitable r, the solution to problem (1.1) belongs to $L^\infty(\Omega)$.

Let u be the solution for problem (1.1). Taking $(u-\delta)^+$ as a test function in (1.1) with $\delta > 0$, we obtain

$$\sum_{i=1}^{N}\int_\Omega |\partial_i u|^{p_i-2}\partial_i u\partial_i(u-\delta)^+dx + \sum_{i=1}^{N}\int_\Omega |\partial_i u|^{q_i-2}\partial_i u\partial_i(u-\delta)^+dx = \int_\Omega \frac{f(u-\delta)^+}{(u)^{\gamma(x)}}dx,$$

from which it follows that

$$\sum_{i=1}^{N}\int_\Omega |\partial_i(u-\delta)^+|^{p_i}dx + \sum_{i=1}^{N}\int_\Omega |\partial_i(u-\delta)^+|^{q_i}dx \le \int_\Omega f(u-\delta)^+\left(kd(x)\right)^{-\gamma(x)}dx,$$

thus, we obtain

$$\sum_{i=1}^{N}\int_\Omega |\partial_i(u-\delta)^+|^{p_i}dx \le \int_\Omega f(u-\delta)^+\left(kd(x)\right)^{-\gamma(x)}dx,$$

which leads to the following inequality

$$\left(\int_\Omega |\partial_i(u-\delta)^+|^{p_i}dx\right)^{\frac{1}{p_i}} \le \left(\int_\Omega f(u-\delta)^+\left(kd(x)\right)^{-\gamma(x)}dx\right)^{\frac{1}{p_i}}. \tag{4.11}$$

Additionally, we employ the following Sobolev-type inequality, as shown in [13], there exists a constant $C > 0$, dependent only on the domain Ω, such that for any $s \in [1, \bar{p}^*]$:

$$\|\xi\|_{L^s(\Omega)} \le C\prod_{i=1}^{N}\|\partial_i\xi\|^{\frac{1}{N}}_{L^{p_i}(\Omega)}, \tag{4.12}$$

where $\xi \in W_0^{1,\vec{p}}(\Omega)$, and $p_i \ge 2$ for each $i = 1,2,\dots,N$. From (4.11)–(4.12) and noting that $\min\{\bar{p}^*,\bar{q}^*\} \le \bar{p}^*$, we obtain

$$\begin{aligned}\|(u-\delta)^+\|_{L^{\min\{\bar{p}^*,\bar{q}^*\}}(\Omega)} &\le C_0\prod_{i=1}^{N}\left(\int_\Omega f(u-\delta)^+\left(kd(x)\right)^{-\gamma(x)}dx\right)^{\frac{1}{Np_i}},\\ &\le C_0\left(\int_\Omega f(u-\delta)^+\left(kd(x)\right)^{-\gamma(x)}dx\right)^{\frac{1}{\bar{p}}}.\end{aligned}$$

Applying Hölder's inequality, we have

$$\begin{aligned}&\|(u-\delta)^+\|_{L^{\min\{\bar{p}^*,\bar{q}^*\}}(\Omega)}\\ &\le C_0\left(\left[\int_{\Lambda_\delta}|f|^{\min\{\bar{p}^*,\bar{q}^*\}'}\left(kd(x)\right)^{-\gamma(x)}dx\right]^{\frac{1}{\min\{\bar{p}^*,\bar{q}^*\}'}}\|(u-\delta)^+\|_{L^{\min\{\bar{p}^*,\bar{q}^*\}}(\Omega)}\right)^{\frac{1}{\bar{p}}}\\ &\le C_0\left(\left[\int_{\Lambda_\delta}|f|^{\min\{\bar{p}^*,\bar{q}^*\}'}dx\right]^{\frac{1}{\min\{\bar{p}^*,\bar{q}^*\}'}}\|\left(kd(\cdot)\right)^{-\gamma(\cdot)}\|^{\frac{1}{\min\{\bar{p}^*,\bar{q}^*\}'}}_{L^\infty(\Lambda_\delta)}\|(u-\delta)^+\|_{L^{\min\{\bar{p}^*,\bar{q}^*\}}(\Omega)}\right)^{\frac{1}{\bar{p}}},\end{aligned}$$

with $\Lambda_\delta = \{x \in \Omega : \ u(x) > \delta\}$.

Now, applying Young's inequality to the right-hand side of the last inequality, let $\epsilon > 0$, we have

$$\begin{aligned}
&\|(u-\delta)^+\|_{L^{\min\{\bar{p}^*,\bar{q}^*\}}(\Omega)} \\
&\quad \leq C_\epsilon \| \left(kd(\cdot)\right)^{-\gamma(\cdot)} \|_{L^\infty(\Lambda_\delta)}^{\frac{\bar{p}'}{\bar{p}\min\{\bar{p}^*,\bar{q}^*\}'}} \left[\int_{\Lambda_\delta} |f|^{\min\{\bar{p}^*,\bar{q}^*\}'} dx\right]^{\frac{1}{\min\{\bar{p}^*,\bar{q}^*\}'}\times\frac{\bar{p}'}{\bar{p}}} + \frac{\epsilon}{\bar{p}}\|(u-\delta)^+\|_{L^{\min\{\bar{p}^*,\bar{q}^*\}}(\Omega)},
\end{aligned}$$

which implies

$$\left(1-\frac{\epsilon}{\bar{p}}\right)\|(u-\delta)^+\|_{L^{\min\{\bar{p}^*,\bar{q}^*\}}(\Omega)} \leq C_1 \| \left(kd(\cdot)\right)^{-\gamma(\cdot)} \|_{L^\infty(\Lambda_\delta)}^{\frac{\bar{p}'}{\bar{p}\min\{\bar{p}^*,\bar{q}^*\}'}} \left[\int_{\Lambda_\delta} |f|^{\min\{\bar{p}^*,\bar{q}^*\}'} dx\right]^{\frac{1}{\min\{\bar{p}^*,\bar{q}^*\}'}\times\frac{\bar{p}'}{\bar{p}}}.$$

By choosing ϵ such that $1-\frac{\epsilon}{\bar{p}} > 0$, we can infer that

$$\|(u-\delta)^+\|_{L^{\min\{\bar{p}^*,\bar{q}^*\}}(\Omega)} \leq C_2 \| \left(kd(\cdot)\right)^{-\gamma(\cdot)} \|_{L^\infty(\Lambda_\delta)}^{\frac{\bar{p}'}{\bar{p}\min\{\bar{p}^*,\bar{q}^*\}'}} \left[\int_{\Lambda_\delta} |f|^{\min\{\bar{p}^*,\bar{q}^*\}'} dx\right]^{\frac{1}{\min\{\bar{p}^*,\bar{q}^*\}'}\times\frac{1}{\bar{p}-1}}.$$

Applying Hölder's inequality with exponents $\frac{r}{\min\{\bar{p}^*,\bar{q}^*\}'}$ and $\left(\frac{r}{\min\{\bar{p}^*,\bar{q}^*\}'}\right)'$ yields

$$\begin{aligned}
&\|(u-\delta)^+\|_{L^{\min\{\bar{p}^*,\bar{q}^*\}}(\Omega)} \\
&\quad \leq C_3 \| \left(kd(\cdot)\right)^{-\gamma(\cdot)} \|_{L^\infty(\Lambda_\delta)}^{\frac{\bar{p}'}{\bar{p}\min\{\bar{p}^*,\bar{q}^*\}'}} \left[\|f\|_{L^r(\Omega)}^{\frac{\min\{\bar{p}^*,\bar{q}^*\}'}{r}} \operatorname{meas}(\Lambda_\delta)^{1-\frac{\min\{\bar{p}^*,\bar{q}^*\}'}{r}}\right]^{\frac{1}{\min\{\bar{p}^*,\bar{q}^*\}'}\times\frac{1}{\bar{p}-1}}.
\end{aligned}$$

Utilizing the fact that $f \in L^r(\Omega)$, where $r \geq \frac{\min\{\bar{p}^*,\bar{q}^*\}}{\min\{\bar{p}^*,\bar{q}^*\}-\max\{\bar{p},\bar{q}\}}$, we have

$$\|(u-\delta)^+\|_{L^{\min\{\bar{p}^*,\bar{q}^*\}}(\Omega)} \leq C_4 \| \left(kd(\cdot)\right)^{-\gamma(\cdot)} \|_{L^\infty(\Lambda_\delta)}^{\frac{\bar{p}'}{\bar{p}\min\{\bar{p}^*,\bar{q}^*\}'}} \operatorname{meas}(\Lambda_\delta)^{\left[1-\frac{\min\{\bar{p}^*,\bar{q}^*\}'}{r}\right]\frac{1}{\min\{\bar{p}^*,\bar{q}^*\}'}\times\frac{1}{\bar{p}-1}}. \quad (4.13)$$

Again, by Hölder's inequality, the following inequality holds

$$\int_\Omega (u-\delta)^+ dx \leq \|(u-\delta)^+\|_{L^{\min\{\bar{p}^*,\bar{q}^*\}}(\Omega)} \operatorname{meas}(\Lambda_\delta)^{1-\frac{1}{\min\{\bar{p}^*,\bar{q}^*\}}}. \quad (4.14)$$

Thus, from (4.13) and (4.14), we get

$$\begin{aligned}
&\int_\Omega (u-\delta)^+ dx \\
&\quad \leq C_5 \| \left(kd(\cdot)\right)^{-\gamma(\cdot)} \|_{L^\infty(\Lambda_\delta)}^{\frac{\bar{p}'}{\bar{p}\min\{\bar{p}^*,\bar{q}^*\}'}} \operatorname{meas}(\Lambda_\delta)^{\left[1-\frac{\min\{\bar{p}^*,\bar{q}^*\}'}{r}\right]\frac{1}{\min\{\bar{p}^*,\bar{q}^*\}'}\times\frac{1}{\bar{p}-1}} \operatorname{meas}(\Lambda_\delta)^{1-\frac{1}{\min\{\bar{p}^*,\bar{q}^*\}}}, \\
&\quad \leq C_5 \| \left(kd(\cdot)\right)^{-\gamma(\cdot)} \|_{L^\infty(\Lambda_\delta)}^{\frac{\bar{p}'}{\bar{p}\min\{\bar{p}^*,\bar{q}^*\}'}} \operatorname{meas}(\Lambda_\delta)^{1-\frac{1}{\min\{\bar{p}^*,\bar{q}^*\}}+\left[1-\frac{\min\{\bar{p}^*,\bar{q}^*\}'}{r}\right]\frac{1}{\min\{\bar{p}^*,\bar{q}^*\}'}\times\frac{1}{\bar{p}-1}}. \quad (4.15)
\end{aligned}$$

Defining $\tau(\delta) = \int_\Omega (u-\delta)^+ dx$, it follows straightforwardly that $\tau'(\delta) = -\operatorname{meas}(\Lambda_\delta)$, where $\Lambda_\delta = \{x \in \Omega : \ u(x) > \delta\}$ (see [26]). Thus, the previous inequality becomes

$$\tau(\delta)^{\frac{1}{1-\frac{1}{\min\{\bar{p}^*,\bar{q}^*\}}+\left[1-\frac{\min\{\bar{p}^*,\bar{q}^*\}'}{r}\right]\frac{1}{\min\{\bar{p}^*,\bar{q}^*\}'}\times\frac{1}{\bar{p}-1}}} \leq -C_6\tau'(\delta). \quad (4.16)$$

Let us define the parameter α as follows

$$\alpha = \frac{1}{1-\frac{1}{\min\{\bar{p}^*,\bar{q}^*\}} + \left[1-\frac{\min\{\bar{p}^*,\bar{q}^*\}'}{r}\right]\frac{1}{\min\{\bar{p}^*,\bar{q}^*\}'}\times\frac{1}{\bar{p}-1}}.$$

Using this expression in combination with equation (4.16), we can further deduce

$$1 \leq -C_6\tau'(\delta)\tau(\delta)^{-\alpha} = \frac{-C_6}{1-\alpha}\left(\tau(\delta)^{1-\alpha}\right)'.$$

Observe that assumption $r \geq \frac{\min\{\bar{p}^*,\bar{q}^*\}}{\min\{\bar{p}^*,\bar{q}^*\}-\max\{\bar{p},\bar{q}\}}$ implies $1-\alpha > 0$. Integrating the last inequality from 0 to δ, we obtain

$$\delta \leq -C_6\left(\tau(\delta)^{1-\alpha} - \tau(0)^{1-\alpha}\right),$$

which leads to

$$C_6\tau(\delta)^{1-\alpha} \leq -\delta + C_6\|u\|_{L^1(\Omega)}^{1-\alpha}.$$

From this inequality and the fact that $\tau(\delta)$ is a non-negative and decreasing, there exists a δ_0 such that $\tau(\delta_0) = 0$. Therefore, $\|u\|_{L^\infty(\Omega)} \leq C_{\delta_0}$, which implies $u \in L^\infty(\Omega)$. □

5 Concluding remarks and future directions

As kindly suggested by one of the referees of this paper, we intend to further develop and extend the analysis presented here to singular double anisotropic variable exponent problems of the form:

$$\begin{cases} -\sum_{i=1}^{N}\partial_i\left(|\partial_i u|^{p_i(x)-2}\partial_i u\right) - \sum_{i=1}^{N}\partial_i\left(|\partial_i u|^{q_i(x)-2}\partial_i u\right) = \dfrac{f(x)}{u^{\gamma(x)}} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.1}$$

where $\Omega \subset \mathbb{R}^N$ is a rectangular domain with dimension $N \geq 3$, $\gamma(x)$ is a positive function in $C(\overline{\Omega})$, and $f(x)$ belongs to $L^r(\Omega)$ for some $r \geq 1$.

The vector functions $\vec{p}, \vec{q} : \bar{\Omega} \to \mathbb{R}^N$ are defined as

$$\vec{p}(x) = (p_1(x), \ldots, p_N(x)), \qquad \vec{q}(x) = (q_1(x), \ldots, q_N(x)),$$

where p_i and q_i belong to $C_+(\overline{\Omega})$ and satisfy certain conditions.

The energy functional associated with this model contains the unbalanced variational integral

$$u \longmapsto \sum_{i=1}^{N}\int_\Omega \frac{1}{p_i(x)}|\partial_i u|^{p_i(x)}\,dx + \sum_{i=1}^{N}\int_\Omega \frac{1}{q_i(x)}|\partial_i u|^{q_i(x)}\,dx. \tag{5.2}$$

This functional provides a sharper version of the classical energy

$$u \longmapsto \sum_{i=1}^{N}\frac{1}{p_i}\int_\Omega |\partial_i u|^{p_i}\,dx + \sum_{i=1}^{N}\frac{1}{q_i}\int_\Omega |\partial_i u|^{q_i}\,dx.$$

These problems allow for a more precise description of heterogeneous and anisotropic physical phenomena in fields such as nonlinear elasticity, material science, and homogenization. They are particularly useful for modeling composite materials with spatially varying properties, where the material behavior depends on both location and direction.

In nonlinear elasticity and material science, composite materials with locally varying hardening exponents $\vec{p}(x)$ and $\vec{q}(x)$ can be effectively characterized using the energy functional in (5.2). Furthermore, such problems have important applications in elasticity, homogenization, the modeling of strongly anisotropic materials, the Lavrentiev phenomenon, and other related areas.

Inspired by the seminal work of Chems Eddine and Repovš [17], we observe that the main results in this study can be extended to more general anisotropic problems involving variable exponents as in (5.1). Specifically, we extend the regularity of solutions to anisotropic Hölder continuous function spaces $C^{0,\overrightarrow{\beta}(\cdot)}(\overline{\Omega})$ over rectangular domains. In their work, Chems Eddine and Repovš introduced a novel framework for embedding anisotropic variable exponent Sobolev spaces into anisotropic variable exponent Hölder continuous function spaces in rectangular domains. Their results generalize classical Sobolev embedding theorems by incorporating anisotropic settings with variable exponents, providing deeper insights into function regularity in different spatial directions. Under log-Hölder continuity conditions on the variable exponents $\overrightarrow{p}(\cdot)$.

A promising new research direction focuses on anisotropic operators with variable exponents and singular reactions. Within this framework, we aim to extend the qualitative analysis conducted in this paper to singular nonlinear boundary value problems with variable exponents, modeled by the following system:

$$\begin{cases} -\sum_{i=1}^{N} \partial_i \left(|\partial_i u|^{p_i(x)-2} \partial_i u \right) = \dfrac{f(x)}{u^{\gamma(x)}} + \mathfrak{F}_u(x,u,v) & \text{in } \Omega, \\ -\sum_{i=1}^{N} \partial_i \left(|\partial_i v|^{q_i(x)-2} \partial_i v \right) = \dfrac{g(x)}{v^{\beta(x)}} + \mathfrak{F}_v(x,u,v) & \text{in } \Omega, \\ u, v > 0 & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.3}$$

This anisotropic system, which features unbalanced growth, was introduced by Figueiredo and Silva [22], where the vectors $\vec{p}$ and $\vec{q}$ are independent of x.

We conclude by noting that a key characteristic of nonlinear problems with variable exponents is the possibility of a subcritical-critical-supercritical multiple regime. A particularly intriguing open problem involves analyzing the singular case of the anisotropic system described in problem (5.3) within this multiple regime. A closely related and highly intriguing research direction involves anisotropic systems with variable exponent operators and singular reactions. These systems exhibit complex behaviors, particularly in the presence of unbalanced growth conditions and multiple regime transitions. Exploring their qualitative properties, existence results, and regularity aspects presents significant mathematical challenges and potential applications in nonlinear elasticity, material science, and phase transition models.

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Some results concerning a system of two max difference equations

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Abstract. In this paper we investigate the behaviour of the solutions of the following close-to-cyclic system of two difference equations with maximum:

$$x_{n+1} = \max\left\{A, \frac{y_n^p}{x_{n-1}^q}\right\},$$
$$y_{n+1} = \max\left\{B, \frac{x_n^p}{y_{n-1}^q}\right\}$$

where $n = 0, 1, \ldots$, the coefficients A, B, are positive real numbers, the exponents p, q are positive real numbers such that $p > q + 1$, and the initial values x_{-1}, x_0, y_{-1}, y_0, are positive real numbers.

Keywords: difference equations with maximum, close-to-cyclic system, equilibrium, solution eventually equal to equilibrium, periodic solutions.

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1 Introduction

Difference equations and systems of difference equations have gained much attention during the last decades due to their wide applications. These equations can be used as mathematical models to describe the biological, population and generally physical as well as economic phenomena (several models can be found, for example, in [11] and [29]). Despite that difference equations have, usually, simple forms, it is extremely difficult to fully describe precisely the global behavior of their solutions (see, for instance, [1]–[15], [17]–[21], [23]–[60] and the related references therein). For some differential equations with maximum see, for example, [16, 22].

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In addition, max operators are used in the study of problems concerning automatic control (see [16], [22] and the references cited therein). Therefore, there exist many papers concerning max difference equations, that is, max-type difference equations or difference equations with maximum (see, e.g., [1,4–7,9,10,12–15,17,18,21,23–27,35–45,47–51,56,58–60] and the related references therein).

At the beginning, difference equations with maximum containing linear terms were studied and are special cases of the difference equation

$$x_{n+1} = \max\left\{\frac{A_n^{(0)}}{x_n}, \frac{A_n^{(1)}}{x_{n-1}}, \ldots, \frac{A_n^{(k)}}{x_{n-k}}\right\}, \qquad n = 0, 1, \ldots,$$

see for example [4,5,7,10,14,15,21,23,57–60].

In 2003 S. Stević started a systematic study of the difference equations containing powers of the terms; see [32] where the difference equation

$$x_{n+1} = \alpha + \frac{x_{n-1}^p}{x_n^p}, \qquad n = 0, 1, \ldots,$$

was studied (for some later results see [3]). The equation with $p = 1$ or $p = -1$, and related ones had been considerably investigated before it (see, for example, [11,28,30,31] and the references therein). For some later generalizations of the equation in these cases see [2].

The investigation in [32] have been continued by him in a series of papers. In [34] beside the equation

$$x_{n+1} = A + \frac{x_n^p}{x_{n-1}^r}, \qquad n = 0, 1, \ldots,$$

(for some generalizations of the equation see, for instance, [20,33,36]), it was also studied its max-type counterpart

$$x_{n+1} = \max\left\{A, \frac{x_n^p}{x_{n-1}^r}\right\}, \qquad n = 0, 1, \ldots,$$

where the parameters A, p, q and the initial values x_{-1} and x_0 are positive numbers. This is one of the first papers where a difference equation with maximum containing powers of the terms was studied. The case $p = r$, was also studied in [35], where, among other results, it was proved that if $A, p \in (0, 1)$ all positive solutions converge to one. This result was later generalized in [9]. For some other results on convergence of solutions to related difference equations with maximum see also [12,13,37,38].

A complete picture regarding the boundedness character of positive solutions to the following difference equation

$$x_n = \max\left\{A, \frac{x_{n-1}^p}{x_{n-k}^p}\right\}, \qquad n = 0, 1, \ldots,$$

where $k \geq 2$, and the parameters A, p and the initial values are positive numbers, was given in [36]. The results on the boundedness therein were later generalized in [41]. For a quite general result in this direction see [49].

A study of a general difference equation with maximum, which includes above mentioned ones, was suggested in [40] (see difference equation (1) therein). Periodicity of quite general classes of difference equation with maximum was studied in [40,42,43,48,50]. Some results

are devoted to solvability of max type equations such as [44] (for some recent results on the topic see, for instance, [46,52,54,55] and the references therein).

In [8], studying cyclic systems of difference equations was initiated. The study was continued, for instance, in [19,46,47,49,52,55–57]. Paper [8] essentially suggested investigating all cyclic systems of difference equations, so also those corresponding to above-mentioned difference equations.

In [45] investigated the following max-type system

$$x_{n+1} = \max\left\{\alpha, \frac{y_n^p}{x_{n-1}^p}\right\}, \quad y_{n+1} = \max\left\{\alpha, \frac{x_n^p}{y_{n-1}^p}\right\}, \qquad n = 0, 1, \ldots,$$

where the parameters α, p and the initial values are positive numbers, whereas its generalization

$$x_{n+1} = \max\left\{\alpha, \frac{y_n^p}{x_{n-1}^q}\right\}, \quad y_{n+1} = \max\left\{\alpha, \frac{x_n^p}{y_{n-1}^q}\right\}, \qquad n = 0, 1, \ldots,$$

was studied in [51]. The corresponding cyclic system was studied in [56].

In [6] the authors studied the periodic character of the solutions of the system of difference equations with maximum

$$x_{n+1} = \max\left\{A, \frac{y_n}{x_{n-1}}\right\},$$

$$y_{n+1} = \max\left\{B, \frac{x_n}{y_{n-1}}\right\}$$

where A, B are positive constants and the initial values x_{-1}, x_0, y_{-1}, y_0, are positive real numbers.

Motivated by above mentioned studies, in this paper we continue the investigation of close-to-cyclic systems of difference equations by studying the behaviour of the solutions of the following close-to-cyclic system of difference equations with maximum:

$$\begin{aligned} x_{n+1} &= \max\left\{A, \frac{y_n^p}{x_{n-1}^q}\right\}, \\ y_{n+1} &= \max\left\{B, \frac{x_n^p}{y_{n-1}^q}\right\} \end{aligned} \tag{1.1}$$

where $n = 0, 1, \ldots$, the coefficients A, B, are positive real numbers, the exponents p, q are positive real numbers such that $p > q + 1$, and the initial values x_{-1}, x_0, y_{-1}, y_0, are positive real numbers.

2 Main results

In the first lemma we study the existence of the positive equilibriums of (1.1).

Lemma 2.1. *Consider the system of difference equations (1.1), where the coefficients A, B, the exponents p, q, and the initial values x_{-1}, x_0, y_{-1}, y_0, are positive real numbers. Assume that*

$$p > 1 + q, \ q > 0, \tag{2.1}$$

then, the following statements are true:

(i) *If*

$$0 < A^{\frac{p}{q+1}} < B < A^{\frac{q+1}{p}} < 1, \tag{2.2}$$

then, system (1.1) *has two equilibria*

$$(\bar{x},\ \bar{y}) = (A, B), \quad (\bar{x},\ \bar{y}) = (1, 1).$$

(ii) *If*

$$0 < B \leq A^{\frac{p}{q+1}} \leq 1, \tag{2.3}$$

then, system (1.1) *has*

(iia) *two equilibria*

$$(\bar{x},\ \bar{y}) = (1, 1), \quad (\bar{x}, \bar{y}) = (A,\ A^{\frac{p}{q+1}}),$$

if $A \neq 1$, *and*

(iib) *a unique equilibrium*

$$(\bar{x},\ \bar{y}) = (1, 1),$$

if $A = 1$.

(iii) *If*

$$0 < A^{\frac{q+1}{p}} \leq B \leq 1, \tag{2.4}$$

then, system (1.1) *has*

(iiia) *two equilibria*

$$(\bar{x},\ \bar{y}) = (1, 1), \quad (\bar{x}, \bar{y}) = (B^{\frac{p}{q+1}}, B),$$

if $B \neq 1$, *and*

(iiib) *a unique equilibrium*

$$(\bar{x},\ \bar{y}) = (1, 1),$$

if $B = 1$.

(iv) *If*

$$A > 1,\ B > 0 \quad or \quad B > 1,\ A > 0 \tag{2.5}$$

then, system (1.1) *has no equilibria.*

Proof. We consider the system of algebraic equations

$$x = \max\Big\{A, \frac{y^p}{x^q}\Big\}, \quad y = \max\Big\{B, \frac{x^p}{y^q}\Big\}. \tag{2.6}$$

Then, one of the following relations is satisfied

$$(i)\ x = A,\ y = B, \quad (ii)\ x = A,\ y = \frac{x^p}{y^q}, \quad (iii)\ x = \frac{y^p}{x^q},\ y = B,$$
$$(iv)\ x = \frac{y^p}{x^q},\ y = \frac{x^p}{y^q},$$

and these relations are equivalent to

$$(i)\ x = A,\ y = B, \quad (ii)\ x = A,\ y = A^{\frac{p}{q+1}}, \quad (iii)\ x = B^{\frac{p}{q+1}},\ y = B,$$

and, since $x, y, p, q > 0$ and $p \neq q+1$,

$$(iv)\ x = 1,\ y = 1.$$

(i). From (2.1) and (2.2), it is obvious that, $0 < A < 1$.

Let $x = A,\ y = B$. From (2.1), (2.2), and since $0 < A \leq 1$, we get

$$\frac{y^p}{x^q} = \frac{B^p}{A^q} < A,$$

and

$$\frac{x^p}{y^q} = \frac{A^p}{B^q} < B.$$

Hence, $(\bar{x}, \bar{y}) = (A, B)$, is an equilibrium of (1.1).

Now, suppose that $x = A,\ y = A^{\frac{p}{q+1}}$, then from (2.6) we get

$$y = A^{\frac{p}{q+1}} \geq B,$$

which contradicts with (2.2). Hence, $(\bar{x}, \bar{y}) = (A, A^{\frac{p}{q+1}})$, is not an equilibrium of (1.1).

To continue, we assume that, $x = B^{\frac{p}{q+1}},\ y = B$, then from (2.6) we have

$$x = B^{\frac{p}{q+1}} \geq A,$$

which contradicts with (2.2), since $p,\ q > 0$. So, $(B^{\frac{p}{q+1}},\ B)$, is not an equilibrium of (1.1).

Finally, since $0 < A, B < 1$, from (2.6), we have that, $(\bar{x}, \bar{y}) = (1, 1)$ is an equilibrium of system (1.1).

(ii). From (2.1) and (2.3), it is obvious that, $0 < A \leq 1$.

Let $x = A,\ y = B$. Then, from (2.6), we get $B \geq \frac{A^p}{B^q}$, and so, $B \geq A^{\frac{p}{q+1}}$, which contradicts with (2.3), except if $B = A^{\frac{p}{q+1}}$, and then, $x = A,\ y = A^{\frac{p}{q+1}}$, case that considered below. Hence, (A, B), $B \neq A^{\frac{p}{q+1}}$, is not an equilibrium of (1.1).

Now, suppose that $x = A,\ y = A^{\frac{p}{q+1}}$. From (2.1), (2.3), and since $0 < A \leq 1$, we get

$$\frac{y^p}{x^q} = \frac{A^{\frac{p^2}{q+1}}}{A^q} \leq A,$$

and

$$\frac{x^p}{y^q} = \frac{A^p}{A^{\frac{qp}{q+1}}} = A^{\frac{p}{q+1}} \geq B.$$

Hence, $(\bar{x}, \bar{y}) = (A, A^{\frac{p}{q+1}})$, is an equilibrium of (1.1). Obviously, if $A = 1$, the equilibrium of (1.1) is $(\bar{x}, \bar{y}) = (1, 1)$.

To continue, we assume that, $x = B^{\frac{p}{q+1}},\ y = B$. Then, from (2.1), (2.3) and (2.6), we have

$$B^{\frac{p}{q+1}} \geq A \geq B^{\frac{q+1}{p}}. \tag{2.7}$$

If $0 < B < 1$, relation (2.7) can not be true, since $p > q+1 > 0$. So, $(B^{\frac{p}{q+1}},\ B)$, is not an equilibrium of (1.1), if $0 < B < 1$. On the other hand, relation (2.7) is true, if $A = B = 1$, and then, $(\bar{x}, \bar{y}) = (1, 1)$ is an equilibrium of (1.1).

Finally, since $0 < A, B \leq 1$, from (2.6), we have that, $(\bar{x}, \bar{y}) = (1,1)$ is an equilibrium of system (1.1).

From all the above, it is obvious that if $A = 1$ and $0 < B \leq 1$, then, $(\bar{x}, \bar{y}) = (1,1)$ is the unique equilibrium of (1.1). This completes the proof of (ii).

(iii). Using (2.4) and (2.6), and arguing as in (ii), we can easily prove (iii).

(iv). From (2.1) and (2.6), we get

$$x \geq y^{\frac{p}{q+1}}, \quad y \geq x^{\frac{p}{q+1}},$$

and so,

$$x \geq x^{\left(\frac{p}{q+1}\right)^2}. \tag{2.8}$$

If $x > 1$, then, from (2.8), $q + 1 \geq p$, which contradicts with (2.1). Hence, $0 < x \leq 1$, and similarly, $0 < y \leq 1$. Therefore, in order to have solutions for (2.6), it is necessary $0 < A, B \leq 1$, since (2.1) holds. So, system (1.1) has no equilibria, if (2.1) and (2.5) holds. This completes the proof of the lemma. □

In the following proposition we study the asymptotic behavior of the positive solutions of (1.1). We need the following lemma on a product-type difference equation, for which related results and methods can be found in the literature (see, for instance, [32, Remark 1], [41, Theorem 1], [54] and [55]).

Lemma 2.2. *Consider the difference equation of the form*

$$z_{n+1} = \frac{z_n^a}{z_{n-1}^b}, \qquad n \geq 0, \tag{2.9}$$

where z_{-1}, z_0 are positive real numbers, and a, b are real numbers, such that

$$a > 1 + b, \ b > 0. \tag{2.10}$$

Then, the following statements are true:

(i) *If*

$$\frac{z_0^{\lambda_1}}{z_{-1}^b} > 1, \tag{2.11}$$

where

$$\lambda_1 = \frac{a + \sqrt{a^2 - 4b}}{2},$$

then

$$\lim_{n\to\infty} z_n = \infty. \tag{2.12}$$

(ii) *If*

$$\frac{z_0^{\lambda_1}}{z_{-1}^b} = 1, \tag{2.13}$$

then

$$\lim_{n\to\infty} z_n = 1. \tag{2.14}$$

(iii) *If*

$$\frac{z_0^{\lambda_1}}{z_{-1}^{b}} < 1, \tag{2.15}$$

then

$$\lim_{n\to\infty} z_n = 0. \tag{2.16}$$

Proof. We prove that the solution z_n of (2.9) is given by

$$z_n = \frac{z_0^{f_n}}{z_{-1}^{g_n}}, \qquad n \geq 1, \tag{2.17}$$

where f_n satisfies the initial value problem

$$f_{n+1} = af_n - bf_{n-1}, \quad n = 2,3,\ldots, \quad f_1 = a, \quad f_2 = a^2 - b, \tag{2.18}$$

and g_n satisfies the initial value problem

$$g_{n+1} = ag_n - bg_{n-1}, \quad n = 2,3,\ldots, \quad g_1 = b, \quad g_2 = ab. \tag{2.19}$$

From (2.9), (2.18) and (2.19), we have

$$z_1 = \frac{z_0^{a}}{z_{-1}^{b}} = \frac{z_0^{f_1}}{z_{-1}^{g_1}}, \tag{2.20}$$

and

$$z_2 = \frac{z_1^{a}}{z_0^{b}} = \frac{z_0^{af_1-b}}{z_{-1}^{ag_1}} = \frac{z_0^{a^2-b}}{z_{-1}^{ab}} = \frac{z_0^{f_2}}{z_{-1}^{g_2}}. \tag{2.21}$$

In addition, from (2.9), (2.18), (2.19), (2.20) and (2.21), we get

$$z_3 = \frac{z_2^{a}}{z_1^{b}} = \frac{z_0^{af_2}}{z_{-1}^{ag_2}} \frac{z_0^{-bf_1}}{z_{-1}^{-bg_1}} = \frac{z_0^{f_3}}{z_{-1}^{g_3}}. \tag{2.22}$$

From (2.9), (2.18), (2.19), (2.21) and (2.22), and working inductively, we can easily prove (2.17).

From relations (2.10), (2.18) and (2.19), we get

$$f_n = c_1\lambda_1^n + c_2\lambda_2^n, \qquad n = 1,2,\ldots, \tag{2.23}$$

where

$$\begin{aligned} \lambda_1 &= \frac{a+\sqrt{a^2-4b}}{2}, & \lambda_2 &= \frac{a-\sqrt{a^2-4b}}{2}, \\ c_1 &= \frac{\lambda_1}{\sqrt{a^2-4b}}, & c_2 &= \frac{-\lambda_2}{\sqrt{a^2-4b}}, \end{aligned} \tag{2.24}$$

and

$$g_n = d_1\lambda_1^n + d_2\lambda_2^n, \qquad n = 1,2,\ldots, \tag{2.25}$$

where

$$d_1 = \frac{b}{\sqrt{a^2-4b}}, \qquad d_2 = -\frac{b}{\sqrt{a^2-4b}}. \tag{2.26}$$

From (2.23) and (2.24), we get

$$f_n = \frac{1}{\sqrt{a^2-4b}}\lambda_1^{n+1} - \frac{1}{\sqrt{a^2-4b}}\lambda_2^{n+1}. \tag{2.27}$$

So, from relations (2.17), (2.25), (2.26) and (2.27), we have

$$\begin{aligned} z_n = \frac{z_0^{f_n}}{z_{-1}^{g_n}} &= \frac{z_0^{\frac{1}{\sqrt{a^2-4b}}\lambda_1^{n+1} - \frac{1}{\sqrt{a^2-4b}}\lambda_2^{n+1}}}{z_{-1}^{\frac{b}{\sqrt{a^2-4b}}\lambda_1^{n} - \frac{b}{\sqrt{a^2-4b}}\lambda_2^{n}}} \\ &= \left(\frac{z_0^{\lambda_1}}{z_{-1}^{b}}\right)^{\frac{1}{\sqrt{a^2-4b}}\lambda_1^n} \left(\frac{z_0^{\lambda_2}}{z_{-1}^{b}}\right)^{-\frac{1}{\sqrt{a^2-4b}}\lambda_2^n}. \end{aligned} \tag{2.28}$$

(i). First, suppose that (2.11) is satisfied. Since, from (2.10),

$$\lambda_1 > 1, \qquad 0 < \lambda_2 < 1, \tag{2.29}$$

then, from (2.28), we get (2.12).

(ii). Now, suppose that (2.13) hold. Then, from (2.28), we get

$$z_n = \left(\frac{z_0^{\lambda_2}}{z_{-1}^{b}}\right)^{-\frac{1}{\sqrt{a^2-4b}}\lambda_2^n},$$

and since $0 < \lambda_2 < 1$, we get (2.14).

(iii). Finally, suppose that (2.15) is satisfied. Then, from (2.28) and (2.29), we get (2.16). This completes the proof of the lemma. □

Lemma 2.3. *Consider the system of difference equations (1.1), where relations (2.1) and (2.2) hold, and the initial values x_{-1}, x_0, y_{-1}, y_0, are positive real numbers. Then, the following statements are true:*

I. *If there exists a positive integer $m \geq 2$, such that*

$$x_m \leq B^{\frac{q+1}{p}}, \tag{2.30}$$

then

$$x_{m+2k} = A, \quad y_{m+2k-1} = B, \quad k \geq 1. \tag{2.31}$$

II. *Suppose that*

$$x_{2n} > B^{\frac{q+1}{p}}, \quad \text{for any } n \geq 1. \tag{2.32}$$

(a). *If*

$$\frac{x_2^{\lambda}}{x_0^{q^2}} > 1, \qquad \lambda = \frac{p^2 - 2q + \sqrt{(p^2-2q)^2 - 4q^2}}{2}, \tag{2.33}$$

then

$$\lim_{n\to\infty} x_{2n} = \infty, \qquad \lim_{n\to\infty} y_{2n+1} = \infty. \tag{2.34}$$

(b). *If*

$$\frac{x_2^{\lambda}}{x_0^{q^2}} = 1, \tag{2.35}$$

then

$$\lim_{n\to\infty} x_{2n} = 1, \qquad \lim_{n\to\infty} y_{2n+1} = 1. \tag{2.36}$$

III. *Suppose that*

$$x_{2n+1} > B^{\frac{q+1}{p}}, \quad \text{for any } n \geq 1. \tag{2.37}$$

(a). *If*

$$\frac{x_3^{\lambda}}{x_1^{q^2}} > 1, \tag{2.38}$$

then

$$\lim_{n\to\infty} x_{2n+1} = \infty, \qquad \lim_{n\to\infty} y_{2n} = \infty. \tag{2.39}$$

(b). *If*

$$\frac{x_3^{\lambda}}{x_1^{q^2}} = 1, \tag{2.40}$$

then

$$\lim_{n\to\infty} x_{2n+1} = 1, \qquad \lim_{n\to\infty} y_{2n} = 1. \tag{2.41}$$

Proof. From (2.2) we have

$$\text{(i) } 0 < A < 1, \quad 0 < B < 1, \quad \text{(ii) } A > B^{\frac{p}{q+1}}, \quad \text{(iii) } B > A^{\frac{p}{q+1}}, \tag{2.42}$$

I. From (1.1) and (2.30), we get

$$\frac{x_m^p}{y_{m-1}^q} \leq \frac{B^{q+1}}{B^q} = B,$$

and so, from (1.1),

$$y_{m+1} = B. \tag{2.43}$$

Then, from (1.1), (i) and (ii) of (2.42) and (2.43), we get

$$\frac{y_{m+1}^p}{x_m^q} \leq \frac{B^p}{A^q} < A,$$

and so, from (1.1),

$$x_{m+2} = A. \tag{2.44}$$

In addition, from (iii) of (2.42) and (2.44), it follows that

$$x_{m+2} < B^{\frac{q+1}{p}}. \tag{2.45}$$

Using (2.45) and working as above we have

$$y_{m+3} = B, \qquad x_{m+4} = A,$$

and so, working inductively, we get (2.31).

II. From (1.1), (iii) of (2.42) and (2.32), we have

$$x_{2n} = \frac{y_{2n-1}^p}{x_{2n-2}^q}, \qquad n \geq 1. \tag{2.46}$$

From (2.1), (i) of (2.42), (2.32) and (2.46), we have, for $n \geq 2$,

$$y_{2n-1} = x_{2n}^{1/p} x_{2n-2}^{q/p} > B^{\frac{q+1}{p}\frac{1}{p}} B^{\frac{q+1}{p}\frac{q}{p}} = B^{(\frac{q+1}{p})^2} > B,$$

and so, from (1.1),

$$y_{2n-1} = \frac{x_{2n-2}^p}{y_{2n-3}^q}, \qquad n \geq 2. \tag{2.47}$$

Relations (2.46) and (2.47) imply that

$$x_{2n} = \frac{x_{2n-2}^{p^2-q}}{y_{2n-3}^{pq}}, \qquad n \geq 2,$$

and then, since from (2.46),

$$y_{2n-3}^p = x_{2n-2} x_{2n-4}^q, \qquad n \geq 2,$$

we get the following product-type difference equation (see, e.g., [54, 55])

$$x_{2n} = \frac{x_{2n-2}^a}{x_{2n-4}^b}, \quad n \geq 2, \quad a = p^2 - 2q, \quad b = q^2, \tag{2.48}$$

with interlacing indices (see [53])

Since (2.1) holds, we can easily prove that (2.10) is true. In (2.48), we set

$$x_{2n-4} = z_{n-1}, \tag{2.49}$$

(see [53]), and we get the equation

$$z_{n+1} = \frac{z_n^a}{z_{n-1}^b}, \qquad n \geq 2. \tag{2.50}$$

We consider the difference equation

$$w_{n+1} = \frac{w_n^a}{w_{n-1}^b}, \qquad n \geq 0. \tag{2.51}$$

Let w_n be the solution of (2.51), such that

$$w_{-1} = z_1, \qquad w_0 = z_2, \tag{2.52}$$

II(a). Using (2.33), (2.49) and (2.52) we have

$$\frac{w_0^\lambda}{w_{-1}^{q^2}} = \frac{z_2^\lambda}{z_1^{q^2}} = \frac{x_2^\lambda}{x_0^{q^2}} > 1. \tag{2.53}$$

Since for equation (2.51), relations (2.10) and (2.53) hold, from (i) of Lemma 2.2, we get

$$\lim_{n\to\infty} w_n = \infty,$$

and using (2.50), (2.51) and (2.52), we have

$$\lim_{n\to\infty} z_n = \infty,$$

and so, from (2.49),

$$\lim_{n\to\infty} x_{2n} = \infty.$$

Then, from (2.46), we get

$$\lim_{n\to\infty} y_{2n+1} = \infty.$$

Hence, (2.34) is true.

II(b). Since (2.35), (2.49) and (2.52) hold, we have

$$\frac{w_0^{\lambda}}{w_{-1}^{q^2}} = \frac{z_2^{\lambda}}{z_1^{q^2}} = \frac{x_2^{\lambda}}{x_0^{q^2}} = 1. \tag{2.54}$$

Since for equation (2.51), relations (2.10) and (2.54) hold, from (ii) of Lemma 2.2, we get

$$\lim_{n\to\infty} w_n = 1,$$

and using (2.50), (2.51) and (2.52), we have

$$\lim_{n\to\infty} z_n = 1,$$

and so, from (2.49) ,

$$\lim_{n\to\infty} x_{2n} = 1.$$

From (2.46), we get

$$\lim_{n\to\infty} y_{2n+1} = 1.$$

Hence, (2.36) is true.

We mention that, difference equation (2.48) is with interlacing indices, and it was expected that its behavior is similar to (2.9) (for more details see, for example, [53]).

Arguing as in cases **II(a)** and **II(b)**, we can easily prove **III(a)** and **III(b)**. □

Remark 2.4. In statement **II** of Lemma 2.3, it is impossible to have

$$\frac{x_2^{\lambda}}{x_0^{q^2}} < 1. \tag{2.55}$$

Indeed, if, on the contrary, we assume that (2.55) holds, then, arguing as in **II(a)** of Lemma 2.3, for the solution w_n of (2.51) we get

$$\frac{w_0^{\lambda}}{w_{-1}^{q^2}} < 1,$$

and so, from statement **(iii)** of Lemma 2.2,

$$\lim_{n\to\infty} w_n = 0,$$

which means that

$$\lim_{n\to\infty} x_{2n} = 0,$$

which contradicts with the fact that $x_n \geq A$, for any $n \geq 1$. So, (2.55) cannot be satisfied.

Similarly, it is impossible to have

$$\frac{x_3^{\lambda}}{x_1^{q^2}} < 1,$$

in statement **III** of Lemma 2.3.

Proposition 2.5. *Consider the system of difference equations (1.1), where relations (2.1) and (2.2) hold, and the initial values x_{-1}, x_0, y_{-1}, y_0, are positive real numbers. Let (x_n, y_n) be a solution of (1.1). Then, the following statements are true:*

I. *Suppose that there exist integers $l, r \geq 1$, such that*

$$x_{2l+1} \leq B^{\frac{q+1}{p}}, \qquad x_{2r} \leq B^{\frac{q+1}{p}}. \tag{2.56}$$

Then, (x_n, y_n) is eventually equal to the positive equilibrium (A, B) of (1.1).

II. *Suppose that there exists an integer $l \geq 1$, such that the first inequality of (2.56) and relations (2.32), (2.33) hold. Then, relation (2.34) and*

$$x_{2n+1} = A, \quad y_{2n} = B, \quad n \geq l+1, \tag{2.57}$$

are satisfied.

III. *Suppose that there exists an integer $l \geq 1$, such that the first inequality of (2.56) and relations (2.32), (2.35) hold. Then, relations (2.36) and (2.57) are satisfied.*

IV. *Suppose that there exists an integer $r \geq 1$, such that the second inequality of (2.56) and relations (2.37), (2.38) hold. Then, relation (2.39) and*

$$x_{2n} = A, \quad y_{2n+1} = B, \quad n \geq r+1, \tag{2.58}$$

are satisfied.

V. *Suppose that there exists an integer $r \geq 1$, such that the second inequality of (2.56) and relations (2.37), (2.40) hold. Then, relations (2.41) and (2.58) are satisfied.*

VI. *If relations (2.32), (2.33), (2.37) and (2.38) hold, then,*

$$\lim_{n\to\infty} x_n = \infty, \qquad \lim_{n\to\infty} y_n = \infty. \tag{2.59}$$

VII. *If relations (2.32), (2.33), (2.37) and (2.40) hold, then, relations (2.34) and (2.41) are satisfied.*

VIII. *If relations (2.32), (2.35), (2.37) and (2.38) hold, then, relations (2.36) and (2.39) are satisfied.*

IX. *If relations (2.32), (2.35), (2.37) and (2.40) hold, then,*

$$\lim_{n\to\infty} x_n = 1, \qquad \lim_{n\to\infty} y_n = 1. \tag{2.60}$$

Proof. **I.** From (2.56) and statement I of Lemma 2.3, we have

$$x_{2l+1+2k} = A, \quad y_{2l+2k} = B, \quad x_{2r+2k} = A, \quad y_{2r+2k-1} = B, \quad k = 1, 2, \ldots.$$

So, for $n \geq \max\{2l+3, 2r+2\}$, we get $x_n = A$, $y_n = B$. This implies that, (x_n, y_n) is eventually equal to (A, B).

II. From the first inequality of (2.56) and statement **I** of Lemma 2.3, we have

$$x_{2l+1+2k} = A, \qquad y_{2l+2k} = B, \qquad k = 1, 2, \ldots,$$

and so, (2.57) is true. Moreover, from (2.32), (2.33) and statement **II(a)** of Lemma 2.3, we have (2.34).

Arguing as above, using the corresponding relations and the appropriate statements of Lemma 2.3, we can easily prove statements **III–IX** of the proposition. This completes the proof of the proposition. □

Lemma 2.6. *Consider the system of difference equations* (1.1)*, where* (2.1) *and* (2.3) *hold, and the initial values* x_{-1}, x_0, y_{-1}, y_0*, are positive real numbers. Then, the following statements are true:*

I. *Suppose that there exists an integer* $m \geq 2$*, such that*

$$y_m = A^{\frac{p}{q+1}}, \tag{2.61}$$

then

$$x_{m+2k+1} = A, \quad y_{m+2k} = A^{\frac{p}{q+1}}, \quad k \geq 0. \tag{2.62}$$

II. *Suppose that*

$$y_{2n} > A^{\frac{p}{q+1}}, \quad \text{for any } n \geq 1. \tag{2.63}$$

(a). *If*

$$\frac{y_2^{\lambda}}{y_0^{q^2}} > 1, \tag{2.64}$$

where λ *was defined in* (2.33)*, then* (2.39) *holds.*

(b). *If*

$$\frac{y_2^{\lambda}}{y_0^{q^2}} = 1, \tag{2.65}$$

then (2.41) *holds.*

III. *Suppose that*

$$y_{2n+1} > A^{\frac{p}{q+1}}, \quad \text{for any } n \geq 1. \tag{2.66}$$

(a). *If*

$$\frac{y_3^{\lambda}}{y_1^{q^2}} > 1, \tag{2.67}$$

then (2.34) *holds.*

(b). *If*

$$\frac{y_3^{\lambda}}{y_1^{q^2}} = 1, \tag{2.68}$$

then (2.36) *holds.*

IV. *Suppose that there exists an integer $s \geq 2$, such that*

$$y_s < A^{\frac{p}{q+1}}. \tag{2.69}$$

Assume that

$$A^{\frac{p^2-q-1}{pq}} \leq B, \tag{2.70}$$

then

$$x_{s+2k-1} = A, \quad k \geq 1, \tag{2.71}$$

and, in addition,

(a). *If*

$$q < 1, \tag{2.72}$$

then

$$\lim_{k\to\infty} y_{s+2k} = A^{\frac{p}{q+1}}. \tag{2.73}$$

(b). *If*

$$q = 1, \tag{2.74}$$

then

$$y_{s+4k-2} = \frac{A^p}{y_s}, \quad y_{s+4k} = y_s, \quad k \geq 1. \tag{2.75}$$

(c). *If*

$$q > 1, \tag{2.76}$$

then there exists a positive integer k_0, such that

$$y_{s+4k-2} = \frac{A^p}{B^q}, \quad y_{s+4k-4} = B, \quad k \geq k_0 + 1. \tag{2.77}$$

Proof. From relations (2.1) and (2.3) we have $0 < A \leq 1$.

I. Since, from (2.1) and (2.3),

$$A^{\frac{p}{q+1}} \leq A^{\frac{q+1}{p}}, \tag{2.78}$$

from (1.1) and (2.61), we get

$$\frac{y_m^p}{x_{m-1}^q} \leq \frac{A^{q+1}}{A^q} = A,$$

and so, from (1.1),

$$x_{m+1} = A. \tag{2.79}$$

Then, from (2.3), (2.61) and (2.79), we get

$$\frac{x_{m+1}^p}{y_m^q} = \frac{A^p}{\left(A^{\frac{p}{q+1}}\right)^q} = A^{\frac{p}{q+1}} \geq B,$$

and so, from (1.1),

$$y_{m+2} = A^{\frac{p}{q+1}},$$

and working inductively, we get (2.62).

Arguing as in statements **II** and **III** of Lemma 2.3, we can easily prove **II** and **III** of this Lemma.

IV. From (1.1), (2.69) and (2.78), we get

$$\frac{y_s^p}{x_{s-1}^q} \leq \frac{A^{q+1}}{A^q} = A,$$

and so, from (1.1),

$$x_{s+1} = A. \tag{2.80}$$

Then, from (2.3), (2.69) and (2.80), we get

$$\frac{x_{s+1}^p}{y_s^q} > \frac{A^p}{\left(A^{\frac{p}{q+1}}\right)^q} = A^{\frac{p}{q+1}} \geq B,$$

and so, from (1.1),

$$y_{s+2} = \frac{A^p}{y_s^q} > A^{\frac{p}{q+1}}. \tag{2.81}$$

From (1.1), (2.70), (2.80) and (2.81), we get

$$\frac{y_{s+2}^p}{x_{s+1}^q} = \frac{A^{p^2-q}}{y_s^{pq}} \leq \frac{A^{p^2-q}}{B^{pq}} \leq \frac{A^{p^2-q}}{A^{p^2-q-1}} = A,$$

and so, from (1.1),

$$x_{s+3} = A. \tag{2.82}$$

From (2.81) and (2.82), we have

$$\frac{x_{s+3}^p}{y_{s+2}^q} = A^{p-pq} y_s^{q^2},$$

and so, from (1.1),

$$y_{s+4} = \max\left\{B,\ A^{p-pq} y_s^{q^2}\right\}. \tag{2.83}$$

In addition, from (2.81) and (2.82), we have

$$\frac{x_{s+3}^p}{y_{s+2}^q} < \frac{A^p}{\left(A^{\frac{p}{q+1}}\right)^q} = A^{\frac{p}{q+1}},$$

and so, from (1.1) and (2.3), we get

$$y_{s+4} < A^{\frac{p}{q+1}}. \tag{2.84}$$

From (2.69), (2.80), (2.81), (2.82), (2.83), (2.84), and working inductively, we get (2.71), and

$$y_{s+4k-2} = \frac{A^p}{y_{s+4k-4}^q} > A^{\frac{p}{q+1}}, \qquad k \geq 1, \tag{2.85}$$

and

$$y_{s+4k} = \max\{B,\ A^{p-pq} y_{s+4k-4}^{q^2}\} < A^{\frac{p}{q+1}}, \qquad k \geq 1. \tag{2.86}$$

IV(a). First, suppose that (2.72) holds, then, from (1.1) and (2.3), we get

$$A^{p-pq} y_{s+4k-4}^{q^2} \geq A^{p-pq} B^{q^2} \geq B,$$

and so, from (2.86), we have

$$y_{s+4k} = A^{p-pq} y_{s+4k-4}^{q^2}, \qquad k \geq 1. \tag{2.87}$$

From (2.87), and by induction, (see also [55] and note that the equation is with interlacing indices [53]), it is easy to prove that

$$y_{s+4k} = A^{p\frac{1-(q^2)^k}{q+1}} y_s^{(q^2)^k}, \qquad k \geq 1. \tag{2.88}$$

From (2.88) and (2.72), we have

$$\lim_{k\to\infty} y_{s+4k} = A^{\frac{p}{q+1}}. \tag{2.89}$$

Using (2.85) and (2.89), we have

$$\lim_{k\to\infty} y_{s+4k-2} = A^{\frac{p}{q+1}}. \tag{2.90}$$

From (2.89) and (2.90), it is obvious that (2.73) is true.

IV(b). Now, suppose that (2.74) holds, then from (1.1) and (2.86), we have

$$y_{s+4k} = \max\{B,\ y_{s+4k-4}\} = y_{s+4k-4}, \qquad k \geq 1,$$

and so, the second relation of (2.75) is true. From the second relation of (2.75) and (2.85), we have that the first relation of (2.75) is also true.

IV(c). Finally, suppose that (2.76) holds. We prove that, there exists a positive integer k_0, such that,

$$y_{s+4k_0} = B. \tag{2.91}$$

On the contrary, we assume that

$$y_{s+4k} > B, \qquad k \geq 0,$$

then, from (2.86), we get (2.87), and so, (2.88) is true. Then, from (2.88), we get

$$y_{s+4k} = A^{\frac{p}{q+1}} \left(\frac{y_s}{A^{\frac{p}{q+1}}} \right)^{(q^2)^k},$$

and so, from (2.69) and (2.76),

$$\lim_{k\to\infty} y_{s+4k} = 0,$$

which is a contradiction, since from (1.1),

$$y_{s+4k} \geq B > 0, \qquad k \geq 0.$$

So, there exists a positive integer k_0, such that (2.91) holds.
From (2.1), (2.3), (2.76), (2.85), (2.86) and (2.91), we have

$$y_{s+4k_0+2} = \frac{A^p}{B^q}, \qquad y_{s+4k_0+4} = \max\{B, A^{p-pq} B^{q^2}\} = B,$$

and working inductively, we get (2.77). This completes the proof of the lemma. □

Using Lemma 2.6 and arguing as in Proposition 2.5, we can easily prove the following proposition.

Proposition 2.7. *Consider the system of difference equations* (1.1)*, where relations* (2.1) *and* (2.3) *hold, and the initial values* x_{-1}, x_0, y_{-1}, y_0*, are positive real numbers. Let* (x_n, y_n) *be a solution of* (1.1)*. Then, the following statements are true:*

I. *Suppose that there exist integers* $l, r \geq 1$*, such that*

$$y_{2l+1} = A^{\frac{p}{q+1}}, \qquad y_{2r} = A^{\frac{p}{q+1}}. \tag{2.92}$$

Then, (x_n, y_n) *is eventually equal to the positive equilibrium* $(A, A^{\frac{p}{q+1}})$ *of* (1.1)*.*

II. *Suppose that there exists an integer* $l \geq 1$*, such that the first equality of* (2.92) *and relations* (2.63), (2.64) *hold. Then, relation* (2.39) *and*

$$x_{2n} = A, \quad y_{2n-1} = A^{\frac{p}{q+1}}, \quad n \geq l+1, \tag{2.93}$$

are satisfied.

III. *Suppose that there exists an integer* $l \geq 1$*, such that the first equality of* (2.92) *and relations* (2.63), (2.65) *hold. Then, relations* (2.41) *and* (2.93) *are satisfied.*

IV. *Suppose that there exists an integer* $r \geq 1$*, such that the second equality of* (2.92) *and relations* (2.66), (2.67) *hold. Then, relation* (2.34) *and*

$$x_{2n+1} = A, \quad y_{2n} = A^{\frac{p}{q+1}}, \quad n \geq r, \tag{2.94}$$

are satisfied.

V. *Suppose that there exists an integer* $r \geq 1$*, such that the second equality of* (2.92) *and relations* (2.66), (2.68) *hold. Then, relations* (2.36) *and* (2.94) *are satisfied.*

VI. *If relations* (2.63), (2.64), (2.66) *and* (2.67) *hold, then* (2.59) *is satisfied.*

VII. *If relations* (2.63), (2.64), (2.66) *and* (2.68) *hold, then, relations* (2.36) *and* (2.39) *are satisfied.*

VIII. *If relations* (2.63), (2.65), (2.66) *and* (2.67) *hold, then, relations* (2.34) *and* (2.41) *are satisfied.*

IX. *If relations* (2.63), (2.65), (2.66) *and* (2.68) *hold, then* (2.60) *is satisfied.*

If, in addition, relation (2.70) *holds, then the following statements are true:*

X. *Suppose that there exists an integer* $l \geq 1$*, such that the first equality of* (2.92) *holds and an integer* $t \geq 1$*, such that*

$$y_{2t} < A^{\frac{p}{q+1}}. \tag{2.95}$$

Then

(a). *if* $q < 1$*, then,*

$$x_n = A, \quad n \geq \max\{2l+2, 2t+1\},$$
$$y_{2n+1} = A^{\frac{p}{q+1}}, \quad n \geq l, \quad \lim_{n\to\infty} y_{2n} = A^{\frac{p}{q+1}},$$

(b). *if* $q = 1$*, then,*

$$x_n = A, \quad n \geq \max\{2l+2, 2t+1\}, \quad y_{2n+1} = A^{\frac{p}{q+1}}, \quad n \geq l,$$
$$y_{2t+4n-2} = \frac{A^p}{y_{2t}}, \quad y_{2t+4n} = y_{2t}, \quad n \geq 1,$$

(c). *if $q > 1$, then, there exists a positive integer n_0, such that*

$$x_n = A, \quad n \geq \max\{2l+2, 2t+1\}, \quad y_{2n+1} = A^{\frac{p}{q+1}}, \quad n \geq l,$$
$$y_{2t+4n-2} = \frac{A^p}{B^q}, \quad y_{2t+4n-4} = B, \quad n \geq n_0 + 1,$$

XI. *Suppose that there exists an integer $r \geq 1$, such that the second equality of (2.92) holds and an integer $v \geq 1$, such that*

$$y_{2v+1} < A^{\frac{p}{q+1}}. \tag{2.96}$$

Then

(a). *if $q < 1$, then,*

$$x_n = A, \quad n \geq \max\{2r+1, 2v+2\},$$
$$y_{2n} = A^{\frac{p}{q+1}}, \quad n \geq r, \quad \lim_{n\to\infty} y_{2n+1} = A^{\frac{p}{q+1}},$$

(b). *if $q = 1$, then,*

$$x_n = A, \quad n \geq \max\{2r+1, 2v+2\}, \quad y_{2n} = A^{\frac{p}{q+1}}, \quad n \geq r,$$
$$y_{2v+4n-1} = \frac{A^p}{y_{2v+1}}, \quad y_{2v+4n+1} = y_{2v+1}, \quad n \geq 1,$$

(c). *if $q > 1$, then, there exists a positive integer n_1, such that*

$$x_n = A, \quad n \geq \max\{2r+1, 2v+2\}, \quad y_{2n} = A^{\frac{p}{q+1}}, \quad n \geq r,$$
$$y_{2v+4n-1} = \frac{A^p}{B^q}, \quad y_{2v+4n-3} = B, \quad n \geq n_1 + 1.$$

XII. *If relations (2.63), (2.64) and (2.96) hold, then*

(a). *if $q < 1$, then,*

$$x_{2n} = A, \quad n \geq v+1, \quad \lim_{n\to\infty} x_{2n+1} = \infty,$$
$$\lim_{n\to\infty} y_{2n} = \infty, \quad \lim_{n\to\infty} y_{2n+1} = A^{\frac{p}{q+1}},$$

(b). *if $q = 1$, then,*

$$x_{2n} = A, \quad n \geq v+1, \quad \lim_{n\to\infty} x_{2n+1} = \infty,$$
$$\lim_{n\to\infty} y_{2n} = \infty, \quad y_{2v+4n-1} = \frac{A^p}{y_{2v+1}}, \quad y_{2v+4n+1} = y_{2v+1}, \quad n \geq 1,$$

(c). *if $q > 1$, then, there exists a positive integer n_1, such that*

$$x_{2n} = A, \quad n \geq v+1, \quad \lim_{n\to\infty} x_{2n+1} = \infty,$$
$$\lim_{n\to\infty} y_{2n} = \infty, \quad y_{2v+4n-1} = \frac{A^p}{B^q}, \quad y_{2v+4n-3} = B, \quad n \geq n_1 + 1.$$

XIII. *If relations (2.63), (2.65) and (2.96) hold, then*

(a). *if* $q < 1$, *then,*

$$x_{2n} = A, \quad n \geq v+1, \quad \lim_{n\to\infty} x_{2n+1} = 1,$$
$$\lim_{n\to\infty} y_{2n} = 1, \quad \lim_{n\to\infty} y_{2n+1} = A^{\frac{p}{q+1}},$$

(b). *if* $q = 1$, *then,*

$$x_{2n} = A, \quad n \geq v+1, \quad \lim_{n\to\infty} x_{2n+1} = 1,$$
$$\lim_{n\to\infty} y_{2n} = 1, \quad y_{2v+4n-1} = \frac{A^p}{y_{2v+1}}, \quad y_{2v+4n+1} = y_{2v+1}, \quad n \geq 1,$$

(c). *if* $q > 1$, *then, there exists a positive integer* n_1, *such that*

$$x_{2n} = A, \quad n \geq v+1, \quad \lim_{n\to\infty} x_{2n+1} = 1,$$
$$\lim_{n\to\infty} y_{2n} = 1, \quad y_{2v+4n-1} = \frac{A^p}{B^q}, \quad y_{2v+4n-3} = B, \quad n \geq n_1 + 1.$$

XIV. *If relations* (2.66), (2.67) *and* (2.95) *hold, then*

(a). *if* $q < 1$, *then,*

$$x_{2n+1} = A, \quad n \geq t, \quad \lim_{n\to\infty} x_{2n} = \infty,$$
$$\lim_{n\to\infty} y_{2n+1} = \infty, \quad \lim_{n\to\infty} y_{2n} = A^{\frac{p}{q+1}},$$

(b). *if* $q = 1$, *then,*

$$x_{2n+1} = A, \quad n \geq t, \quad \lim_{n\to\infty} x_{2n} = \infty,$$
$$\lim_{n\to\infty} y_{2n+1} = \infty, \quad y_{2t+4n-2} = \frac{A^p}{y_{2t}}, \quad y_{2t+4n} = y_{2t}, \quad n \geq 1,$$

(c). *if* $q > 1$, *then, there exists a positive integer* n_0, *such that*

$$x_{2n+1} = A, \quad n \geq t, \quad \lim_{n\to\infty} x_{2n} = \infty,$$
$$\lim_{n\to\infty} y_{2n+1} = \infty, \quad y_{2t+4n-2} = \frac{A^p}{B^q}, \quad y_{2t+4n-4} = B, \quad n \geq n_0 + 1.$$

XV. *If relations* (2.66), (2.68) *and* (2.95) *hold, then*

(a). *if* $q < 1$, *then,*

$$x_{2n+1} = A, \quad n \geq t, \quad \lim_{n\to\infty} x_{2n} = 1,$$
$$\lim_{n\to\infty} y_{2n+1} = 1, \quad \lim_{n\to\infty} y_{2n} = A^{\frac{p}{q+1}},$$

(b). *if* $q = 1$, *then,*

$$x_{2n+1} = A, \quad n \geq t, \quad \lim_{n\to\infty} x_{2n} = 1,$$
$$\lim_{n\to\infty} y_{2n+1} = 1, \quad y_{2t+4n-2} = \frac{A^p}{y_{2t}}, \quad y_{2t+4n} = y_{2t}, \quad n \geq 1,$$

(c). *if $q > 1$, then, there exists a positive integer n_0, such that*

$$x_{2n+1} = A, \quad n \geq t, \quad \lim_{n\to\infty} x_{2n} = 1,$$
$$\lim_{n\to\infty} y_{2n+1} = 1, \quad y_{2t+4n-2} = \frac{A^p}{B^q}, \quad y_{2t+4n-4} = B, \quad n \geq n_0 + 1.$$

XVI. *Suppose that there exist integers $t, v \geq 1$, such that (2.95) and (2.96) hold, then*

(a). *if $q < 1$, then,*

$$x_n = A, \quad n \geq \max\{2t+1, 2v+2\}, \quad \lim_{n\to\infty} y_n = A^{\frac{p}{q+1}},$$

(b). *if $q = 1$, then,*

$$x_n = A, \quad n \geq \max\{2t+1, 2v+2\},$$
$$y_{2t+4n-2} = \frac{A^p}{y_{2t}}, \quad y_{2t+4n} = y_{2t}, \quad n \geq 1,$$
$$y_{2v+4n-1} = \frac{A^p}{y_{2v+1}}, \quad y_{2v+4n+1} = y_{2v+1}, \quad n \geq 1,$$

(c). *if $q > 1$, then, there exist positive integers n_0, n_1, such that*

$$x_n = A, \ n \geq \max\{2t+1, 2v+2\},$$
$$y_{2t+4n-2} = \frac{A^p}{B^q}, \quad y_{2t+4n-4} = B, \quad n \geq n_0 + 1,$$
$$y_{2v+4n-1} = \frac{A^p}{B^q}, \quad y_{2v+4n-3} = B, \quad n \geq n_1 + 1.$$

Remark 2.8. If for the system of difference equations (1.1) relations (2.1), (2.3) and (2.70) hold and the initial values x_{-1}, x_0, y_{-1}, y_0, are positive real numbers, then it is impossible to exist integers $l, v \geq 1$, such that the first equality of (2.92) and (2.96) to be valid simultaneously.

Indeed, if the first equality of (2.92) holds then, from statement I of Lemma 2.6, we have

$$y_{2n+1} = A^{\frac{p}{q+1}}, \quad n \geq l,$$

and so, (2.96) can not be true for $v \geq l$.

Now, suppose that $v < l$. From statement IV of Lemma 2.6 and relations (2.69), (2.85), (2.86) and (2.96), we get

$$y_{2v+4k-1} > A^{\frac{p}{q+1}}, \quad k \geq 1,$$

and

$$y_{2v+4k+1} < A^{\frac{p}{q+1}}, \quad k \geq 1,$$

which contradicts with the first equality of (2.92), since we assume that $v < l$.

Similarly, it is impossible to exist integers $r, t \geq 1$, such that the second equality of (2.92) and (2.95) to be valid simultaneously.

Remark 2.9. If for the system of difference equations (1.1) relations (2.1) and (2.4) hold and the initial values x_{-1}, x_0, y_{-1}, y_0, are positive real numbers, then the behavior of its solutions is similar to the behavior we have in Proposition 2.7. For this reason, we omit the reference.

Proposition 2.10. *Consider the system of difference equations* (1.1)*, where relations* (2.1) *and* (2.5) *hold, and the initial values* x_{-1}, x_0, y_{-1}, y_0*, are positive real numbers. Then, all the solutions of* (1.1) *are unbounded from above.*

Proof. Suppose that $A > 1$ and (x_n, y_n) is a solution of (1.1). We prove that x_n is unbounded from above. On the contrary, we assume that there exists a positive real number M, such that

$$x_n \leq M, \quad \text{for any } n \geq 1. \tag{2.97}$$

From (1.1) and (2.97), we get

$$\frac{y_n^p}{x_{n-1}^q} \leq M, \quad \text{for any } n \geq 0,$$

and so, from (2.97), and since $q > 0$, we have

$$y_n^p \leq M x_{n-1}^q \leq M M^q = M^{q+1}, \quad \text{for any } n \geq 2,$$

which means that

$$y_n \leq M^{\frac{q+1}{p}}, \quad \text{for any } n \geq 2. \tag{2.98}$$

From (1.1) and (2.98), we get

$$\frac{x_n^p}{y_{n-1}^q} \leq M^{\frac{q+1}{p}}, \quad \text{for any } n \geq 2,$$

and so, from (2.1) and (2.98),

$$x_n^p \leq M^{\frac{q+1}{p}} y_{n-1}^q \leq M^{\frac{q+1}{p}} M^{q\frac{q+1}{p}} = M^{\frac{(q+1)^2}{p}}, \quad \text{for any } n \geq 3,$$

which means that

$$x_n \leq M^{(\frac{q+1}{p})^2}, \quad \text{for any } n \geq 3.$$

Working inductively, we get

$$\begin{aligned} x_n &\leq M^{(\frac{q+1}{p})^{2k}}, \quad \text{for any} \quad n \geq 2k+1, \quad k = 1, 2, \ldots, \\ y_n &\leq M^{(\frac{q+1}{p})^{2k-1}}, \quad \text{for any} \quad n \geq 2k, \quad k = 1, 2, \ldots \end{aligned} \tag{2.99}$$

From (2.1), it is obvious that

$$\lim_{k\to\infty} M^{(\frac{q+1}{p})^{2k}} = 1, \qquad \lim_{k\to\infty} M^{(\frac{q+1}{p})^{2k-1}} = 1. \tag{2.100}$$

Since $A > 1$, from (1.1) we have

$$x_n \geq A > 1, \quad \text{for any } n \geq 1,$$

and so, there exists a positive real number ϵ, such that

$$x_n \geq A > 1 + \epsilon, \quad \text{for any } n \geq 1. \tag{2.101}$$

From the first inequality of (2.99) and the first limit of (2.100), we get that there exists a positive integer n_0 such that

$$x_n < 1 + \epsilon, \quad \text{for any } n \geq n_0,$$

which contradicts with (2.101). So, x_n is unbounded from above, if $A > 1$.

Similarly, we can prove that y_n is unbounded from above, if $B > 1$.

We mention that, under some initial values, we can find solutions such that

$$\lim_{n\to\infty} x_n = \infty \quad \text{or} \quad \lim_{n\to\infty} y_n = \infty.$$

Indeed, we consider the difference equation (2.9) of Lemma 2.2, where $a = p$ and $b = q$. Since, from statement (i) of Lemma 2.2, if

$$\frac{z_0^{\lambda_1}}{z_{-1}^q} > 1, \qquad \lambda_1 = \frac{p + \sqrt{p^2 - 4q}}{2},$$

then

$$\lim_{n\to\infty} z_n = \infty, \tag{2.102}$$

there exists a positive integer n_1, such that

$$z_{n+1} > \max\{A, B\}, \quad \text{for any } n \geq n_1. \tag{2.103}$$

If we set

$$x_{-1} = z_{n_1+j-1}, \qquad y_0 = z_{n_1+j}, \tag{2.104}$$

$$y_{-1} = z_{n_1+w-1}, \qquad x_0 = z_{n_1+w}, \tag{2.105}$$

where j, w are arbitrary integers in $\{0, 1, 2, \dots\}$, then, from (1.1), (2.103) and (2.104),

$$x_1 = \max\left\{A, \frac{z_{n_1+j}^p}{z_{n_1+j-1}^q}\right\} = \max\{A, z_{n_1+j+1}\} = z_{n_1+j+1},$$

$$y_2 = \max\left\{B, \frac{z_{n_1+j+1}^p}{z_{n_1+j}^q}\right\} = \max\{B, z_{n_1+j+2}\} = z_{n_1+j+2},$$

and from (1.1), (2.103) and (2.105),

$$y_1 = \max\left\{B, \frac{z_{n_1+w}^p}{z_{n_1+w-1}^q}\right\} = \max\{B, z_{n_1+w+1}\} = z_{n_1+w+1},$$

$$x_2 = \max\left\{A, \frac{z_{n_1+w+1}^p}{z_{n_1+w}^q}\right\} = \max\{A, z_{n_1+w+2}\} = z_{n_1+w+2},$$

and working inductively,

$$x_{2n+1} = z_{n_1+j+2n+1}, \quad y_{2n+2} = z_{n_1+j+2n+2}, \quad \text{for any } n \geq 0, \tag{2.106}$$

and

$$y_{2n+1} = z_{n_1+w+2n+1}, \quad x_{2n+2} = z_{n_1+w+2n+2}, \quad \text{for any } n \geq 0. \tag{2.107}$$

So, if (2.104) holds, from (2.102) and (2.106) we get

$$\lim_{n\to\infty} x_{2n+1} = \infty, \qquad \lim_{n\to\infty} y_{2n} = \infty,$$

and if (2.105) holds, from (2.102) and (2.107) we get

$$\lim_{n\to\infty} x_{2n} = \infty, \qquad \lim_{n\to\infty} y_{2n+1} = \infty.$$

This completes the proof of the proposition. □

Open Problem 2.1. *Consider the system of difference equations* (1.1)*, where* (2.1) *and* (2.3) *hold, and the initial values* x_{-1}, x_0, y_{-1}, y_0*, are positive real numbers. Suppose that there exists an integer* $l \geq 2$*, such that*

$$y_l < A^{\frac{p}{q+1}}.$$

Assume that

$$A^{\frac{p^2-q-1}{pq}} > B.$$

Study the behavior of the solution (x_n, y_n) *of this system.*

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On a special case of a difference equation with powers

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Abstract. Investigation of the long-term behaviour of solutions to the nonlinear difference equation

$$x_{n+1} = A + \frac{x_{n-m}^p}{x_{n-k}^r}, \quad n \in \mathbb{N}_0,$$

where $A, p, q \in \mathbb{R}$, $k, m \in \mathbb{N}_0$, $k \neq m$, was proposed by S. Stević about twenty years ago. A very special case of the equation ($p = 1$, $r = 2$, $m = 0$) has been recently considered in [*J. Appl. Math. Comput.* **67**(2021), 423–437]. We show that the main results therein are known or have some inaccuracies. Among other things, we show that the boundedness result therein is a consequence of some known results and using one of our previous methods we give a better upper bound for positive solutions to the equation, show that the proof of the global convergence result therein is not correct and provide a complete proof of a generalization, and also show that the results on semi-cycles of positive solutions are not correct and present some correct ones. Several comments are also given and some analyses are conducted.

Keywords: difference equation, positive solutions, boundedness, convergence, semi-cycles.

2020 Mathematics Subject Classification: 39A20.

1 Introduction

There has been a considerable recent interest in investigating concrete nonlinear difference equations and systems of difference equations (see, for instance, [1,3–19,29–61] and the related references therein).

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1.1 Notation

The set of positive natural numbers is denoted by $\mathbb{N}$, the set of whole numbers is denoted by $\mathbb{Z}$, the set of positive numbers $(0,+\infty)$ is denoted by $\mathbb{R}_+$, whereas the set of real numbers is denoted by $\mathbb{R}$. If $l \in \mathbb{Z}$ is fixed, then for the set $\{j \in \mathbb{Z} : j \geq l\}$ we use the notation $\mathbb{N}_l$. If $k, l \in \mathbb{Z}$ and $k \leq l$, then we use the notation $j = \overline{k,l}$ instead of writing the following expression: $k \leq j \leq l$, $j \in \mathbb{Z}$. By C_j^n where $n \in \mathbb{N}_0$, $0 \leq j \leq n$, we denote the binomial coefficients, that is, $C_j^n = \frac{n!}{j!(n-j)!}$, where $n \in \mathbb{N}_0$, $0 \leq j \leq n$, and where, as usual, we regard that $0! = 1$. For some basic facts on the binomial coefficients consult, for instance, [2, 20, 21, 25, 26]. Let us mention that in a part of the literature, such as [2, 21], is used the notation C_n^j for the binomial coefficients. Another standard notation, that is, $\binom{n}{j}$ is also frequently used (see, e.g., [20, 25, 26]), but it is more robust, so, as usual, we use here the above notation.

1.2 Two important classes of difference equations with powers

Investigation of the long-term behaviour of positive solutions to the nonlinear difference equation

$$x_{n+1} = A + \frac{x_{n-m}^p}{x_{n-k}^r}, \quad n \in \mathbb{N}_0, \tag{1.1}$$

where $A > 0$, $\min\{p, q\} \geq 0$, $k, m \in \mathbb{N}_0$, $k \neq m$, as well as of its max-type counterpart

$$x_{n+1} = \max\left\{A, \frac{x_{n-m}^p}{x_{n-k}^r}\right\}, \quad n \in \mathbb{N}_0, \tag{1.2}$$

was suggested by S. Stević in January 2004, soon after acceptance of [42] where the boundedness, global attractivity, oscillations, and asymptotic periodicity of positive solutions to the equation (1.1) in the case $A \geq 0$, $p = r$, $m = 1$, $k = 0$ was investigated. This was probably the first paper with some new and non-trivial results on an equation of the form in (1.1) which is not rational (the case had been previously considered in [12], but the results therein were essentially known or incorrect). It should be mentioned that [42] also contains a result on convergence of solutions to equation (1.1) for the case of an arbitrary $m \in \mathbb{N}$.

Equations (1.1) and (1.2) are good prototypes for developing the theory of nonlinear difference equations, because of which S. Stević proposed their investigation. Since the time of his proposal for the investigation, the two classes of difference equations, as well as their various generalizations, including some systems of difference equations, have been studied a lot.

Remark 1.1. If $A = 0$ then equations (1.1) and (1.2) are reduced to the following product-type difference equation

$$x_{n+1} = \frac{x_{n-m}^p}{x_{n-k}^r}, \quad n \in \mathbb{N}_0. \tag{1.3}$$

If only positive solutions of equation (1.3) are considered, then employing the change of variables

$$y_n = \ln x_n$$

for $n \geq -\max\{k, m\}$, which can be found in old literature [22] (see, for instance, also [2, 42]), the difference equation is transformed to the linear one with constant coefficients

$$z_{n+1} - pz_{n-m} + rz_{n-k} = 0, \quad n \in \mathbb{N}_0,$$

which is (theoretically) solvable in closed form ([22–24, 26–28]). Bearing in mind this fact and many known results on the linear difference equations with constant coefficients, the equations (1.1) and (1.2) with $A = 0$ are usually not of a special interest. Hence, although we have considered some special cases of the equation (1.3) with $A = 0$, as it was the case in [42], we frequently skipped considering the case in our papers.

Remark 1.2. It should be also mentioned that solvability of product-type systems of difference equations on the domain of complex numbers, as well as the difference equations and systems which are transformed to them by some changes of variables, has attracted some recent attention. See, for example, [59, 60] and the related references therein.

Remark 1.3. Note that equations (1.1) and (1.2) are some kind of perturbations of equation (1.3), which is one of the reasons why they are good prototypes for some detailed investigations. Although, they have similar forms, the long-term behavior of their positive solutions can be quite different.

1.3 A bit on the history of investigations of equations (1.1) and (1.2)

In what follows we briefly mention some of the previous investigations of equations (1.1) and (1.2), and some of their generalizations. As we have already mentioned, in [42] was considered equation (1.1) in the case $A \geq 0$, $p = r > 0$, $m = 1$, $k = 0$. The case was also studied later in [5]. The boundedness, global attractivity, and periodicity of positive solutions to equations (1.1) and (1.2) in the case $A > 0$, $p = r > 0$, $m = 0$, $k = 1$ were investigated in [46] and [49], respectively. Equations (1.1) and (1.2) in the case $\min\{A, p, q\} > 0$, $m = 0$, $k = 1$ were considered in [47] (this was probably the first paper which also considered the case $p \neq r$). The boundedness character of positive solutions to equations (1.1) and (1.2) in the case $\min\{A, p, q\} > 0$, $m = 0$, $k = 2$ was studied in [52], where S. Stević used several new ideas and methods, among others, a new method that we called "oachkatzlschwoif/squirrel-tail" method. Equation (1.2) in the case $A > 0$, $p = r > 0$, $m = 0$, $k = 3$ was considered first in [51] and later in [16].

The boundedness character of positive solutions to the following extensions of the equations in [47]

$$x_{n+1} = A + \frac{x_n^p}{x_{n-1}^q x_{n-2}^r}, \quad n \in \mathbb{N}_0,$$

$$x_{n+1} = \max\left\{A, \frac{x_n^p}{x_{n-1}^q x_{n-2}^r}\right\}, \quad n \in \mathbb{N}_0,$$

was considered in [17] and [54], respectively.

The following generalization of equation (1.1) in the case $p = r$

$$x_n = A + \left(\frac{\sum_{i=1}^k \alpha_i x_{n-2p_i}}{\sum_{j=1}^m \beta_j x_{n-q_j}}\right)^p, \quad n \in \mathbb{N}_0,$$

where $\min\{p, A\} > 0$, $k, m \in \mathbb{N}$, $p_i \in \mathbb{N}$, $i = \overline{1,k}$, q_j, $j = \overline{1,m}$, are odd natural numbers such that $p_1 < p_2 < \cdots < p_k$, $q_1 < q_2 < \cdots < q_m$, $\min_{1\leq i\leq k, 1\leq j\leq m}\{\alpha_i, \beta_j\} > 0$ such that $\sum_{i=1}^k \alpha_i = \sum_{j=1}^m \beta_j$, was considered in [53].

The boundedness character of the equations (1.1) and (1.2) in the case $A > 0$, $p = r > 0$, $m = 0$, $k \in \mathbb{N}$ was completely characterized in [50]. Equation (1.2) in the case $p \neq q$ was

considered in [55], whereas the corresponding results for equation (1.1) in the case were presented in [15]. The case $\min\{p,q\} > 0$, $m = 1$, $k = 0$ was considered in [35], whereas the case where the constant A is replaced by a sequence A_n was considered in [36]. For some special cases of equations (1.1) and (1.2) in rational form see [1,3,4,40] and the references therein, and with a variable coefficient A_n, see [31,32,41] and the related references therein. A cyclic system of difference equations corresponding to the equation (1.1) in the case $A > 0$, $m = 0$, $k = 1$, was considered in [57], whereas the corresponding cyclic system to the equation (1.2) in the case $A > 0$, $m = 0$, $k = 1$, was studied in [61]. A quite general boundedness result for a generalization of equation (1.1) was given in [58]. For some related difference equations and systems of difference equations, some of which also contain powers of dependent variables, see also [18,29,37,38,56].

1.4 A recent investigation

Recently, in [62], E. Tasdemir considered the following difference equation

$$x_{n+1} = A + B\frac{x_n}{x_{n-m}^2}, \quad n \in \mathbb{N}_0, \tag{1.4}$$

where $m \in \mathbb{N}$ and $\min\{A,B\} > 0$.

Note that by using the change of variables

$$x_n = \sqrt{B}\, y_n, \quad n \in \mathbb{N}_0, \tag{1.5}$$

equation (1.4) is transformed to the equation

$$y_{n+1} = \frac{A}{\sqrt{B}} + \frac{y_n}{y_{n-m}^2}, \quad n \in \mathbb{N}_0, \tag{1.6}$$

which is a very special case of equation (1.1). This immediately implies that equation (1.4) is equivalent with a special case of equation (1.1), and consequently it is not a new equation at all.

Instead of using the change of variables in (1.5) the author of [62] used the change of variables

$$y_n = \frac{x_n}{A}, \quad n \in \mathbb{N}_0, \tag{1.7}$$

and obtained the equation

$$y_{n+1} = 1 + p\frac{y_n}{y_{n-m}^2}, \quad n \in \mathbb{N}_0, \tag{1.8}$$

where $p = B/A^2$, which is, of course, equivalent with the special case of equation (1.1) mentioned above.

The equilibria of equation (1.8) satisfy the algebraic equation

$$\bar{y} = 1 + \frac{p}{\bar{y}}$$

and are equal to

$$\frac{1+\sqrt{1+4p}}{2} \quad \text{and} \quad \frac{1-\sqrt{1+4p}}{2}.$$

From this and since $p > 0$, we see that

$$\bar{y} = \frac{1+\sqrt{1+4p}}{2} \tag{1.9}$$

is a unique positive equilibrium of equation (1.8).

A simple calculation shows that the linearized equation associated with (1.8) about the positive equilibrium $\bar{y}$ is given by

$$z_{n+1} - \frac{p}{\bar{y}^2} z_n + \frac{2p}{\bar{y}^2} z_{n-m} = 0, \quad n \in \mathbb{N}_0. \tag{1.10}$$

It is well-known that

$$\sum_{j=0}^{m} |a_j| < 1 \tag{1.11}$$

is a sufficient condition for the asymptotic stability of the difference equation

$$z_{n+1} + a_0 z_n + a_1 z_{n-1} + \cdots + a_m z_{n-m} = 0, \tag{1.12}$$

from which it follows immediately that if $\frac{3p}{\bar{y}^2} < 1$ and $p > 0$, equation (1.10) is asymptotically stable. A simple calculation shows that this is satisfied for $p \in (0, 3/4)$. This implies that the equilibrium point $\bar{y}$ of equation (1.8) is locally asymptotically stable for $p \in (0, 3/4)$, which was noted in [62].

Remark 1.4. Condition (1.11) is a practical condition which guarantees asymptotic stability of equation (1.12), but his disadvantage is that it is not necessary. Moreover, there are better sufficient conditions guaranteeing asymptotic stability of the equation. Unfortunately, many authors use only this one and obtain pretty narrow sets of parameters which guarantee the stability.

The following claims should be the main results in [62].

Theorem 1.5. *Every solution $(y_n)_{n \in \mathbb{N}_{-m}}$ of equation (1.8) is bounded and persist and satisfies the following estimates*

$$1 < y_n < 1 + p(1+p)^m, \tag{1.13}$$

for $n \geq 2m + 2$.

Theorem 1.6. *Let $0 < p < 3/4$. Then, the positive equilibrium point $\bar{y}$ of equation (1.8) is globally asymptotically stable.*

Theorem 1.7. *Every solution of equation (1.8) satisfies the following statements:*

(a) Every solution of equation (1.8) has semi-cycles of length at most $2m + 1$.

(b) If every solution of equation (1.8) has a semi-cycle of length at least k, then, the following semi-cycle has at least $k + 1$ terms.

Beside these three claims, and the observation related to the local asymptotic stability of the equilibrium $\bar{y}$, [62] also gives a few other simple results, which are easily proved by well known methods and simple calculations such as the nonexistence of two periodic solutions and a growth rate result for solutions converging to the equilibrium $\bar{y}$.

Remark 1.8. At the end of [62] is proposed studying two special cases of equation (1.1) as 'Open Problems' despite the fact that these, as well as some general difference equations have been studied considerably so far, as we have mentioned in the previous subsection. The author of [62] might have not seen any of the papers: [3, 15–17, 31, 32, 35, 36, 42, 46, 47, 49–55, 57, 58, 61], which have been published in several popular journals.

1.5 Our aim

Here we show that Theorem 1.5 directly follows form some of our previous results, and that employing one of known methods can be obtained a better upper estimate than the one given in (1.13). Further, we show that the proof of Theorem 1.6 is not correct and provide a complete proof of a generalization. We also show that the proof of Theorem 1.7 is not correct, and present some correct ones related to it. We also conduct some analyses and present some additional comments.

2 Main results, counterexamples and comments

In this section we give several comments concerning the claims and results in [62], conduct some analyses, correct some of the proofs therein, and extend some of the claims and results therein.

2.1 On Theorem 1.5 and relating results

As far as concerning Theorem 1.5, the fact that all positive solutions to equation (1.8) are bounded follows from several results in the literature. For example, by using one of the methods presented in [50], which essentially originates from [47], in [15] was proved the following generalization of the boundedness results in [47] and [52]:

Theorem 2.1. *Assume $p, r > 0$ and $k \in \mathbb{N}_2$. Then every positive solution of the equation*

$$x_n = A + \frac{x_{n-1}^p}{x_{n-k}^r}, \quad n \in \mathbb{N}_0, \tag{2.1}$$

is bounded if

$$0 < p < \left(\frac{rk^k}{(k-1)^{k-1}} \right)^{1/k}. \tag{2.2}$$

Now note that in equation (1.4) or in the equivalent equation (1.8), we have $p = 1$ and $r = 2$ for which the condition in (2.2) is satisfied for any $k \in \mathbb{N}_2$, from which the boundedness part of Theorem 1.5 immediately follows.

Regarding the upper bound for positive solutions to equation (1.8) given in (1.13), we must say that there is a procedure, which nowadays can be regarded as standard one, for getting upper bounds for positive solutions of related difference equations. The procedure was essentially used, e.g., in [40] and it was a starting point for getting the squirrel-tail method. Namely, by iterating the numerators of the fractions in equation (1.8) m-times, and employing a simple inductive argument we have

$$\begin{aligned} y_{n+1} &= 1 + p\frac{y_n}{y_{n-m}^2} = 1 + \frac{p}{y_{n-m}^2}\left(1 + p\frac{y_{n-1}}{y_{n-m-1}^2}\right) \\ &= 1 + \frac{p}{y_{n-m}^2} + p^2\frac{y_{n-1}}{y_{n-m}^2 y_{n-m-1}^2} = 1 + \frac{p}{y_{n-m}^2} + \frac{p^2}{y_{n-m}^2 y_{n-m-1}^2}\left(1 + p\frac{y_{n-2}}{y_{n-m-2}^2}\right) \\ &= 1 + \frac{p}{y_{n-m}^2} + \frac{p^2}{y_{n-m}^2 y_{n-m-1}^2} + \frac{p^3 y_{n-2}}{y_{n-m}^2 y_{n-m-1}^2 y_{n-m-2}^2} \end{aligned}$$

$$\vdots$$

$$\begin{aligned}
&= 1 + \frac{p}{y_{n-m}^2} + \frac{p^2}{y_{n-m}^2 y_{n-m-1}^2} + \cdots + \frac{p^m y_{n-(m-1)}}{y_{n-m}^2 y_{n-m-1}^2 \cdots y_{n-2m+1}^2} \\
&= 1 + \frac{p}{y_{n-m}^2} + \frac{p^2}{y_{n-m}^2 y_{n-m-1}^2} + \cdots + \frac{p^m}{y_{n-m}^2 y_{n-m-1}^2 \cdots y_{n-2m+1}^2} \left(1 + p\frac{y_{n-m}}{y_{n-2m}^2}\right) \\
&= 1 + \frac{p}{y_{n-m}^2} + \cdots + \frac{p^m}{y_{n-m}^2 y_{n-m-1}^2 \cdots y_{n-2m+1}^2} + \frac{p^{m+1} y_{n-m}}{y_{n-m}^2 y_{n-m-1}^2 \cdots y_{n-2m}^2} \\
&= 1 + \frac{p}{y_{n-m}^2} + \cdots + \frac{p^m}{y_{n-m}^2 y_{n-m-1}^2 \cdots y_{n-2m+1}^2} + \frac{p^{m+1}}{y_{n-m} y_{n-m-1}^2 \cdots y_{n-2m}^2}. \qquad (2.3)
\end{aligned}$$

From (2.3) and since

$$y_n > 1, \quad n \in \mathbb{N},$$

we have

$$y_{n+1} < 1 + p + \cdots + p^m + p^{m+1} = \frac{1 - p^{m+2}}{1 - p}, \qquad (2.4)$$

for $n \geq 2m + 1$, when $p \neq 1$ and

$$y_{n+1} < m + 2, \qquad (2.5)$$

for $n \geq 2m + 1$, when $p = 1$.

Let

$$S_m(p) := \sum_{j=0}^{m-1} p^j, \quad m \in \mathbb{N}.$$

Since

$$S_{m+2}(p) = 1 + p\sum_{j=0}^{m} p^j < 1 + p\sum_{j=0}^{m} C_j^m p^j = 1 + p(1+p)^m$$

we have that the upper bound given in (2.4) is better than the corresponding one in (1.13). Hence, the following result holds.

Theorem 2.2. *Let $p > 0$ and $m \in \mathbb{N}$. Then, every positive solution $(y_n)_{n \in \mathbb{N}_{-m}}$ to equation (1.8) is bounded and persistent and satisfies the following estimates*

$$1 < y_n < S_{m+2}(p),$$

for $n \geq 2m + 2$.

Remark 2.3. Some other methods which can be used in showing the boundedness of positive solutions to nonlinear difference equations, including applications of invariants, can be found, for instance, in [6,15–17,30,33,34,39,46,47,49–51,53–55,57,58], as well as in the related references therein.

2.2 On Theorem 1.6 and some related results

The proof of Theorem 1.6 in [62] is not correct. Namely, to prove it, beside some nonessential conventional inaccuracies, the author of [62] only claims that from

$$m = 1 + p\frac{m}{M^2}, \quad M = 1 + p\frac{M}{m^2} \qquad (2.6)$$

follows $m = M$ and apply a known result (i.e, [13, Theorem 1.15]) from which Theorem 1.6 immediately follows, but only if it is also employed Theorem 1.5 (this was not mentioned in [62], what can be regarded as an oversight). However, the above claim was not proved therein. It should be pointed out that its proof is neither obvious nor quite simple, although it uses only elementary mathematics (see the proof of Theorem 2.4 below). Moreover, it is possible to prove a better convergence result than Theorem 1.6. Namely, the following result holds.

Theorem 2.4. *Let $m \in \mathbb{N}$ and $p \in (0, 3/4] \cup [1, \infty)$. Then, the unique positive equilibrium point $\bar{y}$ of equation (1.8) is global attractor of all positive solutions.*

Proof. Since

$$I_p := [1, S_{m+2}(p)]$$

is an invariant interval for positive solutions to equation (1.8), as mentioned above, to apply [13, Theorem 1.15], it is enough to prove that the algebraic system (2.6) has only the solution $M = m = \bar{y}$ on the set I_p^2.

Assume that there is a solution to system (2.6) such that

$$M \neq m, \tag{2.7}$$

where $m, M \in I_p$.

From (2.6) we have

$$m - M = p\left(\frac{m}{M^2} - \frac{M}{m^2}\right) = \frac{p(m^3 - M^3)}{(mM)^2}$$

from which along with (2.7) it follows that

$$(Mm)^2 = p(m^2 + mM + M^2). \tag{2.8}$$

Further, from (2.6) we have

$$Mm = M + p\frac{m}{M} = m + p\frac{M}{m}$$

from which along with (2.7) it follows that

$$Mm = p(M + m). \tag{2.9}$$

From (2.6) we also have

$$M^2 m = M^2 + pm, \quad Mm^2 = m^2 + pM$$

from which along with (2.7) it follows that

$$Mm = M + m - p. \tag{2.10}$$

From (2.8) and (2.9) we have

$$p(M + m)^2 = m^2 + mM + M^2 = (M + m)^2 - Mm$$

and consequently

$$Mm = (1 - p)(M + m)^2. \tag{2.11}$$

If it were $p = 1$, then from (2.11) we would have $Mm = 0$, which contradicts to the assumption $m, M \in I_p$.

Now assume that $p \neq 1$. From (2.10) and (2.11) we have

$$(1-p)(M+m)^2 - (M+m) + p = (M+m-1)(M+m-p(M+m+1)) = 0. \tag{2.12}$$

Since $m, M \in I_p$ it is not possible to be $M + m = 1$. Hence, it must be $M + m = p(M + m + 1)$, that is

$$M + m = \frac{p}{1-p}. \tag{2.13}$$

From (2.13) and since $M + m > 0$ it follows that it must be $p \in (0, 1)$.

Using this fact, (2.10) and (2.13) we have

$$Mm = \frac{p}{1-p} - p = \frac{p^2}{1-p}. \tag{2.14}$$

From (2.13) and (2.14) we see that M and m are the two positive distinct roots of the quadratic polynomial

$$\widehat{P}_2(t) := t^2 - \frac{p}{1-p}t + \frac{p^2}{1-p}. \tag{2.15}$$

By direct calculation we see that the roots of polynomial (2.15) are

$$t_1 = \frac{p(1+\sqrt{4p-3})}{2(1-p)} \quad \text{and} \quad t_2 = \frac{p(1-\sqrt{4p-3})}{2(1-p)}. \tag{2.16}$$

From (2.16) we see that it is not possible to be $p \in (0, 3/4]$, since then $t_{1,2}$ will be complex numbers if $p \in (0, 3/4)$, or equal if $p = 3/4$.

Hence, the assumption $m \neq M$ is not possible, from which it follows that $M = m$, completing the proof of the theorem. □

Remark 2.5. From Theorem 2.4 and since the equilibrium $\bar{y}$ is locally asymptotically stable for $p \in (0, 3/4)$, the claim in Theorem 1.6 follows.

Remark 2.6. It is interesting to note that if $p \in (3/4, 1)$, then $t_2 > 1$. Indeed, we have

$$t_2 = \frac{2p}{1+\sqrt{4p-3}}$$

and consequently

$$t_2 - 1 = \frac{2p - 1 - \sqrt{4p-3}}{1+\sqrt{4p-3}}.$$

From this and since $2p - 1 - \sqrt{4p-3} > 0$ for $p \in (3/4, 1)$ is equivalent to $4(p-1)^2 > 0$, the claim follows.

To prove that Theorem 2.4 holds also for $p \in (3/4, 1)$ it will be enough to prove that

$$t_1 > \frac{1-p^{m+2}}{1-p}, \tag{2.17}$$

which is equivalent to

$$p(1+\sqrt{4p-3}) + 2p^{m+2} > 2.$$

Note that the function

$$f_m(t) := 2t^{m+2} + t(\sqrt{4t-3}+1) \tag{2.18}$$

is increasing for $t \geq 3/4$, and consequently on the interval $(3/4, 1)$.

On the other hand, we have

$$f_m\left(\frac{3}{4}\right) = 2\left(\frac{3}{4}\right)^{m+2} + \frac{3}{4} \leq \frac{51}{32} < 2$$

for any $m \in \mathbb{N}$, and

$$f_m(1) = 4 > 2.$$

Hence, for each $m \in \mathbb{N}$, there is a unique $\widehat{t}_m \in (3/4, 1)$, so that

$$f_m(\widehat{t}_m) = 2.$$

This with the monotonicity of the function f_m implies that inequality (2.17) holds when $p > \widehat{t}_m$.

This analysis shows that the following extension of Theorem 2.4 holds.

Theorem 2.7. *Let $m \in \mathbb{N}$, f_m be the function defined in (2.18), and $p \in (0, 3/4] \cup (\widehat{t}_m, +\infty)$, where $\widehat{t}_m$ is the unique solution to the equation $f_m(t) = 2$ on the interval $(3/4, 1)$. Then, the unique positive equilibrium point $\bar{y}$ of equation (1.8) is global attractor of all positive solutions.*

Remark 2.8. For each $m \in \mathbb{N}$ the value of $\widehat{t}_m$ can be approximated by a numeric calculation, but it seems cannot be expressed in closed form.

2.3 On Theorem 1.7 and relating results

The proof of the claim *(a)* in Theorem 1.7 is not correct. Before we present the arguments for the claim, let us mention that the constant/equilibrium solution

$$y_n = \bar{y}, \quad n \in \mathbb{N}_{-m},$$

should have been excluded from the consideration therein, since for the solution the claim is obviously not true. Namely, the solution has only one semi-cycle, which can be regarded as positive, but also it can be regarded as negative, depending on a definition of semi-cycles.

To prove the claim (for the nonconstant solutions), the author of [62] unsuccessfully copied a known method, which can be found, for example, in [1]. Namely, he chooses a positive solution $(y_n)_{n \geq \mathbb{N}_{-m}}$ to equation (1.8) and considers one of its negative semi-cycles (it is claimed that a similar argument proves the claim also for positive semi-cycles), say

$$y_N, y_{N+1}, \ldots, y_{N+2m} < \bar{y}$$

where y_N is the first term in the negative semi-cycle.

It is claimed that

$$\begin{aligned} y_{N+m+i} &= 1 + p\frac{y_{N+m+i-1}}{y_{N+i-1}^2} > 1 + p\frac{y_{N+m+i-1}}{\bar{y}^2} \\ &> \left(1 + \frac{p}{\bar{y}^2}\right) y_{N+m+i-1} > y_{N+m+i-1} \end{aligned} \tag{2.19}$$

for $i = \overline{0, m-1}$, from which it is concluded that

$$y_{N+m} < y_{N+m+1} < \cdots < y_{N+2m-1} < y_{N+2m},$$

and consequently

$$y_{N+2m+1} = 1 + p\frac{y_{N+2m}}{y_{N+m}^2} > \bar{y}.$$

However, the second inequality in (2.19) is not correct, since if it was true, then it would be $y_{N+m+i-1} < 1$, which is not possible for any N satisfying the condition $N + m + i - 1 \geq 1$.

The above proof can be applied to positive solutions to the difference equation

$$y_{n+1} = 1 + p\frac{y_n}{y_{n-m}}, \tag{2.20}$$

however, as we have just noticed, it does not hold for the case of equation (1.8).

One of the interesting problems is the existence of solutions of difference equations which do not oscillate around an equilibrium. The characteristic polynomials associated to their linearizations can be of some help in the study. Namely, if a characteristic polynomial has a zero in the interval $(0, 1)$, then it suggests the existence of solutions of the original difference equation which do not oscillate around an equilibrium. Some methods for showing the existence of such solutions, which use the idea, have been developed in the last few decades (see, e.g., [6–11, 14, 43–45, 48] and the related references therein; for another method see [19]).

These facts suggest to consider the problem of the existence of zeros of the associated characteristic polynomial

$$P_{m+1}(t) = t^{m+1} - \frac{p}{\bar{y}^2}t^m + \frac{2p}{\bar{y}^2} = 0, \quad m \in \mathbb{N}_0. \tag{2.21}$$

to equation (1.10), in the interval $(0, 1)$.

First note that

$$P_{m+1}(0) = \frac{2p}{\bar{y}^2} = \frac{4p}{1 + 2p + \sqrt{1 + 4p}} \in (0, 2) \tag{2.22}$$

and

$$P_{m+1}(1) = 1 + \frac{p}{\bar{y}^2} = 1 + \frac{2p}{1 + 2p + \sqrt{1 + 4p}} \in (1, 2) \tag{2.23}$$

when $p > 0$.

We have

$$P'_{m+1}(t) = (m + 1)t^{m-1}\left(t - \frac{2mp}{(m + 1)(1 + 2p + \sqrt{1 + 4p})}\right), \quad m \in \mathbb{N}_0. \tag{2.24}$$

Hence, the polynomial $P'_{m+1}(t)$ has two zeros $\widetilde{t}_1 = 0$ and

$$\widetilde{t}_2 = \frac{2mp}{(m + 1)(1 + 2p + \sqrt{1 + 4p})} \in (0, 1).$$

From (2.24) we have that the polynomial $P_{m+1}(t)$ has a local minimum at $\widetilde{t}_2$, which is equal to

$$P_{m+1}(\widetilde{t}_2) = \frac{p}{\bar{y}^2}\left(2 - \frac{m^m}{(m + 1)^{m+1}}\left(\frac{2p}{1 + 2p + \sqrt{1 + 4p}}\right)^m\right). \tag{2.25}$$

From (2.25) it is easy to see that

$$P_{m+1}(\widetilde{t}_2) > \frac{p}{\bar{y}^2} > 0.$$

Thus, for any $m \in \mathbb{N}$, the polynomial $P_{m+1}(t)$ does not have a zero on the interval $[0,1]$.

If m is odd, i.e., $m = 2l-1$ for some $l \in \mathbb{N}$, than t_2 is a unique local and global extremum (minimum).

If m is even, i.e., $m = 2l$ for some $l \in \mathbb{N}$, then $P_{2l+1}(t)$ has a local maximum at $\widetilde{t}_1$, and increases on the interval $(-\infty, 0]$. Hence, it has a unique negative zero. From this, (2.22) and since

$$P_{2l+1}(-1) = -1 + \frac{p}{\bar{y}^2} = -\frac{1+\sqrt{1+4p}}{1+2p+\sqrt{1+4p}} < 0 \tag{2.26}$$

when $p > 0$, we have that the polynomial $P_{2l+1}(t)$ has a unique zero t_0 in the interval $(-1,0)$. This fact suggests the existence of oscillatory solutions to equation (1.8) in this case, which is not so surprising. It also suggests that the asymptotics of some oscillatory solutions can be find by using some of the methods presented in [6–11, 14, 43–45, 48]. We will not pursue the investigation in the direction here, since the above conducted analysis also suggests that all non-equilibrium solutions are oscillatory. Indeed, the following result on the oscillatory character of positive solutions to equation (1.8) holds.

Theorem 2.9. *Let $m \in \mathbb{N}$ and $p > 0$. Then, equation (1.8) does not have positive non-equilibrium solutions which are eventually bigger or smaller than the positive equilibrium $\bar{y}$.*

Proof. Assume that $(y_n)_{n \geq \mathbb{N}_{-m}}$ is a positive non-equilibrium solution to equation (1.8) such that

$$y_n \geq \bar{y}, \tag{2.27}$$

for $n \geq n_0$, for some $n_0 \in \mathbb{N}_{-m}$. Since equation (1.8) is autonomous we may assume that $n_0 = -m$.

From (1.8) and (2.27) we have

$$y_{n+1} - \bar{y} = p\frac{y_n}{y_{n-m}^2} - \frac{p}{\bar{y}} = p\frac{y_n\bar{y} - y_{n-m}^2}{y_{n-m}^2\bar{y}} \geq 0,$$

for $n \in \mathbb{N}_0$, from which along with (2.27) it follows that

$$y_n \geq \frac{y_{n-m}^2}{\bar{y}} \geq y_{n-m}, \tag{2.28}$$

for $n \in \mathbb{N}_0$.

From (2.28) we see that the subsequences y_{mk+i}, $k \in \mathbb{N}_{-1}$, $i = \overline{0, m-1}$, are nondecreasing. From this and Theorem 1.5 it follows that there are finite limits

$$\lim_{k \to +\infty} y_{mk+i} = \bar{y}_i, \quad i = \overline{0, m-1}. \tag{2.29}$$

Note also that it must be

$$\bar{y}_i \geq \bar{y}, \quad i = \overline{0, m-1}. \tag{2.30}$$

Letting $k \to +\infty$ in the relations

$$y_{mk+i+1} = 1 + p\frac{y_{mk+i}}{y_{m(k-1)+i}^2}, \quad i = \overline{0, m-1}$$

we get that $(\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_m)$ is a solution to the algebraic system

$$\bar{y}_{i+1} = 1 + \frac{p}{\bar{y}_i}, \quad i = \overline{0, m-1}, \tag{2.31}$$

where we use the standard convention $\bar{y}_i = \bar{y}_j$ when $i \equiv j \,(\bmod m)$.

If $\bar{y}_{i_0+1} = \bar{y}_{i_0}$, for some $i_0 \in \{0, 1, \ldots, m-1\}$, then from (2.31) we see that it must be $\bar{y}_{i_0+1} = \bar{y}_{i_0} = \bar{y}$, from which together with (2.31) we easily get $\bar{y}_i = \bar{y}$, for $i = \overline{0, m-1}$, which contradicts to (2.30), the monotonicity of the subsequences $(y_{mk+i})_{k\in\mathbb{N}_{-1}}$, $i = \overline{0, m-1}$, and the assumption that the solution is non-equilibrium.

Now we assume that

$$\bar{y}_i \neq \bar{y}_{i+1}, \quad i = \overline{0, m-1}. \tag{2.32}$$

From (2.31) we have

$$\bar{y}_{i+1} - \bar{y}_i = \frac{p}{\bar{y}_i} - \frac{p}{\bar{y}_{i-1}} = \frac{p(\bar{y}_{i-1} - \bar{y}_i)}{\bar{y}_{i-1}\bar{y}_i}, \quad i = \overline{0, m-1}, \tag{2.33}$$

and

$$\bar{y}_i\bar{y}_{i+1} = \bar{y}_i + p, \quad i = \overline{0, m-1}. \tag{2.34}$$

From (2.33) it follows that

$$\prod_{i=0}^{m-1}(\bar{y}_{i+1} - \bar{y}_i)\left(\prod_{i=0}^{m-1}\bar{y}_i\right)^2 = (-1)^m p^m \prod_{i=0}^{m-1}(\bar{y}_{i+1} - \bar{y}_i), \tag{2.35}$$

from which along with (2.32) we immediately get a contradiction when m is an odd number. If m is an even number we get

$$\left(\prod_{i=0}^{m-1}\bar{y}_i\right)^2 = p^m. \tag{2.36}$$

On the other hand, from (2.34) we have

$$\left(\prod_{i=0}^{m-1}\bar{y}_i\right)^2 = \prod_{i=0}^{m-1}(\bar{y}_i + p). \tag{2.37}$$

From (2.36) and (2.37) and since

$$\prod_{i=0}^{m-1}(\bar{y}_i + p) \geq \prod_{i=0}^{m-1}(\bar{y} + p) > p^m \tag{2.38}$$

we get a contradiction.

If we assume that the inequality

$$y_n \leq \bar{y},$$

holds eventually, then as above is obtained that the subsequences y_{mk+i}, $k \in \mathbb{N}_{-1}$, $i = \overline{0, m-1}$, are nonincreasing and convergent, whereas the proof that the algebraic system (2.31) in this case, does not have a nontrivial solution follows by using the same argument and the inequalities

$$\prod_{i=0}^{m-1}(\bar{y}_i + p) \geq \prod_{i=0}^{m-1}(p+1) > p^m$$

first of which follows from the first inequality in (1.13). □

Concerning the proof of the claim *(b)* in Theorem 1.7, it is quite confusing. Namely, the number k is not specified at all. It does not have any connection with the claim of the theorem and the proof given therein is incorrect.

Here we formulate a claim which should generalize what should have been the claim of Theorem 1.7 *(b)* and give a detailed and complete proof.

Before it, we quote the following convention: a positive semicycle of a solution $(y_n)_{n\in\mathbb{N}_{-m}}$ to a difference equation is a finite or infinite subsequence/string $y_l, y_{l+1}, \dots, y_{N-1}$ satisfying the condition

$$y_j \geq \bar{y}, \quad j = \overline{l, N-1},$$

with $l \geq -m$ and $N \leq +\infty$, such that either $l = -m$, or $l > -m$ and $y_{l-1} < \bar{y}$, and either $N = +\infty$, or $N < +\infty$ and $y_N < \bar{y}$; a negative semicycle of a solution $(y_n)_{n\in\mathbb{N}_{-m}}$ to the difference equation is a finite or infinite subsequence/string $y_l, y_{l+1}, \dots, y_{N-1}$ satisfying the condition

$$y_j < \bar{y}, \quad j = \overline{l, N-1},$$

with $l \geq -m$ and $N \leq +\infty$, such that either $l = -m$, or $l > -m$ and $y_{l-1} \geq \bar{y}$, and either $N = +\infty$, or $N < +\infty$ and $y_N \geq \bar{y}$.

Theorem 2.10. *Let $A > 0$, $m \in \mathbb{N}$ and $g : \mathbb{R}_+ \to \mathbb{R}_+$ be a nondecreasing continuous function. Consider the difference equation*

$$y_{n+1} = A + \frac{y_n}{y_{n-m} g(y_{n-m})}, \quad n \in \mathbb{N}_0. \tag{2.39}$$

If a solution to equation (2.39) has a semi-cycle of length at least m, then every semi-cycle after that has at least $m+1$ terms.

Proof. First note that equation (2.39) has a unique positive equilibrium $\bar{y}$ which satisfies the relation

$$\bar{y} = A + \frac{1}{g(\bar{y})}. \tag{2.40}$$

Indeed the function

$$f(t) := A + \frac{1}{g(t)}$$

is a positive nonincreasing continuous function, from which the claim follows.

Let $y_{N-m}, y_{N-m+1}, \dots, y_{N-1}$ be a semi-cycle of length at least m. We may assume that it is a negative one and that $y_N \geq \bar{y}$. So, we have

$$\max\{y_{N-m}, y_{N-m+1}, \dots, y_{N-1}\} < \bar{y} \leq y_N. \tag{2.41}$$

We may also assume that $N - m \geq 1$.

From (2.40), (2.41), the monotonicity of the function g, and since

$$y_n > A, \quad n \in \mathbb{N},$$

we have

$$y_{N+1} = A + \frac{y_N}{y_{N-m} g(y_{N-m})} \geq A + \frac{y_N}{y_{N-m} g(\bar{y})} > A + \frac{1}{g(\bar{y})} = \bar{y}. \tag{2.42}$$

If we have proved that

$$y_{N+i} \geq \bar{y}$$

for some $0 \leq i < m$, then similar to (2.42) we have

$$y_{N+i+1} = A + \frac{y_{N+i}}{y_{N+i-m}g(y_{N+i-m})} \geq A + \frac{y_{N+i}}{y_{N+i-m}g(\bar{y})} > A + \frac{1}{g(\bar{y})} = \bar{y}. \tag{2.43}$$

From the relations in (2.42) and (2.43), and employing the method of mathematical induction, we obtain that

$$y_{N+i} \geq \bar{y},$$

for $i = \overline{0, m}$, proving the claim for the case of the first semi-cycle after the chosen one.

If $y_{N-m}, y_{N-m+1}, \ldots, y_{N-1}$ is a positive semi-cycle of length at least m and $y_N < \bar{y}$, that is

$$\min\{y_{N-m}, y_{N-m+1}, \ldots, y_{N-1}\} \geq \bar{y} > y_N,$$

the proof is similar/dual, so is omitted.

Repeating the same procedure/arguments for the each next semi-cycle the theorem follows. □

A historical remark. A slightly different version of this paper was submitted to the Journal of Applied Mathematics and Computing on February 19, 2023. However, the paper has not been accepted for publication in the journal, and we have not received a real scientific explanation for the unexpected (at least for us) decision.

Author contributions. S. Stević initiated the investigation, proposed many ideas and methods, and conducted detailed investigations. B. Iričanin and W. Kosmala analyzed the proposed ideas, made some calculations, and gave some comments. All authors read and approved the final manuscript.

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Positive periodic solutions for a second-order damped differential equation with an indefinite singularity

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Abstract. We prove the existence of at least one positive periodic solution for a second-order damped nonlinear differential equation with an indefinite singularity by assuming the corresponding linear equation have a positive Green's function. Our results are applicable to a weak singularity as well as a strong singularity. The proof is based on the Leray–Schauder alternative principle.

Keywords: Leray–Schauder alternative principle, Green's function, indefinite singularity, positive periodic solution.

2020 Mathematics Subject Classification: 34B16, 34B18, 34C25.

1 Introduction

The main purpose of this paper is to investigate the existence of a positive periodic solution for the following damped indefinite singular equation

$$x'' + p(t)x' + q(t)x = \frac{b(t)}{x^\rho} + e(t), \tag{1.1}$$

where ρ is a real constant and $\rho > 0$, q and $e \in C(\mathbb{R}/\omega\mathbb{Z})$ are positive, $p \in C(\mathbb{R}/\omega\mathbb{R})$, $b \in C^1(\mathbb{R}/\omega\mathbb{Z})$, the weight term b may have zero, even it may change sign. Moreover, note that when $p(t) \equiv 0$, equation (1.1) reduces to

$$x'' + q(t)x = \frac{b(t)}{x^\rho} + e(t). \tag{1.2}$$

According to the relevant literature [21], the singular term $\frac{b(t)}{x^\rho}$ represents a singularity of repulsive type in the case that $b(t) > 0$ for all $t \in [0, \omega]$, and a singularity of attractive type in the case that $b(t) < 0$ for all $t \in [0, \omega]$. Additionally, equation (1.1) is said to satisfy the strong force condition if $\rho \geq 1$ and the weak force condition if $0 < \rho < 1$.

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In recent years, Schauder's fixed point theorem [10,23,26], the method of lower and upper solutions [15,17,27,28], the Leray–Schauder alternative principle [9,20,22], Krasnoselskii's fixed point theorem in cones [29,30], and coincidence degree theory [5,7,32] have been employed to investigate the existence of positive periodic solutions for equations (1.1) and (1.2) with the weight term $b(t) > 0$ or $b(t) < 0$ for all $t \in [0,\omega]$(i.e. singularity of repulsive or attractive type).

Recently, several intriguing results have been obtained on differential equations with an indefinite singularity [1,8,13,18,24], which are significant both in theory and in practice. In this paper, we aim to establish the existence of a positive periodic solution for equation (1.1) by applying the Leray–Schauder alternative principle.

Depending on the sign of the weight term b, equation (1.1) may exhibit both attractive and repulsive singularities depending on the variable t, even the singularity can vanish in certain subintervals where the weight term b equals zero. Since the weight term b of the above papers [7–10,20,22,23,26,27,29,30,32] does not change sign, the methods employed in these works are no longer applicable to proving the existence of a positive periodic solution for equation (1.1) with an indefinite singularity. Thus, an alternative approach is required to overcome these difficulties. To address the issue of an indefinite singularity (i.e., the singular term that may change sign), we introduce the change of variable $x = u^\alpha$ with $\alpha = \frac{1}{\rho+1}$ and $\alpha < 1$. This transformation simplifies equation (1.1) into a more easily treatable singular equation

$$u'' + p(t)u' + \frac{1}{\alpha}q(t)u = \frac{e(t)}{\alpha}u^{1-\alpha} + (1-\alpha)\frac{u'^2}{u} + \frac{b(t)}{\alpha}. \tag{1.3}$$

It is readily seen that the existence of a positive periodic solution to equation (1.1) reduces to proving the existence of a positive periodic solution for equation (1.3). A similar transformation method has also appeared in the literature [12].

Our purpose is to show that the Leray–Schauder alternative principle can be applied to indefinite singular equation (1.1). The rest of this paper is organized as follows. In section 2, Green's function is provided, and its positivity is obtained. In section 3, in order to facilitate the study of equation (1.3), we first study the following damped singular differential equation

$$u'' + p(t)u' + \ell(t)u = f(t,u,u') + m(t), \tag{1.4}$$

where $\ell(t) := \frac{1}{\alpha}q(t)$, $m(t) := \frac{b(t)}{\alpha}$, $f \in C(\mathbb{R}\times(0,+\infty)\times\mathbb{R},\mathbb{R})$ is ω-periodic with respect to t and exhibits a singularity of repulsive type at $u = 0$. By applying the Leray–Schauder alternative principle, we prove that equation (1.4) has at least one positive periodic solution. Afterwards, we obtain the existence of a positive periodic solution for equation (1.1), and the results are applicable to both strong and weak singularities.

2 Positivity of Green's function

We consider the following nonhomogeneous linear differential equation

$$\begin{cases} u'' + p(t)u' + \ell(t)u = h(t), \\ u(0) = u(\omega),\ u'(0) = u'(\omega), \end{cases} \tag{2.1}$$

where $h \in C(\mathbb{R}/\omega\mathbb{Z})$. Equation (2.1) has a unique ω-periodic solution which can be written as

$$u(t) = \int_0^\omega G(t,s)h(s)ds,$$

where $G(t,s)$ is the Green's function of equation (2.1). Throughout this paper, we assume that

(A) The Green's function $G(t,s)$ of equation (2.1) is positive for all $(t,s) \in [0,\omega] \times [0,\omega]$.

The rest part of this section is to make a brief on some known sufficient conditions to guarantee condition (A) is satisfied. We will discuss the following three cases.

Case I. The general case $p \in C(\mathbb{R}/\omega\mathbb{R})$.

Define functions

$$\varsigma(p)(t) = \exp\left(\int_0^t p(s)ds\right)$$

and

$$\varsigma_1(p)(t) = \varsigma(p)(\omega)\int_0^t \varsigma(p)(s)ds + \int_t^\omega \varsigma(p)(s)ds.$$

Lemma 2.1 (see [16]). *Assume that $\ell(t) \geq 0$ and $\ell \not\equiv 0$ and the following two inequalities are satisfied*

$$\int_0^\omega \ell(s)\varsigma(p)(s)\varsigma_1(-p)(s)ds \geq 0 \tag{2.2}$$

and

$$\sup_{0\leq t\leq\omega}\left\{\int_t^{t+\omega}\varsigma(-p)(s)ds, \int_t^{t+\omega}\ell(s)\varsigma(p)(s)ds\right\} \leq 4. \tag{2.3}$$

Then condition (A) holds.

Case II. Special case $\bar{p} := \frac{1}{\omega}\int_0^\omega p(t)dt > 0$.

In 2005, Wang et al. [31, Lemma 2.4] discussed the positivity of Green's function $G(t,s)$ for all $(t,s) \in [0,\omega] \times [0,\omega]$ if the following conditions are satisfied:

(A_1) There are continuous ω-periodic functions $a_1(t)$ and $a_2(t)$ such that $\int_0^\omega a_1(t)dt > 0$, $\int_0^\omega a_2(t)dt > 0$ and

$$a_1(t) + a_2(t) = p(t), \qquad a_1'(t) + a_1(t)a_2(t) = \ell(t), \quad \text{for } t \in [0,\omega].$$

(A_2) $\bar{p}^2 \geq 4\omega^2 \exp\left(\frac{1}{\omega}\int_0^\omega \ln \ell(s)ds\right)$.

Obviously, condition (A_2) is hard restrictive for the positivity of the Green's function. After that, Cheng and Ren [7] in 2018 discussed the positivity of the Green's function for all $(t,s) \in [0,\omega] \times [0,\omega]$ if only condition (A_1) is satisfied.

Case III. Special case $\bar{p} = 0$.

Define

$$D(\iota) = \begin{cases} \left(\frac{2\pi}{\iota}\right)^{1/2}\left(\frac{2}{2+\iota}\right)^{1/2-1/\iota}\frac{\Gamma(1/\iota)}{\Gamma(1/2+1/\iota)}, & \text{if } 1 \leq \iota < \infty, \\ 2, & \text{if } \iota = \infty, \end{cases}$$

where Γ is the Gamma function with $\Gamma(t) = \int_0^{+\infty} x^{t-1}e^{-x}dx$.

Lemma 2.2 (see [3, Theorem 5.1]). *Assume that $\bar{p} = 0$ and $\int_0^\omega \varsigma(p)(t)\ell(t)dt > 0$. Suppose further that there exists $1 \leq \xi \leq \infty$ such that*

$$(\Upsilon_1(\omega))^{1+1/\iota}\|\Upsilon_2\|_\xi < D^2(2\iota), \tag{2.4}$$

where $\Upsilon_1(\omega) = \int_0^\omega \varsigma(-p)(t)dt$, $\Upsilon_2(t) = \ell_+(t)(\varsigma(p)(t))^{2-1/\xi}$, $\|\Upsilon_2\|_\xi = \left(\int_0^\omega |\Upsilon_2(t)|^\xi dt\right)^{\frac{1}{\xi}}$.
Then condition (A) holds.

In follows what, we consider the case that $p(t) \equiv 0$.

Remark 2.3 (see [25, Corollary 2.3]). In the case $p(t) \equiv 0$, define

$$\widetilde{D}(\tilde{\iota}) = \begin{cases} \frac{2\pi}{\tilde{\iota}\omega^{1+2/\tilde{\iota}}} \left(\frac{2}{2+\tilde{\iota}}\right)^{1-2/\tilde{\iota}} \left(\frac{\Gamma(1/\tilde{\iota})}{\Gamma(1/2+1/\tilde{\iota})}\right)^2, & \text{if } 1 \le \tilde{\iota} < \infty, \\ \frac{4}{\omega}, & \text{if } \tilde{\iota} = \infty, \end{cases}$$

Assume that $\ell \in L^{\zeta}(0,\omega)$ for $1 \le \zeta \le \infty$ and $\ell(t) \ge 0$ for almost every $t \in [0,\omega]$. If

$$\|\ell\|_{\zeta} := \left(\int_0^{\omega} |\ell(t)|^{\zeta} dt\right)^{\frac{1}{\zeta}} < \widetilde{D}(2\zeta^*),$$

where $\zeta^* = \frac{\zeta}{\zeta-1}$ if $1 \le \zeta < \infty$ and $\zeta^* = 1$ if $\zeta = +\infty$, then the Green's function $G(t,s)$ is positive for all $(t,s) \in [0,\omega] \times [0,\omega]$.

Remark 2.4 (see [19, Lemma 2.5]). In the case $p(t) \equiv 0$ and $\ell(t) = \delta^2$ with $\delta > 0$ and $\delta \ne \frac{2k\pi}{\omega}$ for any natural k, the Green's function has the form

$$G_1(t,s) = \begin{cases} \dfrac{\cos\delta(t-s-\frac{\omega}{2})}{2\delta \sin\frac{\delta\omega}{2}}, & 0 \le s \le t \le \omega, \\ \dfrac{\cos\delta(t-s+\frac{\omega}{2})}{2\delta \sin\frac{\delta\omega}{2}}, & 0 \le t < s \le \omega. \end{cases}$$

If $\delta < \frac{\pi}{\omega}$, then Green's function $G_3(t,s)$ is positive for all $(t,s) \in [0,\omega] \times [0,\omega]$.

In this context, the general mechanism for the construction of Green functions in this context is described in [2,4].

3 Main results

In this section, we prove the existence of a positive periodic solution for equation (1.1). First, defined $X := \{u \in C^1 : u(t+\omega) \equiv u(t), \text{ for all } t \in (R)\}$ with the norm $\|u\| := \max_{t\in\mathbb{R}} |u(t)|$, X is a normed linear space (not complete). Our proof is based on the following Leray–Schauder alternative principle, which can be found in [14, p. 120-130].

Lemma 3.1. *Assume that Ω is an open subset of a convex set K in a normed linear space X and $\beta \in \Omega$. Let $T : \bar{\Omega} \to K$ be a compact map. Then one of the following two conclusions holds:*

(I) T has at least one fixed point in $\bar{\Omega}$.

(II) There exist $u \in \partial\Omega$ and $0 < \lambda < 1$ such that $u = \lambda Tu + (1-\lambda)\beta$.

From condition (A), we denote

$$\mathcal{A} := \min_{0\le s,t\le\omega} G(t,s), \qquad \mathcal{B} := \max_{0\le s,t\le\omega} G(t,s), \qquad \sigma := \frac{\mathcal{A}}{\mathcal{B}}, \qquad \iota := \frac{\max_{0\le s,t\le\omega} \left|\frac{\partial G(t,s)}{\partial t}\right|}{\mathcal{A}}. \tag{3.1}$$

It is clear that $0 < \mathcal{A} \le \mathcal{B}$ and $0 < \sigma \le 1$.

Define the function $\gamma : \mathbb{R} \to \mathbb{R}$ by

$$\gamma(t) := \int_0^{\omega} G(t,s)m(s)ds,$$

which is the unique ω-periodic solution of the following equation

$$u''(t) + p(t)u'(t) + \ell(t)u(t) = m(t).$$

Denote

$$\gamma_* := \min_{t\in\mathbb{R}} \gamma(t) \quad \text{and} \quad \gamma^* := \max_{t\in\mathbb{R}} \gamma(t).$$

3.1 The case $\gamma_* \geq 0$

Theorem 3.2. *Assume that condition (A) and $0 \leq \bar{p}\omega < 1$ hold. Furthermore, assume that the following conditions are satisfied:*

(H_1) *For each constant $L > 0$, there exists a continuous function $\phi_L \succ 0$ such that $f(t,u,y) \geq \phi_L(t)$ for all $(t,u,y) \in [0,\omega] \times (0,L] \times \mathbb{R}$, where $\phi_L(t) \succ 0$ represents $\phi_L(t) \geq 0$ for almost every $t \in [0,\omega]$.*

(H_2) *There exist continuous non-negative functions k, φ, ϱ and continuous positive function g such that*

$$0 \leq f(t,u,y) \leq g(u)\varrho(|y|) + k(t)\varphi(u) \quad \text{for all } (t,u,y) \in [0,\omega] \times (0,\infty) \times \mathbb{R},$$

and $g(u)$ is non-increasing, $\varphi(u)/g(u)$ is non-decreasing and $\varrho(\cdot)$ is non-decreasing in $(0,+\infty)$.

(H_3) *There exists a positive real constant r such that*

$$\frac{r}{g(\sigma r + \gamma_*)\left(\varrho\left(\left|\frac{r\bar{\ell}\omega}{1-\bar{p}\omega} + \gamma'^*\right|\right) + \frac{\|k\|\varphi(r+\gamma^*)}{g(r+\gamma^*)}\right)} > \Lambda^*,$$

where $\Lambda(t) := \int_0^\omega G(t,s)ds$ and $\|k\| = \max_{t\in[0,\omega]} k(t)$.

If $\gamma_* \geq 0$, then equation (1.4) has at least one positive ω-periodic solution u with $u(t) > \gamma(t)$ for all $t \in [0,\omega]$ and $0 < \|u - \gamma\| < r$.

Proof. Consider equation

$$u''(t) + p(t)u'(t) + \ell(t)u(t) = f(t, u(t) + \gamma(t), u'(t) + \gamma'(t)). \tag{3.2}$$

It is easy to see that if equation (3.2) has a positive ω-periodic solution u satisfying $u(t) + \gamma(t) > 0$ for $t \in [0,\omega]$ and $0 < \|u\| < r$, then $v(t) := u(t) + \gamma(t)$ is a positive ω-periodic solution of equation (1.4) with $0 < \|v - \gamma\| < r$. So we just need to consider equation (3.2).

Since condition (H_3) holds, we choose $n_0 \in \{1,2,\ldots\}$ such that $\frac{1}{n_0} < \sigma r + \gamma_*$ and

$$\Lambda^* g(\sigma r + \gamma_*)\left(\varrho\left(\left|\frac{r\bar{\ell}\omega}{1-\bar{p}\omega} + \gamma'^*\right|\right) + \frac{\|k\|\varphi(r+\gamma^*)}{g(r+\gamma^*)}\right) + \frac{1}{n_0} < r.$$

Let $N_0 = \{n_0, n_0+1, \ldots\}$. Consider the family of equations

$$u'' + p(t)u' + \ell(t)u = \lambda f_n(t, u(t) + \gamma(t), u'(t) + \gamma'(t)) + \frac{\ell(t)}{n}, \tag{3.3}$$

where $\lambda \in [0,1]$, and

$$f_n(t,u,y) = \begin{cases} f(t,u,y) & \text{if } u \geq \frac{1}{n}, \\ f(t,\frac{1}{n},y) & \text{if } u < \frac{1}{n}. \end{cases}$$

An ω-periodic solution of equation (3.3) is just a fixed point of operator equation

$$u = \lambda T_n u + (1-\lambda)\beta, \tag{3.4}$$

where $\beta := \frac{1}{n}$ and T_n is a continuous operator defined by

$$(T_n u)(t) := \int_0^\omega G(t,s) f_n(s, u(s)+\gamma(s), u'(s)+\gamma'(s))ds + \frac{1}{n}, \tag{3.5}$$

where we used the fact $\int_0^\omega G(t,s)\ell(s)ds \equiv 1$ from [25, Corollary 2.3] and $G(t,s) > 0$ for all $(t,s) \in [0,\omega]\times[0,\omega]$ from condition (A).

Define

$$K := \{u \in X : u(t) > 0 \text{ for all } t \in \mathbb{R}\},$$

and

$$B_r := \{u \in K : \|u\| < r\},$$

where X and r are defined in Section 3 and Theorem 3.2, respectively. Obviously, B_r is an open subset in K. Further, for any $u \in \overline{B_r}$, it follows from (A) and (H_1) that

$$(T_n u)(t) = \int_0^\omega G(t,s) f_n(s, u(s)+\gamma(s), u'(s)+\gamma'(s))ds + \frac{1}{n} \geq \frac{1}{n} > 0,$$

which implies $T_n(\overline{B_r}) \subset K$. Besides, by using the Arzelà–Ascoli Theorem, it is easy to verify that $T_n : \overline{B_r} \to K$ is completely continuous.

Next, we claim that any fixed point u of equation (3.4) for any $\lambda \in (0,1)$ must satisfy $\|u\| \neq r$. Otherwise, assume that u is a fixed point of equation (3.4) for $\lambda \in (0,1)$ such that $\|u\| = r$. Then, we give

$$\begin{aligned}
u(t) - \frac{1}{n} &= \lambda \int_0^\omega G(t,s) f_n(s, u(s)+\gamma(s), u'(s)+\gamma'(s))ds \\
&\geq \lambda\mathcal{A} \int_0^\omega f_n(s, u(s)+\gamma(s), u'(s)+\gamma'(s))ds \\
&= \lambda\sigma\mathcal{B} \int_0^\omega f_n(s, u(s)+\gamma(s), u'(s)+\gamma'(s))ds \\
&\geq \sigma \max_{t\in\mathbb{R}} \left\{ \lambda \int_0^\omega G(t,s) f_n(s, u(s)+\gamma(s), u'(s)+\gamma'(s))ds \right\} \\
&= \sigma \left\| u - \frac{1}{n} \right\|.
\end{aligned}$$

Therefore, we obtain

$$u(t) \geq \sigma\|u - \frac{1}{n}\| + \frac{1}{n} \geq \sigma\left(\|u\| - \frac{1}{n}\right) + \frac{1}{n} \geq \sigma r,$$

and then

$$u(t) + \gamma(t) \geq \sigma r + \gamma_* > \frac{1}{n},$$

since $\frac{1}{n} \leq \frac{1}{n_0} < \sigma r + \gamma_*$.

Further, we claim that

$$\|u'\| \leq \frac{r\bar{\ell}\omega}{1 - \bar{p}\omega} \tag{3.6}$$

for any ω-periodic solution $u(t)$ of equation (3.3). Integrating equation (3.3) over the interval $[0,\omega]$, we see that

$$\int_0^\omega p(t)u'(t)dt + \int_0^\omega \ell(t)u(t)dt = \lambda \int_0^\omega f_n(t,u(t)+\gamma(t),u'(t)+\gamma'(t))dt + \int_0^\omega \frac{\ell(t)}{n}dt. \quad (3.7)$$

Since $u(0) = u(\omega)$, we know that there exists a point $t_1 \in (0,\omega)$ such that $u'(t_1) = 0$. Therefore, we get

$$\begin{aligned}
\|u'\| &= \max_{t\in\mathbb{R}} \left|\frac{1}{2}(u'(t)+u'(t-\omega))\right| \\
&= \max_{t\in\mathbb{R}} \frac{1}{2}\left|\int_{t_1}^{t} u''(s)ds - \int_{t-\omega}^{t_1} u''(s)ds\right| \\
&\le \frac{1}{2}\int_0^\omega |u''(s)|ds \qquad (3.8)\\
&= \frac{1}{2}\left(\int_0^\omega \left|\lambda f_n(t,u(t)+\gamma(t),u'(t)+\gamma'(t)) + \frac{\ell(t)}{n} - \ell(t)u(t) - p(t)u'(t)\right| dt\right) \\
&\le \frac{1}{2}\int_0^\omega \left(\lambda f_n(t,u(t)+\gamma(t),u'(t)+\gamma'(t)) + \frac{\ell(t)}{n} + \ell(t)u(t)\right) dt + \frac{1}{2}\|u'\|\bar{p}\omega
\end{aligned}$$

since $f(t,u,u') > 0$, $\ell(t) > 0$ and $\bar{p} \ge 0$. Applying (3.7) to (3.8), we obtain

$$\begin{aligned}
\|u'\| \le& \int_0^\omega \ell(t)u(t)dt + \frac{1}{2}\int_0^\omega p(t)u'(t)dt + \frac{1}{2}\|u'\|\int_0^\omega p(t)dt \\
=& r\bar{\ell}\omega + \|u'\|\bar{p}\omega.
\end{aligned}$$

Since $1-\bar{p}\omega > 0$, then we deduce

$$\|u'\| \le \frac{r\bar{\ell}\omega}{1-\bar{p}\omega},$$

which proves that (3.6) holds.

On the other hand, it follows from conditions (H_2) and (H_3) that

$$\begin{aligned}
u(t) &= \lambda\int_0^\omega G(t,s)f_n(s,u(s)+\gamma(s),u'(s)+\gamma'(s))ds + \frac{1}{n} \\
&= \lambda\int_0^\omega G(t,s)f(s,u(s)+\gamma(s),u'(s)+\gamma'(s))ds + \frac{1}{n} \\
&\le \int_0^\omega G(t,s)(g(u(s)+\gamma(s))\varrho(|u'(s)+\gamma'(s)|) + k(s)\varphi(u(s)+\gamma(s)))ds + \frac{1}{n} \\
&= \int_0^\omega G(t,s)g(u(s)+\gamma(s))\left(\varrho(|u'(s)+\gamma'(s)|) + \frac{k(s)\varphi(u(s)+\gamma(s))}{g(u(s)+\gamma(s))}\right)ds + \frac{1}{n} \\
&\le g(\sigma r+\gamma_*)\left(\varrho\left(\left|\frac{r\bar{\ell}\omega}{1-\bar{p}\omega}+\gamma'^*\right|\right) + \frac{\|k\|\varphi(r+\gamma^*)}{g(r+\gamma^*)}\right)\Lambda^* + \frac{1}{n_0},
\end{aligned}$$

which implies

$$r = \|u\| \le g(\sigma r+\gamma_*)\left(\varrho\left(\left|\frac{r\bar{\ell}\omega}{1-\bar{p}\omega}+\gamma'^*\right|\right) + \frac{\|k\|\varphi(r+\gamma^*)}{g(r+\gamma^*)}\right)\Lambda^* + \frac{1}{n_0}.$$

This is a contradiction with the choice of n_0 and the claim is proved.

From this claim, Lemma 3.1 guarantees that

$$u = T_n u$$

has a fixed point, denoted by u_n in B_r. Then, we get

$$u'' + p(t)u' + \ell(t)u = f_n(t, u(t) + \gamma(t), u'(t) + \gamma'(t)) + \frac{\ell(t)}{n} \tag{3.9}$$

has an ω-periodic solution u_n with $\|u_n\| \le r$. Since $u_n(t) + \gamma(t) \ge \frac{1}{n} > 0$ for all $n > n_0$, $t \in \mathbb{R}$, u_n is actually a positive ω-periodic solution of equation (3.9).

From condition (H_1), there exists a continuous function $\phi_r(t) \ge 0$ such that $f(t,u,y) \ge \phi_r(t)$ for all $(t,u,y) \in [0,\omega] \times (0,r] \times \mathbb{R}$. Since $u_n(t) + \gamma(t) \ge \frac{1}{n} > 0$ for all $n > n_0$, $t \in \mathbb{R}$ and $\gamma_* \ge 0$, we have

$$\begin{aligned} u_n(t) + \gamma(t) &= \int_0^\omega G(t,s) f_n(s, u_n(s) + \gamma(s), u_n'(s) + \gamma'(s)) ds + \gamma(t) + \frac{1}{n} \\ &= \int_0^\omega G(t,s) f(s, u_n(s) + \gamma(s), u_n'(s) + \gamma'(s)) ds + \gamma(t) + \frac{1}{n} \\ &\ge \int_0^\omega G(t,s) \phi_{r+\gamma^*} ds + \gamma(t) \\ &\ge \Phi_* + \gamma_* =: \vartheta, \end{aligned} \tag{3.10}$$

where $\Phi(t) = \int_0^\omega G(t,s)\phi_{r+\gamma^*}(s)ds$. Since $G(t,s)$ is regular and $\phi_{r+\gamma^*}(s) > 0$, it follows that $\Phi_* > 0$. So we have $u_n(t) + \gamma(t) \ge \vartheta$.

Similar to the proof of equation (3.6), we get

$$\|u_n'\| \le \frac{r\bar{\ell}\omega}{1 - \bar{p}\omega}, \quad \text{for all } n \ge n_0. \tag{3.11}$$

Further, it follows from (3.9) and (3.11) that

$$\begin{aligned} \|u_n''\| &\le \left| p(t)u_n'(t) + \ell(t)u_n(t) + \lambda f_n(t, u_n(t) + \gamma(t), u_n'(t) + \gamma'(t)) + \frac{\ell(t)}{n} \right| \\ &\le \frac{r\|p\|\bar{\ell}\omega}{1 - \bar{p}\omega} + \|\ell\| r + \|f\| + \frac{\|\ell\|}{n_0} := M_1, \end{aligned}$$

where $\|f\| := \max_{\vartheta \le u_n + \gamma \le r + \gamma^*, \|u_n'\| \le \frac{r\bar{\ell}\omega}{1-\bar{p}\omega}} |f(t, u_n(t) + \gamma(t), u_n'(t) + \gamma'(t))|$.

In consequence, $\{u_n\}_{n\in N_0}$ and $\{u_n'\}_{n\in N_0}$ are bounded and equi-continuous family in C_ω^1. Now the Arzelà–Ascoli theorem guarantees that $\{u_n\}_{n\in N_0}$ has a subsequence $\{u_{n_j}\}_{j\in N_0}$, converging uniformly on $\mathbb{R}$ to a function $u \in C_\omega^1$. From the fact $\|u_n\| \le r$ and $\vartheta \le u_n + \gamma$, u satisfies $\vartheta \le u(t) + \gamma(t) \le r + \gamma^*$ for all t. Moreover, u_{n_j} satisfies the integral equation

$$u_{n_j}(t) = \int_0^\omega G(t,s) f(s, u_{n_j}(s) + \gamma(s), u_{n_j}'(s) + \gamma'(s)) ds + \frac{1}{n_j}.$$

Letting $j \to \infty$, we get

$$u(t) = \int_0^\omega G(t,s) f(s, u(s) + \gamma(s), u'(s) + \gamma'(s)) ds.$$

Therefore, u is a positive periodic solution of equation (3.2) and satisfies $0 < \|u\| \le r$. Besides, it is not difficult to show that $\|u\| < r$, by noting that if $\|u\| = r$, the argument similar to the proof of the first claim will lead to a contradiction. □

By Theorem 3.2, we get the following conclusion.

Corollary 3.3. *Assume that condition* (A), $0 \le \bar{p}\omega < 1$ *and* $\gamma_* \ge 0$ *hold. Furthermore, assume that the nonlinear term* f *satisfies the following condition:*

(F_1) *There exist positive constants* κ, ν, μ, η, ϵ *with* $\epsilon \le \nu + 1$ *such that*

$$f(t,u,y) = \kappa \frac{(|y|^\epsilon + 1)}{u^\nu} + \mu e(t) u^\eta, \quad \text{for all } (t,u,y) \in [0,\omega] \times (0,+\infty) \times \mathbb{R},$$

where $e(t) \in C(\mathbb{R}/\omega\mathbb{Z})$ *is positive.*

(i) If $\eta < 1$, *then equation* (1.4) *has at least one positive* ω*-periodic solution for each* $\mu > 0$.

(ii) If $\eta \ge 1$, *then equation* (1.4) *has at least one positive* ω*-periodic solution for each* $0 < \mu < \mu_1$, *where* μ_1 *is a positive constant.*

Proof. We apply Theorem 3.2. Take

$$\phi_L(t) = \frac{\kappa}{L^\nu}, \quad g(u) = \frac{1}{u^\nu}, \quad \varphi(u) = u^\eta, \quad k(t) = \mu e(t), \quad \varrho(|y|) = \kappa(|y|^\epsilon + 1).$$

Then conditions (H_1) and (H_2) are satisfied and the existence condition (H_3) becomes

$$\mu < \frac{r(\sigma r + \gamma_*)^\nu - \Lambda^* \kappa \left(\left| \frac{r\bar{\ell}\omega}{1-\bar{p}\omega} + \gamma'^* \right|^\epsilon + 1 \right)}{\Lambda^* \|e\| (r + \gamma^*)^{\eta+\nu}}$$

for $r > 0$. Therefore, equation (1.4) has at least one positive periodic solution for

$$0 < \mu < \mu_1 := \sup_{r>0} \frac{r(\sigma r + \gamma_*)^\nu - \Lambda^* \kappa \left(\left| \frac{r\bar{\ell}\omega}{1-\bar{p}\omega} + \gamma'^* \right|^\epsilon + 1 \right)}{\Lambda^* \|e\| (r + \gamma^*)^{\eta+\nu}}.$$

Note that $\mu_1 = \infty$ if $\eta < 1$ and $\mu_1 < \infty$ if $\eta \ge 1$, we have (i) and (ii). □

Theorem 3.4. *Assume that* $\bar{p} = 0$, $\int_0^\omega \varsigma(p)(t)\ell(t)dt > 0$, *inequality* (2.4) *and conditions* (H_1), (H_2) *hold. Furthermore, assume that the following condition is satisfied:*

(H_4) *There exists a positive real constant* r *such that*

$$\frac{r}{g(\sigma r + \gamma_*) \left(\varrho(|Qr + \gamma'^*|) + \frac{\|k\| \varphi(r+\gamma^*)}{g(r+\gamma^*)} \right)} > \Lambda^*,$$

where $Q := \frac{\int_0^\omega \varsigma(p)(t)\ell(t)dt}{\min_{t\in\mathbb{R}} \varsigma(p)(t)}$.

If $\gamma_* \ge 0$, *then equation* (1.4) *has at least one positive* ω*-periodic solution* u *with* $u(t) > \gamma(t)$ *for all* $t \in [0,\omega]$ *and* $0 < \|u - \gamma\| < r$.

Proof. We follow the same strategy and notations as in the proof of Theorem 3.2. Next, we claim

$$\|u'\| \le Qr \tag{3.12}$$

for any ω-periodic solution u of equation (3.3). Multiplying both sides of equation (3.3) by $\varsigma(p)(t)$, we get

$$(\varsigma(p)(t)u'(t))' + \varsigma(p)(t)\ell(t)u(t) = \varsigma(p)(t) \left(\lambda f_n(t, u(t) + \gamma(t), u'(t) + \gamma'(t)) + \frac{\ell(t)}{n} \right). \tag{3.13}$$

Integrating equation (3.13) over the interval $[0,\omega]$, we obtain

$$\int_0^\omega \varsigma(p)(t)\ell(t)u(t)dt = \int_0^\omega \varsigma(p)(t)\left(\lambda f_n(t,u(t)+\gamma(t),u'(t)+\gamma'(t))+\frac{\ell(t)}{n}\right)dt$$

since $\bar{p}=0$. Because $u(0)=u(\omega)$, we know that there exists a point $t_2\in(0,\omega)$ such that $u'(t_2)=0$. Therefore, it is clear that

$$\begin{aligned}
&|\varsigma(p)(t)u'(t)|\\
&\quad=\max_{t\in\mathbb{R}}\left|\frac{1}{2}(\varsigma(p)(t)u'(t)+\varsigma(p)(t)u'(t-\omega))\right|\\
&\quad=\max_{t\in\mathbb{R}}\frac{1}{2}\left|\int_{t_2}^{t}(\varsigma(p)(s)u'(s))'ds-\int_{t-\omega}^{t_2}(\varsigma(p)(s)u'(s))'ds\right|\\
&\quad\le\frac{1}{2}\int_0^\omega|(\varsigma(p)(t)u'(t))'|dt\\
&\quad\le\frac{1}{2}\left(\int_0^\omega\left|\varsigma(p)(t)\left(\lambda f_n(t,u(t)+\gamma(t),u'(t)+\gamma'(t))+\frac{\ell(t)}{n}\right)-\varsigma(p)(t)\ell(t)u(t)\right|dt\right)\\
&\quad=\frac{1}{2}\int_0^\omega\varsigma(p)(t)\left(\lambda f(t,u(t)+\gamma(t),u'(t)+\gamma'(t))+\frac{\ell(t)}{n}+\ell(t)u(t)\right)dt\\
&\quad=\int_0^\omega\varsigma(p)(t)\ell(t)u(t)dt\\
&\quad=r\int_0^\omega\varsigma(p)(t)\ell(t)dt,
\end{aligned}$$

where we used the assumption

$$\int_0^\omega\varsigma(p)(t)\ell(t)dt>0.$$

Therefore, we obtain

$$\min_{t\in\mathbb{R}}\varsigma(p)(t)u'(t)\le r\int_0^\omega\varsigma(p)(t)\ell(t)dt,$$

which implies (3.12) holds.

The remaining part of the proof the same as in Theorem 3.2. □

By Theorem 3.2 and Corollary 3.3, we obtain the following conclusion.

Corollary 3.5. *Assume that* $\bar{p}=0$, $\int_0^\omega\varsigma(p)(t)\ell(t)dt>0$, *inequality* (2.4), *condition* (F_1) *and* $\gamma_*\ge 0$ *hold.*

(i) If $\eta<1$, *then equation* (1.4) *has at least one positive* ω*-periodic solution for each* $\mu>0$.

(ii) If $\eta\ge 1$, *then equation* (1.4) *has at least one positive* ω*-periodic solution for each* $0<\mu<\mu_1$, *where* μ_1 *is a positive constant.*

Remark 3.6. If the nonlinear term in equation (1.3) is

$$f(t,u,y)=(1-\alpha)\frac{y^2}{u}+\frac{e(t)}{\alpha}u^{1-\alpha}.$$

From Corollary 3.3, it is easy to verify that there is no continuous function $\phi_L(t)$ such that $f(t,u,y)\ge\phi_L(t)$, for all $(t,u,y)\in[0,\omega]\times\mathbb{R}^+\times\mathbb{R}$. Hence, condition (H_1) is not satisfied. In order to get around condition (H_1), we have to study the case $\gamma_*>0$ by applications of Theorem 3.2.

Remark 3.7. It is worth mentioning that, Cheng and Ren [6] investigated the following damped equation:

$$x'' + p(t)x' + q(t)x = f(t,x) + e(t), \tag{3.14}$$

where f has a repulsive singularity at the origin. In this paper, the singular term of equation (1.1) is sign-changing, which exhibits an indefinite singularity. This makes its study more complex compared to equation (3.14). Therefore, the results of this paper can be regarded as a generalization and refinement of those in [6].

3.2 The case $\gamma_* > 0$

Theorem 3.8. *Assume that conditions* (A), (H_2), (H_3) *and* $0 \le \bar{p}\omega < 1$ *hold. If* $\gamma_* > 0$, *then equation* (1.4) *has at least one positive* ω*-periodic solution* u *with* $u(t) > \gamma(t)$ *for all* $t \in [0,\omega]$ *and* $0 < \|u - \gamma\| < r$.

Proof. We follow the same strategy and notations as in the proof of Theorem 3.2. Next, we consider that $u_n(t) + \gamma(t)$ have a uniform positive lower bound, i.e., there exists a constant $\vartheta_1 > 0$, independent of $n \in N_0$, such that

$$\min_{t\in[0,\omega]} \{u_n(t) + \gamma(t)\} \ge \vartheta_1,$$

for all $n \in N_0$.

Because $u_n(t) + \gamma(t) \ge \frac{1}{n} > 0$ for all $n > n_0$, $t \in \mathbb{R}$, it follows from (H_2) and (3.10) that

$$\begin{aligned} u_n(t) + \gamma(t) &= \int_0^\omega G(t,s) f_n(s, u_n(s) + \gamma(s), u_n'(s) + \gamma'(s))ds + \gamma(t) + \frac{1}{n} \\ &\ge \int_0^\omega G(t,s) f(s, u_n(s) + \gamma(s), u_n'(s) + \gamma'(s))ds + \gamma(t) \\ &\ge \gamma_* := \vartheta_1, \end{aligned}$$

since $\gamma_* > 0$. So we have $u_n(t) + \gamma(t) \ge \vartheta_1$. The proof left is the same as Theorem 3.2. □

By Theorem 3.8, we get the following conclusion.

Corollary 3.9. *Assume that condition* (A), $0 \le \bar{p}\omega < 1$ *and* $\gamma_* > 0$ *hold. Furthermore, assume that the nonlinear term* f *satisfies the following condition:*

(F_2) *There exist positive constants* κ, ν, μ', η, ϵ *with* $\epsilon \le \nu + 1$ *such that*

$$f(t,u,y) = \kappa \frac{|y|^\epsilon}{u^\nu} + \mu' e(t) u^\eta, \quad \text{for all } (t,u,y) \in [0,\omega] \times (0,+\infty) \times \mathbb{R}.$$

(i) If $\eta < 1$, *then equation* (1.4) *has at least one positive* ω*-periodic solution for each* $\mu' > 0$.

(ii) If $\eta \ge 1$, *then equation* (1.4) *has at least one positive* ω*-periodic solution for each* $0 < \mu' < \mu_1'$, *where* μ_1' *is a positive constant.*

Proof. We apply Theorem 3.8, and follow the same strategy and notations as in the proof of Corollary 3.3. Take

$$g(u) = \frac{1}{u^\nu}, \quad \varphi(u) = u^\eta, \quad k(t) = \mu' e(t), \quad \varrho(|y|) = \kappa |y|^\epsilon.$$

Then condition (H_2) is satisfied and the existence condition (H_3) becomes

$$\mu' < \frac{r(\sigma r+\gamma_*)^\nu - \Lambda^*\kappa\left|\frac{r\bar{\ell}\omega}{1-\bar{p}\omega}+\gamma'^*\right|^\epsilon}{\Lambda^*\|e\|(r+\gamma^*)^{\eta+\nu}}$$

for $r>0$. Therefore, equation (1.4) has at least one positive ω-periodic solution for

$$0<\mu'<\mu_1' := \sup_{r>0}\frac{r(\sigma r+\gamma_*)^\nu - \Lambda^*\kappa\left|\frac{r\bar{\ell}\omega}{1-\bar{p}\omega}+\gamma'^*\right|^\epsilon}{\Lambda^*\|e\|(r+\gamma^*)^{\eta+\nu}}.$$ □

Similar to the proof of Theorem 3.8, we get the following conclusions.

Theorem 3.10. *Assume that $\bar{p}=0$, $\int_0^\omega \varsigma(p)(t)\ell(t)dt>0$, inequality (2.4), conditions (H_2), (H_4) hold. If $\gamma_*>0$, then equation (1.4) has at least one positive ω-periodic solution u with $u(t)>\gamma(t)$ for all $t\in[0,\omega]$ and $0<\|u-\gamma\|<r$.*

By Corollary 3.9 and Theorem 3.10, we obtain the following conclusion.

Corollary 3.11. *Assume that $\bar{p}=0$, $\int_0^\omega \varsigma(p)(t)\ell(t)dt>0$, inequality (2.4), condition (F_2) and $\gamma_*>0$ hold.*

(i) If $\eta<1$, then equation (1.4) has at least one positive ω-periodic solution for each $\mu'>0$.

(ii) If $\eta\geq 1$, then equation (1.4) has at least one positive ω-periodic solution for each $0<\mu'<\mu_1'$, where μ_1' is a positive constant.

Corollary 3.12. *Let the nonlinear term in equation (1.3) be*

$$f(t,u,y)=(1-\alpha)\frac{y^2}{u}+\frac{e(t)}{\alpha}u^{1-\alpha}.$$

Assume that condition (A), $0\leq\bar{p}\omega<1$ and $\gamma_*>0$ hold. Then equation (1.1) has at least one positive ω-periodic solution.

Proof. We apply Corollary 3.9. Take

$$\kappa=1-\alpha,\quad \epsilon=2,\quad \nu=1,\quad \mu'=\frac{1}{\alpha},\quad \eta=1-\alpha.$$

Then condition (F_2) is satisfied. Since $\eta=1-\alpha<1$, equation (1.1) has at least one positive ω-periodic solution. □

Corollary 3.13. *Let the nonlinear term in equation (1.3) be*

$$f(t,u,y)=(1-\alpha)\frac{y^2}{u}+\frac{e(t)}{\alpha}u^{1-\alpha}.$$

Assume that $\bar{p}=0$, $\int_0^\omega \varsigma(p)(t)\ell(t)dt>0$, inequality (2.4) and $\gamma_*>0$ hold. Then equation (1.1) has at least one positive ω-periodic solution.

3.3 The case $p(t) \equiv 0$

Next, we prove the existence of a positive periodic solution for equation (1.2). We write equation (1.2) as

$$u'' + \ell(t)u = f(t, u, u') + m(t). \tag{3.15}$$

By Theorem 3.2, we obtain the following conclusion.

Theorem 3.14. *Assume that inequality (2.4), conditions (H_1), (H_2) and $\gamma_* \geq 0$ hold. Furthermore, assume that the following condition is satisfied:*

(H_5) *there exists a positive number $r > 0$ such that*

$$\frac{r}{g(\sigma r + \gamma_*)\left(\varrho\left(|r\bar{\ell}\omega + \gamma'^*|\right) + \frac{\|k\|\varphi(r+\gamma^*)}{g(r+\gamma^*)}\right)} > \Lambda^*.$$

Then equation (3.15) has at least one positive ω-periodic solution u with $u(t) > \gamma(t)$ for all $t \in [0, \omega]$ and $0 < \|u - \gamma\| < r$.

Proof. Consider the following equation

$$u''(t) + \ell(t)u(t) = f(t, u(t) + \gamma(t), u'(t) + \gamma'(t)). \tag{3.16}$$

We follow the same strategy and notations as the proof of Theorems 3.2 and 3.4. Now, we claim

$$\|u'\| \leq r\bar{\ell}\omega. \tag{3.17}$$

It follows from (3.8) and (3.16) that

$$\begin{aligned}
\|u'\| &\leq \frac{1}{2}\int_0^\omega |u''(t)|dt \\
&\leq \frac{1}{2}\int_0^\omega \left(\lambda f(t, u(t) + \gamma(t), u'(t) + \gamma'(t)) + \frac{\ell(t)}{n} + \ell(t)u(t)\right) dt \\
&\leq \int_0^\omega \ell(t)u(t)dt \\
&\leq r\bar{\ell}\omega,
\end{aligned}$$

since $f(t, u, u') > 0$ and $\ell(t) > 0$.

The proof left is the same as Theorem 3.2. □

By Theorems 3.8–3.14, Corollaries 3.3–3.13, we get the following conclusions.

Corollary 3.15. *Assume that inequality (2.4), condition (F_1) and $\gamma_* \geq 0$ hold.*

(i) If $\eta < 1$, then equation (3.15) has at least one positive ω-periodic solution for each $\mu > 0$.

(ii) If $\eta \geq 1$, then equation (3.15) has at least one positive ω-periodic solution for each

$$0 < \mu < \sup_{r>0} \frac{r(\sigma r + \gamma_*)^\nu - \Lambda^*\kappa(|r\bar{\ell}\omega + \gamma'^*|^\epsilon + 1)}{\Lambda^*\|e\|(r + \gamma^*)^{\eta+\nu}}.$$

Theorem 3.16. *Assume that inequality (2.4), conditions (H_2) and (H_5) hold. If $\gamma_* > 0$, then equation (3.15) has at least one positive ω-periodic solution u with $u(t) > \gamma(t)$ for all $t \in [0, \omega]$ and $0 < \|u - \gamma\| < r$.*

Corollary 3.17. *Assume that inequality* (2.4), *condition* (F_2) *and* $\gamma_* > 0$ *hold.*

(i) If $\eta < 1$, *then equation* (3.15) *has at least one positive* ω*-periodic solution for each* $\mu' > 0$.

(ii) If $\eta \geq 1$, *then equation* (3.15) *has at least one positive* ω*-periodic solution for each*

$$0 < \mu' < \sup_{r>0} \frac{r(\sigma r + \gamma_*)^\nu - \Lambda^* \kappa |r\bar{\ell}\omega + \gamma'^*|^\epsilon}{\Lambda^* \|e\| (r + \gamma^*)^{\eta+\nu}}.$$

Corollary 3.18. *Let the nonlinear term in equation* (3.15) *be*

$$f(t, u, y) = (1 - \alpha)\frac{y^2}{u} + \frac{e(t)}{\alpha} u^{1-\alpha}.$$

Assume that inequality (2.4) *and* $\gamma_* > 0$ *hold. Then equation* (1.2) *has at least one positive* ω*-periodic solution.*

Remark 3.19. It is worth mentioning that when $p(t) \equiv 0$, Chu et al. [11] studied the following equation:

$$x'' + p(t)x' = f(t, x) + e(t),$$

where f has a repulsive singularity at the origin. When $p(t) \equiv 0$, equation (1.1) exhibits an indefinite singularity, and it is evident that our results encompass those of Reference [11].

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On a coupled nonlocal Schrödinger–Kirchhoff system with singular exponential nonlinearity in $\mathbb{R}^N$

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Abstract. This paper is concerned with the existence of solutions for parameters dependent Schrödinger–Kirchhoff system driven by nonlocal integro-differential operators with singular Trudinger–Moser nonlinearity in the whole Euclidean space $\mathbb{R}^N$. These parameters have a major impact on the produced analysis. It is noted that, we also study the asymptotic behaviour of solutions depending upon these parameters. The proofs of the existence results to the aforementioned system rely on the mountain pass theorem, the Ekeland variational principle, the classical deformation lemma, and the Krasnoselskii genus theory. The salient feature and novelty of this paper is that it also covers the so-called degenerate case of the Kirchhoff function, that is, it could vanish at zero.

Keywords: nonlocal integro-differential operator, mountain pass theorem, variational principle, genus theory.

2020 Mathematics Subject Classification: 35A15, 35A23, 35D30, 35R11, 35J60.

1 Introduction and main results

In this paper, we study the following nonlocal Schrödinger–Kirchhoff type system:

$$\begin{cases} M(\|(u,v)\|^p)(\mathcal{L}_p^s(u) + V(x)|u|^{p-2}u) = \dfrac{F_u(x,u,v)}{|x|^\gamma} + \lambda h(x)|u|^{q-2}u & \text{in } \mathbb{R}^N, \\ M(\|(u,v)\|^p)(\mathcal{L}_p^s(v) + V(x)|v|^{p-2}v) = \dfrac{F_v(x,u,v)}{|x|^\gamma} + \mu h(x)|v|^{q-2}v & \text{in } \mathbb{R}^N, \end{cases} \tag{$\mathcal{S}_{\lambda,\mu}$}$$

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where $N \geq 1$, $0 < s < 1$, $sp = N$, $1 < q < \infty$, $\gamma \in [0, N)$, λ and μ are two positive parameters, and the norm

$$\|(u,v)\| = \left(\|u\|^p_{W^{s,p}_{K,V}} + \|v\|^p_{W^{s,p}_{K,V}} \right)^{\frac{1}{p}},$$

where for the singular kernel $K : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}^+$ with $\mathbb{R}^+ = (0, \infty)$ and $w \in \{u, v\}$, we define

$$\|w\|_{W^{s,p}_{K,V}} = \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |w(x) - w(y)|^p K(x-y) \, \mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^N} V(x)|w|^p \, \mathrm{d}x \right)^{\frac{1}{p}}.$$

Consequently, $M : \mathbb{R}^+_0 \to \mathbb{R}^+_0$ with $\mathbb{R}^+_0 = [0, \infty)$ is a Kirchhoff function, $V : \mathbb{R}^N \to \mathbb{R}$ is a scalar potential, $h : \mathbb{R}^N \to \mathbb{R}^+$ is a measurable function, the functions F_u and F_v are partial derivatives of a Carathéodory function F, of exponential type and $\mathcal{L}^s_p$ is the nonlocal fractional operator which, up to a normalization constant, is defined by

$$\mathcal{L}^s_p \phi(x) = 2 \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\epsilon(x)} |\phi(x) - \phi(y)|^{p-2} (\phi(x) - \phi(y)) K(x-y) \, \mathrm{d}y, \qquad \forall \, x \in \mathbb{R}^N,$$

along any $\phi \in C^\infty_0(\mathbb{R}^N)$, where $B_\epsilon(x) = \{y \in \mathbb{R}^N : |x - y| < \epsilon\}$. The singular kernel $K : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}^+$ is a measurable function satisfying the following properties for $sp = N$ and $0 < s < 1$:

(a) $\eta K \in L^1(\mathbb{R}^N)$, where $\eta(x) = \min\{|x|^p, 1\}$.

(b) there exists $K_0 > 0$ such that $K(x) \geq K_0 |x|^{-(N+sp)}$ for any $x \in \mathbb{R}^N \setminus \{0\}$.

In addition, we make a note that from here onwards, $\cdot$ stands for the natural inner product in any Euclidean space $\mathbb{R}^d$ for any dimension $d \geq 1$ and $|\cdot|$ denotes the corresponding Euclidean norm.

Throughout the paper, without further mentioning, we have the following assumptions on the scalar potential V, and the Kirchhoff function M.

(V) The function $V : \mathbb{R}^N \to \mathbb{R}$ is assumed to be continuous and to satisfy:

- (V_1) There exists a constant $V_0 > 0$ such that $\inf_{\mathbb{R}^N} V \geq V_0$.
- (V_2) There exists $h > 0$ such that $\lim_{|y| \to \infty} \text{ meas} \left(\{x \in B_h(y) : V(x) \leq c\} \right) = 0$, $\forall \, c > 0$, where we denote the Lebesgue measure of any set $E \subset \mathbb{R}^N$ by meas(E).

(M) The function $M : \mathbb{R}^+_0 \to \mathbb{R}^+_0$ is assumed to be continuous and to satisfy:

- (M_1) For any $\tau > 0$, there exists $\kappa = \kappa(\tau) > 0$ such that

$$M(t) \geq \kappa, \qquad \forall \, t \geq \tau.$$

- (M_2) There exists $\theta \geq 1$ such that

$$tM(t) \leq \theta \widehat{M}(t), \qquad \forall \, t \in \mathbb{R}^+_0,$$

where

$$\widehat{M}(t) = \int_0^t M(\xi) \, \mathrm{d}\xi.$$

Remark 1.1. The condition (V_2) is weaker than the coercivity assumption, that is, $V(x) \to \infty$ as $|x| \to \infty$.

On concerns about the nonlinearity term F, we have the following assumptions.

(F) The function $F : \mathbb{R}^N \times \mathbb{R}^2 \to \mathbb{R}$ is a Carathéodory function and to satisfy:

(F_1) $F(x,\cdot,\cdot) \in C^1(\mathbb{R}^2)$ for a.e. $x \in \mathbb{R}^N$, $F(x,\cdot,\cdot) \geq 0$ in $\mathbb{R}^2$, $F(x,0,0) = 0$ for a.e. $x \in \mathbb{R}^N$, $F_u(x,u,v) = 0$ for all $u \leq 0$ and $v \in \mathbb{R}$, $F_v(x,u,v) = 0$ for all $u \in \mathbb{R}$ and $v \leq 0$, $F_u(x,u,0) = 0$ for all $u \in \mathbb{R}$ and $F_v(x,0,v) = 0$ for all $v \in \mathbb{R}$. Moreover, for all $(u,v) \in \mathbb{R}^+ \times \mathbb{R}^+$, the function $F(\cdot,u,v)$ is strictly positive for a.e. $x \in \mathbb{R}^N$.

(F_2) There exists $\alpha_0 > 0$ with the property that for all $\varepsilon > 0$, there exists $\kappa_\varepsilon > 0$ such that for a.e. $x \in \mathbb{R}^N$ and all $z = (u,v) \in \mathbb{R}^2$ with $|z| = \sqrt{u^2+v^2}$, $\nabla F = (F_u, F_v)$ and $j_p = \min\{j \in N : \ j \geq p\}$, we have

$$|\nabla F(x,u,v)| \leq \varepsilon |z|^{\theta p - 1} + \kappa_\varepsilon \Phi(\alpha_0 |z|^{p'}),$$

where

$$\Phi(t) = \exp(t) - \sum_{j=0}^{j_p - 2} \frac{t^j}{j!} \quad \text{and} \quad p' = \frac{N}{N-s}.$$

(F_3) There holds

$$\nabla F(x,u,v) = o(|z|^{\theta p - 1}) \quad \text{as } |z| \to 0^+ \text{ uniformly for } x \in \mathbb{R}^N.$$

(F_4) There exists $\sigma > \theta p$ such that

$$0 \leq \sigma F(x,u,v) \leq \nabla F(x,u,v).(u,v), \qquad \forall\ (x,u,v) \in \mathbb{R}^N \times \mathbb{R}^2.$$

In recent years, studying elliptic partial differential equations involving fractional Laplacian or more general nonlocal integro-differential operators has become a very interesting area of nonlinear analysis. Such types of operators occur naturally in several real-world applications, such as finance, optimization, game theory, image processing, multiple scattering, phase transition phenomena, population dynamics, continuum mechanics, ultra-relativistic limits of quantum mechanics, soft thin films, minimal surfaces, and the stochastic stability of Lévy processes. In this regard, we refer the readers to study [6, 12–14, 29, 31] and related references. Indeed, the literature on nonlocal fractional operators and their applications is somewhat vast, as shown by the new monographs [2, 34], the comprehensive work [19] and the references included therein.

In the context of fractional order Sobolev space, the Sobolev embedding states that for $sp < N$ with $s \in (0,1)$, the continuous embedding $W^{s,p}(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N)$ holds for any $r \in [p, p_s^*]$, where $p_s^* = \frac{Np}{N-sp}$ is called the critical Sobolev exponent. In conclusion, to study the variational problems with subcritical and critical growth, the nonlinearity cannot exceed the polynomial of degree p_s^*. Despite this, in the Sobolev limiting case (commonly known as the Trudinger–Moser case), that is, $sp = N$, the continuous embedding $W^{s,p}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ holds for any $q \in [p, \infty)$ but we cannot assume $q = \infty$ for such an embedding. Further, in this case, every polynomial growth is allowed. In this scenario, to deal with variational problems, many authors proved separately that the maximal growth of the nonlinearity is of exponential type, for a detailed study, we refer to [28, 36, 45]. The Trudinger–Moser inequalities have many applications, including extremal problems for determinants and zeta functions under conformal deformation of metric. For instance, on the four-dimensional sphere, the determinant of the conformal Laplacian is extremized under conformal deformation with fixed area by the

standard metric. The most important aspect of the Moser–Trudinger inequality has been its connection to the Polyakov–Onofri log determinant variation formula, as well as its later development in conformal geometry and geometric analysis of conformally invariant operators on higher-dimensional manifolds. In this regard, we refer to see [8, 16]. It is worth noting that there has been significant progress in the Trudinger–Moser inequalities to analyse the existence, nonexistence, and multiplicity of solutions to nonlinear PDEs in the context of the factional Sobolev space in $\mathbb{R}^N$. For a detailed study, one may go through [10,46,47,50–52] and references therein.

During the last few decades, there has been a tremendous amount of attention focused by many authors towards the study of Kirchhoff-type problems driven by nonlocal fractional Laplacian operators due to their applicability in various models of physical and biological systems. A typical model of Kirchhoff function M can be considered by $M(t) = a + b\theta t^{\theta-1}$ for all $t \geq 0$, where $a, b \geq 0$, $a + b > 0$ and $\theta > 1$. We say that M is of degenerate type if $a = 0$, while it is called non-degenerate type if $a > 0$. Obviously, one can easily notice that assumptions (M_1)–(M_2) in this paper also cover the degenerate case that corresponds to $M(0) = 0$. In addition, we make a note that in the study of Kirchhoff-type problems, many authors often used M as a nondecreasing function on $\mathbb{R}_0^+$, for instance, see [24, 37]. But however, in view of the assumption (M_1), one can consider that M is not monotone in nature as M can be chosen $M(t) = (1+t)^k + (1+t)^{-1}$ for $t \geq 0$ with $0 < k < 1$. Moreover, it is worth mentioning that the degenerate case in Kirchhoff's theory is more interesting and significant than the non-degenerate case. From a physical point of view, M measures the change of the tension on the string generated by the change of its length during the vibration, while $M(0) = 0$ indicates that the base tension of the string is zero. The presence of the nonlinear coefficient M is crucial to be considered when the changes in tension during the motion cannot be neglected. The existence of solutions for non-degenerate fractional Kirchhoff stationary problems are discussed in [3–5, 24, 26, 30], whereas degenerate problems are addressed in [22, 40, 48] and the relevant references. To the best of our knowledge, most of the works on Kirchhoff-type problems are driven by nonlocal fractional Laplacian or more general integro-differential operators involve the nonlinear terms satisfying some polynomial growth, but there are only a few papers dealing with nonlinear terms satisfying exponential type of growth, this is one of the key motivation towards the study of this paper. In this context, we recommend the readers to study [9, 32, 33, 43, 49].

Nowadays, the study of elliptic systems involving fractional Laplacian or more general nonlocal integro-differential operators has gained much attention due to a wide range of applications in applied sciences. Indeed, if $w = (u, v)^T$ denotes a vector of concentration variable, $H = (f(x, u, v), g(x, u, v))^T$ describes a local reaction term related to source and loss process, then for $M \equiv 1$, $K(x) = |x|^{-(N+sp)}$ and $s = 1$, the system $(\mathcal{S}_{\lambda,\mu})$ derive from the following p-Laplacian reaction-diffusion elliptic system

$$z_t = \operatorname{div}(|\nabla z|^{p-2}\nabla z) + H(x, z)$$

with

$$\operatorname{div}(|\nabla z|^{p-2}\nabla z) = \begin{pmatrix} \operatorname{div}(|\nabla u|^{p-2}\nabla u) \\ \\ \operatorname{div}(|\nabla v|^{p-2}\nabla v) \end{pmatrix}.$$

Such equations can be frequently seen in physics, plasma physics, biophysics, chemical reaction design, etc. In many situations, $H = (f(x, u, v), g(x, u, v))^T$ has both components of polynomial type with variable coefficients, but for the Liouville–Bratu–Gelfand and the

Frank–Kamenetsky models H has exponential growth at infinity. We refer to [15,17] for other physical examples of such problems. On concerning nonlocal elliptic systems with critical and subcritical growth one can see to [23,30,38] and references therein. However, concerning about nonlocal elliptic systems with exponential growth the literature is very limited (see [18,20,35]). This is another key motivation towards the study of this paper.

Motivated by the above-cited works, especially by [35,45,49], we study for the first time in the literature to solve a coupled Schrödinger–Kirchhoff elliptic system with singular exponential growth driven by the nonlocal integro-differential operator in $\mathbb{R}^N$.

To establish our main results, we first define the weak solution of the system $(\mathcal{S}_{\lambda,\mu})$.

Definition 1.2. We say that $(u,v) \in \mathbf{X}$ (see (2.1) for its definition) is a (weak) solution for the system $(\mathcal{S}_{\lambda,\mu})$, if for all $(\varphi,\psi) \in \mathbf{X}$, we have

$$M(\|(u,v)\|^p)\Big(\langle u,\varphi\rangle_{K_p,V} + \langle v,\psi\rangle_{K_p,V}\Big) = \int_{\mathbb{R}^N} \frac{\nabla F(x,u,v).(\varphi,\psi)}{|x|^\gamma}\,\mathrm{d}x + \lambda\int_{\mathbb{R}^N} h(x)|u|^{q-2}u\varphi\,\mathrm{d}x + \mu\int_{\mathbb{R}^N} h(x)|v|^{q-2}v\psi\,\mathrm{d}x,$$

where for any w_1 and w_2, we define

$$\langle w_1,w_2\rangle_{K_p,V} = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |w_1(x)-w_1(y)|^{p-2}(w_1(x)-w_1(y))(w_2(x)-w_2(y))K(x-y)\,\mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^N} V(x)|w_1|^{p-2}w_1w_2\,\mathrm{d}x.$$

Our main results for this paper are listed below.

Theorem 1.3. *Suppose V satisfies $(V_1)-(V_2)$, M satisfies $(M_1)-(M_2)$ and (F) fulfills $(F_1)-(F_4)$. If $1<q<p$ and $h\in L^\eta(\mathbb{R}^N)$ with $\eta=\frac{N}{N-sq}$, then there exists $\widetilde{\lambda}>0$ such that for all $(\lambda,\mu)\in(0,\widetilde{\lambda})\times(0,\widetilde{\lambda})$, the system $(\mathcal{S}_{\lambda,\mu})$ admits at least one nontrivial nonnegative solution $(u_{\lambda,\mu},v_{\lambda,\mu})$ in $\mathbf{X}$. In addition, there holds*

$$\lim_{(\lambda,\mu)\to(0^+,0^+)} \|(u_{\lambda,\mu},v_{\lambda,\mu})\| = 0.$$

Theorem 1.4. *Suppose V satisfies (V_1)–(V_2), M satisfies (M_1)–(M_2) and (F) fulfills (F_1)–(F_4). If $q>\theta p$ and $h\in L^\infty(\mathbb{R}^N)$, then there exists $\widehat{\lambda}>0$ such that for all $(\lambda,\mu)\in(\widehat{\lambda},\infty)\times(\widehat{\lambda},\infty)$, the system $(\mathcal{S}_{\lambda,\mu})$ admits at least one nontrivial nonnegative solution $(u_{\lambda,\mu},v_{\lambda,\mu})$ in $\mathbf{X}$. Further, there holds*

$$\lim_{(\lambda,\mu)\to(\infty,\infty)} \|(u_{\lambda,\mu},v_{\lambda,\mu})\| = 0.$$

Theorem 1.5. *Suppose V satisfies (V_1)–(V_2) and (F) fulfills (F_2)–(F_4). In addition, we assume that F and M satisfy the following conditions:*

- (F_1') *$F(x,\cdot,\cdot)\in C^1(\mathbb{R}^2)$ for a.e. $x\in\mathbb{R}^N$, $F(x,\cdot,\cdot)\geq 0$ in $\mathbb{R}^2$ and $F(x,0,0)=0$ for a.e. $x\in\mathbb{R}^N$. Consequently, we assume that $F(x,u,v)=F(x,-u,-v)$ for all $(x,u,v)\in\mathbb{R}^N\times\mathbb{R}^2$.*
- (M') *the Kirchhoff function is of type $M(t)=a+b\theta t^{\theta-1}$ for all $t\geq 0$, where $a,b\geq 0$, $a+b>0$ and $\theta>1$.*

Then for $1<q<p$ and $h\in L^\eta(\mathbb{R}^N)$ with $\eta=\frac{N}{N-sq}$, there exists $\bar{\lambda}>0$ such that for all $(\lambda,\mu)\in(0,\bar{\lambda})\times(0,\bar{\lambda})$, the system $(\mathcal{S}_{\lambda,\mu})$ has infinitely many solutions in $\mathbf{X}$.

The rest of the paper is organized as follows: In Section 2, we discuss some preliminary results useful for the next main sections and the variational structure of the system ($\mathcal{S}_{\lambda,\mu}$). Section 3 is devoted to proving Theorem 1.3 via the Ekeland variational principle and using the standard topological tools. In Section 4, we prove Theorem 1.4 with the help of the mountain pass theorem. Finally, in Section 5, by introducing a truncated functional and using the deformation lemma along with the Krasnoselskii genus theory, we prove Theorems 1.5.

Notations. From now on in this paper, we have the following notations:

- For any Banach space $(X, \|\cdot\|_X)$, we denote its continuous dual by $(X^*, \|\cdot\|_{X^*})$.
- $o_n(1)$ denotes the real sequence such that $o_n(1) \to 0$ as $n \to \infty$.
- $\rightharpoonup$ means weak convergence and $\to$ means strong convergence.
- $u^+ = \max\{u, 0\}$ and $u^- = \max\{-u, 0\}$.

2 Preliminary results

In this section, we shall discuss about some basic properties of fractional Sobolev spaces and related lemmas, which are used in the sequel of this paper. Note that throughout this paper, we use $N \geq 1$, $s \in (0,1)$ and $sp = N$.

Let $r \in (1,\infty)$ and $L^r(\mathbb{R}^N)$ denotes the standard Lebesgue space with the norm $\|\cdot\|_r$. Moreover, for nonnegative measurable function $V : \mathbb{R}^N \to \mathbb{R}$, the space $L^r_V(\mathbb{R}^N)$, consisting of all real-valued measurable functions, with $V(x)|u|^r \in L^1(\mathbb{R}^N)$, equipped with the seminorm

$$\|u\|_{r,V} = \left(\int_{\mathbb{R}^N} V(x)|u|^r \, \mathrm{d}x \right)^{\frac{1}{r}},$$

which is a norm, thanks to (V_1). The space $\left(L^r_V(\mathbb{R}^N), \|\cdot\|_{r,V}\right)$ is a uniform convex Banach space (see [39]), thanks to $(V1)$. Consequently, under the assumption of (V_1), the embedding $L^r_V(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N)$ is continuous.

For $\gamma \in [0, N)$, we define the space $L^r(\mathbb{R}^N, |x|^{-\gamma}\mathrm{d}x)$, consisting of all real-valued measurable functions, with $|u|^r|x|^{-\gamma} \in L^1(\mathbb{R}^N)$, equipped with the norm

$$\|u\|_{r,\gamma} = \left(\int_{\mathbb{R}^N} |u|^r|x|^{-\gamma} \, \mathrm{d}x \right)^{\frac{1}{r}}.$$

Define the fractional Sobolev space $W^{s,p}(\mathbb{R}^N)$ by

$$W^{s,p}(\mathbb{R}^N) = \{u \in L^p(\mathbb{R}^N) : [u]_{s,p} < \infty\},$$

where $[u]_{s,p}$ denotes the Gagliardo seminorm, defined by

$$[u]_{s,p} = \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}} \, \mathrm{d}x\mathrm{d}y \right)^{\frac{1}{p}}.$$

Under the following norm

$$\|u\|_{W^{s,p}} = (\|u\|_p^p + [u]_{s,p}^p)^{\frac{1}{p}},$$

the space $\left(W^{s,p}(\mathbb{R}^N), \|\cdot\|_{W^{s,p}}\right)$ is a uniformly convex Banach space and hence a reflexive Banach space (see [39]).

Due to (V_1), the weighted fractional Sobolev space $W^{s,p}_V(\mathbb{R}^N)$ makes sense and defined by

$$W^{s,p}_V(\mathbb{R}^N) = \{u \in W^{s,p}(\mathbb{R}^N) : u \in L^p_V(\mathbb{R}^N)\},$$

endowed with the norm

$$\|u\|_{W^{s,p}_V} = \left(\|u\|^p_{p,V} + [u]^p_{s,p}\right)^{\frac{1}{p}}.$$

It is well-known that the space $\left(W^{s,p}_V(\mathbb{R}^N), \|\cdot\|_{W^{s,p}_V}\right)$ is a uniformly convex Banach space and $C^\infty_0(\mathbb{R}^N)$ is dense in $W^{s,p}_V(\mathbb{R}^N)$. Consequently, in virtue of (V_1), the embedding $W^{s,p}_V(\mathbb{R}^N) \hookrightarrow W^{s,p}(\mathbb{R}^N)$ is continuous and there holds $\min\{1, V_0\}\|u\|^p_{W^{s,p}} \leq \|u\|^p_{W^{s,p}_V}$ for all $u \in W^{s,p}_V(\mathbb{R}^N)$ (see [39]).

Let $K : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}^+$ be the singular kernel, which is stated in the introductory part of this paper. Now we define the generalized fractional Sobolev space $W^{s,p}_K(\mathbb{R}^N)$ by

$$W^{s,p}_K(\mathbb{R}^N) = \{u \in L^p(\mathbb{R}^N) : [u]_{s,K_p} < \infty\},$$

equipped with the following norm

$$\|u\|_{W^{s,p}_K} = \left(\|u\|^p_p + [u]^p_{s,K_p}\right)^{\frac{1}{p}},$$

where

$$[u]_{s,K_p} = \left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |u(x) - u(y)|^p K(x-y)\, \mathrm{d}x\mathrm{d}y\right)^{\frac{1}{p}}.$$

The space $\left(W^{s,p}_K(\mathbb{R}^N), \|\cdot\|_{W^{s,p}}\right)$ is a uniformly convex Banach space and hence a reflexive Banach space (see [45]).

Under the assumption (V_1), the weighted fractional generalized Sobolev space $W^{s,p}_{V,K}(\mathbb{R}^N)$ makes sense and defined by

$$W^{s,p}_{K,V}(\mathbb{R}^N) = \{u \in W^{s,p}_K(\mathbb{R}^N) : u \in L^p_V(\mathbb{R}^N)\},$$

endowed with the norm

$$\|u\|_{W^{s,p}_{K,V}} = \left(\|u\|^p_{p,V} + [u]^p_{s,K_p}\right)^{\frac{1}{p}}.$$

It is easy to see that the space $\left(W^{s,p}_{K,V}(\mathbb{R}^N), \|\cdot\|_{W^{s,p}_{K,V}}\right)$ is a uniformly convex Banach space and hence reflexive (see [45]).

The natural function space to study the system $(\mathcal{S}_{\lambda,\mu})$ is the generalized vectorial fractional Sobolev space $\mathbf{X}$, defined by

$$\mathbf{X} = W^{s,p}_{V,K}(\mathbb{R}^N) \times W^{s,p}_{V,K}(\mathbb{R}^N), \tag{2.1}$$

endowed with the norm

$$\|(u,v)\| = \left([(u,v)]^p_{s,K_p} + \|(u,v)\|^p_{p,V}\right)^{\frac{1}{p}},$$

where

$$[(u,v)]_{s,K_p} = \left([u]^p_{s,K_p} + [v]^p_{s,K_p}\right)^{\frac{1}{p}} \quad \text{and} \quad \|(u,v)\|_{p,V} = \left(\|u\|^p_{p,V} + \|v\|^p_{p,V}\right)^{\frac{1}{p}}.$$

Consequently, we note that the space $\left(\mathbf{X}, \|(\cdot,\cdot)\|\right)$ is a uniform convex Banach space, and thus it is reflexive. Similarly, we can define the norm of the Banach space $W^{s,p}(\mathbb{R}^N) \times W^{s,p}(\mathbb{R}^N)$. Now we list some technical lemmas that will be used later in this paper. The following first three lemmas are direct consequences of [45, Lemma 2, 4 and 6].

Lemma 2.1. *Under the assumption of (V_1), the following chain of embeddings*

$$\mathbf{X} \hookrightarrow W^{s,p}(\mathbb{R}^N) \times W^{s,p}(\mathbb{R}^N) \hookrightarrow L^\nu(\mathbb{R}^N) \times L^\nu(\mathbb{R}^N)$$

are continuous for all $\nu \in [p, \infty)$ and there holds $\min\{V_0, K_0\}\|(u,v)\|^p_{W^{s,p}\times W^{s,p}} \leq \|(u,v)\|^p$ for all $(u,v) \in \mathbf{X}$. Moreover, if $\nu \in [1,\infty)$, then the embeddings $\mathbf{X} \hookrightarrow L^\nu(B_R) \times L^\nu(B_R)$ is compact for any $R > 0$.

Lemma 2.2. *Suppose (V_1) and (V_2) holds. Then the embedding $\mathbf{X} \hookrightarrow L^\nu(\mathbb{R}^N) \times L^\nu(\mathbb{R}^N)$ is compact for all $\nu \in [p,\infty)$. Consequently, there holds $\|(u,v)\|_{L^\nu(\mathbb{R}^N)\times L^\nu(\mathbb{R}^N)} \leq \mathcal{A}_\nu^{-1}\|(u,v)\|$ for all $(u,v) \in \mathbf{X}$, where $\mathcal{A}_\nu$ is the best constant in the embedding $\mathbf{X} \hookrightarrow L^\nu(\mathbb{R}^N) \times L^\nu(\mathbb{R}^N)$ and defined by*

$$\mathcal{A}_\nu = \inf_{(u,v)\in\mathbf{X}\setminus\{(0,0)\}} \frac{\|(u,v)\|}{\|(u,v)\|_{L^\nu(\mathbb{R}^N)\times L^\nu(\mathbb{R}^N)}} \text{ with } \|(u,v)\|_{L^\nu(\mathbb{R}^N)\times L^\nu(\mathbb{R}^N)} = \left(\|u\|_\nu^\nu + \|v\|_\nu^\nu\right)^{\frac{1}{\nu}}.$$

Lemma 2.3. *For any $\nu \in [p,\infty)$ and $\gamma \in [0,N)$, the embedding $\mathbf{X} \hookrightarrow L^\nu(\mathbb{R}^N, |x|^{-\gamma}\,\mathrm{d}x) \times L^\nu(\mathbb{R}^N, |x|^{-\gamma}\,\mathrm{d}x)$ is compact. In addition, there holds*

$$\|(u,v)\|_{L^\nu(\mathbb{R}^N,|x|^{-\gamma}\,\mathrm{d}x)\times L^\nu(\mathbb{R}^N,|x|^{-\gamma}\,\mathrm{d}x)} \leq \mathcal{B}_{\nu,\gamma}^{-1}\|(u,v)\|, \qquad \forall\ (u,v) \in \mathbf{X},$$

where $\mathcal{B}_{\nu,\gamma}$ is the best constant in the embedding $\mathbf{X} \hookrightarrow L^\nu(\mathbb{R}^N, |x|^{-\gamma}\,\mathrm{d}x) \times L^\nu(\mathbb{R}^N, |x|^{-\gamma}\,\mathrm{d}x)$, which is defined by

$$\mathcal{B}_{\nu,\gamma} = \inf_{(u,v)\in\mathbf{X}\setminus\{(0,0)\}} \frac{\|(u,v)\|}{\|(u,v)\|_{L^\nu(\mathbb{R}^N,|x|^{-\gamma}\,\mathrm{d}x)\times L^\nu(\mathbb{R}^N,|x|^{-\gamma}\,\mathrm{d}x)}}$$

with

$$\|(u,v)\|_{L^\nu(\mathbb{R}^N,|x|^{-\gamma}\,\mathrm{d}x)\times L^\nu(\mathbb{R}^N,|x|^{-\gamma}\,\mathrm{d}x)} = \left(\|u\|_{\nu,\gamma}^\nu + \|v\|_{\nu,\gamma}^\nu\right)^{\frac{1}{\nu}}.$$

Lemma 2.4. *Let $1 < q < p$ and $h \in L^\eta(\mathbb{R}^N)$ with $\eta = \frac{N}{N-sq}$, then the embedding $L^p(\mathbb{R}^N) \hookrightarrow L_h^q(\mathbb{R}^N)$ is continuous and there holds*

$$\|u\|_{q,h} \leq \|h\|_\eta^{\frac{1}{q}}\|u\|_p, \quad \forall\ u \in L^p(\mathbb{R}^N) \quad \text{with} \quad \|u\|_{q,h}^q = \int_{\mathbb{R}^N} h(x)|u|^q\,\mathrm{d}x.$$

Further, the embedding $W^{s,p}(\mathbb{R}^N) \hookrightarrow L_h^q(\mathbb{R}^N)$ is compact. In addition, due to [45, Lemma 4], the embedding $W_{K,V}^{s,p}(\mathbb{R}^N) \hookrightarrow L_h^q(\mathbb{R}^N)$ is compact.

Proof. Let $q \in (1,p)$ and $h \in L^\eta(\mathbb{R}^N)$ with $\eta = \frac{N}{N-sq}$. Observe that $\frac{1}{\eta} + \frac{q}{p} = 1$. Consequently, by the Hölder's inequality, we have

$$\|u\|_{q,h} \leq \|h\|_\eta^{\frac{1}{q}}\|u\|_p, \qquad \forall\ u \in L^p(\mathbb{R}^N).$$

It follows that the embedding $L^p(\mathbb{R}^N) \hookrightarrow L_h^q(\mathbb{R}^N)$ is continuous. To complete the proof, we only have to show that if $u_n \rightharpoonup u$ in $W^{s,p}(\mathbb{R}^N)$ as $n \to \infty$, then $u_n \to u$ in $L_h^q(\mathbb{R}^N)$ as $n \to \infty$. For this, first we assume that $u_n \rightharpoonup u$ in $W^{s,p}(\mathbb{R}^N)$ as $n \to \infty$. Due to [45, Lemma 4], the sequence $\{u_n - u\}_n$ is bounded in $L^p(\mathbb{R}^N)$ and thus there exists $m > 0$ such that $\|u_n - u\|_p \leq m$ for all $n \in \mathbb{N}$. Further, since $h \in L^\eta(\mathbb{R}^N)$ and using the fact that every integrable function is tight, we can assume that for every $\epsilon > 0$, there exists $R_\epsilon > 0$ large enough such that

$$\int_{\mathbb{R}^N\setminus B_{R_\epsilon}} |h(x)|^\eta\,\mathrm{d}x < \left(\frac{\epsilon}{2m^q}\right)^\eta.$$

This together with the Hölder's inequality implies at once that

$$\int_{\mathbb{R}^N\setminus B_{R_\epsilon}} h(x)|u_n-u|^q\,\mathrm{d}x \le \left(\int_{\mathbb{R}^N\setminus B_{R_\epsilon}} |h(x)|^\eta\,\mathrm{d}x\right)^{\frac{1}{\eta}} \|u_n-u\|_p^q < \frac{\epsilon}{2}.$$

In virtue of [45, Lemma 4], we also have $u_n \to u$ in $L^p(B_{R_\epsilon})$. Therefore, using the the Hölder's inequality, we get

$$\int_{B_{R_\epsilon}} h(x)|u_n-u|^q\,\mathrm{d}x \le \|h\|_\eta \left(\int_{B_{R_\epsilon}} |u_n-u|^p\,\mathrm{d}x\right)^{\frac{1}{p}} = o_n(1) \quad \text{as } n\to\infty.$$

In conclusion, there exists $n_0\in\mathbb{N}$ and $\epsilon>0$ such that $\int_{B_{R_\epsilon}} h(x)|u_n-u|^q\,\mathrm{d}x < \frac{\epsilon}{2}$ for all $n\ge n_0$. Now gathering all the above information, we obtain for all $n\ge n_0$ that

$$\|u_n-u\|_{q,h}^q = \int_{\mathbb{R}^N\setminus B_{R_\epsilon}} h(x)|u_n-u|^q\,\mathrm{d}x + \int_{B_{R_\epsilon}} h(x)|u_n-u|^q\,\mathrm{d}x < \epsilon,$$

and thus we conclude the proof. □

An immediate byproduct of the above lemma, we have the following result.

Lemma 2.5. *Let $1<q<p$ and $h\in L^\eta(\mathbb{R}^N)$ with $\eta=\frac{N}{N-sq}$, then the embedding*

$$L^p(\mathbb{R}^N)\times L^p(\mathbb{R}^N) \hookrightarrow L_h^q(\mathbb{R}^N)\times L_h^q(\mathbb{R}^N)$$

is continuous. In addition, the embedding $W^{s,p}(\mathbb{R}^N)\times W^{s,p}(\mathbb{R}^N)\hookrightarrow L_h^q(\mathbb{R}^N)\times L_h^q(\mathbb{R}^N)$ is compact. Further, due to Lemma 2.1, the embedding $\mathbf{X}\hookrightarrow L_h^q(\mathbb{R}^N)\times L_h^q(\mathbb{R}^N)$ is compact and there holds $\|(u,v)\|_{L_h^q(\mathbb{R}^N)\times L_h^q(\mathbb{R}^N)} \le \mathcal{S}_{q,h}^{-1}\|(u,v)\|$ for all $(u,v)\in\mathbf{X}$, where $\mathcal{S}_{q,h}$ is the best constant in the embedding $\mathbf{X}\hookrightarrow L_h^q(\mathbb{R}^N)\times L_h^q(\mathbb{R}^N)$, which is defined by

$$\mathcal{S}_{q,h} = \inf_{(u,v)\in\mathbf{X}\setminus\{(0,0)\}} \frac{\|(u,v)\|}{\|(u,v)\|_{L_h^q(\mathbb{R}^N)\times L_h^q(\mathbb{R}^N)}} \quad \text{with} \quad \|(u,v)\|_{L_h^q(\mathbb{R}^N)\times L_h^q(\mathbb{R}^N)} = \left(\|u\|_{q,h}^q+\|v\|_{q,h}^q\right)^{\frac{1}{q}}.$$

Lemma 2.6 (cf. [11,35])**.** *The function $\Phi(t)=\exp(t)-\sum_{j=0}^{j_p-2}\frac{t^j}{j!}$ is increasing and convex in $[0,\infty)$. Moreover, for any $\alpha>0$ and $r>1$, there exists a constant $C=C(r)$ such that for $j_p=\min\{j\in N:\ j\ge p\}$, we have*

$$\left(\exp(\alpha|s|^{p'})-\sum_{j=0}^{j_p-2}\frac{\alpha^j|s|^{jp'}}{j!}\right)^r \le C\left(\exp(\alpha r|s|^{p'})-\sum_{j=0}^{j_p-2}\frac{\alpha^j r^j|s|^{jp'}}{j!}\right), \qquad \forall\, s\in\mathbb{R}.$$

The following theorem was proved by Nguyen in [45], which is called singular Trudinger–Moser inequality in the fractional Sobolev space $W^{s,p}(\mathbb{R}^N)$. It can be read as follows:

Theorem 2.7 (Singular Trudinger–Moser inequality in $\mathbb{R}^N$)**.** *For any $\alpha>0$, $\gamma\in[0,N)$ and $u\in W^{s,p}(\mathbb{R}^N)$ with $s\in(0,1)$ and $sp=N$, there holds*

$$\frac{\Phi(\alpha|u|^{p'})}{|x|^\gamma}\in L^1(\mathbb{R}^N),$$

where Φ is defined as in (F_2). Moreover, there exists $\beta_*>0$ such that for any $0\le\alpha\le\beta_*<\alpha_*$ with $\alpha_*\le\alpha_{s,N}^*$, the following inequality holds true:

$$\sup_{u\in W^{s,p}(\mathbb{R}^N),\|u\|_{W^{s,p}}\le 1}\int_{\mathbb{R}^N}\frac{\Phi(\alpha|u|^{p'})}{|x|^\gamma}\,\mathrm{d}x<+\infty,$$

where

$$\alpha^*_{s,N} = N\left(\frac{2(N\omega_N)^2\Gamma(p+1)}{N!}\sum_{k=0}^{+\infty}\frac{(N+k-1)!}{k!}\frac{1}{(N+2k)^p}\right)^{\frac{s}{N-s}} \quad \text{with } \omega_N = \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N+2}{2}\right)}.$$

In addition, the above inequality is sharp for $\alpha > \alpha^*_{s,N}$, that is, the supremum is infinity.

Since we are interested in studying the nonnegative solutions of the system $(\mathcal{S}_{\lambda,\mu})$, we define the associated Euler–Lagrange variational functional $J_{\lambda,\mu} : \mathbf{X} \to \mathbb{R}$ by

$$J_{\lambda,\mu}(u,v) = \frac{1}{p}\widehat{M}(\|(u,v)\|^p) - \int_{\mathbb{R}^N}\frac{F(x,u,v)}{|x|^\gamma}\,\mathrm{d}x - \frac{1}{q}(\lambda\|u^+\|^q_{q,h} + \mu\|v^+\|^q_{q,h}), \quad \forall\ (u,v)\in\mathbf{X}. \tag{2.2}$$

In virtue of the assumption (F_2) and Theorem 2.7, one can easily verify that $J_{\lambda,\mu}$ is well-defined, of class $C^1(\mathbf{X},\mathbb{R})$ and its Gâteaux derivative is given by

$$\begin{aligned}\langle J'_{\lambda,\mu}(u,v),(\varphi,\psi)\rangle &= M(\|(u,v)\|^p)\left(\langle u,\varphi\rangle_{K_p,V} + \langle v,\psi\rangle_{K_p,V}\right) - \int_{\mathbb{R}^N}\frac{\nabla F(x,u,v).(\varphi,\psi)}{|x|^\gamma}\,\mathrm{d}x\\ &\quad - \int_{\mathbb{R}^N} h(x)\{\lambda(u^+)^{q-1}\varphi + \mu(v^+)^{q-1}\psi\}\,\mathrm{d}x, \qquad \forall\ (\varphi,\psi)\in\mathbf{X},\end{aligned} \tag{2.3}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $\mathbf{X}^*$ and $\mathbf{X}$. It follows that the critical points of $J_{\lambda,\mu}$ are exactly the weak solutions of the system $(\mathcal{S}_{\lambda,\mu})$.

3 Proof of Theorem 1.3

In this section, for the sake of simplicity, we assume without further mentioning that the structural assumptions required in Theorem 1.3 hold.

The following lemma shows that every nontrivial (weak) solution of the system $(\mathcal{S}_{\lambda,\mu})$ is nonnegative.

Lemma 3.1. *For all $\lambda > 0$ and $\mu > 0$, any nontrivial solution of the system $(\mathcal{S}_{\lambda,\mu})$ is nonnegative in $\mathbb{R}^N$.*

Proof. Suppose $\lambda > 0$ and $\mu > 0$ are fixed and $(u,v)\in\mathbf{X}\setminus\{(0,0)\}$ is a solution of the system $(\mathcal{S}_{\lambda,\mu})$. Notice that $u = u^+ - u^-$ and $v = v^+ - v^-$ and thus testing (2.3) by $(-u^-,-v^-)$, we have

$$\langle J'_{\lambda,\mu}(u,v),(-u^-,-v^-)\rangle = 0.$$

Due to (F_1), we obtain the following estimates

$$\int_{\mathbb{R}^N}\frac{\nabla F(x,u,v).(u^-,v^-)}{|x|^\gamma}\,\mathrm{d}x = 0$$

and

$$\int_{\mathbb{R}^N} h(x)\{\lambda(u^+)^{q-1}u^- + \mu(v^+)^{q-1}v^-\}\,\mathrm{d}x = 0.$$

On the other hand, we also have

$$\int_{\mathbb{R}^N} V(x)\left(|u|^{p-2}u(-u^-) + |v|^{p-2}v(-v^-)\right)\mathrm{d}x = \int_{\mathbb{R}^N} V(x)\left(|u^-|^p + |v^-|^p\right)\mathrm{d}x.$$

Invoking [32, Lemma 2.1], for a.e. $x, y \in \mathbb{R}^N$ and $w \in \{u, v\}$, we infer that

$$|w^-(x) - w^-(y)|^p \le |w(x) - w(y)|^{p-2}(w(x) - w(y))(-w^-(x) + w^-(y)).$$

Gathering all the above information, we obtain from $\langle J'_{\lambda,\mu}(u,v), (-u^-, -v^-)\rangle = 0$ that

$$M(\|(u,v)\|^p)(\|(u^-, v^-)\|^p) \le 0.$$

Hence by using the fact that $\|(u,v)\| > 0$ and (M_1), we get $u^- = v^- = 0$ a.e. in $\mathbb{R}^N$. This shows that $u = u^+$ and $v = v^+$ a.e. in $\mathbb{R}^N$, and we finish the proof. □

To apply the minimization argument, we first prove the following geometrical structures of the functional $J_{\lambda,\mu}$.

Lemma 3.2. *There exists $\rho \in (0,1]$ and two positive numbers λ_* and j, depending upon ρ, such that $J_{\lambda,\mu}(u,v) \ge j$ for all $(u,v) \in \mathbf{X}$ with $\|(u,v)\| = \rho$ and for all $\lambda > 0$ and $\mu > 0$ such that $\max\{\lambda, \mu\} \le \lambda_*$.*

Proof. It follows from (M_2) that

$$\widehat{M}(t) \ge \widehat{M}(1)t^\theta, \qquad \forall\, t \in [0,1].$$

By using (F_3), it follows that for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that

$$|\nabla F(x,u,v)| \le \varepsilon |z|^{\theta p - 1}, \qquad \forall\, x \in \mathbb{R}^N \text{ and } |z| \le \delta. \tag{3.1}$$

Further, by using (F_2) and Lemma 2.6, we get for any $\vartheta > \theta p$ that

$$|\nabla F(x,u,v)| \le \tilde{\kappa}_\varepsilon |z|^{\vartheta - 1}\Phi(\alpha_0 |z|^{p'}), \qquad \forall\, x \in \mathbb{R}^N \text{ and } |z| \ge \delta, \tag{3.2}$$

where there exists $r > 0$ such that $\vartheta = \theta p + r$ and $\tilde{\kappa}_\varepsilon = \frac{\varepsilon}{\delta^r \Phi(\alpha_0 \delta^{p'})} + \frac{\kappa_\varepsilon}{\delta^{\vartheta - 1}}$. Combining (3.1) and (3.2), we obtain

$$|\nabla F(x,u,v)| \le \varepsilon |z|^{\theta p - 1} + \tilde{\kappa}_\varepsilon |z|^{\vartheta - 1}\Phi(\alpha_0 |z|^{p'}), \qquad \forall\, (x,z) \in \mathbb{R}^N \times \mathbb{R}^2 \text{ and } \vartheta > \theta p. \tag{3.3}$$

Now we obtain from (3.3) and Lemma 2.6 that

$$\begin{aligned} |F(x,u,v)| &= \left| \int_0^1 \frac{d}{dt} F(x,tu,tv)\, \mathrm{d}t \right| = \left| \int_0^1 \nabla F(x,tu,tv).(u,v)\, \mathrm{d}t \right| \\ &\le \varepsilon |z|^{\theta p} \int_0^1 t^{\theta p - 1}\, \mathrm{d}t + \tilde{\kappa}_\varepsilon |z|^\vartheta \int_0^1 t^{\vartheta - 1}\Phi(\alpha_0 t^{p'} |z|^{p'})\, \mathrm{d}t \\ &\le \varepsilon |z|^{\theta p} + \tilde{\kappa}_\varepsilon |z|^\vartheta \Phi(\alpha_0 |z|^{p'}). \end{aligned}$$

This yields at once that

$$|F(x,u,v)| \le \varepsilon |z|^{\theta p} + \tilde{\kappa}_\varepsilon |z|^\vartheta \Phi(\alpha_0 |z|^{p'}), \qquad \forall\, (x,z) \in \mathbb{R}^N \times \mathbb{R}^2 \text{ and } \vartheta > \theta p. \tag{3.4}$$

Suppose $\bar{\delta} \in (0,1]$ is sufficiently small enough and there holds $0 < \|(u,v)\| \le \bar{\delta}$. It is easy to see that $|z| = \sqrt{u^2 + v^2} \le \psi := |u| + |v|$ and thus by using Lemma 2.1, we get

$$\begin{aligned} \|\psi\|_{W^{s,p}} &\le \|u\|_{W^{s,p}} + \|v\|_{W^{s,p}} \le 2\big(\|u\|^p_{W^{s,p}} + \|v\|^p_{W^{s,p}}\big)^{\frac{1}{p}} \\ &= 2\|(u,v)\|_{W^{s,p}\times W^{s,p}} \le 2(\min\{V_0, K_0\})^{-\frac{1}{p}}\|(u,v)\|. \end{aligned} \tag{3.5}$$

Choose $t, t' > 1$ satisfying $\frac{1}{t} + \frac{1}{t'} = 1$ and $\widetilde{\psi} = \psi/\|\psi\|_{W^{s,p}}$. Due to (3.5) and the fact that $\bar{\delta} \in (0,1]$ is small enough such that $0 < \|(u,v)\| \leq \bar{\delta}$, we can assume that $\alpha_0 t' \|\psi\|^{p'}_{W^{s,p}} \leq (2\bar{\delta})^{p'}(\min\{V_0, K_0\})^{-\frac{1}{p-1}}\alpha_0 t' \leq \beta_* < \alpha_*$ with $\alpha_* \leq \alpha^*_{s,N}$. Therefore, by applying the Hölder's inequality, Lemma 2.3 and Lemma 2.6, we obtain from (3.4) that

$$\begin{aligned}
&\int_{\mathbb{R}^N} \frac{|F(x,u,v)|}{|x|^\gamma}\,\mathrm{d}x \\
&\quad\leq \varepsilon \int_{\mathbb{R}^N} \frac{|z|^{\theta p}}{|x|^\gamma}\,\mathrm{d}x + \tilde{\kappa}_\varepsilon \int_{\mathbb{R}^N} \frac{|z|^\vartheta \Phi(\alpha_0|z|^{p'})}{|x|^\gamma}\,\mathrm{d}x \\
&\quad\leq 2^{\theta p}\mathcal{B}^{-\theta p}_{\theta p,\gamma}\varepsilon\|(u,v)\|^{\theta p} + \tilde{\kappa}_\varepsilon\left(\int_{\mathbb{R}^N} \frac{|z|^{\vartheta t}}{|x|^\gamma}\,\mathrm{d}x\right)^{\frac{1}{t}}\left(\int_{\mathbb{R}^N} \frac{C\,\Phi(\alpha_0 t'|z|^{p'})}{|x|^\gamma}\,\mathrm{d}x\right)^{\frac{1}{t'}} \\
&\quad\leq 2^{\theta p}\mathcal{B}^{-\theta p}_{\theta p,\gamma}\varepsilon\|(u,v)\|^{\theta p} + 2^\vartheta C^{\frac{1}{t'}}\mathcal{B}^{-\vartheta}_{\vartheta t,\gamma}\tilde{\kappa}_\varepsilon\|(u,v)\|^\vartheta\left(\int_{\mathbb{R}^N} \frac{\Phi(\alpha_0 t'\|\psi\|^{p'}_{W^{s,p}}|\widetilde{\psi}|^{p'})}{|x|^\gamma}\,\mathrm{d}x\right)^{\frac{1}{t'}} \\
&\quad\leq 2^{\theta p}\mathcal{B}^{-\theta p}_{\theta p,\gamma}\varepsilon\|(u,v)\|^{\theta p} + 2^\vartheta D C^{\frac{1}{t'}}\mathcal{B}^{-\vartheta}_{\vartheta t,\gamma}\tilde{\kappa}_\varepsilon\|(u,v)\|^\vartheta,
\end{aligned}$$

where $D > 0$ is a constant, thanks to Theorem 2.7. Gathering all the above information and using Lemma 2.5, we obtain for $0 < \|(u,v)\| \leq \bar{\delta}$ that

$$J_{\lambda,\mu}(u,v) \geq \left(\frac{\widehat{M}(1)}{p} - 2^{\theta p}\mathcal{B}^{-\theta p}_{\theta p,\gamma}\varepsilon\right)\|(u,v)\|^{\theta p} - \widetilde{C}\|(u,v)\|^\vartheta - \frac{1}{q}\max\{\lambda,\mu\}\mathcal{S}^{-q}_{q,h}\|(u,v)\|^q, \quad (3.6)$$

where $\widetilde{C} = 2^\vartheta D C^{\frac{1}{t'}}\mathcal{B}^{-\vartheta}_{\vartheta t,\gamma}\tilde{\kappa}_\varepsilon$. Take $\varepsilon = \widehat{M}(1)/2^{\theta p+1}p\mathcal{B}^{-\theta p}_{\theta p,\gamma}$, and consider the function $g : [0,\bar{\delta}] \to \mathbb{R}$ as follows:

$$g(\ell) = \frac{\widehat{M}(1)}{2(p+1)^2}\ell^{\theta p} - \widetilde{C}\ell^\vartheta, \qquad \forall\ \ell \in [0,\bar{\delta}].$$

Notice that g admits a positive maximum j in $[0,\bar{\delta}]$ at a point $\rho \in (0,\bar{\delta}]$. This shows that for all $(u,v) \in \mathbf{X}$ with $\|(u,v)\| = \rho$, one has

$$J_{\lambda,\mu}(u,v) \geq \frac{\widehat{M}(1)}{2p}\rho^{\theta p} - \widetilde{C}\rho^\vartheta - \frac{1}{q}\max\{\lambda,\mu\}\mathcal{S}^{-q}_{q,h}\rho^q \geq g(\rho) = j > 0,$$

for all $\lambda > 0$ and $\mu > 0$ with $\max\{\lambda,\mu\} \leq \lambda_*$, where λ_* is given as follows

$$\lambda_* = \frac{q\widehat{M}(1)}{2(p+1)\mathcal{S}^{-q}_{q,h}}\rho^{\theta p - q}.$$

This finishes the proof. □

Lemma 3.3. *For all $\lambda > 0$ and $\mu > 0$ with $\max\{\lambda,\mu\} \leq \lambda_*$, there holds*

$$m_{\lambda,\mu} = \inf\{J_{\lambda,\mu}(u,v) : (u,v) \in \overline{B}_\rho\} < 0.$$

where

$$B_\rho = \{(u,v) \in \mathbf{X} : \|(u,v)\| < \rho\}.$$

Finally, there exists a nonnegative sequence $\{(u_n,v_n)\}_n$ and some nonnegative function $(u_{\lambda,\mu}, v_{\lambda,\mu})$ in $\overline{B}_\rho$ such that for all $n \in \mathbb{N}$, we have

$$\begin{aligned}
&\|(u_n,v_n)\| < \rho,\ m_{\lambda,\mu} \leq J_{\lambda,\mu}(u_n,v_n) \leq m_{\lambda,\mu} + \frac{1}{n},\ (u_n,v_n) \rightharpoonup (u_{\lambda,\mu}, v_{\lambda,\mu}) \quad \text{in } \mathbf{X}, \\
&(u_n,v_n) \to (u_{\lambda,\mu}, v_{\lambda,\mu}) \quad \text{a.e. in} \quad \mathbb{R}^N \quad \text{and} \quad J'_{\lambda,\mu}(u_n,v_n) \to 0 \quad \text{in } \mathbf{X}^* \quad \text{as } n \to \infty.
\end{aligned} \quad (3.7)$$

Proof. Fix $\lambda > 0$ and $\mu > 0$ with $\max\{\lambda, \mu\} \leq \lambda_*$ and a pair $(u, v) \in \mathbf{X}$ such that $\|(u, v)\| \leq \rho$. Let $\tau \in [0, 1]$, then by using (F_1), we have

$$\begin{aligned} J_{\lambda,\mu}(\tau u, \tau v) &\leq \frac{1}{p}\widehat{M}(\|\tau(u, v)\|^p) - \frac{\tau^q}{q}\big(\lambda\|u^+\|_{q,h}^q + \mu\|v^+\|_{q,h}^q\big) \\ &\leq \tau^p\left(\frac{\rho^p}{p}\max_{s\in[0,\rho^p]} M(s)\right) - \frac{\tau^q}{q}\big(\lambda\|u^+\|_{q,h}^q + \mu\|v^+\|_{q,h}^q\big). \end{aligned}$$

Since $1 < q < p$, it follows that $J_{\lambda,\mu}(\tau u, \tau v) < 0$ for $\tau \in (0, 1]$ sufficiently small enough. This together with (3.6) implies that

$$-\infty < m_{\lambda,\mu} = \inf\{J_{\lambda,\mu}(u, v) : (u, v) \in \overline{B}_\rho\} < 0. \tag{3.8}$$

This shows that the functional $J_{\lambda,\mu}$ is bounded from below and of class C^1 on $\overline{B}_\rho$. In addition, we also know that $\overline{B}_\rho$ is a complete metric space with the metric defined by the norm of $\mathbf{X}$. Due to Lemma 3.2, we obtain that

$$\inf_{\partial B_\rho} J_{\lambda,\mu}(u, v) \geq j > 0. \tag{3.9}$$

In virtue of (3.8) and (3.9), for n large enough, we can assume that

$$\frac{1}{n} \in \left(0, \inf_{\partial B_\rho} J_{\lambda,\mu}(u, v) - \inf_{\overline{B}_\rho} J_{\lambda,\mu}(u, v)\right). \tag{3.10}$$

Now applying Ekeland variational principle [21] to the functional $J_{\lambda,\mu} : \overline{B}_\rho \to \mathbb{R}$, we can find a sequence $\{(u_n, v_n)\}_n$ in $\overline{B}_\rho$ such that

$$m_{\lambda,\mu} \leq J_{\lambda,\mu}(u_n, v_n) \leq m_{\lambda,\mu} + \frac{1}{n} \quad \text{and} \quad J_{\lambda,\mu}(u_n, v_n) \leq J_{\lambda,\mu}(u, v) + \frac{1}{n}\|(u_n, v_n) - (u, v)\| \tag{3.11}$$

for all $(u, v) \in \overline{B}_\rho$ satisfying $(u, v) \neq (u_n, v_n)$ for each $n \in \mathbb{N}$. Consequently, we obtain from (3.10) and (3.11) that

$$J_{\lambda,\mu}(u_n, v_n) \leq m_{\lambda,\mu} + \frac{1}{n} = \inf_{\overline{B}_\rho} J_{\lambda,\mu}(u, v) + \frac{1}{n} < \inf_{\partial B_\rho} J_{\lambda,\mu}(u, v).$$

It follows that $\{(u_n, v_n)\}_n \subset B_\rho$, that is, $\|(u_n, v_n)\| < \rho$ for all $n \in \mathbb{N}$. Suppose $(\varphi, \psi) \in S$, where $S = \{(\varphi, \psi) \in \mathbf{X} : \|(\varphi, \psi)\| = 1\}$, and $t > 0$ sufficiently small such that $(u_n + t\varphi, v_n + t\psi) \in \overline{B}_\rho$ for all $n \in \mathbb{N}$. By using (3.11), we can notice that

$$\langle J'_{\lambda,\mu}(u_n, v_n), (\varphi, \psi)\rangle = \lim_{t\to 0^+} \frac{J_{\lambda,\mu}(u_n + t\varphi, v_n + t\psi) - J_{\lambda,\mu}(u_n, v_n)}{t} \geq -\frac{1}{n}, \qquad \forall\ (\varphi, \psi) \in S.$$

The arbitrariness of $(\varphi, \psi) \in S$ infers that

$$|\langle J'_{\lambda,\mu}(u_n, v_n), (\varphi, \psi)\rangle| \leq \frac{1}{n}, \qquad \forall\ (\varphi, \psi) \in S.$$

This together with (3.11) implies at once that

$$J_{\lambda,\mu}(u_n, v_n) \to m_{\lambda,\mu} \quad \text{and} \quad J'_{\lambda,\mu}(u_n, v_n) \to 0 \quad \text{in } \mathbf{X}^* \quad \text{as } n \to \infty. \tag{3.12}$$

On the other hand, it is easy to see that the sequence $\{(u_n, v_n)\}_n$ is bounded, and thus up to a subsequence still denoted itself and $(u_{\lambda,\mu}, v_{\lambda,\mu}) \in \overline{B}_\rho$ such that $(u_n, v_n) \rightharpoonup (u_{\lambda,\mu}, v_{\lambda,\mu})$ in $\mathbf{X}$ as

$n \to \infty$. In view of Lemma 2.2, we have $(u_n, v_n) \to (u_{\lambda,\mu}, v_{\lambda,\mu})$ in $L^\nu(\mathbb{R}^N) \times L^\nu(\mathbb{R}^N)$ as $n \to \infty$ for all $\nu \in [p, \infty)$. Further, since the maps $u \mapsto u^\pm$ are continuous from $L^\nu(\mathbb{R}^N)$ into itself, therefore we conclude that $u_n \to u_{\lambda,\mu}, u_n^\pm \to u_{\lambda,\mu}^\pm, v_n \to v_{\lambda,\mu}$, and $v_n^\pm \to v_{\lambda,\mu}^\pm$ a.e. in $\mathbb{R}^N$ as $n \to \infty$. In addition, from (3.12), we infer that

$$\langle J'_{\lambda,\mu}(u_n, v_n), (-u_n^-, -v_n^-)\rangle \to 0 \quad \text{as } n \to \infty.$$

Now using a similar strategy developed in Lemma 3.1, we obtain

$$M(\|(u_n, v_n)\|^p)(\|(u_n^-, v_n^-)\|^p) = o_n(1) \quad \text{as } n \to \infty. \tag{3.13}$$

Hence we have two possibilities, that is, either $\inf_{n\in\mathbb{N}}\|(u_n, v_n)\| = d > 0$ or $\inf_{n\in\mathbb{N}}\|(u_n, v_n)\| = 0$.

Case 1. Let $\inf_{n\in\mathbb{N}} \|(u_n, v_n)\| = d > 0$. Denote $\kappa = \kappa(\tau)$ as the number corresponding to $\tau = d^p$ in (M_1) such that

$$M(\|(u_n, v_n)\|^p) \geq \kappa, \qquad \forall\, n \in \mathbb{N}. \tag{3.14}$$

In virtue of (3.13) and (3.14), we conclude that $(u_n^-, v_n^-) \to (0,0)$ in $\mathbf{X}$. It follows that $u_{\lambda,\mu}^- = v_{\lambda,\mu}^- = 0$ a.e. in $\mathbb{R}^N$ and thus the pair $(u_{\lambda,\mu}, v_{\lambda,\mu})$ is nonnegative in $\mathbb{R}^N$. Consequently, without loss of generality, we can assume that $(u_n, v_n) = (u_n^+, v_n^+)$, thanks to $(u_n^-, v_n^-) \to (0,0)$ in $\mathbf{X}$. This shows that the sequence $\{(u_n, v_n)\}_n$ is nonnegative.

Case 2. Let $\inf_{n\in\mathbb{N}} \|(u_n, v_n)\| = 0$, then either $(0,0)$ is an accumulation point of the sequence $\{(u_n, v_n)\}_n$ or $(0,0)$ is an isolated point of the sequence $\{(u_n, v_n)\}_n$. If $(0,0)$ is an accumulation point of $\{(u_n, v_n)\}_n$, then up to a subsequence still denoted by itself such that it strongly converges to $(u_{\lambda,\mu}, v_{\lambda,\mu}) = (0,0)$. This situation is impossible. Indeed, if not, then $0 = J_{\lambda,\mu}(0,0) = m_{\lambda,\mu} < 0$, which is a contradiction. On the other hand, if $(0,0)$ is an isolated point of $\{(u_n, v_n)\}_n$, then there exists a subsequence still denoted by itself such that $\inf_{n\in\mathbb{N}} \|(u_n, v_n)\| > 0$. In this situation, we can proceed as in Case 1 to conclude the proof.

From the above discussions, we infer that the sequence $\{(u_n, v_n)\}_n$ and $(u_{\lambda,\mu}, u_{\lambda,\mu})$ are nonnegative. This finishes the proof. □

Proof of Theorem 1.3. Due to Lemma 3.3, there exists a nonnegative sequence $\{(u_n, v_n)\}_n$ in $\overline{B}_\rho$ such that (3.7) holds, that is, we have

$$J_{\lambda,\mu}(u_n, v_n) \to m_{\lambda,\mu}\ (< 0) \quad \text{and} \quad J'_{\lambda,\mu}(u_n, v_n) \to 0 \quad \text{in } \mathbf{X}^* \quad \text{as } n \to \infty.$$

Evidently, we have two situations according to the behaviour of the Kirchhoff function M: either $\inf_{n\in\mathbb{N}} \|(u_n, v_n)\| = d > 0$ or $\inf_{n\in\mathbb{N}} \|(u_n, v_n)\| = 0$. Hence we divide the proof into two parts.

Case 1. Let $\inf_{n\in\mathbb{N}} \|(u_n, v_n)\| = d > 0$. Denote $\kappa = \kappa(\tau)$ as the number corresponding to $\tau = d^p$ in (M_1) such that

$$M(\|(u_n, v_n)\|^p) \geq \kappa, \qquad \forall\, n \in \mathbb{N}. \tag{3.15}$$

Let $\lambda > 0$ and $\mu > 0$ be such that $\max\{\lambda, \mu\} < \widetilde{\lambda}$ with $\widetilde{\lambda} = \min\{\lambda_*, \lambda_0\}$, where λ_* as in Lemma 3.2, while λ_0 is defined by

$$\lambda_0 = \frac{q\kappa(\sigma - \theta p)\big(\min\{V_0, K_0\}\big)^{\frac{p-q}{p}}}{2^{p-q}(\sigma - q)\theta p \mathcal{S}_{q,h}^{-q}} \left(\frac{\beta_*}{2\alpha_0}\right)^{\frac{p-q}{p'}} > 0. \tag{3.16}$$

By using (3.7) and Lemma 2.2, Lemma 2.3 and Lemma 2.5, one has

$$\begin{cases} (u_n, v_n) \rightharpoonup (u_{\lambda,\mu}, v_{\lambda,\mu}) & \text{in } \mathbf{X}; \\ (u_n, v_n) \to (u_{\lambda,\mu}, v_{\lambda,\mu}) & \text{in } L^{\nu}(\mathbb{R}^N) \times L^{\nu}(\mathbb{R}^N),\ \forall\ \nu \in [p, \infty); \\ (u_n, v_n) \to (u_{\lambda,\mu}, v_{\lambda,\mu}) & \text{in } L^{\xi}(\mathbb{R}^N, |x|^{-\gamma}\,\mathrm{d}x) \times L^{\xi}(\mathbb{R}^N, |x|^{-\gamma}\,\mathrm{d}x),\ \forall\ \xi \in [p, \infty); \\ (u_n, v_n) \to (u_{\lambda,\mu}, v_{\lambda,\mu}) & \text{in } L_h^q(\mathbb{R}^N) \times L_h^q(\mathbb{R}^N); \\ (u_n, v_n) \to (u_{\lambda,\mu}, v_{\lambda,\mu}) & \text{a.e. in } \mathbb{R}^N \quad \text{as} \quad n \to \infty. \end{cases} \tag{3.17}$$

For each $(u, v) \in \mathbf{X}$, we define the functional $L(u, v) : \mathbf{X} \to \mathbb{R}$ by

$$\big\langle L(u, v), (\varphi, \psi) \big\rangle = \big\langle u, \varphi \big\rangle_{K_p, V} + \big\langle v, \psi \big\rangle_{K_p, V}, \qquad \forall\ (\varphi, \psi) \in \mathbf{X}.$$

Recalling the elementary inequality as follows

$$a^{\alpha} c^{1-\alpha} + b^{\alpha} d^{1-\alpha} \le (a + b)^{\alpha} (c + d)^{1-\alpha}, \qquad \forall\ a, b, c, d \ge 0 \quad \text{and} \quad \alpha \in (0, 1). \tag{3.18}$$

By applying the Hölder's inequality and (3.18), it is not difficult to show that

$$\big|\big\langle L(u, v), (\varphi, \psi) \big\rangle\big| \le \|(u, v)\|^{p-1} \|(\varphi, \psi)\|, \qquad \forall\ (\varphi, \psi) \in \mathbf{X}.$$

Thus, the definition of L implies that for each $(u, v) \in \mathbf{X}$, $L(u, v)$ is a continuous linear functional on $\mathbf{X}$. Consequently, the weak convergence of $\{(u_n, v_n)\}_n$ in $\mathbf{X}$ gives that

$$\big\langle L(u_{\lambda,\mu}, v_{\lambda,\mu}), (u_n - u_{\lambda,\mu}, v_n - v_{\lambda,\mu}) \big\rangle = o_n(1) \quad \text{as } n \to \infty. \tag{3.19}$$

Furthermore, due to (3.17), there exists $l_p \ge 0$ such that up to a subsequence still denoted by same symbol, we have $\|(u_n, v_n)\| \to l_p$ as $n \to \infty$. In virtue of (3.7), we can also deduce that

$$\big\langle J'_{\lambda,\mu}(u_n, v_n), (u_n - u_{\lambda,\mu}, v_n - v_{\lambda,\mu}) \big\rangle = o_n(1) \quad \text{as } n \to \infty.$$

On simplifying the above convergence, we have

$$\begin{aligned} & M(\|(u_n, v_n)\|^p) \langle L(u_n, v_n), (u_n - u_{\lambda,\mu}, v_n - v_{\lambda,\mu}) \rangle \\ & \quad - \int_{\mathbb{R}^N} \frac{\nabla F(x, u_n, v_n).(u_n - u_{\lambda,\mu}, v_n - v_{\lambda,\mu})}{|x|^{\gamma}}\,\mathrm{d}x - \lambda \int_{\mathbb{R}^N} h(x) u_n^{q-1} (u_n - u_{\lambda,\mu})\,\mathrm{d}x \\ & \quad - \mu \int_{\mathbb{R}^N} h(x) v_n^{q-1} (v_n - v_{\lambda,\mu})\,\mathrm{d}x = o_n(1) \quad \text{as } n \to \infty. \end{aligned} \tag{3.20}$$

Using the notation of Lemma 2.5, (F_4), (M_2) and (3.15), we obtain from (3.7) that as $n \to \infty$

$$\begin{aligned} 0 > m_{\lambda,\mu} &= J_{\lambda,\mu}(u_n, v_n) - \frac{1}{\sigma} \langle J'_{\lambda,\mu}(u_n, v_n), (u_n, v_n) \rangle + o_n(1) \\ &\ge \frac{1}{p} \widehat{M}(\|(u_n, v_n)\|^p) - \frac{1}{\sigma} M(\|(u_n, v_n)\|^p) \|(u_n, v_n)\|^p \\ &\quad - \max\{\lambda, \mu\} \left(\frac{1}{q} - \frac{1}{\sigma}\right) \mathcal{S}_{q,h}^{-q} \|(u_n, v_n)\|^q + o_n(1) \\ &\ge \left(\frac{1}{\theta p} - \frac{1}{\sigma}\right) \kappa \|(u_n, v_n)\|^p - \max\{\lambda, \mu\} \left(\frac{1}{q} - \frac{1}{\sigma}\right) \mathcal{S}_{q,h}^{-q} \|(u_n, v_n)\|^q + o_n(1). \end{aligned}$$

Define $z_n = (u_n, v_n)$ and $z = (u_{\lambda,\mu}, v_{\lambda,\mu})$. Then it is easy to see that $|z_n| = \sqrt{u_n^2 + v_n^2} \le \psi_n := |u_n| + |v_n|$. By direct calculation, one has

$$\|\psi_n\|_{W^{s,p}} \le 2(\min\{V_0, K_0\})^{-\frac{1}{p}} \|(u_n, v_n)\|.$$

Hence from the above two inequalities and (3.16), we can easily deduce that

$$\limsup_{n\to\infty} \|\psi_n\|^{p'}_{W^{s,p}} \leq \left(\frac{2^{p-q}(\sigma-q)\theta p \mathcal{S}_{q,h}^{-q} \max\{\lambda,\mu\}}{q\kappa(\sigma-\theta p)\big(\min\{V_0,K_0\}\big)^{\frac{p-q}{p}}} \right)^{\frac{p'}{p-q}} < \frac{\beta_*}{2\alpha_0}. \tag{3.21}$$

By using the Hölder's inequality, (3.17) and (3.18), we have

$$\begin{aligned}
&\left| \int_{\mathbb{R}^N} h(x)\{\lambda u_n^{q-1}(u_n - u_{\lambda,\mu}) + \mu v_n^{q-1}(v_n - v_{\lambda,\mu})\}\, \mathrm{d}x \right| \\
&\quad \leq \max\{\lambda,\mu\}\left[\|u_n\|_{q,h}^{q-1} \|u_n - u_{\lambda,\mu}\|_{q,h} + \|v_n\|_{q,h}^{q-1} \|v_n - v_{\lambda,\mu}\|_{q,h} \right] \\
&\quad \leq \max\{\lambda,\mu\} \|(u_n,v_n)\|^{q-1}_{L_h^q(\mathbb{R}^N)\times L_h^q(\mathbb{R}^N)} \|(u_n - u_{\lambda,\mu}, v_n - v_{\lambda,\mu})\|_{L_h^q(\mathbb{R}^N)\times L_h^q(\mathbb{R}^N)} \\
&\quad \leq \max\{\lambda,\mu\} \mathcal{S}_{q,h}^{-(q-1)} \sup_{n\in\mathbb{N}} \|(u_n,v_n)\|^{q-1} \|(u_n - u_{\lambda,\mu}, v_n - v_{\lambda,\mu})\|_{L_h^q(\mathbb{R}^N)\times L_h^q(\mathbb{R}^N)} \\
&\quad \to 0 \quad \text{as } n \to \infty,
\end{aligned}$$

thanks to the boundedness of the sequence $\{(u_n, v_n)\}_n$ in $\mathbf{X}$. It follows at once that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} h(x)\{\lambda u_n^{q-1}(u_n - u_{\lambda,\mu}) + \mu v_n^{q-1}(v_n - v_{\lambda,\mu})\}\, \mathrm{d}x = 0. \tag{3.22}$$

Set

$$I_1 = \varepsilon \int_{\mathbb{R}^N} \frac{|z_n|^{\theta p-1}|z_n - z|}{|x|^\gamma}\, \mathrm{d}x \quad \text{and} \quad I_2 = \kappa_\varepsilon \int_{\mathbb{R}^N} \frac{\Phi(\alpha_0|z_n|^{p'})|z_n - z|}{|x|^\gamma}\, \mathrm{d}x.$$

Invoking the Hölder's inequality, we obtain by using (3.17) and the boundedness of $\{(u_n, v_n)\}_n$ in $\mathbf{X}$ that

$$\begin{aligned}
I_1 &\leq \varepsilon \left(\int_{\mathbb{R}^N} \frac{|z_n|^{\theta p}}{|x|^\gamma}\, \mathrm{d}x \right)^{\frac{\theta p-1}{\theta p}} \left(\int_{\mathbb{R}^N} \frac{|z_n - z|^{\theta p}}{|x|^\gamma}\, \mathrm{d}x \right)^{\frac{1}{\theta p}} \\
&\leq 2^{\theta p} \varepsilon \|(u_n,v_n)\|^{\theta p-1}_{L^{\theta p}(\mathbb{R}^N, |x|^{-\gamma}\,\mathrm{d}x)\times L^{\theta p}(\mathbb{R}^N, |x|^{-\gamma}\,\mathrm{d}x)} \\
&\qquad \times \|(u_n - u_{\lambda,\mu}, v_n - v_{\lambda,\mu})\|_{L^{\theta p}(\mathbb{R}^N, |x|^{-\gamma}\,\mathrm{d}x)\times L^{\theta p}(\mathbb{R}^N, |x|^{-\gamma}\,\mathrm{d}x)} \\
&\leq 2^{\theta p} \varepsilon \mathcal{B}_{\theta p,\gamma}^{-(\theta p-1)} \sup_{n\in\mathbb{N}} \|(u_n,v_n)\|^{\theta p-1} \\
&\qquad \times \|(u_n - u_{\lambda,\mu}, v_n - v_{\lambda,\mu})\|_{L^{\theta p}(\mathbb{R}^N, |x|^{-\gamma}\,\mathrm{d}x)\times L^{\theta p}(\mathbb{R}^N, |x|^{-\gamma}\,\mathrm{d}x)} \\
&\to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

Define $\tilde{\psi}_n = \psi_n / \|\psi_n\|_{W^{s,p}}$. Suppose $t \geq p$ and $t' = \frac{t}{t-1} > 1$ such that $\frac{1}{t} + \frac{1}{t'} = 1$. Due to (3.21), there exist $m > 0$ and $n_0 \in \mathbb{N}$ such that $\|\psi_n\|^{p'}_{W^{s,p}} < m < \beta_*/2\alpha_0$ for all $n \geq n_0$. Now choose $t' > 1$ close to 1 in such a way that there holds $\alpha_0 t' \|\psi_n\|^{p'}_{W^{s,p}} < t'\beta_*/2 < \alpha_*$ for all $n \geq n_0$ with

$\alpha_* \leq \alpha^*_{s,N}$. Using the Hölder's inequality, Lemma 2.6 and (3.17), we get

$$\begin{aligned}
I_2 &\leq \kappa_\varepsilon \left\| \frac{\Phi\big(\alpha_0 |z_n|^{p'}\big)}{|x|^{\frac{\gamma}{t'}}} \right\|_{t'} \left(\int_{\mathbb{R}^N} \frac{|z_n - z|^t}{|x|^\gamma}\, \mathrm{d}x \right)^{\frac{1}{t}} \\
&\leq C^{\frac{1}{t'}} \kappa_\varepsilon \left(\int_{\mathbb{R}^N} \frac{\Phi\big(\alpha_0 t' |\psi_n|^{p'}\big)}{|x|^\gamma}\, \mathrm{d}x \right)^{\frac{1}{t'}} \left(\int_{\mathbb{R}^N} \frac{|z_n - z|^t}{|x|^\gamma}\, \mathrm{d}x \right)^{\frac{1}{t}} \\
&\leq 2C^{\frac{1}{t'}} \kappa_\varepsilon \left(\int_{\mathbb{R}^N} \frac{\Phi\big(\alpha_0 t' \|\psi_n\|^{p'}_{W^{s,p}} |\tilde{\psi}_n|^{p'}\big)}{|x|^\gamma}\, \mathrm{d}x \right)^{\frac{1}{t'}} \\
&\quad \times \|(u_n - u_{\lambda,\mu}, v_n - v_{\lambda,\mu})\|_{L^t(\mathbb{R}^N, |x|^{-\gamma}\, \mathrm{d}x) \times L^t(\mathbb{R}^N, |x|^{-\gamma}\, \mathrm{d}x)} \\
&\leq \hat{C} \|(u_n - u_{\lambda,\mu}, v_n - v_{\lambda,\mu})\|_{L^t(\mathbb{R}^N, |x|^{-\gamma}\, \mathrm{d}x) \times L^t(\mathbb{R}^N, |x|^{-\gamma}\, \mathrm{d}x)} \to 0 \quad \text{as } n \to \infty,
\end{aligned}$$

where

$$\hat{C} = 2C^{\frac{1}{t'}} \kappa_\varepsilon \left(\sup_{n \geq n_0} \int_{\mathbb{R}^N} \frac{\Phi\big(\alpha_0 t' \|\psi_n\|^{p'}_{W^{s,p}} |\tilde{\psi}_n|^{p'}\big)}{|x|^\gamma}\, \mathrm{d}x \right)^{\frac{1}{t'}} < +\infty,$$

thanks to Theorem 2.7. Consequently, we obtain from (F_2) that

$$\left| \int_{\mathbb{R}^N} \frac{\nabla F(x, u_n, v_n).(u_n - u_{\lambda,\mu}, v_n - v_{\lambda,\mu})}{|x|^\gamma}\, \mathrm{d}x \right| \leq I_1 + I_2 \to 0 \quad \text{as } n \to \infty.$$

It follows that

$$\lim_{n \to \infty} \int_{\mathbb{R}^N} \frac{\nabla F(x, u_n, v_n).(u_n - u_{\lambda,\mu}, v_n - v_{\lambda,\mu})}{|x|^\gamma}\, \mathrm{d}x = 0. \tag{3.23}$$

Passing limit $n \to \infty$ in (3.20) and using (3.22) as well as (3.23), we get

$$M(\|(u_n, v_n)\|^p) \langle L(u_n, v_n), (u_n - u_{\lambda,\mu}, v_n - v_{\lambda,\mu}) \rangle = o_n(1) \quad \text{as } n \to \infty.$$

It follows from $\inf_{n \in \mathbb{N}} \|(u_n, v_n)\| = d > 0$ that $l_p > 0$. This shows that $M(\|(u_n, v_n)\|^p) \to M(l_p^p) > 0$ as $n \to \infty$, thanks to the continuity of M. In conclusion, from the above convergence, we deduce that

$$\big\langle L(u_n, v_n), (u_n - u_{\lambda,\mu}, v_n - v_{\lambda,\mu}) \big\rangle = o_n(1) \quad \text{as} \quad n \to \infty. \tag{3.24}$$

On combining (3.19) and (3.23), we obtain

$$\big\langle L(u_n, v_n) - L(u_{\lambda,\mu}, v_{\lambda,\mu}), (u_n - u_{\lambda,\mu}, v_n - v_{\lambda,\mu}) \big\rangle = o_n(1) \quad \text{as } n \to \infty. \tag{3.25}$$

Recall the well-known Simon inequality [42] as follows:

$$|\xi - \eta|^p \leq \begin{cases} C_p (|\xi|^{p-2}\xi - |\eta|^{p-2}\eta).(\xi - \eta) & \text{for } p \geq 2, \\ \widetilde{C}_p [(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta).(\xi - \eta)]^{\frac{p}{2}} (|\xi|^p + |\eta|^p)^{\frac{2-p}{2}} & \text{for } 1 < p < 2, \end{cases} \tag{3.26}$$

where $\xi, \eta \in \mathbb{R}^N$ and C_p as well as $\widetilde{C}_p$ are positive constants depending only upon p. In virtue of the Simon inequality, we distinguish the following two situations :

Situation 1. Let $p \geq 2$, then we obtain from (3.25) and (3.26) that

$$\begin{aligned}
\|(u_n, v_n) - (u_{\lambda,\mu}, v_{\lambda,\mu})\|^p &\leq C_p \big\langle L(u_n, v_n) - L(u_{\lambda,\mu}, v_{\lambda,\mu}), (u_n - u_{\lambda,\mu}, v_n - v_{\lambda,\mu}) \big\rangle \\
&\to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

It follows that $(u_n, v_n) \to (u_{\lambda,\mu}, v_{\lambda,\mu})$ in $\mathbf{X}$ as $n \to \infty$ for any $p \geq 2$.

Situation 2. Let $1 < p < 2$. First, recall the following elementary inequality:

$$(a+b)^{\frac{2-p}{2}} \leq a^{\frac{2-p}{2}} + b^{\frac{2-p}{2}}, \qquad \forall\, a, b \geq 0 \quad \text{and} \quad 1 < p < 2. \tag{3.27}$$

Using the Hölder's inequality, (3.25), (3.26) and (3.27), we obtain

$$\begin{aligned}
&\|(u_n, v_n) - (u_{\lambda,\mu}, v_{\lambda,\mu})\|^p \\
&\quad \leq \widetilde{C}_p[\langle L(u_n, v_n) - L(u_{\lambda,\mu}, v_{\lambda,\mu}), (u_n - u_{\lambda,\mu}, v_n - v_{\lambda,\mu})\rangle]^{\frac{p}{2}} \\
&\qquad \times \left(\|(u_n, v_n)\|^p + \|(u_{\lambda,\mu}, v_{\lambda,\mu})\|^p\right)^{\frac{2-p}{2}} \\
&\quad \leq \widetilde{C}_p[\langle L(u_n, v_n) - L(u_{\lambda,\mu}, v_{\lambda,\mu}), (u_n - u_{\lambda,\mu}, v_n - v_{\lambda,\mu})\rangle]^{\frac{p}{2}} \\
&\qquad \times \left(\|(u_n, v_n)\|^{\frac{(2-p)p}{2}} + \|(u_{\lambda,\mu}, v_{\lambda,\mu})\|^{\frac{(2-p)p}{2}}\right) \\
&\quad \leq C[\langle L(u_n, v_n) - L(u_{\lambda,\mu}, v_{\lambda,\mu}), (u_n - u_{\lambda,\mu}, v_n - v_{\lambda,\mu})\rangle]^{\frac{p}{2}} \to 0 \quad \text{as } n \to \infty,
\end{aligned}$$

where

$$C = \widetilde{C}_p \left(\sup_{n\in\mathbb{N}} \|(u_n, v_n)\|^{\frac{(2-p)p}{2}} + \|(u_{\lambda,\mu}, v_{\lambda,\mu})\|^{\frac{(2-p)p}{2}} \right) < +\infty,$$

thanks to the uniform boundedness of the sequence

$$\left\{ \|(u_n, v_n)\|^{\frac{(2-p)p}{2}} + \|(u_{\lambda,\mu}, v_{\lambda,\mu})\|^{\frac{(2-p)p}{2}} \right\}_n \quad \text{in } \mathbb{R}.$$

This shows at once that $(u_n, v_n) \to (u_{\lambda,\mu}, v_{\lambda,\mu})$ in $\mathbf{X}$ as $n \to \infty$ for any $1 < p < 2$.

From the above discussions, we conclude that $(u_n, v_n) \to (u_{\lambda,\mu}, v_{\lambda,\mu})$ in $\mathbf{X}$ as $n \to \infty$. Now we shall discuss about the situation when $\inf_{n\in\mathbb{N}} \|(u_n, v_n)\| = 0$.

Case 2. Let $\inf_{n\in\mathbb{N}} \|(u_n, v_n)\| = 0$. If $(0,0)$ is an isolated point of the sequence $\{(u_n, v_n)\}_n$, then there exists a subsequence still denoted by the same symbol such that

$$\inf_{n\in\mathbb{N}} \|(u_n, v_n)\| = d > 0.$$

In this scenario, we can proceed as before. On the other hand, if $(0,0)$ is an accumulation point of $\{(u_n, v_n)\}_n$, then up to a subsequence still denoted by itself such that it strongly converges to $(u_{\lambda,\mu}, v_{\lambda,\mu}) = (0,0)$. This situation is impossible. Indeed, if not, then $0 = J_{\lambda,\mu}(0,0) = m_{\lambda,\mu} < 0$, which is a contradiction.

By using $(u_n, v_n) \to (u_{\lambda,\mu}, v_{\lambda,\mu})$ in $\mathbf{X}$ as $n \to \infty$, we deduce that $J_{\lambda,\mu}(u_{\lambda,\mu}, v_{\lambda,\mu}) = m_{\lambda,\mu}$ and $J'_{\lambda,\mu}(u_{\lambda,\mu}, v_{\lambda,\mu}) = 0$, thanks to $J_{\lambda,\mu} \in C^1(\mathbf{X}, \mathbb{R})$. This shows that $(u_{\lambda,\mu}, v_{\lambda,\mu}) \in \overline{B}_\rho$ and thus the solution $(u_{\lambda,\mu}, v_{\lambda,\mu})$ of the system $(\mathcal{S}_{\lambda,\mu})$ is also a minimizer of the functional $J_{\lambda,\mu}$ in $\overline{B}_\rho$. Consequently, we deduce that

$$J_{\lambda,\mu}(u_{\lambda,\mu}, v_{\lambda,\mu}) = m_{\lambda,\mu} < 0 < j \leq J_{\lambda,\mu}(u, v), \qquad \forall\, (u, v) \in \partial B_\rho,$$

thanks to Lemma 3.2. It follows that $(u_{\lambda,\mu}, v_{\lambda,\mu}) \in B_\rho$ and hence $(u_{\lambda,\mu}, v_{\lambda,\mu})$ is a nontrivial nonnegative solution of the system $(\mathcal{S}_{\lambda,\mu})$ for all $\lambda > 0$ and $\mu > 0$ such that $\max\{\lambda, \mu\} < \widetilde{\lambda}$.

On the other hand, one can see that $\{(u_{\lambda,\mu}, v_{\lambda,\mu})\}_{(\lambda,\mu)\in(0,\widetilde{\lambda})\times(0,\widetilde{\lambda})}$ is uniformly bounded in $\mathbf{X}$, thanks to the fact that $\rho > 0$, which is independent of $\lambda > 0$ and $\mu > 0$, as specified in

Lemma 3.2. In virtue of (F_4), (M_2), (3.15) and using $(u_n, v_n) \to (u_{\lambda,\mu}, v_{\lambda,\mu})$ in $\mathbf{X}$ as $n \to \infty$, one has

$$\begin{aligned} 0 > m_{\lambda,\mu} &= \lim_{n\to\infty} \left[J_{\lambda,\mu}(u_n, v_n) - \frac{1}{\sigma} \langle J'_{\lambda,\mu}(u_n, v_n), (u_n, v_n) \rangle \right] \\ &\geq \frac{1}{p} \widehat{M}(\|(u_{\lambda,\mu}, v_{\lambda,\mu})\|^p) - \frac{1}{\sigma} M(\|(u_{\lambda,\mu}, v_{\lambda,\mu})\|^p) \|(u_{\lambda,\mu}, v_{\lambda,\mu})\|^p \\ &\quad - \max\{\lambda, \mu\} \left(\frac{1}{q} - \frac{1}{\sigma} \right) \mathcal{S}_{q,h}^{-q} \|(u_{\lambda,\mu}, v_{\lambda,\mu})\|^q \\ &\geq \left(\frac{1}{\theta p} - \frac{1}{\sigma} \right) \kappa \|(u_{\lambda,\mu}, v_{\lambda,\mu})\|^p - \max\{\lambda, \mu\} \mathcal{D}_{q,h}, \end{aligned}$$

where

$$\mathcal{D}_{q,h} = \left(\frac{1}{q} - \frac{1}{\sigma} \right) \mathcal{S}_{q,h}^{-q} \sup_{(\lambda,\mu)\in(0,\widetilde{\lambda})\times(0,\widetilde{\lambda})} \|(u_{\lambda,\mu}, v_{\lambda,\mu})\|^q < +\infty.$$

This yields at once that

$$0 \geq \limsup_{(\lambda,\mu)\to(0^+,0^+)} m_{\lambda,\mu} \geq \limsup_{(\lambda,\mu)\to(0^+,0^+)} \left(\frac{1}{\theta p} - \frac{1}{\sigma} \right) \kappa \|(u_{\lambda,\mu}, v_{\lambda,\mu})\|^p \geq 0.$$

This implies that

$$\lim_{(\lambda,\mu)\to(0^+,0^+)} \|(u_{\lambda,\mu}, v_{\lambda,\mu})\| = 0,$$

and thus we conclude the proof. □

4 Proof of Theorem 1.4

Throughout this section, we assume without further mentioning that the structural assumptions required in Theorem 1.4 hold. To prove Theorem 1.4, we need the mountain pass theorem and some basic definitions, as stated below.

Definition 4.1 (Palais–Smale compactness condition)**.** Let X be a Banach space and $\mathcal{I} : X \to \mathbb{R}$ be a functional of class $C^1(X, \mathbb{R})$. We say that $\mathcal{I}$ satisfies the Palais–Smale compactness condition at a suitable level $c \in \mathbb{R}$, if for any sequence $\{u_n\}_n \subset X$ such that

$$\mathcal{I}(u_n) \to c \quad \text{and} \quad \sup_{\|\varphi\|_X=1} |\langle \mathcal{I}'(u_n), \varphi \rangle| \to 0 \quad \text{as } n \to \infty \tag{4.1}$$

has a strongly convergent subsequence in X. Note that if this strongly convergent subsequence exists only for some values of c, we say that $\mathcal{I}$ satisfies a local Palais–Smale condition. We also remark that the sequence $\{u_n\}_n \subset X$ satisfying (4.1) is known as a Palais–Smale sequence at level $c \in \mathbb{R}$ [$(PS)_c$ sequence in short].

Theorem 4.2 (The mountain pass theorem, cf. [41, 44])**.** *Let X be a real Banach space and $I \in C^1(X, \mathbb{R})$. Suppose $I(0) = 0$ and*

(a) there exist two constants $\alpha, \rho > 0$ such that $I(u) \geq \alpha$ for all $u \in X$ with $\|u\| = \rho$;

(b) there exists $e \in X$ satisfying $\|e\| > \rho$ such that $I(e) < 0$.

Define

$$\Gamma = \{\gamma_* \in C^1([0,1], X) : \ \gamma_*(0) = 1, \ \gamma_*(1) = e\}.$$

Then

$$c = \inf_{\gamma_* \in \Gamma} \max_{t \in [0,1]} I(\gamma_*(t)) \geq \alpha$$

and there exists a $(PS)_c$ sequence $\{u_n\}_n$ for I in X. Consequently, if I satisfies the Palais–Smale condition, then I possesses a critical value $c \geq \alpha$.

Now we shall verify the validity of the mountain pass geometrical structure of the functional $J_{\lambda,\mu}$.

Lemma 4.3 (Mountain Pass Geometry-I)**.** *For all $\lambda > 0$ and $\mu > 0$, there exist numbers $j_0 > 0$ and $\rho_0 \in (0,1]$ very small enough such that $J_{\lambda,\mu}(u,v) \geq j_0$ for all $(u,v) \in \mathbf{X}$ with $\|(u,v)\| = \rho_0$.*

Proof. The assumption (M_2) gives that

$$\widehat{M}(t) \geq \widehat{M}(1)t^\theta, \qquad \forall\ t \in [0,1].$$

Suppose $\bar{\delta} \in (0,1]$ small enough and there holds $0 < \|(u,v)\| \leq \bar{\delta}$. Then by using the notation of Lemma 3.2, we can easily deduce that

$$\int_{\mathbb{R}^N} \frac{|F(x,u,v)|}{|x|^\gamma}\, \mathrm{d}x \leq 2^{\theta p}\mathcal{B}_{\theta p,\gamma}^{-\theta p}\varepsilon\|(u,v)\|^{\theta p} + 2^{\vartheta} DC^{\frac{1}{t'}}\mathcal{B}_{\vartheta t,\gamma}^{-\vartheta}\tilde{\kappa}_\varepsilon\|(u,v)\|^{\vartheta}$$

for any $0 < \|(u,v)\| \leq \bar{\delta}$. Consequently, from the above information, we obtain for any $0 < \|(u,v)\| \leq \bar{\delta}$ that

$$J_{\lambda,\mu}(u,v) \geq A\|(u,v)\|^{\theta p} - B\|(u,v)\|^{\vartheta} - C\|(u,v)\|^{q}, \tag{4.2}$$

where

$$A = \frac{\widehat{M}(1)}{p} - 2^{\theta p}\mathcal{B}_{\theta p,\gamma}^{-\theta p}\varepsilon, \quad B = 2^{\vartheta} DC^{\frac{1}{t'}}\mathcal{B}_{\vartheta t,\gamma}^{-\vartheta}\tilde{\kappa}_\varepsilon \quad \text{and} \quad C = \frac{1}{q}\max\{\lambda,\mu\}\mathcal{A}_q^{-q}\|h\|_\infty.$$

Choose $\varepsilon > 0$ sufficiently small such that $A > 0$. Suppose $\xi = \min\{\vartheta, q\}$, then using the fact that $0 < \|(u,v)\| \leq \bar{\delta} \leq 1$, we obtain from (4.2) that

$$J_{\lambda,\mu}(u,v) \geq A\|(u,v)\|^{\theta p} - (B+C)\|(u,v)\|^{\xi}, \quad \forall\ (u,v) \in \mathbf{X} \text{ with } 0 < \|(u,v)\| \leq \bar{\delta}. \tag{4.3}$$

Define the function

$$f(\ell) = A\ell^{\theta p} - (B+C)\ell^{\xi}, \qquad \forall\ \ell \in [0,\bar{\delta}].$$

Observe that f admits a positive maximum j_0 in $[0,\bar{\delta}]$ at a point $\rho_0 \in (0,\bar{\delta}]$. This shows that for all $(u,v) \in \mathbf{X}$ with $\|(u,v)\| = \rho_0$, we obtain from (4.3) that

$$J_{\lambda,\mu}(u,v) \geq A\rho_0^{\theta p} - (B+C)\rho_0^{\xi} = f(\rho_0) = j_0 > 0.$$

This completes the proof. □

Lemma 4.4 (Mountain Pass Geometry-II)**.** *For all $\lambda > 0$ and $\mu > 0$, there exist a nonnegative nontrivial couple $(e_1, e_2) \in C_0^\infty(\mathbb{R}^N) \times C_0^\infty(\mathbb{R}^N)$ with both components nontrivial, and also independent of λ and μ such that $J_{\lambda,\mu}(e_1,e_2) < 0$ for all $\|(e_1,e_2)\| > \rho_0$, where ρ_0 is stated as in Lemma 4.3.*

Proof. Due to the assumption (M_2), we obtain

$$\widehat{M}(t) \leq \widehat{M}(1)t^{\theta}, \qquad \forall\, t \in [1,\infty).$$

Let $(u,v) \in C_0^\infty(\mathbb{R}^N) \times C_0^\infty(\mathbb{R}^N)$ be a nonnegative nontrivial couple with both components nontrivial such that $\|(u,v)\| = 1$ and there holds $\int_{\mathbb{R}^N} \frac{F(x,u,v)}{|x|^\gamma}\,\mathrm{d}x > 0$, thanks to (F_1).

Define the map

$$\zeta : [1,\infty) \to \mathbb{R}$$

by

$$\zeta(t) = t^{-\sigma}F(x,tu,tv) - F(x,u,v),\ \forall\, t \geq 1 \quad \text{and} \quad u,v \in \mathbb{R}^+.$$

By direct computation, we have

$$\begin{aligned}\zeta'(t) &= t^{-\sigma}F_u(x,tu,tv)u + t^{-\sigma}F_v(x,tu,tv)v - \sigma t^{-\sigma-1}F(x,tu,tv)\\ &= t^{-\sigma-1}[F_u(x,tu,tv)tu + F_v(x,tu,tv)tv - \sigma F(x,tu,tv)]\\ &= t^{-\sigma-1}[\nabla F(x,tu,tv).(tu,tv) - \sigma F(x,tu,tv)].\end{aligned}$$

This together with (F_4) implies at once that $\zeta'(t) \geq 0$ for all $t \geq 1$ and $u,v \in \mathbb{R}^+$. It follows that the map $[1,\infty) \ni t \mapsto \zeta(t)$ is monotonically increasing. In conclusion, we have

$$F(x,tu,tv) \geq t^{\sigma}F(x,u,v), \quad \forall\, t \geq 1 \quad \text{and} \quad u,v \in \mathbb{R}^+.$$

Choosing $t > \rho_0$ sufficiently large enough, we get by using the above information that

$$\begin{aligned}J_{\lambda,\mu}(tu,tv) &\leq \frac{\widehat{M}(1)}{p}t^{\theta p}\|(u,v)\|^{\theta p} - \int_{\mathbb{R}^N}\frac{F(x,tu,tv)}{|x|^\gamma}\,\mathrm{d}x\\ &\leq \frac{\widehat{M}(1)}{p}t^{\theta p} - t^{\sigma}\int_{\mathbb{R}^N}\frac{F(x,u,v)}{|x|^\gamma}\,\mathrm{d}x \to -\infty \quad \text{as } t \to \infty,\end{aligned}$$

thanks to $\sigma > \theta p$. Hence by taking $(e_1,e_2) = (tu,tv)$ with $t > \rho_0$ sufficiently large enough, we get $J_{\lambda,\mu}(e_1,e_2) < 0$ for all $\|(e_1,e_2)\| > \rho_0$ and we conclude the proof. □

In view of Lemma 4.3 and Lemma 4.4, we notice that the functional $J_{\lambda,\mu}$ satisfies the geometrical assumptions required in Theorem 4.2. Hence we infer that there exists a $(PS)_{c_{\lambda,\mu}}$ sequence $\{(u_n,v_n)\}_n \subset \mathbf{X}$ such that

$$J_{\lambda,\mu}(u_n,v_n) \to c_{\lambda,\mu} \quad \text{and} \quad J'_{\lambda,\mu}(u_n,v_n) \to 0 \quad \text{in } \mathbf{X}^* \quad \text{as } n \to \infty,$$

where

$$c_{\lambda,\mu} = \inf_{\gamma_* \in \Gamma}\max_{t\in[0,1]} I(\gamma_*(t)) \geq j_0$$

with

$$\Gamma = \{\gamma_* \in C^1([0,1],\mathbf{X}) :\ \gamma_*(0) = (0,0),\ \gamma_*(1) = (e_1,e_2)\}.$$

It is obvious that $c_{\lambda,\mu} > 0$, thanks to Lemma 4.3. Now we will discuss about the asymptotic behaviour of the mountain pass level $c_{\lambda,\mu}$.

Lemma 4.5. *There holds*

$$\lim_{(\lambda,\mu)\to(\infty,\infty)} c_{\lambda,\mu} = 0.$$

Proof. Let (e_1, e_2) be a couple as in Lemma 4.4 and thus $J_{\lambda,\mu}(te_1, te_2) \to -\infty$ as $t \to \infty$. So, there exists $t_{\lambda,\mu} > 0$ such that

$$J_{\lambda,\mu}(t_{\lambda,\mu}e_1, t_{\lambda,\mu}e_2) = \max_{t\geq 0} J_{\lambda,\mu}(te_1, te_2).$$

Consequently, we have

$$\left\langle J'_{\lambda,\mu}(t_{\lambda,\mu}e_1, t_{\lambda,\mu}e_2), (e_1, e_2)\right\rangle = 0.$$

On simplifying, we deduce by using (F_4) that

$$\begin{aligned} M(\|t_{\lambda,\mu}(e_1,e_2)\|^p)\|t_{\lambda,\mu}(e_1,e_2)\|^p &= \int_{\mathbb{R}^N} \frac{\nabla F(x, t_{\lambda,\mu}e_1, t_{\lambda,\mu}e_2).(t_{\lambda,\mu}e_1, t_{\lambda,\mu}e_2)}{|x|^\gamma}\,\mathrm{d}x \\ &\quad + t^q_{\lambda,\mu}\int_{\mathbb{R}^N} h(x)\{\lambda e_1^q + \mu e_2^q\}\,\mathrm{d}x \\ &\geq \sigma \int_{\mathbb{R}^N} \frac{F(x, t_{\lambda,\mu}e_1, t_{\lambda,\mu}e_2)}{|x|^\gamma}\,\mathrm{d}x. \end{aligned} \tag{4.4}$$

From this we conclude that $\{t_{\lambda,\mu}\}_{(\lambda,\mu)}$ is a bounded sequence of real numbers. Indeed, if not, let there exists a subsequence of $\{t_{\lambda,\mu}\}_{(\lambda,\mu)}$ still denoted by itself such that $t_{\lambda,\mu} \to \infty$ as $(\lambda,\mu) \to (\infty,\infty)$. Hence without loss of generality, we can assume $t_{\lambda,\mu} \geq 1$ for sufficiently large values of λ and μ. In addition, due to (M_2) and using similarly approach as in Lemma 4.4, we can deduce that

$$\widehat{M}(t) \leq \widehat{M}(1)t^\theta,\ \forall\ t \geq 1 \quad \text{and} \quad F(x, t_{\lambda,\mu}e_1, t_{\lambda,\mu}e_2) \geq t^\sigma_{\lambda,\mu}F(x, e_1, e_2). \tag{4.5}$$

In virtue of (M_2) and using (4.4) as well as (4.5), we get

$$\begin{aligned} \theta\widehat{M}(1)t^\theta_{\lambda,\mu}\|(e_1,e_2)\|^p &\geq \theta\widehat{M}(\|t_{\lambda,\mu}(e_1,e_2)\|^p) \geq M(\|t_{\lambda,\mu}(e_1,e_2)\|^p)\|t_{\lambda,\mu}(e_1,e_2)\|^p \\ &\geq \sigma t^\sigma_{\lambda,\mu}\int_{\mathbb{R}^N} \frac{F(x,e_1,e_2)}{|x|^\gamma}\,\mathrm{d}x. \end{aligned}$$

This yields at once that

$$\frac{1}{\sigma}\theta\widehat{M}(1)t^{\theta p-\sigma}_{\lambda,\mu}\|(e_1,e_2)\|^p \geq \int_{\mathbb{R}^N} \frac{F(x,e_1,e_2)}{|x|^\gamma}\,\mathrm{d}x.$$

Letting $(\lambda,\mu) \to (\infty,\infty)$ in the above inequality and using the fact that

$$\sigma > \theta p \quad \text{and} \quad \int_{\mathbb{R}^N} \frac{F(x,e_1,e_2)}{|x|^\gamma}\,\mathrm{d}x > 0,$$

we arrive at a contradiction, thanks to (F_1). This shows that $\{t_{\lambda,\mu}\}_{(\lambda,\mu)}$ is a bounded sequence of real numbers. Now, we fix a sequence $\{(\lambda_n,\mu_n)\}_n \subset \mathbb{R}^+ \times \mathbb{R}^+$ such that $(\lambda_n,\mu_n) \to (\infty,\infty)$ as $n \to \infty$. From the above arguments, we know that $\{t_{\lambda_n,\mu_n}\}_n$ is a bounded sequence of real numbers. This shows that there exists $t_0 \geq 0$ and a subsequence of $\{(\lambda_n,\mu_n)\}_n$ still denoted by itself such that $t_{\lambda_n,\mu_n} \to t_0$ as $n \to \infty$. In conclusion, due to the continuity of M, we infer that $\{M(\|t_{\lambda_n,\mu_n}(e_1,e_2)\|^p)\|t_{\lambda_n,\mu_n}(e_1,e_2)\|^p\}_n$ is bounded. So, there exists a constant $C > 0$ such that

$$M(\|t_{\lambda_n,\mu_n}(e_1,e_2)\|^p)\|t_{\lambda_n,\mu_n}(e_1,e_2)\|^p \leq C, \qquad \forall\ n \in \mathbb{N}.$$

We claim that $t_0 = 0$. Indeed, if not, let $t_0 > 0$. Consequently, we obtain from the above inequality and (4.4) that for all $n \in \mathbb{N}$, the following estimate holds

$$\int_{\mathbb{R}^N} \frac{\nabla F(x, t_{\lambda_n,\mu_n} e_1, t_{\lambda_n,\mu_n} e_2).(t_{\lambda_n,\mu_n} e_1, t_{\lambda_n,\mu_n} e_2)}{|x|^\gamma}\, \mathrm{d}x + t^q_{\lambda_n,\mu_n}\big(\lambda_n \|e_1\|^q_{q,h} + \mu_n \|e_2\|^q_{q,h}\big) \le C. \tag{4.6}$$

Due to (3.3) and Theorem 2.7, we obtain from the Lebesgue dominated convergence theorem that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \frac{\nabla F(x, t_{\lambda_n,\mu_n} e_1, t_{\lambda_n,\mu_n} e_2).(t_{\lambda_n,\mu_n} e_1, t_{\lambda_n,\mu_n} e_2)}{|x|^\gamma}\, \mathrm{d}x = \int_{\mathbb{R}^N} \frac{\nabla F(x, t_0 e_1, t_0 e_2).(t_0 e_1, t_0 e_2)}{|x|^\gamma}\, \mathrm{d}x.$$

The above convergence together with $\|e_i\|^q_{q,h} > 0$ for $i = 1, 2$ implies that

$$\lim_{n\to\infty} \left[\int_{\mathbb{R}^N} \frac{\nabla F(x, t_{\lambda_n,\mu_n} e_1, t_{\lambda_n,\mu_n} e_2).(t_{\lambda_n,\mu_n} e_1, t_{\lambda_n,\mu_n} e_2)}{|x|^\gamma}\, \mathrm{d}x + t^q_{\lambda_n,\mu_n}\big(\lambda_n \|e_1\|^q_{q,h} + \mu_n \|e_2\|^q_{q,h}\big) \right] = \infty.$$

In virtue of (4.6), we arrive at a contradiction. It follows that $t_0 = 0$ and $t_{\lambda,\mu} \to 0$ as $(\lambda, \mu) \to (\infty, \infty)$, thanks to the arbitrariness of $\{(\lambda_n, \mu_n)\}_n$. Now consider the path $\xi(t) = t(e_1, e_2)$ with $t \in [0, 1]$, belongs to Γ. Using the continuity of $\widehat{M}$, we get

$$\begin{aligned} 0 < c_{\lambda,\mu} &\le \max_{t\ge 0} J_{\lambda,\mu}(\xi(t)) = J_{\lambda,\mu}(t_{\lambda,\mu} e_1, t_{\lambda,\mu} e_2) \\ &\le \frac{1}{p} \widehat{M}(\|t_{\lambda,\mu}(e_1, e_2)\|^p) \to 0 \quad \text{as } (\lambda, \mu) \to (\infty, \infty), \end{aligned}$$

thanks to the couple (e_1, e_2), which does not depend on λ and μ. This completes the proof. □

Lemma 4.6. *There exists $\widehat{\lambda} > 0$ such that for all $(\lambda, \mu) \in (\widehat{\lambda}, \infty) \times (\widehat{\lambda}, \infty)$, the functional $J_{\lambda,\mu}$ satisfies the $(PS)_{c_{\lambda,\mu}}$ condition on $\mathbf{X}$.*

Proof. Let $\{(u_n, v_n)\}_n \subset \mathbf{X}$ be a $(PS)_{c_{\lambda,\mu}}$ for $J_{\lambda,\mu}$. It follows at once that

$$\left\langle J'_{\lambda,\mu}(u_n, v_n), \frac{(u_n, v_n)}{\|(u_n, v_n)\|} \right\rangle = o_n(1) \quad \text{and} \quad J_{\lambda,\mu}(u_n, v_n) = c_{\lambda,\mu} + o_n(1) \quad \text{as } n \to \infty. \tag{4.7}$$

Possibly, we have two situations based on the nature of the Kirchhoff function M: either $\inf_{n\in\mathbb{N}} \|(u_n, v_n)\| = d > 0$ or $\inf_{n\in\mathbb{N}} \|(u_n, v_n)\| = 0$. Hence we have to discuss these situations separately.

Case 1. Let $\inf_{n\in\mathbb{N}} \|(u_n, v_n)\| = d > 0$. Denote $\kappa = \kappa(\tau)$ as the number corresponding to $\tau = d^p$ in (M_1) such that

$$M(\|(u_n, v_n)\|^p) \ge \kappa, \qquad \forall\ n \in \mathbb{N}. \tag{4.8}$$

Now we claim to prove $\{(u_n, v_n)\}_n$ is bounded in $\mathbf{X}$. For this, we first consider the case when $\theta p < \sigma \le q$. In this case, we obtain from (4.7), (4.8), (F_4) and (M_2) that as $n \to \infty$

$$\begin{aligned} c_{\lambda,\mu} + o_n(1) + o_n(1)\|(u_n, v_n)\| &= J_{\lambda,\mu}(u_n, v_n) - \frac{1}{\sigma}\langle J'_{\lambda,\mu}(u_n, v_n), (u_n, v_n)\rangle \\ &\ge \frac{1}{p}\widehat{M}(\|(u_n, v_n)\|^p) - \frac{1}{\sigma} M(\|(u_n, v_n)\|^p)\|(u_n, v_n)\|^p \\ &\ge \left(\frac{1}{\theta p} - \frac{1}{\sigma}\right)\kappa \|(u_n, v_n)\|^p. \end{aligned}$$

Since $p > 1$, we conclude from the above inequality that $\{(u_n, v_n)\}_n$ is bounded in $\mathbf{X}$ and thus we have

$$\|(u_n, v_n)\|^{p'} \leq \left(\frac{\sigma\theta p \; c_{\lambda,\mu}}{(\sigma - \theta p)\kappa} + o_n(1) \right)^{p'}. \tag{4.9}$$

Define $z_n = (u_n, v_n)$ and $z = (u_{\lambda,\mu}, v_{\lambda,\mu})$. Then it is easy to see that $|z_n| = \sqrt{u_n^2 + v_n^2} \leq \psi_n := |u_n| + |v_n|$. By direct calculation, we can deduce that

$$\|\psi_n\|_{W^{s,p}} \leq 2(\min\{V_0, K_0\})^{-\frac{1}{p}} \|(u_n, v_n)\|. \tag{4.10}$$

In virtue of (4.9) and (4.10), we infer that

$$\limsup_{n\to\infty} \|\psi_n\|^{p'}_{W^{s,p}} \leq \frac{2^{p'}}{(\min\{V_0, K_0\})^{\frac{1}{p-1}}} \left[\frac{\sigma\theta p \; c_{\lambda,\mu}}{(\sigma - \theta p)\kappa} \right]^{p'}. \tag{4.11}$$

Similarly, when $\sigma > q > \theta p$, we obtain from (4.7), (4.8), (F_4) and (M_2) that as $n \to \infty$

$$\begin{aligned} c_{\lambda,\mu} + o_n(1) + o_n(1)\|(u_n, v_n)\| &= J_{\lambda,\mu}(u_n, v_n) - \frac{1}{q}\langle J'_{\lambda,\mu}(u_n, v_n), (u_n, v_n)\rangle \\ &\geq \frac{1}{p}\widehat{M}(\|(u_n, v_n)\|^p) - \frac{1}{q}M(\|(u_n, v_n)\|^p)\|(u_n, v_n)\|^p \\ &\geq \left(\frac{1}{\theta p} - \frac{1}{q}\right)\kappa\|(u_n, v_n)\|^p. \end{aligned}$$

This shows that $\{(u_n, v_n)\}_n$ is bounded in $\mathbf{X}$. Using a similar procedure as discussed above, we have

$$\limsup_{n\to\infty} \|\psi_n\|^{p'}_{W^{s,p}} \leq \frac{2^{p'}}{(\min\{V_0, K_0\})^{\frac{1}{p-1}}} \left[\frac{q\theta p \; c_{\lambda,\mu}}{(q - \theta p)\kappa} \right]^{p'}. \tag{4.12}$$

On combining (4.11) and (4.12), we get

$$\limsup_{n\to\infty} \|\psi_n\|^{p'}_{W^{s,p}} \leq \frac{(2c_{\lambda,\mu})^{p'}}{(\min\{V_0, K_0\})^{\frac{1}{p-1}}} \left(\left[\frac{\sigma\theta p}{(\sigma - \theta p)\kappa} \right]^{p'} + \left[\frac{q\theta p}{(q - \theta p)\kappa} \right]^{p'} \right). \tag{4.13}$$

By using Lemma 4.5, there exists $\widehat{\lambda} > 0$ such that for all $(\lambda, \mu) \in (\widehat{\lambda}, \infty) \times (\widehat{\lambda}, \infty)$, we can assume

$$0 < c_{\lambda,\mu} < \left[\frac{\beta_*(\min\{V_0, K_0\})^{\frac{1}{p-1}}}{2^{p'+1}\alpha_0} \right]^{\frac{1}{p'}} \left(\left[\frac{\sigma\theta p}{(\sigma - \theta p)\kappa} \right]^{p'} + \left[\frac{q\theta p}{(q - \theta p)\kappa} \right]^{p'} \right)^{-\frac{1}{p'}}.$$

From the above inequality and (4.13), we obtain

$$\limsup_{n\to\infty} \|\psi_n\|^{p'}_{W^{s,p}} < \frac{\beta_*}{2\alpha_0}, \qquad \forall\ (\lambda, \mu) \in (\widehat{\lambda}, \infty) \times (\widehat{\lambda}, \infty).$$

Due to the boundedness of $\{(u_n, v_n)\}_n$ in $\mathbf{X}$, there exists $(u_{\lambda,\mu}, v_{\lambda,\mu}) \in \mathbf{X}$ such that up to a subsequence still denoted by itself, we can assume that $(u_n, v_n) \rightharpoonup (u_{\lambda,\mu}, v_{\lambda,\mu})$ in $\mathbf{X}$ as $n \to \infty$. In addition, by using the notations used in the proof of Theorem 1.3, it is easy to see that (3.17) holds, there exists $l_{\lambda,\mu} \geq 0$ such that up to a subsequence, we have $\|(u_n, v_n)\| \to l_{\lambda,\mu}$ as $n \to \infty$ and also we can easily obtain from (4.7) that

$$\left\langle J'_{\lambda,\mu}(u_n, v_n), (u_n - u_{\lambda,\mu}, v_n - v_{\lambda,\mu}) \right\rangle = o_n(1) \quad \text{as } n \to \infty,$$

thanks to the boundedness of $\{(u_n, v_n)\}_n$ in $\mathbf{X}$. Moreover, by using the Hölder's inequality, (3.17) and (3.18), one has

$$\begin{aligned}
&\left| \int_{\mathbb{R}^N} h(x)\{\lambda(u_n^+)^{q-1}(u_n - u_{\lambda,\mu}) + \mu(v_n^+)^{q-1}(v_n - v_{\lambda,\mu})\}\, \mathrm{d}x \right| \\
&\quad \le \max\{\lambda,\mu\}\|h\|_\infty \left[\|u_n\|_q^{q-1}\|u_n - u_{\lambda,\mu}\|_q + \|v_n\|_q^{q-1}\|v_n - v_{\lambda,\mu}\|_q \right] \\
&\quad \le \max\{\lambda,\mu\}\|h\|_\infty \|(u_n, v_n)\|^{q-1}_{L^q(\mathbb{R}^N)\times L^q(\mathbb{R}^N)} \|(u_n - u_{\lambda,\mu}, v_n - v_{\lambda,\mu})\|_{L^q(\mathbb{R}^N)\times L^q(\mathbb{R}^N)} \\
&\quad \le \max\{\lambda,\mu\}\|h\|_\infty \mathcal{A}_{q,h}^{-(q-1)} \sup_{n\in\mathbb{N}} \|(u_n, v_n)\|^{q-1} \|(u_n - u_{\lambda,\mu}, v_n - v_{\lambda,\mu})\|_{L^q(\mathbb{R}^N)\times L^q(\mathbb{R}^N)} \\
&\quad \to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

It follows at once that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} h(x)\{\lambda(u_n^+)^{q-1}(u_n - u_{\lambda,\mu}) + \mu(v_n^+)^{q-1}(v_n - v_{\lambda,\mu})\}\, \mathrm{d}x = 0. \tag{4.14}$$

Using all the above information and proceeding likewise as in the proof of Theorem 1.3, we can prove that $(u_n, v_n) \to (u_{\lambda,\mu}, v_{\lambda,\mu})$ in $\mathbf{X}$ as $n \to \infty$. This together with $\inf_{n\in\mathbb{N}} \|(u_n, v_n)\| = d > 0$ implies that $l_{\lambda,\mu} = \|(u_{\lambda,\mu}, v_{\lambda,\mu})\| > 0$. This shows that $M(\|(u_n, v_n)\|^p) \to M(l^p_{\lambda,\mu}) > 0$ as $n \to \infty$, thanks to the continuity of M. Next, we claim that

$$\lim_{(\lambda,\mu)\to(\infty,\infty)} \|(u_{\lambda,\mu}, v_{\lambda,\mu})\| = 0.$$

Indeed, if not, let there exists a sequence $\{(\lambda_k, \mu_k)\}_k$ with $(\lambda_k, \mu_k) \to (\infty, \infty)$ as $k \to \infty$ such that $l_{\lambda_k,\mu_k} \to l_0 > 0$ as $k \to \infty$. It is easy to see that for $\varrho \in \{\sigma, q\}$ (according to the situations: either $\theta p < \sigma \le q$ or $\theta p < q < \sigma$), one has

$$c_{\lambda_k,\mu_k} = \lim_{n\to\infty} \left[J_{\lambda,\mu}(u_n, v_n) - \frac{1}{\varrho}\langle J'_{\lambda,\mu}(u_n, v_n), (u_n, v_n)\rangle \right] \ge \left(\frac{1}{\theta p} - \frac{1}{\varrho}\right) M(l^p_{\lambda_k,\mu_k}) l^p_{\lambda_k,\mu_k}.$$

Letting $k \to \infty$ in the above inequality and using Lemma 4.5, we get

$$0 \ge \left(\frac{1}{\theta p} - \frac{1}{\varrho}\right) M(l_0^p) l_0^p > 0,$$

which is not possible. It follows that

$$\lim_{(\lambda,\mu)\to(\infty,\infty)} \|(u_{\lambda,\mu}, v_{\lambda,\mu})\| = \lim_{(\lambda,\mu)\to(\infty,\infty)} l_{\lambda,\mu} = 0.$$

Now we shall discuss about the situation when $\inf_{n\in\mathbb{N}} \|(u_n, v_n)\| = 0$.

Case 2. Let $\inf_{n\in\mathbb{N}} \|(u_n, v_n)\| = 0$. If $(0,0)$ is an isolated point of the sequence $\{(u_n, v_n)\}_n$, then there exists a subsequence still denoted by the same symbol such that

$$\inf_{n\in\mathbb{N}} \|(u_n, v_n)\| = d > 0.$$

In this situation, we can proceed as in Case 1. Moreover, if $(0,0)$ is an accumulation point of $\{(u_n, v_n)\}_n$, then up to a subsequence still denoted by itself such that it strongly converges to $(u_{\lambda,\mu}, v_{\lambda,\mu}) = (0,0)$. This situation is impossible. Indeed, if not, then $0 = J_{\lambda,\mu}(0,0) = c_{\lambda,\mu} > 0$, which is a contradiction.

By using $(u_n, v_n) \to (u_{\lambda,\mu}, v_{\lambda,\mu})$ in $\mathbf{X}$ as $n \to \infty$ and the fact that $J_{\lambda,\mu} \in C^1(\mathbf{X}, \mathbb{R})$, we get $J_{\lambda,\mu}(u_{\lambda,\mu}, v_{\lambda,\mu}) = c_{\lambda,\mu} > 0$ and $J'_{\lambda,\mu}(u_{\lambda,\mu}, v_{\lambda,\mu}) = 0$. It follows from Lemma 3.1 that $(u_{\lambda,\mu}, v_{\lambda,\mu})$ is a nontrivial nonnegative solution of the system $(\mathcal{S}_{\lambda,\mu})$. This completes the proof. □

Proof of Theorem 1.4. Due to Lemma 4.3–4.6, we infer that $J_{\lambda,\mu}$ enjoys all the assumptions of Theorem 4.2. In addition, by Lemma 4.6, there exists $\widehat{\lambda} > 0$ such that for all $(\lambda,\mu) \in (\widehat{\lambda},\infty)\times(\widehat{\lambda},\infty)$, the functional $J_{\lambda,\mu}$ admits a nontrivial nonnegative critical point $(u_{\lambda,\mu}, v_{\lambda,\mu}) \in \mathbf{X}$, which is a mountain pass solution of the system $(\mathcal{S}_{\lambda,\mu})$. Consequently, there also holds $\|(u_{\lambda,\mu}, v_{\lambda,\mu})\| \to 0$ as $(\lambda,\mu) \to (\infty,\infty)$. This finishes the proof. □

5 Proof of Theorem 1.5

In this section, we shall prove Theorem 1.5 using the Krasnoselskii genus theory. For the sake of simplicity, we assume that the structural assumptions required in Theorem 1.5 hold. For the convenience of the readers, we first summarize some basic properties and definition of the genus.

Let X be a Banach space and A be a subset of X. A is said to be symmetric w.r.t. the origin if $u \in A$ implies $-u \in A$. Define

$$\mathbf{\Sigma} = \{A \subset X \setminus \{0\} : \ A \text{ closed and symmetric w.r.t. the origin}\}.$$

For $A \in \mathbf{\Sigma}$, we denote the genus of A by $\gamma(A)$, which is defined by

$$\gamma(A) = \min\{k \in \mathbb{N} : \ \exists\ \varphi \in C(A, \mathbb{R}^k \setminus \{0\}),\ \varphi(x) = -\varphi(-x)\}.$$

Moreover, we define $\gamma(\emptyset) = 0$. In addition, if such k does not exist, we set $\gamma(A) = \infty$. The following properties of the genus can be found in [7,25,27,41,49].

Proposition 5.1. *Let $A, B \in \mathbf{\Sigma}$. Then the following results hold:*

(a) If there exists an odd map $f \in C(A,B)$, then $\gamma(A) = \gamma(B)$;

(b) If $A \subset B$, then $\gamma(A) \le \gamma(B)$;

(c) If there exists an odd homeomorphism from A onto B, then $\gamma(A) = \gamma(B)$;

(d) If $\gamma(A) \ge 2$, then A has infinitely many points;

(e) If $\mathbb{S}$ is a sphere centered at the origin in $\mathbb{R}^k$, then $\gamma(\mathbb{S}) = k$;

(f) $\gamma(A \cup B) \le \gamma(A) + \gamma(B)$;

(g) If $\gamma(B) < \infty$, then $\gamma(\overline{A \setminus B} \ge \gamma(A) - \gamma(B)$;

(h) If A is compact, then $\gamma(A) < \infty$, and $\exists\ \delta > 0$ such that $N_\delta(A) \in \mathbf{\Sigma}$ and $\gamma(A) = \gamma(N_\delta(A))$, where

$$N_\delta(A) = \{x \in X : \ d(x,A) \le \delta\} \quad \text{and} \quad d(x,A) = \inf\{\|x-u\| : \ u \in A\};$$

(i) If X_0 is a subspace of X with codimension k, and $\gamma(A) > k$, then $A \cap X_0 \neq \emptyset$.

Next, we recall the classical deformation lemma, established by A. Ambrosetti and P. H. Rabinowitz in [1]. It can be read as follows:

Proposition 5.2. *Let X be a Banach space and $I \in C^1(X, \mathbb{R})$ satisfying the (PS) condition. If $c \in \mathbb{R}$ and N is any neighbourhood of*

$$K_c = \{u \in X : \ I(u) = c, \ I'(u) = 0\},$$

then there exist $\eta(t,u) = \eta_t(u) \in C([0,1] \times X, X)$ and constants $\bar{\epsilon} > \epsilon > 0$ such that

(a) $\eta_0(u) = u$ for all $u \in X$;

(b) $\eta_t(u) = u$ for all $u \notin I^{-1}[c - \bar{\epsilon}, c + \bar{\epsilon}]$ and all $t \in [0,1]$;

(c) η_t is a homeomorphism of X onto X for all $t \in [0,1]$;

(d) $I(\eta_t(u)) \leq I(u)$ for all $u \in X$ and all $t \in [0,1]$;

(e) $\eta_1(I^{c+\epsilon} \backslash N) \subset I^{c-\epsilon}$, where $I^c = \{u \in X : \ I(u) \leq c\}$ for all $c \in \mathbb{R}$;

(f) If $K_c = \emptyset$, then there holds $\eta_1(I^{c+\epsilon}) \subset I^{c-\epsilon}$;

(g) If I is even, then η_t is odd in u.

Remark 5.3. We note that Proposition 5.2 is also valid if I satisfies the $(PS)_c$ condition for $c < c_0$ for some $c_0 \in \mathbb{R}$.

To study the infinitely many solutions of the system $(\mathcal{S}_{\lambda,\mu})$, we define the functional $J : \mathbf{X} \to \mathbb{R}$ by

$$J(u,v) = \frac{1}{p}\widehat{M}(\|(u,v)\|^p) - \int_{\mathbb{R}^N} \frac{F(x,u,v)}{|x|^\gamma}\, \mathrm{d}x - \frac{1}{q}(\lambda\|u\|_{q,h}^q + \mu\|v\|_{q,h}^q), \qquad \forall\ (u,v) \in \mathbf{X}. \quad (5.1)$$

By using (F_2) and Theorem 2.7, it follows that J is well-defined, of class $C^1(\mathbf{X}, \mathbb{R})$ and the critical points are weak solutions of the system $(\mathcal{S}_{\lambda,\mu})$.

Note that by arguing similar arguments as in Lemma 4.4, we can prove that J is not bounded from below. Hence we have some mathematical difficulty in studying the multiplicity of critical points of the functional J. To avoid this difficulty, we use the technique used in [25]. For this, we shall construct the truncated functional I corresponding to the functional J and study the behaviour of solutions to the system $(\mathcal{S}_{\lambda,\mu})$.

Suppose that $\|(u,v)\| \leq 1$. Observe that for $\ell \in [0,1]$, we obtain from the definition of $\widehat{M}$ that

$$\widehat{M}(\ell) \geq \widehat{M}(1)\ell^\theta = (a+b)\ell^\theta.$$

Using the notations as stated in Lemma 3.2, for $\|(u,v)\| \leq \bar{\delta}$ with $\bar{\delta} \in (0,1]$, we have

$$J(u,v) \geq \left(\frac{a+b}{p} - 2^{\theta p}\mathcal{B}_{\theta p,\gamma}^{-\theta p}\varepsilon\right)\|(u,v)\|^{\theta p} - \widetilde{C}\|(u,v)\|^\vartheta - \frac{1}{q}\max\{\lambda,\mu\}\mathcal{S}_{q,h}^{-q}\|(u,v)\|^q,$$

where $\widetilde{C} = 2^\vartheta D C^{\frac{1}{\vartheta'}} \mathcal{B}_{\vartheta t,\gamma}^{-\vartheta}\tilde{\kappa}_\varepsilon$. Choosing $\varepsilon = a + b/2^{\theta p+1} p\mathcal{B}_{\theta p,\gamma}^{-\theta p}$, we obtain from the above inequality that

$$J(u,v) \geq \left(\frac{a+b}{2p}\right)\|(u,v)\|^{\theta p} - \widetilde{C}\|(u,v)\|^\vartheta - \frac{1}{q}\max\{\lambda,\mu\}\mathcal{S}_{q,h}^{-q}\|(u,v)\|^q \quad (5.2)$$

for all $(u,v) \in \mathbf{X}$ with $\|(u,v)\| \leq 1$. Define the map $H_{\lambda,\mu} : [0,\infty) \to \mathbb{R}$ as follows

$$H_{\lambda,\mu}(\ell) = \left(\frac{a+b}{2p}\right)\ell^{\theta p} - \widetilde{C}\ell^\vartheta - \frac{1}{q}\max\{\lambda,\mu\}\mathcal{S}_{q,h}^{-q}\ell^q, \qquad \forall\ \ell \in [0,\infty)$$

and write

$$H_{\lambda,\mu}(\ell) = \ell^q \widetilde{H}_{\lambda,\mu}(\ell),$$

where

$$\widetilde{H}_{\lambda,\mu}(\ell) = \left(\frac{a+b}{2p}\right)\ell^{\theta p-q} - \widetilde{C}\ell^{\vartheta-q} - \frac{1}{q}\max\{\lambda,\mu\}\mathcal{S}_{q,h}^{-q}.$$

Hence one can easily notice that $J(u,v) \geq H_{\lambda,\mu}(\|(u,v)\|)$ for all $(u,v) \in \mathbf{X}$ with $\|(u,v)\| \leq 1$. In addition, we have

$$\widetilde{H}_{\lambda,\mu}(0) < 0 \quad \text{and} \quad \widetilde{H}_{\lambda,\mu}(\ell) \to -\infty \quad \text{as } \ell \to \infty. \tag{5.3}$$

By direct calculation, one has

$$\widetilde{H}'_{\lambda,\mu}(\ell) = 0 \implies \ell = \left[\frac{(a+b)(\theta p-q)}{2p\widetilde{C}(\vartheta-q)}\right]^{\frac{1}{\vartheta-\theta p}} =: T.$$

Further, it is easy to see that $\widetilde{H}'_{\lambda,\mu}(\ell) > 0$ for all $\ell \in (0,T)$ and $\widetilde{H}'_{\lambda,\mu}(\ell) < 0$ for all $\ell \in (T,\infty)$. This shows that the map $\ell \mapsto \widetilde{H}_{\lambda,\mu}(\ell)$ is strictly increasing on $(0,T)$ and strictly decreasing on (T,∞). In virtue of (5.3), we deduce that, if if $\widetilde{H}_{\lambda,\mu}(T) > 0$, then

$$\widetilde{H}_{\lambda,\mu}(0)\widetilde{H}_{\lambda,\mu}(T) < 0 \quad \text{and} \quad \widetilde{H}_{\lambda,\mu}(T)\widetilde{H}_{\lambda,\mu}(\ell) < 0 \quad \text{for } \ell \text{ large enough.}$$

Consequently, by the intermediate value theorem, there exists at least one root of $\widetilde{H}_{\lambda,\mu}$ in between 0 and T. Since the map $\ell \mapsto \widetilde{H}_{\lambda,\mu}(\ell)$ is strictly increasing on $(0,T)$, we infer that there exists a unique root $T_0(\lambda,\mu)$ in between 0 and T for $\widetilde{H}_{\lambda,\mu}$. Similarly, we can prove that there exists a unique root $T_1(\lambda,\mu)$ in between T and ∞ for $\widetilde{H}_{\lambda,\mu}$. In conclusion, we can say, if $\widetilde{H}_{\lambda,\mu}(T) > 0$, then there exist two unique real numbers $T_0(\lambda,\mu)$ and $T_1(\lambda,\mu)$ with

$$0 < T_0(\lambda,\mu) < T < T_1(\lambda,\mu) < \infty \quad \text{such that} \quad \widetilde{H}_{\lambda,\mu}(T_0(\lambda,\mu)) = \widetilde{H}_{\lambda,\mu}(T_1(\lambda,\mu)) = 0.$$

Define

$$\boldsymbol{\lambda}^1 = \frac{q(\vartheta-\theta p)}{\mathcal{S}_{q,h}^{-q}}\left(\frac{a+b}{2p(\vartheta-q)}\right)^{\frac{\vartheta-q}{\vartheta-\theta p}}\left(\frac{\theta p-q}{\widetilde{C}}\right)^{\frac{\theta p-q}{\vartheta-\theta p}},$$

then if $\max\{\lambda,\mu\} < \boldsymbol{\lambda}^1$, there holds

$$\widetilde{H}_{\lambda,\mu}(T) = (\vartheta-\theta p)\left(\frac{a+b}{2p(\vartheta-q)}\right)^{\frac{\vartheta-q}{\vartheta-\theta p}}\left(\frac{\theta p-q}{\widetilde{C}}\right)^{\frac{\theta p-q}{\vartheta-\theta p}} - \frac{1}{q}\max\{\lambda,\mu\}\mathcal{S}_{q,h}^{-q} > 0,$$

so that, since we have $H_{\lambda,\mu}(T_0(\lambda,\mu)) = H_{\lambda,\mu}(T_1(\lambda,\mu)) = 0$, thanks to $H_{\lambda,\mu}(\ell) = \ell^q\widetilde{H}_{\lambda,\mu}(\ell)$. In addition, there holds

$$\begin{aligned} &H_{\lambda,\mu}(\ell) > 0, \ \forall\ \ell \in \big(T_0(\lambda,\mu), T_1(\lambda,\mu)\big) \quad \text{and} \\ &H_{\lambda,\mu}(\ell) \leq 0, \ \forall\ \ell \in \big[0, T_0(\lambda,\mu)\big] \cup \big[T_1(\lambda,\mu), \infty\big). \end{aligned} \tag{5.4}$$

Corollary 5.4. *The following holds*

$$\lim_{(\lambda,\mu)\to(0^+,0^+)} T_0(\lambda,\mu) = 0.$$

Proof. It follows from $H_{\lambda,\mu}(T_0(\lambda,\mu)) = 0$ and $H'_{\lambda,\mu}(T_0(\lambda,\mu)) > 0$ that

$$\left(\frac{a+b}{2p}\right)(T_0(\lambda,\mu))^{\theta p} = \widetilde{C}(T_0(\lambda,\mu))^{\vartheta} + \frac{1}{q}\max\{\lambda,\mu\}\mathcal{S}_{q,h}^{-q}(T_0(\lambda,\mu))^{q} \tag{5.5}$$

and

$$\left(\frac{a+b}{2p}\right)\theta p(T_0(\lambda,\mu))^{\theta p-1} > \widetilde{C}\vartheta(T_0(\lambda,\mu))^{\vartheta-1} + \max\{\lambda,\mu\}\mathcal{S}_{q,h}^{-q}(T_0(\lambda,\mu))^{q-1}. \tag{5.6}$$

On solving (5.5) and (5.6), we obtain

$$T_0(\lambda,\mu) < \left[\frac{(a+b)(\theta p-q)}{2p\widetilde{C}(\vartheta-q)}\right]^{\frac{1}{\vartheta-\theta p}}.$$

This shows that $T_0(\lambda,\mu)$ is uniformly bounded w.r.t. λ and μ. Let us choose a sequence $\{(\lambda_n,\mu_n)\}_n$ with $(\lambda_n,\mu_n) \to (0^+,0^+)$ as $n \to \infty$. Using the fact that $\{T_0(\lambda_n,\mu_n)\}_n$ is uniformly bounded, therefore up to a subsequence still denoted by itself such that

$$T_0(\lambda_n,\mu_n) \to T_0\ (\geq 0)\ \text{(say)} \quad \text{as } n \to \infty.$$

Consequently, by replacing λ_n in place of λ and μ_n in place of μ in (5.5) and (5.6), respectively and letting $n \to \infty$, we have

$$\left(\frac{a+b}{2p}\right)T_0^{\theta p} = \widetilde{C}T_0^{\vartheta} \quad \text{and} \quad \left(\frac{a+b}{2p}\right)\theta p T_0^{\theta p-1} \geq \widetilde{C}\vartheta T_0^{\vartheta-1}.$$

This directly implies at once that

$$\widetilde{C}(\vartheta-\theta p)T_0^{\vartheta-1} \leq 0.$$

Hence we deduce that $T_0 = 0$. Due to the arbitrariness of $\{(\lambda_n,\mu_n)\}_n$, we conclude that $T_0(\lambda,\mu) \to 0$ as $(\lambda,\mu) \to (0^+,0^+)$ and thus we conclude the proof. □

By Corollary 5.4, we can find $\boldsymbol{\lambda}^2 > 0$ small enough such that there holds $T_0(\lambda,\mu) < 1$ for all $(\lambda,\mu) \in (0,\boldsymbol{\lambda}^2)\times(0,\boldsymbol{\lambda}^2)$. It follows that $T_0(\lambda,\mu) < \min\{T_1(\lambda,\mu),1\}$ for all $(\lambda,\mu) \in (0,\boldsymbol{\lambda}^2)\times(0,\boldsymbol{\lambda}^2)$. Suppose that $(\lambda,\mu) \in (0,\min\{\boldsymbol{\lambda}^1,\boldsymbol{\lambda}^2\})\times(0,\min\{\boldsymbol{\lambda}^1,\boldsymbol{\lambda}^2\})$ and take a nonincreasing cut-off function $\Psi \in C_0^\infty([0,\infty),[0,1])$, which is defined by

$$\Psi(\ell) = \begin{cases} 1 & \text{if } \ell \in [0,T_0(\lambda,\mu)], \\ 0 & \text{if } \ell \in [\min\{T_1(\lambda,\mu),1\},\infty). \end{cases}$$

Define the truncated energy functional $I : \mathbf{X} \to \mathbb{R}$ by

$$I(u,v) = \frac{1}{p}\widehat{M}(\|(u,v)\|^p) - \Psi(\|(u,v)\|)\int_{\mathbb{R}^N}\frac{F(x,u,v)}{|x|^\gamma}\,\mathrm{d}x - \frac{1}{q}(\lambda\|u\|_{q,h}^q + \mu\|v\|_{q,h}^q),\ \forall\ (u,v) \in \mathbf{X}.$$

By the regularity of Ψ and J, we conclude that $I \in C^1(\mathbf{X},\mathbb{R})$. In addition, one can notice that I is coercive and bounded from below on $\mathbf{X}$. We also mention that the following results hold:

$$J(u,v) = I(u,v),\ \forall\ (u,v) \in \mathbf{X} \quad \text{with} \quad \|(u,v)\| \leq T_0(\lambda,\mu) < \min\{T_1(\lambda,\mu),1\}$$

and

$$J(u,v) \geq G_{\lambda,\mu}(\|(u,v)\|),\ \forall\ (u,v) \in \mathbf{X} \quad \text{with} \quad \|(u,v)\| \leq 1,$$

where

$$G_{\lambda,\mu}(\ell) = \left(\frac{a+b}{2p}\right)\ell^{\theta p} - \widetilde{C}\Psi(\ell)\ell^{\vartheta} - \frac{1}{q}\max\{\lambda,\mu\}\mathcal{S}_{q,h}^{-q}\ell^{q}.$$

Lemma 5.5. *There exists $\bar{\lambda} > 0$ such that for all $(\lambda, \mu) \in (0, \bar{\lambda}) \times (0, \bar{\lambda})$, we have*

(a) If $I(u,v) < 0$, then $\|(u,v)\| < T_0(\lambda,\mu)$ and $I(\varphi,\psi) = J(\varphi,\psi)$ for all (φ,ψ) in a small neighbourhood of (u,v);

(b) For all $c < 0$, the functional I satisfies a local $(PS)_c$ condition.

Proof. Suppose that $\|(u,v)\| \geq 1$, then we have $\Psi(\|(u,v)\|) = 0$. In addition, we obtain

$$\begin{aligned} I(u,v) &= \frac{1}{p}\widehat{M}(\|(u,v)\|^p) - \frac{1}{q}(\lambda\|u\|^q_{q,h} + \mu\|v\|^q_{q,h}) \geq \left(\frac{a+b}{p}\right)\|(u,v)\|^p \\ &\quad - \frac{1}{q}\max\{\lambda,\mu\}\mathcal{S}^{-q}_{q,h}\|(u,v)\|^q \\ &=: \widetilde{g}_{\lambda,\mu}(\|(u,v)\|) \end{aligned}$$

for all $\|(u,v)\| \geq 1$, where $\widetilde{g}_{\lambda,\mu} : [0,\infty) \to \mathbb{R}$ is given by

$$\widetilde{g}_{\lambda,\mu}(\ell) = \left(\frac{a+b}{p}\right)\ell^p - \frac{1}{q}\max\{\lambda,\mu\}\mathcal{S}^{-q}_{q,h}\ell^q.$$

By direct computation, one can notice that $\widetilde{g}_{\lambda,\mu}$ has a global minimum point at

$$\ell_0 = \left(\frac{\max\{\lambda,\mu\}\mathcal{S}^{-q}_{q,h}}{a+b}\right)^{\frac{1}{p-q}}$$

and

$$\widetilde{g}_{\lambda,\mu}(\ell_0) = \left(\frac{\max\{\lambda,\mu\}\mathcal{S}^{-q}_{q,h}}{a+b}\right)^{\frac{q}{p-q}}\max\{\lambda,\mu\}\mathcal{S}^{-q}_{q,h}\left(\frac{1}{p} - \frac{1}{q}\right) < 0,$$

thanks to $1 < q < p$. Further, it is easy to see that

$$\widetilde{g}_{\lambda,\mu}(\ell) \geq 0 \iff \ell \geq \left(\frac{\max\{\lambda,\mu\}p\mathcal{S}^{-q}_{q,h}}{q(a+b)}\right)^{\frac{1}{p-q}} =: \ell_1.$$

This shows that

$$I(u,v) \geq \widetilde{g}_{\lambda,\mu}(\|(u,v)\|) \geq 0, \qquad \forall\ \|(u,v)\| \geq 1$$

with $\ell_1 < 1$, that is, we have

$$\max\{\lambda,\mu\} < \frac{q(a+b)}{p\mathcal{S}^{-q}_{q,h}} =: \boldsymbol{\lambda}^3.$$

Next, we define a positive constant $\boldsymbol{\lambda}^4$ as follows

$$\boldsymbol{\lambda}^4 = \frac{qb(\sigma-\theta p)\big(\min\{V_0,K_0\}\big)^{\frac{\theta p-q}{p}}}{2^{\theta p-q}(\sigma-q)p\mathcal{S}^{-q}_{q,h}}\left(\frac{\beta_*}{2\alpha_0}\right)^{\frac{\theta p-q}{p'}}. \tag{5.7}$$

Pick $\bar{\lambda} = \min\{\boldsymbol{\lambda}^1, \boldsymbol{\lambda}^2, \boldsymbol{\lambda}^3, \boldsymbol{\lambda}^4\}$, then we conclude that for all $(\lambda,\mu) \in (0,\bar{\lambda}) \times (0,\bar{\lambda})$, we have $I(u,v) \geq 0$ for any $\|(u,v)\| \geq 1$. This implies at once that if

$$I(u,v) < 0, \text{ then } \|(u,v)\| < 1,\ \forall\ (\lambda,\mu) \in (0,\bar{\lambda}) \times (0,\bar{\lambda}). \tag{5.8}$$

Consequently, from the definitions of $H_{\lambda,\mu}$, $G_{\lambda,\mu}$ and Ψ, we get for all $(\lambda,\mu)\in(0,\bar{\lambda})\times(0,\bar{\lambda})$ with $I(u,v)<0$ that

$$H_{\lambda,\mu}(\|(u,v)\|\leq G_{\lambda,\mu}(\|(u,v)\|)\leq I(u,v)<0.$$

This together with (5.4) implies that for all $(\lambda,\mu)\in(0,\bar{\lambda})\times(0,\bar{\lambda})$ with $I(u,v)<0$, we have

$$\|(u,v)\|\in(0,1)\cap[(0,T_0(\lambda,\mu))\cup(T_1(\lambda,\mu),\infty)]. \tag{5.9}$$

Now we have following possibilities according to the nature of $T_0(\lambda,\mu)$ and $T_1(\lambda,\mu)$.

Situation 1. Let either $0<T_0(\lambda,\mu)<1<T_1(\lambda,\mu)$ or $0<T_0(\lambda,\mu)<T_1(\lambda,\mu)=1$. In both cases, for all $(\lambda,\mu)\in(0,\bar{\lambda})\times(0,\bar{\lambda})$ with $I(u,v)<0$, it follows directly from (5.9) that $\|(u,v)\|\in(0,T_0(\lambda,\mu))$ and we are done.

Situation 2. Let $0<T_0(\lambda,\mu)<T_1(\lambda,\mu)<1$. Then for all $(\lambda,\mu)\in(0,\bar{\lambda})\times(0,\bar{\lambda})$ with $I(u,v)<0$, we deduce from (5.9) that $\|(u,v)\|\in(0,T_0(\lambda,\mu))\cup(T_1(\lambda,\mu),1)$. Next, we claim that $\|(u,v)\|\notin(T_1(\lambda,\mu),1)$. Indeed, if not, let $T_1(\lambda,\mu)<\|(u,v)\|<1$. By the definition of Ψ, we have $\Psi(\|(u,v)\|)=0$. Consequently, one has

$$\begin{aligned}I(u,v)&=\frac{1}{p}\widehat{M}(\|(u,v)\|^p)-\frac{1}{q}(\lambda\|u\|_{q,h}^q+\mu\|v\|_{q,h}^q)\\&\geq\left(\frac{a+b}{p}\right)\|(u,v)\|^{\theta p}-\frac{1}{q}\max\{\lambda,\mu\}\mathcal{S}_{q,h}^{-q}\|(u,v)\|^q\\&=:h_{\lambda,\mu}(\|(u,v)\|)\end{aligned}$$

for all $T_1(\lambda,\mu)<\|(u,v)\|<1$, where $h_{\lambda,\mu}:[0,\infty)\to\mathbb{R}$ is given by

$$h_{\lambda,\mu}(\ell)=\left(\frac{a+b}{p}\right)\ell^{\theta p}-\frac{1}{q}\max\{\lambda,\mu\}\mathcal{S}_{q,h}^{-q}\ell^q.$$

By simple computation, we can deduce that $h_{\lambda,\mu}$ has a global minimum point at

$$\widehat{\ell}=\left(\frac{\max\{\lambda,\mu\}\mathcal{S}_{q,h}^{-q}}{(a+b)\theta}\right)^{\frac{1}{\theta p-q}}$$

and

$$h_{\lambda,\mu}(\widehat{\ell})=\left(\frac{\max\{\lambda,\mu\}\mathcal{S}_{q,h}^{-q}}{(a+b)\theta}\right)^{\frac{q}{\theta p-q}}\max\{\lambda,\mu\}\mathcal{S}_{q,h}^{-q}\left(\frac{1}{\theta p}-\frac{1}{q}\right)<0,$$

we thank to $1<q<\theta p$. In addition, one sees that

$$h_{\lambda,\mu}(\ell)\geq 0\iff\ell\geq\left(\frac{\max\{\lambda,\mu\}p\mathcal{S}_{q,h}^{-q}}{q(a+b)}\right)^{\frac{1}{\theta p-q}}=:\ell_2.$$

Consequently, we infer that

$$I(u,v)\geq h_{\lambda,\mu}(\|(u,v)\|)\geq 0,\qquad\forall\ \|(u,v)\|\in(T_1(\lambda,\mu),1)$$

with $\ell_2<T_1(\lambda,\mu)$, that is, we conclude that $\max\{\lambda,\mu\}<\boldsymbol{\lambda}^3$, which is a contradiction. This completes the proof of the claim. Hence in this case, we also have $\|(u,v)\|\in(0,T_0(\lambda,\mu))$.

From the above discussions, we deduce that $\|(u,v)\| < T_0(\lambda,\mu)$ and thus $I(u,v) = J(u,v)$. In addition, it follows that $I(\varphi,\psi) = J(\varphi,\psi)$ for all (φ,ψ) with $\|(\varphi,\psi)-(u,v)\| < T_0(\lambda,\mu) - \|(u,v)\|$. This shows that $I(\varphi,\psi) = J(\varphi,\psi)$ for all (φ,ψ) in a small neighbourhood of (u,v) and thus we finish the proof of part (a). Now our aim is to prove part (b). For this, first we take $c < 0$ and a $(PS)_c$ sequence $\{(u_n,v_n)\}_n \subset \mathbf{X}$ for the functional I. Therefore, we can assume that $I(u_n,v_n) < 0$ and $I'(u_n,v_n) = o_n(1)$ as $n \to \infty$ and thus by part (a), for any $(\lambda,\mu) \in (0,\bar{\lambda}) \times (0,\bar{\lambda})$, there holds

$$I(u_n,v_n) = J(u_n,v_n),\ I'(u_n,v_n) = J'(u_n,v_n) \quad \text{and} \quad \|(u_n,v_n)\| < T_0(\lambda,\mu).$$

In addition, since I is coercive on $\mathbf{X}$, we deduce that the $(PS)_c$ sequence $\{(u_n,v_n)\}_n \subset \mathbf{X}$ is bounded. Hence up to a subsequence still denoted by itself such that $(u_n,v_n) \rightharpoonup (u,v)$ in $\mathbf{X}$ as $n \to \infty$ for some couple $(u,v) \in \mathbf{X}$. Further, by using (F_4), we can notice that the following estimates hold as $n \to \infty$

$$\begin{aligned}
0 > c = J(u_n,v_n) &- \frac{1}{\sigma}\langle J'(u_n,v_n),(u_n,v_n)\rangle + o_n(1)\\
&\geq \frac{1}{p}\widehat{M}(\|(u_n,v_n)\|^p) - \frac{1}{\sigma}M(\|(u_n,v_n)\|^p)\|(u_n,v_n)\|^p\\
&\quad - \max\{\lambda,\mu\}\left(\frac{1}{q}-\frac{1}{\sigma}\right)\mathcal{S}_{q,h}^{-q}\|(u_n,v_n)\|^q + o_n(1)\\
&\geq \left(\frac{1}{p}-\frac{\theta}{\sigma}\right)b\|(u_n,v_n)\|^{\theta p} - \max\{\lambda,\mu\}\left(\frac{1}{q}-\frac{1}{\sigma}\right)\mathcal{S}_{q,h}^{-q}\|(u_n,v_n)\|^q + o_n(1).
\end{aligned}$$

Define $z_n = (u_n,v_n)$, then we can see that $|z_n| = \sqrt{u_n^2+v_n^2} \leq \psi_n := |u_n| + |v_n|$. By direct calculation, we have

$$\|\psi_n\|_{W^{s,p}} \leq 2(\min\{V_0,K_0\})^{-\frac{1}{p}}\|(u_n,v_n)\|.$$

It follows from the above two inequalities and (5.7) that

$$\limsup_{n\to\infty}\|\psi_n\|_{W^{s,p}}^{p'} \leq \left(\frac{2^{\theta p-q}(\sigma-q)p\mathcal{S}_{q,h}^{-q}\max\{\lambda,\mu\}}{qb(\sigma-\theta p)\big(\min\{V_0,K_0\}\big)^{\frac{\theta p-q}{p}}}\right)^{\frac{p'}{\theta p-q}} < \frac{\beta_*}{2\alpha_0},$$

thanks to the fact that $\max\{\lambda,\mu\} < \bar{\lambda} \leq \boldsymbol{\lambda}^4$, since we have

$$(\lambda,\mu) \in (0,\bar{\lambda}) \times (0,\bar{\lambda}) \quad \text{and} \quad \bar{\lambda} = \min\{\boldsymbol{\lambda}^1,\boldsymbol{\lambda}^2,\boldsymbol{\lambda}^3,\boldsymbol{\lambda}^4\}.$$

Using all of the above information, and arguing similarly as in the proof of Theorem 1.3, we can easily deduce that $(u_n,v_n) \to (u,v)$ in $\mathbf{X}$ as $n \to \infty$. This completes the proof of part (b). Hence the lemma is well established. □

It follows immediately from the above lemma that the following result holds.

Corollary 5.6. *For all $(\lambda,\mu) \in (0,\bar{\lambda}) \times (0,\bar{\lambda})$, the set K_c, which is given by*

$$K_c = \{(u,v) \in \mathbf{X} :\ I(u,v) = c < 0,\ I'(u,v) = 0\},\ \textit{is compact.}$$

For $\epsilon > 0$, we define

$$I^{-\epsilon} = \{(u,v) \in \mathbf{X} :\ I(u,v) \leq -\epsilon\}.$$

Lemma 5.7. *For any $k \in \mathbb{N}$, there exists $\epsilon_k > 0$ such that there holds $\gamma(I^{-\epsilon_k}) \geq k$.*

Proof. Suppose that $\mathbf{X}_k$ be a k-dimensional subspace of $\mathbf{X}$. For any $(u,v) \in \mathbf{X}_k \setminus \{(0,0)\}$, we define $(u,v) = r_k(\varphi,\psi)$ with $(\varphi,\psi) \in \mathbf{X}_k$, $\|(\varphi,\psi)\| = 1$ and $r_k = \|(u,v)\|$. Under the assumption of h, we known that $\|(\varphi,\psi)\|_{L_h^q(\mathbb{R}^N)\times L_h^q(\mathbb{R}^N)}$ is a norm of $\mathbf{X}_k$ for all $(\varphi,\psi) \in \mathbf{X}_k$. Since all the norms are equivalent in a finite-dimensional Banach space, therefore we infer that for each $(\varphi,\psi) \in \mathbf{X}_k$, there exists $d_k > 0$ such that

$$\|(\varphi,\psi)\|^q_{L_h^q(\mathbb{R}^N)\times L_h^q(\mathbb{R}^N)} \geq d_k.$$

Consequently, for $r_k \in (0, T_0(\lambda,\mu))$, we can easily deduce that

$$I(u,v) = J(u,v) \leq \left(\frac{a+b}{p}\right) r_k^p - \frac{1}{q}\min\{\lambda,\mu\} d_k r_k^q.$$

In virtue of $q < p$, without loss of generality, we can choose $r_k \in (0, T_0(\lambda,\mu))$ sufficiently small enough such that $I(u,v) \leq -\epsilon_k < 0$. Define $S_{r_k} = \{(u,v) \in \mathbf{X} : \ \|(u,v)\| = r_k\}$, then one has $S_{r_k} \cap \mathbf{X}_k \subset I^{-\epsilon_k}$. Now by using Proposition 5.1, we have $\gamma(I^{-\epsilon_k}) \geq \gamma(S_{r_k} \cap \mathbf{X}_k) = k$. This finishes the proof. □

Define

$$\mathbf{\Sigma}_k = \{A \in \mathbf{\Sigma} : \ \gamma(A) \geq k\}$$

and

$$c_k = \inf_{A\in\mathbf{\Sigma}_k} \sup_{(u,v)\in A} I(u,v).$$

It is obvious that the sequence $\{c_k\}_k$ is monotonically increasing in nature and there holds

$$-\infty < c_k \leq -\epsilon_k < 0 \quad \text{for each } k \in \mathbb{N},$$

thanks to the fact that I is bounded from below and $I^{-\epsilon_k} \in \mathbf{\Sigma}_k$. Due to Lemma 5.5, the functional I satisfies the $(PS)_c$ condition for $c < 0$, and we have $c_k < 0$ for each $k \in \mathbb{N}$, therefore by a standard argument, we infer that all c_k are critical values of I.

The next lemma shows that the set K_c, defined in Corollary 5.6, contains infinitely many critical points of I.

Lemma 5.8. *Let $(\lambda,\mu) \in (0,\bar{\lambda}) \times (0,\bar{\lambda})$. Then for $c = c_k = c_{k+1} = \cdots = c_{k+m}$ with some $m \in \mathbb{N}$, we have $\gamma(K_c) \geq m+1$.*

Proof. It will be proven by using the method of contradiction. For this, we first claim that $\gamma(K_c) \geq m+1$ holds. Indeed, if not, let $\gamma(K_c) \leq m$. In virtue of Corollary 5.6, the set K_c is compact and $K_c \in \mathbf{\Sigma}$. Consequently, by using Proposition 5.1, one has $\gamma(K_c) < \infty$ and there exists $\delta > 0$ such that $N_\delta(K_c) \in \mathbf{\Sigma}$ and $\gamma(K_c) = \gamma(N_\delta(K_c)) \leq m$. On the other hand, by using the assumption (F_1'), we infer that the functional I is even. Thus, due to Proposition 5.2, there exists an odd homeomorphism $\eta(t,(u,v)) = \eta_t(u,v) \in C([0,1]\times\mathbf{X},\mathbf{X})$ such that

$$\eta_1(I^{c+\epsilon} \setminus N_\delta(K_c)) \subset I^{c-\epsilon} \quad \text{for some } \epsilon \in (0,-c).$$

From the hypothesis, we have $c = c_{k+m}$ and hence there exists a set $A \in \mathbf{\Sigma}_{k+m}$ such that $\sup_{(u,v)\in A} I(u,v) < c+\epsilon$, that is, $A \subset I^{c+\epsilon}$. This shows that $\eta_1(A \setminus N_\delta(K_c)) \subset \eta_1(I^{c+\epsilon} \setminus N_\delta(K_c)) \subset I^{c-\epsilon}$. Consequently, we deduce that

$$\sup_{(u,v)\in\eta_1(A\setminus N_\delta(K_c))} I(u,v) \leq c-\epsilon. \tag{5.10}$$

Due to Proposition 5.2, we have

$$\gamma(\overline{A \setminus N_\delta(K_c)}) \geq \gamma(A) - \gamma(N_\delta(K_c)) \geq k \quad \text{and} \quad \gamma(\overline{\eta_1(A \setminus N_\delta(K_c))}) \geq k.$$

It follows that $\overline{\eta_1(A \setminus N_\delta(K_c))} \in \mathbf{\Sigma}_k$ and thus we have

$$\sup_{(u,v)\in\overline{\eta_1(A\setminus N_\delta(K_c))}} I(u,v) \geq c_k = c,$$

which contradicts (5.10). This completes the proof of the claim and we finish the proof. □

Proof of Theorem 1.5. Let $(\lambda,\mu) \in (0,\bar{\lambda}) \times (0,\bar{\lambda})$. Notice that if we have $-\infty < c_1 < c_2 < \cdots < c_k < \cdots < 0$, then since c_k are critical values of I, we infer that I has infinitely many critical points. In virtue of Lemma 5.5, we have $J \equiv I$ if $I < 0$. This shows that the system $(\mathcal{S}_{\lambda,\mu})$ has infinitely many weak solutions.

On the other hand, if there exists $c_k = c_{k+m}$ for some $m \in \mathbb{N}$, then $c = c_k = c_{k+1} = \cdots = c_{k+m}$. In virtue of Lemma 5.8, we get $\gamma(K_c) \geq m+1 \geq 2$ and thus by Proposition 5.1, one can notice that the set K_c has infinitely many points. In conclusion, we deduce that the system $(\mathcal{S}_{\lambda,\mu})$ has infinitely many weak solutions. This finishes the proof. □

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Rational solutions and limit cycles of polynomial and trigonometric Abel equations

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Abstract. We study the Abel differential equation $x' = A(t)x^3 + B(t)x^2 + C(t)x$. Specifically, we find bounds on the number of its rational solutions when $A(t), B(t)$ and $C(t)$ are polynomials with real or complex coefficients; and on the number of rational limit cycles when $A(t), B(t)$ and $C(t)$ are trigonometric polynomials with real coefficients.

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1 Introduction

The Abel differential equation

$$x' = A(t)x^3 + B(t)x^2 + C(t)x \tag{1.1}$$

with $A(t)$, $B(t)$, $C(t)$ continuous, and the generalized Abel equations

$$x' = \sum_{i=1}^{n} A_i(t)x^i \tag{1.2}$$

with $n > 3$ and $A_i(t)$ continuous for all $i \in \{1, \dots, n\}$ have been widely studied in the context of Hilbert's 16th Problem, and because of their utility to model real-world models [4] and its own mathematical interest [12].

In the field of the qualitative theory of differential equations, the two most studied problems regarding Abel equations are the Smale–Pugh problem and the Poincaré Centre-Focus problem.

The Smale–Pugh problem [33] consists on bounding the number of limit cycles (isolated periodic solutions) of (1.1) when its coefficients with respect to t are periodic. Lins Neto [23] showed that there is no bound for the number of limit cycles of (1.1) when $A(t)$, $B(t)$ and $C(t)$ are trigonometric polynomials. This problem is related to Hilbert's 16th problem through the

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transformation of Cherkas [10]. Some results in this line can be consulted in, for example, [1–3, 13, 14, 19–22, 27, 29, 31, 37].

The other problem is the adaptation of the classical Poincaré Centre-Focus problem to this setting, proposed by Briskin, Françoise and Yondim [7, 8]. Here, as $x(t) \equiv 0$ is always a periodic solution of (1.1), we will say that the equation has a center if every solution in a neighbourhood of $x(t) \equiv 0$ is also periodic. For more details on the state of art of this problem, see for example [12, 18, 30].

Another natural question regarding an Abel equation is related to how many solutions of a certain type the equation can have. For example, Giné et al. [17] showed that the generalized Abel equations of degree n in x with polynomial coefficients have at most n polynomial solutions. Other works about similar topics are [9, 15, 24, 25, 28, 32, 34].

In [26, 36], the authors study how many rational closed solutions can the Abel equation

$$x' = A(t)x^3 + B(t)x^2 \tag{1.3}$$

have, both in the polynomial (this is, when $A(t)$ and $B(t)$ are polynomials and the rational solutions are quotients of polynomials) and in the trigonometric cases (this is, when $A(t)$ and $B(t)$ are trigonometric polynomials, and the rational solutions are quotients of trigonometric polynomials). Note that, in the trigonometric case, this problem is related with the Smale–Pugh problem, as, if the equation has not a center, the rational solutions are periodic and, consequently, limit cycles of the equation. Thus, the number of certain types of limit cycles of the equation is being determined.

In [5], we studied the polynomial case for the equation (1.3), for polynomials with real or complex coefficients, but looking for all rational solutions of the equation, regardless of whether they are closed or not. In particular, conditions are given for the equation to have at most two rational solutions; a general bound for the number of rational solutions is given; and Darboux's Integrability Theory (see more in [16] for example) is used to determine how many rational solutions can coexist before the equation is integrable.

Furthermore, in [6] the trigonometric case of the equation (1.3) is studied and bounds are found for the number of rational limit cycles, using Darboux's Integrability Theory to improve the bounds given in [36]. This work is only performed in the real context, since the techniques developed in the polynomial case do not translate adequately in the complex environment.

Clearly, there are various ways to extend this problem, for example, by searching results about the number of rational solutions of generalized Abel equations. However, the most immediate extension consists on adding the linear term to the equation, this is, to look for rational solutions of

$$x' = A(t)x^3 + B(t)x^2 + C(t)x + D(t). \tag{1.4}$$

Without loss of generality, we assume $D(t) \equiv 0$. This problem has been studied in [35], but only for closed rational solutions in the polynomial case.

The objective of this work is to extend the techniques and results in [5,6] to the "complete" Abel differential equation.

In precise terms, throughout this paper, we consider the Abel equation

$$x' = A(t)x^3 + B(t)x^2 + C(t)x \tag{1.5}$$

with $A(t)$, $B(t)$, and $C(t) \in R$, where R is either the polynomial ring in t with coefficients in $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, in symbols, $R = \mathbb{K}[t]$, or the ring of trigonometric polynomials with real coefficients, in symbols, $R = \mathbb{R}[\cos(t), \sin(t)]$. Note that we exclude the study over $\mathbb{C}[\cos(t), \sin(t)]$ because, as mentioned above, the techniques can not be applied appropriately in this setting.

Moreover, since the case $C(t) \equiv 0$ is completely studied in [5, 6], in what follows we consider $C(t) \not\equiv 0$, and we also assume that $A(t) \not\equiv 0$ and $B(t) \not\equiv 0$, since otherwise (1.5) would be a Ricatti or Bernoulli equation, respectively. We will also assume that $A(t)$ is non constant, as this case has been studied by Pliss in [31], in the trigonometric case, and by Giné, Grau and Llibre [17] in the polynomial case (see Remark 2.4).

Whenever possible, we will treat both the polynomial case and the trigonometric case as a whole. For this reason, it is convenient to note that $\deg(-)$ indistinguishable means in the polynomial case the usual degree of an univariate polynomial; while in the trigonometric case it means the highest integer n such that $a_n \neq 0$ or $b_n \neq 0$ in the expression

$$a_0 + \sum_{n=1}^{N} a_n \cos(nt) + b_n \sin(nt).$$

Moreover, to facilitate the cases integration, we recall that the problem of finding rational solutions of the equation (1.5), in the polynomial case, and rational solutions (that will be automatically periodic) of (1.5), in the trigonometric case, is equivalent to studying the invariant curves of (1.5) of the form $p(t)x + q(t) = 0$, with $p(t), q(t) \in R$ and $p(t) \neq 0$ for every t. Therefore, we use the term invariant curve of degree one in x, (or simply invariant curve, since there can be no confusion here) to unify the terminology, and the results will be given, always that is possible, in terms of invariant curves and not rational solutions. For more details on invariant curves see [16].

We are now ready to present the main result in this paper. Note that it is written in terms of invariant curves, with the exception of the case (b.2.2), where we can improve the bound by studying instead the number of rational limit cycles, which is equivalent to the number of invariant curves when the equation does not have a center.

Theorem 1.1. *With the above notation and conventions, the following claims are satisfied.*

(a) If $\deg(A) - \deg(C) \leq 1$, *then* (1.5) *has at most one invariant curve.*

(b) If $\deg(A) - \deg(C) > 1$ *and we are either in the polynomial case, or in the trigonometric one with* $C(t)$ *non-constant, then*

(b.1) If $\deg(A) - \deg(C)$ *is odd or* $\deg(A) + \deg(C) < 2\deg(B)$, *then* (1.5) *has at most two invariant curves.*

(b.2) If $\deg(A) - deg(C)$ *is even and* $\deg(A) + \deg(C) \geq 2\deg(B)$, *then,*

(b.2.1) in the polynomial case, the number of invariant curves of (1.5) *is at most*

$$\binom{\deg(A)}{(\deg(A)+1)/2} + 1.$$

Moreover, in this case, (1.5) *is Darboux integrable, if the number of invariant curves of* (1.5) *is greater that* $\deg(A) + \deg(C) + 4$.

(b.2.2) in the trigonometric case, an upper bound for the number of rational limit cycles of (1.5) *is* $2\deg(A) + 2\deg(C) + 4$.

(c) If $\deg(A) - \deg(C) > 1$, *in the trigonometric case and with* $C(t) \equiv c$ *a non zero constant, then an upper bound for the number of invariant curves is* $4\deg(A)$.

Note that we have assumed that $C(t) \not\equiv 0$, and so this result does not cover the results of [5,6]. However, the conditions of (b.1) are very similar to their equivalent versions in [5,6] for the case $C(t) \equiv 0$, and the general bound of (b.2.1) for the polynomial case is exactly the same. On the other hand, when we apply Darboux's Theory of Integrability in this context, because of details of the proof the bounds of (b.2.2) and (c) are less sharp than their equivalent ones in [5,6].

In [5] using computational numerical and algebraic tools we were able to develop a method to study thoroughly the number of invariant curves that equation (1.3) has, in the polynomial case. Its computational effort grows sharply as $\deg(A)$ rises though, so we only studied the cases up to $\deg(A) = 5$.

Using this method, in [5] for low values of $\deg(A)$ we could study the sharpness of the general bound

$$\binom{\deg(A)}{(\deg(A)+1)/2} + 1$$

for the number of invariant curves of the equation (1.3). We did show that this bound was sharp when $\deg(A)$ was one or three, but it was not sharp when it was five.

On the other hand, we did not study the sharpness of the bounds provided by Darboux's Theory of Integrability when $C(t) \equiv 0$ in [5,6], although some numerical experiments seemed to show that they were sharp for low values of $\deg(A)$. In general, the sharpness of these bounds is an open question.

As this paper covers the polynomial and the trigonometric case at the same time, we have decided to not develop an equivalent exhaustive method for (1.5), although we offer a lemma related with the parametrization of the equation in terms of its invariant curves, which is very helpful to make numerical experiments in this direction. Consequently, the sharpness of the bounds provided in Theorem 1.1 are left as an open question.

The structure of the paper is as follows: first we remind the factorization properties of the ring $\mathbb{R}[\cos t, \sin t]$ and in Section 2.1 we show the characterization of the invariant curves of (1.5); then, in Section 2.2, we suppose that (1.5) has two or more different invariant curves, and we prove (a) and (b.1); next, in Section 2.3, we focus on the polynomial case to provide the general bound of (b.2.1); and finally in Section 3 we apply Darboux's Theory of Integrability to prove the remaining parts of (b.2.1), together with (b.2.2) and (c).

2 Invariant curves of degree one in x

We remind that we have set R to be equal to $\mathbb{K}[t]$, $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, or equal to $\mathbb{R}[\cos(t), \sin(t)]$. While in the first case R is a Euclidean domain and everything works fine in terms of divisibility or factorization of elements, in the second case R is not even a unique factorization domain. So special care must be taken when using it with factorization arguments. Nevertheless, $\mathbb{R}[\cos(t), \sin(t)]$ has somewhat stable factorization properties for being a half-factorial Dedekind domain. We summarize those extensively used along this paper and refer the reader to [6, Appendix A] for more details.

Proposition 2.1. *Let $p(t) \in \mathbb{R}[\cos(t), \sin(t)]$. If $p(t) \notin \mathbb{R}$ and $p(t) \neq 0$ for every t, then the following holds.*

(a) The decomposition of $p(t)$ into irreducibles of $\mathbb{R}[\cos(t), \sin(t)]$ is unique, up to the order of the factors and the product by invertible elements.

(b) If $p(t) = h(t)r(t)$ and none of the irreducible factors of $p(t)$ divides $h(t)$, then $r(t)$ divides $p(t)$.

(c) If $q(t) \in \mathbb{R}[\cos(t), \sin(t)] \setminus \{0\}$ is non-constant, then $p(t)$ and $q(t)$ are relatively prime if and only if they have not irreducible factors in common.

As mentioned before, we are interested in studying the invariant curves of degree one in x (that from now on we call simply invariant curves) of (1.5), that is, the invariant curves (1.5) of the form $p(t)x + q(t) = 0$, with $p(t), q(t) \in R$, $p(t)$ non-constant and $p(t) \neq 0$ for every t. Because $p(t) \neq 0$ we always assume that $p(t)$ and $q(t)$ are relatively prime.

Recall that $p(t)x + q(t) = 0$ is invariant for (1.5) if and only if there exists $K(t,x) \in R[x]$, called cofactor of $p(t)x + q(t)$, such that

$$p'(t)x + q'(t) + p(t)\left(A(t)x^3 + B(t)x^2 + C(t)x\right) = (p(t)x + q(t))K(t,x). \tag{2.1}$$

For degree reasons, $K(t,x) = K_2(t)x^2 + K_1(t)x + K_0(x)$ with $K_i(t) \in R$, $i = 0,1,2$.

2.1 One invariant curve

A characterization of the invariant curves of the equation (1.3) can be found in [24], which was then adapted to our case, and the trigonometric case with $C(t) \equiv 0$ in [35, 36], respectively. We include its proof here for the sake of completeness.

Proposition 2.2. *The curve $p(t)x + q(t) = 0$ is invariant for (1.5) if and only if $q(t) = c$ is a nonzero constant and*

$$c^2 A(t) = \left(cB(t) - \left(p'(t) + p(t)C(t)\right)\right)\, p(t). \tag{2.2}$$

Proof. By (2.1), $p(t)x + q(t) = 0$ is invariant for (1.5) if and only if

$$\begin{aligned} &p'(t)x + q'(t) + p(t)\left(A(t)x^3 + B(t)x^2 + C(t)x\right) \\ &\quad = (p(t)x + q(t))\left(K_2(t)x^2 + K_1(t)x + K_0(t)\right), \end{aligned}$$

with $K_i(t) \in R$, $i = 0,1,2$. So, equating the coefficients of the powers of x and omitting the arguments, we have

$$\begin{aligned} q' &= q\,K_0, \\ p' + p\,C &= K_0\,p + K_1\,q, \\ p\,B &= K_1\,p + K_2\,q, \\ p\,A &= K_2\,p \iff A = K_2 \quad (\text{since } p(t) \not\equiv 0). \end{aligned}$$

From the first equality, we obtain both $K_0(t) \equiv 0$ and $q(t) = c$ for a nonzero constant c. Now, substituting this conveniently into the second equation, we get

$$p'(t) + p(t)C(t) = K_1(t)c,$$

that is, $K_1(t) = (p'(t) + p(t)C(t))/c$. So the third equation yields

$$p(t)B(t) = (p'(t) + p(t)C(t))p(t)/c + K_2(t)c.$$

Therefore, $K_2(t) = p(t)B(t)/c - (p'(t) + p(t)C(t))p(t)/c^2$ and, consequently,

$$c^2\,A(t) = \left(cB(t) - \left(p'(t) + p(t)C(t)\right)\right)\, p(t),$$

as claimed. □

Note that, without loss of generality, we can assume $c = -1$ in Proposition 2.2. We will assume this from now until the end of the paper. Thus, our invariant curves will always be of the form $p(t)x - 1 = 0$.

Although the following result is a direct consequence of Proposition 2.2 and its proof, we state it because we will use it extensively throughout the paper.

Corollary 2.3. *If $p(t)x - 1 = 0$ is an invariant curve of (1.5), then $p(t)$ divides $A(t)$. In particular,*

$$-B(t) = A(t)/p(t) + p'(t) + p(t)C(t) \tag{2.3}$$

and the cofactor of $p(t)x - 1 = 0$ is equal to $A(t)x^2 - (p'(t) + p(t)C(t))\, x$.

Remark 2.4. As mentioned in the introduction, in the trigonometric case, the case where $A(t)$ is constant is extensively studied in [31]. In the other case, since, by Corollary 2.3, $A(t)$ being constant implies that $p(t)$ is also constant (in fact, regardless of whether we are in the polynomial or the trigonometric case), we have that the corresponding invariant curve is polynomial; a case that has been extensively studied in [17]. Thus, in the rest of the paper, we always assume that $\deg(A) > 0$

2.2 Two or more invariant curves

Now suppose (1.5) has at least two different invariant curves.

Proposition 2.5. *If $p_1(t)x - 1 = 0$ and $p_2(t)x - 1 = 0$ are two different invariant curves of (1.5), then*

$$A(t) = C(t)p_1(t)p_2(t) + \frac{(p_2(t) - p_1(t))'p_1(t)p_2(t)}{p_2(t) - p_1(t)}. \tag{2.4}$$

In particular, $\deg(A) - \deg(C) \leq \deg(p_1) + \deg(p_2)$.

Proof. By (2.3),

$$-A(t) = \left(B(t) + \left(p_1'(t) + p_1(t)C(t)\right)\right)\, p_1(t).$$

and

$$-A(t) = \left(B(t) + \left(p_2'(t) + p_2(t)C(t)\right)\right)\, p_2(t).$$

If we multiply the first expression by $p_2(t)$ and the second expression by $p_1(t)$ and then take the difference between them, we get

$$-(p_2(t) - p_1(t))A(t) = (p_1'(t) - p_2'(t))p_1(t)p_2(t) + (p_1(t) - p_2(t))C(t)p_1(t)p_2(t).$$

Now, since $p_1(t) \not\equiv p_2(t)$, (2.4) follows.

Finally, for the second part of the statement, it suffices to observe that by (2.4) we have that

$$\begin{aligned} \deg(A) &\leq \max\left\{\deg(C(t)p_1(t)p_2(t)), \deg\left(\frac{(p_2(t) - p_1(t))'p_1(t)p_2(t)}{p_2(t) - p_1(t)}\right)\right\} \\ &= \deg(C(t)p_1(t)p_2(t)) = \deg(C) + \deg(p_1) + \deg(p_2). \end{aligned} \tag{2.5}$$

Therefore, $\deg(A) - \deg(C) \leq \deg(p_1) + \deg(p_2)$ as claimed. □

Proposition 2.6. *Let $C(t)$ be non-constant. If $p_1(t)x - 1 = 0$ and $p_2(t)x - 1 = 0$ are two different invariant curves of (1.5), then*

$$\deg(A) - \deg(C) = \deg(p_1) + \deg(p_2).$$

In particular, $\deg(A) - \deg(C) > 1$.

Proof. Let

$$r(t) := \frac{(p_2(t) - p_1(t))' p_1(t) p_2(t)}{p_2(t) - p_1(t)}.$$

By (2.4), $r(t)$ is equal to $A(t) - C(t)p_1(t)p_2(t)$. Therefore, $r(t) \in R$ and $\deg(r) = \deg(p_1) + \deg(p_2) - 1$, in the polynomial case, and $\deg(r) = \deg(p_1) + \deg(p_2)$, in the trigonometric case.

Now, since $C(t)$ is not constant, in particular, $\deg(C) > 0$ and, consequently, $\deg(Cp_1p_2) > \deg(p_1) + \deg(p_2) \geq \deg(r)$. Therefore, since by (2.5) $\deg(A) \leq \deg(Cp_1p_2)$ and $\deg(A - Cp_1p_2) = \deg(r) < \deg(Cp_1p_2)$, we conclude that $\deg(A) = \deg(Cp_1p_2)$, that is, $\deg(A) - \deg(C) = \deg(p_1) + \deg(p_2)$, as claimed.

The last statement is as simple as noting that $p_1(t)$ and $p_2(t)$ are non-constant, and so $\deg(p_1) + \deg(p_2) > 1$. □

Remark 2.7. Note that, in the polynomial case, $C(t)$ can be chosen constant in Proposition 2.6 and everything remains the same because, in this case, $\deg(r) = \deg(p_1) + \deg(p_2) - 1 < \deg(Cp_1p_2)$.

Remark 2.8. Observe that, in the polynomial case, (2.4) can be interpreted in terms of Euclidean divisibility as follows: if $p_1(t)x - 1 = 0$ and $p_2(t)x - 1 = 0$ are two different invariant curves of (1.5), then the quotient and the remainder of the division of $A(t)$ by $p_1(t)p_2(t)$ are

$$C(t) \quad \text{and} \quad r(t) = \frac{(p_2(t) - p_1(t))' p_1(t) p_2(t)}{p_2(t) - p_1(t)},$$

respectively. Giving rise to a computable necessary condition for two factors of $A(t)$ to define invariant curves.

Let us derive some consequences from Proposition 2.6.

Corollary 2.9. *If* $\deg(A) - \deg(C) \leq 1$, *then* (1.5) *has at most one invariant curve.*

Proof. If $C(t)$ is not constant, then, by Proposition 2.6, we are done. This also extends to the polynomial case, when $C(t)$ is constant (see Remark 2.7). Thus, suppose $A(t)$ and $B(t)$ are non-constant real trigonometric polynomials and $C(t) = c \in \mathbb{R} \setminus \{0\}$; in particular, $\deg(A) = 1$ by hypothesis. Therefore, $p_1(t) = a\,A(t)$ and $p_2(t) = b\,A(t)$ for some $a, b \in \mathbb{R} \setminus \{0\}$ with $a \neq b$. Thus, by (2.4),

$$A(t) = abA(t)(cA(t) + A'(t)).$$

Therefore, $cA(t) + A'(t) = \frac{1}{ab}$. Now, since $A(t) = a_0 + a_1\cos(t) + b_1\sin(t)$, for some a_0, a_1 and $b_1 \in \mathbb{R}$, we obtain that

$$ca_0 + (ca_1 + b_1)\cos(t) + (cb_1 - a_1)\sin(t) = \frac{1}{ab}.$$

Therefore,

$$ca_0ab = 1, \quad ca_1 + b_1 = 0, \quad cb_1 - a_1 = 0.$$

From the last two equalities, $a_1 = b_1 = 0$, in contradiction with $A(t)$ being non constant. □

Lemma 2.10. *Let* $p_1(t)x - 1 = 0$ *and* $p_2(t)x - 1 = 0$ *be two different invariant curves of* (1.5). *If* $\deg(p_1) \neq \deg(p_2)$, *and*

(a) $C(t)$ *is not constant, then has no more invariant curves.*

(b) $C(t)$ is constant, then $\deg(A) = \deg(p_1) + \deg(p_2)$.

Proof. (a) Suppose on contrary, that $h(t,x) = 0$ is an invariant curve of (1.5) different from $p_1(t)x - 1 = 0$ and $p_2(t)x - 1 = 0$. By Proposition 2.2, there exists a polynomial $p_3(t) \in R$ such that $h(t,x) = p_3(t)x - 1$. Since, by Proposition 2.6, $\deg(p_i) + \deg(p_j) = \deg(A) - \deg(C)$ for every $1 \leq i < j \leq 3$, we conclude that $\deg(p_i) = \deg(p_j) = \frac{\deg(A)-\deg(C)}{2}$ for every $1 \leq i < j \leq 3$, in contradiction with the hypothesis.

(b) Note that by Remark 2.7 the result is immediate in the polynomial case, so we focus on the trigonometric one. Let $c \in \mathbb{R} \setminus \{0\}$ be such that $C(t) = c$. Then, by (2.3),

$$-B(t) = A(t)/p_1(t) + p_1'(t) + p_1(t)c = A(t)/p_2(t) + p_2'(t) + p_2(t)c.$$

Suppose that $\deg(A) \neq \deg(p_1) + \deg(p_2)$. Then, by Proposition 2.5, $\deg(A) < \deg(p_1) + \deg(p_2)$. Therefore $\deg(A/p_1) < \deg(p_2)$ and $\deg(A/p_2) < \deg(p_1)$.

Now, since $\deg(p_1) \neq \deg(p_2)$, we can assume $\deg(p_1) < \deg(p_2)$, then the higher degree terms of $p_2'(t)$ and $p_2(t)c$ must cancel each other. Thus, if $p_2(t) = a_n \cos(nt) + b_n \sin(nt) +$ *lower degree terms*, where a_n and b_n are not both zero, then

$$ca_n \cos(nt) + cb_n \sin(nt) = -na_n \sin(t) + nb_n \cos(t).$$

Therefore, $ca_n - nb_n = cb_n + na_n = 0$ and, since $c \in \mathbb{R} \setminus \{0\}$, we obtain $a_n = b_n = 0$, a contradiction. Hence, $\deg(p_1) = \deg(p_2)$. □

Remark 2.11. Note that according to Remark 2.7 nothing changes if $C(t)$ is constant in (a) of the previous lemma in the polynomial case.

Observe that as an immediate consequence of the proof of the above result we have the following corollary.

Corollary 2.12. *Suppose that we are in the polynomial case or in the trigonometric case when $C(t)$ is not constant. If (1.5) has three invariant curves, say, $p_i(t)x - 1 = 0$, $i = 1,2,3$, then*

$$\deg(p_i) = \frac{\deg(A) - \deg(C)}{2},$$

for every $i \in \{1,2,3\}$.

Proposition 2.13. *Suppose that we are in the polynomial case or in the trigonometric case when $C(t)$ is not constant. If $\deg(A) - \deg(C)$ is odd or $2\deg(B) > \deg(A) + \deg(C)$, then (1.5) has at most two invariant curves.*

Proof. Corollary 2.12 clearly implies that if $\deg(A) - \deg(C)$ is odd, then (1.5) can have at most two invariant curves.

Suppose that $2\deg(B) > \deg(A) + \deg(C)$. By (2.3), if $p_1(t)x - 1 = 0$ is an invariant curve, then

$$-B(t) = A(t)/p_1(t) + p_1'(t) + p_1(t)C(t),$$

and thus, $\deg(B) \leq \max\{\deg(A) - \deg(p_1), \deg(p_1), \deg(p_1) + \deg(C)\}$. Now, if (1.5) has more than two invariant curves, by Corollary 2.12, $\deg(p_1) = \frac{\deg(A)-\deg(C)}{2}$, and we conclude that

$$\deg(B) \leq \max\left\{\frac{\deg(A) + \deg(C)}{2}, \frac{\deg(A) - \deg(C)}{2}\right\} = \frac{\deg(A) + \deg(C)}{2}$$

getting a contradiction. □

Let us parameterize the family of equations (1.5) with at least two invariant curves, $p_1(t)x - 1 = 0$ and $p_2(t)x - 1 = 0$.

First, we will introduce some notation. Let $q(t) = \gcd(p_1(t), p_2(t))$, this is also well defined in the trigonometric case by Proposition 2.1(c), and let $s_1(t)$, $s_2(t)$ and $s(t)$ be the polynomials (resp. trigonometric polynomials) such that

$$p_1(t) = q(t)s_1(t), \quad p_2(t) = q(t)s_2(t), \quad A(t) = q(t)s_1(t)s_2(t)s(t). \tag{2.6}$$

Note that $\gcd(s_1(t), s_2(t)) = 1$.

Lemma 2.14. *Let $p_1(t)$ and $p_2(t)$ be two non-constant polynomials in t with coefficients in $\mathbb{K}$ (resp. two non-constant trigonometric polynomials with real coefficients) such that $p_i(t) \neq 0$, for every $t \in \mathbb{R}$ and $i \in \{1, 2\}$. With the previous notation, if*

$$\frac{(p_2(t) - p_1(t))' p_1(t) p_2(t)}{p_2(t) - p_1(t)}$$

is a polynomial (resp. a trigonometric), then $s_2(t) = s_1(t) + k\,\widehat{q}(t)$, where k is a nonzero constant and $\widehat{q}(t)$ is either constant or a product of not necessarily distinct irreducible monic factors of $q(t)$. As a consequence, if $p_1(t)x - 1 = 0$ and $p_2(t)x - 1 = 0$ are two invariant curves of (1.5), the claim must also be true. Furthermore,

$$s(t) = -q'(t) + q(t)\left(C(t) + \frac{\widehat{q}'(t)}{\widehat{q}(t)}\right).$$

Proof. By (2.6), we have

$$\frac{(p_2(t) - p_1(t))' p_1(t) p_2(t)}{p_2(t) - p_1(t)} = q(t)s_1(t)s_2(t)\left(q'(t) + q(t)\frac{(s_2(t) - s_1(t))'}{s_2(t) - s_1(t)}\right).$$

So, by hypothesis,

$$q(t)^2 \frac{(s_2(t) - s_1(t))'}{s_2(t) - s_1(t)}$$

is a polynomial, because there is no factor of $s_2(t) - s_1(t)$ dividing $s_1(t)s_2(t)$ as $\gcd(s_1(t), s_2(t)) = 1$.

Let us decompose $s_2(t) - s_1(t) = q_1(t)q_2(t)$ where all the factors of $q_2(t)$ divide $q(t)$ and no factor of $q_1(t)$ divides $q(t)$; in particular $\gcd(q_1(t), q_2(t)) = 1$. Then $q_1(t)$ divides $s_2'(t) - s_1'(t) = q_1'(t)q_2(t) + q_1(t)q_2'(t)$. So, it divides $q_1'(t)q_2(t)$ by Proposition 2.1. Therefore, $q_1(t)$ must be a nonzero constant. Now, by simply taking $k = q_1(t)$ and $\widehat{q}(t) = q_2(t)$, we are done.

On the other hand, as by (2.3),

$$A(t)/p_1(t) + p_1'(t) + p_1(t)C(t) = -B(t) = A(t)/p_2(t) + p_2'(t) + p_2(t)C(t),$$

then by (2.6),

$$\begin{aligned} &s_2(t)s(t) + q'(t)s_1(t) + q(t)s_1'(t) + q(t)s_1(t)C(t) \\ &\quad = s_1(t)s(t) + q'(t)s_2(t) + q(t)s_2'(t) + q(t)s_2(t)C(t) \end{aligned}$$

or, equivalently,

$$q(t)((s_2'(t) - s_1'(t)) + C(t)(s_2(t) - s_1(t))) = (s_2(t) - s_1(t))(s(t) + q'(t))$$

and

$$s(t) = -q'(t) + q(t)\left(C(t) + \frac{s_2'(t) - s_1'(t)}{s_2(t) - s_1(t)}\right).$$

As $s_2(t) - s_1(t) = k\,\widehat{q}(t)$, the previous expression reduces to

$$s(t) = -\left(q'(t) + q(t)\left(C(t) + \frac{\widehat{q}'(t)}{\widehat{q}(t)}\right)\right). \qquad \square$$

The converse also holds by direct verification using Corollary 2.3. Therefore, we obtain a parameterization of all cases of equation (1.5) having at least two invariant curves.

2.3 Bounds for the invariant curves in the polynomial case

In the polynomial case, it is interesting to look for a general bound on the number of invariant curves of (1.5). We recall that this is not interesting in the trigonometric case, since we can get a much better bound on the number of rational limit cycles using the Darboux theory of integrability (see Section 3). Thus, the rest of this section applies only to the polynomial case, and then we are in a Euclidean domain setting.

Given that if $p(t)x - 1 = 0$ is an invariant curve of (1.5), then $p(t)$ is a divisor of $A(t)$ in $\mathbb{C}[t]$, the number of divisors of $A(t)$ would be an upper bound for the number of such invariant curves, up to proportionality. To search for this bound, we must first study the invariant curves of the equation whose polynomials are proportional.

Proposition 2.15. *The curves $p(t)x - 1 = 0$, $Kp(t)x - 1 = 0$ with $K \in \mathbb{K} \setminus \{0, 1\}$ are invariant curves of (1.5) if and only if*

$$A(t) = Kp(t)(p'(t) + p(t)C(t)), \quad B(t) = -(K+1)(p'(t) + p(t)C(t)).$$

Proof. As $p(t)x - 1 = 0$ is an invariant curve of (1.5), then by Corollary 2.3 there must exist a polynomial $q(t)$ such that

$$A(t) = p(t)q(t), \quad -B(t) = p'(t) + q(t) + p(t)C(t).$$

Moreover, as $Kp(t)x - 1 = 0$ is also an invariant curve of (1.5), again by Corollary 2.3 there must also exist a polynomial $\bar{q}(t)$ such that

$$A(t) = Kp(t)\bar{q}(t), \quad -B(t) = Kp'(t) + \bar{q}(t) + Kp(t)C(t).$$

Thus, $q(t) = K\bar{q}(t)$ and, equating both expressions of $B(t)$, one gets that $\bar{q}(t) = p'(t) + p(t)C(t)$, and,

$$A(t) = Kp(t)(p'(t) + p(t)C(t)), \quad B(t) = -(K+1)(p'(t) + p(t)C(t)).$$

Conversely, taking $q(t) = K(p'(t) + p(t)C(t))$, one has that $A(t) = p(t)q(t)$, $B(t) = p'(t) + q(t) + p(t)C(t)$ and, consequently, that $p(t)x - 1 = 0$ is invariant. Picking $q(t) = (p'(t) + p(t)C(t))$, one would similarly prove that $Kp(t)x - 1 = 0$ is also invariant. $\square$

Proposition 2.16. *Equation (1.5) has at most two invariant curves whose polynomials are proportional.*

Proof. Because of Proposition 2.15, we know that if $p(t)x-1=0$ is an invariant curve of (1.5), then there must exists an unique $K \in \mathbb{C}\setminus\{0,1\}$ such that $Kp(t)x-1=0$, and

$$A(t) = Kp(t)(p'(t)+p(t)C(t)), \quad B(t) = -(K+1)(p'(t)+p(t)C(t)).$$

This, if $K \neq -1$ implies that

$$p(t) = -\frac{K+1}{K}\frac{A(t)}{B(t)}.$$

Now, if there exists a different pair of invariant curves with proportional polynomials, say $\bar{p}(t)x-1=0$, $\bar{K}\bar{p}(t)x-1=0$, then if $\bar{K}\neq -1$ also

$$\bar{p}(t) = -\frac{\bar{K}+1}{\bar{K}}\frac{A(t)}{B(t)} = \frac{\bar{K}+1}{\bar{K}}\frac{K}{K+1}p(t)$$

and we are done.

Finally, note that if $p(t)x-1=0$ and $-p(t)x-1=0$ are both invariant curves of (1.5), then $B(t)\equiv 0$ and we have excluded that case. □

Proposition 2.17. *If* $\deg(A)-\deg(C)\leq 1$, (1.5) *has at most one invariant curve. In another case, if* $\deg(A)-\deg(C)$ *is odd or* $\deg(A)+\deg(C)<2\deg(B)$, *then* (1.5) *has at most two invariant curves. In another case, an upper bound for the number of invariant curves of* (1.5) *is*

$$\binom{\deg(A)}{(\deg(A)+1)/2}+1.$$

Proof. It is enough to remember that the number of divisors of $A(t)$, up to proportionality, is $\binom{\deg(A)}{(\deg(A)+1)/2}$ and that at most two invariant curves have proportional polynomials. □

3 Darboux's theory of integrability

Now let us take advantage of Darboux's Integrability Theory to obtain results related to the integrability of the equation, in the polynomial case, and upper bounds on the number of rational limit cycles, in the trigonometric case.

We first recall the basic result of this theory from the original work of Darboux [11], adapted both to the polynomial and trigonometric settings.

Proposition 3.1. *If* (1.5) *has algebraic irreducible invariant curves* $h_1(t,x)=0,\dots,h_r(t,x)=0$ *with respective cofactors* $K_1(t,x),\dots,K_r(t,x)$ *and there exist not all null numbers* $\alpha_1,\dots,\alpha_r$ *such that*

$$\sum_{i=1}^{r}\alpha_i K_i(t,x)=0,$$

then

$$h(t,x) := \prod_{i=1}^{r} h_i(t,x)^{\alpha_i}$$

is a first integral of (1.5). *In this case, we say that* (1.5) *is Darboux integrable.*

Proposition 3.2. *If (1.5) has invariant curves $p_1(t)x-1=0,\dots,p_r(t)x-1=0$ with respective cofactors $K_1(t,x),\dots,K_r(t,x)$, and we write $K_0(t,x)=A(t)x^2+B(t)x+C(t)$ for the cofactor of the curve $x=0$, then there exist numbers $\alpha_0,\dots,\alpha_r$ not all zero such that*

$$\sum_{i=0}^{r}\alpha_i K_i(t,x)=0$$

if and only if

$$\sum_{i=1}^{r}\alpha_i A(t)/p_i(t)=0 \quad \textit{and} \quad 0=\alpha_0=\sum_{i=1}^{r}\alpha_i.$$

Proof. First, we recall that, by Corollary 2.3, the invariant curves $p_i(t)x-1=0$ have cofactors $K_i(t,x)=A(t)x^2-(p_i'(t)+p_i(t)C(t))x=A(t)x^2+(B(t)+A(t)/p_i(t))x$, $i=1,\dots,r$. Thus,

$$\begin{aligned}\sum_{i=0}^{r}\alpha_i K_i(t,x)&=\alpha_0K_0(t,x)+\sum_{i=1}^{r}\alpha_i K_i(t,x)\\&=\alpha_0\left(A(t)x^2+B(t)x+C(t)\right)+\sum_{i=1}^{r}\alpha_i(A(t)x^2+(B(t)+A(t)/p_i(t))x)\\&=\left(A(t)\sum_{i=0}^{r}\alpha_i\right)x^2+\left(\alpha_0B(t)+\sum_{i=1}^{r}\alpha_i(B(t)+A(t)/p_i(t))\right)x+\alpha_0C(t),\end{aligned}$$

and our claim follows. □

We can now prove the main result of the paper.

Proof of Theorem 1.1. The claims (a) and (b.1) have been proved in Corollaries 2.9 and Proposition 2.13, respectively. Moreover, the first part of claim (b.2.1) has been proven in Proposition 2.17.

Let us now prove the rest of claim (b.2.1). By Proposition 2.6 and Corollary 2.12, if (1.5) has $r\geq 3$ invariant curves, say, $p_i(t)x-1=0$, $i=1,\dots,r$, then $\deg(p_i)=(\deg(A)-\deg(C))/2$, for every $i\in\{1,\dots,r\}$. Moreover, by Corollary 2.3, $p_i(t)$ divides $A(t)$, for every $i\in\{1,\dots,r\}$. Therefore $A(t)/p_i(t)$ is a polynomial of degree $d:=(\deg(A)+\deg(C))/2$, for every $i\in\{1,\dots,r\}$.

Now, since the dimension of the $\mathbb{K}-$vector space of polynomials in t of degree up to d is equal to $d+1$, if $r\geq 2(d+2)$, then there exist $\lambda_1,\dots,\lambda_{d+2}$ and $\mu_{d+3},\dots,\mu_r$ such that

$$\sum_{i=1}^{d+2}\lambda_i A(t)/p_i(t)=\sum_{i=d+3}^{r}\mu_i A(t)/p_i(t)=0.$$

Thus, taking

$$\begin{aligned}\alpha_0&=0,\\\alpha_i&=\left(\sum_{j=d+3}^{r}\mu_j\right)\lambda_i,\ i=1,\dots,d+2,\\\alpha_i&=-\left(\sum_{j=1}^{d+2}\lambda_j\right)\mu_i,\ i=d+3,\dots,r,\end{aligned}$$

we have that $0=\alpha_0=\sum_{i=1}^{r}\alpha_i$ and that $\sum_{i=1}^{r}\alpha_i A(t)/p_i(t)=0$. So, by Proposition 3.2 and as $2(d+2)=\deg(A)+\deg(C)+4$, we are done.

We prove now (b.2.2). Following the same steps of the polynomial case, one gets that if equation (1.5) has $r \geq 3$ invariant curves, say $p_i(t)x - 1 = 0$, $i = 1, \dots, r$, then $A(t)/p_i(t)$ is a trigonometric polynomial of degree $\hat{d} := (\deg(A) + \deg(C))/2$, for every $i \in \{1, \dots, r\}$.

We can repeat the rest of the argument of the polynomial case verbatim, and in view of the dimension of the $\mathbb{R}$-vector space of trigonometric polynomials in t of the degree up to $\hat{d}$ is $2\hat{d} + 1$, we conclude that if $r \geq 2(2\hat{d} + 2)$, (1.5) has a Darboux first integral, and also a center. Thus, it can not have rational limit cycles. Consequently, a bound for the number of rational limit cycles is $2(2\hat{d} + 2) = 2\deg(A) + 2\deg(C) + 4$.

Finally, we prove (c). On the one hand, if $p_i(t)x - 1 = 0$, $i = 1, \dots, r$, are invariant curves of (1.5), we know that $\deg(A/p_i) \leq \deg(A) - 1$ for all $i \in \{1, \dots, r\}$, as $\deg(p_i) \geq 1$. Thus, all the polynomials belong to the $\mathbb{R}$-vector space of trigonometric polynomials in t of degree up to $\bar{d} = \deg(A) - 1$, whose dimension is $2\bar{d} + 1$. Thus, if $r \geq 2(2\bar{d} + 2)$, the equation would have a center.

However, on the other hand, the Abel equation

$$x' = A(t)x^3 + B(t)x^2 + cx$$

can never have a center, as its displacement function $d(x)$, whose isolated zeros correspond one to one with the limit cycles of the equation (see [27]), has derivative

$$d'(x) = \exp\left(\int_0^{2\pi} (3A(t)x^2 + 2B(t)x + c)dt\right) - 1.$$

So $d'(0) = \exp(2\pi c) - 1 \neq 0$, and $d(0) = 0$ because $x(t) \equiv 0$ is a solution of the equation. Therefore, $d(x) \not\equiv 0$ and the equation can never have a center.

To conclude, in this case equation (1.5) must have less that $2(2\bar{d} + 2) = 4\deg(A)$ invariant curves. □

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Prolongation of solutions and Lyapunov stability for Stieltjes dynamical systems

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Abstract. In this article, we present Lyapunov-type results to study the stability of an equilibrium of a Stieltjes dynamical system. We utilize prolongation results to establish the global existence of the maximal solution. Using Lyapunov's second method, we establish results of stability (resp. uniform stability) and asymptotic stability (resp. asymptotic uniform stability). Finally, we present examples and real-life applications to study asymptotic stability of equilibria in two population dynamics models.

Keywords: Stieltjes differential equations, Lyapunov stability, Lyapunov function, stable equilibrium, asymptotic stability, dynamical system.

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1 Introduction

In recent years, the field of differential equations has witnessed a notable surge in interest surrounding Stieltjes differential equations. This renewed focus is largely driven by the pursuit of results that not only unify existing findings but also extend those related to classical derivatives [8, 9, 18, 19, 24, 27–30] through the Stieltjes derivative.

Unlike the classical derivative, the Stieltjes derivative consists of differentiating with respect to a derivator $g : \mathbb{R} \to \mathbb{R}$, assumed to be left-continuous and nondecreasing. Stieltjes differential equations permit to obtain a broader range of applications, particularly in contexts where certain processes may exhibit discontinuities and/or stationary periods [1, 8, 9, 18–21, 24, 25]. Such situations are common in various fields, including population dynamics, and physics, where classical differentiation has limitations in capturing the complexities of real-world phenomena.

Typically, investigations into first-order Stieltjes differential equations and systems focus on solutions defined on bounded intervals. However, Larivière in [17, Chapter 4] turned his attention to Stieltjes differential equations on the positive real half-line. In doing so, he provided results related to the prolongation of solutions and the existence of the maximal

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solution. The motivation behind exploring these equations on the positive real half-line lies in the observation that many natural processes evolve over time without any inherent time limit, while some phenomena can exhibit finite-time blow-up, leading to abrupt changes or singularities. By considering Stieltjes differential equations, we aim to capture these nuanced behaviors and enhance our understanding of more complex dynamical systems from a unified perspective [22]. The reader is also referred to [5] for interesting results on prolongation of solutions and the existence of maximal solutions to generalized ordinary equations on Banach spaces relying on Kurzweil integration.

In the study of dynamical systems, the stability of *equilibria* holds significant importance. Here, the term *equilibrium* refers to a state that does not change dynamically, in the sense that if a system starts at an equilibrium point, it will stay in that state indefinitely. In order to maintain the interactions between species within many ecosystems, stability may be suitable for their functioning, and resilience. Although stability is typically desirable, there are some scenarios where stable equilibrium at zero can be critical and raise concerns for several reasons. For instance, in ecological systems, this concern arises from the vulnerability to perturbations of certain species, which may increase the risk of their extinction.

In this paper, we extend Lyapunov stability results from the classical literature to the Stieltjes dynamical system:

$$\mathbf{x}'_g(t) = \mathbf{f}(t, \mathbf{x}(t)) \quad \text{for } g\text{-almost all } t \geq 0, \tag{1.1}$$

where $\mathbf{f} = (f_1, \dots, f_n) : [0, +\infty) \times B \to \mathbb{R}^n$, $B \subset \mathbb{R}^n$.

Within the realm of dynamical systems theory, Lyapunov's Second Method [23] stands as a fundamental approach for assessing the stability properties of a system near an equilibrium. This stability analysis provides insights into whether small perturbations around an equilibrium lead to convergence or divergence of solutions. The core of this method lies in the concept of the *Lyapunov function* V depending on time and state. This function can be understood as an energy representation of the system (1.1), since in numerous applications, the function considered is the total energy of the system (1.1) through time, see for instance [3] for an example of an energy-based Lyapunov function for physical systems. This function was the subject of numerous works in the classical literature starting from the works [12,13,15,16,23,35] and references therein. In our context, the derivator g takes into account the relevance of each moment during the process by means of the changes of the slopes of g accordingly. Put differently, g amplifies an alternative measurement for time, which may differ from the linear time line typically used in the classical case where $g \equiv \mathrm{id}_{\mathbb{R}}$, see for instance the works [18–21,24,25] where g represents the life cycle of some populations, also we refer to [1,9] for more applications. Nevertheless, in the context of Stieltjes differentiation, we still can rely on Lyapunov's function which will then permit a better understanding of how the energy of the system (1.1) changes, but with respect to this new observed time described by g. Our stability study is inspired by results from classical theory and works such as [14, 16, 33, 34], which address dynamic equations on time scales and impulsive differential equations. To the best of our knowledge, this is the first work to introduce Lyapunov's method adapted to Stieltjes differential equations.

This paper is organized as follows: we present the theoretical framework and some preliminaries in Section 2. In Section 3, we focus on prolongation of solutions and the characterization of the maximal solution. Then, based on a generalized version of the Grönwall lemma [11, 17] for the Stieltjes derivative, we establish the existence of global solutions over the whole positive real half-line. Section 4 is devoted to Lyapunov-like stability results using

Lyapunov's second method. We establish stability results inspired by the works [6, 10], to extend some classical results from [14, 16].

In the last section of this paper, we present two applications to dynamics of population to study the asymptotic stability of some critical equilibria. In the first application, we model the dynamics of a population with Allee's effect [2, 31] negatively impacted by train vibrations. The second application concerns the dynamics of a population of *Cyanobacteria* in a cultured environment, keeping track of ammonia levels in the process.

2 Preliminaries

For $[a,b] \subset \mathbb{R}$, and $\mathbf{u} : [a,b] \to \mathbb{R}^n$ a regulated function, the symbols $\mathbf{u}(t^+)$ and $\mathbf{u}(t^-)$ will be used to denote

$$\mathbf{u}(t^+) = \lim_{\varepsilon \to 0^+} \mathbf{u}(t+\varepsilon), \text{ for all } t \in [a,b),$$
$$\mathbf{u}(t^-) = \lim_{\varepsilon \to 0^-} \mathbf{u}(t-\varepsilon), \text{ for all } t \in (a,b].$$

Throughout this work, we will consider $g : \mathbb{R} \to \mathbb{R}$ a monotone, nondecreasing and left-continuous function, also known as a *derivator*. We denote the set of discontinuity points of g by

$$D_g = \{t \in \mathbb{R} : g(t^+) - g(t) > 0\}.$$

In addition, we denote

$$C_g = \{s \in \mathbb{R} : g \text{ is constant on } (s-\varepsilon, s+\varepsilon) \text{ for some } \varepsilon > 0\}.$$

The set C_g is an open in the usual topology and it can be written as a countable union of disjoint intervals

$$C_g = \bigcup_{n \in \Lambda} (a_n, b_n),$$

with $\Lambda \subset \mathbb{N}$, $a_n, b_n \in \mathbb{R}$. We set $N_g = \{a_n, b_n : n \in \Lambda\} \setminus D_g$.

The function g defines a Lebesgue–Stieltjes measure μ_g such that $\mu_g([a,b)) = g(b) - g(a)$ for any interval $[a,b)$, so that $\mu_g(\{t\}) = g(t^+) - g(t)$ for all $t \in \mathbb{R}$, and $\mu_g(C_g) = \mu_g(N_g) = 0$. The reader is referred to [22] for more details. We refer to the measurability with respect to μ_g by g-measurability. We denote by $\mathcal{L}^1_g([a,b),\mathbb{R})$ the space of μ_g-integrable real-valued functions on $[a,b)$ endowed with the norm

$$\|f\|_{\mathcal{L}^1_g([a,b),\mathbb{R})} := \int_{[a,b)} |f(t)|\, d\mu_g(t), \quad \text{for every } f \in \mathcal{L}^1_g([a,b),\mathbb{R}).$$

Given an interval $I \subset \mathbb{R}$, we set

$$\mathcal{L}^1_g(I,\mathbb{R}^n) := \prod_{i=1}^n \mathcal{L}^1_g(I,\mathbb{R}),$$

and

$$\mathcal{L}^1_{g,\mathrm{loc}}(I,\mathbb{R}^n) = \{\mathbf{u} : I \to \mathbb{R}^n : \mathbf{u}|_{[a,b]} \in \mathcal{L}^1_g([a,b),\mathbb{R}^n) \text{ for every } [a,b] \subset I\}.$$

The derivator g defines a pseudometric $\rho : \mathbb{R} \times \mathbb{R} \to [0,\infty)$ given by

$$\rho(s,t) = |g(s) - g(t)|, \quad \text{for every } s,t \in \mathbb{R}.$$

We denote τ_g the topology induced by the pseudometric ρ. Notice that the interval $(t-\varepsilon, t]$ is open in τ_g for all $t \in D_g$ and all $\varepsilon > 0$. The reader is referred to [9, Section 2] for more properties of the topology τ_g.

Throughout this paper, let $\|\cdot\|$ denotes the maximum norm in $\mathbb{R}^n$ defined by

$$\|\mathbf{x}\| = \max\{|x_1|, \dots, |x_n|\} \quad \text{for } \mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n,$$

and $B_{\mathbb{R}^n}(\mathbf{x}, \delta)$ denotes the open ball centered in $\mathbf{x}$ of radius $\delta > 0$.

Now, we recall the notion of g-continuity introduced in [9].

Definition 2.1. Let $\mathbf{u} : [a, b] \to \mathbb{R}^n$. We say that $\mathbf{u}$ is *g-continuous* at $t \in [a, b]$ if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\forall s \in [a, b], \qquad |g(s) - g(t)| < \delta \implies \|\mathbf{u}(s) - \mathbf{u}(t)\| < \varepsilon.$$

The following proposition relates the regularity of f and g, the reader is referred to [9, Proposition 3.2].

Proposition 2.2. *If $\mathbf{u} : [a, b] \to \mathbb{R}^n$ is g-continuous on $[a, b]$, then the following statements hold:*

(1) *$\mathbf{u}$ is left-continuous at every $t \in (a, b]$.*

(2) *If g is continuous at $t \in [a, b)$, then so is $\mathbf{u}$.*

(3) *If g is constant on some $[c, d] \subset [a, b]$, then so is $\mathbf{u}$.*

Let $\mathcal{BC}_g([a, b], \mathbb{R}^n)$ denotes the Banach space of the bounded, g-continuous functions defined on $[a, b]$ with values in $\mathbb{R}^n$, endowed with the supremum norm.

Now, we define the g-derivative of a real-valued function.

Definition 2.3. Let $u : [a, b] \to \mathbb{R}$ be a function. The *derivative of u with respect to g at a point $t \in [a, b] \setminus C_g$* is defined by:

$$u'_g(t) = \begin{cases} \lim\limits_{s \to t} \dfrac{u(s) - u(t)}{g(s) - g(t)} & \text{if } t \notin D_g, \\ \dfrac{u(t^+) - u(t)}{g(t^+) - g(t)} & \text{if } t \in D_g, \end{cases}$$

provided that the limit exists. In this case, u is said to be *g-differentiable at t* and $u'_g(t)$ is also called the *g-derivative of u at t*.

In the next proposition, we recall the g-derivative of the composition of two functions established in [26, Proposition 3.15]. Another version of this formula can be found in [7, Proposition 4.1].

Proposition 2.4. *Let $t \in \mathbb{R} \setminus C_g$, $f : \mathbb{R} \to \mathbb{R}$ and h a real function defined on a neighborhood of $f(t)$. We assume that there exist $h'(f(t))$, $f'_g(t)$ and that the function h is continuous at $f(t^+)$. Then, the composition $h \circ f$ is g-differentiable in t and*

$$(h \circ f)'_g(t) = \begin{cases} h'(f(t)) f'_g(t) & \text{if } t \notin D_g, \\ \dfrac{h(f(t^+)) - h(f(t))}{f(t^+) - f(t)} f'_g(t) & \text{if } t \in D_g. \end{cases}$$

In the particular case where the derivator $g : \mathbb{R} \to \mathbb{R}$ is increasing and continuous on an interval $[a, b] \subset \mathbb{R}$, we obtain easily a generalized version of the Mean Value theorem in the context of Stieltjes differentiation.

Theorem 2.5. *Let $g : \mathbb{R} \to \mathbb{R}$ be a left-continuous and nondecreasing function, continuous and increasing on an interval $[a, b] \subset \mathbb{R}$. Let $f : [a, b] \to \mathbb{R}$ be g-continuous on $[a, b]$ and g-differentiable on (a, b). Then, there exists $c \in (a, b)$ such that $f'_g(c) = \frac{f(b)-f(a)}{g(b)-g(a)}$.*

Proof. Let us consider the function $F : [a, b] \to \mathbb{R}$ defined by

$$F(t) = f(t) - \left(\frac{f(b) - f(a)}{g(b) - g(a)} (g(t) - g(a)) + f(a) \right), \quad \text{for all } t \in [a, b].$$

Clearly F is g-continuous on $[a, b]$, g-differentiable on (a, b), and

$$F'_g(t) = f'_g(t) - \frac{f(b) - f(a)}{g(b) - g(a)} \quad \text{for all } t \in (a, b).$$

It satisfies also $F(a) = F(b)$.

Proposition 2.2 implies that F is continuous on $[a, b]$. We set

$$m = \min_{t \in [a,b]} F(t) \quad \text{and} \quad M = \max_{t \in [a,b]} F(t).$$

If $m = M$, then F is constant on $[a, b]$ and $F'_g(t) = 0$ for all $t \in (a, b)$. Otherwise if $m < M$, we have $F(a) \neq m$ or $F(a) \neq M$. As $F(a) = F(b)$, without loss of generality, we assume that $F(a) = F(b) \neq M$. Thus, there exists $c \in (a, b)$ such that $F(c) = M$. Therefore, there exists $\delta > 0$ such that $(c - \delta, c + \delta) \subset (a, b)$ and $F(s) \leq F(c) = M$ for all $s \in (c - \delta, c + \delta)$. Since g is increasing and continuous on (a, b),

$$0 \leq F'_g(c) = \lim_{s \to c^+} \frac{F(s) - F(c)}{g(s) - g(c)} = \lim_{s \to c^-} \frac{F(s) - F(c)}{g(s) - g(c)} \leq 0.$$

We deduce that $F'_g(c) = 0$. Hence, there exists $c \in (a, b)$ such that $f'_g(c) = \frac{f(b)-f(a)}{g(b)-g(a)}$. □

Now, we recall the notion of g-absolute continuity.

Definition 2.6. A map $F : [a, b] \to \mathbb{R}$ is *g-absolutely continuous*, if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that, for any family $\{(a_i, b_i)\}_{i=1}^{i=n}$ of pairwise disjoint open subintervals of $[a, b]$,

$$\sum_{i=1}^{n} g(b_i) - g(a_i) < \delta \Rightarrow \sum_{i=1}^{n} |F(b_i) - F(a_i)| < \varepsilon.$$

We denote by $\mathcal{AC}_g([a, b], \mathbb{R})$ the space of g-absolutely continuous functions $F : [a, b] \to \mathbb{R}$ on the interval $[a, b]$ and

$$\mathcal{AC}_g([a, b], \mathbf{B}) = \{\mathbf{u} = (u_1, \ldots, u_n) : [a, b] \to \mathbf{B} \subset \mathbb{R}^n : u_i \in \mathcal{AC}_g([a, b], \mathbb{R}) \quad \text{for } i = 1, \ldots, n\}.$$

In [22, Theorem 5.4], a Fundamental Theorem of Calculus for Lebesgue–Stieltjes integrals was established.

Theorem 2.7 (Fundamental Theorem of Calculus for the Lebesgue–Stieltjes integral)**.** *Let $a, b \in \mathbb{R}$ be such that $a < b$, and let $F : [a, b] \to \mathbb{R}$. The following assumptions are equivalent.*

(1) *The function F is g-absolutely continuous,*

(2) *The function F satisfies the following conditions:*

(a) *there exists $F'_g(t)$ for g-almost all $t \in [a,b)$;*

(b) $F'_g \in \mathcal{L}^1_g([a,b),\mathbb{R})$;

(c) *for each $t \in [a,b]$, we have*

$$F(t) = F(a) + \int_{[a,t)} F'_g(s)\, d\mu_g(s).$$

The following lemma provides conditions ensuring the g-absolute continuity of the composition of two functions, the proof is based on arguments as in [9, Proposition 5.3].

Lemma 2.8. *Let $\mathbf{u} : [a,b] \to B \subset \mathbb{R}^n$ be a g-absolutely continuous function, and let $v : B \to \mathbb{R}$ be a Lipschitz continuous function on B. Then, the composition $v \circ \mathbf{u} \in \mathcal{AC}_g([a,b],\mathbb{R})$.*

Now, we define the partial Stieltjes derivative as follows.

Definition 2.9. Given a function $V : [a,b] \times B_{\mathbb{R}^n}(\mathbf{0}, r_0) \to \mathbb{R}$. The *partial g-derivative of V with respect to the first argument* at a point $(t,\mathbf{x}) \in ([a,b) \setminus C_g) \times B_{\mathbb{R}^n}(\mathbf{0}, r_0)$ is defined as:

$$\frac{\partial V}{\partial_g t}(t,\mathbf{x}) = \begin{cases} \lim\limits_{s \to t} \frac{V(s,\mathbf{x}) - V(t,\mathbf{x})}{g(s) - g(t)}, & \text{if } t \in [a,b] \setminus D_g, \\ \dfrac{V(t^+,\mathbf{x}) - V(t,\mathbf{x})}{g(t^+) - g(t)}, & \text{if } t \in [a,b] \cap D_g, \end{cases}$$

provided that the limit exists.

The following proposition gives a formula related to the g-derivative of the composition involving a function with two variables. Formulae of this fashion were stated without proof in [29, Lemma 11] for $t \notin D_g$. Here, we derive formulae in the case where D_g and N_g do not have accumulation points.

Proposition 2.10. *Let $g : \mathbb{R} \to \mathbb{R}$ be a left-continuous and nondecreasing function such that D_g and N_g do not have accumulation points. Given a function $V : [a,b] \times B_{\mathbb{R}^n}(\mathbf{0}, r_0) \to \mathbb{R}$ satisfying the following assumptions:*

(1) *$V(\cdot,\mathbf{u})$ is g-differentiable on $[a,b] \setminus (D_g \cup C_g)$ for all $\mathbf{u} \in B_{\mathbb{R}^n}(\mathbf{0}, r_0)$;*

(2) *$V(t,\cdot) \in C^1(B_{\mathbb{R}^n}(\mathbf{0}, r_0),\mathbb{R})$ for all $t \in [a,b]$;*

(3) *$\frac{\partial V}{\partial_g t}(\cdot,\mathbf{u})$ is continuous on $[a,b] \setminus (D_g \cup C_g)$ for all $\mathbf{u} \in B_{\mathbb{R}^n}(\mathbf{0}, r_0)$;*

(4) *$V(\cdot,\mathbf{x}(\cdot)) \in \mathcal{AC}_g([a,b],\mathbb{R})$ for every $\mathbf{x} \in \mathcal{AC}_g\big([a,b], B_{\mathbb{R}^n}(\mathbf{0}, r_0)\big)$.*

Then, for every $\mathbf{x} \in \mathcal{AC}_g\big([a,b], B_{\mathbb{R}^n}(\mathbf{0}, r_0)\big)$ and for g-almost all $t \in [a,b] \setminus (D_g \cup C_g)$, one has that

$$V'_g(t,\mathbf{x}(t)) = \frac{\partial V}{\partial_g t}(t,\mathbf{x}(t)) + \sum_{i=1}^{n} \frac{\partial V}{\partial x_i}(t,\mathbf{x}(t))(x_i)'_g(t). \tag{2.1}$$

Moreover, if $t \in [a,b) \cap D_g$, then

$$V'_g(t,\mathbf{x}(t)) = \frac{V\big(t^+, \mathbf{x}(t) + \mu_g(\{t\})\mathbf{x}'_g(t)\big) - V(t,\mathbf{x}(t))}{g(t^+) - g(t)}. \tag{2.2}$$

Proof. Let $\mathbf{x} \in \mathcal{AC}_g\big([a,b], B_{\mathbb{R}^n}(\mathbf{0}, r_0)\big)$. By (4), $V(\cdot, \mathbf{x}(\cdot)) \in \mathcal{AC}_g([a,b], \mathbb{R})$. For g-almost every $t \in [a,b] \setminus (D_g \cup C_g)$:

$$\begin{aligned} V_g'(t, \mathbf{x}(t)) &= \lim_{s \to t} \frac{V(s, \mathbf{x}(s)) - V(t, \mathbf{x}(t))}{g(s) - g(t)} \\ &= \lim_{s \to t} \frac{V(s, \mathbf{x}(s)) - V(t, \mathbf{x}(s))}{g(s) - g(t)} + \frac{V(t, \mathbf{x}(s)) - V(t, \mathbf{x}(t))}{g(s) - g(t)} \\ &= \lim_{s \to t} \frac{V(s, \mathbf{x}(s)) - V(t, \mathbf{x}(s))}{g(s) - g(t)} \\ &\quad + \sum_{i=1}^{n} \Bigg(\frac{V(t, (x_1(t), \dots, x_{i-1}(t), x_i(s), \dots, x_n(s)))}{g(s) - g(t)} \\ &\qquad - \frac{V(t, (x_1(t), \dots, x_i(t), x_{i+1}(s), \dots, x_n(s)))}{g(s) - g(t)} \Bigg). \end{aligned}$$

For s sufficiently close to t, and since D_g and N_g do not have accumulation points, g is continuous and increasing on the interval with endpoint points s and t. By applying Theorem 2.5 to the function $V(\cdot, \mathbf{x}(s))$, we obtain that there exists c between s and t such that

$$\frac{V(s, \mathbf{x}(s)) - V(t, \mathbf{x}(s))}{g(s) - g(t)} = \frac{\partial V}{\partial_g t}(c, \mathbf{x}(s)).$$

As $s \to t$, $c \to t$, and using Condition (3) we obtain

$$\frac{V(s, \mathbf{x}(s)) - V(t, \mathbf{x}(s))}{g(s) - g(t)} = \frac{\partial V}{\partial_g t}(c, \mathbf{x}(s)) \to \frac{\partial V}{\partial_g t}(t, \mathbf{x}(t)).$$

Therefore,

$$V_g'(t, \mathbf{x}(t)) = \frac{\partial V}{\partial_g t}(t, \mathbf{x}(t)) + \sum_{i=1}^{n} \frac{\partial V}{\partial x_i}(t, \mathbf{x}(t)) (x_i)_g'(t).$$

For $t \in [a,b) \cap D_g$, we obtain immediately that

$$\begin{aligned} V_g'(t, \mathbf{x}(t)) &= \frac{V(t^+, \mathbf{x}(t^+)) - V(t, \mathbf{x}(t))}{g(t^+) - g(t)} \\ &= \frac{V\big(t^+, \mathbf{x}(t) + \mu_g(\{t\}) \mathbf{x}_g'(t)\big) - V(t, \mathbf{x}(t))}{g(t^+) - g(t)}. \end{aligned}$$

□

In [9, Definition 6.1], an exponential function was introduced.

Definition 2.11. Let $p \in \mathcal{L}_g^1([a,b), \mathbb{R})$ be such that

$$1 + p(t)\big(g(t^+) - g(t)\big) > 0 \text{ for every } t \in [a,b) \cap D_g. \tag{2.3}$$

The *exponential function* $e_p(\cdot, a) : [a,b] \to (0, \infty)$ is defined by

$$e_p(t,a) = e^{\int_{[a,t)} \tilde{p}(s)\, d\mu_g(s)} \quad \text{for every } t \in [a,b],$$

where

$$\tilde{p}(t) = \begin{cases} p(t) & \text{if } t \in [a,b] \setminus D_g, \\ \dfrac{\log\big(1 + p(t)(g(t^+) - g(t))\big)}{g(t^+) - g(t)} & \text{if } t \in [a,b) \cap D_g. \end{cases} \tag{2.4}$$

In particular, given $p \in \mathcal{L}^1_g([a,b),\mathbb{R})$ a function satisfying Condition (2.3), then $\tilde{p} \in \mathcal{L}^1_g([a,b),\mathbb{R})$ and $e_p(\cdot,a) \in \mathcal{AC}_g([a,b),\mathbb{R})$. The reader is referred to [9, Lemmas 6.2 and 6.3] and [27, Theorem 3.2] for more details.

Now, we recall the generalization of the Grönwall Lemma to the Stieltjes derivative obtained by Larivière in [17, Proposition 4.1.4], and further generalized in [11, Theorem 5.4]. This lemma will play a crucial role in establishing global solutions defined on the positive real half-line as we shall prove in the following section.

Lemma 2.12. *Let $u \in \mathcal{AC}_g([a,b],\mathbb{R})$. Assume that there exist functions $k, p \in \mathcal{L}^1_g([a,b),\mathbb{R})$, satisfying $1 + p(t)\mu_g(\{t\}) > 0$ for all $t \in [a,b) \cap D_g$, such that*

$$u'_g(t) \le k(t) + p(t)u(t) \quad \text{for } g\text{-almost all } t \in [a,b).$$

Then,

$$u(t) \le e_p(t,a)\left(\int_{[a,t)} \frac{e_p^{-1}(s,a)k(s)}{1+p(s)\mu_g(\{s\})}\, d\mu_g(s) + u(a) \right), \quad t \in [a,b].$$

3 Prolongation of solutions and maximal interval of existence

Let O be a nonempty open set of $\mathbb{R}^n$ and $I \subset \mathbb{R}$ an open interval in the topology τ_g. We set $\Omega := I \times O$. Let $(t_0, \mathbf{x}_0) \in \Omega$ be such that $t_0 < \sup I$.

Let us consider the Stieltjes dynamical system:

$$\begin{aligned} &\mathbf{x}'_g(t) = \mathbf{f}(t, \mathbf{x}(t)) \quad \text{for } g\text{-almost all } t \ge t_0,\ t \in I, \\ &\mathbf{x}(t_0) = \mathbf{x}_0, \end{aligned} \tag{3.1}$$

where $\mathbf{f} = (f_1, \dots, f_n) : \Omega \cap \big([t_0,\infty) \times \mathbb{R}^n\big) \to \mathbb{R}^n$ satisfies the following assumptions:

($\mathrm{H}_{\mathbf{x}_0,t_0}$) $\big(t_0, \mathbf{x}_0 + \mu_g(\{t_0\})\mathbf{f}(t_0,\mathbf{x}_0)\big) \in \Omega$.

($\mathrm{H}_{\mathbf{f},t_0}$)
- (a) for all $\mathbf{u} \in O$, $\mathbf{f}(\cdot,\mathbf{u})$ is g-measurable;
- (b) $\mathbf{f}(\cdot,\mathbf{u}_0) \in \mathcal{L}^1_{g,\mathrm{loc}}([t_0,\infty),\mathbb{R}^n)$ for some $\mathbf{u}_0 \in O$;
- (c) $\mathbf{f}$ is *g-integrally locally Lipschitz continuous*, i.e. for every $r > 0$, there exists a function $L_r \in \mathcal{L}^1_{g,\mathrm{loc}}([t_0,\infty),[0,\infty))$ such that

$$\|\mathbf{f}(t,\mathbf{u}) - \mathbf{f}(t,\mathbf{v})\| \le L_r(t)\|\mathbf{u}-\mathbf{v}\|,$$

for g-almost all $t \in I \cap [t_0,\infty)$ and all $\mathbf{u},\mathbf{v} \in \overline{B_{\mathbb{R}^n}(\mathbf{x}_0,r)} \cap O$.

(H_{Ω,t_0}) For every $(t,\mathbf{u}) \in \Omega$ with $t \ge t_0$,

- (a) one of the following conditions hold:
  - (a1) there exists $\delta > 0$ such that $(t-\delta, t+\delta) \times B_{\mathbb{R}^n}(\mathbf{u},\delta) \subset \Omega$;
  - (a2) if for every $\delta > 0$, $(t-\delta,t+\delta) \times B_{\mathbb{R}^n}(\mathbf{u},\delta) \not\subset \Omega$, then $t \in D_g$ and there exists $\varepsilon > 0$ such that $(t-\varepsilon, t] \times B_{\mathbb{R}^n}(\mathbf{u},\delta) \subset \Omega$;
- (b) if $(t,\mathbf{u}) \in \Omega \cap (D_g \times \mathbb{R}^n)$ is such that $(t,\mathbf{u}^+_{\mathbf{f},t}) \in \Omega$, then $(t,\mathbf{u}^+_{\mathbf{f},t})$ satisfies Condition (H_{Ω,t_0})(a)(a1), where

$$\mathbf{u}^+_{\mathbf{f},t} := \mathbf{u} + \mu_g(\{t\})\mathbf{f}(t,\mathbf{u}).$$

We recall the local existence result established in [9, Theorem 7.4].

Theorem 3.1. *Assume that* $(\mathrm{H}_{\mathbf{x}_0,t_0})$ *and* $(\mathrm{H}_{\mathbf{f},t_0})$ *holds. Then, there exists* $\delta > 0$ *such that the system* (3.1) *has a unique solution* $\mathbf{x} \in \mathcal{AC}_g([t_0, t_0+\delta], \mathbb{R}^n)$.

It should be noticed that Condition $(\mathrm{H}_{\mathbf{x}_0,t_0})$ permits to consider the problem (3.1) even in the case where $t_0 \in D_g$.

In the sequel, as shown in [17, Section 4.2], the solution given by Theorem 3.1 can be extended up to a maximal interval of existence if Condition $(\mathrm{H}_{\Omega,t_0})$ is satisfied. Indeed, this condition ensures that the solution $\mathbf{x}$ can be extended on some larger interval $[t_0, t_0+\varepsilon]$, $\varepsilon > \delta$ if

$$t_0 + \delta \in D_g \quad \text{and} \quad \big(\mathbf{x}(t_0+\delta)\big)^+_{\mathbf{f},t_0+\delta} \in O.$$

Let us define the set

$$\begin{aligned}\mathcal{S}(t_0,\mathbf{x}_0) := \{\mathbf{x} : I_{\mathbf{x}} = J_{\mathbf{x}} \cap [t_0,\infty) \to \mathbb{R}^n : \mathbf{x} \text{ is a solution of (3.1)},\\ J_{\mathbf{x}} \text{ is an open interval of } \tau_g \text{ such that } \sup J_{\mathbf{x}} > t_0 \in J_{\mathbf{x}}\}.\end{aligned}$$

For $\mathbf{x} \in \mathcal{S}(t_0,\mathbf{x}_0)$, it is worth mentioning that $\mathbf{x} \in \mathcal{AC}_g([a,b],\mathbb{R}^n)$ for every interval $[a,b] \subset I_{\mathbf{x}}$. In the sequel, for $\mathbf{x} \in \mathcal{S}(t_0,\mathbf{x}_0)$, we denote $\bar{t}_{\mathbf{x}} := \sup I_{\mathbf{x}}$.

Definition 3.2. Let $\mathbf{x}, \mathbf{y} \in \mathcal{S}(t_0,\mathbf{x}_0)$.

(1) We say that $\mathbf{x}$ is *smaller* than $\mathbf{y}$ (and we denote $\mathbf{x} \prec \mathbf{y}$), if and only if

(i) $I_{\mathbf{x}} \subset I_{\mathbf{y}}$;

(ii) $\sup I_{\mathbf{y}} > \sup I_{\mathbf{x}}$;

(iii) $\mathbf{y}|_{I_{\mathbf{x}}} = \mathbf{x}$.

In this case, we say that $\mathbf{x}$ is *extendible to the right* and $\mathbf{y}$ is *a prolongation to the right* of $\mathbf{x}$.

(2) We write that $\mathbf{x} \preceq \mathbf{y} \iff \mathbf{x} \prec \mathbf{y}$ or $\mathbf{x} = \mathbf{y}$.

Remark 3.3. It is worth mentioning that given a solution $\mathbf{x} : [t_0, t_1) \to \mathbb{R}^n$ of (3.1) such that

$$\mathbf{x}(t_1^-) := \lim_{t \to t_1^-} \mathbf{x}(t) \in O \quad \text{and} \quad t_1 \in D_g,$$

to not increase the notation, we will replace $\mathbf{x}$ by the function $\mathbf{x} : [t_0, t_1] \to \mathbb{R}^n$ defined by

$$\mathbf{x} = \begin{cases} \mathbf{x}(t) & \text{if } t \in [t_0, t_1), \\ \mathbf{x}(t_1^-) & \text{if } t = t_1. \end{cases}$$

The next result asserts that two prolongations to the right of a solution are equal on the common interval of existence, the reader is referred to [17, Theorem 4.2.3] for the proof.

Theorem 3.4. *Assume that* $(\mathrm{H}_{\mathbf{x}_0,t_0})$ *and* $(\mathrm{H}_{\mathbf{f},t_0})$ *hold. Let* $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathcal{S}(t_0,\mathbf{x}_0)$ *be such that* $\mathbf{y}$ *and* $\mathbf{z}$ *are two prolongations of* $\mathbf{x}$*, then* $\mathbf{y} = \mathbf{z}$ *on* $I_{\mathbf{y}} \cap I_{\mathbf{z}}$.

In [17, Theorem 4.2.4], extendible solutions to the right were characterized as follows.

Theorem 3.5. *Assume that* $(\mathrm{H}_{\mathbf{x}_0,t_0})$ *and* $(\mathrm{H}_{\mathbf{f},t_0})$ *hold. Let* $\mathbf{x} \in \mathcal{S}(t_0,\mathbf{x}_0)$. *The following assumptions are equivalent:*

(1) $\mathbf{x}$ *is extendible to the right;*

(2) (i) $Graph(\mathbf{x}) := \{(t, \mathbf{x}(t)) : t \in I_{\mathbf{x}}\}$ *is bounded;*

(ii) $A \cup A^+ \subset \Omega$ *where*

$$A = \{(\bar{t}_{\mathbf{x}}, \mathbf{u}) \in [t_0, \infty) \times \mathbb{R}^n : \exists \{t_n\}_n \subset I_{\mathbf{x}},\ t_n \nearrow \bar{t}_{\mathbf{x}} \text{ and } \mathbf{u} = \lim_{n\to\infty} \mathbf{x}(t_n)\},$$
$$A^+ = \{(\bar{t}_{\mathbf{x}}, \mathbf{u}^+_{\mathbf{f},\bar{t}_{\mathbf{x}}}) : (\bar{t}_{\mathbf{x}}, \mathbf{u}) \in A\}.$$

Definition 3.6. Let $\mathbf{x} \in \mathcal{S}(t_0, \mathbf{x}_0)$. We say that $\mathbf{x}$ is a *maximal solution* of (3.1) defined on an interval $I_{\mathbf{x}}$ if, for every $\mathbf{y} \in \mathcal{S}(t_0, \mathbf{x}_0)$ satisfying $\mathbf{x} \preceq \mathbf{y}$, we have $\mathbf{x} = \mathbf{y}$. $I_{\mathbf{x}}$ is referred to as the *maximal interval of existence*.

As shown in [17, Theorem 4.2.5], the existence of the maximal solution holds. In addition, Theorem 3.4 guarantees the uniqueness of the maximal solution, and we obtain the following theorem.

Theorem 3.7. *Assume that* $(H_{\mathbf{x}_0,t_0})$, $(H_{\mathbf{f},t_0})$ *and* (H_{Ω,t_0}) *hold. Then, there exists a unique maximal solution* $\mathbf{x} \in \mathcal{S}(t_0, \mathbf{x}_0)$ *such that* $\omega(t_0, \mathbf{x}_0) := \sup I_{\mathbf{x}} \leq \infty$.

The next theorem highlights three alternative cases that occur, the reader is referred to [17, Theorem 4.2.6] for the proof.

Theorem 3.8. *Assume that* $(H_{\mathbf{x}_0,t_0})$, $(H_{\mathbf{f},t_0})$ *and* (H_{Ω,t_0}) *hold. Let* $\mathbf{x} \in \mathcal{S}(t_0, \mathbf{x}_0)$ *be the maximal solution of* (3.1)*, then one of the alternatives holds:*

(A1) $\omega(t_0, \mathbf{x}_0) = \infty$;

(A2) $\omega(t_0, \mathbf{x}_0) < \infty$, *and for every* $\{t_n\}_n \subset I_{\mathbf{x}}$ *such that* $t_n \nearrow \omega(t_0, \mathbf{x}_0)$, $\{\mathbf{x}(t_n)\}_n$ *is not bounded;*

(A3) $\omega(t_0, \mathbf{x}_0) < \infty$, *and there exists* $\{t_n\}_n \subset I_{\mathbf{x}}$ *satisfying* $t_n \nearrow \omega(t_0, \mathbf{x}_0)$ *and* $\{\mathbf{x}(t_n)\}_n$ *is a bounded sequence such that, for every subsequence* $\{t_{n_k}\}_k$ *verifying* $\mathbf{x}(t_{n_k}) \to \mathbf{u}$, *we have that*

$$\{(\omega(t_0, \mathbf{x}_0), \mathbf{u}), (\omega(t_0, \mathbf{x}_0), \mathbf{u}^+_{\mathbf{f},\omega(t_0,\mathbf{x}_0)})\} \not\subset \Omega.$$

Remark 3.9. Notice that, if Alternative (A1) holds, then $I_{\mathbf{x}} = [t_0, \infty)$, while if Alternative (A2) or (A3) holds, then $I_{\mathbf{x}} = [t_0, \omega(t_0, \mathbf{x}_0))$ or $I_{\mathbf{x}} = [t_0, \omega(t_0, \mathbf{x}_0)]$. In [5], the assumptions made on the Kurzweil integral of $\mathbf{f}$ do not allow the existence of a maximal solution on an interval of the form $[t_0, \omega(t_0, \mathbf{x}_0)]$.

Here is a corollary of Theorem 3.8 which ensures the global existence of the maximal solution over the whole interval $[t_0, \infty)$.

Corollary 3.10. *Let* I *be an interval containing* $[t_0, \infty)$. *Assume that* $(H_{\mathbf{x}_0,t_0})$, $(H_{\mathbf{f},t_0})$ *and* (H_{Ω,t_0}) *hold. Let* $\mathbf{x} \in \mathcal{S}(t_0, \mathbf{x}_0)$ *be the maximal solution of* (3.1)*. If*

$$\Omega^- := \left\{(t, \mathbf{u}) \in \Omega : \left(t, \mathbf{u}^+_{\mathbf{f},t}\right) \in \Omega\right\} = \Omega,$$

and there exists a compact set $D \subset O$ *such that* $\mathbf{x}(t) \in D$ *for every* $t \in I_{\mathbf{x}}$, *then* $\omega(t_0, \mathbf{x}_0) = \infty$.

Proof. Let us assume that $\omega(t_0,\mathbf{x}_0) < \infty$. Since D is compact, for all $\{t_n\}_n \subset I_{\mathbf{x}}$ such that $t_n \nearrow \omega(t_0,\mathbf{x}_0)$, $\{\mathbf{x}(t_n)\}_n$ is bounded. Thus, it results from Theorem 3.8 that Alternative (A3) holds. Therefore, there exists $\{t_n\}_n \subset I_{\mathbf{x}}$ with $t_n \nearrow \omega(t_0,\mathbf{x}_0)$ and $\{(t_n,\mathbf{x}(t_n))\}_n$ is bounded and such that, for a fixed subsequence $\{t_{n_k}\}_k$ verifying $\mathbf{x}(t_{n_k}) \to \mathbf{u}$, we have that

$$\left\{(\omega(t_0,\mathbf{x}_0),\mathbf{u}),\left(\omega(t_0,\mathbf{x}_0),\mathbf{u}^+_{\mathbf{f},\omega(t_0,\mathbf{x}_0)}\right)\right\} \not\subset \Omega. \tag{3.2}$$

On the other hand, D being a compact set yields that

$$(t_{n_k},\mathbf{x}(t_{n_k})) \to (\omega(t_0,\mathbf{x}_0),\mathbf{u}) \in [t_0,\omega(t_0,\mathbf{x}_0)] \times D \subset \Omega.$$

Now, since $\Omega^- = \Omega$, we deduce that $\left(\omega(t_0,\mathbf{x}_0),\mathbf{u}^+_{\mathbf{f},\omega(t_0,\mathbf{x}_0)}\right) \in \Omega$, which contradicts (3.2). Hence, $\omega(t_0,\mathbf{x}_0) = \infty$. □

Now, based on Theorems 3.5 and 3.8, we provide a characterization of the maximal solution of the problem (3.1).

Theorem 3.11. *Assume that* $(\mathrm{H}_{\mathbf{x}_0,t_0})$, $(\mathrm{H}_{\mathbf{f},t_0})$ *and* $(\mathrm{H}_{\Omega,t_0})$ *hold. Let* $\mathbf{x} \in \mathcal{S}(t_0,\mathbf{x}_0)$. *The following assumptions are equivalent:*

(1) $\mathbf{x}$ *is maximal;*

(2) *for every compact set* $K \subset \Omega$, *there exists* $t_K \in I_{\mathbf{x}}$ *such that*

$$\left\{(t,\mathbf{x}(t)),\left(t,(\mathbf{x}(t))^+_{\mathbf{f},t}\right)\right\} \not\subset K,$$

for all $t \geq t_K$, $t \in I_{\mathbf{x}}$.

Proof. Assume that $\mathbf{x}$ is maximal. By Theorem 3.8, two cases may occur:

Case 1: If $\omega(t_0,\mathbf{x}_0) = \sup I_{\mathbf{x}} = \infty$, by contradiction, assume that there exist a compact set $K \subset \Omega$ and a sequence $\{t_n\}_n \subset I_{\mathbf{x}}$ such that

$$t_n \nearrow \omega(t_0,\mathbf{x}_0) \quad \text{and} \quad \left\{(t_n,\mathbf{x}(t_n)),\left(t_n,(\mathbf{x}(t_n))^+_{\mathbf{f},t_n}\right)\right\} \subset K \quad \text{for all } n \in \mathbb{N}.$$

This implies that $\{(t_n,\mathbf{x}(t_n))\}$ is bounded. Therefore, there exists a convergent subsequence $\{(t_{n_k},\mathbf{x}(t_{n_k}))\}$ such that

$$(t_{n_k},\mathbf{x}(t_{n_k})) \to (\tau,\mathbf{u}) \in K \subset [t_0,\infty) \times O.$$

This contradicts that $t_n \nearrow \omega(t_0,\mathbf{x}_0) = \infty$.

Case 2: If $\omega(t_0,\mathbf{x}_0) < \infty$, then, by Theorem 3.8, there are two subcases:

Subcase 1: if Alternative (A2) holds, then

$$I_{\mathbf{x}} = [t_0,\omega(t_0,\mathbf{x}_0)) \quad \text{and} \quad \|\mathbf{x}(t)\| \to \infty \ \text{ as } t \nearrow \omega(t_0,\mathbf{x}_0).$$

Thus, for every $M > 0$, there exists $t^* \in I_{\mathbf{x}}$ such that $\|\mathbf{x}(t)\| \geq M$ for all $t \geq t^*$ with $t \in I_{\mathbf{x}}$. Hence, for every compact $K \subset \Omega$, there exists $t_K \in I_{\mathbf{x}}$ such that for all $t \geq t_K$ with $t \in I_{\mathbf{x}}$, we have $(t,\mathbf{x}(t)) \notin K$. In particular,

$$\left\{(t,\mathbf{x}(t)),\left(t,(\mathbf{x}(t))^+_{\mathbf{f},t}\right)\right\} \not\subset K.$$

Subcase 2: If Alternative (A3) holds, then we distinguish two cases.

If $I_{\mathbf{x}} = [t_0, \omega(t_0, \mathbf{x}_0))$, then the distance between $\operatorname{Graph}(\mathbf{x})$ and the boundary of Ω is zero, so

$$\big(\omega(t_0, \mathbf{x}_0), \mathbf{x}(\omega(t_0, \mathbf{x}_0)^-)\big) \notin \Omega.$$

Thus, for every compact $K \subset \Omega$, there exists $t_K \in [t_0, \omega(t_0, \mathbf{x}_0))$ such that, for all $t \geq t_K$ with $t \in I_{\mathbf{x}}$, $(t, \mathbf{x}(t)) \notin K$, which yields

$$\big\{(t, \mathbf{x}(t)), \big(t, (\mathbf{x}(t))^+_{\mathbf{f},t}\big)\big\} \not\subset K.$$

Now, if $I_{\mathbf{x}} = [t_0, \omega(t_0, \mathbf{x}_0)]$, then $\omega(t_0, \mathbf{x}_0) \in D_g$, $(\omega(t_0, \mathbf{x}_0), \mathbf{x}(\omega(t_0, \mathbf{x}_0))) \in \Omega$ and

$$\Big(\omega(t_0, \mathbf{x}_0), \big(\mathbf{x}(\omega(t_0, \mathbf{x}_0))\big)^+_{\mathbf{f},\omega(t_0,\mathbf{x}_0)}\Big) \notin \Omega.$$

Thus, for every compact $K \subset \Omega$, and for $t_K = \omega(t_0, \mathbf{x}_0)$,

$$\big(t_K, (\mathbf{x}(t_K))^+_{\mathbf{f},t_K}\big) \notin K, \quad \text{hence} \quad \big\{(t_K, \mathbf{x}(t_K)), \big(t_K, (\mathbf{x}(t_K))^+_{\mathbf{f},t_K}\big)\big\} \not\subset K.$$

Conversely, by contradiction, let us assume that $\mathbf{x} : I_{\mathbf{x}} \to \mathbb{R}^n$ is not maximal. Thus, $\bar{t}_{\mathbf{x}} = \sup I_{\mathbf{x}} < \infty$ and $\mathbf{x}$ is extendible to the right. From Theorem 3.5, it follows that $\operatorname{Graph}(\mathbf{x})$ is bounded and $A \cup A^+ \subset \Omega$. Thus, $[t_0, \bar{t}_{\mathbf{x}}] \times \overline{\operatorname{Graph}(\mathbf{x})}$ is compact,

$$\mathbf{x}(\bar{t}_{\mathbf{x}}^-) = \mathbf{x}(\bar{t}_{\mathbf{x}}) \in O \quad \text{and} \quad \Big(\mathbf{x}(\bar{t}_{\mathbf{x}})\Big)^+_{\mathbf{f},\bar{t}_{\mathbf{x}}} \in O.$$

Therefore, for the compact

$$K = \overline{\operatorname{Graph}(\mathbf{x})} \cup \Big\{\Big(\bar{t}_{\mathbf{x}}, \big(\mathbf{x}(\bar{t}_{\mathbf{x}})\big)^+_{\mathbf{f},\bar{t}_{\mathbf{x}}}\Big)\Big\} \subset \Omega,$$

we have that

$$\big\{(t, \mathbf{x}(t)), \big(t, (\mathbf{x}(t))^+_{\mathbf{f},t}\big)\big\} \subset K \quad \text{for all } t \in I_{\mathbf{x}},$$

which yields a contradiction. Hence, $\mathbf{x}$ is maximal. □

The negation of Theorem 3.11 provides also an interesting characterization of extendible solutions.

Corollary 3.12. *Assume that* $(\mathrm{H}_{\mathbf{x}_0,t_0})$, $(\mathrm{H}_{\mathbf{f},t_0})$ *and* $(\mathrm{H}_{\Omega,t_0})$ *hold. Let* $\mathbf{x} \in \mathcal{S}(t_0, \mathbf{x}_0)$. *The following statements are equivalent:*

(1) $\mathbf{x}$ *extendible to the right;*

(2) *there exist a compact set* $K \subset \Omega$ *and a sequence* $\{t_n\} \subset I_{\mathbf{x}}$ *with* $t_n \nearrow \omega(t_0, \mathbf{x}_0)$ *such that*

$$\Big\{(t_n, \mathbf{x}(t_n)), \Big(t_n, \big(\mathbf{x}(t_n)\big)^+_{\mathbf{f},t_n}\Big)\Big\} \subset K \quad \textit{for all } n \in \mathbb{N}.$$

Given the generalization of the Grönwall lemma, Lemma 2.12, we can state the next theorem which provides the global existence of the solution over $[t_0, \infty)$ and a bound of the solution.

Theorem 3.13. *Let* $\Omega \supset [t_0, \infty) \times \mathbb{R}^n$. *Assume that* $(\mathrm{H}_{\mathbf{x}_0,t_0})$, $(\mathrm{H}_{\mathbf{f},t_0})$ *and* $(\mathrm{H}_{\Omega,t_0})$ *hold. In addition, assume that*

(H_{LG}) *there exist* $k \in \mathcal{L}^1_{g,\mathrm{loc}}([t_0,\infty),\mathbb{R})$, *and* $p \in \mathcal{L}^1_{g,\mathrm{loc}}([t_0,\infty),[0,\infty))$ *such that*

$$\|\mathbf{f}(t,\mathbf{u})\| \leq k(t) + p(t)\|\mathbf{u}\| \quad \text{for } g\text{-almost all } t \in [t_0,\infty) \text{ and all } \mathbf{u} \in \mathbb{R}^n.$$

If $\mathbf{x} \in \mathcal{S}(t_0,\mathbf{x}_0)$ *is the maximal solution of* (3.1), *then*

$$\|\mathbf{x}(t)\| \leq e_p(t,t_0)\left(\int_{[t_0,t)} \frac{e_p^{-1}(s,t_0)k(s)}{1+p(s)\mu_g(\{s\})}\,d\mu_g(s) + \|\mathbf{x}_0\|\right), \quad \text{for all } t \in [t_0,T] \text{ with } T \in I_{\mathbf{x}}. \tag{3.3}$$

Moreover, we have $I_{\mathbf{x}} = [t_0,\infty)$.

Proof. Let $\mathbf{x} \in \mathcal{S}(t_0,\mathbf{x}_0)$ be the maximal solution of (3.1) defined on the maximal interval $I_{\mathbf{x}}$. By (H_{LG}), for $T > t_0$ with $T \in I_{\mathbf{x}}$, we have that

$$(\|\mathbf{x}\|)'_g(t) \leq \|\mathbf{x}'_g(t)\| = \|\mathbf{f}(t,\mathbf{x}(t))\| \leq k(t) + p(t)\|\mathbf{x}(t)\| \quad \text{for } g\text{-almost all } t \in [t_0,T).$$

Observe that $\|\cdot\|$ is Lipschitz continuous on $\mathbb{R}^n$. Thus, by Lemma 2.8, $\|\mathbf{x}(\cdot)\| \in \mathcal{AC}_g([t_0,T],\mathbb{R})$. Using the generalized version of the Grönwall Lemma, Lemma 2.12, we obtain

$$\|\mathbf{x}(t)\| \leq e_p(t,t_0)\left(\int_{[t_0,t)} \frac{e_p^{-1}(s,t_0)k(s)}{1+p(s)\mu_g(\{s\})}\,d\mu_g(s) + \|\mathbf{x}_0\|\right) \quad \text{for all } t \in [t_0,T].$$

Assume by contradiction that $\omega(t_0,\mathbf{x}_0) < \infty$. Since $p \in \mathcal{L}^1_{g,\mathrm{loc}}([t_0,\infty),[0,\infty))$, then

$$e_p(t,t_0)(1+p(t)\mu_g(\{t\})) \geq 1 \quad \text{for all } t \in [t_0,\infty).$$

Consequently,

$$\begin{aligned}\|\mathbf{x}(t)\| &\leq \sup_{t\in[t_0,T]} e_p(t,t_0)\Big(\int_{[t_0,t)} |k(s)|\,d\mu_g(s) + \|\mathbf{x}_0\|\Big)\\ &\leq e_p(\omega(t_0,\mathbf{x}_0),t_0)\Big(\|k\|_{\mathcal{L}^1_g([t_0,\omega(t_0,\mathbf{x}_0)),\mathbb{R})} + \|\mathbf{x}_0\|\Big)\\ &= M_1.\end{aligned}$$

As $T \nearrow \omega(t_0,\mathbf{x}_0) < \infty$, $\mathbf{x}(t) \in \overline{B_{\mathbb{R}^n}(\mathbf{0},M_1)}$ for all $t \in I_{\mathbf{x}}$. As $\mathbf{x}(\omega(t_0,\mathbf{x}_0)^-) = \mathbf{u} \in \mathbb{R}^n$, then

$$\mathbf{u}^+_{\mathbf{f},\omega(t_0,\mathbf{x}_0)} \in \mathbb{R}^n.$$

Therefore, for the compact set

$$K = [t_0,\omega(t_0,\mathbf{x}_0)] \times \overline{B_{\mathbb{R}^n}(\mathbf{0},M)} \quad \text{with} \quad M = \max\left\{M_1, \left\|\mathbf{u}^+_{\mathbf{f},\omega(t_0,\mathbf{x}_0)}\right\|\right\},$$

we have that

$$\left\{(t,\mathbf{x}(t)), \left(t,(\mathbf{x}(t))^+_{\mathbf{f},t}\right)\right\} \subset K \quad \text{for all } t \in I_{\mathbf{x}}.$$

By Corollary 3.12, we obtain that $\mathbf{x}$ is extendible to the right which is a contradiction. Hence, $\omega(t_0,\mathbf{x}_0) = \infty$. □

4 Lyapunov-like stability results

In this section, we assume that $[0,\infty) \subset I$ and $\Omega = I \times B_{\mathbb{R}^n}(\mathbf{0}, r)$. We present Lyapunov-type results in the context of Stieltjes dynamical systems, based on the classical Lyapunov's second method, considering Stieltjes dynamical systems of the form:

$$\mathbf{x}'_g(t) = \mathbf{f}(t, \mathbf{x}(t)) \quad \text{for } g\text{-almost all } t \geq \theta \geq 0,\ t \in I, \tag{4.1$_\theta$}$$

where $\mathbf{f} : \Omega \to \mathbb{R}^n$ satisfies $\mathbf{f}(t, \mathbf{0}) = \mathbf{0}$ for all $t \geq 0$. This study permits to draw conclusions about the behavior of solutions of the dynamical system (4.1$_\theta$) around the *equilibrium* $\mathbf{x} = \mathbf{0}$, which is also called the *trivial solution*.

Notice that, under hypotheses ($\mathrm{H}_{\Omega,0}$) and ($\mathrm{H}_{\mathbf{f},0}$) and for $(t_0, \mathbf{x}_0) \in [0,\infty) \times B_{\mathbb{R}^n}(\mathbf{0}, r)$, if ($\mathrm{H}_{\mathbf{x}_0,t_0}$) is satisfied, it follows from Theorem 3.7 that there exists a unique maximal solution $\mathbf{x} = \mathbf{x}(\cdot, t_0, \mathbf{x}_0) \in \mathcal{S}(t_0, \mathbf{x}_0) \cap \mathcal{AC}_{g,\mathrm{loc}}(I_{t_0,\mathbf{x}_0}, \mathbb{R}^n)$ of (4.1$_\theta$) with $\theta = t_0$ and satisfying $\mathbf{x}(t_0) = \mathbf{x}_0$ which is defined on a maximal interval of existence $I_{t_0,\mathbf{x}_0}$. As before, we denote $\omega(t_0, \mathbf{x}_0) = \sup I_{t_0,\mathbf{x}_0} \leq \infty$.

4.1 Lyapunov stability notions

In this subsection, we present stability concepts within the framework of Stieltjes' differentiation. Through illustrative examples, we highlight the influence of the sets C_g and D_g on the change of stability properties.

Definition 4.1. The trivial solution $\mathbf{x} = \mathbf{0}$ of the system (4.1$_\theta$) with $\theta = 0$ is said to be

- *stable* if, for all $\varepsilon > 0$ and $t_0 \in [0,\infty)$, there exists $\delta = \delta(\varepsilon, t_0) \in (0, r)$ such that

$$\|\mathbf{x}_0\| < \delta \quad \text{implies that} \quad \|\mathbf{x}(t, t_0, \mathbf{x}_0)\| < \varepsilon \quad \text{for all } t \in I_{t_0,\mathbf{x}_0};$$

- *uniformly stable* if, for all $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) \in (0, r)$ such that for all $t_0 \in [0,\infty)$,

$$\|\mathbf{x}_0\| < \delta \quad \text{implies that} \quad \|\mathbf{x}(t, t_0, \mathbf{x}_0)\| < \varepsilon \quad \text{for all } t \in I_{t_0,\mathbf{x}_0}.$$

Remark 4.2. In the case where the trivial solution $\mathbf{x} = \mathbf{0}$ of the system (4.1$_\theta$) with $\theta = 0$ is stable, observe that, for $\varepsilon = r$ and $t_0 \geq 0$, there exists $\delta = \delta(\varepsilon, t_0) \in (0, r)$ such that, for all $\mathbf{x}_0 \in B_{\mathbb{R}^n}(\mathbf{0}, \delta)$,

$$\|\mathbf{x}(t, t_0, \mathbf{x}_0)\| < \varepsilon \text{ for all } t \in I_{t_0,\mathbf{x}_0}.$$

Using Theorem 3.8 under ($\mathrm{H}_{\Omega,0}$), ($\mathrm{H}_{\mathbf{f},0}$) and ($\mathrm{H}_{\mathbf{x}_0,t_0}$), we deduce that $I_{t_0,\mathbf{x}_0} = [t_0,\infty)$. Furthermore, observe that if the stability of the trivial solution $\mathbf{x} = \mathbf{0}$ is uniform, then δ do not depend on t_0.

The following definition is a notion of *asymptotic stability*. This concerns the behavior of solutions as $t \to \infty$.

Definition 4.3. The trivial solution $\mathbf{x} = \mathbf{0}$ of the system (4.1$_\theta$) with $\theta = 0$ is said to be

- *asymptotically stable* if it is *stable* and, for every $t_0 \in [0,\infty)$, there exists $\delta = \delta(t_0) \in (0, r)$ such that, for all $\mathbf{x}_0 \in B_{\mathbb{R}^n}(\mathbf{0}, \delta)$ and $\varepsilon > 0$, there exists $\sigma = \sigma(t_0, \mathbf{x}_0, \varepsilon) > 0$ such that

$$\|\mathbf{x}(t, t_0, \mathbf{x}_0)\| < \varepsilon \quad \text{for all } t \in [t_0 + \sigma, \infty) \cap I_{t_0,\mathbf{x}_0};$$

- *uniformly asymptotically stable* if it is *uniformly stable* and there exists $\delta \in (0, r)$ such that, for every $\varepsilon > 0$, there exists $\sigma = \sigma(\varepsilon) > 0$ such that, for all $t_0 \in [0, \infty)$ and $\mathbf{x}_0 \in B_{\mathbb{R}^n}(\mathbf{0}, \delta)$,

$$\|\mathbf{x}(t, t_0, \mathbf{x}_0)\| < \varepsilon \text{ for all } t \in [t_0 + \sigma, \infty) \cap I_{t_0, \mathbf{x}_0}.$$

In the following example, we compare the stability properties of the trivial solution of a linear Stieltjes dynamical system to the ones in the classical case, observing the change of the stability properties depending on the sets C_g and D_g. The resolution of linear Stieltjes differential equations has been studied in the literature, see for instance [7, 9, 17, 26, 27].

Example 4.4. Let us consider the linear Stieltjes dynamical system

$$x'_g(t) = cx(t) \quad \text{for } g\text{-almost all } t \geq \theta \geq 0, \tag{4.2$_\theta$}$$

for $c \in \mathbb{R}$. In the classical case of derivation where $g \equiv \mathrm{id}_{\mathbb{R}}$, for $c > 0$, the equilibrium $x = 0$ is not stable given that the solutions of (4.2$_\theta$) with $\theta = t_0$ and satisfying $x(t_0) = x_0$ are $x(t, t_0, x_0) = x_0 e^{c(t-t_0)}$ for every $x_0 \in \mathbb{R}$ and $t_0 \geq 0$, and they are not bounded on $[t_0, \infty)$ for $x_0 \neq 0$.

However, for $c < 0$, the equilibrium $x = 0$ is asymptotically stable since stability holds and $x(t, t_0, x_0) = x_0 e^{c(t-t_0)} \xrightarrow{t \to \infty} 0$ for all $x_0 \in \mathbb{R}$. In the case where $c = 0$, every constant $z \in \mathbb{R}$ is a uniformly stable equilibrium.

Now, let us reconsider the dynamical system (4.2$_\theta$), where $g : \mathbb{R} \to \mathbb{R}$ is defined by $g(t) = t$ for $t \leq 1$ and $g(t) = 1$ for $t \geq 1$. Thus, the solution of the problem (4.2$_\theta$) with $\theta = t_0$ and satisfying $x(t_0) = x_0$ is $x(t, t_0, x_0) = x_0 e^{c(g(t)-g(t_0))}$, for every $x_0 \in \mathbb{R}$ and $t_0 \geq 0$. Now, we show that the stability properties of the trivial solution $x = 0$ of the system (4.2$_\theta$) with $\theta = 0$ differ from the classical case, for $c \neq 0$. Indeed, for $c > 0$, we deduce the stability of the trivial solution $x = 0$ since, for all $\varepsilon > 0$ and $t_0 \geq 0$, one can take $\delta = \varepsilon / \left(e^{c(g(1)-g(t_0))}\right)$ which is such that

$$|x_0| < \delta \quad \text{implies that} \quad |x_0 e^{c(g(t)-g(t_0))}| < \varepsilon \quad \text{for all } t \geq t_0.$$

Whereas, for $c < 0$ the equilibrium $x = 0$ is uniformly stable since, for all $\varepsilon > 0$, with $\delta = \varepsilon > 0$, one has that

$$|x_0| < \delta \quad \text{implies that} \quad |x_0 e^{c(g(t)-g(t_0))}| < \varepsilon \quad \text{for all } t \geq t_0.$$

Observe that asymptotic stability does not hold for any $c \in \mathbb{R}$, since

$$x_0 e^{c(g(t)-g(t_0))} \to x_0 e^{c(g(1)-g(t_0))} \nrightarrow 0,$$

as $t \to \infty$ for all $t_0 \geq 0$ and $x_0 \neq 0$.

Next, we define the derivator g_1 by $g_1(t) = t + \sum_{n \in \mathbb{N}} \chi_{[n,\infty)}(t)$ for all $t \in \mathbb{R}$. We consider the dynamical system with g_1:

$$x'_{g_1}(t) = \begin{cases} cx(t) & \text{if } t \geq \theta \geq 0, \text{ with } t \notin D_{g_1} = \mathbb{N}, \\ \nu x(t), & \text{if } t \geq \theta \geq 0, \text{ with } t \in D_{g_1}, \end{cases} \tag{4.3$_\theta$}$$

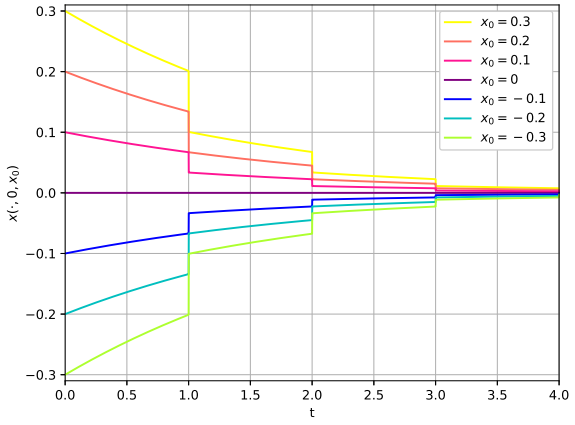
where $c, \nu \in \mathbb{R}$ with $\nu \in (-1, \infty) \setminus \{0\}$. Again the solution of (4.3$_\theta$) with $\theta = t_0$ and satisfying $x(t_0) = x_0$ has the form

$$x(t, t_0, x_0) = x_0 e_{c_*}(t, t_0) = x_0 e^{\int_{[t_0, t)} c_*(s)\, d\mu_{g_1}(s)},$$

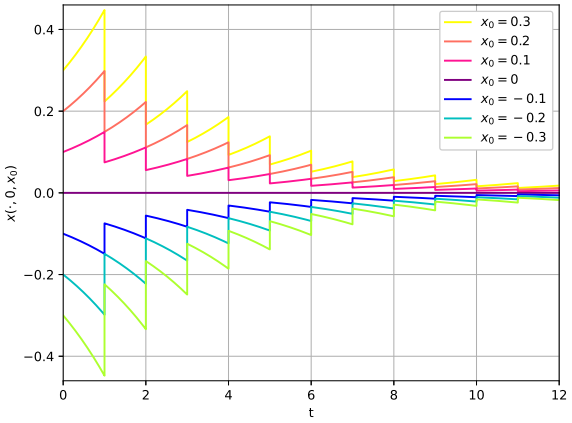
where

$$c_*(t) = \begin{cases} c & \text{if } t \in [t_0, \infty) \setminus D_{g_1}, \\ \log(1+\nu) & \text{if } t \in [t_0, \infty) \cap D_{g_1}. \end{cases}$$

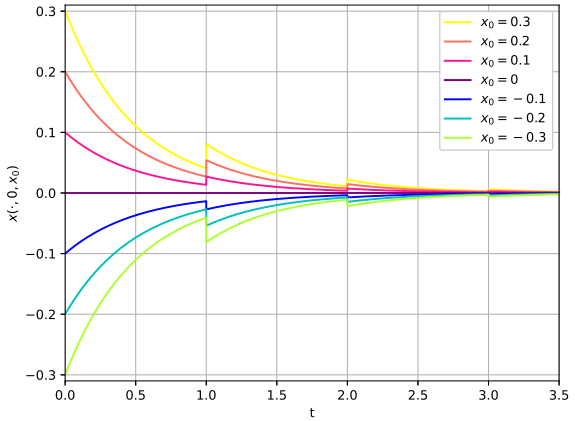
In Figure 4.1, we can observe different patterns depending on the values of c and ν. This implies that the presence of discontinuities can destabilize or restore the stability properties of a dynamical system.



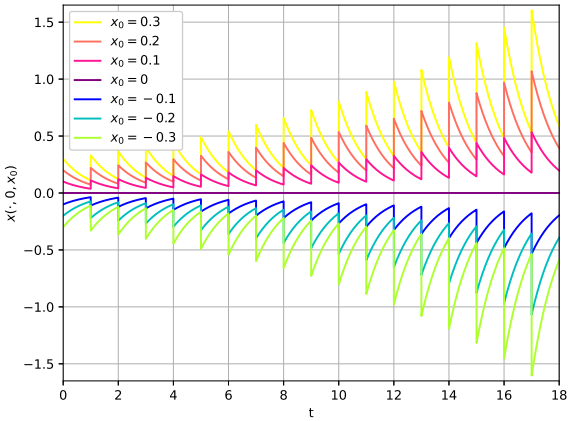
(a) The case of $c = -0.4$, and $\nu = -0.5$.



(b) The case of $c = 0.4$, and $\nu = -0.5$.



(c) The case of $c = -2$, and $\nu = 1$.



(d) The case of $c = -1$, and $\nu = 2$.

Figure 4.1: Behaviour of solutions of (4.3_θ) with $\theta = 0$ in a neighborhood of $x = 0$.

4.2 Stability results based on Lyapunov's function

In order to establish sufficient conditions for different types of stability of the trivial solution $\mathbf{x} = \mathbf{0}$ of the system (4.1_θ) with $\theta = 0$, we introduce specific sets of functions.

Definition 4.5. A function $V : [0, \infty) \times B_{\mathbb{R}^n}(\mathbf{0}, r_0) \to \mathbb{R}$ is said to belong to class $\mathcal{V}_1^g$ if it satisfies the following conditions:

1. $V(t, \cdot)$ is continuous for all $t \geq 0$;

2. $V(\cdot, \mathbf{x}(\cdot)) \in \mathcal{AC}_{g,\mathrm{loc}}(I_{t_0,\mathbf{x}_0}, \mathbb{R})$ for every function $\mathbf{x} : I_{t_0,\mathbf{x}_0} \to B_{\mathbb{R}^n}(\mathbf{0}, r_0)$ of $\mathcal{AC}_{g,\mathrm{loc}}(I_{t_0,\mathbf{x}_0}, \mathbb{R}^n)$ maximal solution of the system (4.1_θ) with $\theta = 0$;
3. $V(t, \mathbf{0}) = 0$ for all $t \geq 0$.

Definition 4.6. A function $\varphi : [0, \infty) \to [0, \infty)$ belongs to the class $\mathcal{K}$ if it fulfills the following assumptions:

1. φ is continuous;
2. $\varphi(0) = 0$;
3. φ is increasing.

In the following, we simply write $\varphi \in \mathcal{K}$ to mean that φ belongs to the class $\mathcal{K}$. Now, we state the first stability result.

Theorem 4.7. *Assume that Conditions* ($\mathrm{H}_{\Omega,0}$) *and* ($\mathrm{H}_{\mathbf{f},0}$) *hold. If there exist functions* $V \in \mathcal{V}_1^g$ *and* $a \in \mathcal{K}$ *such that*

(a) $a(\|\mathbf{u}\|) \leq V(t, \mathbf{u})$ *for all* $(t, \mathbf{u}) \in [0, \infty) \times B_{\mathbb{R}^n}(\mathbf{0}, r_0)$;

(b) *there exists* $r \in (0, r_0]$ *such that, for every* $(t_0, \mathbf{x}_0) \in [0, \infty) \times B_{\mathbb{R}^n}(\mathbf{0}, r)$, ($\mathrm{H}_{\mathbf{x}_0,t_0}$) *holds and, for* $\mathbf{x} : I_{t_0,\mathbf{x}_0} \to B_{\mathbb{R}^n}(\mathbf{0}, r_0)$ *the maximal solution of the system* (4.1_θ) *with* $\theta = t_0$ *and satisfying* $\mathbf{x}(t_0) = \mathbf{x}_0$, *one has that* $V_g'(t, \mathbf{x}(t)) \leq 0$ *for* g*-almost all* $t \in I_{t_0,\mathbf{x}_0}$.

Then, the trivial solution of the system (4.1_θ) *with* $\theta = 0$ *is*

(1) *stable,*

(2) *uniformly stable if there exists* $b \in \mathcal{K}$ *such that*

$$V(t, \mathbf{u}) \leq b(\|\mathbf{u}\|) \quad \text{for all } (t, \mathbf{u}) \in [0, \infty) \times B_{\mathbb{R}^n}(\mathbf{0}, r_0). \tag{4.4}$$

Proof. (1) Since $V \in \mathcal{V}_1^g$, then, for all $\varepsilon > 0$ and $t_0 \in [0, \infty)$, there exists $\delta = \delta(\varepsilon, t_0) \in (0, r)$ such that

$$\sup_{\|\mathbf{u}\| < \delta} V(t_0, \mathbf{u}) < a(\varepsilon).$$

For $\mathbf{x}_0 \in B_{\mathbb{R}^n}(\mathbf{0}, \delta)$, let $\mathbf{x} : I_{t_0,\mathbf{x}_0} \to B_{\mathbb{R}^n}(\mathbf{0}, r_0)$ be the maximal solution of the system (4.1_θ) with $\theta = t_0$ and $\mathbf{x}(t_0) = \mathbf{x}_0$. It follows from Conditions (a) and (b) that

$$a(\|\mathbf{x}(t)\|) \leq V(t, \mathbf{x}(t)) \leq V(t_0, \mathbf{x}_0) < a(\varepsilon).$$

Thus, for all $t \in I_{t_0,\mathbf{x}_0}$, we have that

$$\|\mathbf{x}(t, t_0, \mathbf{x}_0)\| < \varepsilon.$$

Therefore, the trivial solution $\mathbf{x} = \mathbf{0}$ is stable.

(2) Arguing as in (1), we can choose a $\delta = \delta(\varepsilon) \in (0, r)$ independent of t_0 such that

$$b(\delta) < a(\varepsilon).$$

Thus, using (4.4), we obtain for all $t \in I_{t_0,\mathbf{x}_0}$,

$$a(\|\mathbf{x}(t)\|) \leq V(t, \mathbf{x}(t)) \leq V(t_0, \mathbf{x}_0) \leq b(\|\mathbf{x}_0\|) < b(\delta) < a(\varepsilon).$$

This yields that the trivial solution $\mathbf{x} = \mathbf{0}$ is uniformly stable. □

In the next theorem, we impose additional assumptions which will permit to ensure the asymptotic stability of the trivial solution $\mathbf{x} = \mathbf{0}$ to the system (4.1_θ) with $\theta = 0$.

Theorem 4.8. *Assume that Conditions* ($\mathrm{H}_{\Omega,0}$) *and* ($\mathrm{H}_{\mathbf{f},0}$) *hold. Let* $V \in \mathcal{V}_1^g$, $\phi : [0,\infty) \to [0,\infty)$ *continuous,* $a, b \in \mathcal{K}$, *and a* g*-measurable function* $v : [0,\infty) \to [0,\infty)$ *be such that*

(a) $a(\|\mathbf{u}\|) \le V(t,\mathbf{u})$ *for every* $(t,\mathbf{u}) \in [0,\infty) \times B_{\mathbb{R}^n}(\mathbf{0}, r_0)$;

(b) $\phi(s) = 0$ *if and only if* $s = 0$;

(c) *there exists* $r \in (0, r_0]$ *such that, for every* $(t_0, \mathbf{x}_0) \in [0,\infty) \times B_{\mathbb{R}^n}(\mathbf{0}, r)$, ($\mathrm{H}_{\mathbf{x}_0,t_0}$) *holds and, for* $\mathbf{x} : I_{t_0,\mathbf{x}_0} \to B_{\mathbb{R}^n}(\mathbf{0}, r_0)$ *the maximal solution of the system* (4.1_θ) *with* $\theta = t_0$ *and satisfying* $\mathbf{x}(t_0) = \mathbf{x}_0$, *one has that*

$$V_g'(t, \mathbf{x}(t)) \le -v(t)\phi(\|\mathbf{x}(t)\|) \quad \text{for } g\text{-almost all } t \in I_{t_0,\mathbf{x}_0};$$

(d) $\inf_{t_0 \in [0,\infty)} \lim_{t \to +\infty} \int_{[t_0, t_0+t)} v(s)\, d\mu_g(s) = \infty$.

If the trivial solution $\mathbf{x} = \mathbf{0}$ *of the system* (4.1_θ) *with* $\theta = 0$ *is uniformly stable, then* $\mathbf{x} = \mathbf{0}$ *is asymptotically stable.*

Proof. The stability of the trivial solution $\mathbf{x} = \mathbf{0}$ holds from uniform stability. Let us choose $\delta_0 \in (0, r)$ associated to an $\varepsilon_0 \le r$ given by the uniform stability. Now, for a fixed $t_0 \ge 0$, let $\varepsilon > 0$. Again, by the uniform stability, there exists $\delta \in (0, \delta_0)$ such that, for all $\hat{t} \in [0,\infty)$ and every $\hat{\mathbf{x}}_0$ satisfying $\|\hat{\mathbf{x}}_0\| < \delta$, one has

$$\|\hat{\mathbf{x}}(t, \hat{t}, \hat{\mathbf{x}}_0)\| < \varepsilon \quad \text{for all } t \in [\hat{t}, \infty) \cap I_{\hat{t},\hat{\mathbf{x}}_0}.$$

We denote

$$M = \inf_{s \in [\delta, r_0)} |\phi(s)|. \tag{4.5}$$

By Condition (b), observe that $M > 0$.

Let $\mathbf{x}_0 \in B_{\mathbb{R}^n}(\mathbf{0}, \delta_0)$. Since

$$\lim_{t \to +\infty} \int_{[t_0, t_0+t)} v(s)\, d\mu_g(s) = \infty,$$

we can choose $\sigma > 0$ such that

$$\int_{[t_0, t_0+\sigma)} v(s)\, d\mu_g(s) > \frac{V(t_0, \mathbf{x}_0)}{M}.$$

Let $\mathbf{x} : I_{t_0,\mathbf{x}_0} \to B_{\mathbb{R}^n}(\mathbf{0}, r_0)$ be the maximal solution of (4.1_θ) with $\theta = t_0$ and satisfying $\mathbf{x}(t_0) = \mathbf{x}_0$. Using Remark 4.2, notice that $I_{t_0,\mathbf{x}_0} = [t_0, \infty)$. Now, if there exists $\hat{t} \in [t_0, t_0 + \sigma]$ such that $\|\mathbf{x}(\hat{t})\| < \delta$, then, by the uniform stability

$$\|\hat{\mathbf{x}}(t)\| < \varepsilon \quad \text{for all } t \in [\hat{t}, \infty) \cap I_{\hat{t},\mathbf{x}(\hat{t})},$$

where $\hat{\mathbf{x}} : I_{\hat{t},\mathbf{x}(\hat{t})} \to B_{\mathbb{R}^n}(\mathbf{0}, r_0)$ is the maximal solution of (4.1_θ) with $\theta = \hat{t}$ satisfying the initial condition $\hat{\mathbf{x}}(\hat{t}) = \mathbf{x}(\hat{t})$. By the uniqueness of the maximal solution, one has

$$\omega(t_0, \mathbf{x}_0) = \omega(\hat{t}, \mathbf{x}(\hat{t})) = \infty \quad \text{and} \quad \mathbf{x}(t) = \hat{\mathbf{x}}(t) \quad \text{for all } t \in [\hat{t}, \infty).$$

Hence,

$$\|\mathbf{x}(t)\| < \varepsilon \quad \text{for all } t \in [t_0+\sigma, \infty) \cap I_{t_0,\mathbf{x}_0}.$$

On the other hand, if $\|\mathbf{x}(t)\| \geq \delta$ for all $t \in [t_0, t_0+\sigma]$, then using Conditions (a), (c), Theorem 2.7, and (4.5), we obtain

$$\begin{aligned} a(\|\mathbf{x}(t_0+\sigma)\|) &\leq V(t_0+\sigma, \mathbf{x}(t_0+\sigma)) \\ &= V(t_0, \mathbf{x}(t_0, t_0, \mathbf{x}_0)) + \int_{[t_0,t_0+\sigma)} V'_g(s, \mathbf{x}(s))\, d\mu_g(s) \\ &\leq V(t_0, \mathbf{x}_0) - \int_{[t_0,t_0+\sigma)} v(s)\phi(\|\mathbf{x}(s)\|)\, d\mu_g(s) \\ &\leq V(t_0, \mathbf{x}_0) - M \int_{[t_0,t_0+\sigma)} v(s)\, d\mu_g(s) \\ &< 0. \end{aligned}$$

This is a contradiction. Therefore, $\mathbf{x} = \mathbf{0}$ is asymptotically stable. □

In the example below, we present an application of Theorem 4.8.

Example 4.9. Let us consider the Stieltjes dynamical system

$$x'_g(t) = f(t, x(t)) \quad \text{for } g\text{-almost all } t \geq \theta \geq 0, \tag{4.6$_\theta$}$$

with $g : \mathbb{R} \to \mathbb{R}$ defined by $g(t) = t$ for all $t \leq 1$, and $g(t) = t+1$ for $t > 1$, and where $f : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ is a function defined by

$$f(t,x) = \begin{cases} -\dfrac{xt}{1+t^2} & \text{if } t \in [0,\infty) \setminus D_g, \\ \nu x & \text{if } t \in [0,\infty) \cap D_g, \end{cases}$$

for some $\nu \in \mathbb{R} \setminus \{-1, 0\}$. The function f satisfies conditions of Theorem 3.13, thus, for every $(t_0, x_0) \in [0,\infty) \times \mathbb{R}$, the problem (4.6$_\theta$) with $\theta = t_0$ has a maximal solution $x : [t_0, +\infty) \to \mathbb{R}$ satisfying $x(t_0) = x_0$. Observe that $x = 0$ is an equilibrium of the dynamical system (4.6$_\theta$) with $\theta = 0$.

Let us define the function $V : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ for every $(t,x) \in [0,\infty) \times \mathbb{R}$ by

$$V(t,x) = \begin{cases} x^2 & \text{if } t \in [0,1], \\ \dfrac{x^2}{(1+\nu)^2} & \text{if } t > 1. \end{cases}$$

Clearly $V \in \mathcal{V}_g^1$, and $a(|u|) \leq V(t,u) \leq b(|u|)$ for all $(t,u) \in [0,\infty) \times \mathbb{R}$, where $a, b \in \mathcal{K}$ are given by

$$a(s) = \min\left\{s^2, \frac{s^2}{(1+\nu)^2}\right\} \quad \text{and} \quad b(s) = \max\left\{s^2, \frac{s^2}{(1+\nu)^2}\right\} \quad \text{for all } s \in [0,\infty).$$

In addition, for all $(t,x) \in [0,\infty) \times \mathbb{R}$:

$$\begin{aligned} \frac{\partial V}{\partial_g t}(t,x) &= \begin{cases} 0 & \text{if } t \in [0,1) \cup (1,\infty), \\ \dfrac{\frac{x^2}{(1+\nu)^2} - x^2}{g(1^+) - g(1)} & \text{if } t = 1, \end{cases} \\ &= \begin{cases} 0 & \text{if } t \in [0,1) \cup (1,\infty), \\ \dfrac{1-(1+\nu)^2}{(1+\nu)^2} x^2 & \text{if } t = 1. \end{cases} \end{aligned}$$

Thus, by means of Proposition 2.10, for $t \in [t_0, \infty) \setminus D_g$, we obtain

$$\begin{aligned} V_g'(t, x(t)) &= \frac{\partial V}{\partial_g t}(t, x(t)) + \frac{\partial V}{\partial x}(t, x(t)) f(t, x(t)) \\ &= \begin{cases} -\dfrac{2t}{1+t^2} x(t)^2 & \text{if } t \in [t_0, \infty) \cap [0, 1), \\ -\dfrac{2t}{(1+t^2)(1+\nu)^2} x(t)^2 & \text{if } t \in [t_0, \infty) \cap (1, \infty). \end{cases} \end{aligned}$$

For $t \in [t_0, \infty) \cap D_g$, if $t_0 \le 1$, then $t = 1$ and we have that

$$\begin{aligned} V_g'(1, x(1)) &= \frac{V(1^+, x(1^+)) - V(1, x(1))}{g(1^+) - g(1)} \\ &= \frac{V(1^+, x(1) + \mu_g(\{1\}) f(1, x(1))) - V(1, x(1))}{g(1^+) - g(1)} \\ &= \frac{\frac{(1+\nu)^2 x(1)^2}{(1+\nu)^2} - x(1)^2}{g(1^+) - g(1)} \\ &= 0. \end{aligned}$$

This implies that

$$V_g'(t, x(t)) \le -v(t) \phi(|x(t)|), \text{ for every } t \ge t_0,$$

with $v : [0, \infty) \to [0, \infty)$, defined by

$$v(t) = \begin{cases} \dfrac{2t}{1+t^2} & \text{if } t \in [0, 1), \\ 0 & \text{if } t = 1, \\ \dfrac{2t}{(1+t^2)(1+\nu)^2} & \text{if } t \in (1, \infty), \end{cases}$$

and $\phi \in \mathcal{K}$, defined by $\phi(y) = y^2$ for all $y \in [0, \infty)$. Moreover, for every $t_0 \in [0, \infty)$, we have

$$\begin{aligned} \lim_{t \to \infty} \int_{[t_0, t_0+t)} v(s)\, d\mu_g(s) &= \lim_{t \to \infty} \int_{[t_0, t_0+t) \cap ([0,1) \cup \{1\} \cup (1, \infty))} v(s)\, d\mu_g(s) \\ &\ge \lim_{t \to \infty} \frac{1}{(1+\nu)^2} \int_{[t_0, t_0+t) \cap (1, \infty)} \frac{2s}{1+s^2}\, ds \\ &= \lim_{t \to \infty} \frac{1}{(1+\nu)^2} \big(\log(1 + (t_0+t)^2) - \sup\{\log(2), \log(1+t_0^2)\} \big) \\ &= \infty. \end{aligned}$$

Hence,

$$\inf_{t_0 \in [0, \infty)} \lim_{t \to +\infty} \int_{[t_0, t_0+t)} v(s)\, d\mu_g(s) = \infty.$$

Thus, Condition (d) holds. Therefore, by means of Theorem 4.8, we deduce that $x = 0$ is an asymptotically stable equilibrium. Figure 4.2 illustrates the asymptotic behavior of solutions of the dynamical system (4.6_θ) with $\theta = 0$.

In the following example, we consider the case where g has an infinite number of discontinuities.

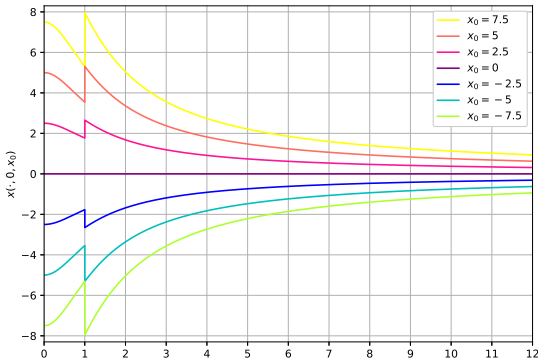


Figure 4.2: Asymptotic behavior of solutions of the dynamical system (4.6_θ) with $\theta = 0$ obtained with a time-discretization step-size of 10^{-3}, with $\nu = 0.5$.

Example 4.10. Let us consider the dynamical system

$$x_g'(t) = f(t, x(t)) \quad \text{for } g\text{-almost all } t \geq \theta \geq 0, \tag{4.7_θ}$$

where $g : \mathbb{R} \to \mathbb{R}$ is the derivator defined by

$$g(t) = t + \sum_{k \in \mathbb{N}} \chi_{[t_k, +\infty)}(t) \quad \text{for all } t \in \mathbb{R}, \tag{4.8}$$

where $\{t_k\}$ is an unbounded, increasing sequence in $(0, \infty)$, and where the function $f : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ is defined by

$$f(t, x) = \begin{cases} -\dfrac{xt}{1 + t^2} & \text{if } t \in [0, \infty) \setminus D_g, \\ \nu x & \text{if } t \in [0, \infty) \cap D_g, \end{cases}$$

for some $\nu \in [-2, -1)$. The function f satisfies conditions of Theorem 3.13. Thus, for every $(t_0, x_0) \in [0, \infty) \times \mathbb{R}$, the problem ($4.6_\theta$) with $\theta = t_0$ has a maximal solution $x : [t_0, +\infty) \to \mathbb{R}$ satisfying $x(t_0) = x_0$. Observe that $x = 0$ is an equilibrium of the dynamical system (4.7_θ) with $\theta = 0$.

Let us define the function $V : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ by $V(t, x) = x^2$. Clearly, $V \in \mathcal{V}_g^1$ and

$$a(|u|) \leq V(t, u) \leq b(|u|) \quad \text{for all } (t, u) \in [0, \infty) \times \mathbb{R},$$

where $a, b \in \mathcal{K}$ are given by

$$a(s) = s^2 \quad \text{and} \quad b(s) = \frac{s^2}{(1 + \nu)^2} \quad \text{for all } s \in [0, \infty).$$

In addition,

$$\frac{\partial V}{\partial_g t}(t, x) = 0 \quad \text{for all } (t, x) \in [0, \infty) \times \mathbb{R}.$$

Thus, by means of Proposition 2.10, for g-almost every $t \in [t_0, \infty) \setminus D_g$, we obtain

$$\begin{aligned} V_g'(t, x(t)) &= \frac{\partial V}{\partial_g t}(t, x(t)) + \frac{\partial V}{\partial x}(t, x(t)) f(t, x(t)) \\ &= -\frac{2t}{1 + t^2} x(t)^2. \end{aligned}$$

For $t_k \in [t_0, \infty) \cap D_g$, we have that

$$\begin{aligned} V_g'(t_k, x(t_k)) &= \frac{V(t_k^+, x(t_k^+)) - V(t_k, x(t_k))}{g(t_k^+) - g(t_k)} \\ &= \frac{V(t_k^+, x(t_k) + \mu_g(\{t_k\}) f(t_k, x(t_k))) - V(t_k, x(t_k))}{g(t_k^+) - g(t_k)} \\ &= \frac{((1+\nu)x(t_k))^2 - x(t_k)^2}{g(t_k^+) - g(t_k)} \\ &= \nu(2+\nu)x(t_k)^2. \end{aligned}$$

This implies that

$$V_g'(t, x(t)) \le -v(t)\phi(|x(t)|) \quad \text{for } g\text{-almost every } t \ge t_0,$$

with $v : [0, \infty) \to [0, \infty)$ defined by

$$v(t) = \begin{cases} \dfrac{2t}{1+t^2} & \text{if } t \in [0, t_1) \cup (t_k, t_{k+1}),\ k \in \mathbb{N}, \\ -\nu(2+\nu) & \text{if } t \in \{t_k\}_{k \in \mathbb{N}}, \end{cases}$$

and $\phi \in \mathcal{K}$ defined by $\phi(y) = y^2$ for all $y \in [0, \infty)$. For $t_0, t \in [0, \infty)$, observe that v satisfies:

$$\begin{aligned} \int_{[t_0, t_0+t)} v(s)\, d\mu_g(s) &= \sum_{s \in [t_0, t_0+t) \cap D_g} -\nu(2+\nu)\mu_g(\{s\}) + \int_{t_0}^{t_0+t} \frac{2s}{1+s^2}\, ds \\ &= \sum_{s \in [t_0, t_0+t) \cap D_g} -\nu(2+\nu)\mu_g(\{s\}) + \log\left(\frac{1+(t_0+t)^2}{1+t_0^2}\right). \end{aligned}$$

Thus,

$$\inf_{t_0 \in [0,\infty)} \lim_{t \to +\infty} \int_{[t_0, t_0+t)} v(s)\, d\mu_g(s) = +\infty.$$

Consequently, Condition (d) holds. Therefore, by means of Theorem 4.8, we deduce that $x = 0$ is an asymptotically stable equilibrium. Figure 4.3 illustrates the asymptotic behavior of solutions of the dynamical system (4.7_θ) with $\theta = 0$.

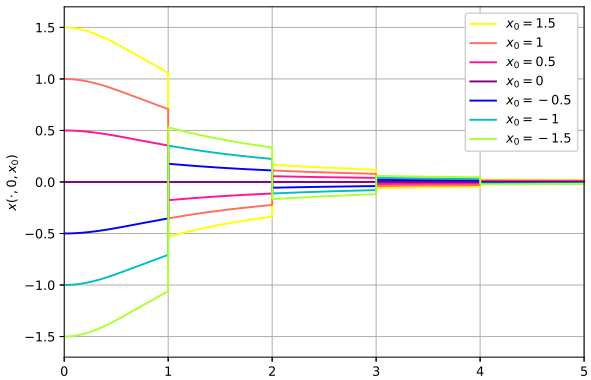


Figure 4.3: Asymptotic behavior of solutions of the dynamical system (4.7_θ) with $\theta = 0$ obtained with a time-discretization step-size of 10^{-3}, with $\nu = -3/2$.

The following result gives conditions to ensure the uniform asymptotic stability of the trivial solution of (4.1_θ) with $\theta = 0$.

Theorem 4.11. *Assume that Conditions* ($H_{\Omega,0}$), ($H_{\mathbf{f},0}$) *hold. If there exist* $V \in \mathcal{V}_1^g$, $a, b \in \mathcal{K}$, $\phi : [0,\infty) \to [0,\infty)$ *continuous, and a g-measurable function* $v : [0,\infty) \to [0,\infty)$ *such that*

(a) $a(\|\mathbf{u}\|) \leq V(t,\mathbf{u}) \leq b(\|\mathbf{u}\|)$ *for every* $(t,\mathbf{u}) \in [0,\infty) \times B_{\mathbb{R}^n}(\mathbf{0}, r_0)$;

(b) $\phi(s) = 0$ *if and only if* $s = 0$;

(c) *there exists* $r \in (0, r_0]$ *such that, for every* $(t_0, \mathbf{x}_0) \in [0,\infty) \times B_{\mathbb{R}^n}(\mathbf{0}, r)$, ($H_{\mathbf{x}_0,t_0}$) *holds and, for* $\mathbf{x} : I_{t_0,\mathbf{x}_0} \to B_{\mathbb{R}^n}(\mathbf{0}, r_0)$ *the maximal solution of the system* (4.1_θ) *with* $\theta = t_0$ *and satisfying* $\mathbf{x}(t_0) = \mathbf{x}_0$, *one has that*

$$V_g'(t, \mathbf{x}(t)) \leq -v(t)\phi(\|\mathbf{x}(t)\|) \quad \text{for } g\text{-almost all } t \in I_{t_0,\mathbf{x}_0}.$$

(d) $\lim_{t\to+\infty} \inf_{t_0\in[0,\infty)} \int_{[t_0,t_0+t)} v(s)\, d\mu_g(s) = +\infty$.

Then, the trivial solution $\mathbf{x} = \mathbf{0}$ *of the system* (4.1_θ) *with* $\theta = 0$ *is uniformly asymptotically stable.*

Proof. By (2) of Theorem 4.7, the trivial solution $\mathbf{x} = \mathbf{0}$ is uniformly stable. Thus, let us choose $\delta_0 \in (0, r)$ associated to an $\varepsilon_0 \leq r$. Let $\varepsilon > 0$. Again, by uniform stability, there exists $\delta \in (0, r)$ such that, for all $\hat{t} \in [0,\infty)$ and every $\hat{\mathbf{x}}_0$ such that $\|\hat{\mathbf{x}}_0\| < \delta$, one has

$$\|\hat{\mathbf{x}}(t, \hat{t}, \hat{\mathbf{x}}_0)\| < \varepsilon \quad \text{for all } t \in [\hat{t},\infty) \cap I_{\hat{t},\hat{\mathbf{x}}_0}.$$

Let M be as defined in (4.5). Since

$$\lim_{t\to+\infty} \inf_{t_0\in[0,\infty)} \int_{[t_0,t_0+t)} v(s)\, d\mu_g(s) = \infty,$$

we can choose $\sigma > 0$ such that

$$\int_{[t_0,t_0+\sigma)} v(s)\, d\mu_g(s) > \frac{b(\delta_0)}{M} \quad \text{for all } t_0 \in [0,\infty).$$

Let $(t_0, \mathbf{x}_0) \in [0,\infty) \times B_{\mathbb{R}^n}(\mathbf{0}, \delta_0)$ and $\mathbf{x} : I_{t_0,\mathbf{x}_0} \to B_{\mathbb{R}^n}(\mathbf{0}, r_0)$ a maximal solution of (4.1_θ) with $\theta = t_0$ and satisfying $\mathbf{x}(t_0) = \mathbf{x}_0$.

If there exists $\hat{t} \in [t_0, t_0 + \sigma] \cap I_{t_0,\mathbf{x}_0}$ such that $\|\mathbf{x}(\hat{t})\| < \delta$, then

$$\|\hat{\mathbf{x}}(t)\| < \varepsilon \quad \text{for all } t \in [\hat{t},\infty) \cap I_{\hat{t},\mathbf{x}(\hat{t})},$$

where $\hat{\mathbf{x}} : I_{\hat{t},\mathbf{x}(\hat{t})} \to B_{\mathbb{R}^n}(\mathbf{0}, r_0)$ is the maximal solution of (4.1_θ) with $\theta = \hat{t}$ satisfying the initial condition $\hat{\mathbf{x}}(\hat{t}) = \mathbf{x}(\hat{t})$. By the uniqueness of the maximal solution, one has

$$\omega(t_0, \mathbf{x}_0) = \omega(\hat{t}, \mathbf{x}(\hat{t})) \quad \text{and} \quad \mathbf{x}(t) = \hat{\mathbf{x}}(t) \quad \text{for all } t \in [\hat{t},\infty) \cap I_{t_0,\mathbf{x}_0}.$$

Hence,

$$\|\mathbf{x}(t)\| < \varepsilon \quad \text{for all } t \in [t_0 + \sigma,\infty) \cap I_{t_0,\mathbf{x}_0}.$$

On the other hand, if $\|\mathbf{x}(t)\| \geq \delta$ for all $t \in [t_0, t_0+\sigma]$, then, using Conditions (a), (c), Theorem 2.7, and (4.5), we obtain

$$\begin{aligned} a(\|\mathbf{x}(t_0+\sigma)\|) &\leq V(t_0+\sigma, \mathbf{x}(t_0+\sigma)) \\ &= V(t_0, \mathbf{x}(t_0, t_0, \mathbf{x}_0)) + \int_{[t_0,t_0+\sigma)} V_g'(s, \mathbf{x}(s))\, d\mu_g(s) \\ &\leq V(t_0, \mathbf{x}_0) - \int_{[t_0,t_0+\sigma)} v(s)\phi(\|\mathbf{x}(s)\|)\, d\mu_g(s) \\ &\leq V(t_0, \mathbf{x}_0) - M\int_{[t_0,t_0+\sigma)} v(s)\, d\mu_g(s) \\ &< b(\|\mathbf{x}_0\|) - b(\|\mathbf{x}_0\|) \\ &= 0. \end{aligned}$$

This is a contradiction. Hence, we conclude that $\mathbf{x} = \mathbf{0}$ is uniformly asymptotically stable. □

In the next example, we provide an application of Theorem 4.11 for a system subject to a derivator having an infinite number of discontinuities and for which the trivial solution $\mathbf{x} = \mathbf{0}$ is uniformly asymptotically stable.

Example 4.12. Let us consider the dynamical system

$$x_g'(t) = f(t, x(t)) \quad \text{for } g\text{-almost all } t \geq \theta \geq 0, \tag{4.9$_\theta$}$$

where $g : \mathbb{R} \to \mathbb{R}$ is given in (4.8), and $f : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ is the function defined by

$$f(t,x) = \begin{cases} -t\arctan(x) & \text{if } t \in [0,\infty) \setminus D_g, \\ \nu_k x & \text{if } t = t_k,\ k \in \mathbb{N}, \end{cases}$$

where $\{\nu_k\}_k \subset \mathbb{R}_+^*$ is a sequence satisfying

$$\lim_{k\to\infty} \frac{1}{\prod_{i=1}^k (1+\nu_i)^2} = a_0 > 0.$$

The map f satisfies conditions of Theorem 3.13. Thus, for every $(t_0, x_0) \in [0,\infty) \times \mathbb{R}$, the problem (4.9$_\theta$) with $\theta = t_0$ has a maximal solution $x : [t_0, +\infty) \to \mathbb{R}$ satisfying $x(t_0) = x_0$. Observe that $x = 0$ is an equilibrium of the Stieltjes dynamical system (4.9$_\theta$) with $\theta = 0$.

Let us define the function $V : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ by

$$V(t,x) = \begin{cases} x^2 & \text{if } t \in [0, t_1], \\ \dfrac{x^2}{\prod_{i=1}^k (1+\nu_i)^2} & \text{if } t \in (t_k, t_{k+1}],\ k \in \mathbb{N}. \end{cases}$$

Clearly $V \in \mathcal{V}_g^1$, and $a(|x|) \leq V(t,x) \leq b(|x|)$ for all $(t,x) \in [0,\infty) \times \mathbb{R}$, where $a, b \in \mathcal{K}$ are functions defined by $a(s) = a_0 s^2$ and $b(s) = s^2$ for all $s \in [0,\infty)$. In addition, for all $(t,x) \in [0,\infty) \times \mathbb{R}$:

$$\begin{aligned} \frac{\partial V}{\partial_g t}(t,x) &= \begin{cases} 0 & \text{if } t \in [0,t_1] \cup (t_k, t_{k+1}),\ k \in \mathbb{N}, \\ \dfrac{\frac{x^2}{\prod_{i=1}^k (1+\nu_i)^2} - \frac{x^2}{\prod_{i=1}^{k-1} (1+\nu_i)^2}}{g(t_k^+) - g(t_k)} & \text{if } t = t_k,\ k \in \mathbb{N}, \end{cases} \\ &= \begin{cases} 0 & \text{if } t \in [0,t_1) \cup (t_k, t_{k+1}),\ k \in \mathbb{N}, \\ \dfrac{1-(1+\nu_k)^2}{\prod_{i=1}^k (1+\nu_i)^2} x^2 & \text{if } t = t_k,\ k \in \mathbb{N}. \end{cases} \end{aligned}$$

For g-almost every $t \in [t_0, \infty) \setminus D_g$, we have that $t \in [0, t_1)$ or there exists $k \in \mathbb{N}$ such that $t \in (t_k, t_{k+1})$. Thus, by means of Proposition 2.10, we obtain if $t \in [0, t_1)$,

$$\begin{aligned} V_g'(t, x(t)) &= \frac{\partial V}{\partial_g t}(t, x(t)) + \frac{\partial V}{\partial x}(t, x(t)) f(t, x(t)) \\ &= -2x(t)t \arctan(x(t)) \\ &\leq -2t \frac{x(t)^2}{1 + x(t)^2}, \end{aligned}$$

where the last inequality follows from the Mean Value Theorem. While if there exists $k \in \mathbb{N}$ such that $t \in (t_k, t_{k+1})$, then

$$\begin{aligned} V_g'(t, x(t)) &= \frac{\partial V}{\partial_g t}(t, x(t)) + \frac{\partial V}{\partial x}(t, x(t)) f(t, x(t)) \\ &= \frac{-2x(t)}{\prod_{i=1}^{k}(1+\nu_i)^2} t \arctan(x(t)) \\ &\leq \frac{-2tx(t)^2}{(1 + x(t)^2)\left(\prod_{i=1}^{k}(1+\nu_i)^2\right)}. \end{aligned}$$

For $t_k \in [t_0, \infty) \cap D_g$, we have that

$$\begin{aligned} V_g'(t_k, x(t_k)) &= \frac{V(t_k^+, x(t_k^+)) - V(t_k, x(t_k))}{g(t_k^+) - g(t_k)} \\ &= \frac{V(t_k^+, x(t_k) + \mu_g(\{t_k\}) f(t_k, x(t_k))) - V(t_k, x(t_k))}{g(t_k^+) - g(t_k)} \\ &= \frac{\frac{(1+\nu_k)^2 x(t_k)^2}{\prod_{i=1}^{k}(1+\nu_i)^2} - \frac{x(t_k)^2}{\prod_{i=1}^{k-1}(1+\nu_i)^2}}{g(t_k^+) - g(t_k)} \\ &= 0. \end{aligned}$$

Therefore, we conclude that

$$V_g'(t, x(t)) \leq -v(t)\phi(|x(t)|) \quad \text{for } g\text{-almost all } t \geq t_0,$$

where $v : [0, \infty) \to [0, \infty)$ is the function defined for every $t \in [0, \infty)$ by

$$v(t) = \begin{cases} 2t & \text{if } t \in [0, t_1), \\ 0 & \text{if } t = t_k,\ k \in \mathbb{N}, \\ \dfrac{2t}{\prod_{i=1}^{k}(1+\nu_i)^2} & \text{if } t \in (t_k, t_{k+1}),\ k \in \mathbb{N}, \end{cases}$$

and $\phi \in \mathcal{K}$ the function given by $\phi(y) = y^2/(1 + y^2)$ for all $y \in [0, \infty)$.

Observe that the function v satisfies

$$\begin{aligned} \lim_{t \to +\infty} \inf_{t_0 \in [0,\infty)} \int_{[t_0, t_0+t)} v(s)\, d\mu_g(s) &\geq \lim_{t \to +\infty} \inf_{t_0 \in [0,\infty)} \int_{t_0}^{t_0+t} a_0 2s\, ds \\ &= a_0 \lim_{t \to +\infty} \inf_{t_0 \in [0,\infty)} 2tt_0 + t^2 \\ &= \infty. \end{aligned}$$

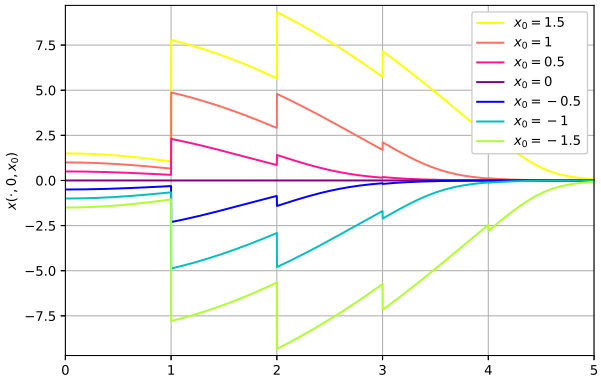


Figure 4.4: Asymptotic behavior of solutions of the dynamical system (4.9_θ) with $\theta = 0$ and $D_g = \mathbb{N}$, obtained with a time-discretization step-size of 10^{-3}, with $\nu_k = e^{\frac{2}{k^2}} - 1$ for all $k \in \mathbb{N}$.

By means of Theorem 4.11, we deduce that $x = 0$ is uniformly asymptotically stable equilibrium. Figure 4.4 illustrates the asymptotic behavior of solutions of the dynamical system (4.9_θ) with $\theta = 0$ and $D_g = \mathbb{N}$.

Remark 4.13. Observe that, in Example 4.9, the function v satisfies

$$\lim_{t\to+\infty} \inf_{t_0\in[0,\infty)} \int_{[t_0,t_0+t)} v(s)\, d\mu_g(s) = 0 \neq \infty.$$

Thus, Condition (d) of Theorem 4.11 does not hold. Consequently, uniform asymptotic stability cannot be deduced.

In the classical case where $g \equiv \mathrm{id}_{\mathbb{R}}$, corollary results [13, Theorem 4.2] and [23] are well-known when Conditions (b) of Theorem 4.8 is replaced by $V_g'(t, \mathbf{x}(t))$ being negative definite along each maximal solution $\mathbf{x}$ for every $t \geq t_0$. However, to present an analogous statement, we require an additional assumption to avoid the case when $\lim_{t\to\infty} g(t) = l < \infty$, and in particular, when there exists $T \geq 0$ such that $(T, \infty) \subset C_g$. Example 4.4 provides an interesting illustration of attractivity lack for asymptotic stability.

Corollary 4.14. *Assume that g is not bounded from above and Conditions ($H_{\Omega,0}$) and ($H_{\mathbf{f},0}$) hold. If there exist $V : [0,\infty) \times B_{\mathbb{R}^n}(\mathbf{0}, r_0) \to \mathbb{R}$; $V \in \mathcal{V}_1^g$ and $a, b, \phi \in \mathcal{K}$ such that*

(a) *$a(\|\mathbf{u}\|) \leq V(t,\mathbf{u}) \leq b(\|\mathbf{u}\|)$, for every $(t,\mathbf{u}) \in [0,\infty) \times B_{\mathbb{R}^n}(\mathbf{0}, r_0)$;*

(b) *there exists $r \in (0, r_0]$ such that, for every $(t_0, \mathbf{x}_0) \in [0,\infty) \times B_{\mathbb{R}^n}(\mathbf{0}, r)$, ($H_{\mathbf{x}_0,t_0}$) holds and, for $\mathbf{x} : I_{t_0,\mathbf{x}_0} \to B_{\mathbb{R}^n}(\mathbf{0}, r_0)$ the maximal solution of the system (4.1_θ) with $\theta = t_0$ and satisfying $\mathbf{x}(t_0) = \mathbf{x}_0$, one has that*

$$V_g'(t, \mathbf{x}(t)) \leq -\phi(\|\mathbf{x}(t)\|), \quad \text{for } g\text{-almost all } t \geq t_0.$$

Then, the trivial solution $\mathbf{x} = \mathbf{0}$ of the system (4.1_θ) with $\theta = 0$ is asymptotically stable. Furthermore, if

$$\lim_{t\to+\infty} \inf_{t_0\in[0,\infty)} g(t + t_0) - g(t_0) = \infty,$$

then, the trivial solution $\mathbf{x} = \mathbf{0}$ of the system (4.1_θ) with $\theta = 0$ is uniformly asymptotically stable.

Proof. Observe that Conditions of Theorem 4.8 hold for $v \equiv 1$ as

$$\inf_{t_0\in[0,\infty)} \lim_{t\to+\infty} \int_{[t_0,t+t_0)} \omega(s)\, d\mu_g(s) = \inf_{t_0\in[0,\infty)} \lim_{t\to+\infty} g(t+t_0) - g(t_0) = \infty.$$

Hence, the trivial solution is $\mathbf{x} = \mathbf{0}$ of the system (4.1_θ) with $\theta = 0$ is asymptotically stable. Moreover, if

$$\lim_{t\to+\infty} \inf_{t_0\in[0,\infty)} g(t+t_0) - g(t_0) = \infty, \tag{4.10}$$

then, Theorem 4.11 ensures that $\mathbf{x} = \mathbf{0}$ is uniformly asymptotically stable. □

Remark 4.15. In Corollary 4.14, it is worth mentioning that the assumption that the derivator g is not bounded from above does not necessarily imply that (4.10) holds. Indeed, let us consider the increasing sequence $\{t_n\}_{n\in\mathbb{N}}$ defined by

$$\begin{cases} t_n = 2t_{n-1} - t_{n-2} + 1, & n \in \mathbb{N},\ n \geq 3, \\ t_2 = 3,\ t_1 = 1. \end{cases}$$

Now, consider the derivator $g : \mathbb{R} \to \mathbb{R}$ defined by

$$g(t) = \begin{cases} t & \text{if } t \leq 0, \\ 0 & \text{if } t \in [0, t_1], \\ n & \text{if } t \in (t_n, t_{n+1}],\ n \in \mathbb{N}. \end{cases}$$

Observe that

$$D_g = \{t_n\}_{n\in\mathbb{N}} = \{1, 3, 6, 10, 15, \dots\} \quad \text{and} \quad C_g = (0, t_1) \cup \bigcup_{n\in\mathbb{N}} (t_n, t_{n+1}).$$

Notice that for every $t > 0$, there exists $t_0 \in [0, \infty)$ such that $[t_0, t + t_0] \subset C_g$. Thus,

$$\lim_{t\to+\infty} \inf_{t_0\in[0,\infty)} g(t+t_0) - g(t_0) = 0 \neq \infty.$$

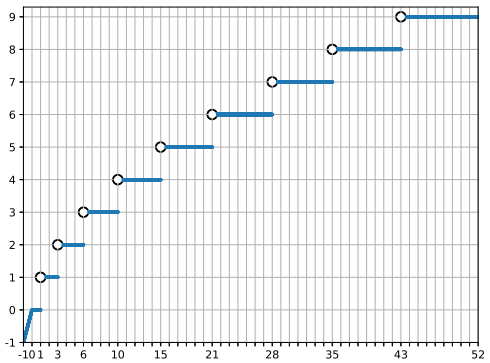


Figure 4.5: Graph of the derivator g in Remark 4.15.

5 Applications to dynamics of population

5.1 Stable equilibrium of a population subject to train vibrations

In this subsection, by means of a system of Stieltjes differential equations, we study the long-term impact of high-speed train vibrations and noise pollution on a population of animals living near railways. Depending on the species and their sensitivity to vibrations, various implications can be observed, we mention for instance:

- Hearing damage resulting in from the significant noise and vibrations that can potentially harm animals with sensitive hearing such as certain small mammals, birds, and bats which rely heavily on their hearing for communication, navigation, detection of predators, and finding food, thus, prolonged exposure to train vibrations may lead to hearing impairment or damage, disrupting their normal behaviors and increasing their death rate.

- Increased stress levels since some animals may be startled by the vibrations. This can affect their feeding patterns, reduce their reproduction rate, or lead to emigration resulting in a loss of suitable habitat and altering the composition and diversity of the local ecosystem

- Ecological interactions disruption which can affect pollination for instance if the vibrations deter insects that are important pollinators, disturb ground-dwelling organisms (insects, reptiles, and small mammals...) which can implicitly impact other species that rely on them as a food source.

In particular, an Allee effect can be observed in this regard, especially when the survival of the population depends on a minimum threshold size $M > 0$. In the follows, we denote by $x(t)$ the number of individuals of a population living in a region near a railway with a carrying capacity $K > M$.

Let us assume that a certain number $m > 0$ of trains pass through the area every day. We refer to $\{\delta_i\}_{i=1}^{i=m}$ as the moments when trains pass in a single day. Once a train pass by the area, its impact is significant for a proportion of individuals living near the railway. They may experience vibrations or direct injuries. In the following analysis, we use a Stieltjes differential equation to model the dynamics of this population affected by train vibrations, and we study the asymptotic behavior of its solutions. In doing so, we require a derivator $g : \mathbb{R} \to \mathbb{R}$ presenting discontinuities for $t \in \{\delta_i\}_{i=1}^{i=m} + 24\mathbb{N}$ such that $\mu_g(\{t\})$ quantifies the rate at which the risk of damage varies. Depending on the specific δ_i, this rate can either increase or decrease, reflecting the varying impact of vibrations during daylight and nighttime hours. For simplicity, we can take for instance:

$$g(t) = t + \sum_{k\in\mathbb{N}} \sum_{i=1}^{m} \chi_{[\delta_i+24k,\infty)}(t) \quad \text{for all } t \in \mathbb{R},$$

with $\mu_g(\{\cdot\}) \equiv 1$ on $D_g = \{\delta_i : i = 1, \dots, m\} + 24\mathbb{N}$.

In the sequel, we suggest to analyse the asymptotic behaviour of the dynamics of this population, through the study of asymptotic stability of the zero equilibrium of the Stieltjes dynamical system:

$$x'_g(t) = f(t, x(t)) \quad \text{for } g\text{-almost every } t \geq \theta \geq 0, \tag{5.1$_\theta$}$$

where $f : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ is defined by

$$f(t,x) = \begin{cases} \rho x \left(1 - \dfrac{x}{K}\right)\left(\dfrac{x}{M} - 1\right), & \text{if } t \notin \{\delta_i + 24k\}_{i=1}^{m},\ k = 0,1,2\ldots \\ -dx, & \text{if } t \in \{\delta_i + 24k\}_{i=1}^{m},\ k = 0,1,2\ldots \end{cases} \tag{5.2}$$

The parameters of the model can be understood as:

$K > M$**:** the carrying capacity of the environment;

$\rho > 0$**:** the intrinsic rate of reproduction of the population;

$d \in (0,1)$**:** a constant related to the impact induced by trains, either a migration rate immediately following the passage of trains or a mortality rate for certain populations that live in close proximity to the railway.

To simplify the analysis, we make the assumption that a train passes every hour over a 24-hour period. Thus, we consider

$$g(t) = t + \sum_{n\in\mathbb{N}} \chi_{[n,\infty)}(t) \quad \text{for all } t \in \mathbb{R},$$

which is such that $D_g = \mathbb{N}$ and $\mu_g(\{t\}) = 1$ for all $t \in D_g$.

Now, we will focus on the zero equilibrium to study its local asymptotic stability within a region $(-r_0, r_0)$ for some $r_0 > 0$ that will be determined to enhance the impact of environmental factors threatening this population.

Since the trivial solution $x = 0$ is an equilibrium of the dynamical system (5.1_θ) with $\theta = 0$, let us consider $r_0 \leq M$. For all $(t_0, x_0) \in ([0,\infty) \cap D_g) \times (-r_0, r_0)$, if x is a solution of (5.1_θ) with $\theta = t_0$ and satisfying $x(t_0) = x_0$, then

$$x_0 + \mu_g(\{t_0\}) f(t_0, x_0) = (1-d)x_0 \in (-r_0, r_0).$$

Thus, (H_{x_0,t_0}) holds for $r = r_0$. One can also verify that f satisfies ($\mathrm{H}_{f,0}$) and ($\mathrm{H}_{\Omega,0}$). Hence, Theorem 3.7, yields the existence of a maximal solution $x : I_{t_0,x_0} \to \mathbb{R}$ of (5.1_θ) with $\theta = t_0$ and satisfying $x(t_0) = x_0$. Let us show that $\omega(t_0, x_0) = \infty$. First of all, let $\tau \in (t_0, \omega(t_0, x_0))$ be such that $x : [t_0, \tau] \to (-r_0, r_0)$. Observe that, if $x(\tau) = 0$, by uniqueness of the solution, we deduce that $x \equiv 0$ which lies in $(-r_0, r_0)$. Thus, two other cases occur when $x(\tau) \neq 0$.

Case 1: If $\tau \in D_g$, then $x(\tau^+) = x(\tau) + \mu_g(\{\tau\}) f(\tau, x(\tau)) = (1-d)x(\tau) \in (-r_0, r_0)$.

Case 2: If $\tau \notin D_g$, and if we assume that $x(\tau) \in (0, r_0)$, then $x'_g(\tau) = f(\tau, x(\tau)) < 0$. Thus, by g-continuity of x at τ, there exists $\tau_1 \in (\tau, \omega(t_0, x_0))$ such that

$$x : [t_0, \tau_1] \to (0, x(\tau)] \subset (0, r_0).$$

Similarly, if $x(\tau) \in (-r_0, 0)$ then $x'_g(\tau) = f(\tau, x(\tau)) > 0$. Thus, there exists $\tau_2 > 0$ such that

$$x : [\tau, \tau_2] \to [x(\tau), 0) \subset (-r_0, 0).$$

Repeating the same argument, we deduce that

$$x(t) \in [-\lambda_{x_0,t_0}, \lambda_{x_0,t_0}] \subset (-r_0, r_0) \quad \text{for all } t \in I_{t_0,x_0}$$

where

$$\lambda_{x_0,t_0} = \sup_{t\in[t_0,\tau]} |x(t, t_0, x_0)|.$$

By Corollary 3.10, we deduce that $\omega(t_0, x_0) = \infty$.

Now, we define the function $V : [0,\infty) \times (-r_0, r_0) \to \mathbb{R}$ by

$$V(t,x) = x^2 \quad \text{for all } (t,x) \in [0,\infty) \times (-r_0, r_0).$$

Clearly, $V \in \mathcal{V}_g^1$. For all $(t_0, x_0) \in [0,\infty) \times (-r_0, r_0)$, let $x : [t_0,\infty) \to (-r_0, r_0)$ be the maximal solution of (5.1_θ) with $\theta = t_0$ and satisfying $x(t_0) = x_0$. Thus, using Proposition 2.10, we obtain, for $t_0 \in [0,\infty)$ and for g-almost all $t \in [t_0,\infty) \setminus D_g$,

$$\begin{aligned} V_g'(t,x(t)) &= \frac{\partial V}{\partial_g t}(t,x(t)) + \frac{\partial V}{\partial x}(t,x(t))x_g'(t) \\ &= \frac{\partial V}{\partial_g t}(t,x(t)) + \frac{\partial V}{\partial x}(t,x(t))f(t,x(t)) \\ &= 2\rho x(t)^2 \left(1 - \frac{x(t)}{K}\right)\left(\frac{x(t)}{M} - 1\right). \end{aligned}$$

While for $t \in [t_0,\infty) \cap D_g$, we obtain

$$\begin{aligned} V_g'(t,x(t)) &= \frac{V(t^+, x(t^+)) - V(t,x(t))}{g(t^+) - g(t)} \\ &= \frac{V(t^+, x(t) + \mu_g(\{t\})f(t,x(t))) - V(t,x(t))}{g(t^+) - g(t)} \\ &= (1-d)^2 x(t)^2 - x(t)^2 \\ &= (-2d + d^2)x(t)^2. \end{aligned}$$

As $(-2d + d^2) < 0$, it follows that $V_g'(t,x(t))$ is negative definite. Since

$$a(|x|) \le V(t,x) \le b(|x|) \quad \text{for all } (t,x) \in [0,\infty) \times (-r_0, r_0),$$

with $a, b \in \mathcal{K}$ defined by $a(s) = b(s) = s^2$ for all $s \in [0,\infty)$, Corollary 4.14 ensures that the trivial solution $x = 0$ is uniformly asymptotically stable.

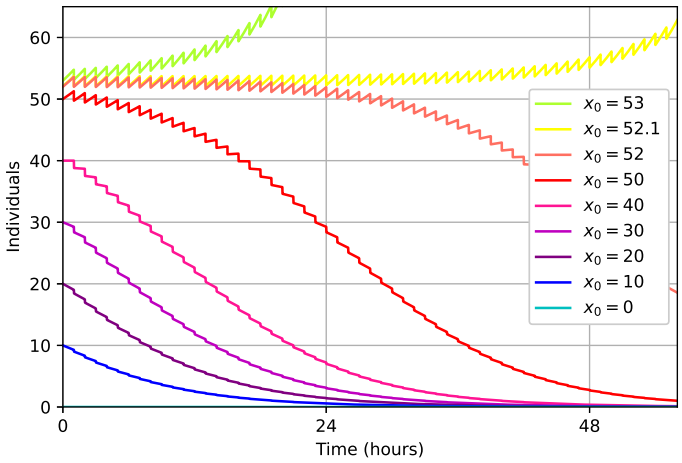


Figure 5.1: Asymptotic behavior of solutions of the dynamical system (5.1_θ) with $\theta = 0$ obtained with a time-discretization step-size of 10^{-3}, with $\rho = 0.1$, $K = 1000$, $M = 40$, and $d = 0.03$.

The asymptotic behavior of solutions is illustrated in Figure 5.1. It illustrates the long-term effect of the high-speed trains vibrations. The outcome raises alarm about the overall stability and the persistence of the whole ecosystem in this region. Specifically, the impact of the vibrations shifts the minimum threshold size M to a new threshold $M' > M$, as shown in Figure 5.1 (where we observe that $M' = 53$ individuals whereas $M = 40$ individuals). This shift indicates a heightened likelihood of population extinction if there is no measure ensuring that the initial population exceeds M' at the onset of trains activity.

5.2 Stable equilibrium of bacteria–ammonia dynamics

Cyanobacteria, similar to plants, participate in oxygenic photosynthesis as they are photosynthetic bacteria. *Photosynthesis* is the biochemical process through which organisms convert light energy into chemical energy in the form of glucose or other organic compounds resulting oxygen release. The energy captured is then used to fuel the synthesis of organic molecules such as glucose, serving as an energy source for the bacteria.

Cyanobacteria are found in diverse habitats, including freshwater, marine environments, and terrestrial ecosystems. In this subsection, we consider a species of cyanobacteria that has also the ability to fix nitrogen such as *Anabaena cyanobacteria*. Through nitrogen fixation, atmospheric nitrogen gas (N_2) is converted into a form that can be used by plants and other organisms by means the enzyme *nitrogenase*. This enzyme catalyzes the conversion of nitrogen gas (N_2) into ammonium ions (NH_4^+) based on a considerable amount of energy obtained from photosynthesis, these ammonium ions are assimilated into amino acids and proteins, which are essential for the growth and survival of Cyanobacteria. As a by product of nitrogen fixation, ammonia gas (NH_3) can be released. Some of the assimilated ammonium ions (NH_4^+) are released back into the environment providing neighboring vegetative cells with a source of nitrogen.

To avoid losing valuable nitrogen nutrients and optimizing nitrogen utilization efficiency, this population has mechanisms allowing ammonium ions (NH_4^+) reabsorption, and ammonia gas (NH_3) assimilation through converting ammonia gas (NH_3) into ammonium ions (NH_4^+). In what follows, the term "ammonia" refers to both the protonated and unprotonated forms, which are denoted as (NH_3) and (NH_4^+) respectively [4]. It is worth mentioning that ammonia is commonly used in cleaning products, fertilizers. Beyond that, ammonia's cooling properties make it an essential refrigerant in air conditioning systems and refrigerators.

To optimize resource utilization and adapt to the varying environmental conditions, the population undergoes a day-night cycling of nitrogen fixation and carbon consumption [32]. This is due to the sensitivity of the nitrogenase enzyme responsible for nitrogen fixation to oxygen. Thus, nitrogen fixation is carried out during daylight hours when the photosynthesis can provide the necessary energy and oxygen levels are relatively low. During the nights, since oxygen levels within the cells would be higher, this population of cyanobacteria reduces their metabolic activity and growth and relies on stored carbon compounds to fulfill their energy needs. Our objective in the sequel is to observe the dynamics of this population which thrives in the presence of ammonia in a culture room without exposure to artificial light, tracking the levels of the ammonia during this process. Here, we assume that the carbon dioxide (CO_2), the nitrogen gas (N_2) and nutrients supply are maintained steady as well as the PH level. Since the population undergoes dormancy phases during the nights, we identify the days with intervals of the form $[2k, 2k+1]$, $k = 0, 1, 2, \ldots$, and the night with intervals $[2k+1, 2k+2]$, $k = 0, 1, 2, \ldots$ In our example, we differentiate with respect to a derivator g whose slopes

describe the intensity of light, which is necessary for the photosynthesis process. We require that g presents smaller slopes at the beginning and at the end of the daylight hours, with a maximal slope of 1 representing the peak light intensity at middays where $t = 2k + 1/2$, and remains constant during the dormancy phases in the night $[2k + 1, 2k + 2]$, $k = 0, 1, 2, \dots$ For instance, we consider $g : \mathbb{R} \to \mathbb{R}$ defined by

$$g(t) = \begin{cases} \dfrac{\sin(\pi(t - 1/2)) + 1}{\pi} & \text{if } t \in [0, 1] \\ 2/\pi & \text{if } t \in (1, 2], \end{cases}$$

and $g(t) = 2/\pi + g(t - 2)$ for $t \geq 2$.

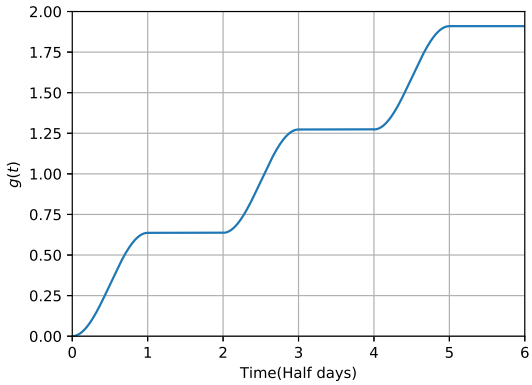


Figure 5.2: Graph of the derivator g.

We denote by $N(t)$ the biomass of cyanobacteria (grams per liter), and $A(t)$ the ammonia concentration in the environment at time $t \geq 0$. Since the population thrives in the presence of ammonia, we can assume that the growth rate is proportional to the level of ammonia. We denote ρ the maximal intrinsic coefficient of reproduction that the population can reach in the presence of one unit of ammonia with maximal sunlight intensity. Thus, the dynamics can be modeled using the autonomous system of Stieltjes differential equations:

$$\mathbf{u}'_g(t) = \mathbf{F}(\mathbf{u}(t)), \quad \text{for } g\text{-almost every } t \geq \theta \geq 0, \tag{5.3$_\theta$}$$

where $\mathbf{u} = (N, A)$ and $\mathbf{F} = (F_1, F_2) : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by

$$\mathbf{F}(N, A) = \left(\rho AN\left(1 - \frac{N}{K}\right), (\alpha N - \beta AN)\right), \tag{5.4}$$

and where the parameters of the model can be understood as:

$K > 0$: the carrying capacity of the culture room, which forms a spacial constraint for growth;

$\alpha > 0$: a constant related to the production of ammonia through nitrogen fixation.

$\beta > 0$: a constant related to the proportion of the reabsorption of ammonia by the population depending on the level of ammonia in the environment.

Observe that $\mathbf{u}^* = (K, \alpha/\beta)$ is an equilibrium of the system (5.3$_\theta$) with $\theta = 0$ among other equilibria. Its asymptotic stability would guarantee the persistence of the population with nonzero ammonia production. Therefore, we study local asymptotic stability in a neighborhood $B_{\mathbb{R}^2}(\mathbf{u}^*, r_0)$ for some $r_0 > 0$ of this equilibrium. To this aim, we translate our study in a

neighborhood of $\mathbf{x} = \mathbf{0} = (0,0) \in \mathbb{R}^2$ with the change of variables $\mathbf{x} = (x_1, x_2) := \mathbf{u} - \mathbf{u}^*$, we obtain the system

$$\mathbf{x}'_g(t) = \mathbf{f}(\mathbf{x}(t)) \quad \text{for } g\text{-almost every } t \geq \theta \geq 0, \tag{5.5$_\theta$}$$

where $\mathbf{f} = (f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by

$$\mathbf{f}(x_1, x_2) = \left(-\rho \frac{x_1}{K} \left(x_2 + \frac{\alpha}{\beta} \right) (x_1 + K), -\beta x_2 (x_1 + K) \right). \tag{5.6}$$

So, $\mathbf{x} = \mathbf{0} = (0,0) \in \mathbb{R}^2$ is an equilibrium of the dynamical system (5.5$_\theta$) with $\theta = 0$. Next, we prove that $\mathbf{x} = \mathbf{0}$ is asymptotically stable, and hence, $\mathbf{u}^* = \left(K, \alpha/\beta\right)$ is an asymptotically stable equilibrium of the system (5.3$_\theta$) with $\theta = 0$.

Let us consider $r_0 = \min\{K, \alpha/\beta\}$, and let $(t_0, \mathbf{x}_0) \in [0, \infty) \times B_{\mathbb{R}^2}(\mathbf{0}, r_0)$. Arguing as in the previous subsection, since the function $\mathbf{f}$ satisfies conditions of Theorem 3.7, there exists, $\mathbf{x} = (x_1, x_2) : I_{t_0,\mathbf{x}_0} \to \mathbb{R}^2$, the maximal solution of the system (5.5$_\theta$) with $\theta = t_0$ and satisfying $\mathbf{x}(t_0) = \mathbf{x}_0$. Let $\tau \in (0, \omega(t_0, \mathbf{x}_0)) \setminus C_g$ be such that $\mathbf{x} = (x_1, x_2) : [t_0, \tau] \to B_{\mathbb{R}^2}(\mathbf{0}, r_0)$. In what follows, we analyze the possible cases:

Case 1: If $x_1(\tau), x_2(\tau) \in (0, r_0)$ (resp. $x_1(\tau), x_2(\tau) \in (-r_0, 0)$), then, for $i = 1, 2$, we have $(x_i)'_g(\tau) = f_i(\mathbf{x}(\tau)) < 0$ (resp. $(x_i)'_g(\tau) = f_i(\mathbf{x}(\tau)) > 0$). Therefore, by the g-continuity of $\mathbf{x}$ at τ, there exists $\tau_1 \in (\tau, \omega(t_0, \mathbf{x}_0))$ (τ_1 can be chosen such that $\tau_1 \notin C_g$) such that

$$x_i : [t_0, \tau_1] \to (0, x_i(\tau)] \subset (0, r_0) \quad (\text{resp. } x_i : [t_0, \tau_1] \to [x_i(t_0), 0) \subset (-r_0, 0)).$$

Case 2: If $x_1(\tau) \in (0, r_0)$ and $x_2(\tau) \in (-r_0, 0)$, then

$$(x_1)'_g(\tau) = f_1(\mathbf{x}(\tau)) < 0 \quad \text{and} \quad (x_2)'_g(\tau) = f_2(\mathbf{x}(\tau)) > 0.$$

Therefore, there exists $\tau_2 \in (\tau, \omega(t_0, \mathbf{x}_0)) \setminus C_g$ such that

$$x_1 : [\tau, \tau_2] \to (0, x_1(\tau)] \subset (0, r_0) \quad \text{and} \quad x_2 : [\tau, \tau_2] \to [x_2(\tau), 0) \subset (-r_0, 0).$$

Case 3: If $x_1(\tau) \in (-r_0, 0)$ and $x_2(\tau) \in (0, r_0)$, then, similarly to Case 2, we deduce that there exists $\tau_3 \in (\tau, \omega(t_0, \mathbf{x}_0))$ such that

$$x_1 : [\tau, \tau_3] \to [x_1(\tau), 0) \subset (-r_0, 0) \quad \text{and} \quad x_2 : [\tau, \tau_3] \to (0, x_2(\tau)] \subset (0, r_0).$$

From this argument, we deduce that the solution

$$\mathbf{x}(t) \in [-\lambda_{\mathbf{x}_{0,1},t_0}, \lambda_{\mathbf{x}_{0,1},t_0}] \times [-\lambda_{\mathbf{x}_{0,2},t_0}, \lambda_{\mathbf{x}_{0,2},t_0}] \subset B_{\mathbb{R}^2}(\mathbf{0}, r_0) \quad \text{for all } t \in I_{t_0,\mathbf{x}_0},$$

where

$$\lambda_{\mathbf{x}_{0,i},t_0} = \sup_{t \in [t_0,\tau]} |x_i(t, t_0, \mathbf{x}_0)| \quad \text{for } i = 1, 2, \text{ and } \mathbf{x}_0 = (\mathbf{x}_{0,1}, \mathbf{x}_{0,2}).$$

Using Corollary 3.10, we deduce that $\omega(t_0, \mathbf{x}_0) = \infty$.

Now, let us consider the function $V : [0, \infty) \times B_{\mathbb{R}^2}(\mathbf{0}, r_0) \to \mathbb{R}$ defined by $V(t, \mathbf{x}) = x_1^2 + x_2^2$ for all $t \in [0, \infty)$ and $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$. It is clear that $V \in \mathcal{V}_1^g$. Let $(t_0, \mathbf{x}_0) \in [0, \infty) \times B_{\mathbb{R}^2}(\mathbf{0}, r_0)$,

and $\mathbf{x} : [t_0, \infty) \to B_{\mathbb{R}^2}(\mathbf{0}, r_0)$ the maximal solution of (5.5_θ) with $\theta = t_0$ and satisfying $\mathbf{x}(t_0) = \mathbf{x}_0$. By means of Proposition 2.10, for g-almost all $t \in [t_0, \infty)$, we obtain

$$\begin{aligned} V_g'(t, \mathbf{x}(t)) &= \frac{\partial V}{\partial_g t}(t, \mathbf{x}(t)) + \sum_{i=1}^{2} \frac{\partial V}{\partial x_i}(t, \mathbf{x}(t))(x_i)_g'(t) \\ &= \frac{\partial V}{\partial_g t}(t, \mathbf{x}(t)) + \sum_{i=1}^{2} \frac{\partial V}{\partial x_i}(t, \mathbf{x}(t)) f_i(t, \mathbf{x}(t)) \\ &= -2\rho \frac{x_1(t)^2}{K}(x_2(t) + \alpha/\beta)(x_1(t) + K) - 2\beta x_2(t)^2(x_1(t) + K). \end{aligned} \tag{5.7}$$

Thus, $V_g'(t, \mathbf{x}(t))$ is negative definite. Since

$$a(\|\mathbf{z}\|) \le V(t, \mathbf{z}) \le b(\|\mathbf{z}\|) \quad \text{for all } (t, \mathbf{z}) \in [0, \infty) \times B_{\mathbb{R}^2}(\mathbf{0}, r_0),$$

where $a, b \in \mathcal{K}$ are defined by $a(s) = s^2$ and $b(s) = 2s^2$ for all $s \in [0, \infty)$. It follows from Corollary 4.14 that $\mathbf{x} = \mathbf{0}$ is uniformly asymptotically stable. Hence, $\mathbf{u}^* = (K, \alpha/\beta)$ is a uniformly asymptotically stable equilibrium of the system (5.3_θ). The graph of asymptotic behavior of solutions near the equilibrium $\mathbf{u}^* = (K, \alpha/\beta)$ is given in Figure 5.3 with $\theta = 0$.

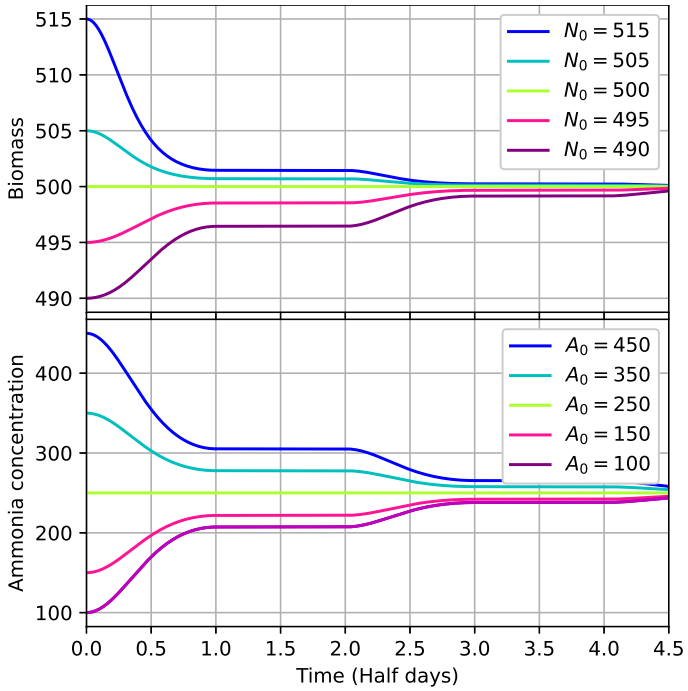


Figure 5.3: Asymptotic behavior of solutions of the dynamical system (5.3_θ) with $\theta = 0$ obtained with a time-discretization step-size of 10^{-3}, with different initial densities N_0 and ammonia concentrations A_0 (g/L), where $\rho = 0.01$, $K = 500$, $\alpha = 1$, and $\beta = 0.004$.

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Existence and convergence of sign-changing solutions for Kirchhoff-type p-Laplacian problems involving critical exponent in $\mathbb{R}^N$

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Abstract. We investigate the existence of sign-changing solutions for Kirchhoff-type problems with p-Laplacian involving critical exponent:

$$-\left(1+b|\nabla v|_p^p\right)\Delta_p v + a(x)|v|^{p-2}v = |v|^{p^*-2}v + \lambda f(v), \quad x \in \mathbb{R}^N,$$

where b and λ are positive parameters, $\Delta_p v = \operatorname{div}(|\nabla v|^{p-2}\nabla v)$, $p^* = \frac{Np}{N-p}$, $1 < p < N$, and $|\cdot|_p$ is the Lebesgue L^p-norm. For sufficiently large λ, employing minimization techniques, quantitative deformation lemma and the constrained variational method, we demonstrate the existence of a least-energy sign-changing solution, whose energy is greater than twice that of the ground state solution. Additionally, we show the convergence behavior of the solution as the parameter $b \searrow 0$. Our findings generalize and extend upon recent results in the literature.

Keywords: sign-changing solution, variational methods, p-Laplacian, Kirchhoff-type problem.

2020 Mathematics Subject Classification: 35J92, 35J60, 35J20.

1 Introduction and main result

We are interested in the existence of sign-changing solutions to the following class of Kirchhoff-type problem with the p-Laplacian involving a critical exponent:

$$-\left(1+b|\nabla v|_p^p\right)\Delta_p v + a(x)|v|^{p-2}v = |v|^{p^*-2}v + \lambda f(v), \quad x \in \mathbb{R}^N, \tag{$\mathcal{QKP}$}$$

where b and λ are positive parameters, $\Delta_p v = \operatorname{div}(|\nabla v|^{p-2}\nabla v)$ is the p-Laplacian, $p^* = \frac{Np}{N-p}$, $1 < p < N$, and $f \in \mathcal{C}^1(\mathbb{R},\mathbb{R})$. The term p-Laplacian has emerged as a crucial concept in various physics backgrounds and nonlinear analysis. This functional has found applications in describing the movement of sandpiles [2], modeling Game Theory [11], and is employed in

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image inpainting [6]. A variety of intriguing results have been obtained for problem ($\mathcal{QKP}$) and similar problems, which has been investigated widely, see [7,8] and the reference therein.

Problem ($\mathcal{QKP}$) is derived from the following Kirchhoff equation

$$\begin{cases} -\left(a + b\int_\Omega |\nabla v|^2\right)\Delta v = f(v), & x \in \Omega, \\ v = 0, & x \in \partial\Omega. \end{cases} \tag{1.1}$$

Problem (1.1) corresponds to the following stationary form of a Kirchhoff type equation

$$v_{tt} - \left(a + b\int_\Omega |\nabla v|^2 dx\right)\Delta v = f(v), \tag{1.2}$$

where a and b are positive constants. In one and two dimensions, equation (1.2) has been employed to model various phenomena in physics, engineering, and other scientific disciplines, effectively approximating the nonlinear vibrations of beams or plates. In [13], Kirchhoff originally introduced (1.2) as an extension of the classical D'Alembert wave equation, which models the free vibrations of elastic strings

$$\rho\frac{\partial^2 v}{\partial t^2} - \left(\frac{p_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial v}{\partial x}\right|^2 dx\right)\frac{\partial^2 v}{\partial x^2} = f(v),$$

with constants ρ, p_0, h, E, and L.

We introduce a mathematical model that characterizes the behavior of a compressible fluid within a homogeneous, isotropic, rigid porous medium. In this context, $\vec{a} = \vec{a}(x,t)$ represents the seepage velocity, φ is the volumetric moisture content, and $v = v(x,t)$ is the density. The continuity equation is given by:

$$\varphi\frac{\partial v}{\partial t} + \operatorname{div}(v\vec{a}) = 0. \tag{1.3}$$

In the case of laminar flow regime through the porous medium, the relationship between the pressure $\pi = \pi(x,t)$ and the momentum velocity $\rho\vec{a}$ is described by the following Darcy's law:

$$\rho\vec{a} = -\lambda \operatorname{grad}\pi. \tag{1.4}$$

The flow rate is not the same in turbulent flow regimes, leading various authors to propose a nonlinear alternative to (1.4). A commonly considered form of the nonlinear Darcy law can be found in works such as Wu et al. [21]:

$$\rho\vec{a} = -\lambda|\operatorname{grad}\pi|^{p-2}\operatorname{grad}\pi, \tag{1.5}$$

where $p > 1$. While considering the polytropic gas equation of state

$$\pi = cv, \tag{1.6}$$

where $c > 0$ is a positive constant of proportionality. By (1.3), (1.5) and (1.6), we obtain

$$\varphi\frac{\partial v}{\partial t} = c^{p-1}\lambda \operatorname{div}\left(|\operatorname{grad} v|^{p-2}\operatorname{grad} v\right). \tag{1.7}$$

Following the variable substitution, we obtain

$$\frac{\partial v}{\partial t} = \operatorname{div}\left(|\nabla v|^{p-2}\nabla v\right), \tag{1.8}$$

with p being a real number greater than 1.

$$\Delta_p v := \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(|\nabla v|^{p-2}\frac{\partial v}{\partial x_i}\right) = \operatorname{div}\left(|\nabla v|^{p-2}\nabla v\right),$$

and

$$v \longmapsto \Delta_p v, \tag{1.9}$$

it is commonly called the p-Laplace operator or the p-Laplacian. In the case of one dimension, one has

$$\Delta_p v = \left(|v'|^{p-2}v'\right)'.$$

Definition 1.1. A solution v is nontrivial provided that $v \neq 0$. A nontrivial solution is called a ground state (or least energy) solution if its energy is the lowest among all nontrivial solutions. Additionally, if v is a nontrivial solution and both v^+ and v^- are nonzero, then v is referred to as sign-changing solution, where $v^+ := \max\{v, 0\}$, $v^- := \min\{v, 0\}$.

Baldelli and Filippucci [4] studied the existence and multiplicity of nontrivial solutions to the following generalized quasilinear Schrödinger equation:

$$-\Delta_p v - \frac{\alpha}{2}\Delta_p\left(|v|^{\alpha}\right)|v|^{\alpha-2}v = \lambda V(x)|v|^{k-2}v + \beta K(x)|v|^{\alpha p^*-2}v \quad \text{in } \mathbb{R}^N,$$

where $\alpha > 1$, β, $\lambda > 0$, and $\alpha < k < \alpha p^*$. The weights are nontrivial and satisfy $0 \leqslant V \in L^{\alpha p^*/(\alpha p^*-k)}(\mathbb{R}^N) \cap \mathcal{C}(\mathbb{R}^N)$, $K \in L^{\infty}(\mathbb{R}^N) \cap \mathcal{C}(\mathbb{R}^N)$. For $N \geqslant 3$ and $1 < p < N$, using the concentration compactness principle, the symmetric Mountain Pass Theorem, the truncation of the energy functional, and the theory of Krasnosel'skii genus, they established the existence of infinitely many nontrivial solutions $v_n \in \mathcal{D}^{1,p}(\mathbb{R}^N) = \{u \in L^{p^*}(\mathbb{R}^N) : \nabla u \in (L^p(\mathbb{R}^N))^N\}$, provided that $1 < \alpha < k < p$.

Li [15] addressed the existence of solutions for the following equation:

$$\begin{aligned}-\operatorname{div}\left(g^p(v)|\nabla v|^{p-2}\nabla v\right) + g^{p-1}(v)g'(u)|\nabla v|^p + V(x)|v|^{p-2}v \\ = K(x)f(v) + Q(x)g(v)|G(v)|^{p^*-2}G(v),\end{aligned}$$

where $1 < p \leqslant N$, $g \in \mathcal{C}^1\left(\mathbb{R}, \mathbb{R}^+\right)$, $V(x)$ and $K(x)$ are positive continuous functions, $Q(x)$ is a bounded continuous function, and $G(v) = \int_0^v g(t)dt$. The author obtained the existence of positive solutions when $N \geqslant 3$. Their proof is based on the truncation of the energy functional and the Mountain Pass Theorem.

In [14], the p-Laplacian Kirchhoff-type equation with logarithmic nonlinearity was investigated:

$$\begin{cases} -\left(a + b|\nabla v|_p^p\right)\Delta_p v = |v|^{q-2}v\ln v^2, & x \in \Omega, \\ v = 0, & x \in \partial\Omega. \end{cases} \tag{1.10}$$

Using the constraint variation method, topological degree theory, and the quantitative deformation lemma, the authors established the existence of ground state sign-changing solutions for problem (1.10). In problem (1.10), they assumed that $N > p$, a and b are positive, and $4 \leqslant 2p < q < p^*$ for some $q > p$. Chen et al. [10] investigated a modified version of problem ($\mathcal{QKP}$) with $\lambda = 1$, and the nonlinearity $f(v)$ replaced by $R(x)g(v)$ without the critical term. The modified problem is given by

$$-\left(1 + b|\nabla v|_p^p\right)\Delta_p v + a(x)|v|^{p-2}v = R(x)g(v),$$

in which $R(x)$ and $a(x)$ represent continuous functions that are positive and $v \in \mathcal{D}^{1,p}(\mathbb{R}^N)$. The authors establish the existence of a ground state sign-changing solution by employing the non-Nehari manifold approach (Unlike the Nahari manifold method, the main idea of the non-Nehari manifold approach lies on finding a minimizing sequence for the energy functional outside the manifold). Motivated by the studies mentioned above, in this paper, we suppose that f belongs to $\mathcal{C}^1(\mathbb{R},\mathbb{R})$ and a belongs to $\mathcal{C}\left(\mathbb{R}^N,\mathbb{R}\right)$ fulfills the following assumptions:

$(\mathcal{A})$ $\inf_{\mathbb{R}^N} a(x) > 0$ and there is R, such that

$$\lim_{|y|\to\infty} m\left\{x \in \mathbb{R}^N :\ x \in B_R(y),\ a(x) \leqslant A\right\} = 0, \quad \forall A > 0,$$

here m denotes the Lebesgue measure.

$(\mathcal{P}_1)$ There holds $\lim_{t\to 0} \frac{f(t)}{|t|^{p-1}} = 0$;

$(\mathcal{P}_2)$ There is $\vartheta \in (2p, p^*)$ such that

$$\lim_{s\to\infty} \frac{f(t)}{|t|^{\vartheta-1}} = 0;$$

$(\mathcal{P}_3)$ There holds $\lim_{t\to\infty} \frac{F(t)}{|t|^{2p}} = +\infty$;

$(\mathcal{P}_4)$ The map $t \mapsto \frac{f(t)}{|t|^{2p-1}}$ is strictly increasing, for all $t \in \mathbb{R}\backslash\{0\}$,

From the assumptions $(\mathcal{P}_3)$ and $(\mathcal{P}_4)$, one obtains

$$f(t)t > 0, \quad t \neq 0; \quad F(t) \geqslant 0, \quad t \in \mathbb{R}. \tag{1.11}$$

Furthermore, from the assumptions $(\mathcal{P}_3)$ and $(\mathcal{P}_4)$, one gets

$$f'(t)t - (2p-1)f(t) < (>)0, \quad \forall t < (>)0. \tag{1.12}$$

For simplicity, we will denote distinct positive constants by c_i, C_i, and C, and $\int$ always stand for $\int_{\mathbb{R}^N}$. Define

$$\mathcal{BC}(\mathbb{R}^N) := \left\{ v \in \mathcal{C}(\mathbb{R}^N) :\ |v|_\infty := \sup_{x\in\mathbb{R}^N} |v(x)| < \infty \right\}.$$

The closure of $\mathcal{C}_c(\mathbb{R}^N)$ in $\mathcal{BC}(\mathbb{R}^N)$ with respect to the uniform norm is the space $\mathcal{C}_0(\mathbb{R}^N)$, where $\mathcal{C}_c(\mathbb{R}^N)$ is space of continuous functions with compact support in $\mathbb{R}^N$. The space of finite measures on $\mathbb{R}^N$ is referred to as $\mathcal{M}(\mathbb{R}^N)$. A sequence $\{\nu_n\}$ converges weakly* to ν in $\mathcal{M}(\mathbb{R}^N)$, written

$$\nu_n \rightharpoonup^* \nu,$$

provided

$$\langle \nu_n, \varphi\rangle \to \langle \nu, \varphi\rangle,$$

$\forall \varphi \in \mathcal{C}_0(\mathbb{R}^N)$. For $1 \leqslant s \leqslant p^*$, we introduce the following norms:

$$\|v\|_{L^s(\mathbb{R}^N)} = |v|_s = |v|_{L^s(\mathbb{R}^N)} := \left(\int |v(x)|^s \, \mathrm{d}x\right)^{\frac{1}{s}}.$$

In order to state our principal results, let

$$W = \left\{ v \in W^{1,p} : \int a(x)|v|^p \, \mathrm{d}x + |\nabla v|_p^p < +\infty \right\},$$

endowed with the following norm

$$\|v\| = \left[\int a(x)|v|^p \, \mathrm{d}x + |\nabla v|_p^p \right]^{\frac{1}{p}}.$$

It is clear that W is continuously embedded into $W^{1,p}(\mathbb{R}^N)$ and consequently the embedding into $L^s(\mathbb{R}^N)$ is also continuous for $s \in [p, p^*]$. This implies that there is a $\gamma_s > 0$ such that

$$|v|_s \leqslant \gamma_s \|v\|, \quad \forall v \in W. \tag{1.13}$$

Lemma 1.2 ([12]). *Assuming that $(\mathcal{A})$ holds. For $p \leqslant s < p^*$, W is compactly embedded into $L^s(\mathbb{R}^N)$.*

Let η_1 denote the first eigenvalue of $-\Delta_p v$. Thus, η_1 represents the smallest value of η such that the problem

$$\begin{cases} -\Delta_p v = \eta |v|^{p-2} v, & \text{in } \Omega, \\ v = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.14}$$

possesses a nontrivial solution. Let the functional $\mathcal{K} : W_0^{1,p} \to \mathbb{R}$ be defined as follows:

$$\mathcal{K}(v) = \frac{1}{p} \int_\Omega |\nabla v|^p \, \mathrm{d}x - \frac{\eta}{p} \int_\Omega |v|^p \, \mathrm{d}x.$$

For problem ($\mathcal{QKP}$), the energy functional $\mathcal{K}_b^\lambda : W \to \mathbb{R}$ is given by:

$$\mathcal{K}_b^\lambda = \frac{1}{p}\|v\|^p + \frac{b}{2p}|\nabla v|_p^{2p} - \frac{1}{p^*}|v|_{p^*}^{p^*} \, \mathrm{d}x - \lambda \int F(v) \, \mathrm{d}x, \tag{1.15}$$

for every $v \in W$. Furthermore, $\mathcal{K}_b^\lambda \in \mathcal{C}^1(W, \mathbb{R})$, and its Fréchet derivative is defined as follows:

$$\begin{aligned} \left\langle (\mathcal{K}_b^\lambda)'(v), \varphi \right\rangle = & \left(1 + b \int |\nabla v|^p \, \mathrm{d}x \right) \left(\int |\nabla v|^{p-2} \nabla v \cdot \nabla \varphi \, \mathrm{d}x \right) \\ & + \int a(x)|v|^{p-2} v\varphi \, \mathrm{d}x - \int |v|^{p^*-2} v\varphi \, \mathrm{d}x - \lambda \int f(v)\varphi \, \mathrm{d}x, \end{aligned} \tag{1.16}$$

for every $v, \varphi \in W$. It is widely acknowledged that there exist several intriguing studies concerning the existence of sign-changing solution to problem (1.14). Nevertheless, the approaches employed to seek such solutions are greatly reliant on the subsequent decompositions:

$$\langle \mathcal{K}'(v), v^- \rangle = \langle \mathcal{K}'(v^-), v^- \rangle, \quad \langle \mathcal{K}'(v), v^+ \rangle = \langle \mathcal{K}' \left(v^+ \right), v^+ \rangle, \quad \mathcal{K}(v) = \mathcal{K}(v^+) + \mathcal{K}(v^-). \tag{1.17}$$

However, when $b > 0$, the functional $\mathcal{K}_b^\lambda$ lacks the same decompositions as in equation (1.17). Specifically, if $v^\pm \neq 0$, it can be seen that

$$\begin{aligned} \mathcal{K}_b^\lambda(v) &= \mathcal{K}_b^\lambda(v^+) + \mathcal{K}_b^\lambda(v^-) + \frac{b}{p}|\nabla v^+|_p^p |\nabla v^-|_p^p > \mathcal{K}_b^\lambda(v^+) + \mathcal{K}_b^\lambda(v^-), \\ \langle (\mathcal{K}_b^\lambda)'(v), v^+ \rangle &= \langle (\mathcal{K}_b^\lambda)'(v^+), v^+ \rangle + b|\nabla v^+|_p^p |\nabla v^-|_p^p > \langle (\mathcal{K}_b^\lambda)'(v^+), v^+ \rangle, \\ \langle (\mathcal{K}_b^\lambda)'(v), v^- \rangle &= \langle (\mathcal{K}_b^\lambda)'(v^-), v^- \rangle + b|\nabla v^+|_p^p |\nabla v^-|_p^p > \langle (\mathcal{K}_b^\lambda)'(v^-), v^- \rangle. \end{aligned}$$

So, the conventional techniques cannot be employed to obtain a sign-changing solutions for problem ($\mathcal{QKP}$). Rather, we use the strategy proposed in [5], which requires defining a constrained set as follows:

$$\mathcal{M}_b^\lambda = \left\{ v \in W : \ v^\pm \neq 0 \text{ and } \langle (\mathcal{K}_b^\lambda)'(v), v^\pm \rangle = 0 \right\}, \tag{1.18}$$

and investigate the following minimization problem

$$c_b^\lambda = \inf_{v \in \mathcal{M}_b^\lambda} \mathcal{K}_b^\lambda(v).$$

The inclusion of the nonlocal term in problem ($\mathcal{QKP}$) creates numerous obstacles. Specifically, when compared to the general problem (1.14), in equation (1.17), the decompositions for $\mathcal{K}_b^\lambda$ are much more intricate, which results in some technical challenges while demonstrating the nonemptiness of $\mathcal{M}_b^\lambda$. Furthermore, observe that implicit theorem and the parametric method cannot be employed to address problem ($\mathcal{QKP}$) because of the complex nature of the involved nonlocal term. Consequently, our demonstration follows an alternative approach inspired by [1], specifically by utilizing a modified version of Miranda's theorem (refer to [17]). Additionally, by utilizing degree theory and the quantitative deformation lemma, the minimizer of the constrained problem is shown to be a sign-changing solution.

Now, we proceed to outline our primary results.

Theorem 1.3. *Assume that conditions $(\mathcal{P}_1)$–$(\mathcal{P}_4)$ and $(\mathcal{A})$ are satisfied. Then, there is a positive constant $\lambda^\star$ such that for every λ with $\lambda^\star \leqslant \lambda < \infty$, problem ($\mathcal{QKP}$) possesses a ground state sign-changing solution v_b.*

One purpose of this study is to demonstrate the energy doubling property, which ensures that the energy of any sign-changing solution to problem ($\mathcal{QKP}$) is strictly greater than two times the ground state energy. Although the result is straightforward for equation (1.14), we seek to determine whether it also holds for problem ($\mathcal{QKP}$). The theorem presented below affirms this result for our problem.

Theorem 1.4. *Assume that conditions $(\mathcal{P}_1)$–$(\mathcal{P}_4)$ and $(\mathcal{A})$ are satisfied. Then there is $\lambda^{\star\star} > 0$ such that for all $\lambda^{\star\star} \leqslant \lambda < \infty$, the infimum $c_{b,\lambda}^* > 0$ is attained and the following inequality holds*

$$\mathcal{K}_b^\lambda(v) > 2c_{b,\lambda}^*,$$

where $c_{b,\lambda}^* = \inf_{v \in \mathcal{N}_\lambda} \mathcal{K}_b^\lambda(v)$, $\mathcal{N}_b^\lambda = \{v \in W : \ v \neq 0 \text{ and } \langle (\mathcal{K}_b^\lambda)'(v), v \rangle = 0\}$, and v is the solution obtained in Theorem 1.3.

Finally, we demonstrate the convergence behavior of the solution as $b \searrow 0$.

Theorem 1.5. *Assume that conditions $(\mathcal{P}_1)$–$(\mathcal{P}_4)$ and $(\mathcal{A})$ are satisfied. Then, for any sequence $\{b_n\}$ satisfies $b_n \searrow 0$, there is a subsequence of $\{b_n\}$, such that $\{v_{b_n}\}$ converges strongly to v_0 in W, where v_0 is a least energy nodal solution for the problem below*

$$-\Delta_p v + a(x)|v|^{p-2}v = |v|^{p^*-2}v + \lambda f(v), \quad x \in \mathbb{R}^N. \tag{$\mathcal{QKP}_0$}$$

Remark 1.6. We would like to highlight that Theorem 1.3 represents the first result demonstrating the existence of sign-changing solutions for p-Laplacian Kirchhoff-type problems involving critical exponent in $\mathbb{R}^N$.

Remark 1.7. The primary challenges stem from dealing with a degenerate quasi-linear elliptic operator and the increased intricacy of the calculations involved. To surmount these challenges, we employ quantitative deformation lemma and degree theory to derive our findings.

2 Some preliminary results

Let $v \in W$ be fixed with $v^{\pm} \neq 0$, let $g_v : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ and $G_v : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^2$ defined as follows

$$g_v(\alpha, \beta) = \mathcal{K}_b^{\lambda}(\alpha v^+ + \beta v^-), \tag{2.1}$$

$$G_v(\alpha, \beta) = \left(\langle (\mathcal{K}_b^{\lambda})'(\alpha v^+ + \beta v^-), \alpha v^+ \rangle, \langle (\mathcal{K}_b^{\lambda})'(\alpha v^+ + \beta v^-), \beta v^- \rangle \right). \tag{2.2}$$

Lemma 2.1. *Let f and a satisfy $(\mathcal{A})$ and $(\mathcal{P}_1)$–$(\mathcal{P}_4)$. Suppose that $v \in W$ with $v^{\pm} \neq 0$, the following properties hold:*

(A_1) *$\alpha v^+ + \beta v^- \in \mathcal{M}_b^{\lambda}$ if and only if g_v has a critical point $(\alpha, \beta) \in \mathbb{R}_*^+ \times \mathbb{R}_*^+$;*

(A_2) *The pair (α_v, β_v) is the unique critical point of g_v on $\mathbb{R}_*^+ \times \mathbb{R}_*^+$, as well as it is the unique maximum of g_v on $\mathbb{R}^+ \times \mathbb{R}^+$;*

(A_3) *If $\langle (\mathcal{K}_b^{\lambda})'(v), v^{\pm} \rangle \leqslant 0$ then $0 < \alpha_v, \beta_v \leqslant 1$.*

Proof. Proof of (A_1): Note that if $v^{\pm} \neq 0$, one has

$$\begin{aligned}
\left|\nabla(\alpha v^+ + \beta v^-)\right|^p =& \left|\nabla(\alpha v^+ + \beta v^-)\right|^2 \left|\nabla(\alpha v^+ + \beta v^-)\right|^{p-2} \\
=& \left|\nabla(\alpha v^+ + \beta v^-)\right|^{p-2} \nabla(\alpha v^+ + \beta v^-)\nabla(\alpha v^+) \\
&+ \left|\nabla(\alpha v^+ + \beta v^-)\right|^{p-2} \nabla(\alpha v^+ + \beta v^-)\nabla(\beta v^-) \\
=& |\nabla(\alpha v^+)|^{p-2}\nabla(\alpha v^+)\nabla(\alpha u^+) + \left|\nabla(\beta v^-)\right|^{p-2} \nabla(\beta v^-)\nabla(\beta u^-) \\
=& |\alpha|^p \left|\nabla v^+\right|^p + |\beta|^p \left|\nabla v^-\right|^p.
\end{aligned}$$

Therefore, a direct computation yields

$$\begin{aligned}
\nabla g_v(\alpha, \beta) &= \left(\frac{\partial g_v}{\partial \alpha}(\alpha, \beta), \frac{\partial g_v}{\partial \beta}(\alpha, \beta) \right) \\
&= \left(\langle (\mathcal{K}_b^{\lambda})'(\alpha v^+ + \beta v^-), v^+ \rangle, \langle (\mathcal{K}_b^{\lambda})'(\alpha v^+ + \beta v^-), v^- \rangle \right) \\
&= \left(\frac{1}{\alpha} \langle (\mathcal{K}_b^{\lambda})'(\alpha v^+ + \beta v^-), \alpha v^+ \rangle, \frac{1}{\beta} \langle (\mathcal{K}_b^{\lambda})'(\alpha v^+ + \beta v^-), \beta v^- \rangle \right),
\end{aligned} \tag{2.3}$$

where

$$\begin{aligned}
\langle (\mathcal{K}_b^{\lambda})'(\alpha v^+ + \beta v^-), \alpha v^+ \rangle =& b\alpha^p \beta^p |\nabla v^+|_p^p |\nabla v^-|_p^p - \lambda \int f(\alpha v^+)\alpha v^+ \, \mathrm{d}x \\
&+ \alpha^p \|v^+\|^p + b\alpha^{2p} |\nabla v^+|_p^{2p} - \alpha^{p^*} |v^+|_{p^*}^{p^*},
\end{aligned}$$

$$\begin{aligned}
\langle (\mathcal{K}_b^{\lambda})'(\alpha v^+ + \beta v^-), \beta v^- \rangle =& b\beta^p \alpha^p |\nabla v^-|_p^p |\nabla v^+|_p^p - \lambda \int f(\beta v^-)\beta v^- \, \mathrm{d}x \\
&+ \beta^p \|v^-\|^p + b\beta^{2p} |\nabla v^-|_p^{2p} - \beta^{p^*} |v^-|_{p^*}^{p^*}.
\end{aligned}$$

From (2.3) and the definition of $\mathcal{M}_b^{\lambda}$, it's straightforward to show that (A_1) is satisfied.

Proof of (A_2): From $(\mathcal{P}_1)$ and $(\mathcal{P}_2)$, for any $\varepsilon > 0$, there is $C_\varepsilon > 0$ such that

$$|f(t)| \leqslant \varepsilon |t|^{p-1} + C_\varepsilon |t|^{\vartheta - 1}, \quad \forall t \in \mathbb{R}. \tag{2.4}$$

From (1.13) and (2.4), we obtain

$$\begin{aligned}\langle (\mathcal{K}_b^\lambda)'(\alpha v^+ + \beta v^-), \alpha v^+\rangle &\geqslant \alpha^p \|v^+\|^p + b\alpha^{2p}|\nabla v^+|_p^{2p} + b\alpha^p\beta^p|\nabla v^+|_p^p|\nabla v^-|_p^p - \alpha^{p*}|v^+|_{p^*}^{p^*}\\ &\quad - \lambda\varepsilon\alpha^p|v^+|_p^p - \lambda C_\varepsilon \alpha^\vartheta |v^+|_\vartheta^\vartheta\\ &\geqslant \alpha^p\left(1-\lambda\varepsilon\gamma_p\right)\|v^+\|^p + b\alpha^{2p}|\nabla v^+|_p^{2p} - \lambda C_\varepsilon\alpha^\vartheta\gamma_\vartheta\|v^+\|^\vartheta\\ &\quad - \alpha^{p*}\gamma_{p*}\|v^+\|^{p^*},\end{aligned}$$

and

$$\begin{aligned}\langle (\mathcal{K}_b^\lambda)'(\alpha v^+ + \beta v^-), \beta v^-\rangle &\geqslant \beta^p \|v^-\|^p + b\beta^{2p}|\nabla v^-|_p^{2p} + b\beta^p\alpha^p|\nabla v^-|_p^p|\nabla v^+|_p^p - \beta^{p*}|v^-|_{p^*}^{p^*}\\ &\quad - \lambda\varepsilon\beta^p|v^-|_p^p - \lambda C_\varepsilon \beta^\vartheta |v^-|_\vartheta^\vartheta\\ &\geqslant \beta^p\left(1-\lambda\varepsilon\gamma_p\right)\|v^-\|^p + b\beta^{2p}|\nabla v^-|_p^{2p} - \lambda C_\varepsilon\beta^\vartheta\gamma_\vartheta\|v^-\|^\vartheta\\ &\quad - \beta^{p*}\gamma_{p*}\|v^-\|^{p^*}.\end{aligned}$$

Select ε in such a manner that $\left(1-\lambda\varepsilon\gamma_p\right) > 0$. Given that $\vartheta \in (2p, p^*)$, one has

$$\langle (\mathcal{K}_b^\lambda)'(\alpha v^+ + \beta v^-), \alpha v^+\rangle > 0, \quad \text{for } \alpha \text{ small enough and all } \beta \geqslant 0, \tag{2.5}$$

and

$$\langle (\mathcal{K}_b^\lambda)'(\alpha v^+ + \beta v^-), \beta v^-\rangle > 0, \quad \text{for } \beta \text{ small enough and all } \alpha \geqslant 0. \tag{2.6}$$

Hence, there is a positive constant τ_1 such that

$$\langle (\mathcal{K}_b^\lambda)'(\tau_1 v^+ + \beta v^-), \tau_1 v^+\rangle > 0, \quad \langle (\mathcal{K}_b^\lambda)'(\alpha v^+ + \tau_1 v^-), \tau_1 v^-\rangle > 0, \tag{2.7}$$

for all $\alpha, \beta \geqslant 0$. Conversely, choosing $\alpha = \tau_2' > \tau_1$. If $\beta \in [\tau_1, \tau_2']$ and τ_2' is large enough, using (1.11), one has that

$$\begin{aligned}\langle (\mathcal{K}_b^\lambda)'(\tau_2' v^+ + \beta v^-), \tau_2' v^+\rangle &\leqslant (\tau_2')^p\|v^+\|^p + b(\tau_2')^{2p}|\nabla v^+|_p^{2p} - (\tau_2')^{p^*}|v^+|_{p*}^{p*}\\ &\quad + b(\tau_2')^{2p}|\nabla v^+|_p^p|\nabla v^-|_p^p\\ &\leqslant 0,\end{aligned}$$

and

$$\begin{aligned}\langle (\mathcal{K}_b^\lambda)'(\alpha v^+ + \tau_2' v^-), \tau_2' v^-\rangle &\leqslant (\tau_2')^p\|v^-\|^p + b(\tau_2')^{2p}|\nabla v^-|_p^{2p} - (\tau_2')^{p^*}|v^-|_{p*}^{p*}\\ &\quad + b(\tau_2')^{2p}|\nabla v^-|_p^p|\nabla v^+|_p^p\\ &\leqslant 0.\end{aligned}$$

Let $\tau_2 > \tau_2'$ be large enough, we get

$$\langle (\mathcal{K}_b^\lambda)'(\alpha v^+ + \tau_2 v^-), \tau_2 v^-\rangle < 0, \qquad \langle (\mathcal{K}_b^\lambda)'(\alpha v^+ + \tau_2 v^-), \tau_2 v^-\rangle < 0, \tag{2.8}$$

for all $\alpha, \beta \in [\tau_1, \tau_2]$. Combining equations (2.7) and (2.8) with Miranda's Theorem [17], we find that there is $(\alpha_v, \beta_v) \in \mathbb{R}_*^+ \times \mathbb{R}_*^+$ such that $G_v(\alpha, \beta) = (0, 0)$, that is, $\alpha v^+ + \beta v^- \in \mathcal{M}_b^\lambda$.

Next, we demonstrate the pair (α_v, β_v) is unique. To finalize the proof, we will investigate two separate cases.

Case 1) $v \in \mathcal{M}_b^\lambda$.

For any $v \in \mathcal{M}_b^\lambda$, one has that

$$\|v^+\|^p + b|\nabla v^+|_p^p|\nabla v^-|_p^p + b|\nabla v^+|_p^{2p} = |v^+|_{p*}^{p^*} + \lambda\int f(v^+)v^+ \, \mathrm{d}x, \tag{2.9}$$

and

$$\|v^-\|^p + b|\nabla v^-|_p^p|\nabla v^+|_p^p + b|\nabla v^-|_p^{2p} = |v^-|_{p*}^{p^*} + \lambda \int f(v^-)v^- \, \mathrm{d}x. \tag{2.10}$$

Consider a pair of numbers (α_0, β_0) satisfying $\alpha_0 v^+ + \beta_0 v^- \in \mathcal{M}_b^\lambda$ and $0 < \alpha_0 \leqslant \beta_0$. So, we have that

$$\alpha_0^p \|v^+\|^p + b\alpha_0^p \beta_0^p |\nabla v^+|_p^p |\nabla v^-|_p^p + b\alpha_0^{2p} |\nabla v^+|_p^{2p} = \alpha_0^{p^*} |v^+|_{p^*}^{p^*} + \lambda \int f(\alpha_0 v^+)\alpha_0 v^+ \, \mathrm{d}x,$$

and

$$\beta_0^p \|v^-\|^p + b\beta_0^p \alpha_0^p |\nabla v^-|_p^p |\nabla v^+|_p^p + b\beta_0^{2p} |\nabla v^-|_p^{2p} = \beta_0^{p^*} |v^-|_{p^*}^{p^*} + \lambda \int f(\beta_0 v^-)\beta_0 v^- \, \mathrm{d}x.$$

Hence, thanks to $0 < \alpha_0 \leqslant \beta_0$, one has that

$$\frac{1}{\alpha_0^p}\|v^+\|^p + b|\nabla v^+|_p^p|\nabla v^-|_p^p + b|\nabla v^+|_p^{2p} \leqslant \alpha_0^{p^*-2p}|v^+|_{p^*}^{p^*} + \lambda \int \frac{f(\alpha_0 v^+)}{(\alpha_0 v^+)^{2p-1}}(v^+)^{2p} \, \mathrm{d}x, \tag{2.11}$$

and

$$\frac{1}{\beta_0^p}\|v^-\|^p + b|\nabla v^-|_p^p|\nabla v^+|_p^p + b|\nabla v^-|_p^{2p} \geqslant \beta_0^{p^*-2p}|v^-|_p^{p^*} + \lambda \int \frac{f(\beta_0 v^-)}{(\beta_0 v^-)^{2p-1}}(v^-)^{2p} \, \mathrm{d}x. \tag{2.12}$$

Combining (2.10) with (2.12), one has that

$$\left(\frac{1}{\beta_0^p} - 1\right)\|v^-\|^p \geqslant \left(\beta_0^{p^*-2p} - 1\right)|v^-|_{p^*}^{p^*} + \lambda \int \left[\frac{f(\beta_0 v^-)}{(\beta_0 v^-)^{2p-1}} - \frac{f(v^-)}{(v^-)^{2p-1}}\right](v^-)^{2p} \, \mathrm{d}x.$$

If $\beta_0 > 1$, the left-hand side of the aforementioned inequality becomes less than zero, thereby contradicting condition $(\mathcal{P}_4)$ that guarantees the positivity of the right-hand side. Consequently, if follows that $0 < \alpha_0 \leqslant \beta_0 \leqslant 1$. Similarly, combining (2.9) with (2.11), thanks to $0 < \alpha_0 \leqslant \beta_0$, we obtain

$$\left(\frac{1}{\alpha_0^p} - 1\right)\|v^+\|^p \leqslant \left(\alpha_0^{p^*-2p} - 1\right)|v^+|_{p^*}^{p^*} + \lambda \int \left[\frac{f(\alpha_0 v^+)}{(\alpha_0 v^+)^{2p-1}} - \frac{f(v^+)}{(v^+)^{2p-1}}\right](v^+)^{2p} \, \mathrm{d}x.$$

In view of $(\mathcal{P}_4)$, we have $\alpha_0 \geqslant 1$. Due to this, $\alpha_0 = \beta_0 = 1$. So, we conclude that if $v \in \mathcal{M}_b^\lambda$, the unique pair such that $\alpha_v v^+ + \beta_v v^- \in \mathcal{M}_b^\lambda$ is $(\alpha_v, \beta_v) = (1, 1)$.

Case 2) $v \notin \mathcal{M}_b^\lambda$.

Assume the existence of two pairs $(\alpha_1, \beta_1), (\alpha_2, \beta_2)$, satisfy the following condition

$$u_1 = \alpha_1 v^+ + \beta_1 v^- \in \mathcal{M}_b^\lambda, \qquad u_2 = \alpha_2 v^+ + \beta_2 v^- \in \mathcal{M}_b^\lambda.$$

Therefore, one has

$$u_2 = \left(\frac{\alpha_2}{\alpha_1}\right)\alpha_1 v^+ + \left(\frac{\beta_2}{\beta_1}\right)\beta_1 v^- = \left(\frac{\alpha_2}{\alpha_1}\right)u_1^+ + \left(\frac{\beta_2}{\beta_1}\right)u_1^- \in \mathcal{M}_b^\lambda.$$

By $u_1 \in \mathcal{M}_b^\lambda$, we have that

$$\frac{\beta_2}{\beta_1} = \frac{\alpha_2}{\alpha_1} = 1.$$

Hence, $\alpha_1 = \alpha_2, \beta_1 = \beta_2$.

Lastly, we show that the unique maximum of g_v on $\mathbb{R}^+ \times \mathbb{R}^+$ is (α_v, β_v).
Indeed, using (1.11), it can be seen that

$$\begin{aligned} g_v(\alpha,\beta) &= \mathcal{K}_b^\lambda(\alpha v^+ + \beta v^-) \\ &= \frac{1}{p}\|\alpha v^+ + \beta v^-\|^p + \frac{b}{2p}|\nabla \alpha v^+ + \beta v^-|_p^{2p} - \frac{1}{p^*}|\alpha v^+ + \beta v^-|_{p^*}^{p^*} \\ &\quad - \lambda \int F(\alpha v^+ + \beta v^-)\, \mathrm{d}x \\ &\leqslant \frac{\alpha^p}{p}\|v^+\|^p + \frac{\beta^p}{p}\|v^-\|^p - \frac{\beta^{p^*}}{p^*}|v^-|_{p^*}^{p^*} - \frac{\alpha^{p^*}}{p^*}|v^+|_{p^*}^{p^*} + \frac{b\alpha^{2p}}{2p}|\nabla v^+|_p^{2p} \\ &\quad + \frac{b\beta^{2p}}{2p}|\nabla v^-|_p^{2p} + \frac{b\alpha^p\beta^p}{p}|\nabla v^+|_p^p|\nabla v^-|_p^p. \end{aligned} \tag{2.13}$$

Thus, for any $v \in W$ with $v^\pm \neq 0$, in light of equation (2.13) and the condition $p^* > 2p$, one has

$$\lim_{|(\alpha,\beta)|\to\infty} g_v(\alpha,\beta) = -\infty.$$

Thus, (α_v, β_v) represents the unique critical point of g_v within the domain $\mathbb{R}^+ \times \mathbb{R}^+$. Hence, our task is to establish that it is impossible to reach a maximum point on the boundary $(0, \beta_0)$ and $(\alpha_0, 0)$. For the sake of contradiction, let us assume that $(0, \beta_0)$ is a maximum of g_v with $\beta_0 \geqslant 0$. Consequently, it follows that

$$\begin{aligned} g_v(\alpha,\beta_0) &= \frac{\alpha^p}{p}\|v^+\|^p + \frac{\beta_0^p}{p}\|v^-\|^p - \frac{\beta_0^{p^*}}{p^*}|v^-|_{p^*}^{p^*} - \frac{\alpha^{p^*}}{p^*}|v^+|_{p^*}^{p^*} + \frac{b\beta_0^{2p}}{2p}|\nabla v^-|_p^{2p} \\ &\quad + \frac{b\alpha^{2p}}{2p}|\nabla v^+|_p^{2p} + \frac{b\alpha^p\beta_0^p}{p}|\nabla v^+|_p^p|\nabla v^-|_p^p - \lambda\int F(\alpha v^+) - \lambda\int F(\beta_0 v^-). \end{aligned}$$

Thus, it can be seen that

$$\begin{aligned} \frac{\partial g_v}{\partial \alpha}(\alpha,\beta_0) &= \alpha^{p-1}\|v^+\|^p - \alpha^{p^*-1}|v^+|_{p^*}^{p^*} + b\alpha^{2p-1}|\nabla v^+|_p^{2p} + b\alpha^{p-1}\beta_0^p|\nabla v^+|_p^p|\nabla v^-|_p^p \\ &\quad - \lambda\int f(\alpha v^+)v^+\, \mathrm{d}x \\ &> 0, \end{aligned}$$

if α is sufficiently small. This implies that g_v is increasing with respect to α if α is sufficiently small. This leads to a contradiction. Likewise, g_v can not attain its global maximum on $(\alpha_0, 0)$ with $\alpha_0 \geqslant 0$.

Proof of (A_3): Suppose $\alpha_v \geqslant \beta_v > 0$. By $\alpha_v v^+ + \beta_v v^- \in \mathcal{M}_b^\lambda$, we have that

$$\alpha_v^p\|v^+\|^p + b\alpha_v^{2p}|\nabla v^+|_p^{2p} + b\alpha_v^{2p}|\nabla v^+|_p^p|\nabla v^-|_p^p \geqslant \lambda\int f(\alpha_v v^+)\alpha_v v^+\, \mathrm{d}x + \alpha_v^{p^*}|v^+|_{p^*}^{p^*}. \tag{2.14}$$

Conversely, from the inequality $\langle (\mathcal{K}_b^\lambda)'(v), v^+\rangle \leqslant 0$, we deduce that

$$\lambda\int f(v^+)v^+\, \mathrm{d}x + |v^+|_{p^*}^{p^*} \geqslant \|v^+\|^p + b|\nabla v^+|_p^p|\nabla v^-|_p^p + b|\nabla v^+|_p^{2p}. \tag{2.15}$$

Combining (2.14) and (2.15), we obtain

$$\left(\frac{1}{\alpha_v^p} - 1\right)\|v^+\|^p \geqslant \left(\alpha_v^{p^*-2p} - 1\right)|v^+|_{p^*}^{p^*} + \lambda\int\left[\frac{f(\alpha_v v^+)}{(\alpha_v v^+)^{2p-1}} - \frac{f(v^+)}{(v^+)^{2p-1}}\right](v^+)^{2p}\, \mathrm{d}x.$$

Hence, due to condition $(\mathcal{P}_4)$, we can infer that $\alpha_v \leqslant 1$. Therefore, we establish that $0 < \alpha_v, \beta_v \leqslant 1$. Consequently, Lemma 2.1 has been proved. □

The following result plays a crucial role to prove that c_b^λ is achieved.

Lemma 2.2. *Let $c_b^\lambda = \inf_{v\in\mathcal{M}_b^\lambda} \mathcal{K}_b^\lambda(v)$, then we have that*

$$\lim_{\lambda\to\infty} c_b^\lambda = 0.$$

Proof. For every $v \in \mathcal{M}_\lambda$, it holds that

$$\|v^+\|^p + b|\nabla v^+|_p^{2p} + b|\nabla v^+|_p^p|\nabla v^-|_p^p = |v^+|_{p^*}^{p^*} + \lambda\int f(v^+)v^+ \,\mathrm{d}x,$$

and

$$\|v^-\|^p + b|\nabla v^+|_p^{2p} + b|\nabla v^-|_p^p|\nabla v^+|_p^p = |v^-|_{p^*}^{p^*} + \lambda\int f(v^-)v^- \,\mathrm{d}x.$$

Then by (1.13) and (2.4), we get

$$\left(1-\lambda\varepsilon\gamma_p\right)\|v^+\|^p \leqslant \lambda C_\varepsilon\gamma_\vartheta\|v^+\|^\vartheta + \gamma_{p*}\|v^+\|^{p^*},$$

and

$$\left(1-\lambda\varepsilon\gamma_p\right)\|v^-\|^p \leqslant \lambda C_\varepsilon\gamma_\vartheta\|v^-\|^\vartheta + \gamma_{p*}\|v^-\|^{p^*}.$$

Taking $\varepsilon < \frac{1}{\lambda\gamma_p}$ such that $1-\lambda\varepsilon\gamma_p > 0$. Given that $\vartheta \in (2p, p^*)$, there is $\rho > 0$ such that

$$\|v^+\| \geqslant \rho, \quad \|v^-\| \geqslant \rho, \quad \forall v \in \mathcal{M}_b^\lambda. \tag{2.16}$$

Conversely, for any $v \in \mathcal{M}_b^\lambda$, it becomes evident that

$$\begin{aligned}\langle(\mathcal{K}_b^\lambda)'(v), v\rangle &= \langle(\mathcal{K}_b^\lambda)'(v), v^+\rangle + \langle(\mathcal{K}_b^\lambda)'(v), v^-\rangle\\ &= 0.\end{aligned}$$

From (1.12), we can deduce that

$$f(t)t - 2pF(t) \geqslant 0, \tag{2.17}$$

is increasing when $t > 0$ and decreasing when $t < 0$. Thus, we get

$$\begin{aligned}\mathcal{K}_b^\lambda(v) &= \mathcal{K}_b^\lambda(v) - \frac{1}{2p}\langle(\mathcal{K}_b^\lambda)'(v), v\rangle\\ &= \frac{(p-1)}{2p}\|v\|^p + \frac{p^*-2p}{2pp^*}|v|_{p^*}^{p^*} + \frac{\lambda}{2p}\int\left[f(v)v - 2pF(v)\right]\,\mathrm{d}x\\ &\geqslant \frac{(p-1)}{2p}\|v\|^p\\ &> \frac{(p-1)}{2p}\rho^p.\end{aligned}$$

Therefore, $\mathcal{K}_b^\lambda$ is bounded below on $\mathcal{M}_b^\lambda$. Hence, $c_b^\lambda = \inf_{v\in\mathcal{M}_b^\lambda}\mathcal{K}_b^\lambda(v)$ is well defined.

Fix $v \in W$ with $v^\pm \neq 0$. Lemma 2.1 guarantees that for every $\lambda > 0$, there is $\alpha_\lambda, \beta_\lambda > 0$ such that $\alpha_\lambda v^+ + \beta_\lambda v^- \in \mathcal{M}_b^\lambda$. From Lemma 2.1, one has

$$\begin{aligned}0 \leqslant c_b^\lambda &\leqslant \mathcal{K}_b^\lambda(\alpha_\lambda v^+ + \beta_\lambda v^-)\\ &\leqslant \frac{1}{p}\|\alpha_\lambda v^+ + \beta_\lambda v^-\|^p + \frac{b}{2p}|\nabla(\alpha_\lambda v^+ + \beta_\lambda v^-)|_p^{2p}\\ &= \alpha_\lambda^p\frac{1}{p}\|v^+\|^p + \beta_\lambda^p\frac{1}{p}\|v^-\|^p + \alpha_\lambda^{2p}\frac{b}{2p}|\nabla v^+|_p^{2p} + \beta_\lambda^{2p}\frac{b}{2p}|\nabla v^-|_p^{2p}\\ &\quad + \alpha_\lambda^p\beta_\lambda^p\frac{b}{p}|\nabla v^+|_p^p|\nabla v^-|_p^p.\end{aligned}$$

We need only verify that $\alpha_\lambda \to 0$ and $\beta_\lambda \to 0$, as $\lambda \to \infty$.
Define

$$\mathcal{G}_v = \left\{(\alpha_\lambda, \beta_\lambda) \in \mathbb{R}^+ \times \mathbb{R}^+ : G_v(\alpha_\lambda, \beta_\lambda) = (0,0), \lambda > 0\right\},$$

In light of (2.4), we have that

$$\begin{aligned}\alpha_\lambda^{p^*}|v^+|_{p^*}^{p^*} + \beta_\lambda^{p^*}|v^-|_{p^*}^{p^*} &\leqslant \lambda \int f(\alpha_\lambda v^+)\alpha_\lambda v^+ \,\mathrm{d}x + \lambda \int f(\beta_\lambda v^-)\beta_\lambda v^- \,\mathrm{d}x \\ &\quad + \alpha_\lambda^{p^*}|v^+|_{p^*}^{p^*} + \beta_\lambda^{p^*}|v^-|_{p^*}^{p^*} \\ &= \|\alpha_\lambda v^+ + \beta_\lambda v^-\|^p + b|\nabla(\alpha_\lambda v^+ + \beta_\lambda v^-)|_p^{2p}.\end{aligned}$$

Therefore, $\mathcal{G}_v$ is bounded, since $p^* > 2p$. Assume that $\{\lambda_n\}$ is a sequence in $(0,\infty)$ with $\lambda_n \to \infty$, as n tends to infinity. Then there is α_0 and β_0 such that $(\alpha_{\lambda_n}, \beta_{\lambda_n}) \to (\alpha_0, \beta_0)$, as $n \to \infty$. We assert that $\alpha_0 = \beta_0 = 0$. Suppose, the contrary $\alpha_0 > 0$ or $\beta_0 > 0$. Given that $\alpha_{\lambda_n} v^+ + \beta_{\lambda_n} v^- \in \mathcal{M}_b^{\lambda_n}$, then for any $n \in \mathbb{N}$, we have that

$$\begin{aligned}&\|\alpha_{\lambda_n} v^+ + \beta_{\lambda_n} v^-\|^p + b|\nabla(\alpha_{\lambda_n} v^+ + \beta_{\lambda_n} v^-)|_p^{2p} \\ &\quad = \lambda_n \int f(\alpha_{\lambda_n} v^+ + \beta_{\lambda_n} v^-)(\alpha_{\lambda_n} v^+ + \beta_{\lambda_n} v^-) \,\mathrm{d}x + |\alpha_{\lambda_n} v^+ + \beta_{\lambda_n} v^-|_{p^*}^{p^*}.\end{aligned} \tag{2.18}$$

Then, invoking $\alpha_{\lambda_n} v^+ \to \alpha_0 v^+, \beta_{\lambda_n} v^- \to \beta_0 v^-$ in W along with the Lebesgue dominated convergence theorem, we obtain that

$$\int f(\alpha_{\lambda_n} v^+ + \beta_{\lambda_n} v^-)(\alpha_{\lambda_n} v^+ + \beta_{\lambda_n} v^-) \to \int f(\alpha_0 v^+ + \beta_0 v^-)(\alpha_0 v^+ + \beta_0 v^-) > 0,$$

as $n \to \infty$. This leads to a contradiction with equation (2.18), because $\lambda_n \to \infty$, as $n \to \infty$ and and the sequence $\{\alpha_{\lambda_n} v^+ + \beta_{\lambda_n} v^-\}$ is bounded in W. Hence, we conclude that $\alpha_0 = \beta_0 = 0$, from which we infer that $\lim_{\lambda\to\infty} c_b^\lambda = 0$. □

Let ϕ be a standard cut off function, namely ϕ belongs to the space $\mathcal{C}_c^\infty(\mathbb{R}^N)$, satisfying $0 \leqslant \phi \leqslant 1$, ϕ being equal to 1 on $B_{1/2}(0)$ and $\operatorname{supp}\phi \subset B_1(0)$. For any $\epsilon > 0$, let $\phi_\epsilon(x) = \phi\left(\frac{x}{\epsilon}\right)$.

Lemma 2.3. *Let $1 < p < N$, for every $x_i \in \mathbb{R}^N$ and for $u \in L^p(\mathbb{R}^N)$, we have*

$$|v(x)\nabla\phi_\epsilon(x - x_i)|_{L^p(\mathbb{R}^N)}^p \leqslant |\nabla\phi|_{L^{\frac{p^*p}{p^*-p}}(\mathbb{R}^N)}^p |v|_{L^{p^*}(B_\epsilon(x_i))}^p.$$

Proof. Using the Hölder inequality, one has

$$\begin{aligned}\left|v(x)\nabla\phi_\epsilon(x - x_i)\right|_{L^p(\mathbb{R}^N)}^p &\leqslant \int \epsilon^{-p}\left|\nabla\phi\left(\frac{x - x_i}{\epsilon}\right)\right|^p |v(x)|^p \,dx \\ &\leqslant \epsilon^{-p}\left(\int_{B_\epsilon(x_i)} \left|\nabla\phi\left(\frac{x - x_i}{\epsilon}\right)\right|^{\frac{p^*p}{p^*-p}} dx\right)^{\frac{p^*-p}{p^*}} |v|_{L^{p^*}(B_\epsilon(x_i))}^p \\ &\leqslant \epsilon^{-p}\epsilon^{N\left(\frac{p^*-p}{p^*}\right)} |\nabla\phi|_{L^{\frac{p^*p}{p^*-p}}(\mathbb{R}^N)}^p |v|_{L^{p^*}(B_\epsilon(x_i))}^p.\end{aligned}$$

The result follows by noting that $p = \left(\frac{p^*-p}{p^*}\right)N$. □

With the above results, we will now undertake a three-step process to obtain that $c_b^\lambda = \inf_{v\in\mathcal{M}_b^\lambda} \mathcal{K}_b^\lambda(v)$ is achieved.

Lemma 2.4. *There is $\lambda^\star > 0$ such that for all $\lambda \in [\lambda^\star, \infty)$, the infimum c_b^λ is attained.*

Proof. As defined by c_b^λ, there is a sequence $\{v_n\} \subset \mathcal{M}_b^\lambda$ satisfies

$$\lim_{n\to\infty} \mathcal{K}_b^\lambda(v_n) = c_b^\lambda.$$

On the other hand, by using (2.17) and $\langle (\mathcal{K}_b^\lambda)'(v_n), v_n \rangle = 0$, we get

$$\begin{aligned} c_b^\lambda + o(1) &= \mathcal{K}_b^\lambda(v_n) - \frac{1}{2p}\langle (\mathcal{K}_b^\lambda)'(v_n), u_n\rangle \\ &= \frac{(p-1)}{2p}\|v\|^p + \frac{p^*-2p}{2pp^*}|v|_{p^*}^{p^*} + \frac{\lambda}{2p}\int [f(v)v - 2pF(v)] \\ &\geqslant \frac{(p-1)}{2p}\|v\|^p. \end{aligned} \tag{2.19}$$

Therefore, $\{v_n\}$ is bounded in W, i.e., there is a constant $M > 0$ such that

$$\|v_n\| \leqslant M, \quad \forall n \in \mathbb{N}. \tag{2.20}$$

From (2.16), (2.19), and Ekeland's variational principle [20], $\{v_n\}$ represents a $(\mathrm{PS})_{c_b^\lambda}$ sequence for $\mathcal{K}_b^\lambda|_{\mathcal{M}_b^\lambda}$, i.e.

$$\mathcal{K}_b^\lambda(v_n) \to c_b^\lambda, \quad (\mathcal{K}_b^\lambda)'(v_n) \to 0. \tag{2.21}$$

Since v_n is bounded in W, there existence a $v \in W$ and a subsequence $\{v_n\}$ where

$$\begin{aligned} v_n &\rightharpoonup v && \text{in } W, \\ v_n &\to v && \text{in } L^s \text{ for } p \leqslant s < p^*, \\ v_n &\to v && \text{a.e. in } \mathbb{R}^N, \\ v_n &\rightharpoonup v && \text{in } L^{p^*}, \\ f(v_n) &\rightharpoonup f(v) && \text{in } (L^{p^*})', \\ |v_n|^{p^*-2}v_n &\rightharpoonup |v|^{p^*-2}v && \text{in } (L^{p^*})'. \end{aligned} \tag{2.22}$$

We migh suppose that

$$\begin{aligned} |\nabla v_n|^p &\rightharpoonup^* \mu \text{ in } \mathcal{M}(\mathbb{R}^N), \\ |v_n|^{p^*} &\rightharpoonup^* \nu \text{ in } \mathcal{M}(\mathbb{R}^N). \end{aligned} \tag{2.23}$$

for some measures μ and ν. Then, according to Concentration-Compactness principle [16], there exists an at most countable set $\mathfrak{J}$ and $\{x_i\}_{i\in\mathfrak{J}} \subset \mathbb{R}^N$, $\{\mu_i\}_{i\in\mathfrak{J}} \subset (0,+\infty)$, $\{\nu_i\}_{i\in\mathfrak{J}} \subset (0,+\infty)$ such that

$$\begin{aligned} \nu &= \sum_{i\in\mathfrak{J}} \nu_i \delta_{x_i} + |v|^{p^*}, \\ \mu &\geqslant \sum_{i\in\mathfrak{J}} \mu_i \delta_{x_i} + |\nabla v|^p, \end{aligned} \tag{2.24}$$

with $S\nu_i^{\frac{p}{p^*}} \leqslant \mu_i$, where

$$S := \inf_{\substack{v\in W\setminus\{0\}, \\ |v|_{p^*}=1}} \frac{\|v\|^p}{|v|_{p^*}^p}.$$

Our claim is that $\nabla v_n^\pm \to \nabla v^\pm$ almost everywhere in $\mathbb{R}^N$. Following the approach in [3,9], we will carry out the proof in two steps.

Step a) The set $\mathfrak{J}$ is either finite or empty. Let's take a fixed $i \in \mathfrak{J}$ and let $\epsilon > 0$ be chosen small enough. Utilizing Lemma 2.3, we can conclude that

$$|v_n\phi_\epsilon(x-x_i)|_p^p \leqslant |v_n|_p^p,$$

and

$$\begin{aligned}|\nabla\left(v_n\phi_\epsilon(x-x_i)\right)|_p^p &\leqslant 2^{p-1}\left||\nabla\phi_\epsilon(x-x_i)|^p\,|v_n|^p\right|_1 + 2^{p-1}\left||\phi_\epsilon(x-x_i)|^p|\nabla v_n|^p\right|_1\\ &\leqslant 2^{p-1}|v_n|_{p^*}^p|\nabla\phi|^p_{L^{\frac{p^*p}{p^*-p}}(\mathbb{R}^N)} + 2^{p-1}\|v_n\|^p.\end{aligned}$$

Consequently, a constant C_1 exists such that, for any $\epsilon > 0$ and $i \in \mathfrak{J}$, we have

$$\|v_n\phi_\epsilon\left(x-x_i\right)\| \leqslant C_1.$$

It is clear that $\{v_n\phi_\epsilon\left(x-x_i\right)\}_{n=1}^{+\infty} \subset W$. Therefore, $\{v_n\phi_\epsilon\left(x-x_i\right)\}$ is bounded in W. For any fixed $x_i \in \mathfrak{J}$, from (2.21), we obtain

$$\left\langle\left(\mathcal{K}_b^\lambda\right)'\left(v_n\right), v_n\phi_\epsilon(x-x_i)\right\rangle = o(1).$$

Thus, one gets

$$\begin{aligned}&(1+b\|v_n\|^p)\int|\nabla v_n|^p\,\phi_\epsilon(x-x_i)\,\mathrm{d}x + (1+b\|v_n\|^p)\int|\nabla v_n|^{p-2}\,\nabla v_n\cdot v_n\nabla\phi_\epsilon\left(x-x_i\right)\,\mathrm{d}x\\ &\quad= \int|v_n|^{p^*}\,\phi_\epsilon\left(x-x_i\right)\,\mathrm{d}x + \lambda\int f(v_n)v_n\phi_\epsilon(x-x_i)\,\mathrm{d}x + o(1).\end{aligned} \tag{2.25}$$

By means of the Young inequality, for any $\delta > 0$, there is C_δ such that

$$\left|\int|\nabla v_n|^{p-2}\nabla v_n\cdot v_n\nabla\phi_\epsilon(x-x_i)\,\mathrm{d}x\right| \leqslant C_\delta|v_n\nabla\phi_\epsilon(x-x_i)|_p^p + \delta\|v_n\|_p^p.$$

Applying the Strauss Lemma [19], we obtain

$$\lim_{n\to\infty}|v_n\nabla\phi_\epsilon(x-x_i)|_p^p = |v\nabla\phi_\epsilon(x-x_i)|_p^p.$$

Therefore, utilizing Lemma 2.3, we derive

$$\begin{aligned}\limsup_{n\to\infty}\left|\int|\nabla v_n|^{p-2}\nabla v_n\cdot v_n\nabla\phi_\epsilon(x-x_i)\,\mathrm{d}x\right| &\leqslant C_\delta|v\nabla\phi_\epsilon(x-x_i)|_p^p + \delta\limsup_{n\to\infty}\|v_n\|^p\\ &\leqslant \delta C_2 + C_\delta C_3|v|^p_{L^{p^*}(B_\epsilon(x_i))},\end{aligned} \tag{2.26}$$

where $C_3 = |\nabla\phi|^p_{\frac{p^*p}{p^*-p}}$ is constant and independent of ϵ and δ. Using (2.4) along with the Strauss Lemma [19] and (2.22), we deduce

$$\lim_{n\to\infty}\int f(v_n)v_n\phi_\epsilon(x-x_i)\,\mathrm{d}x = \int f(v)v\phi_\epsilon(x-x_i)\,\mathrm{d}x. \tag{2.27}$$

Combining (2.25)–(2.27) and using (2.20), we obtain

$$\begin{aligned}\lim_{n\to\infty}\int|\nabla v_n|^p\,\phi_\epsilon(x-x_i)\,\mathrm{d}x \leqslant \lambda\int f(v)v\phi_\epsilon(x-x_i)\,\mathrm{d}x + \lim_{n\to\infty}\int|v_n|^{p^*}\,\phi_\epsilon(x-x_i)\,\mathrm{d}x\\ + \delta(1+bM^p)\limsup_{n\to\infty}\|v_n\|^p + C_\delta C_3(1+bM^p)|v|^p_{L^{p^*}(B_\epsilon(x_i))}.\end{aligned}$$

Therefore, since $\{v_n\}$ is bounded in W and by (2.23), we can conclude that

$$\int \phi_\epsilon(x-x_i)\,\mathrm{d}\mu \leqslant \lambda \int f(v)v\phi_\epsilon(x-x_i)\,\mathrm{d}x + \int \phi_\epsilon(x-x_i)\,\mathrm{d}\nu + \delta C_4 + C_\delta C_5 |v|^p_{L^{p^*}(B_\epsilon(x_i))},$$

where C_4 and C_5 are independent of ϵ and δ. Thus, by the conditions $0 \leqslant \phi \leqslant 1$ and $\phi \equiv 1$ on $B_{1/2}(0)$, one gets

$$\begin{aligned}\mu\left(B_{\epsilon/2}(x_i)\right) &\leqslant \int \phi_\epsilon(x-x_i)\,\mathrm{d}\nu \\ &\leqslant \nu\left(B_\epsilon(x_i)\right) + \lambda \int\limits_{B_\epsilon(x_i)} f(v)v\,\mathrm{d}x + \delta C_4 + C_\delta C_5 |v|^p_{L^{p^*}(B_\epsilon(x_i))}.\end{aligned} \tag{2.28}$$

From (2.24), we have that

$$\begin{aligned}\nu_i^{\frac{p}{p^*}} S &\leqslant \mu_i \\ &\leqslant \mu\left(B_{\epsilon/2}(x_i)\right),\end{aligned}$$

and

$$\lim_{\epsilon\to 0^+} \nu\left(B_\epsilon(x_i)\right) = \nu_i.$$

Taking the limit as $\epsilon \to 0^+$ in (2.28) yields

$$a\nu_i^{\frac{p}{p^*}} S \leqslant \nu_i + \delta C_4.$$

As $\delta > 0$ is arbitrary, we conclude that

$$a\nu_i^{\frac{p}{p^*}} S \leqslant \nu_i,$$

i.e.,

$$\nu_i \geqslant (aS)^{\frac{p^*}{p^*-p}} > 0.$$

Since ν is a bounded positive measure, it follows that $\mathfrak{J}$ must be either a empty set or finite. The proof of step a is now finished.

Step b) The convergence $\nabla v_n \to \nabla v$ holds a.e. in $\mathbb{R}^N$. First, let's assume that $\mathfrak{J}$ is a finite nonempty set, namely $\mathfrak{J} = \{1, 2, \ldots, m\}$. For any $\epsilon > 0$, we define an open set $\Omega_\epsilon \subset \mathbb{R}^N$ as follows

$$\Omega_\epsilon = B_{1/2\epsilon}(0) \setminus \bigcup_{i=1}^m B_\epsilon(x_i).$$

There is $\epsilon_0 > 0$ such that

$$B_{\epsilon_0}(x_{i_1}) \cap B_{\epsilon_0}(x_{i_2}) = \emptyset,$$

for any $i_1, i_2 \in \mathfrak{J}$ with $i_1 \neq i_2$, which are possible since $\mathfrak{J}$ is a finite set. Now, let $0 < \zeta < \epsilon_0$. Our claim is

$$\int_{\Omega_\zeta} \left(|\nabla v_n|^{p-2}\nabla v_n - |\nabla v|^{p-2}\nabla v\right)\cdot\nabla(v_n - v)\,\mathrm{d}x \to 0. \tag{2.29}$$

Indeed, Define, for $x \in \mathbb{R}^N$ with $0 < \epsilon < \zeta$,

$$\psi_\epsilon(x) = \phi_{1/\epsilon}(x) - \sum_{i=1}^m \phi_\epsilon\left(x - x_i\right),$$

where ϕ was given in Lemma (2.3). Therefore, we have

$$\begin{aligned}
&\psi_\epsilon(x) \in \mathcal{C}_c^\infty\left(\mathbb{R}^N\right),\\
&\psi_\epsilon(x) = 0 \text{ on } \bigcup_{i=1}^m B_{\epsilon/2}(x_i),\\
&\psi_\epsilon(x) = 1 \text{ on } B_{1/2\epsilon}(0) \setminus \bigcup_{i=1}^m B_\epsilon(x_i).
\end{aligned}$$

From Lemma 2.3 and the fact that $\{v_n\}$ is bounded in W for all $n \in \mathbb{N}$, It is straightforward to verify that $\{\psi_\epsilon v_n\}$ is bounded in W for all n. Using (2.21), we deduce that as $n \to \infty$, the following holds

$$\begin{aligned}
(1+b\|v_n\|^p)\left(\int |\nabla v_n|^p \psi_\epsilon \,\mathrm{d}x + \int |\nabla v_n|^{p-2}\nabla v_n \cdot v_n \nabla \psi_\epsilon \,\mathrm{d}x\right)&\\
= \lambda \int f(v_n) v_n \psi_\epsilon \,\mathrm{d}x + \int |v_n|^{p^*}\psi_\epsilon \,\mathrm{d}x + o(1).&
\end{aligned}$$

As in step a, we conclude that

$$\begin{aligned}
\lim_{n\to\infty}(1+b\|v_n\|^p)\int |\nabla v_n|^p \,\psi_\epsilon \,\mathrm{d}x = \lim_{n\to\infty}\left(-(1+b\|v_n\|^p)\int |\nabla v_n|^{p-2}\nabla v_n \cdot v_n \nabla\psi_\epsilon \,\mathrm{d}x\right)&\\
+ \lim_{n\to\infty}\int |v_n|^{p^*}\psi_\epsilon \,\mathrm{d}x + \lambda\int f(v) v \psi_\epsilon \,\mathrm{d}x.&
\end{aligned}$$

Given that $\psi_\epsilon \equiv 0$ on $\bigcup_{i=1}^m B_{\epsilon/2}(x_i)$ and $\nu = |v|^{p^*} + \sum_{i=1}^m \nu_i \delta_{x_i}$, we obtain

$$\begin{aligned}
\lim_{n\to\infty}\int |v_n|^{p^*}\,\psi_\epsilon \,\mathrm{d}x &= \int \psi_\epsilon \,\mathrm{d}\nu\\
&= \int |v|^{p^*}\psi_\epsilon \,\mathrm{d}x.
\end{aligned}$$

Thus, one has

$$\begin{aligned}
\lim_{n\to\infty}(1+b\|v_n\|^p)\int |\nabla v_n|^p \,\psi_\epsilon \,\mathrm{d}x = \lim_{n\to\infty}\left(-(1+b\|v_n\|^p)\int |\nabla v_n|^{p-2}\nabla v_n \cdot u_n \nabla\psi_\epsilon \,\mathrm{d}x\right)&\\
+ \lambda\int f(v) v \psi_\epsilon \,\mathrm{d}x + \int |v|^{p^*}\psi_\epsilon \,\mathrm{d}x.&
\end{aligned} \tag{2.30}$$

Similarly, in light of (2.21), we conclude that as $n \to \infty$, the following holds

$$\begin{aligned}
(1+b\|v_n\|^p)\int |\nabla v_n|^{p-2}\,\nabla v_n \cdot \psi_\epsilon \nabla v \,\mathrm{d}x = -(1+b\|v_n\|^p)\int |\nabla v_n|^{p-2}\nabla v_n \cdot v\nabla\psi_\epsilon \,\mathrm{d}x&\\
+ \int |v_n|^{p^*-2} v_n v \psi_\epsilon \,\mathrm{d}x + \lambda\int f(v_n) v\psi_\epsilon \,\mathrm{d}x + o(1).&
\end{aligned} \tag{2.31}$$

From Lemma 2.3 and the Young inequality, one has

$$\begin{aligned}
\limsup_{n\to\infty}\left|\int |\nabla v_n|^{p-2}\nabla v_n \cdot v\nabla\psi_\epsilon \,\mathrm{d}x\right| &\leqslant \delta \limsup_{n\to\infty}\|v_n\|^p + C_\delta |v\nabla\psi_\epsilon|_p^p\\
&\leqslant \delta C_2 + C_\delta C_3 |v|^p_{L^{p^*}(B_\epsilon(x_i))}.
\end{aligned} \tag{2.32}$$

Utilizing the fact that $h:\mathbb{R}^N\to\mathbb{R}$ defined by $h(x)=|x|^p$ is strictly convex, one has

$$0\leqslant\left(|\nabla v_n|^{p-2}\nabla v_n-|\nabla v|^{p-2}\nabla v\right)\cdot\nabla\left(v_n-v\right), \tag{2.33}$$

On the other hand, since $\nabla v_n\rightharpoonup\nabla v$ in $L^p(\mathbb{R}^N)$, one has

$$\int\psi_\epsilon|\nabla v|^{p-2}\nabla v\cdot\left(\nabla v_n-\nabla v\right)\,\mathrm{d}x\to 0. \tag{2.34}$$

Then, taking into account that $\psi_\epsilon\equiv 1$ on Ω_ϵ, and combining (2.16), (2.22), (2.30)-(2.31), (2.33)-(2.34), along with the boundedness of $\{v_n\}$, it follows that as n tends to infinity, we deduce that

$$\begin{aligned}
&\int_{\Omega_\zeta}\left(|\nabla v_n|^{p-2}\nabla v_n-|\nabla v|^{p-2}\nabla v\right)\cdot\nabla(v_n-v)\,\mathrm{d}x\\
&\quad\leqslant(1+b\|v_n\|^p)\int\psi_\epsilon\left(|\nabla v_n|^{p-2}\nabla v_n-|\nabla v|^{p-2}\nabla v\right)\cdot\nabla(v_n-v)\,\mathrm{d}x\\
&\quad=(1+b\|v_n\|^p)\int\Big(\psi_\epsilon|\nabla v_n|^{p-2}\nabla v_n\cdot\nabla v_n-\psi_\epsilon|\nabla v_n|^{p-2}\nabla v_n\cdot\nabla v\\
&\qquad-\psi_\epsilon|\nabla v|^{p-2}\nabla v\cdot\nabla v_n+\psi_\epsilon|\nabla v|^{p-2}\nabla v\cdot\nabla v\Big)\,\mathrm{d}x\\
&\quad=\lim_{n\to\infty}(1+b\|v_n\|^p)\int|\nabla v_n|^{p-2}\nabla v_n\cdot v\nabla\psi_\epsilon\,\mathrm{d}x\\
&\qquad-\lim_{n\to\infty}(1+b\|v_n\|^p)\int|\nabla v_n|^{p-2}\nabla v_n\cdot v_n\nabla\psi_\epsilon\,\mathrm{d}x+o(1).
\end{aligned}$$

Given that $0\leqslant\psi_\varepsilon\leqslant 1$, and $\psi_\epsilon=1$ on Ω_ϵ for $0<\epsilon<\epsilon_0$, by (2.26) and (2.32), as n tends to infinity, we conclude that

$$\begin{aligned}
\int_{\Omega_\zeta}\left(|\nabla v_n|^{p-2}\nabla v_n-|\nabla v|^{p-2}\nabla v\right)\cdot\nabla(v_n-v)\,\mathrm{d}x\leqslant 2(1+bM^p)C_\delta C_3|v|^p_{L^{p^*}(B_\epsilon(x_i))}\\
+2(1+bM^p)\delta C_2+o(1),
\end{aligned} \tag{2.35}$$

where constants C_2, C_3 are independent of δ and ϵ. Therefore, in (2.35), first let $n\to\infty$, then let $\epsilon\to 0^+$, and finally let $\delta\to 0^+$, yielding

$$\lim_{n\to\infty}\int_{\Omega_\zeta}\left(|\nabla v_n|^{p-2}\nabla v_n-|\nabla v|^{p-2}\nabla v\right)\cdot\nabla(v_n-v)\,\mathrm{d}x=0. \tag{2.36}$$

Using an elementary inequality (see [18])

$$\left(a-b,|a|^{p-2}a-|b|^{p-2}b\right)\geqslant\begin{cases}C_p|a-b|^p & \text{if } p\geqslant 2,\\ C_p\frac{|a-b|^2}{(|a|+|b|)^{2-p}} & \text{if } 1<p<2,\end{cases}$$

where $a,b\in\mathbb{R}^N$. If $p\geqslant 2$, the convergence of (2.36) to zero as $n\to\infty$ implies that

$$\lim_{n\to\infty}C_p\int_{\Omega_\zeta}|\nabla v_n-\nabla v|^p\,\mathrm{d}x=0. \tag{2.37}$$

Moreover, when $1<p<2$, one has

$$\lim_{n\to\infty}C_p\int_{\Omega_\zeta}\frac{|\nabla v_n-\nabla v|^2}{\left(|\nabla v|+|\nabla v_n|\right)^{2-p}}\,\mathrm{d}x=0.$$

Therefore, applying the Hölder inequality, one gets

$$\begin{aligned}\int_{\Omega_\zeta} |\nabla (v_n - v)|^p \, \mathrm{d}x &= \int_{\Omega_\zeta} \frac{|\nabla (v_n - v)|^p \left(|\nabla v_n| + |\nabla v|\right)^{p(p-2)/2}}{\left(|\nabla v_n| + |\nabla v|\right)^{p(p-2)/2}} \, \mathrm{d}x \\ &\leqslant \left(\int_{\Omega_\zeta} \frac{|\nabla (v_n - v)|^2}{\left(|\nabla v_n| + |\nabla v|\right)^{2-p}} \, \mathrm{d}x \right)^{p/2} \\ &\quad \times \left(\int_{\Omega_\zeta} \left(|\nabla v_n| + |\nabla v|\right)^p \, \mathrm{d}x \right)^{(2-p)/2}. \end{aligned} \tag{2.38}$$

Hence, from (2.37) and (2.38), one can conclude

$$\nabla v_n \to \nabla v \quad \text{a.e. in } \Omega_\zeta.$$

Because ζ is arbitrary, by the Cantor diagonal argument, a subsequence $\{v_n\}$ can be chosen such that

$$\nabla v_n \to \nabla v \quad \text{a.e. on } \mathbb{R}^N, \tag{2.39}$$

as $n \to \infty$. If $\mathfrak{J}$ is empty, then $v_n \to v$ in L^{p^*}, it is easy to see that (2.36) is true for $\mathbb{R}^N$ in place of Ω_ζ. Hence (2.39) holds. From the convergences in (2.22) and (2.39), we have

$$\begin{aligned} v_n^\pm &\rightharpoonup v^\pm && \text{in } W, \\ v_n^\pm &\to v^\pm && \text{in } L^s \text{ for } p \leqslant s < p^*), \\ v_n^\pm &\to v^\pm && \text{a.e. in } \mathbb{R}^N, \\ \nabla v_n^\pm &\to \nabla v^\pm && \text{a.e. in } \mathbb{R}^N. \end{aligned}$$

Using Fatou's lemma, the weak lower semicontinuity of norm, and the Brézis–Lieb lemma, one gets

$$\begin{aligned} &\liminf_{n\to\infty} \mathcal{K}_b^\lambda(\alpha v_n^+ + \beta v_n^-) \\ &\quad \geqslant \frac{\alpha^p}{p} \lim_{n\to\infty} \left(\|v^+\|^p + \|v_n^+ - v^+\|^p\right) + \frac{\beta^p}{p} \lim_{n\to\infty} \left(\|v^-\|^p + \|v_n^- - v^-\|^p\right) \\ &\qquad + \frac{b\alpha^{2p}}{2p} \left[|\nabla v^+|_p^p + \lim_{n\to\infty} |\nabla v_n^+ - \nabla v^+|_p^p\right]^2 + \frac{b\beta^{2p}}{2p} \left[|\nabla v^-|_p^p + \lim_{n\to\infty} |\nabla v_n^- - \nabla v^-|_p^p\right]^2 \\ &\qquad + \frac{b\alpha^p\beta^p}{p} \liminf_{n\to\infty} |\nabla v_n^+|_p^p |\nabla v_n^-|_p^p - \lambda \int F(\alpha v^+) \, \mathrm{d}x - \lambda \int F(\beta v^-) \, \mathrm{d}x \\ &\qquad - \frac{\alpha^{p^*}}{p^*} \left[|v^+|_{p^*}^{p^*} + \lim_{n\to\infty} |v_n^+ - v^+|_{p^*}^{p^*}\right] - \frac{\beta^{p^*}}{p^*} \left[|v^-|_{p^*}^{p^*} + \lim_{n\to\infty} |v_n^- - v^-|_{p^*}^{p^*}\right] \\ &\quad \geqslant \mathcal{K}_b^\lambda(\alpha v^+ + \beta v^-) + \frac{\alpha^p}{p} A_1 + \frac{b\alpha^{2p}}{2p} A_3^2 + \frac{b\alpha^{2p}}{p} A_3 |\nabla v^+|_p^p - \frac{\alpha^{p^*}}{p^*} B_1 + \frac{\beta^p}{p} A_2 \\ &\qquad + \frac{b\beta^{2p}}{2p} A_4^2 + \frac{b\beta^{2p}}{p} A_4 |\nabla v^-|_p^p - \frac{\beta^{p^*}}{p^*} B_2, \end{aligned}$$

where

$$\begin{aligned} A_1 &= \lim_{n\to\infty} \|v_n^+ - v^+\|^p, & A_2 &= \lim_{n\to\infty} \|v_n^- - v^-\|^p, & A_3 &= \lim_{n\to\infty} |\nabla v_n^+ - \nabla v^+|_p^p, \\ A_4 &= \lim_{n\to\infty} |\nabla v_n^- - \nabla v^-|_p^p, & B_1 &= \lim_{n\to\infty} |v_n^+ - v^+|_{p^*}^{p^*}, & B_2 &= \lim_{n\to\infty} |v_n^- - v^-|_{p^*}^{p^*}. \end{aligned}$$

In view of Lemma 2.1 (A_2), one obtains

$$\mathcal{K}_b^\lambda(v_n) \geqslant \mathcal{K}_b^\lambda(\alpha v_n^+ + \beta v_n^-),$$

for all $\alpha, \beta \geqslant 0$. That is,

$$\begin{aligned} c_b^\lambda \geqslant \mathcal{K}_b^\lambda(\alpha v^+ + \beta v^-) + \frac{\alpha^p}{p}A_1 + \frac{b\alpha^{2p}}{2p}A_3^2 + \frac{b\alpha^{2p}}{p}A_3|\nabla v^+|_p^p - \frac{\alpha^{p^*}}{p^*}B_1 + \frac{\beta^p}{p}A_2 \\ + \frac{b\beta^{2p}}{2p}A_4^2 + \frac{b\beta^{2p}}{p}A_4|\nabla v^-|_p^p - \frac{\beta^{p^*}}{p^*}B_2, \end{aligned} \tag{2.40}$$

for all $\alpha, \beta \geqslant 0$.

Step 1: We claim that $v^\pm \neq 0$. Suppose, for the sake of contradiction, that $v^+ = 0$. Letting $\beta = 0$ in (2.40), one has

$$c_b^\lambda \geqslant \frac{\alpha^p}{p}A_1 + \frac{b\alpha^{2p}}{2p}A_3^2 - \frac{\alpha^{p^*}}{p^*}B_1, \tag{2.41}$$

for all $\alpha \geqslant 0$.

Firstly, we consider the case where $B_1 = 0$. If $A_1 = 0$, then $v_n^+ \to v^+$ in W. From (2.16), it follows that $\|v^+\| > 0$. This contradicts the fact that we assume $v^+ = 0$. If $A_1 > 0$, one has

$$c_b^\lambda \geqslant \frac{\alpha^p}{p}A_1 > 0,$$

for all $\alpha \geqslant 0$, which gives a contradiction with Lemma 2.2.

Finally, we consider the case where $B_1 > 0$. Lemma 2.2 ensures the existence of a $\lambda^\star > 0$ suth that the following inequality holds

$$c_b^\lambda < \frac{p}{N}S^{N/p^2}, \quad \forall \lambda \geqslant \lambda^\star. \tag{2.42}$$

From (1.13) and $B_1 > 0$, we conclude $A_1 > 0$. Using (2.41), one has

$$\begin{aligned} \frac{p}{N}S^{N/p^2} &\leqslant \frac{p}{N}\left[\frac{A_1}{B_1^{p/p^*}}\right]^{N/p^2} \\ &\leqslant \max_{\alpha\geqslant 0}\left\{\frac{\alpha^p}{p}A_1 - \frac{\alpha^{p^*}}{p^*}B_1\right\} \\ &\leqslant \max_{\alpha\geqslant 0}\left\{\frac{\alpha^p}{p}A_1 + \frac{b\alpha^{2p}}{2p}A_3^2 - \frac{\alpha^{p^*}}{p^*}B_1\right\} \\ &\leqslant c_b^\lambda. \end{aligned}$$

Thanks to (2.42), by (2.41), ones has

$$\begin{aligned} c_b^\lambda &< \frac{p}{N}S^{N/p^2} \\ &\leqslant \max_{\alpha\geqslant 0}\left\{\frac{\alpha^p}{p}A_1 + \frac{b\alpha^{2p}}{2p}A_3^2 - \frac{\alpha^{p^*}}{p^*}B_1\right\} \\ &\leqslant c_b^\lambda, \end{aligned}$$

which is a contradiction. Consequently, $v^+ \neq 0$. Likewise, we deduce that $v^- \neq 0$ in a similar manner.

Step 2: We will prove that $B_1 = B_2 = 0$. We just prove $B_1 = 0$ (in an analogous manner, we demonstrate that $B_2 = 0$). Suppose, for the sake of contradiction, that $B_1 > 0$.

First case $B_2 > 0$. As both $B_1 > 0$, and $B_2 > 0$, it follows A_1, and $A_2 > 0$. Let $\widetilde{\alpha}$ and $\widetilde{\beta}$ satisfy

$$\frac{\widetilde{\alpha}^p}{p}A_1 + \frac{b\widetilde{\alpha}^{2p}}{2p}A_3^2 - \frac{\widetilde{\alpha}^{p^*}}{p^*}B_1 = \max_{\alpha \geqslant 0}\left\{\frac{\alpha^p}{p}A_1 + \frac{b\alpha^{2p}}{2p}A_3^2 - \frac{\alpha^{p^*}}{p^*}B_1\right\},$$

and

$$\frac{\widetilde{\beta}^p}{p}A_2 + \frac{b\widetilde{\beta}^{2p}}{2p}A_4^2 - \frac{\widetilde{\beta}^{p^*}}{p^*}B_2 = \max_{\beta \geqslant 0}\left\{\frac{a\beta^p}{p}A_2 + \frac{b\beta^{2p}}{2p}A_4^2 - \frac{\beta^{p^*}}{p^*}B_2\right\}.$$

Due to the compactness of the set $[0,\widetilde{\alpha}] \times [0,\widetilde{\beta}]$ is compact and the continuity of g_v, there is a pair$(\alpha_v, \beta_v) \in [0,\widetilde{\alpha}] \times [0,\widetilde{\beta}]$ satisfying

$$g_v(\alpha_v, \beta_v) = \max_{(\alpha,\beta)\in[0,\widetilde{\alpha}]\times[0,\widetilde{\beta}]} g_v(\alpha, \beta).$$

Therefore, it suffices to verify that a maximum cannot be attained on the boundary, i.e., $(\alpha_v, \beta_v) \in (0,\widetilde{\alpha}) \times (0,\widetilde{\beta})$. Observe that if β is sufficiently small, one has

$$\begin{aligned} g_v(\alpha, 0) &= \mathcal{K}_b^\lambda(\alpha v^+) \\ &< \mathcal{K}_b^\lambda(\alpha v^+) + \mathcal{K}_b^\lambda(\beta v^-) \\ &\leqslant \mathcal{K}_b^\lambda(\alpha v^+ + \beta v^-) \\ &= g_v(\alpha, \beta), \end{aligned}$$

for all $\alpha \in [0,\widetilde{\alpha}]$. Thus, there is $\beta_0 \in [0,\widetilde{\beta}]$ such that $g_v(\alpha, 0) \leqslant g_v(\alpha, \beta_0)$, for all $\alpha \in [0,\widetilde{\alpha}]$, i.e., $(\alpha_v, \beta_v) \notin [0,\widetilde{\alpha}] \times \{0\}$. Using a similar approach, we can demonstrate that $(\alpha_v, \beta_v) \notin \{0\} \times [0,\widetilde{\beta}]$.

It is straightforward to observe that

$$\frac{\alpha^p}{p}A_1 + \frac{b\alpha^{2p}}{2p}A_3^2 + \frac{b\alpha^{2p}}{p}A_3|\nabla v^+|_p^p - \frac{\alpha^{p^*}}{p^*}B_1 > 0, \quad \alpha \in (0,\widetilde{\alpha}], \tag{2.43}$$

and

$$\frac{\beta^p}{p}A_2 + \frac{b\beta^{2p}}{2p}A_4^2 + \frac{b\beta^{2p}}{p}A_4|\nabla v^-|_p^p - \frac{\beta^{p^*}}{p^*}B_2 > 0, \quad \beta \in (0,\widetilde{\beta}]. \tag{2.44}$$

Therefore, one gets

$$\begin{aligned} \frac{p}{N}S^{N/p^2} \leqslant \frac{\widetilde{\alpha}^p}{p}A_1 + \frac{b\widetilde{\alpha}^{2p}}{2p}A_3^2 + \frac{b\widetilde{\alpha}^{2p}}{p}A_3\int|\nabla v^+|^p\,\mathrm{d}x - \frac{\widetilde{\alpha}^{p^*}}{p^*}B_1 + \frac{\beta^p}{p}A_2 \\ + \frac{b\beta^{2p}}{2p}A_4^2 + \frac{b\beta^{2p}}{p}A_4|\nabla v^-|_p^p - \frac{\beta^{p^*}}{p^*}B_2, \end{aligned}$$

and

$$\begin{aligned} \frac{p}{N}S^{N/p^2} \leqslant \frac{\alpha^p}{p}A_1 + \frac{b\alpha^{2p}}{2p}A_3^2 + \frac{b\alpha^{2p}}{p}A_3|\nabla v^+|_p^p - \frac{\alpha^{p^*}}{p^*}B_1 + \frac{\widetilde{\beta}^p}{p}A_2 \\ + \frac{b\widetilde{\beta}^{2p}}{2p}A_4^2 + \frac{b\widetilde{\beta}^{2p}}{p}A_4|\nabla v^-|_p^p - \frac{\widetilde{\beta}^{p^*}}{p^*}B_2, \end{aligned}$$

for all $\alpha \in [0,\widetilde{\alpha}], \beta \in [0,\widetilde{\beta}]$. Combining this with (2.40), one obtains

$$g_v(\alpha,\widetilde{\beta}) \leqslant 0, \quad \forall \alpha \in [0,\widetilde{\alpha}],$$

and, one also has

$$g_v(\widetilde{\alpha},\beta) \leqslant 0, \quad \forall \beta \in [0,\widetilde{\beta}].$$

That is, $(\alpha_v,\beta_v) \notin \{\widetilde{\alpha}\} \times [0,\widetilde{\beta}]$ and $(\alpha_v,\beta_v) \notin \times[0,\widetilde{\alpha}] \times \{\widetilde{\beta}\}$.

In conclusion, it follows that $(\alpha_v,\beta_v) \in (0,\widetilde{\alpha}) \times (0,\widetilde{\beta})$. Thus, the pair (α_v,β_v) is a critical point of g_v. Hence, $\alpha_v v^+ + \beta_v v^- \in \mathcal{M}_b^\lambda$. So, combining (2.40), (2.43) and (2.44), one has that

$$\begin{aligned} c_b^\lambda &\geqslant \mathcal{K}_b^\lambda(\alpha_v v^+ + \beta_v v^-) + \frac{\alpha_v^p}{p}A_1 + \frac{b\alpha_v^{2p}}{2p}A_3^2 + \frac{b\alpha_v^{2p}}{p}A_3|\nabla v^+|_p^p - \frac{\alpha_v^{p^*}}{p^*}B_1 + \frac{\beta_v^p}{p}A_2 \\ &\quad + \frac{b\beta_v^{2p}}{2p}A_4^2 + \frac{b\beta_v^{2p}}{p}A_4|\nabla v^-|_p^p - \frac{\beta_v^{p^*}}{p^*}B_2 \\ &> \mathcal{K}_b^\lambda(\alpha_v v^+ + \beta_v v^-) \\ &\geqslant c_b^\lambda. \end{aligned}$$

This leads to a contradiction. Hence $B_1 = 0$.

Second case $B_2 = 0$. In this situation, maximization can be carried out in $[0,\widetilde{\alpha}] \times [0,\infty)$. In fact, it can be shown that there is $\beta_0 \in [0,\infty)$ satisfying

$$\mathcal{K}_b^\lambda(\alpha v^+ + \beta v^-) \leqslant 0,$$

for all $(\alpha,\beta) \in [0,\widetilde{\alpha}] \times [\beta_0,\infty)$. Hence, there is $(\alpha_v,\beta_v) \in [0,\widetilde{\alpha}] \times [0,\infty)$ that satisfies

$$g_v(\alpha_v,\beta_v) = \max_{(\alpha,\beta)\in[0,\widetilde{\alpha}]\times[0,\infty)} g_v(\alpha,\beta).$$

Next, we show that $(\alpha_v,\beta_v) \in (0,\widetilde{\alpha}) \times (0,\infty)$. One has

$$g_v(\alpha,0) < g_v(\alpha,\beta), \quad \forall \alpha \in [0,\widetilde{\alpha}] \text{ and } \beta \text{ sufficiently small},$$

thus, $(\alpha_v,\beta_v) \notin [0,\widetilde{\alpha}] \times \{0\}$. At the same time, one has

$$g_v(0,\beta) < g_v(\alpha,\beta), \quad \forall \beta \in [0,\infty) \text{ and } \alpha \text{ sufficiently small}.$$

Therefore, we get that $(\alpha_v,\beta_v) \notin \{0\} \times [0,\infty)$.

Conversely, for all $\beta \in [0,\infty)$, we have

$$\begin{aligned} \frac{p}{N}S^{N/p^2} &\leqslant \frac{\widetilde{\alpha}^p}{p}A_1 + \frac{b\widetilde{\alpha}^{2p}}{2p}A_3^2 + \frac{b\widetilde{\alpha}^{2p}}{p}A_3|\nabla v^+|_p^p - \frac{\widetilde{\alpha}^{p^*}}{p^*}B_1 + \frac{\beta^p}{p}A_2 \\ &\quad + \frac{b\beta^{2p}}{2p}A_4^2 + \frac{b\beta^{2p}}{p}A_4|\nabla v^-|_p^p. \end{aligned}$$

Thus, one has

$$g_v(\widetilde{\alpha},\beta) \leqslant 0, \quad \forall \beta \in [0,\infty).$$

Therefore, $(\alpha_v,\beta_v) \notin \{\widetilde{\alpha}\} \times [0,\infty)$. Then, $(\alpha_v,\beta_v) \in (0,\widetilde{\alpha}) \times (0,\infty)$. Hence, (α_v,β_v) is an inner maximizer of g_v in $[0,\widetilde{\alpha}) \times [0,\infty)$. And so $\alpha_v v^+ + \beta_v v^- \in \mathcal{M}_b^\lambda$.

Consequently, based on (2.43), we get

$$\begin{aligned} c_b^\lambda &\geqslant \mathcal{K}_b^\lambda(\alpha_v v^+ + \beta_v v^-) + \frac{\alpha_v^p}{p}A_1 + \frac{b\alpha_v^{2p}}{2p}A_3^2 + \frac{b\alpha_v^{2p}}{p}A_3|\nabla v^+|_p^p - \frac{\alpha_v^{p^*}}{p^*}B_1 + \frac{\beta_v^p}{p}A_2 \\ &\quad + \frac{b\beta_v^{2p}}{2p}A_4^2 + \frac{b\beta_v^{2p}}{p}A_4|\nabla v^-|_p^p \\ &> \mathcal{K}_b^\lambda(\alpha_v v^+ + \beta_v v^-) \\ &\geqslant c_b^\lambda, \end{aligned}$$

which is absurd.

Hence, based on the preceding discussion, it follows that $B_1 = B_2 = 0$.

Step 3: Lastly, we establish the attainment of c_b^λ. Given $v^\pm \neq 0$, according to Lemma 2.1, there is $\alpha_v, \beta_v > 0$ such that $\widetilde{v} := \alpha_v v^+ + \beta_v v^- \in \mathcal{M}_b^\lambda$. Moreover, we have that $\langle (\mathcal{K}_b^\lambda)'(v), v^\pm \rangle \leqslant 0$. Lemma (2.1) implies that $0 < \alpha_v, \beta_v \leqslant 1$.
Combining $v_n \in \mathcal{M}_b^\lambda$ and Lemma 2.1, one gets

$$\begin{aligned} \mathcal{K}_b^\lambda(v_n^+ + v_n^-) &= \mathcal{K}_b^\lambda(v_n) \\ &\geqslant \mathcal{K}_b^\lambda(\alpha_v v_n^+ + \beta_v v_n^-). \end{aligned}$$

Taking into consideration $B_1 = B_2 = 0$ and the semicontinuity of the norm, one has

$$\begin{aligned} c_b^\lambda &\leqslant \mathcal{K}_b^\lambda(\widetilde{v}) = \mathcal{K}_b^\lambda(\widetilde{v}) - \frac{1}{2p}\left\langle (\mathcal{K}_b^\lambda)'(\widetilde{v}), \widetilde{v} \right\rangle \\ &= \frac{(p-1)}{2p}\|\widetilde{v}\|^p + \frac{p^*-2p}{2pp^*}|\widetilde{v}|_{p^*}^{p^*} + \frac{\lambda}{2p}\int [f(\widetilde{v})\widetilde{v} - 2pF(\widetilde{v})]\,\mathrm{d}x \\ &= \frac{(p-1)}{2p}\left(\|\alpha_v v^+\|^p + \|\beta_v v^-\|^p\right) + \frac{p^*-2p}{2pp^*}\left[|\alpha_v v^+|_{p^*}^{p^*} + |\beta_v v^-|_{p^*}^{p^*}\right] \\ &\quad + \frac{\lambda}{2p}\left[\int f(\alpha_v v^+)\alpha_v v^+ - 2pF(\alpha_v v^+) + \int f(\beta_v v^-)\beta_v v^- - 2pF(\beta_v v^-)\right] \\ &\leqslant \frac{(p-1)}{2p}\|v\|^p + \frac{p^*-2p}{2pp^*}|v|_{p^*}^{p^*} + \frac{\lambda}{2p}\int f(v)v - 2pF(v) \\ &\leqslant \liminf_{n\to\infty}\left[\mathcal{K}_b^\lambda(v_n) - \frac{1}{2p}\left\langle (\mathcal{K}_b^\lambda)'(v_n), v_n \right\rangle\right] \\ &= \liminf_{n\to\infty}\mathcal{K}_b^\lambda(v_n) \\ &= c_b^\lambda. \end{aligned}$$

Therefore, it follows that $\alpha_v = \beta_v = 1$, and c_b^λ is attained by $v_b := v^+ + v^- \in \mathcal{M}_b^\lambda$. □

3 Proof of theorems

We begin by proving Theorem 1.3.

Proof of Theorem 1.3. By applying Lemma 2.4, we can demonstrate that the minimizer v_b for c_b^λ is a sign-changing solution to problem ($\mathcal{QKP}$). As $v_b \in \mathcal{M}_b^\lambda$, one has

$$\langle (\mathcal{K}_b^\lambda)'(v_b), v_b^\pm \rangle = 0.$$

For $(\alpha,\beta)\in(\mathbb{R}^+\times\mathbb{R}^+)\backslash(1,1)$, using Lemma 2.1 and Lemma 2.4, we can conclude

$$\mathcal{K}_b^\lambda(\alpha v_b^+ + \beta v_b^-) < \mathcal{K}_b^\lambda(v_b^+ + v_b^-) = c_b^\lambda. \tag{3.1}$$

At this point, we aim to demonstrate that $(\mathcal{K}_b^\lambda)'(v_b)=0$. Suppose, the contrary $(\mathcal{K}_b^\lambda)'(v_b)\neq 0$. It follows that there is $\delta>0$ and $\theta>0$ such that for all $\|u-v_b\|\geqslant 3\delta$, one has

$$\|(\mathcal{K}_b^\lambda)'(u)\|\geqslant\theta.$$

Select $\sigma\in\left(0,\min\left\{\frac{1}{2^{(p-1)}},\frac{\delta}{2^{(p-1)/p}\|v_b\|}\right\}\right)$. Define $D:=(1-\sigma,1+\sigma)\times(1-\sigma,1+\sigma)$, and

$$k(\alpha,\beta)=\alpha v_b^+ + \beta v_b^-,\quad \forall(\alpha,\beta)\in D.$$

From (3.1), one has that

$$\widetilde{c}_\lambda := \max_{\alpha,\beta\in\partial D}\mathcal{K}_b^\lambda\circ k < c_b^\lambda.$$

Let $S_\delta := B_\delta(v_b)$ and $\varepsilon := \min\left\{\frac{(c_b^\lambda-\widetilde{c}_\lambda)}{2},\frac{\theta\delta}{8}\right\}$, it follows from [20, Lemma 2.3] that there is a deformation $\eta\in\mathcal{C}([0,1]\times W,W)$ such that

(a) If $u\notin(\mathcal{K}_b^\lambda)^{-1}\left(\left[c_b^\lambda-2\varepsilon,c_b^\lambda+2\varepsilon\right]\cap S_{2\delta}\right)$, then $\eta(1,u)=u$,

(b) $\eta\left(1,(\mathcal{K}_b^\lambda)^{c_b^\lambda+\varepsilon}\cap S_\delta\right)\subset(\mathcal{K}_b^\lambda)^{c_b^\lambda-\varepsilon}$,

(c) $\mathcal{K}_b^\lambda(\eta(1,u))\leqslant\mathcal{K}_b^\lambda(u),\quad\forall u\in W$.

To begin, we must demonstrate that

$$\max_{(\alpha,\beta)\in\bar{D}}\mathcal{K}_b^\lambda(\eta(1,k(\alpha,\beta)))<c_b^\lambda. \tag{3.2}$$

Indeed, by Lemma 2.1, one has

$$\mathcal{K}_b^\lambda(k(\alpha,\beta))\leqslant c_b^\lambda<c_b^\lambda+\varepsilon,$$

i.e,

$$k(\alpha,\beta)\in\left(\mathcal{K}_b^\lambda\right)^{c_b^\lambda+\varepsilon}.$$

Conversely, one has

$$\begin{aligned}\|k(\alpha,\beta)-v_b\|^p &= \|(\beta-1)v_b^- + (\alpha-1)v_b^+\|^p\\ &\leqslant 2^{p-1}\left((\beta-1)^p\|v_b^-\|^p+(\alpha-1)^p\|v_b^+\|^p\right)\\ &\leqslant 2^{p-1}\sigma^p\|v_b\|^p\\ &<\delta^p.\end{aligned}$$

It follows that

$$k(\alpha,\beta)\in S_\delta\quad\forall(\alpha,\beta)\in\bar{D}.$$

Thus, in light of (b), one gets

$$\mathcal{K}_b^\lambda(\eta(1,k(\alpha,\beta)))<c_b^\lambda-\varepsilon.$$

Hence, we confirm that (3.2) holds.

Our next step is to show that $\eta(1,k(D)) \cap \mathcal{M}_b^\lambda \neq \emptyset$, leading to a contradiction with the definition of c_b^λ. Define $h(\alpha,\beta) := \eta(1,k(\alpha,\beta))$,

$$\begin{aligned}\Psi_0(\alpha,\beta) :=& \left(\langle (\mathcal{K}_b^\lambda)'(k(\alpha,\beta)), v_b^+\rangle, \langle (\mathcal{K}_b^\lambda)'(k(\alpha,\beta)), v_b^-\rangle\right)\\ =& \left(\langle (\mathcal{K}_b^\lambda)'(\alpha v_b^+ + \beta v_b^-), v_b^+\rangle, \langle (\mathcal{K}_b^\lambda)'(\alpha v_b^+ + \beta v_b^-), v_b^-\rangle\right)\\ :=& \left(\varphi^1_{v_b}(\alpha,\beta), \varphi^2_{v_b}(\alpha,\beta)\right),\end{aligned}$$

and

$$\Psi_1(\alpha,\beta) := \left(\frac{1}{\alpha}\langle (\mathcal{K}_b^\lambda)'(h(\alpha,\beta)), (h(\alpha,\beta))^+\rangle, \frac{1}{\beta}\langle (\mathcal{K}_b^\lambda)'(h(\alpha,\beta)), (h(\alpha,\beta))^-\rangle\right).$$

A straightforward computation yields

$$\begin{aligned}\left.\frac{\varphi^1_{v_b}(\alpha,\beta)}{\partial\alpha}\right|_{(1,1)} &= (p-1)\|v_b^+\|^p - (p^*-1)|v_b^+|_{p^*}^{p^*} + b(2p-1)|\nabla v_b^+|_p^{2p}\\ &\quad - \lambda\int \partial_\alpha f(v_b^+)(v_b^+)^2\,\mathrm{d}x + b(p-1)|\nabla v_b^+|_p^p|\nabla v_b^-|_p^p,\\ \left.\frac{\varphi^2_{v_b}(\alpha,\beta)}{\partial\beta}\right|_{(1,1)} &= (p-1)\|v_b^-\|^p - (p^*-1)|v_b^-|_{p^*}^{p^*} + b(2p-1)|\nabla v_b^-|_p^{2p}\\ &\quad - \lambda\int \partial_\alpha f(v_b^-)(v_b^-)^2\,\mathrm{d}x + b(p-1)|\nabla v_b^-|_p^p|\nabla v_b^+|_p^p,\end{aligned}$$

$$\left.\frac{\varphi^1_{v_b}(\alpha,\beta)}{\partial\beta}\right|_{(1,1)} = bp|\nabla v_b^+|_p^p|\nabla v_b^-|_p^p, \qquad \left.\frac{\varphi^2_{v_b}(\alpha,\beta)}{\partial\alpha}\right|_{(1,1)} = bp|\nabla v_b^-|_p^p|\nabla v_b^+|_p^p.$$

Let

$$M = \begin{bmatrix} \left.\frac{\varphi^1_{v_b}(\alpha,\beta)}{\partial\alpha}\right|_{(1,1)} & \left.\frac{\varphi^2_{v_b}(\alpha,\beta)}{\partial\alpha}\right|_{(1,1)} \\ \left.\frac{\varphi^1_{v_b}(\alpha,\beta)}{\partial\beta}\right|_{(1,1)} & \left.\frac{\varphi^2_{v_b}(\alpha,\beta)}{\partial\beta}\right|_{(1,1)} \end{bmatrix}.$$

Employing condition $(\mathcal{P}_4)$ with $t \neq 0$, it follows that

$$\partial_t f(t)t^2 - (2p-1)f(t)t > 0.$$

Thus, since $v_b \in \mathcal{M}_b^\lambda$, one has

$$\begin{aligned}\det M =& \left.\frac{\varphi^1_{v_b}(\alpha,\beta)}{\partial\alpha}\right|_{(1,1)} \times \left.\frac{\varphi^2_{v_b}(\alpha,\beta)}{\partial\beta}\right|_{(1,1)} - \left.\frac{\varphi^1_{v_b}(\alpha,\beta)}{\partial\beta}\right|_{(1,1)} \times \left.\frac{\varphi^2_{v_b}(\alpha,\beta)}{\partial\alpha}\right|_{(1,1)}\\ =& \left[p\|v_b^+\|^p + bp|\nabla v_b^+|_p^p|\nabla v_b^-|_p^p + (p^*-2p)|v_b^+|_{p^*}^{p^*}\right.\\ &\left. + \lambda\int (\partial_\alpha f(v_b^+)(v_b^+)^2 - (2p-1)f(v_b^+)(v_b^+)\,\mathrm{d}x\right]\\ &\times \left[p\|v_b^-\|^p + bp|\nabla v_b^+|_p^p|\nabla v_b^-|_p^p + (p^*-2p)|v_b^-|_{p^*}^{p^*}\right.\\ &\left. + \lambda\int (\partial_\beta f(v_b^-)(v_b^-)^2 - (2p-1)f(v_b^-)(v_b^-)\,\mathrm{d}x\right]\\ &- (bp)^2|\nabla v_b^+|_p^{2p}|\nabla v_b^-|_p^{2p}\\ >& 0.\end{aligned}$$

As $(1,1)$ is the unique isolated zero of the $\mathcal{C}^1$ function $\Psi_0(\alpha,\beta)$, it follows from degree theory that $\deg(\Psi_0, D, 0) = 1$. Consequently, taking both (3.2) and (a) into account, one has

$$h(\alpha,\beta) = k(\alpha,\beta), \quad \text{on } \partial D.$$

Thus, it follows that $\deg(\Psi_1, D, 0) = 1$. Hence, $\Psi_1(\alpha_0,\beta_0) = 0$ for some $(\alpha_0,\beta_0) \in D$ so that

$$\eta(1, k(\alpha_0,\beta_0)) = h(\alpha_0,\beta_0) \in \mathcal{M}_b^\lambda.$$

Thereby contradicting (3.2). Thus, $(\mathcal{K}_b^\lambda)'(v_b) = 0$, which implies v_b is a critical point of $\mathcal{K}_b^\lambda$. Therefore, we conclude that v_b is a sign-changing solution for problem ($\mathcal{QKP}$). To conclude our proof, we show that v has exactly two nodal domains. We proceed by assuming, for contradiction, that

$$v = v_1 + v_2 + v_3, \quad \text{with } v_i \neq 0, v_1 \geqslant 0, v_2 \leqslant 0,$$

$$\operatorname{supp}(v_i) \cap \operatorname{supp}(v_j) = \emptyset, \quad \text{for } i \neq j,\ i,j = 1,2,3,$$

and

$$\langle (\mathcal{K}_b^\lambda)'(v), v_i \rangle = 0, \quad \text{for } i = 1,2,3.$$

Let $u := v_1 + v_2$, we observe that $u^+ = v_1$ and $u^- = v_2$. Thus, $u^\pm \neq 0$. Therefore, there is a unique (α_u, β_u) of positive numbers such that $\alpha_u v_1 + \beta_u v_2 \in \mathcal{M}_b^\lambda$.

Hence,

$$\mathcal{K}_b^\lambda(\alpha_u v_1 + \beta_u v_2) \geqslant c_b^\lambda.$$

Additionally, by noting that $\langle (\mathcal{K}_b^\lambda)'(v), v_i \rangle = 0$, one gets $\langle (\mathcal{K}_b^\lambda)'(u), u^\pm \rangle < 0$. From Lemma 2.1, it follows that $(\alpha_u, \beta_u) \in (0,1] \times (0,1]$. Conversely, one has

$$\begin{aligned}
0 &= \frac{1}{2p}\langle (\mathcal{K}_b^\lambda)'(v), v_3 \rangle \\
&= \frac{1}{2p}\|v_3\|^p + \frac{b}{2p}|\nabla v_3|_p^{2p} + \frac{b}{2p}|\nabla v_3|_p^p |\nabla v_1|_p^p + \frac{b}{2p}|\nabla v_3|_p^p |\nabla v_2|_p^p \\
&\quad - \frac{\lambda}{2p}\int f(v_3)v_3 \, \mathrm{d}x - \frac{1}{2p}|v|_{p^*}^{p^*} \\
&< \mathcal{K}_b^\lambda(v_3) + \frac{b}{2p}|\nabla v_3|_p^p |\nabla v_1|_p^p + \frac{b}{2p}|\nabla v_3|_p^p |\nabla v_2|_p^p.
\end{aligned}$$

Hence, by (2.17) we obtain

$$\begin{aligned}
c_b^\lambda &\leqslant \mathcal{K}_b^\lambda(\alpha_u v_1 + \beta_u v_2) \\
&= \mathcal{K}_b^\lambda(\alpha_u v_1 + \beta_u v_2) - \frac{1}{2p}\langle (\mathcal{K}_b^\lambda)'(\alpha_u v_1 + \beta_u v_2), (\alpha_u v_1 + \beta_u v_2) \rangle \\
&= \frac{1}{2p}\left(\|\alpha_u v_1\|^p + \|\beta_u v_2\|^p\right) + \frac{\lambda}{2p}\int \left[f(\alpha_u v_1)(\alpha_u v_1) - 2pF(\alpha_u v_1)\right] \, \mathrm{d}x \\
&\quad + \frac{\lambda}{2p}\int \left[f(\beta_u v_2)(\beta_u v_2) - 2pF(\beta_u v_2)\right] \, \mathrm{d}x \\
&\quad + \frac{p^* - 2p}{2pp^*}\alpha_u^{p^*}|v_1|_{p^*}^{p^*} x + \frac{p^* - 2p}{2pp^*}\beta_u^{p^*}|v_2|_{p^*}^{p^*} \\
&\leqslant \mathcal{K}_b^\lambda(v_1 + v_2) - \frac{1}{2p}\langle (\mathcal{K}_b^\lambda)'(v_1 + v_2), (v_1 + v_2) \rangle
\end{aligned}$$

$$
\begin{aligned}
&= \mathcal{K}_b^\lambda(v_1+v_2) + \frac{1}{2p}\langle(\mathcal{K}_b^\lambda)'(v), v_3\rangle + \frac{b}{2p}|\nabla v_3|_p^p|\nabla v_1|_p^p + \frac{b}{2p}|\nabla v_3|_p^p|\nabla v_2|_p^p \\
&< \mathcal{K}_b^\lambda(v_1) + \mathcal{K}_b^\lambda(v_2) + \mathcal{K}_b^\lambda(v_3) + \frac{b}{2p}\left(|\nabla v_3|_p^p + |\nabla v_2|_p^p\right)|\nabla v_1|_p^p \\
&\quad + \frac{b}{2p}\left(|\nabla v_3|_p^p + |\nabla v_1|_p^p\right)|\nabla v_2|_p^p + \frac{b}{2p}\left(|\nabla v_1|_p^p + |\nabla v_2|_p^p\right)|\nabla v_3|_p^p \\
&= \mathcal{K}_b^\lambda(v) \\
&= c_b^\lambda,
\end{aligned}
$$

this leads to a contradiction, which implies that $v_3 = 0$. Hence, v has precisely two nodal domains. Utilizing Theorem 1.3, we prove the existence of a least energy sign-changing solution v_b to problem ($\mathcal{QKP}$). □

Next, we provide a proof for Theorem 1.4, asserting that the energy of v_b is strictly greater than twice the ground state energy.

Proof of Theorem 1.4. Similarly to the proof of Lemma 2.2, there is $\lambda_1^\star > 0$ such that for all $\lambda \geqslant \lambda_1^\star$, there is $v \in \mathcal{N}_b^\lambda$ such that $\mathcal{K}_b^\lambda(v) = c_{b,\lambda}^* > 0$. Using standard arguments the critical points of the functional $\mathcal{K}_b^\lambda$ on $\mathcal{N}_b^\lambda$ are critical points of $\mathcal{K}_b^\lambda$ in W. As a result, $(\mathcal{K}_b^\lambda)'(v) = 0$, which implies that v is a ground state solution of problem ($\mathcal{QKP}$). Theorem 1.3 indicates that for all $\lambda \geqslant \lambda^\star$, there exists a least energy sign-changing solution v_b to problem ($\mathcal{QKP}$) that changes its sign only once. Define $\lambda^{\star\star} = \max\{\lambda_1^\star, \lambda^\star\}$. Assume that $v_b = v_b^+ + v_b^-$. Following the approach in the proof of Lemma 2.1, there are $\alpha_{v_b^+} > 0$ and $\beta_{v_b^-} > 0$ such that

$$
\alpha_{v_b^+} v_b^+ \in \mathcal{N}_b^\lambda, \qquad \beta_{v_b^-} v_b^- \in \mathcal{N}_b^\lambda.
$$

Moreover, according to Lemma 2.1, $\alpha_{v_b^+}, \beta_{v_b^-} \in (0,1)$. Consequently, by applying Lemma 2.1 again, one obtains

$$
\begin{aligned}
2c_{b,\lambda}^* &\leqslant \mathcal{K}_b^\lambda(\alpha_{v_b^+} v_b^+) + \mathcal{K}_b^\lambda(\beta_{v_b^-} v_b^-) \\
&\leqslant \mathcal{K}_b^\lambda(\alpha_{v_b^+} v_b^+ + \beta_{v_b^-} v_b^-) \\
&< \mathcal{K}_b^\lambda(v_b^+ + v_b^-) \\
&= c_b^\lambda.
\end{aligned}
$$

Hence, $c_{b,\lambda}^* > 0$ cannot be attained by any sign-changing function. □

Lastly, we provide the proof for Theorem 1.5.

Proof of Theorem 1.5. **Step 1:** We prove that for any sequence $\{b_n\}$, $\{v_{b_n}\}$ is bounded in W if $b_n \searrow 0$. Let $\chi \in C_0^\infty(\mathbb{R}^N)$ be a nonzero function with fixed $\chi^\pm \neq 0$. Similar to the argument presented in Lemma 2.1, for any $b \in [0,1]$, there is (τ_1, τ_2) that are independent of b, such that

$$
\langle(\mathcal{K}_b^\lambda)'(\tau_1\chi^+ + \tau_2\chi^-), \tau_1\chi^+\rangle < 0, \quad \text{and} \quad \langle(\mathcal{K}_b^\lambda)'(\tau_1\chi^+ + \tau_2\chi^-), \tau_2\chi^-\rangle < 0.
$$

As per Lemma 2.1, for any $b \in [0,1]$, there is a unique $(\alpha_\chi(b), \beta_\chi(b)) \in (0,1] \times (0,1]$ such that $\bar{\chi} := \alpha_\chi(b)\tau_1\chi^+ + \beta_\chi(b)\tau_2\chi^- \in \mathcal{M}_b^\lambda$.

Hence, using (2.4), we can deduce that for any $b \in [0,1]$,

$$\begin{aligned}
\mathcal{K}_b^\lambda(v_b) &\leqslant \mathcal{K}_b^\lambda(\bar{\chi}) \\
&= \mathcal{K}_b^\lambda(\bar{\chi}) - \frac{1}{2p}\langle (\mathcal{K}_b^\lambda)'(\bar{\chi}), \bar{\chi}\rangle \\
&= \frac{(p-1)}{2p}\|\bar{\chi}\|^p + \frac{p^*-2p}{2pp^*}|\bar{\chi}|_{p^*}^{p^*} + \frac{\lambda}{2p}\int [f(\bar{\chi})\bar{\chi} - 2pF(\bar{\chi})]\,\mathrm{d}x \\
&\leqslant \frac{(p-1)}{2p}\|\bar{\chi}\|^p + \frac{p^*-2p}{2pp^*}|\bar{\chi}|_{p^*}^{p^*} + \frac{\lambda}{2p}\left(C_\varepsilon|\bar{\chi}|_\vartheta^\vartheta + \varepsilon|\bar{\chi}|_p^p\right) \\
&\leqslant \frac{(p-1)}{2p}\left(\|\tau_1\chi^+\|^p + \|\tau_2\chi^-\|^p\right) + \frac{p^*-2p}{2pp^*}\left[|\tau_1\chi^+|_{p^*}^{p^*} + |\tau_2\chi|_{p^*}^{p^*}\right] \\
&\quad + \frac{\lambda}{2p}\left[\left(\varepsilon|\tau_1\chi^+|_p^p + C_\varepsilon|\tau_1\chi^+|_\vartheta^\vartheta\right) + \left(\varepsilon|\tau_2\chi^-|_p^p + C_\varepsilon|\tau_2\chi^-|_\vartheta^\vartheta\right)\right] \\
&= C_\chi.
\end{aligned}$$

Therefore, as $n \to \infty$, one gets

$$\begin{aligned}
C_\chi + 1 &\geqslant \mathcal{K}_{b_n}^\lambda(v_{b_n}) \\
&= \mathcal{K}_{b_n}^\lambda(v_{b_n}) - \frac{1}{2p}\langle (\mathcal{K}_{b_n}^\lambda)'(v_{b_n}), v_{b_n}\rangle \\
&\geqslant \frac{(p-1)}{2p}\|v_{b_n}\|^p.
\end{aligned}$$

Hence, $\{v_{b_n}\}$ is bounded in W.

Step 2: We demonstrate the existence of a nodal solution v_0 to problem ($\mathcal{QKP}$). As the sequence $\{v_{b_n}\}$ is bounded in W, up to a subsequence, there is $v_0 \in W$ such that

$$\begin{aligned}
v_{b_n} &\rightharpoonup v_0 && \text{in } W, \\
v_{b_n} &\to v_0 && \text{in } L^s(\mathbb{R}^N), \text{ for } s \in [p, p^*), \\
v_{b_n} &\to v_0 && \text{a.e. in } \mathbb{R}^N, \\
\nabla v_{b_n} &\to \nabla v_0 && \text{a.e. in } \mathbb{R}^N.
\end{aligned} \tag{3.3}$$

Since $\{v_{b_n}\}$ is a weak solution of problem ($\mathcal{QKP}$) with $b = b_n$, one obtains

$$\begin{aligned}
\int (|\nabla v_{b_n}|^{p-2}\nabla v_{b_n}\cdot\nabla\varphi + a(x)|v_{b_n}|^{p-2}\varphi)\,\mathrm{d}x + b_n|\nabla v_{b_n}|_p^p\int |v_{b_n}|^{p-2}\nabla v_{b_n}\cdot\nabla\varphi\,\mathrm{d}x \\
= \lambda\int f(v_{b_n})\varphi dx + \int |v_{b_n}|^{p^*-2}v_{b_n}\varphi\,\mathrm{d}x,
\end{aligned} \tag{3.4}$$

for all $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^N)$. By combining equations (3.3) and (3.4) with Step 1, we can conclude that

$$\begin{aligned}
\int (|\nabla v_0|^{p-2}\nabla v_0\cdot\nabla\varphi + a(x)|v_0|^{p-2}\varphi)\,\mathrm{d}x + b_n|\nabla v_0|_p^p\int_{\mathbb{R}^N} |v_0|^{p-2}\nabla v_0\cdot\nabla\varphi\,\mathrm{d}x \\
= \lambda\int f(v_0)\varphi\,\mathrm{d}x + \int |v_0|^{p^*-2}v_0\varphi\,\mathrm{d}x,
\end{aligned}$$

for all $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^N)$. Thus, v_0 is a weak solution to problem ($\mathcal{QKP}_0$). In a manner akin to Lemma 2.2, we can infer that $v_0^+ \neq 0$ and $v_0^- \neq 0$.

Step 3: Here, we demonstrate that problem ($\mathcal{QKP}_0$) has a least energy nodal solution u_0, and that there is a unique pair $(\alpha_{b_n}, \beta_{b_n}) \in \mathbb{R}^+ \times \mathbb{R}^+$ satisfying $\alpha_{b_n} u_0^+ + \beta_{b_n} u_0^- \in \mathcal{M}_{b_n}^\lambda$. Furthermore, we show that as $n \to \infty$, $(\alpha_{b_n}, \beta_{b_n})$ converges to $(1,1)$.

Using an approach similar to the one in the proof of Theorem 1.3, it follows that problem ($\mathcal{QKP}_0$) possesses a nodal solution u_0 with least energy, where $\mathcal{K}_0^\lambda(u_0) = c_0^\lambda$ and $(\mathcal{K}_0^\lambda)'(u_0) = 0$. Then, following Lemma 2.1, we can easily obtain the existence and uniqueness of a pair $(\alpha_{b_n}, \beta_{b_n})$ such that $\alpha_{b_n} u_0^+ + \beta_{b_n} u_0^- \in \mathcal{M}_{b_n}^\lambda$, with $\alpha_{b_n} > 0$ and $\beta_{b_n} > 0$. In order to complete the proof, it is enough to prove that $(\alpha_{b_n}, \beta_{b_n}) \to (1,1)$ as $n \to \infty$. Indeed, as $\alpha_{b_n} u_0^+ + \beta_{b_n} u_0^- \in \mathcal{M}_{b_n}^\lambda$, one has that

$$\begin{aligned} \alpha_{b_n}^p \|u_0^+\|^p + b_n \alpha_{b_n}^{2p} |\nabla u_0^+|_p^{2p} + b_n \alpha_{b_n}^p \beta_{b_n} |\nabla u_0^+|_p^p |\nabla u_0^-|_p^p \\ = \lambda \int f(\alpha_{b_n} u_0^+) \alpha_{b_n} u_0^+ \, \mathrm{d}x + \alpha_{b_n}^{p^*} |u_0^+|_{p^*}^{p^*}, \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} \beta_{b_n}^p \|u_0^-\|^p + b_n \beta_{b_n}^{2p} |\nabla u_0^-|_p^{2p} + b_n \beta_{b_n}^p \alpha_{b_n} |\nabla u_0^-|_p^p |\nabla u_0^+|_p^p \\ = \lambda \int f(\alpha_{b_n} u_0^-) \beta_{b_n} u_0^- \, \mathrm{d}x + \beta_{b_n}^{p^*} |u_0^-|_{p^*}^{p^*}. \end{aligned} \tag{3.6}$$

using the fact that $b_n \searrow 0$, it follows that $\{\alpha_{b_n}\}$ and $\{\beta_{b_n}\}$ are bounded. Suppose, up to a subsequence, $\alpha_{b_n} \to \alpha_0$ and $\beta_{b_n} \to \beta_0$. Then by (3.5) and (3.6), we have

$$\alpha_0^p \|u_0^+\|^p = \lambda \int f(\alpha_0 u_0^+) \alpha_0 u_0^+ \, \mathrm{d}x + \alpha_0^{p^*} |u_0^+|_{p^*}^{p^*}, \tag{3.7}$$

and

$$\beta_0^p \|u_0^-\|^p = \lambda \int f(\beta_0 u_0^-) \beta_0 u_0^- \, \mathrm{d}x + \beta_0^{p^*} |u_0^-|_{p^*}^{p^*}. \tag{3.8}$$

Noticing that v_0 is a nodal solution to problem ($\mathcal{QKP}_0$), one has

$$\|u_0^+\|^p = \lambda \int f(u_0^+) u_0^+ \, \mathrm{d}x + |u_0^+|_{p^*}^{p^*}, \tag{3.9}$$

and

$$\|u_0^+\|^p = \lambda \int f\left(u_0^-\right) u_0^- \, \mathrm{d}x + |u_0^-|_{p^*}^{p^*}. \tag{3.10}$$

Therefore, from (3.7)–(3.10), we can easily obtain that $(\alpha_0, \beta_0) = (1,1)$. We can now complete the proof of Theorem 1.5. We claim that v_0 obtained in Step 2 is a least energy solution to problem ($\mathcal{QKP}_0$). In fact, according to Step 3 and Lemma 2.1, we see that

$$\begin{aligned} \mathcal{K}_0^\lambda(u_0) \leqslant \mathcal{K}_0^\lambda(v_0) &= \lim_{n\to\infty} \mathcal{K}_{b_n}^\lambda(v_{b_n}) \\ &\leqslant \lim_{n\to\infty} \mathcal{K}_{b_n}^\lambda(\alpha_{b_n} u_0^+ + \beta_{b_n} u_0^-) \\ &= \lim_{n\to\infty} \mathcal{K}_0^\lambda(u_0^+ + u_0^-) \\ &= \mathcal{K}_0^\lambda\left(u_0\right), \end{aligned}$$

which completes the proof of Theorem 1.5. □

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Statements and Declarations

Conflict of interest

The authors declare that they have no conflict of interest.

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Existence of solutions for singular quasilinear elliptic problems with dependence of the gradient

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Abstract. In this paper we establish existence of solutions to the following boundary value problem involving a p-gradient term

$$-\Delta_p u + g(u)|\nabla u|^p = \lambda u^\sigma + \Psi(x), \quad u > 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega,$$

where $\Delta_p := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is p-Laplacian operator, $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a bounded domain with smooth boundary, $1 < p < N$, $0 < \sigma < p^* - 1$ with $p^* := pN/(N-p)$, Ψ is a measurable function and $g(s) \geq 0$ is a continuous function on the interval $(0, +\infty)$ which may have a singularity at the origin, i.e. $g(s) \to +\infty$ as $s \to 0$. Using the topological degree theory, under certain assumptions on Ψ, we prove the existence of a continuum of positive solutions.

Keywords: p-gradient term, singular equations, elliptic equations.

2020 Mathematics Subject Classification: 35J92, 35J75.

1 Introduction

In this paper we establish existence of a continuum of positive solutions to the following class of singular quasilinear elliptic equations with a p-gradient term,

$$\begin{cases} -\Delta_p u + g(u)|\nabla u|^p = \lambda u^\sigma + \Psi(x) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \qquad (P)_{\lambda\sigma}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a bounded domain with smooth boundary, $1 < p < N$, $0 < \sigma < p^* - 1$ with $p^* := pN/(N-p)$, $g : (0, \infty) \to \mathbb{R}^+$ is a continuous measurable function in a neighborhood of zero and $\Psi : \Omega \to \mathbb{R}^+$ is a L^q integrable function with $q \in \left[\frac{pN}{N(p-1)+p}, \frac{pN}{N-p}\right)$.

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This type of equations involving singular nonlinearities appears in the models of several physical phenomena, such as in theory of electric conductivity [19], in study of pseudoplastic fluids [14], in minimal surfaces with isolated singularities [13] and several other models.

The classic references involving the problem $(P)_{\lambda\sigma}$ were published by Leray and Lions [29] in 1965 and by Ladyzenskaya and Uraltseva [26] in 1968.

For this class of equations involving the term gradient we quote [1–5,7–9,22,31] and without the gradient term we quote [16–18,20,21,23,24,27,30,36]. For existence results involving quasilinear and parabolic elliptic problems with quadratic gradient term we quote [12] and for existence of a continuum of solutions for a quasilinear singular elliptic problem we quote [15].

In 2009, Arcoya, Barile and Martínez-Aparicio [3] studied the quasilinear elliptic boundary value problem

$$\begin{cases} -\Delta u + g(x,u)|\nabla u|^2 = a(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a bounded domain with smooth boundary and $g(x,s)$ is a Carathéodory function on $\Omega \times (0,\infty)$ which may have a singularity at $s = 0$ and may change of sign. Assuming that $a \in L^q(\Omega)$, with $q > N/2$, satisfies the following inequality

$$\inf\{a(x) \mid x \in \Omega_0\} > 0; \qquad \forall \Omega_0 \subset\subset \Omega$$

they proved that if there exist a increasing function $b : (0,+\infty) \to (0,+\infty)$ and a parameter $\mu \in (0,1)$ such that

$$-\mu \leq sg(x,s) \leq b(s); \qquad \forall s > 0; \ \ a.e. \ \ x \in \Omega,$$

then the previous problem has at least one positive solution.

In 2015, Y. Wang and M. Wang [35] extended the result obtained by Arcoya, Barile and Martínez-Aparicio [3] to the case involving the p-Laplacian operator.

Arcoya, Carmona and Martínez-Aparicio [5] studied the boundary value problem with a power type nonlinearity

$$\begin{cases} -\Delta u + g(u)|\nabla u|^2 = \lambda u^p + f_0(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is an open and bounded domain, $\lambda \geq 0$, $0 \leq p < \frac{N+2}{N-2}$, $0 \lneqq f_0 \in L^{\frac{2N}{N+2}}(\Omega)$ and $g \geq 0$ is continuous in $[0,+\infty)$ or $g \geq 0$ is continuous in $(0,+\infty)$, decreasing and integrable in a neighborhood of zero with $\lim_{s\to 0} g(s) = +\infty$. Using the Leray–Schauder degree, the authors showed the existence of "continua of solutions" of (1.2), i.e., connected and closed subsets in the solution set

$$Q := \left\{(\lambda,u) \in \mathbb{R} \times H_0^1(\Omega) : u = K(\lambda,u)\right\}$$

where $K : \mathbb{R} \times H_0^1(\Omega) \to H_0^1(\Omega)$ is an operator such that, for every $\lambda \in \mathbb{R}$ and for every $w \in H_0^1(\Omega)$, $K(\lambda,w)$ is the unique solution $u \in H_0^1(\Omega)$ of an auxiliary problem.

In this paper, we generalize the equation studied by Arcoya, Carmona and Martínez-Aparicio [5] for the p-Laplacian operator $\Delta_p := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, with a non negative continuous function g which may have a singularity at the origin and a measurable function Ψ. To

state our results, we say that $u \in W_0^{1,p}(\Omega)$ is a positive solution for $(P)_{\lambda\sigma}$ if $u > 0$ a.e $x \in \Omega$, $g(u)\,|\nabla u|^p \in L^1(\Omega)$ and

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla \varphi dx + \int_\Omega g(u)|\nabla u|^p \varphi dx = \lambda \int_\Omega u^\sigma \varphi dx + \int_\Omega \Psi \varphi dx \tag{1.3}$$

for all $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

Our main results read as follows.

Theorem 1.1. *Let $g : (0, +\infty) \to \mathbb{R}^+$ be a continuous and integrable function in a neighborhood of zero such that $\lim_{s\to 0} g(s) = +\infty$. If $\Psi \in L^q(\Omega)$, be a function not identically zero with $q \in \left[\frac{pN}{N(p-1)+p}, \frac{pN}{N-p}\right)$, then problem $(P)_{\lambda\sigma}$ has a unique solution $u \in W_0^{1,p}(\Omega)$ for $\lambda = 0$.*

For $q = \frac{pN}{N(p-1)+p}$, Y. Wang and M. Wang [35, Theorem 3.1] proved the existence of a solution to Theorem 1.1. Before we establish the next result we need some definitions. In this way, we consider the auxiliary problem

$$\begin{cases} -\Delta_p u + g(u)|\nabla u|^p = \lambda^+ w^+(x)^\sigma + \Psi(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where $\lambda \geq 0$, $w \in W_0^{1,p}(\Omega)$ and $w^+ := \max\{0, w\}$.

By Theorem 1.1, for every $(\lambda, w) \in [0, +\infty) \times W_0^{1,p}(\Omega)$, the problem (1.4) has a unique solution $u = T(\lambda, w) \in W_0^{1,p}(\Omega)$. Thus, by following ideas of Arcoya, Carmona and Martínez-Aparicio [5], we define an operator $K : \times W_0^{1,p}(\Omega) \to W_0^{1,p}(\Omega)$ such that

$$K(\lambda, w) := \begin{cases} T(\lambda, w), & \text{if } \lambda \geq 0; \\ T(0, w), & \text{if } \lambda < 0, \end{cases}$$

and a set

$$S := \left\{(\lambda, u) \in \mathbb{R} \times W_0^{1,p}(\Omega) : u = K(\lambda, u)\right\}.$$

Notice that the function $\lambda w^+(x)^\sigma + \Psi(x)$ is in $L^q(\Omega)$, for $q = \frac{Np}{N(p-1)+p}$. Thus $K(\lambda, w)$ is well defined. Indeed,

$$\int_\Omega |\lambda w^+(x)^\sigma + \Psi(x)|^q dx \leq 2^{q-1} \int_\Omega |\lambda w^+(x)^\sigma|^q dx + 2^{q-1} \int_\Omega |\Psi(x)|^q dx,$$

and $\sigma q = \frac{N(p-1)+p}{N-p} \frac{pN}{N(p-1)+p} = \frac{pN}{N-p} = p^*$.

Therefore, with this notation, $(P)_{\lambda\sigma}$ can be rewritten as a fixed point problem, namely,

$$u = K(\lambda, u). \tag{1.5}$$

The next result is related to the case $\Psi \gneqq 0$ and it states the existence of global continua in solution set S emanating from the unique solution of Theorem 1.1.

Theorem 1.2. *Consider $0 \lneqq \Psi \in L^{\frac{pN}{N(p-1)+p}}(\Omega)$ and assume that $g \geq 0$ is continuous in $[0, +\infty)$ or $g \geq 0$ is continuous in $(0, +\infty)$ and integrable in an neighborhood of zero with $\lim_{s\to 0} g(s) = +\infty$. Then there exists an unbounded continuum $\Sigma \subset S$ of positive solutions which contains $(0, u_0)$, where u_0 is the unique solution of $(P)_{\lambda\sigma}$ for $\lambda = 0$.*

To prove Theorem 1.1 we follow some general ideas of [3,5], i.e., we construct a infinite sequence of auxiliary problems $(P)_n$ with $n \in \mathbb{N}$, such that $(P)_{\lambda\sigma}$ for $\lambda = 0$ has at least one solution as $n \to +\infty$. To prove the uniqueness of the solution we follows some general ideas of [6]. Finally, to prove Theorem 1.2 we show that K is a compact operator and using the Leray–Schauder degree theory we prove the existence of "continua of solutions" of $(P)_{\lambda\sigma}$, i.e., connected and closed subsets in the solution set S.

The main difficulties found in the proof of these results are the existence of a singularity g, the presence of the term gradient and the non-linearity of the operator in the case where $p \in (1, N)$. The case where $p = 2$ has been extensively studied by several researches. For example, in the case where g is continuous at zero the existence is due to [11] and the uniqueness to [4]. Moreover, in the case where g is singular at zero the existence is due to [10] and the uniqueness to [4].

This paper is organized as follows: in section 2 we introduce an approximated problem, whose solutions are also solutions to problem (1.4); and we prove some auxiliary lemmas. In section 3, we prove the integrability of $g(u)\,|\nabla u|^p\,\varphi$ and the compactness for the operator K defined in (1.5). After, we prove the Theorem 1.2 using the topological Leray–Schauder degree.

2 Proof of Theorem 1.1

To prove Theorem 1.1 first we give some preliminary considerations and lemmas. In this way, motivated by [2,3], we define

$$g_n(s) = \begin{cases} 0, & \text{if } s \leq 0; \\ n^p s^p T_n(g(s)), & \text{if } 0 < s < \frac{1}{n}; \\ T_n(g(s)), & \text{if } \frac{1}{n} \leq s; \end{cases} \tag{2.1}$$

where $T_n(s)$ is the truncate function given by

$$T_n(s) = \begin{cases} s, & \text{if } |s| < n; \\ -n, & \text{if } s \leq -n; \\ n, & \text{if } s \geq n. \end{cases} \tag{2.2}$$

It is easy to verify that g_n satisfies the following properties

(a) $|g_n(s)| \leq \min\{n, g(s)\}$;

(b) $g_n(s) \leq n^p s^{p-1}$, for all $s > 0$;

(c) $\lim_{n\to\infty} g_n(s) = g(s)$, for all $s > 0$.

Consider the following approximated problem

$$\begin{cases} -\Delta_p w + \frac{g_n(w)|\nabla w|^p}{1+\frac{1}{n}|\nabla w|^p} = \Psi_n(x) & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases} \tag{P}_n$$

where $\Psi_n := T_n(\Psi)$.

Lemma 2.1. *There exists at least one solution $w_n \in W_0^{1,p}(\Omega) \cap C^1(\overline{\Omega})$ of the approximated problem $(P)_n$.*

Proof. Notice that, since the operator Δ_p^{-1} is an homeomorphism, using the Browder–Minty Theorem we obtain that

$$-\Delta_p : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega) \quad \text{is a homeomorphism.} \tag{2.4}$$

Furthermore, for every $w \in C^1(\overline{\Omega})$ we define

$$F_n(w) = \Psi_n - \frac{g_n(w)|\nabla w|^p}{1 + \frac{1}{n}|\nabla w|^p},$$

and for every $u \in C^1(\overline{\Omega})$ we define the problem

$$\begin{cases} -\Delta_p w = F_n(u) & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.5}$$

Let $G : W_0^{1,p}(\Omega) \to \mathbb{R}$ be the functional defined by

$$G(\varphi) = \int_\Omega F_n(u)\varphi dx.$$

Since $F_n(u)$ is bounded independent of u, then G is well defined, linear and $|G(\varphi)| \leq C_n\|\varphi\|_{1,p}$, for some $C_n > 0$, i.e., $G \in W^{-1,p'}(\Omega)$. Hence, by (2.4) there exists a unique function $w \in W_0^{1,p}(\Omega)$ such that $-\Delta_p w = G$, i.e.,

$$\int_\Omega |\nabla w|^{p-2}\nabla w \nabla \varphi dx = \int_\Omega F_n(u)\varphi dx.$$

Thus, since all the assumptions of the regularity result obtained by Hai (see [24, Lemma 3.1]) are satisfied, there are constants $\alpha \in (0,1)$ and $M > 0$ such that $w \in C^{1,\alpha}(\overline{\Omega})$ and $\|w\|_{C^{1,\alpha}(\overline{\Omega})} < M$.

Let $K : C^1(\overline{\Omega}) \to C^1(\overline{\Omega})$ be the operator defined by $K(u) = w$, where w is the unique solution of (2.5). Since $K(C^1(\overline{\Omega})) \subset C^{1,\alpha}(\overline{\Omega})$ and $\|K(u)\|_{1,\alpha} \leq M$ for every $u \in C^1(\overline{\Omega})$, then K is compact. Furthermore, K is continuous. Indeed, let $\{u_k\} \subset C^1(\overline{\Omega})$ be a sequence such that $u_k \to u$ in $C^1(\overline{\Omega})$. Define now

$$w_k = K(u_k) \quad \text{and} \quad w = K(u).$$

By definition of K, we have

$$-\Delta_p w_k - (-\Delta_p w) = F_n(u_k) - F_n(u) \quad \text{in } \Omega,$$

and consequently

$$\begin{aligned} \int_\Omega \left(|\nabla u_k|^{p-2}\nabla u_k - |\nabla u|^{p-2}\nabla u\right).\nabla(w_k - w)dx &= \int_\Omega \big(F_n(u_k(x)) - F_n(u(x))\big)(w_k - w)dx \\ &\leq 2M \int_\Omega |F_n(u_k) - F_n(u)|dx. \end{aligned} \tag{2.6}$$

Since $F_n(u_k)$ is bounded and $F_n(u_k(x)) \to F_n(u(x))$ a.e. in Ω, then by the Dominated Convergence Theorem we have

$$\int_\Omega |F_n(u_k) - F_n(u)|dx \to 0,$$

as $k \to \infty$.

Thus, applying in (2.6) the inequality (see [33])

$$\langle |x|^{p-2}x - |y|^{p-2}y, x-y\rangle_e \geq C \begin{cases} \dfrac{|x-y|^2}{(|x|+|y|)^{2-p}} & \text{if } 1 < p < 2; \\ |x-y|^p & \text{if } p \geq 2, \end{cases} \tag{2.7}$$

where $x, y \in^N$ and $C := C(p)$ is a constant, we obtain

$$w_k \to w \quad \text{in } W_0^{1,p}(\Omega).$$

On the other hand, since $\|w_k\|_{C^1(\overline{\Omega})} < M$, by going to subsequence if necessary, there exists $w_0 \in C^1(\overline{\Omega})$ such that

$$w_k \to w_0 \quad \text{in } C^1(\overline{\Omega}).$$

Hence, by uniqueness of limits, we conclude $K(u_k) \to K(u)$ in $C^1(\overline{\Omega})$; i.e., K is continuous.

How K is a continuous and compact operator and $K(C^1(\overline{\Omega})) \subset B_{\tilde{M}}$, where $B_{\tilde{M}}$ is a ball centered at the origin with radius $\tilde{M}$ in $C^1(\overline{\Omega})$, then by Schauder's Fixed Point Theorem, there exists $u \in B_{\tilde{M}}$, such that $K(u) = u$. Therefore, by definition of K, we have that u is solution of $(P)_n$. □

Lemma 2.2. *If $\Psi \in L^q(\Omega)$ with $q \in \left[\frac{pN}{N(p-1)+p}, \frac{pN}{N-p}\right)$, then the solution w_n of $(P)_n$ satisfies the following statements,*

(i) *$\{w_n\}$ is bounded independent of n in $W_0^{1,p}(\Omega)$;*

(ii) *$w_n(x) > 0$, for all $x \in \Omega$.*

Proof. (i) First, we will prove that $w_n \geq 0$ in Ω. Indeed, multiplying $(P)_n$ by w_n^- and integrating in Ω, we obtain

$$\int_\Omega |\nabla w_n|^{p-2}\nabla w_n.\nabla w_n^- dx + \int_\Omega \frac{g_n(w_n)|\nabla w_n|^p}{1+\frac{1}{n}|\nabla w_n|^p} w_n^- dx = \int_\Omega \Psi_n w_n^- dx. \tag{2.8}$$

Since $w_n = w_n^+ - w_n^-$ and $\nabla w_n = \nabla w_n^+ - \nabla w_n^-$, then

$$\begin{aligned} \nabla w_n \nabla w_n^- &= (\nabla w_n^+ - \nabla w_n^-)\nabla w_n^- \\ &= \nabla w_n \nabla w_n^- - (\nabla w_n^-)^2 \\ &\quad - (\nabla w_n^-)^2. \end{aligned} \tag{2.9}$$

Furthermore, we have

$$g_n(w_n)w_n^- = 0, \quad \text{for all } w_n \in W_0^{1,p}(\Omega). \tag{2.10}$$

Then, by relations (2.8), (2.9) and (2.10), we have

$$-\int_\Omega |\nabla w_n^-|^p dx = \int_\Omega \Psi_n w_n^- dx \geq 0,$$

i.e., $w_n^- \equiv 0$. Therefore, $w_n = w_n^+ \geq 0$ in Ω.

Taking w_n as test function in $(P)_n$, we obtain

$$\begin{aligned} \int_\Omega |\nabla w_n|^p dx + \int_\Omega \frac{g_n(w_n)|\nabla w_n|^p}{1+\frac{1}{n}|\nabla w_n|^p} w_n dx &= \int_\Omega \Psi_n w_n dx \\ &\leq \int_\Omega \Psi w_n dx. \end{aligned}$$

Since $w_n \geq 0$, we have that $\frac{g_n(w_n)|\nabla w_n|^p}{1+\frac{1}{n}|\nabla w_n|} w_n \geq 0$. Thus, using Hölder's inequality, we obtain

$$\begin{aligned} \int_\Omega |\nabla w_n|^p dx &\leq \int_\Omega \Psi_n w_n dx \\ &\leq \|\Psi\|_q \|w_n\|_{q'}, \end{aligned} \tag{2.11}$$

where $q' := \frac{q}{q-1}$. Since $q \geq \frac{pN}{N(p-1)+p}$, then $q' \leq \frac{pN}{N-p}$ and $L^{p*}(\Omega) \subset L^{q'}(\Omega)$. Hence, there are constants $c_1, c_2 > 0$ such that

$$\|w_n\|_{q'} \leq c_1 \|w_n\|_{p*} \leq c_2 \|w_n\|_{1,p}.$$

By applying the previous inequality in the relation (2.11), we conclude that $\{w_n\}$ is bounded in $W_0^{1,p}(\Omega)$.

(ii) Since $w_n \geq 0$ in Ω, by item (b) of properties of g_n listed previously, we obtain

$$\frac{g_n(w_n)|\nabla w_n|^p}{1+\frac{1}{n}|\nabla w_n|^p} \leq n^p |w_n|^{p-1} n = n^{p+1} |w_n|^{p-1}. \tag{2.12}$$

Let the function $\beta : [0, +\infty) \to \mathbb{R}$ be defined by

$$\beta(s) = n^{p+1}(s)^{p-1}.$$

Notice that β is continuous, non-decreasing, $\beta(0) = 0$ and $\beta(s) > 0$, for all $s > 0$. Furthermore,

$$\int_0^1 (\beta(s)s)^{-\frac{1}{p}} ds = \infty.$$

Hence, by relation (2.12) we have

$$-\Delta_p w_n + \beta(w_n) \geq \Psi_n \geq 0 \quad \text{a.e. in } \Omega,$$

i.e., $\Delta_p w_n \leq \beta(w_n)$ a.e. in Ω.

Since $\Psi_n \not\equiv 0$, then by the Strong Principle of Maximum (see [34, Theorem 5]) we conclude that $w_n > 0$ in Ω. □

Lemma 2.3. *There are $w \in W_0^{1,p}(\Omega)$ and $q \in (1, p)$ such that*

(i) $\nabla w_n \to \nabla w$ *in* $L^q(\Omega)$;

(ii) $w > 0$ *in* Ω.

Proof. (i) Since the sequence $\{w_n\}$ is bounded in $W_0^{1,p}(\Omega)$, going if necessary to a subsequence, there exists $w \in W_0^{1,p}(\Omega)$ such that

(a) $w_n \rightharpoonup w$ in $W_0^{1,p}(\Omega)$;

(b) $w_n \to w$ in $L^p(\Omega)$;

(c) $w_n(x) \to w(x)$ a.e. in Ω.

By fixing compact set $K \subset \Omega$, we take $\phi_K \in C_0^\infty(\Omega)$ with $0 \le \phi_K \le 1$ and $\phi_K = 1$ in K. Thus, taking v_n as test function in $(P)_n$ defined by

$$v_n := \phi_K [T_\eta(w_n - w)]^+ \in W_0^{1,p}(\Omega),$$

we have

$$\int_\Omega |\nabla w_n|^{p-2}\nabla w_n \nabla v_n dx + \int_\Omega \frac{g_n(w_n)|\nabla w_n|^p}{1+\frac{1}{n}|\nabla w_n|^p} v_n dx = \int_\Omega \Psi_n v_n dx. \tag{2.13}$$

By applying in (2.13) the relations

$$\nabla v_n = \nabla \phi_K [T_\eta(w_n - w)]^+ + \phi_K \nabla [T_\eta(w_n - w)]^+$$

and

$$\frac{g_n(w_n)|\nabla w_n|^p}{1+\frac{1}{n}|\nabla w_n|^p} v_n \ge 0,$$

we obtain

$$\begin{aligned}\int_\Omega |\nabla w_n|^{p-2}\nabla w_n \phi_K \nabla [T_\eta(w_n - w)]^+ dx \\ \le \int_\Omega \Psi_n v_n dx - \int_\Omega |\nabla w_n|^{p-2}\nabla w_n \nabla \phi_K [T_\eta(w_n - w)]^+ dx.\end{aligned}$$

Hence, we have

$$\begin{aligned}\int_\Omega \phi_K \big[|\nabla w_n|^{p-2}\nabla w_n - |\nabla w|^{p-2}\nabla w\big] \nabla [T_\eta(w_n - w)]^+ dx \\ \le \int_\Omega \Psi_n v_n dx - \int_\Omega |\nabla w_n|^{p-2}\nabla w_n \nabla \phi_K [T_\eta(w_n - w)]^+ dx \\ - \int_\Omega \phi_K |\nabla w|^{p-2}\nabla w \nabla [T_\eta(w_n - w)]^+ dx.\end{aligned} \tag{2.14}$$

Since $w_n \rightharpoonup w$ in $W_0^{1,p}(\Omega)$, then we have $[T_\eta(w_n - w)]^+ \rightharpoonup 0$, i.e.,

$$\langle \phi, [T_\eta(w_n - w)]^+ \rangle \to 0 \quad \text{for all } \ \phi \in W_0^{1,p}(\Omega)^*, \tag{2.15}$$

where $W_0^{1,p}(\Omega)^*$ is the dual space of $W_0^{1,p}(\Omega)$.

Note that, by the Dominated Convergence Theorem and by relation (2.15) we obtain

$$\int_\Omega |\nabla w_n|^{p-2}\nabla w_n \nabla \phi_K [T_\eta(w_n - w)]^+ dx \to 0 \quad \text{as } n \to \infty \tag{2.16}$$

and

$$\int_\Omega \phi_K |\nabla w|^{p-2}\nabla w \nabla [T_\eta(w_n - w)]^+ dx \to 0 \quad \text{as } n \to \infty. \tag{2.17}$$

Indeed, by Hölder's inequality and by (2.15) we have

$$\left| \int_\Omega |\nabla w_n|^{p-2}\nabla w_n \nabla \phi_K [T_\eta(w_n - w)]^+ dx \right| \le \|w_n\|_{1,p}^{p-1} \left(\int_\Omega |\nabla \phi_K [T_\eta(w_n - w)]^+|^p dx \right)^{\frac{1}{p}},$$

where $\|w_n\|_{1,p}$ and $\nabla \phi_K [T_\eta(w_n - w)]^+$ are bounded and $[T_\eta(w_n - w)]^+ \to 0$ a.e. in Ω. Thus, by the Dominated Convergence Theorem the relation (2.16) holds true.

Again, by Hölder's inequality we have

$$\begin{aligned}
&\left| \int_\Omega \phi_K |\nabla w|^{p-2}\nabla w \nabla [T_\eta(w_n - w)]^+ dx \right| \\
&\qquad \leq \|w_n\|_{1,p}^{p-1} \left[\int_\Omega |\phi_k [\nabla T_\eta(w_n - w)]^+|^p dx \right]^{\frac{1}{p}} \\
&\qquad = \|w_n\|_{1,p}^{p-1} \left[\int_\Omega |\phi_k|^p . \chi_{\{x\in\Omega : |w_n - w| < \eta\}} |\nabla (w_n - w)]^+|^p dx \right]^{\frac{1}{p}}.
\end{aligned}$$

Since $|\phi_k|^p . \chi_{\{x\in\Omega:|w_n-w|<\eta\}}$ is bounded in $L^p(\Omega)^*$ and $|\phi_k|^p . \chi_{\{x\in\Omega:|w_n-w|<\eta\}} \to |\phi_k|^p$ a.e. in Ω, then by Vitali's Convergence Theorem we have

$$|\phi_K|^p . \chi_{\{x\in\Omega:|w_n-w|<\eta\}} \to |\phi_K|^p \quad \text{in } L^p(\Omega)^*.$$

Hence, since $\left|\nabla (w_n - w)^+\right|^p \rightharpoonup 0$ in $L^p(\Omega)$, then the relation (2.17) holds true.

For fixed η, combining the relations (2.14), (2.16) and (2.17), we have

$$\limsup_{n\to\infty} \int_K \left(|\nabla w_n|^{p-2}\nabla w_n - |\nabla w|^{p-2}\nabla w \right) \nabla [T_\eta(w_n - w)]^+ dx \leq C_\Psi \eta. \tag{2.18}$$

Let H_n^+ defined by

$$H_n^+(x) := \left[|\nabla w_n|^{p-2}\nabla w_n - |\nabla w|^{p-2}\nabla w \right] \nabla [T_\eta(w_n - w)]^+(x).$$

By relation (2.18) we have that H_n^+ is bounded in $L^1(K)$. Moreover, by inequality (2.7) and by definition of T_η, we have that $H_n^+ \geq 0$.

By defining the sets

$$A_n^\eta := \{x \in K : |w_n(x) - w(x)| \leq \eta\} \quad \text{and} \quad B_n^\eta := \{x \in K : |w_n(x) - w(x)| > \eta\};$$

and fixing $\nu \in (0,1)$, we obtain

$$\int_K (H_n{}^+)^\nu dx \leq \left(\int_{A_n^\eta} (H_n^+) dx \right)^\nu |A_n^\eta|^{1-\nu} + \left(\int_{B_n^\eta} (H_n^+) dx \right)^\nu |B_n^\eta|^{1-\nu}.$$

For fixed η, we have that $|B_n^\eta| \to 0$ as $n \to \infty$. Moreover, since H_n^+ is bounded in $L^1(K)$, we have

$$\limsup_{n\to\infty} \int_K (H_n^+)^\nu dx \leq (C_\Psi \eta)^\nu |\Omega|^{1-\nu}. \tag{2.19}$$

By letting $\eta \to 0$ in the previous inequality, we obtain

$$(H_n^+)^\nu \to 0 \quad \text{in } L^1(K).$$

Now, choose v_n as test function in $(P)_n$ defined by

$$v_n := \phi_K [T_\eta(w_n - w)]^- \in W_0^{1,p}(\Omega),$$

where $s^- := \max\{-s, 0\}$. Hence, repeating the arguments previously used, we can conclude that

$$(H_n^-)^\nu \to 0 \quad \text{in } L^1(K),$$

where

$$H_n^-(x) := \Big[|\nabla w_n|^{p-2}\nabla w_n - |\nabla w|^{p-2}\nabla w\Big]\nabla[(w_n - w)]^-(x).$$

Therefore, if $H_n := H_n^+ - H_n^-$, then

$$H_n(x) = \Big[|\nabla w_n|^{p-2}\nabla w_n - |\nabla w|^{p-2}\nabla w\Big]\nabla[(w_n - w)](x)$$

and $H_n \to 0$ a.e. in K.

Consider $\Omega = \bigcup_{j=1}^{\infty}\Omega_j$, such that $\Omega_j \subset\subset \Omega_{j+1} \subset\subset \Omega$.

Thus, for $K = \overline{\Omega}_1$, we have

$$H_1^1(x), H_2^1(x), H_3^1(x), \ldots, H_n^1(x) \to 0 \quad \text{a.e. in } \overline{\Omega}_1.$$

Analogously, for $K = \overline{\Omega}_2$, we have

$$H_1^2(x), H_2^2(x), H_3^2(x), \ldots, H_n^2(x) \to 0 \quad \text{a.e. in } \overline{\Omega}_2.$$

Repeating the previous process, we obtain

$$\begin{array}{ccccccccc}
H_1^1(x) & H_2^1(x) & H_3^1(x) & \ldots & H_n^1(x) & \longrightarrow 0 & \text{a.e. in} & \overline{\Omega}_1 \\
H_1^2(x) & H_2^2(x) & H_3^2(x) & \ldots & H_n^2(x) & \longrightarrow 0 & \text{a.e. in} & \overline{\Omega}_2 \\
H_1^3(x) & H_3^3(x) & H_3^3(x) & \ldots & H_n^3(x) & \longrightarrow 0 & \text{a.e. in} & \overline{\Omega}_3 \\
\vdots & \vdots & \vdots & \ddots & \vdots & & & \vdots \\
H_1^j(x) & H_2^j(x) & H_3^j(x) & \ldots & H_n^j(x) & \longrightarrow 0 & \text{a.e. in} & \overline{\Omega}_j \\
\vdots & \vdots & \vdots & & \vdots & & & \vdots
\end{array}$$

Hence, taking the diagonal sequence $\widehat{H}_j = H_j^j$, we have

$$\widehat{H}_j(x) \to 0 \quad \text{a.e. in } \Omega.$$

So, for the sequence of compact sets Ω_j, there exists a subsequence $\{H_{n'}\}$ such that

$$H_{n'}(x) \to 0 \quad \text{a.e. in } \Omega.$$

By applying again the inequality (2.7), we obtain

$$\nabla w_{n'}(x) \to \nabla w(x) \quad \text{a.e. in } \Omega.$$

Thus, since $\{\nabla w_n\}$ is bounded independent of n, by Vitali's Convergence Theorem we have

$$\nabla w_n \to \nabla w \quad \text{in } L^q(\Omega), \quad q < p.$$

(ii) Now, we will prove that w is strictly positive in Ω. Indeed, we have $w_n > 0$ in Ω with $w_n \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$. In analogy to the proof of (i), we have

$$\int_\Omega |\nabla w_n|^{p-2}\nabla w_n \nabla \varphi dx + \int_\Omega \frac{g_n(w_n)|\nabla w_n|^p}{1+\frac{1}{n}|\nabla w_n|^p}\varphi dx = \int_\Omega \Psi_n \varphi dx, \qquad \varphi \in W_0^{1,p}(\Omega). \tag{2.20}$$

Thus, taking v_n as test function in (2.20) defined by

$$v_n := e^{-\tilde{H}_n(w_n)}\varphi, \qquad \varphi \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega), \qquad \varphi \geq 0,$$

where $\tilde{H}_n(t) := \int_0^t g_n(s)ds$ and $\tilde{H}_n{}'(t) := g_n(t)$, with $g_n(s) \leq g(s)$, we obtain

$$\begin{aligned} &-\int_\Omega |\nabla w_n|^p \tilde{H}_n{}'(w_n)e^{-\tilde{H}_n(w_n)}\varphi dx + \int_\Omega |\nabla w_n|^{p-2}\nabla w_n e^{-\tilde{H}_n(w_n)}\nabla\varphi dx \\ &\quad = \int_\Omega \Psi_n e^{-\tilde{H}_n(w_n)}\varphi dx - \int_\Omega \frac{g_n(w_n)|\nabla w_n|^p}{1+\frac{1}{n}|\nabla w_n|^p}e^{-\tilde{H}_n(w_n)}\varphi dx. \end{aligned}$$

By applying in the previous equation the following inequality

$$\frac{|y|^p}{1+\frac{1}{n}|y|^p} \leq |y|^p \quad \text{for every } \ y\in\mathbb{R}^n,$$

we obtain

$$\begin{aligned} &\int_\Omega |\nabla w_n|^{p-2}\nabla w_n e^{-\tilde{H}_n(w_n)}\nabla\varphi dx - \int_\Omega \Psi_n e^{-\tilde{H}_n(w_n)}\varphi dx \\ &\quad = \int_\Omega |\nabla w_n|^p \tilde{H}_n{}'(w_n)e^{-\tilde{H}_n(w_n)}\varphi dx - \int_\Omega \frac{g_n(w_n)|\nabla w_n|^p}{1+\frac{1}{n}|\nabla w_n|^p}e^{-\tilde{H}_n(w_n)}\varphi dx \\ &\quad \geq \int_\Omega g_n(w_n)|\nabla w_n|^p e^{-\tilde{H}_n(w_n)}\varphi dx - \int_\Omega g_n(w_n)|\nabla w_n|^p e^{-\tilde{H}_n(w_n)}\varphi dx \\ &\quad = 0. \end{aligned}$$

Hence,

$$\int_\Omega |\nabla w_n|^{p-2}\nabla w_n \nabla\varphi e^{-\tilde{H}_n(w_n)}dx \geq \int_\Omega \Psi_n e^{-\tilde{H}_n(w_n)}\varphi dx. \tag{2.21}$$

Define $\tilde{H}(w) = \lim_{n\to\infty}\tilde{H}_n(w_n)$. Taking the limit in (2.21) as $n\to\infty$, since $w_n > 0$ and $e^{-\tilde{H}_n(w_n)} < 1$ in Ω, we obtain

$$\begin{aligned} \int_\Omega |\nabla w|^{p-2}\nabla w\nabla\varphi e^{-\tilde{H}(w)}dx &\geq \int_\Omega \Psi e^{-\tilde{H}(w)}\varphi dx \\ &\geq \int_\Omega T_1(\Psi)e^{-\tilde{H}(w)}\varphi dx. \end{aligned} \tag{2.22}$$

Define $v(x) := \psi(w(x)) = \int_0^{w(x)} \left(e^{-\tilde{H}(s)}\right)^{\frac{1}{p-1}}dt$, where $\psi(s) := \int_0^s \left(e^{-\tilde{H}(s)}\right)^{\frac{1}{p-1}}dt$ is strictly increasing. Let z be a solution of problem

$$\begin{cases} -\Delta_p z = \dfrac{T_1(\Psi)}{e^{\tilde{H}(w)}} & \text{in } \Omega, \\ z = 0 & \text{on } \partial\Omega. \end{cases}$$

Since $\frac{T_1(\Psi)}{e^{\tilde{H}(w)}} \in L^\infty(\Omega)$, by a result of Lieberman (see [28, Theorem 1]), we have that $z \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha\in(0,1)$. Moreover, by strong maximum principle (see [34]), we conclude that $z > 0$ in Ω.

By applying in (2.22) the relation $\nabla v = \nabla w\left(e^{-\tilde{H}(w)}\right)^{\frac{1}{p-1}}$, we obtain

$$\begin{aligned} \int_\Omega |\nabla v|^{p-2}\nabla v\nabla\varphi dx &= \int_\Omega |\nabla w|^{p-2}\nabla w\nabla\varphi e^{-\tilde{H}(w)}dx \\ &\geq \int_\Omega T_1(\Psi)e^{-\tilde{H}(w)}\varphi dx. \end{aligned}$$

Hence, by weak comparison principle (see [34]), we have that $v(x) \geq z(x) > 0$ in Ω. Finally, since $\psi(w(x)) = v(x) > 0$ and ψ is strictly increasing in Ω, then $w(x) > 0$ in Ω. □

The following lemmas concerning with the uniqueness of solution will be useful in the sequel and they can be deduced by using ideas of Bénilan, Boccardo, Gallouët, Gariepy, Pierre and Vazquez [6].

Lemma 2.4. *If w is a solution of $(P)_{\lambda\sigma}$ for $\lambda = 0$, then for every $a, k > 0$*

(i) $\frac{1}{k}\int_{\{|w|<k\}} |\nabla w|^p dx \leq \int_\Omega \Psi dx$;

(ii) $\frac{1}{a}\int_{\{k<|w|<k+a\}} |\nabla w|^p dx \leq \int_\Omega T_{k,a}(w)\Psi dx \leq \int_{\{|w|>k\}} \Psi dx$,

where $T_{k,a}(s) := T_a\big(s - T_k(s)\big)$.

Lemma 2.5. *Let $1 < p < N$. If Ω is a bounded domain in $\mathbb{R}^N$ and $w \in W_0^{1,p}(\Omega)$ satisfies*

$$\frac{1}{k}\int_{\{|w|<k\}} |\nabla w|^p dx \leq M \tag{2.23}$$

for every $k > 0$, then there exists $C = C(N,p)$ such that

$$meas\,\{x \in \Omega \,:\, |w| > k\} \leq C M^{\frac{N}{N-p}} k^{-p_1}, \tag{2.24}$$

where $p_1 = \frac{N(p-1)}{N-p}$.

Completing the proof of Theorem 1.1: Let u and v solutions of $(P)_{\lambda\sigma}$ for $\lambda = 0$, so

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla \varphi dx + \int_\Omega g(u)|\nabla u|^p \varphi dx = \int_\Omega \Psi \varphi dx \tag{2.25}$$

and

$$\int_\Omega |\nabla v|^{p-2}\nabla v \nabla \varphi dx + \int_\Omega g(v)|\nabla v|^p \varphi dx = \int_\Omega \Psi \varphi dx, \tag{2.26}$$

for all $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

For every $h \geq 0$, choosing $\varphi = T_k(u - T_h v)^+$ and $\varphi = T_k(v - T_h u)^+$ in (2.25) and (2.26), respectively, we obtain

$$\int_{\{|u-T_h v|<k\}} \langle |\nabla u|^{p-2}\nabla u, \nabla(u - T_h v)^+\rangle dx \leq \int_\Omega T_k(u - T_h v)^+ \Psi dx$$

and

$$\int_{\{|v-T_h u|<k\}} \langle |\nabla v|^{p-2}\nabla v, \nabla(v - T_h u)^+\rangle dx \leq \int_\Omega T_k(v - T_h u)^+ \Psi dx.$$

Thus, if we define

$$I := \int_{\{|u-T_h v|<k\}} \langle |\nabla u|^{p-2}\nabla u, \nabla(u - T_h v)^+\rangle dx + \int_{\{|v-T_h u|<k\}} \langle |\nabla v|^{p-2}\nabla v \nabla(v - T_h u)^+\rangle dx, \tag{2.27}$$

the conclusion $u = v$ will be reached after passing to the limit $h \to \infty$ in the previous relations and disregarding some positive terms. We will to split the previous integrals into the contributions corresponding to different integration sets.

Consider the following set

$$A_0 := \{x \in \Omega \,:\, |u - v| < k,\ |u| < h,\ |v| < h\}.$$

Thus, when restricted to A_0 the first member of (2.27) gives the following main contribution

$$\begin{aligned} I_0 &:= \int_{A_0} \langle |\nabla u|^{p-2}\nabla u, \nabla (u-v)^+ \rangle dx + \int_{A_0} \langle |\nabla v|^{p-2}\nabla v, \nabla (v-u)^+ \rangle dx \\ &= \int_{A_0} \langle |\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v, \nabla (u-v)^+ \rangle dx. \end{aligned}$$

The remaining first member of (2.27) is estimated taking the first term on the set

$$A_1 := \{x \in \Omega \ : \ |u - T_h v| < k, \ |v| > h\},$$

i.e.,

$$\int_{A_1} \langle |\nabla u|^{p-2}\nabla u, \nabla (u - T_h v)^+ \rangle dx = \int_{A_1} |\nabla u|^p dx \geq 0.$$

On the remaining set

$$A_2 := \{x \in \Omega \ : \ |u - T_h v| < k, \ |v| < h, \ |u| \geq h\}$$

we have

$$\begin{aligned} \int_{A_2} \langle |\nabla u|^{p-2}\nabla u, \nabla (u - T_h v)^+ \rangle dx &= \int_{A_2} \langle |\nabla u|^{p-2}\nabla u, \nabla (u - v)^+ \rangle dx \\ &\geq - \int_{A_2} |\nabla u|^{p-2}\nabla u \nabla v dx. \end{aligned}$$

Now, we estimate the second member of (2.27) in the sets A_1' where $|u| \geq h$, and A_2', where $|u| < h$ and $|v| \geq h$. Notice that all these sets and integrals depend of k and h.

Summing up we estimate the first member of (2.27) as follows

$$I \geq I_0 - I_3,$$

where

$$I_3 := \int_{A_2} |\nabla u|^{p-2}\nabla u \nabla v dx + \int_{A_2'} |\nabla v|^{p-2}\nabla v \nabla u dx.$$

Now, we will check that $I_3 \to 0$ as $h \to \infty$. Indeed, the first term of I_3 can be estimated by

$$\begin{aligned} \int_{A_2} |\nabla u|^{p-2}\nabla u \nabla v dx &\leq \left(\int_{A_2} |\nabla u|^{(p-1)\frac{p}{p-1}} dx \right)^{\frac{p-1}{p}} \left(\int_{A_2'} |\nabla v|^p dx \right)^{\frac{1}{p}} \\ &= \left(\int_{\{h \leq |u| \leq h+k\}} |\nabla u|^p dx \right)^{\frac{p-1}{p}} \left(\int_{\{h-k \leq |v| \leq h\}} |\nabla v|^p dx \right)^{\frac{1}{p}} \\ &= \|u\|^{p-1}_{L^p(\{h \leq |u| \leq h+k\})} \|v\|_{L^p(\{h \leq |u| \leq h+k\})}, \end{aligned}$$

which converges to 0 as $h \to \infty$ due to Lemmas 2.4 and 2.5. The treatment of the second term is analogous.

Now, we will estimate

$$\int_\Omega \Psi \left[T_k(u - T_h v)^+ - T_k(v - T_h u)^+ \right] dx.$$

The previous integral on the set $B_0 := \{x \in \Omega \ : \ |u| < h, \ |v| < h\}$ gives

$$J_0 := \int_{B_0} \Psi \left[T_k(u - T_h v)^+ - T_k(v - T_h u)^+ \right] dx = 0.$$

The integral on the set $B_1 := \{x \in \Omega \ : \ |u| \geq h\}$ is estimate by

$$\begin{aligned} |J_1| &:= \left| \int_{B_1} \Psi \left[T_k(u - T_h v)^+ - T_k(v - T_h u)^+ \right] dx \right| \\ &= \left| \int_{B_1} \Psi \left[T_k(u - T_h v)^+ - T_k(v - h)^+ \right] dx \right| \\ &\leq 2k \int_{B_1} |\Psi| dx, \end{aligned}$$

while on $B_2 := \{x \in \Omega \ : \ |v| \geq h\}$ we get

$$\begin{aligned} |J_2| &:= \left| \int_{B_2} \Psi \left[T_k(u - T_h v)^+ - T_k(v - T_h u)^+ \right] dx \right| \\ &= \left| \int_{B_2} \Psi \left[T_k(u - h)^+ - T_k(v - Thu)^+ \right] dx \right| \\ &\leq 2k \int_{B_2} |\Psi| dx. \end{aligned}$$

Since the measure of both sets $B_1(h,k)$ and $B_2(h,k)$ converges to zero as $h \to \infty$ for fixed $k > 0$, then $J_1 + J_2 \to 0$ as $h \to \infty$.

Combining the previous estimates, for fixed $k > 0$, we get from (2.27)

$$\int_{A_0(h,k)} \langle |\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v, \nabla (u - v)^+ \rangle dx \leq \varpi(h),$$

where $\lim_{h \to \infty} \varpi(h) = 0$.

Since the set $A_0(h,k)$ converges to $\{x \in \Omega \ : \ |u - v| < k\}$, then

$$\int_{\{x \in \Omega \ : \ |u-v| < k\}} \langle |\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v, \nabla (u - v)^+ \rangle dx \leq 0, \qquad k > 0 \text{ fixed.}$$

Since the previous inequality is true for all $k > 0$, we conclude by (2.7) that $\nabla u(x) = \nabla v(x)$ a.e. in Ω. Thus, since $u, v \in W_0^{1,p}(\Omega)$ then $u(x) = v(x)$ a.e. in Ω.

Now, we will prove that w satisfies

$$\int_\Omega |\nabla w|^{p-2} \nabla w \nabla \varphi dx + \int_\Omega g(w) |\nabla w|^p \varphi dx = \int_\Omega \Psi \varphi dx, \tag{2.28}$$

for all $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

Indeed, we have

$$\int_\Omega |\nabla w_n|^{p-2} \nabla w_n \nabla \varphi dx + \int_\Omega \frac{g(w_n) |\nabla w_n|^p}{1 + \frac{1}{n} |\nabla w_n|^p} \varphi dx = \int_\Omega \Psi_n \varphi dx \leq \int_\Omega \Psi \varphi dx, \tag{2.29}$$

for all $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ with $\varphi \geq 0$.

For every $\epsilon > 0$, taking $\varphi = \frac{1}{\epsilon} T_\epsilon(w_n)$ as test function in the previous relation, we have

$$\begin{aligned} \frac{1}{\epsilon} \int_\Omega |\nabla w_n|^{p-2} \nabla w_n \nabla T_\epsilon(w_n) dx + \int_\Omega \frac{g(w_n) |\nabla w_n|^p}{1 + \frac{1}{n} |\nabla w_n|^p} \frac{1}{\epsilon} T_\epsilon(w_n) dx &= \int_\Omega \Psi_n \frac{1}{\epsilon} T_\epsilon(w_n) dx \\ &\leq \int_\Omega \Psi_n dx; \end{aligned}$$

hence,

$$\int_\Omega \frac{g(w_n)|\nabla w_n|^p}{1+\frac{1}{n}|\nabla w_n|^p}\frac{T_\epsilon(w_n)}{\epsilon}dx \le \int_\Omega \Psi_n dx. \tag{2.30}$$

Since $\frac{T_\epsilon(w_n(x))}{\epsilon} = \frac{1}{\epsilon}w_n(x)\chi_{\{x\in\Omega\,:\,w_n\le\epsilon\}} + \chi_{\{x\in\Omega\,:\,w_n>\epsilon\}}$ for every $x\in\Omega$, then $w_n(x) > \epsilon$ and $T_\epsilon(w_n(x)) = \epsilon$ as $\epsilon\to 0$. Taking the limit in (2.30) as $\epsilon\to 0$, by the Dominated Convergence Theorem, we have

$$\int_\Omega \frac{g(w_n)|\nabla w_n|^p}{1+\frac{1}{n}|\nabla w_n|^p}dx \le \int_\Omega \Psi_n dx, \tag{2.31}$$

i.e.,

$$\frac{g(w_n)|\nabla w_n|^p}{1+\frac{1}{n}|\nabla w_n|^p}\varphi \in L^1(\Omega), \qquad \forall \varphi \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega).$$

Define $A_n := |\nabla w_n|^{p-2}\nabla w_n\nabla\varphi$ and $B_n := \frac{g(w_n)|\nabla w_n|^p}{1+\frac{1}{n}|\nabla w_n|^p}\varphi$. So, using Fatou's lemma in (2.29), we obtain

$$\begin{aligned}\int_\Omega |\nabla w|^{p-2}\nabla w\nabla\varphi dx + \int_\Omega g(w)|\nabla w|^p\varphi dx &\le \liminf_{n\to\infty}\left(\int_\Omega (A_n+B_n)\varphi dx\right)\\ &\le \int_\Omega \Psi\varphi dx,\end{aligned}$$

i.e.,

$$\int_\Omega |\nabla w|^{p-2}\nabla w\nabla\varphi dx + \int_\Omega g(w)|\nabla w|^p\varphi dx \le \int_\Omega \Psi\varphi dx, \tag{2.32}$$

for all $\varphi\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ with $\varphi\ge 0$.

Now, define $S(t) := \int_0^t \beta(s)ds$, $\beta\ge 0$ measurable, and take $\varphi\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$, $\varphi\ge 0$ and $k>0$. Thus, taking v_n as test function in $(P)_n$ defined by

$$v_n := e^{-S(w_n)}e^{S(T_k(w_n))}\varphi,$$

we obtain

$$\begin{aligned}&\int_\Omega |\nabla w_n|^{p-2}\nabla w_n\nabla\varphi e^{-S(w_n)}e^{S(T_k(w_n))}dx\\ &\quad+\int_\Omega |\nabla w_n|^{p-2}\nabla w_n\nabla T_k(w_n)\beta(T_k(w_n))e^{-S(w_n)}e^{S(T_k(w_n))}\varphi dx\\ &=\int_\Omega |\nabla w_n|^{p-2}\nabla w_n\nabla w_n\beta(w_n)e^{-S(w_n)}e^{S(w_n)}\varphi dx\\ &\quad-\int_\Omega \frac{g(w_n)|\nabla w_n|^p}{1+\frac{1}{n}|\nabla w_n|^p}e^{-S(w_n)}e^{S(T_k(w_n))}\varphi dx\\ &\quad+\int_\Omega \Psi_n e^{-S(w_n)}e^{S(T_k(w_n))}\varphi dx\\ &\ge 0,\end{aligned}$$

because

$$\int_\Omega \frac{g(w_n)|\nabla w_n|^p}{1+\frac{1}{n}|\nabla w_n|^p}e^{-S(w_n)}e^{S(T_k(w_n))}\varphi dx \le \int_\Omega \Psi_n e^{-S(w_n)}e^{S(T_k(w_n))}\varphi dx.$$

Again, by Fatou's lemma, we have

$$\begin{aligned}
&\int_\Omega |\nabla w|^{p-2}\nabla w\nabla\varphi e^{-S(w)}e^{S(T_k(w))}dx \\
&\quad + \int_\Omega |\nabla w|^{p-2}\nabla w\nabla T_k(w)\beta(T_k(w))e^{-S(w)}e^{S(T_k(w))}\varphi dx \\
&\geq \int_\Omega |\nabla w|^{p-2}\nabla w\nabla w\beta(w)e^{-S(w)}e^{S(w)}\varphi dx \\
&\quad - \int_\Omega g(w)|\nabla w|^p e^{-S(w)}e^{S(T_k(w))}\varphi dx \\
&\quad + \int_\Omega \Psi e^{-S(w)}e^{S(T_k(w))}\varphi dx,
\end{aligned}$$

as $n \to \infty$. Since $0 \leq e^{-S(w)}e^{S(T_k(w))} \leq 1$, by letting $k \to \infty$, it follows immediately from the previous inequality that

$$\int_\Omega |\nabla w|^{p-2}\nabla w\nabla\varphi dx + \int_\Omega g(w)|\nabla w|^p\varphi dx \geq \int_\Omega \Psi\varphi dx, \tag{2.33}$$

for all $\varphi \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ with $\varphi \geq 0$.

Hence, using the relations (2.32) and (2.33), we conclude that the equality (2.28) holds for all $\varphi \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ with $\varphi \geq 0$.

Thus, since $\varphi := \varphi^+ - \varphi^-$ and $\varphi^+, \varphi^- \geq 0$, we obtain

$$\int_\Omega |\nabla w|^{p-2}\nabla w\nabla\varphi dx + \int_\Omega g(w)|\nabla w|^p\varphi dx = \int_\Omega \Psi\varphi dx$$

for all $\varphi \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$.

Therefore, $(P)_{\lambda\sigma}$ has unique solution in $W_0^{1,p}(\Omega)$ for $\lambda = 0$. □

3 Proof of Theorem 1.2

In this section, first we prove some results which are used in the proof of our main theorem. Notice that our definition of solution of $(P)_{\lambda\sigma}$ includes the integrability of $g(u)|\nabla u|^p$. Using some ideas of Arcoya, Carmona and Martínez-Aparicio [5], we will see in the following result that a consequence is the integrability of $g(u)|\nabla u|^p\varphi$ for all $\varphi \in W_0^{1,p}(\Omega)$.

Lemma 3.1. *If $0 < u \in W_0^{1,p}(\Omega)$ is a solution for $(P)_{\lambda\sigma}$, then $g(u)|\nabla u|^p\varphi$ is integrable in Ω for all $\varphi \in W_0^{1,p}(\Omega)$. Moreover, we have*

$$\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla\varphi dx + \int_\Omega g(u)|\nabla u|^p\varphi dx = \lambda\int_\Omega u^\sigma\varphi dx + \int_\Omega \Psi\varphi dx. \tag{3.1}$$

Proof. Since $\sigma \leq p^* - 1$, note that u^σ, $u^\sigma\varphi \in L^1(\Omega)$. Indeed, since $u \in L^p(\Omega)$ and $W_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ for $p < N$, we have

$$\int_\Omega u^\sigma dx \leq \int_\Omega |u|^\sigma dx < \infty.$$

On the other hand, we have

$$\sigma\frac{p^*}{p^*-1} \leq (p^*-1)\frac{p^*}{p^*-1} = p^*;$$

thus, by Hölder's inequality we obtain

$$\begin{aligned}\int_\Omega u^\sigma \varphi dx &\le \left[\left(\int_\Omega |u|^{\sigma\frac{p^*}{p^*-1}}dx\right)^{\frac{p^*-1}{\sigma p^*}}\right]^\sigma \left(\int_\Omega |\varphi|^{p^*}dx\right)^{\frac{1}{p^*}} \\ &\le C\|u\|_{1,p}^\sigma \|\varphi\|.\end{aligned} \tag{3.2}$$

Hence, by previous relations, we conclude that $u^\sigma,\ u^\sigma\varphi \in L^1(\Omega)$.

By taking $T_k(\varphi^+)$ as test function in (1.3) and using Hölder's inequality we have

$$\begin{aligned}\int_\Omega g(u)|\nabla u|^p T_k(\varphi^+)dx &= -\int_\Omega |\nabla u|^{p-2}\nabla u \nabla T_k(\varphi^+)dx + \int_\Omega (\lambda u^\sigma + \Psi)T_k(\varphi^+)dx \\ &\le \int_\Omega |\nabla u|^{p-1}|\nabla T_k(\varphi^+)|dx + \int_\Omega (\lambda u^\sigma + \Psi)T_k(\varphi^+)dx \\ &\le \left(\int_\Omega (|\nabla u|^{p-1})^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}} \left(\int_\Omega |\nabla T_k(\varphi^+)|^p dx\right)^{\frac{1}{p}} \\ &\quad + \int_\Omega (\lambda u^\sigma + \Psi)T_k(\varphi^+)dx \\ &\le \|u\|_{1,p}^{p-1}\|\varphi\|_{1,p} + \int_\Omega (\lambda u^\sigma + \Psi)T_k(\varphi^+)dx.\end{aligned}$$

Now, by taking limit as $k \to \infty$ and using Fatou's lemma, we deduce that $g(u)|\nabla u|^p\varphi^+ \in L^1(\Omega)$ with

$$\int_\Omega g(u)|\nabla u|^p\varphi^+ dx \le \|u\|_{1,p}\|\varphi\|_{1,p} + \int_\Omega (\lambda u^\sigma + \Psi)\varphi^+ dx. \tag{3.3}$$

Similarly, by taking $T_k(-\varphi^-)$ as test function in (1.3), we obtain that $g(u)|\nabla u|^p\varphi^- \in L^1(\Omega)$ with

$$-\int_\Omega g(u)|\nabla u|^p\varphi^- dx \le \|u\|_{1,p}\|\varphi^-\|_{1,p} + \int_\Omega (\lambda u^\sigma - \Psi)\varphi^- dx. \tag{3.4}$$

By combining the relations (3.3) and (3.4), we conclude that $g(u)|\nabla u|^p\varphi \in L^1(\Omega)$ for all $\varphi \in W_0^{1,p}(\Omega)$ with

$$\int_\Omega g(u)|\nabla u|^p\varphi dx \le \|u\|_{1,p}\|\varphi\|_{1,p} + \int_\Omega (\lambda u^\sigma + \Psi)\varphi dx.$$

Lastly, note that this integrability of $g(u)|\nabla u|^p\varphi$ allows to use a density argument to conclude (3.1) from (1.3). □

The next result will be related with the compactness for the operator $K(\lambda, w)$ defined in (1.5).

Lemma 3.2. *Assume that $\Psi \in L^q(\Omega)$ with $q = \frac{pN}{N(p-1)+p}$, $g \ge 0$ is continuous in $[0,+\infty)$ or $g \ge 0$ is continuous in $(0,+\infty)$ and integrable in an neighborhood of zero with $\lim_{s\to 0} g(s) = +\infty$. If the sequences $\{t_n\} \subset [0,1]$ and $\{\lambda_n\} \subset (0,\infty)$ are convergent, respectively, to t^* and λ, and $\{w_n\} \subset W_0^{1,p}(\Omega)$ weakly convergent to w, then the sequence of (uniquely defined) solutions $\{u_n\} \subset W_0^{1,p}(\Omega)$ of*

$$\begin{cases} -\Delta_p u_n + t_n g(u_n)|\nabla u_n|^p = \lambda_n (w_n^+(x))^\sigma + \Psi(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.5}$$

is strongly convergent in $W_0^{1,p}(\Omega)$ to the solution u of

$$\begin{cases} -\Delta_p u + t^* g(u)|\nabla u|^p = \lambda (w^+(x))^\sigma + \Psi(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.6}$$

Proof. Since the functions $\lambda_n(w_n^+(x))^\sigma + \Psi(x)$ and $\lambda(w^+(x))^\sigma + \Psi(x)$ are in $L^q(\Omega)$ with $q = \frac{pN}{N(p-1)+p}$, then by Theorem 1.1 the existence of an unique solution of (3.5) and (3.6) it is holds. Indeed, we have that

$$\int_\Omega |\lambda w_n^+(x)^\sigma + \Psi(x)|^q dx \leq 2^{q-1}\int_\Omega |\lambda w_n^+(x)^\sigma|^q dx + 2^{q-1}\int_\Omega |\Psi(x)|^q dx$$

and $\sigma q \leq \frac{N(p-1)+p}{N-P}\frac{pN}{N(p-1)+p} = \frac{pN}{N-p} = p^*$.

In order to prove the compactness of $K(\lambda, u)$ is suffices to prove that every subsequence of $\{u_n\}$ possesses a subsequence converging to the unique solution $u \in W_0^{1,p}(\Omega)$ of (3.6). First we will prove that $\{u_n\}$ is a bounded sequence in $W_0^{1,p}(\Omega)$. Indeed, choosing u_n as test function in (3.5) and using that t_n and $g(u_n)$ are nonnegative, we have

$$\begin{aligned}\int_\Omega |\nabla u_n|^p dx &\leq \int_\Omega |\nabla u_n|^p dx + t_n\int_\Omega g(u_n)|\nabla u_n|^p u_n dx \\ &= \lambda_n\int_\Omega (w_n^+(x))^\sigma u_n dx + \int_\Omega \Psi u_n dx.\end{aligned} \tag{3.7}$$

Since $W_0^{1,p}(\Omega) \hookrightarrow L^{p*}(\Omega)$, by Hölder's inequality we obtain

$$\int_\Omega \Psi u_n dx \leq C\|\Psi\|_{\frac{pN}{N(p-1)+p}}\|u_n\|_{1,p}. \tag{3.8}$$

Furthermore, since $\sigma\frac{N(p-1)+p}{pN} = \frac{p^*}{p^*-1} \leq p^*$, again by Hölder's inequality we have

$$\begin{aligned}\lambda_n\int_\Omega (w_n^+(x))^\sigma u_n dx &\leq C\left[\left(\int_\Omega |w_n|^{\sigma\frac{N(p-1)+p}{pN}} dx\right)^{\frac{pN}{\sigma[N(p-1)+p]}}\right]^\sigma \left(\int_\Omega |u_n|^{\frac{pN}{N-p}} dx\right)^{\frac{N-p}{pN}} \\ &\leq \overline{C}\|w_n\|_{p^*}^\sigma \|u_n\|_{p^*} \\ &\leq \overline{\overline{C}}\|w_n\|_{1,p}^\sigma \|u_n\|_{1,p}.\end{aligned} \tag{3.9}$$

By combining the relations (3.7), (3.8) and (3.9) we conclude that

$$\|u_n\|_{1,p} \leq \left[C\|w_n\|_{1,p}^\sigma + \|\Psi\|_{\frac{pN}{N(p-1)+p}}\right]^{\frac{1}{p-1}}.$$

Therefore, $\{u_n\}$ is a bounded sequence in $W_0^{1,p}(\Omega)$. Thus, going if necessary to a subsequence, still denoted by $\{u_n\}$, there exists $\overline{u} \in W_0^{1,p}(\Omega)$ such that $u_n \rightharpoonup \overline{u}$ weakly in $W_0^{1,p}(\Omega)$.

Repeating the arguments used in the proof Lemma 2.3, we obtain that

$$\nabla u_n(x) \to \nabla\overline{u}(x) \quad \text{a.e in } \Omega \quad \text{and} \quad \nabla u_n \to \nabla\overline{u} \quad \text{in } L^q(\Omega), \quad q < p.$$

Now, we will prove that $\overline{u}$ satisfies the following equality

$$\int_\Omega |\nabla\overline{u}|^{p-2}\nabla\overline{u}\cdot\nabla\varphi dx + t^*\int_\Omega g(\overline{u})|\nabla\overline{u}|^p\varphi dx = \int_\Omega (\lambda(w^+)^\sigma + \Psi)\varphi dx, \quad \varphi \in W_0^{1,p}(\Omega). \tag{3.10}$$

First, we will show that $\overline{u} > 0$ in Ω. Indeed, we have

$$\int_\Omega |\nabla u_n|^{p-2}\nabla u_n\cdot\nabla\varphi dx + t_n\int_\Omega g(u_n)|\nabla u_n|^p\varphi dx = \int_\Omega (\lambda_n(w_n^+)^\sigma + \Psi)\varphi dx, \quad \varphi \in W_0^{1,p}(\Omega). \tag{3.11}$$

Thus, choosing v_n as test function in the previous equality such that

$$v_n := e^{-H(u_n)}\varphi, \quad \varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega), \quad \varphi \geq 0,$$

we obtain

$$\begin{aligned} -\int_\Omega |\nabla u_n|^p H'(u_n) e^{-H(u_n)}\varphi dx + \int_\Omega |\nabla u_n|^{p-2}\nabla u_n e^{-H(u_n)}\nabla\varphi dx \\ = \int_\Omega (\lambda_n (w_n^+)^\sigma + \Psi) e^{-H(u_n)}\varphi dx - t_n \int_\Omega g(u_n)|\nabla u_n|^p e^{-H(u_n)}\varphi dx, \end{aligned}$$

where $H(t) := \int_0^t g(s)ds$.

By ordering the terms of the previous equation, by using $H_n'(t) = g(t)$, we obtain

$$\begin{aligned} &\int_\Omega |\nabla u_n|^{p-2}\nabla u_n e^{-H(u_n)}\nabla\varphi dx - \int_\Omega (\lambda_n (w_n^+)^\sigma + \Psi) e^{-H(u_n)}\varphi dx \\ &\quad = (1 - t_n)\int_\Omega g(u_n)|\nabla u_n|^p e^{-H(u_n)}\varphi dx \\ &\quad \geq 0, \end{aligned}$$

i.e.,

$$\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla\varphi e^{-H(u_n)} dx \geq \int_\Omega (\lambda_n (w_n^+)^\sigma + \Psi) e^{-H(u_n)}\varphi dx.$$

Thus, by taking limit as $n \to \infty$, we have

$$\int_\Omega |\nabla \overline{u}|^{p-2}\nabla \overline{u}\nabla\varphi e^{-H(\overline{u})} dx \geq \int_\Omega (\lambda (w^+)^\sigma + \Psi) e^{-H(\overline{u})}\varphi dx. \tag{3.12}$$

Define $v(x) := \psi(\overline{u}(x)) = \int_0^{\overline{u}(x)} \left(e^{-H(s)}\right)^{\frac{1}{p-1}} dt$, where $\psi(s) := \int_0^s \left(e^{-H(s)}\right)^{\frac{1}{p-1}} dt$ is strictly increasing. Let z be a solution of problem

$$\begin{cases} -\Delta_p z = \dfrac{T_1((w^+)^\sigma + \Psi)}{e^{H(\overline{u})}} & \text{in } \Omega, \\ z = 0 & \text{on } \partial\Omega. \end{cases}$$

Since $\frac{T_1(\Psi)}{e^{H(\overline{u})}} \in L^\infty(\Omega)$, by a result of Lieberman (see [28, Theorem 1]), we have that $z \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$. Furthermore, by strong maximum principle, we conclude that $z > 0$ in Ω. By applying in (3.12) the relation $\nabla v = \nabla \overline{u}\left(e^{-H(\overline{u})}\right)^{\frac{1}{p-1}}$, we obtain

$$\int_\Omega |\nabla v|^{p-2}\nabla v\nabla\varphi dx \geq \int_\Omega T_1(\Psi) e^{-H(\overline{u})}\varphi dx.$$

Thus, by weak comparison principle, we have $v(x) \geq z(x) > 0$ in Ω. However, since $\psi(\overline{u}(x)) := v(x) > 0$ and ψ is strictly increasing in Ω, then $\overline{u}(x) > 0$ in Ω.

Now, we resume the proof of (3.10). For every $\epsilon > 0$, taking $\varphi := \frac{1}{\epsilon} T_\epsilon(u_n)$ as test function in (3.10), we obtain

$$\begin{aligned} \frac{1}{\epsilon}\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla T_\epsilon(u_n) dx + t_n \int_\Omega g(u_n)|\nabla u_n|^p \frac{1}{\epsilon} T_\epsilon(u_n) dx &= \int_\Omega (\lambda_n (w_n^+)^\sigma + \Psi)\frac{1}{\epsilon} T_\epsilon(u_n) dx \\ &\leq \int_\Omega (\lambda_n (w_n^+)^\sigma + \Psi) dx; \end{aligned}$$

hence,

$$t_n \int_\Omega g(u_n)|\nabla u_n|^p \frac{T_\epsilon(u_n)}{\epsilon} dx \leq \int_\Omega (\lambda_n (w_n^+)^\sigma + \Psi) dx. \tag{3.13}$$

Since $\frac{T_\epsilon(u_n(x))}{\epsilon} = \frac{1}{\epsilon}u_n(x)\chi_{\{x\in\Omega\,:\,w_n\le\epsilon\}} + \chi_{\{x\in\Omega\,:\,u_n>\epsilon\}}$ for every $x\in\Omega$, then $u_n(x) > \epsilon$ and $T_\epsilon(u_n(x)) = \epsilon$ as $\epsilon\to 0$. Taking the limit in (3.13) as $\epsilon\to 0$, by the Dominated Convergence Theorem, we have

$$t_n\int_\Omega g(u_n)|\nabla u_n|^p dx \le \int_\Omega(\lambda_n(w_n^+)^\sigma + \Psi)dx, \tag{3.14}$$

i.e.,

$$g(u_n)|\nabla u_n|^p\varphi \in L^1(\Omega), \qquad \forall\varphi\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega).$$

Define $A_n := |\nabla u_n|^{p-2}\nabla u_n\nabla\varphi$ and $B_n := g(u_n)|\nabla u_n|^p\varphi$. Hence, using Fatou's lemma in (3.14), we have

$$\begin{aligned}\int_\Omega|\nabla\overline{u}|^{p-2}\nabla\overline{u}\nabla\varphi dx + t^*\int_\Omega g(\overline{u})|\nabla\overline{u}|^p\varphi dx &\le \liminf_{n\to\infty}\left(\int_\Omega(A_n+B_n)\varphi dx\right)\\ &= \int_\Omega(\lambda(w^+)^\sigma+\Psi)\varphi dx,\end{aligned}$$

i.e.,

$$\int_\Omega|\nabla\overline{u}|^{p-2}\nabla\overline{u}\nabla\varphi dx + t^*\int_\Omega g(\overline{u})|\nabla\overline{u}|^p\varphi dx \le \int_\Omega(\lambda(w^+)^\sigma+\Psi)\varphi dx, \tag{3.15}$$

for all $\varphi\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ with $\varphi\ge 0$.

Now, define $S(t) := \int_0^t\beta(s)ds$, $\beta\ge 0$ measurable, and take $\varphi\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$, $\phi\ge 0$ and $k>0$. Hence, taking v_n as test function in (3.12) defined by

$$v_n := e^{-S(u_n)}e^{S(T_k(u_n))}\varphi,$$

we obtain

$$\begin{aligned}&\int_\Omega|\nabla u_n|^{p-2}\nabla u_n\nabla\varphi e^{-S(u_n)}e^{S(T_k(u_n))}dx\\ &\quad+\int_\Omega|\nabla u_n|^{p-2}\nabla u_n\nabla T_k(u_n)\beta(T_k(u_n))e^{-S(u_n)}e^{S(T_k(u_n))}\varphi dx\\ &=\int_\Omega|\nabla u_n|^{p-2}\nabla u_n\nabla u_n\beta(u_n)e^{-S(u_n)}e^{S(u_n)}\varphi dx\\ &\quad-t_n\int_\Omega g(u_n)|\nabla u_n|^p e^{-S(u_n)}e^{S(T_k(u_n))}\varphi dx\\ &\quad+\int_\Omega(\lambda(w_n^+)^\sigma+\Psi)\varphi e^{-S(u_n)}e^{S(T_k(u_n))}\varphi dx\\ &\ge 0,\end{aligned}$$

because

$$t_n\int_\Omega g(u_n)|\nabla u_n|^p e^{-S(u_n)}e^{S(T_k(u_n))}\varphi dx \le \int_\Omega(\lambda(w_n^+)^\sigma+\Psi)\varphi e^{-S(u_n)}e^{S(T_k(u_n))}\varphi dx.$$

Again, by Fatou's lemma, we have

$$\begin{aligned}&\int_\Omega|\nabla\overline{u}|^{p-2}\nabla\overline{u}\nabla\varphi e^{-S(\overline{u})}e^{S(T_k(\overline{u}))}dx\\ &\quad+\int_\Omega|\nabla\overline{u}|^{p-2}\nabla\overline{u}\nabla T_k(\overline{u})\beta(T_k(\overline{u}))e^{-S(\overline{u})}e^{S(T_k(\overline{u}))}\varphi dx\\ &\ge\int_\Omega|\nabla\overline{u}|^{p-2}\nabla\overline{u}\nabla\overline{u}\beta(\overline{u})e^{-S(\overline{u})}e^{S(\overline{u})}\varphi dx\\ &\quad-t^*\int_\Omega g(\overline{u})|\nabla\overline{u}|^p e^{-S(\overline{u})}e^{S(T_k(\overline{u}))}\varphi dx\\ &\quad+\int_\Omega(\lambda(w^+)^\sigma+\Psi)\varphi e^{-S(\overline{u})}e^{S(T_k(\overline{u}))}\varphi dx.\end{aligned} \tag{3.16}$$

as $n \to \infty$. Since $0 \leq e^{-S(\overline{u})}e^{S(T_k(\overline{u}))} \leq 1$, by letting $k \to \infty$, it follows from (3.16) that

$$\int_\Omega |\nabla \overline{u}|^{p-2}\nabla \overline{u}\nabla \varphi dx + t^* \int_\Omega g(\overline{u})|\nabla \overline{u}|^p \varphi dx \geq \int_\Omega (\lambda (w^+)^\sigma + \Psi)\varphi dx, \tag{3.17}$$

for all $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ with $\varphi \geq 0$.

Thus, by (3.15) and (3.17), we conclude that (3.10) holds for all for all $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ with $\varphi \geq 0$.

Hence, since $\varphi := \varphi^+ - \varphi^-$ and $\varphi^+, \varphi^- \geq 0$, we have

$$\int_\Omega |\nabla \overline{u}|^{p-2}\nabla \overline{u}\nabla \varphi dx + t^* \int_\Omega g(\overline{u})|\nabla \overline{u}|^p \varphi dx = \int_\Omega (\lambda (w^+)^\sigma + \Psi)\varphi dx, \tag{3.18}$$

for all $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

Since problem (3.6) has an unique solution, then $u = \overline{u}$.

We still need to prove that $u_n \to u$ in $W_0^{1,p}(\Omega)$. For fixed $k > 0$, by taking $u_n = G_k(u_n) + T_k(u_n)$, we have

$$\begin{aligned} \|u_n - u\|_{1,p} &= \|u_n - T_k(u) + T_k(u) - u\|_{1,p} \\ &\leq \|u_n - T_k(u)\|_{1,p} + \|T_k(u) - u\|_{1,p} \\ &= \|G_k(u_n) + T_k(u_n) - T_k(u)\|_{1,p} + \|T_k(u) - u\|_{1,p} \\ &\leq \|G_k(u_n)\|_{1,p} + \|T_k(u_n) - T_k(u)\|_{1,p} + \|T_k(u) - u\|_{1,p}. \end{aligned} \tag{3.19}$$

Hence, the strong convergence of $\{u_n\}$ in $W_0^{1,p}(\Omega)$ is stated provided that we show the strong convergence of $\{T_k(u_n)\}$ to $T_k(u)$ in $W_0^{1,p}(\Omega)$ and that for every $\delta > 0$ there exists $k_0 = k_0(\delta)$ such that $k \geq k_0$ implies

$$\|G_k(u_n)\|_{1,p} < \delta, \qquad \text{for all } n \in \mathbb{N}.$$

This is done in two steps.

Step 1. For fixed $k > 0$, we have that $T_k(u_n) \to T_k(u)$ in $W_0^{1,p}(\Omega)$. Indeed, by fixing compact set $K \subset \Omega$ we take $\varphi_K \in C_0^\infty(\Omega)$ with $0 \leq \varphi_K \leq 1$ and $\varphi = 1$ in K. Thus, taking v_n as test function in (3.5) defined by

$$v_n := \big(T_k(u_n) - T_k(u)\big)^+ \varphi_K \in W_0^{1,p}(\Omega),$$

we have

(i) $\nabla v_n = \nabla \varphi_K \big(T_k(u_n) - T_k(u)\big)^+ + \nabla \big(T_k(u_n) - T_k(u)\big)^+ \varphi_K$;

(ii) $t_n g(u_n)|\nabla u_n|^p v_n \geq 0$;

(iii) $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$.

In addition, we also have

$$\begin{aligned} &\int_\Omega \varphi_K |\nabla u_n|^{p-2}\nabla u_n \nabla \big(T_k(u_n) - T_k(u)\big)^+ dx \\ &\quad \leq \int_\Omega (\lambda_(w_n^+)^\sigma + \Psi) v_n dx - \int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla \phi_K \big(T_k(u_n) - T_k(u)\big)^+ dx. \end{aligned} \tag{3.20}$$

Thus, by Kavian (see [25, Lemma 4.8]), we conclude that

$$T_k(u_n) \rightharpoonup T_k(u) \quad \text{in } W_0^{1,p}(\Omega). \tag{3.21}$$

For fixed $k > 0$, combining the relations (3.20) and (3.21), we obtain

$$\int_\Omega \varphi_K |\nabla u_n|^{p-2}\nabla u_n \nabla \big(T_k(u_n) - T_k(u)\big)^+ dx \to 0, \quad \text{as} \quad n \to \infty. \tag{3.22}$$

By define the sets

$$S_n := \{x \in \Omega \ : \ |u_n(x)| \le k\}| \quad \text{and} \quad G_n := \{x \in \Omega \ : \ |u_n(x)| > k\},$$

and denote by χ_{G_n} the characteristic function of G_n. Moreover, take

$$\begin{aligned} E_n^+ &:= \int_\Omega \varphi_K \big[|\nabla T_k(u_n)|^{p-2}\nabla T_k(u_n) - |\nabla T_k(u)|^{p-2}\nabla T_k(u)\big]\nabla \big(T_k(u_n) - T_k(u)\big)^+ dx \\ &= \int_\Omega \varphi_K |\nabla u_n|^{p-2}\nabla u_n \nabla \big(T_k(u_n) - T_k(u)\big)^+ dx \\ &\quad - \int_\Omega \varphi_K \big[|\nabla u_n|^{p-2}\nabla u_n - |\nabla T_k(u)|^{p-2}\nabla T_k(u)\big]\nabla \big(T_k(u_n) - T_k(u)\big)^+ dx \\ &\quad - \int_\Omega \varphi_K |\nabla T_k(u)|^{p-2}\nabla T_k(u) \nabla \big(T_k(u_n) - T_k(u)\big)^+ dx. \end{aligned} \tag{3.23}$$

By using the relations (3.21) and (3.22), we conclude that the first and third term of (3.23) tends to zero as $n \to \infty$.

With respect to the second term of (3.23), we have

$$\begin{aligned} &- \int_\Omega \varphi_K \big[|\nabla u_n|^{p-2}\nabla u_n - |\nabla T_k(u)|^{p-2}\nabla T_k(u)\big]\nabla \big(T_k(u_n) - T_k(u)\big)^+ dx \\ &\quad = - \int_{G_n} \varphi_K \big[|\nabla u_n|^{p-2}\nabla u_n - |\nabla T_k(u)|^{p-2}\nabla T_k(u)\big]\nabla \big(T_k(u_n) - T_k(u)\big)^+ dx \\ &\qquad - \int_{S_n} \varphi_K \big[|\nabla u_n|^{p-2}\nabla u_n - |\nabla T_k(u)|^{p-2}\nabla T_k(u)\big]\nabla \big(T_k(u_n) - T_k(u)\big)^+ dx \\ &\quad = - \int_{G_n} \varphi_K |\nabla u_n|^{p-2}\nabla u_n \nabla \big(T_k(u_n) - T_k(u)\big)^+ dx. \end{aligned}$$

Furthermore, by the Dominated Convergence Theorem, we have

$$\begin{aligned} \int_{G_n} \varphi_K |\nabla u_n|^{p-2}\nabla u_n \nabla \big(T_k(u_n) - T_k(u)\big)^+ dx &= \int_\Omega \varphi_K |\nabla u_n|^{p-2}\nabla u_n . \chi_{G_n} \big(\nabla \big(T_k(u_n) - T_k(u)\big)^+\big) dx \\ &\le \|u_n\|_{1,p}^{p-1} \left[\int_\Omega \chi_{G_n} |\nabla T_k(u)|^p dx\right]^{\frac{1}{p}} \to 0, \end{aligned}$$

as $n \to \infty$.

Therefore, we conclude that $E_n^+ \to 0$ as $n \to \infty$ and thus,

$$\int_K \big[|\nabla T_k(u_n)|^{p-2}\nabla T_k(u_n) - |\nabla T_k(u)|^{p-2}\nabla T_k(u)\big]\nabla \big(T_k(u_n) - T_k(u)\big)^+ dx \to 0. \tag{3.24}$$

Now, taking v_n as test function in (3.5) defined by

$$v_n = \big(T_k(u_n) - T_k(u)\big)^- \varphi_K \in W_0^{1,p}(\Omega)$$

and repeating the previous arguments, we obtain

$$\int_K \big[|\nabla T_k(u_n)|^{p-2}\nabla T_k(u_n) - |\nabla T_k(u)|^{p-2}\nabla T_k(u)\big]\nabla \big(T_k(u_n) - T_k(u)\big)^- dx \to 0. \tag{3.25}$$

By combining the relations (3.24) and (3.25), we have

$$\int_K \big[|\nabla T_k(u_n)|^{p-2}\nabla T_k(u_n) - |\nabla T_k(u)|^{p-2}\nabla T_k(u)\big]\nabla\big(T_k(u_n) - T_k(u)\big)dx \to 0.$$

Thus, by inequality (2.7) we conclude that $\nabla T_k(u_n) \to \nabla T_k(u)$ in $L^p(K)$. Since $T_k(u_n) \in W_0^{1,p}(\Omega)$, then

$$\nabla T_k(u_n) \to \nabla T_k(u) \quad \text{in } L^p(\Omega).$$

Therefore, the sequence $\{T_k(u_n)\}$ converges strongly to $T_k(u)$ in $W_0^{1,p}(\Omega)$.

Step 2. By taking $v_n := G_k(u_n)$ as test function in (3.5) we have

$$\begin{aligned}\int_\Omega |\nabla G_k(u_n)|^p dx &= \int_{\{u_n \geq k\}} |\nabla u_n|^{p-2}\nabla u_n \nabla G_k(u_n)dx \\ &\leq \left(\int_{\{u_n \geq k\}} (\lambda_n (w_n^+)^\sigma + \Psi)dx\right)^{\frac{pN}{N(p-1)+p}} \|G_k(u_n)\|_{p^*}.\end{aligned}$$

Since $\frac{pN}{N(p-1)+p}\sigma < p^*$, $\{w_n\}$ is strongly convergent in $L^{\frac{pN}{N(p-1)+p}\sigma}(\Omega)$, $\{\lambda_n\}$ is bounded and $\Psi \in L^{\frac{pN}{N(p-1)+p}}(\Omega)$, the right-hand side of the previous inequality tends uniformly in n to zero as k_0 diverges, i.e., for every $\delta > 0$ there exists $k_0 = k_0(\delta)$ such that $k \geq k_0$ implies

$$\|\nabla G_k(u_n)\|_{1,p} < \delta, \quad \text{for all } n \in \mathbb{N}.$$

Therefore, by step 1 and 2 and by inequality (3.19), we conclude that $\{u_n\}$ converges strongly to u in $W_0^{1,p}(\Omega)$. □

Completing the proof of Theorem 1.2: Let $u_0 \in W_0^{1,p}(\Omega)$ be solution of $(P)_{\lambda\sigma}$ for $\lambda = 0$. For every isolated solution $u_\lambda \in W_0^{1,p}(\Omega)$ of $(P)_{\lambda\sigma}$ for some $\lambda \in \mathbb{R}$, we denote by $i(K_\lambda, , u_\lambda)$ the index of such a solution, that is, the topological Leray–Schauder degree $\deg(I - K_\lambda, B_\epsilon(u_\lambda), 0)$ of the operator $I - K_\lambda$ in a ball $B_\epsilon(u_\lambda)$ centered at u_λ with radius $\epsilon > 0$ small enough.

We will prove that $\deg(I - K_\lambda, B_\epsilon(u_\lambda), 0) \neq 0$ for $\lambda = 0$. Indeed, we denote by $U(t)$ the unique solution for

$$\begin{cases} -\Delta_p u + tg(u)|\nabla u|^p = \Psi(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

and we define the following homotopy

$$\begin{aligned} H : [0,1] \times W_0^{1,p}(\Omega) &\longrightarrow W_0^{1,p}(\Omega) \\ (t, w) &\longmapsto H(t,w) := U(t). \end{aligned}$$

Hence,

$$H(1,w) = U(1) = K_0(w) = K(0,w) = u_0$$

and

$$H(0,w) = U(0) = (-\Delta_p^{-1})(\Psi(x)).$$

Since $i\big((-\Delta_p^{-1})(\Psi(x)), U(0)\big) \neq 0$, by Lemma 3.2 we deduce that H is compact. Observing the first part of the proof of Lemma 3.2 we obtain $R > 0$ such that

$$\|U(t)\|_{1,p} < R, \quad \text{for all } t \in [0,1].$$

If $u \in W_0^{1,p}(\Omega)$ and $\|u\|_{1,p} \geq R$, then $u \neq H(t,u)$. Thus, by the homotopy invariance of the degree, we conclude that

$$\begin{aligned} i(K_0,u_0) &= i\big(H(1,\cdot),U(1)\big) \\ &= i\big(H(0,\cdot),U(0)\big) \\ &= i\big((-\Delta_p^{-1})(\Psi(x)),U(0)\big) \\ &\neq 0. \end{aligned} \tag{3.26}$$

Hence, we have that $K : \times\overline{B}_R \to B_R$ is continuous and compact and u_0 is an isolated solution of $(P)_{\lambda\sigma}$ in the ball $B_\epsilon(u_\lambda)$ for $\lambda = 0$. Thus, for $\lambda_0 > 0$ small enough, we have

$$K \ : \ [0,\lambda_0] \times \overline{B_\epsilon(u_0)} \longrightarrow B_\epsilon(u_0).$$

If $\Phi(\lambda,u) := u - K(\lambda,u)$, then $\deg\big(\Phi(\lambda,\cdot),B_\epsilon(u_0),0\big)$ is well defined for $\lambda \leq \lambda_0$. Hence, by applying the homotopy invariance of the degree, we have

$$\deg\big(\Phi(\lambda,\cdot),B_\epsilon(u_0),0\big) = \text{constant}, \qquad \lambda \leq \lambda_0.$$

Thus, by relation (3.26), we conclude that

$$\deg\big(\Phi(\lambda,\cdot),B_\epsilon(u_0),0\big) \neq 0, \qquad |\lambda| \leq \lambda_0.$$

The theorem follows now from the Rabinowitz Theorem 3.2 in [32]. □

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Bifurcation in two parameters for a quasilinear Schrödinger equation

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Abstract. This paper deals with existence and multiplicity of positive solutions for the quasilinear Schrödinger equation

$$\begin{cases} -\Delta u - \lambda m(x) u \Delta(u^2) = f(\mu, x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

where Ω is a bounded open domain in $\mathbb{R}^N$ with smooth boundary and m is a bounded non negative continuous function. Under suitable assumptions on the asymptotically linear f, we use bifurcation theory to analyze the set of positive solutions.

Keywords: bifurcation theory, Schrödinger equation.

2020 Mathematics Subject Classification: 35J10, 35B32, 34K18.

1 Introduction

The quasilinear Schrödinger equation

$$i\partial_t z = -\Delta z + V_0(x)z - z(\Delta|z|^2) - \theta|z|^{p-1}z, \tag{1.1}$$

known as the superfluid film equation in plasma physics due to Kurihara ([9]), has been studied by Poppenberg, Schmitt and Wang in [10]. Here the potential V_0 is a function $V_0\colon \mathbb{R}^N \to \mathbb{R}$, while $\theta > 0$, $p > 1$ are constants. Using the change of variables $z(t,x) = \exp\{-i\kappa t\}u(x)$ for $\kappa \in \mathbb{R}$ and putting $V = V_0 - \kappa$, they obtain the elliptic equation

$$-\Delta u - u(\Delta|u|^2) = \theta|u|^{p-1}u - V(x)u, \tag{1.2}$$

corresponding to the standing wave solutions of (1.1). In that work, variational techniques are applied to prove the existence of standing wave solutions. Also, readers can find more references in [10] about the physical meanings of the equation.

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In this work we study positive solutions of

$$\begin{cases} -\Delta u - \lambda m(x) u \Delta(u^2) = f(\mu, x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{P_μ^λ}$$

where Ω is a bounded open domain with smooth boundary, λ and μ are real constants, $m(x)$ is a non negative bounded continuous function in $\overline{\Omega}$ and $f\colon \mathbb{R} \times \overline{\Omega} \times [0,\infty) \to \mathbb{R}$ is a continuous function such that $f(\mu, x, 0) = 0$ for every $x \in \overline{\Omega}$ and $\mu \in \mathbb{R}$.

We emphasize that the equation in (P_μ^λ) is more general than (1.2). Observe also that when $\lambda = 0$ the problem (P_μ^0) is just a classical semilinear problem which has been extensively studied. For the case of asymptotically linear datum f at 0 see [2]. The case of the quasilinear problem (P_μ^λ) with $m(x) = 1$ has been studied for $\lambda > 0$ in several works made subsequently after [10], see for instance [6–8] and references therein. In all these works, the problem (with $m(x) = 1$) is variational and one of the main ideas is to reduce the problem to a semilinear one by using the change of variables $u = g(v)$ where g is the unique solution of the differential equation

$$g' = \frac{1}{\sqrt{1 + 2\lambda g^2}}, \qquad g(0) = 0. \tag{1.3}$$

Contrary to the cited previous works, here λ can be either positive or negative, implying an accelerating ($\lambda > 0$) or slowdown ($\lambda < 0$) process for the non linear diffusion term. In addition, since we are considering nonconstant density $m(x)$, this term also increases or decreases this accelerating-slowdown process depending on the point of the domain. Note that even in the simplest case $m(x) = 1$, the uniform ellipticity of the operator (which may be rewritten in divergence form as $-\operatorname{div}((1 + 2\lambda u^2)\nabla u) + 2\lambda u|\nabla u|^2$) can be lost when λ is negative.

Moreover, although the change of variables (1.3) is still available for negative λ when $m(x) = 1$, it leads to a semilinear problem of a completely different nature that in the case positive λ. Indeed, in that case, the nonlinear term has an asymptote and the main properties of this nonlinear term, as used in [6–8] and references therein, are no longer satisfied for negative λ. In the general case where $m(x)$ is not constant, no similar change of variable becomes the problem semilinear.

To overcome these difficulties, for every fixed $\lambda \in \mathbb{R}$ we see that problem (P_μ^λ) is equivalent to

$$\Phi_\mu^\lambda(u) = 0, \quad u \in \mathcal{U}_\lambda = \{u \in C_0^1(\overline{\Omega}) : 1 + 2\lambda m(x) u^2(x) > 0, x \in \Omega\};$$

i.e., to find the ceros u of an operator Φ_μ^λ defined in the open subset $\mathcal{U}_\lambda$ of the Banach space $C_0^1(\overline{\Omega})$ of the differentiable functions with continuous derivative in $\overline{\Omega}$ which vanishes on the boundary $\partial\Omega$. Notice that $\mathcal{U}_\lambda = C_0^1(\overline{\Omega})$ if $\lambda \geq 0$, while $\mathcal{U}_\lambda \subsetneq C_0^1(\overline{\Omega})$ when $\lambda < 0$. Because of this we prove in Theorem 3.1 an extension of the global bifurcation theorem of Rabinowitz ([11]) for operators (depending on the parameter μ) which are defined in open subsets of a Banach space.

Next, assuming that $f(\mu, x, s)$ is asymptotically linear at $s = 0$; i.e.,

$$\lim_{s \to 0^+} \frac{f(\mu, x, s)}{s} = \mu f'_+(x, 0) \in L^\infty(\Omega),$$

uniformly in (μ, x), with $f'_+(x, 0)$ nonnegative and not identically zero and denoting by $\mu_1(f'_+)$ the first eigenvalue of the eigenvalue problem

$$\begin{cases} -\Delta\varphi = \mu f'_+(x, 0)\varphi & \text{in } \Omega, \\ \varphi = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

we apply the cited extension theorem to obtain in Theorem 3.3 that the closure $\overline{\Sigma}$ of the nontrivial solution set of all $(\lambda,\mu,u)\in\mathbb{R}\times\mathbb{R}\times C_0^1(\overline{\Omega})$ such that $u\not\equiv 0$ solves (P_μ^λ); i.e.,

$$\overline{\Sigma}=\mathrm{cl}\{(\lambda,\mu,u)\in\mathbb{R}\times\mathbb{R}\times C_0^1(\overline{\Omega}) : u \text{ solves } (P_\mu^\lambda), u\not\equiv 0\}$$

intersects the plane $\pi_0=\{(\lambda,\mu,u)\in\mathbb{R}\times\mathbb{R}\times C_0^1(\overline{\Omega}) : u\equiv 0\}$ in the line

$$\overline{\Sigma}\cap\pi_0=r_0\equiv\{(\lambda,\mu_1(f'_+),0):\lambda\in\mathbb{R}\}.$$

Moreover, there exists a continuum $C\subset\overline{\Sigma}$ such that $r_0\subset C$ and, for every $\lambda\in\mathbb{R}$, $C\setminus\{(\lambda,\mu_1(f'_+),0)\}$ contains a connected subset $\{\lambda\}\times C_\lambda$ of nontrivial solutions.

Afterwards, for each fixed λ in $\mathbb{R}$ we study in Theorem 4.1 the laterality of the branch C_λ of positive solutions emanating from $(\mu_1(f'_+),0)$. Following [3, 4], the behavior of $f(\mu,x,s)$ when (μ,s) is close to $(\mu_1(f'_+),0)$ determines the laterality. Specifically, if we suppose the existence of the limit

$$\lim_{(\mu,s)\to(\mu_1(f'_+),0^+)}\frac{f(\mu,x,s)-\mu f'_+(x,0)s}{s^3}=G(x)\quad\text{uniformly in } x\in\overline{\Omega},$$

then the bifurcation of positive solutions from $(\mu_1(f'_+),0)$ is subcritical (i.e., to the left of $\mu=\mu_1(f'_+)$) provided that

- either $G(x)=+\infty$ a.e. $x\in\Omega$
- or $0<\int_\Omega G(x)\varphi_1(x)^4<\infty$, with φ_1 the positive eigenfunction of norm 1 for (1.4), and $\int_\Omega m(x)\varphi_1(x)^2[|\nabla\varphi_1(x)|^2-\mu_1(f'_+)f'_+(x,0)\varphi_1(x)^2]=0$.

A similar supercriticality laterality, i.e., to the right, is also shown when either $G(x)=-\infty$ a.e. $x\in\Omega$ or $\int_\Omega m(x)\varphi_1(x)^2[|\nabla\varphi_1(x)|^2-\mu_1(f'_+)\varphi_1(x)^2]=0$ and $-\infty<\int_\Omega G(x)\varphi_1(x)^4<0$. Roughly speaking, in these cases the laterality is the same that the one of the semilinear problem corresponding to the case $m(x)=0$ and does not depends on the value of λ.

The case $G\varphi_1^4\in L^1(\Omega)$, $\int_\Omega m(x)\varphi_1(x)^2[|\nabla\varphi_1(x)|^2-\mu_1(f'_+)f'_+(x,0)\varphi_1(x)^2]\neq 0$ is more delicate due to the key role played by the quasilinear term $\lambda m(x)u\Delta(u^2)$. Indeed, if

$$\lambda^*=-\frac{1}{2}\int_\Omega G(x)\varphi_1(x)^4\left[\int_\Omega m(x)\varphi_1(x)^2[|\nabla\varphi_1(x)|^2-\mu_1(f'_+)f'_+(x,0)\varphi_1(x)^2]\right]^{-1},\qquad(1.5)$$

and $\int_\Omega m(x)\varphi_1(x)^2[|\nabla\varphi_1(x)|^2-\mu_1(f'_+)f'_+(x,0)\varphi_1(x)^2]>0$, then the bifurcation of positive solution at $\mu_1(f'_+)$ is

- subcritical for $\lambda>\lambda^*$ and
- supercritical for $\lambda<\lambda^*$.

A similar result is also shown when the reverse inequality is satisfied, i.e. in the case $\int_\Omega m(x)\varphi_1(x)^2[|\nabla\varphi_1(x)|^2-\mu_1(f'_+)f'_+(x,0)\varphi_1(x)^2]<0$. We observe that λ^* may be positive, negative or zero.

For our knowledge this new phenomenon on the dependence of the laterality according to the value of λ is new and may cause significant changes in the existence of solutions of (P_λ^μ). As an example of the consequences of this phenomenon we study the simpler case $m(x)=1$ and $f(\mu,x,u)=\mu u-u^p$ with $p>1$ (logistic case). Specifically, we recover the previous results obtained by [8] proving uniqueness of positive solution when $\lambda>0$. In contrast, we handle

also the new case $\lambda < 0$ and prove the multiplicity of positive solutions when $\lambda < \lambda^* < 0$ and close to λ^*.

The outline of the paper is as follows. In Section 2, we study some properties of the open sets $\mathcal{U}_\lambda$ and extend the function $f(\mu,x,s)$ for $s < 0$ to ensure that every solution of the problem (P_μ^λ) is nonnegative. In Section 3, we prove the extension of the Rabinowitz's global bifurcation theorem for operators defined in open subsets and we apply it to show the existence of the continuum of positive solutions. In Section 4, we study the laterality of the branches of positive solutions obtained in Section 3. Finally, in Section 5, we discuss the logistic case.

2 Preliminaries

Let Ω be a bounded open domain with smooth boundary. Assume that $m(x)$ is a non negative bounded continuous function in $\overline{\Omega}$ and

(f_0) $\quad f\colon \mathbb{R}\times\overline{\Omega}\times[0,+\infty)\to\mathbb{R}$ is a continuous function such that

$$f(\mu,x,0)=0 \quad \text{for every } x\in\overline{\Omega} \text{ and } \mu\in\mathbb{R}.$$

Extending f to $\mathbb{R}\times\overline{\Omega}\times\mathbb{R}$ by taking

$$f(\mu,x,s)=f(\mu,x,0)=0, \quad \text{for every } \mu\in\mathbb{R},\ x\in\overline{\Omega} \text{ and } s<0, \tag{2.1}$$

we deduce that every weak solution of (P_μ^λ) is non negative.

Lemma 2.1. *Let f be a continuous function in $\mathbb{R}\times\overline{\Omega}\times[0,+\infty)$ satisfying (f_0) and extended by zero as (2.1). Let $u\in H_0^1(\Omega)$ be a solution of (P_μ^λ) for certain $\lambda,\mu\in\mathbb{R}$. Then, the following holds.*

1. *If $u\in\mathcal{U}_\lambda$, then $u(x)\geq 0$ for every $x\in\Omega$.*
2. *Moreover, if $u\in C^2(\Omega)\cap C(\overline{\Omega})$, $f(\mu,\cdot,\cdot)$ is C^1 in $\overline{\Omega}\times\mathbb{R}$ and u is a nonzero solution of (P_μ^λ) with $1+2\lambda m(x)u(x)^2>0$, $x\in\overline{\Omega}$, then*

$$u(x)>0, \quad \forall x\in\Omega \quad \text{and} \quad \frac{\partial u}{\partial n}(x)<0, \quad \forall x\in\partial\Omega, \tag{2.2}$$

where $\frac{\partial u}{\partial n}$ denotes the outward normal derivative of u at the point $x\in\partial\Omega$.

Proof. Let us prove part *(1)*. Observe first that (P_μ^λ) reduces to

$$\begin{cases} -\Delta u=\dfrac{f(\mu,x,u)+2\lambda m(x)u|\nabla u|^2}{1+2\lambda m(x)u^2} & \text{in } \Omega,\\ \quad u=0 & \text{on } \partial\Omega. \end{cases} \tag{2.3}$$

Since $u\in\mathcal{U}_\lambda$ we can take $\varphi=u^-=\min\{u,0\}\in H_0^1(\Omega)$ as a test function in (2.3) and using that $f(\mu,x,s)=0$ for every $\mu\in\mathbb{R}$, $x\in\Omega$ and $s\leq 0$ we have

$$\int_\Omega|\nabla u^-|^2=\int_\Omega\frac{2\lambda m(x)(u^-)^2}{1+2\lambda m(x)(u^-)^2}|\nabla u^-|^2=\int_\Omega|\nabla u^-|^2-\int_\Omega\frac{|\nabla u^-|^2}{1+2\lambda m(x)(u^-)^2}.$$

Then, we obtain that $u^-=0$; i.e., $u\geq 0$.

For the proof of the part *(2)*, since f is C^1 in $\overline{\Omega}\times\mathbb{R}$, there exists $C>0$ such that $f(\mu,x,s)+Cs$ is increasing in $s\in[0,\|u\|_\infty]$. Denoting $h(x)=f(\mu,x,u(x))+Cu(x)$, we have that $0\lneq h(x)\in C_0^1(\overline{\Omega})$.

Now, we distinguish cases depending on the sign of λ. If $\lambda\geq 0$ then u satisfies

$$\begin{cases} -\Delta u+\dfrac{C}{1+2\lambda m(x)u^2}u=\dfrac{h(x)+2\lambda m(x)u|\nabla u|^2}{1+2\lambda m(x)u^2}\gneq 0 & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega. \end{cases}$$

Since $\frac{C}{1+2\lambda m(x)u^2}$ is positive and bounded by the regularity of u and m, we can apply the strong maximum principle to conclude that u belongs to the interior of the cone of nonnegative solutions

$$P=\left\{u\in C_0^1(\overline{\Omega})\,:\,u \text{ is a nonnegative solution of (2.3)}\right\};$$

i.e., u satisfies (2.2).

In case $\lambda<0$, then u satisfies

$$\begin{cases} -\Delta u+\left(\dfrac{C}{1+2\lambda m(x)u^2}+\dfrac{2(-\lambda)m(x)|\nabla u|^2}{1+2\lambda m(x)u^2}\right)u=\dfrac{h(x)}{1+2\lambda m(x)u^2}\gneq 0 & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega. \end{cases}$$

Since $\frac{C}{1+2\lambda m(x)u^2}+\frac{2(-\lambda)m(x)|\nabla u|^2}{1+2\lambda m(x)u^2}$ is positive and bounded by the regularity of u and m, we can apply the strong maximum principle to conclude that u belongs to the interior of P. □

3 Continuum of positive solutions

Let $(X,\|\cdot\|)$ be a real Banach space, let $\mathcal{U}$ be an open set of X with $0\in\mathcal{U}$ and let $K\colon\mathbb{R}\times\mathcal{U}\to\mathbb{R}$ be a compact operator. For $\mu\in\mathbb{R}$, we consider the equation

$$\Phi(\mu,u)=u-K(\mu,u)=0,\quad u\in\mathcal{U}. \tag{3.1}$$

The operator K can be viewed as a family of compact operators $\{K_\mu:\mu\in\mathbb{R}\}$ by taking

$$K_\mu(u)=K(\mu,u),\quad u\in\mathcal{U}.$$

Similarly, we denote $\Phi_\mu=I-K_\mu$.

We also define

$$\Sigma=\{(\mu,u)\in\mathbb{R}\times\mathcal{U}\colon\Phi(\mu,u)=0,u\neq 0\},$$

and denote by Σ_μ the μ-slide of Σ, that is, $\Sigma_\mu=\{u\in\mathcal{U}\colon(\mu,u)\in\Sigma\}$.

Before showing the next result, let us recall that μ^* is a bifurcation point of (3.1) if there exists a sequence $(\mu_n,u_n)\in\mathbb{R}\times\mathcal{U}$ with $u_n\neq 0$, such that $\mu_n\to\mu^*$ and $\Phi(\mu_n,u_n)=0$.

The following theorem is a slight improvement of the global bifurcation theorem of Rabinowitz of [11] (see Theorem 1.3 and Corollary 1.12). In fact, with our assumptions it will be easy to arrive at the same point in which the proof of Theorem 6.3.1 of [1] is applicable.

Theorem 3.1. *Let $\mu^*\in\mathbb{R}$ and $\varepsilon_0>0$ be such that the set $(\mu^*-\varepsilon_0,\mu^*+\varepsilon_0)\backslash\{\mu^*\}$ does not contain any bifurcation point from (3.1). Assume that the Leray–Schauder topological index verifies*

$$i(\Phi_{\underline{\mu}},0)\neq i(\Phi_{\overline{\mu}},0),\quad \text{for some } \underline{\mu}\in(\mu^*-\varepsilon_0,\mu^*),\ \overline{\mu}\in(\mu^*,\mu^*+\varepsilon_0).$$

Then the connected component C of $\overline{\Sigma}$ that contains $(\mu^*,0)$ satisfies at least one of the following properties:

(a) $(\mathbb{R}\times\partial\mathcal{U})\cap C\neq\emptyset$.

(b) There exists $\mu^{\#}\neq\mu^{*}$ *such that* $(\mu^{\#},0)\in C$, *i.e.* $\mu^{\#}$ *is another bifurcation point from zero.*

(c) C *is unbounded.*

Proof. First, it is well known that a change of the index in μ^{*} implies that μ^{*} is a bifurcation point. Let C be the connected component of $\overline{\Sigma}$ that contains $(\mu^{*},0)$. Proceeding by contradiction, let us assume that C verifies neither (a) nor (b) nor (c). Denying property (a) implies that there is a neighborhood U_δ of C such that $(\mathbb{R}\times\partial\mathcal{U})\cap U_\delta=\emptyset$. Then, knowing that C is far from $\mathbb{R}\times\partial\mathcal{U}$ the proof follows same as Theorem 6.3.1 of [1]. □

For fixed $\lambda\in\mathbb{R}$, we define the set

$$\mathcal{U}_\lambda=\{u\in C_0^1(\overline{\Omega}):1+2\lambda m(x)u^2(x)\geq m_0, x\in\Omega,\ \text{for some } m_0>0\}.$$

The following result summarizes the main properties of this set.

Lemma 3.2. *For every* $\lambda\in\mathbb{R}$ *the set* $\mathcal{U}_\lambda$ *is open in* $C_0^1(\overline{\Omega})$.

Proof. Observe that $\mathcal{U}_\lambda=C_0^1(\overline{\Omega})$ for $\lambda\geq 0$. Assume now that $\lambda<0$, $u\in\mathcal{U}_\lambda$ and $\|u-v\|_1\leq r$. Then

$$|v(x)|-|u(x)|\leq||u(x)|-|v(x)||\leq\|u-v\|_\infty\leq r,\quad x\in\overline{\Omega},$$

and thus

$$1+2\lambda m(x)v^2(x)\geq 1+2\lambda m(x)(r+u^2(x))\geq m_0+2\lambda\|m\|_\infty r,$$

where $m_0>0$ is the minimum that attains the continuous function $1+2\lambda m(x)u^2(x)$ in $\overline{\Omega}$. Taking $r<\frac{m_0}{-2\lambda\|m\|_\infty}$ we have that $v\in\mathcal{U}_\lambda$ for every $v\in C_0^1(\overline{\Omega})$ with $\|u-v\|_1\leq r$. □

Now, we consider the Nemitski operator $N^\lambda\colon\mathbb{R}\times\mathcal{U}_\lambda\to C_0(\overline{\Omega})$ given by

$$N^\lambda(\mu,u)=\frac{f(\mu,x,u)+2\lambda m(x)u|\nabla u|^2}{1+2\lambda m(x)u^2}.$$

Since f and m are continuous and $u\in C_0^1(\overline{\Omega})$ we have that N^λ is a continuous operator. Also, we recall the compactness of the inverse of the Laplacian $(-\Delta)^{-1}\colon C_0(\overline{\Omega})\to C_0^1(\overline{\Omega})$.

Thus, the operator $K^\lambda\colon\mathbb{R}\times\mathcal{U}_\lambda\to C_0^1(\overline{\Omega})$ given by

$$K^\lambda(\mu,u)=(-\Delta)^{-1}\left[N^\lambda(\mu,u)\right],\quad \mu\in\mathbb{R},\ u\in\mathcal{U}_\lambda,$$

is compact.

We define $\Phi_\mu^\lambda\colon\mathcal{U}_\lambda\to C_0^1(\overline{\Omega})$ given by

$$\Phi_\mu^\lambda(u)=u-K^\lambda(\mu,u),\quad\forall u\in\mathcal{U}_\lambda.$$

In this way, the solutions of (P_μ^λ) are zeros of Φ_μ^λ.

Finally, we denote by $\mu_1(f'_+)$ the first eigenvalue and φ_1 the positive eigenfunction of norm 1 of the eigenvalue problem (1.4).

Theorem 3.3. *Let* f *be a continuous function in* $\mathbb{R}\times\overline{\Omega}\times[0,+\infty)$ *satisfying* (f_0), *and*

(f_1) *For each bounded set* Γ *in* $\mathbb{R}\setminus\{0\}$ *and* $\mu\in\Gamma$, *we have*

$$\lim_{s\to 0^+}\frac{f(\mu,x,s)}{s}=\mu f'_+(x,0),\quad \text{uniformly in } (\mu,x)\in\Gamma\times\Omega,$$

with $0\le f'_+(x,0)\in L^\infty(\Omega)$ *not identically zero.*

Then, for every $\lambda\in\mathbb{R}$, $\mu_1(f'_+)$ *is a bifurcation point from zero of the problem* (P^λ_μ). *Moreover, there exists a continuum* C *of positive solutions, i.e., a closed and connected subset of the closure* $\overline{\Sigma}$ *of the set*

$$\Sigma=\{(\lambda,\mu,u)\in\mathbb{R}\times\mathbb{R}\times C^1_0(\overline{\Omega})\colon\ u\in\mathcal{U}_\lambda,\ \Phi^\lambda_\mu(u)=0,\ u\neq 0\}$$

with $(\lambda,\mu_1(f'_+),0)\in C$ *for every* $\lambda\in\mathbb{R}$.

Remark 3.4. In addition, the continuum C provided by Theorem 3.3 satisfies that the slice

$$C_\lambda=\{(\mu,u)\in\mathbb{R}\times C^1_0(\overline{\Omega})\ :\ (\lambda,\mu,u)\in C\}$$

satisfies at least one of the following conditions:

- C_λ is unbounded (which is always the case for $\lambda\ge 0$).
- $C_\lambda\cap\mathbb{R}\times\partial\mathcal{U}_\lambda\neq\emptyset$.

Proof. We divide the proof into the following steps.

- Step 1. For every compact set $\Gamma\subset\mathbb{R}^2\setminus\{(\lambda,\mu_1(f'_+))\colon\lambda\in\mathbb{R}\}$ there exists $\varepsilon>0$ such that

$$\Phi^\lambda_\mu(u)\neq 0,\quad \forall(\lambda,\mu)\in\Gamma,\ \forall u\in\mathcal{U}_\lambda \text{ with } 0<\|u\|_1<\varepsilon,$$

  where $\|u\|_1$ denotes the norm of u in $C^1(\Omega)$.

- Step 2. If $\mu<\mu_1(f'_+)$, it is satisfied

$$i(\Phi^\lambda_\mu,0)=1,\quad\forall\lambda\in\mathbb{R}.$$

- Step 3. If $\mu>\mu_1(f'_+)$, it is satisfied

$$i(\Phi^\lambda_\mu,0)=0,\quad\forall\lambda\in\mathbb{R}.$$

To prove Step 1, we proceed by contradiction. Let Γ be a compact subset of $\mathbb{R}^2\setminus\{(\lambda,\mu_1(f'_+))\ :\ \lambda\in\mathbb{R}\}$ and let us assume that there exists a sequence (λ_n,μ_n,u_n) with $(\lambda_n,\mu_n)\in\Gamma$ and $u_n\in\mathcal{U}_{\lambda_n}$ satisfying

$$\lambda_n\to\lambda,\quad \mu_n\to\mu\neq\mu_1(f'_+),\quad \|u_n\|_1\to 0,$$

$$\Phi^{\lambda_n}_{\mu_n}(u_n)=0,\quad u_n\ge 0.$$

Since $\Phi^{\lambda_n}_{\mu_n}(u_n)=0$, we have that

$$u_n=(-\Delta)^{-1}\left[\frac{f(\mu_n,x,u_n)+2\lambda_n m(x)u_n|\nabla u_n|^2}{1+2\lambda_n m(x)u_n^2}\right].$$

Dividing by $\|u_n\|_1$ we come to the fact that

$$\frac{u_n}{\|u_n\|_1} = (-\Delta)^{-1}\left[\frac{f(\mu_n, x, u_n) + 2\lambda_n m(x) u_n |\nabla u_n|^2}{\|u_n\|_1 (1 + 2\lambda_n m(x) u_n^2)}\right]. \tag{3.2}$$

Since $\|u_n\|_1 \to 0$, we have that $u_n \to 0$. Then, there exists $n_0 > 0$ such that

$$\frac{1}{2} < 1 + 2\lambda_n m(x) u_n^2, \quad \forall n \geq n_0.$$

Thus, we conclude

$$\frac{|f(\mu_n, x, u_n) + 2\lambda_n m(x) u_n |\nabla u_n|^2|}{\|u_n\|_1 (1 + 2\lambda_n m(x) u_n^2)} < 2\left(\frac{|f(\mu_n, x, u_n)|}{\|u_n\|_1} + \frac{|2\lambda_n m(x) u_n |\nabla u_n|^2|}{\|u_n\|_1}\right)$$

and each term of the right hand side is bounded.

- Using (f_1) and that $f'_+(x,0) \in L^\infty(\Omega)$, $\frac{f(\mu_n, x, u_n)}{\|u_n\|_1}$ is bounded.
- Since λ_n is convergent, m is bounded, $|\nabla u_n|^2$ converges to zero and $\frac{u_n(x)}{\|u_n\|_1}$ is bounded by 1, the term $\frac{2\lambda_n m(x) u_n |\nabla u_n|^2}{\|u_n\|_1}$ is bounded.

Then, since the operator $(-\Delta)^{-1}$ is compact, up to a subsequence denoted the same way, $\frac{u_n}{\|u_n\|_1}$ converges strongly to a non negative function $\varphi \in C_0^1(\overline{\Omega})$. Taking limit in (3.2), we arrive to

$$\varphi = (-\Delta)^{-1}[\mu f'_+(x,0)\varphi].$$

So, φ is a positive solution with norm 1 of the problem (1.4). Therefore $\varphi = \varphi_1$ and $\mu = \mu_1(f'_+)$, which is a contradiction.

As a consequence of step 1, we have that for every $\lambda \in \mathbb{R}$ the unique possible bifurcation point from zero of Φ_μ^λ is $\mu_1(f'_+)$.

Moreover, further consequences of step 1 are

$$i(\Phi_\mu^\lambda, 0) = c_1, \quad \forall \mu < \mu_1(f'_+), \quad \forall \lambda \in \mathbb{R}$$

and

$$i(\Phi_\mu^\lambda, 0) = c_2, \quad \forall \mu > \mu_1(f'_+), \quad \forall \lambda \in \mathbb{R},$$

where c_1 and c_2 are constants.

To prove step 2, for every $t \in [0,1]$ we claim that, for each $t \in [0,1]$, there exists $\varepsilon > 0$ satisfying

$$u - (-\Delta)^{-1}[tf(0,x,u)] \neq 0, \quad \forall 0 < \|u\|_1 < \varepsilon.$$

Arguing by contradiction, let us suppose that there exists a sequence $u_n \in C_0^1(\overline{\Omega})$ such that

$$\|u_n\|_1 \to 0, \quad u_n \geq 0$$

and

$$u_n - t(-\Delta)^{-1}[f(0,x,u_n)] = 0. \tag{3.3}$$

Dividing (3.3) by $\|u_n\|_1$ we have that

$$\frac{u_n}{\|u_n\|_1} = t(-\Delta)^{-1}\left[\frac{f(0,x,u_n)}{\|u_n\|_1}\right]$$

Now, using (f_1) as before, we have that, up to a subsequence noted the same way, $\frac{u_n}{\|u_n\|_1}$ converges strongly to a non negative function $v \in C_0^1(\overline{\Omega})$ with norm 1. Taking limits in (3.3), we conclude that

$$v = t(-\Delta)^{-1}[0],$$

but, since $(-\Delta)^{-1}[0] = 0$, then $v = 0$ and we have a contradiction.

As a consequence of this fact, defining $\Psi_t(u) = u - (-\Delta)^{-1}[tf(0,x,u)]$, we have proved that

$$i(\Psi_t, 0) = \text{constant}, \quad \forall t \in [0,1].$$

Using that $\Psi_1 = \Phi_0^0$ and $\Psi_0 = \mathrm{Id}$, we deduce that

$$i(\Phi_\mu^\lambda, 0) = i(\Phi_0^0, 0) = i(\mathrm{Id}, 0) = 1, \quad \forall \mu < \mu_1(f'_+),\ \forall \lambda \in \mathbb{R}.$$

Finally we prove step 3. To do so, we will prove that for fixed $\lambda \in \mathbb{R}$ and $\mu > \mu_1(f'_+)$, there exists $\varepsilon > 0$ such that

$$\Phi_\mu^\lambda(u) \neq \tau\varphi_1, \quad \forall \tau > 0,\ \forall 0 < \|u\|_1 < \varepsilon.$$

Let us assume by contradiction that there exists a sequence $u_n \in C_0^1(\overline{\Omega})$ and $\tau_n \geq 0$ satisfying $u_n > 0$ in Ω, $\|u_n\|_1 \to 0$ and

$$\Phi_\mu^\lambda(u_n) = \tau_n\varphi_1.$$

Equivalently,

$$u_n = (-\Delta)^{-1}\left[\frac{f(\mu,x,u_n) + 2\lambda m(x)u_n|\nabla u_n|^2}{1 + 2\lambda m(x)u_n^2}\right] + \tau_n\varphi_1.$$

Dividing by $\|u_n\|_1$ and using the compactness of $(-\Delta)^{-1}$ we have that

$$(-\Delta)^{-1}\left[\frac{f(\mu,x,u_n) + 2\lambda m(x)u_n|\nabla u_n|^2}{\|u_n\|_1(1 + 2\lambda m(x)u_n^2)}\right]$$

is convergent. Adding that $\frac{u_n}{\|u_n\|_1}$ is bounded by 1, the sequence $\frac{\tau_n}{\|u_n\|_1}$ is bounded. Thus, taking another subsequence if it's necessary, we have that $\frac{\tau_n}{\|u_n\|_1} \to \tau \geq 0$ and $\frac{u_n}{\|u_n\|_1} \to v$ with $v \in C_0^1(\overline{\Omega})$ satisfying

$$\begin{cases} -\Delta v = \mu f'_+(x,0)v + \tau\mu_1(f'_+)f'_+(x,0)\varphi_1 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \\ \|v\|_1 = 1. \end{cases}$$

Using the weak formulation and taking φ_1 as test function, we have that

$$\int_\Omega \nabla\varphi_1\nabla v = \int_\Omega \mu f'_+(x,0)v\varphi_1 + \tau\mu_1(f'_+)f'_+(x,0)\varphi_1^2.$$

Now, using the definition of φ_1 we have that

$$\int_\Omega \mu_1(f'_+)f'_+(x,0)v\varphi_1 = \int_\Omega \mu f'_+(x,0)v\varphi_1 + \tau\mu_1(f'_+)f'_+(x,0)\varphi_1^2.$$

Equivalently, we arrive to

$$\int_\Omega (\mu_1(f'_+) - \mu)f'_+(x,0)v\varphi_1 = \int_\Omega \tau\mu_1(f'_+)f'_+(x,0)\varphi_1^2,$$

where the term in the right hand side is non-positive and the corresponding one in the left hand side is non-negative. So, each term is zero and conclude that $\tau = 0$ and $v = 0$, giving a contradiction.

Therefore, we have that

$$i(\Phi_\mu^\lambda, 0) = i(\Phi_\mu^\lambda - \tau\varphi_1, 0), \quad \forall \tau > 0.$$

However, the problem

$$\begin{cases} -\Delta w = \mu f'_+(x,0)w + \tau\mu_1(f'_+)f'_+(x,0)\varphi_1 & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases}$$

has no trivial solution and neither solutions with small norm. Hence,

$$i(\Phi_\mu^\lambda, 0) = i(\Phi_\mu^\lambda - \tau\varphi_1, 0) = 0.$$

Fixed λ^* in $\mathbb{R}$, there is a change of index in $\mu_1(f'_+)$, generating a continuum of solutions emanating from the point $(\lambda^*, \mu_1(f'_+), 0)$ and that it is contained in the plane $\lambda = \lambda^*$. Because of the arbitrariness of λ^*, a continuum of solutions emanates from each point on the line $\{(\lambda, \mu, u) : \mu = \mu_1(f'_+), u = 0\}$. So, since every branch is connected by the straight line $\{(\lambda, \mu, u) : \mu = \mu_1(f'_+), u = 0\}$ a general continuum is generated. □

Remark 3.5. For the case $f'_+(x,0) = +\infty$ Theorem 3.3 also holds considering $\mu_1(f'_+) = 0$, see Remark 4.2 of [3].

4 Laterality

For a fixed $\lambda \in \mathbb{R}$, we have proved in Theorem 3.3, see also Remark 3.4, that a continuum $C_\lambda = \{(\mu, u) : (\lambda, \mu, u) \in C\}$ of positive solutions emanates from $(\mu_1(f'_+), 0)$. In this section, we study the behavior of this continuum. For λ fixed, let us recall that:

- the bifurcation at zero is subcritical if, when taking a sequence $\{(\mu_n, u_n)\}$ with $\mu_n \to \mu_1(f'_+)$, $u_n \to 0$ in $C_0^1(\overline{\Omega})$ and with $\Phi_{\mu_n}^\lambda(u_n) = 0$, we have that $\mu_n \leq \mu_1(f'_+)$ for n large enough.
- the bifurcation at zero is supercritical if, when taking a sequence $\{(\mu_n, u_n)\}$ with $\mu_n \to \mu_1(f'_+)$, $u_n \to 0$ in $C_0^1(\overline{\Omega})$ and with $\Phi_{\mu_n}^\lambda(u_n) = 0$, we have that $\mu_n \geq \mu_1(f'_+)$ for n large enough.

While the parameter λ did not play any role in the existence of positive solutions found in Theorem 3.3, we see here that the bifurcation is subcritical or supercritical depending on the value and sign of λ.

Theorem 4.1. *Let f be a continuous function in $\mathbb{R} \times \overline{\Omega} \times [0, +\infty)$ satisfying (f_0), (f_1) and*

$$\lim_{(\mu,s)\to(\mu_1(f'_+),0^+)} \frac{f(\mu,x,s) - \mu f'_+(x,0)s}{s^3} = G(x) \quad \text{uniformly in } x \in \overline{\Omega}. \tag{f_2}$$

1. *If $G(x) = +\infty$ a.e. $x \in \Omega$, then for every $\lambda \in \mathbb{R}$ the bifurcation of positive solutions is subcritical. Respectively, if $G(x) = -\infty$ a.e. $x \in \Omega$, then for every $\lambda \in \mathbb{R}$ the bifurcation of positive solutions is supercritical.*

2. *If* $G\varphi_1^4 \in L^1(\Omega)$ *and* $\int_\Omega m(x)\varphi_1(x)^2[|\nabla\varphi_1(x)|^2 - \mu_1(f'_+)f'_+(x,0)\varphi_1(x)^2] = 0$, *then:*
   (a) *If* $\int_\Omega G(x)\varphi_1(x)^4 > 0$, *then for every* $\lambda \in \mathbb{R}$ *the bifurcation of positive solutions is subcritical.*
   (b) *If* $\int_\Omega G(x)\varphi_1(x)^4 < 0$, *then for every* $\lambda \in \mathbb{R}$ *the bifurcation of positive solutions is supercritical.*
3. *If* $G\varphi_1^4 \in L^1(\Omega)$ *and* $\int_\Omega m(x)\varphi_1(x)^2[|\nabla\varphi_1(x)|^2 - \mu_1(f'_+)f'_+(x,0)\varphi_1(x)^2] \neq 0$, *then taking* λ^* *defined as* (1.5) *we have that:*
   (a) *If* $\int_\Omega m(x)\varphi_1(x)^2[|\nabla\varphi_1(x)|^2 - \mu_1(f'_+)f'_+(x,0)\varphi_1(x)^2] > 0$, *then the bifurcation of positive solutions is subcritical for* $\lambda > \lambda^*$ *and supercritical for* $\lambda < \lambda^*$.
   (b) *If* $\int_\Omega m(x)\varphi_1(x)^2[|\nabla\varphi_1(x)|^2 - \mu_1(f'_+)f'_+(x,0)\varphi_1(x)^2] < 0$, *then the bifurcation of positive solutions is supercritical for* $\lambda > \lambda^*$ *and subcritical for* $\lambda < \lambda^*$.

Proof. For fixed $\lambda \in \mathbb{R}$, using (f_0), (f_1) and (f_2) we can apply Theorem 3.3 to deduce that $\mu_1(f'_+)$ is a bifurcation point of positive solutions from zero of the problem (P_μ^λ). Thus, we can take a sequence (μ_n, u_n) verifying

$$u_n \neq 0, \quad \mu_n \to \mu_1(f'_+), \quad \|u_n\|_1 \to 0, \quad \Phi^\lambda_{\mu_n}(u_n) = 0.$$

Since u_n solves (2.3) with $\mu = \mu_n$, multiplying by φ_1 and integrating by parts, we have that

$$\int_\Omega \nabla u_n \nabla\varphi_1 = \int_\Omega \frac{f(\mu_n, x, u_n)}{1+2\lambda m(x)u_n^2}\varphi_1 + \int_\Omega \frac{2\lambda m(x)u_n|\nabla u_n|^2}{1+2\lambda m(x)u_n^2}\varphi_1. \tag{4.1}$$

Observe that, using u_n as test function in (1.4), we obtain that

$$\int_\Omega \nabla u_n \nabla\varphi_1 = \int_\Omega \mu_1(f'_+)f'_+(x,0)u_n\varphi_1.$$

Thus, by (4.1), we deduce

$$\begin{aligned}(\mu_1(f'_+) - \mu_n)\int_\Omega f'_+(x,0)u_n\varphi_1 &= \int_\Omega \nabla u_n\nabla\varphi_1 - \int_\Omega \mu_n f'_+(x,0)u_n\varphi_1 \\ &= \int_\Omega \frac{(f(\mu_n,x,u_n) - \mu_n f'_+(x,0)u_n)\varphi_1 + 2\lambda m(x)u_n\varphi_1[|\nabla u_n|^2 - \mu_n f'_+(x,0)u_n^2]}{1+2\lambda m(x)u_n^2}.\end{aligned}$$

If we multiply by $\|u_n\|_1^{-3}$ and denote $z_n = u_n/\|u_n\|_1$, we have that

$$\begin{aligned}(\mu_1(f'_+) - \mu_n)\int_\Omega \frac{f'_+(x,0)\varphi_1 z_n}{\|u_n\|_1^2} &= \int_\Omega \frac{(f(\mu_n,x,u_n) - \mu_n f'_+(x,0)u_n)\varphi_1 z_n^3}{u_n^3(1+2\lambda m(x)u_n^2)} \\ &\quad + \int_\Omega \frac{2\lambda m(x)\varphi_1 z_n}{1+2\lambda m(x)u_n^2}\left[|\nabla z_n|^2 - \mu_n f'_+(x,0)z_n^2\right].\end{aligned} \tag{4.2}$$

Now, taking into account that the sign of the left hand side coincides with the sign of $(\mu_1(f'_+) - \mu_n)$, the sign of the right hand side of (4.2) determines if the bifurcation is subcritical or supercritical.

In order to prove *(1)*, using that z_n converges to φ_1 and the dominated convergence theorem we obtain that the second term of the right hand side of (4.2) converges to

$$2\lambda\int_\Omega m(x)\varphi_1(x)^2[|\nabla\varphi_1(x)|^2 - \mu_1(f'_+)f'_+(x,0)\varphi_1(x)^2].$$

Applying Fatou's lemma to the first term of the right hand side, we deduce that

$$+\infty = \int_\Omega G(x)\varphi_1^4 \le \liminf_{(\mu,s)\to(\mu_1(f'_+),0)} \int_\Omega \frac{(f(\mu_n,x,u_n)-\mu_n f'_+(x,0)u_n)\varphi_1 z_n^3}{u_n^3(1+2\lambda m(x)u_n^2)}.$$

Thus, the right hand side diverges to $+\infty$ (analogously diverges to $-\infty$ when $G(x) = -\infty$) and for every $\lambda \in \mathbb{R}$ the bifurcation is subcritical (respectively supercritical).

Now we assume that $G\varphi_1^4 \in L^1(\Omega)$. In this case, using the dominated convergence theorem and that z_n converges to φ_1, the right hand side of (4.2) converges to

$$\int_\Omega G(x)\varphi_1^4 + 2\lambda \int_\Omega m(x)\varphi_1^2 \left[|\nabla\varphi_1|^2 - \mu_1(f'_+)f'_+(x,0)\varphi_1^2\right].$$

Therefore, the proof of item *(2)* follows since, in that case the sign of the right hand side of (4.2) coincides with that of $\int_\Omega G(x)\varphi_1(x)^4$ for every $\lambda \in \mathbb{R}$.

Finally, in the case of item *(3)*, we observe that

$$\begin{aligned}\int_\Omega G(x)\varphi_1^4 + 2\lambda \int_\Omega m(x)\varphi_1^2 &\left[|\nabla\varphi_1|^2 - \mu_1(f'_+)f'_+(x,0)\varphi_1^2\right] \\ &= 2(\lambda-\lambda^*)\int_\Omega m(x)\varphi_1^2\left[|\nabla\varphi_1|^2 - \mu_1(f'_+)f'_+(x,0)\varphi_1^2\right].\end{aligned}$$

Therefore, in the case of *(a)*, the right hand side of (4.2) is positive (respectively negative) and the bifurcation is subcritical (respectively supercritical) if, $\lambda > \lambda^*$ (respectively $\lambda < \lambda^*$). Analogously we conclude in the case of *(b)*. □

5 Multiplicity of solutions for the logistic datum

In this section, for $p > 1$, we study the problem (P_μ^λ) with $f(\mu,x,u) = \mu u - u^p$ and $m(x) = 1$; i.e.,

$$\begin{cases} -\Delta u - \lambda u\Delta(u^2) = \mu u - u^p & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.1}$$

Condition (f_2) becomes

$$\lim_{s\to 0^+} \frac{\mu s - s^p}{s} = \mu,$$

and $f'_+(x,0) = 1$. Thus, in this case (1.4) becomes

$$\begin{cases} -\Delta\varphi = \mu\varphi & \text{in } \Omega, \\ \varphi = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.2}$$

First, we are able to apply Theorem 3.3, giving the existence of a continuum of positive solutions. For $\lambda > 0$, we can make the change of variables $u = g(v)$ with g given by (1.3). Thus, the problem turn into the semilinear equation

$$\begin{cases} -\Delta v = \mu g(v)g'(v) - g(v)^p g'(v) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

The datum satisfies that

$$\frac{\mu g(v)g'(v) - g(v)^p g'(v)}{v} \quad \text{is decreasing for } v \in (0,+\infty).$$

Then, by the classical result of Brezis and Oswald ([5]), the problem has a unique positive solution. Now, let us see that for the case $\lambda < 0$ we can find a range of values in which we have at least two solutions. To do this, observe that the condition (f_2) for the problem (5.1) becomes

$$\lim_{(\mu,u)\to(\mu_1,0)} \frac{\mu u - u^p - \mu u}{u^3} = \lim_{(\mu,u)\to(\mu_1,0)} -u^{p-3} = \begin{cases} -\infty & \text{if } p \in (1,3), \\ -1 & \text{if } p = 3, \\ 0 & \text{if } p \in (3,+\infty). \end{cases}$$

Consequently, (1.5) becomes

$$\lambda^* = \frac{\displaystyle\int_\Omega \varphi_1^4}{\displaystyle\int_\Omega 2\varphi_1^2[|\nabla\varphi_1(x)|^2 - \mu_1\varphi_1(x)^2]}.$$

if $p = 3$ and $\lambda^* = 0$ if $p > 3$.

Moreover, taking φ_1^3 as test function in the weak formulation of (5.2) we have that

$$\int_\Omega 3\varphi_1^2|\nabla\varphi_1|^2 = \int_\Omega \mu_1\varphi_1^4,$$

which implies that

$$\int_\Omega 2\varphi_1^2[|\nabla\varphi_1(x)|^2 - \mu_1\varphi_1(x)^2] < 0.$$

Therefore, $\lambda^* < 0$.

Corollary 5.1. *For a fixed $\lambda \in \mathbb{R}$, the bifurcation of positive solutions occurs according to the following cases:*

1. *If $p \in (1,3)$ then the bifurcation is supercritical for every $\lambda \in \mathbb{R}$.*
2. *If $p \geq 3$ then the bifurcation is subcritical if $\lambda < \lambda^*$ and supercritical if $\lambda > \lambda^*$.*

Now, we see that, in the case $p > 3$, there is a range of values of λ for which we can obtain two solutions. To do this, we first need the following result about a priori bounds on the solution.

Lemma 5.2. *Let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ be a positive solution of (5.1) with $\lambda < 0$. Then*

1. *if $1 + 2\lambda\|u\|_\infty^2 > 0$, then $\|u\|_\infty \leq \mu^{\frac{1}{p-1}}$.*
2. *if $\mu > \mu_1$ and $\lambda < 0$, then $\mu - \mu_1 \leq \|u\|_\infty^{p-1}$.*

Proof. First, since u is a solution of (P_μ^λ), we have that

$$\begin{aligned}(1 + 2\lambda u(x)^2)(-\Delta u(x)) &= \mu u(x) - u(x)^p + 2\lambda u(x)|\nabla u(x)|^2 \\ &= u(x)(\mu - u(x)^{p-1}) + 2\lambda u(x)|\nabla u(x)|^2, \quad \forall x \in \Omega.\end{aligned}$$

However, taking x_0 the point in which u attains the maximum, we obtain that

$$(1 + 2\lambda\|u\|_\infty^2)(-\Delta u(x_0)) = \|u\|_\infty(\mu - \|u\|_\infty^{p-1}).$$

with $\Delta u(x_0) \leq 0$. Hence, $\|u\|_\infty \leq \mu^{\frac{1}{p-1}}$ and *(1)* is proved.

To prove (2), we check that

$$-\mathrm{div}[(1+2\lambda u^2)\nabla u] + 2\lambda u|\nabla u|^2 = -2\lambda u|\nabla u|^2 + (1+2\lambda u^2)(-\Delta u) = \mu u - u^p.$$

Thus $-\mathrm{div}[(1+2\lambda u^2)\nabla u] + 2\lambda u|\nabla u|^2 + u^p = \mu u$ and, defining the operator

$$-\mathrm{div}[(1+2\lambda u^2)\nabla(\cdot)] + 2\lambda|\nabla u|^2 + u^{p-1},$$

by the positivity of u, we have that

$$\sigma_1(-\mathrm{div}[(1+2\lambda u^2)\nabla(\cdot)] + 2\lambda|\nabla u|^2 + u^{p-1}) = \mu,$$

where $\sigma_1(\cdot)$ denotes the first eigenvalue of the operator under zero Dirichlet boundary conditions. Using the monotonicity of the first eigenvalue and that $\lambda \leq 0$, we have that

$$\mu \leq \sigma_1\left[-\mathrm{div}[\nabla(\cdot)] + u^{p-1}\right] \leq \sigma_1\left[-\mathrm{div}[\nabla(\cdot)] + \|u\|_\infty^{p-1}\right] = \mu_1 + \|u\|_\infty^{p-1}.$$

Finally, we conclude that

$$\|u\|_\infty^{p-1} \geq \mu - \mu_1. \qquad \square$$

Theorem 5.3. *If $\lambda \in \left(\frac{-1}{2}\mu_1^{2/(1-p)}, 0\right)$ and $p > 3$, then there is $\mu < \mu_1$ such that the problem (5.1) has at least two positive solutions.*

Proof. Let $\lambda \in \left(\frac{-1}{2}\mu_1^{2/(1-p)}, 0\right)$ be fixed, Theorem 3.3 implies that μ_1 is a bifurcation point from zero of positive solutions. Then, there exist a continuum C_λ that emanates from $(\mu_1, 0)$, subset of $\overline{\Sigma}_\lambda$ where

$$\Sigma_\lambda = \{(\mu, u) \in \mathbb{R} \times C_0^1(\overline{\Omega}) \,:\, u \in \mathcal{U}_\lambda, \Phi_\mu^\lambda(u) = 0, u \neq 0\}.$$

By Theorem 3.1 there are three possibilities of behavior (not incompatibles) for C_λ. However, by the Step 1 of Theorem 3.3 there is no other bifurcation point from zero. Also, the condition $\lambda \in \left(\frac{-1}{2}\mu_1^{2/(1-p)}, 0\right)$ implies that $\mu_1^{\frac{1}{p-1}} < \frac{1}{\sqrt{-2\lambda}}$ and combining with Lemma (5.2) we deduce that

$$\|u\|_\infty \leq \mu^{\frac{1}{p-1}}, \quad \|u\|_\infty < \frac{1}{\sqrt{-2\lambda}}, \quad \forall (\mu, u) \in C_\lambda. \tag{5.3}$$

$$(\mu - \mu_1)^{\frac{1}{p-1}} \leq \|u\|_\infty, \quad \forall (\mu, u) \in C_\lambda \quad \text{with } \mu > \mu_1. \tag{5.4}$$

Combining the two inequalities of (5.3), see Figure 5.1, and that $(s - \mu_1)^{\frac{1}{p-1}}$ diverges positively when s tends to $+\infty$, we have that there is no solution for $\mu \geq \mu_1 + (-2\lambda)^{\frac{1-p}{2}}$. Thus, we have that C_λ is bounded and $\overline{C}_\lambda \cap (\mathbb{R} \times \partial\mathcal{U}_\lambda) \neq \emptyset$. Again, by (5.3) the points $(\mu, u) \in \overline{C}_\lambda \cap (\mathbb{R} \times \partial\mathcal{U}_\lambda)$ verify that

$$(-2\lambda)^{\frac{1-p}{2}} \leq \mu \leq \mu_1 + (-2\lambda)^{\frac{1-p}{2}}, \quad \forall (\mu, u) \in \overline{C}_\lambda \cap (\mathbb{R} \times \partial\mathcal{U}_\lambda).$$

Now, notice that condition $\lambda \in \left(\frac{-1}{2}\mu_1^{2/(1-p)}, 0\right)$ implies that $\mu_1 < (-2\lambda)^{\frac{1-p}{2}}$. Therefore, we have proved that there are positive solutions with $\mu > \mu_1$.

Nevertheless, since $p > 3$ and $\lambda < 0$, Corollary 5.1 implies that the bifurcation is subcritical. Then, the continuum C_λ emanates from zero at μ_1 with solutions with $\mu < \mu_1$ and turn around connecting with solutions with $\mu > \mu_1$. Consequently, there is at least one value of $\mu < \mu_1$ such that the problem (5.1) has two positive solutions.

$\square$

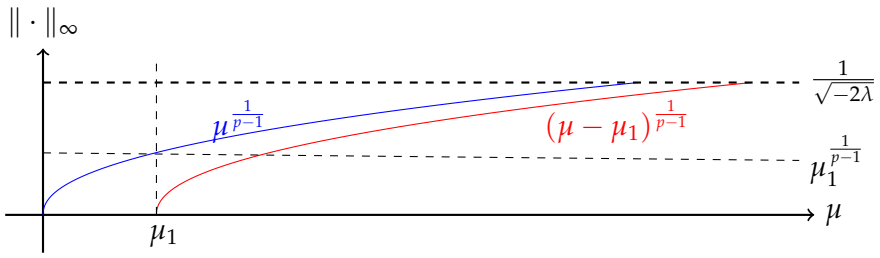


Figure 5.1: Bounds (5.3) of the continuum C_λ.

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Study of singular elliptic equations with mixed boundary conditions and nonlocal source terms

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Abstract. In this article, we study the generalized solutions for singular elliptic equations with mixed boundary conditions, a nonlocal source term and a Hardy potential. More precisely, we use the variational methods to establish the existence of at least three generalized solutions for this problem.

Keywords: double-phase Laplacian operator, Hardy potential, mixed boundary condition, variable exponent space.

2020 Mathematics Subject Classification: 35J15, 35J20, 35J25.

1 Introduction

The study of the singular elliptic equations involving double-phase $(p_1(x), p_2(x))$-Laplacian operators with Hardy potential in variable exponent spaces has attracted the interest of many authors in different fields such as fluid dynamics, structural design and electrical systems [19]. In the recent years, several papers demonstrate the existence and multiplicity of solutions of this type of singular elliptic problems, for more details, see [2,4,5,7,14–18]. Recently Jian Liu et al. [16] established the existence of one weak solution, respectively two weak solutions to the following singular elliptic problem with mixed boundary conditions in variable exponent spaces

$$\begin{cases} -\Delta_{p(x)}u - \Delta_{q(x)}u + \frac{a(x)|u|^{h-2}u}{|x|^h} = \lambda f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_1, \\ \left(|\nabla u|^{p(x)-2}\nabla u + |\nabla u|^{q(x)-2}\nabla u\right)\cdot \nu + c(x)|u|^{t(x)-2}u = 0 & \text{on } \Gamma_2, \end{cases}$$

where Ω is a bounded subset in $\mathbb{R}^N (N \geq 2)$ with smooth boundary $\partial\Omega, \nu$ is the outward normal vector field on $\partial\Omega, \Gamma_1$ and Γ_2 are two smooth $(N-1)$-dimensional submanifolds of $\partial\Omega$ such that $\Gamma_1 \cap \Gamma_2 = \emptyset, \overline{\Gamma_1} \cup \overline{\Gamma_2} = \partial\Omega, \overline{\Gamma_1} \cap \overline{\Gamma_2}$ is a $(N-2)$-dimensional submanifold

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of $\partial\Omega, \lambda > 0, 0 < a(x) \in L^{\infty}(\Omega), \Delta_{p(x)}u = \operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right)$ is $p(x)$-Laplacian operator, $\Delta_{q(x)}u = \operatorname{div}\left(|\nabla u|^{q(x)-2}\nabla u\right)$ is $q(x)$-Laplacian operator, $p(x), q(x) \in C(\bar{\Omega})$ with $1 < q(x) < p(x) < p^*(x)$, where

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)}, & p(x) < N \\ +\infty, & p(x) \geq N \end{cases}$$

$1 < h < p(x)$, $t(x) \in C\left(\overline{\Gamma_2}\right)$ with $1 < t(x) < p_*(x)$, where

$$p_*(x) = \begin{cases} \frac{(N-1)p(x)}{N-p(x)}, & p(x) < N \\ +\infty, & p(x) \geq N \end{cases}$$

and $0 < c(x) \in L^{e(x)}(\Gamma_2)$, with $e(x) > \frac{p_*(x)t(x)}{p_*(x)-t(x)}$. The nonlinearity $f(x,u)$ is assumed to be a Carathéodory function such that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ which satisfy the following growth condition

$$|f(x,u)| \leq M_1(x) + M_2|u|^{s(x)-1}, \quad \text{a.e. } (x,u) \in \Omega \times \mathbb{R}$$

where $M_1(x) > 0$, $M_1(x) \in L^{\frac{s(x)}{s(x)-1}}(\Omega)$, $M_2 > 0$, $s(x) \in C_+(\bar{\Omega}) := \{\xi : \xi \in C(\overline{\Omega}), \xi(x) > 1, \text{ for all } x \in \overline{\Omega}\}$ with $s(x) < p^*(x)$. Moreover an Ambrosetti–Rabinowitz condition was imposed to guaranty the existence of two weak solutions when λ is in an appropriate range (See Theorem 3.1 in [16]).

In mathematical physics, the distinction between local and nonlocal source terms plays a key role in understanding the nature of interactions within a system and in developing appropriate models. A local source term depends only on the solution and its derivatives at a single point in the domain, capturing interactions that occur in a localized region. These terms are typically encountered in problems where effects are confined to immediate neighborhoods, such as local forces or reactions. However, in many physical systems, interactions can extend over longer distances or be influenced by global effects. In such cases, a nonlocal source term is more appropriate. This term involves the solution values from multiple points, often integrated over the domain, reflecting long-range interactions or global influences. Nonlocal source terms are commonly found in models of nonlocal elasticity, diffusion with memory, or biological systems with spatial coupling. This distinction motivates our study of a singular elliptic equation in variable exponent spaces, where we consider a more complex nonlocal source term to explore the existence of multiple weak solutions under mixed boundary conditions.

Motivated by the above advantages of a nonlocal source term and the previous work of Liu et al. [16], we shall study the singular elliptic equation with mixed boundary conditions in variable exponent spaces

$$\begin{cases} \sum_{i=1}^{2} -\Delta_{p_i(x)}u + \dfrac{a(x)|u|^{q-2}u}{|x|^q} = \lambda f(x,u)\Big(\int_{\Omega} F(x,u)\,dx\Big)^r & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_1, \\ \left(\sum_{i=1}^{2} |\nabla u|^{p_i(x)-2}\nabla u\right)\nu + b(x)|u|^{h(x)-2}u = 0, & \text{on } \Gamma_2, \end{cases} \tag{P}$$

where $F(x,u) = \int_0^u f(x,t)\,dt$, for all $(x,u) \in \Omega \times \mathbb{R}$, r is a positive constant. In this given context, Ω is assumed to be an open bounded subset of $\mathbb{R}^N$, $(N > 2)$. The boundary of Ω

is denoted by $\partial\Omega$, is a smooth surface. The outward normal vectors on this boundary are represented by the vector field ν.

Additionally, there are two distinct smooth submanifolds, Γ_1 and Γ_2, which are parts of the boundary $\partial\Omega$. These submanifolds have a dimension of $N-1$ and do not intersect. The union of these submanifolds, denoted by $\overline{\Gamma_1} \cup \overline{\Gamma_2}$, covers the entire boundary, while their intersection, denoted by $\overline{\Gamma_1} \cap \overline{\Gamma_2}$, forms a submanifold of dimension $N-2$ within the boundary $\partial\Omega$. Moreover λ is a positive parameter and the function $a(x)$ is chosen to satisfy the assertion $0 < a(x) \in L^\infty(\Omega)$.

In the whole paper and for $i = 1,2$, $-\Delta_{p_i(x)}u = \operatorname{div}(|\nabla u|^{p_i(x)-2}\nabla u)$ denotes the $p_i(x)$-Laplacian operator, where $p_i(x) \in C_+(\overline{\Omega})$. In the sequel, we denote by $\tilde{p}(x) = \max\{p_1(x), p_2(x)\}$ such that $1 < \tilde{p}(x) < \tilde{p}^*(x)$, where

$$\tilde{p}^*(x) = \begin{cases} \dfrac{N\tilde{p}(x)}{N-\tilde{p}(x)}, & \text{if } \tilde{p}(x) < N, \\ +\infty, & \text{if } \tilde{p}(x) \geq N, \end{cases}$$

$1 < q < \min\{N, \tilde{p}(x)\}$, $h(x) \in C(\overline{\Gamma_2})$ with $1 < h(x) < \tilde{p}_*(x)$, where

$$\tilde{p}_*(x) = \begin{cases} \dfrac{(N-1)\tilde{p}(x)}{N-\tilde{p}(x)}, & \text{if } \tilde{p}(x) < N, \\ +\infty, & \text{if } \tilde{p}(x) \geq N, \end{cases}$$

the function $b(x) \in L^{e(x)}(\Gamma_2)$, with $e(x) > \frac{\tilde{p}_*(x)h(x)}{\tilde{p}_*(x)-h(x)}$. We will assume in the next of this paper that $f : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function satisfying

(H) $$m_1|u|^{\alpha(x)-1} \leq f(x,u) \leq m_2|u|^{\beta(x)-1},$$

where, m_1, m_2 are positive constants. The functions $\alpha(x)$ and $\beta(x)$ belongs to the set $C_+(\overline{\Omega})$ and satisfy $\alpha(x) \leq \beta(x) < \tilde{p}(x)$, for all $x \in \Omega$.

The key differences between the problem of Liu et al. [16] and Problem (*P*) lie primarily in the structure of the source term, the conditions required for the existence of solutions, and the role of $b(x)$. In [16], the source term is a local nonlinearity $\lambda f(x,u)$, where $f(x,u)$ is a Carathéodory function satisfying specific growth conditions. The existence of two weak solutions relies on the Ambrosetti–Rabinowitz condition, which ensures the existence of solutions via variational methods. Additionally, the boundary condition on Γ_2 requires the positivity of $c(x)$.

In contrast, our work involves a more complex source term, which includes a nonlocal term $\left(\int_\Omega F(x,u)\,dx\right)^r$, where $F(x,u)$ is the primitive of $f(x,u)$. This nonlocal factor complicates the analysis but allows for the existence of three weak solutions without the need for the Ambrosetti-Rabinowitz condition. Moreover, the boundary condition on Γ_2 in our work involves $b(x)$, which is not required to be positive, providing more flexibility in the boundary conditions. These differences demonstrate how the second work handles a more intricate problem setup and proves the existence of multiple solutions with fewer restrictions compared to the first.

The rest of this paper is structured as follows. In the next section we give some basic facts and variational structure concerning our problem. In Section 3, we state and prove the main result of this paper.

2 Mathematical backgrounds

In this section, we recall some definitions and basic properties of the generalized Lebesgue Sobolev spaces.

For any $p \in C_+(\overline{\Omega})$, let $p^+ = \max_{x\in\overline{\Omega}} p(x)$ and $p^- = \min_{x\in\overline{\Omega}} p(x)$. The generalized Lebesgue Sobolev space is defined as

$$L^{p(x)}(\Omega) = \left\{ u : \Omega \to \mathbb{R}, u \text{ measurable} : \int_\Omega |u(x)|^{p(x)}\, dx < \infty \right\}.$$

The so-called Luxemburg norm on this spaces defined by

$$\|u\|_{L^{p(x)}(\Omega)} = \|u\|_{p(x)} = \inf\left\{ \eta > 0 : \int_\Omega \left|\frac{u(x)}{\eta}\right|^{p(x)} dx \le 1 \right\}.$$

The Lebesgue–Sobolev space $W^{1,p(x)}(\Omega)$ is defined as follows

$$W^{1,p(x)}(\Omega) := \left\{ u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega) \right\},$$

and equipped with the norm

$$\|u\|_{W^{1,p(x)}(\Omega)} = \|\nabla u\|_{p(x)} + \|u\|_{p(x)},$$

where

$$|\nabla u| = \left(\sum_{i=1}^{N} \left|\frac{\partial u}{\partial x_i}\right|^2 dx \right)^{\frac{1}{2}}.$$

Let $W_0^{1,p(x)}(\Omega)$ be the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$. In the following, let

$$E = W_0^{1,\tilde{p}(x)}(\Omega).$$

The set E is a closed subspace of $W^{1,\tilde{p}(x)}(\Omega)$, then E is a reflexive, separable and uniformly convex Banach space. So, we can get that $\|u\|_{W^{1,\tilde{p}(x)}(\Omega)}$ is equivalent to the norm

$$\|u\| = \|\nabla u\|_{\tilde{p}(x)} = \inf\left\{ \eta > 0 : \int_\Omega \left|\frac{\nabla u(x)}{\eta}\right|^{\tilde{p}(x)} dx \le 1 \right\}.$$

Lemma 2.1 ([11]). *The continuous embedding $E \hookrightarrow L^{\alpha(x)}$ holds for any $\alpha \in C_+(\overline{\Omega})$ such that $\alpha(x) \le \tilde{p}(x)$ a.e. on Ω. This leads to the inequality*

$$\|u\|_{\alpha(x)} \le c_\alpha \|\nabla u\|_{\tilde{p}(x)}, \tag{2.1}$$

where c_α is a positive constant.

Lemma 2.2 ([12]). *If $\tilde{p} \in C_+(\overline{\Omega})$, then*

$$\min\left\{ \|\nabla u\|_{\tilde{p}(x)}^{\tilde{p}^-}, \|\nabla u\|_{\tilde{p}(x)}^{\tilde{p}^+} \right\} \le \int_\Omega |\nabla u(x)|^{\tilde{p}(x)} dx \le \max\left\{ \|\nabla u\|_{\tilde{p}(x)}^{\tilde{p}^-}, \|\nabla u\|_{\tilde{p}(x)}^{\tilde{p}^+} \right\},$$

for any $u \in L^{\tilde{p}(x)}(\Omega)$ and for a.e. $x \in \Omega$.

Lemma 2.3 ([10]). *If $s_1, s_2 \in C_+(\overline{\Omega})$ such that $s_1(x) \le s_2(x)$ a.e. $x \in \Omega$, then exists the continuous embedding $W^{1,s_2(x)}(\Omega) \hookrightarrow W^{1,s_1(x)}(\Omega)$.*

Lemma 2.4 ([10]). *Let Ω having the cone property on its boundary, and let $\tilde{p} \in C_+(\overline{\Omega})$. Suppose that $s \in C_+(\overline{\Omega})$ and $s(x) < \tilde{p}^*(x)$ for all $x \in \overline{\Omega}$. Under these conditions, the embedding*

$$W^{1,\tilde{p}(x)}(\Omega) \hookrightarrow W^{1,s(x)}(\Omega)$$

is both continuous and compact.

Denote

$$C_+(\partial\Omega) := \{\xi : \xi \in C(\partial\Omega), \xi(x) > 1, \text{ on } \partial\Omega\},$$

and the surface measure on $\partial\Omega$ by $d\sigma$.

$$L^{\tilde{p}(x)}(\partial\Omega) := \left\{ u \mid u : \partial\Omega \to \mathbb{R} \text{ is measurable, and } \int_{\partial\Omega} |u(x)|^{\tilde{p}(x)}\, d\sigma < \infty \right\},$$

where $\tilde{p}(x)$ denotes a variable exponent that may vary with the point $x \in \partial\Omega$.

The norm associated with this space is given by the Luxemburg norm

$$\|u\|_{L^{\tilde{p}(x)}(\partial\Omega)} = \|u\|_{\tilde{p}(x)(\partial\Omega)} = \inf \left\{ \eta > 0 \mid \int_{\partial\Omega} \left| \frac{u(x)}{\eta} \right|^{\tilde{p}(x)} d\sigma \leq 1 \right\}.$$

For a measurable function v on a subset $\Gamma_2 \subset \partial\Omega$, the space $L^{\tilde{p}(x)}(\Gamma_2)$ consists of functions that are measurable on Γ_2 and can be extended to some function $u \in L^{\tilde{p}(x)}(\partial\Omega)$ such that $u = v$ on Γ_2. Specifically, we define

$$L^{\tilde{p}(x)}(\Gamma_2) = \left\{ v \mid v \text{ is measurable on } \Gamma_2, \exists u \in L^{\tilde{p}(x)}(\partial\Omega) \text{ such that } u = v \text{ on } \Gamma_2 \right\}.$$

The corresponding norm on $L^{\tilde{p}(x)}(\Gamma_2)$ is

$$\|v\|_{L^{\tilde{p}(x)}(\Gamma_2)} = \|v\|_{\tilde{p}(x)(\Gamma_2)} = \inf \left\{ \|u\|_{\tilde{p}(x)(\partial\Omega)} \mid u = v \text{ on } \Gamma_2 \right\}.$$

Lemma 2.5 ([8]). *Assume that the boundary of Ω possesses the cone property and $\tilde{p} \in C_+(\overline{\Omega})$, if $h \in C_+(\partial\Omega)$ and $h(x) < \tilde{p}_*(x)$ for all $x \in \partial\Omega$ then the trace embedding*

$$W^{1,\tilde{p}(x)}(\Omega) \hookrightarrow L^{h(x)}(\partial\Omega)$$

is compact and continous.

Lemma 2.6 ([12]). *Let $G \subset \mathbb{R}^N$ be a measurable subset with $0 < meas(G) < +\infty$. Consider a Carathéodory function $f : G \times \mathbb{R} \to \mathbb{R}$ satisfying the estimate*

$$|f(x,u)| \leq M_1(x) + M_2|u|^{\frac{p(x)}{q(x)}}, \quad \text{for almost every } (x,u) \in G \times \mathbb{R},$$

where $p(x), q(x) \in C_+(\overline{\Omega})$, $0 < M_1(x) \in L^{q(x)}(G)$, and $M_2 > 0$. In this case, the Nemytskii operator

$$N_f(u)(x) = f(x, u(x))$$

maps the space $L^{p(x)}(G)$ into $L^{q(x)}(G)$, and it is both continuous and bounded.

Lemma 2.7 (Hölder type inequality [1, 9]). *Let p, q, and s be measurable functions defined on Ω with the condition*

$$\frac{1}{s(x)} = \frac{1}{p(x)} + \frac{1}{q(x)}, \quad \text{for almost every } x \in \Omega.$$

If $f \in L^{p(x)}(\Omega)$ and $g \in L^{q(x)}(\Omega)$, then the product fg belongs to $L^{s(x)}(\Omega)$, and the following inequality holds

$$\|fg\|_{s(x)} \leq 2\|f\|_{p(x)}\|g\|_{q(x)}.$$

Furthermore, if the condition

$$\frac{1}{s(x)} + \frac{1}{p(x)} + \frac{1}{q(x)} = 1, \quad \text{for almost every } x \in \Omega,$$

is satisfied, then it follows that

$$\int_\Omega |f(x)g(x)h(x)|\,dx \leq 3\|f\|_{s(x)}\|g\|_{p(x)}\|h\|_{q(x)}.$$

Hereinafter, for $M > 0$ and $r(x) \in C_+(\overline{\Omega})$, we use the following notations

$$[M]^r = \max\{M^{r^-}, M^{r^+}\} \quad \text{and} \quad [M]_r = \min\{M^{r^-}, M^{r^+}\}.$$

In order to formulate the variational approach of the problem (P), we define the functional $J_\lambda : E \longrightarrow \mathbb{R}$ by

$$J_\lambda = J_1 - \lambda J_2,$$

where

$$J_1(u) = \sum_{i=1}^{2}\left(\int_\Omega \frac{1}{p_i(x)}|\nabla u|^{p_i(x)}dx\right) + \int_\Omega \frac{a(x)|u|^q}{q|x|^q}\,dx,$$
$$J_2(u) = \frac{1}{r+1}\left(\int_\Omega F(x,u)\,dx\right)^{r+1} - \frac{1}{\lambda}\int_{\Gamma_2}\frac{b(x)|u|^{h(x)}}{h(x)}\,d\sigma,$$

where $F(x,u) = \int_0^u f(x,t)\,dt$, $\forall\ (x,u) \in \Omega \times \mathbb{R}$.

It is clear that $u \in E$ is a weak solution of the problem (P) if

$$J'_\lambda(u)(v) = J'_1(u)(v) - \lambda J'_2(u)(v) = 0, \qquad \forall\ v\ \in E,$$

where J_1, J_2 are continuously Gâteaux differentiable and we have

$$J'_1(u)(v) = \sum_{i=1}^{2}\left(\int_\Omega |\nabla u|^{p_i(x)-2}\nabla u \nabla v\,dx\right) + \int_\Omega \frac{a(x)|u|^{q-2}uv}{|x|^q}\,dx,$$

and

$$J'_2(u)(v) = \left(\int_\Omega F(x,u)\,dx\right)^r \int_\Omega f(x,u)v\,dx - \frac{1}{\lambda}\int_{\Gamma_2} b(x)|u|^{h(x)-2}uv\,d\sigma,$$

Lemma 2.8 ([16]). *The operator J'_1 verifies the following properties*

(i) *J'_1 is coercive and strictly monotone in E.*

(ii) *J'_1 is an homeomorphism.*

Lemma 2.9. *The operator $J'_2 : E \longrightarrow E^*$ is compact.*

Proof. Let $J_2' = \varphi_1'(u) - \frac{1}{\lambda}\varphi_2'(u)$, where

$$\varphi_1'(u)(v) = \left(\int_\Omega F(x,u)\,dx\right)^r \int_\Omega f(x,u)v\,dx, \qquad \varphi_2'(u)(v) = \int_{\Gamma_2} b(x)|u|^{h(x)-2}uv\,d\sigma, \quad \forall v \in E.$$

From condition (**H**), the compact embeddings $E \hookrightarrow L^{\beta(x)}(\Omega)$, where $1 < \beta(x) < \tilde{p}^*(x)$, and $E \hookrightarrow L^{h(x)}(\partial\Omega)$, where $1 < h(x) < \tilde{p}_*(x)$, imply the compactness of $\varphi_1'(u)$ and $\varphi_2'(u)$. Consequently, the functional $J_2'(u) : E \to E^*$ is compact.

To demonstrate this, let $(u_k)_k \subset E$ be a sequence such that $u_k \rightharpoonup u$ weakly in E. Due to the compactness of the embeddings, there exists a subsequence, still denoted by $(u_k)_k$, such that $u_k \to u$ strongly in $L^{\beta(x)}(\Omega)$, and $u_k(x) \to u(x)$ for almost every $x \in \Omega$. The continuity of $F(x,u)$ with respect to u ensures that

$$F(x,u_k) \to F(x,u) \quad \text{for almost every } x.$$

Also, there exists $C > 0$ such that

$$|F(x,u_k)| < C|u_k|^{\beta(x)}.$$

By using the dominated convergence theorem, we can write

$$\int_\Omega F(x,u_k)\,dx \to \int_\Omega F(x,u_k)\,dx \quad \text{as } k \to +\infty. \tag{2.2}$$

In the subsequent part of the proof, we employ the approach introduced by Liu et al. [16]. Specifically, we assert that the Nemytskii operator $N_f(u)(x) = f(x,u(x))$ is continuous, as $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function that satisfies condition (**H**). Consequently, it follows that $N_f(u_k) \to N_f(u)$ in $L^{\frac{\beta(x)}{\beta(x)-1}}(\Omega)$.

Next, utilizing Lemma 2.1 and the Hölder inequality, for any $v \in E$, we have the estimate

$$\begin{aligned}\left|\int_\Omega f(x,u_k)v\,dx - \int_\Omega f(x,u)v\,dx\right| &\le \int_\Omega |(f(x,u_k) - f(x,u))v|\,dx \\ &\le 2\|N_f(u_k) - N_f(u)\|_{\frac{\beta(x)}{\beta(x)-1}}\|v\|_{\beta(x)}, \\ &\le 2c_\beta\|N_f(u_k) - N_f(u)\|_{\frac{\beta(x)}{\beta(x)-1}}\|\nabla v\|_{\tilde{p}(x)},\end{aligned} \tag{2.3}$$

where c_β is the constant associated with the embedding $E \hookrightarrow L^{\beta(x)}(\Omega)$ with $1 < \beta(x) < \tilde{p}^*(x)$.

By combining the results from equations (2.2) and (2.3), we deduce that $\varphi_1'(u_k) \to \varphi_1'(u)$ in E, which implies that φ_1' is completely continuous. Therefore, φ_1' is a compact operator.

By similar argument for φ_2' on Γ_2, The Nemytskii operator $N_h(u) = |u|^{h(x)-2}u$ is continuous and $N_h(u_k) \longrightarrow N_h(u)$ in $L^{\frac{h(x)}{h(x)-1}}(\Gamma_2)$. According to the Hölder type inequality in Lemma 2.7 and the compact embedding $E \hookrightarrow L^{h(x)}(\partial\Omega)$, $1 < h(x) < \tilde{p}_*(x)$, thus for all $v \in E$, we have

$$\begin{aligned}\left|\varphi_2'(u_k)(v) - \varphi_2'(u)(v)\right| &= \left|\int_{\Gamma_2} b(x)|u_k|^{h(x)-2}u_k v\,d\sigma - \int_{\Gamma_1} b(x)|u|^{h(x)-2}uv\,d\sigma\right| \\ &\le 3\|b\|_{e(x)}\|N_h(u_k) - N_h(u)\|_{\frac{h(x)}{h(x)-1}(\Gamma_2)}\|v\|_{\frac{e(x)h(x)}{e(x)-h(x)}(\Gamma_2)} \\ &\le 3\|b\|_{e(x)}\|N_h(u_k) - N_h(u)\|_{\frac{h(x)}{h(x)-1}(\Gamma_2)}\|v\|_{\frac{e(x)h(x)}{e(x)-h(x)}(\partial\Omega)} \\ &\le 3b_h\|b\|_{e(x)}\|N_h(u_k) - N_h(u)\|_{\frac{h(x)}{h(x)-1}(\Gamma_2)}\|v\|,\end{aligned}$$

where b_h is the embedding constant of the trace embedding $W^{1,\tilde{p}(x)} \hookrightarrow L^{\frac{e(x)h(x)}{e(x)-h(x)}}(\partial\Omega)$. Thus $\varphi_2'(u_k) \longrightarrow \varphi_2'(u)$ in E^* as $n \longrightarrow +\infty$, i.e. φ_2' is completely continuous, so φ_2' is compact. Therefore J_2' is compact. □

To obtain our result, we use the following critical points theorem.

Theorem 2.10 ([6, Theoem 3.6]). *Consider a reflexive real Banach space E, and $J_1 : E \to \mathbb{R}$ be a coercive, continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional. Assume that the Gâteaux derivative of J_1 has a continuous inverse on E^*. Furthermore, let $J_2 : E \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact, assume that*

$$\inf_E J_1 = J_1(0) = J_2(0) = 0.$$

Suppose that there exist $d > 0$ and $\overline{x} \in E$, with $d < J_1(\overline{x})$, such that

(i) $\frac{\sup_{J_1(x)<d} J_2(x)}{d} < \frac{J_2(\overline{x})}{J_1(\overline{x})}$,

(ii) for every $\lambda \in \Psi_d := \left]\frac{J_1(\overline{x})}{J_2(\overline{x})}, \frac{d}{\sup_{J_1(x)\leq d} J_2(x)}\right[$, $J_\lambda := J_1 - \lambda J_2$ *is coercive.*

Then, for every $\lambda \in \Psi_d$, $J_1 - \lambda J_2$ has at least three distinct critical points in E.

3 Main results

Before giving our result, we recall the Hardy inequality (for more details, see [13]), if $1 < q < N$, then

$$\int_\Omega \frac{|u(x)|^q}{|x|^q}\,dx \leq \frac{1}{H}\int_\Omega |\nabla u|^q\,dx, \qquad \forall u \in W_0^{1,q}(\Omega) = \{u \in W^{1,q}(\Omega) : u|_{\Gamma_1} = 0\},$$

where $H = \left(\frac{N-q}{q}\right)^q$.

Using (2.2) and due to $q < \tilde{p}(x)$ for any $x \in \Omega$, we conclude that $E \hookrightarrow W_0^{1,q}(\Omega)$, thus there is a positive constant k such that

$$\int_\Omega \frac{|u(x)|^q}{|x|^q}\,dx \leq \frac{k^q}{H}\|u\|^q, \qquad \forall\, u \in E. \tag{3.1}$$

To present our main result, we define

$$D(x) = \sup\{D > 0 \mid B(x, D) \subseteq \Omega\},$$

for all $x \in \Omega$, here $B(x, D)$ represents a ball centered at x with radius D. It is simple to see that there exists a point $x_0 \in \Omega$ such that $B(x_0, R) \subseteq \Omega$, where $R = \sup_{x\in\Omega} D(x)$.

Theorem 3.1. *Assume that $\tilde{p}^- > \beta^+(r+1)$, and that there exists $d, \delta > 0$ such that*

$$\left(\frac{1}{p_1^-}\left[\frac{2\delta}{R}\right]^{p_1} + \frac{1}{p_2^-}\left[\frac{2\delta}{R}\right]^{p_2}\right)\left|B(x_0, R) \setminus \overline{B}\left(x_0, \frac{R}{2}\right)\right| = d,$$

then for every $\lambda \in]A_\delta, B_d[$, where

$$A_\delta = \frac{\left(\frac{1}{p_1^-}\left[\frac{2\delta}{R}\right]^{p_1} + \frac{1}{p_2^-}\left[\frac{2\delta}{R}\right]^{p_2}\right)\left|B(x_0, R) \setminus \overline{B}(x_0, \frac{R}{2})\right| + \frac{k^q\|a\|_\infty}{qH}\tilde{M}^q}{\frac{m_1^{r+1}\left([\delta]_\alpha\right)^{r+1}}{(r+1)(\alpha^+)^{r+1}}\left|B(x_0, \frac{R}{2})\right|^{r+1}},$$

and

$$B_d = \frac{d}{\frac{m_2^{r+1}([c_\beta]^\beta)^{r+1}(\tilde{p}^+)^{\frac{\beta^+(r+1)}{\tilde{p}^-}}}{(r+1)(\beta^-)^{r+1}}\left(\left[\left[d\right]^{\frac{1}{\tilde{p}}}\right]^\beta\right)^{r+1}},$$

problem (P) *admits at least three weak solutions.*

Proof. It is worth noting that J_1 and J_2 satisfy the regularity assumptions outlined in Theorem 2.10. We will now establish the fulfillment of conditions (i) and (ii). To this end, let's consider

$$\left(\frac{1}{p_1^-}\left[\frac{2\delta}{R}\right]^{p_1} + \frac{1}{p_2^-}\left[\frac{2\delta}{R}\right]^{p_2}\right)\left|B(x_0,R)\setminus\overline{B}\left(x_0,\frac{R}{2}\right)\right| = d,$$

and consider $v_\delta \in E$ such that

$$v_\delta\begin{cases}0 & x\in\Omega\setminus B(x_0,R),\\ \frac{2\delta}{R}(R-|x-x_0|) & \in B(x_0,R)\setminus B\left(x_0,\frac{R}{2}\right),\\ \delta & x\in B\left(x_0,\frac{R}{2}\right).\end{cases}$$

Then, from Lemma 2.2 and by a direct calculation, shows that $\nabla v_\delta(x) = \frac{2\delta}{R}$, thus

$$\begin{aligned}&\|\nabla v_\delta(x)\|_{\tilde{p}(x)}\\ &\quad\le \max\{\|\nabla v_\delta\|^{\tilde{p}^-}_{\tilde{p}(x)}, \|\nabla v_\delta\|^{\tilde{p}^+}_{\tilde{p}(x)}\},\\ &\quad\le \max\left\{\left(\left[\frac{2\delta}{R}\right]^{\tilde{p}}\right)^{\tilde{p}^-}\left|B(x_0,R)\setminus\overline{B}\left(x_0,\frac{R}{2}\right)\right|^{\tilde{p}^-}, \left(\left[\frac{2\delta}{R}\right]^{\tilde{p}}\right)^{\tilde{p}^+}\left|B(x_0,R)\setminus\overline{B}\left(x_0,\frac{R}{2}\right)\right|^{\tilde{p}^+}\right\},\\ &\quad= \tilde{M}.\end{aligned}$$

By equation (3.1), we have

$$\begin{aligned}J_1(v_\delta) &= \sum_{i=1}^{2}\left(\int_\Omega \frac{1}{p_i(x)}|\nabla v_\delta|^{p_i(x)}\,dx\right) + \int_\Omega \frac{a(x)|v_\delta|^q}{q|x|^q}\,dx,\\ &\le \left(\frac{1}{p_1^-}\left[\frac{2\delta}{R}\right]^{p_1} + \frac{1}{p_2^-}\left[\frac{2\delta}{R}\right]^{p_2}\right)\left|B(x_0,R)\setminus\overline{B}\left(x_0,\frac{R}{2}\right)\right| + \frac{k^q\|a\|_\infty}{qH}\|v_\delta\|^q,\\ &= \left(\frac{1}{p_1^-}\left[\frac{2\delta}{R}\right]^{p_1} + \frac{1}{p_2^-}\left[\frac{2\delta}{R}\right]^{p_2}\right)\left|B(x_0,R)\setminus\overline{B}\left(x_0,\frac{R}{2}\right)\right| + \frac{k^q\|a\|_\infty}{qH}\|\nabla v_\delta(x)\|^q_{\tilde{p}(x)},\\ &\le \left(\frac{1}{p_1^-}\left[\frac{2\delta}{R}\right]^{p_1} + \frac{1}{p_2^-}\left[\frac{2\delta}{R}\right]^{p_2}\right)\left|B(x_0,R)\setminus\overline{B}\left(x_0,\frac{R}{2}\right)\right| + \frac{k^q\|a\|_\infty}{qH}\tilde{M}^q.\end{aligned}$$

Therefore, $J_1(v_\delta) > d$. however, it is important to consider the following

$$\begin{aligned}J_2(v_\delta) \ge \frac{1}{r+1}\left(\int_\Omega F(x,v_\delta)dx\right)^{r+1} &\ge \frac{m_1^{r+1}}{(r+1)(\alpha^+)^{r+1}}\left(\int_{B(x_0,\frac{R}{2})}|\delta|^{\alpha(x)}\,dx\right)^{r+1},\\ &\ge \frac{m_1^{r+1}([\delta]_\alpha)^{r+1}}{(r+1)(\alpha^+)^{r+1}}\left|B(x_0,\frac{R}{2})\right|^{r+1}.\end{aligned} \tag{3.2}$$

In addition, for every $u \in J_1^{-1}(]-\infty, d[)$, one has the following

$$\frac{1}{\tilde{p}^+}\left[\|\nabla u\|_{\tilde{p}(x)}\right]_{\tilde{p}} \leq d, \tag{3.3}$$

therefore,

$$\|\nabla u\|_{\tilde{p}(x)} \leq \left[\tilde{p}^+ J_1(u)\right]^{\frac{1}{\tilde{p}}} < \left[\tilde{p}^+ d\right]^{\frac{1}{\tilde{p}}}.$$

Under the assumption (**H**), we conclude the following

$$\begin{aligned} J_2(u) &\leq \frac{1}{r+1}\left(\int_\Omega F(x,u)\,dx\right)^{r+1}, \\ &\leq \frac{m_2^{r+1}}{(r+1)(\beta^-)^{r+1}}\left(\int_\Omega |u|^{\beta(x)}\,dx\right)^{r+1}, \\ &\leq \frac{m_2^{r+1}}{(r+1)(\beta^-)^{r+1}}\left(\left[\|u\|_{\beta(x)}\right]^{\beta}\right)^{r+1}, \\ &\leq \frac{m_2^{r+1}}{(r+1)(\beta^-)^{r+1}}\left(\left[c_\beta\|\nabla u\|_{\tilde{p}(x)}\right]^{\beta}\right)^{r+1}. \end{aligned} \tag{3.4}$$

This leads to the following result

$$\sup_{J_1(u)<d} J_2(u) \leq \frac{m_2^{r+1}([c_\beta]^\beta)^{r+1}(\tilde{p}^+)^{\frac{\beta^+(r+1)}{\tilde{p}^-}}}{(r+1)(\beta^-)^{r+1}}\left(\left[[d]^{\frac{1}{\tilde{p}}}\right]^{\beta}\right)^{r+1},$$

and

$$\frac{1}{d}\sup_{J_1(u)<d} J_2(u) < \frac{1}{\lambda}.$$

Furthermore, we can establish the coerciveness of J_λ for any positive value of λ by employing inequality (3.4) once more. This yields the following result

$$J_2(u) \leq \frac{m_2^{r+1}([c_\beta]^\beta)^{r+1}}{(r+1)(\beta^-)^{r+1}}\left(\left[\|\nabla u\|_{\tilde{p}(x)}\right]^{\beta}\right)^{r+1}.$$

When $\|u\| > 1$, the following can be inferred

$$J_1(u) - \lambda J_2(u) \geq \frac{1}{\tilde{p}^+}\|\nabla u\|_{\tilde{p}(x)}^{\tilde{p}^-} - \lambda\frac{m_2^{r+1}([c_\beta]^\beta)^{r+1}}{(r+1)(\beta^-)^{r+1}}\left(\left[\|\nabla u\|_{\tilde{p}(x)}\right]^{\beta}\right)^{r+1}.$$

By considering the fact that $\tilde{p}^- > \beta^+(r+1)$, we can reach the desired conclusion. In conclusion, considering the aforementioned fact that

$$\Psi_d^- = (A_\delta, B_d) \subseteq \left(\frac{J_1(v)}{J_2(v)}, \frac{d}{\sup_{J_1(u)<d} J_2(u)}\right).$$

Based on Theoem 2.10, it can be deduced that for $\lambda \in \Psi_d^-$, the function $J_1 - \lambda J_2$ possesses at least three critical points in E. These critical points correspond to weak solutions of problem (P). □

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Boundedness in a chemotaxis-consumption model with singular sensitivity

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Abstract. This paper deals with the chemotaxis system with singular sensitivity and nonlocal logistic source

$$\begin{cases} u_t = u_{xx} - \left(\frac{u}{v}v_x\right)_x + u^\alpha\left(\gamma - \mu\int_\Omega u^\beta\right), & x\in\Omega, t>0,\\ v_t = v_{xx} - uv, & x\in\Omega, t>0, \end{cases}$$

under Neumann boundary conditions in a bounded open interval $\Omega\subset\mathbb{R}$, where α,β,γ,μ are positive constants. It is shown that for any $\beta > 1+\frac{(\alpha-3)_+}{2}$ and $\alpha\geq 1$, the system possesses a global and bounded classical solution. Moreover, we establish uniform-in-time boundedness of $\frac{v_x}{v}$.

Keywords: chemotaxis, nonlocal source, boundedness, singular sensitivity.

2020 Mathematics Subject Classification: Primary: 35B65; Secondary: 35J25, 35J60, 35D40.

1 Introduction

In this paper, we consider the one-dimensional chemotaxis model with singular sensitivity and nonlocal term

$$\begin{cases} u_t = u_{xx} - \left(\frac{u}{v}v_x\right)_x + u^\alpha\left(\gamma - \mu\int_\Omega u^\beta\right), & x\in\Omega, t>0,\\ v_t = v_{xx} - uv, & x\in\Omega, t>0,\\ u_x = v_x = 0, & x\in\partial\Omega, t>0,\\ (u,v)(x,0) = \big(u_0(x), v_0(x)\big), & x\in\Omega, \end{cases} \tag{1.1}$$

in a bounded open interval $\Omega\subseteq\mathbb{R}$, where α, β, γ, μ are positive constants and where $u = u(x,t)$ denotes the cell density and where $v = v(x,t)$ represents the oxygen concentration.

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The initial data in (1.1) satisfy

$$\begin{cases} u_0 \in C^0(\bar{\Omega}),\ u_0 \geq 0 \text{ and } u_0 \not\equiv 0, x \in \bar{\Omega}, \\ v_0 \in W^{1,q}(\Omega)\ (q > 2),\ v_0 > 0 \text{ in } \bar{\Omega} \text{ and } (v_0)_x = 0,\ x \in \partial\Omega. \end{cases} \tag{1.2}$$

Chemotaxis processes are known to play an important role in various biological contexts. There are also many works on various central aspects like global existence, lager time behaviors, finite time blow-up, and so on (see [4,6,7,11,28,32] and the references therein). When placed at one end of a capillary tube that contains oxygen and an energy source, bacteria of the species E. coli that have a gradient of nutrient concentration form bands that are visible to the unaided eye. They migrate at a constant speed due to a chemotactic mechanism [1]. In order to describe the consumption of the critical substrate and the change in bacterial density by random motion and chemotaxis, Keller and Segel proposed a phenomenological model of wave-like solution behavior without any type of cell kinetics [20], a prototypical version of which is given by:

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot \left(\frac{u}{v} \nabla v\right), & x \in \Omega, t > 0, \\ v_t = \Delta v - uv, & x \in \Omega, t > 0, \end{cases} \tag{1.3}$$

where the second equation models consumption of the signal upon contact with cells, and where in the first equation it is assumed that the chemotactic stimulus is perceived following the Weber–Fechner law, thus requiring the chemotactic sensitivity $\chi \frac{u}{v}$ to be chosen proportional to the reciprocal signal density. When $n = 2$, there exists a global generalized solution to (1.3) with $v \to 0$ in $L^p(\Omega)$ as $t \to \infty$ [33], and the solution becomes eventually smooth and converges to the homogeneous steady state if the initial mass $\int_\Omega u_0$ is small [35]. In particular, under an explicit smallness condition on $u_0 \ln u_0 \in L^1(\Omega)$ and $v_0^{-1} \nabla v_0 \in L^2(\Omega)$, the system (1.3) possesses a global classical solution [35]. Moreover, when $n \geq 3$, for all suitably regular and radially symmetric initial data (u_0, v_0), the existence of a globally defined pair (u, v) of radially symmetric functions in an appropriate generalized sense was established in [36]. When uv in (1.3) is replaced by $f(u)v$, $f \in C^1(\mathbb{R})$ essentially behaving like u^β, $\beta \in (0,1)$, $\chi \in (0,1)$, Lankeit and Viglialoro [25] showed that system has a global classical solution with any sufficiently regular initial data. Moreover, if additionally $\int_\Omega u(x,0)$ is sufficiently small, then also their boundedness is achieved. Replacing the first equation in (1.3) by $u_t = \Delta u - \chi \nabla \left(\frac{u}{v} \nabla v\right) + \gamma u - \mu u^k$, the authors in [23] showed that system (1.3) has a generalized global solution for any $\chi, \gamma, \mu > 0$ in the case $k = 2$. It is proved that as $\frac{1}{v}$ is replaced by $\phi(v) \in C^1(0,\infty)$ satisfying $\phi(v) \to \infty$ as $v \to 0$, this system possesses a global classical solution if $k > 1$ for $n = 1$ or $k > 1 + n/2$ for $n \geq 2$ [45]. The asymptotic behavior of solutions with $n = 2$ is determined [45]. For the local reaction term, a sequence of recent results on the existence of global-in-time solution, long-time behavior, vanishing coefficient limit and optimal time decay rates of the solution in the case $n = 1$ were obtained by Zeng and Zhao [41,42]. Recently, many variants of (1.3) have been proposed for applications under different frameworks, see for instance [2].

In those Keller–Segel models (cf. [21]) where v does not stand for a nutrient to be consumed but a signaling substance produced by the bacteria themselves, i.e., the second equation of (1.3) is replaced by $\tau v_t = \epsilon \Delta v - v + u$. For this system, the global solutions are known to exist if χ is sufficiently small, where the precise condition depends on the dimension as well as on $\tau = 0$ [15,29] or $\tau = 1$ [13,38,39]. As to the corresponding case with logistic source $\gamma u - \mu u^k$, the global existence of classical solutions and convergence to constant states were established in [9,16,18,22,27,40,43,44,47,48]. For a generalized solution concept, we refer to [12,26].

Nonlocal term in mathematical models can forecast how a disassociated sticky cell population will aggregate and react to the adhesive pressures produced by binding during cell-cell adhesion. See [3,17,30,31] for the reference. Bian et al. were devoted to the analysis of nonnegative solutions for the chemotaxis model with nonlocal nonlinear source in bounded domain in [5]

$$\begin{cases} u_t = \Delta u - \chi\nabla\cdot(u\nabla v) + u^\alpha\Big(1-\int_\Omega u^\beta\Big), & x\in\Omega, t>0,\\ 0 = \Delta v - v + u, & x\in\Omega, t>0, \end{cases} \tag{1.4}$$

where $\alpha \geq 1$, $\beta > 1$. They proved that if spatial dimension $n \geq 3$, either $2 \leq \alpha < 1 + \frac{2\beta}{n}$ or $\alpha < 2$ and $\frac{(2+n)(2-\alpha)}{n} < 1 + \frac{2\beta}{n} - \alpha$, the model admits a classical solution which is uniformly bounded. The same conclusion is established for the full-parabolic version in [8]. Recently, when the second equation of (1.4) is replaced by $v_t = \epsilon\Delta v - v + u$ and $-\chi\nabla\cdot(u\nabla v)$ is replaced by $-\chi\nabla\cdot(\frac{u}{v}\nabla v)$, the authors in [10] provide corresponding conditions to ensure the existence and boundedness of solutions, respectively.

In this paper, we consider the chemotaxis-consumption system with singular sensitivity and nonlocal logistic source. It is the purpose of this work to investigate the question of global existence and boundedness to (1.1) when $n = 1$. We first consider a non-singular chemotaxis system for (u, w) based on the transformation $w = -\ln\frac{v}{\|v_0\|_{L^\infty(\Omega)}}$ (see [33]). We obtain the key estimate on $\|w_x\|_{L^2(\Omega)}$ by deriving a subtle estimate for $\int_\Omega w_x^2 + \int_\Omega u^p$ with $1 < p < 2$. This is crucial to imply any further useful global regularity information on the quantity u itself. Before stating our results, we state that $x_+ = \max\{x, 0\}$.

Theorem 1.1. *Let $\Omega \subseteq \mathbb{R}$ be a bounded open interval. Suppose that $\gamma, \mu > 0$ and that*

$$\beta > 1 + \frac{(\alpha-3)_+}{2}, \quad \alpha \geq 1. \tag{1.5}$$

Then for any choice of u_0 and v_0 complying with (1.2), the problem (1.1) possesses a global bounded classical solution. Moreover, there exists some $\bar{M}$ such that

$$\Big\|\frac{v_x}{v}\Big\|_{L^\infty(\Omega)} \leq \bar{M}, \quad t > 0. \tag{1.6}$$

This paper is structured as follows. In Section 2, we present some preliminaries. In Section 3, we establish the global boundedness for $\frac{v_x}{v}$ and u in L^∞-norm.

2 Preliminaries

We need the Neumann heat semigroup in $\{e^{t\Delta}\}_{t\geq 0}$ estimates in $\Omega \subset \mathbb{R}^n$ to prove the local existence and the global existence of solutions. The following lemma below could be found in the vast existing literature, for instance, [14,34].

Lemma 2.1 ([14,34]). *Let $n \geq 1$, $\{e^{t\Delta}\}_{t\geq 0}$ be the Neumann heat semigroup in Ω, and $\lambda_1 > 0$ denotes the first nonzero eigenvalue of $-\Delta$ in Ω with respect to the Neumann boundary condition. Then there exist $K_1, \ldots, K_4 > 0$ depending on Ω only such that the following properties:*

(i) If $1 \leq p_2 \leq p_1 \leq \infty$, then

$$\|e^{t\Delta}z\|_{L^{p_1}(\Omega)} \leq K_1\left(1 + t^{-\frac{n}{2}(\frac{1}{p_2}-\frac{1}{p_1})}\right)e^{-\lambda_1 t}\|z\|_{L^{p_2}(\Omega)}, \quad t \in (0, T) \tag{2.1}$$

for all $z \in L^{p_2}(\Omega)$ satisfying $\int_\Omega z = 0$.

(ii) If $1 \le p_2 \le p_1 \le \infty$, *then*

$$\|\nabla e^{t\Delta} z\|_{L^{p_1}(\Omega)} \le K_2 \left(1 + t^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{p_2}-\frac{1}{p_1})}\right) e^{-\lambda_1 t} \|z\|_{L^{p_2}(\Omega)}, \quad t \in (0,T) \tag{2.2}$$

for all $z \in L^{p_2}(\Omega)$.

(iii) If $2 \le p_2 \le p_1 \le \infty$, *then*

$$\|\nabla e^{t\Delta} z\|_{L^{p_1}(\Omega)} \le K_3 \left(1 + t^{-\frac{n}{2}(\frac{1}{p_2}-\frac{1}{p_1})}\right) e^{-\lambda_1 t} \|\nabla z\|_{L^{p_2}(\Omega)}, \quad t \in (0,T) \tag{2.3}$$

for all $z \in W^{1,p_2}(\Omega)$.

(iv) If $1 < p_2 \le p_1 < \infty$ *or* $1 < p_2 < \infty$ *and* $p_1 = \infty$, *then*

$$\|e^{t\Delta} \nabla z\|_{L^{p_1}(\Omega)} \le K_4 \left(1 + t^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{p_2}-\frac{1}{p_1})}\right) e^{-\lambda_1 t} \|z\|_{L^{p_2}(\Omega)}, \quad t \in (0,T) \tag{2.4}$$

for all $z \in (L^{p_2}(\Omega))^n$.

Let us state a basic result on local existence and extensibility of classical solutions, which can be proved by well-established fixed point arguments (see [19,37,46]). However, we could not find a precise reference that covers our model, therefore, we show a short proof here.

Lemma 2.2. *Let* $\alpha, \beta \ge 1$ *and* $\gamma, \mu > 0$. *Suppose that* u_0 *and* v_0 *satisfy* (1.2). *Then there exist a maximal* $T \in (0, \infty]$ *and a nonnegative classical solution* (u, v) *of functions*

$$\begin{cases} u \in C^0\Big(\bar{\Omega} \times [0,T)\Big) \cap C^{2,1}\Big(\bar{\Omega} \times (0,T)\Big), \\ v \in C^0\Big(\bar{\Omega} \times [0,T)\Big) \cap C^{2,1}\Big(\bar{\Omega} \times (0,T)\Big) \cap L^\infty_{loc}\Big([0,T), W^{1,q}(\Omega)\Big) \end{cases}$$

satisfying (1.1) *in the classical sense in* $\Omega \times (0,T)$. *Moreover, we have the following alternative:*

$$\textit{Either} \quad T = \infty, \quad \textit{or} \quad \|u\|_{L^\infty(\Omega)} + \|v\|_{W^{1,q}(\Omega)} \to \infty \quad \textit{as } t \to T.$$

Proof. We first fix

$$K > 3 \max\{\|u_0\|_{L^\infty(\Omega)}, \|v_0\|_{L^q(\Omega)}, 6K_3 \|v_0'\|_{L^q(\Omega)}\}, \tag{2.5}$$

where K_3 is as in (2.3). For small $T_* \in (0,1)$ to be specified below, in the Banach space

$$X =: C^0([0,T_*]; C^0(\bar{\Omega})) \times C^0([0,T_*]; W^{1,q}(\Omega))$$

we consider the closed set

$$\begin{aligned} S := \{(u,v) \in X \mid \|u\|_{L^\infty((0,T_*);L^\infty(\Omega))} \le K, \ \|v\|_{L^\infty((0,T_*);W^{1,q}(\Omega))} \le K, \\ \inf_{x\in\Omega} v(\cdot,t) \ge \frac{1}{e} \inf_{x\in\Omega} v_0(x), \ \textit{for a.e. } t \in (0,T_*)\}. \end{aligned} \tag{2.6}$$

For $(u,v) \in S$ and $t \in (0,T_*)$, we let

$$\begin{aligned} \Phi(u,v)(t) &:= \begin{pmatrix} \phi_1(u,v)(t) \\ \phi_2(u,v)(t) \end{pmatrix} \\ &:= \begin{pmatrix} e^{t\Delta}u_0 - \int_0^t e^{(t-s)\Delta}(\frac{u}{v}v_x)_x + \gamma \int_0^t e^{(t-s)\Delta} u^\alpha - \mu \int_0^t e^{(t-s)\Delta}\left(u^\alpha \int_\Omega u^\beta\right) \\ e^{t\Delta}v_0 - \int_0^t e^{(t-s)\Delta} uv \end{pmatrix}. \end{aligned}$$

On dropping a nonpositive term, we use (2.4) and (2.6) to estimate

$$\begin{aligned}\|\phi_1(u,v)(t)\|_{L^\infty(\Omega)} \le\ & \|u_0\|_{L^\infty(\Omega)} + \frac{K_4 e}{\inf\limits_{x\in\Omega} v_0(x)} \int_0^t (t-s)^{-\frac{1}{2}-\frac{1}{2q}} e^{-\lambda_1(t-s)} \|u\|_{L^\infty(\Omega)} \|v_x\|_{L^q(\Omega)} \\ & + \gamma \int_0^t \sup_{t\in(0,T)} \|u\|^\alpha_{L^\infty(\Omega)} \\ \le\ & \frac{K}{3} + CT_*^{\frac{1}{2}-\frac{1}{2q}} + \gamma K^\alpha T_*, \quad t\in(0,T_*),\end{aligned} \tag{2.7}$$

where we use C to denote a generic positive constant independent of T^* here and below. Here when we deal with the first and third term we have used that $\|e^{\sigma\Delta}z\|_{L^{r_3}(\Omega)} \le \|z\|_{L^{r_3}(\Omega)}$, for all $z\in L^{r_3}(\Omega)$ with $1\le r_3\le\infty$.

We next derive the $W^{1,q}$-estimate on $\phi_2(u,v)$. On the one hand, we drop a nonpositive term to get

$$\|\phi_2(u,v)(t)\|_{L^q(\Omega)} = \left\|e^{t\Delta}v_0 - \int_0^t e^{(t-s)\Delta}uv\right\|_{L^q(\Omega)} \le \|v_0\|_{L^q(\Omega)} \le \frac{K}{3} \tag{2.8}$$

for all $t\in(0,T_*)$. On the other hand, it follows from (2.2), (2.3), (2.5) and (2.6) that

$$\begin{aligned}\|(\phi_2(u,v)(t))_x\|_{L^q(\Omega)} =\ & \left\|(e^{t\Delta}v_0)_x - \int_0^t (e^{(t-s)\Delta}uv)_x\right\|_{L^q(\Omega)} \\ \le\ & 2K_3\|(v_0)_x\|_{L^q(\Omega)} + K_2\int_0^t (t-s)^{-\frac{1}{2}} e^{-\lambda_1(t-s)}\|uv\|_{L^q(\Omega)} \\ \le\ & \frac{K}{3} + CT_*^{\frac{1}{2}}, \quad t\in(0,T_*).\end{aligned} \tag{2.9}$$

A combination of (2.8) and (2.9) entails

$$\|\phi_2(u,v)(t)\|_{W^{1,q}(\Omega)} \le \frac{2K}{3} + CT_*^{\frac{1}{2}}, \quad t\in(0,T_*). \tag{2.10}$$

The comparison principle to the second equation of (1.1) leads to $\|v\|_{L^\infty(\Omega)} \le \|v_0\|_{L^\infty(\Omega)}$ for all $t\in(0,T_*)$. By the order preserving of the Neumann heat semigroup, we find

$$\begin{aligned}\phi_2(u,v)(t) \ge e^{t\Delta}v_0 - \int_0^t e^{(t-s)\Delta}uv \ge & \inf_{x\in\Omega} v_0(x) - \int_0^t \|e^{(t-s)\Delta}uv\|_{L^\infty(\Omega)} \\ \ge & \inf_{x\in\Omega} v_0(x) - K\|v_0\|_{L^\infty(\Omega)}T_*, \quad t\in(0,T_*).\end{aligned} \tag{2.11}$$

If we take T_* small enough, then it follows from (2.7), (2.10) and (2.11) that $\Phi S\subseteq S$.

Moreover, with T_* still at our disposal, we proceed to check that for all $(u_1,v_1),(u_2,v_2)$ belonging to S,

$$\begin{aligned}\|\phi_1(u_1,v_1)(t) - \phi_1(u_2,v_2)(t)\|_{L^\infty(\Omega)} \le\ & \int_0^t \left\|e^{(t-s)\Delta}\left(\frac{u_1(v_1)_x}{v_1} - \frac{u_2v_2'}{v_2}\right)_x\right\|_{L^\infty(\Omega)} \\ & + \gamma\int_0^t \|e^{(t-s)\Delta}(u_1^\alpha - u_2^\alpha)\|_{L^\infty(\Omega)} \\ & + \mu\int_0^t \left\|e^{(t-s)\Delta}u_2^\alpha\int_\Omega (u_2^\beta - u_1^\beta)\right\|_{L^\infty(\Omega)} \\ & + \mu\int_0^t \left\|e^{(t-s)\Delta}(u_2^\alpha - u_1^\alpha)\int_\Omega u_1^\beta\right\|_{L^\infty(\Omega)}\end{aligned} \tag{2.12}$$

for all $t \in (0, T_*)$. For the first term of the right-hand side of (2.12), in accordance with (2.6) and (2.4), we can estimate

$$\begin{aligned}
&\int_0^t \left\| e^{(t-s)\Delta} \left(\frac{u_1(v_1)_x}{v_1} - \frac{u_2(v_2)_x}{v_2} \right)_x \right\|_{L^\infty(\Omega)} \\
&\quad \le K_2 \int_0^t (t-s)^{-\frac{1}{2}-\frac{1}{2q}} e^{-\lambda_1(t-s)} \left\| \frac{u_1(v_1)_x}{v_1} - \frac{u_2(v_2)_x}{v_2} \right\|_{L^q(\Omega)} \\
&\quad \le K_2 T_*^{\frac{1}{2}-\frac{1}{2q}} \|(u_1, v_1) - (u_2, v_2)\|_X.
\end{aligned} \tag{2.13}$$

Since $\alpha, \beta \ge 1$, from the mean value theorem and (2.6), we know

$$\|u_2^\alpha - u_1^\alpha\|_{L^\infty(\Omega)} \le \alpha K^{\alpha-1} \|u_2 - u_1\|_{L^\infty(\Omega)}, \qquad \|u_2^\beta - u_1^\beta\|_{L^\infty(\Omega)} \le \beta K^{\beta-1} \|u_2 - u_1\|_{L^\infty(\Omega)}.$$

We thus gain that

$$\begin{aligned}
&\gamma \int_0^t \|e^{(t-s)\Delta}(u_1^\alpha - u_2^\alpha)\|_{L^\infty(\Omega)} + \mu \int_0^t \left\| e^{(t-s)\Delta} u_2^\alpha \int_\Omega (u_2^\beta - u_1^\beta) \right\|_{L^\infty(\Omega)} \\
&\qquad + \mu \int_0^t \left\| e^{(t-s)\Delta}(u_2^\alpha - u_1^\alpha) \int_\Omega u_1^\beta \right\|_{L^\infty(\Omega)} \\
&\quad \le C T_* \|u_1 - u_2\|_{L^\infty(\Omega)}, \quad t \in (0, T_*).
\end{aligned} \tag{2.14}$$

Thus (2.12), (2.13), (2.14) provide

$$\|\phi_1(u_1, v_1)(t) - \phi_1(u_2, v_2)(t)\|_{L^\infty(\Omega)} \le C T_*^{\frac{1}{2}-\frac{1}{2q}} \|(u_1, v_1) - (u_2, v_2)\|_X, \quad t \in (0, T_*). \tag{2.15}$$

Similar process to (2.8)-(2.10), we can conclude that

$$\begin{aligned}
&\|\phi_2(u_1, v_1)(t) - \phi_2(u_2, v_2)(t)\|_{W^{1,q}(\Omega)} \\
&\quad \le \int_0^t \|e^{(t-s)\Delta} u_1(v_1 - v_2)\|_{W^{1,q}(\Omega)} + \int_0^t \|e^{(t-s)\Delta} v_2(u_1 - u_2)\|_{W^{1,q}(\Omega)} \\
&\quad \le C T_*^{\frac{1}{2}} \|(u_1, v_1) - (u_2, v_2)\|_X, \quad t \in (0, T_*),
\end{aligned} \tag{2.16}$$

so that Φ is shown to be a contraction if T_* is sufficiently small. Accordingly, the Banach fixed point theorem asserts the existence of some $(u, v) \in S$ such that $\Phi(u, v) = (u, v)$. Relying on straightforward regularity arguments including standard semigroup techniques, it can easily be checked that in fact (u, v) lies in the asserted regularity class and is a classical solution of (1.1) in $\Omega \times (0, T_*)$.

Finally, the nonnegative of (u, v) can be obtained by the strong maximum principle, which can be derived as that in [24]. It is sufficient to justify the conditions of Theorem B.1 in [24] in the following. Firstly, the equation

$$v_t = v_{xx} - uv, \quad x \in \Omega, t \in (0, T)$$

is considered. We interpret u as given function from above, i.e. $u \in C^0\big(\bar{\Omega} \times [0, T_*)\big) \cap C^{2,1}\big(\bar{\Omega} \times (0, T_*)\big)$. For given $T_1 < T_*$ one can fine $C > 0$ such that

$$u \le C, \quad \bar{\Omega} \times [0, T_1]. \tag{2.17}$$

With $b \equiv 0 \in C^1(\bar{\Omega}\times(0,T_1),\mathbb{R}) \cap L^\infty_{loc}(\Omega\times[0,T_1),\mathbb{R})$ fulfilling $b_x = 0$ on $\partial\Omega\times(0,T_1)$ and $f(x,t,u) := -u(x,t)v \in C^0(\Omega\times[0,T_1)\times\mathbb{R})$ satisfying

$$|f(x,t,v_1) - f(x,t,v_2)| = |u(v_1-v_2)| \le C|v_1-v_2|, \quad x\in\bar{\Omega}, t\in[0,T), v_1,v_2\in K$$

with compact K. Define $\underline{v} = \inf_{x\in\Omega} v_0(x)e^{-Ct}$. A direct calculation shows that

$$\begin{cases} \underline{v}_t = -C\underline{v} = \underline{v}_{xx} - C\underline{v} \le \underline{v}_{xx} - u\underline{v}, & x\in\Omega,\ t\in(0,T_1),\\ v_x = 0, & x\in\partial\Omega,\ t\in(0,T_1),\\ \underline{v}(\cdot,0) \le v_0(x), & x\in\Omega. \end{cases}$$

which by comparison implies

$$v(x,t) \ge \inf_{x\in\Omega} v_0(x)e^{-Ct} > 0 \quad \text{on } \Omega\times(0,T_1). \tag{2.18}$$

For given function $v \in C^0\big(\bar{\Omega}\times[0,T_*)\big)\cap C^{2,1}\big(\bar{\Omega}\times(0,T_*)\big)$ satisfying (2.18), we can study the first equation of (1.1)

$$u_t = u_{xx} - \Big(\frac{u}{v}v_x\Big)_x + u^\alpha\Big(\gamma - \mu\int_\Omega u^\beta\Big).$$

In order to apply Theorem B.1 in [24] to $b(x,t) = \frac{v_x}{v}$, $f(x,t,u) = u^\alpha\big(\gamma-\mu\int_\Omega u^\beta\big)$, we now verify the conditions of the comparison theorem. To avoid repetition, we just point out the differences, which are the local Lipschitz estimation of $f(x,t,u)$. For every compact K,

$$\begin{aligned} |f(x,t,u_1) - f(x,t,u_2)| &= \Big|u_1^\alpha\Big(\gamma-\mu\int_\Omega u_1^\beta\Big) - u_2^\alpha\Big(\gamma-\mu\int_\Omega u_2^\beta\Big)\Big| \\ &\le \gamma|u_1^\alpha - u_2^\alpha| + \mu\Big|u_2^\alpha\int_\Omega(u_2^\beta-u_1^\beta)\Big| + \mu\Big|(u_2^\alpha-u_1^\alpha)\int_\Omega u_1^\beta\Big| \\ &\le L(K,\alpha,\beta)|u_1-u_2|, \quad x\in\bar{\Omega}, t\in[0,T_1), u_1,u_2\in K, \end{aligned}$$

where $L(K,\alpha,\beta)$ is a positive constant. Therefore, the comparision theorem becomes applicable to suitable subsolutions and supersolutions of this equation. Moreover, since 0 is a subsolution of the equation, we have $u \ge 0$. □

A first basic property of this solution is immediate.

Lemma 2.3. *Let $\alpha,\beta \ge 1$ and $\gamma,\mu > 0$. Then*

$$\int_\Omega u \le M_0 := \max\left\{\int_\Omega u_0, |\Omega|^{1-\frac{1}{\beta}}\left(\frac{\gamma}{\mu}\right)^{\frac{1}{\beta}}\right\}, \quad t\in(0,T). \tag{2.19}$$

Proof. Integrating the first equation of (1.1), using the Hölder inequality, we obtain

$$\frac{d}{dt}\int_\Omega u \le \int_\Omega u^\alpha\left[\gamma - |\Omega|^{1-\beta}\mu\left(\int_\Omega u\right)^\beta\right], \quad t\in(0,T),$$

which implies (2.19) by a straightforward ODE analysis. □

Now following a standard procedure of changing variables in (1.1) we substitute

$$w = -\ln \frac{v}{\|v_0\|_{L^\infty(\Omega)}}$$

and thus infer that $w \geq 0$ in $\Omega \times (0,T)$ and

$$(u,w) \in \left(C^0\left(\bar{\Omega} \times [0,T)\right) \cap C^{2,1}\left(\bar{\Omega} \times (0,T)\right)\right)^2$$

solves

$$\begin{cases} u_t = u_{xx} + (uw_x)_x + u^\alpha\left(\gamma - \mu \int_\Omega u^\beta\right), & x \in \Omega, t \in (0,T), \\ w_t = w_{xx} - w_x^2 + u, & x \in \Omega, t \in (0,T), \\ u_x = w_x = 0, & x \in \partial\Omega, t \in (0,T), \\ u(x,0) = u_0,\ w(x,0) = -\ln \frac{v_0}{\|v_0\|_{L^\infty(\Omega)}}, & x \in \Omega. \end{cases} \tag{2.20}$$

We now prepare an estimate on $\int_\Omega u^k$ for all $k > 1$.

Lemma 2.4. *Let $\gamma, \mu > 0$. Suppose that u_0 and v_0 satisfy (1.2). Assume that α, β satisfy (1.5). Then for all $k > 1$, there exists $M_1(k) > 0$ such that*

$$\frac{d}{dt}\int_\Omega u^k + \int_\Omega u^k \leq -\frac{2(k-1)}{k}\int_\Omega \left|\left(u^{\frac{k}{2}}\right)_x\right|^2 - k(k-1)\int_\Omega u^{k-1}u_x w_x + M_1, \quad t \in (0,T). \tag{2.21}$$

Proof. Testing the first equation of (1.1) by ku^{k-1} and using Young's inequality, we obtain

$$\begin{aligned} \frac{d}{dt}\int_\Omega u^k + \int_\Omega u^k = & -\frac{4(k-1)}{k}\int_\Omega \left|\left(u^{\frac{k}{2}}\right)_x\right|^2 - k(k-1)\int_\Omega u^{k-1}u_x w_x + \int_\Omega u^k \\ & + k\gamma\int_\Omega u^{k+\alpha-1} - k\mu\int_\Omega u^{k+\alpha-1}\int_\Omega u^\beta, \quad t \in (0,T). \end{aligned} \tag{2.22}$$

We apply the Gagliardo–Nirenberg inequality and Young's inequality to find positive constants $C_0(\Omega), C_1(k,\Omega)$ such that

$$\begin{aligned} \int_\Omega u^k &\leq C_0\left(\int_\Omega \left|\left(u^{\frac{k}{2}}\right)_x\right|^2\right)^{\frac{k-1}{k+1}} \|u\|_{L^1(\Omega)}^{\frac{2k}{k+1}} + C_0\|u\|_{L^1(\Omega)}^k \\ &\leq \frac{k-1}{k}\int_\Omega \left|\left(u^{\frac{k}{2}}\right)_x\right|^2 + C_1\|u\|_{L^1(\Omega)}^k, \quad t \in (0,T). \end{aligned} \tag{2.23}$$

Let us fix β_0 satisfying

$$\beta_0 < \beta \tag{2.24}$$

and

$$\frac{\alpha-1}{2} < \beta_0 < k + \alpha - 1, \tag{2.25}$$

which is possible because $\beta > \frac{\alpha-1}{2}$. By virtue of Hölder's inequality and (2.24) we see that

$$-k\mu\int_\Omega u^{k+\alpha-1}\int_\Omega u^\beta \leq -k\mu|\Omega|^{1-\frac{\beta}{\beta_0}}\int_\Omega u^{k+\alpha-1}\left(\int_\Omega u^{\beta_0}\right)^{\frac{\beta}{\beta_0}}. \tag{2.26}$$

Due to the condition (2.25), we may invoke the Gagliardo–Nirenberg inequality along with the Young inequality, which says that there exist positive constants $C_2(\alpha,\gamma,k,\Omega)$ and $C_3(\alpha,\beta,\gamma,k,\Omega)$ satisfying

$$\begin{aligned} k\gamma\int_\Omega u^{k+\alpha-1}dx &\le C_2\left(\int_\Omega |(u^{\frac{k}{2}})_x|^2dx\right)^{\frac{a(k+\alpha-1)}{k}}\|u\|^{(1-a)(k+\alpha-1)}_{L^{\frac{Mk}{2}}(\Omega)}+C_2\|u\|^{k+\alpha-1}_{L^1(\Omega)}\\ &\le \frac{k-1}{k}\int_\Omega |(u^{\frac{k}{2}})_x|^2+C_3\|u\|^{\frac{(1-a)k(k+\alpha-1)}{k-a(k+\alpha-1)}}_{L^{\frac{Mk}{2}}(\Omega)}+C_2\|u\|^{k+\alpha-1}_{L^1(\Omega)}, \end{aligned} \tag{2.27}$$

where

$$M=2\beta_0(k+\alpha-1+\beta)k^{-1}(\beta+\beta_0)^{-1}$$

and where

$$a=\beta k(k+\alpha-1-\beta_0)(k+\alpha-1)^{-1}[(\beta+\beta_0)k+(k+\alpha-1+\beta)\beta_0]^{-1}\in(0,1),$$

thanks to (2.25). Moreover, making full use of (2.25) we see that actually

$$\beta_0<\frac{Mk}{2}<k+\alpha-1, \tag{2.28}$$

and that

$$\frac{k(k+\alpha-1)(1-a)}{[k-a(k+\alpha-1)](k+\alpha-1+\beta)}<1. \tag{2.29}$$

Indeed, (2.29) is possible since the definition of a ensures that

$$\begin{aligned} &\frac{k(k+\alpha-1)(1-a)}{[k-a(k+\alpha-1)](k+\alpha-1+\beta)}\\ &\quad=\frac{(k+\alpha-1)[\beta_0k+(k+\alpha-1+\beta)\beta_0]+\beta\beta_0k}{(k+\alpha-1)[\beta_0k+(k+\alpha-1+\beta)\beta_0]+\beta\beta_0k+\beta(k+\alpha-1+\beta)[2\beta_0-(\alpha-1)]}. \end{aligned}$$

Therefore, in view of (2.28) and (2.29), one further application of Young's inequality and the interpolation inequality provides $C_4(\alpha,\beta,\gamma,\mu,k,\Omega)$ such that

$$\begin{aligned} C_3\|u\|^{\frac{(1-a)k(k+\alpha-1)}{k-a(k+\alpha-1)}}_{L^{\frac{Mk}{2}}(\Omega)} &\le C_3\left[\int_\Omega u^{k+\alpha-1}\left(\int_\Omega u^{\beta_0}\right)^{\frac{\beta}{\beta_0}}\right]^{\frac{k(k+\alpha-1)(1-a)}{[k-a(k+\alpha-1)](k+\alpha-1+\beta)}}\\ &\le k\mu|\Omega|^{1-\frac{\beta}{\beta_0}}\int_\Omega u^{k+\alpha-1}\left(\int_\Omega u^{\beta_0}\right)^{\frac{\beta}{\beta_0}}+C_4. \end{aligned} \tag{2.30}$$

From (2.22), (2.23), (2.26), (2.27) and (2.30), we have

$$\begin{aligned} \frac{d}{dt}\int_\Omega u^k+\int_\Omega u^k \le &-\frac{2(k-1)}{k}\int_\Omega\left|\left(u^{\frac{k}{2}}\right)_x\right|^2-k(k-1)\int_\Omega u^{k-1}u_xw_x+C_1\|u\|^k_{L^1(\Omega)}\\ &+C_2\|u\|^{k+\alpha-1}_{L^1(\Omega)}+C_4,\quad t\in(0,T). \end{aligned} \tag{2.31}$$

Therefore, (2.21) results from (2.31) by taking $M_1=C_1M_0^k+C_2M_0^{k+\alpha-1}+C_4$. □

3 Global boundedness of solutions

This section is devoted to the derivation of estimates for w, so as to finally obtain the global boundedness of solutions. For this purpose, we establish L^2 estimate on w_x and L^4 estimate on w_x for our present setting.

Lemma 3.1. *Let $\gamma, \mu > 0$, and suppose that u_0 and v_0 satisfy (1.2). Assume that α, β satisfy (1.5). Then there exists constant $M_2 > 0$ depend on $\alpha, \beta, \gamma, \mu, \Omega$ such that*

$$\int_\Omega w_x^2 \le M_2, \quad t \in (0,T). \tag{3.1}$$

Proof. Recalling $(w_0)_x = 0$ on $\partial\Omega$, we multiply the second equation in (2.20) by $-w_{xx}$ and integrate by parts to see, using Young's inequality, that

$$\frac{d}{dt}\int_\Omega w_x^2 = -2\int_\Omega w_{xx}^2 - 2\int_\Omega u w_{xx} \le -\int_\Omega w_{xx}^2 + \int_\Omega u^2, \quad t \in (0,T). \tag{3.2}$$

For fixed $1 < p < 2$, applying $k = p$ to (2.21) takes the form

$$\begin{aligned}\frac{d}{dt}\int_\Omega u^p + \int_\Omega u^p &\le -\frac{2(p-1)}{p}\int_\Omega \left|\left(u^{\frac{p}{2}}\right)_x\right|^2 - p(p-1)\int_\Omega u^{p-1}u_x w_x + M_1 \\ &= -\frac{2(p-1)}{p}\int_\Omega \left|\left(u^{\frac{p}{2}}\right)_x\right|^2 + (p-1)\int_\Omega u^p w_{xx} + M_1, \quad t \in (0,T),\end{aligned}$$

which along with Young's inequality derives

$$\frac{d}{dt}\int_\Omega u^p + \int_\Omega u^p \le -\frac{2(p-1)}{p}\int_\Omega |(u^{\frac{p}{2}})_x|^2 + \frac{1}{2}\int_\Omega w_{xx}^2 + \frac{(p-1)^2}{2}\int_\Omega u^{2p} + M_1 \tag{3.3}$$

for all $t \in (0,T)$. According to Poincaré's inequality and the fact $(w_0)_x = 0$ on $\partial\Omega$, we know there is constant $C_5(\Omega)$ such that

$$-\frac{1}{2}\int_\Omega w_{xx}^2 \le -\frac{1}{2C_5}\int_\Omega w_x^2. \tag{3.4}$$

Recalling $p > 1$, Young's inequality provide $C_7(p,\Omega) > 0$ such that

$$\int_\Omega u^2 \le \int_\Omega u^{2p} + C_7. \tag{3.5}$$

Collecting (3.2)–(3.5) we see that

$$\begin{aligned}&\frac{d}{dt}\left(\int_\Omega w_x^2 + \int_\Omega u^p\right) + C_6\left(\int_\Omega w_x^2 + \int_\Omega u^p\right) \\ &\quad\le -\frac{2(p-1)}{p}\int_\Omega |(u^{\frac{p}{2}})_x|^2 + \left[1 + \frac{(p-1)^2}{2}\right]\int_\Omega u^{2p} + C_8, \quad t \in (0,T),\end{aligned} \tag{3.6}$$

where $C_6 = \min\{1, \frac{1}{2C_5}\}$ and $C_8 = C_7 + M_1$. Here our assumption for $1 < p < 2$ warrants that

$$0 < \frac{2p-1}{p+1} < 1. \tag{3.7}$$

We may once more rely on Young's inequality together with Gagliardo–Nirenberg inequality to see that there exist constants $C_9(p,\Omega), C_{10}(p,\Omega) > 0$ such that

$$\begin{aligned}\left[1+\frac{(p-1)^2}{2}\right]\int_\Omega u^{2p} &= \frac{(p-1)^2}{2}\|u^{\frac{p}{2}}\|^4_{L^4(\Omega)}\\ &\le C_9\left(\int_\Omega \left|(u^{\frac{p}{2}})_x\right|^2\right)^{\frac{2p-1}{p+1}}\|u\|^{\frac{3p}{p+1}}_{L^1(\Omega)} + C_9\|u\|^{2p}_{L^1(\Omega)}\\ &\le \frac{p-1}{p}\int_\Omega |(u^{\frac{p}{2}})_x|^2 + C_{10}\|u\|^{\frac{3p}{2-p}}_{L^1(\Omega)} + C_9\|u\|^{2p}_{L^1(\Omega)}\end{aligned} \tag{3.8}$$

for all $t \in (0,T)$, which together with (3.6) and (3.8) asserts

$$\frac{d}{dt}\left(\int_\Omega w_x^2 + \int_\Omega u^p\right) + C_6\left(\int_\Omega w_x^2 + \int_\Omega u^p\right) \le C_8 + C_9\|u\|^{2p}_{L^1(\Omega)} + C_{10}\|u\|^{\frac{3p}{2-p}}_{L^1(\Omega)} \tag{3.9}$$

for all $t \in (0,T)$. Thereupon, (3.1) can be derived by (3.9) and (2.19). □

Lemma 3.2. *Let $\gamma, \mu > 0$, and suppose that u_0 and v_0 satisfy (1.2). Assume that α, β satisfy (1.5). Then there exists $M_3(\alpha,\beta,\gamma,\mu,\Omega) > 0$ such that*

$$\int_\Omega w_x^4 + \int_\Omega u^2 \le M_3, \quad t \in (0,T). \tag{3.10}$$

Proof. By (2.21) with $k = 2$, we get

$$\frac{d}{dt}\int_\Omega u^2 + \int_\Omega u^2 \le -\int_\Omega u_x^2 - 2\int_\Omega u u_x w_x + M_1.$$

Fixing $\theta > 1$, we use Young's inequality and Hölder's inequality to obtain

$$\frac{d}{dt}\int_\Omega u^2 + \int_\Omega u^2 \le -\frac{1}{2}\int_\Omega u_x^2 + 2\|u\|^2_{L^{2\theta}(\Omega)}\|w_x^2\|_{L^{\frac{\theta}{\theta-1}}(\Omega)} + M_1, \quad t \in (0,T). \tag{3.11}$$

Now another application of Young's inequality, followed by Hölder's inequality, reveals that

$$\begin{aligned}\frac{d}{dt}\int_\Omega w_x^4 &= -12\int_\Omega w_x^2 w_{xx}^2 - 12\int_\Omega w_x^2 w_{xx} u\\ &\le -12\int_\Omega w_x^2 w_{xx}^2 + 8\int_\Omega w_x^2 w_{xx}^2 + \frac{9}{2}\int_\Omega w_x^2 u^2\\ &\le -\int_\Omega |(w_x^2)_x|^2 + \frac{9}{2}\|u\|^2_{L^{2\theta}(\Omega)}\|w_x^2\|_{L^{\frac{\theta}{\theta-1}}(\Omega)}\end{aligned} \tag{3.12}$$

for all $t \in (0,T)$. Adding (3.11) and (3.12), we write

$$\begin{aligned}&\frac{d}{dt}\left(\int_\Omega w_x^4 + \int_\Omega u^2\right) + \left(\int_\Omega w_x^4 + \int_\Omega u^2\right)\\ &\quad\le -\int_\Omega |(w_x^2)_x|^2 - \frac{1}{2}\int_\Omega u_x^2 + \|w_x^2\|^2_{L^2(\Omega)} + \frac{13}{2}\|u\|^2_{L^{2\theta}(\Omega)}\|w_x^2\|_{L^{\frac{\theta}{\theta-1}}(\Omega)} + M_1\end{aligned} \tag{3.13}$$

for all $t \in (0,T)$. Using the Gagliardo–Nirenberg inequality, we deduce $C_{11}(\Omega) > 0$ such that

$$\|u\|^2_{L^{2\theta}(\Omega)} \le 4C_{11}^2\left(\int_\Omega u_x^2\right)^{\frac{2\theta-1}{3\theta}}\|u\|^{\frac{2(\theta+1)}{3\theta}}_{L^1(\Omega)} + 4C_{11}^2\|u\|^2_{L^1(\Omega)}, \quad t \in (0,T) \tag{3.14}$$

and

$$\|w_x^2\|_{L^{\frac{\theta}{\theta-1}}(\Omega)} \le C_{11}\left(\int_\Omega |(w_x^2)_x|^2\right)^{\frac{1}{3\theta}} \|w_x^2\|_{L^1(\Omega)}^{\frac{3\theta-2}{3\theta}} + C_{11}\|w_x^2\|_{L^1(\Omega)}, \quad t\in(0,T) \tag{3.15}$$

as well as

$$\|w_x^2\|_{L^2(\Omega)}^2 \le 4C_{11}^2\left(\int_\Omega |(w_x^2)_x|^2\right)^{\frac{1}{3}} \|w_x^2\|_{L^1(\Omega)}^{\frac{4}{3}} + 4C_{11}^2\|w_x^2\|_{L^1(\Omega)}^2, \quad t\in(0,T). \tag{3.16}$$

Therefore, in view of $\frac{2\theta-1}{3\theta}+\frac{1}{3\theta}=\frac{2}{3}<1$, we apply Young's inequality and recall (3.14)–(3.16) to estimate

$$\begin{aligned}
&\|w_x^2\|_{L^2(\Omega)}^2 + \frac{13}{2}\|u\|_{L^{2\theta}(\Omega)}^2\|w_x^2\|_{L^{\frac{\theta}{\theta-1}}(\Omega)} \\
&\le \frac{1}{2}\int_\Omega |u_x|^2 + \frac{3}{4}\int_\Omega |(w_x^2)_x|^2 + C_{11}\|u\|_{L^1(\Omega)}^{\frac{2(\theta+1)}{\theta}}\|w_x^2\|_{L^1(\Omega)}^{\frac{3\theta-2}{\theta}} + C_{11}\|u\|_{L^1(\Omega)}^2\|w_x^2\|_{L^1(\Omega)}^{\frac{3\theta}{\theta+1}} \\
&\quad + C_{11}\|u\|_{L^1(\Omega)}^{\frac{6\theta}{3\theta-1}}\|w_x^2\|_{L^1(\Omega)}^{\frac{3\theta-2}{3\theta-1}} + C_{11}\|u\|_{L^1(\Omega)}^2\|w_x^2\|_{L^1(\Omega)} + C_{11}\|w_x^2\|_{L^1(\Omega)}^2
\end{aligned} \tag{3.17}$$

for all $t\in(0,T)$. Here C_{11} is a constant that depends on Ω. In view of (2.19), (3.1), (3.13) and (3.17), this immediately leads to (3.10), upon an ODE comparison principle. □

With the estimate (3.10) at hand, we now turn to establish uniform-in-time boundedness of u and w_x.

Proof of Theorem 1.1. In view of Lemma 2.2, it is routine to check that for any fixed $t_0\in(0,T)$, $\sup_{0\le t\le t_0}\|u(\cdot,t)\|_{L^\infty(\Omega)}$, $\sup_{0\le t\le t_0}\|w_x(\cdot,t)\|_{L^q(\Omega)}$ are bounded. It is sufficient to justify it for $t\in(t_0,T)$.

Employing the Neumann heat semigroup $\{e^{\Delta t}\}_{t\ge0}$, we represent w_x according to

$$w_x(t) = \left(e^{t\Delta}w_0\right)_x - \int_0^t \left(e^{(t-s)\Delta}w_x^2\right)_x + \int_0^t \left(e^{(t-s)\Delta}u\right)_x, \quad t\in(t_0,T),$$

recalling (2.2), (2.3) and (3.10), we obtain $C_{12}(\Omega)>0$ such that

$$\begin{aligned}
\|w_x\|_{L^\infty(\Omega)} &\le C_{12}[1+t^{-\frac{1}{2q}}]e^{-\lambda_1 t}\|w_0'\|_{L^q(\Omega)} + C_{12}\int_0^t [1+(t-s)^{-\frac{3}{4}}]e^{-\lambda_1(t-s)}\sup_{t\in(0,T)}\|w_x\|_{L^4(\Omega)}^2 \\
&\quad + C_{12}\int_0^t [1+(t-s)^{-\frac{3}{4}}]e^{-\lambda_1(t-s)}\sup_{t\in(0,T)}\|u\|_{L^2(\Omega)} \\
&\le C_{13}, \quad t\in(t_0,T),
\end{aligned} \tag{3.18}$$

where $C_{13} = C_{12}[1+t_0^{-\frac{1}{2q}}]\|w_0'\|_{L^q(\Omega)} + 2C_{12}M_3^{\frac{1}{2}}\int_0^t[1+(t-s)^{-\frac{3}{4}}]e^{-\lambda_1(t-s)}$ is a bounded constant because $\int_0^t[1+(t-s)^{-\frac{3}{4}}]e^{-\lambda_1(t-s)}$ is finite. In consequence, (3.18) and the definition of w hence establish (1.6). Using (2.21) and Young's inequality along with (3.18), we obtain

$$\begin{aligned}
\frac{d}{dt}\int_\Omega u^k + \int_\Omega u^k &\le -\frac{k-1}{k}\int_\Omega \left|\left(u^{\frac{k}{2}}\right)_x\right|^2 - \frac{k(k-1)}{4}\int_\Omega u^{k-2}u_x^2 \\
&\quad - k(k-1)\int_\Omega u^{k-1}u_x w_x + M_1 \\
&\le -\frac{k-1}{k}\int_\Omega \left|\left(u^{\frac{k}{2}}\right)_x\right|^2 + k(k-1)C_{13}^2\int_\Omega u^k + M_1,
\end{aligned} \tag{3.19}$$

for all $t \in (t_0, T)$ and $k > 2$. Following procedures analogous to (2.23) we can obtain $C_{14}(k, \alpha, \beta, \gamma, \mu, \Omega) > 0$ such that

$$k(k-1)C_{13}^2 \int_\Omega u^k \le \frac{(k-1)}{k} \int_\Omega \left| \left(u^{\frac{k}{2}} \right)_x \right|^2 + C_{14} \|u\|_{L^1(\Omega)}^k, \quad t \in (t_0, T). \tag{3.20}$$

A combination of the above estimates (3.19) and (3.20) yields that

$$\frac{d}{dt} \int_\Omega u^k + \int_\Omega u^k \le M_1 + C_{14} \|u\|_{L^1(\Omega)}^k, \quad t \in (t_0, T).$$

It holds true that for any $2 < k < \infty$, there exists $C_{15}(k, \alpha, \beta, \gamma, \mu, \Omega) > 0$ such that

$$\int_\Omega u^k \le C_{15}, \quad t \in (t_0, T), \tag{3.21}$$

here we use the ordinary differential equations comparison argument together with (2.19). By the variation-of-constants formula, u can be represented as

$$\begin{aligned} u = e^{t\Delta} u_0 + \int_0^t e^{(t-s)\Delta} (u w_x)_x + \int_0^t e^{(t-s)\Delta} \left(\frac{\gamma}{|\Omega|} \int_\Omega u^\alpha \right) \\ + \gamma \int_0^t e^{(t-s)\Delta} \left(u^\alpha - \frac{1}{|\Omega|} \int_\Omega u^\alpha \right) - \mu \int_0^t e^{(t-s)\Delta} \left(u^\alpha \int_\Omega u^\beta \right), \quad t \in (t_0, T). \end{aligned}$$

On dropping a nonpositive term, we have

$$\begin{aligned} u \le e^{t\Delta} u_0 + \int_0^t e^{(t-s)\Delta} (u w_x)_x + \int_0^t e^{(t-s)\Delta} \left(\frac{\gamma}{|\Omega|} \int_\Omega u^\alpha \right) \\ + \gamma \int_0^t e^{(t-s)\Delta} \left(u^\alpha - \frac{1}{|\Omega|} \int_\Omega u^\alpha \right), \quad t \in (t_0, T). \end{aligned}$$

With this representation and standard smoothing estimates (2.1) and (2.4), we arrive at

$$\begin{aligned} \|u\|_{L^\infty(\Omega)} \le \|u_0\|_{L^\infty(\Omega)} + C_{16} \int_0^t [1 + (t-s)^{-\frac{3}{4}}] e^{-\lambda_1 (t-s)} \sup_{t \in (0,T)} \|u\|_{L^2(\Omega)} \|w_x\|_{L^\infty(\Omega)} \\ + \left[\frac{\gamma}{|\Omega|} + 2C_{16}\gamma \int_0^t [1 + (t-s)^{-\frac{1}{2}}] e^{-\lambda_1 (t-s)} \right] \left(\sup_{t \in (0,T)} \|u\|_{L^\alpha(\Omega)} \right)^\alpha, \quad t \in (t_0, T) \end{aligned}$$

with $C_{16}(\Omega) > 0$, whence observing (3.18) and (3.21), we therefore obtain the desired global boundedness of u. Consequently, Theorem 1.1 is completed. □

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Existence and nonexistence of solutions for generalized quasilinear Kirchhoff–Schrödinger–Poisson system

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Abstract. In this paper, we consider the existence and nonexistence of solutions for a class of modified Schrödinger–Poisson system with Kirchhoff-type perturbation by use of variational methods. When nonlinear term $h(u) = |u|^{p-2}u$, $1 \leq p < \infty$, the nonexistence of nontrivial solutions of system is demonstrated through Pohožaev identity. When nonlinear term $h(u)$ satisfies appropriate assumptions, taking advantage of critical point theorem, we obtain a positive radial solution and a nontrivial one of system when $g(u)$ satisfies different conditions. Moreover, some convergence properties are established as the parameter $b \to 0$. What is more, the nonexistence of nontrivial solutions in critical case is also proved by use of Pohožaev identity.

Keywords: variational methods, Pohožaev identity, critical point theorem, nonexistence, radial solution.

2020 Mathematics Subject Classification: 35A01, 35A15, 35J10.

1 Introduction

In this paper, we studied the following generalized quasilinear Schrödinger–Poisson system with a Kirchhoff-type perturbation, which is an innovative research topic.

$$\begin{cases} \left(1 + b\int_{\mathbb{R}^3} g^2(u)|\nabla u|^2 dx\right)[-\operatorname{div}(g^2(u)\nabla u) + g(u)g'(u)|\nabla u|^2] \\ +V(x)u + \phi u = h(u), & x \in \mathbb{R}^3, \\ -\Delta\phi = u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $b \geq 0$, $V : \mathbb{R}^3 \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ are continuous functions, $g \in C^1(\mathbb{R}, \mathbb{R}^+)$ satisfies the following assumption:

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(g_1) g is even with $g'(t) \leq 0$ for all $t \geq 0$ and $g(0) = 1$, $\lim_{t\to+\infty} g(t) = a$, $a \in (0, 1)$.

When $\phi = 0$, the system (1.1) has become the following Kirchhoff–Schrödinger equation:

$$\left(1 + b\int_{\mathbb{R}^3} g^2(u)|\nabla u|^2 dx\right)[-\mathrm{div}(g^2(u)\nabla u) + g(u)g'(u)|\nabla u|^2] + V(x)u = h(x, u), \quad x \in \mathbb{R}^N. \tag{1.2}$$

Problem (1.2) is related to the stationary analogue of the Kirchhoff–Schrödinger type equation, which was proposed by Kirchhoff as an extension of classical D'Alembert's wave equation for free vibrations of elastic strings [13, 18]. Since the Kirchhoff-type problems arise in various models of physical and biological systems, numerous scholars have conducted research on problem (1.2). References [3] and [20] considered the existence and nonexistence of solutions for problem (1.2). For $h(x, u) = \lambda f(u) + g(u)G^5(u)$ as considered in [3], where $G(u) = \int_0^u g(t)dt$. Under some suitable assumptions on $f(u)$, problem (1.2) admits at least one positive ground state solution for $\lambda > \lambda^* > 0$. If $\lambda = 0$, the corresponding equation had no nontrivial solution. For $h(x, u) = f(u)$ and $f(u)$ satisfied appropriate conditions, then problem (1.2) had at least one radial ground state solution. In [20], when $h(x, u)$ satisfied critical or supercritical growth at infinity, the nonexistence result for (1.2) was proved via Pohožaev identity. If $h(x, u)$ showed asymptotically cubic growth at infinity, the existence of positive radial solutions for (1.2) was obtained by use of variational methods. Moreover, some properties were established as the parameter $b \to 0$. Both of the existence of ground state and sign-changing ground state solutions about (1.2) were testified in [23]. Chen et al. [6] applied some new analytical techniques and non-Nehari manifold method to obtain one ground state sign-changing solution $v_b = G^{-1}(u_b)$. Moreover, they illustrated that the energy of $v_b = G^{-1}(u_b)$ is strictly larger than twice of the Nehari type ground state solution. They also established the convergence properties of $v_b = G^{-1}(u_b)$ as the parameter $b \to 0$.

For $g(u) = 1$, problem (1.2) transforms to the following classical Kirchhoff equation:

$$-\left(1 + b\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)\Delta u + V(x)u = h(x, u), \quad x \in \mathbb{R}^3,$$

which takes into account the changes in length of the string produced by transverse vibrations, hence the nonlocal term appeared. If $b = 0$, problem (1.2) takes the following form:

$$-\mathrm{div}(g^2(u)\nabla u) + g(u)g'(u)|\nabla u|^2 + V(x)u = h(x, u), \quad x \in \mathbb{R}^3. \tag{1.3}$$

This equation is related to the existence of solitary wave solutions for quasilinear Schrödinger equations:

$$i\partial_t z = -\Delta z + W(x)z - k(x, |z|) - \Delta l(|z|^2)l'(|z|^2)z, \tag{1.4}$$

where $z : \mathbb{R} \times \mathbb{R}^N \to \mathbb{C}$, $W : \mathbb{R}^N \to \mathbb{R}$ is a given potential, $l : \mathbb{R} \to \mathbb{R}$ and $k : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ are suitable functions. Equation (1.4) appeared in plasma physics and fluid mechanics [25], dissipative quantum mechanics [12] and condensed matter theory [24]. The existing results for equation (1.3) such as nontrivial solutions [15], ground state solutions [4, 31, 33], positive solutions [9,10,29], nonexistence of solutions [14], high-energy solutions [22], multiple solutions [21] and infinitely many solutions [30] were obtained respectively.

System (1.1) is the so called quasilinear Kirchhoff–Schrödinger–Poisson system. If $b = 0$, (1.1) reduces to the following quasilinear Schrödinger–Poisson system:

$$\begin{cases} -\mathrm{div}(g^2(u)\nabla u) + g(u)g'(u)|\nabla u|^2 + V(x)u + \phi u = h(u), & x \in \mathbb{R}^3, \\ -\Delta\phi = u^2, & x \in \mathbb{R}^3. \end{cases} \tag{1.5}$$

In [27], the authors proved the following problem admits at least a ground state solution

$$\begin{cases} -\mathrm{div}(g^2(u)\nabla u) + g(u)g'(u)|\nabla u|^2 + V(x)u + \phi G(u)g(u) = h(x,u), & x \in \mathbb{R}^3, \\ -\Delta\phi = G^2(u), & x \in \mathbb{R}^3, \end{cases}$$

for $h(x,u) = b(x)|G(u)|^{p-2}G(u)g(u) - c(x)|G(u)|^{q-2}G(u)g(u)$, $2 < q < 4 < p < 6$. Recently, Zhang and Liu obtained a nontrivial ground state solution for $h(x,u) = \lambda f(u) + g(u)G^5(u)$ in reference [38]. By setting $g^2(u) = 1 + 2u^2$ in (1.5), we get the following quasilinear Schrödinger–Poisson system:

$$\begin{cases} -\Delta u + V(x)u + \phi u - ku\Delta(u^2) = h(u), & \text{in } \mathbb{R}^3, \\ -\Delta\phi = u^2, & \text{in } \mathbb{R}^3. \end{cases} \tag{1.6}$$

There are a variety of excellent results for system (1.6). References [11, 34, 35] studied the existence of nontrivial solutions respectively for $k = \frac{1}{2}$ or $k = 1$. The authors in [32] obtained the existence of ground state solution and infinitely many geometrically distinct solutions. The sign-changing solutions can be referred to the references [2, 37].

By setting $g^2(u) = 1 + 2u^2$ in (1.1), we get the following modified quasilinear Kirchhoff–Schrödinger–Poisson system:

$$\begin{cases} -(1 + b\int_{\mathbb{R}^3}|\nabla u|^2dx)\Delta u + V(x)u - \frac{1}{2}u\Delta(u^2) + \phi u = h(u), & x \in \mathbb{R}^3, \\ -\Delta\phi = u^2, & x \in \mathbb{R}^3. \end{cases} \tag{1.7}$$

System (1.7) was introduced in [1] very recently, the authors proved that problem (1.7) has at least three solutions: one is positive, one is negative, and one changes its sign. Furthermore, if h is odd with respect to u, they also obtained unbounded sequence of sign-changing solutions. Under appropriate conditions, the authors in [5] discussed the existence of nontrivial nonpositive and nonnegative solutions, a sequence of high-energy solutions via perturbation method. Combining perturbation method with discontinuous finite element method, a series of weak solutions of system (1.7) were gained in [7] for $h(x,u) = K(x)u^{p-2}u$, where $K(x) \in L^{\frac{2}{2-p}}(\mathbb{R}^3)$, $1 < p < 2$ and $K(x) > 0$ for $x \in \mathbb{R}^3$. When system (1.7) involving a nonlocal term and an integral constraint, infinitely many sign-changing solutions were obtained in [8] according to the method of invariant sets of the descending flow combined with the genus theory. The authors in [36] proved that the system (1.7) has a sign-changing solution u_0, which has precisely two nodal domains. The same conclusion was gained in [17] as in [36] for critical case that $h(u) = \lambda|u|^{q-2}u\ln|u|^2 + |u|^4u$.

Nevertheless, there are relatively few achievements on the generalized quasilinear Kirchhoff–Schrödinger–Poisson system (1.1), so the discussion in this paper makes innovations of the pioneering work. Based on the existing results, we extend the results of references [3] and [20] to the generalized quasilinear Kirchhoff–Schrödinger–Poisson system, then we obtain the existence and nonexistence of solutions for system (1.1).

We make some assumptions on $V(x)$ and $h(u)$:

(V_1) $V \in C(\mathbb{R}^3, \mathbb{R})$, $V(x) = V(|x|)$, $0 < V_0 \le V(x) \le V_\infty := \lim_{|x|\to\infty} V(x) < \infty$;

(h_1) $h \in C(\mathbb{R}, \mathbb{R})$, $h(t) = 0$, $\forall t \le 0$ and $\lim_{t\to 0}\frac{h(t)}{t} = 0$;

(h_2) $|h(t)| \le C(1 + |t|^{q-1})$ for some $C > 0$ and $q \in (4, 6)$;

(h_3) there exists $\tau > 4$ such that

$$0 < \tau g(t)H(t) \le h(t)G(t), \quad \forall t > 0,$$

where $H(t) = \int_0^t h(s)ds$, $G(t) = \int_0^t g(s)ds$.

Clearly, (h_1) and (h_2) show that for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|h(u)| \le \varepsilon|u| + C_\varepsilon|u|^{q-1}, \quad \forall u \in \mathbb{R}. \tag{1.8}$$

Now we state our main results.

Theorem 1.1. *For $h(u) = |u|^{p-2}u$ in (1.1), assume that (g_1) holds. If $V(x) + 2\langle \nabla V(x), x\rangle \le 0$, problem (1.1) has no nontrivial solutions when $p \le \frac{12a}{5}$. If $\frac{2}{5} \le a < 1$ and $V(x) + 2\langle \nabla V(x), x\rangle \ge 0$, problem (1.1) has no nontrivial solutions when $p \ge 6$.*

Theorem 1.2. *Assume that (V_1), (g_1) and (h_1)–(h_3) are satisfied, then problem (1.1) has a positive radial solution.*

Theorem 1.3. *Assume that (V_1), (g_1) and (h_1)–(h_3) are satisfied, $\{u_{b_n}\} \subset \mathcal{H}$ are the positive radial solutions obtained in Theorem 1.2 for each $n \in \mathbb{N}$. Then, $u_{b_n} \to u_0$ in $\mathcal{H}$ as $b_n \to 0$, $n \to \infty$, where u_0 is a positive radial solution for problem (1.5).*

The following condition of g is necessary to obtain the next two important results.

(g') $g \in C^1(\mathbb{R}, \mathbb{R}^+)$ is even with $g'(t) \ge 0$ for all $t \ge 0$ and $g(0) = 1$, $tg'(t) < g(t)$ for all $t \in \mathbb{R}$.

Theorem 1.4. *Assume that (V_1), (g') and (h_1)–(h_3) are satisfied, then problem (1.1) has a nontrivial radial solution.*

Theorem 1.5. *Assume that (V_1), (g') and (h_1)–(h_3) are satisfied, $\{u'_{b_n}\} \subset \mathcal{H}$ are the nontrivial radial solutions obtained in Theorem 1.4 for each $n \in \mathbb{N}$. Then, $u'_{b_n} \to u'_0$ in $\mathcal{H}$ as $b_n \to 0$, $n \to \infty$, where u'_0 is a nontrivial radial solution for problem (1.5).*

Finally, we consider the following generalized quasilinear Schrödinger–Poisson system involving a Kirchhoff-type perturbation and critical Sobolev exponent

$$\begin{cases} \left(1 + b\int_{\mathbb{R}^3} g^2(u)|\nabla u|^2\right)[-\operatorname{div}(g^2(u)\nabla u) + g(u)g'(u)|\nabla u|^2] & \\ +V(x)u + \phi u = g(u)G^{p-1}(u), & \text{in } \mathbb{R}^3, \\ -\Delta\phi = u^2, & \text{in } \mathbb{R}^3. \end{cases} \tag{1.9}$$

Theorem 1.6. *Suppose that (g') holds. If*

$$2V(x) + \langle \nabla V(x), x\rangle \ge 0, \quad \forall x \in \mathbb{R}^3,$$

then problem (1.9) has no nontrivial solutions when $p \ge 6$.

Remark 1.7. Throughout the paper we denote by C, C_i $(i = 1, 2, \dots) > 0$ various positive constants which may vary from line to line and are not essential to the problem.

2 Preliminary results

In this section, we introduce the variational framework associated with problem (1.1). Let $L^p(\mathbb{R}^3)$ be the usual Lebesgue space with the norm $|u|_p = \left(\int_{\mathbb{R}^3} |u|^p dx\right)^{\frac{1}{p}}$.

Define the space H given by

$$H^1 := H^1(\mathbb{R}^3) = \{u \in L^2(\mathbb{R}^3) : \nabla u \in L^2(\mathbb{R}^3)\},$$

with the norm:

$$\|u\| = \left(\int_{\mathbb{R}^3} (|\nabla u|^2 + u^2) dx\right)^{\frac{1}{2}},$$

and the corresponding inner is

$$\langle u, v\rangle = \int_{\mathbb{R}^3} (\nabla u \cdot \nabla v + uv) dx.$$

Let

$$D^{1,2}(\mathbb{R}^3) = \{u \in L^6(\mathbb{R}^3) : \nabla u \in L^2(\mathbb{R}^3)\}.$$

According to the assumption (V_1), we use the space

$$\mathcal{H} := \left\{u \in H^1(\mathbb{R}^3) : u(x) = u(|x|), \int_{\mathbb{R}^N} V(x)u^2 dx < \infty\right\},$$

with the norm:

$$\|u\|_{\mathcal{H}} = \left(\int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2) dx\right)^{\frac{1}{2}},$$

then the embedding $\mathcal{H} \hookrightarrow L^p(\mathbb{R}^3)$ is compact for $2 < p < 6$.

According to [3], the energy functional associated with (1.1) is

$$\begin{aligned} I_b(u) = {} & \frac{1}{2}\int_{\mathbb{R}^3} g^2(u)|\nabla u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^3} V(x)|u|^2 dx + \frac{b}{4}\left(\int_{\mathbb{R}^3} g^2(u)|\nabla u|^2 dx\right)^2 \\ & + \frac{1}{4}\int_{\mathbb{R}^3} \phi|u|^2 dx - \int_{\mathbb{R}^3} H(u) dx, \end{aligned} \tag{2.1}$$

where $H(t) = \int_0^t h(s)ds$. Since the term $\int_{\mathbb{R}^3} g^2(u)|\nabla u|^2 dx$ is not well defined in $\mathcal{H}$, to overcome this difficulty, a change of variable constructed in [28] is very helpful to us. For any $v \in \mathcal{H}$, let

$$u = G^{-1}(v) \quad \text{and} \quad G(u) = \int_0^u g(t)dt,$$

then

$$\int_{\mathbb{R}^3} g^2(u)|\nabla u|^2 dx = \int_{\mathbb{R}^3} g^2(G^{-1}(v))|\nabla G^{-1}(v)|^2 dx =: |\nabla v|_2^2 < \infty,$$

and $I_b(u)$ can be reduced to

$$\begin{aligned} J_b(v) = {} & \frac{1}{2}\int_{\mathbb{R}^3} |\nabla v|^2 dx + \frac{1}{2}\int_{\mathbb{R}^3} V(x)|G^{-1}(v)|^2 dx + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla v|^2 dx\right)^2 \\ & + \frac{1}{4}\int_{\mathbb{R}^3} \phi_{G^{-1}(v)}|G^{-1}(v)|^2 dx - \int_{\mathbb{R}^3} H(G^{-1}(v)) dx. \end{aligned} \tag{2.2}$$

Then $v \in \mathcal{H}$ is a solution of (1.1) if

$$\langle J_b'(v), \eta\rangle = \int_{\mathbb{R}^3} \nabla v \cdot \nabla \eta dx + \int_{\mathbb{R}^3} V(x)\frac{G^{-1}(v)}{g(G^{-1}(v))}\eta dx + b\int_{\mathbb{R}^3} |\nabla v|^2 dx \int_{\mathbb{R}^3} \nabla v \cdot \nabla \eta dx + \int_{\mathbb{R}^3} \phi_{G^{-1}(v)}\frac{G^{-1}(v)}{g(G^{-1}(v))}\eta dx - \int_{\mathbb{R}^3} \frac{h(G^{-1}(v))}{g(G^{-1}(v))}\eta dx = 0,$$

for all $\eta \in H^1(\mathbb{R}^3)$.

We observe that by the Lax–Milgram theorem, for given $u \in H^1(\mathbb{R}^3)$, there exists a unique solution $\phi = \phi_u \in D^{1,2}(\mathbb{R}^3)$ satisfying $-\Delta\phi_u = u^2$ in a weak sense. The function ϕ_u is represented by

$$\phi_u(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|}dy,$$

and it has the following properties.

Lemma 2.1 ([11,26]). *The following properties hold:*

(*i*) *there exists $C > 0$ such that for any $u \in H^1(\mathbb{R}^3)$,*

$$\|\phi_u\|_{D^{1,2}} \le C|u|^2_{\frac{12}{5}}, \quad \int_{\mathbb{R}^3} |\nabla\phi_u|^2 dx = \int_{\mathbb{R}^3} \phi_u u^2 dx \le C\|u\|^4_{\mathcal{H}};$$

(*ii*) *$\phi_u \ge 0$ for all $u \in H^1(\mathbb{R}^3)$;*

(*iii*) *$\phi_{tu} = t^2\phi_u$ for all $t > 0$ and $u \in H^1(\mathbb{R}^3)$;*

(*iv*) *if $u_j \rightharpoonup u$ weakly in $H^1(\mathbb{R}^3)$, then, up to a subsequence, $\phi_{u_j} \to \phi_u$ in $D^{1,2}(\mathbb{R}^3)$ and*

$$\int_{\mathbb{R}^3} \phi_{u_j} u_j^2 dx \to \int_{\mathbb{R}^3} \phi_u u^2 dx.$$

Lemma 2.2 ([19,22]). *For the function g, G, and G^{-1}, the following properties hold under the condition (g_1):*

(1) $\frac{t}{g(t)}g'(t) \le 0$ *for all* $t \ge 0$;

(2) $|t| \le |G^{-1}(t)| \le \frac{|t|}{a}$ *for all* $t \in \mathbb{R}$;

(3) $t^2 \le \frac{t}{g(t)}G(t) \le \frac{t^2}{a}$ *for all* $t \in \mathbb{R}$.

Under condition (g'), the following properties hold:

(4) $|G^{-1}(t)| \le \frac{1}{g(0)}|t| = |t|$ *for all* $t \in \mathbb{R}$;

(5) $\frac{G^{-1}(t)t}{g(G^{-1}(t))} \le |G^{-1}(t)|^2$ *for all* $t \in \mathbb{R}$;

(6) $\lim\limits_{|t|\to 0}\frac{G^{-1}(t)}{t} = \frac{1}{g(0)} = 1$ *and*

$$\lim_{|t|\to\infty}\frac{G^{-1}(t)}{t} = \begin{cases} \frac{1}{g(\infty)}, & \text{if } g \text{ is bounded,} \\ 0, & \text{if } g \text{ is unbounded.} \end{cases}$$

3 Proof of the main results

3.1 Proof of Theorem 1.1

In this section, we will prove the nonexistence of nontrivial solutions for the following system:

$$\begin{cases} \left(1+b\int_{\mathbb{R}^3} g^2(u)|\nabla u|^2dx\right)[-\mathrm{div}(g^2(u)\nabla u)+g(u)g'(u)|\nabla u|^2] \\ +V(x)u+\phi u=|u|^{p-2}u, & x\in\mathbb{R}^3, \\ -\Delta\phi=u^2, & x\in\mathbb{R}^3. \end{cases} \tag{3.1}$$

By a standard argument in [20,35], we can obtain the following Pohožaev identity.

Lemma 3.1 (Pohožaev identity). *If $v\in\mathcal{H}$ is a weak solution of (3.1), then v satisfies*

$$\begin{aligned} &\frac{1}{2}\int_{\mathbb{R}^3}|\nabla v|^2dx+\frac{3}{2}\int_{\mathbb{R}^3}V(x)|G^{-1}(v)|^2dx+\frac{1}{2}\int_{\mathbb{R}^3}\langle\nabla V(x),x\rangle|G^{-1}(v)|^2dx \\ &\quad+\frac{b}{2}\left(\int_{\mathbb{R}^3}|\nabla v|^2dx\right)^2+\frac{5}{4}\int_{\mathbb{R}^3}\phi_{G^{-1}(v)}|G^{-1}(v)|^2dx=\frac{3}{p}\int_{\mathbb{R}^3}|G^{-1}(v)|^pdx. \end{aligned} \tag{3.2}$$

Proof of Theorem 1.1. Indeed, because $(v,\phi_{G^{-1}(v)})\in\mathcal{H}\times D^{1,2}(\mathbb{R}^3)$ is a solution of (3.1), we have the following equation:

$$\begin{aligned} &\int_{\mathbb{R}^3}|\nabla v|^2dx+\int_{\mathbb{R}^3}V(x)\frac{G^{-1}(v)}{g(G^{-1}(v))}vdx+b\left(\int_{\mathbb{R}^3}|\nabla v|^2dx\right)^2 \\ &\quad+\int_{\mathbb{R}^3}\phi_{G^{-1}(v)}\frac{G^{-1}(v)}{g(G^{-1}(v))}vdx=\int_{\mathbb{R}^3}\frac{|G^{-1}(v)|^{p-2}G^{-1}(v)}{g(G^{-1}(v))}vdx. \end{aligned} \tag{3.3}$$

Multiply the equation of the (3.3) by $\frac{5}{4}$ and combined with (3.2), we have

$$\begin{aligned} &\frac{3}{4}\int_{\mathbb{R}^3}|\nabla v|^2dx+\int_{\mathbb{R}^3}V(x)\left(\frac{5}{4}\frac{G^{-1}(v)v}{g(G^{-1}(v))}-\frac{3}{2}|G^{-1}(v)|^2\right)dx+\frac{3}{4}b\left(\int_{\mathbb{R}^3}|\nabla v|^2dx\right)^2 \\ &\quad-\frac{1}{2}\int_{\mathbb{R}^3}\langle\nabla V(x),x\rangle|G^{-1}(v)|^2dx+\frac{5}{4}\int_{\mathbb{R}^3}\phi_{G^{-1}(v)}\left(\frac{G^{-1}(v)v}{g(G^{-1}(v))}-|G^{-1}(v)|^2\right)dx \\ &=\int_{\mathbb{R}^3}|G^{-1}(v)|^{p-2}\left(\frac{5}{4}\frac{G^{-1}(v)v}{g(G^{-1}(v))}-\frac{3}{p}|G^{-1}(v)|^2\right)dx. \end{aligned} \tag{3.4}$$

If $V(x)+2\langle\nabla V(x),x\rangle\le 0$, then by Lemma 2.2-(3) that the left hand of (3.4) is nonnegative, so that the equation (3.4) has no nontrivial solutions when $p\le\frac{12a}{5}$.

Combining (3.2) and (3.3), it can be concluded that

$$\begin{aligned} &\frac{1}{2}\int_{\mathbb{R}^3}V(x)\left(\frac{G^{-1}(v)v}{g(G^{-1}(v))}-3|G^{-1}(v)|^2\right)dx-\frac{1}{2}\int_{\mathbb{R}^3}\langle\nabla V(x),x\rangle|G^{-1}(v)|^2dx \\ &\quad+\frac{1}{2}\int_{\mathbb{R}^3}\phi_{G^{-1}(v)}\left(\frac{G^{-1}(v)v}{g(G^{-1}(v))}-\frac{5}{2}|G^{-1}(v)|^2\right)dx \\ &=\int_{\mathbb{R}^3}|G^{-1}(v)|^{p-2}\left(\frac{G^{-1}(v)v}{2g(G^{-1}(v))}-\frac{3}{p}|G^{-1}(v)|^2\right)dx. \end{aligned} \tag{3.5}$$

Under conditions $\frac{2}{5}\le a<1$, Lemma 2.2-(3) and $V(x)+2\langle\nabla V(x),x\rangle\ge 0$, for $p\ge 6$, the equation (3.5) has no nontrivial solutions. □

3.2 Proof of Theorem 1.2

This section provides the proof of Theorem 1.2. Clearly, as mentioned previously, we will devote to studying the functional J_b. It is hard to prove the boundedness of the (PS) sequence of J_b, so that finding a special bounded (PS) sequence of $J_{b,\mu}$ may provides great help,

$$\begin{aligned} J_{b,\mu}(v) := &\frac{1}{2}\int_{\mathbb{R}^3}|\nabla v|^2dx + \frac{1}{2}\int_{\mathbb{R}^3}V(x)|G^{-1}(v)|^2dx + \frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla v|^2dx\right)^2 \\ &+ \frac{1}{4}\int_{\mathbb{R}^3}\phi_{G^{-1}(v)}|G^{-1}(v)|^2dx - \mu\int_{\mathbb{R}^3}H(G^{-1}(v))dx, \end{aligned} \tag{3.6}$$

where $\mu \in [1, 2]$.

Next, we list a useful critical point theorem proposed by Jeanjean [3,16], which is crucial to obtain our main result.

Theorem 3.2. *Let $(X, \|\cdot\|)$ be a Banach space and $\mathbb{I} \subset \mathbb{R}^+$ an interval. Consider the following family of C^1-functionals on X:*

$$J_{b,\mu}(v) = A(v) - \mu B(v), \quad \forall \mu \in \mathbb{I},$$

where $B(v) \geq 0$, $\forall v \in X$, and either $A(v) \to +\infty$ or $B(v) \to +\infty$ as $\|v\| \to \infty$. Assume that there are two points v_1, v_2 in X, such that

$$c_\mu := \inf_{\gamma\in\Gamma_\mu}\max_{t\in[0,1]} J_{b,\mu}(\gamma(t)) > \max\{J_{b,\mu}(v_1), J_{b,\mu}(v_2)\}, \quad \forall \mu \in \mathbb{I},$$

where

$$\Gamma_\mu = \{\gamma \in \mathcal{C}([0,1], X) : \gamma(0) = v_1, \gamma(1) = v_2\}.$$

Then, for almost every $\mu \in \mathbb{I}$, there is a sequence $\{v_n\} \subset X$, such that

(i) $\{v_n\}$ is bounded;

(ii) $J_{b,\mu}(v_n) \to c_\mu$;

(iii) $J'_{b,\mu}(v_n) \to 0$ in the dual X^{-1} of X.

Moreover, the map $\mu \mapsto c_\mu$ is continuous from the left.

Lemma 3.3. *Assume that (V_1), (g_1), (h_1)–(h_3) are satisfied, then*

(i) for $\mu \in [1, 2]$, there exists $v \in \mathcal{H}\backslash\{0\}$ such that $J_{b,\mu}(v) < 0$;

(ii) there exists $\rho, \alpha > 0$ such that $J_{b,\mu}(v) \geq \alpha$ for $\|v\|_{\mathcal{H}} = \rho$.

Proof. (i) According to (h_1) and (h_3), there exists constant $C > 0$ such that

$$H(G^{-1}(v)) \geq C|v|^\tau \quad \text{for all } v \geq 0 \quad \text{and} \quad H(G^{-1}(v)) \equiv 0 \quad \text{for all } v \leq 0,$$

where τ is defined in (h_3).

For any fixed $\psi \in \mathcal{H}$ with $\psi > 0$, by Lemma 2.2-(2), we have for $\tau > 4$

$$\begin{aligned} J_{b,\mu}(t\psi) = &\frac{1}{2}\int_{\mathbb{R}^3}\left(|\nabla t\psi|^2 + V(x)|G^{-1}(t\psi)|^2\right)dx + \frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla t\psi|^2dx\right)^2 \\ &+ \frac{1}{4}\int_{\mathbb{R}^3}\phi_{G^{-1}(t\psi)}|G^{-1}(t\psi)|^2dx - \mu\int_{\mathbb{R}^3}H(G^{-1}(t\psi))dx \\ \leq\ & Ct^2\|\psi\|^2_{\mathcal{H}} + \frac{bt^4}{4}\left(\int_{\mathbb{R}^3}|\nabla\psi|^2dx\right)^2 + \frac{t^4}{4}C\|\psi\|^4_{\mathcal{H}} - \mu Ct^\tau\int_{\mathbb{R}^3}|\psi|^\tau dx \\ \to\ & -\infty, \end{aligned}$$

as $t \to \infty$, which implies that (i) holds if we take $v = t\psi$ with t sufficiently large.

(ii) Let $\varepsilon \in \left(0, \frac{a^2 V_0}{2\mu}\right)$, by Lemma 2.2-(2) and (1.8), we obtain

$$\begin{aligned} J_{b,\mu}(v) &= \frac{1}{2}\int_{\mathbb{R}^3} |\nabla v|^2 dx + \frac{1}{2}\int_{\mathbb{R}^3} V(x)|G^{-1}(v)|^2 dx + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla v|^2 dx\right)^2 \\ &\quad + \frac{1}{4}\int_{\mathbb{R}^3} \phi_{G^{-1}(v)}|G^{-1}(v)|^2 dx - \mu\int_{\mathbb{R}^3} H(G^{-1}(v))dx \\ &\geq \frac{1}{2}\int_{\mathbb{R}^3} |\nabla v|^2 dx + \frac{1}{2}\int_{\mathbb{R}^3} \left(V_0 - \frac{\mu\varepsilon}{a^2}\right)|v|^2 dx - \frac{C_\varepsilon \mu}{qa^q}\int_{\mathbb{R}^3} |v|^q dx. \\ &\geq C_1\|v\|_{\mathcal{H}}^2 - C_2\|v\|_{\mathcal{H}}^q. \end{aligned}$$

Hence, for $q > 4$ we can choose $\|v\|_{\mathcal{H}} = \rho > 0$ small enough such that $J_{b,\mu}(v) \geq \alpha > 0$. □

Define

$$A(v) := \frac{1}{2}\int_{\mathbb{R}^3}\left[|\nabla v|^2 + V(x)|G^{-1}(v)|^2\right]dx + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla v|^2 dx\right)^2 + \frac{1}{4}\int_{\mathbb{R}^3} \phi_{G^{-1}(v)}|G^{-1}(v)|^2 dx,$$

$$B(v) := \int_{\mathbb{R}^3} H(G^{-1}(v))dx.$$

It is deduced from (V_1) and Lemma 2.2-(2) that

$$A(v) > \frac{1}{2}\|v\|_{\mathcal{H}}^2 \to +\infty, \quad \text{as } \|v\|_{\mathcal{H}} \to +\infty.$$

Moreover, from (h_1), it can be observed that $B(v) = \int_{\mathbb{R}^3} H(G^{-1}(v))dx \geq 0$, $\forall v \in \mathcal{H}$.

Based on the above facts and Lemma 3.3, the conclusion of Theorem 3.2 holds. It shows that for a.e. $\mu \in [1, 2]$, there is a bounded $(PS)_{c_\mu}$ sequence $\{v_n\} \subset \mathcal{H}$, satisfying $J_{b,\mu}(v_n) \to c_\mu$ and $J'_{b,\mu}(v_n) \to 0$, where c_μ is the mountain pass level.

Lemma 3.4. *Suppose (V_1), (g_1), (h_1)–(h_3) hold, $\{v_n\}$ is the sequence obtained above, going if necessary to a subsequence, $v_n \to v$ in $\mathcal{H}$.*

Proof. Since $\{v_n\} \subset \mathcal{H}$ is bounded, up to a subsequence, there exists $v \in \mathcal{H}$ such that

$$\begin{aligned} v_n &\rightharpoonup v \quad \text{in } \mathcal{H}, \\ v_n &\to v \quad \text{in } L^r(\mathbb{R}^3), \ \ 2 < r < 6, \\ v_n(x) &\to v(x) \quad \text{a.e. on } x \in \mathbb{R}^3. \end{aligned}$$

Define $\varphi : \mathbb{R} \to \mathbb{R}$ by $\varphi(t) = \frac{G^{-1}(t)}{g(G^{-1}(t))}$. According to (g_1), then $a < g(t) \leq 1$ for $t \in \mathbb{R}$, jointly with Lemma 2.2-(1), we have

$$\varphi'(t) = \frac{1}{g^2(G^{-1}(t))}\left[1 - \frac{G^{-1}(t)g'(G^{-1}(t))}{g(G^{-1}(t))}\right] \geq \frac{1}{g^2(G^{-1}(t))} \geq 1.$$

According to the mean value theorem, for any $x \in \mathbb{R}^3$, there exists a function $\xi(x)$ between $v(x)$ and $v_n(x)$ such that

$$\begin{aligned} &\int_{\mathbb{R}^3} V(x)\left[\frac{G^{-1}(v_n)}{g(G^{-1}(v_n))} - \frac{G^{-1}(v)}{g(G^{-1}(v))}\right](v_n - v)dx \\ &= \int_{\mathbb{R}^3} V(x)\varphi'(\xi)|v_n - v|^2 dx \\ &\geq \int_{\mathbb{R}^3} V(x)|v_n - v|^2 dx, \end{aligned}$$

and with the help of Lemma 2.1, we have

$$\begin{aligned}
&\int_{\mathbb{R}^3}\left[\phi_{G^{-1}(v_n)}\frac{G^{-1}(v_n)}{g(G^{-1}(v_n))}-\phi_{G^{-1}(v)}\frac{G^{-1}(v)}{g(G^{-1}(v))}\right](v_n-v)dx\\
&\quad=\int_{\mathbb{R}^3}\phi_{G^{-1}(v_n)}\left[\frac{G^{-1}(v_n)}{g(G^{-1}(v_n))}-\frac{G^{-1}(v)}{g(G^{-1}(v))}\right](v_n-v)dx\\
&\qquad+\int_{\mathbb{R}^3}\left(\phi_{G^{-1}(v_n)}-\phi_{G^{-1}(v)}\right)\frac{G^{-1}(v)}{g(G^{-1}(v))}(v_n-v)dx\\
&\quad\geq\int_{\mathbb{R}^3}\phi_{G^{-1}(v_n)}\varphi'(\xi)|v_n-v|^2dx\\
&\quad\geq\int_{\mathbb{R}^3}\phi_{G^{-1}(v_n)}|v_n-v|^2dx\\
&\quad\to 0.
\end{aligned}$$

Choose $A_n=\int_{\mathbb{R}^3}|\nabla v_n|^2dx$ and $A=\int_{\mathbb{R}^3}|\nabla v|^2dx$. By $v_n\rightharpoonup v$ in $\mathcal{H}$, thus $\{A_n-A\}$ is bounded. By the weak convergence of v_n in $\mathcal{H}$, we get $b(A_n-A)\int_{\mathbb{R}^3}\nabla v\nabla(v_n-v)dx=o_n(1)$. So that

$$\begin{aligned}
&b\int_{\mathbb{R}^3}|\nabla v_n|^2dx\int_{\mathbb{R}^3}\nabla v_n\nabla(v_n-v)dx-b\int_{\mathbb{R}^3}|\nabla v|^2dx\int_{\mathbb{R}^3}\nabla v\nabla(v_n-v)dx\\
&\quad\geq b(A_n-A)\int_{\mathbb{R}^3}\nabla v\nabla(v_n-v)dx\\
&\quad\to\ 0,
\end{aligned}$$

as $n\to\infty$. Noting that $a<g(t)\leq 1$, then by Lemma 2.2-(2), (1.8) and Hölder inequality, $v_n\to v$ in $L^r(\mathbb{R}^3)$, $2<r<6$, we obtain

$$\begin{aligned}
&\left|\int_{\mathbb{R}^3}\left[\frac{h(G^{-1}(v_n))}{g(G^{-1}(v_n))}-\frac{h(G^{-1}(v))}{g(G^{-1}(v))}\right](v_n-v)dx\right|\\
&\quad\leq C\int_{\mathbb{R}^3}\left[\varepsilon(|v_n|+|v|)+C_\varepsilon(|v_n|^{q-1}+|v|^{q-1})\right]|v_n-v|dx\\
&\quad\leq C\varepsilon(|v_n|_2+|v|_2)|v_n-v|_2+CC_\varepsilon(|v_n|_q^{q-1}+|v|_q^{q-1})|v_n-v|_q\\
&\quad\to 0,
\end{aligned}$$

as $n\to\infty$. Then

$$\begin{aligned}
o_n(1)&=\langle J'_{b,\mu}(v_n)-J'_{b,\mu}(v),\,v_n-v\rangle\\
&=\int_{\mathbb{R}^3}|\nabla(v_n-v)|^2dx+\int_{\mathbb{R}^3}V(x)\left[\frac{G^{-1}(v_n)}{g(G^{-1}(v_n))}-\frac{G^{-1}(v)}{g(G^{-1}(v))}\right](v_n-v)dx\\
&\quad+b\left[\int_{\mathbb{R}^3}|\nabla v_n|^2dx\int_{\mathbb{R}^3}\nabla v_n\cdot\nabla(v_n-v)dx-\int_{\mathbb{R}^3}|\nabla v|^2dx\int_{\mathbb{R}^3}\nabla v\cdot\nabla(v_n-v)dx\right]\\
&\quad+\int_{\mathbb{R}^3}\left[\phi_{G^{-1}(v_n)}\frac{G^{-1}(v_n)}{g(G^{-1}(v_n))}-\phi_{G^{-1}(v)}\frac{G^{-1}(v)}{g(G^{-1}(v))}\right](v_n-v)dx\\
&\quad-\mu\int_{\mathbb{R}^3}\left[\frac{h(G^{-1}(v_n))}{g(G^{-1}(v_n))}-\frac{h(G^{-1}(v))}{g(G^{-1}(v))}\right](v_n-v)dx\\
&\geq\int_{\mathbb{R}^3}|\nabla(v_n-v)|^2dx+\int_{\mathbb{R}^3}V(x)|v_n-v|^2dx\\
&=\|v_n-v\|^2_{\mathcal{H}}.
\end{aligned}$$

Therefore, $v_n\to v$ in $\mathcal{H}$. □

According to Theorem 3.2, there is a bounded sequence $\{v_n\} \subset \mathcal{H}$ satisfying $J_{b,\mu}(v_n) \to c_\mu$ and $J'_{b,\mu}(v_n) \to 0$. From Lemma 3.4, for every $\mu \in [1, 2]$, we get that $J_{b,\mu}$ has a nontrivial critical point v satisfying $J_{b,\mu}(v) = c_\mu$ and $J'_{b,\mu}(v) = 0$. Therefore, there exists $\{\mu_n\} \subset [1, 2]$ such that $\lim_{n\to\infty} \mu_n = 1$, $\{v_{\mu_n}\} \subset \mathcal{H}$ such that $J_{b,\mu_n}(v_{\mu_n}) = c_{\mu_n} > 0$ and $J'_{b,\mu_n}(v_{\mu_n}) = 0$.

Lemma 3.5. *Suppose that (g_1), (h_1) and (h_3) are satisfied, then sequence $\{v_{\mu_n}\}$ is bounded in $\mathcal{H}$.*

Proof. Since sequence $\{v_{\mu_n}\} \subset \mathcal{H}$ satisfies

$$J_{b,\mu_n}(v_{\mu_n}) = c_{\mu_n} \quad \text{and} \quad J'_{b,\mu_n}(v_{\mu_n}) = 0.$$

Then we have

$$\begin{aligned} J_{b,\mu_n}(v_{\mu_n}) &= \frac{1}{2}\int_{\mathbb{R}^3} |\nabla v_{\mu_n}|^2 dx + \frac{1}{2}\int_{\mathbb{R}^3} V(x)|G^{-1}(v_{\mu_n})|^2 dx + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla v_{\mu_n}|^2 dx\right)^2 \\ &\quad + \frac{1}{4}\int_{\mathbb{R}^3} \phi_{G^{-1}(v_{\mu_n})}|G^{-1}(v_{\mu_n})|^2 dx - \mu_n \int_{\mathbb{R}^3} H(G^{-1}(v_{\mu_n}))dx \\ &= c_{\mu_n} > 0. \end{aligned} \tag{3.7}$$

We also deduce that

$$\begin{aligned} \langle J'_{b,\mu_n}(v_{\mu_n}), v_{\mu_n}\rangle = &\int_{\mathbb{R}^3} |\nabla v_{\mu_n}|^2 dx + \int_{\mathbb{R}^3} V(x)\frac{G^{-1}(v_{\mu_n})}{g(G^{-1}(v_{\mu_n}))} v_{\mu_n} dx \\ &+ b\left(\int_{\mathbb{R}^3} |\nabla v_{\mu_n}|^2 dx\right)^2 + \int_{\mathbb{R}^3} \phi_{G^{-1}(v_{\mu_n})}\frac{G^{-1}(v_{\mu_n})}{g(G^{-1}(v_{\mu_n}))} v_{\mu_n} dx \\ &- \mu_n \int_{\mathbb{R}^3} \frac{h(G^{-1}(v_{\mu_n}))}{g(G^{-1}(v_{\mu_n}))} v_{\mu_n} dx \\ = &\, o(1)\|v_{\mu_n}\|, \end{aligned} \tag{3.8}$$

as $n \to \infty$. It follows from (3.7), (3.8), (h_3) and Lemma 2.2-(2), (3) that

$$\begin{aligned} &c_{\mu_n} + 1 + \|v_{\mu_n}\| \\ &\geq J_{b,\mu_n}(v_{\mu_n}) - \frac{a}{4}\langle J'_{b,\mu_n}(v_{\mu_n}), v_{\mu_n}\rangle \\ &\geq \left(\frac{1}{2} - \frac{a}{4}\right)\int_{\mathbb{R}^3} |\nabla v_{\mu_n}|^2 dx + \int_{\mathbb{R}^3} V(x)G^{-1}(v_{\mu_n})\left[\frac{1}{2}G^{-1}(v_{\mu_n}) - \frac{a v_{\mu_n}}{4g(G^{-1}(v_{\mu_n}))}\right]dx \\ &\quad + \left(\frac{1}{4} - \frac{a}{4}\right) b\left(\int_{\mathbb{R}^3} |\nabla v_{\mu_n}|^2 dx\right)^2 + \int_{\mathbb{R}^3} \phi_{G^{-1}(v_{\mu_n})} G^{-1}(v_{\mu_n})\left[\frac{1}{4}G^{-1}(v_{\mu_n}) - \frac{a v_{\mu_n}}{4g(G^{-1}(v_{\mu_n}))}\right]dx \\ &\quad - \mu_n \int_{\mathbb{R}^3} \left[H(G^{-1}(v_{\mu_n})) - \frac{a}{4}\frac{h(G^{-1}(v_{\mu_n}))}{g(G^{-1}(v_{\mu_n}))} v_{\mu_n}\right] dx \\ &\geq \left(\frac{1}{2} - \frac{a}{4}\right)\int_{\mathbb{R}^3} |\nabla v_{\mu_n}|^2 dx + \frac{1}{4}\int_{\mathbb{R}^3} V(x)|G^{-1}(v_{\mu_n})|^2 dx \\ &\geq \frac{1}{4}\|v_{\mu_n}\|^2_{\mathcal{H}}, \end{aligned}$$

which means that $\{v_{\mu_n}\}$ is bounded in $\mathcal{H}$. □

Proof of Theorem 1.2. Taking a subsequence of $\{v_{\mu_n}\}$ still represented by $\{v_{\mu_n}\}$, because $\{v_{\mu_n}\}$ is bounded in $\mathcal{H}$, similar to the proof of Lemma 3.4, we obtain $v_{\mu_n} \to v_\mu$ in $\mathcal{H}$. From Theorem 3.2,

we know that $\mu \mapsto c_\mu$ is continuous from the left. So

$$\begin{aligned}\lim_{n\to\infty} J_b(v_{\mu_n}) &= \lim_{n\to\infty}\left[J_{b,\mu_n}(v_{\mu_n}) + (\mu_n - 1)\int_{\mathbb{R}^3} H(G^{-1}(v_{\mu_n}))dx\right] \\ &= \lim_{n\to\infty} c_{\mu_n} = \tilde{c}.\end{aligned}$$

In addition,

$$\lim_{n\to\infty}\langle J_b'(v_{\mu_n}), \eta\rangle = \lim_{n\to\infty}\left[\langle J_{b,\mu_n}'(v_{\mu_n}), \eta\rangle + (\mu_n - 1)\int_{\mathbb{R}^3}\frac{h(G^{-1}(v_{\mu_n}))}{g(G^{-1}(v_{\mu_n}))}\eta dx\right] = 0,$$

for any $\eta \in C_0^\infty(\mathbb{R}^3)$, which means that $J_b'(v) = 0$ satisfies $J_b(v) = \tilde{c} > 0$.

Let $v^- = \min\{v, 0\}$. Using Lemma 2.2-(2), (3) and with help of the assumption (h_1) and Lemma 2.1, we have

$$\begin{aligned}0 &= \langle J_b'(v), v^-\rangle \\ &= \int_{\mathbb{R}^3}\left(|\nabla v^-|^2 + V(x)\frac{G^{-1}(v^-)}{g(G^{-1}(v^-))}v^-\right)dx \\ &\geq \int_{\mathbb{R}^3}|\nabla v^-|^2 + V(x)|v^-|^2 dx \\ &\geq \|v^-\|_{\mathcal{H}}^2.\end{aligned}$$

It shows that $v^- \equiv 0$. Applying the strong maximum principle, we obtain $v > 0$. □

4 Asymptotic properties

Now, we are in a situation to give the proof of convergence properties.

Proof of Theorem 1.3. If v_{b_n} is a critical point of J_{b_n}, which is obtained in Theorem 1.2 for each $n \in \mathbb{N}$. Similar to the proof of Lemma 3.3, for $b_n \to 0$, $n \to \infty$, $\{v_{b_n}\}$ is a (PS) sequence, which is bounded in $\mathcal{H}$. There exists a subsequence of $\{b_n\}$, still denoted by $\{b_n\}$, such that $v_{b_n} \rightharpoonup v_0$ in $\mathcal{H}$. It is easy to obtain

$$\begin{aligned}\|v_{b_n} - v_0\|_{\mathcal{H}}^2 \leq{}& \langle J_{b_n}'(v_{b_n}) - J_0'(v_0), v_{b_n} - v_0\rangle - b_n\int_{\mathbb{R}^3}|\nabla v_{b_n}|^2 dx\int_{\mathbb{R}^3}\nabla v_{b_n}\nabla(v_{b_n} - v_0)dx \\ &+ \int_{\mathbb{R}^3}\left[\phi_{G^{-1}(v_0)}\frac{G^{-1}(v_0)}{g(G^{-1}(v_0))} - \phi_{G^{-1}(v_{b_n})}\frac{G^{-1}(v_{b_n})}{g(G^{-1}(v_{b_n}))}\right](v_{b_n} - v_0)dx \\ &+ \int_{\mathbb{R}^3}\left[\frac{h(G^{-1}(v_{b_n}))}{g(G^{-1}(v_{b_n}))} - \frac{h(G^{-1}(v_0))}{g(G^{-1}(v_0))}\right](v_{b_n} - v_0)dx \\ ={}& o_n(1).\end{aligned}$$

On the one hand, for all $\eta \in \mathcal{H}\backslash\{0\}$, in view of Lemma 2.2-(2), we can use the Lebesgue dominated convergence theorem to obtain

$$\lim_{n\to\infty}\int_{\mathbb{R}^3} V(x)\frac{G^{-1}(v_{b_n})\eta}{g(G^{-1}(v_{b_n}))}dx = \int_{\mathbb{R}^3} V(x)\frac{G^{-1}(v_0)\eta}{g(G^{-1}(v_0))}dx,$$

$$\lim_{n\to\infty}\int_{\mathbb{R}^3}\frac{h(G^{-1}(v_{b_n}))\eta}{g(G^{-1}(v_{b_n}))}dx = \int_{\mathbb{R}^3}\frac{h(G^{-1}(v_0))\eta}{g(G^{-1}(v_0))}dx.$$

By Lemma 2.1, we can obtain

$$\lim_{n\to\infty}\int_{\mathbb{R}^3}\phi_{G^{-1}(v_{b_n})}\frac{G^{-1}(v_{b_n})\eta}{g(G^{-1}(v_{b_n}))}dx=\int_{\mathbb{R}^3}\phi_{G^{-1}(v_0)}\frac{G^{-1}(v_0)\eta}{g(G^{-1}(v_0))}dx.$$

On the other hand, we have $\langle J'_{b_n}(v_{b_n}),\eta\rangle=o_n(1)$ and $\langle J'_0(v_0),\eta\rangle=o_n(1)$. Moreover,

$$\lim_{n\to\infty}\int_{\mathbb{R}^3}\nabla v_{b_n}\nabla\eta dx=\int_{\mathbb{R}^3}\nabla v_0\nabla\eta dx,$$

$$\lim_{n\to\infty}b_n\int_{\mathbb{R}^3}|\nabla v_{b_n}|^2dx\int_{\mathbb{R}^3}\nabla v_{b_n}\nabla\eta=0.$$

Thus,

$$\int_{\mathbb{R}^3}\nabla v_0\nabla\eta dx+\int_{\mathbb{R}^3}V(x)\frac{G^{-1}(v_0)}{g(G^{-1}(v_0)}\eta dx+\int_{\mathbb{R}^3}\phi_{G^{-1}(v_0)}\frac{G^{-1}(v_0)}{g(G^{-1}(v_0))}\eta dx=\int_{\mathbb{R}^3}\frac{h(G^{-1}(v_0))}{g(G^{-1}(v_0))}\eta dx.$$

It shows that v_0 is a positive solution of (1.5). □

5 Proof of Theorem 1.4 and Theorem 1.5

In this section, we will prove the existence of nontrivial radial solution for system (1.1) under the condition (g').

Lemma 5.1. *Assume that (V_1), (g'), (h_1) and (h_3) are satisfied, then*

(i) for $\mu\in[1,2]$, there exists $v\in\mathcal{H}\backslash\{0\}$ such that $J_{b,\mu}(v)<0$;

(ii) there exists $\rho_1,\alpha_1>0$ such that $J_{b,\mu}(v)\geq\alpha_1$ and $\|v\|_{\mathcal{H}}=\rho_1$.

Proof. (i) According to (h_1) and (h_3), there exists constant $C>0$ such that

$$H(G^{-1}(v))\geq C|v|^{\tau}\quad\text{for all } v\geq 0\quad\text{and}\quad H(G^{-1}(v))\equiv 0\quad\text{for all } v\leq 0,$$

where τ is defined in (h_3).

For any fixed $\psi_1\in\mathcal{H}$ with $\psi_1>0$, by Lemma 2.2-(4), we have

$$\begin{aligned}J_{b,\mu}(t\psi_1)&=\frac{1}{2}\int_{\mathbb{R}^3}\left(|\nabla t\psi_1|^2+V(x)|G^{-1}(t\psi_1)|^2\right)dx+\frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla t\psi_1|^2dx\right)^2\\&\quad+\frac{1}{4}\int_{\mathbb{R}^3}\phi_{G^{-1}(t\psi_1)}|G^{-1}(t\psi_1)|^2dx-\mu\int_{\mathbb{R}^3}H(G^{-1}(t\psi_1))dx\\&\leq\frac{t^2}{2}\|\psi_1\|^2_{\mathcal{H}}+\frac{b}{4}t^4\left(\int_{\mathbb{R}^3}|\nabla\psi_1|^2dx\right)^2+\frac{t^4}{4}C\|\psi_1\|^4_{\mathcal{H}}-\mu Ct^{\tau}\int_{\mathbb{R}^3}|\psi_1|^{\tau}dx\\&\to-\infty,\end{aligned}$$

as $t\to\infty$ for $\tau>4$, which implies that (i) holds if we take $v=t\psi_1$ with t sufficiently large.

(ii) It follows from (3.6) that

$$\begin{aligned}J_{b,\mu}(v)&=\frac{1}{2}\int_{\mathbb{R}^3}\left(|\nabla v|^2+V(x)|G^{-1}(v)|^2\right)dx+\frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla v|^2dx\right)^2\\&\quad+\frac{1}{4}\int_{\mathbb{R}^3}\phi_{G^{-1}(v)}|G^{-1}(v)|^2dx-\mu\int_{\mathbb{R}^3}H(G^{-1}(v))dx\\&=\frac{1}{2}\int_{\mathbb{R}^3}|\nabla v|^2dx-\int_{\mathbb{R}^3}\left(-\frac{1}{2}V(x)|G^{-1}(v)|^2+\mu H(G^{-1}(v))\right)dx\\&\quad+\frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla v|^2dx\right)^2+\frac{1}{4}\int_{\mathbb{R}^3}\phi_{G^{-1}(v)}|G^{-1}(v)|^2dx.\end{aligned}\tag{5.1}$$

Let $A(x,s) := -\frac{1}{2}V(x)|G^{-1}(s)|^2 + \mu H(G^{-1}(s))$, then by Lemma 2.2-(6), we have

$$\lim_{s\to 0}\frac{A(x,s)}{|s|^2} = \lim_{s\to 0}\left[-\frac{1}{2}V(x)\left|\frac{G^{-1}(s)}{s}\right|^2 + \mu\frac{H(G^{-1}(s))}{|s|^2}\right] = -\frac{1}{2}V(x), \tag{5.2}$$

and

$$\lim_{s\to +\infty}\frac{A(x,s)}{|s|^6} = \lim_{s\to +\infty}\left[-\frac{1}{2}V(x)\left|\frac{G^{-1}(s)}{s}\right|^2\left(\frac{1}{|s|^4}\right) + \mu\frac{H(G^{-1}(s))}{|s|^6}\right] = 0. \tag{5.3}$$

Thus, by (5.2) and (5.3), for $\epsilon > 0$ sufficiently small, there exists a constant $C_\epsilon > 0$ such that

$$A(x,s) \le \left(-\frac{1}{2}V(x) + \epsilon\right)|s|^2 + C_\epsilon|s|^6. \tag{5.4}$$

Then, according to (V_1), (5.1), (5.4), Lemma 2.1 and Sobolev embedding theorem, we have

$$\begin{aligned} J_{b,\mu}(v) &\ge \frac{1}{2}\int_{\mathbb{R}^3}|\nabla v|^2dx - \int_{\mathbb{R}^3}\left(-\frac{1}{2}V(x)+\epsilon\right)|v|^2dx - \int_{\mathbb{R}^3}C_\epsilon|v|^6dx \\ &\quad + \frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla v|^2dx\right)^2 + \frac{1}{4}\int_{\mathbb{R}^3}\phi_{G^{-1}(v)}|G^{-1}(v)|^2dx \\ &\ge \frac{1}{2}\int_{\mathbb{R}^3}|\nabla v|^2dx + \frac{1}{2}\int_{\mathbb{R}^3}V(x)v^2dx - \epsilon\int_{\mathbb{R}^3}|v|^2dx - C_\epsilon\int_{\mathbb{R}^3}|v|^6dx \\ &\ge \frac{1}{2}\|v\|_{\mathcal{H}}^2 - \epsilon C\|v\|_{\mathcal{H}}^2 - C_\epsilon\|v\|_{\mathcal{H}}^6. \end{aligned}$$

It follows that

$$J_{b,\mu}(v) \ge C\|v\|_{\mathcal{H}}^2 - C\|v\|_{\mathcal{H}}^6,$$

if we choose sufficiently small $\rho_1 > 0$, which implies that

$$J_{b,\mu}(v) \ge C\rho_1^2 - C\rho_1^6 := \alpha_1 > 0. \qquad \square$$

Obviously, as defined in Section 3.2, $A(v)$ and $B(v)$ are the same, we can still obtain that $B(v) \ge 0$ for $v \in \mathcal{H}$ and $A(v) \to +\infty$ as $\|v\| \to \infty$. Also by Lemma 5.1 that the conclusion of Theorem 3.2 holds, then there is a bounded $(PS)_{c'_\mu}$ sequence $\{v'_n\} \subset \mathcal{H}$, satisfying $J_{b,\mu}(v'_n) \to c'_\mu$ and $J'_{b,\mu}(v'_n) \to 0$, where c'_μ is the mountain pass level.

Lemma 5.2. *Suppose (V_1), (g'), (h_1)–(h_3) hold, $\{v'_n\}$ is the sequence obtained above, going if necessary to a subsequence, $v'_n \to v'$ in $\mathcal{H}$.*

Proof. Since $\{v'_n\} \subset \mathcal{H}$ is bounded, up to a subsequence, there exists $v' \in \mathcal{H}$ such that

$$\begin{aligned} v'_n &\rightharpoonup v' \quad \text{in } \mathcal{H}, \\ v'_n &\to v' \quad \text{in } L^r(\mathbb{R}^3), \text{ for } 2 < r < 6, \\ v'_n(x) &\to v'(x) \quad \text{a.e. on } x \in \mathbb{R}^3. \end{aligned}$$

Firstly, we claim that there exists $C > 0$ such that

$$\int_{\mathbb{R}^3}\left[|\nabla(v'_n - v')|^2 + V(x)\left(\frac{G^{-1}(v'_n)}{g(G^{-1}(v'_n))} - \frac{G^{-1}(v')}{g(G^{-1}(v'))}\right)(v'_n - v')\right]dx \ge C\|v'_n - v'\|_{\mathcal{H}}^2. \tag{5.5}$$

Indeed, we may assume $v_n' \neq v'$ (otherwise the conclusion is trivial). Set

$$\omega_n' = \frac{v_n' - v'}{\|v_n' - v'\|}$$

and

$$l_n = \frac{1}{v_n' - v'}\left(\frac{G^{-1}(v_n')}{g(G^{-1}(v_n'))} - \frac{G^{-1}(v')}{g(G^{-1}(v'))}\right).$$

Argue by contradiction and assume that

$$\int_{\mathbb{R}^3}(|\nabla\omega_n'|^2 + V(x)l_n(x)\omega_n'^2)dx \to 0.$$

Since

$$\frac{d}{ds}\left(\frac{G^{-1}(s)}{g(G^{-1}(s))}\right) = \frac{1}{g^2(G^{-1}(s))}\left(1 - \frac{G^{-1}(s)g'(G^{-1}(s))}{g(G^{-1}(s))}\right) > 0,$$

$\frac{G^{-1}(s)}{g(G^{-1}(s))}$ is strictly increasing, and for each $C > 0$ there exists $\delta' > 0$ such that

$$\frac{d}{ds}\left(\frac{G^{-1}(s)}{g(G^{-1}(s))}\right) > \delta',$$

when $|s| \leq C$. Hence, we see that $l_n(x)$ is positive. Hence

$$\int_{\mathbb{R}^3}|\nabla\omega_n'|^2dx \to 0, \quad \int_{\mathbb{R}^3}V(x)l_n(x)\omega_n'^2dx \to 0, \quad \text{and} \quad \int_{\mathbb{R}^3}V(x)\omega_n'^2dx \to 1.$$

Similar to Lemma 2.5 in Reference [30], we assert that for each $\varepsilon' > 0$, there exists $C_1 > 0$ independent of n such that $\text{meas}(\Omega_n) < \varepsilon'$, where $\Omega_n := \{x \in \mathbb{R}^3 : |v_n'(x) - v'(x)| \geq C_1\}$, it can be inferred that there exists a constant $C > 0$ such that

$$\int_{\mathbb{R}^3\setminus\Omega_n}V(x)\omega_n'^2dx \leq C\int_{\mathbb{R}^3\setminus\Omega_n}V(x)\frac{|G^{-1}(v_n' - v')|^2}{\|v_n' - v'\|^2}dx \leq C\int_{\mathbb{R}^3}V(x)l_n(x)dx \to 0. \tag{5.6}$$

On the other hand, by the integral absolutely continuity, there exists $\varepsilon' > 0$ such that whenever $\Omega \subset \mathbb{R}^3$ and $\text{meas}(\Omega) < \varepsilon'$,

$$\int_{\Omega}V(x)\omega_n'^2dx \leq \frac{1}{2}. \tag{5.7}$$

Combining (5.6) with (5.7), we have

$$\int_{\mathbb{R}^3}V(x)\omega_n'^2dx = \int_{\mathbb{R}^3\setminus\Omega_n}V(x)\omega_n'^2dx + \int_{\Omega_n}V(x)\omega_n'^2dx \leq \frac{1}{2} + o_n(1),$$

which implies $1 \leq \frac{1}{2}$, a contradiction. This implies that (5.5) holds. What's more, by Lemma 2.1, we have

$$\begin{aligned}
&\int_{\mathbb{R}^3}\left[\phi_{G^{-1}(v_n')}\frac{G^{-1}(v_n')}{g(G^{-1}(v_n'))} - \phi_{G^{-1}(v')}\frac{G^{-1}(v')}{g(G^{-1}(v'))}\right](v_n' - v')dx \\
&\geq \int_{\mathbb{R}^3}\phi_{G^{-1}(v_n')}\left[\frac{G^{-1}(v_n')}{g(G^{-1}(v_n'))} - \frac{G^{-1}(v')}{g(G^{-1}(v'))}\right](v_n' - v')dx \\
&\quad + \int_{\mathbb{R}^3}\left(\phi_{G^{-1}(v_n')} - \phi_{G^{-1}(v')}\right)\frac{G^{-1}(v')}{g(G^{-1}(v'))}(v_n' - v')dx \\
&\to 0.
\end{aligned} \tag{5.8}$$

Similar to Lemma 3.4, we can easily obtain

$$\begin{aligned} & b\int_{\mathbb{R}^3}|\nabla v_n'|^2dx\int_{\mathbb{R}^3}\nabla v_n'\nabla(v_n'-v')dx-b\int_{\mathbb{R}^3}|\nabla v'|^2dx\int_{\mathbb{R}^3}\nabla v'\nabla(v_n'-v')dx \\ & \geq b\int_{\mathbb{R}^3}\left(|\nabla v_n'|^2-|\nabla v'|^2\right)dx\int_{\mathbb{R}^3}\nabla v'\nabla(v_n'-v')dx \\ & \to 0, \end{aligned} \tag{5.9}$$

as $n\to\infty$. From (g'), there is $g(t)\geq 1$, $\forall t\geq 0$. In addition, by Lemma 2.2-(4), (1.8) and Hölder inequality, $v_n\to v$ in $L^r(\mathbb{R}^3)$, $2<r<6$, we obtain

$$\begin{aligned} & \left|\int_{\mathbb{R}^3}\left[\frac{h(G^{-1}(v_n'))}{g(G^{-1}(v_n'))}-\frac{h(G^{-1}(v'))}{g(G^{-1}(v'))}\right](v_n'-v')dx\right| \\ & \leq \int_{\mathbb{R}^3}\left[\varepsilon(|v_n'|+|v'|)+C_\varepsilon(|v_n'|^{q-1}+|v'|^{q-1})\right]|v_n'-v'|dx \\ & \leq \varepsilon(|v_n'|_2+|v'|_2)|v_n'-v'|_2+C_\varepsilon(|v_n'|_q^{q-1}+|v'|_q^{q-1})|v_n'-v'|_q \\ & \to 0. \end{aligned} \tag{5.10}$$

By virtue of (5.5), (5.8)–(5.10), we have

$$\begin{aligned} o_n(1) &= \langle J_{b,\mu}'(v_n')-J_{b,\mu}'(v'),\, v_n'-v'\rangle \\ &= \int_{\mathbb{R}^3}|\nabla(v_n'-v')|^2dx+\int_{\mathbb{R}^3}V(x)\left[\frac{G^{-1}(v_n')}{g(G^{-1}(v_n'))}-\frac{G^{-1}(v')}{g(G^{-1}(v'))}\right](v_n'-v')dx \\ &\quad +b\left[\int_{\mathbb{R}^3}|\nabla v_n'|^2dx\int_{\mathbb{R}^3}\nabla v_n'\cdot\nabla(v_n'-v')dx-\int_{\mathbb{R}^3}|\nabla v'|^2dx\int_{\mathbb{R}^3}\nabla v'\cdot\nabla(v_n'-v')dx\right] \\ &\quad +\int_{\mathbb{R}^3}\left[\phi_{G^{-1}(v_n')}\frac{G^{-1}(v_n')}{g(G^{-1}(v_n'))}-\phi_{G^{-1}(v')}\frac{G^{-1}(v')}{g(G^{-1}(v'))}\right](v_n'-v')dx \\ &\quad -\mu\int_{\mathbb{R}^3}\left[\frac{h(G^{-1}(v_n'))}{g(G^{-1}(v_n'))}-\frac{h(G^{-1}(v'))}{g(G^{-1}(v'))}\right](v_n'-v')dx \\ &\geq C\|v_n'-v'\|_{\mathcal{H}}^2+o_n(1), \end{aligned}$$

which implies that $v_n'\to v'$ in $\mathcal{H}$. □

From Lemma 5.2, for every $\mu\in[1,2]$, we get that $J_{b,\mu}$ has a nontrivial critical point v' satisfying $J_{b,\mu}(v')=c_\mu'$ and $J_{b,\mu}'(v')=0$. Therefore, there exists $\{\mu_n\}\subset[1,2]$ such that $\lim_{n\to\infty}\mu_n=1$, $\{v_{\mu_n}'\}\subset\mathcal{H}$ such that $J_{b,\mu_n}(v_{\mu_n}')=c_{\mu_n}'>0$ and $J_{b,\mu_n}'(v_{\mu_n}')=0$.

Lemma 5.3. *Suppose that (V_1), (g'), (h_1) and (h_3) are satisfied, then any sequence $\{v_{\mu_n}'\}$ is bounded in $\mathcal{H}$.*

Proof. Since $\{v_{\mu_n}'\}\subset\mathcal{H}$ satisfies $J_{b,\mu_n}(v_{\mu_n}')=c_{\mu_n}'>0$, we have

$$\begin{aligned} J_{b,\mu_n}(v_{\mu_n}') &= \frac{1}{2}\int_{\mathbb{R}^3}|\nabla v_{\mu_n}'|^2dx+\frac{1}{2}\int_{\mathbb{R}^3}V(x)|G^{-1}(v_{\mu_n}')|^2dx+\frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla v_{\mu_n}'|^2dx\right)^2 \\ &\quad +\frac{1}{4}\int_{\mathbb{R}^3}\phi_{G^{-1}(v_{\mu_n}')}|G^{-1}(v_{\mu_n}')|^2dx-\mu_n\int_{\mathbb{R}^3}H(G^{-1}(v_{\mu_n}'))dx \\ &= c_{\mu_n}'>0. \end{aligned} \tag{5.11}$$

We also deduce that

$$\begin{aligned}\langle J'_{b,\mu_n}(v'_{\mu_n}), v'_{\mu_n}\rangle &= \int_{\mathbb{R}^3} |\nabla v'_{\mu_n}|^2 dx + \int_{\mathbb{R}^3} V(x)\frac{G^{-1}(v'_{\mu_n})}{g(G^{-1}(v'_{\mu_n}))}v'_{\mu_n}dx \\ &\quad + b\left(\int_{\mathbb{R}^3} |\nabla v'_{\mu_n}|^2 dx\right)^2 + \int_{\mathbb{R}^3} \phi_{G^{-1}(v'_{\mu_n})}\frac{G^{-1}(v'_{\mu_n})}{g(G^{-1}(v'_{\mu_n}))}v'_{\mu_n}dx \\ &\quad - \mu_n \int_{\mathbb{R}^3} \frac{h(G^{-1}(v'_{\mu_n}))}{g(G^{-1}(v'_{\mu_n}))}v'_{\mu_n}dx.\end{aligned} \tag{5.12}$$

It follows from (5.11), (5.12), (h_3) and Lemma 2.2-(5) that

$$\begin{aligned}&2c'_{\mu_n} + 1 + \|v'_{\mu_n}\| \\ &\geq J_{b,\mu_n}(v'_{\mu_n}) - \frac{1}{4}\langle J'_{b,\mu_n}(v'_{\mu_n}), v'_{\mu_n}\rangle \\ &= \frac{1}{4}\int_{\mathbb{R}^3} |\nabla v'_{\mu_n}|^2 dx + \int_{\mathbb{R}^3} V(x)G^{-1}(v'_{\mu_n})\left[\frac{1}{2}G^{-1}(v'_{\mu_n}) - \frac{v'_{\mu_n}}{4g(G^{-1}(v'_{\mu_n}))}\right]dx \\ &\quad + \frac{1}{4}\int_{\mathbb{R}^3} \phi_{G^{-1}(v'_{\mu_n})}G^{-1}(v'_{\mu_n})\left[G^{-1}(v'_{\mu_n}) - \frac{v'_{\mu_n}}{g(G^{-1}(v'_{\mu_n}))}\right]dx \\ &\quad - \mu_n \int_{\mathbb{R}^3}\left[H(G^{-1}(v'_{\mu_n})) - \frac{1}{4}\frac{h(G^{-1}(v'_{\mu_n}))}{g(G^{-1}(v'_{\mu_n}))}v'_{\mu_n}\right]dx \\ &\geq \frac{1}{4}\int_{\mathbb{R}^3} |\nabla v'_{\mu_n}|^2 dx + \frac{1}{4}\int_{\mathbb{R}^3} V(x)|G^{-1}(v'_{\mu_n})|^2 dx.\end{aligned} \tag{5.13}$$

According to the assumption (h_3) and (V_1), we have $H(s) \geq CG(s)^\tau \geq CG(s)^2$ for all $s \geq 1$. Then by Lemma 2.2-(4)

$$\begin{aligned}&\int_{\{x:\,|G^{-1}(v'_{\mu_n})|>1\}} V(x)v'^2_{\mu_n}dx \\ &\leq C\int_{\{x:\,|G^{-1}(v'_{\mu_n})|>1\}} H(G^{-1}(v'_{\mu_n}))dx \\ &\leq C\int_{\mathbb{R}^3} H(G^{-1}(v'_{\mu_n}))dx + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla v'_{\mu_n}|^2 dx\right)^2 + C\int_{\mathbb{R}^3} \phi_{G^{-1}(v'_{\mu_n})}v'^2_{\mu_n}dx \\ &\leq C\left[\frac{1}{2}\int_{\mathbb{R}^3} |\nabla v'_{\mu_n}|^2 dx + \frac{1}{2}\int_{\mathbb{R}^3} V(x)|G^{-1}(v'_{\mu_n})|^2 dx - c_1 + o_n(1)\right].\end{aligned} \tag{5.14}$$

On the other hand, for $x \in \{x : |G^{-1}(v'_n)| \leq 1\}$ we know that

$$\begin{aligned}\frac{1}{g^2(1)}\int_{\{x:\,|G^{-1}(v'_{\mu_n})|\leq 1\}} V(x)v'^2_{\mu_n}dx &\leq C\int_{\{x:\,|G^{-1}(v'_{\mu_n})|\leq 1\}} V(x)|G^{-1}(v'_{\mu_n})|^2 dx \\ &\leq C\int_{\mathbb{R}^3} V(x)|G^{-1}(v'_{\mu_n})|^2 dx.\end{aligned} \tag{5.15}$$

Since (g'), we know that $g(s)$ is nondecreasing. Combining (5.13) and (5.14) with (5.15), we deduce that $\{v'_{\mu_n}\}$ is bounded in $\mathcal{H}$. □

Proof of Theorem 1.4. Similar to the proof of Theorem 1.2, we take a subsequence of $\{v'_{\mu_n}\}$, and still represented by $\{v'_{\mu_n}\}$, because $\{v'_{\mu_n}\}$ is bounded in $\mathcal{H}$, similar to the proof of Lemma 5.2,

we obtain $v'_{\mu_n} \to v'_{\mu}$ in $\mathcal{H}$. From Theorem 3.2, we known that $\mu \mapsto c'_{\mu}$ is continuous from the left. So

$$\begin{aligned} \lim_{n\to\infty} J_b(v'_{\mu_n}) &= \lim_{n\to\infty} \left[J_{b,\mu_n}(v'_{\mu_n}) + (\mu_n - 1) \int_{\mathbb{R}^3} H(G^{-1}(v'_{\mu_n}))dx \right] \\ &= \lim_{n\to\infty} c'_{\mu_n} = \tilde{c}'. \end{aligned} \tag{5.16}$$

In addition,

$$\lim_{n\to\infty} \langle J'_b(v'_{\mu_n}), \eta \rangle = \lim_{n\to\infty} \left[\langle J'_{b,\mu_n}(v'_{\mu_n}), \eta \rangle + (\mu_n - 1) \int_{\mathbb{R}^3} \frac{h(G^{-1}(v'_{\mu_n}))}{g(G^{-1}(v'_{\mu_n}))} \eta dx \right] = 0, \tag{5.17}$$

for any $\eta \in C_0^\infty(\mathbb{R}^3)$, which means that $J'_b(v') = 0$ satisfies $J_b(v') = \tilde{c}' > 0$. □

Proof of Theorem 1.5. The proof process is similar to Theorem 1.3, and specific steps are omitted here. □

6 Nonexistence in critical case

In this section, we prove the nonexistence of nontrivial solutions for system (1.9) by Pohožaev identity.

Lemma 6.1. *(Pohožaev identity) If $v \in \mathcal{H}$ is a weak solution of problem (1.9), then we have the following Pohožaev identity:*

$$\begin{aligned} &\frac{1}{2}\int_{\mathbb{R}^3} |\nabla v|^2 dx + \frac{3}{2}\int_{\mathbb{R}^3} V(x)|G^{-1}(v)|^2 dx + \frac{1}{2}\int_{\mathbb{R}^3} \langle \nabla V(x), x\rangle |G^{-1}(v)|^2 dx \\ &\quad + \frac{b}{2}\left(\int_{\mathbb{R}^3} |\nabla v|^2 dx\right)^2 + \frac{5}{4}\int_{\mathbb{R}^3} \phi_{G^{-1}(v)}|G^{-1}(v)|^2 dx - \frac{3}{p}\int_{\mathbb{R}^3} |v|^p dx = 0. \end{aligned} \tag{6.1}$$

Proof of Theorem 1.6. Suppose that $v \in \mathcal{H}$ is a solution to system (1.9), then v satisfies

$$\begin{aligned} &\int_{\mathbb{R}^3} |\nabla v|^2 dx + \int_{\mathbb{R}^3} V(x)\frac{G^{-1}(v)}{g(G^{-1}(v))} v dx + b\left(\int_{\mathbb{R}^3} |\nabla v|^2 dx\right)^2 \\ &\quad + \int_{\mathbb{R}^3} \phi_{G^{-1}(v)}\frac{G^{-1}(v)}{g(G^{-1}(v))} v dx - \int_{\mathbb{R}^3} |v|^p dx = 0. \end{aligned} \tag{6.2}$$

Combining (6.1) and (6.2), we can get

$$\begin{aligned} &\int_{\mathbb{R}^3} V(x)\left(\frac{1}{2}\frac{G^{-1}(v)v}{g(G^{-1}(v))} - \frac{3}{2}|G^{-1}(v)|^2\right) dx - \frac{1}{2}\int_{\mathbb{R}^3} \langle \nabla V(x), x\rangle |G^{-1}(v)|^2 dx \\ &\quad + \int_{\mathbb{R}^3} \phi_{G^{-1}(v)}\left(\frac{1}{2}\frac{G^{-1}(v)v}{g(G^{-1}(v))} - \frac{5}{4}|G^{-1}(v)|^2\right) dx = \left(\frac{1}{2} - \frac{3}{p}\right)\int_{\mathbb{R}^3} |v|^p dx. \end{aligned} \tag{6.3}$$

Since Lemma 2.2-(5) that $\frac{G^{-1}(v)}{g(G^{-1}(v))} v \le |G^{-1}(v)|^2$, if $2V(x) + \langle \nabla V(x), x\rangle \ge 0$, then (6.3) implies that $v = 0$ for $p \ge 6$. So that $0 = u = G^{-1}(v)$. Then (1.9) has no nontrivial solutions. □

Declarations

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The authors declare that they have equal contributions.

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Asymptotic stability analysis on a neutral Nicholson's blowflies equation with time-varying delays

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Abstract. This study is devoted to the stability issue of a time-varying delayed Nicholson's blowflies equation of neutral type. By the aid of the Lyapunov stability theory and novel analysis techniques, two sharp conditions ensuring separately the global asymptotic stability of the zero equilibrium and positive equilibrium are derived, which completely cover the corresponding ones in the associated non-neutral equation. In addition, a numerical example is taken to support the availability of the theoretical results.

Keywords: Nicholson's blowflies equation, neutral type, time-varying delay, global asymptotic stability.

MSC 2020 Subject Classification: 34K40, 34D23.

1 Introduction

The following Nicholson's blowflies equation

$$w'(t) = -\delta w(t) + \beta w(t-\tau)e^{-aw(t-\tau)}, \tag{1.1}$$

was proposed by Gurney et al. in [6] to describe the dynamics of the Australian sheep-blowfly. Here $w(t)$ labels the size of blowflies population at time t, δ, β, a, τ are positive constants with distinct biological significance, that is, δ signifies the average daily mortality rate among adult blowflies, β quantifies the maximal mean rate of oviposition per day, $\frac{1}{a}$ means the size at which the blowflies population reproduces at its maximum rate, and τ stands for the maturation delay. In the past four decades, extensive research has elucidated the qualitative dynamics and stability characteristics of Eq. (1.1) and its generalized forms (see [1, 8, 9, 11, 14] and the references therein). In particular, if the biological parameters satisfy $\frac{\beta}{\delta} \leq 1$ and $1 < \frac{\beta}{\delta} \leq e^2$, the global asymptotic stability has been respectively proven for the trivial equilibrium and positive equilibrium to Eq. (1.1) in [1, 8, 9, 11, 14]. Furthermore, Ref. [17] revealed that the

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positive equilibrium of Eq. (1.1) admits global attractivity if $\frac{\beta}{\delta} > 1$ and the delay τ is small. Meanwhile, the authors in [19] and [5] established the global asymptotical stability for the positive equilibrium of Eq. (1.1) if $1 < \frac{\beta}{\delta} \leq e^2$. It is noteworthy that Yang and So in [18] showed that the positive equilibrium is unstable and Hopf bifurcation appears when $\frac{\beta}{\delta} > e^2$ and the delay τ is large. It is demonstrated from aforementioned findings that $\frac{\beta}{\delta} \leq 1$ and $1 < \frac{\beta}{\delta} \leq e^2$ are two sharp conditions which respectively assure the global asymptotic stability of trivial equilibrium and positive equilibrium of Eq. (1.1).

In the realistic world, biological populations have complex dynamic characteristics, and the existing population models cannot accurately describe the properties of a population evolution process. Consequently, it is both natural and interesting that biological models should incorporate some information about the derivative of past states to better depict and model the dynamics of such complex population evolution processes. This has drawn extensive attention to neutral type delayed population dynamic models [3, 7, 15, 21]. In particular, due to the effect of the population age distribution, Eq. (1.1) can be commonly generalized to the following neutral functional differential equation (NFDE),

$$(w(t) - cw(t-\alpha))' = -\delta w(t) + c\delta w(t-\alpha) + \beta w(t-\tau)e^{-aw(t-\tau)}, \quad t \in [0, +\infty), \tag{1.2}$$

where $a, \delta, \beta, \alpha, \tau \in (0, +\infty), c \in [0, 1)$. For more details on the biological ecology background and the derivation of Eq. (1.2), one can refer to [2, 12]. It should be pointed out that, if the population function $w(t)$ is continuously differentiable, Eq. (1.2) can be rewritten as the following special case,

$$w'(t) - cw'(t-\alpha) = -\delta w(t) + c\delta w(t-\alpha) + \beta w(t-\tau)e^{-aw(t-\tau)}, \quad t \in [0, +\infty). \tag{1.3}$$

Recently, the dynamics including Hopf bifurcation, the stability of (almost) periodic solutions of Eq. (1.3) and its generalizations have been extensively studied in [10, 12, 13, 15, 20, 22]. Meanwhile, delays in population and ecology models may depend on the time-varying environmental conditions and climate, and hence the neutral delayed Nicholson's blowflies equation (1.2) can be naturally extrapolated to the following non-autonomous form:

$$(w(t) - cw(t-\tau_1(t)))' = -\delta w(t) + c\delta w(t-\tau_1(t)) + \beta w(t-\tau_2(t))e^{-aw(t-\tau_2(t))}, \tag{1.4}$$

where $c \in [0, 1)$, $t \geq t_0 \in \mathbb{R}$, the delay functions $\tau_i(t)$ $(i = 1, 2)$ possess continuity and boundedness. In addition, according to the definition of initial set [23], we always assume that there exists $r_i \in (0, +\infty)$ satisfying

$$\{t - \tau_i(t) : t - \tau_i(t) \leq t_0,\ t \geq t_0\} \cup \{t_0\} = [t_0 - r_i,\ t_0], \quad i = 1, 2, \tag{1.5}$$

and

$$r = r_2 \geq r_1, \quad \text{and} \quad \sigma := \min\left\{\inf_{t\in\mathbb{R}} \tau_1(t), \inf_{t\in\mathbb{R}} \tau_2(t)\right\} > 0. \tag{1.6}$$

Obviously, when $\tau_i(t)$ $(i = 1, 2)$ are constants, Eqs. (1.1)–(1.3) are special cases of Eq. (1.4). From the basic theory of functional differential equations (FDEs), it is evident that the initial value conditions for Eqs. (1.2) and (1.4) are fundamentally different, which yields theoretical difficulties in analyzing the dynamic behavior of Eq. (1.4). In addition, the existence of both neutral structure and time-varying delays also brings technical challenges to explore the kinetic properties of the non-autonomous NFDE. Therefore, a natural question is whether one could establish the sharp criteria separately ensuring the global asymptotic stability of trivial

and positive equilibria of neutral time-varying delayed Nicholson's blowflies equation (1.4), and such a problem has not been reported up to now.

Inspired by the aforementioned discussions, the primary object of this paper is to establish sharp global asymptotic stability criteria for Eq. (1.4). Specifically, the main contributions of this study can be summarized as follows:

1) A kind of neutral Nicholson's blowflies equation with time-varying delays is proposed, the well-posedness including existence, uniqueness, boundedness and positiveness of solutions to the addressed model is first proved.

2) Global asymptotic stability criteria on the trivial and positive equilibria to Eq. (1.4) are established, respectively, which are sharp and substantially extend the existing stability results of the corresponding non-neutral ones.

3) A numerical example and some comparative analyses are presented to explicate the correctness and innovation of the theoretical findings.

The structure of this paper is outlined below. Section 2 introduces some preliminaries. Sections 3 and 4 establish respectively the global asymptotic stability of zero equilibrium and positive equilibrium. Section 5 affords a numerical example to exhibit effectiveness of the theoretical results. At last, the conclusion is conducted in Section 6.

2 Preliminaries

Firstly, we present some notations which are needed later. For convenience, denote the Banach space supplemented with supremum norm $\|\cdot\|$ by $\Gamma = C([-r,\ 0],\ \mathbb{R})$. Given that $\zeta \geq 0$, $t_0 \in \mathbb{R}$, and $W \in C([t_0 - r, t_0 + \zeta], \mathbb{R})$, then, for any $t \in [t_0, t_0 + \zeta]$, $W_t \in \Gamma$ is interpreted by $W_t(\nu) = W(t + \nu)$, $-r \leq \nu \leq 0$. For $t_0 \in \mathbb{R}$, denote

$$\Gamma_+ = \Big\{\varphi \in \Gamma \big| \varphi(0) - c\varphi(-\tau_1(t_0)) \geq 0,\ \varphi(\theta) \geq 0 \ \text{ for all } \ \theta \in [-r,\ 0]\Big\},$$

and let $w_t(t_0, \varphi)(w(t; t_0, \varphi))$ be the solution of Eq. (1.4) under the admissible initial conditions

$$w_{t_0} = \varphi, \qquad \varphi \in \Gamma_+. \tag{2.1}$$

Clearly, Eq. (1.4) only possesses a trivial equilibrium $N_0 = 0$ when $\frac{\beta}{\delta(1-c)} \leq 1$ and a unique positive equilibrium $N_* = \frac{1}{a} \ln \frac{\beta}{\delta(1-c)} \leq \frac{2}{a}$ when $1 < \frac{\beta}{\delta(1-c)} \leq e^2$. For simplicity of notation, we designate

$$w(t) = w(t; t_0, \varphi), \quad x(t) = w(t) - cw(t - \tau_1(t)),$$

and

$$y(t) = x(t) - (1-c)N_* = (w(t) - N_*) - c(w(t - \tau_1(t)) - N_*).$$

Moreover, for $\varphi \in \Gamma_+$, we extend the initial value function as follows:

$$\varphi(\theta - \tau_1(\theta)) = \varphi(-r) \quad \text{for all } \theta \in [-r,\ 0] \text{ and } \theta - \tau_1(\theta) < -r. \tag{2.2}$$

Now, we investigate some basic properties of the solutions for Eq. (1.4).

Lemma 2.1. *Let $\varphi \in \Gamma_+$. Then Eq. (1.4) has a unique, non-negative, ultimately bounded solution $w_t(t_0, \varphi)$ on $[t_0,\ +\infty)$. Meanwhile, for $\varphi \in \Gamma_+ \setminus \{0\}$, $w_t(t_0, \varphi)$ is ultimately positive.*

Proof. For any $t \in [t_0, t_0+\sigma]$, multiplying both sides of Eq. (1.4) by $e^{\delta t}$ and integrating from t_0 to t, together with (2.1), gives us

$$x(t) = (\varphi(0) - c\varphi(-\tau_1(t_0)))e^{-\delta(t-t_0)} + \beta e^{-\delta t}\int_{t_0}^{t} \varphi(s-\tau_2(s)-t_0)e^{-a\varphi(s-\tau_2(s)-t_0)}e^{\delta s}ds \geq 0$$

and

$$w(t) = x(t) + c\varphi(t-\tau_1(t)-t_0) \geq 0.$$

Similarly, applying the step-by-step method, the case $x(t) \geq 0$ and $w(t) \geq 0$ on $[t_0+\sigma, t_0+2\sigma]$ holding as follows. Therefore, $x(t) \geq 0$ and $w(t) \geq 0$ for all $t \geq t_0$. Consequently, $w_t(t_0, \varphi)$ exists and is unique on $[t_0, +\infty)$. In addition, $w_t(t_0, \varphi) \geq 0$ for all $t \in [t_0, +\infty)$.

On the other hand, the fact $\sup_{u\geq 0} ue^{-u} = \frac{1}{e}$ leads to

$$\begin{aligned} x(t) =& (\varphi(0) - c\varphi(-\tau_1(t_0)))e^{-\delta(t-t_0)} \\ &+ e^{-\delta t}\int_{t_0}^{t} \beta w(s-\tau_2(s))e^{-aw(s-\tau_2(s))}e^{\delta s}ds \\ \leq& (\varphi(0) - c\varphi(-\tau_1(t_0)))e^{-\delta(t-t_0)} + \frac{\beta}{ae}e^{-\delta t}\int_{t_0}^{t} e^{\delta s}ds \\ =& (\varphi(0) - c\varphi(-\tau_1(t_0)))e^{-\delta(t-t_0)} + \frac{\beta}{\delta ae}\left[1 - e^{-\delta(t-t_0)}\right] \quad \text{for all } t \in [t_0, +\infty), \end{aligned}$$

and hence

$$\limsup_{t\to+\infty} x(t) \leq \frac{\beta}{\delta ae},$$

which, together with the following fact

$$\limsup_{t\to+\infty} w(t) = \limsup_{t\to+\infty}[x(t) + cw(t-\tau_1(t))] \leq \limsup_{t\to+\infty} x(t) + c\limsup_{t\to+\infty} w(t),$$

gives us

$$\limsup_{t\to+\infty} w(t) \leq \frac{1}{1-c}\limsup_{t\to+\infty} x(t) \leq \frac{1}{1-c}\frac{\beta}{\delta ae}.$$

This implies that $w(t)$ is ultimately uniformly bounded.

Now, let $\varphi \in \Gamma_+ \setminus \{0\}$. We state that there is a $t_1 \in [t_0, +\infty)$ satisfying $x(t_1) > 0$. Otherwise, $x(t) \equiv 0$ on $[t_0, +\infty)$. Consequently, $\varphi \not\equiv 0$, this allows one can choose $t^* \in [t_0 - r, t_0)$ satisfying $w(t^*) = \varphi(t^* - t_0) > 0$. This, together with (1.5) and (1.6), shows that there is $\xi^{**} \in (t_0, +\infty)$ such that

$$t^* = \xi^{**} - \tau_2(\xi^{**}) \quad \text{and} \quad w(\xi^{**} - \tau_2(\xi^{**})) = w(t^*) = \varphi(t^* - t_0) > 0,$$

which yields

$$\begin{aligned} 0 =& x'(\xi^{**}) \\ =& -\delta x(\xi^{**}) + \beta w(\xi^{**} - \tau_2(\xi^{**}))e^{-aw(\xi^{**}-\tau_2(\xi^{**}))} \\ =& \beta w(\xi^{**} - \tau_2(\xi^{**}))e^{-aw(\xi^{**}-\tau_2(\xi^{**}))} \\ >& 0. \end{aligned}$$

This is a contradiction, and hence

$$\begin{aligned} w(t) &\geq x(t) \\ &= x(t_1)e^{-\delta(t-t_1)} + e^{-\delta t}\int_{t_1}^{t} \beta w(s-\tau_2(s))e^{-aw(s-\tau_2(s))}e^{\delta s}ds \\ &> 0 \quad \text{for all } \ t \in [t_1,\ +\infty). \end{aligned}$$

In other words, $x(t)$ and $w(t)$ are ultimately positive. □

To establish the next main theorems, we make use of the following results.

Lemma 2.2. *If $W \in C([t_0,\ +\infty),[0,\ +\infty))$ is bounded, and there exist constants $p,q \in \mathbb{R}$ satisfying that $0 \leq q < 1$ and*

$$\lim_{t\to+\infty}\left[W(t) - qW(t-\tau_1(t))\right] = p,$$

then

$$\lim_{t\to+\infty} W(t) = \frac{p}{1-q}.$$

Proof. Let $A = \liminf_{t\to+\infty} W(t)$ and $B = \limsup_{t\to+\infty} W(t)$, then $0 \leq A \leq B$, and there exist two monotonically increasing sequences $\{Z_n^1\}_{n\geq1}$ and $\{Z_n^2\}_{n\geq1}$ such that

$$\lim_{n\to+\infty} Z_n^1 = +\infty, \quad \lim_{n\to+\infty} Z_n^2 = +\infty, \quad \lim_{n\to+\infty} W(Z_n^1) = A, \quad \lim_{n\to+\infty} W(Z_n^2) = B,$$

and

$$\lim_{n\to+\infty} W(Z_n^i - \tau_1(Z_n^i)) = C_i \in [A,\ B], \quad i = 1,2.$$

Therefore,

$$p = \lim_{n\to+\infty}\left[W(Z_n^1) - qW(Z_n^1 - \tau_1(Z_n^1))\right] = A - qC_1 \leq A(1-q)$$

and

$$p = \lim_{n\to+\infty}[W(Z_n^2) - qW(Z_n^2 - \tau_1(Z_n^2))] = B - qC_2 \geq B(1-q),$$

which assures that $\lim_{t\to+\infty} W(t) = A = B = \frac{p}{1-q}$. □

Lemma 2.3 (see [4, Lemma 2.3])**.** *If $r^* \in (0,2]$, then*

$$\left|\mu e^{-\mu} - r^*e^{-r^*}\right| < e^{-r^*}|\mu - r^*| \quad \text{for all } \mu > 0 \text{ and } \mu \neq r^*. \tag{2.3}$$

3 Global asymptotic stability of $N_0 = 0$ when $\beta \leq \delta(1-c)$

In this section, we deduce a new criterion for global asymptotic stability on $N_0 = 0$ of Eq. (1.4), which extends the previous stability findings in its special cases. This result can be stated in the following theorem.

Theorem 3.1. *If $\beta \leq \delta(1-c)$, then the trivial equilibrium N_0 of Eq. (1.4) is globally asymptotically stable. Particularly, when $\beta < \delta(1-c)$, N_0 is globally exponentially stable.*

Proof. First, we demonstrate the stability of $N_0 = 0$. For any $\varepsilon > 0$, denote $H = (1-c)\varepsilon$ and $\|\varphi\| < H$ with $\varphi \in \Gamma_+$. We shall demonstrate that $w(t) < \varepsilon$ for all $t \in [t_0 - r,\ +\infty)$. Then, by (2.1), (2.2), and Lemma 2.1, we have

$$x(t) \geq 0 \quad \text{for any } t \geq t_0,$$

and

$$x(t) = w(t) - cw(t - \tau_1(t)) = \varphi(t - t_0) - c\varphi(t - \tau_1(t) - t_0) < H \text{ for any } t \in [t_0 - r,\ t_0].$$

Now, we claim that

$$0 \leq x(t) < H \quad \text{for any } t > t_0. \tag{3.1}$$

Otherwise, there exists $G^* > t_0$ such that

$$x(G^*) = H \quad \text{and} \quad x(t) < H \quad \text{for any } t \in [t_0 - r,\ G^*), \tag{3.2}$$

it follows that

$$\begin{aligned} w(\theta) &= x(\theta) + cw(\theta - \tau_1(\theta)) \\ &\leq x(\theta) + c \sup_{\min\limits_{s\in[t_0-r,\theta]}(s-\tau_1(s))\leq u\leq\theta} w(u - \tau_1(u)) \\ &\leq x(\theta) + c \sup_{t_0-r\leq u\leq t} w(u) \\ &< H + c \sup_{t_0-r\leq u\leq t} w(u), \end{aligned}$$

for any $\theta \in [t_0 - r,\ t]$ and $t \in [t_0 - r,\ G^*)$. Hence

$$w(t) \leq \sup_{t_0-r\leq u\leq t} w(u) < \frac{H}{1-c} \quad \text{for any } t \in [t_0 - r,\ G^*). \tag{3.3}$$

Moreover, it indicates from Eqs. (1.4) and (3.3) that

$$\begin{aligned} 0 &\leq x'(G^*) \\ &= -\delta x(G^*) + \beta w(G^* - \tau_2(G^*))e^{-aw(G^*-\tau_2(G^*))} \\ &\leq -\delta x(G^*) + \beta w(G^* - \tau_2(G^*)) \\ &< H\Big(-\delta + \frac{\beta}{1-c}\Big) \\ &\leq 0, \end{aligned}$$

we yield the contradiction and find that assertion (3.1) is true. Utilizing a similar approach as in the identification of (3.3), we obtain

$$w(t) \leq \sup_{t_0-r\leq u\leq t} w(u) < \frac{H}{1-c} = \varepsilon \quad \text{for any } t \geq t_0,$$

which shows that N_0 is stable.

Second, we prove the global attractivity of N_0. Since

$$\limsup_{t\to+\infty} w(t) \leq \limsup_{t\to+\infty} x(t) + c \limsup_{t\to+\infty} w(t - \tau_1(t)) \leq \limsup_{t\to+\infty} x(t) + c \limsup_{t\to+\infty} w(t),$$

one has

$$\limsup_{t\to+\infty} w(t) \le \frac{1}{1-c}\limsup_{t\to+\infty} x(t). \tag{3.4}$$

It suffices to verify $\limsup_{t\to+\infty} x(t) = 0$. We find from the fluctuation lemma [16, Lemma A.1.] that there exists a monotonically increasing sequence $\{Z_n\}_{n\ge1}$ agreeing with

$$\lim_{n\to+\infty} Z_n = +\infty, \quad \lim_{n\to+\infty} x'(Z_n) = 0, \quad \text{and} \quad \lim_{n\to+\infty} x(Z_n) = \limsup_{t\to+\infty} x(t),$$

and $\lim_{n\to+\infty} w(Z_n - \tau_2(Z_n)) = \eta$ holds. Then (3.4) yields

$$\eta = \lim_{n\to+\infty} w(Z_n - \tau_2(Z_n)) \le \limsup_{t\to+\infty} w(t) \le \frac{1}{1-c}\limsup_{t\to+\infty} x(t). \tag{3.5}$$

Taking the limit on both sides of the following equation:

$$x'(Z_n) = -\delta x(Z_n) + \beta w(Z_n - \tau_2(Z_n))e^{-aw(Z_n-\tau_2(Z_n))}$$

produces

$$0 = -\delta\limsup_{t\to+\infty} x(t) + \beta\eta e^{-a\eta}. \tag{3.6}$$

For the sake of contradiction, assume that $\limsup_{t\to+\infty} x(t) > 0$, (3.5) and (3.6) give us

$$0 < \eta \le \frac{1}{1-c}\limsup_{t\to+\infty} x(t),$$

and the contradiction

$$0 = -\delta\limsup_{t\to+\infty} x(t) + \beta\eta e^{-a\eta} < \left(-\delta + \beta\frac{1}{1-c}\right)\limsup_{t\to+\infty} x(t) \le 0,$$

and thus

$$\lim_{t\to+\infty} w(t) = N_0 = 0.$$

Finally, we show the global exponential stability of N_0 when $\beta < \delta(1-c)$, we only need to prove its globally exponential attractivity. To do this, we pick a sufficiently small $\lambda > 0$ obeying

$$1 - ce^{\lambda r} > 0 \quad \text{and} \quad -(\delta-\lambda) + \frac{\beta e^{\lambda r}}{1-ce^{\lambda r}} < 0. \tag{3.7}$$

Since $\varphi(\theta - \tau_1(\theta)) = \varphi(-r)$ for all $\theta \in [-r,\ 0]$ with $\theta - \tau_1(\theta) < -r$, we can denote

$$x(t)e^{\lambda t} = x(t_0 - r)e^{\lambda(t_0-r)} \quad \text{for any } t \in (-\infty,\ t_0 - r],$$

$$z(t) = x(t)e^{\lambda t} \quad \text{for any } t \in [t_0 - r,\ +\infty),$$

and

$$M_\varphi = \max_{t_0-r\le t\le t_0} x(t)e^{\lambda t} = \max_{t_0-r\le t\le t_0} \left[\varphi(t-t_0) - c\varphi(t-\tau_1(t)-t_0)\right]e^{\lambda t}.$$

Clearly, $z(t) < M_\varphi + 1$ for any $t \in [t_0 - r,\ t_0]$. We hence claim

$$z(t) < M_\varphi + 1 \quad \text{for all } t \ge t_0. \tag{3.8}$$

Otherwise, there exists $t_2 > t_0$ satisfying

$$z(t_2) = M_\varphi + 1, \quad 0 \le z(t) < M_\varphi + 1 \quad \text{for all } t \in [t_0 - r, t_2). \tag{3.9}$$

For any $\theta \in [t_0 - r,\ t]$ and $t \in [t_0 - r,\ t_2)$, we have

$$\begin{aligned} w(\theta)e^{\lambda\theta} &= x(\theta)e^{\lambda\theta} + cw(\theta - \tau_1(\theta))e^{\lambda(\theta-\tau_1(\theta))}e^{\lambda\tau_1(\theta)} \\ &\le z(\theta) + c \sup_{\min\limits_{s\in[t_0-r,\theta]}(s-\tau_1(s))\le u\le\theta} w(u - \tau_1(u))e^{\lambda(u-\tau_1(u))}e^{\lambda r} \\ &\le z(\theta) + c \sup_{t_0-r\le u\le t} w(u)e^{\lambda u}e^{\lambda r} \\ &< M_\varphi + 1 + c \sup_{t_0-r\le u\le t} w(u)e^{\lambda u}e^{\lambda r}, \end{aligned}$$

and thus

$$w(t)e^{\lambda t} \le \sup_{t_0-r\le u\le t} w(u)e^{\lambda u} < \frac{M_\varphi + 1}{1 - ce^{\lambda r}} \quad \text{for any } t \in [t_0 - r,\ t_2). \tag{3.10}$$

It follows immediately from (1.4), (3.7), (3.9) and (3.10) that

$$\begin{aligned} 0 &\le z'(t_2) \\ &= x'(t_2)e^{\lambda t_2} + \lambda x(t_2)e^{\lambda t_2} \\ &= -(\delta - \lambda)x(t_2)e^{\lambda t_2} + \beta w(t_2 - \tau_2(t_2))e^{\lambda(t_2-\tau_2(t_2))}e^{\lambda\tau_2(t_2)}e^{-aw(t_2-\tau_2(t_2))} \\ &< \Big[-(\delta - \lambda) + \frac{\beta e^{\lambda r}}{1 - ce^{\lambda r}}\Big](M_\varphi + 1) \\ &< 0, \end{aligned}$$

which results in a contradiction, and claim (3.8) holds. Furthermore, by the same manner as those in establishing (3.10), it follows from (3.8) that

$$w(t) \le \frac{M_\varphi + 1}{1 - ce^{\lambda r}}e^{-\lambda t} \quad \text{for any } t \ge t_0,$$

which confirms the global exponential attractivity of N_0. The evidence of Theorem 3.1 is finished. □

4 Global asymptotic stability of $N_* = \frac{1}{a}\ln\frac{\beta}{\delta(1-c)}$ when $1 < \frac{\beta}{\delta(1-c)} \le e^2$

We are now ready to derive the conditions under which the positive equilibrium N_* of Eq. (1.4) has global asymptotic stability.

Theorem 4.1. *If $1 < \frac{\beta}{\delta(1-c)} \le e^2$, then Eq. (1.4) admits a unique positive equilibrium $N_* = \frac{1}{a}\ln\frac{\beta}{\delta(1-c)}$, which possesses global asymptotic stability.*

Proof. Clearly, Eq. (1.4) admits a unique positive equilibrium $N_* = \frac{1}{a}\ln\frac{\beta}{\delta(1-c)} \le \frac{2}{a}$. Now, we reveal the stability of N_*. For any $\varepsilon > 0$, let $\bar{H} = \frac{(1-c)}{1+c}\varepsilon$ and $\|\varphi - N_*\| < \bar{H}$ with $\varphi \in \Gamma_+$, it suffices to check that

$$|w(t) - N_*| < \varepsilon \quad \text{for all } t \in [t_0 - r,\ +\infty).$$

In fact, one finds from (2.1) and (2.2) that

$$\begin{aligned}|y(t)| &= |x(t) - (1-c)N_*| \\ &= |w(t) - N_* - c(w(t-\tau_1(t)) - N_*)| \\ &= |\varphi(t-t_0) - N_* - c(\varphi(t-\tau_1(t)-t_0) - N_*)| \\ &< (1+c)\bar{H} \quad \text{for all } t \in [t_0 - r,\ t_0],\end{aligned}$$

which allows us to assert that

$$|y(t)| < (1+c)\bar{H} \quad \text{for any } t > t_0. \tag{4.1}$$

Otherwise, there exists $S_2 > t_0$ satisfying

$$|y(S_2)| = (1+c)\bar{H} \quad \text{and} \quad |y(t)| < (1+c)\bar{H} \quad \text{for any } t \in [t_0 - r,\ S_2), \tag{4.2}$$

which gives us

$$\begin{aligned}|w(\theta) - N_*| &= |y(\theta) + c(w(\theta - \tau_1(\theta)) - N_*)| \\ &\le |y(\theta)| + c \sup_{\min\limits_{s\in[t_0-r,\theta]}(s-\tau_1(s)) \le u \le \theta} |w(u - \tau_1(u)) - N_*| \\ &\le |y(\theta)| + c \sup_{t_0 - r \le u \le t} |w(u) - N_*| \\ &< (1+c)\bar{H} + c \sup_{t_0 - r \le u \le t} |w(u) - N_*|,\end{aligned} \tag{4.3}$$

for any $\theta \in [t_0 - r,\ t]$ and $t \in [t_0 - r,\ S_2)$. A simple calculation yields

$$|w(t) - N_*| \le \sup_{t_0 - r \le u \le t} |w(u) - N_*| < \frac{(1+c)\bar{H}}{1-c} \quad \text{for any } t \in [t_0 - r,\ S_2). \tag{4.4}$$

In view of Eq. (1.4), $y(t)$ satisfies

$$y'(t) = -\delta y(t) + \beta\Big[w(t-\tau_2(t))e^{-aw(t-\tau_2(t))} - N_* e^{-aN_*}\Big], \quad t \ge t_0 \in \mathbb{R}. \tag{4.5}$$

By calculating the Dini derivative of $|y(t)|$, (4.2), (4.4) and (4.5) and Lemma 2.3 imply that

$$\begin{aligned}0 &\le D^-|y(t)|\big|_{t=S_2} \\ &\le -\delta|y(S_2)| + \beta|w(S_2 - \tau_2(S_2))e^{-aw(S_2-\tau_2(S_2))} - N_* e^{-aN_*}| \\ &\le -\delta|y(S_2)| + \beta e^{-aN_*}|w(S_2 - \tau_2(S_2)) - N_*| \\ &< \Big[-\delta + \beta e^{-aN_*}\frac{1}{1-c}\Big](1+c)\bar{H} \\ &= 0,\end{aligned}$$

it follows a contradiction and implies that (4.1) is true. One can follow a similar argument as above to deduce that (4.4) holds, we obtain

$$|w(t) - N_*| \le \sup_{t_0 - r \le u \le t} |w(u) - N_*| < \frac{(1+c)\bar{H}}{1-c} = \varepsilon \quad \text{for any } t \ge t_0.$$

Hence, the positive equilibrium N_* admits stability.

Next, we prove the global attractivity of N_*. Label

$$L = \limsup_{t\to+\infty} y(t) \quad \text{and} \quad l = \liminf_{t\to+\infty} y(t).$$

In view of Lemma 2.2, the global attractivity of N_* is equivalent to discover

$$\max\{|L|, |l|\} = 0. \tag{4.6}$$

To prove (4.6), we argue by contradiction. Assume $\max\{|L|, |l|\} = L > 0$ (the case of $\max\{|L|, |l|\} = -l > 0$ is similar). Based on the fluctuation lemma [16, Lemma A.1.], we know that there exists a sequence $\{F_k\}_{k=1}^{+\infty}$ satisfying

$$\lim_{k\to+\infty} F_k = +\infty, \quad \lim_{k\to+\infty} y(F_k) = L, \quad \text{and} \quad \lim_{k\to+\infty} y'(F_k) = 0. \tag{4.7}$$

In general, we may suppose that $\lim_{k\to+\infty} w(F_k - \tau_2(F_k))$ exists as well.

If $\lim_{k\to+\infty} w(F_k - \tau_2(F_k)) \neq 0$, it follows from (2.3), (4.5) and (4.7) that

$$\begin{aligned}
0 =& \lim_{k\to+\infty} y'(F_k) \\
=& -\delta \lim_{k\to+\infty} y(F_k) + \beta\Big[\lim_{k\to+\infty} w(F_k - \tau_2(F_k))e^{-a\lim\limits_{k\to+\infty} w(F_k-\tau_2(F_k))} - N_* e^{-aN_*}\Big] \\
\leq& -\delta \lim_{k\to+\infty} y(F_k) + \frac{\beta}{a}\Big|a\lim_{k\to+\infty} w(F_k - \tau_2(F_k))e^{-a\lim\limits_{k\to+\infty} w(F_k-\tau_2(F_k))} - aN_* e^{-aN_*}\Big| \\
<& -\delta \lim_{k\to+\infty} y(F_k) + \beta e^{-aN_*}\lim_{k\to+\infty}\big|w(F_k - \tau_2(F_k)) - N_*\big| \\
\leq& L\Big[-\delta + \beta\frac{1}{1-c}e^{-aN_*}\Big] \\
=& 0.
\end{aligned}$$

This is a contradiction.

Likewise, if $\lim_{k\to+\infty} w(F_k - \tau_2(F_k)) = 0$, one sees readily the above contradiction. Accordingly

$$\max\{|L|, |l|\} = 0,$$

this completes the proof of Theorem 4.1. □

Remark 4.2. In references [12, 20], the stability analysis of (1.3) requires the solution $w(t)$ to be a differentiable function, with its derivative $w'(t-\alpha)$ existing. This imposes restrictions on the range of admissible initial functions and consequently narrows the scope of potential solutions. Moreover, the methodologies and strategies proposed in the aforementioned literature cannot be directly applied to analyze the stability and attractiveness of the non-autonomous neutral delayed Nicholson's blowflies equation (1.4) under the initial condition (2.1). Additionally, in contrast to the initial function set Γ in [3], which depends on the coefficient of the death term, the set Γ_+ introduced in this manuscript is independent of this coefficient. This distinction offers a more comprehensive and versatile framework for understanding the stability and attractiveness of solutions across different initial value ranges, providing deeper insights into the problem.

Remark 4.3. Evidently, when $c = 0$, Eq. (1.4) reduces to the standard Nicholson's blowflies equation Eq. (1.1), which has received widespread attention and in-depth research. Especially,

the sharp criteria $\frac{\beta}{\delta} \leq 1$ and $1 < \frac{\beta}{\delta} \leq e^2$ for respectively ensuring the global asymptotic stability of trivial and positive equilibria were rigidly established in [5, 11, 17–19], which are special cases of the corresponding ones in Theorems 3.1 and 4.1 if $c = 0$. In this sense, global asymptotic stability criteria $\frac{\beta}{\delta(1-c)} \leq 1$ and $1 < \frac{\beta}{\delta(1-c)} \leq e^2$ established are also sharp to time-varying delayed Nicholson's blowflies equation of neutral type (1.4). On the other hand, since the delays are time varying, Eq. (1.4) is non-autonomous, which has brought theoretical and technical difficulties to the dynamic study of Eq. (1.4), and the stability problem for such a neutral model has not received attention until now. Hence, the theoretical results established in this paper are substantially new and enrich the theory of functional differential equations to some extent.

5 Numerical examples

In this section, numerical examples with simulation are afforded to illustrate the main theoretical findings.

Example 5.1. Consider a neutral time-varying delayed Nicholson's blowflies model with different parameters,

$$\begin{aligned}\Big[w(t) - \frac{1}{2}w(t - e^{-|\sin 2t|})\Big]' = & -4(w(t) - \frac{1}{2}w(t - e^{-|\sin 2t|})) \\ & + w(t - 2e^{-|\sin 2t|})e^{-2w(t-2e^{-|\sin 2t|})}, \quad t \geq t_0 = 0, \end{aligned} \tag{5.1}$$

$$\begin{aligned}\Big[w(t) - \frac{1}{2}w(t - e^{-|\sin 2t|})\Big]' = & -2(w(t) - \frac{1}{2}w(t - e^{-|\sin 2t|})) \\ & + ew(t - 2e^{-|\sin 2t|})e^{-2w(t-2e^{-|\sin 2t|})}, \quad t \geq t_0 = 0, \end{aligned} \tag{5.2}$$

and

$$\begin{aligned}\Big[w(t) - \frac{1}{2}w(t - e^{-|\sin 2t|})\Big]' = & -2(w(t) - \frac{1}{2}w(t - e^{-|\sin 2t|})) \\ & + 100w(t - 2e^{-|\sin 2t|})e^{-2w(t-2e^{-|\sin 2t|})}, \quad t \geq t_0 = 0. \end{aligned} \tag{5.3}$$

Obviously, $\tau_1(t) = e^{-|\sin 2t|}$, $\tau_2(t) = 2e^{-|\sin 2t|}$, $2 = r = r_2 > r_1 = 1$, let

$$\Gamma_+ = \big\{\varphi \in C([-2,\ 0],\ \mathbb{R}) \mid \varphi(0) - c\varphi(-\tau_1(0)) \geq 0,\ \varphi(\theta) \geq 0 \text{ for all } \theta \in [-2,\ 0]\big\}.$$

One can easily verify that $\frac{\beta}{\delta(1-c)} = \frac{1}{4(1-\frac{1}{2})} = \frac{1}{2} < 1$ and $1 < \frac{\beta}{\delta(1-c)} = \frac{e}{2\times(1-\frac{1}{2})} = e < e^2$ hold for Eqs. (5.1) and (5.2), respectively. It is concluded from Theorems 3.1 and 4.1 that the zero equilibrium of Eq. (5.1) and the positive equilibrium of Eq. (5.2) are all globally asymptotically stable (see Figs. 5.1 and 5.2). However, the stability criteria $\frac{\beta}{\delta(1-c)} \leq 1$ and $1 < \frac{\beta}{\delta(1-c)} \leq e^2$ are invalid for Eq. (5.3), which means that the positive equilibrium of Eq. (5.3) is not globally asymptotically stable, Fig. 5.3 indicates this fact.

Remark 5.2. It is not difficult to check that $\tau_1(t) = e^{-|\sin 2t|}$ and $\tau_2(t) = 2e^{-|\sin 2t|}$ in Eqs. (5.1)–(5.3) are time dependent, which do not satisfy the basic constraints of the delay terms in [2, 3, 7, 12, 21]. In addition, the sharp asymptotic stability conditions have not been touched in [10, 13, 20, 22]. Consequently, the results in above-mentioned literatures and their references could not directly applied to this example. This shows that the established theoretical results in this paper extend and improve existing ones.

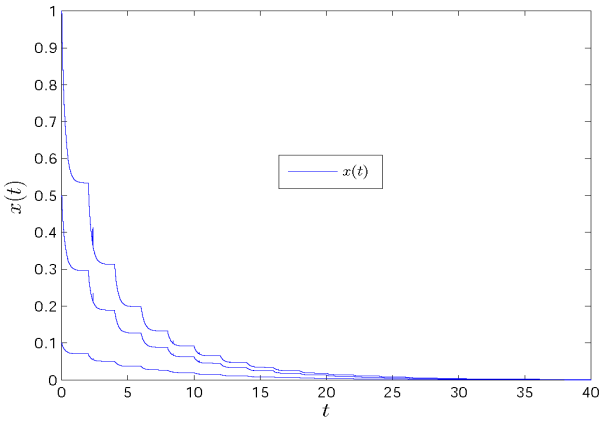


Figure 5.1: The state trajectories to Eq. (5.1) with initial values $0.1, 0.5, 1$.

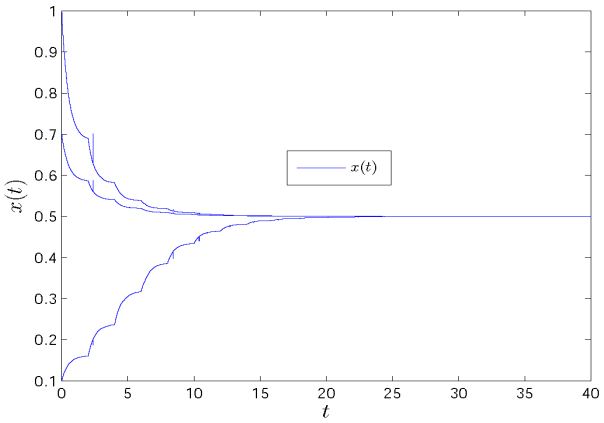


Figure 5.2: The state trajectories to Eq. (5.2) with initial values $0.1, 0.7, 1$.

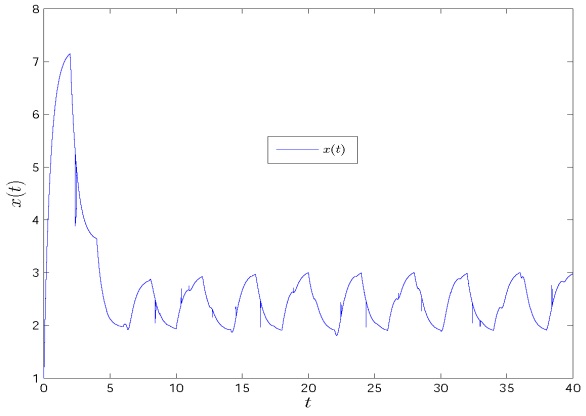


Figure 5.3: The state trajectory to Eq. (5.3) with initial value 1.

6 Conclusion

In this paper, the stability and attraction of the neutral Nicholson's blowflies equation with time-varying delays are addressed. By applying the analytical method such as Dini derivative and differential inequality techniques, sharp sufficient criteria substantially extend the existing results on global asymptotic stability, comparative analysis and simulations are also given to support the availability of the theoretical results.

It is also worth noting that the authors in [17] evidenced that the positive equilibrium point of the classical non-neutral delayed Nicholson's blowflies model attracts all solutions with nonnegative initial values under the delay-dependent condition $(e^{\delta\tau}-1)\ln\frac{P}{\delta}\leq 1$. Whether or not our methods used in this paper are available to find delay-dependent criterion to ensure the global attraction of positive equilibrium to Eq. (1.4), it is an interesting problem and we leave it as an important topic of future researchers.

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On the existence of ground state solutions for the logarithmic Schrödinger–Bopp–Podolsky system

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Abstract. This paper deals with the following logarithmic Schrödinger–Bopp–Podolsky system

$$\begin{cases} -\Delta u + V(x)u - \phi u = u\log u^2 & \text{in } \mathbb{R}^3, \\ -\Delta\phi + \Delta^2\phi = 4\pi u^2 & \text{in } \mathbb{R}^3, \end{cases}$$

where $V(x) \in C(\mathbb{R}^3, \mathbb{R})$. By using the variational method developed by Szulkin for the functional which is the sum of a smooth and a convex lower semicontinuous term, we study the properties of the solutions for the above system under different potential conditions. When the potential is coercive, we discuss the existence of a ground state solution. Moreover, we also consider the cases where $V(x)$ is periodic or asymptotically periodic, and obtain a ground state solution in each scenario, respectively.

Keywords: Schrödinger–Bopp–Podolsky system, logarithmic nonlinearity, variational methods.

2020 Mathematics Subject Classification: 35J20, 35J50, 35J40.

1 Introduction and main results

In the past few decades, the following nonlinear Schrödinger–Bopp–Podolsky system

$$\begin{cases} -\Delta u + V(x)u + \lambda\phi u = f(u) & \text{in } \mathbb{R}^3, \\ -\Delta\phi + a^2\Delta^2\phi = 4\pi u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $\lambda, a \in \mathbb{R}$ are parameters, has been studied by many researchers. This system is closely related to the Bopp–Podolsky electromagnetic theory and arises when coupling a Schrödinger field $\psi = \psi(t,x)$ with its electromagnetic field in the Bopp–Podolsky electromagnetic theory. Among them, the Bopp–Podolsky theory is called a second-order gauge theory of electromagnetic fields, which was established by Bopp [3] and later developed by Podolsky [22] to solve

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the so-called infinity problem in the classical Maxwell theory. From the viewpoint of electromagnetic fields, the Bopp–Podolsky theory can be explained as an effective theory for short distances, while for large distances it is experimentally indistinguishable from the Maxwell theory (see [9]).

To our knowledge, d'Avenia and Siciliano in [7] first proved the existence and non-existence of nontrivial solutions for system (1.1) with constant potential via using the variational method and splitting lemma, where the nonlinear term is denoted as $f(u) = |u|^{p-2}u$. Moreover, in the radial case, they found that solutions tend to those of the classical Schrödinger–Poisson system as $a \to 0$. In [13], Li, Pucci and Tang generalized the existing results for system (1.1) to the critical case and used the Pohožaev–Nehari manifold method to divide the equation into two cases, the constant potential and the asymptotic constant potential, proving that there is a ground state solution when the nonlinear term increases critically. Then in [4], the authors improved the nonlinear term in [13] to a general nonlinear term, and adopted some new analytical techniques and new inequalities to prove the existence of solutions in different cases. Yang, Chen and Liu [27] applied a cut-off function, the mountain pass theorem and Moser iteration to prove the existence of nontrivial solution for system (1.1) with critical growth. Zhu, Chen and Chen [30] studied the existence of different solutions for system (1.1) under nonlinearity effects.

Later, Jia, Li and Chen [12] established the existence of ground state solutions for nonautonomous Schrödinger–Bopp–Podolsky system. Liu and Chen [18] studied the existence, nonexistence and asymptotic behavior of ground state solutions for problem (1.1) with critical Sobolev exponent. Peng [20] proved the existence and multiplicity of solutions for the system (1.1). Zhang [28] investigated the existence of sign-changing solutions for system (1.1) with general nonlinearity. Yang, Yuan and Liu [26] were concerned with the existence of ground states for a nonlinear Schrödinger–Bopp–Podolsky system with asymptotically periodic potentials. Li and Zhang [14] found the existence of normalized solution for the system (1.1). For more information on the results of a system like (1.1), the readers can refer to [1, 5, 10, 19, 29] and the references therein.

Recently, the Schrödinger problem with logarithmic terms given by

$$i\varepsilon\frac{\partial \Phi}{\partial t} = -\varepsilon^2\Delta\Phi + W(x)\Phi - \Phi\log|\Phi|^2, \quad N \geq 3, \tag{1.2}$$

where $\Phi\colon [0,+\infty) \times \mathbb{R}^N \to \mathbb{C}$, has also received extensive attention due to its physical influence, such as quantum mechanics, quantum optics, nuclear physics, effective quantum, and Bose–Einstein condensation (see [31]). The standing wave solutions for problem (1.2) have the ansatz form $\Phi(t,x) = u(x)\mathrm{e}^{-i\omega t/\varepsilon}$, which leads to the following equation

$$-\varepsilon^2\Delta u + V(x)u = u\log u^2, \quad \text{in } \mathbb{R}^N, \tag{1.3}$$

where $V(x) = W(x) - \omega$ and $\omega \in \mathbb{R}$. Its associated energy functional is as follows

$$\tilde{\mathcal{J}}_\varepsilon(u) = \frac{1}{2}\int_{\mathbb{R}^N}\left(\varepsilon^2|\nabla u|^2 + (V(x)+1)u^2\right)\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N} u^2\log u^2\,\mathrm{d}x. \tag{1.4}$$

From a mathematical point of view, problem like (1.3) is very interesting because it is not like the general Schrödinger equation whose energy functional is C^1 class, while the energy functional $\tilde{\mathcal{J}}_\varepsilon(u)$ of the logarithmic Schrödinger equation is non-smooth. Therefore, it cannot be solved by the general critical point theory, and new technical means must be employed for

research, which brings many difficulties to the research process. In fact, there have been some results in this direction.

In [6], d'Avenia, Montefusco and Squassina used the non-smooth critical point theory to demonstrate the existence of infinitely many weak solutions for the logarithmic Schrödinger equation (1.3) as $\varepsilon = 1$ and $V(x) = 1$. Furthermore, they have proven that there exists a unique positive solution that is radially symmetric and nondegenerate. In [8], d'Avenia, Squassina and Zenari adopted the same way to show the existence of infinitely many solutions of a fractional Schrödinger equation with logarithmic nonlinearity.

Squassina and Szulkin [23] combined the method of the minimax principles for lower semicontinuous functionals proposed by Szulkin [24] to study the following problem

$$-\Delta u + V(x)u = Q(x)u \log u^2, \quad \text{in } \mathbb{R}^N, \tag{1.5}$$

where $V, Q\colon \mathbb{R}^N \to \mathbb{R}$ are 1-periodic continuous functions for x verifying

$$\min_{x\in\mathbb{R}^N} Q(x) > 0 \quad \text{and} \quad \min_{x\in\mathbb{R}^N} (V+Q)(x) > 0.$$

As a consequence, they obtained the existence of infinitely many geometrically distinct solutions for problem (1.5). In [11], Ji and Szulkin, inspired by [23], discussed the existence of multiple solutions and a ground state solution for equation (1.3), where $\varepsilon = 1$ and the potential $V(x)$ satisfies

$$V(x) \in C\left(\mathbb{R}^N, \mathbb{R}\right), \quad \lim_{|x|\to+\infty} V(x) = V_\infty \quad \text{and} \quad V_\infty + 1 \in (0, +\infty].$$

In the recent paper [2], Alves and de Morais Filho investigated the existence of positive solutions for the logarithmic elliptic equation (1.3). By using the variational method developed by Szulkin, they got the existence and concentration of solutions as $\varepsilon \to 0$, and they also considered the cases when the potential $V(x)$ is periodic or asymptotically periodic.

In particular, Peng and Jia [21] studied the logarithmic Schrödinger–Bopp–Podolsky system as follows

$$\begin{cases} -\varepsilon^2\Delta u + V(x)u - \phi u = u\log u^2 & \text{in } \mathbb{R}^3, \\ -\varepsilon^2\Delta\phi + \varepsilon^4\Delta^2\phi = 4\pi u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.6}$$

where V satisfies the following global condition:

$$(V)V(x) \in C\left(\mathbb{R}^3, \mathbb{R}\right) \quad \text{and} \quad V_\infty := \lim_{|x|\to+\infty} V(x) > \inf_{x\in\mathbb{R}^3} V(x) = V_0 > -1.$$

Borrowing an idea from [2], they proved the existence and concentration behavior of positive solution for equation (1.6).

Inspired by the above literatures, in this paper, we establish the existence of ground state solutions for the following logarithmic Schrödinger–Bopp–Podolsky system under different potentials

$$\begin{cases} -\Delta u + V(x)u - \phi u = u\log u^2 & \text{in } \mathbb{R}^3, \\ -\Delta\phi + \Delta^2\phi = 4\pi u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.7}$$

where $V(x) \in C\left(\mathbb{R}^3, \mathbb{R}\right)$ satisfies the following global conditions:

(V_1) $V(x)$ is coercive and $-1 < V_0 = \inf_{x\in\mathbb{R}^3} V(x)$.

(V_2) $V(x)$ is a continuous $\mathbb{Z}^3$-periodic function, i.e.,

$$V(x+y)=V(x), \quad \forall x\in\mathbb{R}^3, \forall y\in\mathbb{Z}^3 \quad \text{and} \quad -1<V_0=\inf_{x\in\mathbb{R}^3} V(x).$$

(V_3) $V(x)$ is a continuous asymptotically periodic, that is, there is a continuous $\mathbb{Z}^3$-periodic function V_p such that

$$-1<V_0=\inf_{x\in\mathbb{R}^3} V(x)\le V(x)<V_p(x), \quad \text{for all } x\in\mathbb{R}^3$$

and

$$\left|V(x)-V_p(x)\right|\to 0 \quad \text{as } |x|\to+\infty.$$

We will use the Variational Method developed by Szulkin [24] to get our results. Compared with previous papers, the problem we study is more complex due to the interaction between logarithmic and non-local terms. To our knowledge, there is currently only one article [21] in this direction, but in that literature the authors considered the case of asymptotic constant potential, which is different from the problem we are studying. Therefore, it is necessary to employ different methods to address our research question.

Definition 1.1. A weak solution for system (1.7) means a pair of $(u,\phi)\in E_V\times\mathcal{D}$ satisfying $u^2\log u^2\in L^1\left(\mathbb{R}^3\right)$ (i.e., $I(u)<+\infty$) and

$$\int_{\mathbb{R}^3}(\nabla u\nabla\zeta+V(x)u\zeta-\phi u\zeta)\mathrm{d}x=\int_{\mathbb{R}^3}u\zeta\log u^2\,\mathrm{d}x, \quad \text{for any } \zeta\in C_0^\infty\left(\mathbb{R}^3\right),$$

$$\int_{\mathbb{R}^3}\nabla\phi\nabla\xi\mathrm{d}x+\int_{\mathbb{R}^3}\Delta\phi\Delta\xi\mathrm{d}x=4\pi\int_{\mathbb{R}^3}\xi u^2\,\mathrm{d}x, \quad \text{for any } \xi\in\mathcal{D},$$

where the definitions of $I(u)$, E_V and $\mathcal{D}$ will be given in Sect. 2.

We derive the following results.

Theorem 1.2. *Assume that the condition (V_1) holds. Then, system (1.7) has one ground state solution.*

Theorem 1.3. *Assume that the condition (V_2) holds. Then, system (1.7) has one ground state solution.*

Theorem 1.4. *Assume that the condition (V_3) holds. Then, system (1.7) has one ground state solution.*

Remark 1.5. In this article, we just consider the case that $V_p<+\infty$ in the condition (V_3). Indeed, the case $V_p=+\infty$ is simpler since the embedding $E_V\hookrightarrow L^s(\mathbb{R}^3)$ is compact for $s\in[2,6)$, where E_V see Sect. 2. Moreover, we should note that $V_0>-1$ is assumed instead of the general condition $V_0>0$ because the nonlinear term we consider is logarithmic. In fact, this question can be explained from (1.4), and readers can also refer to [25] for more details.

Remark 1.6. When considering the logarithmic equation, we will cite the useful logarithmic Sobolev inequality found in [15]:

$$\int_{\mathbb{R}^3}u^2\log u^2\,\mathrm{d}x\le\frac{a^2}{\pi}\|\nabla u\|_2^2+\left(\log\|u\|_2^2-3(1+\log a)\right)\|u\|_2^2, \quad \text{for all } a>0. \tag{1.8}$$

Let us give a sketch of the proof of the results and explain the difficulties encountered in the process of solving them.

- Due to the existence of the logarithmic term, the energy functional $I(u)$ [see (2.3)] related to system (1.7) may take the value $+\infty$ since there is a function $u \in H^1(\mathbb{R}^3)$ such that $\int_{\mathbb{R}^3} u^2 \log u^2 \, \mathrm{d}x = -\infty$. Therefore, the functional $I(u)$ is not of class C^1, which makes it impossible to be solved by the general critical point theory.
- In order to find solutions of system (1.7), similar to [23], we will use a technical decomposition of $I(u)$ [see (2.4)]. In this case, we can apply the Mountain Pass Theorem without (PS) condition for a functional that is the sum of a smooth and a convex lower semicontinuous term, which was first mentioned in [2].
- Because of the interaction between the logarithmic term and the non-local term, the boundedness of (PS) sequences is more difficult to obtain than in [2], so we solve this problem with the help of literature [21].
- For the proof of Theorem 1.2, since the compactness can be directly obtained under the coercive potential condition, we only need to prove the boundness of (PS) sequence and combine the Mountain Pass Theorem without (PS) condition to complete the proof.
- The lack of compactness makes the proofs of Theorems 1.3 and 1.4 more difficult. Under the periodic potential condition, we still begin by decomposing the functional and defining the Nehari manifold $\mathcal{M}_V$ [see (3.6)]. Subsequently, we derive the required conclusions via combining the Mountain Pass Theorem without (PS) condition. During this process, it is necessary to refer to [17] to return to the solution of the original problem [see Lemma 5.1]. The key steps in the proof involve verifying that the mountain pass value is equal to the infimum on the Nehari manifold, i.e., $c = c_V := \inf_{u \in \mathcal{M}_V} I(u)$ [see Lemma 3.7], and $u_n \rightharpoonup u_0 \neq 0$. However, the difference of asymptotic periodic potential is that the proof of $u_n \rightharpoonup u_0 \neq 0$ is obtained via comparing c_V with c_{V_p} [see Lemma 6.2], and other proofs are similar to the periodic potential case.

This paper is organized as follows. In Section 2, we recall some lemmas which we will use in the paper. In Section 3, we introduce the Mountain Pass Theorem without (PS) condition. In Section 4, we give the proof of Theorem 1.2. Subsequently, Theorems 1.3 and 1.4 will be proved in Sections 5 and 6, respectively.

Throughout this paper, we use the following notations:

- $L^s(\mathbb{R}^3)$ $(2 \leq s < \infty)$ denotes the Lebesgue space with the norm $\|u\|_s = \left(\int_{\mathbb{R}^3} |u|^s \, \mathrm{d}x\right)^{1/s}$;
- $B_r(x)$ denotes the ball centered at x with radius r;
- C, C', $\tilde{C}$, C_i $(i = 1, 2, \dots)$ denote positive constants possibly different in different places;
- $o_n(1)$ denotes a real sequence with $o_n(1) \to 0$ as $n \to +\infty$;
- $H_c^1\left(\mathbb{R}^3\right) = \left\{u \in H^1\left(\mathbb{R}^3\right) \mid u \text{ has a compact support } \right\}$.

2 Preliminaries

In this section, we present the variational setting and give a special decomposition of the functional $I(u)$, which needs to be adjusted since $I(u)$ may not be well defined in $H^1(\mathbb{R}^3)$. According to the research techniques in [23], we decompose $I(u)$ into a sum of a C^1 functional plus a convex lower semicontinuous functional.

2.1 Variational setting

Let $H^1(\mathbb{R}^3)$ denote the Sobolev space equipped with the norm

$$\|u\|_H = \left(\int_{\mathbb{R}^3} \left(|\nabla u|^2 + u^2 \right) \mathrm{d}x \right)^{\frac{1}{2}},$$

and $\mathcal{D}^{1,2}\left(\mathbb{R}^3\right) = \left\{ u \mid u \in L^6\left(\mathbb{R}^3\right), \nabla u \in L^2\left(\mathbb{R}^3\right) \right\}$ be endowed with the norm

$$\|u\|_{\mathcal{D}^{1,2}} = \left(\int_{\mathbb{R}^3} |\nabla u|^2 \, \mathrm{d}x \right)^{\frac{1}{2}}.$$

Let $\mathcal{D}$ be the completion of $C_0^\infty\left(\mathbb{R}^3\right)$ with respect to the norm $\|\cdot\|_{\mathcal{D}}$ induced by the scalar product

$$\langle \phi, \xi \rangle_{\mathcal{D}} := \int_{\mathbb{R}^3} \nabla\phi \nabla\xi \mathrm{d}x + \int_{\mathbb{R}^3} \Delta\phi \Delta\xi \mathrm{d}x.$$

Clearly, $\mathcal{D}$ is a Hilbert space continuously embedded into $\mathcal{D}^{1,2}\left(\mathbb{R}^3\right)$ and consequently in $L^6\left(\mathbb{R}^3\right)$. Moreover, the space $\mathcal{D}$ is continuously embedded in $L^\infty\left(\mathbb{R}^3\right)$ (see Lemma 3.1 in [7]).

Next, we present the basic lemma.

Lemma 2.1 ([7, Lemma 3.2]). *The space $C_0^\infty\left(\mathbb{R}^3\right)$ is dense in*

$$\mathcal{A} := \left\{ \phi \in \mathcal{D}^{1,2}\left(\mathbb{R}^3\right) : \Delta\phi \in L^2\left(\mathbb{R}^3\right) \right\}$$

normed by $\sqrt{\langle \phi, \phi \rangle_{\mathcal{D}}}$ and, therefore, $\mathcal{D} = \mathcal{A}$.

In view of the Riesz Theorem, for every fixed $u \in H^1\left(\mathbb{R}^3\right)$, there exists a unique solution $\phi_u \in \mathcal{D}$ of the second equation in (1.7), namely,

$$-\Delta\phi + \Delta^2\phi = 4\pi u^2$$

and ϕ_u can be represented by

$$\phi_u = \int_{\mathbb{R}^3} \frac{1 - \mathrm{e}^{-|x-y|}}{|x-y|} u^2(y) \mathrm{d}y. \tag{2.1}$$

Then we have the following fundamental properties.

Lemma 2.2 ([7, Lemma 3.4] and [27, Lemma 2.2]). *For $u \in H^1\left(\mathbb{R}^3\right) \setminus \{0\}$, we have:*

(1) $\phi_u \geq 0$ in $\mathbb{R}^3$;

(2) $\phi_u \in L^s\left(\mathbb{R}^3\right) \cap C_0\left(\mathbb{R}^3\right)$, $\forall s \in (3, +\infty]$;

(3) $\phi_{u(\cdot+y)} = \phi_u(\cdot + y)$, $\forall y \in \mathbb{R}^3$;

(4) $\|\phi_u\|_{\mathcal{D}} \leq C\|u\|_H^2$, $\int_{\mathbb{R}^3} \phi_u u^2 \, \mathrm{d}x \leq C\|u\|_{12/5}^4$;

(5) if $u_n \rightharpoonup u$ in $H^1\left(\mathbb{R}^3\right)$, then $\phi_{u_n} \rightharpoonup \phi_u$ in $\mathcal{D}$;

(6) if $u_n \to u$ in $L^{\frac{12}{5}}\left(\mathbb{R}^3\right)$, then $\phi_{u_n} \to \phi_u$ in $\mathcal{D}$ and $\int_{\mathbb{R}^3} \phi_{u_n} u_n^2 \, \mathrm{d}x \to \int_{\mathbb{R}^3} \phi_u u^2 \, \mathrm{d}x$;

(7) ϕ_u is the unique minimizer of the functional

$$X(\phi) = \frac{1}{2}\|\nabla\phi\|_2^2 + \frac{1}{2}\|\Delta\phi\|_2^2 - 4\pi\int_{\mathbb{R}^3}\phi u^2\,\mathrm{d}x, \quad \phi \in \mathcal{D}.$$

Now we introduce the working space,

$$E_V := \left\{u \in H^1\left(\mathbb{R}^3\right) : \int_{\mathbb{R}^3} V(x)u^2\,\mathrm{d}x < +\infty\right\}$$

with the norm $\|u\|_V$ induced by the scalar product

$$\langle u, v\rangle_V = \int_{\mathbb{R}^3}(\nabla u\nabla v + (V(x)+1)uv)\,\mathrm{d}x.$$

Note that E_V is a Hilbert space, and the embedding $E_V \hookrightarrow H^1\left(\mathbb{R}^3\right)$ is continuous.

For any $(u,\phi) \in E_V \times \mathcal{D}$, the associated energy functional of system (1.7) is given by

$$\begin{aligned}\mathcal{S}(u,\phi) = &\frac{1}{2}\int_{\mathbb{R}^3}\left(|\nabla u|^2 + (V(x)+1)u^2\right)\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^3}\phi u^2\,\mathrm{d}x + \frac{1}{16\pi}\int_{\mathbb{R}^3}|\nabla\phi|^2\,\mathrm{d}x\\ &+ \frac{1}{16\pi}\int_{\mathbb{R}^3}|\Delta\phi|^2\,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^3}u^2\log u^2\,\mathrm{d}x.\end{aligned} \tag{2.2}$$

To avoid the difficulties caused by the strong uncertainty of the functional, we use the usual reduction function procedure introduced in [7]. From this, we can obtain that the reduced functional has the form as follows

$$\begin{aligned}I(u) := \mathcal{S}(u,\phi_u) = &\frac{1}{2}\int_{\mathbb{R}^3}\left(|\nabla u|^2 + (V(x)+1)u^2\right)\mathrm{d}x\\ &- \frac{1}{4}\int_{\mathbb{R}^3}\phi_u u^2\,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^3}u^2\log u^2\,\mathrm{d}x.\end{aligned} \tag{2.3}$$

Remark 2.3. The following conclusions are equivalent:

(i) the pair $(u,\phi) \in E_V \times \mathcal{D}$ is a critical point of $\mathcal{S}$, namely, (u,ϕ) is a solution of (1.7);

(ii) u is a critical point of $I(u)$ and $\phi = \phi(u)$.

2.2 Decomposition of the functional $I(u)$

For $\delta > 0$ small, let us define the following functions:

$$F(s) = \begin{cases}0, & s = 0,\\ -\frac{1}{2}s^2\log s^2, & 0 < |s| < \delta,\\ -\frac{1}{2}s^2\left(\log\delta^2 + 3\right) + 2\delta|s| - \frac{1}{2}\delta^2, & |s| \geq \delta\end{cases}$$

and

$$G(s) = \begin{cases}0, & |s| < \delta,\\ \frac{1}{2}s^2\log\left(s^2/\delta^2\right) + 2\delta|s| - \frac{3}{2}s^2 - \frac{1}{2}\delta^2, & |s| \geq \delta.\end{cases}$$

Then,

$$G(s) - F(s) = \frac{1}{2}s^2\log s^2, \quad \forall s \in \mathbb{R}$$

and the functional $I : E_V \to (-\infty, +\infty]$ may be denoted as

$$I(u) = \Phi(u) + \Psi(u), \quad u \in E_V, \tag{2.4}$$

where

$$\Phi(u) := \frac{1}{2}\int_{\mathbb{R}^3} \left(|\nabla u|^2 + (V(x)+1)u^2\right)\, \mathrm{d}x - \frac{1}{4}\int_{\mathbb{R}^3} \phi_u u^2\, \mathrm{d}x - \int_{\mathbb{R}^3} G(u)\, \mathrm{d}x$$

and

$$\Psi(u) := \int_{\mathbb{R}^3} F(u)\, \mathrm{d}x.$$

As proven in [11], we can list some properties of F and G as follows:

- $F, G \in C^1(\mathbb{R}, \mathbb{R})$.
- If $\delta > 0$ is fixed and small enough, F is a nonnegative, convex, even function and $F'(s)s \geq 0$, for all $s \in \mathbb{R}$.
- For every fixed $p \in (2, 6)$, there exists $C > 0$ such that for any $s \in \mathbb{R}$,

$$\left|G'(s)\right| \leq C\,|s|^{p-1}. \tag{2.5}$$

Therefore, referring to [11], we can get that Ψ is nonnegative, convex and lower semicontinuous, and $\Phi \in C^1(E_V, \mathbb{R})$. Next, we will review some definitions and results of convex analysis that first appeared in [24].

Definition 2.4. Let E be a Banach space, E' be the dual space of E and $\langle \cdot, \cdot \rangle$ be the duality pairing between E and E'. Let $I : E \to \mathbb{R}$ be a functional and $I(u) = \Phi(u) + \Psi(u)$, where $\Phi \in C^1(E, \mathbb{R})$ and Ψ is convex and lower semicontinuous. Then the following results are given:

(i) The set $D(I) := \{u \in E \colon I(u) < +\infty\}$ is called the effective domain of I.

(ii) The sub-differential $\partial I(u)$ of the functional I at a point $u \in E$ is the following set

$$\left\{w \in E' : \left\langle \Phi'(u), v - u \right\rangle + \Psi(v) - \Psi(u) \geq \langle w, v - u \rangle,\ \forall v \in E\right\}.$$

(iii) $u \in E$ is a critical point of I such that $u \in D(I)$ and $0 \in \partial I(u)$, i.e.,

$$\left\langle \Phi'(u), v - u \right\rangle + \Psi(v) - \Psi(u) \geq 0, \quad \forall v \in E.$$

(iv) A Palais–Smale sequence at level c for I is a sequence $\{u_n\} \subset E$ such that $I(u_n) \to c$ and there is a numerical sequence $\sigma_n \to 0^+$ with

$$\left\langle \Phi'(u_n), v - u_n \right\rangle + \Psi(v) - \Psi(u_n) \geq -\sigma_n \|v - u_n\|, \quad \forall v \in E. \tag{2.6}$$

(v) The functional I satisfies the Palais–Smale condition at level c ($(PS)_c$ condition, for short) if each Palais–Smale sequence of I has a convergent subsequence in E.

To further advance the analysis, for any $u \in D(I)$, we define $I'(u) : H_c^1(\mathbb{R}^3) \to \mathbb{R}$ given by

$$\langle I'(u), z\rangle = \langle \Phi'(u), z\rangle + \int_{\mathbb{R}^3} F'(u)z \, \mathrm{d}x, \ \forall z \in H_c^1(\mathbb{R}^3)$$

and

$$\|I'(u)\| := \sup\left\{\langle I'(u), z\rangle : z \in H_c^1(\mathbb{R}^3) \text{ with } \|z\|_V \le 1\right\}.$$

If $\|I'(u)\| < +\infty$, then $I'(u)$ may be extended to a bounded operator in E_V, and it can also be considered as an element of E_V'.

From Lemma 3.3 of reference [21], we state some useful results that can help us solve the considered problem.

Lemma 2.5. *Assume that $I(u)$ satisfies (2.4). Then*

(i) if $u \in D(I)$ is a critical point of I, then for any $v \in E_V$,

$$\langle \Phi'(u), v-u\rangle + \Psi(v) - \Psi(u) \ge 0,$$

namely,

$$\begin{aligned}\int_{\mathbb{R}^3} \nabla u \nabla(v-u)\mathrm{d}x + \int_{\mathbb{R}^3} (V(x)+1)u(v-u)\mathrm{d}x - \int_{\mathbb{R}^3} \phi_u u(v-u)\,\mathrm{d}x \\ + \int_{\mathbb{R}^3} F(v)\mathrm{d}x - \int_{\mathbb{R}^3} F(u)\mathrm{d}x \ge \int_{\mathbb{R}^3} G'(u)(v-u)\mathrm{d}x;\end{aligned}$$

(ii) for every $u \in D(I)$ such that $\|I'(u)\| < +\infty$, we have $\partial I(u) \neq \emptyset$, i.e., there is $w \in E_V'$, which is denoted by $w = I'(u)$, such that for any $v \in E_V$,

$$\langle \Phi'(u), v-u\rangle + \int_{\mathbb{R}^3} F(v)\mathrm{d}x - \int_{\mathbb{R}^3} F(u)\mathrm{d}x \ge \langle w, v-u\rangle;$$

(iii) if $u \in D(I)$ is a critical point of I, then (u, ϕ_u) is one solution of system (1.7);

(iv) if $\{u_n\} \subset E_V$ is a Palais–Smale sequence, then for any $z \in H_c^1(\mathbb{R}^3)$,

$$\langle I'(u_n), z\rangle = o_n(1)\|z\|_V;$$

(v) if Λ is a bounded domain with regular boundary, then Ψ (and hence I) is of class C^1 in $H^1(\Lambda)$. Precisely, for any $u \in H^1(\Lambda)$, the functional

$$\Psi(u) = \int_\Lambda F(u)\mathrm{d}x$$

belongs to $C^1\left(H^1(\Lambda), \mathbb{R}\right)$.

According to the above proprieties, we can directly get the following consequences.

Lemma 2.6. *If $u \in D(I)$ and $\|I'(u)\| < +\infty$, then $F'(u)u \in L^1(\mathbb{R}^3)$, where I satisfies (2.4).*

Proof. This proof process relies on Lemma 2.5-(ii) and (v), and for more details, readers can refer to the Lemma 2.1 in [2]. For brevity, we will omit the specifics here. □

In what follows, for each $u \in D(I)$, we can set the functional $I'(u) : E_V \to \mathbb{R}$ given by

$$I'(u)u = \int_{\mathbb{R}^3} \left(|\nabla u|^2 + (V(x)+1)u^2 - \phi_u u^2 - G'(u)u\right)\,\mathrm{d}x + \int_{\mathbb{R}^3} F'(u)u\,\mathrm{d}x \tag{2.7}$$

$$= \int_{\mathbb{R}^3} \left(|\nabla u|^2 + V(x)u^2 - \phi_u u^2\right)\mathrm{d}x - \int_{\mathbb{R}^3} u^2 \log u^2\,\mathrm{d}x. \tag{2.8}$$

3 Mountain pass theorem without (PS) condition

In this section, we will introduce an abstract theorem for the functional of the type $I(u) = \Phi(u) + \Psi(u)$, where $\Phi \in C^1$ and Ψ is convex and lower semicontinuous. This method was proposed by Alves and de Morais Filho [2] under the influence of Szulkin [24].

Proposition 3.1 ([2]). *Let E be a Banach space and $I\colon E \to (-\infty, +\infty]$ be a functional such that:*

(i) $I(u) = \Phi(u) + \Psi(u)$, *where* $\Phi(u) \in C^1(E, \mathbb{R})$, *and* $\Psi\colon E \to (-\infty, +\infty]$ *is convex, lower semicontinuous and* $\Psi(u) \not\equiv +\infty$;

(ii) $I(0) = 0$ *and* $I|_{\partial B_\rho(0)} \geq \alpha$, *for some* $\rho, \alpha > 0$;

(iii) $I(e) \leq 0$, *for some* $e \notin \overline{B}_\rho(0)$.

Then for fixed $\varepsilon > 0$, there is $u_\varepsilon \in E$ satisfying

$$\left\langle \Phi'(u_\varepsilon), w - u_\varepsilon \right\rangle + \Psi(w) - \Psi(u_\varepsilon) \geq -3\varepsilon \|w - u_\varepsilon\|, \quad \forall w \in E$$

and

$$I(u_\varepsilon) \in [c - \varepsilon, c + \varepsilon],$$

where

$$c := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t))$$

and

$$\Gamma = \{\gamma \in C([0,1], E) : \gamma(0) = 0, I(\gamma(1)) < 0\}.$$

Corollary 3.2. *From Proposition 3.1, it is clear that there exists a (PS) sequence $\{u_n\} \subset E$ for I, namely, $I(u_n) \to c$ and*

$$\left\langle \Phi'(u_n), w - u_n \right\rangle + \Psi(w) - \Psi(u_n) \geq -\sigma_n \|w - u_n\|, \quad \forall w \in E$$

with $\sigma_n \to 0^+$.

In the sequel, we will apply Proposition 3.1 and Corollary 3.2 to obtain our results. The most crucial point is to prove that the (PS) sequence has a convergent subsequence. First, we prove that I possesses the Mountain Pass Geometry.

Lemma 3.3. *Assume that (V_i) holds, $i = 1, 2, 3$. Then*

(i) there exist $b_0, r_0 > 0$ such that $I(u) \geq b_0$ with $\|u\|_V = r_0$;

(ii) there exists $\tilde{e} \in \mathbb{R}^3 \setminus B_{r_0}(0)$ with $\|\tilde{e}\|_V > r_0$ such that $I(\tilde{e}) < 0$.

Proof. (i) It is clear that $I(0) = 0$. According to (2.4) and (2.5) for $p \in (2, 6)$, together with $F \geq 0$ and Lemma 2.2-(4), we get

$$I(u) \geq \frac{1}{2}\|u\|_V^2 - C_1\|u\|_V^4 - C_2\|u\|_V^p \geq b_0 > 0$$

for some $b_0 > 0$ and $\|u\|_V = r_0$ small enough.

(ii) First fix $u \in D(I)\setminus\{0\}$ and $t>0$, then according to (2.3) and Lemma 2.2-(1), one can conclude

$$\begin{aligned} I(tu) &\leq \frac{t^2}{2}\|u\|_V^2 - \frac{1}{2}\int_{\mathbb{R}^3} t^2u^2\log\left(|tu|^2\right)\mathrm{d}x \\ &= t^2\left(I(u) + \frac{1}{4}\int_{\mathbb{R}^3}\phi_u u^2\,\mathrm{d}x - \log t\int_{\mathbb{R}^3}u^2\,\mathrm{d}x\right)\to -\infty, \end{aligned}$$

as $t\to+\infty$. Choosing $\tilde{e}=t_*u$ with $\|\tilde{e}\|_V > r_0$ for $t_*>0$ large enough, then we obtain $I(\tilde{e})<0$. □

Remark 3.4. According to Proposition 3.1 and Lemma 3.3, there exists a (PS) sequence $\{u_n\}\subset E_V$ of $I(u)$ at the level $c>0$, where

$$c := \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I(\gamma(t)) \tag{3.1}$$

and

$$\Gamma = \{\gamma\in C([0,1],E_V) : \gamma(0)=0, \gamma(1)=\tilde{e}\}.$$

Lemma 3.5. *Assume that (V_i) holds, $i=1,2,3$. If $\{u_n\}\subset E_V$ is a (PS) sequence of $I(u)$ at the level c, then $\{u_n\}$ is bounded in E_V, where c is defined in (3.1).*

Proof. By (2.3) and (2.8), for some $C>0$, we deduce

$$\begin{aligned} \int_{\mathbb{R}^3} u_n^2\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^3}\phi_{u_n}u_n^2\,\mathrm{d}x &= 2I(u_n) - I'(u_n)u_n \\ &= 2c + o_n(1) + o_n(1)\|u_n\|_V \\ &\leq C + o_n(1)\|u_n\|_V. \end{aligned}$$

Consequently,

$$\|u_n\|_2^2 \leq C + o_n(1)\|u_n\|_V. \tag{3.2}$$

Next, we will use the logarithmic Sobolev inequality (1.8) for a convenient small $a>0$. Fixing $\frac{a^2}{\pi}=\frac{1}{4}$, it follows from (1.8) and (3.2) that

$$\begin{aligned} \int_{\mathbb{R}^3} u_n^2\log u_n^2\,\mathrm{d}x &\leq \frac{1}{4}\|\nabla u_n\|_2^2 + \left(\log\|u_n\|_2^2 - 3\left(1+\log\sqrt{\frac{4}{\pi}}\right)\right)\|u_n\|_2^2 \\ &\leq \frac{1}{4}\|\nabla u_n\|_2^2 + C_1\left(\log\|u_n\|_2^2+1\right)\|u_n\|_2^2 \\ &\leq \frac{1}{4}\|\nabla u_n\|_2^2 + C_1\left(\log\left(C+o_n(1)\|u_n\|_V\right)+1\right)\left(C+o_n(1)\|u_n\|_V\right) \\ &\leq \frac{1}{4}\|\nabla u_n\|_2^2 + C_2\left(\log\|u_n\|_V\right)\|u_n\|_V. \end{aligned} \tag{3.3}$$

Using the fact that given $\theta\in(0,1)$ there is $A>0$ satisfying

$$|t\log t| \leq A\left(1+|t|^{1+\theta}\right), \quad t\geq 0,$$

we obtain, together with (3.3), the inequalities below

$$\begin{aligned} \int_{\mathbb{R}^3} u_n^2\log u_n^2\,\mathrm{d}x &\leq \frac{1}{4}\|\nabla u_n\|_2^2 + \tilde{C}\left(1+\|u_n\|_V^{1+\theta}\right) \\ &\leq \frac{1}{4}\|\nabla u_n\|_2^2 + \tilde{C}\left(1+\|u_n\|_V\right)^{1+\theta} \end{aligned} \tag{3.4}$$

for $\theta \in (0,1)$ and n large enough. Then by (3.4), we have

$$\begin{aligned} c + o_n(1)\,\|u_n\|_V &= I(u_n) - \frac{1}{4} I'(u_n)\,u_n \\ &\geq \frac{1}{4}\left(\|u_n\|_V^2 - \int_{\mathbb{R}^3} u_n^2 \log u_n^2 \,\mathrm{d}x\right) \\ &\geq C'\left(\|u_n\|_V^2 - (1+\|u_n\|_V)^{1+\theta}\right) \end{aligned}$$

for some $C' > 0$ independent of n. From the above discussion, we get that $\{u_n\}$ is bounded in E_V. □

Remark 3.6. According to Lemma 3.3, we consider the fiber mapping $t \to f(t) := I(tu)$ given by

$$\begin{aligned} f(t) = \frac{t^2}{2}\int_{\mathbb{R}^3} \left(|\nabla u|^2 + (V(x)+1)u^2\right)\mathrm{d}x \\ - \frac{t^4}{4}\int_{\mathbb{R}^3} \phi_u u^2 \,\mathrm{d}x - \frac{t^2}{2}\int_{\mathbb{R}^3} u^2 \log|tu|^2 \,\mathrm{d}x. \end{aligned} \tag{3.5}$$

Then we can easily infer that $f(t)$ has a maximum value at the unique critical point $t_\mu > 0$. Indeed, from the expression of $f(t)$ we can observe

$$\frac{t^2}{2}\int_{\mathbb{R}^3} u^2 \log|tu|^2 \,\mathrm{d}x = \frac{t^2}{2}\int_{\mathbb{R}^3} u^2 \log|u|^2 \,\mathrm{d}x + \frac{t^2}{2}\log t^2 \int_{\mathbb{R}^3} |u|^2 \,\mathrm{d}x.$$

Hence, for given $u \in E_V$, we can denote $f(t)$ as

$$f(t) := C_1 t^2 - C_2 t^4 - C_3 t^2 \log t^2,$$

correspondingly,

$$\begin{aligned} f'(t) &= C_4 t - C_5 t^3 - C_6 t \log t^2 \\ &= t g(t), \end{aligned}$$

where

$$g(t) := C_4 - C_5 t^2 - C_6 \log t^2.$$

From the expression of g, we know that g is a monotonically decreasing function for $t > 0$, and g has the unique zero point t_μ. In other words, $g(t) > 0$ for $t \in (0, t_\mu)$ and $g(t) < 0$ for $t \in (t_\mu, +\infty)$. Hence, it's easy to conclude $f'(t) > 0$ for $t \in (0, t_\mu)$ and $f'(t) < 0$ for $t \in (t_\mu, +\infty)$, i.e., f achieves a positive maximum at the unique critical point $t_\mu > 0$.

Next, we define

$$\mathcal{M}_V = \left\{u \in D(I)\setminus\{0\} \,\middle|\, I'(u)u = 0\right\}. \tag{3.6}$$

In fact, for any $u \in D(I)\setminus\{0\}$, every ray $\{tu \mid t > 0\}$ intersects the set (3.6) at exactly the unique point $t_\mu u$. In this way, $t_\mu = 1$, if and only if, $u \in \mathcal{M}_V$.

Lemma 3.7. *If c denotes the mountain level associated with $I(u)$, by Remark 3.6, it is possible to prove the equality*

$$0 < c = c_V,$$

where $c_V := \inf_{u\in\mathcal{M}_V} I(u)$.

Proof. Since $u \in \mathcal{M}_V$, by Lemma 3.3 and Remark 3.6, one has

$$0 < c \leq \max_{t\geq 0} I(tu) = I(t_\mu u) = I(u),$$

namely,

$$c \leq \inf_{u\in\mathcal{M}_V} I(u).$$

Next, we need to prove the reverse inequality. Let $\{u_n\} \subset E_V$ be a $(PS)_c$ sequence of $I(u)$, then Lemma 3.5 implies that $\{u_n\}$ is bounded in E_V. Now, we claim that

$$\|u_n\|_2 \nrightarrow 0. \tag{3.7}$$

In fact, according to the contradictory argument and interpolation inequality, we have that $u_n \to 0$ in $L^p(\mathbb{R}^3)$, $\forall p \in [2,6)$. Then, by (2.5) we get

$$\int_{\mathbb{R}^3} G'(u_n)\, u_n \,\mathrm{d}x \leq C \int_{\mathbb{R}^3} |u_n|^p \,\mathrm{d}x \to 0.$$

On the other hand, from (2.7), Lemma 2.2-(4) and the fact that $F'(s)s \geq 0$, we obtain

$$\begin{aligned}\|u_n\|_V^2 + \int_{\mathbb{R}^3} F'(u_n)\, u_n \,\mathrm{d}x &= I'(u_n)u_n + \int_{\mathbb{R}^3} G'(u_n)\, u_n \,\mathrm{d}x + \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 \,\mathrm{d}x \\ &= o_n(1),\end{aligned}$$

from where it follows that $u_n \to 0$ in E_V and $F'(u_n)u_n \to 0$ in $L^1(\mathbb{R}^3)$ as $n \to +\infty$. Furthermore, from the definition of $F(s)$, we can directly calculate that $0 \leq F(s) \leq F'(s)s$ for all $s \in \mathbb{R}$. Hence, $F(u_n) \to 0$ in $L^1(\mathbb{R}^3)$, and so, $I(u_n) \to 0$, which is contradictory to $I(u_n) \to c > 0$. This proves (3.7). Naturally, we may assume that there exist constants $a, b > 0$ such that

$$0 < a \leq \|u_n\|_2 \leq b, \quad \forall n \in \mathbb{N}.$$

Moreover, by Remark 3.6, for every $u_n \in E_V$, we can let $s_n > 0$ be such that $s_n u_n \in \mathcal{M}_V$. From the definition of $\mathcal{M}_V$, we can see

$$I(s_n u_n) = I(s_n u_n) - \frac{1}{2} I'(s_n u_n)\, s_n u_n, \tag{3.8}$$

which means

$$\begin{aligned}&\frac{s_n^2}{2}\int_{\mathbb{R}^3} \left(|\nabla u_n|^2 + (V(x)+1)u_n^2\right)\mathrm{d}x - \frac{s_n^4}{4}\int_{\mathbb{R}^3} \phi_{u_n} u_n^2 \,\mathrm{d}x \\ &\quad - \frac{s_n^2}{2}\int_{\mathbb{R}^3} u_n^2 \log|s_n u_n|^2 \,\mathrm{d}x = \frac{s_n^2}{2}\int_{\mathbb{R}^3} |u_n|^2 \,\mathrm{d}x + \frac{s_n^4}{4}\int_{\mathbb{R}^3} \phi_{u_n} u_n^2 \,\mathrm{d}x,\end{aligned}$$

namely,

$$\begin{aligned}&\int_{\mathbb{R}^3} \left(|\nabla u_n|^2 + V(x)u_n^2\right)\mathrm{d}x - 2\log s_n \int_{\mathbb{R}^3} u_n^2 \,\mathrm{d}x \\ &\quad - \int_{\mathbb{R}^3} u_n^2 \log u_n^2 \,\mathrm{d}x = s_n^2 \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 \,\mathrm{d}x.\end{aligned} \tag{3.9}$$

Recalling that $\{u_n\}$ is a bounded $(PS)_c$ sequence of $I(u)$, and based on (2.8), we have

$$\begin{aligned}I'(u_n)\, u_n &= \int_{\mathbb{R}^3} \left(|\nabla u_n|^2 + V(x)u_n^2 - \phi_{u_n} u_n^2\right)\mathrm{d}x - \int_{\mathbb{R}^3} u_n^2 \log u_n^2 \,\mathrm{d}x \\ &= o_n(1).\end{aligned} \tag{3.10}$$

It follows from (3.9) and (3.10) that

$$o_n(1) = 2\|u_n\|_2^2 \log s_n + (s_n^2 - 1)\int_{\mathbb{R}^3} \phi_{u_n} u_n^2 \, \mathrm{d}x,$$

then combining (3.7) and Lemma 2.2-(4), we derive that $s_n \to 1$ as $n \to +\infty$. From this information, we arrive at

$$\inf_{u\in\mathcal{M}_V} I(u) \leq I(s_n u_n) \to c \quad \text{as } n \to +\infty.$$

The proof has been completed. □

4 Proof of Theorem 1.2

Theorem 4.1 ([11]). *Under the condition (V_1), E_V can be compactly embedded into $L^p(\mathbb{R}^3)$, $p \in [2,6)$.*

Proof of Theorem 1.2. From Lemma 3.5, it is known that the sequence $\{u_n\}$ is bounded. Passing to a subsequence, $u_n \rightharpoonup u$ in E_V for some u and by Theorem 4.1 with (3.7), we have $u_n \to u \neq 0$ in $L^p(\mathbb{R}^3)$ for $p \in [2,6)$. Then by the Hölder inequality, combining Lemma 2.2-(4) and (2.5), respectively, we can obtain the following conclusions

$$\int_{\mathbb{R}^3} \phi_{u_n} u_n (u - u_n) \, \mathrm{d}x \to 0, \tag{4.1}$$

$$\int_{\mathbb{R}^3} G'(u_n)(u - u_n) \, \mathrm{d}x \to 0. \tag{4.2}$$

Since $\{u_n\}$ is the (PS) sequence, depending on Corollary 3.2 and taking $w = u$, we derive that

$$\langle \Phi'(u_n), u - u_n \rangle + \Psi(u) - \Psi(u_n) \geq -\sigma_n \|u - u_n\|_V,$$

namely,

$$\int_{\mathbb{R}^3} \nabla u_n \nabla (u - u_n) \, \mathrm{d}x + \int_{\mathbb{R}^3} (V(x) + 1) u_n (u - u_n) \, \mathrm{d}x - \int_{\mathbb{R}^3} \phi_{u_n} u_n (u - u_n) \, \mathrm{d}x - \int_{\mathbb{R}^3} G'(u_n)(u - u_n) \, \mathrm{d}x + \Psi(u) - \Psi(u_n) \geq -\sigma_n \|u - u_n\|_V,$$

equivalently,

$$\langle u_n, u - u_n \rangle_V - \int_{\mathbb{R}^3} \phi_{u_n} u_n (u - u_n) \, \mathrm{d}x - \int_{\mathbb{R}^3} G'(u_n)(u - u_n) \, \mathrm{d}x + \Psi(u) - \Psi(u_n) \geq -\sigma_n \|u - u_n\|_V.$$

Hence, one has

$$\overline{\lim_{n\to+\infty}} \left(\langle u_n, u \rangle_V - \|u_n\|_V^2 \right) - \lim_{n\to+\infty} \left(\int_{\mathbb{R}^3} \phi_{u_n} u_n (u - u_n) \, \mathrm{d}x + \int_{\mathbb{R}^3} G'(u_n)(u - u_n) \, \mathrm{d}x \right) + \overline{\lim_{n\to+\infty}} \left(\Psi(u) - \Psi(u_n) \right) \geq 0.$$

Combining $u_n \rightharpoonup u$ in E_V with (4.1) and (4.2), the above inequality becomes

$$\overline{\lim_{n\to+\infty}}\left(\|u\|_V^2-\|u_n\|_V^2\right)+\overline{\lim_{n\to+\infty}}\left(\Psi(u)-\Psi\left(u_n\right)\right)\geq 0, \tag{4.3}$$

namely,

$$\|u\|_V^2-\underline{\lim_{n\to+\infty}}\|u_n\|_V^2+\Psi(u)-\underline{\lim_{n\to+\infty}}\Psi\left(u_n\right)\geq 0. \tag{4.4}$$

On the other hand, Ψ is lower semicontinuous, i.e., $\Psi(u)\leq \underline{\lim}_{n\to+\infty}\Psi\left(u_n\right)$, together with (4.4), we deduce

$$\|u\|_V^2-\underline{\lim_{n\to+\infty}}\|u_n\|_V^2\geq 0.$$

Then according to the weak lower semi-continuity of the norm $\|u\|_V^2\leq \underline{\lim}_{n\to+\infty}\|u_n\|_V^2$, we have $\|u_n\|_V\to\|u\|_V$, which implies $u_n\to u$ in E_V. Furthermore, by combining this result with Lemma 3.7, we can conclude that $I(u)=c=c_V$, i.e., u is a ground state solution for equation (1.7). The proof is completed. □

5 Proof of Theorem 1.3

In this section, we give the proofs of Theorem 1.3. Since the defined manifold $\mathcal{M}_V$ lacks C^1 regularity, we will adopt an indirect approach by borrowing the method from [17] to obtain our results.

Lemma 5.1. *Assume that (V_i) holds, $i=2,3$. If $u\in\mathcal{M}_V$ and $I(u)=c$, then u is a solution of Eq.(1.7).*

Proof. Suppose to the contrary, there exists u such that $I(u)=c$ and $I'(u)\neq 0$. Then there exists $\eta\in C_0^\infty\left(\mathbb{R}^3\right)$ such that $\langle I'(u),\eta\rangle<-1$. Choose a constant $\epsilon\in(0,1)$ small enough such that for all $|t-1|\leq\epsilon$ and $|\sigma|\leq\epsilon$,

$$\left\langle I'(tu+\sigma\eta),\eta\right\rangle\leq-\frac{1}{2}. \tag{5.1}$$

Define a cut-off function $0\leq\chi\leq 1$ such that $\chi(t)=1$ for $|t-1|\leq\frac{\epsilon}{2}$ and $\chi(t)=0$ for $|t-1|\geq\epsilon$. For $t>0$, we introduce a curve $\gamma(t)=tu$ for $|t-1|\geq\epsilon$ and $\gamma(t)=tu+\epsilon\chi(t)\eta$ for $|t-1|<\epsilon$. Clearly, $\gamma(t)$ is a continuous curve, and for $|t-1|<\epsilon$, $\|\gamma(t)\|>0$ holds when ϵ small enough. Next, we claim $I(\gamma(t))<c$, for all $t>0$. Indeed, if $|t-1|\geq\epsilon$, together with Remark 3.6, $I(\gamma(t))=I(tu)<I(u)=c$. If $|t-1|<\epsilon$, then by Lemma 2.5-(v), the mapping $[0,\epsilon]\ni\sigma\mapsto I(tu+\sigma\chi(t)\eta)$ is of C^1. Consequently, together with (5.1), there exists $\tilde{\sigma}\in(0,\epsilon)$ such that

$$I(tu+\epsilon\chi(t)\eta)=I(tu)+\left\langle I'(tu+\tilde{\sigma}\chi(t)\eta),\epsilon\chi(t)\eta\right\rangle\leq I(tu)-\frac{\epsilon}{2}\chi(t)<c.$$

Let $\omega(u)=\langle I'(u),u\rangle$. By the definition of $\gamma(t)$, we have $\omega(\gamma(1-\epsilon))=\omega((1-\epsilon)u)>0$ and $\omega(\gamma(1+\epsilon))=\omega((1+\epsilon)u)<0$. Since the mapping $t\to\omega(\gamma(t))$ is continuous, there exists $\tilde{t}\in(1-\epsilon,1+\epsilon)$ such that $\omega\left(\gamma\left(\tilde{t}\right)\right)=0$. Thus, $\gamma\left(\tilde{t}\right)\in\mathcal{M}_V$ and $I\left(\gamma\left(\tilde{t}\right)\right)<c$, which is a contradiction. The proof is complete. □

Proof of Theorem 1.3. From Proposition 3.1, Corollary 3.2 and Lemma 3.3, there exists a $(PS)_c$ sequence $\{u_n\}$ of $I(u)$ such that

$$I\left(u_n\right)\to c$$

and

$$\left\langle \Phi'(u_n), u - u_n \right\rangle + \Psi(u) - \Psi(u_n) \geq -\sigma_n \|u - u_n\|_V, \quad \forall u \in E_V,$$

with $\sigma_n \to 0^+$. As in the previous section $\{u_n\}$ is a bounded sequence in E_V, then there is $u_0 \in E_V$, and a subsequence of $\{u_n\}$, still denoted $\{u_n\}$, such that

$$u_n \rightharpoonup u_0 \quad \text{in } E_V, \tag{5.2}$$
$$u_n \to u_0 \quad \text{in } L^2_{\text{loc}}(\mathbb{R}^3), \tag{5.3}$$
$$u_n \to u_0 \quad \text{a.e. in } \mathbb{R}^3. \tag{5.4}$$

Next, taking any test function $\eta \in C_0^\infty(\mathbb{R}^3)$, from (2.5) and (5.2) together with Lemmas 2.2-(4) and 2.5-(v), we have

$$\begin{aligned}
0 &= \lim_{n\to+\infty} \left\langle I'(u_n), \eta \right\rangle \\
&= \lim_{n\to+\infty} \left[\langle u_n, \eta \rangle_V - \int_{\mathbb{R}^3} \phi_{u_n} u_n \eta \, \mathrm{d}x + \int_{\mathbb{R}^3} F'(u_n)\eta \, \mathrm{d}x - \int_{\mathbb{R}^3} G'(u_n)\eta \, \mathrm{d}x \right] \\
&= \langle u_0, \eta \rangle_V - \int_{\mathbb{R}^3} \phi_{u_0} u_0 \eta \, \mathrm{d}x + \int_{\mathbb{R}^3} F'(u_0)\eta \, \mathrm{d}x - \int_{\mathbb{R}^3} G'(u_0)\eta \, \mathrm{d}x \\
&= \left\langle I'(u_0), \eta \right\rangle,
\end{aligned}$$

which means that u_0 is a weak solution to equation (1.7). To complete this proof of Theorem 1.3, the key is to prove that $u_0 \neq 0$ in E_V. In fact, combined with (3.7), the Lions Concentration Compactness Principle [16] implies that there are parameters $r, \beta > 0$, and a sequence $\{y_n\} \subset \mathbb{Z}^3$ such that

$$\lim_{n\to+\infty} \int_{B_r(y_n)} |u_n|^2 \, \mathrm{d}x \geq \beta > 0.$$

Now, setting $v_n(x) = u_n(x + y_n)$, $\{y_n\} \subset \mathbb{Z}^3$, it follows that

$$\int_{B_r(0)} |v_n|^2 \, \mathrm{d}x = \int_{B_r(0)} |u_n(x + y_n)|^2 \, \mathrm{d}x = \int_{B_r(y_n)} |u_n|^2 \, \mathrm{d}x \geq \frac{\beta}{2} > 0. \tag{5.5}$$

Since $V(x)$ satisfies (V_2), there hold $\|v_n\|_V = \|u_n\|_V$ and

$$I(v_n) \to c, \; I'(v_n) \to 0, \quad \text{as } n \to +\infty,$$

so $\{v_n\}$ is also a bounded $(PS)_c$ sequence of I. Therefore, if v_0 denotes the weak limit of $\{v_n\}$ in E_V, for some subsequence, we have that

$$v_n \rightharpoonup v_0 \quad \text{in } E_V, \tag{5.6}$$
$$v_n \to v_0 \quad \text{a.e. in } \mathbb{R}^3, \tag{5.7}$$
$$v_n \to v_0 \quad \text{in } L^q(B_r(0)), \quad \forall\, r > 0 \;\text{ and }\; q \in [1, 6). \tag{5.8}$$

From (5.8), we derive that

$$\int_{B_r(0)} |v_0|^2 \, \mathrm{d}x \geq \frac{\beta}{2} > 0,$$

which shows that $v_0 \neq 0$. Using the standard argument, one has $I'(v_0)\eta = 0$, $\forall \eta \in C_0^\infty(\mathbb{R}^3)$, i.e., v_0 is a non trivial weak solution for equation (1.7).

Further, because $v_0 \in \mathcal{M}_V$, it follows from (2.3), (2.8) and the Fatou lemma that

$$\begin{aligned} 2c \leq 2I(v_0) &= 2I(v_0) - I'(v_0)v_0 \\ &= \int_{\mathbb{R}^3} {v_0}^2 \,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^3} \phi_{v_0}{v_0}^2 \,\mathrm{d}x \\ &\leq \varliminf_{n\to+\infty} \left(\int_{\mathbb{R}^3} v_n^2 \,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^3} \phi_{v_n} v_n^2 \,\mathrm{d}x\right) \\ &\leq \varlimsup_{n\to+\infty} \left(\int_{\mathbb{R}^3} v_n^2 \,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^3} \phi_{v_n} v_n^2 \,\mathrm{d}x\right) \\ &= \varlimsup_{n\to+\infty} \left(2I(v_n) - I'(v_n)v_n\right) = 2c, \end{aligned}$$

that is, $c_V = c = I(v_0)$ and consequently, v_0 is a ground state solution of equation (1.7). □

6 Proof of Theorem 1.4

In this section, we modify some notations.

Consider the vector space $H^1\left(\mathbb{R}^3\right)$ endowed with the norm

$$\|u\|_H = \left(\int_{\mathbb{R}^3} \left(|\nabla u|^2 + \left(V_p(x)+1\right)u^2\right)\mathrm{d}x\right)^{\frac{1}{2}},$$

where V_p is defined in (V_3). By replacing V by V_p, we have a periodic problem as in the following problem

$$\begin{cases} -\Delta u + V_p(x)u - \phi u = u\log u^2 & \text{in } \mathbb{R}^3, \\ -\Delta\phi + \Delta^2\phi = 4\pi u^2 & \text{in } \mathbb{R}^3. \end{cases} \tag{6.1}$$

The underlying energy functional $I_p : H^1\left(\mathbb{R}^3\right) \to (-\infty, +\infty]$ associated with problem (6.1) can be defined as

$$I_p(u) = \Phi_p(u) + \Psi(u), \quad \forall u \in H^1\left(\mathbb{R}^3\right), \tag{6.2}$$

where

$$\Phi_p(u) := \frac{1}{2}\int_{\mathbb{R}^3} \left(|\nabla u|^2 + \left(V_p(x)+1\right)u^2\right)\mathrm{d}x - \frac{1}{4}\int_{\mathbb{R}^3} \phi_u u^2 \,\mathrm{d}x - \int_{\mathbb{R}^3} G(u)\,\mathrm{d}x,$$

and

$$\Psi(u) := \int_{\mathbb{R}^3} F(u)\,\mathrm{d}x.$$

Remark 6.1. Note that we would like to point out that the proof of Lemma 3.7 is independent of potential conditions. Therefore, in the case of asymptotically periodic potentials, the following items are valid:

(i) if c denotes the mountain level associated with $I(u)$, we have

$$0 < c = c_V := \inf_{u\in\mathcal{M}_V} I(u), \tag{6.3}$$

where $\mathcal{M}_V$ is defined in (3.6).

(ii) If d denotes the mountain level associated with $I_p(u)$, we have that

$$0 < d = c_{V_p} := \inf_{u \in \mathcal{M}_p} I_p(u), \tag{6.4}$$

where

$$\mathcal{M}_p = \left\{ u \in D(I) \setminus \{0\} : I_p'(u)u = 0 \right\}.$$

Lemma 6.2. *Assume that $V(x)$ satisfies (V_3). Then*

(i) $c_V < c_{V_p}$.

(ii) *If $I(u_n) \to c \in (0, c_{V_p})$ and $I'(u_n) \to 0$, then $u_n \rightharpoonup u_0 \neq 0$ after passing to a subsequence, u_0 is a critical point of $I(u)$ and $I(u_0) \leq c$.*

Proof. (i) Similar to Remark 3.6, we know that for any $u \in \mathcal{M}_p$, one has $I_p(u) = \max_{t>0} I_p(tu)$, and there exists $t_\mu > 0$ such that $t_\mu u \in \mathcal{M}_V$ and $I(t_\mu u) = \max_{t>0} I(tu)$. By (V_3),

$$c_V = \inf_{u \in \mathcal{M}_V} I(u) \leq I(t_\mu u) = \max_{t>0} I(tu) < \max_{t>0} I_p(tu) = I_p(u).$$

Because of the arbitrariness of u, we get that $c_V < c_{V_p}$.

(ii) As in Lemma 3.5, $\{u_n\}$ is bounded in E_V with the level c. Up to a subsequence, we can assume that

$$\begin{aligned} &u_n \rightharpoonup u_0 \quad \text{in } E_V, \\ &u_n \to u_0 \quad \text{in } L^2_{\mathrm{loc}}\left(\mathbb{R}^3\right), \\ &u_n \to u_0 \quad \text{a.e. in } \mathbb{R}^3. \end{aligned}$$

Similar to the proof of Theorem 1.3, taking any $\varphi \in C_0^\infty(\mathbb{R}^3)$, one has $I'(u_0)\varphi = 0$, i.e., u_0 is a weak solution of equation (1.7). By Fatou's lemma, one has

$$\begin{aligned} I(u_0) = I(u_0) - \frac{1}{2} I'(u_0)u_0 &= \int_{\mathbb{R}^3} {u_0}^2 \, \mathrm{d}x + \frac{1}{2} \int_{\mathbb{R}^3} \phi_{u_0} {u_0}^2 \, \mathrm{d}x \\ &\leq \varliminf_{n \to +\infty} \left(\int_{\mathbb{R}^3} u_n^2 \, \mathrm{d}x + \frac{1}{2} \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 \, \mathrm{d}x \right) \\ &= \varliminf_{n \to +\infty} \left(I(u_n) - \frac{1}{2} I'(u_n)u_n \right) = c, \end{aligned} \tag{6.5}$$

so, $I(u_0) \leq c$. Next, we claim that $u_0 \neq 0$ in E_V. Suppose, by contradiction, that $u_0 = 0$. For any $\epsilon > 0$, there exists $R(\epsilon) > 0$ such that

$$\left| V(x) - V_p(x) \right| < \epsilon, \quad \forall |x| > R.$$

Since $u_n \to u_0 = 0$ in $L^2_{\mathrm{loc}}\left(\mathbb{R}^3\right)$ and $\{u_n\}$ is bounded, we obtain

$$\int_{\mathbb{R}^3} \left| V(x) - V_p(x) \right| u_n^2 \, \mathrm{d}x \leq \int_{B_R(0)} \left| V(x) - V_p(x) \right| u_n^2 \, \mathrm{d}x + \epsilon \int_{\mathbb{R}^3 \setminus B_R(0)} u_n^2 \, \mathrm{d}x = o_n(1),$$

which yields, as $n \to +\infty$

$$I\left(u_n\right) - I_p\left(u_n\right) = \frac{1}{2} \int_{\mathbb{R}^3} \left| V(x) - V_p(x) \right| u_n^2 \, \mathrm{d}x \to 0.$$

Using the Hölder inequality and taking w with $\|w\| = 1$, we obtain, as $n \to +\infty$

$$\begin{aligned}\left|\left\langle I'(u_n) - I_p'(u_n), w\right\rangle\right| &\leq \int_{\mathbb{R}^3} \left|V(x) - V_p(x)\right| |u_n| |w| \, \mathrm{d}x \\ &\leq C\left(\int_{\mathbb{R}^3} \left|V(x) - V_p(x)\right| u_n^2 \, \mathrm{d}x\right)^{\frac{1}{2}} \\ &\to 0.\end{aligned}$$

In summary, we can immediately conclude

$$I_p(u_n) \to c \quad \text{and} \quad I_p'(u_n) \to 0,$$

namely, $\{u_n\}$ is a bounded $(PS)_c$ sequence for I_p. From (3.7) and Lions lemma [16], it follows that there exist $R, \kappa > 0$, $\{z_n\} \subset \mathbb{Z}^3$ satisfying

$$\int_{B_R(z_n)} |u_n|^2 \, \mathrm{d}x \geq \kappa > 0, \tag{6.6}$$

for all $n \in \mathbb{N}$. Since $u(x) = 0$ for all $x \in \mathbb{R}^3$, we have $|z_n| \to +\infty$. Taking $\omega_n(x) = u_n(x + z_n)$, then according to condition (V_3), $\{\omega_n\}$ is also a bounded $(PS)_c$ sequence for I_p. Therefore, we can assume there exists $\omega_0 \in E_V$ such that $\omega_n \rightharpoonup \omega_0$ in E_V, $\omega_n \to \omega_0$ in $L^2_{\mathrm{loc}}\left(\mathbb{R}^3\right)$ and $\omega_n \to \omega_0$ a.e. in $\mathbb{R}^3$ up to a subsequence. By (6.6), we have

$$\int_{B_R(0)} |\omega_n|^2 \, \mathrm{d}x = \int_{B_R(z_n)} |u_n|^2 \, \mathrm{d}x \geq \kappa > 0,$$

so, $\omega_0 \neq 0$. Then, using the standard argument, we get that for any $v \in C_0^\infty\left(\mathbb{R}^3\right)$, $I_p'(\omega_0)\, v = 0$, i.e., ω_0 is a nontrivial week solution of Eq. (6.1). Following the proof of Theorem 1.3, we know that ω_0 is a ground-state solution for equation (6.1), i.e.,

$$c_{V_p} = I_p(\omega_0). \tag{6.7}$$

By repeating the method of (6.5), this inequality $I_p(\omega_0) \leq c$ holds. Therefore, together with (6.7) and the fact that $c \in (0, c_{V_p})$, we obtain the following conclusions

$$I_p(\omega_0) \leq c < c_{V_p} = I_p(\omega_0),$$

which is a contradiction. This completes the proof. □

Proof of Theorem 1.4. From Lemmas 3.3 and 3.5, there exists a bounded (PS) sequence $\{u_n\}$ for I with the level $c \in (0, c_{V_p})$. By Lemma 6.2-(ii), we obtain a critical point $u_0 \neq 0$ of I such that $I(u_0) \leq c$. So, we have $I'(u_0)u_0 = 0$, i.e., $u_0 \in \mathcal{M}_V$. Then, arguing again as in the periodic case, it is possible to prove that u_0 is a ground state solution for Eq.(1.7). This completes the proof of Theorem 1.4. □

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A family of planar differential systems with hyperbolic algebraic limit cycles

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Abstract. In this paper, we characterize a family of planar polynomial differential systems of degree greater or equal than $n+1$, by presenting polynomial curves of degree n, which generally contain closed components. These systems admit precisely the bounded components of the curve as hyperbolic limit cycles.

Keywords: invariant curve, periodic solution, algebraic limit cycle, hyperbolic limit cycle.

2020 Mathematics Subject Classification: 34C25, 34C05, 34C07.

1 Introduction and the main result

The study of limit cycles in planar polynomial differential systems has been a central topic in the qualitative theory of ordinary differential equations for decades. Algebraic limit cycles have also received notable attention owing to their clear algebraic structure and geometric significance. The existence, classification, and stability of such cycles remain one of the most challenging problems, which is related to Hilbert's 16th problem and its generalizations. In this context, hyperbolic algebraic limit cycles are of particular interest, as their stability properties persist under small perturbations, making them a critical object of study in both theoretical and applied dynamical systems.

In this article, we consider the autonomous planar polynomial system of ordinary differential equations

$$\begin{aligned} \dot{x} &= \frac{dx}{dt} = f(x,y), \\ \dot{y} &= \frac{dy}{dt} = g(x,y), \end{aligned} \tag{1.1}$$

where f and g are two polynomials of $R[x,y]$ with no common factor, the derivatives are performed with respect to the time variable t. By definition, the degree of the system (1.1) is $n = \max(\deg(f), \deg(g))$. There is a huge literature about the analysis of the non-existence, existence, number, and stability of limit cycles of the system (1.1), see for instance [2, 7, 10, 6, 8] and the references therein.

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Starting with Colin Christopher [4] proved that if the line $D(x,y) = ux + vy + w = 0$ lies outside all non-singular compact components (ovals) of the algebraic curve $U = 0$ and the constants α and β are chosen such that $\alpha u + \beta v \neq 0$, then the system

$$\begin{aligned} \dot{x} &= \alpha U(x,y) + D(x,y)U_y(x,y), \\ \dot{y} &= \beta U(x,y) - D(x,y)U_x(x,y) \end{aligned}$$

admits all the bounded components of the curve $U = 0$ as hyperbolic limit cycles. Furthermore, the vector field (1.1) has no other limit cycles.

S. Benyoucef in his paper [3] analyses the existence of hyperbolic limit cycles of the differential system

$$\begin{aligned} \dot{x} &= \alpha U(x,y) + (ax + by + \Phi(\beta x - \alpha y))\, U_y(x,y), \\ \dot{y} &= \beta U(x,y) - (ax + by + \Phi(\beta x - \alpha y))\, U_x(x,y), \end{aligned}$$

where α and β are real numbers, Φ is a polynomial function of degree n and $U(x;y)$ is a polynomial of degree m.

As a continuation of [3], J. Llibre and C. Valls in [9] described a new class of polynomial differential systems that has the ovals of the algebraic curve as hyperbolic limit cycles

$$\begin{aligned} \dot{x} &= P(y)U(x,y) + (R(x,y) + \Phi(w(x,y)))\, U_y(x,y), \\ \dot{y} &= Q(x)U(x,y) - (R(x,y) + \Phi(w(x,y)))\, U_x(x,y), \end{aligned}$$

where $R(x,y) = \alpha x + \beta y + \int Q(x)dx - \int P(y)dy$, with $\alpha, \beta \in \mathbb{R}$ and the function $w = w(x,y) = \int Q(x)dx - \int P(y)dy$.

Our work builds on and extends earlier results by [3, 4, 9]. Specifically, we provide a new way to construct a new family of systems by presenting polynomial curves which generally contain closed components, and we investigate the conditions under which such systems admit bounded components of these algebraic curves as hyperbolic limit cycles.

Consider $U(x,y) = 0$ be a curve of degree n with $n \geq 2$. We consider a polynomial differential system of degree m with $m \geq 3$:

$$\begin{aligned} \dot{x} &= \frac{\partial Q}{\partial y}(x,y)U(x,y) + \gamma \frac{\partial U}{\partial y}(x,y), \\ \dot{y} &= \frac{\partial Q}{\partial x}(x,y)U(x,y) - \gamma \frac{\partial U}{\partial x}(x,y), \end{aligned} \tag{1.2}$$

with $Q \in R[x,y]$, $\gamma \in \mathbb{R} - \{0\}$ and $(\cdot)_x, (\cdot)_y$ denoting the partial derivatives with respect to the x and y, respectively.

Our main result is the following.

Theorem 1.1. *Let $U = 0$ be a non-singular algebraic curve of degree n and Q a polynomial of degree $m \geq 2$, chosen so that the curve $Q_{xy}(x,y) = 0$ lies outside all oval components of $U = 0$ or $Q_{xy}(x,y) \neq 0$, then the polynomial differential equation (1.2) has all the oval components of $U = 0$ as hyperbolic limit cycles. Furthermore, the vector field has no other limit cycles.*

The paper is organized as follows. In Section 2, we introduce some preliminary results. Theorem 1.1 is proved in Section 3. The last part of Section 3 is devoted to providing some examples satisfying all the conditions of Theorem 1.1.

2 Preliminary results

In this section, we will present the basic concepts that we need to prove the main results of this paper.

As usual, the vector field

$$\mathcal{X} = f(x,y)\frac{\partial}{\partial x} + g(x,y)\frac{\partial}{\partial y},$$

is associated to the differential system (1.1).

Invariant algebraic curves

Let $U : \mathbb{R}^2 \to \mathbb{R}$ be a real polynomial. We say that U is an invariant for $\mathcal{X}$ if it satisfies

$$f(x,y)\frac{\partial U(x,y)}{\partial x} + g(x,y)\frac{\partial U(x,y)}{\partial y} = K(x,y)\,U(x,y), \tag{2.1}$$

for all $(x,y) \in \mathbb{R}^2$. Here, $K : \mathbb{R}^2 \to \mathbb{R}$, which is the cofactor of U. Simple analysis of equation (2.1) shows that the degree of the cofactor is at most $n-1$ where n represents the maximum of the degrees of f and g, and that the curve $U = 0$ is formed by trajectories of the system (1.1). Also, if the curve $U = 0$ is non-singular, the equilibrium points of the system are contained either in its non-bounded components or are located on the curve $K = 0$.

Note that if $K = 0$, then if $\mathcal{X}U = 0$, that is, $U(x,y) = 0$ is an invariant curve of (1.1) with zero cofactor. Equivalently, $U(x,y) = 0$ is constant if $(x(t), y(t))$ is a solution of (1.1).

Hyperbolic Limit Cycle

In the phase plane, a limit cycle $\Gamma = \{(x(t), y(t)), t \in [0,T]\}$ is a T-periodic solution isolated in the set of all periodic solutions of the system. If the limit cycle is contained in the zero of an invariant algebraic curve of the plane, then it is said to be algebraic; otherwise, it is said to be non-algebraic.

To see that all the oval components of Γ are hyperbolic limit cycles of system (1.1), we will use a classic result characterizing limit cycles among other periodic orbits, which states that: a T-periodic solution Γ is a hyperbolic limit cycle of system (1.1) if

$$\int_0^T \mathrm{Div}(\mathcal{X})(\Gamma(t))\,dt \neq 0,$$

where $\mathrm{Div}(\mathcal{X}) = \frac{\partial f(x,y)}{\partial x} + \frac{\partial g(x,y)}{\partial y}$, (see for instance [11] for more details).

The curve $U = 0$ is non-singular of system (1.1), i.e., it must not contain any singular point that satisfies

$$f(x,y) = 0, \qquad g(x,y) = 0.$$

Green's formula

In order to confirm the hyperbolicity of the limit cycles, we will make use of the following Green's theorem; see for instance [1].

Suppose Ω is a domain in $\mathbb{R}^2$ whose positively oriented boundary Γ is a finite collection of pairwise disjoint piecewise continuous simple closed curves. Suppose f and g are continuous functions defined on a larger open set, which contains both Ω and Γ, and suppose f and g have continuous first partial derivatives on this larger open set. Then

$$\oint_\Gamma f(x,y)\,dx + g(x,y)\,dy = \int\int_\Omega \left(\frac{\partial g(x,y)}{\partial x} - \frac{\partial f(x,y)}{\partial y}\right) dxdy.$$

3 Proof of Theorem 1.1

We assume that all bounded components of $U = 0$ are non-singular of system (1.2). Let $\Gamma = \{(x(t), y(t)), t \in [0, T]\}$ is an oval of the curve $U = 0$, corresponding to the periodic solution $(x(t), y(t))$ with period T. To see that all the oval components of $U = 0$ are hyperbolic limit cycles, we will prove that $U = 0$ is an invariant curve of the system (1.2), and $\int_0^T \mathrm{Div}(\mathcal{X})(\Gamma(t))\, dt \neq 0$.

(i) It is easy to see that $U = 0$ is an invariant algebraic curve with cofactor

$$K(x,y) = Q_x(x,y)\, U_y(x,y) + Q_y(x,y)\, U_x(x,y),$$

because

$$\begin{aligned}\frac{dU(x,y)}{dt} &= \dot{x}\frac{\partial U(x,y)}{\partial x} + \dot{y}\frac{\partial U(x,y)}{\partial y} \\ &= \left(Q_x(x,y)\, U_y(x,y) + Q_y(x,y)\, U_x(x,y)\right) U(x,y).\end{aligned}$$

(ii) The divergence of system (1.2) is

$$\mathrm{Div}(x,y) = Q_x(x,y)\, U_y(x,y) + Q_y(x,y)\, U_x(x,y) + 2Q_{xy}(x,y)\, U(x,y)$$

Therefore, on the curve $U = 0$, we have

$$\begin{aligned}\int_0^T \mathrm{Div}(x,y)\, dt &= \int_0^T Q_x(x,y)U_y(x,y) + Q_y(x,y)U_x(x,y)\, dt \\ &= \oint_\Gamma \frac{Q_x(x,y)U_y(x,y)}{\gamma U_y(x,y)}\, dx + \oint_\Gamma \frac{Q_y(x,y)U_x(x,y)}{-\gamma U_x(x,y)}\, dy \\ &= \frac{1}{\gamma}\left(\oint_\Gamma Q_x(x,y)\, dx - \oint_\Gamma Q_y(x,y)\, dy\right).\end{aligned}$$

Now, we apply the Green formula to get

$$\begin{aligned}\oint_\Gamma Q_x(x,y)\, dx - \oint_\Gamma Q_y(x,y)\, dy &= \iint_{\mathrm{int}(\Gamma)} \left(\frac{\partial(-Q_y(x,y))}{\partial x} - \frac{\partial Q_x(x,y)}{\partial y}\right) dx\, dy \\ &= -2\iint_{\mathrm{int}(\Gamma)} Q_{xy}(x,y)\, dx\, dy.\end{aligned}$$

where $\mathrm{int}(\Gamma)$ denotes the interior of the bounded region limited by Γ, thus

$$\int_0^T \mathrm{Div}(x,y)\, dt = -\frac{2}{\gamma}\iint_{\mathrm{int}(\Gamma)} Q_{xy}(x,y)\, dxdy.$$

Since by assumption $Q_{xy}(x,y) = 0$ lies outside all oval components of $U = 0$ or $Q_{xy}(x,y) \neq 0$, then $\int\int_{\mathrm{int}(\Gamma)} Q_{xy}(x,y)\, dxdy$ is non-zero, and the system (1.2) has all closed components of $U = 0$ as hyperbolic limit cycles.

Remark 3.1.

1. According to the hypotheses $Q_{xy}(x,y) \neq 0$, the polynomial $Q(x,y)$ must satisfy $\deg Q \geq 2$ and $Q(x,y) \neq h(x) + l(y)$, with $h \in R[x]$ and $l \in R[y]$.

2. In the case when $Q(x,y) = h(x) + l(y) + \alpha xy$, with $\alpha \neq 0$, the integral

$$\int_0^T \mathrm{Div}(x,y)\,dt = -\frac{2a}{\gamma}\iint_{\mathrm{int}(\Gamma)} dx\,dy.$$

So, we have the particular case; if $\alpha \neq 0$, the system (1.2) admits all bounded components of Γ as hyperbolic algebraic limit cycles.

Remark 3.2. If we take $Q(x,y) = yP(x)$, with $P \in \mathbb{R}[x]$, system (1.2) becomes

$$\begin{aligned} \dot{x} &= P(x)U(x,y) + \gamma\frac{\partial U}{\partial y}(x,y), \\ \dot{y} &= yP'(x)U(x,y) - \gamma\frac{\partial U}{\partial x}(x,y), \end{aligned} \tag{3.1}$$

then, the integral

$$\int_0^T \mathrm{Div}\,(x,y)\,dt = -\frac{2}{\gamma}\iint_{\mathrm{int}(\Gamma)} P'\,(x)\,dxdy.$$

So, if we choose that the curve $P'\,(x) = 0$ lies outside all oval components of $U = 0$, then the polynomial differential system (3.1) has all the oval components of $U = 0$ as hyperbolic limit cycles. Furthermore, the vector field has no other limit cycle.

By the change of variables $(x,y) \to (u,v)$ with $x = u$, $y = v + h(u)$, the system (3.1) is transformed into

$$\begin{aligned} \dot{u} &= -\left(h(u)+v\right)^2 \frac{\partial}{\partial v}\left(\frac{P(u)}{h(u)+v}\right)\tilde{U}(u,v) + \gamma\frac{\partial \tilde{U}}{\partial v}(u,v), \\ \dot{v} &= \left(h(u)+v\right)^2 \frac{\partial}{\partial u}\left(\frac{P(u)}{h(u)+v}\right)\tilde{U}(u,v) - \gamma\frac{\partial \tilde{U}}{\partial u}(u,v), \end{aligned} \tag{3.2}$$

where $h, P \in R[x]$, with $R(0) = 0, \tilde{U}\,(u,v) = \tilde{U}\,(u, v + h(u))$ and $\gamma \in \mathbb{R} - \{0\}$. For this last system we can choose $P'(u) = 0$ lies outside all oval components of $\tilde{U} = 0$, then in this case all the oval components of $\tilde{U} = 0$ are hyperbolic limit cycles.

Now we provide some examples of polynomial differential systems of the form (1.2) satisfying all the assumptions of Theorem 1.1 and admitting all oval components of $U = 0$ as hyperbolic limit cycles.

Example 3.3. Let $\gamma = 1$ and $Q\,(x,y) = y\left(x^2 - 4x\right)$. The system

$$\begin{aligned} \dot{x} &= x(x-4)(3x^2+5y^2-7) + 10y, \\ \dot{y} &= 2y(x-2)(3x^2+5y^2-7) - 6x, \end{aligned} \tag{3.3}$$

has an invariant algebraic curve given by $3x^2 + 5y^2 - 7 = 0$. Since $2\,(x-2) = 0$ does not intersect the curve $3x^2 + 5y^2 - 7 = 0$, this oval is an algebraic limit cycle hyperbolic of system (3.3) (see Figure 3.1).

Example 3.4. Let $\gamma = 1$ and $Q\,(x,y) = yx^3 + y^3x$. The system

$$\begin{aligned} \dot{x} &= x\left(x^2+3y^2\right)\left(x^4 - 2x^2 + y^2 + \frac{1}{4}\right) + 2y^2, \\ \dot{y} &= y\left(3x^2+y^2\right)\left(x^4 - 2x^2 + y^2 + \frac{1}{4}\right) - 4x\left(x^2-1\right) \end{aligned} \tag{3.4}$$

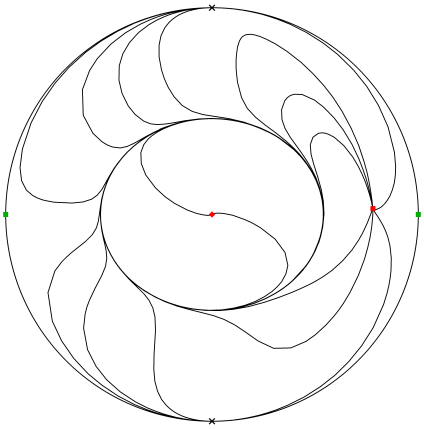


Figure 3.1: The limit cycle of (3.3).

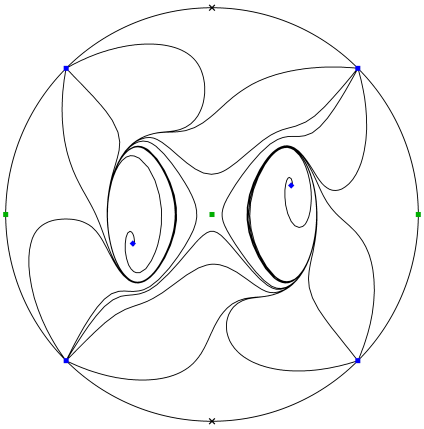


Figure 3.2: The two limit cycles of (3.4).

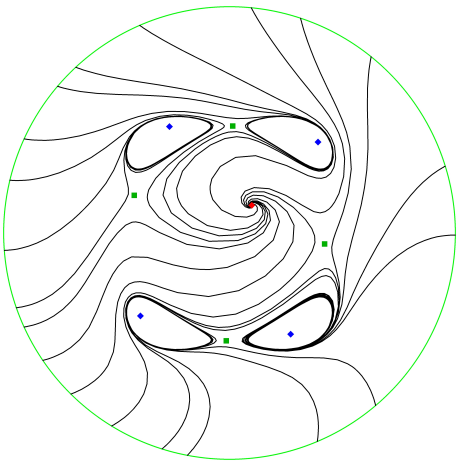
has an invariant algebraic curve given by $x^4-2x^2+y^2+\frac{1}{4}=0$. Since $Q_{xy}(x,y)=3\left(x^2+y^2\right)\neq 0$. So the two ovals of $x^4-2x^2+y^2+\frac{1}{4}=0$ are hyperbolic algebraic limit cycles of system (3.4) (see Figure 3.2).

Example 3.5. Let $\gamma=-2$ and $Q(x)=\alpha xy-\beta y+\gamma, \alpha\neq 0$. The family

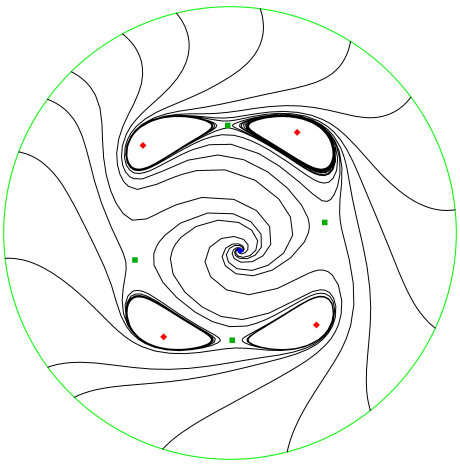
$$\begin{aligned}\dot{x}&=(\alpha x-\beta)\left(y^4+x^4-3y^2-2x^2+\frac{7}{3}\right)-4y\left(2y^2-3\right),\\ \dot{y}&=\alpha y\left(y^4+x^4-3y^2-2x^2+\frac{7}{3}\right)+8x\left(x^2-1\right)\end{aligned}\tag{3.5}$$

has an invariant algebraic curve given by $y^4+x^4-3y^2-2x^2+\frac{7}{3}=0$. Since $Q_{x,y}(x,y)=\alpha\neq 0$, the four ovals of $y^4+x^4-3y^2-2x^2+\frac{7}{3}=0$ are hyperbolic algebraic limit cycles of system (3.5) (see Figure 3.3).

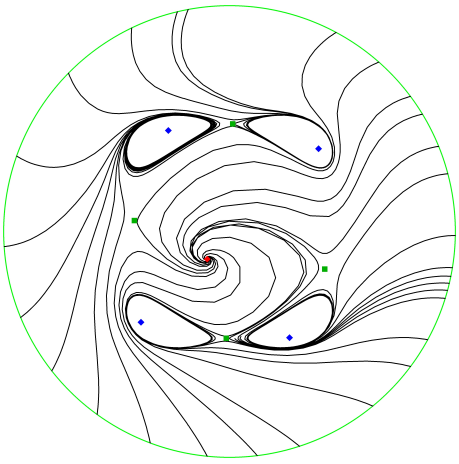
Note: All figures are plotted on the Poincaré disk by using a program for polynomial planar phase portraits, such as the one described in [5, pp. 233–257].



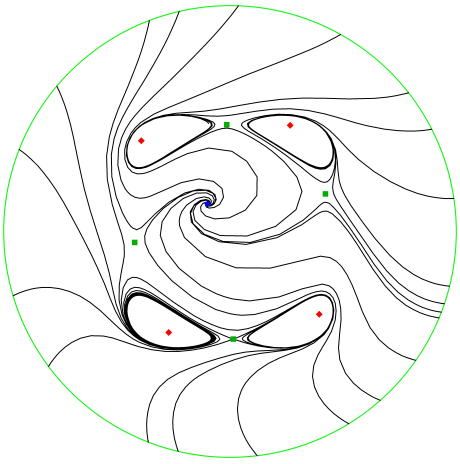
(a) $\alpha = 3$ and $\beta = 2$



(b) $\alpha = -2$ and $\beta = -1$



(c) $\alpha = 3$ and $\beta = -2$



(d) $\alpha = -3$ and $\beta = 2$

Figure 3.3: The four limit cycles of (3.5) with different values of α and β.

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On Shilnikov's scenario in 3D: Topological chaos for vectorfields of class C^1

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Abstract. Shilnikov's scenario in $\mathbb{R}^3$ means that the equation $x' = V(x) \in \mathbb{R}^3$ with $V(0) = 0$ has a homoclinic solution and the eigenvalues of $DV(0)$ are $u > 0$ and $\sigma \pm i\mu$ with $\sigma < 0 < \mu$ and $0 < u + \sigma$. For V once continuously differentiable we consider a flow which is equivalent to the flow of V and prove that topological chaos exists for a planar return map which describes flowline behaviour near the homoclinic orbit: For every sequence in 2 symbols $0, 1$ there are trajectories of the return map which take values in disjoint sets M_0, M_1 according to the symbol sequence. The proof is by the analysis of the action of the return map on curves and does not involve covering relations for 2-dimensional sets.

Keywords: Shilnikov's scenario, topological chaos, vectorfield, flow.

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1 Introduction

In his seminal paper [5] Shilnikov considered a differential equation

$$x'(t) = V(x(t)) \in \mathbb{R}^3 \tag{1.1}$$

with $V(0) = 0$ so that there is a homoclinic solution $h_V : \mathbb{R} \to \mathbb{R}^3$, $0 \neq h_V(t) \to 0$ for $|t| \to \infty$, and the derivative $DV(0)$ has eigenvalues $u > 0$ and $\sigma \pm i\,\mu \in \mathbb{C}$, $\sigma < 0 < \mu$, with

(H) $\qquad 0 < \sigma + u.$

Shilnikov's result in [5] is that for V analytic there exist countably many periodic orbits close to the homoclinic orbit $h_V(\mathbb{R})$. These periodic orbits arise from fixed points of a planar return map which is given by intersections of solutions with a transversal to the homoclinic orbit.

A related, stronger statement about complicated motion is conjugacy of the return map with the shift $(s_n)_{n=-\infty}^{\infty} \mapsto (s_{n+1})_{n=-\infty}^{\infty}$ in two symbols $s_n \in \{0, 1\}$. For work on the verification

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of this property see the monographs [1,2,9,10] and their references. A detailed presentation for V linear near the origin is contained in [2].

Shilnikov-type results on complicated motion close to a homoclinic orbit have also been obtained in infinite-dimensional spaces, for semiflows of solution operators which are linear close to equilibrium. See for example [4,7] on delay differential equations. These results are related to Shilnikov's scenario in $\mathbb{R}^4$ [6] with pairs of complex conjugate eigenvalues of $DV(0)$ in either halfplane.

The present paper deals with Shilnikov's scenario in $\mathbb{R}^3$ under the minimal smoothness assumption that V is once continuously differentiable – corresponding to the setting of a homoclinic solution combined with a spectral condition the statement of which does not require differentiability of higher order. Apart from this and with generalizations in mind, let us mention that for differential equations with state-dependent delay solution operators are in general not better than once continuously differentiable [3]. For a planar return map analogous to Shilnikov's in [5] we work out a proof that *topological chaos* exists, which means that for every given sequence $(s_n)_{n=-\infty}^{\infty}$, $s_n \in \{0,1\}$, there are trajectories $(x_n)_{n=-\infty}^{\infty}$ of the return map which take values in disjoint sets M_0, M_1 according to the rule $x_n \in M_{s_n}$. Notice that this is complementary to Shilnikov's result on periodic orbits [5] and weaker than conjugacy of the return map with a shift in two symbols.

The main results of the present paper are stated precisely in Section 8 below. They address return maps which are given by flowlines, as opposed to solutions of differential equations. Accordingly the subsequent sections 2–7 deal with flows and flowlines, and not with differential equations. Let us explain why. From the recent preprint [8] we know how to verify topological chaos for Shilnikov's scenario in $\mathbb{R}^3$ with the vectorfield V being twice continuously differentiable: The very first step is a transformation of V to a vectorfield whose local stable and unstable manifolds at the origin are flat, by a diffeomorphism which is twice continuously differentiable. The transformation reduces the order of smoothness for the vectorfields but preserves the smoothness of the flow F_V of Eq. (1.1). In the present situation, with V only continuously differentiable, the analogous transformation yields a vectorfield which is in general only continuous and thereby not good enough for arguments as used in [8]. The transformed flow F, however, is continuously differentiable with flat local invariant manifolds and further properties (F1)–(F5), from which we can proceed in Section 2 below. For more about the reduction by transformations see Section 9.

In Section 2 we immediately turn to scaled flows given by $F_\epsilon(t,x) = \frac{1}{\epsilon}F(t,\epsilon\, x)$, $\epsilon > 0$, all of which are equivalent to F. In the sequel we investigate the behaviour of these scaled flows inside and outside of a fixed neighbourhood of the origin, instead of studying F with respect to a family of shrinking neighbourhoods. As in [8] (and following Shilnikov [5]) we introduce a return map, now only for a sequence of small $\epsilon > 0$. The domain of the return map in a transversal to the homoclinic flowline is situated on one side of the flat local stable manifold, as shown at the top of Figure 1.1. Expressed in suitable coordinates the return map becomes a map from a rectangle into the plane. Section 7 shows how this map turns curves which connect certain horizontal levels in the rectangle into spirals around the origin. This suffices for the proof of Proposition 8.1 about one-directional topological chaos (with forward symbol sequences $(s_n)_{n=0}^{\infty}$). Theorem 8.2 extends the result of Proposition 8.1 to entire symbol sequences $(s_n)_{n=-\infty}^{\infty}$, by means of familiar compactness arguments.

The choice of Δ_2 in Section 7 shows that actually we obtain a countable family of sets of complicated trajectories of the return map. Another aspect which may be of interest is that the proofs of Proposition 8.1 and Theorem 8.2 do not involve covering relations for 2-dimensional

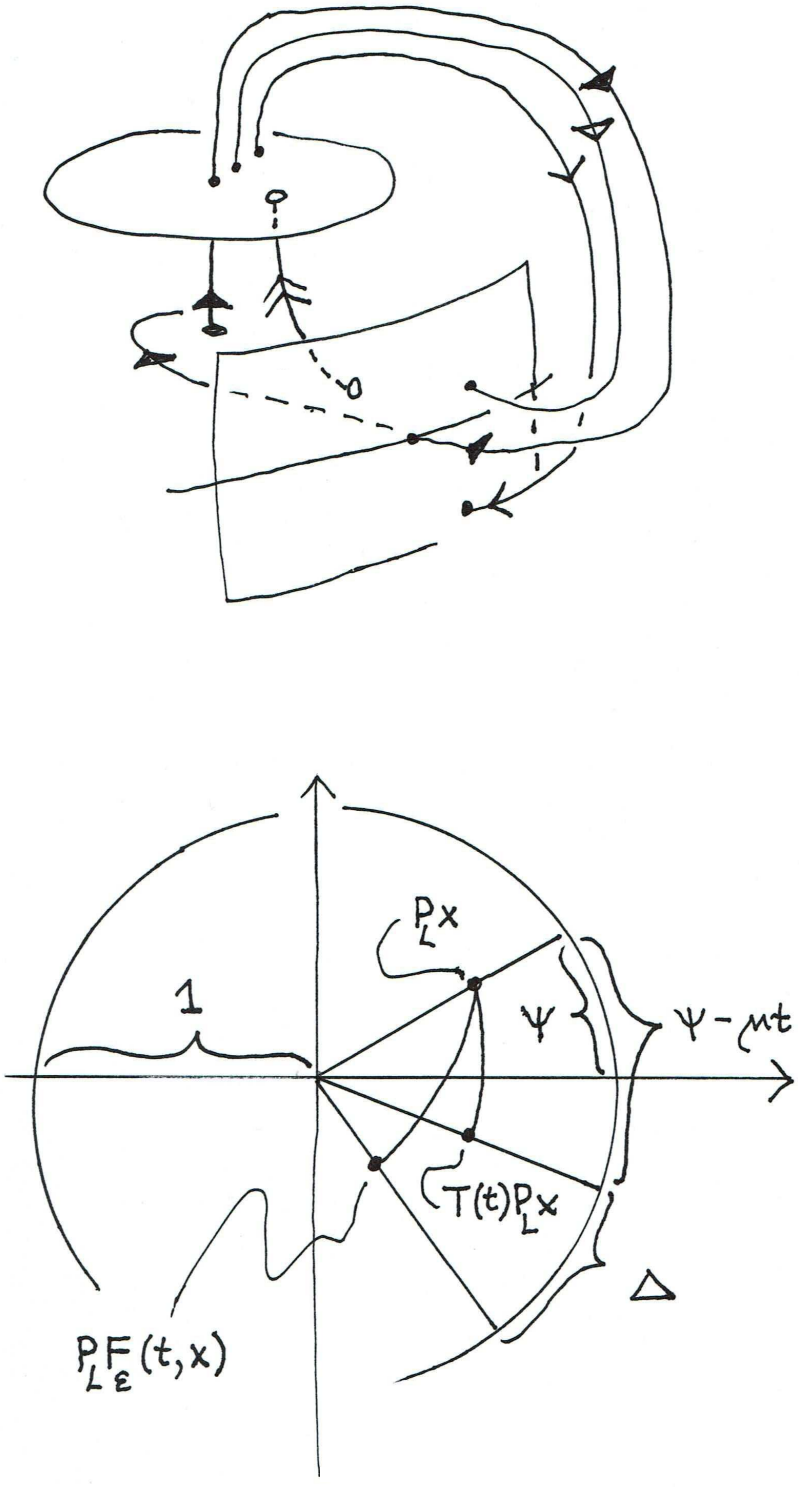


Figure 1.1: Top: The return map as a composition of the inner map with the exterior map. Bottom: Angles from Proposition 3.1.

sets.

Beyond results the emphasis in the present paper is on providing detailed proofs which may serve as a basis for future work. The content of Sections 2–6 parallels its counterpart in the preceding preprint [8], but working with the flows instead of differential equations necessitates modifications, among them in Section 3 another access to angles along projections of flowlines into the stable plane, and a rearrangement of arguments in Sections 4–6. Sections 7 and 8 are almost the same as their counterparts in the preprint [8]. We include all arguments in order to keep the paper self-contained.

What remains open, among others, is existence of periodic orbits corresponding to periodic symbol sequences. Also of interest might be a version of the present approach for Shilnikov's scenario in $\mathbb{R}^4$ [6].

Notation, preliminaries. A forward trajectory of a map $f : M \supset \text{dom} \to M$ is a sequence $(x_j)_{j=0}^{\infty}$ in dom with $x_{j+1} = f(x_j)$ for all integers $j \geq 0$. Entire trajectories are defined analogously, with all integers as indices.

For a vectorspace X, $x \in X$, and $M \subset X$, we set $x \pm M = \{y \in X : y \mp x \in M\}$. Similarly, for $A \subset \mathbb{R}$ and $x \in X$, $Ax = \{y \in X : \text{For some } a \in A, y = ax\}$.

The interior, the boundary, and the closure of a subset of a topological space are denoted by int $M, \partial M$, and cl M, respectively.

A curve is a continuous map from an interval $I \subset \mathbb{R}$ into a topological space.

Components of vectors in Euclidean spaces $\mathbb{R}^n$ are indicated by lower indices.The inner product on $\mathbb{R}^n$ is written as $\langle x, y\rangle = \sum_{i=1}^n x_i y_i$, and we use the Euclidean norm given by $|x| = \sqrt{\langle x, x\rangle}$. The vectors of the canonical orthonormal basis on $\mathbb{R}^n$ are denoted by e_j, $j = 1, \ldots, n$, $e_{j,j} = 1$ and $e_{j,k} = 0$ for $j \neq k$. In $\mathbb{R}^3$ we write $L = \mathbb{R}e_1 \oplus \mathbb{R}e_2$ and $U = \mathbb{R}e_3$. The associated projections $\mathbb{R}^3 \to \mathbb{R}^3$ onto L and onto U are denoted by P_L and P_U, respectively. For every $x \in \mathbb{R}^3$, $|P_U x|^2 + |P_L x|^2 = |x|^2$, and each of the projections has norm 1 in the space $L_c(\mathbb{R}^3, \mathbb{R}^3)$ of linear (continuous) maps $\mathbb{R}^3 \to \mathbb{R}^3$.

For a function $f : \mathbb{R}^n \supset \text{dom} \to \mathbb{R}^k$ on an open subset derivatives as linear maps $\mathbb{R}^n \to \mathbb{R}^k$ are denoted by $Df(x)$. For $n = k = 1$, $f'(x) = Df(x)1$. For partial derivatives in case $k = 1$, $\partial_j f(x) = Df(x)e_j$ for $j = 1, \ldots, n$.

Let $M \subset R^n$ be a continuously differentiable submanifold. For $x \in M$ the tangent space $T_x M$ is the set of tangent vectors $v = c'(0)$ of continuously differentiable curves $c : I \to \mathbb{R}^n$ with I an interval, not a singleton, $c(I) \subset M$, $0 \in I$, $c(0) = x$. A continuously differentiable map $f : M \supset \text{dom} \to N$, dom open in M and N a continuously differentiable submanifold of $\mathbb{R}^k$, is locally given by restrictions of continuously differentiable maps $g : \mathbb{R}^n \supset U \to \mathbb{R}^k$. For such U and g, and for $x \in \text{dom} \cap U$, the derivative of f at x is the linear map $T_x f : T_x M \to T_{f(x)} N$ given by $T_x f(v) = (g \circ c)'(0) = Dg(x)v$ for $v = c'(0)$ and c and g as above (with $c(I) \subset \text{dom} \cap U$).

The flow $\mathcal{F}$ generated by a vectorfield $\mathcal{V} : \mathbb{R}^n \supset \mathcal{U} \to \mathbb{R}^n$ which is locally Lipschitz continuous is the map $\mathbb{R} \times \mathbb{R}^n \supset \text{dom}_{\mathcal{F}} \to \mathbb{R}^n$ which is given by $(t, x) \in \text{dom}_{\mathcal{F}}$ if and only if t belongs to the domain of the maximal solution $y : I_x \to \mathbb{R}^n$ of the differential equation $x'(t) = \mathcal{V}(x(t))$ with initial value $y(0) = x$, and for such y, $\mathcal{F}(t, x) = y(t)$. $\mathcal{F}$ is of the same order of differentiability as $\mathcal{V}$. A subset $M \subset U$ is invariant under $\mathcal{F}$ if $x \in M$ implies $\mathcal{F}(t, x) \in M$ for all $t \in \mathbb{R}$ with $(t, x) \in \text{dom}_{\mathcal{F}}$. For a further subset $N \subset U$ the set M is called invariant under $\mathcal{F}$ in N if for every $x \in M \cap N$ and for every interval $I \ni 0$ with $\mathcal{F}(I \times \{x\}) \subset N$ we have $\mathcal{F}(I \times \{x\}) \subset M$. A flowline $\xi : \mathbb{R} \to \mathbb{R}^n$ of a flow $\mathcal{F}$ on $\text{dom}_{\mathcal{F}} = \mathbb{R} \times \mathbb{R}^n$ satisfies $\xi(t + s) = \mathcal{F}(t, \xi(s))$ for all reals s, t.

2 Transformation, scaling, projected flowlines

In the appendix Section 9 we describe how the flow F_V and the homoclinic flowline h_V can be transformed to a continuously differentiable flow $F : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$ and a flowline $h : \mathbb{R} \to \mathbb{R}^3$ of F with the following properties.

(F1) For all $t \in \mathbb{R}$, $F(t, 0) = 0$.

(F2) $h(t) \neq 0 \neq h'(t)$ for all $t \in \mathbb{R}$, and $\lim_{|t|\to\infty} h(t) = 0$.

(F3) Every linear map $T(t) : \mathbb{R}^3 \ni x \mapsto D_2F(t, 0)x \in \mathbb{R}^3$, $t \in \mathbb{R}$, satisfies $T(t)L \subset L$ and $T(t)U \subset U$. For $t \in \mathbb{R}, x \in \mathbb{R}^3$, and $y = T(t)x$,

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = e^{\sigma t} \begin{pmatrix} \cos(\mu t) & \sin(\mu t) \\ -\sin(\mu t) & \cos(\mu t) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \text{and} \quad y_3 = e^{ut} x_3.$$

(F4) There exists $r_F > 0$ such that L and U are invariant under F in $\{x \in \mathbb{R}^3 : |x| < r_F\}$.

(F5) There exist reals $t_U < t_L$ with $h(t) \in U$ for all $t \leq t_U$ and $h(t) \in L$ for all $t \geq t_L$. Either $h(t) \in (0, \infty)e_3$ for all $t \leq t_U$, or $h(t) \in (-\infty, 0)e_3$ for all $t \leq t_U$.

In the sequel we focus on the case $h(t) \in (0, \infty)e_3$ for all $t \leq t_U$, the other case being analogous. We define

$$\begin{aligned} B_1 &= \{x \in \mathbb{R}^3 : |P_L x| \leq 1, |P_U x| \leq 1\}, \\ r_B &= 2\left(\max_{0\leq t\leq 1} |T(t)| + e^u + 2\right), \quad \text{and} \\ B &= \{x \in \mathbb{R}^3 : |x| \leq r_B\}. \end{aligned}$$

Obviously, $B_1 \subset B$. For $\epsilon > 0$ we consider the scaled flows given by $F_\epsilon(t, x) = \frac{1}{\epsilon}F(t, \epsilon x)$. If x is a flowline of F_ϵ then ϵx is a flowline of F, and conversely, if y is a flowline of F, then $\frac{1}{\epsilon}y$ is a flowline of F_ϵ. Observe that for all $\epsilon > 0, x \in \mathbb{R}^3, y \in \mathbb{R}^3, t \in \mathbb{R}$,

$$D_2F_\epsilon(t, x)y = \frac{1}{\epsilon} D_2F(t, \epsilon x)\epsilon y = D_2F(t, \epsilon x)y.$$

Proposition 2.1.

(i) For every $\eta > 0$ there exists $\epsilon(\eta) > 0$ such that for all $\epsilon \in (0, \epsilon(\eta))$,

$$|D_2F_\epsilon(t, x) - T(t)| < \eta \quad \textit{for all } x \in B_1.$$

(ii) There exists $\epsilon_B > 0$ such that for $0 < \epsilon < \epsilon_B$,

$$F_\epsilon([0, 1] \times B_1) \subset B,$$

and L and U are invariant under F_ϵ in B.

Proof. 1. On (i). By compactness and continuity there exists $\epsilon(\eta) > 0$ such that for $0 < \epsilon < \epsilon(\eta)$ and $x \in B_1$ and for all $t \in [0, 1]$,

$$\eta \geq |D_2F(t, \epsilon x) - D_2F(t, 0)| = |D_2F_\epsilon(t, x) - T(t)|.$$

2. Let $0 \le t \le 1$. For $0 < \epsilon < \epsilon(1)$ assertion (i) yields $|D_2F_\epsilon(t,y)| \le \max_{0\le s\le 1}|T(s)|+1 \le r_B/2$ for all $y \in B_1$. For $x \in B_1$ we have $|x| \le 2$, and we infer

$$|F_\epsilon(t,x)| = |F_\epsilon(t,x)-0| = |F_\epsilon(t,x)-F_\epsilon(t,0)| = \left|\int_0^1 D_2F_\epsilon(t,sx)xds\right| \le r_B|x|/2 \le r_B,$$

hence $F_\epsilon(t,x) \in B$.

3. On invariance. Let $0 < \epsilon < r_F/r_B$. Assume $x \in L$ and $F_\epsilon(s,x) \in B$ for s between 0 and $t \in \mathbb{R}$. Then we have $F(s,\epsilon x) \in \epsilon B$, or $|F(s,\epsilon x)| \le \epsilon r_B < r_F$. This yields $F(t,\epsilon x) \in L$. Consequently, $F_\epsilon(t,x) = \frac{1}{\epsilon}F(t,\epsilon x) \in \frac{1}{\epsilon}L = L$. Analogously for $x \in U$. □

The next proposition expresses closeness of the flow to its linearization at the origin in terms of components in U and L, for $0 \le t \le 1$ and $x \in B_1$.

Proposition 2.2. *For $\eta > 0$, $0 < \epsilon < \min\{\epsilon(\eta), \epsilon_B\}$, $x \in B_1$, and $0 \le t \le 1$,*

$$|P_UF_\epsilon(t,x) - T(t)P_Ux)| \le \eta|P_Ux| \quad \textit{and} \quad |P_LF_\epsilon(t,x) - T(t)P_Lx)| \le \eta|P_Lx|.$$

Moreover,

$$|P_UF_\epsilon(t,x)| \in (e^{ut}+[-\eta,\eta])|P_Ux| \quad \textit{and} \quad |P_LF_\epsilon(t,x)| \in (e^{\sigma t}+[-\eta,\eta])|P_Lx|.$$

Proof. Let $x \in B_1$ and $0 \le t \le 1$. We apply Proposition 2.1(ii) to $P_Ux \in B_1$ and to $P_Lx \in B_1$ and obtain $F_\epsilon(t,P_Ux) \in U$ and $F_\epsilon(t,P_Lx) \in L$, hence $P_LF_\epsilon(t,P_Ux) = 0$ and $P_UF_\epsilon(t,P_Lx) = 0$. It follows that

$$\begin{aligned}
|P_UF_\epsilon(t,x) - T(t)P_Ux| &= |P_UF_\epsilon(t,x) - T(t)P_Ux - (P_UF_\epsilon(t,P_Lx) - P_UT(t)P_Lx)| \\
&= \left|\int_0^1 (P_UD_2F_\epsilon(t,P_Lx+s(x-P_Lx))[x-P_Lx] - P_UT(t)[x-P_Lx])ds\right| \\
&= \left|\int_0^1 P_U\{D_2F_\epsilon(t,P_Lx+s(x-P_Lx))[P_Ux] - P_UT(t)[P_Ux]\})ds\right| \\
&\le |P_U|\max_{y\in B_1}|D_2F_\epsilon(t,y)-T(t)||P_Ux| \le \eta|P_Ux| \\
&\qquad \text{(with Proposition 2.1(i) and } |P_U| = 1).
\end{aligned}$$

We have $(P_Ux)_3 = x_3$, and for $z = T(t)P_Ux \in U$, $|z| = |z_3|$ with $z_3 = e^{ut}x_3$. Using the previous estimate we obtain

$$\begin{aligned}
(e^{ut}-\eta)|x_3| = |z| - \eta|x_3| = |T(t)P_Ux| - \eta|P_Ux| &\le |P_UF_\epsilon(t,x)| \\
&\le |T(t)P_Ux| + \eta|P_Ux| = |z| + \eta|x_3| = (e^{ut}+\eta)|x_3|
\end{aligned}$$

which yields

$$|P_UF_\epsilon(t,x)| \in (e^{ut}+[-\eta,\eta])|P_Ux|.$$

The remaining assertions are shown analogously. □

Proposition 2.3. *Assume $0 < \eta < e^\sigma$, $e^\sigma + \eta < 1$, and $0 < \epsilon < \min\{\epsilon(\eta), \epsilon_B\}$. Let $n \in \mathbb{N}$ and $x \in B_1$ be given with $|P_UF_\epsilon(j+1,x)| \le 1$ for $j = 0,\dots,n-1$. Then we have $F_\epsilon(j,x) \in B_1$ for $j = 0,\dots,n$, and for $0 \le t \le 1$,*

$$\begin{aligned}
e^{\sigma(t+n)}(1-\eta e^{-\sigma})^{n+1}|P_Lx| \le |P_LF_\epsilon(t+n,x)| \le e^{\sigma(t+n)}(1+\eta e^{-\sigma})^{n+1}|P_Lx|, \\
e^{u(t+n)}(1-\eta)^{n+1}|P_Ux| \le |P_UF_\epsilon(t+n,x)| \le e^{u(t+n)}(1+\eta)^{n+1}|P_Ux|.
\end{aligned}$$

Proof. 1. From Proposition 2.2 for $t = 1$,

$$e^u(1-\eta)|P_U x| \le (e^u-\eta)|P_U x| \le |P_U F_\epsilon(1,x)| \le (e^u+\eta)|P_U x| \le e^u(1+\eta)|P_U x|$$

and

$$e^\sigma(1-\eta e^{-\sigma})|P_L x| = (e^\sigma-\eta)|P_L x| \le |P_L F_\epsilon(1,x)| \le (e^\sigma+\eta)|P_L x| = e^\sigma(1+\eta e^{-\sigma})|P_L x|.$$

Using $x \in B_1$ and $e^\sigma + \eta < 1$ we infer $|P_L F_\epsilon(1,x)| \le 1$. With $|P_U F_\epsilon(1,x)| \le 1$, we get $F_\epsilon(1,x) \in B_1$.

By induction we obtain $F_\epsilon(j,x) \in B_1$ for $j = 1,\dots,n$, with

$$e^{\sigma j}(1-\eta e^{-\sigma})^j|P_L x| \le |P_L F_\epsilon(j,x)| \le e^{\sigma j}(1+\eta e^{-\sigma})^j|P_L x| \quad \text{and}$$
$$e^{uj}(1-\eta)^j|P_U x| \le |P_U F_\epsilon(j,x)| \le e^{uj}(1+\eta)^j|P_U x|.$$

2. On the lower estimates of the assertion. For $j = n$ and $0 \le t \le 1$ Proposition 2.2 yields

$$(e^{ut}-\eta)|P_U F_\epsilon(n,x)| \le |P_U F_\epsilon(t+n,x)|.$$

Use $e^{ut}(1-\eta) \le e^{ut} - \eta$ and the lower estimate for $|P_U F_\epsilon(n,x)|$ in order to get

$$e^{u(t+n)}(1-\eta)^{n+1}|P_U x| \le |P_U F_\epsilon(t+n,x)|.$$

In the same way, now using $e^{\sigma t}(1-\eta e^{-\sigma}) \le e^{\sigma t} - \eta$, one finds

$$e^{\sigma(t+n)}(1-\eta e^{-\sigma})^{n+1}|P_L x| \le |P_L F_\epsilon(t+n,x)|.$$

3. The upper estimates of the assertion are shown analogously. □

For later use we turn to exponential estimates for flowlines which stay sufficiently long in B_1.

Proposition 2.4. *Let $\tilde\eta > 0$ be given. Assume $0 < \eta < e^\sigma$ and $e^\sigma + \eta < 1$ as in Proposition 2.3, and in addition*

$$\log(1+\eta e^{-\sigma}) < \tilde\eta \quad \textit{and} \quad \log\left(\frac{1}{1-\eta e^{-\sigma}}\right) < \tilde\eta.$$

Let $n \in \mathbb{N}$ be given with

$$\frac{n+1}{n}\log(1+\eta e^{-\sigma}) < \tilde\eta \quad \textit{and} \quad \frac{n+1}{n}\log\left(\frac{1}{1-\eta e^{-\sigma}}\right) < \tilde\eta.$$

Let $0 < \epsilon < \min\{\epsilon(\eta), \epsilon_B\}$ and consider $x \in B_1$ with $|P_U F_\epsilon(j+1,x)| \le 1$ for $j = 0,\dots,n-1$ as in Proposition 2.3. Then we have, for $0 \le t \le 1$,

$$e^{(\sigma-\tilde\eta)(t+n)}|P_L x| \le |P_L F_\epsilon(t+n,x)| \le e^{(\sigma+\tilde\eta)(t+n)}|P_L x|,$$
$$e^{(u-\tilde\eta)(t+n)}|P_U x| \le |P_U F_\epsilon(t+n,x)| \le e^{(u+\tilde\eta)(t+n)}|P_U x|.$$

Proof. In view of Proposition 2.3 the estimates of $|P_U F_\epsilon(t+n,x)|$ follow from the estimates

$$e^{(u-\tilde\eta)(t+n)} \le e^{u(t+n)}(1-\eta)^{n+1} \quad \text{and} \quad e^{u(t+n)}(1+\eta)^{n+1} \le e^{(u+\tilde\eta)(t+n)},$$

or equivalently,

$$-(t+n)\tilde{\eta} \le (n+1)\log(1-\eta) \quad \text{and} \quad (n+1)\log(1+\eta) \le (t+n)\tilde{\eta}.$$

Sufficient for the latter are

$$-n\tilde{\eta} \le (n+1)\log(1-\eta) \quad \text{and} \quad (n+1)\log(1+\eta) \le n\tilde{\eta},$$

which are a consequence of the hypotheses on η and n in combination with

$$\log\left(\frac{1}{1-\eta}\right) < \log\left(\frac{1}{1-\eta e^{-\sigma}}\right) \quad \text{and} \quad \log(1+\eta) < \log(1+\eta e^{-\sigma}).$$

Similarly the estimates of $|P_L F_\epsilon(t+n,x)|$ follow from the estimates

$$e^{(\sigma-\tilde{\eta})(t+n)} \le e^{\sigma(t+n)}(1-\eta e^{-\sigma})^{n+1} \quad \text{and} \quad e^{\sigma(t+n)}(1+\eta e^{-\sigma})^{n+1} \le e^{(\sigma+\tilde{\eta})(t+n)}$$

which are equivalent to

$$-\tilde{\eta}(t+n) \le (n+1)\log(1-\eta e^{-\sigma}) \quad \text{and} \quad (n+1)\log(1+\eta e^{-\sigma}) \le \tilde{\eta}(t+n).$$

Sufficient for the latter are

$$-\tilde{\eta}n \le (n+1)\log(1-\eta e^{-\sigma}) \quad \text{and} \quad (n+1)\log(1+\eta e^{-\sigma}) \le \tilde{\eta}n,$$

which are obvious from the hypotheses on η and n. □

3 Angles

This section deals with angles in the plane L, along projected flowlines $t \mapsto P_L F_\epsilon(t,x)$, for $x \in B_1 \setminus U$. The first step computes such angles for $0 \le t \le 1$.

Proposition 3.1. *Assume* $0 < \eta < \frac{e^\sigma}{2}$, $0 < \epsilon < \min\{\epsilon(\eta), \epsilon_B\}$, $x \in B_1 \setminus U$, *and*

$$\frac{1}{|P_L x|} P_L x = \begin{pmatrix} \cos(\psi) \\ \sin(\psi) \\ 0 \end{pmatrix} \quad \textit{for some } \psi \in \mathbb{R}.$$

Let $0 \le t \le 1$. *For the unique* $\Delta = \Delta(t,x,\psi) \in [-\pi, \pi)$ *with*

$$\frac{1}{|P_L F_\epsilon(t,x)|} P_L F_\epsilon(t,x) = \begin{pmatrix} \cos(\psi - \mu t + \Delta) \\ \sin(\psi - \mu t + \Delta) \\ 0 \end{pmatrix}$$

we have

$$\Delta(t,x,\psi) = \arcsin(\langle v, w^\perp \rangle) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

where

$$v = \frac{1}{|P_L F_\epsilon(t,x)|} P_L F_\epsilon(t,x), \quad w = \frac{1}{|T(t) P_L x|} T(t) P_L x, \quad \textit{and} \quad w^\perp = \begin{pmatrix} -w_2 \\ w_1 \\ 0 \end{pmatrix}.$$

In particular, $\Delta(0,x,\psi) = 0$. *Moreover,*

$$|\langle v, w^\perp \rangle| \le \frac{\eta}{e^\sigma - \eta} \quad \textit{and} \quad |\Delta| \le \arcsin\left(\frac{\eta}{e^\sigma - \eta}\right).$$

Compare Figure 1.1, bottom.

Proof. 1. Using $F_\epsilon(B_1 \times \{x\}) \subset B$ and invariance of U under F_ϵ in B from Proposition 2.1 (ii) we infer $F_\epsilon(t,x) \in B \setminus U$, hence $P_L F_\epsilon(t,x) \neq 0$. Also, $T(t)P_L x \neq 0$.

2. We have

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} \cos(\mu t) & \sin(\mu t) \\ -\sin(\mu t) & \cos(\mu t) \end{pmatrix} \cdot \begin{pmatrix} \cos(\psi) \\ \sin(\psi) \end{pmatrix} = \begin{pmatrix} \cos(\psi - \mu t) \\ \sin(\psi - \mu t) \end{pmatrix}$$
$$= \begin{pmatrix} \cos(\psi - \mu t) & -\sin(\psi - \mu t) \\ \sin(\psi - \mu t) & \cos(\psi - \mu t) \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

hence

$$\begin{pmatrix} -w_2 \\ w_1 \end{pmatrix} = \begin{pmatrix} \cos(\psi - \mu t) & -\sin(\psi - \mu t) \\ \sin(\psi - \mu t) & \cos(\psi - \mu t) \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

It follows that multiplication with the matrix

$$\begin{pmatrix} \cos(\psi - \mu t) & -\sin(\psi - \mu t) & 0 \\ \sin(\psi - \mu t) & \cos(\psi - \mu t) & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \cos(\psi - \mu t) & \sin(\psi - \mu t) & 0 \\ -\sin(\psi - \mu t) & \cos(\psi - \mu t) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

defines a linear map $\rho : L \to L$ which satisfies $\rho w = e_1$ and $\rho w^\perp = e_2$. The map ρ preserves the inner product and the norm. We obtain

$$\begin{pmatrix} \cos \\ \sin \\ 0 \end{pmatrix}(\Delta) = \begin{pmatrix} \cos(\psi - \mu t + \Delta - (\psi - \mu t)) \\ \sin(\psi - \mu t + \Delta - (\psi - \mu t)) \\ 0 \end{pmatrix}$$
$$= \begin{pmatrix} \cos(\psi - \mu t) & \sin(\psi - \mu t) & 0 \\ -\sin(\psi - \mu t) & \cos(\psi - \mu t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos(\psi - \mu t + \Delta) \\ \sin(\psi - \mu t + \Delta) \\ 0 \end{pmatrix}$$
$$= \rho v = \langle \rho v, e_1 \rangle e_1 + \langle \rho v, e_2 \rangle e_2 = \langle \rho v, \rho w \rangle e_1 + \langle \rho v, \rho w^\perp \rangle e_2$$
$$= \langle v, w \rangle e_1 + \langle v, w^\perp \rangle e_2,$$

hence $\cos(\Delta) = \langle v, w \rangle$ and $\sin(\Delta) = \langle v, w^\perp \rangle$. Let $\tilde{v} = P_L F_\epsilon(t,x) \neq 0$ and $\tilde{w} = T(t)P_L x \neq 0$. Then

$$\langle v, w \rangle = \frac{1}{|\tilde{v}||\tilde{w}|} \langle \tilde{v}, \tilde{w} \rangle.$$

Proposition 2.2 shows that $\tilde{z} = \tilde{v} - \tilde{w}$ satisfies $|\tilde{z}| \leq \eta |P_L x|$. Recall $|\tilde{w}| = e^{\sigma t}|P_L x|$. We infer

$$\langle \tilde{v}, \tilde{w} \rangle = \langle \tilde{w} + \tilde{z}, \tilde{w} \rangle \ \geq |\tilde{w}|^2 - |\tilde{z}||\tilde{w}| \geq |P_L x|^2 e^{\sigma t}(e^{\sigma t} - \eta) \geq |P_L x|^2 e^{\sigma}(e^{\sigma} - \eta) > 0,$$

which yields $\cos(\Delta) = \langle v, w \rangle > 0$. Recall $-\pi \leq \Delta < \pi$. It follows that $|\Delta| < \frac{\pi}{2}$. Consequently,

$$\Delta = \arcsin(\sin(\Delta)) = \arcsin(\langle v, w^\perp \rangle).$$

The equation $\Delta(0, x, \psi) = 0$ follows from $P_L F_\epsilon(0, x) = P_L x = T(0)P_L x$, which yields $v = w$, hence $\langle v, w^\perp \rangle = 0$.

3. Proof of the estimates of $\langle v, w^\perp \rangle$ and Δ. We have

$$|\langle v, w^\perp \rangle| = \frac{1}{|\tilde{v}||\tilde{w}|} |\langle \tilde{v}, \tilde{w}^\perp \rangle| = \frac{1}{|\tilde{v}||\tilde{w}|} |\langle \tilde{z}, \tilde{w}^\perp \rangle| \leq \frac{|\tilde{z}|}{|\tilde{v}|}.$$

Using $|\tilde{w}| = e^{\sigma t}|P_L x|$ and $|\tilde{z}| \le \eta |P_L x|$ we get

$$|\tilde{v}| \ge |\tilde{w}| - |\tilde{z}| = e^{\sigma t}|P_L x| - |\tilde{z}| \ge (e^{\sigma t} - \eta)|P_L x| \ge (e^{\sigma} - \eta)|P_L x|$$

and obtain

$$\frac{|\tilde{z}|}{|\tilde{v}|} \le \frac{\eta}{e^{\sigma} - \eta} < 1$$

where the last inequality holds by the hypothesis on η. Finally,

$$|\Delta| = |\arcsin(\langle v, w^{\perp}\rangle)| = \arcsin(|\langle v, w^{\perp}\rangle|) \le \arcsin\left(\frac{\eta}{e^{\sigma} - \eta}\right). \qquad \square$$

The next result describes the angles along projected flowlines $P_L F_\epsilon(\cdot, x)$, $x \in B_1 \setminus U$, by continuous functions, a little longer than the projected values $P_U F_\epsilon(\nu, x)$, $\nu \in \mathbb{N}$, remain in $B_1 \setminus U$. For later use we restrict attention to flowlines which start from the strip on the cylinder

$$M_I = \{x \in \mathbb{R}^3 : |P_L x| = 1\}$$

which is given by $0 < x_3 \le 1$,

$$x = \begin{pmatrix} \cos(\psi) \\ \sin(\psi) \\ \delta \end{pmatrix} \quad \text{with } \psi \in \mathbb{R} \text{ and } 0 < \delta \le 1. \tag{3.1}$$

See Figure 3.1, top.

For integers $n \ge 0$ we define

$$\begin{aligned} \mathrm{dom}_n = \big\{(t, \psi, \delta) \in [0, n+1] \times \mathbb{R} \times (0, 1] : \\ |P_U F_\epsilon(\nu, x)| \le 1 \text{ for } x \text{ given by (3.1) and } \nu = 0, \dots, n\big\}. \end{aligned}$$

Proposition 3.2. *Assume* $0 < \eta < \frac{e^{\sigma}}{2}$, $0 < \epsilon < \min\{\epsilon(\eta), \epsilon_B\}$.

(i) For every integer $n \ge 0$ *there exists a continuous function* $\phi^{(n)} : \mathrm{dom}_n \to \mathbb{R}$ *so that for each* $(t, \psi, \delta) \in \mathrm{dom}_n$, *with* x *given by* (3.1), *we have*

$$\frac{1}{|P_L F_\epsilon(t, x)|} P_L F_\epsilon(t, x) = \begin{pmatrix} \cos(\phi^{(n)}(t, \psi, \delta)) \\ \sin(\phi^{(n)}(t, \psi, \delta)) \\ 0 \end{pmatrix}, \tag{3.2}$$

and in case $n \le t \le n+1$,

$$\psi - t\mu - (n+1)\arcsin\left(\frac{\eta}{e^{\sigma} - \eta}\right) \le \phi^{(n)}(t, \psi, \delta) \le \psi - t\mu + (n+1)\arcsin\left(\frac{\eta}{e^{\sigma} - \eta}\right). \tag{3.3}$$

(ii) For every $n \in \mathbb{N}$, $([0, n] \times \mathbb{R} \times (0, 1]) \cap \mathrm{dom}_n \subset \mathrm{dom}_{n-1}$, *and on this set,* $\phi^{(n)}(t, \psi, \delta) = \phi^{(n-1)}(t, \psi, \delta)$.

Proof. 1. Obviously, $([0, n] \times \mathbb{R} \times (0, 1]) \cap \mathrm{dom}_n \subset \mathrm{dom}_{n-1}$ for all integers $n \ge 1$.

2. Proof of assertion (i). We construct the functions $\phi^{(n)}$ recursively.

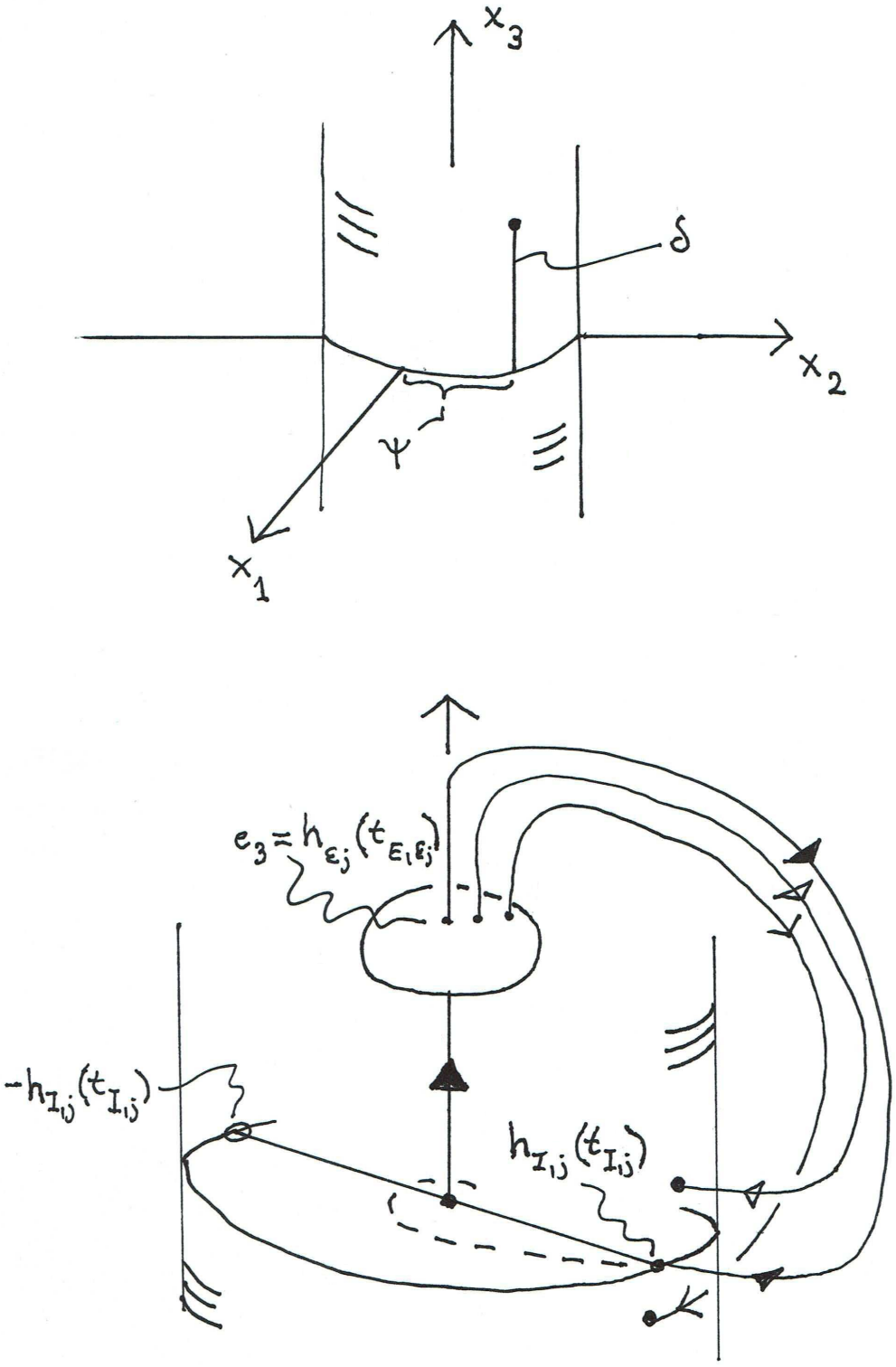


Figure 3.1: Top: The relations (3.1). Bottom: The exterior map.

2.1. For $n = 0$ Proposition 3.1 shows that the continuous function $\phi^{(0)} : \mathrm{dom}_0 \to \mathbb{R}$ defined by

$$\phi^{(0)}(t,\psi,\delta) = \psi - t\mu + \arcsin(\Delta) \quad \text{for } 0 \le t \le 1,$$

with $\Delta = \Delta(t,x,\psi)$ from Proposition 3.1, for x given by (3.1), satisfies (3.2) and (3.3).

2.2. Suppose now that for some integer $n \ge 0$ the continuous function $\phi^{(n)} : \mathrm{dom}_n \to \mathbb{R}$ satisfies (3.2) and (3.3). We define $\phi^{(n+1)} : \mathrm{dom}_{n+1} \to \mathbb{R}$ as follows.

For $(t,\psi,\delta) \in \mathrm{dom}_{n+1}$ with $0 \le t \le n+1$ we have $(t,\psi,\delta) \in \mathrm{dom}_n$ due to Part 1, and we set $\phi^{(n+1)}(t,\psi,\delta) = \phi^{(n)}(t,\psi,\delta)$.

For $(t,\psi,\delta) \in \mathrm{dom}_{n+1}$ with $n+1 \le t \le n+2$, observe first that for x given by (3.1) the property $|P_U F_\epsilon(\nu,x)| \le 1$ for $\nu = 0,\dots,n+1$ yields $F_\epsilon(\nu,x) \in B_1$ for $\nu = 1,\dots,n+1$, by means of Proposition 2.3. By Proposition 2.1 (ii), $F_\epsilon(s,x) \in B$ for $0 \le s \le n+2$. Using this in combination with $x \in B_1 \setminus U \subset B \setminus U$ we get $F_\epsilon(n+1,x) \in B \setminus U$, by invariance of U from Proposition 2.1 (ii). Altogether, $y = F_\epsilon(n+1,x)$ is contained in $B_1 \setminus U$, and

$$\frac{1}{|P_L y|} P_L y = \begin{pmatrix} \cos(\phi^{(n)}(n+1,\psi,\delta)) \\ \sin(\phi^{(n)}(n+1,\psi,\delta)) \\ 0 \end{pmatrix}.$$

An application of Proposition 3.1 to $y \in B_1 \setminus U$ shows that the continuous function

$$\phi^* : \mathrm{dom}_0 \to \mathbb{R}, \quad \phi^*(t,\psi,\delta) = \phi^{(n)}(n+1,\psi,\delta) - t\mu + \arcsin(\Delta) \quad \text{for } 0 \le t \le 1,$$

with $\Delta = \Delta(t,y,\phi^{(n)}(n+1,\psi,\delta))$ according to Proposition 3.1, satisfies (3.2) and (3.3) with y and $\phi^{(n)}(n+1,\psi,\delta)$ in place of x and ψ, respectively. Also,

$$\phi^{(n)}(n+1,\psi,\delta) - t\mu - \arcsin\left(\frac{\eta}{e^\sigma - \eta}\right) \le \phi^*(t,\psi,\delta) \le \phi^{(n)}(n+1,\psi,\delta) - t\mu + \arcsin\left(\frac{\eta}{e^\sigma - \eta}\right).$$

For $(t,\psi,\delta) \in \mathrm{dom}_{n+1}$ with $n+1 \le t \le n+2$ we have $(t-(n+1),\psi,\delta) \in \mathrm{dom}_0$. We complete the definition of $\phi^{(n+1)}$ by

$$\phi^{(n+1)}(t,\psi,\delta) = \phi^*(t-(n+1),\psi,\delta) \quad \text{for } (t,\psi,\delta) \in \mathrm{dom}_{n+1} \text{ with } n+1 < t \le n+2.$$

Proof that $\phi^{(n+1)}$ is continuous. Let $(t,\psi,\delta) \in \mathrm{dom}_{n+1}$ and a sequence $(t_m,\psi_m,\delta_m)_{m\in\mathbb{N}}$ in dom_{n+1} with $\lim_{m\to\infty}(t_m,\psi_m,\delta_m) = (t,\psi,\delta)$ be given. It is enough to show that for a subsequence we have $\phi^{(n+1)}(t_{m_\mu},\psi_{m_\mu},\delta_{m_\mu}) \to \phi^{(n+1)}(t,\psi,\delta)$ as $\mu \to \infty$. In case $t < n+1$, $(t,\psi,\delta) \in \mathrm{dom}_n$ and $\phi^{(n+1)}(t,\psi,\delta) = \phi^{(n)}(t,\psi,\delta)$, and for m sufficiently large, $t_m < n+1$, hence $(t_m,\psi_m,\delta_m) \in \mathrm{dom}_n$ and $\phi^{(n+1)}(t_m,\psi_m,\delta_m) = \phi^{(n)}(t_m,\psi_m,\delta_m)$ for such m, and continuity of $\phi^{(n)}$ yields $\phi^{(n+1)}(t_m,\psi_m,\delta_m) \to \phi^{(n+1)}(t,\psi,\delta)$ for $m \to \infty$. In case $n+1 < t \le n+2$ a similar argument with $\phi^*(\cdot-(n+1),\cdot,\cdot)$ in place of $\phi^{(n)}$ yields $\phi^{(n+1)}(t_m,\psi_m,\delta_m) \to \phi^{(n+1)}(t,\psi,\delta)$ for $m \to \infty$ as well. In case $t = n+1$ we distinguish the subcases that the set of indices m with $t_m \le n+1$ is bounded or unbounded. If it is unbounded then we can argue for a subsequence as in case $t < n+1$ above. If it is bounded we can argue as in case $n+1 < t \le n+2$ above and find

$$\begin{aligned} \lim_{m\to\infty} \phi^{(n+1)}(t_m,\psi_m,\delta_m) &= \lim_{m\to\infty} \phi^*(t_m-(n+1),\psi_m,\delta_m) = \phi^*(0,\psi,\delta) \\ &= \phi^{(n)}(n+1,\psi,\delta) = \phi^{(n+1)}(n+1,\psi,\delta). \end{aligned}$$

2.3. Proof of (3.2) for $\phi^{(n+1)}$. By the properties of $\phi^{(n)}$, for $(t,\psi,\delta) \in \mathrm{dom}_{n+1}$ with $0 \le t \le n+1$, obviously

$$\frac{1}{|P_L F_\epsilon(t,x)|} P_L F_\epsilon(t,x) = \begin{pmatrix} \cos(\phi^{(n)}(t,\psi,\delta)) \\ \sin(\phi^{(n)}(t,\psi,\delta)) \\ 0 \end{pmatrix} = \begin{pmatrix} \cos(\phi^{(n+1)}(t,\psi,\delta)) \\ \sin(\phi^{(n+1)}(t,\psi,\delta)) \\ 0 \end{pmatrix}.$$

For $(t,\psi,\delta) \in \mathrm{dom}_{n+1}$ with $n+1 < t \le n+2$ let $s = t-(n+1)$ and $y = F_\epsilon(n+1,x)$ with x given by (3.1). Using the version of (3.2) for ϕ^* we have

$$\begin{aligned} \frac{1}{|P_L F_\epsilon(t,x)|} P_L F_\epsilon(t,x) &= \frac{1}{|P_L F_\epsilon(s,y)|} P_L F_\epsilon(s,y) = \begin{pmatrix} \cos(\phi^*(s,\psi,\delta)) \\ \sin(\phi^*(s,\psi,\delta)) \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} \cos(\phi^*(t-(n+1),\psi,\delta)) \\ \sin(\phi^*(t-(n+1),\psi,\delta)) \\ 0 \end{pmatrix} = \begin{pmatrix} \cos(\phi^{(n+1)}(t,\psi,\delta)) \\ \sin(\phi^{(n+1)}(t,\psi,\delta)) \\ 0 \end{pmatrix}. \end{aligned}$$

2.4. Proof of (3.3) for $\phi^{(n+1)}$. For $(n+1+t,\psi,\delta) \in \mathrm{dom}_{n+1}$ with $0 < t \le 1$ we have

$$\begin{aligned} \phi^{(n+1)}(n+1+t,\psi,\delta) &= \phi^*(t,\psi,\delta) \le \phi^{(n)}(n+1,\psi,\delta) - t\mu + \arcsin\left(\frac{\eta}{e^\sigma - \eta}\right) \\ &\le \left(\psi - (n+1)\mu + (n+1)\arcsin\left(\frac{\eta}{e^\sigma-\eta}\right)\right) - t\mu + \arcsin\left(\frac{\eta}{e^\sigma-\eta}\right) \\ &\quad \text{(by (3.3) for } \phi^{(n)}\text{)} \\ &= \psi - (n+1+t)\mu + (n+2)\arcsin\left(\frac{\eta}{e^\sigma-\eta}\right). \end{aligned}$$

Analogously we get the lower estimate

$$\psi - (n+1+t)\mu - (n+2)\arcsin\left(\frac{\eta}{e^\sigma-\eta}\right) \le \phi^{(n+1)}(n+1+t,\psi,\delta)$$

for $0 < t \le 1$. By continuity both estimates hold also at $t = n+1$.

2.5. The proof of assertion (i) is complete. Assertion (ii) is obvious from the previous construction which includes the relation

$$\phi^{(n)}(t,\psi,\delta) = \phi^{(n-1)}(t,\psi,\delta)$$

for $(t,\psi,\delta) \in ([0,n] \times \mathbb{R} \times (0,1]) \cap \mathrm{dom}_n \subset \mathrm{dom}_{n-1}$. □

We proceed to estimates of $\phi^{(n)}$ from Proposition 3.2 on the interval $[n,n+1]$ in terms of affine linear maps with slopes close to $\pm\mu$.

Corollary 3.3. *Let $\tilde\eta > 0$ and $\eta > 0$ be given with $\eta < \frac{e^\sigma}{2}$ and*

$$\arcsin\left(\frac{\eta}{e^\sigma-\eta}\right) < \tilde\eta.$$

Let $0 < \epsilon < \min\{\epsilon(\eta), \epsilon_B\}$ and let $n \in \mathbb{N}$ be given with

$$\frac{n+1}{n}\arcsin\left(\frac{\eta}{e^\sigma-\eta}\right) < \tilde\eta.$$

Then the function $\phi^{(n)}$ obtained in Proposition 3.2 satisfies

$$\psi-(n+t)(\mu+\tilde{\eta})\le\phi^{(n)}(n+t,\psi,\delta)\le\psi-(n+t)(\mu-\tilde{\eta})$$

for every $(n+t,\psi,\delta)\in\mathrm{dom}_n$ with $0\le t\le 1$.

Proof. 1. The estimate (3.3) in Proposition 3.2 shows that the upper estimate of the assertion follows from

$$-(n+t)\mu+(n+1)\arcsin\left(\frac{\eta}{e^\sigma-\eta}\right)\le-(n+t)(\mu-\tilde{\eta})$$

which is equivalent to

$$(n+1)\arcsin\left(\frac{\eta}{e^\sigma-\eta}\right)\le(n+t)\tilde{\eta}.$$

The preceding estimate follows from

$$(n+1)\arcsin\left(\frac{\eta}{e^\sigma-\eta}\right)\le n\tilde{\eta}$$

which is obvious from the hypotheses on η and n.

2. Analogously the estimate (3.3) shows that the lower estimate of the assertion follows from

$$-(n+t)\mu-(n+1)\arcsin\left(\frac{\eta}{e^\sigma-\eta}\right)\ge-(n+t)(\mu+\tilde{\eta})$$

which is equivalent to

$$-(n+1)\arcsin\left(\frac{\eta}{e^\sigma-\eta}\right)\ge-(n+t)\tilde{\eta}.$$

The preceding estimate follows from

$$-(n+1)\arcsin\left(\frac{\eta}{e^\sigma-\eta}\right)\ge-n\tilde{\eta}$$

which is obvious from the hypotheses on η and n. □

4 Transversality, and exterior maps

For every $\epsilon>0$ the flowline $h_\epsilon=\frac{1}{\epsilon}h$ of F_ϵ is homoclinic with $h_\epsilon(t)\ne 0\ne h_\epsilon'(t)$ everywhere and $h_\epsilon(t)\to 0$ as $|t|\to\infty$.

Proposition 4.1.

(i) For $0<\epsilon<|h(t_U)|$ there are reals $t_{E,\epsilon}\le t_U$ with

$$h_\epsilon(t_{E,\epsilon})=e_3\quad\text{and}\quad h_\epsilon(t)\in(0,\infty)e_3\quad\text{for all } t\le t_{E,\epsilon}.$$

(ii) There are strictly montonic sequences $(t_{I,j})_{j\in\mathbb{N}}$ in $[t_L,\infty)$ and $(\epsilon_j)_{j\in\mathbb{N}}$ in $(0,\infty)$ with $t_{I,j}\to\infty$ and $\epsilon_j\to 0$ as $j\to\infty$ such that for every $j\in\mathbb{N}$,

$$\epsilon_j<|h(t_U)|,\quad|h_{\epsilon_j}(t_{I,j})|=1,\quad(|h_{\epsilon_j}|^2)'(t_{I,j})<0,\quad h_{\epsilon_j}(t)\in L\quad\text{for } t\ge t_{I,j},\ h'_{\epsilon_j}(t_{I,j})\in L.$$

Proof. 1. On (i). Recall that for all $t \le t_U$, $h(t) \in (0,\infty)e_3$. Let $0 < \epsilon < |h(t_U)|$. The relation $\lim_{t\to-\infty} h(t) = 0$ shows that for each $\epsilon \in (0, |h(t_U)|)$ there exists $t_{E,\epsilon} \le t_U$ with $|h(t_{E,\epsilon})| = \epsilon$. It follows that $h(t_{E,\epsilon}) = \epsilon e_3$, hence $h_\epsilon(t_{E,\epsilon}) = e_3$. For $0 < \epsilon < |h(t_U)|$ and for all $t \le t_{E,\epsilon} \le t_U$ we get $h_\epsilon(t) = \frac{1}{\epsilon}h(t) \in \frac{1}{\epsilon}(0,\infty)e_3 = (0,\infty)\epsilon$.

2. On (ii). From $h(t) \neq 0$ everywhere and $h(t) \to 0$ as $t \to \infty$ we get a strictly increasing sequence $(t_{I,j})_{j\in\mathbb{N}}$ in $[t_L, \infty)$ with $t_{I,j} \to \infty$ for $j \to \infty$ so that $|h(t_{I,j})| < |h(t_U)|$ and $(|h|^2)'(t_{I,j}) < 0$ for all $j \in \mathbb{N}$, and the sequence given by $\epsilon_j = |h(t_{I,j})|$ is strictly decreasing. It follows that

$$|h_{\epsilon_j}(t_{I,j})| = \frac{1}{\epsilon_j}|h(t_{I,j})| = 1 \quad \text{and} \quad (|h_{\epsilon_j}|^2)'(t_{I,j}) = \frac{1}{\epsilon_j^2}|(|h|^2)'(t_{I,j}) < 0 \quad \text{for all } j \in \mathbb{N}.$$

Also, for $t \ge t_{I,j} \ge t_L$, $h_{\epsilon_j}(t) = \frac{1}{\epsilon_j}h(t) \in \frac{1}{\epsilon_j}L = L$, which yields $h'_{\epsilon_j}(t_{I,j}) \in L$. □

We want to describe the behaviour of flowlines close to the homoclinic loop $h_\epsilon(\mathbb{R}) \cup \{0\}$ for small $\epsilon > 0$. This will be done in terms of a return map which is given by the return of flowlines $F_\epsilon(\cdot, x)$ from points x in the cylinder

$$M_I = \{z \in \mathbb{R}^3 : |P_L z| = 1\}$$

with $0 < x_3$ slightly above the plane L, to targets in M_I. The return map will be obtained as a composition of an *inner map*, which follows the flow until it reaches the plane

$$M_E = e_3 + L,$$

parallel to L , with an *exterior map* following the flow from a neighbourhood of e_3 in M_E until it reaches M_I.

The constructions and continuous differentiability of the inner and exterior maps requires that the homoclinic flowline intersects the smooth 2-dimensional submanifolds M_E and M_I of $\mathbb{R}^3$ transversally. Obviously, $T_x M_E = L$ for all $x \in M_E$, and at $x \in M_I \cap L$,

$$T_x M_I = \mathbb{R}x^\perp \oplus \mathbb{R}e_3, \quad \text{with } x^\perp = \begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix}.$$

Proposition 4.2.

(i) For $0 < \epsilon < |h(t_U)|$, $\partial_1 F_\epsilon(0, e_3) = D_1 F_\epsilon(0, e_3)1 = h'_\epsilon(t_{E,\epsilon}) \notin L = T_{e_3} M_E$.

(ii) Let $j \in \mathbb{N}$ be given and $x = h_{\epsilon_j}(t_{I,j})$. Then

$$\partial_1 F_{\epsilon_j}(0,x) = h'_{\epsilon_j}(t_{I,j}) = \partial_1 F_{\epsilon_j}(t_{I,j} - t_{E,\epsilon_j}, e_3), \quad \langle \partial_1 F_{\epsilon_j}(0,x), x\rangle < 0, \quad \text{and} \quad \partial_1 F_{\epsilon_j}(0,x) \notin T_x M_I.$$

Proof. 1. On (i). From

$$F_\epsilon(t, e_3) - F_\epsilon(0, e_3) = F_\epsilon(t, h_\epsilon(t_{E,\epsilon})) - e_3 = h_\epsilon(t + t_{E,\epsilon}) - h_\epsilon(E, t_\epsilon)$$

we have $\partial_1 F_\epsilon(0, e_3) = h'_\epsilon(t_{E,\epsilon}) \neq 0$. Using $h_\epsilon(t) \in U$ for $t \le t_{E,\epsilon}$ we get $h'_\epsilon(t_{E,\epsilon}) \in U$. It follows that $\partial_1 F_\epsilon(0, e_3) \in U \setminus \{0\} \subset \mathbb{R}^3 \setminus L$.

2. On (ii). Let $j \in \mathbb{N}$ be given. Let $x = h_{\epsilon_j}(t_{I,j}) \in M_I \cap L$. Arguing as in Part 1 we get $\partial_1 F_{\epsilon_j}(0,x) = h'_{\epsilon_j}(t_{I,j}) = \partial_1 F_{\epsilon_j}(t_{I,j} - t_{E,\epsilon_j}, e_3)$. From Proposition 4.1 (ii),

$$0 > (|h_{\epsilon_j}|^2)'(t_{I,j}) = 2\langle h'_{\epsilon_j}(t_{I,j}), h_{\epsilon_j}(t_{I,j})\rangle = 2\langle \partial_1 F_{\epsilon_j}(0,x), x\rangle$$

and $h'_{\epsilon_j}(t_{I,j}) \in L$. From the preceding relations, $\partial_1 F_{\epsilon_j}(0,x) \notin \mathbb{R}x^\perp \oplus \mathbb{R}e_3 = T_x M_I$. □

For $r > 0$ let

$$M_E(r) = \{y \in M_E : |y - e_3| < r\}.$$

Corollary 4.3. *There is a decreasing sequence $(r_j)_{j\in\mathbb{N}}$ with $r_j \to 0$ as $j \to \infty$ so that for every $j \in \mathbb{N}$ there exists a continuously differentiable map*

$$t_j : M_E(r_j) \to (0, \infty)$$

with $t_j(e_3) = t_{I,j} - t_{E,\epsilon_j}$, $F_{\epsilon_j}(t_j(y), y) \in \{z \in M_I : P_L z \neq -h_{\epsilon_j}(t_{I,j})\}$ for all $y \in M_E(r_j)$, and

$$\partial_1 F_{\epsilon_j}(0, y) \notin L \quad \text{for all } y \in M_E(r_j).$$

Proof. Let $j \in \mathbb{N}$ be given. With the function $G : \mathbb{R}^3 \to \mathbb{R}$, $G(z) = z_1^2 + z_2^2 - 1$, the relation $F_{\epsilon_j}(t, y) \in M_I$ is equivalent to the equation $G(F_{\epsilon_j}(t, y)) = 0$. Using Proposition 4.2 (ii) with $x = F_{\epsilon_j}(t_{I,j} - t_{E,\epsilon_j}, e_3)$ we obtain

$$\partial_1(G \circ F_{\epsilon_j}(t_{I,j} - t_{E,\epsilon_j}, e_3) = 2\langle \partial_1 F_{\epsilon_j}(t_{I,j} - t_{E,\epsilon_j}, e_3), F_{\epsilon_j}(t_{I,j} - t_{E,\epsilon_j}, e_3)\rangle = 2\langle \partial_1 F_{\epsilon_j}(0, x), x)\rangle < 0.$$

Therefore the Implicit Function Theorem applies and yields a continuously differentiable positive function t_j^* on a neighbourhood N_j of e_3 in $\mathbb{R}^3$ which satisfies $t_j^*(e_3) = t_{I,j} - t_{E,\epsilon_j}$ and $F_{\epsilon_j}(t_j^*(y), y) \in M_I$ for all $y \in N_j$. Given any real $r > 0$ it follows from Proposition 4.2 (i) and continuity that there exists $r_j \in (0, r)$ with $M_E(r_j) \subset N_j$ and $\partial_1 F_{\epsilon_j}(0, y) \notin L$ for all $y \in M_E(r_j)$. Using $F_{\epsilon_j}(t_j^*(e_3), e_3) = h_{\epsilon_j}(t_{I,j}) \in L$ and continuity we also achieve $P_L F_{\epsilon_j}(t_j^*(y), y) \neq -h_{\epsilon_j}(t_{I,j}))$ for all $y \in M_E(r_j)$. Let t_j be the restriction of t_j^* to $M_E(r_j)$.

The desired decreasing sequence $(r_j)_{j\in\mathbb{N}}$ can be obtained recursively. □

For every $j \in \mathbb{N}$ the *exterior map*

$$E_j : M_E(r_j) \to \{z \in M_I : P_L z \neq -h_{\epsilon_j}(t_{I,j})\}, \quad E_j(y) = F_{\epsilon_j}(t_j(y), y),$$

into the open subset $\{z \in M_I : P_L z \neq -h_{\epsilon_j}(t_{I,j})\} = \{z \in M_I : P_L z \neq -E_j(e_3))\}$ of the manifold M_I is continuously differentiable. Compare Figure 3.1, bottom.

Corollary 4.4. *Let $j \in \mathbb{N}$ be given and $x = E_j(e_3)$. Then $T_{e_3}E_j(T_{e_3}M_E) = T_x M_I$.*

Proof. Let $v = \partial_1 F_{\epsilon_j}(0, e_3)$ and $w = h'_{\epsilon_j}(t_{I,j}) = \partial_1 F_{\epsilon_j}(t_j(e_3), e_3) = \partial F_{\epsilon_j}(0, x)$. Then $v \notin L = T_{e_3}M_E$ and $w \notin T_x M_I$. The map $T_{e_3}E_j : T_{e_3}M_E \to T_x M_I$ is given by

$$T_{e_3}E_j(z) = P_j D_2 F_{\epsilon_j}(t_j(e_3), e_3)z$$

with the projection $P_j : \mathbb{R}^3 \to \mathbb{R}^3$ along $\mathbb{R}w$ onto $T_x M_I$. The isomorphism $D_2 F_{\epsilon_j}(t_j(e_3), e_3)$ sends v to w and maps $T_{e_3}M_E$ onto a 2-dimensional space Q which is complementary to $\mathbb{R}w$. The projection P_j maps the complementary space Q onto the complementary space $T_x M_I$. □

5 Inner maps

We begin with the travel time from the strip $\{x \in M_I : 0 < x_3 < 1\}$ to the plane M_E.

Proposition 5.1. *Assume $0 < \eta < e^\sigma$, $e^\sigma + \eta < 1$, and $0 < \epsilon < \min\{\epsilon(\eta), \epsilon_B\}$. Let $x \in M_I$ with $0 < x_3 < 1$ be given. Then there exists $t = \tau_\epsilon(x) > 0$ with $0 < F_{\epsilon,3}(s, x) < 1$ for $0 \le s < t$ and $F_{\epsilon,3}(t, x) = 1$.*

Proof. 1. We show $|F_{\epsilon,3}(s,x)| > 1$ for some $s > 0$. Suppose this is false. Then Proposition 2.3 yields $F_\epsilon(n,x) \in B_1$ for all integers $n \geq 0$. Using an estimate from Proposition 2.3 and $1-\eta > e^\sigma$ we get

$$|F_{\epsilon,3}(n,x)| = |P_U F_\epsilon(n,x)| \geq e^{un}(1-\eta)^{n+1}|P_U x| \geq e^{(u+\sigma)n}e^\sigma|x_3| > 0$$

for all integers $n \geq 0$, and the hypothesis $u+\sigma > 0$ yields a contradiction to the assumption that $|F_{\epsilon,3}(s,x)|$, $s \geq 0$, is bounded.

2. It follows that

$$0 < \inf\{s \geq 0 : |F_{\epsilon,3}(s,x)| \geq 1\} < \infty.$$

Set $t = \tau_\epsilon(x) = \inf\{s \geq 0 : |F_{\epsilon,3}(s,x)| \geq 1\}$. Then $|F_{\epsilon,3}(s,x)| < 1$ for $0 \leq s < t$, and by continuity, $|F_{\epsilon,3}(t,x)| = 1$. Let $n = n_\epsilon(x)$ denote the largest integer in $[0,t]$. Proposition 2.3 yields $F_\epsilon(j,x) \in B_1$ for $j = 0,\ldots,n$. By $0 < x_3$, $x \in B_1 \setminus L$. Using Proposition 2.1 (ii) we infer $F_\epsilon(s,x) \in B \setminus L$ on $[0,t]$, hence $|F_{\epsilon,3}(s,x)| > 0$ on $[0,t]$. Finally, $0 < x_3$ and continuity combined yield $0 < F_{\epsilon,3}(s,x)$ on $[0,t]$. □

In order to use the travel time $\tau_\epsilon(x)$ of Proposition 5.1 for an *inner map* with values $F_\epsilon(\tau_\epsilon(x),x)$ in the domain $M_E(r_j)$ of an exterior map E_j we observe that due to hyperbolic behavior of flowlines close to the stationary point 0 the relation $F_\epsilon(\tau_\epsilon(x),x) \in M_E(r_j)$ should hold for $x \in M_I$ with $0 < x_3$ sufficiently small. Recall ϵ_j from Proposition 4.1 and r_j from Corollary 4.3.

Proposition 5.2. *Assume $0 < \eta < e^\sigma$ and $1 + \eta e^{-\sigma} < e^{-\sigma/2}$. Consider an integer j so large that*

$$\epsilon_j < \min\{\epsilon(\eta), \epsilon_B\}.$$

For every $x \in M_I$ with $0 < x_3 < 1$ the largest integer $n = n_j(x)$ in $[0, \tau_{\epsilon_j}(x))$ satisfies

$$n > \frac{1}{u+\log(2)} \log\left(\frac{1}{x_3}\right) - 1. \tag{5.1}$$

For $\delta_j^* \in (0,1)$ so small that

$$\left(1 + \frac{2}{\sigma}\log(r_j)\right) < \frac{1}{u+\log(2)} \log\left(\frac{1}{\delta_j^*}\right) - 1. \tag{5.2}$$

we have

$$|F_{\epsilon_j}(\tau_{\epsilon_j}(x),x) - e_3| = |P_L F_{\epsilon_j}(\tau_{\epsilon_j}(x),x)| < r_j \quad \text{for all } x \in M_I \text{ with } 0 < x_3 < \delta_j^*.$$

Proof. 1. On (5.1). Notice that we have $e^\sigma + \eta < 1$, so that the hypothesis concerning $\eta > 0$ in Proposition 5.1 is satisfied. Let $x \in M_I$ with $0 < x_3 < 1$ be given and let $t = \tau_{\epsilon_j}(x)$. We derive an estimate of the largest integer $n = n_j(x)$ in $[0,t)$. Proposition 5.1 in combination with Proposition 2.3, both for $\epsilon = \epsilon_j$, yield

$$\begin{aligned} 1 = |P_U F_{\epsilon_j}(t,x)| &\leq e^{ut}(1+\eta)^{n+1}|P_U x| = e^{ut}(1+\eta)^{n+1}x_3 \\ &< (2e^u)^{n+1}x_3 \quad (\text{with } t \leq n+1 \text{ and } \eta < 1), \end{aligned}$$

hence

$$n > \frac{1}{u+\log(2)} \log\left(\frac{1}{x_3}\right) - 1.$$

2. Now consider $\delta_j^* \in (0,1)$ which satisfies the inequality (5.1), and let $x \in M_I$ with $0 < x_3 < \delta_j^*$ be given. Using $|P_L x| = 1$ from $x \in M_I$ and an upper estimate from Proposition 2.3 we have, with $n = n_j(x)$,

$$|P_L F_{\epsilon_j}(t,x)| \le e^{\sigma t}(1+\eta e^{-\sigma})^{n+1}|P_L x| \le e^{\sigma\cdot n} e^{(-\sigma/2)(n+1)} = e^{(\sigma/2)(n-1)}.$$

Thereby the desired inequality $|P_L F_{\epsilon_j}(t,x)| < r_j$ follows from $e^{(\sigma/2)(n-1)} < r_j$ which is equivalent to

$$n > 1 + \frac{2}{\sigma}\log(r_j).$$

The preceding equation follows from the lower estimate (5.1) of $n = n_j(x)$ in Part 1 in combination with $x_3 < \delta_j^*$ and with the smallness assumption (5.2) on δ_j^*. □

Next we use transversality of the flow at points of $M_E(r_j)$ as prepared in Corollary 4.3 in order to obtain smoothness of the travel time.

Corollary 5.3. *Let η, j, and δ_j^* satisfy the hypotheses of Proposition 5.2. Then the map*

$$\tau_j^* : \{x \in M_I : 0 < x_3 < \delta_j^*\} \to (0,\infty), \quad \tau_j^*(x) = \tau_{\epsilon_j}(x),$$

is continuously differentiable.

Proof. 1. Continuity. Let $x \in M_I$ with $0 < x_3 < \delta_j^*$ be given, and let $t = \tau_j^*(x) = \tau_{\epsilon_j}(x)$. Let $y = F_{\epsilon_j}(t,x)$ and observe that because of $F_{\epsilon_j,3}(s,x) < 1 = F_{\epsilon_j,3}(t,x)$ for $0 \le s < t$ we have $\partial_1 F_{\epsilon_j,3}(0,y) = \partial_1 F_{\epsilon_j,3}(t,x) \ge 0$. Recall $y \in M_E(r_j)$ from Proposition 5.2. The relation $\partial_1 F_{\epsilon_j}(0,y) \notin L$ from Corollary 4.3 yields $\partial_1 F_{\epsilon_j,3}(0,y) \ne 0$. We conclude that $\partial_1 F_{\epsilon_j,3}(t,x) = \partial_1 F_{\epsilon_j,3}(0,y) > 0$. Now let $\rho \in (0,t)$ be given. Then for some $s \in (t, t+\rho)$, $F_{\epsilon_j,3}(s,x) > 1$. By continuity there is a neighbourhood N_1 of x in $\mathbb{R}^3$ with $F_{\epsilon_j,3}(s,z) > 1$ for all $z \in N_1$. By continuity and compactness we also find a neighbourhood $N \subset N_1$ of x in $\mathbb{R}^3$ so that for all $z \in N$ and for all $s \in [0, t-\rho]$ we have $0 < F_{\epsilon_j,3}(s,z) < 1$. It follows that for all $z \in N \cap \{\xi \in M_I : 0 < \xi_3 < \delta_j^*\}$ we have $t - \rho < \tau_{\epsilon_j}(z) < s < t + \rho$, which yields continuity of the map τ_j^* at x.

2. We show that locally the map τ_j^* is given by continuously differentiable maps. Let $x \in M_I$ with $0 < x_3 < \delta_j^*$ be given and let $t = \tau_j^*(x)$. Then $F_{\epsilon_j,3}(t,x) = 1$, and $\partial_1 F_{\epsilon_j,3}(t,x) > 0$, see Part 1. The Implicit Function Theorem yields an open neighbourhood N of x in $\mathbb{R}^3$ and $\rho > 0$ and a continuously differentiable map $\tau : N \to (t-\rho, t+\rho)$ with $F_{\epsilon_j,3}(\tau(z),z) = 1$ for all $z \in N$, and on $(t-\rho, t+\rho) \times N$,

$$F_{\epsilon_j,3}(s,z) = 1 \quad \text{if and only if} \quad s = \tau(z).$$

By continuity according to Part 1, there is an open neighbourhood $N_1 \subset N$ of x in $\mathbb{R}^3$ so that for all $z \in N_1 \cap \{\xi \in M_I : 0 < \xi_3 < \delta_j^*\}$ we have $t - \rho < \tau_j^*(z) < t + \rho$. Recall $F_{\epsilon_j,3}(\tau_j^*(z),z) = F_{\epsilon_j,3}(\tau_{\epsilon_j}(z),z) = 1$ for all $z \in M_I$ with $0 < z_3 < \delta_j^*$. It follows that on $N_1 \cap \{\xi \in M_I : 0 < \xi_3 < \delta_j^*\}$ we have $\tau_j^*(z) = \tau(z)$. The restriction of τ to $N_1 \cap \{\xi \in M_I : 0 < \xi_3 < \delta_j^*\}$ is a continuously differentiable function on the open subset $N_1 \cap \{\xi \in M_I : 0 < \xi_3 < \delta_j^*\}$ of the submanifold M_I. □

In the sequel we arrange for an inner map on a subset of $\{x \in M_I : 0 < x_3 < \delta_j^*\}$ which can be estimated in the same way as its counterpart in [8]. Upon that we will be able to follow [8] in proving existence of chaotic motion.

For $0 < \eta < \frac{e^\sigma}{2}$ we abbreviate

$$m(\eta,\sigma) = \max\left\{\log(1+\eta e^{-\sigma}), \log\left(\frac{1}{1-\eta e^{-\sigma}}\right), \arcsin\left(\frac{\eta}{e^\sigma - \eta}\right)\right\}$$

and consider the following hypotheses.

For $\tilde\eta > 0$ given,

$$0 < \eta < \frac{e^\sigma}{2} \quad \text{and} \quad 1 + \eta e^{-\sigma} < e^{-\sigma/2} \tag{5.3}$$

(observe that this yields $\eta < e^\sigma$ and $e^\sigma + \eta < e^{\sigma/2} < 1$)
and

$$m(\eta,\sigma) < \tilde\eta. \tag{5.4}$$

$j \in \mathbb{N}$ is so large that

$$\epsilon_j < \min\{\epsilon(\eta), \epsilon_B\}. \tag{5.5}$$

$\delta_j^* \in (0,1)$ satisfies (5.2) and $\delta_j \in (0, \delta_j^*)$ is so small that

$$\frac{1}{u+\log(2)} \log\left(\frac{1}{\delta_j}\right) - 1 > \frac{m(\eta,\sigma)}{\tilde\eta - m(\eta,\sigma)}. \tag{5.6}$$

Proposition 5.4. *Let $\tilde\eta > 0$ be given. Assume that the relations (5.3)–(5.6) hold for $\eta > 0$, $j \in \mathbb{N}$, and δ_j. For every $x \in M_I$ with $0 < x_3 < \delta_j$ the largest integer $n = n_j(x)$ in $[0, \tau_j^*(x))$ satisfies*

$$\frac{n+1}{n} m(\eta,\sigma) < \tilde\eta.$$

Proof. Using the relations (5.1) and (5.6) we get

$$n > \frac{1}{u+\log(2)} \log\left(\frac{1}{x_3}\right) - 1 > \frac{1}{u+\log(2)} \log\left(\frac{1}{\delta_j}\right) - 1 > \frac{m(\eta,\sigma)}{\tilde\eta - m(\eta,\sigma)}$$

which yields $\frac{n+1}{n} m(\eta,\sigma) < \tilde\eta$. □

We are ready for the definition of the inner map I_j, given $\tilde\eta > 0$ and $\eta > 0$, $j \in \mathbb{N}$, and δ_j which satisfy the relations (5.3)–(5.6). As a domain for I_j we take

$$M_{I,j} = \{x \in M_I : 0 < x_3 < \delta_j \quad \text{and} \quad P_L x \neq -E_j(e_3)\}.$$

Here the line given by $P_L x = -E_j(e_3) = -h_{\epsilon_j}(t_{I,j})$ is excluded for later use, in order to have a global parametrization of $M_{I,j}$ available. With $\tau_j(x) = \tau_j^*(x)$ on $M_{I,j}$ we set

$$I_j(x) = F_{\epsilon_j}(\tau_j(x), x)$$

and obtain a continuously differentiable map $I_j : M_{I,j} \to M_E$, with values in $M_E(r_j)$ according to Proposition 5.1. Compare Figure 5.1, top.

Corollary 5.5. *Let $\tilde\eta > 0$ and assume that $\eta > 0$, $j \in \mathbb{N}$, and δ_j satisfy the relations (5.3)–(5.6). Let $x \in M_{I,j}$. Then we have*

$$e^{(\sigma-\tilde\eta)\tau_j(x)} \le |P_L I_j(x)| \le e^{(\sigma+\tilde\eta)\tau_j(x)},$$
$$e^{(u-\tilde\eta)\tau_j(x)} x_3 \le 1 \le e^{(u+\tilde\eta)\tau_j(x)} x_3.$$

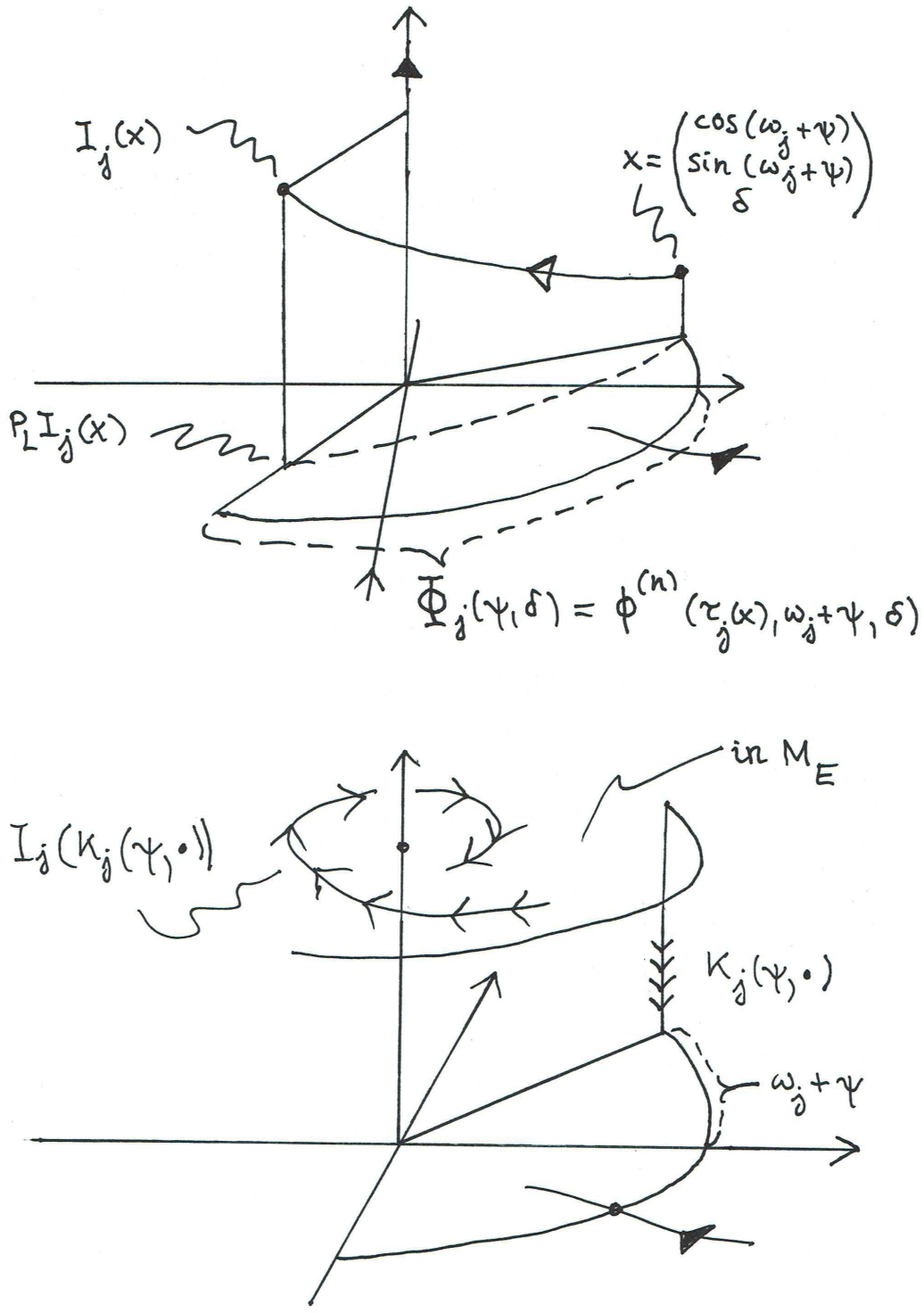


Figure 5.1: Top: The inner map and related angles. Bottom: The inner map along vertical line segments.

For $\psi \in \mathbb{R}$ with

$$x = \begin{pmatrix} \cos(\psi) \\ \sin(\psi) \\ x_3 \end{pmatrix}$$

and $\delta = x_3$ the function $\phi^{(n)}$ obtained in Proposition 3.2, with $n = n_j(x)$, satisfies

$$\psi - \tau_j(x)(\mu + \tilde{\eta}) \le \phi^{(n)}(\tau_j(x), \psi, \delta) \le \psi - \tau_j(x)(\mu - \tilde{\eta}).$$

Proof. Proposition 5.4 shows that the hypotheses on the integer n in Proposition 2.4 and Corollary 3.3 are satisfied for $n = n_j(x)$. Apply Proposition 2.4 to $t = \tau_j(x) - n_j(x) \in [0,1]$, with $|P_L x| = 1$ and $|P_U x| = x_3$ and $P_U I_j(x) = e_3$. This yields the estimates of $P_L I_j(x)$ and $P_U I_j(x)$.

From $|P_L x| = 1$, $x \in B_1 \setminus U$. An application of Corollary 3.3 to $t = \tau_j(x) - n_j(x) \in [0,1]$ yields the estimate of the angle function $\phi^{(n)}$ with $n = n_j(x)$. □

Corollary 5.6. *Let $\tilde{\eta} > 0$ be given with $\tilde{\eta} < u$ and assume that $\eta > 0$, $j \in \mathbb{N}$, and δ_j satisfy the relations (5.3)–(5.6). For every $x \in M_{I,j}$,*

$$\frac{1}{u + \tilde{\eta}} \log\left(\frac{1}{x_3}\right) \le \tau_j(x) \le \frac{1}{u - \tilde{\eta}} \log\left(\frac{1}{x_3}\right).$$

6 The return map in the plane

We begin with parametrizations of the open subsets $\{x \in M_I : P_L x \neq -E_j(e_3)\}$ of the submanifold M_I. Let $j \in \mathbb{N}$ be given. Consider $\omega_j \in [-\pi, \pi)$ determined by

$$\begin{pmatrix} \cos(\omega_j) \\ \sin(\omega_j) \\ 0 \end{pmatrix} = E_j(e_3) \quad (= h_{\epsilon_j}(t_{I,j})).$$

The map

$$K_j : (-\pi, \pi) \times \mathbb{R} \to \{x \in M_I : P_L x \neq -E_j(e_3)\}, \quad K_j(\psi, \delta) = \begin{pmatrix} \cos(\omega_j + \psi) \\ \sin(\omega_j + \psi) \\ \delta \end{pmatrix},$$

is a continuously differentiable diffeomorphism with $K_j(0,0) = E_j(e_3)$. The return map $R_j = E_j \circ I_j$ sends its domain $M_{I,j} = \{x \in M_I : 0 < x_3 < \delta_j, P_L x \neq -E_j(e_3)\}$ into the set $\{z \in M_I : P_L z \neq -E_j(e_3)\}$ which equals the image $K_j((-\pi, \pi) \times \mathbb{R})$. By the *return map in the plane* we mean the continuously differentiable map

$$Q_j : (-\pi, \pi) \times (0, \delta_j) \to (-\pi, \pi) \times \mathbb{R}$$

given by $Q_j(\psi, \delta) = K_j^{-1}(R_j(K_j(\psi, \delta)))$.

We also need information about a coordinate representation of the exterior map E_j alone. Corollary 4.4 yields that the derivative $T_{e_3}(K_j^{-1} \circ E_j)$ is an isomorphism from $L = T_{e_3} M_E$ onto the plane $\mathbb{R}^2$. So it sends basis vectors v_j and w_j of L to the vectors $\binom{1}{0}$ and $\binom{0}{1}$, respectively.

Let κ_j denote the isomorphism $L \to \mathbb{R}^2$ given by

$$\kappa_j(\xi v_j + \eta w_j) = \begin{pmatrix} \xi \\ \eta \end{pmatrix}.$$

The restriction P_j of $\kappa_j \circ P_L$ to the open subset $M_E(r_j)$ of the submanifold M_E defines a continuously differentiable diffeomorphism onto an open neighbourhood of $0 = P_j(e_3)$ in $\mathbb{R}^2$. Obviously,

$$P_j(e_3 + \xi v_j + \eta w_j) = \begin{pmatrix} \xi \\ \eta \end{pmatrix} \quad \text{for all reals } \xi, \eta.$$

As $T_{e_3}P_j$ and $T_{e_3}(K_j^{-1} \circ E_j)$ act the same on the basis v_j, w_j of $T_{e_3}M_E = L$ the *exterior map in coordinates*

$$K_j^{-1} \circ E_j \circ P_j^{-1} : \mathbb{R}^2 \supset P_j(M_E(r_j)) \to \mathbb{R}^2$$

satisfies

$$D(K_j^{-1} \circ E_j \circ P_j^{-1})(0) = \mathrm{id}_{\mathbb{R}^2}. \tag{6.1}$$

Corollary 6.1. *Let $j \in \mathbb{N}$ and $\beta > 0$ be given. There exists $\alpha_j = \alpha_j(\beta) \in (0, \pi)$ so that for all $(\psi, \delta) \in [-\alpha_j, \alpha_j] \times [-\alpha_j, \alpha_j]$ we have $(\psi, \delta) \in P_j(M_E(r_j))$, and*

$$|(K_j^{-1} \circ E_j \circ P_j^{-1})(\psi, \delta) - (\psi, \delta)| \le \beta |(\psi, \delta)|, \tag{6.2}$$

$$|D(K_j^{-1} \circ E_j \circ P_j^{-1})(\psi, \delta) - id_{\mathbb{R}^2}| \le \beta. \tag{6.3}$$

We proceed to estimates of the range of the *inner map in coordinates*

$$(-\pi, \pi) \times (0, \delta_j) \ni (\psi, \delta) \mapsto P_j(I_j(K_j(\psi, \delta))) \in \mathbb{R}^2$$

and of the return map in the plane Q_j.

Proposition 6.2. *Assume $0 < \tilde{\eta} < -\sigma/2$ and consider $\eta > 0, j \in \mathbb{N}, \delta_j$ so that the relations (5.3)–(5.6) hold. Let $\beta \in (0, \frac{1}{2}]$ be given. Consider $\alpha_j = \alpha_j(\beta) > 0$ according to Corollary 6.1, with $\alpha_j < \delta_j$. Then*

$$\delta_{\beta,j} = \left(\frac{2}{3(|\kappa_j| + 1)} \alpha_j \right)^{\frac{3u}{-\sigma}}$$

satisfies $\delta_{\beta,j} \le \frac{2}{3}\alpha_j$, and for all $(\psi, \delta) \in [-\alpha_j, \alpha_j] \times (0, \delta_{\beta,j}]$ we have

$$|P_j(I_j(K_j(\psi, \delta)))| \le \frac{2}{3}\alpha_j, \tag{6.4}$$

$$Q_j(\psi, \delta) \in [-\alpha_j, \alpha_j] \times [-\alpha_j, \alpha_j]. \tag{6.5}$$

Proof. 1. In order to show $\delta_{\beta,j} \le \frac{2}{3}\alpha_j$ notice first that

$$\delta_{\beta,j} < \alpha_j < \delta_j < 1.$$

The hypothesis on $\tilde{\eta}$ yields

$$\frac{u + \tilde{\eta}}{-\sigma - \tilde{\eta}} < \frac{\frac{3u}{2}}{\frac{-\sigma}{2}} = \frac{3u}{-\sigma}.$$

It follows that

$$\delta_{\beta,j} = \left(\frac{2}{3(|\kappa_j| + 1)} \alpha_j \right)^{\frac{3u}{-\sigma}} \le \left(\frac{2}{3(|\kappa_j| + 1)} \alpha_j \right)^{\frac{u+\tilde{\eta}}{-\sigma-\tilde{\eta}}},$$

hence

$$\delta_{\beta,j}^{\frac{-\sigma-\tilde{\eta}}{u+\tilde{\eta}}} \le \frac{2}{3(|\kappa_j| + 1)} \alpha_j.$$

Consequently, with $0 < -\sigma - \tilde{\eta} < u + \tilde{\eta}$ and $\delta_{\beta,j} < 1$,

$$\begin{aligned} \delta_{\beta,j} \le \delta_{\beta,j}^{\frac{-\sigma-\tilde{\eta}}{u+\tilde{\eta}}} &\le \frac{2}{3(|\kappa_j|+1)}\alpha_j \\ &\le \frac{2}{3}\alpha_j. \end{aligned} \tag{6.6}$$

2. From Corollary 6.1 with $0 < \beta \le 1/2$ we get $\alpha_j < \delta_j$ so that for all $(\tilde{\psi}, \tilde{\delta}) \in [-\alpha_j, \alpha_j] \times [-\alpha_j, \alpha_j]$, we have

$$|(K_j^{-1} \circ E_j \circ P_j^{-1})(\tilde{\psi}, \tilde{\delta})| \le (1+\beta)|(\tilde{\psi}, \tilde{\delta})| \le \frac{3}{2}|(\tilde{\psi}, \tilde{\delta})|.$$

3. Proof of (6.4). For $(\psi, \delta) \in [-\alpha_j, \alpha_j] \times (0, \delta_{\beta,j}]$ we have $K_{j,3}(\psi, \delta) = \delta \in (0, \delta_{\beta,j}] \subset (0, \delta_j)$. With $x = K_j(\psi, \delta)$, Corollary 5.5 yields

$$|P_j(I_j(K_j(\psi, \delta)))| = |\kappa_j P_L I_j(x)| \le |\kappa_j| e^{(\sigma+\tilde{\eta})\tau_j(x)}.$$

Notice that $\tilde{\eta} < u$. Using the lower estimate of $\tau_j(x)$ from Corollary 5.6 we infer

$$\begin{aligned} |P_j(I_j(K_j(\psi, \delta)))| &\le |\kappa_j| \left(\frac{1}{x_3}\right)^{\frac{\sigma+\tilde{\eta}}{u+\tilde{\eta}}} \\ &\le |\kappa_j| \delta_{\beta,j}^{\frac{-\sigma-\tilde{\eta}}{u+\tilde{\eta}}} \le \frac{2}{3}\alpha_j \quad \text{(with (6.6))}. \end{aligned}$$

4. Proof of (6.5). For (ψ, δ) as in Part 3 let $(\tilde{\psi}, \tilde{\delta}) = (P_j(I_j(K_j)))(\psi, \delta)$. Then

$$|(\tilde{\psi}, \tilde{\delta})| \le \frac{2}{3}\alpha_j,$$

hence $(\tilde{\psi}, \tilde{\delta}) \in [-\alpha_j, \alpha_j] \times [-\alpha_j, \alpha_j]$, which according to Part 2 yields

$$|(K_j^{-1} \circ E_j \circ P_j^{-1})(\tilde{\psi}, \tilde{\delta})| \le \frac{3}{2}|(\tilde{\psi}, \tilde{\delta})|.$$

It follows that

$$\begin{aligned} |Q_j(\psi, \delta)| &= |K_j^{-1}(R_j(K_j(\psi, \delta)))| = |(K_j^{-1} \circ E_j \circ P_j^{-1} \circ P_j \circ I_j)(K_j(\psi, \delta))| \\ &= |(K_j^{-1} \circ E_j \circ P_j^{-1})(\tilde{\psi}, \tilde{\delta})| \\ &\le \frac{3}{2}|(\tilde{\psi}, \tilde{\delta})| \le \alpha_j. \end{aligned}$$

Finally, use that the disk of radius α_j and center $0 \in \mathbb{R}^2$ is contained in the square $[-\alpha_j, \alpha_j] \times [-\alpha_j, \alpha_j]$. □

The last result of this section concerns continuity of the angle corresponding to $P_L I_j(K_j(\psi, \delta))$, as a function of $(\psi, \delta) \in (-\pi, \pi) \times (0, \delta_j)$.

Proposition 6.3. *Assume $0 < \tilde{\eta} < -\sigma/2$ and consider $\eta > 0, j \in \mathbb{N}, \delta_j$ so that the relations (5.3)–(5.6) hold. Then the function $\Phi_j : (-\pi, \pi) \times (0, \delta_j) \to \mathbb{R}$ given by*

$$\Phi_j(\psi, \delta) = \phi^{(n)}(\tau_j(x), \omega_j + \psi, \delta)$$

with $\phi^{(n)}$ according to Proposition 3.2, $n = n_j(x)$ the largest integer in $[0, \tau_j(x))$ and $x = K_j(\psi, \delta)$, satisfies

$$\frac{1}{|P_L I_j(K_j(\psi,\delta))|} P_L I_j(K_j(\psi,\delta)) = \begin{pmatrix} \cos(\Phi_j(\psi,\delta)) \\ \sin(\Phi_j(\psi,\delta)) \\ 0 \end{pmatrix} \quad \textit{for all} \quad (\psi,\delta) \in (-\pi,\pi) \times (0,\delta_j) \quad (6.7)$$

and is continuous.

Proof. 1. The definition of Φ_j makes sense because for $(\psi,\delta) \in (-\pi,\pi) \times (0,\delta_j)$ and $x = K_j(\psi,\delta)$ we have $|P_U F_{\epsilon_j}(t,x)| < 1$ on $[0,\tau_j(x))$, hence $|P_U F_{\epsilon_j}(\nu,x)| < 1$ for $\nu = 0,\dots,n_j(x)$, which in combination with $n_j(x) < \tau_j(x) \le n_j(x)+1$ yields $(\tau_j(x), \omega_j + \psi, \delta) \in \mathrm{dom}_{n_j(x)}$.

Eq. (6.7) holds because for $\phi = \phi^{n_j(x)}(\tau_j(x), \omega_j + \psi, \delta) = \Phi_j(\psi,\delta)$ we have

$$\frac{1}{|P_L I_j(K_j(\psi,\delta))|} P_L I_j(x) = \frac{1}{|P_L F_{\epsilon_j}(\tau_j(x),x)|} P_L F_{\epsilon_j}(\tau_j(x),x) = \begin{pmatrix} \cos(\phi) \\ \sin(\phi) \\ 0 \end{pmatrix}$$

due to Proposition 3.2 (i).

2. Proof that Φ_j is continuous at $(\psi,\delta) \in (-\pi,\pi) \times (0,\delta_j)$. Let $x = K_j(\psi,\delta)$. We have $n_j(x) < \tau_j(x) \le n_j(x)+1 < n_j(x)+2$. Let a sequence $(\psi_m, \delta_{(m)})_{m\in\mathbb{N}}$ in $(-\pi,\pi) \times (0,\delta_j)$ be given which converges to (ψ,δ). It is enough to find a subsequence so that $\Phi_j(\psi_{m_\mu}, \delta_{(m_\mu)}) \to \Phi_j(\psi,\delta)$ as $\mu \to \infty$.

2.1. In case $\tau_j(x) < n+1$ the continuity of τ_j yields an integer $m_x \ge 0$ so that for all indices $m \ge m_x$ we have $n_j(x) < \tau_j(x_m) < n_j(x)+1$ for $x_m = K_j(\psi_m, \delta_{(m)})$. Hence $n_j(x) = n_j(x_m)$ for $m \ge m_x$. Consequently, $(\tau_j(x_m), \omega_j + \psi_m, \delta_{(m)}) \in \mathrm{dom}_{n_j(x_m)} = \mathrm{dom}_{n_j(x)}$ and

$$\Phi_j(\psi_m,\delta_{(m)}) = \phi^{(n_j(x_m))}(\tau_j(x_m), \omega_j + \psi_m, \delta_{(m)}) = \phi^{(n_j(x))}(\tau_j(x_m), \omega_j + \psi_m, \delta_{(m)}) \quad \text{for} \quad m \ge m_x.$$

As $\phi^{(n_j(x))}$ is continuous according to Proposition 3.2 we arrive at

$$\lim_{m\to\infty} \Phi_j(\psi_m,\delta_{(m)}) = \lim_{m\to\infty} \phi^{(n_j(x))}(\tau_j(x_m), \omega_j + \psi_m, \delta_{(m)}) = \phi^{(n_j(x))}(\tau_j(x), \omega_j + \psi, \delta) = \Phi_j(\psi,\delta).$$

2.2. In case $\tau_j(x) = n_j(x)+1$ we have $|P_U F_{\epsilon_j}(n_j(x)+1,x)| = |P_U F_{\epsilon_j}(\tau_j(x),x)| = 1$ in addition to $|P_U F_{\epsilon_j}(\nu,x)| < 1$ for $\nu = 0,\dots,n_j(x)$ and conclude that $(\tau_j(x), \omega_j + \psi, \delta) \in \mathrm{dom}_{n_j(x)+1}$.

By Proposition 3.2 (ii), $\phi^{(n_j(x)+1)}(n_j(x)+1, \omega_j + \psi, \delta) = \phi^{(n_j(x))}(n_j(x)+1, \omega_j + \psi, \delta)$.

We distinguish the subcases that the indices m with $\tau_j(x_m) \le n_j(x)+1$ are bounded or not.

2.2.1. If the indices with $\tau_j(x_m) \le n_j(x)+1$ are unbounded then there is a strictly increasing sequence $(m_\mu)_{\mu\in\mathbb{N}}$ of positive integers with $\tau_j(x_{m_\mu}) \le n_j(x)+1$ for all $\mu \in \mathbb{N}$. As in Part 2.1 we find $\Phi_j(\psi_{m_\mu}, \delta_{(m_\mu)}) \to \Phi_j(\psi,\delta)$ for $\mu \to \infty$.

2.2.2. If there is an upper bound $m_x \in \mathbb{N}$ for the indices m with $\tau_j(x_m) \le n_j(x)+1$ then $n_j(x)+1 < \tau_j(x_m)$ for all indices $m > m_x$. In addition we may assume $\tau_j(x_m) < n_j(x)+2$ for all $m > m_x$. It follows that $n_j(x)+1 = n_j(x_m)$ and

$$\Phi_j(\psi_m,\delta_{(m)}) = \phi^{(n_j(x_m))}(\tau_j(x_m), \omega_j + \psi_m, \delta_{(m)}) = \phi^{(n_j(x)+1)}(\tau_j(x_m), \omega_j + \psi_m, \delta_{(m)})$$

for $m > m_x$. Using continuity of $\phi^{(n_j(x)+1)}$ from Proposition 3.2 we find

$$\begin{aligned} \lim_{m\to\infty} \Phi_j(\psi_m,\delta_{(m)}) &= \lim_{m\to\infty} \phi^{(n_j(x)+1)}(\tau_j(x_m), \omega_j + \psi_m, \delta_{(m)}) \\ &= \phi^{(n_j(x)+1)}(\tau_j(x), \omega_j + \psi, \delta) = \phi^{(n_j(x)+1)}(n_j(x)+1, \omega_j + \psi, \delta) \\ &= \phi^{(n_j(x))}(n_j(x)+1, \omega_j + \psi, \delta) = \phi^{(n_j(x))}(\tau_j(x), \omega_j + \psi, \delta) = \Phi_j(\psi,\delta). \quad \square \end{aligned}$$

7 Curves expanded by the return map in the plane

In this section we consider curves $g : [a,b] \to (-\pi,\pi) \times (0,\delta_j)$ which connect level sets $(-\pi,\pi) \times \{\Delta_1\}$ and $(-\pi,\pi) \times \{\Delta_2\}$ with $0 < \Delta_1 < \Delta_2 < \delta_j$. We find subintervals $[a_0,b_0]$ and $[a_1,b_1]$ so that the angle function Φ_j sends $g((a_0,b_0))$ and $g((a_1,b_1))$ into disjoint sets M_0 and M_1, and transport by the return map in the plane Q_j yields two curves which again connect the said level sets. Be aware that the angle function Φ_j is not directly related to the return map in the plane but only to the inner map in coordinates (depicted in Figure 5.1, bottom), which is the first composite of the return map in the plane.

Throughout this section we assume $0 < \tilde{\eta} < \min\{\mu, -\sigma/2\}$, and that $\eta > 0$, $j \in \mathbb{N}$, and $\delta_j \in (0,1)$ satisfy the relations (5.3)–(5.6). We begin with a comparison of angles $\Phi_j(\psi,\delta)$ for arguments in level sets as above. Let

$$\begin{aligned} c = c_{\tilde{\eta}} &= \frac{(u+\tilde{\eta})(\mu+\tilde{\eta})}{(u-\tilde{\eta})(\mu-\tilde{\eta})} \\ &> 1, \\ k = k_{\tilde{\eta}} &= e^{-6\pi\frac{u+\tilde{\eta}}{\mu-\tilde{\eta}}} \\ &< 1. \end{aligned}$$

For $\Delta_2 \in (0,\delta_j)$ we set

$$\begin{aligned} \Delta_1 = \Delta_1(\Delta_2) &= k\Delta_2^c \\ &< \Delta_2. \end{aligned}$$

Proposition 7.1. *Let $\Delta_2 \in (0,\delta_j)$ be given and consider $\Delta_1 = \Delta_1(\Delta_2) = k\Delta_2^c$. Then*

$$4\,\pi \le \Phi_j(\psi,\Delta_2) - \Phi_j(\gamma,\Delta_1) \quad \textit{for all } \psi,\gamma \textit{ in } (-\pi,\pi).$$

Proof. Assume $-\pi < \psi < \pi$, $-\pi < \gamma < \pi$. Recall the definition of Φ_j in Proposition 6.3 and apply the last estimate in Corollary 5.5, and the estimate of the travel time in Corollary 5.6. This yields the inequalities

$$\Phi_j(\psi,\Delta_2) - (\omega_j+\psi) \ge (-\mu-\tilde{\eta}) \cdot \frac{1}{u-\tilde{\eta}} \log\left(\frac{1}{\Delta_2}\right)$$

and

$$\Phi_j(\gamma,\Delta_1) - (\omega_j+\gamma) \le (-\mu+\tilde{\eta}) \cdot \frac{1}{u+\tilde{\eta}} \log\left(\frac{1}{\Delta_1}\right)$$

from which we obtain

$$\begin{aligned} \Phi_j(\psi,\Delta_2) - \Phi_j(\gamma,\Delta_1) &\ge -2\pi + \frac{\mu+\tilde{\eta}}{u-\tilde{\eta}}\log(\Delta_2) - \frac{\mu-\tilde{\eta}}{u+\tilde{\eta}}\log(\Delta_1) \\ &\ge -2\pi + \frac{\mu+\tilde{\eta}}{u-\tilde{\eta}}\log(\Delta_2) - \frac{\mu-\tilde{\eta}}{u+\tilde{\eta}}[\log(k) + c\log(\Delta_2)] \\ &= -2\pi + 6\pi + \left(\frac{\mu+\tilde{\eta}}{u-\tilde{\eta}} - c\frac{\mu-\tilde{\eta}}{u+\tilde{\eta}}\right)\log(\Delta_2) = 4\pi. \end{aligned}$$

□

From here on let also $\beta \in (0,1/2]$ be given and consider $\alpha_j = \alpha_j(\beta) \in (0,\delta_j)$ according to Proposition 6.2. Proposition 7.1 shows that the quantities

$$m_1 = m_1(j,\Delta_1) = \max_{|\gamma|\le\alpha_j} \Phi_j(\gamma,\Delta_1) \quad \text{and} \quad m_2 = m_2(j,\Delta_2) = \min_{|\psi|\le\alpha_j} \Phi_j(\psi,\Delta_2)$$

satisfy $m_1 + 4\pi \le m_2$. Also, there exists $\psi_j \in [m_1+\pi, m_2-\pi]$ with

$$\begin{pmatrix} \cos(\psi_j) \\ \sin(\psi_j) \\ 0 \end{pmatrix} = \frac{1}{|w_j|} w_j.$$

Proposition 7.2 (Angles along curves connecting vertical levels)**.** *Consider $\Delta_2 \in (0,\delta_j)$ and $\Delta_1 = \Delta_1(\Delta_2)$ as above. Let a curve $g : [a,b] \to (-\pi,\pi)\times(0,\delta_j)$ be given with $g_2(b) = \Delta_2$ and $g_2(a) = \Delta_1$. Then there exist $a_0' < b_0' \le a_1' < b_1'$ in $[a,b]$ such that*

$$\Phi_j(g(t)) \in (\psi_j - \pi, \psi_j) \quad \text{on} \quad (a_0', b_0'), \quad \Phi_j(g(a_0')) = \psi_j - \pi, \quad \Phi_j(g(b_0')) = \psi_j,$$

$$\Phi_j(g(t)) \in (\psi_j, \psi_j + \pi) \quad \text{on} \quad (a_1', b_1'), \quad \Phi_j(g(a_1')) = \psi_j, \quad \Phi_j(g(b_1')) = \psi_j + \pi.$$

Compare Figure 7.1.

Proof. 1. We construct a_1' and b_1'. From

$$\Phi_j(g(a)) \le m_1 < m_1 + \pi \le \psi_j \le m_2 - \pi < m_2 \le \Phi_j(g(b))$$

we have

$$\Phi_j(g(a)) \le \psi_j - \pi < \psi_j < \psi_j + \pi \le \Phi_j(g(b)).$$

By continuity, $\psi_j = \Phi_j(g(t))$ for some $t \in (a,b)$. Again by continuity there exists $b_1' \in (t,b]$ with $\Phi_j(g(s)) < \psi_j + \pi$ on $[t,b_1')$ and $\Phi_j(g(b_1')) = \psi_j + \pi$. Upon that, there exists $a_1' \in [t,b_1')$ with $\psi_j < \Phi_j(g(s))$ on $(a_1',b_1']$ and $\Phi_j(g(a_1')) = \psi_j$.

2. The construction of a_0' and b_0' with $b_0' \le a_1'$ is analogous. □

We turn to the position of $Q_j(\psi,\delta)$ for arguments $(\psi,\delta) \in (-\pi,\pi)\times[\Delta_1,\Delta_2]$. A look at Eq. (6.7) in Proposition 6.3 confirms that in the cases

$$\Phi_j(\psi,\delta) = \psi_j - \pi, \quad \Phi_j(\psi,\delta) = \psi_j, \quad \Phi_j(\psi,\delta) = \psi_j + \pi$$

the value $P_L I_j(K_j(\psi,\delta))$ belongs to the rays

$$(0,\infty)(-w_j), \quad (0,\infty)w_j, \quad (0,\infty)(-w_j), \quad \text{respectively,}$$

hence $P_j(I_j(K_j(\psi,\delta)))$ is on the vertical axis in $\mathbb{R}^2$.

Proposition 7.3 (From angles to vertical levels)**.** *Assume in addition to the hypotheses made in the present section up to here that $\tilde\eta$ satisfies*

$$c_{\tilde\eta} \frac{-\sigma+\tilde\eta}{u-\tilde\eta} < 1. \tag{7.1}$$

Let $\beta \in (0,1/2]$ be given and choose reals $\alpha_j = \alpha_j(\beta) \in (0,\delta_j)$ and $\delta_{\beta,j} \in (0,2\alpha_j/3]$ according to Proposition 6.2. Consider $\Delta_2 \in (0,\delta_{\beta,j})$ so small that

$$2\sqrt{2}\Delta_2 < \frac{1}{|\kappa_j^{-1}|} k^{\frac{-\sigma+\tilde\eta}{u-\tilde\eta}} \Delta_2^{c\frac{-\sigma+\tilde\eta}{u-\tilde\eta}}, \quad \text{with} \quad c = c_{\tilde\eta} \quad \text{and} \quad k = k_{\tilde\eta}. \tag{7.2}$$

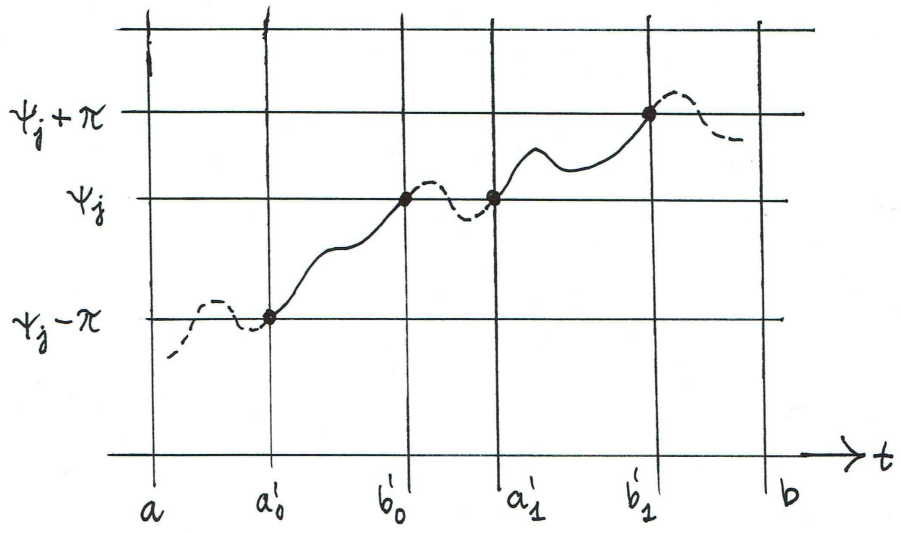


Figure 7.1: The map $\Phi_j \circ g$.

Let $\Delta_1 = \Delta_1(\Delta_2)$, $(\psi, \delta) \in [-\alpha_j, \alpha_j] \times [\Delta_1, \Delta_2]$, *and* $z = Q_j(\psi, \delta)$. *Then*

(i) $|z| > \sqrt{2}\Delta_2$.

(ii) In the cases $\Phi_j(\psi, \delta) = \psi_j - \pi$, $\Phi_j(\psi, \delta) = \psi_j$, $\Phi_j(\psi, \delta) = \psi_j + \pi$, *we have*

$$z_2 < -\Delta_2, \quad z_2 > \Delta_2, \quad z_2 < -\Delta_2, \quad \textit{respectively.}$$

Compare Figure 7.2.

Proof. 1. On assertion (i). Observe first that due to Proposition 6.2 the rectangle $[-\alpha_j, \alpha_j] \times [\Delta_1, \Delta_2]$ is contained in the domain of definition of the maps $P_j(I_j(K_j(\cdot,\cdot)))$ and Q_j. Let $(\psi, \delta) \in [-\alpha_j, \alpha_j] \times [\Delta_1, \Delta_2]$ be given, and let

$$x = P_j(I_j(K_j(\psi, \delta))) \quad (= \kappa_j P_L I_j(K_j(\psi, \delta))).$$

Observe $K_{j,3}(\psi, \delta) = \delta$ and apply Corollaries 5.5 and 5.6. This yields

$$|x| \geq \frac{1}{|\kappa_j^{-1}|}\delta^{\frac{-\sigma+\tilde{\eta}}{u-\tilde{\eta}}} \geq \frac{1}{|\kappa_j^{-1}|}\Delta_1^{\frac{-\sigma+\tilde{\eta}}{u-\tilde{\eta}}} = \frac{1}{|\kappa_j^{-1}|}k^{\frac{-\rho+\tilde{\eta}}{u-\tilde{\eta}}}\Delta_2^{c\frac{-\sigma+\tilde{\eta}}{u-\tilde{\eta}}} > 2\sqrt{2}\Delta_2.$$

We have

$$z = Q_j(\psi, \delta) = (K_j^{-1} \circ E_j \circ I_j)(K_j(\psi, \delta)) = (K_j^{-1} \circ E_j \circ P_j^{-1} \circ P_j \circ I_j)(K_j(\psi, \delta)) = (K_j^{-1} \circ E_j \circ P_j^{-1})(x).$$

The inequality (6.4) in Proposition 6.2 gives us $|x| \leq \frac{2}{3}\alpha_j$, hence $x \in [-\alpha_j, \alpha_j] \times [-\alpha_j, \alpha_j]$. Using the relation (6.2) in Corollary 6.1 we infer $|z - x| \leq \beta|x|$. It follows that

$$|z| \geq |x| - \beta|x| \geq \frac{1}{2}|x| > \sqrt{2}\Delta_2.$$

2. On assertion (ii) for the case $(\psi, \delta) \in [-\alpha_j, \alpha_j] \times [\Delta_1, \Delta_2]$ with $\Phi_j(\psi, \delta) = \psi_j$. Let $z = Q_j(\psi, \delta) \in \mathbb{R}^2$.

2.1. From (6.7) in Proposition 6.3 we have that $P_L I_j(K_j(\psi,\delta))$ is a positive multiple of

$$\begin{pmatrix} \cos(\Phi_j(\psi,\delta)) \\ \sin(\Phi_j(\psi,\delta)) \\ 0 \end{pmatrix} = \begin{pmatrix} \cos(\psi_j) \\ \sin(\psi_j) \\ 0 \end{pmatrix} \in (0,\infty) w_j,$$

hence

$$P_j(I_j(K_j(\psi,\delta))) = \kappa_j P_L I_j(K_j(\psi,\delta)) \in (0,\infty) \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

2.2. From Part 1, $|z| > \sqrt{2}\Delta_2$. For $x = P_j(I_j(K_j(\psi,\delta))) = \kappa_j P_L I_j(K_j(\psi,\delta))$ Part 2.1 yields $x = \left(\begin{smallmatrix} 0 \\ x_2 \end{smallmatrix}\right)$ with $x_2 > 0$.

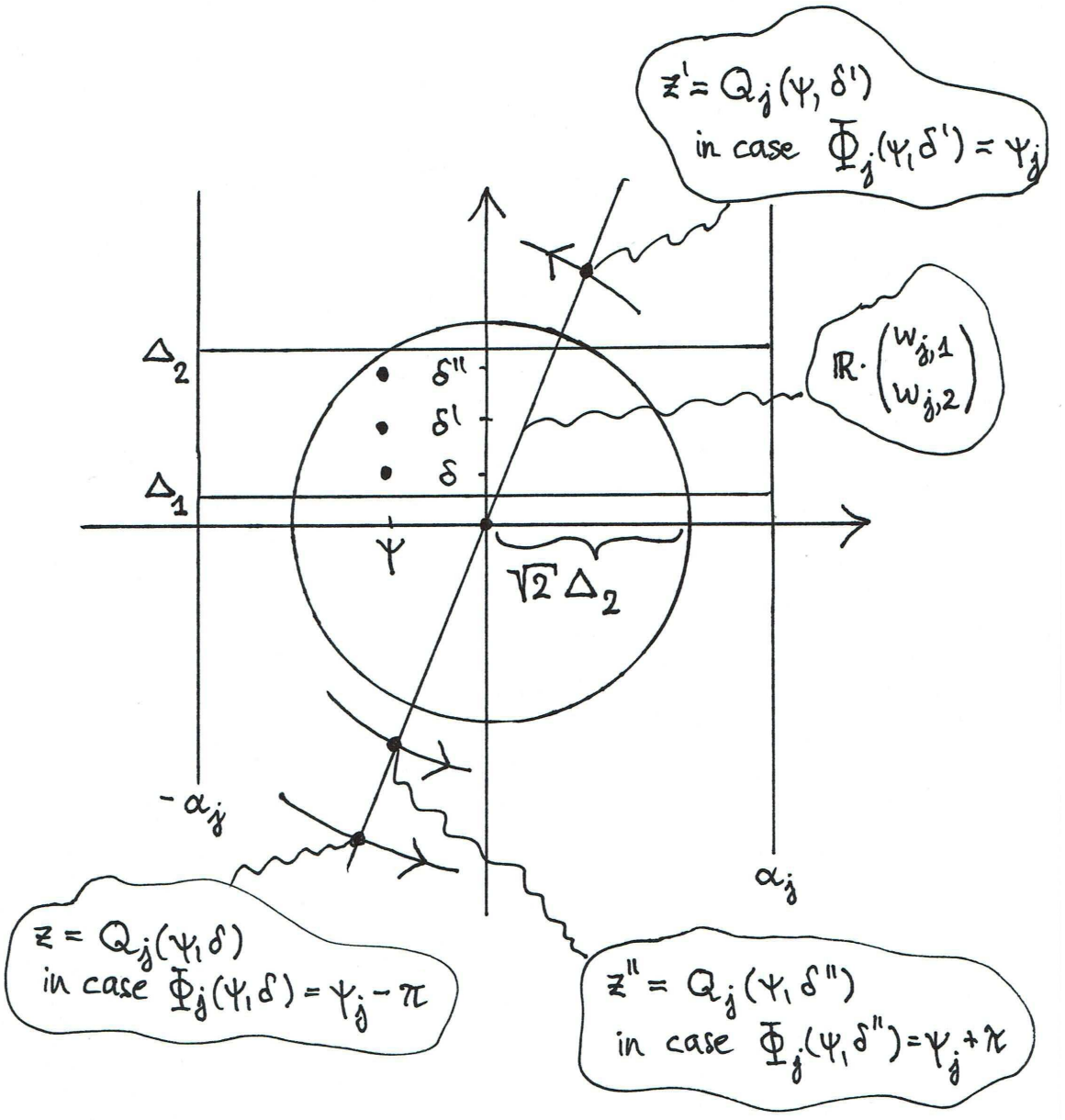


Figure 7.2: Positions of $Q_j(\psi,\delta)$ depending on the angle $\Phi_j(\psi,\delta)$.

2.3. Proof of $|z| \leq \sqrt{2}z_2$: We have $|x| = x_2$. From $x_2 - z_2 \leq |x_2 - z_2| \leq |z - x| \leq \beta|x| = \beta x_2$, $z_2 \geq (1-\beta)x_2 > 0$. Also, from $x_1 = 0$, $|z_1| \leq |x_1| + \beta|x| = \beta x_2$. It follows that

$$|z|^2 = z_1^2 + z_2^2 \leq \beta^2 x_2^2 + z_2^2 \leq \frac{\beta^2}{(1-\beta)^2} z_2^2 + z_2^2 \leq 2z_2^2.$$

2.4. Consequently, $z_2 = |z_2| \geq \frac{1}{\sqrt{2}}|z| > \Delta_2$.

3. The proofs of assertion (ii) in the two remaining cases are analogous, making use of the fact that in both cases we have that $P_j(I_j(K_j(\psi,\delta))) = \kappa_j P_L I_j(K_j(\psi,\delta))$ is a positive multiple of $\left(\begin{smallmatrix} 0 \\ -1 \end{smallmatrix}\right)$. □

The next result makes precise what was briefly announced at the begin of the section. The disjoint sets mentioned there will be given in terms of the angle $\Phi_j(\psi,\delta)$ corresponding to the value $P_j(I_j(K_j(\psi,\delta)))$ of the inner map in coordinates, and not by the position of the value $Q_j(\psi,\delta)$ of the return map in the plane to the left or right of the vertical axis. Our choice of disjoint sets circumvents a discussion how the latter are related to the more accessible angles $\Phi_j(\psi,\delta)$.

Proposition 7.4. *Assume the hypotheses of Proposition 7.3 are satisfied and let $\Delta_1 = \Delta_1(\Delta_2)$. Consider the disjoint sets*

$$M_0 = \{(\psi,\delta) \in [-\alpha_j,\alpha_j] \times [\Delta_1,\Delta_2] : \psi_j - \pi < \Phi_j(\psi,\delta) < \psi_j\}$$

and

$$M_1 = \{(\psi,\delta) \in [-\alpha_j,\alpha_j] \times [\Delta_1,\Delta_2] : \psi_j < \Phi_j(\psi,\delta) < \psi_j + \pi\}.$$

For every curve $g : [a,b] \to [-\alpha_j,\alpha_j] \times [\Delta_1,\Delta_2]$ with $g_2(a) = \Delta_1$ and $g_2(b) = \Delta_2$ there exist $a_0 < b_0 < a_1 < b_1$ in $[a,b]$ such that

$$\begin{aligned} &\text{on} \quad (a_0,b_0), \quad g(t) \in M_0 \quad \text{and} \quad Q_{j,2}(g(t)) \in (\Delta_1,\Delta_2), \\ &\quad \text{with} \quad Q_{j,2}(g(a_0)) = \Delta_1 \quad \text{and} \quad Q_{j,2}(g(b_0)) = \Delta_2, \\ &\text{while on} \quad (a_1,b_1), \quad g(t) \in M_1 \quad \text{and} \quad Q_{j,2}(g(t)) \in (\Delta_1,\Delta_2), \\ &\quad \text{with} \quad Q_{j,2}(g(a_1)) = \Delta_2 \quad \text{and} \quad Q_{j,2}(g(b_1)) = \Delta_1. \end{aligned}$$

Proof. Proposition 7.2 yields $a_0' < b_0' \leq a_1' < b_1'$ in $[a,b]$ such that

$$\begin{aligned} \text{on} \quad &(a_0',b_0'), \quad \Phi_j(g(t)) \in (\psi_j - \pi, \psi_j), \\ &\text{with} \quad \Phi_j(g(a_0')) = \psi_j - \pi \quad \text{and} \quad \Phi_j(g(b_0')) = \psi_j, \\ \text{and on} \quad &(a_1',b_1'), \quad \Phi_j(g(t)) \in (\psi_j, \psi_j + \pi), \\ &\text{with} \quad \Phi_j(g(a_1')) = \psi_j \quad \text{and} \quad \Phi_j(g(b_1'))_2 = \psi_j + \pi. \end{aligned}$$

From Proposition 7.3 (ii),

$$Q_{j,2}(g(a_0')) < -\Delta_2, \quad Q_{j,2}(g(b_0')) > \Delta_2, \quad Q_{j,2}(g(a_1')) > \Delta_2, \quad Q_{j,2}(g(b_1')) < -\Delta_2.$$

As in the proof of Proposition 7.2 one finds $a_0 < b_0$ in (a_0',b_0') and $a_1 < b_1$ in (a_1',b_1') with

$$\begin{aligned} &Q_{j,2}(g(a_0)) = \Delta_1 \quad \text{and} \quad Q_{j,2}(g(b_0)) = \Delta_2, \quad \text{and} \quad Q_{j,2}(g(t)) \in (\Delta_1,\Delta_2) \quad \text{on} \quad (a_0,b_0), \\ &Q_{j,2}(g(a_1)) = \Delta_2 \quad \text{and} \quad Q_{j,2}(g(b_1)) = \Delta_1, \quad \text{and} \quad Q_{j,2}(g(t)) \in (\Delta_1,\Delta_2) \quad \text{on} \quad (a_1,b_1). \end{aligned}$$

Observe that on $(a_0,b_0) \subset (a_0',b_0')$ we have $g(t) \in M_0$ while on $(a_1,b_1) \subset (a_1',b_1')$ we have $g(t) \in M_1$. □

See Figure 7.3.

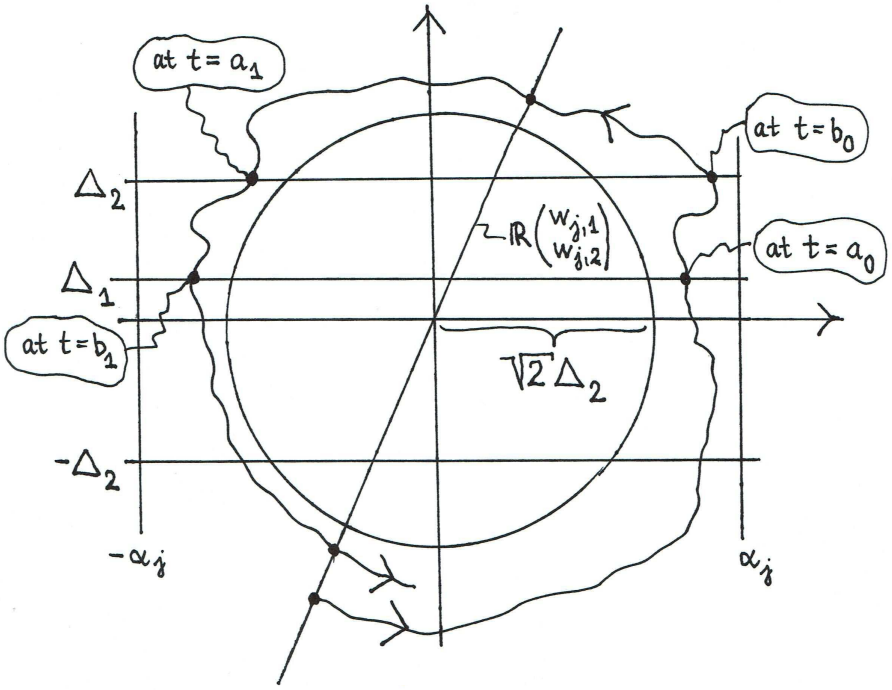


Figure 7.3: The values $Q_j(g(t))$ for $a \le t \le b$.

8 Complicated dynamics

For the results of this section we assume that the hypotheses of Proposition 7.4 are satisfied. It may be convenient to repeat all of these assumption here, beginning with the choice of a real number $\tilde{\eta} > 0$ with

$$\tilde{\eta} < \min\{\mu, -\sigma/2\}$$

which satisfies the inequality (7.1).

The numbers $\eta > 0$, $j \in \mathbb{N}$, and $\delta_j \in (0,1)$ are chosen so that the relations (5.3)–(5.6) hold.

For given $\beta \in (0, 1/2]$, the reals $\alpha_j = \alpha_j(\beta) \in (0, \delta_j)$ and $\delta_{\beta,j} \in (0, 2\alpha_j/3]$ are chosen according to Proposition 6.2.

$\Delta_2 \in (0, \delta_{\beta,j})$ is chosen so that the inequality (7.2) holds, and $\Delta_1 = \Delta_1(\Delta_2)$.

Recall the disjoint sets M_0 and M_1 from Proposition 7.4.

Proposition 8.1. *For every sequence $(s_n)_{n=0}^{\infty}$ in $\{0,1\}$ there are forward trajectories $(x_n)_{n=0}^{\infty}$ of Q_j in $[-\alpha_j, \alpha_j] \times [\Delta_1, \Delta_2]$ with $x_n \in M_{s_n}$ and $\Delta_1 \le Q_{j,2}(x_n) \le \Delta_2$ for all integers $n \ge 0$.*

Proof. 1. Let a sequence $(s_n)_{n=0}^{\infty}$ in $\{0,1\}$ be given. Choose a curve $g : [a,b] \to [-\alpha_j, \alpha_j] \times [\Delta_1, \Delta_2]$ such that $g_2(t) \in (\Delta_1, \Delta_2)$ for $a < t < b$ and $g_2(a) = \Delta_1$, $g_2(b) = \Delta_2$, for example, $g(t) = (0,t)$ for $a = \Delta_1 \le t \le \Delta_2 = b$.

For integers $n \ge 0$ we construct recursively curves $g_n : [A_n, B_n] \to [-\alpha_j, \alpha_j] \times [\Delta_1, \Delta_2]$ with decreasing domains in $[a,b]$ as follows.

1.1. In order to find g_0 we apply Proposition 7.4 to the curve g and obtain $a_0 < b_0 < a_1 < b_1$ in $[a,b]$ with the properties stated in Proposition 7.4. In case $s_0 = 0$ we define g_0 by $A_0 =$

a_0, $B_0 = b_0$, $g_0(t) = g(t)$ for $A_0 \le t \le B_0$. Notice that $g_0(t) \in M_{s_0}$ for all $t \in (A_0, B_0)$, $Q_{j,2}(g_0(t)) \in (\Delta_1, \Delta_2)$ on (A_0, B_0), $Q_{j,2}(g_0(A_0)) = \Delta_1$, and $Q_{j,2}(g_0(B_0)) = \Delta_2$. In case $s_0 = 1$ we define g_0 by $A_0 = a_1$, $B_0 = b_1$, $g_0(t) = g(a_1 + b_1 - t)$ for $A_0 \le t \le B_0$. Notice that also in this case $g_0(t) \in M_{s_0}$ for all $t \in (A_0, B_0)$, $Q_{j,2}(g_0(t)) \in (\Delta_1, \Delta_2)$ on (A_0, B_0), and $Q_{j,2}(g_0(A_0)) = Q_{j,2}(g(a_1 + b_1 - a_1)) = \Delta_1$, $Q_{j,2}(g_0(B_0)) = Q_{j,2}(g(a_1 + b_1 - b_1)) = \Delta_2$.

1.2. For an integer $n \ge 0$ let a curve $g_n : [A_n, B_n] \to [-\alpha_j, \alpha_j] \times [\Delta_1, \Delta_2]$ be given with $g_n(t) \in M_{s_n}$ for all $t \in (A_n, B_n)$ and $Q_{j,2}(g_n(t)) \in (\Delta_1, \Delta_2)$ on (A_n, B_n), $Q_{j,2}(g_n(A_n)) = \Delta_1$, $Q_{j,2}(g_n(B_n)) = \Delta_2$. Proceeding as in Part 1.1, with the curve $[A_n, B_n] \ni t \mapsto Q_j(g_n(t)) \in [-\alpha_j, \alpha_j] \times [\Delta_1, \Delta_2]$ in place of the former curve g, we obtain $A_{n+1} < B_{n+1}$ in $[A_n, B_n]$ and a curve $g_{n+1} : [A_{n+1}, B_{n+1}] \to [-\alpha_j, \alpha_j] \times [\Delta_1, \Delta_2]$ with $g_{n+1}(t) \in M_{s_{n+1}}$ for all $t \in (A_{n+1}, B_{n+1})$ and $Q_{j,2}(g_{n+1}(t)) \in (\Delta_1, \Delta_2)$ on (A_{n+1}, B_{n+1}), $Q_{j,2}(g_{n+1}(A_{n+1})) = \Delta_1$, $Q_{j,2}(g_{n+1}(B_{n+1})) = \Delta_2$.

2. From $A_n \le A_{n+1} < B_{n+1} \le B_n$ for all integers $n \ge 0$ we get $\cap_{n\ge 0}[A_n, B_n] = [A, B]$ with $A = \lim_{n\to\infty} A_n \le \lim_{n\to\infty} B_n = B$. Choose $t \in [A, B]$ and define

$$x_n = g_n(t) \in [-\alpha_j, \alpha_j] \times [\Delta_1, \Delta_2]$$

for every integer $n \ge 0$. This yields a forward trajectory of Q_j with $\Delta_1 \le Q_{j,2}(x_n) \le \Delta_2$ for all integers $n \ge 0$.

Let an integer $n \ge 0$ be given. Proof of $x_n \in M_{s_n}$. We have $x_n = g_n(t)$ with $A_n \le A \le t \le B \le B_n$, and for all $v \in (A_n, B_n)$, $g_n(v) \in M_{s_n}$. By continuity, $x_n \in \operatorname{cl} M_{s_n}$. In case $s_n = 0$ this yields

$$\psi_j - \pi \le \Phi_j(x_n) \le \psi.$$

Assume $\Phi_j(x_n) \in \{\psi - \pi, \psi\}$, Then Proposition 7.3 (ii) gives $|Q_{j,2}(x_n)| > \Delta_2$, which contradicts the previous estimate of $Q_{j,2}(x_n)$. Consequently,

$$\psi_j - \pi < \Phi_j(x_n) < \psi, \quad \text{or,} \quad x_n \in M_0 = M_{s_n}.$$

In case $s_n = 1$ the proof is analogous. □

The final result extends Proposition 8.1 to entire trajectories.

Theorem 8.2. *For every sequence $(s_n)_{n=-\infty}^{\infty}$ in $\{0, 1\}$ there exist entire trajectories $(y_n)_{n=-\infty}^{\infty}$ of Q_j with $y_n \in M_{s_n}$ for all integers n.*

Proof. 1. Let $(s_n)_{n=-\infty}^{\infty}$ in $\{0, 1\}$ be given. Proposition 8.1 guarantees that for every integer k there is a forward trajectory $(y_{k,n})_{n=0}^{\infty}$ of Q_j in $[-\alpha_j, \alpha_j] \times [\Delta_1, \Delta_2]$ so that for each integer $n \ge 0$,

$$y_{k,n} \in M_{s_{n-k}} \quad \text{and} \quad \Delta_1 \le Q_{j,2}(y_{k,n}) \le \Delta_2.$$

For integers k, n with $k \ge -n$ we define

$$z_{k,n} = y_{k,n+k},$$

so that

$$\begin{aligned} z_{k,n} &= y_{k,n+k} \in M_{s_{n+k-k}} = M_{s_n}, \\ z_{k,n+1} &= y_{k,n+1+k} = Q_j(y_{k,n+k}) = Q_j(z_{k,n}), \\ Q_{j,2}(z_{k,n}) &= Q_{j,2}(y_{k,n+k}) \in [\Delta_1, \Delta_2]. \end{aligned}$$

1.1. Choice of subsequences for integers $N \ge 0$.

1.1.1. The case $N = 0$: For every integer $k \geq 0$, $z_{k,0} \in M_{s_0}$. The sequence $(z_{k,0})_{k=0}^{\infty}$ in the compact set $[-\alpha_j, \alpha_j] \times [\Delta_1, \Delta_2]$ has a convergent subsequence $(z_{\kappa_0(k),0})_{k=0}^{\infty}$ given by a strictly increasing map $\kappa_0 : \mathbb{N}_0 \to \mathbb{N}_0$. Let $y_0 = \lim_{k\to\infty} z_{\kappa_0(k),0} \in \text{cl } M_{s_0} \subset [-\alpha_j, \alpha_j] \times [\Delta_1, \Delta_2]$.

1.1.2. The case $N = 1$: We choose a convergent subsequence of $(z_{\kappa_0(k),-1})_{k=0}^{\infty}$ given by a strictly increasing map $\mu_{-1} : \mathbb{N}_0 \to \mathbb{N}_0$, and upon this a convergent subsequence of $(z_{\kappa_0\circ\mu_{-1}(k),1})_{k=0}^{\infty}$ given by a strictly increasing map $\mu_1 : \mathbb{N}_0 \to \mathbb{N}_0$, and define $\kappa_1 = \mu_{-1} \circ \mu_1$. For $k \to \infty$,

$$z_{\kappa_0\circ\kappa_1(k),-0}, \quad z_{\kappa_0\circ\kappa_1(k),-1} \quad \text{and} \quad z_{\kappa_0\circ\kappa_1(k),1}$$

converge to elements $y_0 \in \text{cl } M_{s_0}$, $y_{-1} \in \text{cl } M_{s_{-1}}$, and $y_1 \in \text{cl } M_{s_1}$, respectively.

1.1.3. The general case $N \in \mathbb{N}$: Consecutively choosing further convergent subsequences analogously to Part 1.1.2 we obtain strictly increasing maps $\kappa_n : \mathbb{N}_0 \to \mathbb{N}_0$, $n = 0, \ldots, N$, so that for each $n \in \{-N, \ldots, N\}$ the sequence

$$(z_{\kappa_0\circ\cdots\circ\kappa_N(k),n})_{k=0}^{\infty}$$

converges for $k \to \infty$ to some $y_n \in \text{cl } M_{s_n}$.

2. The *diagonal sequence* $K : \mathbb{N}_0 \to \mathbb{N}_0$ defined by $K(N) = (\kappa_0 \circ \cdots \circ \kappa_N)(N)$ is strictly increasing since for every $N \in \mathbb{N}_0$ we have

$$K(N+1) = (\kappa_0 \circ \cdots \circ \kappa_N)(\kappa_{N+1}(N+1)) > (\kappa_0 \circ \cdots \circ \kappa_N)(\kappa_{N+1}(N)) \geq (\kappa_0 \circ \cdots \circ \kappa_N)(N) = K(N)$$

due to strict monotonicity of all maps involved.

3. Let an integer n be given and set $N = |n|$. Proof that

$$(z_{K(k),n})_{k=N+1}^{\infty} \quad \text{is a subsequence of} \quad (z_{(\kappa_0\circ\cdots\circ\kappa_N)(k),n})_{k=N+1}^{\infty}.$$

Consider the map $\lambda : \{k \in \mathbb{N}_0 : k > N\} \to \mathbb{N}_0$ given by $\lambda(k) = (\kappa_{N+1} \circ \cdots \circ \kappa_k)(k)$. For every integer $k \geq N+1$,

$$K(k) = (\kappa_0 \circ \cdots \circ \kappa_N)(\lambda(k)),$$

and λ is strictly increasing because analogously to Part 2 we have

$$\begin{aligned}\lambda(k+1) &= (\kappa_{N+1} \circ \cdots \circ \kappa_{k+1})(k+1) = (\kappa_{N+1} \circ \cdots \circ \kappa_k)(\kappa_{k+1}(k+1)) \\ &\geq (\kappa_{N+1} \circ \cdots \circ \kappa_k)(k+1) > (\kappa_{N+1} \circ \cdots \circ \kappa_k)(k) = \lambda(k)\end{aligned}$$

for every integer $k \geq N+1$.

Being a subsequence of $(z_{(\kappa_1\circ\cdots\circ\kappa_{|n|})(k),n})_{k=N+1}^{\infty}$ the sequence $(z_{K(k),n})_{k=N+1}^{\infty}$ converges for $k \to \infty$ to $y_n \in \text{cl } M_{s_n}$.

4. We show that $(y_n)_{n=-\infty}^{\infty}$ is an entire trajectory of Q_j. Let an integer n be given and set $N = |n|$. From Part 3 in combination with Part 1.1.3 we get that $(z_{K(k),n})_{k=N+1}^{\infty}$ converges to $y_n \in [-\alpha_j, \alpha_j] \times [\Delta_1, \Delta_2]$ and that $(z_{K(k),n+1})_{k=N+2}^{\infty}$ converges to y_{n+1}. Recall $z_{k,n+1} = Q_j(z_{k,n})$ for all integers k, n with $k \geq -n$. For integers $k > N = |n|$ we have $K(k) \geq k \geq -n$, and the preceding statement yields

$$z_{K(k),n+1} = Q_j(z_{K(k),n}).$$

It follows that

$$y_{n+1} = \lim_{N+2\leq k\to\infty} z_{K(k),n+1} = \lim_{N+1\leq k\to\infty} Q_j(z_{K(k),n}) = Q_j(y_n).$$

5. Proof of $y_n \in M_{s_n}$ for all integers n. Let an integer n be given. We have $y_n \in \text{cl } M_{s_n}$. In case $s_n = 0$ this yields $\psi_j - \pi \leq \Phi_j(y_n) \leq \psi_j$. Therefore the assumption $y_n \notin M_{s_n}$ results in $\Phi_j(y_n) \in \{\psi_j - \pi, \psi_j\}$, which according to Proposition 7.3 (ii) means $|Q_{j,2}(y_n)| > \Delta_2$, in contradiction to $\Delta_1 \leq Q_{j,2}(y_n) \leq \Delta_2$. The proof in case $s_n = 1$ is analogous. □

9 Appendix: From the vectorfield V to the flow F

Consider Shilnikov's scenario according to Section 1, with a continuously differentiable vectorfield $V : \mathbb{R}^3 \supset \mathrm{dom}_V \to \mathbb{R}^3$, dom_V open and $V(0) = 0$, so that there is a homoclinic solution h_V of Eq. (1.1), and the eigenvalues $u > 0$ and $\sigma \pm i\mu$, $\sigma < 0 < \mu$, of $DV(0)$ satisfy

(H) $\qquad 0 < \sigma + u.$

For simplicity assume in addition $\mathrm{dom}_V = \mathbb{R}^3$ and that V is bounded. This is not a severe restriction since we are only interested in solutions close to the compact homoclinic loop $\mathrm{cl}\, h_V(\mathbb{R}) = h_V(\mathbb{R}) \cup \{0\}$, and one can achieve the desired properties by a modification of the vectorfield outside a neighbourhood of $\mathrm{cl}\, h_V(\mathbb{R})$. — Then the solutions of Eq. (1.1) constitute a continuously differentiable flow $F_V : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$. In the sequel we describe how to transform F_V into a flow F with the properties (F1)–(F5) stated in Section 2.

Choose eigenvectors $w \in \mathbb{R}^3$ for the eigenvalue $u > 0$ of $DV(0)$, and $z \in \mathbb{C}^3$ for the eigenvalue $\sigma + i\,\mu$ of the complexification of $DV(0)$. Set $w_1 = \Re z \in \mathbb{R}^3$ and $w_2 = \Im z \in \mathbb{R}^3$. Then $E_s = \mathbb{R}w_1 \oplus \mathbb{R}w_2$ and $E_u = \mathbb{R}w$ are invariant under $DV(0)$. For the isomorphism $B : \mathbb{R}^3 \to \mathbb{R}^3$ given by $Be_1 = w_1, Be_2 = w_2, Be_3 = w$ we have that the linear map $B^{-1}DV(0)B$ is multiplication with the matrix

$$A = \begin{pmatrix} \sigma & \mu & 0 \\ -\mu & \sigma & 0 \\ 0 & 0 & u \end{pmatrix},$$

which leaves $L = B^{-1}E_s$ and $U = B^{-1}E_u$ invariant.

Recall the local stable and unstable manifolds W_s and W_u of Eq. (1.1) at the origin. Given any $\lambda > 0$ we may assume that with some $r = r(\lambda) > 0$ they have the form

$$W_s = \{x + w_s(x) : x \in E_s, |x| < r\} \quad \text{and} \quad W_u = \{y + w_u(y) : y \in E_u, |y| < r\}$$

for continuously differentiable maps

$$w_s : \{x \in E_s : |x| < r\} \to E_u \quad \text{and} \quad w_u : \{y \in E_u : |y| < r\} \to E_s$$

which satisfy $w_s(0) = 0, Dw_s(0) = 0, w_u(0) = 0, Dw_u(0) = 0$, and

$$|Dw_s(x)| < \lambda \quad for \quad x \in E_s \quad \text{with} \quad |x| < r, \quad |Dw_u(y)| < \lambda \quad \text{for} \quad y \in E_u \quad \text{with} \quad |y| < r.$$

For sufficiently small neigbourhoods N of the origin in $\mathbb{R}^3$ the submanifolds W_s and W_u are invariant under F_V in N, and $F_V(t, x) \in N$ for all $t \geq 0$ implies $x \in W_s$ while $F_V(t, x) \in N$ for all $t \leq 0$ implies $x \in W_u$.

There exist $\hat{r} \in (0, r)$ and a continuously differentiable extension $\hat{w}_s : E_s \to E_u$ of the restriction of w_s to $\{x \in E_s : |x| \leq \hat{r}\}$ with $|D\hat{w}_s(x)| < \lambda$ on E_s, and analogously there is a continuously differentiable extension $\hat{w}_u : E_u \to E_s$ of the restriction of w_u to $\{y \in E_u : |y| \leq \hat{r}\}$ with $|D\hat{w}_u(y)| < \lambda$ on E_u. For $\lambda > 0$ sufficiently small the continuously differentiable map

$$S : \mathbb{R}^3 \to \mathbb{R}^3, \quad S(x + y) = x + y - \hat{w}_s(x) - \hat{w}_u(y) \quad \text{for } x \in E_s, y \in E_u,$$

with $S(0) = 0$ and $DS(0) = id_{\mathbb{R}^3}$ satisfies

$$|DS(z) - id_{\mathbb{R}^3}| \leq \frac{1}{2} \quad \text{for all } z \in \mathbb{R}^3.$$

It follows that all derivatives of S are isomorphisms. Using integration along line segments and the previous estimate one shows that S is one-to-one. In order to see that S is onto notice that for every $\zeta \in \mathbb{R}^3$ the map $z \mapsto \zeta - (S(z) - z)$ is a strict contraction, whose fixed point is a preimage of ζ under S. As all derivatives of S are isomorphisms applications of the Inverse Mapping Theorem yield that S^{-1} is continuously differentiable. Altogether, S is a diffeomorphism which maps $\hat{W}_s = \{x + \hat{w}_s(x) : x \in E_s\}$ onto E_s and $\hat{W}_u = \{y + \hat{w}_u(y) : y \in E_u\}$ onto E_u .

Define the continuously differentiable flow $F : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$ by

$$F(t,x) = B^{-1}S(F_V(t, S^{-1}(Bx))).$$

For all $t \in \mathbb{R}$, $F(t,0) = 0$, which is property (F1).

On (F2): Necessarily, $h_V'(t) \neq 0$ for all $t \in \mathbb{R}$. From the properties $h_V(t) \neq 0 \neq h_V'(t)$ for all $t \in \mathbb{R}$ and $\lim_{|t|\to\infty} h_V(t) = 0$ in combination with the fact that $B^{-1} \circ S$ is a diffeomorphism with fixed point 0 we infer that the flowline $h = B^{-1} \circ S \circ h_V$ of F satisfies $h(t) \neq 0 \neq h'(t)$ for all $t \in \mathbb{R}$ and $\lim_{|t|\to\infty} h(t) = 0$.

On (F3): For $t \in \mathbb{R}$ let $T(t) = D_2F(t,0)$. Then

$$T(t) = B^{-1}DS(0)D_2F_V(t, S^{-1}(B0))DS^{-1}(0)B = B^{-1}D_2F_V(t,0)B = B^{-1}e^{t\,DV(0)}B = e^{t\,B^{-1}DV(0)B}.$$

As the linear map $B^{-1}DV(0)B$ is multiplication by the matrix A we get $T(t)x = e^{t\,B^{-1}DV(0)B}x = e^{t\,A} \cdot x$ for all $x \in \mathbb{R}^3$ and $t \in \mathbb{R}$. It follows that $T(t)L \subset L$ and $T(t)U \subset U$ for every $t \in \mathbb{R}$. The representation of $y = T(t)x = e^{t\,A} \cdot x$ described in property (F3) is confirmed by solving the initial value problems

$$x' = Ax, \quad x(0) = e_\nu, \quad \nu = 1,2,3,$$

whose solutions coincide at $t \in \mathbb{R}$ with the columns of the matrix $e^{t\,A}$.

On (F4). We show that for a neighbourhood N of the origin in $\mathbb{R}^3$ so small that W_s is invariant under F_V in N and

$$|x| < \hat{r} \quad \text{for all} \quad x \in E_s \quad \text{and} \quad y \in E_u \quad \text{with} \quad x + y \in N$$

the space L is invariant under F in $\tilde{N} = B^{-1}S(N)$.

Proof. Let $z \in L \cap \tilde{N}$ and $t \in \mathbb{R}$ be given with $F(\tau, z) \in \tilde{N}$ for all $\tau \in t \cdot [0,1]$. As $B^{-1} \circ S$ maps $\hat{W}_s$ onto L we get $x = S^{-1}(Bz) \in \hat{W}_s \cap N$, and for every $\tau \in t \cdot [0,1]$,

$$F_V(\tau, x) = S^{-1}(BF(\tau, B^{-1}S(x))) = S^{-1}(BF(\tau, z)) \in S^{-1}(B\tilde{N}) = N.$$

By the choice of N, $|\xi| < \hat{r}$ for $\xi \in E_s$ with $x = \xi + \hat{w}_s(\xi)$. Hence $\hat{w}_s(\xi) = w_s(\xi)$, and thereby $x \in W_s \cap N$. The invariance of W_s under F_V in N yields $F_V(t,x) \in W_s \cap N$. As above, $|\xi_t| < \hat{r}$ for $\xi_t \in E_s$ with $F_V(t,x) = \xi_t + w_s(\xi_t)$. It follows that $w_s(\xi_t) = \hat{w}_s(\xi_t)$, and thereby

$$F(t,z) = B^{-1}S(F_V(t, S^{-1}(Bz))) = B^{-1}S(F_V(t,x)) = B^{-1}S(\xi_t + \hat{w}_s(\xi_t)) \in L,$$

which shows the desired invariance of L under F in $\tilde{N}$.

It follows easily that for some $r_L > 0$ the space L is invariant under F in $\{z \in \mathbb{R}^3 : |z| < r_L\}$. Analogously U is invariant under F in $\{z \in \mathbb{R}^3 : |z| < r_U\}$, for some $r_U > 0$. Let

$r_F = \min\{r_L, r_U\}$. Then both L and U are invariant under F in $\{z \in \mathbb{R}^3 : |z| < r_F\}$, which is property (F4).

On (F5). From $\lim_{t\to\infty} h_V(t) = 0$, $h_V(t) \in W_s \cap \{x + y : x \in E_s, y \in E_u, |x| < \hat{r}\}$ for $t > 0$ sufficiently large. By $\hat{w}_s(x) = w_s(x)$ for $x \in E_s$ with $|x| < \hat{r}$,

$$B^{-1}S(\{x + w_s(x) : x \in E_s, |x| < \hat{r}\}) \subset B^{-1}S(\hat{W}_s) = L.$$

Consequently, there is $t_L \in \mathbb{R}$ with $h(t) = B^{-1}S(h_V(t)) \in L$ for all $t \geq t_L$. Analogously, $h(t) \in U$ for all $t \leq t_U$, with some $t_u < t_L$. By continuity and $h(t) \neq 0$ everywhere, either $h(t) \in (0, \infty)e_3$ for all $t \leq t_U$, or $h(t) \in (-\infty, 0)e_3$ for all $t \leq t_U$.

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Nonlinear integral inequalities involving Ψ-Hilfer fractional integrals and iterated fractional integrals, with applications to Ψ-Caputo fractional differential equations

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Abstract. In this paper, modifications of the well-known desingularization method proposed by the first author are used to derive nonlinear versions of the Henry–Gronwall inequality for integral inequalities with the Ψ-Hilfer fractional integral, as well as nonlinear integral inequalities with iterated Ψ-Hilfer fractional integrals. The results are applied to obtain bounds for solutions of initial value problems for nonlinear Ψ-Caputo fractional differential equations, and sufficient conditions for the non-existence of blowing-up solutions.

Keywords: Ψ-Hilfer fractional integral inequality, generalized Henry–Gronwall inequality, iterated Ψ-Hilfer fractional integrals, Ψ-Caputo fractional derivative.

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1 Introduction and preliminaries

The monograph [5] by Dan Henry is devoted to the study of semilinear parabolic differential equations, where these equations are studied in the framework of so-called mild solutions of abstract evolution equations. The mild solutions are given as solutions of some Volterra integral equations with weakly singular kernels. One of the basic tools used there is a linear integral inequality with a weakly singular kernel, for which a generalization of the Gronwall inequality is proved (see [5, Lemma 7.1.1]). This result is being frequently quoted as Henry's lemma. Currently, this method is also applied in the theory of fractional differential equations. The first nonlinear version of the Henry's integral inequality is proved in the paper [12] by a

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new method, often referred to as the desingularization method. In this paper, we apply it to the following nonlinear integral inequalities:

$$u(t) \le a(t) + b(t) \int_{t_0}^{t} [\Psi(t) - \Psi(s)]^{\alpha-1} \Psi'(s) F(s) \omega(u(s)) ds, \quad t \in [t_0, T]; \tag{1.1}$$

$$u(t) \le a(t) + \sum_{i=1}^{n} b_i(t) \int_{t_0}^{t} [\Psi(t) - \Psi(s)]^{\alpha_i-1} \Psi'(s) F_i(s) \omega_i(u(s)) ds, \quad t \in [t_0, T]; \tag{1.2}$$

$$\begin{aligned} u(t) \le a(t) &+ b_1(t) \int_{t_0}^{t} [\Psi(t) - \Psi(s)]^{\alpha-1} \Psi'(s) F_1(s) \omega_1(u(s)) ds \\ &+ b_2(t) \int_{t_0}^{t} [\Psi(t) - \Psi(s)]^{\alpha-1} \Psi'(s) c(s) \\ &\times \int_{t_0}^{s} [\Psi(s) - \Psi(\sigma)]^{\beta-1} \Psi'(\sigma) F_2(\sigma) \omega_2(u(\sigma)) d\sigma \, ds, \quad t \in [t_0, T]; \end{aligned} \tag{1.3}$$

$$\begin{aligned} u(t) \le a(t) &+ b_1(t) \int_{t_0}^{t} [\Psi(t) - \Psi(s)]^{\alpha-1} \Psi'(s) F_1(s) \omega_1(u(s)) ds \\ &+ b_2(t) \int_{t_0}^{t} [\Psi(t) - \Psi(s)]^{\alpha-1} \Psi'(s) F_2(s) \\ &\times \omega_2 \left(b_3(s) \int_{t_0}^{s} [\Psi(s) - \Psi(\sigma)]^{\beta-1} \Psi'(\sigma) F_3(\sigma) \omega_3(u(\sigma)) d\sigma \right) ds, \quad t \in [t_0, T]. \end{aligned} \tag{1.4}$$

For the readers' convenience, we recall the following definitions from the tempered Ψ-fractional calculus.

Definition 1.1 ([4,16]). Let $\alpha > 0$, $\lambda \ge 0$, x be a continuous function on $[t_0, T]$, and $\Psi \in C^1[t_0, T]$ satisfy $\Psi'(t) > 0$ for all $t \in [t_0, T]$. Tempered Ψ-Hilfer fractional integral of order $\alpha > 0$ is defined by

$$I_{t_0}^{\alpha,\lambda,\Psi} x(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (\Psi(t) - \Psi(s))^{\alpha-1} \, \mathrm{e}^{-\lambda(\Psi(t)-\Psi(s))} \, \Psi'(s) \, x(s) \, ds$$

for $t \in [t_0, T]$, where $\Gamma(\cdot)$ is the Euler gamma function.

Definition 1.2 ([19]). Let $\Psi \in C^n[t_0, T]$, $n \in \mathbb{N}$ be such that $\Psi'(t) > 0$ for all $t \in [t_0, T]$. For $n - 1 < \alpha < n$, $\lambda \ge 0$, tempered Ψ-Caputo fractional derivative of order α of $x \in C^{n-1}[t_0, T]$ is defined by

$${}^{C}D_{t_0}^{\alpha,\lambda,\Psi} x(t) = {}^{RL}D_{t_0}^{\alpha,\lambda,\Psi} \left[x(t) - \mathrm{e}^{-\lambda\Psi(t)} \sum_{k=0}^{n-1} \frac{x_{\lambda,\Psi}^{[k]}(t_0)}{k!} (\Psi(t) - \Psi(t_0))^k \right]$$

for $t \in [t_0, T]$ whenever the right side makes sense, where

$$x_{\lambda,\Psi}^{[n]}(t) = \left(\frac{1}{\Psi'(t)} \frac{d}{dt} \right)^n \left(\mathrm{e}^{\lambda\Psi(t)} \, x(t) \right)$$

and

$$\begin{aligned} {}^{RL}D_{t_0}^{\alpha,\lambda,\Psi} x(t) &= \mathrm{e}^{-\lambda\Psi(t)} \left(I_{t_0}^{n-\alpha,\lambda,\Psi} x(t) \right)_{\lambda,\Psi}^{[n]} \\ &= \frac{\mathrm{e}^{-\lambda\Psi(t)}}{\Gamma(n-\alpha)} \left(\frac{1}{\Psi'(t)} \frac{d}{dt} \right)^n \int_{t_0}^{t} (\Psi(t) - \Psi(s))^{n-\alpha-1} \Psi'(s) \, \mathrm{e}^{\lambda\Psi(s)} \, x(s) \, ds \end{aligned}$$

is the tempered Ψ-Riemann–Liouville fractional derivative of order α.

It is worth to note that if $x \in C^n[t_0, T]$, then ${}^C D^{\alpha,\lambda,\Psi}_{t_0} x(t)$ may be expressed as

$$ {}^C D^{\alpha,\lambda,\Psi}_{t_0} x(t) = \frac{\mathrm{e}^{-\lambda\Psi(t)}}{\Gamma(n-\alpha)} \int_{t_0}^{t} (\Psi(t) - \Psi(s))^{n-\alpha-1} \Psi'(s) x^{[n]}_{\lambda,\Psi}(s)\, ds $$

(see [16,19]). In this case, since $x^{[n]}_{\lambda,\Psi} \in C[t_0, T]$, Lemma 1 of [20] yields that ${}^C D^{\alpha,\lambda,\Psi}_{t_0} x \in C[t_0, T]$.

We refer the reader to papers [4,16,18–20] for the properties of these operators. The above notions generalize Ψ-Hilfer fractional integral from [7], $I^{\alpha,\Psi}_{t_0} = I^{\alpha,0,\Psi}_{t_0}$, and Ψ-Caputo fractional derivative from [1], ${}^C D^{\alpha,\Psi}_{t_0} = {}^C D^{\alpha,0,\Psi}_{t_0}$, which we use in this paper.

In the paper [18], the following integral inequality with the tempered Ψ-Hilfer fractional integral is studied:

$$ u(t) \le a(t) + b(t) \int_{t_0}^{t} [\Psi(t) - \Psi(s)]^{\alpha-1} \mathrm{e}^{-\lambda[\Psi(t)-\Psi(s)]} \Psi'(s) F(s) \omega(u(s)) ds, \ t \in [t_0, T). $$

The obtained Henry–Gronwall inequality is proved under the assumption $\lambda > 0$. The method of desingularization applied in the proof is not suitable for the inequality with $\lambda = 0$, i.e., for inequality (1.1). In this case, we need to apply a modification of the desingularization method proposed in the paper [12].

In [21, Theorem 3], inequality (1.1) is studied for $\omega(u) = u$, $F(t) \equiv 1$ and an estimation of u is obtained in a form of a series. Here, we investigate nonlinear analogues (see Corollaries 2.5, 2.13). Similarly, Corollaries 2.7, 2.15 generalize e.g. [11, Theorem 1.4], where inequality (1.2) is investigated with $\Psi(t) = t$, $F_i(t) \equiv 1$ and $\omega_i(u) = u$ for each $i = 1, 2, \dots, n$.

Inequality (1.1) with $\alpha \in (0,1)$, $\Psi(t) = t$, $a(t) = at^{-\mu}$, and $b(t) = bt^{-\nu}$ with various values of parameters μ, ν is studied in [6,23–25]. In [9], inequality (1.1) is investigated with $\Psi(t) = t$, $\alpha \in (0,1)$, but instead of non-decreasing ω, $\widehat{\omega}$ is considered such that $\widehat{\omega}^p(u) \le \omega(u^p)$ for $p > 1/\alpha$ and some non-decreasing ω. In [22], (1.1) is studied with $\Psi(t) = t$, $\alpha \in (0,1)$, $F(t) = t^{-\gamma}$, $\gamma \ge 0$ and $1 - \alpha + \gamma < 1$; also the case $a(t) = at^{-\mu}$ is considered.

Other integral inequalities useful for fractional calculus can be found e.g. in [2,19].

Throughout the whole paper, for generality, we consider inequality (1.1) as well as others on an interval $[t_0, T]$ for some $t_0 < T \le \infty$, where the case $T = \infty$ is understood as $[t_0, \infty)$. We use the notation $\mathbb{N}$ for the set of all positive integers, and $\|\cdot\|$ for any vector norm in $\mathbb{R}^N$, $N \in \mathbb{N}$.

The paper is structured as follows. In the next section, we derive the main results of the paper – the nonlinear integral inequalities for weakly singular integrals. We apply two types of desingularization, namely with inserting an auxiliary exponential function and without it. The inequalities are applied in Section 3 to obtain boundedness results for solutions of nonlinear fractional differential equations, and sufficient conditions for the non-existence of solutions blowing-up in a finite time. Lastly, Section 4 concludes the results and sketches directions of future research.

2 Nonlinear integral inequalities and their corollaries

This section is devoted to integral inequalities (1.1)–(1.4). It is split in two subsections according to a method used to desingularize the fractional integral, i.e., to get rid of the singular kernel.

For the case of brevity, we introduce the notation for classes of functions,

$$\mathfrak{P}_M^k := \{\varphi \in C^k(M) \mid \varphi'(t) > 0 \text{ for all } t \in M\},$$
$$\mathfrak{O} := \{\varphi \in C([0,\infty),[0,\infty)) \mid \varphi(t) > 0\ \forall t > 0,\ \varphi \text{ is non-decreasing}\}.$$

In our proofs, we rely on a generalization of the well-known Bihari's inequality [3, 10] that was proved in [9]. Next, we provide its version sufficient for our purposes.

Theorem 2.1. *Let a, b, F be non-negative, continuous functions on $[t_0, T]$. Let $\omega \in \mathfrak{O}$ and u be a non-negative continuous function on $[t_0, T]$, satisfying*

$$u(t) \le a(t) + b(t) \int_{t_0}^{t} F(s)\omega(u(s))\, ds, \quad t \in [t_0, T].$$

Then

$$u(t) \le \Omega^{-1}\left(\Omega(A(t)) + B(t) \int_{t_0}^{t} F(s)\, ds\right), \quad t \in [t_0, t_1],$$

where $t_1 \in (t_0, T]$ is such that

$$\Omega(A(t_1)) + B(t_1) \int_{t_0}^{t_1} F(s)\, ds \in \operatorname{Dom} \Omega^{-1},$$

$\Omega(v) = \int_{v_0}^{v} \frac{1}{\omega(\sigma)} d\sigma$ for $v \ge v_0 > 0$, Ω^{-1} is the inverse of Ω, $A(t) = \sup_{s\in[t_0,t]} a(s)$, $B(t) = \sup_{s\in[t_0,t]} b(s)$.

It should be noted that in [9, Theorem 1] the latter result is proved for positive functions a and b, and with $F \in L^1[t_0, T]$ positive on a set of positive measure. Then, in [9, Remark 1], a is allowed to be non-negative if $\omega(t) > 0$ for all $t > 0$. Since we use continuous functions, if there does not exist a set of positive measure where F is positive, then $F \equiv 0$ and the statement holds trivially. Next, if $b(t_0) > 0$ then $B(t_0) > 0$ which is sufficient for the proof to work. If $b(t) = 0$ for all $t \in [t_0, t_0 + h]$ for some $h > 0$ then, again, the statement clearly holds for all those t. Moreover, if $\int_0 \frac{1}{\omega(\sigma)} d\sigma$ converges, one can set $v_0 = 0$. Therefore, we allow $v_0 \ge 0$ in this paper.

2.1 Parametric desingularization

Here we apply a parameter-dependent modification of the desingularization method from [12], where an exponential function was added and the Hölder inequality was applied to tame the weakly singular integral kernel. So, we are able to prove the following nonlinear Henry–Gronwall type inequality.

Theorem 2.2. *Let $\alpha \in (0,1)$, $\alpha q > 1$, $\mu > 0$, a, b and F be non-negative, continuous functions on $[t_0, T]$, and $\Psi \in \mathfrak{P}^1_{[t_0,T]}$. Let $\omega \in \mathfrak{O}$ and u be a non-negative continuous function on $[t_0, T]$, satisfying inequality (1.1). Then*

$$u(t) \le \left[\Omega^{-1}\left(\Omega(A(t)) + B(t) \int_{t_0}^{t} \Psi'(s) F(s)^q \, \mathrm{e}^{-q\mu\Psi(s)}\, ds\right)\right]^{1/q}, \quad t \in [t_0, t_1], \tag{2.1}$$

where $t_1 \in (t_0, T]$ is such that

$$\Omega(A(t_1)) + B(t_1) \int_{t_0}^{t_1} \Psi'(s) F(s)^q \, \mathrm{e}^{-q\mu\Psi(s)}\, ds \in \operatorname{Dom} \Omega^{-1}, \tag{2.2}$$

$\Omega(v) = \int_{v_0}^{v} \frac{1}{[\omega(\sigma^{1/q})]^q} d\sigma$ *for* $v \geq v_0 \geq 0$, Ω^{-1} *is the inverse of* Ω, *and*

$$A(t) = 2^{q-1} \sup_{s\in[t_0,t]} a(s)^q, \quad B(t) = 2^{q-1} M_{p,\mu,\alpha}^q \sup_{s\in[t_0,t]} \left\{ b(s)^q \, \mathrm{e}^{q\mu\Psi(s)} \right\},$$
$$M_{p,\mu,\alpha} = \left(\frac{\Gamma(p(\alpha-1)+1)}{(p\mu)^{p(\alpha-1)+1}} \right)^{1/p}, \quad p = \frac{q}{q-1}. \tag{2.3}$$

In the proof of this theorem, we will need the following generalization of the inequality from [12, page 354].

Lemma 2.3. *Let* $\alpha \in (0,1)$, $p > 0$, $p(\alpha-1)+1 > 0$, $\mu > 0$, *and* $\Psi(t) \in \mathfrak{P}^1_{[t_0,T]}$. *Then*

$$\int_{t_0}^{t} [\Psi(t) - \Psi(s)]^{p(\alpha-1)} \, \mathrm{e}^{p\mu\Psi(s)} \, \Psi'(s) ds \leq M_{p,\mu,\alpha}^p \, \mathrm{e}^{p\mu\Psi(t)}, \quad t \in [t_0, T], \tag{2.4}$$

where $M_{p,\mu,\alpha}$ *is given by the formula* (2.3).

Proof. Using the substitution $\Psi(t) - \Psi(s) = \sigma$ one can rewrite the left-hand side of inequality (2.4) as

$$\int_{t_0}^{t} [\Psi(t) - \Psi(s)]^{p(\alpha-1)} \, \mathrm{e}^{p\mu\Psi(s)} \, \Psi'(s) ds = \int_{0}^{\Psi(t)-\Psi(t_0)} \sigma^{p(\alpha-1)} \, \mathrm{e}^{p\mu(\Psi(t)-\sigma)} \, d\sigma$$
$$= \mathrm{e}^{p\mu\Psi(t)} \int_{t_0}^{\Psi(t)-\Psi(t_0)} \sigma^{p(\alpha-1)} \, \mathrm{e}^{-p\mu\sigma} \, d\sigma \leq \mathrm{e}^{p\mu\Psi(t)} \int_{0}^{\infty} \sigma^{p(\alpha-1)} \, \mathrm{e}^{-p\mu\sigma} \, d\sigma = M_{p,\mu,\alpha}^p \, \mathrm{e}^{p\mu\Psi(t)}. \square$$

Now, we present the proof of Theorem 2.2.

Proof of Theorem 2.2. First notice that $q > \frac{1}{\alpha} > 1$. Thus, $p(\alpha-1)+1 > 0$ and $M_{p,\mu,\alpha}$ is well defined. Since $\frac{1}{p} + \frac{1}{q} = 1$, we can write $\Psi'(t) = \Psi'(t)^{1/p} \Psi'(t)^{1/q}$. Using the Hölder inequality and Lemma 2.3, we derive

$$\begin{aligned}
&\int_{t_0}^{t} [\Psi(t) - \Psi(s)]^{\alpha-1} \Psi'(s) F(s) \omega(u(s)) ds \\
&\quad = \int_{t_0}^{t} [\Psi(t) - \Psi(s)]^{\alpha-1} \, \mathrm{e}^{\mu\Psi(s)} \, \Psi'(s)^{1/p} F(s) \, \mathrm{e}^{-\mu\Psi(s)} \, \Psi'(s)^{1/q} \omega(u(s)) ds \\
&\quad \leq \left(\int_{t_0}^{t} [\Psi(t) - \Psi(s)]^{p(\alpha-1)} \, \mathrm{e}^{p\mu\Psi(s)} \, \Psi'(s) ds \right)^{1/p} \left(\int_{t_0}^{t} F(s)^q \, \mathrm{e}^{-q\mu\Psi(s)} \, \Psi'(s) \omega(u(s))^q ds \right)^{1/q} \\
&\quad \leq M_{p,\mu,\alpha} \, \mathrm{e}^{\mu\Psi(t)} \left(\int_{t_0}^{t} F(s)^q \, \mathrm{e}^{-q\mu\Psi(s)} \, \Psi'(s) \omega(u(s))^q ds \right)^{1/q}.
\end{aligned} \tag{2.5}$$

Hence, this inequality along with (1.1) yield

$$u(t) \leq a(t) + b(t) M_{p,\mu,\alpha} \, \mathrm{e}^{\mu\Psi(t)} \left(\int_{t_0}^{t} F(s)^q \, \mathrm{e}^{-q\mu\Psi(s)} \, \Psi'(s) \omega(u(s))^q ds \right)^{1/q}. \tag{2.6}$$

Using the inequality $(b_1 + b_2)^q \leq 2^{q-1}(b_1^q + b_2^q)$ valid for any $b_1, b_2 \geq 0$, we obtain

$$v(t) \leq 2^{q-1} a(t)^q + 2^{q-1} b(t)^q M_{p,\mu,\alpha}^q \, \mathrm{e}^{q\mu\Psi(t)} \int_{t_0}^{t} F(s)^q \, \mathrm{e}^{-q\mu\Psi(s)} \, \Psi'(s) \omega(u(s))^q ds$$
$$= 2^{q-1} a(t)^q + 2^{q-1} b(t)^q M_{p,\mu,\alpha}^q \, \mathrm{e}^{q\mu\Psi(t)} \int_{t_0}^{t} F(s)^q \, \mathrm{e}^{-q\mu\Psi(s)} \, \Psi'(s) \omega(v(s)^{1/q})^q ds,$$

where $v(t) = u(t)^q$. Consequently, Theorem 2.1 implies

$$v(t) \le \Omega^{-1}\left(\Omega(A(t)) + B(t)\int_{t_0}^{t} F(s)^q \,\mathrm{e}^{-q\mu\Psi(s)}\,\Psi'(s)ds\right),$$

where A, B, Ω and Ω^{-1} are as in the theorem. This proves inequality (2.1). □

Remark 2.4. Since functions A, B, Ω are non-decreasing and $\Psi'(t)F(t)^q\,\mathrm{e}^{-q\mu\Psi(t)} \ge 0$ for all $t \in [t_0, T]$, condition (2.2) is equivalent to

$$\Omega(A(t)) + B(t)\int_{t_0}^{t} \Psi'(s)F(s)^q\,\mathrm{e}^{-q\mu\Psi(s)}\,ds \in \operatorname{Dom}\Omega^{-1}$$

for all $t \in [t_0, t_1]$.

Some of corollaries of Theorem 2.2 are explicitly stated below.

Corollary 2.5. *Let the assumptions of Theorem 2.2 be fulfilled with $\omega(u) = u$, i.e., let u satisfy*

$$u(t) \le a(t) + b(t)\int_{t_0}^{t} [\Psi(t) - \Psi(s)]^{\alpha-1}\Psi'(s)F(s)u(s)ds, \quad t \in [t_0, T].$$

Then

$$u(t) \le A(t)^{1/q}\exp\left\{\frac{B(t)}{q}\int_{t_0}^{t} F(s)^q\,\mathrm{e}^{-q\mu\Psi(s)}\,\Psi'(s)ds\right\}, \quad t \in [t_0, T],$$

where functions A, B are given by (2.3).

Proof. Putting the particular form of ω in the statement of Theorem 2.2 immediately yields $\Omega(v) = \ln\frac{v}{v_0}$ for $v \ge v_0 > 0$, $\Omega^{-1}(v) = v_0\,\mathrm{e}^v$ for $v \ge 0$, $v_0 > 0$, and the corollary follows from (2.1). □

Corollary 2.6. *Let the assumptions of Theorem 2.2 be fulfilled with $\omega(u) = u^m$ for $m > 0$, $m \ne 1$, i.e., let u satisfy*

$$u(t) \le a(t) + b(t)\int_{t_0}^{t} [\Psi(t) - \Psi(s)]^{\alpha-1}\Psi'(s)F(s)u(s)^m ds, \quad t \in [t_0, T].$$

Then, the following assertions hold:

1. *if $0 < m < 1$, then*

$$u(t) \le \left(A(t)^{1-m} + (1-m)B(t)\int_{t_0}^{t} F(s)^q\,\mathrm{e}^{-q\mu\Psi(s)}\,\Psi'(s)ds\right)^{\frac{1}{q(1-m)}}, \quad t \in [t_0, T];$$

2. *if $m > 1$, then*

$$u(t) \le \frac{A(t)^{1/q}}{\left(1 - (m-1)A(t)^{m-1}B(t)\int_{t_0}^{t} F(s)^q\,\mathrm{e}^{-q\mu\Psi(s)}\,\Psi'(s)ds\right)^{\frac{1}{q(m-1)}}}, \quad t \in [t_0, t_1]$$

for any $t_1 \in (t_0, T]$ such that

$$\int_{t_0}^{t_1} F(s)^q\,\mathrm{e}^{-q\mu\Psi(s)}\,\Psi'(s)ds < \frac{1}{(m-1)A(t_1)^{m-1}B(t_1)};$$

where functions A, B are given by (2.3).

Proof. The statement can be obtained analogously to Corollary 2.5 with $\omega(u) = u^m$. Note that here we use the fact that functions A, B are non-decreasing. □

Theorem 2.2 can also be easily extended to the case of multiple Ψ-Hilfer fractional integrals:

Corollary 2.7. *Let $n \in \mathbb{N}$, $\alpha_i \in (0,1)$ for $i = 1,2,\dots,n$, $q \min_{i=1,2,\dots,n} \alpha_i > 1$, $\mu > 0$, a, b_i and F_i be non-negative, continuous functions on $[t_0,T]$ for each $i = 1,2,\dots,n$, and $\Psi \in \mathfrak{P}^1_{[t_0,T]}$. Let $\omega_i \in \mathfrak{O}$ for each $i = 1,2,\dots,n$ and u be a non-negative continuous function on $[t_0,T]$, satisfying inequality (1.2). Then,*

$$u(t) \le \left[\Omega^{-1}\left(\Omega(A(t)) + B(t)\int_{t_0}^{t} \Psi'(s)F(s)^q \,\mathrm{e}^{-q\mu\Psi(s)}\,ds\right)\right]^{1/q}, \quad t \in [t_0,t_1],$$

where $t_1 \in (t_0,T]$ is such that

$$\Omega(A(t_1)) + B(t_1)\int_{t_0}^{t_1} \Psi'(s)F(s)^q \,\mathrm{e}^{-q\mu\Psi(s)}\,ds \in \operatorname{Dom}\Omega^{-1},$$

$\Omega(v) = \int_{v_0}^{v} \frac{1}{[\omega(\sigma^{1/q})]^q} d\sigma$ for $v \ge v_0 \ge 0$, Ω^{-1} is the inverse of Ω, and

$$F = \max_{i=1,2,\dots,n} F_i, \quad \omega = \max_{i=1,2,\dots,n} \omega_i,$$

$$A(t) = 2^{q-1} \sup_{s\in[t_0,t]} a(s)^q, \quad B(t) = 2^{q-1} \sup_{s\in[t_0,t]} \left(\sum_{i=1}^{n} M_{p,\mu,\alpha_i} b_i(s)\,\mathrm{e}^{\mu\Psi(s)}\right)^q,$$

$$M_{p,\mu,\alpha} = \left(\frac{\Gamma(p(\alpha-1)+1)}{(p\mu)^{p(\alpha-1)+1}}\right)^{1/p}, \quad p = \frac{q}{q-1}.$$

Proof. The right-hand side of inequality (1.2) can be estimated from above by

$$\begin{aligned} &a(t) + \sum_{i=1}^{n} b_i(t)\int_{t_0}^{t} [\Psi(t)-\Psi(s)]^{\alpha_i-1}\Psi'(s)F(s)\omega(u(s))ds \\ &\quad \le a(t) + \sum_{i=1}^{n} b_i(t) M_{p,\mu,\alpha_i}\,\mathrm{e}^{\mu\Psi(t)} \left(\int_{t_0}^{t} F(s)^q\,\mathrm{e}^{-q\mu\Psi(s)}\,\Psi'(s)\omega(u(s))^q ds\right)^{1/q} \end{aligned}$$

as in (2.5). Then the proof is finished as the proof of Theorem 2.2. □

Next, we give an estimation of function u satisfying inequality (1.3).

Theorem 2.8. *Let $\alpha,\beta \in (0,1)$, $\alpha+\beta < 1$, $\alpha q > 1$, $\mu > 0$, a, b_1, b_2, c, F_1 and F_2 be non-negative, continuous functions on $[t_0,T]$, and $\Psi \in \mathfrak{P}^1_{[t_0,T]}$. Let $\omega_1,\omega_2 \in \mathfrak{O}$ and u be a non-negative continuous function on $[t_0,T]$, satisfying inequality (1.3). Then,*

$$u(t) \le \left[\Omega^{-1}\left(\Omega(A(t)) + B(t)\int_{t_0}^{t} \Psi'(s)G(s)^q \,\mathrm{e}^{-q\mu\Psi(s)}\,ds\right)\right]^{1/q}, \quad t \in [t_0,t_1],$$

where $t_1 \in (t_0,T]$ is such that

$$\Omega(A(t_1)) + B(t_1)\int_{t_0}^{t_1} \Psi'(s)G(s)^q \,\mathrm{e}^{-q\mu\Psi(s)}\,ds \in \operatorname{Dom}\Omega^{-1},$$

$\Omega(v) = \int_{v_0}^{v} \frac{1}{[\omega(\sigma^{1/q})]^q} d\sigma$ *for* $v \geq v_0 \geq 0$, Ω^{-1} *is the inverse of* Ω, *and*

$$G = \max\{F_1, F_2\}, \quad \omega = \max\{\omega_1, \omega_2\}, \quad A(t) = 2^{q-1} \sup_{s\in[t_0,t]} a(s)^q,$$

$$B(t) = 2^{q-1} \sup_{s\in[t_0,t]} \left\{ \mathrm{e}^{q\mu\Psi(s)} \left(b_1(s) M_{p,\mu,\alpha} + B(\alpha,\beta) b_2(s) M_{p,\mu,\alpha+\beta} \sup_{\sigma\in[t_0,s]} c(\sigma) \right)^q \right\}, \tag{2.7}$$

$$M_{p,\mu,\alpha} = \left(\frac{\Gamma(p(\alpha-1)+1)}{(p\mu)^{p(\alpha-1)+1}} \right)^{1/p}, \quad p = \frac{q}{q-1},$$

$B(\cdot,\cdot)$ *is the Euler beta function.*

Proof. By a trivial estimation using functions G and ω, and applying (2.5), we get

$$\begin{aligned}
u(t) &\leq a(t) + b_1(t) \int_{t_0}^{t} [\Psi(t) - \Psi(s)]^{\alpha-1} \Psi'(s) G(s) \omega(u(s)) ds \\
&\quad + b_2(t) \int_{t_0}^{t} [\Psi(t) - \Psi(s)]^{\alpha-1} \Psi'(s) c(s) \\
&\quad \times \int_{t_0}^{s} [\Psi(s) - \Psi(\sigma)]^{\beta-1} \Psi'(\sigma) G(\sigma) \omega(u(\sigma)) d\sigma\, ds \\
&\leq a(t) + b_1(t) M_{p,\mu,\alpha}\, \mathrm{e}^{\mu\Psi(t)} \left(\int_{t_0}^{t} G(s)^q\, \mathrm{e}^{-q\mu\Psi(s)}\, \Psi'(s) \omega(u(s))^q ds \right)^{1/q} \\
&\quad + b_2(t) \sup_{s\in[t_0,t]} c(s) \int_{t_0}^{t} [\Psi(t) - \Psi(s)]^{\alpha-1} \Psi'(s) \\
&\quad \times \int_{t_0}^{s} [\Psi(s) - \Psi(\sigma)]^{\beta-1} \Psi'(\sigma) G(\sigma) \omega(u(\sigma)) d\sigma\, ds.
\end{aligned}$$

Changing the order of integration and by the change of variable $\Psi(t) - \Psi(s) = \rho(\Psi(t) - \Psi(\sigma))$, we derive

$$\begin{aligned}
&\int_{t_0}^{t} [\Psi(t) - \Psi(s)]^{\alpha-1} \Psi'(s) \int_{t_0}^{s} [\Psi(s) - \Psi(\sigma)]^{\beta-1} \Psi'(\sigma) G(\sigma) \omega(u(\sigma)) d\sigma\, ds \\
&\quad = \int_{t_0}^{t} \Psi'(\sigma) G(\sigma) \omega(u(\sigma)) \left(\int_{\sigma}^{t} [\Psi(t) - \Psi(s)]^{\alpha-1} \Psi'(s) [\Psi(s) - \Psi(\sigma)]^{\beta-1} ds \right) d\sigma \\
&\quad = \int_{t_0}^{t} \Psi'(\sigma) G(\sigma) \omega(u(\sigma)) [\Psi(t) - \Psi(\sigma)]^{\alpha+\beta-1} \left(\int_{0}^{1} \rho^{\alpha-1} (1-\rho)^{\beta-1} d\rho \right) d\sigma \\
&\quad = B(\alpha,\beta) \int_{t_0}^{t} [\Psi(t) - \Psi(\sigma)]^{\alpha+\beta-1} \Psi'(\sigma) G(\sigma) \omega(u(\sigma)) d\sigma.
\end{aligned}$$

Now, we apply an estimation analogous to (2.5) to estimate the latter right-hand side by

$$B(\alpha,\beta) M_{p,\mu,\alpha+\beta}\, \mathrm{e}^{\mu\Psi(t)} \left(\int_{t_0}^{t} G(s)^q\, \mathrm{e}^{-q\mu\Psi(s)}\, \Psi'(s) \omega(u(s))^q ds \right)^{1/q}.$$

So, we arrive at

$$\begin{aligned}
u(t) &\leq a(t) + \left(b_1(t) M_{p,\mu,\alpha} + B(\alpha,\beta) b_2(t) M_{p,\mu,\alpha+\beta} \sup_{s\in[t_0,t]} c(s) \right) \mathrm{e}^{\mu\Psi(t)} \\
&\quad \times \left(\int_{t_0}^{t} G(s)^q\, \mathrm{e}^{-q\mu\Psi(s)}\, \Psi'(s) \omega(u(s))^q ds \right)^{1/q},
\end{aligned}$$

which is of the form of (2.6). Hence, the proof is finished by the same way as the proof of Theorem 2.2. □

Finally, we investigate inequality (1.4).

Theorem 2.9. *Let $\alpha, \beta \in (0,1)$, $q\min\{\alpha,\beta\} > 1$, $\mu > 0$, a, b_1, b_2, b_3, F_1, F_2, F_3 be non-negative, continuous functions on $[t_0, T]$, and $\Psi \in \mathfrak{P}^1_{[t_0,T]}$. Let $\omega_1, \omega_2, \omega_3 \in \mathfrak{O}$ and u be a non-negative continuous function on $[t_0, T]$, satisfying inequality (1.4). Then,*

$$u(t) \le \left[\Omega^{-1}\left(\Omega(A(t)) + B(t)\int_{t_0}^{t} G(s)^q\, \mathrm{e}^{-q\mu\Psi(s)}\,\Psi'(s)ds \right.\right. \\ \left.\left. + C(t)\int_{t_0}^{t} F_2(s)^q\, \mathrm{e}^{-q\mu\Psi(s)}\,\Psi'(s)ds\right)\right]^{1/q}, \quad t \in [t_0, t_1],$$

where $t_1 \in (t_0, T]$ is such that

$$\Omega(A(t_1)) + B(t_1)\int_{t_0}^{t_1} G(s)^q\, \mathrm{e}^{-q\mu\Psi(s)}\,\Psi'(s)ds + C(t_1)\int_{t_0}^{t_1} F_2(s)^q\, \mathrm{e}^{-q\mu\Psi(s)}\,\Psi'(s)ds \in \operatorname{Dom}\Omega^{-1},$$

$\Omega(v) = \int_{v_0}^{v} \frac{1}{[\omega(\sigma^{1/q})]^q} d\sigma$ for $v \ge v_0 \ge 0$, Ω^{-1} is the inverse of Ω, and

$$\begin{gathered} G = \max\{F_1, F_3\}, \quad \omega = \max\{\omega_1, \omega_2, \omega_3\}, \quad b = \max\{b_1, b_3\}, \\ A(t) = 3^{q-1}\sup_{s\in[t_0,t]} a(s)^q, \quad B(t) = 3^{q-1}M^q_{p,\mu}\sup_{s\in[t_0,t]}\left\{b(s)^q\,\mathrm{e}^{q\mu\Psi(s)}\right\}, \\ C(t) = 3^{q-1}M^q_{p,\mu,\alpha}\sup_{s\in[t_0,t]}\left\{b_2(s)^q\,\mathrm{e}^{q\mu\Psi(s)}\right\}, \quad M_{p,\mu} = \max\{M_{p,\mu,\alpha}, M_{p,\mu,\beta}\}, \\ M_{p,\mu,\alpha} = \left(\frac{\Gamma(p(\alpha-1)+1)}{(p\mu)^{p(\alpha-1)+1}}\right)^{1/p}, \quad p = \frac{q}{q-1}. \end{gathered} \tag{2.8}$$

To prove Theorem 2.9, we need the following auxiliary result.

Lemma 2.10. *Let a, b, c, f, g be non-negative, continuous functions on $[t_0, T]$. Let $\omega \in \mathfrak{O}$ and u be a non-negative continuous function on $[t_0, T]$, satisfying*

$$u(t) \le a(t) + b(t)\int_{t_0}^{t} f(s)\omega(u(s))ds \\ + c(t)\int_{t_0}^{t} g(s)\omega\left(b(s)\int_{t_0}^{s} f(\sigma)\omega(u(\sigma))d\sigma\right)ds, \quad t \in [t_0, T]. \tag{2.9}$$

Then,

$$u(t) \le \Omega^{-1}\left(\Omega(\bar{a}(t)) + \bar{b}(t)\int_{t_0}^{t} f(s)ds + \bar{c}(t)\int_{t_0}^{t} g(s)ds\right), \quad t \in [t_0, t_1], \tag{2.10}$$

where $t_1 \in (t_0, T]$ is such that

$$\Omega(\bar{a}(t_1)) + \bar{b}(t_1)\int_{t_0}^{t_1} f(s)ds + \bar{c}(t_1)\int_{t_0}^{t_1} g(s)ds \in \operatorname{Dom}\Omega^{-1},$$

$\Omega(v) = \int_{v_0}^{v} \frac{d\sigma}{\omega(\sigma)}$ for $v \ge v_0 \ge 0$, Ω^{-1} is the inverse of Ω, and

$$\bar{a}(t) = \sup_{s\in[t_0,t]} a(s), \quad \bar{b}(t) = \sup_{s\in[t_0,t]} b(s), \quad \bar{c}(t) = \sup_{s\in[t_0,t]} c(s).$$

Proof. Let $\widetilde{T} \in (t_0, T]$ be sufficiently small (this is specified later) and fixed. From (2.9), for any $t \in [t_0, \widetilde{T}]$,

$$\begin{aligned} u(t) &\le \bar{a}(\widetilde{T}) + \bar{b}(\widetilde{T}) \int_{t_0}^{t} f(s)\omega(u(s))ds \\ &\quad + \bar{c}(\widetilde{T}) \int_{t_0}^{t} g(s)\omega\left(\bar{b}(\widetilde{T}) \int_{t_0}^{s} f(\sigma)\omega(u(\sigma))d\sigma\right) ds =: z(t). \end{aligned}$$

It is easy to see that function z is non-negative, non-decreasing, C^1, and $z(t_0) = \bar{a}(\widetilde{T})$. Differentiating, we obtain

$$\begin{aligned} z'(t) &\le \bar{b}(\widetilde{T})f(t)\omega(u(t)) + \bar{c}(\widetilde{T})g(t)\omega\left(\bar{b}(\widetilde{T}) \int_{t_0}^{t} f(s)\omega(u(s))ds\right) \\ &\le \bar{b}(\widetilde{T})f(t)\omega(z(t)) + \bar{c}(\widetilde{T})g(t)\omega(z(t)) \end{aligned}$$

for all $t \in [t_0, \widetilde{T}]$. Integrating over $[t_0, t]$ yields

$$z(t) \le \bar{a}(\widetilde{T}) + \int_{t_0}^{t} \left(\bar{b}(\widetilde{T})f(s) + \bar{c}(\widetilde{T})g(s)\right) \omega(z(s))ds, \quad t \in [t_0, \widetilde{T}].$$

Now, applying Theorem 2.1 results in

$$u(t) \le z(t) \le \Omega^{-1}\left(\Omega(\bar{a}(\widetilde{T})) + \int_{t_0}^{t} \bar{b}(\widetilde{T})f(s) + \bar{c}(\widetilde{T})g(s)ds\right)$$

for all $t \in [t_0, \widetilde{T}]$. In particular, this estimation holds for $t = \widetilde{T}$ and we obtain (2.10). Now, one can see that $\widetilde{T}$ at the beginning of the proof had to be smaller than or equal to t_1. □

Proof of Theorem 2.9. Using notation (2.8) and estimation (2.5), inequality (1.4) gives

$$\begin{aligned} u(t) &\le a(t) + b(t) \int_{t_0}^{t} [\Psi(t) - \Psi(s)]^{\alpha-1}\Psi'(s)G(s)\omega(u(s))ds \\ &\quad + b_2(t) \int_{t_0}^{t} [\Psi(t) - \Psi(s)]^{\alpha-1}\Psi'(s)F_2(s) \\ &\quad \times \omega\left(b(s) \int_{t_0}^{s} [\Psi(s) - \Psi(\sigma)]^{\beta-1}\Psi'(\sigma)G(\sigma)\omega(u(\sigma))d\sigma\right) ds \\ &\le a(t) + b(t)M_{p,\mu}\, \mathrm{e}^{\mu\Psi(t)} \left(\int_{t_0}^{t} G(s)^q\, \mathrm{e}^{-q\mu\Psi(s)}\, \Psi'(s)\omega(u(s))^q ds\right)^{1/q} \\ &\quad + b_2(t)M_{p,\mu,\alpha}\, \mathrm{e}^{\mu\Psi(t)} \left(\int_{t_0}^{t} F_2(s)^q\, \mathrm{e}^{-q\mu\Psi(s)}\, \Psi'(s)\right. \\ &\quad \left. \times \omega\left(b(s)M_{p,\mu}\, \mathrm{e}^{\mu\Psi(s)} \left(\int_{t_0}^{s} G(\sigma)^q\, \mathrm{e}^{-q\mu\Psi(\sigma)}\, \Psi'(\sigma)\omega(u(\sigma))^q d\sigma\right)^{1/q}\right)^q ds\right)^{1/q}. \end{aligned}$$

Using the inequality $(a_1 + a_2 + a_3)^q \le 3^{q-1}(a_1^q + a_2^q + a_3^q)$ valid for any $a_1, a_2, a_3 \ge 0$, and the estimation

$$b(s)M_{p,\mu}\, \mathrm{e}^{\mu\Psi(s)} \le \left(3^{q-1}b(s)^q M_{p,\mu}^q\, \mathrm{e}^{q\mu\Psi(s)}\right)^{1/q},$$

we obtain

$$\begin{aligned} v(t) &\le 3^{q-1}a(t)^q + 3^{q-1}b(t)^q M_{p,\mu}^q \,\mathrm{e}^{q\mu\Psi(t)} \int_{t_0}^t G(s)^q \,\mathrm{e}^{-q\mu\Psi(s)}\, \Psi'(s)\omega(v(s)^{1/q})^q ds \\ &\quad + 3^{q-1}b_2(t)^q M_{p,\mu,\alpha}^q \,\mathrm{e}^{q\mu\Psi(t)} \int_{t_0}^t F_2(s)^q \,\mathrm{e}^{-q\mu\Psi(s)}\, \Psi'(s) \\ &\quad \times \omega\left(\left(3^{q-1}b(s)^q M_{p,\mu}^q \,\mathrm{e}^{q\mu\Psi(s)} \int_{t_0}^s G(\sigma)^q \,\mathrm{e}^{-q\mu\Psi(\sigma)}\, \Psi'(\sigma)\omega(v(\sigma)^{1/q})^q d\sigma\right)^{1/q}\right)^q ds \end{aligned}$$

for all $t \in [t_0, T]$, where $v(t) = u(t)^q$. This is of the form of (2.9), and the statement follows from Lemma 2.10. □

2.2 Non-exponential desingularization

In this part, we apply the desingularization method without inserting $\mathrm{e}^{\mu\Psi(s)}\,\mathrm{e}^{-\mu\Psi(s)}$ prior to estimating by the Hölder inequality. This can be considered as a limit case $\mu = 0$. Notice that here, Lemma 2.3 can not be used, since $\lim_{\mu\to 0^+} M_{p,\mu,\alpha} = \infty$. The results are analogous to those of the previous section, therefore the proofs are rather sketchy and only the main differences are mentioned. It is shown in the next section, that both approaches, $\mu > 0$ and $\mu = 0$, might give useful and different estimations.

First, we provide an alternative to Theorem 2.2.

Theorem 2.11. *Let all the assumptions of Theorem 2.2 be satisfied. Then,*

$$u(t) \le \left[\Omega^{-1}\left(\Omega(A(t)) + \mathcal{B}(t)\int_{t_0}^t \Psi'(s)F(s)^q ds\right)\right]^{1/q}, \quad t \in [t_0, t_1], \tag{2.11}$$

where $t_1 \in (t_0, T]$ is such that

$$\Omega(A(t_1)) + \mathcal{B}(t_1)\int_{t_0}^{t_1} \Psi'(s)F(s)^q ds \in \operatorname{Dom}\Omega^{-1},$$

$\Omega(v) = \int_{v_0}^v \frac{1}{[\omega(\sigma^{1/q})]^q} d\sigma$ *for $v \ge v_0 \ge 0$, Ω^{-1} is the inverse of Ω, and*

$$\begin{gathered} A(t) = 2^{q-1}\sup_{s\in[t_0,t]} a(s)^q, \quad \mathcal{B}(t) = 2^{q-1}\sup_{s\in[t_0,t]}\{b(s)^q \mathcal{M}_{p,\alpha}(s)^q\}, \\ \mathcal{M}_{p,\alpha}(t) = \left(\frac{[\Psi(t)-\Psi(t_0)]^{p(\alpha-1)+1}}{p(\alpha-1)+1}\right)^{1/p}, \quad p = \frac{q}{q-1}. \end{gathered} \tag{2.12}$$

Proof. The idea of the proof follows the proof of Theorem 2.2. This time, instead of Lemma 2.3, we employ the change of the dummy variable $\Psi(t) - \Psi(s) = \sigma$ to get the equality

$$\int_{t_0}^t [\Psi(t) - \Psi(s)]^{p(\alpha-1)}\Psi'(s)ds = \int_0^{\Psi(t)-\Psi(t_0)} \sigma^{p(\alpha-1)} d\sigma = \mathcal{M}_{p,\alpha}(t)^p.$$

So, instead of (2.5), we have

$$\begin{aligned} &\int_{t_0}^t [\Psi(t) - \Psi(s)]^{\alpha-1}\Psi'(s)F(s)\omega(u(s))ds \\ &\quad \le \left(\int_{t_0}^t [\Psi(t) - \Psi(s)]^{p(\alpha-1)}\Psi'(s)ds\right)^{1/p} \left(\int_{t_0}^t F(s)^q\Psi'(s)\omega(u(s))^q ds\right)^{1/q} \\ &\quad = \mathcal{M}_{p,\alpha}(t)\left(\int_{t_0}^t F(s)^q\Psi'(s)\omega(u(s))^q ds\right)^{1/q}. \end{aligned} \tag{2.13}$$

Therefore, instead of (2.6), we have

$$u(t) \le a(t) + b(t)\mathcal{M}_{p,\alpha}(t)\left(\int_{t_0}^{t} F(s)^q \Psi'(s)\omega(u(s))^q ds\right)^{1/q}.$$

The proof is then completed as the proof of Theorem 2.2. □

Remark 2.12. Let us assume that $\alpha \ge 1$. Then the integral in (1.1) does not contain a singular kernel, so we have

$$u(t) \le a(t) + b(t)(\Psi(t) - \Psi(t_0))^{\alpha-1} \int_{t_0}^{t} \Psi'(s)F(s)\omega(u(s))ds, \quad t \in [t_0, T]$$

and Theorem 2.1 yields

$$u(t) \le \Omega^{-1}\left(\Omega(A(t)) + B(t)\int_{t_0}^{t} \Psi'(s)F(s)ds\right), \quad t \in [t_0, t_1], \tag{2.14}$$

where $t_1 \in (t_0, T]$ is such that

$$\Omega(A(t_1)) + B(t_1)\int_{t_0}^{t_1} \Psi'(s)F(s)ds \in \operatorname{Dom}\Omega^{-1},$$

$\Omega(v) = \int_{v_0}^{v} \frac{d\sigma}{\omega(\sigma)}$ for $v \ge v_0 \ge 0$, Ω^{-1} is the inverse of Ω, and

$$A(t) = \sup_{s\in[t_0,t]} a(s), \quad B(t) = \sup_{s\in[t_0,t]} \{b(s)(\Psi(s) - \Psi(t_0))^{\alpha-1}\}.$$

On the other side, the proof of Theorem 2.11 now works for any $q > 1$, i.e., estimation (2.11) holds for any $q > 1$. Taking the limit $q \to 1^+$ (i.e., $p \to \infty$) in (2.11) gives exactly (2.14), since

$$\begin{aligned}\lim_{q\to 1^+} \mathcal{M}_{p,\alpha}(s)^q &= \lim_{q\to 1^+} \frac{(\Psi(s) - \Psi(t_0))^{q\alpha-1}}{(p(\alpha-1)+1)^{q-1}} \\ &= \lim_{q\to 1^+} \frac{(\Psi(s) - \Psi(t_0))^{q\alpha-1}}{\left(\frac{q\alpha-1}{q-1}\right)^{q-1}} = (\Psi(s) - \Psi(t_0))^{\alpha-1}\end{aligned}$$

if $\alpha > 1$, and

$$\lim_{q\to 1^+} \mathcal{M}_{p,\alpha}(s)^q = (\Psi(s) - \Psi(t_0))^{q-1} = 1$$

if $\alpha = 1$.

Analogues of Corollaries 2.5, 2.6, and 2.7 are stated below.

Corollary 2.13. *If all the assumptions of Corollary 2.5 are satisfied, then*

$$u(t) \le A(t)^{1/q} \exp\left\{\frac{\mathcal{B}(t)}{q}\int_{t_0}^{t} F(s)^q \Psi'(s)ds\right\}, \quad t \in [t_0, T],$$

where functions A, $\mathcal{B}$ are given by (2.12).

Proof. The statement follows from Theorem 2.11 by setting $\omega(u) = u$ (cf. the proof of Corollary 2.5). □

Corollary 2.14. *If all the assumptions of Corollary 2.6 are satisfied, then the following assertions hold:*

1. *if* $0 < m < 1$, *then*

$$u(t) \le \left(A(t)^{1-m} + (1-m)\mathcal{B}(t) \int_{t_0}^{t} F(s)^q \Psi'(s) ds \right)^{\frac{1}{q(1-m)}}, \quad t \in [t_0, T];$$

2. *if* $m > 1$, *then*

$$u(t) \le \frac{A(t)^{1/q}}{\left(1 - (m-1)A(t)^{m-1}\mathcal{B}(t) \int_{t_0}^{t} F(s)^q \Psi'(s) ds\right)^{\frac{1}{q(m-1)}}}, \quad t \in [t_0, t_1]$$

for any $t_1 \in (t_0, T]$ *such that*

$$\int_{t_0}^{t_1} F(s)^q \Psi'(s) ds < \frac{1}{(m-1)A(t_1)^{m-1}\mathcal{B}(t_1)};$$

where functions A, $\mathcal{B}$ *are given by* (2.12).

Proof. The statement follows from Theorem 2.11 by setting $\omega(u) = u^m$ (cf. the proof of Corollary 2.6). □

Corollary 2.15. *If all the assumptions of Corollary 2.7 are satisfied, then*

$$u(t) \le \left[\Omega^{-1} \left(\Omega(A(t)) + \mathcal{B}(t) \int_{t_0}^{t} \Psi'(s) F(s)^q ds \right) \right]^{1/q}, \quad t \in [t_0, t_1],$$

where $t_1 \in (t_0, T]$ *is such that*

$$\Omega(A(t_1)) + \mathcal{B}(t_1) \int_{t_0}^{t_1} \Psi'(s) F(s)^q ds \in \operatorname{Dom} \Omega^{-1},$$

$\Omega(v) = \int_{v_0}^{v} \frac{1}{[\omega(\sigma^{1/q})]^q} d\sigma$ *for* $v \ge v_0 \ge 0$, Ω^{-1} *is the inverse of* Ω, *and*

$$F = \max_{i=1,2,\dots,n} F_i, \quad \omega = \max_{i=1,2,\dots,n} \omega_i,$$

$$A(t) = 2^{q-1} \sup_{s \in [t_0,t]} a(s)^q, \quad \mathcal{B}(t) = 2^{q-1} \sup_{s \in [t_0,t]} \left(\sum_{i=1}^{n} b_i(s) \mathcal{M}_{p,\alpha_i}(s) \right)^q,$$

$$\mathcal{M}_{p,\alpha}(t) = \left(\frac{[\Psi(t) - \Psi(t_0)]^{p(\alpha-1)+1}}{p(\alpha-1)+1} \right)^{1/p}, \quad p = \frac{q}{q-1}.$$

Proof. Analogously to the proof of Corollary 2.7, but using estimation (2.13) instead of (2.5), the right-hand side of (1.2) can be estimated by

$$a(t) + \sum_{i=1}^{n} b_i(t) \mathcal{M}_{p,\alpha_i}(t) \left(\int_{t_0}^{t} F(s)^q \Psi'(s) \omega(u(s))^q ds \right)^{1/q}.$$

Then the proof is completed as the proof of Theorem 2.2, by taking the q-th power and using Theorem 2.1. □

Next, we give a result on inequality (1.3).

Theorem 2.16. *Let all the assumptions of Theorem 2.8 be satisfied. Then,*

$$u(t) \le \left[\Omega^{-1}\left(\Omega(A(t)) + \mathcal{B}(t)\int_{t_0}^{t}\Psi'(s)G(s)^q ds\right)\right]^{1/q}, \quad t \in [t_0, t_1],$$

where $t_1 \in (t_0, T]$ *is such that*

$$\Omega(A(t_1)) + \mathcal{B}(t_1)\int_{t_0}^{t_1}\Psi'(s)G(s)^q ds \in \operatorname{Dom}\Omega^{-1},$$

$\Omega(v) = \int_{v_0}^{v}\frac{1}{[\omega(\sigma^{1/q})]^q}d\sigma$ *for* $v \ge v_0 \ge 0$, Ω^{-1} *is the inverse of* Ω, *and*

$$\begin{aligned} G &= \max\{F_1, F_2\}, \quad \omega = \max\{\omega_1, \omega_2\}, \quad A(t) = 2^{q-1}\sup_{s\in[t_0,t]} a(s)^q, \\ \mathcal{B}(t) &= 2^{q-1}\sup_{s\in[t_0,t]}\left\{b_1(s)\mathcal{M}_{p,\alpha}(s) + B(\alpha,\beta)b_2(s)\mathcal{M}_{p,\alpha+\beta}(s)\sup_{\sigma\in[t_0,s]}c(\sigma)\right\}^q, \\ \mathcal{M}_{p,\alpha}(t) &= \left(\frac{[\Psi(t)-\Psi(t_0)]^{p(\alpha-1)+1}}{p(\alpha-1)+1}\right)^{1/p}, \quad p = \frac{q}{q-1}. \end{aligned} \tag{2.15}$$

Proof. We follow the proof of Theorem 2.8. Instead of (2.5), we apply estimation (2.13) to arrive at

$$\begin{aligned} u(t) &\le a(t) + b_1(t)\mathcal{M}_{p,\alpha}(t)\left(\int_{t_0}^{t}G(s)^q\Psi'(s)\omega(u(s))^q ds\right)^{1/q} \\ &\quad + B(\alpha,\beta)b_2(t)\sup_{s\in[t_0,t]}c(s)\int_{t_0}^{t}[\Psi(t)-\Psi(\sigma)]^{\alpha+\beta-1}\Psi'(\sigma)G(\sigma)\omega(u(\sigma))d\sigma \\ &\le a(t) + b_1(t)\mathcal{M}_{p,\alpha}(t)\left(\int_{t_0}^{t}G(s)^q\Psi'(s)\omega(u(s))^q ds\right)^{1/q} \\ &\quad + B(\alpha,\beta)b_2(t)\mathcal{M}_{p,\alpha+\beta}(t)\sup_{s\in[t_0,t]}c(s)\left(\int_{t_0}^{t}G(s)^q\Psi'(s)\omega(u(s))^q ds\right)^{1/q}. \end{aligned}$$

The proof is completed as the proof of Theorem 2.2. □

A result on inequality (1.4) follows.

Theorem 2.17. *Let all the assumptions of Theorem 2.9 be satisfied. Then,*

$$\begin{aligned} u(t) \le \Bigg[\Omega^{-1}\Bigg(&\Omega(A(t)) + \mathcal{B}(t)\int_{t_0}^{t}G(s)^q\Psi'(s)ds \\ &+ C(t)\int_{t_0}^{t}F_2(s)^q\Psi'(s)ds\Bigg)\Bigg]^{1/q}, \quad t \in [t_0, t_1], \end{aligned}$$

where $t_1 \in (t_0, T]$ *is such that*

$$\Omega(A(t_1)) + \mathcal{B}(t_1)\int_{t_0}^{t_1}G(s)^q\Psi'(s)ds + C(t_1)\int_{t_0}^{t_1}F_2(s)^q\Psi'(s)ds \in \operatorname{Dom}\Omega^{-1},$$

$\Omega(v) = \int_{v_0}^{v} \frac{1}{[\omega(\sigma^{1/q})]^q} d\sigma$ *for* $v \geq v_0 \geq 0$, Ω^{-1} *is the inverse of* Ω, *and*

$$\begin{gathered} G = \max\{F_1, F_3\}, \quad \omega = \max\{\omega_1, \omega_2, \omega_3\}, \quad b = \max\{b_1, b_3\}, \\ A(t) = 3^{q-1} \sup_{s\in[t_0,t]} a(s)^q, \quad \mathcal{B}(t) = 3^{q-1} \sup_{s\in[t_0,t]} \left\{b(s)\mathcal{M}_p(s)\right\}^q, \\ C(t) = 3^{q-1} \sup_{s\in[t_0,t]} \left\{b_2(s)\mathcal{M}_{p,\alpha}(s)\right\}^q, \quad p = \frac{q}{q-1}, \\ \mathcal{M}_p = \max\{\mathcal{M}_{p,\alpha}, \mathcal{M}_{p,\beta}\}, \quad \mathcal{M}_{p,\alpha}(t) = \left(\frac{[\Psi(t) - \Psi(t_0)]^{p(\alpha-1)+1}}{p(\alpha-1)+1}\right)^{1/p}. \end{gathered} \tag{2.16}$$

Proof. We follow the proof of Theorem 2.9, but now, we apply estimation (2.13) instead of (2.5) to obtain

$$\begin{aligned} u(t) \leq\ & a(t) + b(t)\mathcal{M}_p(t) \left(\int_{t_0}^{t} G(s)^q \Psi'(s) \omega(u(s))^q ds\right)^{1/q} \\ & + b_2(t)\mathcal{M}_{p,\alpha}(t) \left(\int_{t_0}^{t} F_2(s)^q \Psi'(s) \right. \\ & \left. \times\, \omega\left(b(s)\mathcal{M}_p(s) \left(\int_{t_0}^{s} G(\sigma)^q \Psi'(\sigma) \omega(u(\sigma))^q d\sigma\right)^{1/q}\right)^q ds\right)^{1/q}. \end{aligned}$$

Taking the q-th power of this inequality and an application of Lemma 2.10 completes the proof, similarly to the proof of Theorem 2.9. □

3 Applications to fractional differential equations with Ψ-Caputo derivative

In this section, we apply our integral inequalities to various classes of initial value problems (IVPs) for fractional differential equations with Ψ-Caputo derivative, to derive results on the boundedness of the solutions and on the non-existence of blowing-up solutions. We note that in our results, the existence of solutions is assumed. For results on the existence we refer the reader to [1, 20].

Nevertheless, we start with the following theoretical example to show that, in different values of t, each one of Theorems 2.2 and 2.11 might give a better estimation than the other.

Example 3.1. Let a and u be non-negative continuous functions on $[0, T]$ for some $0 < T \leq \infty$, $\omega \in \mathfrak{O}$, and u satisfy

$$u(t) \leq a(t) + \mathrm{e}^{-t} \int_0^t (t-s)^{-\frac{1}{4}} \omega(u(s)) ds, \quad t \in [0, T]. \tag{3.1}$$

In this case, $\alpha = 3/4$, $b(t) = \mathrm{e}^{-t}$, $\Psi(t) = t$, $F(t) \equiv 1$, and $t_0 = 0$. Let us take $q = 2$, implying $p = 2$, and $\mu > 0$. Let Ω, Ω^{-1}, A be as in Theorem 2.2. By (2.3), $M_{p,\mu,\alpha} = \sqrt[4]{\frac{\pi}{2\mu}}$ and

$$B(t) = 2\sqrt{\frac{\pi}{2\mu}} \sup_{s\in[0,t]} \left\{\mathrm{e}^{-2s}\, \mathrm{e}^{2\mu s}\right\} = \begin{cases} \sqrt{\frac{2\pi}{\mu}}, & 0 < \mu \leq 1, \\ \sqrt{\frac{2\pi}{\mu}}\, \mathrm{e}^{2(\mu-1)t}, & \mu > 1. \end{cases}$$

Hence, Theorem 2.2 gives

$$\Omega(u(t)^2) - \Omega(A(t)) \le B(t) \int_0^t \mathrm{e}^{-2\mu s}\, ds = \begin{cases} \sqrt{\frac{\pi}{2\mu}} \frac{1-\mathrm{e}^{-2\mu t}}{\mu}, & 0 < \mu \le 1, \\ \sqrt{\frac{\pi}{2\mu}} \frac{(\mathrm{e}^{2\mu t}-1)\,\mathrm{e}^{-2t}}{\mu}, & \mu > 1 \end{cases} \tag{3.2}$$

for all $t \in [0, t_1]$, where $t_1 \in (0, T]$ is such that

$$\Omega(A(t_1)) + \sqrt{\frac{\pi}{2\mu}} \frac{1-\mathrm{e}^{-2\mu t_1}}{\mu} \in \operatorname{Dom} \Omega^{-1}$$

if $0 < \mu \le 1$, and

$$\Omega(A(t_1)) + \sqrt{\frac{\pi}{2\mu}} \frac{(\mathrm{e}^{2\mu t_1}-1)\,\mathrm{e}^{-2t_1}}{\mu} \in \operatorname{Dom} \Omega^{-1}$$

if $\mu > 1$.

On the other side, in the notation of (2.12), $\mathcal{M}_{p,\alpha}(t) = \sqrt[4]{4t}$ and

$$\mathcal{B}(t) = 2 \sup_{s\in[0,t]} \left\{2\sqrt{s}\,\mathrm{e}^{-2s}\right\} = 4 \sup_{s\in[0,t]} \left\{\sqrt{s}\,\mathrm{e}^{-2s}\right\} = \begin{cases} 4\sqrt{t}\,\mathrm{e}^{-2t}, & 0 \le t \le \frac{1}{4}, \\ \frac{2}{\sqrt{\mathrm{e}}}, & t \ge \frac{1}{4}. \end{cases}$$

Therefore, by Theorem 2.11,

$$\Omega(u(t)^2) - \Omega(A(t)) \le \mathcal{B}(t) \int_0^t 1 ds = \begin{cases} 4t^{\frac{3}{2}}\,\mathrm{e}^{-2t}, & 0 \le t \le \frac{1}{4}, \\ \frac{2t}{\sqrt{\mathrm{e}}}, & t > \frac{1}{4} \end{cases} \tag{3.3}$$

for all $t \in [0, t_1]$, where $t_1 \in (0, T]$ is such that

$$\Omega(A(t_1)) + 4t_1^{\frac{3}{2}}\,\mathrm{e}^{-2t_1} \in \operatorname{Dom} \Omega^{-1}$$

if $t_1 \le \frac{1}{4}$, and

$$\Omega(A(t_1)) + \frac{2t_1}{\sqrt{\mathrm{e}}} \in \operatorname{Dom} \Omega^{-1}$$

if $t_1 > \frac{1}{4}$.

Let us denote $E_1(t)$, $E_2(t)$ the right-hand sides of estimations (3.2), (3.3), respectively. For better illustration, graphs of these functions are given in Figure 3.1.

Having a concrete function $\omega \in \mathfrak{O}$ in (3.1), one obtains an explicit estimation of u. For instance, if $\omega(u) = u$, then function $\Omega(v) = \ln \frac{v}{v_0}$ (see the proof of Corollary 2.5) maps $[v_0, \infty)$ for $v_0 > 0$ onto $[0, \infty)$. Hence,

$$u(t) \le \left(v_0\, \mathrm{e}^{\ln A(t) - \ln v_0 + E_i(t)}\right)^{1/2} = \sqrt{A(t)\,\mathrm{e}^{E_i(t)}}, \quad t \in [0, T]$$

for each $i = 1, 2$. As another example, consider $\omega(u) = \sqrt{u}$. Then, $\Omega(v) = 2(\sqrt{v} - \sqrt{v_0})$ maps $[v_0, \infty)$ for $v_0 \ge 0$ onto $[0, \infty)$. Thus, $\Omega^{-1}(v) = (\sqrt{v_0} + v/2)^2$ and

$$u(t) \le \sqrt{v_0} + \frac{1}{2}\left(2(\sqrt{A(t)} - \sqrt{v_0}) + E_i(t)\right) = \sqrt{A(t)} + \frac{E_i(t)}{2}, \quad t \in [0, T]$$

for each $i = 1, 2$.

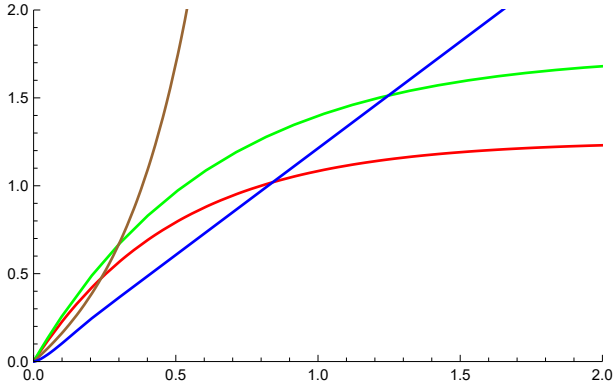


Figure 3.1: Graphs of the right-hand sides of (3.2) ($\mu = 0.8$ – green, $\mu = 1$ – red, $\mu = 3$ – brown) and (3.3) (blue).

Now, we investigate the IVPs with Ψ-Caputo fractional derivative. First, we consider the following one

$$ {}^{C}D^{\alpha,\Psi}_{t_0} x(t) = f(t, x(t)), \quad t > t_0, \tag{3.4} $$

$$ x(t_0) = x_0 \tag{3.5} $$

for some constant $x_0 \in \mathbb{R}^N$, $N \in \mathbb{N}$, where $\alpha \in (0,1)$, $\Psi \in \mathfrak{P}^1_{[t_0,\infty)}$, and $f \in C([t_0,\infty) \times \mathbb{R}^N, \mathbb{R}^N)$.

Definition 3.2. A mapping $x \in C([t_0, t_0+h], \mathbb{R}^N)$, $0 < h \leq \infty$ is a solution of IVP (3.4), (3.5), if ${}^{C}D^{\alpha,\Psi}_{a} x(t)$ exists and is continuous on $(t_0, t_0+h]$, x fulfills equation (3.4) and initial condition (3.5). The solution x is called blowing-up, if it is defined on $[t_0, \tau)$ for some $t_0 < \tau < \infty$ and $\lim_{t\to\tau^-} \|x(t)\| = \infty$.

Theorem 3.3. *Let F be a non-negative continuous function defined on $[t_0, T]$, and $\omega \in \mathfrak{O}$ be such that*

$$ \|f(t,x)\| \leq F(t)\omega(\|x\|), \quad t \in [t_0, T],\ x \in \mathbb{R}^N. $$

If x is a solution of (3.4), (3.5) defined on $[t_0, T]$, $\alpha q > 1$, $\mu > 0$, then it satisfies

1. *the inequality*

$$ \|x(t)\| \leq \left[\Omega^{-1}\left(\Omega(2^{q-1}\|x_0\|^q) + \frac{2^{q-1}M^q_{p,\mu,\alpha}\, \mathrm{e}^{q\mu\Psi(t)}}{\Gamma(\alpha)^q}\int_{t_0}^{t} \Psi'(s)F(s)^q\, \mathrm{e}^{-q\mu\Psi(s)}\, ds\right)\right]^{1/q} $$

for all $t \in [t_0, t_1]$, where $M_{p,\mu,\alpha}$ is given by (2.3) and $t_1 \in (t_0, T]$ is such that

$$ \Omega(2^{q-1}\|x_0\|^q) + \frac{2^{q-1}M^q_{p,\mu,\alpha}\, \mathrm{e}^{q\mu\Psi(t_1)}}{\Gamma(\alpha)^q}\int_{t_0}^{t_1} \Psi'(s)F(s)^q\, \mathrm{e}^{-q\mu\Psi(s)}\, ds \in \operatorname{Dom}\Omega^{-1}; $$

2. *the inequality*

$$ \|x(t)\| \leq \left[\Omega^{-1}\left(\Omega(2^{q-1}\|x_0\|^q) + \frac{2^{q-1}\mathcal{M}_{p,\alpha}(t)^q}{\Gamma(\alpha)^q}\int_{t_0}^{t} \Psi'(s)F(s)^q ds\right)\right]^{1/q} $$

for all $t \in [t_0, t_1]$, where $\mathcal{M}_{p,\alpha}$ is given by (2.12) and $t_1 \in (t_0, T]$ is such that

$$ \Omega(2^{q-1}\|x_0\|^q) + \frac{2^{q-1}\mathcal{M}_{p,\alpha}(t_1)^q}{\Gamma(\alpha)^q}\int_{t_0}^{t_1} \Psi'(s)F(s)^q ds \in \operatorname{Dom}\Omega^{-1}, $$

where $\Omega(v) = \int_{v_0}^{v} \frac{1}{[\omega(\sigma^{1/q})]^q} d\sigma$ *for* $v \geq v_0 \geq 0$, *and* Ω^{-1} *is the inverse of* Ω.

Proof. From [1, Theorem 2] (see also [17, Theorem 2]), we know that x satisfies the integral equation

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} [\Psi(t) - \Psi(s)]^{\alpha-1} \Psi'(s) f(s, x(s)) ds$$

for all $t \in [t_0, T]$. Then, for the norm of the solution, we have

$$\begin{aligned} \|x(t)\| &\leq \|x_0\| + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} [\Psi(t) - \Psi(s)]^{\alpha-1} \Psi'(s) \|f(s, x(s))\| ds \\ &\leq \|x_0\| + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} [\Psi(t) - \Psi(s)]^{\alpha-1} \Psi'(s) F(s) \omega(\|x(s)\|) ds \end{aligned}$$

for any $t \in [t_0, T]$. Applying Theorems 2.2, 2.11 proves the first and the second statement, respectively. □

Theorem 3.4. *Let all the assumptions of Theorem 3.3 be fulfilled with* $T = \infty$ *and*

$$\int_{v_0}^{\infty} \frac{d\sigma}{[\omega(\sigma^{1/q})]^q} = \infty \tag{3.6}$$

for some $v_0 > 0$. *Then* (3.4), (3.5) *does not possess a blowing-up solution.*

Proof. Let x be a blowing-up solution of (3.4), (3.5) which is defined on $[t_0, b)$ for some $t_0 < b < \infty$, i.e., $\lim_{t \to b^-} \|x(t)\| = \infty$. Notice that condition (3.6) means $\Omega(\infty) = \infty$ for Ω as in the Theorem 3.3. In other words, $[0, \infty) \subset \Omega^{-1}$, and Statement 1 of Theorem 3.3 holds for any $t_1 \in [t_0, b)$, $\mu = 1$. But this is equivalent to

$$\Omega(\|x(t)^q\|) \leq \Omega(2^{q-1}\|x_0\|^q) + \frac{2^{q-1} M_{p,1,\alpha}^q \, \mathrm{e}^{q\Psi(t)}}{\Gamma(\alpha)^q} \int_{t_0}^{t} \Psi'(s) F(s)^q \, \mathrm{e}^{-q\Psi(s)} \, ds.$$

Taking the limit $t \to b^-$ gives a contradiction, since the right-hand side is bounded. Therefore, $b = \infty$ and the proof is complete. □

The following example illustrates the use of the latter result.

Example 3.5. Let x be a solution of initial value problem (3.4), (3.5), and the assumptions of Theorem 3.3 be satisfied with

$$\omega(u) = u^{\frac{q-1}{q}} \left(\ln(1+u)\right)^{\frac{1}{q}}.$$

Then x is not blowing-up.

Indeed, for $v_0 > 0$, we have

$$\begin{aligned} \int_{v_0}^{\infty} \frac{d\sigma}{[\omega(\sigma^{1/q})]^q} &= q \int_{v_0^{1/q}}^{\infty} \frac{\tau^{q-1} d\tau}{[\omega(\tau)]^q} = q \int_{v_0^{1/q}}^{\infty} \frac{\tau^{q-1} d\tau}{\tau^{q-1} \ln(1+\tau)} \\ &= q \int_{v_0^{1/q}}^{\infty} \frac{d\tau}{\ln(1+\tau)} \geq q \int_{v_0^{1/q}}^{\infty} \frac{d\tau}{\tau} = \infty. \end{aligned}$$

So, the assumptions of Theorem 3.4 are satisfied.

Next, we consider the IVP with a Ψ-Hilfer fractional integral on the right side,

$$ {}^{C}D_{t_0}^{\alpha,\Psi}x(t) = f(t,x(t)) + c(t)I_{t_0}^{\beta,\Psi}g(t,x(t)), \quad t > t_0, \tag{3.7} $$

$$ x(t_0) = x_0 \tag{3.8} $$

for $x_0 \in \mathbb{R}^N$, $N \in \mathbb{N}$, where $\alpha, \beta \in (0,1)$, $\alpha + \beta < 1$, $\Psi \in \mathfrak{P}^1_{[t_0,\infty)}$, $c \in C([t_0,\infty),\mathbb{R})$, and $f, g \in C([t_0,\infty)\times\mathbb{R}^N,\mathbb{R}^N)$.

Theorem 3.6. *Let F, G be non-negative continuous functions defined on $[t_0,T]$, and $\omega,\omega_2 \in \mathfrak{O}$ be such that*

$$ \begin{aligned} \|f(t,x)\| &\le F(t)\omega_1(\|x\|), \\ \|g(t,x)\| &\le G(t)\omega_2(\|x\|), \end{aligned} \qquad t \in [t_0,T],\ x \in \mathbb{R}^N. $$

If x is a solution of (3.7), (3.8) defined on $[t_0,T]$, $\alpha q > 1$, $\mu > 0$, then it satisfies

1. *the inequality*

$$ \|x(t)\| \le \left[\Omega^{-1}\left(\Omega(2^{q-1}\|x_0\|^q) + B(t)\int_{t_0}^{t}\Psi'(s)H(s)^q\,\mathrm{e}^{-q\mu\Psi(s)}\,ds\right)\right]^{1/q} $$

for all $t \in [t_0,t_1]$, where

$$ B(t) = 2^{q-1}\sup_{s\in[t_0,t]}\left\{\mathrm{e}^{q\mu\Psi(s)}\left(\frac{M_{p,\mu,\alpha}}{\Gamma(\alpha)} + \frac{M_{p,\mu,\alpha+\beta}}{\Gamma(\alpha+\beta)}\sup_{\sigma\in[t_0,s]}|c(\sigma)|\right)^q\right\}, $$

$M_{p,\mu,\alpha}$ is given by (2.7), and $t_1 \in (t_0,T]$ is such that

$$ \Omega(2^{q-1}\|x_0\|^q) + B(t_1)\int_{t_0}^{t_1}\Psi'(s)H(s)^q\,\mathrm{e}^{-q\mu\Psi(s)}\,ds \in \operatorname{Dom}\Omega^{-1}; $$

2. *the inequality*

$$ \|x(t)\| \le \left[\Omega^{-1}\left(\Omega(2^{q-1}\|x_0\|^q) + \mathcal{B}(t)\int_{t_0}^{t}\Psi'(s)H(s)^q ds\right)\right]^{1/q} $$

for all $t \in [t_0,t_1]$, where

$$ \mathcal{B}(t) = 2^{q-1}\sup_{s\in[t_0,t]}\left\{\frac{\mathcal{M}_{p,\alpha}(s)}{\Gamma(\alpha)} + \frac{\mathcal{M}_{p,\alpha+\beta}(s)}{\Gamma(\alpha+\beta)}\sup_{\sigma\in[t_0,s]}|c(\sigma)|\right\}^q, $$

$\mathcal{M}_{p,\alpha}$ is given by (2.15), and $t_1 \in (t_0,T]$ is such that

$$ \Omega(2^{q-1}\|x_0\|^q) + \mathcal{B}(t_1)\int_{t_0}^{t_1}\Psi'(s)H(s)^q ds \in \operatorname{Dom}\Omega^{-1}, $$

where $H = \max\{F,G\}$, $\omega = \max\{\omega_1,\omega_2\}$, $\Omega(v) = \int_{v_0}^{v}\frac{1}{[\omega(\sigma^{1/q})]^q}d\sigma$ for $v \ge v_0 \ge 0$, and Ω^{-1} is the inverse of Ω.

Proof. As in the proof of Theorem 3.3, we use the integral equation for the solution of (3.7), (3.8), and we estimate its norm using the assumptions on f, g to obtain

$$\begin{aligned}\|x(t)\| \le \|x_0\| &+ \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}[\Psi(t)-\Psi(s)]^{\alpha-1}\Psi'(s)F(s)\omega_1(\|x(s)\|)ds\\ &+ \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}[\Psi(t)-\Psi(s)]^{\alpha-1}|c(s)|\\ &\times\frac{1}{\Gamma(\beta)}\int_{t_0}^{s}[\Psi(s)-\Psi(\sigma)]^{\beta-1}G(\sigma)\omega_2(\|x(\sigma)\|)d\sigma\,ds\end{aligned}$$

for any $t\in[t_0,T]$. The application of Theorems 2.8 and 2.16 proves the statements. □

Theorem 3.7. *Let all the assumptions of Theorem 3.6 be fulfilled with $T=\infty$ and $\Omega(\infty)=\infty$ for Ω as in Theorem 3.6. Then (3.7), (3.8) does not possess a blowing-up solution.*

Proof. The statement can be proved exactly as Theorem 3.4 with the use of Theorem 3.6 instead of Theorem 3.3. □

Finally, we consider the more general equation with the right-hand side depending on the fractional integral of the solution,

$$\begin{aligned}{}^{C}D_{t_0}^{\alpha,\Psi}x(t) &= f(t,x(t),I_{t_0}^{\beta,\Psi}g(t,x(t))), \quad t>t_0, &(3.9)\\ x(t_0) &= x_0 &(3.10)\end{aligned}$$

for $x_0\in\mathbb{R}^N$, $N\in\mathbb{N}$, where $\alpha,\beta\in(0,1)$, $\Psi\in\mathfrak{P}^1_{[t_0,\infty)}$, and $f\in C([t_0,\infty)\times\mathbb{R}^N\times\mathbb{R}^N,\mathbb{R}^N)$, $g\in C([t_0,\infty)\times\mathbb{R}^N,\mathbb{R}^N)$.

Theorem 3.8. *Let F, G be non-negative continuous functions defined on $[t_0,T]$, and $\omega_1,\omega_2,\omega_3\in\mathfrak{O}$ be such that*

$$\begin{aligned}\|f(t,x,y)\| &\le F(t)(\omega_1(\|x\|)+\omega_2(\|y\|)), \quad t\in[t_0,T],\ x,y\in\mathbb{R}^N,\\ \|g(t,x)\| &\le G(t)\omega_3(\|x\|), \quad t\in[t_0,T],\ x\in\mathbb{R}^N.\end{aligned}$$

If x is a solution of (3.9), (3.10) defined on $[t_0,T]$, $q\min\{\alpha,\beta\}>1$, $\mu>0$, then it satisfies

1. *the inequality*

$$\begin{aligned}\|x(t)\| \le \Bigg[\Omega^{-1}\Big(&\Omega(3^{q-1}\|x_0\|^q)\\ &+3^{q-1}M_{p,\mu}^q\,\mathrm{e}^{q\mu\Psi(t)}\max\left\{\frac{1}{\Gamma(\alpha)},\frac{1}{\Gamma(\beta)}\right\}^q\int_{t_0}^{t}H(s)^q\,\mathrm{e}^{-q\mu\Psi(s)}\,\Psi'(s)ds\\ &+\frac{3^{q-1}M_{p,\mu,\alpha}^q\,\mathrm{e}^{q\mu\Psi(t)}}{\Gamma(\alpha)^q}\int_{t_0}^{t}F(s)^q\,\mathrm{e}^{-q\mu\Psi(s)}\,\Psi'(s)ds\Big)\Bigg]^{1/q}\end{aligned}$$

for all $t\in[t_0,t_1]$, where $M_{p,\mu}=\max\{M_{p,\mu,\alpha},M_{p,\mu,\beta}\}$, $M_{p,\mu,\alpha}$ is given by (2.8), and $t_1\in(t_0,T]$ is such that

$$\begin{aligned}&\Omega(3^{q-1}\|x_0\|^q)+3^{q-1}M_{p,\mu}^q\,\mathrm{e}^{q\mu\Psi(t_1)}\max\left\{\frac{1}{\Gamma(\alpha)},\frac{1}{\Gamma(\beta)}\right\}^q\int_{t_0}^{t_1}H(s)^q\,\mathrm{e}^{-q\mu\Psi(s)}\,\Psi'(s)ds\\ &+\frac{3^{q-1}M_{p,\mu,\alpha}^q\,\mathrm{e}^{q\mu\Psi(t_1)}}{\Gamma(\alpha)^q}\int_{t_0}^{t_1}F(s)^q\,\mathrm{e}^{-q\mu\Psi(s)}\,\Psi'(s)ds\in\operatorname{Dom}\Omega^{-1};\end{aligned}$$

2. *the inequality*

$$\begin{aligned}\|x(t)\| \leq \Bigg[&\Omega^{-1}\Big(\Omega(3^{q-1}\|x_0\|^q) \\ &+ 3^{q-1}\mathcal{M}_p(t)^q \max\left\{\frac{1}{\Gamma(\alpha)}, \frac{1}{\Gamma(\beta)}\right\}^q \int_{t_0}^t H(s)^q \Psi'(s) ds \\ &+ \frac{3^{q-1}\mathcal{M}_{p,\alpha}(t)^q}{\Gamma(\alpha)^q} \int_{t_0}^t F(s)^q \Psi'(s) ds\Big)\Bigg]^{1/q}\end{aligned}$$

for all $t \in [t_0, t_1]$, *where* $\mathcal{M}_p = \max\{\mathcal{M}_{p,\alpha}, \mathcal{M}_{p,\beta}\}$, $\mathcal{M}_{p,\alpha}$ *is given by* (2.16), *and* $t_1 \in (t_0, T]$ *is such that*

$$\begin{aligned}&\Omega(3^{q-1}\|x_0\|^q) + 3^{q-1}\mathcal{M}_p(t_1)^q \max\left\{\frac{1}{\Gamma(\alpha)}, \frac{1}{\Gamma(\beta)}\right\}^q \int_{t_0}^{t_1} H(s)^q \Psi'(s) ds \\ &+ \frac{3^{q-1}\mathcal{M}_{p,\alpha}(t_1)^q}{\Gamma(\alpha)^q} \int_{t_0}^{t_1} F(s)^q \Psi'(s) ds \in \operatorname{Dom} \Omega^{-1}\end{aligned}$$

where $H = \max\{F, G\}$, $\omega = \max\{\omega_1, \omega_2, \omega_3\}$, $\Omega(v) = \int_{v_0}^v \frac{1}{[\omega(\sigma^{1/q})]^q} d\sigma$ *for* $v \geq v_0 \geq 0$, *and* Ω^{-1} *is the inverse of* Ω.

Proof. Using the estimation

$$\|f(t, x(t), I_{t_0}^{\beta,\Psi} g(t, x(t)))\| \leq F(t)\left(\omega_1(\|x(t)\|) + \omega_2(I_{t_0}^{\beta,\Psi}(G(t)\omega_3(\|x(t)\|)))\right)$$

following from the assumptions on f and g, and an appropriate integral equation (see [17, Theorem 2] or the proof of Theorem 3.3), we get

$$\begin{aligned}\|x(t)\| \leq \|x_0\| &+ \frac{1}{\Gamma(\alpha)} \int_{t_0}^t [\Psi(t) - \Psi(s)]^{\alpha-1} \Psi'(s) F(s) \\ &\times \left(\omega_1(\|x(s)\|) + \omega_2\left(\frac{1}{\Gamma(\beta)} \int_{t_0}^s [\Psi(s) - \Psi(\sigma)]^{\beta-1} G(\sigma)\omega_3(\|x(\sigma)\|) d\sigma\right)\right) ds\end{aligned}$$

for any $t \in [t_0, T]$. The statements then follow from Theorems 2.9 and 2.17, respectively. □

Theorem 3.9. *Let all the assumptions of Theorem 3.8 be fulfilled with* $T = \infty$ *and* $\Omega(\infty) = \infty$ *for* Ω *as in Theorem 3.8. Then* (3.9), (3.10) *does not possess a blowing-up solution.*

Proof. The statement can be proved exactly as Theorem 3.4 with the use of Theorem 3.8 instead of Theorem 3.3. □

4 Conclusions and discussion

In this paper, the parametrized desingularization method and the desingularization method without an exponential function have been applied to derive new estimations of functions satisfying integral inequalities involving integrals with a weakly singular kernel, namely Ψ-Hilfer fractional integrals of order from $(0, 1)$. More precisely, nonlinear Henry–Gronwall integral inequalities (Theorems 2.2, 2.11), integral inequalities with an iterated fractional integral (Theorems 2.8, 2.16), and integral inequalities with a fractional integral of a function of a fractional integral (Theorems 2.9, 2.17) have been proved.

It should be mentioned, that all of the results of Section 2 are valid also for $\alpha \geq 1$, $\beta \geq 1$, or $\alpha + \beta \geq 1$ in Theorems 2.8, 2.16, although, in these cases, better estimations might exist, since the integrals are no more singular.

The integral inequalities have been applied to IVPs for fractional differential equations with Ψ-Caputo derivatives to obtain bounds for the solutions and to prove sufficient conditions for the non-existence of blowing-up solutions.

Further applications of our integral inequalities may be found in the study of stability or existence of solutions of generalizations of evolution equations investigated in [8, 13–15].

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On the set of eigenvalues for some classes of coercive and noncoercive problems involving $(2, p(x))$-Laplacian-like operators

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Abstract. We consider a class of double-phase nonlinear eigenvalue problems driven by a $(2, \phi)$-Laplace-like operator:

$$-\Delta u - \varepsilon \operatorname{div}[\phi(x, |\nabla u|)\nabla u] = \lambda(u + \varepsilon)$$

in a domain Ω, subject to Dirichlet boundary conditions, where Ω is a bounded subset of $\mathbb{R}^N$ with a smooth boundary. Here, $\varepsilon > 0$, and the potential function ϕ exhibits $p(x)$-variable growth.

We establish several results on the existence and concentration of eigenvalues for this problem, focusing on the influence of the growth behavior of the potential function ϕ, specifically through the interaction between the variable growth exponent $p(x)$ and the constant growth exponent 2. The proofs rely on variational arguments based on the Direct Method in the Calculus of Variations, Ekeland's variational principle, and energy estimates.

Keywords: double-phase differential operator, continuous bounded or unbounded spectrum, variable exponent, eigenvalue problem, coercive and noncoercive case.

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1 Introduction

The study of variational problems with nonstandard growth conditions has been extensively developed in recent years. Moreover, as technological advancements in key areas such as robotics, aerospace engineering, and image restoration have accelerated, new mathematical models have emerged to obtain significant results. Notably, $(2, p)$-equations (i.e., equations driven by the sum of a Laplacian and a p-Laplacian) appear in mathematical physics, including quantum physics (see Benci, D'Avenia, Fortunato, Pisani [7]) and plasma physics (see Cherfils, Ilyasov [13]). Since equations driven by the sum of two differential operators with

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different structures (such as $(2,p)$-equations or, in our case, $(2,p(x))$-equations) arise in mathematical models of various physical processes, we refer to the work of Marano and Mosconi [24] for more details.

Problems involving Laplace operators with different homogeneity have been extensively studied over the years. We mention here some relevant works by Barile and Figueiredo [6], Motreanu and Tanaka [29], as well as Papageorgiou and Rădulescu [31]. A significant advancement in the study of problems involving sums of two differential operators has been made through the investigation of problems with the $p(x)$-Laplacian and $p(x)$-Laplacian like operators, which exhibit a more complex structure and lack homogeneity. There is strong motivation for studying equations with variable exponent growth conditions, as they model various phenomena in elastic mechanics (see Zhikov [44]), electrorheological fluids (see Acerbi, Mingione [3], Diening [15]), and image restoration (see Chen, Levine, Rao [11]). Additionally, recent trends in variational fractional analysis allow us to model problems arising in fluid mechanics (see Ragusa [34]), electromagnetism and electrochemistry (see Guariglia [19], Li, Dao, and Guo [23]), viscoelasticity (see Abbas and Ragusa [1], Li, Dao, and Guo [23]), and signal processing (see Guariglia [18]).

In this paper, we extend the results obtained in the aforementioned studies by replacing the p-Laplace operator with a $p(x)$-Laplacian-type operator driven by a potential function ϕ, which may take the following forms:

(1) $\phi(x,z) = z^{p(x)-2}$, which includes the weighted $p(x)$-Laplace operator;

(2) $\phi(x,z) = \sqrt{(1+|z|^2)^{p(x)-2}}$, corresponding to the generalized mean curvature operator;

(3) $\phi(x,z) = \left(1 + \frac{z^{p(x)}}{\sqrt{\left(1+z^{2p(x)}\right)}}\right)$, which describes the capillary phenomenon.

The last case has recently gained significant attention (see Avci [4], Uță [41], Vetro [42], Wang, Zhou [43]). This growing interest is driven not only by the intrinsic fascination with naturally occurring phenomena such as the motion of drops, bubbles, and waves but also by the importance of these studies in applied fields, including industrial, biomedical, pharmaceutical, and microfluidic systems.

Our study is based on newly developed differential operators introduced by Kim and Kim [21], which allow us to analyze problems that may lack uniform convexity. In this paper, we study the operators introduced by Kim and Kim from a different perspective, as our problems involve the $(2,p(x))$-Laplace-like operator, which has a more complex structure. Additionally, the fact that the reaction function, i.e., $f(x,z) = z + \varepsilon$, exhibits constant growth behavior introduces additional technical difficulties, especially due to the presence of the Laplace differential operator embedded in the structure of the differential operator.

To analyze our problem, we consider different growth behaviors for the variable exponent $p(x)$, as follows:

(1) The first case:
$$1 < p^- \leq p(x) \leq p^+ < 2 \quad \text{on } \overline{\Omega}. \tag{1.1}$$

(2) The second case:
$$2 < p^- \leq p(x) \leq p^+ < p^*(x) \quad \text{on } \overline{\Omega}. \tag{1.2}$$

(3) The third case:
$$1 < p^- < 2 < p^+ < p^*(x) \quad \text{on } \overline{\Omega}. \tag{1.3}$$

In both the second and third cases, $p^*(x) = \frac{Np(x)}{N-p(x)}$ represents the critical Sobolev exponent for variable exponent spaces.

We note that a problem similar to the case (1.1) has been studied by Costea and Mihăilescu [14] for a simpler $p(x)$-Laplace-like operator, using Banach's fixed point theorem. For further results on $(p_1(x), p_2(x))$-problems involving similar potentials, we refer to Rădulescu *et al.* [5, 9], and Uță [38] for the coercive case; Rădulescu [12] and Uță [39, 41] for the noncoercive case; Aberqi, Bennouna, Benslimane, and Ragusa [2], and Uță [37] for studies on manifolds; and Repovš [35] and Uță [40] for cases involving a lack of compactness.

The structure of this paper is as follows: In Section 2, we provide a brief description of the functional framework, including constant Lebesgue and Sobolev spaces for the first case and variable exponent Lebesgue and Sobolev spaces for the second and third cases. In Section 3, we introduce the basic hypotheses, some auxiliary results essential to our study, and we state the main results for each case. Section 4 is dedicated to proving the main results. Finally, in Section 5, we present some remarks, future research perspectives related to our methods, and open problems that may arise under different assumptions.

2 The functional framework

In this section, we introduce the basic properties of variable exponent spaces, which constitute the necessary functional framework for the study of problem ($P_{\lambda,\varepsilon}$).

These results are described in the following books: Diening, Hästö, Harjulehto, Ružička [20], Rădulescu, Repovš [33]. We also refer to the survey paper by Rădulescu [32].

Let Ω be a bounded domain in $\mathbb{R}^N$.

For a measurable function $p : \overline{\Omega} \to \mathbb{R}$, we define:

$$p^+ = \sup_{x\in\Omega} p(x), \quad p^- = \inf_{x\in\Omega} p(x).$$

Define:

$$C_+(\Omega) = \left\{p \in C(\overline{\Omega}) : \ p(x) > 1, \forall x \in \overline{\Omega}\right\}.$$

The variable exponent Lebesgue space $L^{p(x)}(\Omega)$ is defined as:

$$L^{p(x)}(\Omega) = \left\{u : \Omega \to \mathbb{R} \text{ measurable} : \int_\Omega |u|^{p(x)} dx < \infty\right\},$$

with the norm:

$$|u|_{p(x)} = \inf\left\{\mu > 0 : \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)} dx \le 1\right\}.$$

Equipped with this norm, $L^{p(x)}(\Omega)$ becomes a Banach space whose dual is $L^{p'(x)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$.

Remark 2.1. If $1 < p(x) < \infty$, then $L^{p(x)}(\Omega)$ is a reflexive Banach space. Moreover, if p is measurable and bounded, then $L^{p(x)}(\Omega)$ is also separable.

Remark 2.2. If $0 < |\Omega| < \infty$ and $h(x)$, $r(x)$ satisfy $h(x) < r(x)$ almost everywhere in Ω, then the following continuous embedding holds:

$$L^{r(x)}(\Omega) \hookrightarrow L^{h(x)}(\Omega).$$

Denoting $L^{p'(x)}(\Omega)$ as the dual space of $L^{p(x)}(\Omega)$, we have the following Hölder-type inequality for all $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$:

$$\left| \int_\Omega uv\,dx \right| \leq \left(\frac{1}{p^-} + \frac{1}{p'^-} \right) |u|_{p(x)} |v|_{p'(x)} \leq 2|u|_{p(x)} |v|_{p'(x)}. \tag{2.1}$$

A key role in studies involving variable exponent Lebesgue spaces is played by the modular of $L^{p(x)}(\Omega)$, denoted as $\rho_{p(x)} : L^{p(x)}(\Omega) \to \mathbb{R}$ and defined by:

$$\rho_{p(x)}(u) = \int_\Omega |u(x)|^{p(x)} dx.$$

Remark 2.3. If $p(x) \not\equiv$ constant in Ω, then for u, $(u_n) \in L^{p(x)}(\Omega)$, the following relations hold:

$$|u|_{p(x)} < 1 \Rightarrow |u|^{p^+}_{p(x)} \leq \rho_{p(x)}(u) \leq |u|^{p^-}_{p(x)}, \tag{2.2}$$

$$|u|_{p(x)} > 1 \Rightarrow |u|^{p^-}_{p(x)} \leq \rho_{p(x)}(u) \leq |u|^{p^+}_{p(x)}, \tag{2.3}$$

$$|u|_{p(x)} = 1 \Rightarrow \rho_{p(x)}(u) = 1, \tag{2.4}$$

$$|u_n - u|_{p(x)} \to 0 \Leftrightarrow \rho_{p(x)}(u_n - u) \to 0. \tag{2.5}$$

The variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ is defined as:

$$W^{1,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega) \right\}$$

with the equivalent norms:

$$\|u\|_{p(x)} = |u|_{p(x)} + |\nabla u|_{p(x)},$$

$$\|u\| = \inf \left\{ \mu : \int_\Omega \left(\left| \frac{\nabla u(x)}{\mu} \right|^{p(x)} + \left| \frac{u(x)}{\mu} \right|^{p(x)} \right) dx \leq 1 \right\}.$$

The space $W^{1,p(x)}_0(\Omega)$ is defined as the closure of $C^\infty_0(\Omega)$ with respect to the norm $\|\cdot\|_{p(x)}$:

$$W^{1,p(x)}_0(\Omega) = \left\{ u : u|_{\partial\Omega} = 0, u \in L^{p(x)}(\Omega), |\nabla u| \in L^{p(x)}(\Omega) \right\}.$$

Taking into account [21], for $p \in C_+(\overline{\Omega})$, we have the $p(\cdot)$-Poincaré-type inequality:

$$|u|_{p(x)} \leq C|\nabla u|_{p(x)}, \tag{2.6}$$

where $C > 0$ is a constant depending on p and Ω.

Remark 2.4. If $p, q : \Omega \to (1, \infty)$ are Lipschitz continuous, with $p^+ < N$ and $p(x) \leq q(x) \leq p^*(x)$ for all $x \in \Omega$, where $p^*(x) = \frac{Np(x)}{N - p(x)}$, then the embedding:

$$W^{1,p(x)}_0(\Omega) \hookrightarrow L^{q(x)}(\Omega)$$

is compact and continuous.

Remark 2.5. If $0 < |\Omega| < \infty$ and $p_2(x) < p_1(x)$ in Ω, then the following continuous embedding holds:

$$W^{1,p_1(x)}_0(\Omega) \hookrightarrow W^{1,p_2(x)}_0(\Omega).$$

Remark 2.6. If $p(x) = p \equiv$ constant for every $x \in \Omega$, then the modular $\rho_{p(x)}(u)$ becomes $\|u\|_p^p$, and thus we work within the framework of constant exponent spaces:

$$L^p(\Omega) = \left\{ u : \Omega \to \mathbb{R} \text{ measurable} : \int_\Omega |u|^p \, dx < \infty \right\},$$

endowed with the norm

$$|u|_p = \left(\int_\Omega |u(x)|^p \, dx \right)^{\frac{1}{p}}.$$

Similarly, the Sobolev space is defined as

$$W_0^{1,p}(\Omega) = \{ u : \Omega \to \mathbb{R} \text{ measurable} : u|_{\partial\Omega} = 0,\ u \in L^p(\Omega),\ |\nabla u| \in L^p(\Omega) \},$$

endowed with the norm

$$\|u\|_p = \left(\int_\Omega |\nabla u(x)|^p \, dx \right)^{\frac{1}{p}}.$$

3 Basic hypotheses

We will study the problem

$$\begin{cases} -\Delta u - \varepsilon \operatorname{div} [\phi(x, |\nabla u|)\nabla u] = \lambda (u + \varepsilon) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{$P_{\lambda,\varepsilon}$}$$

In order to state more precisely our results we have that:

(ϕ_1) $\phi : \Omega \times [0, \infty) \to [0, \infty)$ fulfill the following assumptions:

- $\phi(\cdot, z)$ is measurable on Ω for all $z \geq 0$;
- $\phi(x, \cdot)$ is locally absolutely continuous on $[0, \infty)$ for almost all $x \in \Omega$.

(ϕ_2) There exist a function $\alpha \in L^{p'(x)}(\Omega)$ and a positive constant β such that

$$|\phi(x, |z|)z| \leq \alpha(x) + \beta |z|^{p(x)-1}$$

for almost all $x \in \Omega$ and for all $z \in \mathbb{R}^N$.

(ϕ_3) There is a positive constant c such that the following hypotheses hold for almost all $x \in \Omega$:

- $\phi(x, z) \geq c z^{p(x)-2}$;
- $z \dfrac{\partial \phi}{\partial z}(x, z) + \phi(x, z) \geq c z^{p(x)-2}$, for almost all $z > 0$.

Definition 3.1. We say that $\lambda \in \mathbb{R}$ is an eigenvalue of the problem ($P_{\lambda,\varepsilon}$) if there exists $u \in W \setminus \{0\}$ such that:

$$\int_\Omega \nabla u \nabla \varphi dx + \varepsilon \int_\Omega [\phi(x, |\nabla u|)\nabla u \nabla \varphi] \, dx = \lambda \int_\Omega (u + \varepsilon) \varphi dx$$

for all $\varphi \in W$.

Remark 3.2. As we are interested in finding weak solutions to problem ($P_{\lambda,\varepsilon}$), and our differential operator exhibits a double-phase behavior (being driven by both the Laplace and the ϕ-Laplace operator) the natural function space in which we seek solutions depends on the relationship between the growth behaviors of these two differential operators.

In the first case, we aim to find solutions in the functional space $W_0^{1,2}(\Omega) := H_0^1(\Omega)$, considering that $1 < p^- \leq p(x) \leq p^+ < 2$ on $\overline{\Omega}$. In the second and third cases, the appropriate space for finding solutions is $W_0^{1,p(x)}(\Omega)$, taking into account that either $2 < p^- \leq p(x) \leq p^+$ and $2 < p^*(x)$, or $p^- < 2 < p^+$ and $2 < p^*(x)$ on $\overline{\Omega}$.

Throughout this paper, we denote the appropriate functional framework by W, where $W = W_0^{1,2}(\Omega) := H_0^1(\Omega)$ in the first case, and $W = W_0^{1,p(x)}(\Omega)$ in the second and third cases. For further details, we refer to [22, Theorem 2.8].

In what follows we set:

$$S_0(x,t) = \int_0^t \phi(x,z)z dz.$$

An important role in our variational approach is played by the fact that the following assumption holds true for the potential ϕ:

(ϕ_4) For all $x \in \overline{\Omega}$, all $z \in \mathbb{R}^N$, the following estimate is true:

$$0 \leq \phi(x,|z|)|z|^2 \leq p^+ S_0(x,|z|).$$

In order to identify the eigenvalues of problem ($P_{\lambda,\varepsilon}$), we consider the following energy functional and demonstrate that its critical points correspond to weak solutions of our problem. Let $T_{\lambda,\varepsilon} : W \to \mathbb{R}$ be defined as follows:

$$T_{\lambda,\varepsilon}(u) = S(u) - \lambda R(u),$$

where

$$S(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \varepsilon \int_\Omega S_0(x,|\nabla u|)dx,$$
$$R(u) = \frac{\lambda}{2}\int_\Omega u^2 dx + \lambda\varepsilon \int_\Omega u dx.$$

Now, using [21, Lemmas 3.2, 3.4], by standard arguments we can see that $T_{\lambda,\varepsilon} \in C^1(W,\mathbb{R})$ and

$$\langle T'_{\lambda,\varepsilon}(u), \varphi\rangle = \int_\Omega \nabla u \nabla \varphi dx + \varepsilon \int_\Omega \phi(x,|\nabla u|)\nabla u \nabla \varphi dx - \lambda \int_\Omega (u+\varepsilon)\, dx,$$

hence we can say that the critical points of $T_{\lambda,\varepsilon}$ are weak solutions for the problem ($P_{\lambda,\varepsilon}$).

Remark 3.3. Throughout this paper by

$$\lambda_1(\Delta) := \min_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega u^2 dx}$$

we will denote the principal eigenvalue of the Laplace operator.

We are now ready to state our existence theorems.

Theorem 3.4. *Assume that hypotheses* (ϕ_1)–(ϕ_4), (1.1) *and* $\varepsilon > 0$ *hold true, then every* $\lambda \in (0, \lambda_1(\Delta))$ *is an eigenvalue of the problem* ($P_{\lambda,\varepsilon}$).

Theorem 3.5. *Assume that hypotheses* (ϕ_1)–(ϕ_4), (1.2) *and* $\varepsilon > 0$ *hold true, then there exists a constant* $\lambda_0 > 0$, *such that any* $\lambda \in (0, \lambda_0)$ *is an eigenvalue of the problem* ($P_{\lambda,\varepsilon}$).

Theorem 3.6. *Assume that hypotheses* (ϕ_1)–(ϕ_4), (1.2) *and* $\varepsilon > 0$ *hold true, then every* $\lambda \in \mathbb{R}$, $\lambda > 0$ *is an eigenvalue of the problem* ($P_{\lambda,\varepsilon}$).

Theorem 3.7. *Assume that hypotheses* (ϕ_1)–(ϕ_4), (1.3) *and* $\varepsilon > 0$ *hold true, then every* $\lambda \in (0, \lambda_1(\Delta))$ *is an eigenvalue for the problem* ($P_{\lambda,\varepsilon}$).

Theorem 3.8. *Assume that hypotheses* (ϕ_1)–(ϕ_4), (1.3) *and* $\varepsilon > 0$ *hold true, then there exists a constant* $\lambda^* > 0$, *such that any* $\lambda \in (0, \lambda^*)$ *is an eigenvalue of the problem* ($P_{\lambda,\varepsilon}$).

4 Main results

4.1 Proof of Theorem 3.4

Since (1.1) holds, we have $1 < p^- \leq p(x) \leq p^+ < 2$, which implies the following continuous embeddings hold:

$$\begin{aligned} W_0^{1,p(x)}(\Omega) &\hookrightarrow W_0^{1,2}(\Omega) := H_0^1(\Omega), \\ L^{p(x)}(\Omega) &\hookrightarrow L^2(\Omega), \end{aligned} \tag{4.1}$$

and by W we will denote $H_0^1(\Omega)$. As $\lambda \in (0, \lambda_1(\Delta))$ we can remark the following.

Remark 4.1. There exists a constant $C_\lambda > 0$ such that, for every $u \in W$, the following holds:

$$C_\lambda \int_\Omega |\nabla u|^2 dx \leq \int_\Omega \left(|\nabla u|^2 - \lambda |u|^2\right) dx. \tag{4.2}$$

Therefore we have that:

$$\begin{aligned} T_{\lambda,\varepsilon}(u) &= S(u) - \lambda R(u) \\ &= \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \varepsilon \int_\Omega S_0(x, |\nabla u|) dx - \frac{\lambda}{2}\int_\Omega u^2 dx - \lambda\varepsilon \int_\Omega u dx \\ &\overset{(4.2)}{\geq} \frac{1}{2} C_\lambda \int_\Omega |\nabla u|^2 dx + \varepsilon \int_\Omega S_0(x, |\nabla u|) dx - \lambda\varepsilon \int_\Omega u dx \\ &\overset{(\phi_4)}{\geq} \frac{1}{2} C_\lambda \int_\Omega |\nabla u|^2 dx + \frac{\varepsilon}{p^+} \int_\Omega \phi(x, |\nabla u|) |\nabla u|^2 dx - \lambda\varepsilon \int_\Omega u dx \\ &\overset{(\phi_3)}{\geq} \frac{1}{2} C_\lambda \|u\|_W^2 + \frac{c\varepsilon}{p^+} \int_\Omega |\nabla u|^{p(x)} dx - \lambda\varepsilon |u|_1 \\ &\geq \frac{1}{2} C_\lambda \|u\|_W^2 + \frac{c\varepsilon C^p}{p^+} \min\left\{\|u\|_W^{p^-}, \|u\|_W^{p^+}\right\} - \lambda\varepsilon C_1 \|u\|_W, \end{aligned}$$

where $C^p = \max\{C^{p^-}, C^{p^+}\}$, and $C^{p^\pm} > 0$ are some constants obtained from the continuous embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow H_0^1(\Omega)$, and $C_1 > 0$ is a constant obtained from the continuous embedding $H_0^1(\Omega) \hookrightarrow L^1(\Omega)$.

Now letting $\|u\|_W \to \infty$ we obtain that

$$\lim_{\|u\|_W \to \infty} T_{\lambda,\varepsilon}(u) = \infty,$$

hence we get that the energy functional $T_{\lambda,\varepsilon}$ is coercive. By the properties of the potential function ϕ and using relation (1.1) we have that $T_{\lambda,\varepsilon}$ is weakly lower semicontinuous, i.e. $T_{\lambda,\varepsilon}(u) \leq \liminf_{n\to\infty} T_{\lambda,\varepsilon}(u_n)$, for any sequence $(u_n)_{n\geq 1} \subset W$ for which we have $u_n \rightharpoonup u$ in W (for more details we refer to [21, Lemma 4.3]).

In conclusion, using [36, Theorem 1.2], we get that for each $\lambda \in (0, \lambda_1(\Delta))$ we obtain at least one nontrivial critical point of $T_{\lambda,\varepsilon}$, which is in fact a global minimum point of our energy functional. □

Remark 4.2. In order to enhance clarity and improve the readability of the paper, we present Figure 4.1 illustrating a simple function that exhibits the same geometrical properties as our energy functional. For simplicity, we omit the x-dependence of f and define $f : \mathbb{R} \to \mathbb{R}$ as

$$f(z) = \frac{1}{2}|z|^2 + \varepsilon|z|^{p(z)} - \lambda\left(\frac{1}{2}|z|^2 + \varepsilon \cdot z\right).$$

We consider the following parameters $\varepsilon = 3, \lambda = 0.2, p(x) = 1.5 + 0.49 \cdot \sin(6\pi x)$ for:

(a) the behavior near the origin;

(b) the behavior away from the origin.

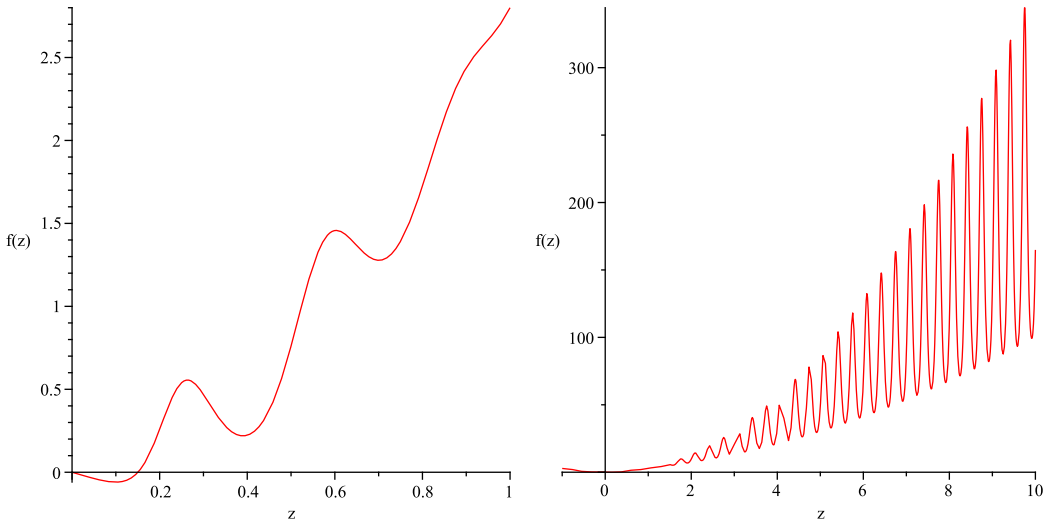


Figure 4.1: For the parameters specified above, the first figure illustrates the existence of a global minimum near the origin, while the second figure demonstrates the coercivity of the function f.

Remark 4.3. It is obvious that the critical point of our energy functional, described as

$$T_{\lambda,\varepsilon}(u_\lambda) = \min_{u\in W} T_{\lambda,\varepsilon}(u)$$

is nontrivial since $u = 0$ is not a solution of the problem $(P_{\lambda,\varepsilon})$.

Remark 4.4. For similar results in the anisotropic case we refer to [8, Theorem 3.2], [26, Theorem 3], [27, Theorem 1.1(c)], [30, Theorem 1] and to [33, Chapter 5].

4.2 Proof of Theorem 3.5

Since from now on (1.2) holds true, we have that $2 < p^- \leq p(x) < p^+$ and $2 < p^*(x) = \frac{Np(x)}{N-p(x)}$.

Given that we are in the second case of study, we will consider $W := W_0^{1,p(x)}(\Omega)$ as the appropriate functional framework.

Since the proof of the present theorem is longer and involves more technical difficulties than Theorem 3.4, we will split it into several steps.

Firstly we shall prove some geometric properties of the energy functional associated to our problem.

Step 1. We will prove the existence of a "mountain" near the origin, that is the existence of some constants $\rho > 0$ and $\theta > 0$ such that

$$T_{\lambda,\varepsilon}(u) \geq \theta > 0$$

for any $u \in W$ with $\|u\|_W = \rho$.

Consider $u \in W$ such that $\|u\|_W < 1$, we have that

$$\begin{aligned}
T_{\lambda,\varepsilon}(u) &= S(u) - \lambda R(u) \\
&= \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \varepsilon \int_\Omega S_0(x, |\nabla u|) dx - \frac{\lambda}{2}\int_\Omega u^2 dx - \lambda\varepsilon \int_\Omega u dx \\
&\geq \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{c\varepsilon}{p^+}\int_\Omega |\nabla u|^{p(x)} dx - \frac{\lambda}{2}\int_\Omega u^2 dx - \lambda\varepsilon \int_\Omega u dx \\
&\quad \text{(using hypotheses } (\phi_3), (\phi_4)) \\
&\geq \frac{1}{2} C_{2p}\|u\|_W^2 + \frac{c\varepsilon}{p^+}\|u\|_W^{p^+} - \frac{\lambda}{2}|u|_2^2 - \lambda\varepsilon|u|_1 \\
&\geq \frac{c\varepsilon}{p^+}\rho^{p^+} - \lambda\left(\frac{C'_{2p}}{2}\rho^2 + \varepsilon C_{1p}\rho\right) \\
&\geq \rho\left[\frac{c\varepsilon}{p^+}\rho^{p^+-1} - \lambda\left(\frac{C'_{2p}}{2} + \varepsilon C_{1p}\right)\right]
\end{aligned}$$

where $C_{2p} > 0$, $C'_{2p} > 0$, $C_{1p} > 0$ are some constants obtained using the following continuous embeddings $W \hookrightarrow H_0^1(\Omega)$, $W \hookrightarrow L^2(\Omega)$, respectively $W \hookrightarrow L^1(\Omega)$.

Therefore, we can set

$$\lambda_0 = \frac{c\varepsilon\rho^{p^+-1}}{p^+} \cdot \frac{2}{C'_{2p} + 2\varepsilon C_{1p}} \tag{4.3}$$

then for any $\lambda \in (0, \lambda_0)$, $\varepsilon > 0$,we may find a constant $\theta = \theta(\lambda_0, \varepsilon) > 0$ such that

$$T_{\lambda,\varepsilon}(u) \geq \theta > 0,$$

for any $u \in W$, with $\|u\|_W = \rho$.

Step 2. We shall prove in what follows the existence of a "valley" at a suitable distance from the origin. Hence, we will highlight the existence of an element $v \in W$ such that

$$T_{\lambda,\varepsilon}(zv) < 0, \tag{4.4}$$

provided that $z > 0$ is sufficiently small.

So, let $v \in C_0^\infty(\Omega)$, $v(x) \geq 0$, then there exists $\overline{\Omega}_0 \subset \Omega$ such that for any $x \in \overline{\Omega}_0$ we have $v(x) > 0$. For more details on similar arguments we refer to [28, Lemma 2.3] and [25, Lemma 2(iii)] (see also [26, proof of Theorem 4], [27, proof of Theorem 1.1(b)]). Then for $z \in (0,1)$ sufficiently small we have that:

$$\begin{aligned} T_{\lambda,\varepsilon}(zv) &= S(zv) - \lambda R(zv) \\ &\leq C_v \varepsilon z^{p^-} + \frac{z^2}{2}\int_\Omega |\nabla v|^2 dx - \lambda \frac{z^2}{2}\int_\Omega |v|^2 dx - \lambda \varepsilon z \int_{\Omega_0} v dx \\ &\leq C_v \varepsilon z^{p^-} + \frac{z^2}{2}\int_\Omega |\nabla v|^2 dx - \lambda \varepsilon z \int_{\Omega_0} v dx \\ &\leq z^2 \left(C_v \varepsilon + \frac{1}{2}\int_\Omega |\nabla v|^2 dx \right) - \lambda \varepsilon z \int_{\Omega_0} v dx \\ &\quad \text{(due to the fact that } 2 < p^- \text{ and } z \in (0,1)), \end{aligned}$$

where $C_v = 2C_\phi |\alpha|_{p'(x)} \|v\|_W^{p^-} + \frac{\beta}{p^-}\|v\|_W^{p^-}$, and $C_\phi > 0$ is a constant which depends on the potential ϕ.

Since z is chosen to be small enough we get that

$$T_{\lambda,\varepsilon}(zv) < 0,$$

provided by any $z < z_0$, where

$$0 < z_0 < \min\left\{1, \frac{\lambda\varepsilon \int_{\Omega_0} v dx}{C_v \varepsilon + \frac{1}{2}\int_\Omega |\nabla v|^2 dx}\right\}.$$

Remark 4.5. We may observe from the results obtained in Step 2 that our energy functional does not have the appropriate geometry for applying the mountain pass theorem, as the valley obtained is not sufficiently far from the origin as required.

Step 3. In what follows, we aim to prove that the geometrical properties of the energy functional established in the previous steps can still guarantee the existence of a nontrivial critical point. Therefore, we seek to find a Palais–Smale sequence that converges to this critical point.

By the assumption that $\lambda \in (0, \lambda_0)$ and taking into account Step 1, we obtain

$$\inf_{u \in \partial B(0,\rho)} T_{\lambda,\varepsilon}(u) > 0, \tag{4.5}$$

where $B(0,\rho)$ denotes the ball centered at 0 with radius ρ in W, and $\partial B(0,\rho)$ represents its boundary.

Moreover, by Step 2, there exists $v \in W$ such that $T_{\lambda,\varepsilon}(zv) < 0$, provided that $z > 0$ is sufficiently small.

Hence, by the fact that $2 < p^*(x)$ and by relations (2.2), (2.3) we obtain the following continuous embeddings

$$W \hookrightarrow L^2(\Omega) \hookrightarrow L^1(\Omega),$$

and

$$T_{\lambda,\varepsilon}(u) \geq \frac{1}{2}C_{2p}\|u\|_W^2 + \frac{c\varepsilon}{p^+}\|u\|_W^{p^+} - \frac{\lambda}{2}C'_{2p}|u|_W^2 - \lambda\varepsilon C_{1p}|u|_W. \tag{4.6}$$

So, we obtain that there exists a constant $\underline{c} < 0$ such that:

$$-\infty < \underline{c} := \inf_{B(0,\rho)} T_{\lambda,\varepsilon} < 0. \tag{4.7}$$

Using (4.5), (4.6), and (4.7), let $\omega > 0$ be a constant such that:

$$\omega < \inf_{\partial B(0,\rho)} T_{\lambda,\varepsilon} - \inf_{B(0,\rho)} T_{\lambda,\varepsilon},$$

and by applying Ekeland's variational principle (see [17]) to the energy functional $T_{\lambda,\varepsilon} : \overline{B(0,\rho)} \to \mathbb{R}$, we obtain the existence of a function $u_\omega \in \overline{B(0,\rho)}$ such that

$$\begin{aligned} &T_{\lambda,\varepsilon}(u_\omega) \leq \inf_{\overline{B(0,\rho)}} T_{\lambda,\varepsilon} + \omega, \\ &T_{\lambda,\varepsilon}(u_\omega) \leq T_{\lambda,\varepsilon}(u) + \omega \|u - u_\omega\|_W, \quad u \neq u_\omega. \end{aligned}$$

Hence,

$$T_{\lambda,\varepsilon}(u_\omega) \leq \inf_{\overline{B(0,\rho)}} T_{\lambda,\varepsilon} + \omega \leq \inf_{B(0,\rho)} T_{\lambda,\varepsilon} + \omega < \inf_{\partial B(0,\rho)} T_{\lambda,\varepsilon},$$

therefore we point out that $\|u_\omega\| < \rho$. In what follows, let $\mathcal{E}$ be an energy functional, defined as:

$$\begin{aligned} &\mathcal{E} : \overline{B(0,\rho)} \to \mathbb{R}, \\ &\mathcal{E}(u) = T_{\lambda,\varepsilon}(u) + \omega \|u - u_\omega\|_W. \end{aligned} \tag{4.8}$$

So, by (4.8) we can say that

$$\begin{aligned} \mathcal{E}(u_\omega) &= T_{\lambda,\varepsilon}(u_\omega) < T_{\lambda,\varepsilon}(u) + \omega \|u - u_\omega\|_W \\ &= \mathcal{E}(u), u \neq u_\omega. \end{aligned} \tag{4.9}$$

Now, using (4.9), one can observe that u_ω is a minimum point for $\mathcal{E}$, so it follows that (using the same arguments as in [5,21,28])

$$\frac{\mathcal{E}(u_\omega + z\gamma) - \mathcal{E}(u_\omega)}{z} \geq 0 \tag{4.10}$$

for $z > 0$ small and every $\gamma \in W$ with $\|\gamma\|_W < 1$.

Therefore, using (4.10) we obtain that

$$\frac{T_{\lambda,\varepsilon}(u_\omega + z\gamma) - T_{\lambda,\varepsilon}(u_\omega)}{z} + \omega \|\gamma\|_W \geq 0.$$

Letting $z \to 0$, it follows that

$$\begin{aligned} &\langle T'_{\lambda,\varepsilon}(u_\omega), \gamma \rangle > -\omega \|\gamma\|_W, \\ &\langle T'_{\lambda,\varepsilon}(u_\omega), \gamma \rangle > -\omega. \end{aligned}$$

Thus, we obtain that $\|T'_{\lambda,\varepsilon}(u_\omega)\|_W \leq \omega$.

By the above relations we obtain the existence of a sequence $(u_n)_{n\geq 1} \subset B(0,\rho)$ such that

$$\begin{aligned} &T_{\lambda,\varepsilon}(u_n) \to \underline{c} \\ &T'_{\lambda,\varepsilon}(u_n) \to 0. \end{aligned} \tag{4.11}$$

By the fact that $(u_n)_{n\geq 1} \subset B(0,\rho)$ we have that

$$\|u_n\|_W \leq \rho, \forall\, n \geq 1,$$

so, $(u_n)_{n\geq 1}$ is bounded in W.

Hence, there exists an element $u_0 \in W$ such that (passing eventually to a subsequence)

$$u_n \rightharpoonup u_0 \quad \text{in } W.$$

Now, by the compact embeddings

$$W \hookrightarrow L^{p(x)}(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow L^1(\Omega),$$

we get that

$$\begin{aligned} \lim_{n\to\infty} R(u_n) &= R(u_0) \\ \lim_{n\to\infty} \langle R'(u_n), u_n - u_0\rangle &= 0. \end{aligned} \tag{4.12}$$

Using relation (4.11) we obtain that

$$\lim_{n\to\infty} \langle T'_{\lambda,\varepsilon}(u_n), u_n - u_0\rangle = 0. \tag{4.13}$$

Using (4.12) and (4.13) we can obtain that

$$\lim_{n\to\infty} \langle S'(u_n) - S'(u_0), u_n - u_0\rangle \leq \lim_{n\to\infty} \langle T_{\lambda,\varepsilon}(u_n), u_n - u_0\rangle = 0,$$

thus (using same arguments as in [21, Lemma 3.4]) we get that

$$u_n \to u_0 \quad \text{in } W. \tag{4.14}$$

Now, combining relations (4.12), (4.14), and (4.11), we obtain that

$$T_{\lambda,\varepsilon}(u_0) = \underline{c} < 0 \quad \text{and} \quad T'_{\lambda,\varepsilon}(u_0) = 0,$$

so u_0 is a nontrivial critical point of the energy functional $T_{\lambda,\varepsilon}$. Hence, it follows that every $\lambda \in (0, \lambda_0)$ is an eigenvalue for the problem $(P_{\lambda,\varepsilon})$, with u_0 as its associated eigenfunction, thus completing our proof. □

4.3 Proof of Theorem 3.6

We now proceed to prove our second existence result for the case (1.2). Given that the relation (1.2) holds, we have that $2 < p^- \leq p(x) < p^+$, $2 < p^*(x) = \frac{Np(x)}{N-p(x)}$, and since we are in the second case of study, we will consider $W := W_0^{1,p(x)}(\Omega)$ as the appropriate functional framework.

In what follows we will show that the energy functional associated to the problem meets the hypotheses of [36, Theorem 1.2]. So, we have that:

$$\begin{aligned} T_{\lambda,\varepsilon}(u) &= \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \varepsilon \int_\Omega S_0(x, |\nabla u|)dx - \frac{\lambda}{2}\int_\Omega u^2 dx - \lambda\varepsilon \int_\Omega u dx \\ &\geq \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{c\varepsilon}{p^+}\int_\Omega |\nabla u|^{p(x)} dx - \frac{\lambda}{2}\int_\Omega u^2 dx - \lambda\varepsilon \int_\Omega u dx \\ &\quad \text{(using hypotheses } (\phi_3) \text{ and } (\phi_4)) \\ &\geq \frac{C_{2p}}{2}\|u\|_W^2 + \frac{c\varepsilon}{p^+}\min\left\{\|u\|_W^{p^-}, \|u\|_W^{p^+}\right\} - \frac{\lambda C'_{2p}}{2}\|u\|_W^2 - \lambda\varepsilon C_{1p}\|u\|_W, \end{aligned}$$

where $C_{1p} > 0$, $C_{2p} > 0$, $C'_{2p} > 0$ are the same constants we used in the previous proofs.

Since $2 < p^- < p^+$, letting $\|u\|_W \to \infty$ we have that

$$\lim_{\|u\|_W \to \infty} T_{\lambda,\varepsilon}(u) = \infty,$$

so the energy functional $T_{\lambda,\varepsilon}$ is coercive.

Now, by the properties of the potential function ϕ and relation (1.2) we have that $T_{\lambda,\varepsilon}$ is weakly lower semicontinuous (see also [21, Lemmas 4.3, 4.4]), i.e.,

$$T_{\lambda,\varepsilon}(u) \leq \liminf_{n\to\infty} T_{\lambda,\varepsilon}(u_n)$$

for any sequence $(u_n)_{n\geq 1} \subset W$, such that $u_n \rightharpoonup u$ in W.

Therefore, as in the proof of Theorem 3.4, we only need to apply [36, Theorem 1.2], and we obtain that $T_{\lambda,\varepsilon}$ has a global minimum point, which is a solution to the problem $(P_{\lambda,\varepsilon})$. Thus, we have proved that for every $\lambda \in \mathbb{R}^*_+$, there exists a solution to the problem that is associated with the eigenvalue λ. □

Remark 4.6. In order to enhance clarity and improve the readability of the paper, we present Figure 4.2 illustrating a simple function that exhibits the same geometrical properties as our energy functional. For simplicity, we omit the x-dependence of f and define $f : \mathbb{R} \to \mathbb{R}$ as

$$f(z) = \frac{1}{2}|z|^2 + \varepsilon|z|^{p(z)} - \lambda\left(\frac{1}{2}|z|^2 + \varepsilon \cdot z\right).$$

We consider the following parameters:

(a) $\varepsilon = 10, \lambda = 0.4, p(x) = 2.51 + 0.5 \cdot \sin(6\pi x)$;

(b) $\varepsilon = 10, \lambda = 10, p(x) = 2.51 + 0.5 \cdot \sin(6\pi x)$.

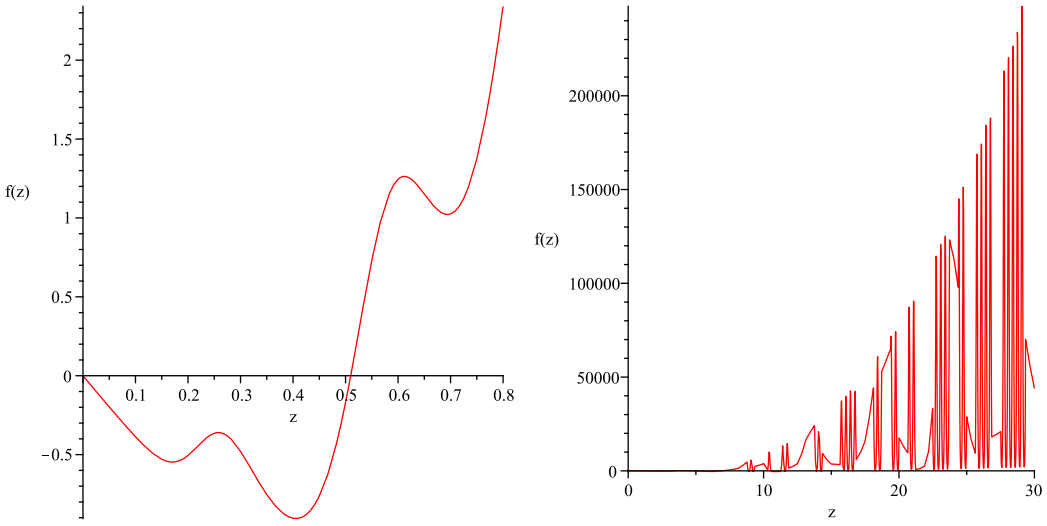


Figure 4.2: For the parameters stated in (a), we highlight the existence of both a local minimum and a global minimum near the origin. For the parameters stated in (b), we highlight the existence of a global minimum point and the coercivity of the function f.

4.4 Proof of Theorem 3.7

We now proceed to prove our first existence result for the case (1.3). Due to the fact that relation (1.3) holds, we have that $1 < p^- < 2 < p^+$ and $2 < p^*(x) = \frac{Np(x)}{N-p(x)}$. We will consider $W := W_0^{1,p(x)}(\Omega)$ as the appropriate functional framework.

In order to identify the critical points of our energy functional, we first aim to prove that $T_{\lambda,\varepsilon}$ is coercive. To this end, we have that

$$\begin{aligned} T_{\lambda,\varepsilon}(u) &\geq \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{c\varepsilon}{p^+}\min\left\{\|u\|_W^{p^+}, \|u\|_W^{p^-}\right\} - \frac{\lambda}{2}\int_\Omega u^2 dx - \lambda\varepsilon\int_\Omega u dx \\ &\geq \frac{C_\lambda}{2}\int_\Omega |\nabla u|^2 dx + \frac{c\varepsilon}{p^+}\min\left\{\|u\|_W^{p^+}, \|u\|_W^{p^-}\right\} - \lambda\varepsilon C_{1p}\|u\|_W, \end{aligned}$$

where $C_{1p} > 0$ is a constant as in previous proofs (we also have used of relation (4.2)). Now, letting $\|u\|_W \to \infty$, since $1 < p^-$ we get that

$$\lim_{\|u\|_W \to \infty} T_{\lambda,\varepsilon}(u) = +\infty$$

so the fact that $T_{\lambda,\varepsilon}$ is coercive holds true. In what follows the existence of a nontrivial critical point is obtained in the same fashion as in the proof of Theorem 3.4. □

4.5 Proof of Theorem 3.8

Since, similarly to the proof of Theorem 3.5, the proof of this theorem is longer, we will split it into several steps. Also, as in the proof of Theorem 3.7, we will consider $W := W_0^{1,p(x)}(\Omega)$ as the appropriate functional framework. Furthermore, since relation (1.3) implies that $2 < p^+ < p^*(x)$ for all $x \in \overline{\Omega}$, throughout this proof, we can use the same embeddings for the functional spaces as in the proof of Theorem 3.5.

Step 1. We will prove the existence of a "mountain" near the origin, that is the existence of some constants $r > 0$ and $\alpha > 0$ such that

$$T_{\lambda,\varepsilon}(u) \geq \alpha > 0,$$

for any $u \in W$ with $\|u\|_W = r$. Without losing generality we may suppose that $\|u\|_W < 1$, then we have that

$$\begin{aligned} T_{\lambda,\varepsilon}(u) &\geq \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{c\varepsilon}{p^+}\|u\|_W^{p^+} - \frac{\lambda}{2}\int_\Omega u^2 dx - \lambda\varepsilon\int_\Omega u dx \\ &\geq \frac{C_{2p}}{2}\|u\|_W^2 + \frac{c\varepsilon}{p^+}\|u\|_W^{p^+} - \frac{\lambda C'_{2p}}{2}\|u\|_W^2 - \lambda\varepsilon C_{1p}\|u\|_W \\ &\geq r\left[\left(\frac{C_{2p}}{2} + \frac{c\varepsilon}{p^+}\right)r^{p^+-1} - \lambda\left(\frac{C'_{2p}}{2} + \varepsilon C_{1p}\right)\right], \end{aligned}$$

where the constants $C_{2p} > 0$, $C'_{2p} > 0$, $C_{1p} > 0$ are the same as in the previous proofs.

So, by the above inequality if we take $\lambda \in (0, \lambda^*)$, with λ^* defined as it follows:

$$\lambda^* = \frac{C_{2p}p^+ + 2c\varepsilon}{C'_{2p} + 2\varepsilon C_{1p}} \cdot r^{p^+-1} \tag{4.15}$$

we can find a constant $\alpha = \alpha\big(\frac{C_{2p}p^+ + 2c\varepsilon}{C'_{2p}+2\varepsilon C_{1p}} \cdot r^{p^+-1}\big) > 0$ such that

$$T_{\lambda,\varepsilon}(u) \geq \alpha > 0.$$

Step 2. We now proceed to prove the existence of a "valley" near the origin. To this end, we aim to show that there exists $e \in W$, with $e > 0$, such that

$$T_{\lambda,\varepsilon}(te) < 0,$$

provided that $t > 0$ is sufficiently small.

In this regard, we approximate our energy functional using similar techniques as in Step 2 of the proof of Theorem 3.5. We obtain

$$T_{\lambda,\varepsilon}(te) \leq \frac{t^2}{2}\int_\Omega |\nabla e|^2 dx + t^{p^-} \cdot C_e\varepsilon - \frac{\lambda t^2}{2}\int_\Omega e^2 dx - \lambda\varepsilon t\int_\Omega e dx,$$

where $C_e = 2C_\phi |\alpha|_{p'(x)}\|e\|_W^{p^-} + \frac{\beta}{p^-}\|e\|_W^{p^-}$, and $C_\phi > 0$ is a constant depending on the potential ϕ.

Since $t \in (0,1)$, the above relation implies that

$$T_{\lambda,\varepsilon}(te) \leq t^{p^-}\left(\frac{1}{2}\int_\Omega |\nabla e|^2 dx + C_e\varepsilon\right) - \lambda\varepsilon t\int_\Omega e dx.$$

Now, since $1 < p^-$, it follows that

$$T_{\lambda,\varepsilon}(te) < 0,$$

provided that

$$t < t_0 = \frac{\lambda\varepsilon\int_\Omega e dx}{\frac{1}{2}\int_\Omega |\nabla e|^2 dx + C_e\varepsilon}.$$

Step 3. To prove that the geometrical properties established in Step 1 and Step 2 lead to the existence of a nontrivial critical point for our energy functional, we proceed as in the proof of Theorem 3.5. Therefore, our goal is to find a Palais–Smale sequence whose limit is the corresponding critical point.

Let $\lambda^* > 0$ be as defined in (4.15). Keeping in mind the results established in Step 1, we have that

$$\inf_{\partial B(0,r)} T_{\lambda,\varepsilon} > 0, \tag{4.16}$$

where $B(0,r)$ denotes the ball centered at 0 with radius r in W.

Now, by the results established in Step 2, we have the existence of $e \in W$ such that $T_{\lambda,\varepsilon}(te) < 0$, provided that $t > 0$ is sufficiently small. Moreover, since $2 < p^*(x)$, it follows that $W \hookrightarrow L^2(\Omega) \hookrightarrow L^1(\Omega)$, and

$$T_{\lambda,\varepsilon}(u) \geq \left(\frac{C_{2p}}{2} + \frac{c\varepsilon}{p^+}\right)\|u\|_W^{p^+} - \left(\frac{\lambda C'_{2p}}{2} + \lambda\varepsilon\right)\|u\|_W,$$

for any $u \in B(0,r)$.

Therefore, we can say that there exists a constant $c_* < 0$ such that

$$-\infty < c_* := \inf_{B(0,r)} T_{\lambda,\varepsilon} < 0.$$

Now, by the above relations, let ϑ be a constant such that:

$$0 < \vartheta < \inf_{\partial B(0,r)} T_{\lambda,\varepsilon} - \inf_{B(0,r)} T_{\lambda,\varepsilon}.$$

Using Ekeland's variational principle (see [17]) for the energy functional $T_{\lambda,\varepsilon} : \overline{B(0,r)} \to \mathbb{R}$ we may find $u_\vartheta \in \overline{B(0,r)}$ such that

$$\begin{aligned} &T_{\lambda,\varepsilon}(u_\vartheta) < \inf_{B(0,r)} T_{\lambda,\varepsilon} + \vartheta \\ &T_{\lambda,\varepsilon}(u_\vartheta) < T_{\lambda,\varepsilon}(u) + \vartheta \|u - u_\vartheta\|_W, u \neq u_\vartheta. \end{aligned}$$

Since

$$T_{\lambda,\varepsilon}(u_\vartheta) \leq \inf_{B(0,r)} T_{\lambda,\varepsilon} + \vartheta \leq \inf_{B(0,r)} T_{\lambda,\varepsilon} + \vartheta < \inf_{\partial B(0,r)} T_{\lambda,\varepsilon},$$

we obtain that $u_\vartheta \in B(0,r)$.

Now, let us define $J_{\lambda,\varepsilon} : \overline{B(0,r)} \to \mathbb{R}$ as

$$J_{\lambda,\varepsilon}(u) = T_{\lambda,\varepsilon}(u) + \vartheta \|u - u_\vartheta\|.$$

Hence, by the same arguments as in the previous proof (Theorem 3.5, see also [5,21,28]), we conclude that u_ϑ is a minimum point of $J_{\lambda,\varepsilon}$, and we have

$$\frac{J_{\lambda,\varepsilon}(u_\vartheta + tw) - J_{\lambda,\varepsilon}(u_\vartheta)}{t} \geq 0, \tag{4.17}$$

for small $t > 0$ and any $w \in B(0,1)$.

Thus, relation (4.17) implies that

$$\frac{T_{\lambda,\varepsilon}(u_\vartheta - tw) - T_{\lambda,\varepsilon}(u_\vartheta)}{t} + \vartheta \|w\|_W \geq 0.$$

Letting $t \to 0$, it follows that

$$\langle T'_{\lambda,\varepsilon}(u_\vartheta), w \rangle + \vartheta \|w\|_W > 0,$$

which implies that $\|T'_{\lambda,\varepsilon}(u_\vartheta)\|_W \leq \vartheta$.

From the above relations, we obtain the existence of a sequence $(u_n)_{n\geq 1} \subseteq B(0,r)$ such that

$$\begin{aligned} &T_{\lambda,\varepsilon}(u_n) \to c_*, \\ &T'_{\lambda,\varepsilon}(u_n) \to 0. \end{aligned}$$

Now, since relation (1.3) implies that $2 < p^*(x)$, using the same arguments as in Step 3 of the proof of Theorem 3.5, we obtain that there exists some $u_* \in W$ such that $u_n \to u_*$ in W and

$$\begin{aligned} &T_{\lambda,\varepsilon}(u_*) = c_* < 0, \\ &T'_{\lambda,\varepsilon}(u_*) = 0, \end{aligned}$$

therefore, u_* is a critical point of the energy functional $T_{\lambda,\varepsilon}$, and thus every $\lambda \in (0, \lambda^*)$ is an eigenvalue for problem $(P_{\lambda,\varepsilon})$. □

Remark 4.7. As in the previous cases in order to enhance clarity and improve the readability of the paper, we present Figure 4.3 illustrating a simple function that exhibits the same geometrical properties as our energy functional. For simplicity, we omit the x-dependence of f and define $f : \mathbb{R} \to \mathbb{R}$ as

$$f(z) = \frac{1}{2}|z|^2 + \varepsilon|z|^{p(z)} - \lambda\left(\frac{1}{2}|z|^2 + \varepsilon \cdot z\right).$$

We consider the following parameters $\varepsilon = 10, \lambda = 0.4, p(x) = 2 + 0.5 \cdot \sin(6\pi x)$ for:

(a) the behavior near the origin;

(b) the behavior away from the origin.

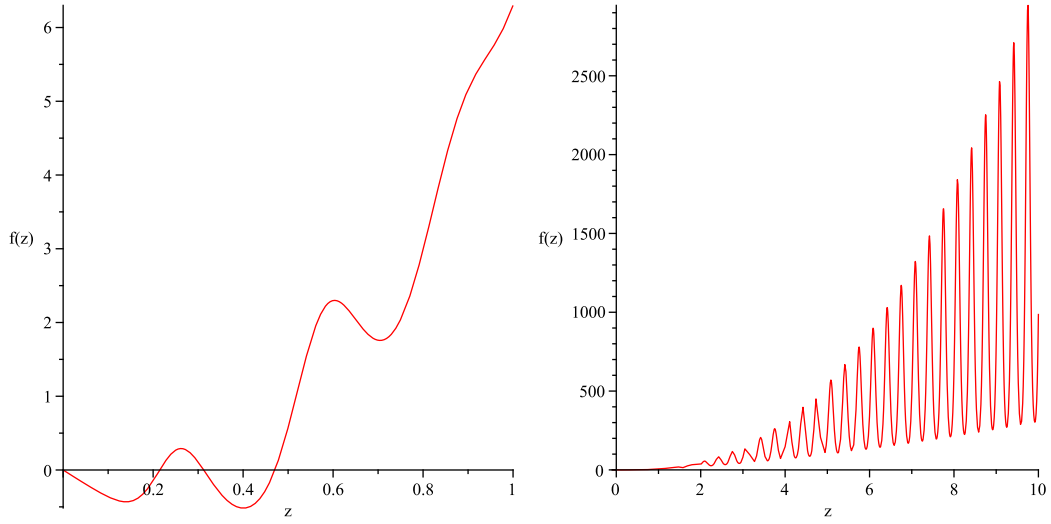


Figure 4.3: For the parameters specified above, the first figure illustrates the existence of a both a local minimum and a global minimum near the origin, while the second figure demonstrates the coercivity of the function f.

5 Final remarks and open questions

Remark 5.1. At this point, we do not know the multiplicity of eigenvalues in any of the cases. For a simpler problem, similar to the problem corresponding to the case (1.1), it has been proved that there exists a unique eigenfunction in the interval $(0, \lambda_1(\Delta))$. For more details, we refer to [14].

Remark 5.2. The only case where we can prove the existence of eigenvalues for large values of λ is in (1.2). Also, in this case, we can obtain a concentration result near the origin. At this point, we do not know whether the solutions given by Theorems 3.5 and 3.6 coincide for $\lambda \in (0, \lambda_0)$.

Remark 5.3. We observe that in (1.3), there is a concentration of the spectrum in the near proximity of the origin. At this point, the order relation between the parameters $\lambda_1(\Delta)$ and λ^* is not known. Furthermore, the multiplicity of eigenvalues in the interval $(0, \lambda_1(\Delta)) \cap (0, \lambda^*)$ is also unknown, as we lack information on whether the solutions given by Theorems 3.7 and 3.8 coincide.

Remark 5.4. Since we only have some information on the qualitative behavior for large values of the parameter λ in (1.2) (we refer here to Theorem 3.6), a more challenging question is to develop an exhaustive analysis for all $\lambda > 0$ in the other two cases.

Remark 5.5. A promising direction for future research emerges when the reaction function is defined as $f(x,z) = |u|^{p^*(x)-1} + \varepsilon$. This formulation enables an investigation into the interplay between the growth behavior of the potential function ϕ and the critical growth behavior of the reaction function f. For related results in this direction, we refer to [10,16].

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Dynamic behavior of a stochastic fast-slow fishery economic model with Allee effect

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Abstract. This article investigates a stochastic fast-slow fishery economic model with Allee effect. The fast-slow dynamic behavior of the deterministic system is discussed by using the theory of singular perturbation. Research shows that singular Hopf bifurcations will occur under the influence of strong Allee effect, while the dynamic behavior of the system will be more complex under the influence of weak Allee effect, resulting in relaxation oscillations. For the stochastic system, the existence of stationary distribution is discussed for both stochastic fast-slow system and system with time scale parameter as ordinary parameter. Then, the stochastic bifurcation behavior of the system is discussed, and it is found that stochastic-P bifurcation and stochastic-D bifurcation will occur. Finally, the correctness of the conclusion is verified through numerical simulation.

Keywords: fast-slow system, canard circle, relaxation oscillation, ergodic stationary distribution, stochastic bifurcation.

2020 Mathematics Subject Classification: 34E13, 34C26, 34C23, 34E15, 34E17, 37H20, 37H30.

1 Introduction

The bioeconomic model of fishery resources is based on the dynamic model of fishery populations, combined with ecological, economic and social factors, to study the relationship between the quantity of fishery resources and human activities. The fishing industry has high social significance, as it directly affects the livelihoods and food supply of fishermen. In order to ensure the stable growth of human needs, it is necessary to study the stability of the dynamic model of fishing resources [20, 34].

In the fishery economy, the human capture operations are a key link and the main way to obtain fishery resources, driving the development of the fishery economy. In recent years, many scholars have incorporated capture as a parameter into biological models, studied bioeconomic models described by differential algebraic equations, and generated interesting and rich dynamic behaviors [2, 15, 25]. Clark [7] incorporated dynamic capture into the model,

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believing that capture would be affected by total revenue and total cost, and treating it as a fixed parameter is unrealistic. Therefore, it is necessary to introduce capture as a variable into the bioeconomic system.

The Allee effect is an important dynamic phenomenon in ecosystems [18], which suggests that when a population is too sparse or too dense, it can inhibit population growth, meaning that each population has its own optimal density for survival. With the deepening of research on population dynamics, scholars have gradually introduced the classification of strong and weak Allee effects in order to more accurately describe the performance of populations under different degrees of low density. For species with strong Allee effect [35], once the population drops to the critical density, the species is likely to become extinct. For species under weak Allee effect [11], when the species density is low, they still have a certain degree of self-sustaining ability, which means that even if the species density is small, they can naturally grow to a certain extent. Understanding and considering the Allee effect is crucial for sustainable fishing in the fishery economy. Considering the Allee effect can help fishery managers develop reasonable fishing quotas and fishing bans to ensure the health of fish populations and the sustainable use of fishery resources. Lin [23] studied a single-species logistic model incorporating the Allee effect and explored how the Allee effect impacts species stability and extinction risk. Ashutosh [27] developed a fish capture model with logistic growth and Allee effect and found periodic fishing to yield higher economic benefits for fish populations. Liu et al. [24] studied the dynamic behavior of the Leslie Power model with weak Allee effect and fear effect. However, there is limited research on the effects of different Allee effects on fisheries economic models. Therefore, it is necessary for this article to consider the impact of different Allee effects on system dynamics.

Fast-slow systems have been widely applied in fields such as physics, chemistry, pharmacology, and ecology. A key feature of these systems is that their different variables vary over two or more time scales and such systems are typically handled by geometric singular perturbation theory [9]. In ecosystems, considering the difference in growth rates between prey and predators, the relationship between the two can be established using a fast-slow system [6, 21, 32]. Saha et al. [31] investigated a predator-prey model with Beddington–DeAngelis response, revealed rich dynamics including relaxation oscillations, canard cycles, and canard explosions. Li and Zhang [22] discussed the dynamic behavior of a fast-slow Leslie–Gower predator-prey model with constant harvesting. As mentioned in reference [3], there is a significant disparity between the rate of change in fishing effort and that in fish species. Therefore, the fast-slow system is also applicable to the fishery economic model, and discussing the fast-slow dynamics of the fishery economic model is also very interesting.

In the natural environment, the number and structure of biological populations are affected by a variety of random factors, such as climate change and random fluctuations in food resources. Stochastic differential equations are frequently employed to simulate random fluctuations in biological systems, as a result, they play an important role in studying the dynamic analysis of systems [14,26,29,42]. Yu [41] proposed and explored a single species model with Allee effect driven by correlated colored noise. Wang [36] studied a population model with Allee effect affected by both additive and multiplicative noise, and the results showed that correlated noise has a complex impact on the final population size distribution. Han and Jiang [12] considered and studied a stochastic predator-prey system with general infinite delay, and provided the necessary conditions for species persistence and extinction. Overall, environmental noise has a complex impact on the final distribution of population size.

Based on the above analysis, although some scholars have proposed that there is a signif-

icant difference between the rate of change in capture effort and the growth rate of species, few have specifically discussed the fast-slow dynamic properties of fast-slow fishery economic models. Therefore, we consider the Allee effect in fish populations and the influence of external factors on both fish populations and catches, and establish a stochastic fast-slow fishery economic model. We use singular perturbation techniques to study the complex fast-slow dynamics of the model and discuss the bifurcation behavior of the random fast-slow system. The structure of this article is as follows: In Section 2, a fast-slow fishery economic model with Allee effect is proposed, and the existence of equilibrium is discussed. In Section 3, we discuss the bifurcation behavior of deterministic system, which can generate transcritical bifurcation and Hopf bifurcation. In Section 4, we use singular perturbation theory to discuss the singular Hopf bifurcation and relaxation oscillation generated by fast-slow system, and investigate the existence of stationary distributions for stochastic fast-slow reduction system. In Section 5, we consider the time scale parameter ε as a general parameter and discuss the existence of stationary distribution for stochastic systems with weak Allee effects, as well as the dynamic behavior generated by the stochastic system. In Section 6, the conclusion and outlook are elaborated.

2 Model establishment and equilibrium analysis

2.1 Model establishment

In this section, we consider a single population fishery economic model, which takes the following form:

$$\begin{cases} \dfrac{dx}{dt} = rx\left(1-\dfrac{x}{K}\right)(x-A) - qEx, \\ \dfrac{dE}{dt} = m\left(pqx-c\right)E, \end{cases} \tag{2.1}$$

where the growth of the fish population follows logistic growth, x represents the density of the fish population at time t, E represents the effort required for capture, which is positively correlated with the total profit. Table 2.1. provides the biological explanations for the parameters in the model.

Parameter	Biological Interpretation
r	Intrinsic growth rate of fish stocks
K	Maximum environmental capacity of fish stocks
A	Allee effect threshold, where $0 < A < K$ indicates strong Allee effect, and $-K < A < 0$ indicates weak Allee effect
m	Adjustment coefficient of E
p	The price per unit of catch of fish stocks
q	Capture coefficient
c	Unit fishing cost of fish stocks

Table 2.1: Definition of Parameters in system (2.1)

Perform a dimensionless transformation on system (2.1),

$$x \to \frac{x}{K}, \quad t \to rt,$$

we have

$$\begin{cases} \dfrac{dx}{dt} = x\left(1-x\right)\left(Kx-A\right) - \alpha E x & =: f(x,E,\varepsilon), \\ \dfrac{dE}{dt} = \varepsilon\left(\beta p x - c\right)E & =: \varepsilon g(x,E,\varepsilon), \end{cases} \tag{2.2}$$

where new dimensionless parameters are $\alpha = \frac{q}{r}$, $\beta = qK$, $\varepsilon = \frac{m}{r}$. These parameters are positive. Since m is an adjustment parameter for E, which is a very small number, there is $m \ll r$, resulting in $0 < \varepsilon \ll 1$. At this point, system (2.2) is a fast-slow fishery economic system.

Let $\tau = \varepsilon t$, the system (2.2) becomes that

$$\begin{cases} \varepsilon\dfrac{dx}{d\tau} = x\left(1-x\right)\left(Kx-A\right) - \alpha E x & =: f(x,E,\varepsilon), \\ \dfrac{dE}{d\tau} = \left(\beta p x - c\right)E & =: g(x,E,\varepsilon). \end{cases} \tag{2.3}$$

Under nonnegative initial conditions, the dynamics of the system will be limited within the first quadrant. We name t as the fast time scale and τ as the slow time scale. When $0 < \varepsilon \ll 1$, x evolves on a fast time scale t and E evolves on a slow time scale τ. Therefore, we refer to x as a fast variable and E as a slow variable.

2.2 Equilibrium analysis

In the section, the existence of equilibrium points and the stability of certain equilibrium points within system (2.2) will be discussed. From a biological perspective, we only focus on the dynamic behavior of system (2.2) in region $\mathbb{R}^2_+ = \{(x,E) : x \geq 0, E \geq 0\}$.

There are three boundary equilibrium points $S_1(0,0)$, $S_2(1,0)$ and $S_3(\frac{A}{K},0)$ in system (2.2). There exists a unique internal equilibrium point $S_*\left(x_*, E_*\right)$ if $\frac{A}{K} < \frac{c}{\beta p} < 1$, where $x_* = \frac{c}{\beta p}$, $E_* = \frac{(1-x_*)(Kx_*-A)}{\alpha}$.

The Jacobian matrix of system (2.2) at (x,E) is

$$J := \begin{pmatrix} -3Kx^2 + 2(K+A)x - A - \alpha E & -\alpha E \\ \varepsilon\beta p E & \varepsilon(\beta p x - c) \end{pmatrix}.$$

The Jacobian matrix at $S_1(0,0)$ is

$$J|_{S_1} = \begin{pmatrix} -A & 0 \\ 0 & -c\varepsilon \end{pmatrix},$$

and its corresponding eigenvalues are $\lambda_1 = -A$ and $\lambda_2 = -c\varepsilon$, where λ_1 is influenced by the Allee effect type. Therefore, under the strong Allee effect $0 < A < K$, S_1 is a stable node, and under the weak Allee effect $A < 0$, S_1 is an unstable node.

The Jacobian matrix at $S_2(1,0)$ is

$$J|_{S_2} = \begin{pmatrix} A-K & -\alpha \\ 0 & \varepsilon(\beta p - c) \end{pmatrix},$$

and its corresponding eigenvalues are $\lambda_1 = A - K$ and $\lambda_2 = \varepsilon(\beta p - c)$. The stability of equilibrium point S_2 is not affected by the Allee effect. When $c > \beta p$, S_2 is stable node, and when $c < \beta p$, S_2 is unstable node.

The Jacobian matrix at $S_3(\frac{A}{K}, 0)$ is

$$J|_{S_3} = \begin{pmatrix} A\left(1 - \frac{A}{K}\right) & -\frac{A\alpha}{K} \\ 0 & \varepsilon\left(\frac{\beta p A}{K} - c\right) \end{pmatrix},$$

and its corresponding eigenvalues are $\lambda_1 = A(1 - \frac{A}{K})$ and $\lambda_2 = \varepsilon(\frac{\beta p A}{K} - c)$. Under the strong Allee effect $0 < A < K$, $\lambda_1 < 0$, so S_3 is a saddle point for $c > \frac{\beta p A}{K}$ and S_3 is an unstable node for $c < \frac{\beta p A}{K}$. Under the weak Allee effect $A < 0$, S_3 is a non positive equilibrium point, we will not discuss it.

The Jacobian matrix at $S_*(x_*, E_*)$ (whose existence condition is $\frac{A}{K} < \frac{c}{\beta p} < 1$) is

$$J|_{S_*} = \begin{pmatrix} \frac{c}{\beta p}\left(A + K - \frac{2Kc}{\beta p}\right) & -\frac{\alpha c}{\beta p} \\ \frac{\varepsilon \beta p}{\alpha}\left(1 - \frac{c}{\beta p}\right)\left(\frac{Kc}{\beta p} - A\right) & 0 \end{pmatrix}.$$

The corresponding characteristic equation is

$$|\lambda E - J|_{S_*} = \lambda^2 + q_1 \lambda + q_2 = 0, \tag{2.4}$$

where $q_1 = -\frac{c}{\beta p}(A + K - \frac{2Kc}{\beta p})$, $q_2 = c\varepsilon(1 - \frac{c}{\beta p})(\frac{Kc}{\beta p} - A)$. Since $\frac{A}{K} < \frac{c}{\beta p} < 1$, we can obtain $q_2 > 0$. By using the Veda theorem, we can conclude that if $q_1 = 0$ i.e. $c = \frac{(A+K)\beta p}{2K}$, its eigenvalues are a pair of pure imaginary roots, so S_* is a center; if $q_1 > 0$, the real parts of its eigenvalues are all negative, so S_* is locally asymptotically stable; and if $q_1 < 0$, the real parts of its eigenvalues are all positive, S_* is unstable.

3 Bifurcation analysis

In this section, we mainly discuss the dynamic behavior of the system (2.2). We delve into the bifurcation behavior of the system (2.2) at the boundary equilibrium and analyze the existence and stability of its Hopf bifurcation at the internal equilibrium $S_*(x_*, E_*)$.

3.1 Transcritical bifurcation

Theorem 3.1. *For system (2.2), a transcritical bifurcation will occur near the equilibrium point S_2 as the bifurcation parameter c passes through $c^* = \beta p$.*

Proof. We know that S_2 is stable with $c > \beta p$ and S_2 is unstable with $c < \beta p$. That is, as the value of parameter c passes through βp, the stability of S_2 changes, so there may be a transcritical bifurcation. Next, we prove the bifurcation behavior at equilibrium point S_2 by using Sotomayor's theorem to .

Denote

$$X = (x, E)^{\mathrm{T}},\ F(X, c) = \begin{pmatrix} x(1 - x)(Kx - A) - \alpha E x \\ \varepsilon(\beta p x - c)E \end{pmatrix},$$

$J|_{S_2} = DF(S_2, \beta p)$ has a zero eigenvalue at $c^* = \beta p$, and its corresponding eigenvector is $\vartheta = (\alpha, A-K)^{\mathrm{T}} = (\vartheta_1, \vartheta_2)^{\mathrm{T}}$. The eigenvector corresponding to the zero eigenvalue of transpose matrix $D^{\mathrm{T}}F(S_2, \beta p)$ is $\omega = (0,1)^{\mathrm{T}}$. After calculation, it is easy to obtain the following formula:

$$F_c(S_2, \beta p) = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$
$$DF_c(S_2, \beta p)\vartheta = \begin{pmatrix} 0 \\ \varepsilon(K-A) \end{pmatrix},$$
$$D^2F_c(S_2, \beta p)(\vartheta, \vartheta) = \begin{pmatrix} \alpha^2(A-3K) \\ \varepsilon\beta p\alpha(A-K) \end{pmatrix}.$$

Furthermore, it can be obtained that

$$\omega^{\mathrm{T}}F_c(S_2, \beta p) = 0,$$
$$\omega^{\mathrm{T}}[DF_c(S_2, \beta p)\vartheta] = \varepsilon(K-A) \neq 0,$$
$$\omega^{\mathrm{T}}[D^2F_c(S_2, \beta p)(\vartheta, \vartheta)] = \varepsilon\beta p\alpha(A-K) \neq 0.$$

Therefore, according to Sotomayor's theorem, system (2.2) undergoes a transcritical bifurcation at equilibrium point S_2 if $c^* = \beta p$. □

3.2 Hopf bifurcation

In two-dimensional systems, singular Hopf bifurcation is a special case of Hopf bifurcation, therefore, it is necessary to discuss the existence and stability of Hopf bifurcation.

According to the analysis in Subsection 2.2, we know that S_* is a center if $c = \frac{(A+K)\beta p}{2K}$, and the stability of S_* changes as the c value passes through $\frac{(A+K)\beta p}{2K}$. Therefore, Hopf bifurcation may occur at this equilibrium point.

Lemma 3.2. *System (2.2) will generate supercritical Hopf bifurcation near the equilibrium point $S_*(x_*, y_*)$, as the bifurcation parameter c passes through $\tilde{c} = \frac{(A+K)\beta p}{2K}$.*

Proof. Select c as the bifurcation parameter, with a bifurcation parameter threshold $\tilde{c} = \frac{(A+K)\beta p}{2K}$ that satisfies $\det(J|_{S_*}) > 0$ and $\mathrm{tr}(J|_{S_*}) = 0$. When $c = \tilde{c} = \frac{(A+K)\beta p}{2K}$, equation (2.4) has a pair of pure imaginary roots $\pm\sqrt{q_2}$, where $q_2 = c\varepsilon(1 - \frac{c}{\beta p})(\frac{Kc}{\beta p} - A)$. Next, we will verify the transversality condition for Hopf bifurcation. Assuming that $\lambda = \varphi(c) + i\psi(c)$ a root of the characteristic equation (2.4), we can obtain that $\varphi(\tilde{c}) = 0$, $\psi(\tilde{c}) = \sqrt{q_2}$. Substituting $\lambda = \varphi(c) + i\psi(c)$ into equation (2.4) and separating the real and imaginary parts, we obtain

$$\begin{cases} \varphi^2 - \psi^2 + q_1\varphi + q_2 = 0, \\ 2\varphi\psi + q_1\psi = 0. \end{cases}$$

After taking the derivative of c at both ends of the above equation, we obtain

$$\begin{cases} -2\psi\psi' + q_1\varphi' + q_2' = 0, \\ 2\varphi' + q_1' = 0. \end{cases}$$

After further calculation, we obtain $\frac{d\varphi}{dc}|_{c=\tilde{c}} = -\frac{A+K}{2\beta p} \neq 0$, which satisfies the transversality condition for Hopf bifurcation. Next, we will discuss the stability of Hopf bifurcation, it can

help determine under what conditions ecosystems can maintain equilibrium. We take the transformation $\bar{x} = x - x_*$, $\bar{E} = E - E_*$ to move (x_*, E_*) to the origin. Then dropping the bars, the system (2.2) performs Taylor expansion at the origin, and we obtain:

$$\begin{cases} \dfrac{dx}{dt} = a_{10}x + a_{01}E + a_{20}x^2 + a_{11}xE + a_{02}E^2 \\ \qquad\quad + a_{30}x^3 + a_{21}x^2E + a_{12}xE^2 + a_{03}E^3 + o(|x,E|^4), \\ \dfrac{dE}{dt} = b_{10}x + b_{01}E + b_{20}x^2 + b_{11}xE + b_{02}E^2 \\ \qquad\quad + b_{30}x^3 + b_{21}x^2E + b_{12}xE^2 + b_{03}E^3 + o(|x,E|^4), \end{cases} \tag{3.1}$$

where

$$\begin{aligned} &a_{10} = x_*(A + K - 2Kx_*), \quad a_{01} = -\alpha x_*, \quad a_{20} = -3Kx_* + A + K, \\ &a_{11} = -\alpha, \quad a_{30} = -K, \quad b_{10} = \frac{\varepsilon\beta p}{\alpha}(1 - x_*)(Kx_* - A), \quad b_{11} = \varepsilon\beta p, \\ &a_{02} = a_{21} = a_{12} = a_{03} = b_{01} = b_{20} = b_{02} = b_{30} = b_{21} = b_{12} = b_{03} = 0. \end{aligned}$$

The Jacobian matrix of system (3.1) at $(0,0)$ is

$$J|_{(0,0)} = \begin{pmatrix} x_*(A + K - 2Kx_*) & -\alpha x_* \\ \dfrac{\varepsilon\beta p}{\alpha}(1 - x_*)(Kx_* - A) & 0 \end{pmatrix}.$$

Through simple calculations, we can obtain $\mathrm{tr}(J|_{(0,0)}) = 0$ and $\det(J|_{(0,0)}) = \varepsilon\beta p x_*(1 - x_*)(Kx_* - A) > 0$ at $c = \tilde{c}$. Thus, $J|_{(0,0)}$ has a pair of pure imaginary eigenvalue

$$\lambda = \pm i\sqrt{\varepsilon\beta p x_*(1 - x_*)(Kx_* - A)} \quad \text{for } c = \tilde{c}.$$

Furthermore, the first Lyapunov coefficient l_1 [30] for determining the stability of the limit cycle is given:

$$l_1 = -\frac{3\pi}{2a_{01}D^{\frac{3}{2}}}\sum_{i=1}^{8}\xi_i,$$

where

$$\begin{aligned} &D = a_{10}b_{01} - a_{01}b_{10}, \quad \xi_1 = \varepsilon\alpha\beta p x_*(1 - x_*)(Kx_* - A)(A + K - 2Kx_*), \\ &\xi_2 = -\varepsilon\alpha\beta p x_*^2(A + K - 2Kx_*)(\varepsilon\beta p - 3Kx_* + A + K), \quad \xi_3 = \xi_4 = 0, \\ &\xi_5 = 2\alpha x_*^2(A + K - 2Kx_*)(A + K - 3Kx_*)^2, \quad \xi_6 = 0, \\ &\xi_7 = \left(\varepsilon\alpha\beta p x_*(1 - x_*)(Kx_* - A) + 2\alpha x_*^2(A + K - 2Kx_*)^2\right)(3Kx_* - A - K), \\ &\xi_8 = 3\alpha K x_*^3(A + K - 2Kx_*)^2 - 3\varepsilon\alpha\beta p K x_*^2(1 - x_*)(Kx_* - A). \end{aligned}$$

After substituting $x_* = \frac{c}{\beta p}$ and $\tilde{c} = \frac{(A+K)\beta p}{2K}$ into the calculation, and since $|A| < K$, we obtain

$$\begin{aligned} l_1 &= \left(\frac{3\pi K}{\alpha(A + K)\left(\frac{1}{2}\right)^{\frac{3}{2}}\left(\frac{\varepsilon\beta p A^3}{4K^2} - \frac{p\beta\varepsilon A^2}{4K} - \frac{A\beta p\varepsilon}{4} + \frac{K\beta p\varepsilon}{4}\right)^{\frac{3}{2}}}\right)\left(-\frac{K^2\alpha\beta p\varepsilon}{8} - \frac{A^4\alpha\beta p\varepsilon}{8K^2} + \frac{A^2\alpha\beta p\varepsilon}{4}\right) \\ &= -\left(\frac{3\pi K}{\alpha(A + K)\left(\frac{1}{2}\right)^{\frac{3}{2}}\left(\frac{\varepsilon\beta p}{4K^2}(K - A)^2(K + A)\right)^{\frac{3}{2}}}\right)\frac{\alpha\beta p\varepsilon}{8K^2}(K^2 - A^2)^2 \\ &= -\frac{6\sqrt{2}\pi K^2}{(K - A)\sqrt{\varepsilon\beta p(K + A)}} < 0. \end{aligned}$$

Therefore, system (2.2) generates a supercritical Hopf bifurcation near the equilibrium point $S_*(x_*, E_*)$. □

3.3 Numerical simulation

In this section, we verify the conclusions given earlier through numerical simulations. For system (2.2) with strong Allee effect, we select parameter $K = 3$, $A = 0.8$, $\alpha = 0.17$, $\varepsilon = 0.5$, $\beta = 1.5$, $p = 3$ (The data source from Table 3.2.).

Parameter	Reference range	Data sources
r	$[0.3, 0.9]$	[4,28]
K	$[2, 10]$	[33]
A	$[-1, 0) \cup (0, 1]$	[38,43]
m	$[0.01, 0.4]$	[10]
p	$[3, 15]$	[5]
q	$[0.1, 1.5]$	[5,39]
c	$[1, 15]$	[5]

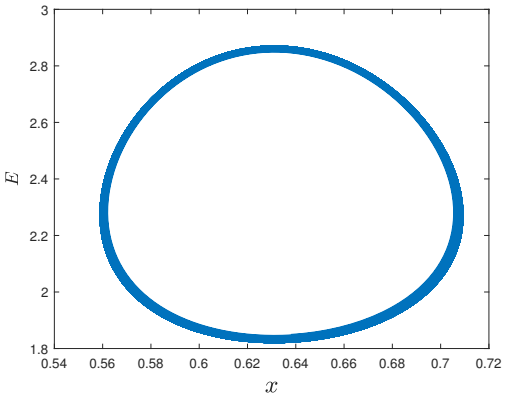
Table 3.1: Parameter selection range in Model (2.1).

Parameter	Reference range
K	$[2, 10]$
A	$[-1, 0) \cup (0, 1]$
$\alpha = \frac{q}{r}$	$[0.1, 1.5]$
$\beta = qK$	$[0.2, 15]$
p	$[3, 15]$
c	$[1, 15]$
$\varepsilon = \frac{m}{r}$	[0.01,1.3]

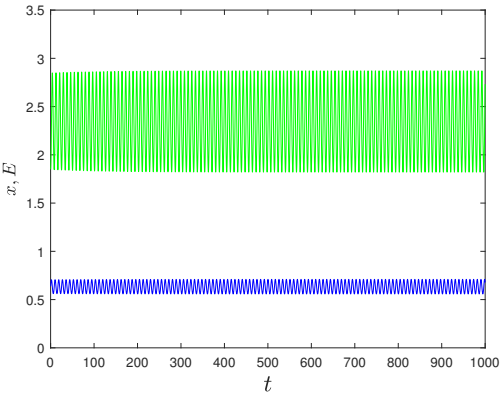
Table 3.2: Range of parameter values in system (2.2).

After calculation, it can be concluded that the bifurcation parameter value $c_H = 2.85$, and $S_*(x_*, E_*) = (0.6311, 2.3725)$. According to Lemma 3.2, it is known that a supercritical Hopf bifurcation will occur. As shown in Figure 3.1(a). Figure 3.1(b) is a time series graph of x and E. Figure 3.1(c) is the phase diagram of system (2.2). Furthermore, selecting parameter $\varepsilon = 0.5$, $c = 2.89 > c_H$, we can obtain that the equilibrium point S_* is locally asymptotically stable. (See Fig. 3.2).

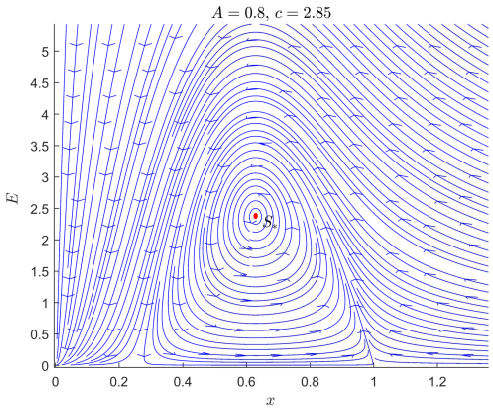
For system (2.2) with weak Allee effect, we select parameter $K = 3$, $A = -0.3$, $\alpha = 0.17$, $\varepsilon = 0.5$, $\beta = 1.5$, $p = 3$ (the data source from Table 3.2.). After calculation, it can be concluded that the bifurcation parameter value $c_H = 2.025$, and $S_*(x_*, E_*) = (0.4444, 5.3377)$. As shown in Figure 3.1(d-f).



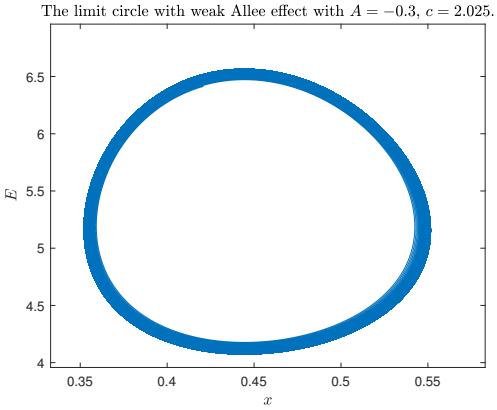
(a)



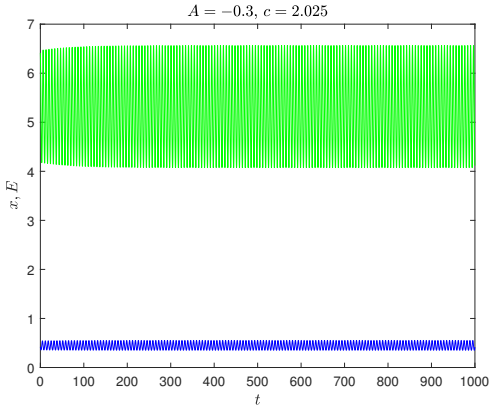
(b)



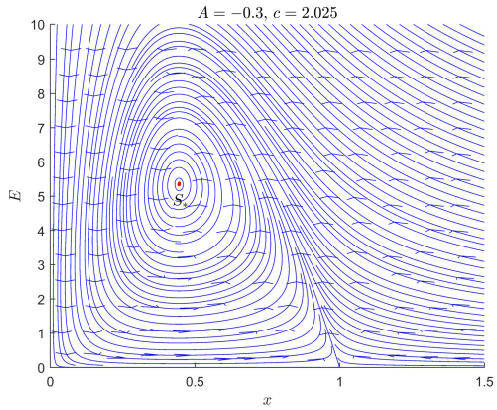
(c)



(d)

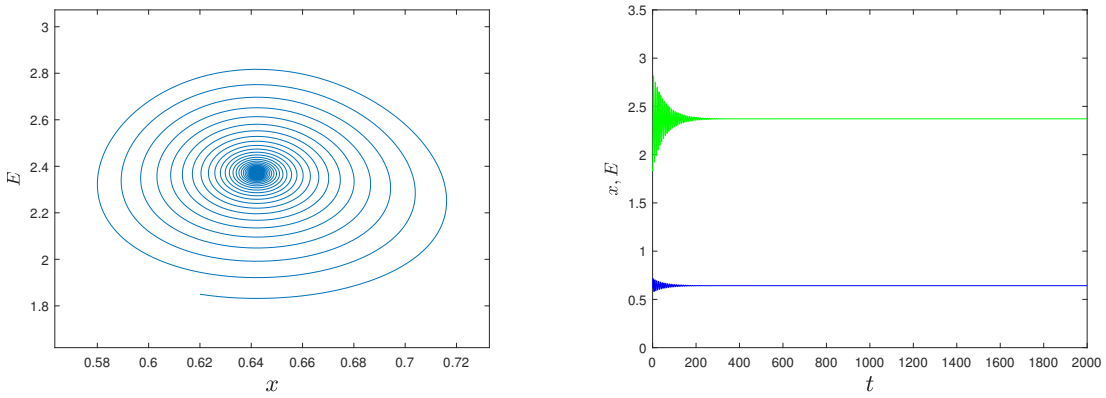


(e)

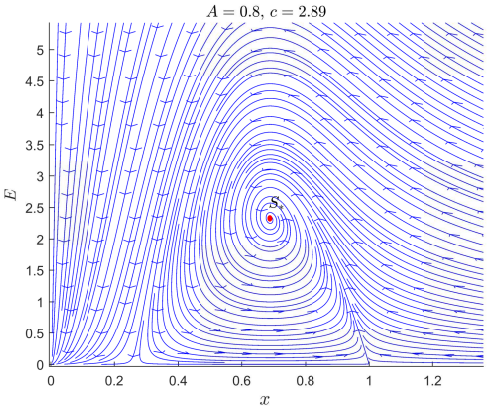


(f)

Figure 3.1: Phase plane analysis of system (2.2). (a, d) The periodic cyclic orbit generated by the system (2.2). (b, d) $x - t$ (blue) and $E - t$ (green) time series diagram. (c, f) The phase diagram of system (2.2).



(a) (b)



(c)

Figure 3.2: Phase plane analysis of system (2.2) for $A = 0.8$ and $c = 2.89$. (a) The equilibrium point S_* is locally asymptotically stable when $K = 3$, $A = 0.8$, $\alpha = 0.17$, $\varepsilon = 0.5$, $\beta = 1.5$, $p = 3$, $c = 2.89$. (b) $x - t$ (blue) and $E - t$ (green) time series diagram. (c) The phase diagram of system (2.2).

4 Singular perturbation analysis

In this section, we use singular perturbation theory [9] to discuss the fast-slow dynamic behavior of system (2.2). Under the strong Allee effect, system (2.2) will generate a singular Hopf bifurcation near the internal equilibrium point. Under the weak Allee effect, the dynamical behavior generated by system (2.2) is more complex, resulting not only in singular Hopf bifurcations but also relaxation oscillations.

4.1 Asymptotic expansion of critical manifolds

Let $\varepsilon \to 0$, system (2.2) becomes

$$\begin{cases} \dfrac{dx}{dt} = f(x, E, 0), \\ \dfrac{dE}{dt} = 0, \end{cases} \tag{4.1}$$

which is called the fast subsystem or layer equation, and the corresponding flow is called fast flow. Let $\varepsilon \to 0$, system (2.3) can be reduced to a differential algebraic system given by

$$\begin{cases} 0 = f(x, E, 0), \\ \dfrac{dE}{d\tau} = g(x, E, 0), \end{cases} \tag{4.2}$$

which is called the slow subsystem, and the flow it generates is called slow flow. The slow flow is constrained in set $C_0 = \left\{(x, E) \in \mathbb{R}^2_+, f(x, E, \varepsilon) = 0\right\}$. C_0 is called the critical manifold, which consists of two parts and is given by

$$\begin{aligned} C_0 = C_{10} \cup C_{20} &= \left\{(x, E) \in \mathbb{R}^2_+,\ x = 0\right\} \\ &\cup \left\{(x, E) \in \mathbb{R}^2_+,\ E = u_0(x),\ u_0(x) = \frac{(1-x)(Kx - A)}{\alpha},\ \frac{A}{K} < x < 1,\ E > 0\right\}, \end{aligned} \tag{4.3}$$

which is the critical manifold of system (2.2) with strong Allee effect($0 < A < K$). The critical manifold C_{20} is similar to a parabola, with its maximum value being the fold point $P(x_f, E_f)$, where $x_f = \frac{A+K}{2K}$ and $E_f = \frac{(1-x_f)(Kx_f - A)}{\alpha}$. The nonhyperbolic fold point divides the critical manifold C_0 into two parts: the attracting sub-manifold $C_0^a = C_{20}^a \cup C_{10} = \{(x, E) \in \mathbb{R}^2_+, x_f < x < 1\} \cup C_{10}$ and the repelling sub-manifold $C_0^r = C_{20}^r = \left\{(x, E) \in \mathbb{R}^2_+, \frac{A}{K} < x < 1\right\}$ (see Figure 4.1(a)).

For weak Allee effect ($A < 0$), the slow flow generated by the slow subsystem is limited to the critical manifold

$$\begin{aligned} C_0 = C_{10} \cup C_{20} &= \left\{(x, E) \in \mathbb{R}^2_+,\ x = 0\right\} \\ &\cup \left\{(x, E) \in \mathbb{R}^2_+,\ E = u_0(x),\ u_0(x) = \frac{(1-x)(Kx - A)}{\alpha},\ 0 < x < 1,\ E > 0\right\}. \end{aligned} \tag{4.4}$$

After calculation, we obtain a maximum point $P(x_f, E_f)$, where $x_f = \frac{A+K}{2K}$, $E_f = \frac{(1-x_f)(Kx_f - A)}{\alpha}$, a transcritical point $B(0, E_B) = (0, -\frac{A}{\alpha})$, and a boundary point $C(x_C, 0) = (1, 0)$. The critical manifold C_0 can be partitioned into two segments:

$$\begin{aligned} C_0^a = C_{10}^a \cup C_{20}^a &= \left\{(x, E) \in \mathbb{R}^2_+, x = 0, E > -\frac{A}{\alpha}\right\} \cup \left\{(x, E) \in \mathbb{R}^2_+, x_f < x < 1\right\}, \\ C_0^r = C_{10}^r \cup C_{20}^r &= \left\{(x, E) \in \mathbb{R}^2_+, x = 0, 0 < E < -\frac{A}{\alpha}\right\} \cup \left\{(x, E) \in \mathbb{R}^2_+, 0 < x < x_f\right\}, \end{aligned}$$

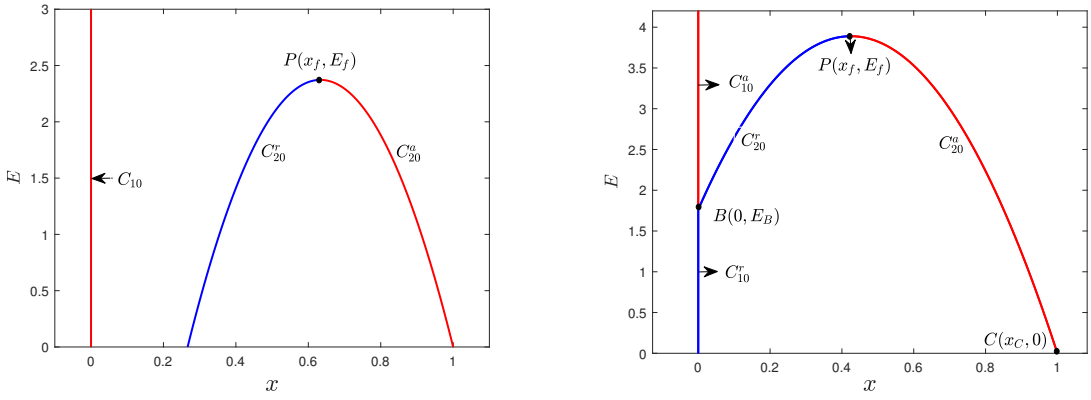
where C_0^a is normally attracting and C_0^r is normally repelling (see Figure 4.1(b)). According to the Fenichel's Theorem [19], there exists a locally invariant manifold C_ε with differential homeomorphism at C_{20}. The Hausdorff distance from C_ε to C_{20} is $O(\varepsilon)$, and the flow on C_ε converges to a slow flow when ε tends to 0. Therefore, C_ε can be seen as a small perturbation of C_{20}, given by:

$$C_\varepsilon := u_0(x) + \varepsilon u_1(x) + \varepsilon^2 u_2(x) + \cdots = C_{20} + O(\varepsilon), \tag{4.5}$$

where $u_0(x)$ is the critical manifold of C_{20} as given by (4.3). In addition, we use C_ε^a and C_ε^r represent the perturbed attracting sub-manifold and repelling sub-manifold, respectively. Due to the local invariance of approximating manifold C_ε, we have

$$\frac{dE}{dt} = \frac{dE}{dx}\frac{dx}{dt} = \left(\frac{du_0}{dx} + \varepsilon\frac{du_1}{dx} + \varepsilon^2\frac{du_2}{dx} + \dots\right)\frac{dx}{dt}. \tag{4.6}$$

Substituting $\frac{dE}{dt}$ and $\frac{dx}{dt}$ into equation (4.6), and by comparing the coefficients of ε, it can be



(a) (b)

Figure 4.1: The critical manifold. (a) The critical manifold of system (2.2) with strong Allee effect. (b) The critical manifold of system (2.2) with weak Allee effect. The blue lines represent the repelling sub-manifold of the critical manifold, and the red lines represent the attracting sub-manifold of the critical manifold.

obtained that

$$u_1(x) = \frac{(c - \beta p x)u_0}{\alpha x \frac{du_0}{dx}}.$$

By comparing the coefficients of ε^2, it can be concluded that

$$u_2(x) = \frac{(c - \beta p x - \alpha x \frac{du_1}{dx})u_1}{\alpha x \frac{du_1}{dx}}.$$

Using the same method, $u_j(x), j = 3, 4, \ldots$ can be obtained. But according to Fenichel's theorem, perturbation manifolds cannot be extended at fold points (x_f, E_f) that lose hyperbolic properties [6].

4.2 Singular Hopf bifurcation

In this section, we discuss the existence of singular Hopf bifurcation of system (2.2). And we do not limit the threshold of the Allee effect, that is, whether it is strong or weak Allee effect, system (2.2) will produce singular Hopf bifurcation.

According to the analysis in Subsection 2.2, the system has an internal equilibrium point (x_*, E_*) for $\frac{A}{K} < \frac{c}{\beta p} < 1$. The Jacobian matrix at this point is

$$J|^t_{S_*} = \begin{pmatrix} \frac{c}{\beta p}\left(A + K - \frac{2Kc}{\beta p}\right) & -\frac{\alpha c}{\beta p} \\ \frac{\varepsilon \beta p}{\alpha}\left(1 - \frac{c}{\beta p}\right)\left(\frac{Kc}{\beta p} - A\right) & 0 \end{pmatrix},$$

where t represents the Jacobian matrix of the system at S_* at a fast time scale.

According to the Hopf bifurcation principle, we need to solve the Hopf bifurcation threshold to satisfy that the trace of $J|^t_{S_*}$ is equal to 0 and the determinant value of $J|^t_{S_*}$ is greater than 0, which means that $q_1 = 0$ and $q_2 > 0$ in (2.4). Taking c as the bifurcation parameter, we obtain the bifurcation threshold

$$c_H = \frac{(A+K)\beta p}{2K},$$

and the equilibrium (x_H, E_H) at $c = c_H$, where $x_H = \frac{c}{\beta p}$, $E_H = \frac{(1-x_*)(Kx_*-A)}{\alpha}$. This point coincides perfectly with the fold point. At the Hopf bifurcation threshold, $J|^t_{S_*}$ has a pair of pure imaginary eigenvalues:

$$\lambda_H(\sqrt{\varepsilon}) = \pm i\sqrt{\frac{(K-A)^2(A+K)\beta p\varepsilon}{8K^2}}.$$

Theorem 4.1. *Set $0 < \varepsilon \ll 1$. Assuming (x_f, E_f) is the generic folded singularity of system (2.2), system (2.2) undergoes a singular Hopf bifurcation in a certain domain of x_f when $c = c_H$, resulting in a stable canard cycles. In addition, there exists a singular Hopf bifurcation curve $\delta = \delta_H(\sqrt{\varepsilon})$ such that (x_f, E_f) is stable when $\delta < \delta_H(\sqrt{\varepsilon})$ and*

$$\delta_H(\sqrt{\varepsilon}) = -\frac{a_1 + a_5}{2}\varepsilon + \mathcal{O}(\varepsilon^{\frac{3}{2}}). \tag{4.7}$$

Proof. First, we discuss that (x_f, E_f) is a generic folded singularity. By calculation, it can be concluded that (x_f, E_f) satisfies the following conditions:

$$\begin{aligned} f(x_f, E_f) &= 0, \qquad f_x(x_f, E_f) = 0, \\ f_{xx}(x_f, E_f) &\neq 0, \qquad f_y(x_f, E_f) \neq 0, \end{aligned}$$

and

$$g(x_f, E_f) = 0,$$

thus (x_f, E_f) is a fold singularity. In addition, it satisfies the condition:

$$g_x(x_f, E_f) \neq 0, \quad g_c(x_f, E_f) \neq 0,$$

thus (x_f, E_f) is a generic fold singularity.

In order to study the local behavior of system (2.2) near $P(x_f, E_f)$, we perform coordinate transformation

$$u = x - x_*, \quad v = E - E_*, \quad \delta = c - c_H,$$

and system (2.2) becomes

$$\begin{cases} \dfrac{du}{dt} = -vh_1(u, v, \delta, \varepsilon) + u^2 h_2(u, v, \delta, \varepsilon) + \varepsilon h_3(u, v, \delta, \varepsilon), \\ \dfrac{dv}{dt} = \varepsilon \left\{ uh_4(u, v, \delta, \varepsilon) - \delta h_5(u, v, \delta, \varepsilon) + vh_6(u, v, \delta, \varepsilon) + o(|u, v, \delta|^3) \right\}, \end{cases} \tag{4.8}$$

where

$$\begin{aligned} h_1(u, v, \delta, \varepsilon) &= \alpha u + \alpha x_*, \\ h_2(u, v, \delta, \varepsilon) &= A + K - 3Kx_* - Ku, \\ h_3(u, v, \delta, \varepsilon) &= 0, \\ h_4(u, v, \delta, \varepsilon) &= \beta p v + \beta p E_*, \\ h_5(u, v, \delta, \varepsilon) &= v + E_*, \\ h_6(u, v, \delta, \varepsilon) &= \beta p x_* - c_H. \end{aligned}$$

According to the theory in [19], we have obtained

$$\begin{aligned} a_1 &= \frac{\partial h_3}{\partial u}(0,0,0,0) = 0, \\ a_2 &= \frac{\partial h_1}{\partial u}(0,0,0,0) = \alpha, \\ a_3 &= \frac{\partial h_2}{\partial u}(0,0,0,0) = -K, \\ a_4 &= \frac{\partial h_4}{\partial u}(0,0,0,0) = 0, \\ a_5 &= h_6(0,0,0,0) = \beta p x_* - c_H. \end{aligned} \tag{4.9}$$

After further calculation, the Hopf bifurcation curve is obtained to be

$$\delta_H(\sqrt{\varepsilon}) = -\frac{a_1 + a_5}{2}\varepsilon + \mathcal{O}(\varepsilon^{\frac{3}{2}}) = \mathcal{O}(\varepsilon^{\frac{3}{2}}),$$

and the maximum canard curve is

$$\delta_c(\sqrt{\varepsilon}) = -\Big(\frac{a_1 + a_2}{2} + \frac{A}{8}\Big)\varepsilon + \mathcal{O}(\varepsilon^{\frac{3}{2}}) = \frac{\alpha + 3K}{8} + \mathcal{O}(\varepsilon^{\frac{3}{2}}). \qquad \square$$

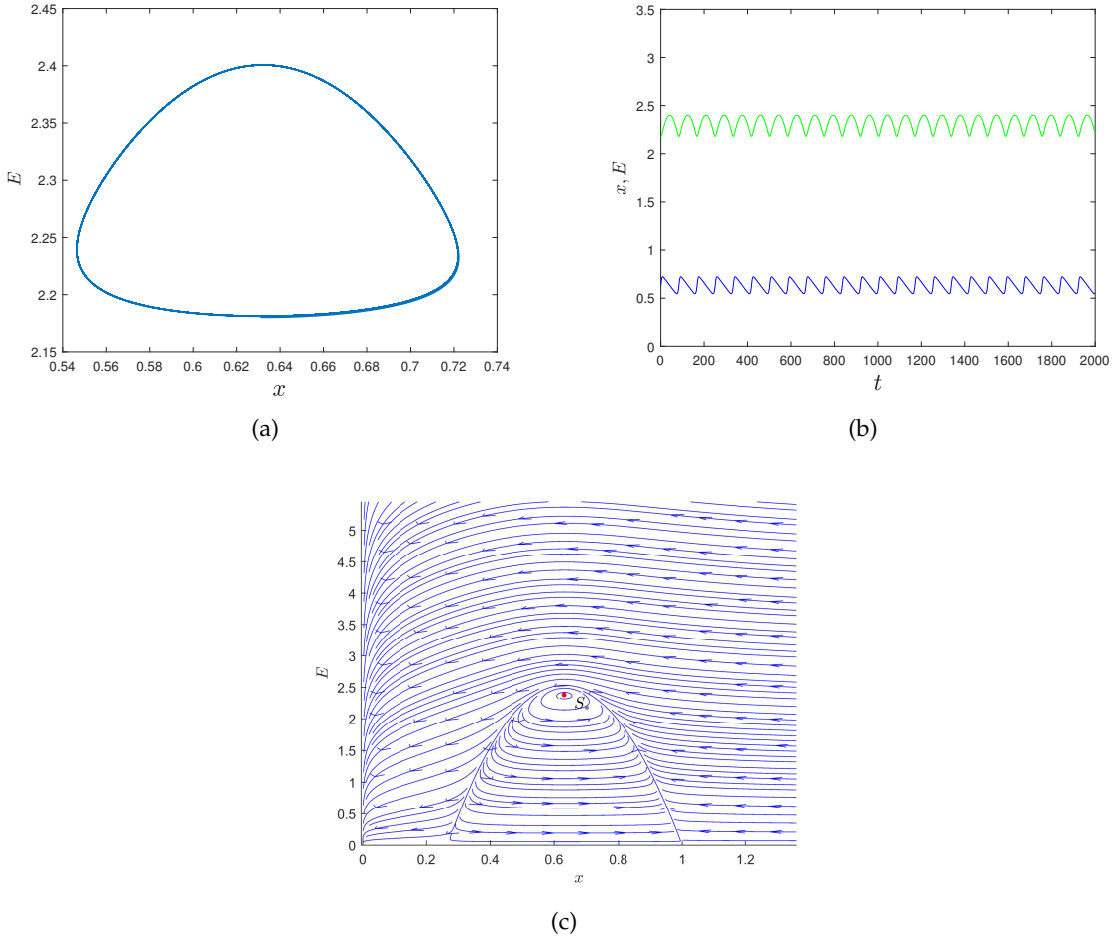
Compared with the Hopf bifurcation in Subsection 3.1, we can find that the singular Hopf bifurcation coincides exactly with the case of the aforementioned Hopf bifurcation. Thus, the stability of periodic solutions generated by singular Hopf bifurcation is the same as that of Hopf bifurcation. According to Lemma 3.2, system (2.2) undergoes a supercritical singular Hopf bifurcation for $l_1 < 0$. According to [19], we can calculate that system (2.2) will generate a singular supercritical Hopf bifurcation by calculating $A_1 = -a_2 + 3a_3 - 2a_4 + 2a_5 = -\alpha - 3K < 0$. The conclusion obtained from Lemma 3.2 is also consistent.

Through numerical simulation, we verify the above conclusion. A canard without head will be generated, when $K = 3$, $A = 0.8$, $\alpha = 0.17$, $\varepsilon = 0.05$, $\beta = 1.5$, $p = 3$, $c = 2.8446$, (see Figure 4.2). Compared with Hopf bifurcation, under the influence of perturbation parameter ε, the bottom of the limit cycle generated by the singular Hopf bifurcation will be flatter. By comparing Figure 3.1(a) with Figure 4.2(a), it can be seen.

Remark 4.2. Fishing effort refers to the amount of work invested in a fishery using the same fishing operation method over a period of time, reflecting the mortality level of the captured resource population. The emergence of Hopf bifurcation and singular Hopf bifurcation indicates that the system will experience periodic fluctuations, which means that population density and capture effort will show periodic changes. This also means that there is a dynamic balance between population density and mortality rate, and the ecosystem is in a state of dynamic equilibrium.

4.3 Relaxation oscillation

Compared with system (2.2) with strong Allee effect, system (2.2) with weak Allee effect produce more complex dynamical behavior, resulting to relaxation oscillations. This section mainly discusses the relaxation oscillations generated by system (2.2) with weak Allee effect ($A < 0$).



(a) (b)

(c)

Figure 4.2: Phase plane analysis of system (2.2) for $\varepsilon = 0.05$ and $c = 2.8446$. (a) The canard without head generated by the system (2.2), when $K = 3$, $A = 0.8$, $\alpha = 0.17$, $\varepsilon = 0.05$, $\beta = 1.5$, $p = 3$, $c = 2.8446$. (b) $x - t$ (blue) and $E - t$ (green) time series diagram. (c) The phase diagram.

To discuss the existence of relaxation oscillations when $0 < x_* < x_f$, system (2.2) is expressed as follows:

$$\begin{cases} \dfrac{dx}{dt} = x f_1(x, E, \varepsilon), \\ \dfrac{dE}{dt} = \varepsilon g(x, E, \varepsilon), \end{cases} \tag{4.10}$$

where $(x, E) \in \mathbb{R}^2_+$, $0 < \varepsilon \ll 1$, $f_1(x, E, \varepsilon) = (1 - x)(Kx - A) - \alpha E$ and $g(x, E, \varepsilon) = (\beta p x - c) E$. f_1 and g are sufficiently smooth and satisfy the following conditions:

$$f_1(x, E, \varepsilon) < 0, \quad g(x, E, \varepsilon) < 0, \quad \text{for } E > E_B > 0,$$
$$f_1(x, E, \varepsilon) > 0, \quad \text{for } E < E_B.$$

For $\varepsilon = 0$, the E-axis consists of the attracting part with $E > E_B$ and the repelling part with $E < E_B$. For a very small ε ($\varepsilon > 0$), the trajectory that starts from the point (x_0, E_0) ($x_0 > 0$ is quite small, $E > E_B$) gets attracted to the E-axis. Subsequently, it drifts downwards, and when it passes through $E = E_B$, it is repelled outside the E-axis. For a very small $\varepsilon(\varepsilon > 0)$, the

trajectory intersects again with the line $x = x_0$ at the point whose E-coordinate value is $m_\varepsilon(E_0)$, such that $\lim_{\varepsilon\to 0} m_\varepsilon(E_0) = m_0(E_0)$, where $m_0(E_0)$ is is calculated by the following integration:

$$\int_{E_0}^{m_0(E_0)} \frac{f_1(0,E,0)}{g(0,E,0)} dE = 0. \tag{4.11}$$

The function $E_0 \to m_0(E_0)$ implicitly defined above is called the entry-exit function [8,37](see Figure 4.3).

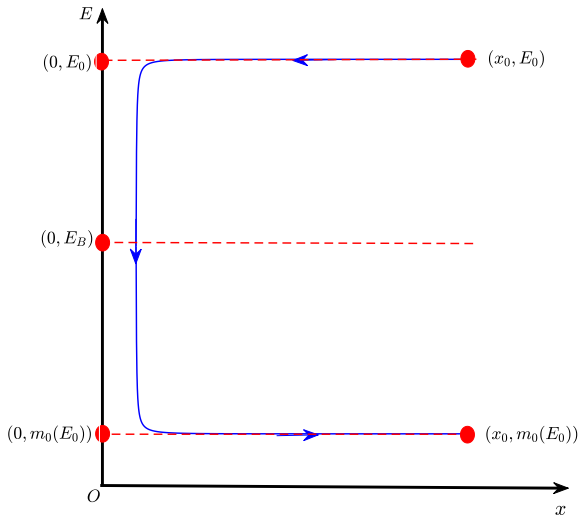


Figure 4.3: The entry-exit function. For system (2.2), when $\varepsilon = 0$, the branch of the E-axis is composed of the attracting part of $E > E_B$ and the repelling part of $E < E_B$. When $0 < \varepsilon \ll 1$, a typical orbit of system (2.2) starts at (x_0, E_0), where $E_0 > E_B$, and $x_0 > 0$ is very small, is attracted to the E-axis (not crossing the E-axis), gradually moves downwards along the E-axis until it reaches near $(0, m_0(E_0))$. Then it is repelled by the E-axis, and ends at point $(x_0, m_0(E_0))$.

On the basis of this, we have obtained the following results:

Lemma 4.3. *There exists a unique $\tilde{E}$ where $0 < \tilde{E} < -\frac{A}{\alpha}$, such that*

$$G(E) = \int_{E_f}^{E} \frac{f_1(0,s,0)}{g(0,s,0)} ds = 0. \tag{4.12}$$

Proof. We have

$$\begin{aligned} G(E) &= \int_{E_f}^{E} \frac{f_1(0,s,0)}{g(0,s,0)} ds \\ &= \int_{E_f}^{E} \frac{A+\alpha s}{cs} ds \\ &= \frac{A}{c} \ln \frac{E}{E_f} + \frac{\alpha}{c}(E - E_f) \to +\infty \quad \text{as } E \to 0^+. \end{aligned} \tag{4.13}$$

Furthermore, it can be calculated that

$$G'(E) = \frac{A+\alpha s}{cs}, \quad \text{for } 0 < E < -\frac{A}{\alpha}. \tag{4.14}$$

Therefore, G_E is strictly monotonically decreasing when $0 < E < -\frac{A}{\alpha}$. Again,

$$G\left(-\frac{A}{\alpha}\right) = \int_{E_f}^{-\frac{A}{\alpha}} \frac{A+\alpha s}{cs} ds < 0. \tag{4.15}$$

Thus there exists a unique $\tilde{E}$ where $0 < \tilde{E} < -\frac{A}{\alpha}$, such that $G(\tilde{E}) = 0$. □

The points $P(x_f, E_f)$ and $B(0, E_B)$ on the critical manifolds C_{10} and C_{20} that lose their hyperbolic properties are fold point and transcritical point, respectively. For fold point $P(x_f, E_f)$, we have

$$\frac{\partial f}{\partial E}|_{P(x_f,E_f)} = -\alpha x_f \neq 0,$$

as $c < \frac{(A+K)\beta p}{2K}$ and $g(x_f, E_f) \neq 0$ as $0 < x_* < x_f$. In addition, we have

$$\frac{\partial f}{\partial x^2}|_{P(x_f,E_f)} = -6Kx_f + 2(A+K) \neq 0,$$

consequently, $P(x_f, E_f)$ is a generic fold point as well as also a jump point, at which point the fast fiber flow leaves the critical manifold C_{20}.

For the transcritical point $B(0, E_B)$, we have

$$\begin{aligned} \frac{\partial f}{\partial x}|_{B(0,E_B)} &= 0, \\ \frac{\partial f}{\partial E}|_{B(0,E_B)} &= 0, \\ g(0, E_B) &= \frac{Ac}{\alpha} < 0, \\ \begin{vmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial E} \\ \frac{\partial^2 f}{\partial x \partial E} & \frac{\partial^2 f}{\partial E^2} \end{vmatrix}_{B(0,E_B)} &= -\alpha^2 < 0, \end{aligned}$$

therefore, $B(0, E_B)$ is a generic transcritical point and jump point, at which point the fast flow is removed from the critical manifold C_{20}.

Define a singular fast-slow cycle Γ_0: the trajectory starts from $A(0, E_f)$, follows the slow flow along the E-axis to $C(0, \tilde{E})$. At $C(0, \tilde{E})$, it is repelled to the right, following the fast flow until it intersects with the attraction branch at point $D(\tilde{x}, \tilde{E})$. Then it follows the slow flow C_{20}^a to point $P(x_f, E_f)$, and finally follows the fast flow to the left back to point $A(0, E_f)$. Therefore, Γ_0 has four singular orbits, where $P(x_f, E_f)$ and $B(0, E_B)$ is the jump point, $A(0, E_f)$ and $D(\tilde{x}, \tilde{E})$ is the drop point (as shown by the red line in Figure 4.4).

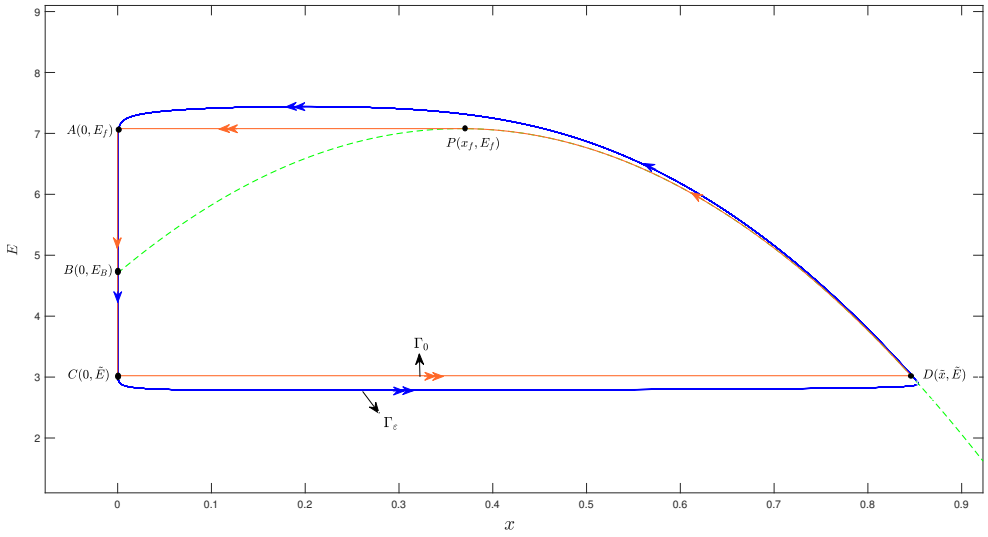


Figure 4.4: Representation the singular orbit Γ_0(red) and the relaxation oscillation orbit Γ_ε(blue) when $\varepsilon > 0$. The critical manifold of system (2.2) is represented by a green dashed line. Point $P(x_f, E_f)$ represents the maximum point of curve $E = u_0(x)$. The intersection point between curve $E = u_0(x)$ and the E-axis is $B(0, E_B)$. The branch on the E-axis is hyperbolic attracting with $E > E_B$ and The branches on the E-axis are hyperbolic repelling with $E < E_B$. The branch between P and B is hyperbolic repelling, while the right half branch of P is hyperbolic attracting. Double arrows denote fast flow. Single arrows denote slow flow.

Theorem 4.4. *If* $\frac{A}{K} < \frac{c}{\beta p} < 1$, $c < \frac{(A+K)\beta p}{2K}$ *and* U *is a small neighborhood of* Γ_0 *for system* (2.2). *Then, for any sufficiently small* $\varepsilon > 0$, *there exists a unique limit cycle* $\Gamma_\varepsilon \in U$. *The limit cycle* Γ_ε *i attracting, and its characteristic multiplier is bounded by* $-C/\varepsilon$ *for a constant* $C > 0$. *Moreover, the limit cycle* Γ_ε *converges to* Γ_0 *in the Hausdoff distance when* ε *approaches 0.*

Proof. According to the Fenichel's theorem [9], $C^a_{10} = \{(x, E) \in \mathbb{R}^2_+, x = 0, E > -\frac{A}{\alpha}\}$ and $C^a_{20} = \{(x, E) \in \mathbb{R}^2_+, x_f < x < 1\}$ are the attracting part of the critical manifold, perturbing the nearby slow manifolds $C^a_{20,\varepsilon}$ and $C^a_{10,\varepsilon}$. By analyzing the dynamics near the fold point, the slow flow $C^a_{20,\varepsilon}$ follows the critical manifold C^a_{20} until it reaches the area near the generic fold point $P(x_f, E_f)$, and then jumps to another attracting branch C^a_{10}.

We consider a small cross-section Δ that intersects to C^a_{10} at a point between P and A, and track two trajectories Γ^1_ε and Γ^2_ε starting from Δ. According to the Fenichel's theorem [9] and Theorem 7.4.1 in [19], Γ^1_ε and Γ^2_ε will be attracted to the slow manifold C^a_{10} at an exponential velocity $\mathcal{O}(e^{\frac{1}{\varepsilon}})$. When passing through the generic transcritical point $B(0, E_B)$, $\Gamma^{1,2}_\varepsilon$ will contract exponentially with each other until they reach a neighborhood of $C^a_{20,\varepsilon}$. Applying Fenichel's theorem again, it is known that Γ^1_ε and Γ^2_ε will be attracted to the branch C^a_{20}, passing through point $P(x_f, E_f)$, and finally returning to A.

Therefore, we obtained a return map Σ: $\Delta \to \Delta$ induced by the flow of system (2.2) with $0 < \varepsilon \ll 1$. Σ is a compressed map when the trajectories contract with each other at a rate $\mathcal{O}(e^{\frac{1}{\varepsilon}})$. According to the principle of compression map, Σ has a unique stable fixed point, which is the expected limit cycle Γ_ε(As shown by the blue line in Figure 4.4). Since contraction

is exponential, the characteristic multiplier of Γ_ε has an upper bound of $-C/\varepsilon$ for $C > 0$. Applying the Fenichel's theorem [9] and Theorem 7.4.1 in [19], it is obtained that the limit cycle Γ_ε converges to Γ_0 in the Hausdoff distance when $\varepsilon \to 0$. □

Remark 4.5. For the fishery economy, relaxation oscillations usually rotate counterclockwise. This is due to the input of food by the aquaculture plant, which leads to an increase in fish density. After the increase in fish density, the workload for catching also increases. Correspondingly, as the amount of capture effort increases, the density of fish schools will also decrease. As the number of fish schools decreases, the workload for catching will also decrease. After going through these steps, the system returned to its initial state and continued to oscillate counterclockwise. This means that the system is in a dynamic equilibrium, with a focus on the capture effort increasing at a slow time scale compared to fish density. For aquaculture, this means that its economy is turnover, allowing for optimized resource allocation and sustained economic activities.

4.4 Stochastic differential system

In nature, biological populations are inevitably subject to random disturbances from the external environment, such as storms, earthquakes, natural enemies, etc. Similarly, the capture of population is also affected by weather and natural disasters. Therefore, a random perturbation term is added to discuss the existence of stationary distribution of the system's solution. We assume that the random disturbance is white noise type, which is proportional to x and E in system (2.2). Therefore, the system with added random disturbance is described as:

$$\begin{cases} dx = [x(1-x)(Kx-A) - \alpha Ex]dt + \sigma_f x dB_1(t) = f(x,E)dt + \sigma_f x dB_1(t), \\ dE = \varepsilon(\beta px - c)Edt + \sqrt{\varepsilon}\sigma_g E dB_2(t) = \varepsilon g(x,E)dt + \sqrt{\varepsilon}\sigma_g E dB_2(t). \end{cases} \tag{4.16}$$

Perform a time scale transformation of $\tau = \varepsilon t$, the above equation is transformed into

$$\begin{cases} dx = \dfrac{1}{\varepsilon}[x(1-x)(Kx-A) - \alpha Ex]d\tau + \dfrac{\sigma_f x}{\sqrt{\varepsilon}} dB_1(\tau) = \dfrac{1}{\varepsilon} f(x,E)dt + \dfrac{\sigma_f x}{\sqrt{\varepsilon}} dB_1(\tau), \\ dE = (\beta px - c)Ed\tau + \sigma_g E dB_2(\tau) = g(x,E)d\tau + \sigma_g E dB_2(\tau), \end{cases} \tag{4.17}$$

where $B_i(\tau)$, $i = 1,2$ is the standard Brownian motion on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, σ_f and σ_g represents noise intensity.

Theorem 4.6. *For any $(x(0), E(0)) \in \mathbb{R}^2_+$, as $\varepsilon \to 0$, solution of (4.17) converges weakly to $E(\cdot)$ that is a solution of the stochastic differential equation*

$$dE(\tau) = \left(\beta pE \int_{x(0)}^{\infty} x\mu_E(x)dx - cE\right) d\tau + \sigma_g E dB_2(\tau). \tag{4.18}$$

Proof. According to [17], we need to calculate the invariant measure $\mu_E(x)$ of equation (4.17)-1, that is, to calculate the Fokker–Planck equation corresponding to equation (4.17)-1:

$$\frac{\partial P(x,\tau|E)}{\partial \tau} = -\frac{\partial}{\partial x}\left\{\frac{1}{\varepsilon}\left(x(1-x)(Kx-A) - \alpha Ex\right)p(x,\tau|E)\right\} + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left(\frac{\sigma_f^2 x^2}{\varepsilon} p(x,\tau|E)\right) = 0.$$

By transforming $\tau = \varepsilon t$, the above equation is transformed into

$$\frac{\partial P(x,t|E)}{\partial t} = -\frac{\partial}{\partial x}\left\{\left(x(1-x)(Kx-A) - \alpha Ex\right)p(x,t|E)\right\} + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left(\sigma_f^2 x^2 p(x,t|E)\right).$$

Solving $\frac{\partial P(x,t|E)}{\partial t} = 0$, we obtain

$$\mu_E(x) = Cx^{-N} \exp\left\{\frac{2(K+A)}{\sigma_f^2}x - \frac{K}{\sigma_f^2}x^2\right\},$$

where $N = \frac{2(A+\alpha E)}{\sigma_f^2} + 2$, C is a normalization constant that satisfies

$$C\int_{x(0)}^{\infty} x^{-N} \exp\left\{\frac{2(K+A)}{\sigma_f^2}x - \frac{K}{\sigma_f^2}x^2\right\} dx = 1.$$

To ensure that C is computable, we prove that the integral $\int_{x(0)}^{\infty} x^{-N} \exp\left\{\frac{2(K+A)}{\sigma_f^2}x - \frac{K}{\sigma_f^2}x^2\right\}dx$ converges. For the proof of integral convergence, please refer to the appendix.

According to the Khasminskii's limit theorem [16] and Theorem 5.1 in [40], in the extreme case $\varepsilon \to 0$, the evolution of the slow variable E can be described by a simplified stochastic differential equation, as follows

$$dE(\tau) = g_{red}(E)d\tau + \sigma_{red}(E)dB_2(\tau), \tag{4.19}$$

where

$$g_{red}(E) = \int_{x(0)}^{\infty} (\beta p x - c)E\mu_E(x)dx,$$
$$\sigma_{red}^2(E) = \int_{x(0)}^{\infty} \sigma_g^2 E^2 \mu_E(x)dx.$$

Since $\int_{x(0)}^{\infty} \mu_E(x)dx = 1$, we obtain that

$$g_{red}(E) = \beta p E \int_{x(0)}^{\infty} x\mu_E(x)dx - cE,$$
$$\sigma_{red}^2(E) = \sigma_g^2 E^2.$$

Therefore, $\sigma_{red}(E) = \sigma_g E$,

$$\int_0^{\infty} x\mu_E(x)dx = \frac{\int_{x(0)}^{\infty} x^{-N+1} \exp\left\{\frac{2(K+A)}{\sigma_f^2}x - \frac{K}{\sigma_f^2}x^2\right\} dx}{\int_{x(0)}^{\infty} x^{-N} \exp\left\{\frac{2(K+A)}{\sigma_f^2}x - \frac{K}{\sigma_f^2}x^2\right\} dx}.$$

Similarly, we also need to prove that $\int_{x(0)}^{\infty} x^{-N+1} \exp\left\{\frac{2(K+A)}{\sigma_f^2}x - \frac{K}{\sigma_f^2}x^2\right\}dx$ is convergent, as shown in the appendix. Furthermore, substituting $g_{red}(E)$ and $\sigma_{red}(E)$ into eq. (4.19), we have

$$dE(\tau) = \left(\beta p E \int_{x(0)}^{\infty} x\mu_E(x)dx - cE\right)d\tau + \sigma_g E dB_2(\tau).$$

□

Theorem 4.7. *System* (4.18) *has a globally unique positive solution, for any initial value* $(x(0), E(0)) \in \mathbb{R}_+^2$.

Proof. We verify the existence and uniqueness of the global positive solution by proving that system(4.18) satisfies the Lipschitz condition and linear growth condition.

Taking $L = \left|\beta p(\int_{x(0)}^{\infty} x\mu_E(x)dx)\right| + |c| + |\sigma_g|$, and we calculate that

$$\begin{aligned}
&|g_{red}(E_1) - g_{red}(E_2)| + |\sigma_{red}(E_1) - \sigma_{red}(E_2)| \\
&= \left|\beta p(\int_{x(0)}^{\infty} x\mu_E(x)dx) - c\right| |E_1 - E_2| + |\sigma_g||E_1 - E_2| \\
&= \left(\left|\beta p(\int_{x(0)}^{\infty} x\mu_E(x)dx) - c\right| + |\sigma_g|\right) |E_1 - R_2| \\
&\leqslant \left(\left|\beta p(\int_{x(0)}^{\infty} x\mu_E(x)dx)\right| + |c| + |\sigma_g|\right) |E_1 - E_2| \leqslant L|E_1 - E_2|,
\end{aligned}$$

and

$$\begin{aligned}
|g_{red}(E)| + |\sigma_{red}(E)| &= \left(\left|\beta p(\int_{x(0)}^{\infty} x\mu_E(x)dx) - c\right| + |\sigma_g|\right) |E| \\
&\leqslant \left(\left|\beta p(\int_{x(0)}^{\infty} x\mu_E(x)dx)\right| + |c| + |\sigma_g|\right) |E| \\
&\leqslant L(1 + |E|).
\end{aligned}$$

Therefore, the Lipschitz condition and linear growth condition are satisfied. □

Theorem 4.8. *For any $(x(0), E(0)) \in \mathbb{R}^2_+$, if $2\beta p \int_{x(0)}^{\infty} x\mu_E(x)dx < \sigma_g^2 + 2c$ is met, the system (4.18) exhibits a stationary distribution.*

Proof. The FPK equation for system (4.18) is

$$\frac{\partial P(E)}{\partial \tau} = \frac{\partial}{\partial E}\left(\beta pE\int_{x(0)}^{\infty} x\mu_E(x)dx - cE\right) p(E) + \frac{1}{2}\frac{\partial^2}{\partial E^2}\sigma_g^2 E^2 p(E).$$

In order to obtain the probability density function of the stationary distribution of $E(\tau)$, we need to solve for $\frac{\partial P(E)}{\partial \tau} = 0$. The calculated result is as follows

$$p(E) = ME^{-\frac{2\sigma_g^2 - 2\left(\beta p\int_{x(0)}^{\infty} x\mu_E(x)dx - c\right)}{\sigma_g^2}},$$

where M is a normalization constant.

Below we prove that $\int_{E(0)}^{+\infty} p(E)dE = 1$ which is holds, it indicates the existence of the probability density function of $E(\tau)$, which also proves the existence of a stationary distribution for system (4.18).

We prove that the integral

$$\int_{E(0)}^{+\infty} E^{-\frac{2\sigma_g^2 - 2\left(\beta p\int_{x(0)}^{\infty} x\mu_E(x)dx - c\right)}{\sigma_g^2}} dE$$

is convergent. Based on the condition

$$2\beta p\int_{x(0)}^{\infty} x\mu_E(x)dx < \sigma_g^2 + 2c,$$

we obtain

$$-\frac{2\sigma_g^2 - 2\left(\beta p\int_{x(0)}^{\infty} x\mu_E(x)dx - c\right)}{\sigma_g^2} > 1.$$

According to the convergence rule of integrals, when $-\frac{2\sigma_g^2-2\left(\beta p\int_{x(0)}^{\infty}x\mu_E(x)dx-c\right)}{\sigma_g^2}>1$, the integral is convergent. At this point, there must exist a constant M that makes $\int_{E(0)}^{+\infty}p(E)dE=1$. Thus, Theorem 4.8 is proven. □

Remark 4.9. Calculating the integral $\int_{x(0)}^{\infty}x^{-N}\exp\left\{\frac{2(K+A)}{\sigma_f^2}x-\frac{K}{\sigma_f^2}x^2\right\}dx$ is too complex, and we have not provided an exact solution for the integral here. We have only proven that the integral is convergent. For more accurate results, one can use Matlab or Maple to find its numerical solution.

5 Stochastic model with weak Allee effect

In this section, we consider the time scale parameter ε as a regular parameter and discuss the stochastic behavior of the system (4.17). We discussed the existence of a stationary distribution for system (4.17) and the stochastic bifurcation of the system (4.17).

5.1 Existence and uniqueness of global positive solutions

When discussing the dynamic behavior of biological systems, the first thing we need to focus on is whether the system has a global positive solution. So we presented the theorem about the existence of a unique global positive solution for system (4.17):

Theorem 5.1. *Let $-K<A<0$, for any initial value taken from $(x(0),E(0))\in\mathbb{R}_+^2$, system (4.17) has a unique global positive solution $(x(\tau),E(\tau))$(where $\tau\geq 0$). And the solution is always within $\mathbb{R}_+^2$ with probability 1, which means that for any $\tau\geq 0$, $(x(\tau),E(\tau))\in\mathbb{R}_+^2$ almost surely (a.s.).*

Proof. For any initial value $(x(0),E(0))\in\mathbb{R}_+^2$, the system always has a solution $(x(\tau),E(\tau))\in\mathbb{R}_+^2$ at $t\in[0,\tau_e)$, τ_e is the moment of explosion. To prove that the solution is a global solution, it is only necessary to prove that $\tau_e=\infty$ a.s. Assuming that $n_0\geq 1$ is sufficiently large such that $(x(0),E(0))\in(\frac{1}{n_0},n_0)$. For any integer $n>n_0$, define the stopping time:

$$\tau_n=\inf\left\{\tau\in(0,\tau_e):\ \min\left\{x(\tau),E(\tau)\right\}\leq\tfrac{1}{n}\ \text{or}\ \max\left\{x(\tau),E(\tau)\right\}\geq n\right\},$$

where we assume that $inf\,\emptyset=\infty$ in this article (as is common, $\emptyset$ denotes the empty set). Obviously, τ_n monotonically increases when $n\to\infty$. Let $\tau_\infty=\lim_{n\to\infty}\tau_n$, where $\tau_\infty\leq\tau_n$ a.s. If $\tau_\infty=\infty$ a.s. holds, then $\tau_e=\infty$ a.s also holds, and there is $(x(\tau),E(\tau))\in\mathbb{R}_+^2$ for all $\tau\geq 0$, so we only need to prove that $\tau_\infty=\infty$ a.s. Otherwise, there will be a pair of constants $T>0$, $\eta\in(0,1)$, such that $P\left\{\tau_n\leq T\right\}>\eta$. Therefore, there is an integer $n_1\geq n_0$ such that $P\left\{\tau_n\leq T\right\}\geq\eta$ for all $n>n_1$.

Define a C^2-function $V:\mathbb{R}_+^2\to\mathbb{R}$ by

$$V(x(\tau),E(\tau))=x-a-a\ln\frac{x}{a}+b(E-1-\ln E),\tag{5.1}$$

where a and b are positive constants, it is easy to obtain that $V(x,E)$ is a nonnegative function. Using Itô's formula, we obtain

$$dV(x,E)=LV(x,E)d\tau+\frac{\sigma_f}{\sqrt{\varepsilon}}(x-a)dB_1(\tau)+b\sigma_g(E-1)dB_2(\tau),\tag{5.2}$$

where $LV(x,E)$: $\mathbb{R}^2_+ \to \mathbb{R}$ is defined as follows:

$$\begin{aligned} LV(x,E) = & -\frac{K}{\varepsilon}x^3 + \frac{A+K}{\varepsilon}x^2 - \frac{A}{\varepsilon}x + \frac{\alpha}{\varepsilon}xE + \frac{\alpha K}{\varepsilon}x^2 - \frac{\alpha(A+K)}{\varepsilon}x \\ & + \frac{aA}{\varepsilon} + \frac{\alpha a}{\varepsilon}E + b\beta pxE - bcE - b\beta px + bc + \frac{a\sigma_f^2}{2\varepsilon} + \frac{b\sigma_g^2}{2} \\ = & -\frac{K}{\varepsilon}x^3 + \frac{A+K+aK}{\varepsilon}x^2 - \frac{A+a(A+K)+b\varepsilon\beta p}{\varepsilon}x \\ & + \left(\frac{\alpha a}{\varepsilon} - bc\right)E + \left(b\beta p - \frac{\alpha}{\varepsilon}\right)xE + \frac{aA}{\varepsilon} + bc + \frac{a\sigma_f^2}{2\varepsilon} + \frac{b\sigma_g^2}{2}, \end{aligned}$$

Take $a = \frac{c}{\beta p}$, $b = \frac{\alpha}{\varepsilon\beta p}$, so that $\frac{\alpha a}{\varepsilon} - bc = 0$ and $b\beta p - \frac{\alpha}{\varepsilon} = 0$, then

$$\begin{aligned} LV(x,E) = & -\frac{K}{\varepsilon}x^3 + \frac{A+K+aK}{\varepsilon}x^2 - \frac{A+a(A+K)+b\varepsilon\beta p}{\varepsilon}x + \frac{aA}{\varepsilon} + bc + \frac{a\sigma_f^2}{2\varepsilon} + \frac{b\sigma_g^2}{2} \\ \le & \, b_1 + \frac{aA}{\varepsilon} + bc + \frac{a\sigma_f^2}{2\varepsilon} + \frac{b\sigma_g^2}{2} =: M, \end{aligned}$$

where $b_1 = \sup_{x\in(0,+\infty)}\left\{-\frac{K}{\varepsilon}x^3 + \frac{A+K+aK}{\varepsilon}x^2 - \frac{A+a(A+K)+b\varepsilon\beta p}{\varepsilon}x\right\}$, M is a constant. Then we can obtain

$$dV(x,E) \le Md\tau + \frac{\sigma_f}{\sqrt{\varepsilon}}(x-a)dB_1(\tau) + b\sigma_g(E-1)dB_2(\tau). \tag{5.3}$$

Integrating from 0 to $\tau_n \wedge T$ at both sides of (5.3), we have:

$$\begin{aligned} V(x(\tau_n\wedge T), E(\tau_n\wedge T)) \le & \, V(x(0),E(0)) + M(x(\tau_n\wedge T)) \\ & + \int_0^{\tau_n\wedge T}\frac{\sigma_f}{\sqrt{\varepsilon}}(x-a)dB_1(\tau) + b\int_0^{\tau_n\wedge T}\sigma_g(E-1)dB_2(\tau). \end{aligned}$$

By taking the expected values from both sides of the above equation, we can obtain

$$\begin{aligned} EV(x(\tau_n\wedge T), E(\tau_n\wedge T)) & \le V(x(0),E(0)) + ME(\tau_n\wedge T) \\ & \le V(x(0),E(0)) + MT. \end{aligned} \tag{5.4}$$

Therefore, let $\Omega_n = \{\omega\in\Omega, \tau_n = \tau_n(\omega) \le T\}$, it can be obtained that $P(\Omega_n) > \eta$ for all $n \ge n_0$. For each $\omega\in\Omega_n$, $x(\tau,\omega)$ and $E(\tau,\omega)$ have at least one equal to n or $\frac{1}{n}$, then we get

$$\begin{aligned} & V(x(0),E(0)) + MT \\ & \quad \ge E(I_{\Omega_n}V(x(\tau_n,\omega),E(\tau_n,\omega)) \\ & \quad \ge \eta\left[b(n-1-\ln n)\wedge b\left(\frac{1}{n}-1-\ln\frac{1}{n}\right)\wedge\left(n-a-a\ln\frac{n}{a}\right)\wedge\left(\frac{1}{n}-a+\ln an\right)\right], \end{aligned}$$

where $I_{\Omega_n(\omega)}$ is the indicative function of Ω_n. If $n\to\infty$, a contradiction arises where $\infty > V(x(0),E(0)) + MY = \infty$. Therefore, there is $\tau_\infty = \infty$ a.s. □

5.2 Stationary distribution and ergodic properties

A stationary distribution is characterized by the fluctuation of the solution of a stochastic system within a certain neighborhood of the positive equilibrium point of the corresponding deterministic system and can be regarded as weak stability. Next, we provide sufficient conditions for the existence of a stationary distribution in system (4.17).

Theorem 5.2. *Assuming* $\frac{\varepsilon(A+\frac{\sigma_f^2}{2})}{\alpha} < \frac{c+\frac{\sigma_g^2}{c}}{c}$, $\varepsilon < \alpha c$ *and* $\beta p < \frac{1}{\varepsilon}$, *the system* (4.17) *has a stationary distribution and it has the ergodic property for any initial value* $(x(0), E(0)) \in \mathbb{R}_+^2$.

Proof. Define C^2-function V_1 as follows:

$$V_1(x,E) = v(x,E) + \alpha kE - v(x_1,E_1),$$

where $v(x,E) = kx - \frac{\varepsilon^2 k}{\alpha}\ln x + \frac{k}{c}\ln E + \frac{1}{E^\rho}$, k and $\rho(0<\rho<1)$ are positive numbers, $(x_1,E_1) = (\frac{\varepsilon^2}{\alpha}, (\frac{k}{c\rho})^{-\frac{1}{\rho}})$ is the minimum value point of $v(x,E)$.

Using Itô's formula, we get

$$\begin{aligned} LV_1 = & \frac{k}{\varepsilon}[x(1-x)(Kx-A) - \alpha xE] - \frac{\varepsilon k}{\alpha}[(1-x)(Kx-A) - \alpha E] \\ & + \frac{k}{c}(\beta px - c) - \rho E^{-\rho}(\beta px - c) + \alpha k(\beta px - c)E + \frac{k\varepsilon\sigma_f^2}{2\alpha} - \frac{k\sigma_g^2}{2c} + \frac{1}{2}\rho(\rho+1)\sigma_g^2 E^{-\rho} \\ = & -\frac{kK}{\varepsilon}x^3 + \frac{\alpha k(A+K) + \varepsilon^2 kK}{\alpha\varepsilon}x^2 - \frac{\varepsilon k(A+K)}{\alpha}x - \frac{kA}{\varepsilon}x + \frac{k\beta p}{c}x \\ & - k(\alpha c - \varepsilon)E - k\left(\frac{\alpha}{\varepsilon} - \alpha\beta p\right)xE - \rho\beta pxE^{-\rho} + \rho E^{-\rho}\left(c + \frac{1}{2}\rho(\rho+1)\sigma_g^2\right) \\ & - k\left(\frac{(c+\frac{\sigma_g^2}{2})}{c} - \frac{\varepsilon(A+\frac{\sigma_f^2}{2})}{\alpha}\right). \end{aligned} \tag{5.5}$$

Considering a bounded closed set

$$U_{\eta_1} = \left\{(x,E) \in \mathbb{R}_+^2, \eta_1 \le x \le \frac{1}{\eta_1}, \eta_1 \le E \le \frac{1}{\eta_1}\right\}.$$

Furthermore, we can obtain

$$\mathbb{R}_+^2 \setminus U_{\eta_1} = U_1 \cup U_2 \cup U_3 \cup U_4,$$

where

$$\begin{aligned} U_1 &= \left\{(x,E) \in \mathbb{R}_+^2, E > \frac{1}{\eta_1}\right\}; \qquad & U_2 &= \left\{(x,E) \in \mathbb{R}_+^2, 0 < x < \eta_1\right\}; \\ U_2 &= \left\{(x,E) \in \mathbb{R}_+^2, 0 < E < \eta_1\right\}; \qquad & U_4 &= \left\{(x,E) \in \mathbb{R}_+^2, x > \frac{1}{\eta_1}\right\}. \end{aligned}$$

We choose $\eta_1 > 0$ as a sufficiently small constant and satisfy the following conditions:

$$B_1 - k(\alpha c - \varepsilon)\frac{1}{\eta_1} + \rho\eta_1^\rho(c + \frac{1}{2}(\rho+1)\sigma_g^2) \le -1, \tag{5.6}$$

$$\frac{2\beta p\eta_1}{c\left(\frac{c+\frac{\sigma_g^2}{2}}{c} - \frac{\varepsilon(A+\frac{\sigma_f^2}{2})}{\alpha}\right)} < \frac{1}{2}, \tag{5.7}$$

$$\frac{2\varepsilon\eta_1}{\frac{c+\frac{\sigma_g^2}{2}}{c} - \frac{\varepsilon(A+\frac{\sigma_f^2}{2})}{\alpha}} < \frac{1}{2}, \tag{5.8}$$

$$D_1 - \frac{\alpha k(A+K) + \varepsilon^2 kK}{\alpha\varepsilon\eta_1} < -1, \tag{5.9}$$

where B_2 and D_2 are positive constants that can be determined from the following inequalities (5.10), (5.12). Next, we prove that LV_1 is negative in regions U_i, $i = 1, 2, 3, 4$.

Case 1: If $(x, E) \in U_1$, we have

$$\begin{aligned} LV_1 \leq & -\frac{kK}{\varepsilon}x^3 + \frac{\alpha k(A+K)+\varepsilon^2 kK}{\alpha\varepsilon}x^2 - \frac{\alpha kA+\varepsilon^2 k(A+K)}{\alpha\varepsilon}x \\ & + \frac{k\beta p}{c}x - k(\alpha c-\varepsilon)E + \rho E^{-\rho}\left(c+\frac{1}{2}\rho(\rho+1)\sigma_g^2\right) \\ \leq & B_1 - k(\alpha c-\varepsilon)E + \rho E^{-\rho}\left(c+\frac{1}{2}\rho(\rho+1)\sigma_g^2\right) \\ \leq & B_1 - k(\alpha c-\varepsilon)\frac{1}{\eta_1} + \rho\eta_1^{\rho}\left(c+\frac{1}{2}\rho(\rho+1)\sigma_g^2\right), \end{aligned} \tag{5.10}$$

where

$$B_1 = \sup_{x\in(0,+\infty)}\left\{-\frac{kK}{\varepsilon}x^3 + \frac{\alpha k(A+K)+\varepsilon^2 kK}{\alpha\varepsilon}x^2 - \frac{\alpha kA+\varepsilon^2 k(A+K)}{\alpha\varepsilon}x + \frac{k\beta p}{c}x\right\}.$$

According to condition (5.6), we can obtain that $LV_1 \leq -1$ on U_1.

Case 2: If $(x, E) \in U_2$, one can see that

$$\begin{aligned} LV_1 \leq & -\frac{kK}{\varepsilon}x^3 + \frac{\alpha k(A+K)+\varepsilon^2 kK}{\alpha\varepsilon}x^2 - \frac{kA}{\varepsilon}x + \frac{k\beta p}{c}x - k(\alpha c-\varepsilon)E \\ & + \rho E^{-\rho}\left(c+\frac{1}{2}\rho(\rho+1)\sigma_g^2\right) - k\left(\frac{(c+\frac{\sigma_g^2}{2})}{c} - \frac{\varepsilon(A+\frac{\sigma_f^2}{2})}{\alpha}\right) + \frac{k\beta p}{c}x \\ \leq & C_1 - k\left(\frac{(c+\frac{\sigma_g^2}{2})}{c} - \frac{\varepsilon(A+\frac{\sigma_f^2}{2})}{\alpha}\right) + \frac{k\beta p}{c}x \\ \leq & C_1 - k\left(\frac{(c+\frac{\sigma_g^2}{2})}{c} - \frac{\varepsilon(A+\frac{\sigma_f^2}{2})}{\alpha}\right) + \frac{k\beta p}{c}\eta_1, \end{aligned} \tag{5.11}$$

where

$$\begin{aligned} C_1 = \sup_{(x,E)\in\mathbb{R}_+^2}\Big\{ & -\frac{kK}{\varepsilon}x^3 + \frac{\alpha k(A+K)+\varepsilon^2 kK}{\alpha\varepsilon}x^2 - \frac{kA}{\varepsilon}x + \frac{k\beta p}{c}x \\ & -k(\alpha c-\varepsilon)E + \rho E^{-\rho}\left(c+\frac{1}{2}\rho(\rho+1)\sigma_g^2\right)\Big\}. \end{aligned}$$

If we choose $k = \dfrac{2C_1}{\frac{(c+\frac{\sigma_g^2}{2})}{c} - \frac{\varepsilon(A+\frac{\sigma_f^2}{2})}{\alpha}}$, we can obtain

$$LV_1 \leq -C_1 + \frac{2\beta p C_1\eta_1}{c\left(\frac{(c+\frac{\sigma_g^2}{2})}{c} - \frac{\varepsilon(A+\frac{\sigma_f^2}{2})}{\alpha}\right)}.$$

It follows from (5.7), we can obtain $LV_1 \leq -\frac{C_1}{2}$ on U_2.

Case 3: If $(x, E) \in U_3$, we have

$$\begin{aligned} LV_1 \leq & -\frac{kK}{\varepsilon}x^3 + \frac{\alpha k(A+K)+\varepsilon^2 kK}{\alpha\varepsilon}x^2 - \frac{kA}{\varepsilon}x + \frac{k\beta p}{c}x - k(\alpha c-\varepsilon)E \\ & + \rho E^{-\rho}\left(c + \frac{1}{2}\rho(\rho+1)\sigma_g^2\right) - k\left(\frac{(c+\frac{\sigma_g^2}{2})}{c} - \frac{\varepsilon(A+\frac{\sigma_f^2}{2})}{\alpha}\right) + k\varepsilon E \\ \leq & C_1 - k\left(\frac{(c+\frac{\sigma_g^2}{2})}{c} - \frac{\varepsilon(A+\frac{\sigma_f^2}{2})}{\alpha}\right) + k\varepsilon E \\ \leq & C_1 - k\left(\frac{(c+\frac{\sigma_g^2}{2})}{c} - \frac{\varepsilon(A+\frac{\sigma_f^2}{2})}{\alpha}\right) + k\varepsilon\eta_1, \end{aligned}$$

where $k = \frac{2C_1}{\frac{(c+\frac{\sigma_g^2}{2})}{c} - \frac{\varepsilon(A+\frac{\sigma_f^2}{2})}{\alpha}}$, C_1 is given by (5.11), then we can obtain

$$LV_1 \leq -C_1 + \frac{2\varepsilon C_1\eta_1}{c\left(\frac{(c+\frac{\sigma_g^2}{2})}{c} - \frac{\varepsilon(A+\frac{\sigma_f^2}{2})}{\alpha}\right)}.$$

According to condition (5.8), we can obtain $LV_1 \leq -\frac{C_2}{2}$ on U_3.

Case 4: If $(x, E) \in U_4$, one can see that

$$\begin{aligned} LV_1 \leq & -\frac{kK}{\varepsilon}x^3 + \frac{\alpha k(A+K)+\varepsilon^2 kK}{\alpha\varepsilon}x^2 - \frac{kA}{\varepsilon}x + \frac{k\beta p}{c}x - k(\alpha c-\varepsilon)E \\ & + \rho E^{-\rho}\left(c + \frac{1}{2}\rho(\rho+1)\sigma_g^2\right) - \frac{\alpha kA+\varepsilon^2 k(A+K)}{\alpha\varepsilon}x \\ \leq & D_1 - \frac{\alpha kA+\varepsilon^2 k(A+K)}{\alpha\varepsilon}x \\ \leq & D_1 - \frac{\alpha kA+\varepsilon^2 k(A+K)}{\alpha\varepsilon\eta_1}, \end{aligned} \tag{5.12}$$

where

$$\begin{aligned} D_1 = \sup_{x\in(0,+\infty)}\Bigg\{ & -\frac{kK}{\varepsilon}x^3 + \frac{\alpha k(A+K)+\varepsilon^2 kK}{\alpha\varepsilon}x^2 - \frac{kA}{\varepsilon}x + \frac{k\beta p}{c}x \\ & -k(\alpha c-\varepsilon)E + \rho E^{-\rho}\left(c + \frac{1}{2}\rho(\rho+1)\sigma_g^2\right)\Bigg\}. \end{aligned}$$

According to condition (5.9), we can conclude that $LV_1 \leq -1$ on U_4. Thus we can obtain that for a enough small η, $LV_1 \leq -\max\left\{1, \frac{C_1}{2}\right\}$ for all $(x, E) \in \mathbb{R}^2_+ \setminus U_{\eta_1}$.

In addition, the diffusion matrix of system (4.17) is

$$\begin{pmatrix} \frac{\sigma_f^2}{\varepsilon}x^2 & 0 \\ 0 & \sigma_g^2 E^2 \end{pmatrix}.$$

Choose

$$M_1 = \min_{(x,E)\in\mathbb{R}^2_+\setminus U_{\eta_1}} \left\{ \frac{\sigma_f^2}{\varepsilon}x^2, \sigma_g^2 E^2 \right\},$$

we have

$$\frac{\sigma_f^2}{\varepsilon}x^2\xi_1^2 + \sigma_g^2 E^2\xi_2^2 \geq M_1|\xi^2|, \ \ (x,E)\in\mathbb{R}^2_+\setminus U_{\eta_1}, \ \ \xi = (\xi_1,\xi_2)\in\mathbb{R}^2_+.$$

Therefore, the system has a stationary distribution and it has the ergodic property. □

Combining Theorem 4.6 and 4.7, we can conclude that both the system (4.17) without random reduction and the system (4.18) after random reduction have globally unique positive solutions. And under certain conditions, they have stationary distribution.

Remark 5.3. The existence of a stationary distribution indicates that under current fishing and environmental conditions, fish populations can maintain a relatively healthy quantity state, which is conducive to the sustainable development of fisheries. For fisheries practitioners and managers, a stationary distribution is an important reference for formulating fishing strategies and fisheries management policies. It can help determine a reasonable fishing intensity to achieve a balance between economic and ecological benefits. If the stable distribution of fish populations indicates relatively abundant resources, then it is appropriate to reduce aquaculture inputs and increase fishing output; On the contrary, if the stable distribution shows that the resources are at an unstable or low level, it is necessary to increase aquaculture efforts and control fishing intensity to ensure the sustainable development of the fishery economy.

5.3 Stochastic stability

In this section, the stability and bifurcation behavior of the stochastic system (4.16) near its internal equilibrium point S_* will be discussed. In order to obtain stochastic differential equations in small parameter form, we introduce the following standard rescalings [26]:

$$K = \varepsilon\tilde{K}, \quad A = \varepsilon\tilde{A}, \quad \alpha = \varepsilon\tilde{\alpha}, \quad \sigma_f = \sqrt{\varepsilon}\tilde{\sigma}_f,$$

where $0 < \varepsilon \ll 1$ is the time scale parameter. And system (4.16) is transformed into:

$$\begin{cases} dx = \varepsilon[x(1-x)(\tilde{K}x - \tilde{A}) - \tilde{\alpha}Ex]dt + \sqrt{\varepsilon}\tilde{\sigma}_f x dB_1(t) =: \varepsilon\tilde{f}(x,E,\varepsilon)dt + \tilde{\sigma}_f x dB_1(t), \\ dE = \varepsilon(\beta p x - c)Edt + \sqrt{\varepsilon}\sigma_g E dB_2(t) =: \varepsilon g(x,E)dt + \sqrt{\varepsilon}\sigma_g E dB_2(t), \end{cases} \tag{5.13}$$

which system (5.13) is a regular stochastic differential equation. In order to better study the dynamic behavior near the equilibrium point $\tilde{S}_*(\tilde{x}_*, \tilde{E}_*)$ where $\tilde{x}_* = \frac{c}{\beta p}$, $w\tilde{E}_* = \frac{(1-\tilde{x}_*)(\tilde{K}\tilde{x}_* - \tilde{A})}{\tilde{\alpha}}$, we translate it to the origin and perform a polar coordinate transformation $x = \tilde{x}_* + r\cos\theta$, $E = \tilde{E}_* + r\sin\theta$, and system (5.13) is transformed into the following form:

$$\begin{cases} dr = \varepsilon f_1(r,\theta,\varepsilon)dt + \sqrt{\varepsilon}g_{11}(r,\theta,\varepsilon)dB_1(t) + \sqrt{\varepsilon}g_{12}(r,\theta,\varepsilon)dB_2(t), \\ d\theta = \varepsilon f_2(r,\theta,\varepsilon)dt + \sqrt{\varepsilon}g_{21}(r,\theta,\varepsilon)dB_1(t) + \sqrt{\varepsilon}g_{22}(r,\theta,\varepsilon)dB_2(t), \end{cases} \tag{5.14}$$

where

$$\begin{aligned} f_1(r,\theta) &= r\left((\tilde{A} + \tilde{K} - 2\tilde{K}\tilde{x}_*)\tilde{x}_*\cos^2\theta - \tilde{\alpha}\tilde{x}_*\sin\theta\cos\theta + \beta p\tilde{E}_*\sin\theta\cos\theta\right) \\ &\quad + r^2\left(-3(\tilde{K}\tilde{x}_* - \tilde{A} - \tilde{K})\cos^3\theta - \tilde{\alpha}\sin\theta\cos^2\theta + \beta p\sin^2\theta\cos\theta\right) - r^3\tilde{K}\cos^4\theta, \\ g_{11}(r,\theta) &= \tilde{\sigma}_f r\cos^2\theta,\ g_{12}(r,\theta) = \sigma_g r\sin^2\theta, \\ f_2(r,\theta) &= \left(\beta p\tilde{E}_*\cos^2\theta - (\tilde{A} + \tilde{K} - 2\tilde{K}\tilde{x}_*)\tilde{x}_*\sin\theta\cos\theta + \tilde{\alpha}\tilde{x}_*\sin^2\theta\right) \\ &\quad + r\left(\beta p\sin\theta\cos^2\theta + (3\tilde{K}\tilde{x}_* - \tilde{A} - \tilde{K})\sin\theta\cos^2\theta + \tilde{\alpha}\sin^2\theta\cos\theta\right) + r^3\tilde{K}\sin\theta\cos^3\theta, \\ g_{21}(r,\theta) &= -\tilde{\sigma}_f\sin\theta\cos\theta,\ g_{22}(r,\theta) = \sigma_g\sin\theta\cos\theta. \end{aligned}$$

According to the Khasminskii's limit theorem in [16,44], when the intensities of noise σ_f and σ_g are small and $\varepsilon \to 0$, on the time interval of ε^{-1} magnitude, the process $\{r(t),\theta(t)\}$ of system (5.14) weakly converges to a two-dimensional Markov diffusion process. Through the application of the stochastic averaging method, a stochastic differential equation as follows is obtained.

$$\begin{cases} dr = f_r(r)dt + \sigma_{11}(r)dB_r(t) + \sigma_{12}(r)dB_\theta(t), \\ d\theta = f_\theta(r)dt + \sigma_{21}(r)dB_r(t) + \sigma_{22}(r)dB_\theta(t), \end{cases} \tag{5.15}$$

where $B_r(t)$ and $B_\theta(t)$ are independent standard Wiener processes, and the coefficients of system (5.15) satisfy

$$\begin{aligned} f_r &= \varepsilon\left(\frac{(\tilde{A}+\tilde{K}-2\tilde{K}\tilde{x}_*)r\tilde{x}_*}{2} + \frac{5r}{8}(\tilde{\sigma}_f^2+\sigma_g^2) - \frac{r}{4}\tilde{\sigma}_f\sigma_g - \frac{3r^3\tilde{K}}{8}\right) \\ &= \frac{(A+K-2Kx_*)rx_*}{2} + \frac{5r}{8}(\sigma_f^2+\varepsilon\sigma_g^2) - \frac{r}{4}\sqrt{\varepsilon}\sigma_f\sigma_g - \frac{3r^3K}{8}, \\ f_\theta &= \frac{\varepsilon}{2}(\tilde{\alpha}\tilde{x}_* + \beta p\tilde{E}_*) = \frac{1}{2}(\alpha x_* + \varepsilon\beta pE_*), \\ \sigma_{12} &= \sigma_{21} = 0, \\ \sigma_{11}^2 &= \frac{3\varepsilon r^2}{8}\left(\tilde{\sigma}_f^2+\sigma_g^2+\frac{2}{3}\tilde{\sigma}_f\sigma_g\right) = \frac{3r^2}{8}\left(\sigma_f^2+\varepsilon\sigma_g^2+\frac{2}{3}\sqrt{\varepsilon}\sigma_f\sigma_g\right), \\ \sigma_{22}^2 &= \frac{\varepsilon}{8}(\tilde{\sigma}_f-\sigma_g)^2 = \frac{1}{8}(\sigma_f-\varepsilon\sigma_g)^2. \end{aligned}$$

We can easily conclude that in system (5.15), the modulus equation and phase equation are decoupled. In order to study the stability and branching phenomena of system (4.16), we only need to discuss the mean modulus equation of system (5.15), which is

$$dr = f_r(r)dt + \sigma_{11}(r)dB_r(t). \tag{5.16}$$

For convenience, we introduce the following notation:

$$c_1 = \frac{(A+K-2Kx_*)x_*}{2},\ c_2 = 5(\sigma_f^2+\varepsilon\sigma_g^2) - 2\sqrt{\varepsilon}\sigma_f\sigma_g, c_3 = 3K,\ c_4 = 3\left(\sigma_f^2+\varepsilon\sigma_g^2+\frac{2}{3}\sqrt{\varepsilon}\sigma_f\sigma_g\right).$$

Therefore, we can rewrite system (5.16) as

$$dr = \left(\left(c_1+\frac{c_2}{8}\right)r - \frac{c_4}{8}r^3\right)dt + \left(\frac{c_4}{8}r^2\right)^{\frac{1}{2}}dB_r(t). \tag{5.17}$$

We know that $r=0$ is a fixed point of system (5.16), corresponding to the trivial solution of the original system (4.17). This section discusses the stability of the fixed points of system (5.16), as well as the stability of the singularity S_* of system (4.17).

We adopt the Lyapunov exponent method to determine the local stability of the trivial solution of system (5.17). Consider the linear Itô stochastic differential equation for system (5.17):

$$dr = \left(c_1+\frac{c_2}{8}\right)rdt + \left(\frac{c_4}{8}r^2\right)^{\frac{1}{2}}dB_r(t). \tag{5.18}$$

By utilizing the solution of Itô's stochastic differential equation, we can obtain a solution of (5.18):

$$r(t) = r(0)\exp\left\{\int_0^t\left(c_1+\frac{c_2}{8}-\frac{c_4}{16}\right)ds + \int_0^t\left(\frac{c_4}{8}\right)^{\frac{1}{2}}dB_r(s)\right\}.$$

We obtained the Lyapunov exponent of system (5.17), which is given

$$\lambda(t) = \lim_{t\to\infty} \frac{1}{t} \ln \|r(t)\| = \lim_{t\to\infty} \frac{1}{2t} \ln r(t) = \frac{1}{2}\left(c + \frac{c_2}{8} - \frac{c_4}{16}\right). \tag{5.19}$$

Therefore, the trivial solution of system (5.18) is asymptotically stable with a probability of 1, when $\lambda < 0$, i.e. $c_1 + \frac{c_2}{8} - \frac{c_4}{16} < 0$. Moreover, it has robustness against high-order random disturbances, which results in the asymptotic stability of the trivial solution of the original nonlinear system (5.17) with probability 1.

Therefore, we can obtain that:

Theorem 5.4. *If $\lambda < 0$, i.e. $c_1 + \frac{c_2}{8} - \frac{c_4}{16} < 0$, the trivial solution of system (5.17) is locally asymptotically stable with probability 1. If $\lambda > 0$, i.e. $c_1 + \frac{c_2}{8} - \frac{c_4}{16} > 0$, the trivial solution of system (5.17) is unstable. If $\lambda = 0$, i.e. $c_1 + \frac{c_2}{8} - \frac{c_4}{16} = 0$, system (5.17) may experience bifurcation.*

We use singular boundary theory to discuss the global stability of S_* of system (4.17). According to the theory of singular boundaries, when $r \to 0^+$, $\sigma_{11} = 0$, therefore $r = 0$ is the first type of singular boundary. After calculation, the diffusion index α_l, drift index β_l, and characteristic value c_l can be obtained that

$$\alpha_l = 2, \quad \beta_l = 1, \quad c_l = \lim_{r\to 0^+} \frac{2f_r(r)r^{\alpha_l - \beta_l}}{\sigma_{11}^2(r)} = \frac{16c_1 + 2c_2}{c_4}.$$

Therefore, it can be concluded that:

(a) When $c_l < 1$, i.e. $c_4 > 16c_1 + 2c_2$, the bound $r = 0$ is attractively natural;

(b) When $c_l > 1$, i.e. $c_4 < 16c_1 + 2c_2$, the bound $r = 0$ is repulsively nature;

(c) When $c_l = 1$, i.e. $c_4 = 16c_1 + 2c_2$, the bound $r = 0$ is strictly natural.

The right boundary $r = \infty$ is the second type of singular boundary, because when $r \to \infty$, $|f_r| \to \infty$. By calculation, the diffusion index $\alpha_r = 2$ and the drift index $\beta_r = 3$ can be obtained. Additionally, since $f_r(\infty) < 0$, the right boundary $r = \infty$ is an entrance boundary.

We know that all solutions of the system will enter from the right boundary and then be attracted to the left boundary, when $r = 0$ is attractively nature and $r = \infty$ is an entrance boundary. This implies that the trivial solution of system (5.17) is globally stable, which coincides with the conclusion derived by employing the Lyapunov exponent method. In other words, the trivial solution of the system is both locally and globally stable.

Therefore, we provide the following theorem:

Theorem 5.5. *The trivial solution of system (5.17) is globally stable for $c_4 > 16c_1 + 2c_2$. The trivial solution of system (5.17) is unstable for $c_4 < 16c_1 + 2c_2$. System (5.17) may undergo bifurcation for $c_4 = 16c_1 + 2c_2$.*

Remark 5.6. We refer to the method in reference [26] and transform system (4.16) into system (5.13) by introducing appropriate transformations. It should be pointed out that reference [26] uses the regular perturbation method to transform the original system into a weakly perturbed system, while the processing method in this paper differs from it. Specifically, since the system (4.16) already contains a small parameter ε, we only implement the transformation on the first equation of system (4.16). Although the transformed system (5.13) has similarities in form with weakly perturbed systems, due to significant differences in magnitude between $\tilde{f}(x, E, \varepsilon)$

and $g(x,E)$ (where $\tilde{f}(x,E,\varepsilon) = O(1/\varepsilon)$ and $g(x,E) = O(1)$), the system (5.13) still maintains typical fast-slow system characteristics. This property is mainly reflected in: There is a clear separation in the time scale of system variables; The evolution rates of fast variable x and slow variable E maintain a difference in magnitude between $O(1)$ and $O(\varepsilon)$; This conclusion differs fundamentally from the weak perturbation system obtained through regular perturbation in reference [26] in terms of mathematical properties, and also reflects the characteristics of the processing method proposed in this paper.

5.4 Stochastic bifurcation

In this section, we will study the bifurcation behavior of the system from a dynamic perspective.

Theorem 5.7. *Stochastic system* (5.17) *will generate a D-bifurcation as the value of bifurcation parameter* c_4 *passed through* $16c_1 + 2c_2$.

Proof. Consider the linear stochastic differential equation (5.18), which is equivalent to a Stratonovich stochastic differential equation:

$$dr = \left(c_1 + \frac{c_2}{8} - \frac{c_4}{16}\right) r dt + \left(\frac{c_4}{8} r^2\right)^{\frac{1}{2}} dB_r(t). \tag{5.20}$$

Let $m_1(r) = (c_1 + \frac{c_2}{8} - \frac{c_4}{16})r$, $\sigma_1(r) = (\frac{c_4}{8}r^2)^{\frac{1}{2}}$, system (5.20) can be rewritten as

$$dr = m_1(r)dt + \sigma_1(r)dB_r(t). \tag{5.21}$$

The continuous dynamic system generated by it is

$$\varphi(t)x = x + \int_0^t m_1(\varphi(s)x)ds + \int_0^t \sigma_1(\varphi(s)x)dB_r(s),$$

where $m_1(r) = 0$ and $\sigma_1(0) = 0$, thus 0 is a fixed point of $\varphi(x)$. Since $m_1(r)$ is bounded, the condition $\sigma_1(0) \neq 0$ is satisfied for any $m_1(r) \neq 0$, ensuring that there exists at most one stationary probability density. We have

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial r}\left(\left(c_1 + \frac{c_2}{8}\right) p\right) + \frac{1}{2}\frac{\partial^2}{\partial r^2}\left(\left(\frac{c_4}{8} r^2\right) p\right). \tag{5.22}$$

which is the FPK equation for system (5.18).

Solving for $\frac{\partial p}{\partial t} = 0$, we obtain

$$p(r) = c \mid \sigma_1^{-1}(r) \mid \exp\left(\int_0^r \frac{2m_1(s)}{\sigma_1^2(s)} ds\right), \tag{5.23}$$

where c is constant. The system (5.22) has two possible equilibrium states: fixed point and non trivial stationary motion. The density of the invariant measure δ_0 for the former is $\delta(x)$, while the density of the invariant measure v for the latter is (5.23). As calculated, the solution for system (5.21) is

$$r(t) = r(0) \exp\left[\int_0^t (m_1'(r) + \frac{\sigma_1(r)\sigma_1''(r)}{2})ds + \int_0^t \sigma_1'(r)dB_r(s)\right]. \tag{5.24}$$

The Lyapunov exponent of a stochastic dynamical system with regard to the measure μ is

$$\lambda_{\varphi}(\mu) = \lim_{t\to\infty} \frac{1}{t} \ln \|r(t)\| . \tag{5.25}$$

Substitute (5.24) into(5.25), we obtain that

$$\begin{aligned}\lambda\varphi(\delta_0) &= \lim_{t\to+\infty} \frac{1}{t}\left[\ln r(0) + m_1'(0)\int_0^t ds + \sigma_1'(r)\int_0^t dB_r(s)\right] \\ &= m_1'(0) + \sigma_1'(0)\lim_{t\to+\infty}\frac{B_r(t)}{t} = m_1'(0) = c_1 + \frac{c_2}{8} - \frac{c_4}{16}.\end{aligned}$$

For the invariant measure v with a density of (5.23), the Lyapunov exponent is

$$\begin{aligned}\lambda_{\varphi}(v) &= \lim_{t\to\infty}\frac{1}{t}\int_0^t \left(m_1' + \frac{\sigma_1\sigma_1''}{2}\right)(r)ds \\ &= -32\sqrt{2}c_4^{\frac{3}{2}}\left(c_1 + \frac{c_2}{8} - \frac{c_4}{16}\right)^2 \exp\left(\frac{16}{c_4}\left(\frac{c_2}{8} - \frac{c_4}{16}\right)\right) < 0.\end{aligned}$$

In conclusion, if $c_4 > 16c_1 + 2c_2$, the fixed-point invariant measure is stable; if $c_4 < 16c_1 + 2c_2$, it is unstable; if $c_4 = 16c_1 + 2c_2$, system (5.17) undergoes a D-bifurcation. □

Theorem 5.8. *Stochastic system* (5.17) *will generate P-bifurcation as the value of bifurcation parameter* c_4 *passed through* $8c_1 + c_2$.

Proof. Based on the amplitude $r(t)$ of the stochastic differential equation, we calculate its FPK equation as follows:

$$\frac{\partial p}{\partial r} = \left\{\left((c_1 + \frac{c_2}{8})r - \frac{c_3}{8}r^3\right)p\right\} + \frac{1}{2}\frac{\partial^2}{\partial r^2}\left((\frac{c_4}{8}r^2)p\right),$$

whose initial condition is

$$p(r,t|r_0,t_0) \to \delta(r - r_0), \quad t \to t_0,$$

where $p(r,t|r_0,t_0)$ represents the transition probability density of the diffusion process $r(t)$. The invariant measure of $r(t)$ is the steady-state probability density function $p_{st}(r)$, which serves as the solution to the following degenerate system:

$$-\frac{\partial}{\partial r}\left\{\left(\left(c_1 + \frac{c_2}{8}\right)r - \frac{c_3}{8}r^3\right)p\right\} + \frac{1}{2}\frac{\partial^2}{\partial r^2}\left(\frac{c_4}{8}r^2\right)p = 0.$$

By calculation, it can be concluded that

$$p_{st}(r) = \begin{cases} \dfrac{r^{\frac{16c_1+2c_2-2c_4}{c_4}} e^{-\frac{c_3}{c_4}r^2}}{\Gamma\left(\frac{8c_1+c_2}{c_4}\right)\left(\frac{c_4}{c_3}\right)^{\frac{8c_1+c_2}{c_4}}}, & c_4 < 16c_1 + 2c_2, \\ \delta_0, & c_4 \geq 16c_1 + 2c_2. \end{cases} \tag{5.26}$$

According to Namachivaya's theory, the extremum of invariant measures can explain the key characteristics of correlated nonlinear stochastic systems. If r_* represents the maximum value point of $p_{st}(r)$, the sample trajectory is likely to remain in the vicinity of r_* with a high probability, suggesting that r_* is stable in a probabilistic sense. Conversely, if $p_{st}(r)$ has a

minimum value, the situation is reversed.

To solve for the extremum of $p_{st}(r)$, we need to solve the following equation:

$$\frac{\left(\frac{16c_1+2c_2-2c_4}{c_4}-\frac{2c_3}{c_4}r^2\right)r^{\frac{16c_1+2c_2-3c_4}{c_4}}e^{-\frac{c_3}{c_4}r^2}}{\Gamma\left(\frac{8c_1+c_2}{c_4}\right)\left(\frac{c_4}{c_3}\right)^{\frac{8c_1+c_2}{c_4}}}=0.$$

By simple calculation, we obtain $r=0$ or $r_*=\sqrt{\frac{8c_1+c_2-c_4}{c_3}}$. The system reaches its maximum value at r_* if $c_4<8c_1+c_2$. Therefore, we can prove the following three situations:

Case (i): If $8c_1+c_2<c_4<16c_1+2c_2$ and $r\to 0^+$, $p_{st}(r)$ tends to infinity. At this time, the sample trajectory of the system is concentrated around $r=0$.

Case (ii): If $\frac{16c_1+2c_2}{3}<c_4<8c_1+c_2$, $p_{st}(r)$ attains its maximum at $r=r_*$ and its minimum at $r=0$. However, $p_{st}(r)$ is not differentiable at $r=0$ in this scenario.

Case (iii): If $c_4<\frac{16c_1+2c_2}{3}$, $p_{st}(r)$ attains its maximum at $r=r_*$ and its minimum at $r=0$. At this time, $p_{st}(r)$ is differentiable at $r=0$, and it is easy to find that $p_{st}(r)$ is smooth at $r=0$.

Based on the above analysis, we know that the shape of $p_{st}(r)$ will change from a monotonic to a unimodal as the c_4 value exceeds $8c_1+c_2$. System (5.17) generates a stochatic P-bifurcation if $c_4=\frac{16c_1+2c_2}{3}$. □

Remark 5.9. The sample trajectory of the system is concentrated around $r=0$ for $8c_1+c_2<c_4<16c_1+2c_2$, indicating a high probability of a small population size, which is not conducive to population growth. As c_4 decreases, the peak value of $p_{st}(r)$ narrows and shifts to the right (see Figure 5.2(a) and Figure 5.2(b)), indicating a higher probability of a larger population size, which is beneficial for population growth.

5.5 Numerical simulation

In this section, we will use data simulation to verify the correctness of the above theory.

We select parameter $K=3$, $A=-0.8$, $\alpha=0.17$, $\varepsilon=0.1$, $\beta=1.5$, $p=3$, $c=1.69$, $\sigma_f=0.001$ and $\sigma_g=0.001$ and $\sigma_g=0.001$ in system (4.17), which satisfy the conditions

$$\frac{\varepsilon(A+\frac{\sigma_f^2}{2})}{\alpha}<\frac{c+\frac{\sigma_g^2}{c}}{c},\quad \varepsilon<\alpha c\quad\text{and}\quad \beta p<\frac{1}{\varepsilon}$$

(the data source from Table 3.2.). According to Theorem 5.2, it is known that system (4.17) has a stationary distribution. In a sense, it also indicates that small-scale noise is beneficial for system stability. As shown in Figure 5.1. The values of $x(t)$ and $E(t)$ in Figure 5.1(a) and Figure 5.1(c) oscillate slightly around the equilibrium point of the deterministic system. The bar charts in Figure 5.1(b) and Figure 5.1(d) show that system (4.17) exhibits a stationary distribution.

Next, we will verify the conclusion of stochastic bifurcation through numerical simulation. We select parameters $c_1=0.25$, $c_2=3$, $c_3=10$ and $c_4=5$ so that they satisfy the case (i) in the proof of Theorem 5.8. Figure 5.2 shows the stationary probability density $p_{st}(r)$ of system (5.17), from which we can see that $p_{st}(r)$ tends towards infinity when $r\to 0^+$.

We select parameters $c_1=0.25$, $c_2=3$, $c_3=10$ and $c_4=3.5$ so that they satisfy the case (ii). Figure 5.3 shows the graph of $p_{st}(r)$, from which we can see that $p_{st}(r)$ reaches its maximum at $r=r_*$ and its minimum at $r=0$.

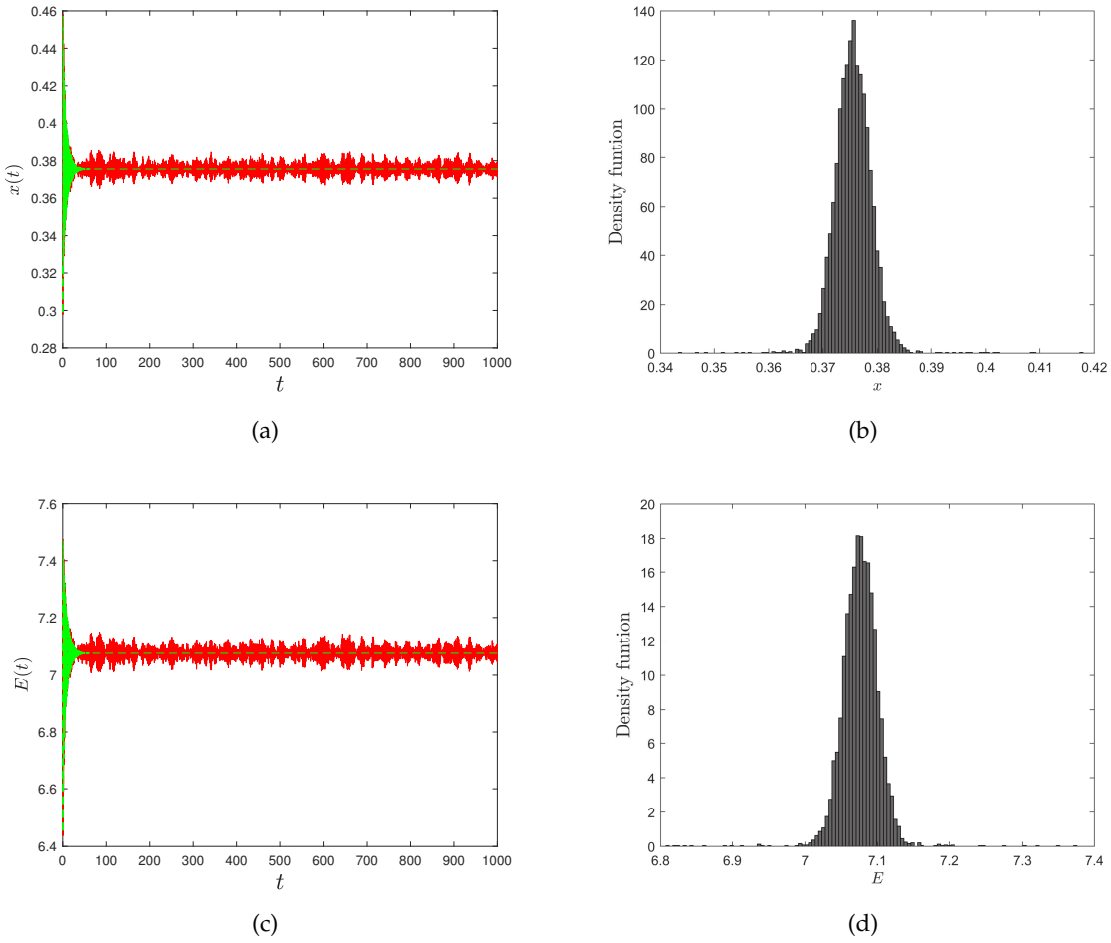
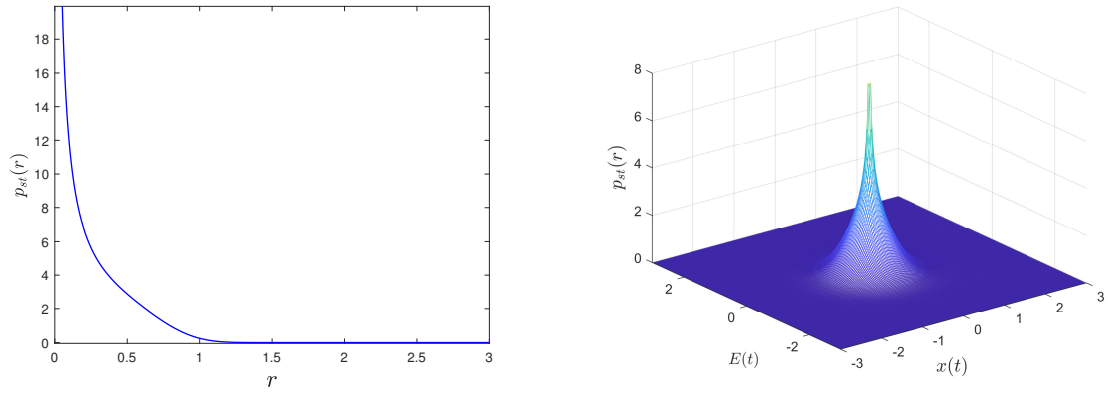
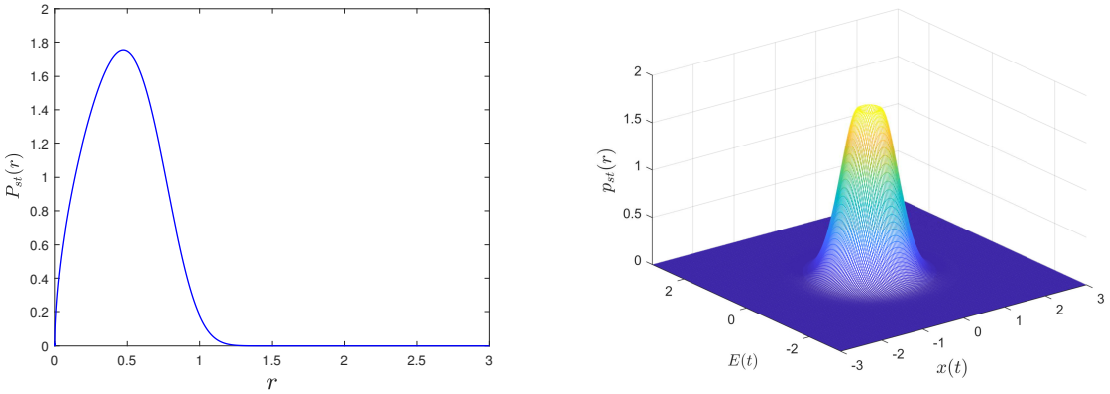


Figure 5.1: (a,c): Comparison of the solutions $x(t)$ and $E(t)$ of system (4.17) with the corresponding deterministic system solutions. The red line stands for the solution of the stochastic system, and the green line stands for the solution of the corresponding deterministic system. (b,d): The histogram of the probability density functions of x, E populations.



(a) (b)

Figure 5.2: The stationary probability density $p_{st}(r)$ of system (5.17) with $c_1 = 0.25$, $c_2 = 3$, $c_3 = 10$ and $c_4 = 5$.

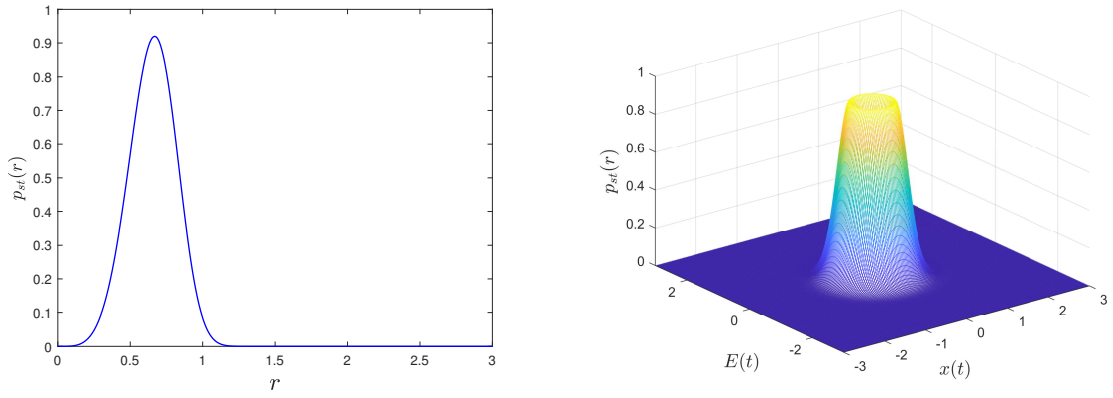


(a) (b)

Figure 5.3: The stationary probability density $p_{st}(r)$ of system (5.17) with $c_1 = 0.25$, $c_2 = 3$, $c_3 = 10$ and $c_4 = 3.5$.

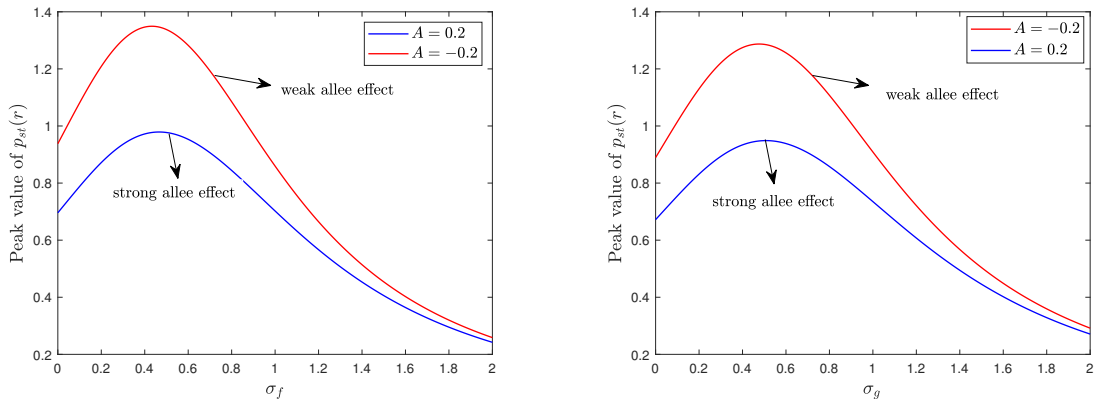
We select parameters $c_1 = 0.25$, $c_2 = 3$, $c_3 = 10$ and $c_4 = 2$ so that they satisfy the case (iii). Figure 5.4 shows the graph of $p_{st}(r)$, from which we can see that $p_{st}(r)$ reaches its maximum at $r = r_*$ and its minimum at $r = 0$.

We compare the effects of noise intensity σ_f and σ_g on the peak value of probability density $p_{st}(r)$ under strong and weak Allee effect. Selecting parameter $K = 3$, $A = 0.2$, $\alpha = 0.2$, $\varepsilon = 0.9$, $\beta = 1.1$, $p = 2.5$, $c = 1$, $\sigma_g = 0.7$ and $K = 3$, $A = -0.2$, $\alpha = 0.2$, $\varepsilon = 0.9$, $\beta = 1.1$, $p = 2.5$, $c = 1$, $\sigma_g = 0.7$, we plotted the graph of peak versus σ_f(see Figure 5.5(a)). Selecting parameter $K = 3$, $A = 0.2$, $\alpha = 0.2$, $\varepsilon = 0.9$, $\beta = 1.1$, $p = 2.5$, $c = 1$, $\sigma_f = 0.7$ and $K = 3$, $A = -0.2$, $\alpha = 0.2$, $\varepsilon = 0.9$, $\beta = 1.1$, $p = 2.5$, $c = 1$, $\sigma_f = 0.7$, we plotted the graph of peak versus σ_g(see Figure 5.5(b)). From the graph, we can observe that as the values of σ_f and σ_g increase, the peak value of $p_{st}(r)$ first increases and then decreases, indicating that the survival probability of the population will be higher under appropriate noise intensity. This is also consistent with reality, indicating that moderate harsh environments can promote



(a) (b)

Figure 5.4: The stationary probability density $p_{st}(r)$ of system (5.17) with $c_1 = 0.25$, $c_2 = 3$, $c_3 = 10$ and $c_4 = 2$.



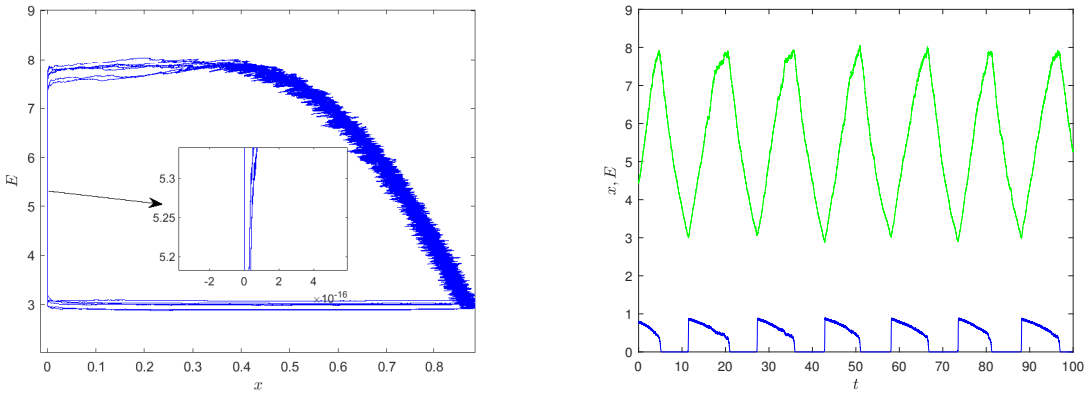
(a) (b)

Figure 5.5: (a) The effect of parameter σ_f on the peak value of $p_{st}(r)$. (b) The effect of parameter σ_g on the peak value of $p_{st}(r)$.

the reproduction of biological populations, while extreme natural environments can lead to the extinction of organisms. From Figure 5.5, it can also be observed that the probability of population survival under weak Allee effects is higher than that under strong Allee effects.

Finally, we discuss the impact of random noise on relaxation oscillations [1] through numerical simulation. Choosing parameter $K = 2$, $A = -0.5$, $\alpha = 0.17$, $\varepsilon = 0.01$, $\beta = 0.4$, $p = 1$, $c = 0.14$, $\sigma_f = 0.02$ and $\sigma_g = 0.02$, We have plotted the influence diagram noise on relaxation oscillation (see Figure 5.6). We found that noise will fluctuate near the relaxation oscillation, that is, the noise will approximately stabilize near the relaxation oscillation of deterministic system.

Remark 5.10. Environmental fluctuations highlight the importance of ecosystem protection. The fishery bioeconomic model can help evaluate the role of ecosystem conservation measures in responding to environmental fluctuations. For example, after setting up a marine protected area in a fishing ground, during environmental fluctuations, the fish population



(a) (b)

Figure 5.6: (a) The influence diagram noise on relaxation oscillation. (b) $x-t$ (blue) and $E-t$ (green) time series diagram.

within the protected area may serve as a "source" to provide juvenile fish supplements to surrounding fishing grounds, thereby maintaining the stability of the entire fishing ecosystem. Fishery managers also need to adjust their fishing strategies based on environmental fluctuations. During periods of significant environmental fluctuations, stricter fishing restrictions may be necessary, such as reducing fishing quotas and extending fishing bans. This is because environmental fluctuations have already caused natural pressure on fish populations, and overfishing may make it difficult for populations to recover, leading to the depletion of fishery resources. The fishery bioeconomic model can simulate the effects of different fishing strategies under environmental fluctuations, providing scientific decision-making basis for managers.

6 Conclusion

This paper investigates the bifurcation behavior of a class of stochastic fast slow single population fishery economic models with Allee effect.

For the deterministic fast-slow system (2.2) with strong Allee effect, we studied the fast-slow dynamic behavior of the system and revealed that as the bifurcation parameters change, the system produces stable canard circle, which happen to coincide with the Hopf bifurcation of the normal system. Compared to the strong Allee effect, for deterministic fast-slow systems (2.2) with weak Allee effect, more complex dynamic properties will be generated, not only canard circle, but also a fast-slow periodic cycle (relaxation oscillation). The existence of canard circle and relaxation oscillations indicates that the system will have a periodic oscillation, which means that the system is in dynamic equilibrium. For the aquaculture industry, it is beneficial for fishermen to optimize resource allocation and sustain economic turnover.

For the stochastic fast-slow fishery economic system, we discussed the existence of a stationary distribution for the system (4.18)after stochastic reduction. And with the time scale parameter as a general parameter, we studied the existence of stationary distribution and stochastic bifurcation behavior of the system (4.17). We conclude that both strong Allee effect and weak Allee effect, the system system (4.18) has a stationary distribution, and as the bifurcation parameter c_4 changes, stochastic P-bifurcation and stochastic D-bifurcation will

occur. Then we compared the changes in peak density of stationary distributions under different Allee effects, and concluded that under the same noise influence, weak Allee effects are more conducive to population survival (see Figure 5.5). Using the image (Figure 5.6) to describe the effect of the time scale parameter c on relaxation oscillations, we found that random fluctuations tend to stabilize near relaxation oscillations.

On the basis of this article, there is still relevant work that can be further studied. In this article, we only considered single population models, and in the future, we can study the relevant properties and bifurcation studies of multiple population models, as well as the bifurcation behavior of systems at multiple time scales.

Appendix

We provide a proof of convergence for the integrals $I_1 = \int_{x(0)}^{+\infty} x^{-N} \exp\left\{\frac{2(K+A)}{\sigma_f^2}x - \frac{K}{\sigma_f^2}x^2\right\}dx$ and $I_2 = \int_{x(0)}^{+\infty} x^{-N+1} \exp\left\{\frac{2(K+A)}{\sigma_f^2}x - \frac{K}{\sigma_f^2}x^2\right\}dx$.

Let $u = \frac{x}{\sigma_f}$, we obtain

$$\begin{aligned} I_1 &= \sigma_f^{-N+1} \int_{x(0)}^{+\infty} u^{-N} \exp\left\{-Ku^2 + \frac{2(K+A)u}{\sigma_f}\right\} du \\ &= \sigma_f^{-N+1} e^{\frac{(K+A)^2}{K\sigma_f^2}} \int_{x(0)}^{+\infty} u^{-N} \exp\left\{-K\left(u - \frac{K+A}{K\sigma_f}\right)^2\right\} du, \quad \text{taking } y = u - \frac{K+A}{K\sigma_f}, \\ &= \sigma_f^{-N+1} e^{\frac{(K+A)^2}{K\sigma_f^2}} \int_{x(0)-\frac{K+A}{K\sigma_f}}^{+\infty} \left(y + \frac{K+A}{K\sigma_f}\right)^{-N} e^{-Ky^2} dy. \end{aligned}$$

By calculation, we can conclude that e^{-Ky^2} is monotonically decreasing and bounded on the interval $(x(0) - \frac{K+A}{K\sigma_f}, +\infty)$. Since $N > 2$, we can obtain that

$$\int_{x(0)-\frac{K+A}{K\sigma_f}}^{+\infty} \left(y + \frac{K+A}{K\sigma_f}\right)^{-N} dy = \int_{x(0)}^{+\infty} u^{-N} du,$$

is convergent. Therefore, according to the Abel convergence criterion, we conclude that the integral I_1 is convergent. Similarly, we can also conclude that I_2 converges.

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Strong fast invertibility and Lyapunov exponents for linear systems

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Abstract. With the goal of deriving the existence of a dominated splitting, Quas, Thieullen and Zarrabi introduced the concept of strong fast invertibility for linear cocycles in 2019. Here, we take a closer look at strongly fast invertible systems with bounded coefficients. By linking the dimensions at which a system admits strong fast invertibility to the multiplicities of Lyapunov exponents, we are able to give a full characterization of regular strongly fast invertible systems similar to that of systems with stable Lyapunov exponents. In particular, we show that the stability of Lyapunov exponents implies strong fast invertibility (even in the absence of regularity). Central to our arguments are certain induced systems on spaces of exterior products that represent the evolution of volumes.

Finally, we derive convergence results for the computation of Lyapunov exponents via Benettin's algorithm using perturbation theory. While the stronger assumption of stable Lyapunov exponents clearly leaves more freedom on how to choose stepsizes, we derive conditions for the stepsizes with which convergence can be ensured even if a system is only strongly fast invertible.

Keywords: strong fast invertibility, Lyapunov exponents, linear dynamical systems, Benettin's algorithm, convergence analysis.

2020 Mathematics Subject Classification: 34D08, 37M25.

1 Introduction

Strong fast invertibility is a new property for dynamical systems introduced by Quas, Thieullen and Zarrabi in 2019 [21]. While it originally served as an ingredient to ensure uniform invertibility of a cocycle along its fastest growing direction (hence the name), strong fast invertibility can be best describes using the evolution of volumes: A linear system is L-dim. strongly fast invertible according to [21] if and only if there is a constant $c > 0$ such that

$$\frac{\| \wedge^L X(t,\tau)\|}{\| \wedge^L X(s,\tau)\|} \leq \| \wedge^L X(t,s)\| \leq c \frac{\| \wedge^L X(t,\tau)\|}{\| \wedge^L X(s,\tau)\|}$$

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for all $t \geq s \geq \tau$, where $\wedge^L X(t,s)$ denotes L-fold exterior product of the Cauchy matrix of the system (see Lemma 3.29). In other words, the maximal growth of L-volumes over a time interval can be computed (up to a constant independent of the interval) by bisecting the interval and computing the maximal growths of L-volumes on the subintervals.

By combining strong fast invertibility with a uniform singular value gap, Quas et al. prove the existence of a *dominated equivariant uniform splitting* of the dynamics, i.e., a splitting into equivariant fast and slow subspaces such that the angle between them is bounded from below and solutions corresponding to the fast subspace grow uniformly exponentially faster than solutions corresponding to the slow subspace. Using an equivalent notion, one may also call the splitting *integrally separated* [22, Definition 2.3].

The latter terminology is usually applied in studies of Lyapunov exponents. In particular, the existence of an integrally separated splitting is necessary to ensure the stability of Lyapunov exponents. Essentially, given an integrally separated splitting, the stability of Lyapunov exponents boils down to the stability on the subspaces of the splitting:

Theorem ([1, Theorem 5.4.9], [3])**.** *Assume a linear system with bounded coefficients and Lyapunov exponents $\lambda_1 > \cdots > \lambda_p$ with multiplicities $d_1 + \cdots + d_p = d$. The Lyapunov exponents are stable if and only if there exists a Lyapunov transformation reducing the system to block diagonal form*

$$\dot{y} = \operatorname{diag}(B_1(t), \dots, B_p(t))y,$$

where $B_i(t) \in \mathbb{R}^{d_i \times d_i}$ is upper triangular, such that the following hold:

(i) all non-trivial solutions of $\dot{y}_i = B_i(t)y_i$ have characteristic exponent λ_i,

(ii) λ_i is stable for $\dot{y}_i = B_i(t)y_i$,

(iii) there are constants $a, b > 0$ such that

$$\|Y_i(t,s)^{-1}\|^{-1} \geq be^{a(t-s)}\|Y_{i+1}(t,s)\|$$

for all $t \geq s$, where $Y_i(t,s)$ denotes the Cauchy matrix of $\dot{y}_i = B_i(t)y_i$.

One of our two main goals is to provide a characterization of strong fast invertibility that allows a direct comparison to stability of Lyapunov exponents via the above theorem. To connect strong fast invertibility, a property that concerns the evolution of volumes, to Lyapunov exponents, we work with certain induced systems on spaces of exterior products. These systems allow us to link Lyapunov exponents to volume growth assuming the original system is regular. Our characterization theorem is the following:

Theorem. *Assume a linear system with bounded coefficients and Lyapunov exponents $\lambda_1 > \cdots > \lambda_p$ with multiplicities $d_1 + \cdots + d_p = d$. If the system is regular and strongly fast invertible at dimensions $d_1 + \cdots + d_l$ for $l = 1, \dots, p$, then there exists a Lyapunov transformation reducing the system to block diagonal form*

$$\dot{y} = \operatorname{diag}(B_1(t), \dots, B_p(t))y,$$

where $B_i(t) \in \mathbb{R}^{d_i \times d_i}$ is upper triangular, such that the following hold:

(i) all non-trivial solutions of $\dot{y}_i = B_i(t)y_i$ have characteristic exponent λ_i,

(ii) there is a constant $b > 0$ such that

$$\|Y_i(t,s)^{-1}\|^{-1} \geq b\|Y_{i+1}(t,s)\|$$

for all $t \geq s$, where $Y_i(t,s)$ denotes the Cauchy matrix of $\dot{y}_i = B_i(t)y_i$.

Conversely, any block diagonal system $\dot{y} = B(t)y$ satisfying (i) and (ii) is strongly fast invertible at dimensions $d_1 + \cdots + d_l$ for $l = 1, \ldots, p$.

In particular, our theorem provides an equivalent characterization of strong fast invertibility for systems that are regular and have bounded coefficients. Moreover, it shows that the stability of Lyapunov exponents implies strong fast invertibility at the respective dimensions. More aspects, such as another characterization in the case of simple Lyapunov spectra, can be found in our article. However, we note that there are still interesting aspects to explore that we did not pursue here.

Our second main goal is to derive convergence results for the computation of Lyapunov exponents. More precisely, we focus on *Benettin's algorithm* [4,5] as it is the most fundamental and common algorithm to compute Lyapunov exponents. Its underlying idea is to propagate a set of linear perturbations that are reorthonormalized periodically. The Lyapunov exponents are then computed as averages of volume expansion via the rescaling factors from the orthonormalization procedure.

While it is not difficult to prove convergence of Benettin's algorithm in the absence of numerical errors, integration errors can accumulate and persist. This happens especially if the stepsizes are kept constant. In practice, it is hard to quantify these error, since the exact Lyapunov exponents are usually unknown.

Major efforts have been made by Dieci, Van Vleck and co-authors. They advocate the use of adaptive stepsizes to bound the local integration error and were able to prove error estimates for computed Lyapunov exponents of linear systems that are regular and have stable Lyapunov exponents [10, 11]. While Dieci and Van Vleck proved that the asymptotic limits of the computed exponents can be made arbitrarily close to the true Lyapunov exponents by decreasing the error tolerance or the fixed stepsize, true convergence requires to simultaneously increase the integration time and decrease the stepsizes. This was already conjectured by Mc Donald and Higham in their error analysis for autonomous linear systems in 2001 [17, Section 5].

By tackling both limits simultaneously, we derive new convergence results that differentiate between systems with stable Lyapunov exponents and systems that are only strongly fast invertible. The main difference between the respective convergence results are the requirements for stepsizes. While stepsizes h_n such that

$$\sum_{n=1}^{\infty} h_n = \infty \quad \text{and} \quad h_n \to 0$$

are enough to achieve convergence for systems with stable Lyapunov exponents, we need stricter assumptions to compute Lyapunov exponents for strongly fast invertible systems:

$$\sum_{n=1}^{\infty} h_n = \infty \quad \text{and} \quad \sum_{n=1}^{\infty} h_n^{p+1} < \infty,$$

where $p > 0$ is the order of consistency of the numerical integrator.

Finally, we provide three numerical examples to highlight the effects of different types of stepsizes (adaptive, constant and varying) on the computation of Lyapunov exponents. While the first example is linear and in accordance with our convergence theory, the other two examples are nonlinear and serve as an outlook suggesting that similar convergence properties may be valid for more general classes of systems.

2 Exterior products and powers

Exterior products are a handy tool when it comes to studying the evolution of volumes and hence also Lyapunov exponents. In this section we briefly introduce them and some of their properties. Our main reference is [2, Section 3.2.3].

For $1 \leq L \leq d$, the L-fold *exterior power* of $\mathbb{R}^d$ is the space $\wedge^L \mathbb{R}^d$ consisting of alternating L-linear forms on the dual space $(\mathbb{R}^d)^* \cong \mathbb{R}^d$. A basis can be obtained by taking *exterior products* of basis elements of $\mathbb{R}^d$. For example, the set

$$\{e_I := e_{i_1} \wedge \cdots \wedge e_{i_L} \mid I = (i_1, \ldots, i_L) \text{ with } 1 \leq i_1 < \cdots < i_L \leq d\},$$

where e_i is the i-th unit vector of $\mathbb{R}^d$, defines a natural basis of $\wedge^L \mathbb{R}^d$. In particular, $\wedge^L \mathbb{R}^d$ has dimension $\binom{d}{L}$.

Not all elements of $\wedge^L \mathbb{R}^d$ are *decomposable*, i.e., of the form $u_1 \wedge \cdots \wedge u_L$. Some elements are *indecomposable* and can only be expressed as linear combinations of decomposable elements.

Given subspaces $U, V \subset \mathbb{R}^d$, we define induced subspaces

$$(\wedge^k U) \wedge (\wedge^{L-k} V) := \operatorname{span}\{u_1 \wedge \cdots \wedge u_k \wedge v_{k+1} \wedge \cdots \wedge v_L \mid u_i \in U, v_i \in V\} \subset \wedge^L \mathbb{R}^d$$

as spans of the corresponding induced decomposable elements.

By bilinear extension from the set of decomposable elements to $\wedge^L \mathbb{R}^d$, the following defines a scalar product on $\wedge^L \mathbb{R}^d$:

$$\langle u_1 \wedge \cdots \wedge u_L, v_1 \wedge \cdots \wedge v_L \rangle := \det(\langle u_i, v_j \rangle)_{ij}.$$

In particular, the induced norm of a decomposable element $u_1 \wedge \cdots \wedge u_L$ is the L-volume of the parallelepiped spanned by $u_1, \ldots, u_L$:

$$\|u_1 \wedge \cdots \wedge u_L\| = \sqrt{\det(\langle u_i, u_j \rangle)_{ij}}.$$

Throughout this article norms without annotation always denote euclidean norms or associated matrix norms.

Lemma 2.1. *We have*

(i) $\|u_1 \wedge \cdots \wedge u_L\| \leq \|u_1 \wedge \cdots \wedge u_k\| \cdot \|u_{k+1} \wedge \cdots \wedge u_L\|$

for $u_1, \ldots, u_L \in \mathbb{R}^d$ *and*

(ii) $\langle \hat{u} \wedge \hat{v}, \hat{u}' \wedge \hat{v}' \rangle = \langle \hat{u}, \hat{u}' \rangle \langle \hat{v}, \hat{v}' \rangle$,

(iii) $\|\hat{u} \wedge \hat{v}\| = \|\hat{u}\| \, \|\hat{v}\|$

for $\hat{u}, \hat{u}' \in \wedge^k U$ *and* $\hat{v}, \hat{v}' \in \wedge^{L-k} V$ *with* $U, V \subset \mathbb{R}^d$ *orthogonal.*

Proof. (i) can be found in [2, Subsection 3.2.3].

Let $U, V \subset \mathbb{R}^d$ with $U \perp V$. Choose bases $(u_i)_i$ of U and $(v_j)_j$ of V. We denote the elements of the induced bases of $\wedge^k U$ and $\wedge^{L-k} V$ by u_I and v_J respectively. Since

$$\langle u_I \wedge v_J, u_{I'} \wedge v_{J'} \rangle$$

is the determinant of a block diagonal matrix, it is equal to the product of the determinants of both blocks, which is

$$\langle u_I, u_{I'} \rangle \langle v_J, v_{J'} \rangle.$$

Thus, (ii) holds on basis elements and by bilinearity of the inner product on arbitrary elements.

Assertion (iii) follows immediately from (ii) by setting $\hat{u} = \hat{u}'$ and $\hat{v} = \hat{v}'$. □

Next, we discuss several constructions for linear maps. The L-fold exterior power of $A \in \mathbb{R}^{d\times d}$ is defined via

$$(\wedge^L A)(u_1 \wedge \cdots \wedge u_L) := Au_1 \wedge \cdots \wedge Au_L.$$

Similarly, one may define

$$((\wedge^k A) \wedge (\wedge^{L-k} B))(u_1 \wedge \cdots \wedge u_L) := Au_1 \wedge \cdots \wedge Au_k \wedge Bu_{k+1} \wedge \cdots \wedge Bu_L.$$

for $A, B \in \mathbb{R}^{d\times d}$. Another helpful construction is

$$\hat{A}^L(u_1 \wedge \cdots \wedge u_L) := \sum_{k=1}^{L} u_1 \wedge \cdots \wedge u_{k-1} \wedge Au_k \wedge u_{k+1} \wedge \cdots \wedge u_L.$$

Since we will make extensive use of these induced maps, we state and derive a list of basic properties for them.

Lemma 2.2. *The following are true for $A, B \in \mathbb{R}^{d\times d}$:*

(i) $\wedge^L I_{\mathbb{R}^d} = I_{\wedge^L \mathbb{R}^d}$,

(ii) $\wedge^L (AB) = (\wedge^L A)(\wedge^L B)$,

(iii) $(\wedge^L A)^{-1} = \wedge^L A^{-1}$ *if* $A \in GL(d, \mathbb{R})$,

(iv) $\widehat{\alpha A + \beta B}^L = \alpha \hat{A}^L + \beta \hat{B}^L$,

(v) $\| \wedge^L A\| = \sigma_1(A) \dots \sigma_L(A)$, *where* $\sigma_1 \geq \cdots \geq \sigma_d$ *denote the singular values,*

(vi) $\| \wedge^L A\| \leq \|A\|^L$,

(vii) $\|\hat{A}^L\| \leq L\|A\|$,

(viii) $\|(\wedge^k A) \wedge (\wedge^{L-k} B)\| \leq \binom{d}{L}^{\frac{1}{2}} \| \wedge^k A\| \, \| \wedge^{L-k} B\|$,

(ix) if $A[u_1, \dots, u_L] = QR$, *then* $\|(\wedge^L A)(u_1 \wedge \cdots \wedge u_L)\| = r_{11} \dots r_{LL}$, *where* $r_{ii} \geq 0$ *denote the diagonal elements of* R *ordered in decreasing size,*

(x) $\det(\wedge^L A) = \det(A)^{\binom{d-1}{L-1}}$.

Proof. (i)–(vii) and (x) can be found in [2, Subsection 3.2.3].

To show $(viii)$, write

$$I_1 = (i_1, \dots, i_k) \text{ and } I_2 = (i_{k+1}, \dots, i_L)$$

for a given tuple $I = (i_1, \dots, i_L)$ and estimate

$$\begin{aligned}
\|(\wedge^k A) \wedge (\wedge^{L-k} B)\| &= \sup_{\sum \alpha_I^2 = 1} \left\| ((\wedge^k A) \wedge (\wedge^{L-k} B)) \left(\sum \alpha_I e_I\right) \right\| \\
&\leq \sup_{\sum \alpha_I^2 = 1} \sum |\alpha_I| \, \|((\wedge^k A) e_{I_1}) \wedge ((\wedge^{L-k} B) e_{I_2})\| \leq \sup_{\sum \alpha_I^2 = 1} \sum |\alpha_I| \, \|(\wedge^k A) e_{I_1}\| \, \|(\wedge^{L-k} B) e_{I_2}\| \\
&\leq \binom{d}{L}^{\frac{1}{2}} \| \wedge^k A\| \, \| \wedge^{L-k} B\|.
\end{aligned}$$

Finally, we prove (ix):

$$\|(\wedge^L A)(u_1 \wedge \cdots \wedge u_L)\|^2 = \det(\langle Au_i, Au_j \rangle)_{ij} = \det((QR)^T (QR)) = \det(R^T R) = r_{11}^2 \dots r_{LL}^2.$$

□

Lemma 2.3. *Let $A = \operatorname{diag}(A_1, A_2) \in \mathbb{R}^{d\times d}$ and set $B := \operatorname{diag}(A_1, 0)$ and $C := \operatorname{diag}(0, A_2)$. The following are true:*

(i) $\wedge^L A = \sum_{k=0}^{L} (\wedge^k B) \wedge (\wedge^{L-k} C)$,

(ii) $\| \wedge^L A\| = \max_k \|(\wedge^k B) \wedge (\wedge^{L-k} C)\|$,

(iii) $\|(\wedge^k B) \wedge (\wedge^{L-k} C)\| \geq \| \wedge^k B\| \, \| \wedge^{L-k} C\|$.

Proof. Let $A_1 \in \mathbb{R}^{d_1\times d_1}$. Given a basis element

$$e_I = e_{i_1} \wedge \cdots \wedge e_{i_L},$$

choose j such that $i_j \leq d_1 < i_{j+1}$. It holds

$$(\wedge^L A)e_I = ((\wedge^j B) \wedge (\wedge^{L-j} C))e_I.$$

Moreover, all summands in

$$\sum_{k=0}^{L} ((\wedge^k B) \wedge (\wedge^{L-k} C))e_I$$

vanish except for $k = j$. Hence, the two maps in (i) coincide on basis elements e_I.

The subspaces

$$W_k := \left(\wedge^k \left(\mathbb{R}^{d_1} \times \{0\}\right)\right) \wedge \left(\wedge^{L-k} \left(\{0\} \times \mathbb{R}^{d-d_1}\right)\right)$$

for $k = 0, \ldots, L$ form an orthogonal decomposition of $\wedge^L \mathbb{R}^d$. Since

$$\operatorname{im}\left((\wedge^k B) \wedge (\wedge^{L-k} C)\right) \subset W_k$$

and

$$W_i \subset \ker\left((\wedge^k B) \wedge (\wedge^{L-k} C)\right)$$

for $i \neq k$, (ii) easily follows by means of this decomposition.

To prove (iii), we use (iii) of Lemma 2.1. It follows that

$$\begin{aligned}\|(\wedge^k A) \wedge (\wedge^{L-k} B)\| &\geq \max_{\|\hat{u}\wedge\hat{v}\|=1} \|(\wedge^k A)\hat{u} \wedge (\wedge^{L-k} B)\hat{v}\| = \max_{\|\hat{u}\|=1} \|(\wedge^k A)\hat{u}\| \max_{\|\hat{v}\|=1} \|(\wedge^{L-k} B)\hat{v}\| \\ &= \| \wedge^k A\| \, \| \wedge^{L-k} B\|,\end{aligned}$$

where maxima are with respect to $\hat{u} \wedge \hat{v} \in W_k$. □

Next, we relate the principle angles between two complementary subspaces of $\mathbb{R}^d$ to a principle angle on $\wedge^L \mathbb{R}^d$.

Proposition 2.4. *Let $\mathbb{R}^d = U \oplus V$ with $\dim U = L$. Set $\hat{U} := \wedge^L U$ and $\hat{V} := \left(\wedge^L V^\perp\right)^\perp$. Then*

$$\prod_{i=1}^{\min(L,d-L)} \sin\alpha_i(U, V) = \sin\alpha_1(\hat{U}, \hat{V}), \tag{2.1}$$

where $0 < \alpha_1 \leq \alpha_2 \leq \cdots \leq \frac{\pi}{2}$ are the principle angles between the respective subspaces.

Proof. First we pass to the orthogonal complement of V in order to work with two subspaces of the same dimension. According to [25] it holds

$$\prod_{i=1}^{\min(L,d-L)} \sin\alpha_i(U,V) = \prod_{i=1}^{L} \sin\left(\frac{\pi}{2} - \alpha_i(U,V^\perp)\right) = \prod_{i=1}^{L} \cos\alpha_i(U,V^\perp).$$

The concept of "higher dimensional angle" coined in [13] helps us to transition to $\wedge^L\mathbb{R}^d$. In fact, the higher dimensional angle $\theta(U,V^\perp)$ between two subspaces U and $V^\perp$ of the same dimension satisfies

$$\cos\theta(U,V^\perp) = \prod_{i=1}^{L} \cos\alpha_i(U,V^\perp)$$

and is defined via

$$\cos\theta(U,V^\perp) := \frac{\langle\alpha,\beta\rangle}{\|\alpha\|\,\|\beta\|},$$

where $\alpha,\beta \in \wedge^L\mathbb{R}^d$ are decomposable elements representing U and $V^\perp$, i.e., $\alpha = u_1\wedge\cdots\wedge u_L$ with $\operatorname{span}(u_1,\dots,u_L) = U$ and $\beta = v_1'\wedge\cdots\wedge v_L'$ with $\operatorname{span}(v_1',\dots,v_L') = V^\perp$. The latter is nothing else than the angle between α and β or the principle angle between their corresponding 1-dimensional subspaces:

$$\cos\theta(U,V^\perp) = \cos\alpha_1(\wedge^L U,\wedge^L V^\perp) = \cos\alpha_1(\hat{U},\hat{V}^\perp) = \sin\alpha_1(\hat{U},\hat{V}). \qquad \square$$

Corollary 2.5. *In Proposition 2.4 it holds*

$$\left(\wedge^L V^\perp\right)^\perp = \oplus_{k=0}^{L-1}\left((\wedge^k U)\wedge(\wedge^{L-k}V)\right).$$

In particular, equation (2.1) is true for

$$\begin{aligned} U &:= \operatorname{span}\{Ae_1,\dots,Ae_L\},\\ V &:= \operatorname{span}\{Ae_{L+1},\dots,Ae_d\},\\ \hat{U} &= \operatorname{span}\{(\wedge^L A)(e_1\wedge\cdots\wedge e_L)\},\\ \hat{V} &= \operatorname{span}\{(\wedge^L A)(e_{i_1}\wedge\cdots\wedge e_{i_L}) \mid \{i_1,\dots,i_L\} \neq \{1,\dots,L\}\} \end{aligned}$$

for any given $A \in GL(d,\mathbb{R})$.

Proof. The identity can be checked by using that both spaces have codimension 1 and by showing that the inner product between elements of the form

$$v_1'\wedge\cdots\wedge v_L' \in \wedge^L V^\perp$$

and

$$u_1\wedge\cdots\wedge u_k\wedge v_{k+1}\wedge\cdots\wedge v_L$$

with $u_i \in U$ and $v_i \in V$ vanishes for $k < L$. $\square$

3 Asymptotic properties of linear systems

Consider the linear differential equation

$$\dot{x} = A(t)x, \quad A \in C(\mathbb{R}_{\geq 0},\mathbb{R}^{d\times d}) \tag{3.1}$$

with bounded coefficients, i.e.,

$$M := \sup_t \|A(t)\| < \infty.$$

We write $X(t)$ for a fundamental matrix of equation (3.1) and $X(t,s) := X(t)X(s)^{-1}$, $t \geq s$, for the Cauchy matrix. By requiring that A is bounded, solutions grow or decay at most exponentially fast:

$$e^{-M(t-s)} \leq \|X(t,s)^{\pm 1}\| \leq e^{M(t-s)}$$

for all $t \geq s$.

In our article, equation (3.1) will be changed by so-called *Lyapunov transformations*. These transformations leave asymptotic properties[1] like boundedness of the system matrix, the Lyapunov spectrum, regularity, the stability of Lyapunov exponents, or strong fast invertibility invariant.

Definition 3.1 ([1, Definition 3.1.1])**.** $x = L(t)y$ is called *Lyapunov transformation* if L is continuously differentiable, everywhere invertible and $L, L^{-1}, \dot{L}$ are bounded. The transformed system $\dot{y} = B(t)y$ is given by

$$B(t) = L(t)^{-1}A(t)L(t) - L(t)^{-1}\dot{L}(t).$$

It is easy to see that Lyapunov transformations form a group with respect to multiplication. A particular Lyapunov transformation is obtained via the Gram–Schmidt procedure.

Proposition 3.2 ([1, Lemma 3.3.1 and Theorem 3.3.1])**.** *There is an orthogonal Lyapunov transformation such that the fundamental matrix $Y(t)$ of the transformed system is upper triangular.*

3.1 Characteristic exponents

In this subsection, we define characteristic exponents and Lyapunov exponents.

Definition 3.3 ([1, Definition 2.1.1])**.** The *characteristic exponent* of $f : [0,\infty) \to \mathbb{R}$ is given by

$$\chi[f] := \limsup_{t\to\infty} \frac{1}{t} \log|f(t)| \in \mathbb{R} \cup \{\pm\infty\}.$$

A few handy properties follow easily from the definition (see [1, Section 2.1] and [2, Lemma 3.2.1]):

(i) $\chi[c] = 0$ for $c \neq 0$

(ii) $\chi[cf] = \chi[f]$ for $c \neq 0$,

(iii) $\chi[|f|^c] = c\chi[f]$ for $c \in \mathbb{R}$ (set $0(\pm\infty) = 0$),

(iv) $\chi[f] \leq \chi[g]$ if $|f| \leq |g|$,

(v) $\chi[f+g] \leq \max(\chi[f], \chi[g])$ with equality if $\chi[f] \neq \chi[g]$,

(vi) $\chi[fg] \leq \chi[f] + \chi[g]$ (if the right-hand side makes sense).

[1] We will define all mentioned properties throughout our article.

In the context of dynamical systems, we apply characteristic exponents to measure the (upper) exponential growth of solutions $x(t)$:

$$\chi[x] := \chi[\|x\|] = \limsup_{t\to\infty} \frac{1}{t} \log \|x(t)\|.$$

Note that $\chi[x]$ is independent of the chosen norm because all norms on $\mathbb{R}^d$ are equivalent.[2]

Since solutions with distinct characteristic exponents are linearly independent, the characteristic exponents can take at most d values

$$\infty > M \geq \lambda_1 > \cdots > \lambda_p \geq -M > -\infty.$$

In particular, there is a filtration of subspaces of the space of solutions called *Lyapunov filtration*

$$\mathbb{R}^d \cong V_1 \supset \cdots \supset V_p \supset V_{p+1} := \{0\}$$

with $V_i := \{x \,|\, \chi[x] \leq \lambda_i\}$ satisfying

$$\chi[x] = \lambda_i \quad \Longleftrightarrow \quad x \in V_i \setminus V_{i+1}.$$

We set $d_i := \dim V_i - \dim V_{i+1}$.

Remark 3.4. Sometimes the Lyapunov filtration is defined on the space of initial vectors. In fact, both versions of filtration spaces are naturally isomorphic:

$$V_i' := \{v \mid \chi[X(t,0)v] \leq \lambda_i\} = \{x(0) \mid x \in V_i\}.$$

One may also define the filtration spaces using a given fundamental matrix $X(t)$:

$$V_{i,X} := \{v \mid \chi[X(t)v] \leq \lambda_i\}.$$

Definition 3.5. The *(forward) Lyapunov spectrum* $(\lambda_i, d_i)_{i=1,\ldots,p}$ of a system consists of its *Lyapunov exponents* λ_i together with their *multiplicities* d_i. Since we sometimes count Lyapunov exponents according to their multiplicities, we define $\Lambda_1, \ldots, \Lambda_d$ via

$$\Lambda_{d_1+\cdots+d_{i-1}+j} := \lambda_i \quad \text{for } j = 1, \ldots, d_i.$$

A Lyapunov exponent λ_i is called *simple* or *nondegenerate* if $d_i = 1$ and otherwise *degenerate*. If all Lyapunov exponents are simple, we call the Lyapunov spectrum *simple*.

Example 3.6. Let $\epsilon > 0$. We set $a_n := n^2 - \log n$ and $b_n := n^2$ and define $f_n \in C(\mathbb{R}_{\geq 0}, \mathbb{R})$ through

$$f_n = \begin{cases} 0, & t \in [0, a_n - \epsilon] \\ \frac{t-(a_n-\epsilon)}{\epsilon}, & t \in (a_n - \epsilon, a_n] \\ 1, & t \in (a_n, b_n] \\ \frac{b_n+\epsilon-t}{\epsilon}, & t \in (b_n, b_n + \epsilon] \\ 0. & t \in (b_n + \epsilon, \infty) \end{cases}$$

[2] In infinite dimensions the characteristic exponents generally depend on the chosen norm, although independence can be achieved on certain scales of Banach spaces [6].

For small ϵ the functions f_n have disjoint supports. Thus, the system matrix

$$A(t) := \begin{pmatrix} 1 & 0 \\ 0 & a_{22}(t) \end{pmatrix}$$

with $a_{22}(t) := \sum_{n\in\mathbb{N}} f_n(t)$ is bounded and continuous. A fundamental matrix is given by

$$X(t) := \begin{pmatrix} e^t & 0 \\ 0 & e^{\int_0^t a_{22}(\tau)\,d\tau} \end{pmatrix}.$$

Since

$$1 \leq e^{\int_0^{b_n+\epsilon} a_{22}(\tau)\,d\tau} \leq e^{\sum_{k=1}^{n}(\log(k)+2\epsilon)} = n!e^{2\epsilon n},$$

we have

$$0 \leq \frac{1}{t}\log e^{\int_0^t a_{22}(\tau)\,d\tau} \leq \frac{2\epsilon(n+1)}{n^2} + \frac{\log((n+1)!)}{n^2}$$

for $t \in [b_n, b_{n+1}+\epsilon]$. In particular, the system exhibits solutions with characteristic exponents:

$$\chi[X(t)e_1] = 1 \quad \text{and} \quad \chi[X(t)e_2] = 0.$$

Since the characteristic exponents are distinct, the columns of $X(t)$ realize the whole Lyapunov spectrum ($\lambda_1 = 1$ and $\lambda_2 = 0$) and can be used to compute the Lyapunov filtration ($V_{1,X} = \mathbb{R}^2$ and $V_{2,X} = \{0\} \times \mathbb{R}$).

We may always choose[3] a basis of solutions that realizes the whole Lyapunov spectrum. In that case

$$\sum_{i=1}^{d} \chi[x_i] = \sum_{i=1}^{p} d_i\lambda_i. \tag{3.2}$$

Definition 3.7 ([1, Definition 2.4.2])**.** A basis $x_1, \dots, x_d$ of solutions is called *normal* if $\sum_i \chi[x_i]$ is minimal, i.e., if equation (3.2) holds. Moreover, we call a fundamental matrix *normal* if its columns form a normal basis.

If we order a normal basis such that $\chi[x_i]$ decreases with i, then $\chi[x_i] = \Lambda_i$.

Proposition 3.8 ([1, Theorem 2.5.1, Corollary 2.5.1 and Remark 2.5.2])**.** *Any fundamental matrix $X(t) = [x_1, \dots, x_d]$ satisfies*

$$\sum_{i=1}^{d} \chi[x_i] \geq \sum_{i=1}^{p} d_i\lambda_i \geq \chi[\det X(t)].$$

If equality holds, then $X(t)$ is normal. However, $X(t)$ being normal does not imply equality in general.

The largest Lyapunov exponent can be expressed as the characteristic exponent of $\|X(t)\|$.

Proposition 3.9. *The largest Lyapunov exponent satisfies*

$$\lambda_1 = \chi[\|X(t)\|].$$

Proof. One inequality is trivial and the other follows from

$$\|X(t)\| \leq \sqrt{d}\max_i \|X(t)e_i\|_1,$$

where e_i is the i-th unit vector of $\mathbb{R}^d$. □

[3] Construct a basis by iteratively choosing d_i solutions in $V_i \setminus V_{i+1}$ for $i = p, \dots, 1$.

3.2 Regularity

We now introduce the notion of regularity, which implies that characteristic exponents of solutions can be obtained as limits instead of limes superiors.

Definition 3.10 ([2, p. 115])**.** We call a system *(forward) regular*[4] if

$$\sum_{i=1}^{p} d_i\lambda_i = \liminf_{t\to\infty} \frac{1}{t} \log|\det X(t)|.$$

In case of triangular systems regularity can be checked via the diagonal elements of the system matrix[5]. Moreover, one may check regularity of general systems through Perron's regularity test, which compares the Lyapunov spectra of the system and its adjoint system $\dot{y} = -A(t)^T y$ [1, Theorem 3.6.1].

Proposition 3.11 ([1, Theorem 3.9.1])**.** *Regular systems have sharp characteristic exponents, i.e., it holds*

$$\chi[x] = \lim_{t\to\infty} \frac{1}{t} \log\|x(t)\|.$$

Example 3.12. The system from Example 3.6 is regular: Indeed, the fundamental matrix

$$X(t) = \begin{pmatrix} e^t & 0 \\ 0 & e^{\int_0^t a_{22}(\tau)\,d\tau} \end{pmatrix}$$

is diagonal and our previous estimates show that the diagonal elements have sharp characteristic exponents:

$$1 + 0 = \lambda_1 + \lambda_2 = \lim_{t\to\infty} \frac{1}{t} \log e^t + \lim_{t\to\infty} \frac{1}{t} \log e^{\int_0^t a_{22}(\tau)\,d\tau} = \lim_{t\to\infty} \frac{1}{t} \log|\det X(t)|.$$

Alternatively, one may show that diagonal elements of the system matrix $A(t)$ have finite mean values.

All Lyapunov exponents of a regular system can be obtained via characteristic exponents of singular values of $X(t)$.

Proposition 3.13. *If equation* (3.1) *is regular, then*

$$\Lambda_i = \lim_{t\to\infty} \frac{1}{t} \log \sigma_i(X(t))$$

for all $i = 1, \dots, d$*, where* $\sigma_1 \geq \cdots \geq \sigma_d$ *denote the singular values.*

[4]The definition from Arnold's book is equivalent to [1, Definition 3.5.1]. This can be checked via [1, Lemma 3.5.1] and the Liouville–Ostrogradski formula

$$\det X(t) = \det X(t_0) e^{\int_{t_0}^t \operatorname{tr}(A(\tau))\,d\tau}.$$

[5][1, Corollary 3.8.1] states that a lower triangular system is regular if and only if its diagonal elements $a_{ii}(t)$ have finite mean values:

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t a_{ii}(s)\,ds < \infty.$$

Since adjoint systems of regular systems are regular ([1, Corollary 3.6.1]), the same holds true for upper triangular systems.

We prove Proposition 3.13 in the next subsection using induced systems on the space of exterior products. Next, let us state a result on singular vectors of $X(t)$.

Proposition 3.14. *If equation (3.1) is regular and $u_1(t), \dots, u_d(t)$ denote the right singular vectors of $X(t)$, then*

$$U_{i,X}(t) := \operatorname{span}(u_{d_1+\cdots+d_{i-1}+1}(t), \dots, u_{d_1+\cdots+d_i}(t))$$

converges to $U_{i,X} := V_{i+1,X}^{\perp} \cap V_{i,X}$ exponentially fast. More precisely, it holds

$$\limsup_{t\to\infty} \frac{1}{t} \log \max_{\substack{u\in U_{i,X}(t),\\ u'\in U_{j,X},\\ \|u\|=\|u'\|=1}} |\langle u, u'\rangle| \leq -|\lambda_i - \lambda_j|$$

for $i \neq j$.

Proof. Since equation (3.1) has bounded coefficients and its induced systems are regular (see Lemma 3.19), the deterministic version of the multiplicative ergodic theorem [2, Proposition 3.4.2] applies. In its proof, Arnold shows that the filtration $F_X(t)$ given by the spaces

$$V_{i,X}(t) := U_{p,X}(t) \oplus \cdots \oplus U_{i,X}(t)$$

converges exponentially fast to the Lyapunov filtration F_X given by the spaces

$$V_{i,X} = U_{p,X} \oplus \cdots \oplus U_{i,X}$$

using the metric δ (see [2, Equation 3.4.10]) on the manifold of filtrations. In particular, he shows that

$$\limsup_{n\to\infty} \frac{1}{n} \log \delta(F_X(n), F_X) \leq -h,$$

where $h > 0$ is a parameter also appearing in the definition of δ. Disentangling the metric yields the desired convergence statement for discrete time.

The version for continuous time follows as described in the proof of [2, Theorem 3.4.1] since

$$\limsup_{n\to\infty} \frac{1}{n} \sup_{0\leq t\leq 1} \log \|X(n+t, n)^{\pm 1}\| \leq 0. \qquad \square$$

If at least two singular values coincide, the singular value decomposition is not unique. However, according to Proposition 3.13 the singular values corresponding to different Lyapunov exponents are distinct for large t. In particular, the spaces $U_{i,X}(t)$ are uniquely defined for large t.

Corollary 3.15. *If equation (3.1) is regular and λ_1 is simple, then*

$$\lim_{t\to\infty} \frac{\|X(t)v\|}{\|X(t)\|} = \|v - P_{V_{2,X}} v\|$$

for $v \in \mathbb{R}^d$, where $P_{V_{2,X}}$ denotes the orthogonal projection onto $V_{2,X}$.

Proof. To prove the corollary, we use the two orthogonal decompositions $\mathbb{R}^d = U_{1,X} \oplus V_{2,X}$ and $\mathbb{R}^d = U_{1,X}(t) \oplus V_{2,X}(t)$ from Proposition 3.14 and its proof. Since $d_1 = 1$, there are unit vectors u_1 and $u_1(t)$ spanning $U_{1,X}$ and $U_{1,X}(t)$. Moreover, Proposition 3.14 implies

$$\lim_{t\to\infty} \|P_{V_{2,X}(t)} u_1\| = 0,$$

where $P_{V_{2,X}(t)}$ denotes the orthogonal projection onto $V_{2,X}(t)$. We estimate $\|X(t)u_1\| \leq \|X(t)\|$ and

$$\begin{aligned}\|X(t)u_1\| &= \|X(t)(\langle u_1, u_1(t)\rangle u_1(t) + P_{V_{2,X}(t)}u_1)\| \geq |\langle u_1, u_1(t)\rangle| \, \|X(t)\| - \|X(t)\| \, \|P_{V_{2,X}(t)}u_1\| \\ &= \left(\sqrt{1 - \|P_{V_{2,X}(t)}u_1\|^2} - \|P_{V_{2,X}(t)}u_1\|\right) \|X(t)\|.\end{aligned}$$

Thus, it holds

$$\lim_{t\to\infty} \frac{\|X(t)u_1\|}{\|X(t)\|} = 1.$$

Now, we decompose $v \in \mathbb{R}^d$ into $v = \langle v, u_1\rangle u_1 + v_2$ according to $\mathbb{R}^d = U_{1,X} \oplus V_{2,X}$. Since the system is regular and the characteristic exponents satisfy $\chi[X(t)v_2] \leq \lambda_2$ and $\chi[\|X(t)\|] = \lambda_1$, we have

$$\lim_{t\to\infty} \frac{\|X(t)v_2\|}{\|X(t)\|} = 0.$$

The claim of the corollary follows from

$$\frac{\|X(t)v\|}{\|X(t)\|} \leq |\langle v, u_1\rangle| \frac{\|X(t)u_1\|}{\|X(t)\|} + \frac{\|X(t)v_2\|}{\|X(t)\|}$$

and

$$\frac{\|X(t)v\|}{\|X(t)\|} \geq |\langle v, u_1\rangle| \frac{\|X(t)u_1\|}{\|X(t)\|} - \frac{\|X(t)v_2\|}{\|X(t)\|}.$$ □

3.3 Induced systems

Equation (3.1) induces a system on $\wedge^L\mathbb{R}^d$ via

$$\dot{\hat{x}} = \widehat{A(t)}^L \hat{x}. \tag{3.3}$$

In the following, we refer to equation (3.3) using the term *induced system*.

A fundamental matrix of the induced system can be obtained by taking the exterior power of a fundamental matrix of the original system:

Proposition 3.16. *It holds*

$$\frac{d}{dt}(\wedge^L X) = \hat{A}^L(\wedge^L X).$$

Proof. Since the determinant is multilinear, we have

$$\begin{aligned}\left\langle \frac{d}{dt}(\wedge^L X)e_I, e_J \right\rangle &= \frac{d}{dt}\langle Xe_{i_1} \wedge \cdots \wedge Xe_{i_L}, e_J\rangle = \sum_{k=1}^{L} \left\langle Xe_{i_1} \wedge \cdots \wedge \frac{d}{dt}(Xe_{i_k}) \wedge \cdots \wedge Xe_{i_L}, e_J \right\rangle \\ &= \left\langle \sum_{k=1}^{L} Xe_{i_1} \wedge \cdots \wedge AXe_{i_k} \wedge \cdots \wedge Xe_{i_L}, e_J \right\rangle = \left\langle \hat{A}^L(\wedge^L X)e_I, e_J \right\rangle.\end{aligned}$$ □

Several properties carry over from the original system to the induced system. For instance, the Lyapunov spectrum $(\lambda_{i,L}, d_{i,L})_{i=1,\ldots,p_L}$ of the induced system can be related to the spectrum of the original system if the system is regular (see also [2, Theorem 5.3.1]).

Theorem 3.17. *Let equation (3.1) be regular. The Lyapunov exponents $\lambda_{i,L}$ of the induced system are the different values given by*

$$\Lambda_{i_1} + \cdots + \Lambda_{i_L}$$

for indices $i_1 < \cdots < i_L$. The corresponding multiplicity $d_{i,L}$ is the number of combinations of indices for which $\lambda_{i,L}$ can be achieved. In particular,

$$\lambda_{1,L} = \Lambda_1 + \cdots + \Lambda_L.$$

If $L = d_1 + \cdots + d_l$, then $\lambda_{1,L}$ is simple and the second space of the Lyapunov filtration $V_{2,L}$ is given by

$$\operatorname{span}\{x_1 \wedge \cdots \wedge x_L \mid \operatorname{span}(x_1, \ldots, x_L) \cap V_{l+1} \neq \{0\}\}.$$

Remark 3.18. If equation (3.1) is not regular, the spectrum of the induced system can differ from what is described in Theorem 3.17.

Proof. In the absence of regularity, the fastest growing solutions do not necessarily span the fastest growing subspace. This can be the case if the characteristic exponents of the fastest growing solutions are only obtainable along distinct subsequences.

Indeed, one readily checks that

$$X(t) := \begin{pmatrix} e^{t\sin(\log(1+t))} & 0 & 0 \\ 0 & e^{t\cos(\log(1+t))} & 0 \\ 0 & 0 & e^{\frac{1}{2}t} \end{pmatrix}$$

is a normal fundamental matrix corresponding to a bounded, continuous system with Lyapunov exponents $\Lambda_1 = \Lambda_2 = 1$ and $\Lambda_3 = 1/2$. Since the characteristic exponents of the first two columns of $X = [x_1, x_2, x_3]$ are not sharp, the system is not regular. Moreover, one computes

$$\begin{aligned} \chi[x_1 \wedge x_2] &= \chi[e^{t(\sin(\log(1+t))+\cos(\log(1+t)))}] = \sqrt{2} \\ \chi[x_1 \wedge x_3] &= \chi[e^{t(\sin(\log(1+t))+\frac{1}{2})}] = \frac{3}{2} \\ \chi[x_2 \wedge x_3] &= \chi[e^{t(\cos(\log(1+t))+\frac{1}{2})}] = \frac{3}{2}. \end{aligned}$$

Since $x_1(t) \wedge x_3(t)$ and $x_2(t) \wedge x_3(t)$ are orthogonal for each t, any nontrivial linear combination of them has characteristic exponent $3/2$. In particular, $\Lambda_{1,2} = \Lambda_{2,2} = 3/2$ and $\Lambda_{3,2} = \sqrt{2}$. Thus, even though x_1 and x_2 have the highest characteristic exponents, their associated volume element has the lowest characteristic exponent. □

We prove Theorem 3.17 using the following lemma:

Lemma 3.19. *If equation (3.1) is regular, then the induced basis of a normal basis is normal and the induced system is regular.*

Proof. Let $x_1, \ldots, x_d$ be a normal basis such that $\chi[x_i] = \Lambda_i$. Since

$$\chi[x_{i_1} \wedge \cdots \wedge x_{i_L}] \leq \chi[x_{i_1}] + \cdots + \chi[x_{i_L}] = \Lambda_{i_1} + \cdots + \Lambda_{i_L},$$

we have

$$\sum_{i_1<\cdots<i_L} \Lambda_{i_1} + \cdots + \Lambda_{i_L} \geq \sum_{i_1<\cdots<i_L} \chi[x_{i_1} \wedge \cdots \wedge x_{i_L}] \geq \chi[\det(\wedge^L X(t))]$$
$$\geq \liminf_{t\to\infty} \frac{1}{t} \log |\det(\wedge^L X(t))| = \binom{d-1}{L-1} \liminf_{t\to\infty} \frac{1}{t} \log |\det X(t)| = \binom{d-1}{L-1} \sum_{i=1}^{d} \Lambda_i.$$

Each index i appears in precisely $\binom{d-1}{L-1}$ combinations of indices $i_1 < \cdots < i_L$. Hence, the above inequalities are actually equalities, proving that the induced basis is a normal basis and that the induced system is regular. □

The beginning of the proof of Lemma 3.19 implies the following:

Proposition 3.20. *It holds*

$$\lambda_{1,L} \leq \Lambda_1 + \cdots + \Lambda_L$$

independent of the regularity of equation (3.1).

Proof of Theorem 3.17. Let $x_1, \ldots, x_d$ be a normal basis such that $\chi[x_i] = \Lambda_i$. Since the induced basis is normal, it realizes the whole Lyapunov spectrum of the induced system. Our claims about the spectrum in Theorem 3.17 now follow from

$$\chi[x_{i_1} \wedge \cdots \wedge x_{i_L}] = \Lambda_{i_1} \wedge \cdots \wedge \Lambda_{i_L}.$$

Moreover, $V_{2,L}$ is spanned by solutions

$$x_{i_1} \wedge \cdots \wedge x_{i_L}$$

such that

$$\Lambda_{i_1} + \cdots + \Lambda_{i_L} < \Lambda_1 + \cdots + \Lambda_L.$$

If $L = d_1 + \cdots + d_l$, the latter is equivalent to $x_{i_L} \in V_{l+1}$, which proves that $V_{2,L}$ is a subset of the set defined in Theorem 3.17. On the other hand, their dimensions must coincide since the codimension of $V_{2,L}$ is $d_{1,L} = 1$ and neither set contains the solution $x_1 \wedge \cdots \wedge x_L$. □

As a direct consequence, we get Proposition 3.13.

Proof of Proposition 3.13. The previous lemma and Proposition 3.9 imply

$$\Lambda_L = (\Lambda_1 + \cdots + \Lambda_L) - (\Lambda_1 + \cdots + \Lambda_{L-1}) = \lim_{t\to\infty} \frac{1}{t} \log \| \wedge^L X(t)\| - \lim_{t\to\infty} \frac{1}{t} \log \| \wedge^{L-1} X(t)\|$$
$$= \lim_{t\to\infty} \frac{1}{t} \log \frac{\sigma_1(X(t)) \ldots \sigma_L(X(t))}{\sigma_1(X(t)) \ldots \sigma_{L-1}(X(t))} = \lim_{t\to\infty} \frac{1}{t} \log \sigma_L(X(t)).$$

□

Corollary 3.21. *If equation* (3.1) *is regular and* $L = d_1 + \cdots + d_l$, *then*

$$\lim_{t\to\infty} \frac{\|(\wedge^L X(t))\hat{v}\|}{\| \wedge^L X(t)\|} > 0$$

for every $\hat{v} \notin V_{2,L,X}$.

Proof. Since the induced system on the space of L-fold exterior products is regular and $\lambda_{1,L}$ is simple, the claim follows from Corollary 3.15. □

3.4 Stability of Lyapunov exponents

The stability of Lyapunov exponents has been widely studied. Here, we mainly state results listed in the overview from [1, Chapter V] and refer to [7] for the relation to integral separation.

Definition 3.22 ([1, Definition 5.2.1])**.** The Lyapunov exponents of equation (3.1) are called *stable* if for any $\epsilon > 0$ there exist a $\delta > 0$ such that the Lyapunov exponents of the continuously perturbed system $\dot{\tilde{x}} = (A(t) + Q(t))\tilde{x}$ with $\sup_{t\geq 0} \|Q(t)\| < \delta$ satisfy

$$|\Lambda_i - \tilde{\Lambda}_i| < \epsilon$$

for $i = 1, \dots, d$.

The stability of Lyapunov exponents can be characterized by a uniform gap between the exponential growth rates of solutions corresponding to different Lyapunov exponents (this property is called *integral separation*; see also [7]).

Theorem 3.23 ([1, Theorem 5.4.8])**.** *The Lyapunov exponents of a system with simple spectrum are stable if and only if there exists a fundamental matrix $X(t) = [x_1(t), \dots, x_d(t)]$ and constants $a, b > 0$ such that*

$$\frac{\|x_i(t)\|}{\|x_i(s)\|} \geq be^{a(t-s)} \frac{\|x_{i+1}(t)\|}{\|x_{i+1}(s)\|}$$

for all $t \geq s$.

In fact, one may use such a fundamental matrix to construct a Lyapunov transformation that reduces equation (3.1) to a diagonal system [1, Theorem 5.3.1].

The stability of Lyapunov exponents for systems with degenerate spectra can be characterized in a similar fashion by transforming the system to block diagonal form.

Theorem 3.24 ([1, Theorem 5.4.9], [3])**.** *The Lyapunov exponents of equation (3.1) are stable if and only if there exists a Lyapunov transformation reducing the system to block diagonal form*

$$\dot{y} = \operatorname{diag}(B_1(t), \dots, B_p(t))y,$$

where $B_i(t) \in \mathbb{R}^{d_i \times d_i}$ is upper triangular, such that the following hold:

(i) all non-trivial solutions of $\dot{y}_i = B_i(t)y_i$ have characteristic exponent λ_i,

(ii) λ_i is stable[6] for $\dot{y}_i = B_i(t)y_i$,

(iii) there are constants $a, b > 0$ such that

$$\|Y_i(t,s)^{-1}\|^{-1} \geq be^{a(t-s)}\|Y_{i+1}(t,s)\| \tag{3.4}$$

for all $t \geq s$, where $Y_i(t,s)$ denotes the Cauchy matrix of $\dot{y}_i = B_i(t)y_i$.

Example 3.25. Let us return to Example 3.6. The system

$$A(t) = \begin{pmatrix} 1 & 0 \\ 0 & a_{22}(t) \end{pmatrix}$$

[6] [1, Theorem 5.4.9] uses the condition $\overline{\omega}_i = \lambda_i = \Omega_i$, which is equivalent to the stability of λ_i on the subsystem (this follows from [1, Theorem 5.4.9] applied to the subsystem). The quantities $\overline{\omega}_i$ and Ω_i can be computed in terms of *lower* and *upper functions* for the subsystem (see [1, Definition 5.1.2, Definition 5.1.3 and p. 171]).

is already in block diagonal form such that the first block corresponds to $\lambda_1 = 1$ and the second block corresponds to $\lambda_2 = 0$. However, it does not satisfy (iii) of Theorem 3.24 since there are no constants $a, b > 0$ such that

$$e^{(b_n - a_n) - \int_{a_n}^{b_n} a_{22}(\tau)d\tau} = e^{\log n - \log n} = 1$$

is bounded from below by $be^{a(b_n - a_n)} = bn^a$ for all $n \in \mathbb{N}$.

Now, any Lyapunov transformation that maps to a system of the same block diagonal form satisfying (i) must be a diagonal transformation

$$L(t) = \begin{pmatrix} l_{11}(t) & 0 \\ 0 & l_{22}(t) \end{pmatrix}.$$

Since $l_{11}(t)$ and $l_{22}(t)$ are bounded from above and below by positive constants, the Lyapunov transformation does not change whether (iii) is satisfied or not. In particular, this proves that the Lyapunov exponents of the system cannot be stable.

Systems with stable Lyapunov exponents retain their spectra under perturbations that tend to zero.

Theorem 3.26 ([3])**.** *Assume equation* (3.1) *has stable Lyapunov exponents. If $Q(t)$ is a bounded, piecewise continuous perturbation such that*

$$\|Q(t)\| \to 0 \text{ as } t \to \infty,$$

then the perturbed system $\dot{\tilde{x}} = (A(t) + Q(t))\tilde{x}$ has the same Lyapunov spectrum.

While L^∞-perturbations of regular systems do not retain regularity in general, we have the following result:

Proposition 3.27. *If equation* (3.1) *is regular, has stable Lyapunov exponents, and $Q(t)$ is a bounded, piecewise continuous perturbation such that*

$$\|Q(t)\| \to 0 \text{ as } t \to \infty,$$

then the perturbed system is regular.

Proof. Theorem 3.26 implies that

$$\Lambda_1 + \cdots + \Lambda_d = \tilde{\Lambda}_1 + \cdots + \tilde{\Lambda}_d.$$

Moreover, due to the Liouville–Ostrogradski formula, we have

$$\det \tilde{X}(t) = \det \tilde{X}(0)\, e^{\int_0^t \operatorname{tr}(A(\tau)+Q(\tau))d\tau} = \det(\tilde{X}(0)X(0)^{-1}) \det X(t)\, e^{\int_0^t \operatorname{tr}(Q(\tau))d\tau}.$$

Since

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \operatorname{tr}(Q(\tau))d\tau = 0,$$

regularity of the original system implies regularity of the perturbed system:

$$\begin{aligned} \Lambda_1 + \cdots + \Lambda_d = \tilde{\Lambda}_1 + \cdots + \tilde{\Lambda}_d &\geq \liminf_{t\to\infty} \frac{1}{t} \log |\det \tilde{X}(t)| = \liminf_{t\to\infty} \frac{1}{t} \log |\det X(t)| \\ &= \Lambda_1 + \cdots + \Lambda_d. \end{aligned}$$

□

3.5 Strong fast invertibility

In this subsection, we introduce *strong fast invertibility*, which is a weaker concept than the stability of Lyapunov exponents but still sufficient for their computation. Our main objectives are a characterization result that allows us to compare strong fast invertibility to the stability of Lyapunov exponents and perturbation results in preparation for the analysis of Benettin's algorithm.

3.5.1 Definition and relation to induced systems

Quas et al. introduce three notions of fast invertibility (weak/standard/strong) to analyze the existence of a dominated splitting for discrete-time systems [21]. Here, we focus solely on the strong version, since weak and standard fast invertibility are trivially satisfied for systems with bounded coefficients.

Definition 3.28 ([21])**.** Equation (3.1) is said to be *L-dimensionally strongly fast invertible*[7] if

$$c_{FI,L} := \sup_{t\geq s\geq \tau} \prod_{k=1}^{L} \frac{\sigma_k(X(t,s))\sigma_k(X(s,\tau))}{\sigma_k(X(t,\tau))} < \infty.$$

There are several equivalent formulations immediately visible from the definition.

Lemma 3.29. *The following are equivalent:*

(i) Equation (3.1) is L-dim. strongly fast invertible.

(ii) The induced system on $\wedge^L\mathbb{R}^d$ is 1-dim. strongly fast invertible.

(iii) There is a constant $c > 0$ such that

$$\frac{\| \wedge^L X(t,\tau)\|}{\| \wedge^L X(s,\tau)\|} \leq \| \wedge^L X(t,s)\| \leq c\frac{\| \wedge^L X(t,\tau)\|}{\| \wedge^L X(s,\tau)\|}$$

for all $t \geq s \geq \tau$.

(iv) $\sup_{t\geq s} \prod_{k=1}^{L} \frac{\sigma_k(X(t,s))\sigma_k(X(s))}{\sigma_k(X(t))} < \infty$ for any fundamental matrix $X(t)$.

(v) $\sup_{t\geq s\geq\tau\geq T} \prod_{k=1}^{L} \frac{\sigma_k(X(t,s))\sigma_k(X(s,\tau))}{\sigma_k(X(t,\tau))} < \infty$ for any $T > 0$.

Proof. The relation between strong fast invertibility and exterior products follows from Lemma 2.2. The third characterization is merely a reformulation of strong fast invertibility for the induced system. Equivalence of strong fast invertibility to the fourth characterization follows from

$$\frac{\| \wedge^L X(t,s)\| \| \wedge^L X(s)\|}{\| \wedge^L X(t)\|} \leq \| \wedge^L X(0)\| \| \wedge^L X(0)^{-1}\| \frac{\| \wedge^L X(t,s)\| \| \wedge^L X(s,0)\|}{\| \wedge^L X(t,0)\|}$$

and

$$\begin{aligned}
&\frac{\| \wedge^L X(t,s)\| \| \wedge^L X(s,\tau)\|}{\| \wedge^L X(t,\tau)\|} \\
&\quad = \frac{\| \wedge^L X(t,s)\| \| \wedge^L X(s)\|}{\| \wedge^L X(t)\|} \cdot \frac{\| \wedge^L X(s,\tau)\| \| \wedge^L X(\tau)\|}{\| \wedge^L X(s)\|} \cdot \frac{\| \wedge^L X(t)\|}{\| \wedge^L X(t,\tau)\| \| \wedge^L X(\tau)\|} \\
&\quad \leq \frac{\| \wedge^L X(t,s)\| \| \wedge^L X(s)\|}{\| \wedge^L X(t)\|} \cdot \frac{\| \wedge^L X(s,\tau)\| \| \wedge^L X(\tau)\|}{\| \wedge^L X(s)\|}.
\end{aligned}$$

[7] The name "fast invertibility" stems from the context of [21] as it ensures that the cocylce is uniformly invertible on the fastest subspace of the respective dimension (without assuming bounded coefficients).

Finally, using

$$\wedge^L X(T+t,t) \wedge^L X(t,s) = \wedge^L X(T+t,T+s) \wedge^L X(T+s,s),$$

and $\sup_t \|\hat{A}^L(t)\| \leq LM$, one may show the final characterization:

$$\frac{\|\wedge^L X(t,s)\| \, \|\wedge^L X(s,\tau)\|}{\|\wedge^L X(t,\tau)\|} \leq e^{6TLM} \frac{\|\wedge^L X(T+t,T+s)\| \, \|\wedge^L X(T+s,T+\tau)\|}{\|\wedge^L X(T+t,T+\tau)\|}. \qquad \square$$

Maybe the most intuitive of the above characterizations of strong fast invertibility is the third. It means that the maximal growth of L-volumes along any interval can be computed (up to a constant independent of the interval) by bisecting the interval and multiplying the maximal growths of L-volumes on the subintervals.

Proposition 3.30. *Every system is d-dim. strongly fast invertible.*

Proof. Since $\dim(\wedge^d \mathbb{R}^d) = 1$, we have $c_{FI,d} = 1$. $\square$

Example 3.31. The system from Example 3.6 is strongly fast invertible at dimensions 1 and 2: Due to the previous proposition, we only need to consider the first dimension. We compute the Cauchy matrix

$$X(t,s) := \begin{pmatrix} e^{t-s} & 0 \\ 0 & e^{\int_s^t a_{22}(\tau)\,d\tau} \end{pmatrix}.$$

Since $0 \leq a_{22}(t) \leq 1$, the first diagonal entry of $X(t,s)$ is always larger than the second for $t \geq s$. In particular, it holds

$$\sigma_1(X(t,s)) = e^{t-s},$$

which implies 1-dim. strong fast invertibility.

3.5.2 Characterization and comparison to stability of Lyapunov exponents

To derive the characterization theorem mentioned in the introduction, we first prove that regularity and strong fast invertibility imply the existence of a Lyapunov transformation that brings the system into block diagonal form.

Proposition 3.32. *Assume equation (3.1) has Lyapunov exponents $\lambda_1 > \cdots > \lambda_p$ with multiplicities $d_1 + \cdots + d_p = d$. If the system is regular and strongly fast invertible at dim. $d_1 + \cdots + d_l$ for $l = 1, \ldots, p$, then there is a Lyapunov transformation reducing the system to block diagonal form:*

$$\dot{y} = \operatorname{diag}(B_1(t), \ldots, B_p(t))y,$$

where $B_i \in \mathbb{R}^{d_i \times d_i}$ is upper triangular, such that all non-trivial solutions of $\dot{y}_i = B_i y_i$ have characteristic exponent λ_i.

We prove the proposition via induced systems using two auxiliary lemmata and the following result:

Theorem 3.33 ([1, Theorems 3.3.3 and 3.3.4])**.** *There exists a Lyapunov transformation reducing equation (3.1) to block diagonal form:*

$$\dot{y} = \operatorname{diag}(B_1(t), \ldots, B_k(t))y,$$

where $B_i \in \mathbb{R}^{n_i \times n_i}$ is upper triangular, if and only if there is a fundamental matrix

$$X = [X_1, \dots, X_k]$$

with $X_i \in \mathbb{R}^{d \times n_i}$ satisfying

$$\inf_t \frac{G(X)}{G(X_1) \dots G(X_k)} > 0, \tag{3.5}$$

where G denotes the Gram determinant.

Remark 3.34. Given a fundamental matrix $X(t)$ as in Theorem 3.33, the Lyapunov transformation is constructed by applying the Gram–Schmidt procedure to each block $X_i(t)$ individually, yielding

$$X(t) = [Q_1(t), \dots, Q_k(t)] \operatorname{diag}(R_1(t), \dots, R_k(t)) =: L(t)Y(t).$$

Lemma 3.35 ([1, Remark 3.3.4])**.** *Let $X = [X_1, X_2]$. It holds*

$$\frac{G(X)}{G(X_1)G(X_2)} = \prod_{i=1}^{l} \sin^2 \alpha_i(\operatorname{im} X_1, \operatorname{im} X_2),$$

where $\alpha_1 \le \dots \le \alpha_l$ are the principle angles.

Lemma 3.36. *If equation (3.1) is 1-dim. strongly fast invertible and $X = [X_1, X_2]$ is a fundamental matrix such that $X_1 \in \mathbb{R}^{d \times 1}$ has a higher characteristic exponent than the columns of $X_2 \in \mathbb{R}^{d \times (d-1)}$, then*

$$\inf_t \sin^2 \alpha_1(\operatorname{im} X_1(t), \operatorname{im} X_2(t)) > 0. \tag{3.6}$$

Proof. Let $X = [X_1, X_2]$ be as in the claim. We first apply the Gram–Schmidt procedure to bring X into upper triangular form. Indeed, the Gram–Schmidt procedure is an orthogonal Lyapunov transformation (Proposition 3.2). Thus, it leaves the fraction in equation (3.5) and hence also $\sin^2 \alpha_1(\operatorname{im} X_1, \operatorname{im} X_2)$ invariant. So, it is sufficient to check equation (3.6) for

$$Y = [Y_1, Y_2] = \begin{pmatrix} y_{11} & Y_{12} \\ 0 & Y_{22} \end{pmatrix}$$

in which y_{11} has a higher characteristic exponent than $\|Y_{12}\|$ and $\|Y_{22}\|$.

Given $s \ge 0$, we always find $t_s \ge s$ such that

$$\max_{\alpha \neq 0} \frac{\left| \frac{y_{11}(s)}{y_{11}(t_s)} Y_{12}(t_s)\alpha \right|}{\|Y_{22}(s)\alpha\|} \le 1$$

and $\|Y(t_s)\| \le 2|y_{11}(t_s)|$. In order to make use of strong fast invertibility, we compute the Cauchy matrix:

$$Y(t,s) = \begin{pmatrix} y_{11}(t,s) & Y_{12}(t,s) \\ 0 & Y_{22}(t,s) \end{pmatrix} = \begin{pmatrix} \frac{y_{11}(t)}{y_{11}(s)} & Y_{12}(t)Y_{22}(s)^{-1} - \frac{y_{11}(t)}{y_{11}(s)} Y_{12}(s)Y_{22}(s)^{-1} \\ 0 & Y_{22}(t)Y_{22}(s)^{-1} \end{pmatrix}.$$

Now, for $s \ge 0$ and $\alpha \neq 0$ it holds

$$\begin{aligned} \frac{|Y_{12}(s)\alpha|}{\|Y_{22}(s)\alpha\|} &\le 1 + \left| \frac{\frac{y_{11}(s)}{y_{11}(t_s)} Y_{12}(t_s)\alpha}{\|Y_{22}(s)\alpha\|} - \frac{Y_{12}(s)\alpha}{\|Y_{22}(s)\alpha\|} \right| = 1 + \frac{\left| \frac{y_{11}(s)}{y_{11}(t_s)} Y_{12}(t_s, s)Y_{22}(s)\alpha \right|}{\|Y_{22}(s)\alpha\|} \\ &\le 1 + \frac{\|Y_{12}(t_s, s)\| \, |y_{11}(s)|}{|y_{11}(t_s)|} \le 1 + 2\frac{\|Y(t_s, s)\| \, \|Y(s)\|}{\|Y(t_s)\|} \le \underbrace{1 + 2c_{FI,L} \, \|Y(0)\| \, \|Y(0)^{-1}\|}_{=:c}. \end{aligned}$$

Since

$$\left(\frac{\langle Y_1(t), Y_2(t)\alpha\rangle}{\|Y_1(t)\|\ \|Y_2(t)\alpha\|}\right)^2 = \left(\frac{\langle e_1, Y_2(t)\alpha\rangle}{\|Y_2(t)\alpha\|}\right)^2 = \frac{|Y_{12}(t)\alpha|^2}{|Y_{12}(t)\alpha|^2 + \|Y_{22}(t)\alpha\|^2} = \frac{1}{1+\left(\frac{|Y_{12}(t)\alpha|}{\|Y_{22}(t)\alpha\|}\right)^{-2}}$$
$$\leq \frac{1}{1+c^{-2}},$$

we have

$$\sin^2\alpha_1(\operatorname{im} Y_1, \operatorname{im} Y_2) = 1 - \cos^2\alpha_1(\operatorname{im} Y_1, \operatorname{im} Y_2) = 1 - \max_{\alpha\neq 0}\left(\frac{\langle Y_1(t), Y_2(t)\alpha\rangle}{\|Y_1(t)\|\ \|Y_2(t)\alpha\|}\right)^2$$
$$\geq 1 - \frac{1}{1+c^{-2}} > 0. \qquad \square$$

Proof of Proposition 3.32. Let $X(t) = [X_1, \dots, X_p]$ be an ordered normal fundamental matrix so that the columns of X_i have characteristic exponent λ_i. We first show that equation (3.5) holds with $n_1 = L$ and $n_2 = d - L$ if the system is strongly fast invertible at dim. $L = d_1 + \cdots + d_l$ and then apply Theorem 3.33 successively to arrive at the desired form.

Fix $L = d_1 + \cdots + d_l$ and set

$$U := \operatorname{im}[X_1, \dots, X_l] \quad \text{and} \quad V := \operatorname{im}[X_{l+1}, \dots, X_p].$$

Lemma 3.35 and Corollary 2.5 imply

$$\frac{G(X)}{G([X_1, \dots, X_l])G([X_{l+1}, \dots, X_p])} = \prod_{i=1}^{\min(L,d-L)} \sin^2\alpha_i(U,V) = \sin^2\alpha_1(\hat{U}, \hat{V}),$$

where

$$\hat{U} = \operatorname{span}\{(\wedge^L X)(e_1 \wedge \cdots \wedge e_L)\}$$

and

$$\hat{V} = \operatorname{span}\{(\wedge^L X)(e_{i_1} \wedge \cdots \wedge e_{i_L}) \mid \{i_1, \dots, i_L\} \neq \{1, \dots, L\}\}.$$

Now, let $Y = [Y_1, Y_2]$ be a matrix representation of $\wedge^L X$ with respect to the standard basis on $\wedge^L\mathbb{R}^d$. Order the induced basis starting with $e_1 \wedge \cdots \wedge e_L$ so that the first column Y_1 of Y represents $(\wedge^L X)(e_1 \wedge \cdots \wedge e_L)$. Then,

$$\sin^2\alpha_1(\hat{U}, \hat{V}) = \sin^2\alpha_1(\operatorname{im} Y_1, \operatorname{im} Y_2).$$

Due to regularity, Y_1 has characteristic exponent $\Lambda_1 + \cdots + \Lambda_L$ and the columns of Y_2 have lower characteristic exponents. Hence, Lemma 3.36 and the above arguments imply

$$\inf_t \frac{G(X)}{G([X_1, \dots, X_l])G([X_{l+1}, \dots, X_p])} > 0.$$

To arrive at the block diagonal form claimed in the proposition, we apply the transformation from Remark 3.34 successively. First, we use strong fast invertibility at dim. d_1 to make the first d_1 columns of X orthogonal to the rest. Assuming orthogonality of the first d_1 columns, strong fast invertibility at dim. $d_1 + d_2$ implies

$$0 < \inf_t \frac{G(X)}{G([X_{d_1}, X_{d_2}])G([X_{d_3}, \dots, X_{d_p}])} = \inf_t \frac{G(X)}{G(X_{d_1})G(X_{d_2})G([X_{d_3}, \dots, X_{d_p}])}.$$

Thus, we may apply another Lyapunov transformation to decouple the next subsystem and so on. $\square$

Using Proposition 3.32, we now may prove our characterization result for strong fast invertibility.

Theorem 3.37. *Assume equation (3.1) has Lyapunov exponents $\lambda_1 > \cdots > \lambda_p$ with multiplicities $d_1 + \cdots + d_p = d$. If the system is regular and strongly fast invertible at dim. $d_1 + \cdots + d_l$ for $l = 1, \dots, p$, then there exists a Lyapunov transformation reducing the system to block diagonal form*

$$\dot{y} = \operatorname{diag}(B_1(t), \dots, B_p(t))y,$$

where $B_i(t) \in \mathbb{R}^{d_i \times d_i}$ is upper triangular, such that the following hold:

(i) all non-trivial solutions of $\dot{y}_i = B_i(t)y_i$ have characteristic exponent λ_i,

(ii) there is a constant $b > 0$ such that

$$\|Y_i(t,s)^{-1}\|^{-1} \geq b\|Y_{i+1}(t,s)\| \tag{3.7}$$

for all $t \geq s$, where $Y_i(t,s)$ denotes the Cauchy matrix of $\dot{y}_i = B_i(t)y_i$.

Conversely, any block diagonal system $\dot{y} = B(t)y$ satisfying (i) and (ii) is strongly fast invertible at dim. $d_1 + \cdots + d_l$ for $l = 1, \dots, p$.

Proof. Let us first assume that the system is regular and strongly fast invertible at the respective dimensions. After applying Proposition 3.32, all that remains is to show that equation (3.7) is satisfied for the reduced system. To this end, fix $L = d_1 + \cdots + d_l$ and set $Y = Z_1 + Z_2$ with

$$Z_1 := \operatorname{diag}(Y_1, \dots, Y_L, 0, \dots, 0)$$

and

$$Z_2 := \operatorname{diag}(0, \dots, 0, Y_{L+1}, \dots, Y_p).$$

Due to Lemma 2.3, we have

$$\| \wedge^L Y\| = \max_k \|(\wedge^k Z_1) \wedge (\wedge^{L-k} Z_2)\|$$

and

$$\| \wedge^L Y\| \geq \| \wedge^{L-1} Z_1\| \, \|Z_2\| = \frac{\| \wedge^L Z_1\|}{\sigma_L(Z_1)}\|Z_2\|.$$

The singular value $\sigma_L(Z_1)$ can be estimated via

$$\sigma_L(Z_1) = \sigma_L(\operatorname{diag}(Y_1, \dots, Y_l)) = \| \operatorname{diag}(Y_1^{-1}, \dots, Y_l^{-1})\|^{-1} \leq \|Y_l^{-1}\|^{-1}.$$

Due to regularity, we have

$$\chi[\| \wedge^L Z_1\|] = \lambda_{1,L} > \lambda_{2,L} \geq \chi[\|(\wedge^k Z_1) \wedge (\wedge^{L-k} Z_2)\|]$$

for $k < L$, which implies

$$\| \wedge^L Y(t)\| = \| \wedge^L Z_1(t)\|$$

for t large enough. Moreover, it holds

$$\| \wedge^L Z_1(t,s)\| = \| \wedge^L \operatorname{diag}(Y_1(t,s), \dots, Y_l(t,s))\| = \frac{\| \wedge^L \operatorname{diag}(Y_1(t), \dots, Y_l(t))\|}{\| \wedge^L \operatorname{diag}(Y_1(s), \dots, Y_l(s))\|} = \frac{\| \wedge^L Z_1(t)\|}{\| \wedge^L Z_1(s)\|}$$

since $\wedge^L \mathbb{R}^L$ is one-dimensional. Combining the previous estimates, we get

$$\begin{aligned}\|Y_l(t,s)^{-1}\|^{-1} &\geq \frac{\|\wedge^L Z_1(t,s)\|}{\|\wedge^L Y(t,s)\|}\|Z_2(t,s)\| \geq \frac{\|\wedge^L Z_1(t,s)\|}{\|\wedge^L Y(t,s)\|}\|Y_{l+1}(t,s)\| \\ &= \frac{\|\wedge^L Z_1(t)\|}{\|\wedge^L Y(t,s)\|\,\|\wedge^L Z_1(s)\|}\|Y_{l+1}(t,s)\| \geq \frac{\|\wedge^L Y(t)\|}{\|\wedge^L Y(t,s)\|\,\|\wedge^L Y(s)\|}\|Y_{l+1}(t,s)\| \\ &\geq \frac{1}{c_{FI,L}\|\wedge^L Y(0)\|\,\|\wedge^L Y(0)^{-1}\|}\|Y_{l+1}(t,s)\|\end{aligned}$$

for large $t > 0$, which proves equation (3.7).

Next, we show that any block diagonal system as in Theorem 3.37 is strongly fast invertible at the respective dimensions. As before, fix $L = d_1 + \cdots + d_l$ and assume the decomposition $Y = Z_1 + Z_2$. Applying equation (3.7) repeatedly yields

$$\|Y_i^{-1}\|^{-1} \geq b\|Y_{i+1}\| \geq b\|Y_{i+1}^{-1}\|^{-1} \geq \cdots \geq b^k\|Y_{i+k}\|$$

and hence

$$\begin{aligned}\sigma_L(Z_1) = \sigma_L(\operatorname{diag}(Y_1,\dots,Y_l)) &= \|\operatorname{diag}(Y_1^{-1},\dots,Y_l^{-1})\|^{-1} = \min_{i=1,\dots,l}\|Y_i^{-1}\|^{-1} \\ &\geq b^{p-1}\max_{i=l+1,\dots,p}\|Y_i\| = b^{p-1}\|Z_2\|\end{aligned}$$

since $b \in (0,1]$ (set $s = t$ in equation (3.7)). Now, we estimate

$$\begin{aligned}\|\wedge^L Y\| &= \max_k \|(\wedge^k Z_1) \wedge (\wedge^{L-k} Z_2)\| \leq \max_k \binom{d}{L}^{\frac{1}{2}} \|\wedge^k Z_1\|\,\|\wedge^{L-k} Z_2\| \\ &\leq \max_k \binom{d}{L}^{\frac{1}{2}} \|\wedge^k Z_1\|\,\|Z_2\|^{L-k} \leq \max_k \binom{d}{L}^{\frac{1}{2}} \|\wedge^k Z_1\| b^{-(p-1)(L-k)}\sigma_L(Z_1)^{L-k} \\ &\leq \underbrace{\max_k \left(\binom{d}{L}^{\frac{1}{2}} b^{-(p-1)(L-k)}\right)}_{=:c} \|\wedge^L Z_1\|.\end{aligned}$$

Hence, we get

$$\sup_{t\geq s}\frac{\|\wedge^L Y(t,s)\|\,\|\wedge^L Y(s)\|}{\|\wedge^L Y(t)\|} \leq c^2 \underbrace{\sup_{t\geq s}\frac{\|\wedge^L Z_1(t,s)\|\,\|\wedge^L Z_1(s)\|}{\|\wedge^L Z_1(t)\|}}_{=1} < \infty. \qquad \square$$

Remark 3.38. When testing for strong fast invertibility (resp. for stability of Lyapunov exponents), it is enough to check the conditions of Theorem 3.37 (resp. Theorem 3.24) after transforming the original system to any block diagonal system of the form

$$\dot{y} = \operatorname{diag}(B_1(t),\dots,B_p(t))y$$

such that the non-trivial solutions of block $B_i(t) \in \mathbb{R}^{d_i\times d_i}$ have characteristic exponent λ_i.

Proof. Any Lyapunov transformation between systems of this form acts as Lyapunov transformations on the individual blocks, i.e., $L(t) = \operatorname{diag}(L_1(t),\dots,L_p(t))$. This implies that equation (3.7) (resp. stability of the Lyapunov exponents of the subsystems and equation (3.4)) holds for one such system if and only if it holds for all of them. $\square$

An immediate consequence of Theorem 3.37 is that stability of Lyapunov exponents implies strong fast invertibility.

Corollary 3.39. *Assume equation (3.1) has Lyapunov exponents $\lambda_1 > \cdots > \lambda_p$ with multiplicities $d_1 + \cdots + d_p = d$. If the Lyapunov exponents are stable, then the system is strongly fast invertible at dim. $d_1 + \cdots + d_l$ for $l = 1, \dots, p$.*

Remark 3.40. Example 3.6 shows that the converse of Corollary 3.39 is false in general (even with the help of regularity). There are regular systems that are strongly fast invertible at every dimension that do not have stable Lyapunov exponents.

In case the Lyapunov spectrum is simple, strong fast invertibility implies the existence of a fundamental system of solutions with uniformly separated growth.

Theorem 3.41. *If equation (3.1) is regular, has simple Lyapunov spectrum and is strongly fast invertible at every dimension, then there exists a fundamental matrix $X(t) = [x_1(t), \dots, x_d(t)]$ and a constant $b > 0$ such that*

$$\frac{\|x_i(t)\|}{\|x_i(s)\|} \geq b \frac{\|x_{i+1}(t)\|}{\|x_{i+1}(s)\|} \tag{3.8}$$

for all $t \geq s$.

Proof. According to Theorem 3.37 there is a Lyapunov transformation $L(t)$ bringing the system to diagonal form with fundamental matrix $Y = \operatorname{diag}(y_{11}, \dots, y_{dd})$ and a constant $b' > 0$ such that

$$\frac{|y_{ii}(t)|}{|y_{ii}(s)|} \geq b' \frac{|y_{(i+1)(i+1)}(t)|}{|y_{(i+1)(i+1)}(s)|}$$

for all $t \geq s$. The fundamental matrix $X = LY$ of the original system satisfies

$$\|x_i(t)\| = \|L(t)e_i\| \, |y_{ii}(t)|.$$

Since $L(t)$ is a Lyapunov transformation, the term $\|L(t)e_i\|$ is uniformly bounded from below and above by positive constants. equation (3.8) follows. □

Remark 3.42. There is a regular system with simple Lyapunov spectrum satisfying equation (3.8) that is not strongly fast invertible at every dimension.

Proof. Let $0 < \epsilon_n \leq 1/e$ be a sequence of numbers converging to zero. We set

$$c_n := \left(\frac{9}{4}\right)^n \prod_{k=1}^{n} \left(\frac{1}{\epsilon_k}\right)^{\frac{1}{\epsilon_k}}.$$

The numbers

$$\begin{aligned}
t_{n,0} &:= c_n, \\
t_{n,1} &:= c_n + \frac{2}{3(1-\epsilon_n)}, \\
t_{n,2} &:= c_n + \frac{2}{3(1-\epsilon_n)} + \frac{\log\left(\frac{1}{\epsilon_n}\right)}{1-\epsilon_n} - \frac{2}{3}, \\
t_{n,3} &:= c_n + \frac{2}{3(1-\epsilon_n)} + \frac{\log\left(\frac{1}{\epsilon_n}\right)}{1-\epsilon_n},
\end{aligned}$$

$$t_{n,4} := c_n + \frac{2}{3(1-\epsilon_n)} + \frac{\log\left(\frac{1}{\epsilon_n}\right)}{\epsilon_n(1-\epsilon_n)},$$
$$t_{n,5} := c_n + \frac{2}{3(1-\epsilon_n)} + \frac{\log\left(\frac{1}{\epsilon_n}\right)}{\epsilon_n(1-\epsilon_n)} + 1,$$
$$t_{n,6} := c_n + \frac{5}{3(1-\epsilon_n)} + \frac{\log\left(\frac{1}{\epsilon_n}\right)}{\epsilon_n(1-\epsilon_n)},$$

satisfy

$$t_{n,0} \leq \cdots \leq t_{n,6} \leq t_{n+1,0}.$$

On the partition introduced by these numbers, we define functions

$$f(t) := \begin{cases} 1, & t \in [0, t_{1,0}) \\ c_{n-1} p_{n,f}(t - t_{n,0}), & t \in [t_{n,0}, t_{n,1}) \\ c_{n-1} \frac{3}{2} e^{(1-\epsilon_n)(t-t_{n,1})}, & t \in [t_{n,1}, t_{n,4}) \\ c_{n-1} \frac{3}{2} \left(\frac{1}{\epsilon_n}\right)^{\frac{1}{\epsilon_n}} q_{n,f}(t - t_{n,4}), & t \in [t_{n,4}, t_{n,6}) \\ c_n, & t \in [t_{n,6}, t_{n+1,0}) \end{cases}$$

and

$$g(t) := \begin{cases} 1, & t \in [0, t_{1,2}) \\ c_{n-1} p_{n,g}(t - t_{n,2}), & t \in [t_{n,2}, t_{n,3}) \\ c_{n-1} \frac{3}{2} e^{t-t_{n,3}}, & t \in [t_{n,3}, t_{n,4}) \\ c_{n-1} \frac{3}{2} \left(\frac{1}{\epsilon_n}\right)^{\frac{1}{\epsilon_n}} q_{n,g}(t - t_{n,4}), & t \in [t_{n,4}, t_{n,5}) \\ c_n, & t \in [t_{n,5}, t_{n+1,2}) \end{cases}$$

using the polynomials

$$p_{n,f}(t) := \frac{9}{8}(1-\epsilon_n)^2 t^2 + 1,$$
$$p_{n,g}(t) := \frac{9}{8} t^2 + 1,$$
$$q_{n,f}(t) := -\frac{1}{2}(1-\epsilon_n)^2 t^2 + (1-\epsilon_n)t + 1,$$
$$q_{n,g}(t) := -\frac{1}{2} t^2 + t + 1,$$

for smoothing. The functions have the following properties:

- $f, g \in C^1(\mathbb{R}_{\geq 0}, \mathbb{R})$,
- f, g are monotonically increasing,
- $|f' - f|/|g|$ bounded,
- $|g'|/|g|$ bounded,
- $1 \leq f(t), g(t) \leq t$ for $t \geq 1$.

In particular,

$$\dot{x} = A(t)x := \begin{pmatrix} 1 & \frac{f'-f}{g} \\ 0 & \frac{g'}{g} \end{pmatrix} x$$

is a continuous, bounded system with fundamental matrix

$$X(t) = [x_1(t), x_2(t)] = \begin{pmatrix} e^t & f \\ 0 & g \end{pmatrix}.$$

The solution $x_2(t)$ has characteristic exponent zero and the system is regular.

To show property equation (3.8), we first remark that

$$\frac{|p'_{n,f}|}{|p_{n,f}|}, \frac{|q'_{n,f}|}{|q_{n,f}|} \leq 1$$

on the respective intervals used in the definition of f. Since

$$(\log f(t))' = \frac{f'(t)}{f(t)} \leq 1,$$

the mean value theorem implies

$$\frac{f(t)}{f(s)} \leq e^{t-s}$$

for $t \geq s$. In combination with $g(t) \leq \frac{3}{2} f(t)$, we get equation (3.8):

$$\frac{\|x_2(t)\|^2}{\|x_2(s)\|^2} = \frac{f^2(t) + g^2(t)}{f^2(s) + g^2(s)} \leq \frac{13}{4} \frac{f^2(t)}{f^2(s)} \leq \frac{13}{4} \frac{\|x_1(t)\|^2}{\|x_1(s)\|^2}$$

for $t \geq s$.

To show that the system is not 1-dim. strongly fast invertible, we prove that there is no Lyapunov transformation bringing the system into diagonal form. For our particular fundamental matrix, it holds

$$\left(\frac{\langle x_1, x_2 \rangle}{\|x_1\| \, \|x_2\|} \right)^2 = \frac{1}{1 + \left(\frac{f}{g}\right)^{-2}}.$$

Since $f(t_{n,3})/g(t_{n,3}) = 1/\epsilon_n \to \infty$, the above expression converges to 1 along the subsequence $(t_{n,3})_n$. Now, let $Y = [y_1, y_2]$ be an arbitrary fundamental matrix. By expanding y_1 and y_2 with respect to our particular solutions and by using

$$\lim_{t \to \infty} \frac{\|x_2(t)\|}{\|x_1(t)\|} = 0,$$

one computes

$$\lim_{n \to \infty} \left(\frac{\langle y_1(t_{n,3}), y_2(t_{n,3}) \rangle}{\|y_1(t_{n,3})\| \, \|y_2(t_{n,3})\|} \right)^2 = 1.$$

Thus,

$$\inf_t \frac{G(Y)}{G(y_1)G(y_2)} = 1 - \sup_t \left(\frac{\langle y_1, y_2 \rangle}{\|y_1\| \, \|y_2\|} \right)^2 = 0,$$

and Theorem 3.33 implies that there is no Lyapunov transformation bringing the system into diagonal form. In particular, the system cannot be 1-dim. strongly fast invertible according to Theorem 3.37. □

3.5.3 Perturbation theory

We now establish several perturbation results for strongly fast invertible systems. Since we plan to use them to derive convergence theorems for Benettin's algorithm and since numerical integration can be represented as a piecewise continuous dynamical system, the perturbation results are formulated for systems

$$\dot{x} = A(t)x \tag{3.9}$$

such that A is bounded and piecewise continuous. Moreover, we assume all perturbations in this subsection to be of the same class, i.e., bounded and piecewise continuous.

While strong fast invertibility is weaker than the stability of Lyapunov exponents, it is enough to ensure upper semicontinuity of $\lambda_{1,L}$.

Theorem 3.43. *Assume equation (3.9) is L-dim. strongly fast invertible. For any perturbation $Q(t)$, the largest Lyapunov exponent of the induced system satisfies*

$$\tilde{\lambda}_{1,L} \leq \lambda_{1,L} + Lc_{FI,L} \limsup_{t\to\infty} \frac{1}{t}\int_0^t \|Q(\tau)\|\,d\tau \leq \lambda_{1,L} + Lc_{FI,L}\|Q\|_{L^\infty}$$

Remark 3.44. Remember that

$$\lambda_{1,L} = \Lambda_1 + \cdots + \Lambda_L$$

for regular systems.

Proof. Let $\tilde{X}(t)$ be a fundamental matrix of the perturbed system. $\wedge^L\tilde{X}$ solves the induced equation

$$\frac{d}{dt}(\wedge^L\tilde{X}) = \widehat{A+Q}^L(\wedge^L\tilde{X}) = \hat{A}^L(\wedge^L X) + \hat{Q}^L(\wedge^L X)$$

and can be seen as a solution of a perturbed system to

$$\frac{d}{dt}(\wedge^L X) = \hat{A}^L(\wedge^L X).$$

The variation of constants formula implies

$$\wedge^L\tilde{X}(t,s) = \wedge^L X(t,s) + \int_s^t (\wedge^L X(t,\tau))\hat{Q}^L(\tau)(\wedge^L\tilde{X}(\tau,s))\,d\tau.$$

Using $\|\hat{Q}^L(t)\| \leq L\|Q(t)\|$, we first estimate

$$\begin{aligned}\frac{\|\wedge^L\tilde{X}(t,s)\|}{\|\wedge^L X(t,s)\|} &\leq 1 + L\int_s^t \frac{\|\wedge^L X(t,\tau)\|\,\|\wedge^L X(\tau,s)\|}{\|\wedge^L X(t,s)\|}\|Q(\tau)\|\frac{\|\wedge^L\tilde{X}(\tau,s)\|}{\|\wedge^L X(\tau,s)\|}\,d\tau\\ &\leq 1 + Lc_{FI,L}\int_s^t \|Q(\tau)\|\frac{\|\wedge^L\tilde{X}(\tau,s)\|}{\|\wedge^L X(\tau,s)\|}\,d\tau\end{aligned}$$

and then apply Grönwall's inequality to get

$$\frac{\|\wedge^L\tilde{X}(t,s)\|}{\|\wedge^L X(t,s)\|} \leq e^{Lc_{FI,L}\int_s^t\|Q(\tau)\|\,d\tau}.$$

The claim now follows from Proposition 3.9. □

Lemma 3.45. *Perturbations on finite intervals do not change the Lyapunov spectrum of a system or its induced systems.*

Proof. Since characteristic exponents are asymptotic quantities and since one may choose fundamental matrices $X(t)$ and $\tilde{X}(t)$ that coincide for large t, the claim follows. □

Theorem 3.46. *Assume equation (3.9) is L-dim. strongly fast invertible. If $Q(t)$ is a perturbation such that*

$$\|Q(t)\| \to 0 \text{ as } t \to \infty,$$

then

$$\tilde{\lambda}_{1,L} \leq \lambda_{1,L}.$$

Proof. The theorem is a direct consequence of Theorem 3.43. □

Equality in Theorem 3.46 can be achieved for L^1-perturbations.

Theorem 3.47. *Assume equation (3.9) is L-dim. strongly fast invertible. If $Q(t)$ is a perturbation such that*

$$\int_0^\infty \|Q(t)\|\, dt < \infty,$$

then

$$\tilde{\lambda}_{1,L} = \lambda_{1,L}.$$

The theorem follows directly from Theorem 3.46 using the following lemma.

Lemma 3.48. *Assume equation (3.9) is L-dim. strongly fast invertible. If $Q(t)$ is a perturbation such that*

$$\int_0^\infty \|Q(t)\|\, dt < \infty,$$

then the perturbed system is L-dim. strongly fast invertible. Moreover, $c_{FI,L}$ is continuous with respect to L^1-perturbations.

Proof. According to Lemma 3.29 strong fast invertibility can be tested on $[T, \infty)$ for any fixed $T > 0$. Thus, by setting $Q(t)$ to zero on a finite interval, we may assume that

$$\int_0^\infty \|Q(t)\|\, dt < \frac{\log(2)}{L c_{FI,L}} \tag{3.10}$$

without affecting whether the perturbed system is strongly fast invertible or not. The variation of constants formula implies

$$\wedge^L \tilde{X}(t,s) - \wedge^L X(t,s) = \int_s^t (\wedge^L X(t,\tau))\hat{Q}^L(\tau)(\wedge^L \tilde{X}(\tau,s))\, d\tau.$$

Thus, it holds

$$\begin{aligned} &1 + \frac{\| \wedge^L \tilde{X}(t,s) - \wedge^L X(t,s)\|}{\| \wedge^L X(t,s)\|} \\ &\quad \leq 1 + L \int_s^t \frac{\| \wedge^L X(t,\tau)\| \, \| \wedge^L X(\tau,s)\|}{\| \wedge^L X(t,s)\|} \|Q(\tau)\| \left(1 + \frac{\| \wedge^L \tilde{X}(\tau,s) - \wedge^L X(\tau,s)\|}{\|(\wedge^L X(\tau,s)\|}\right) d\tau. \end{aligned}$$

Through Grönwall's inequality we get

$$1 + \frac{\| \wedge^L \tilde{X}(t,s) - \wedge^L X(t,s)\|}{\| \wedge^L X(t,s)\|} \leq e^{L c_{FI,L} \int_s^t \|Q(\tau)\|\, d\tau}$$

and due to equation (3.10)

$$\begin{aligned}\| \wedge^L \tilde{X}(t,s)\| &\geq \| \wedge^L X(t,s)\| \left(1 - \frac{\| \wedge^L \tilde{X}(t,s) - \wedge^L X(t,s)\|}{\| \wedge^L X(t,s)\|}\right) \\ &\geq \| \wedge^L X(t,s)\| \left(2 - e^{Lc_{FI,L}\int_s^t \|Q(\tau)\|\, d\tau}\right).\end{aligned}$$

Combining the last estimate with the one from the proof of Theorem 3.43 yields

$$\frac{\| \wedge^L \tilde{X}(t,\tau)\| \, \| \wedge^L \tilde{X}(\tau,s)\|}{\| \wedge^L \tilde{X}(t,s)\|} \leq \frac{\| \wedge^L X(t,\tau)\| \, \| \wedge^L X(\tau,s)\|}{\| \wedge^L X(t,s)\|} \cdot \frac{e^{Lc_{FI,L}\int_s^t \|Q(u)\|\, du}}{2 - e^{Lc_{FI,L}\int_s^t \|Q(u)\|\, du}},$$

which proves that the perturbed system is L-dim. strongly fast invertible.

Moreover, it follows that

$$\tilde{c}_{FI,L} \leq c_{FI,L} \frac{e^{Lc_{FI,L}\|Q\|_{L^1}}}{2 - e^{Lc_{FI,L}\|Q\|_{L^1}}},$$

which implies upper semicontinuity of $c_{FI,L}$ with respect to L^1-perturbations. Lower semicontinuity follows by switching roles, i.e., by viewing the original system as a perturbation of the perturbed system via $-Q(t)$. □

Corollary 3.49. *Assume equation (3.9) is regular and L-dim. strongly fast invertible. For every $\hat{v} \notin V'_{2,L}$, we find $\epsilon > 0$ such that*

$$\lim_{t\to\infty} \frac{1}{t} \log \|(\wedge^L \tilde{X}(t,0))\hat{v}\| = \Lambda_1 + \cdots + \Lambda_L$$

for any perturbation $Q(t)$ with

$$\int_0^\infty \|Q(t)\|\, dt < \epsilon.$$

Proof. Since characteristic exponents are invariant under multiplication of the solution by a nonzero constant, we may assume that $\|\hat{v}\| = 1$. It holds

$$\begin{aligned}\|(\wedge^L \tilde{X}(t,0))\hat{v}\| &\geq \|(\wedge^L X(t,0))\hat{v}\| - \|(\wedge^L \tilde{X}(t,0) - \wedge^L X(t,0))\hat{v}\| \\ &\geq \|(\wedge^L X(t,0))\hat{v}\| \left(1 - \frac{\| \wedge^L \tilde{X}(t,0) - \wedge^L X(t,0)\|}{\| \wedge^L X(t,0)\|} \frac{\| \wedge^L X(t,0)\|}{\|(\wedge^L X(t,0))\hat{v}\|}\right).\end{aligned}$$

Since the system is regular, Corollary 3.21 implies that

$$\lim_{t\to\infty} \frac{\| \wedge^L X(t,0)\|}{\|(\wedge^L X(t,0))\hat{v}\|} < \infty.$$

Now, according to the proof of Lemma 3.48, we may choose $\epsilon > 0$ small enough to ensure

$$\|(\wedge^L \tilde{X}(t,0))\hat{v}\| \geq c\|(\wedge^L X(t,0))\hat{v}\|$$

for some constant $c > 0$ independent of t. Hence, it holds

$$\liminf_{t\to\infty} \frac{1}{t} \log \|(\wedge^L \tilde{X}(t,0))\hat{v}\| \geq \liminf_{t\to\infty} \frac{1}{t} \log \|(\wedge^L X(t,0))\hat{v}\| = \lambda_{1,L}$$

and

$$\limsup_{t\to\infty} \frac{1}{t} \log \|(\wedge^L \tilde{X}(t,0))\hat{v}\| \leq \tilde{\lambda}_{1,L} = \lambda_{1,L}$$

due to Theorem 3.47. The claim follows since regularity implies $\lambda_{1,L} = \Lambda_1 + \cdots + \Lambda_L$. □

4 Computation of Lyapunov exponents

In this section we derive convergence results for the computation of Lyapunov exponents via *Benettin's algorithm* [4,5].

Assume $h_{\max} < \infty$ and let

$$\Phi : \mathbb{R}_{\geq 0} \times [0, h_{\max}] \to GL(d, \mathbb{R})$$

be a (linear) one-step method that is consistent of order $p > 0$.

Definition 4.1. We call Φ *consistent* if there is a constant $c_\Phi > 0$ such that

$$\|\Phi(t,h) - X(t+h,t)\| \leq c_\Phi h^{p+1}$$

for all $t \geq 0$ and $0 \leq h \leq h_{\max}$.

Given stepsizes $0 < h_n \leq h_{\max}$, we shorten our notation by defining

$$\Phi_n := \Phi(t_{n-1}, h_n)$$

and

$$\Phi^n := \Phi_n \dots \Phi_1.$$

A similar notation will be adopted for X.

4.1 Benettin's algorithm

The idea behind Benettin's algorithm can be explained via exterior products: Since $V_{2,L}$ is a proper subspace of the space of solutions of the induced system on $\wedge^L \mathbb{R}^d$, the solution to Lebesgue-almost every initial condition $v_1 \wedge \dots \wedge v_L$ has characteristic exponent $\lambda_{1,L}$. Using $X(t)[v_1, \dots, v_L] = Q(t)R(t)$, it holds

$$\chi[r_{11}(t) \dots r_{LL}(t)] = \chi[(\wedge^L X(t))(v_1 \wedge \dots \wedge v_L)] = \lambda_{1,L}$$

and hence, for regular systems, the Lyapunov exponents can be computed as

$$\Lambda_i = \lim_{t \to \infty} \frac{1}{t} \log r_{ii}(t)$$

for Lebesgue-almost every tuple of initial vectors, which is the core idea behind Benettin's algorithm.

The propagated vectors[8] (columns of V_n) are reorthonormalized periodically to prevent numerical singularities. If not, they could collapse onto the fastest expanding direction rendering them numerically indistinguishable, which makes it impossible to compute their associated volumes. Analytically, however, intermediate orthonormalizations do not play a role since the associated volumes stay the same. Indeed, we get the same output analytically if we perform the QR-decomposition only once at the end:

$$\Phi^n V_0 = \Phi_n \dots \Phi_2 \Phi_1 V_0 = \Phi_n \dots \Phi_2 V_1 R_1 = \dots = V_n (R_n \dots R_1).$$

[8] The propagated vectors from Benettin's algorithm are more than a mere byproduct for Lyapunov exponents. For instance, they have been exploited by Ginelli et al. [12] and by Wolfe–Samelson [24] in their algorithms to compute covariant Lyapunov vectors (see [18,19] for a theoretical analysis).

Algorithm 1: Benettin's algorithm [4,5]

Input: number of integration steps N, stepsizes $(h_n)_{n=1}^N$
Output: computed Lyapunov exponents $\mu_1, \dots, \mu_d$

$V_0 = \text{rand(d)}$ // set random initial vectors
for n = *1:N* **do**
  $W = \Phi_n V_{n-1}$ // evolve
  $[Q_n, R_n] = \text{qr}(W)$ // orthonormalize
  $V_n = Q_n$ // set new vectors
for i = *1:d* **do**
  $\mu_i(N) = \frac{1}{t_N} \sum_{n=1}^N \log(R_n)_{ii}$ // average

One may adjust the frequency of orthonormalizations depending on how fast V_n becomes singular. Furthermore, in practice it makes sense to compute the Lyapunov exponents as a running average during the propagation loop to save memory and to monitor convergence properties.

While the algorithm works perfectly fine analytically, there are some numerical challenges. In practice several types of errors influence the output of Benettin's algorithm. For example, errors are introduced by numerical integration, by limiting the integration time, by finite-precision computing or through errors of the underlying equations[9].

4.2 Convergence results

For all results in this subsection, we assume:

- $0 < h_n \leq h_{\max}$,
- $t_n = h_1 + \cdots + h_n \to \infty$,
- Φ is consistent of order $p > 0$.

Before deriving the desired convergence theorems for Benettin's algorithm, we introduce two auxiliary systems: a piecewise constant approximation of our original system and a piecewise constant system representing the numerical integration.

If $\|X_n - I\| < 1$, the logarithm of X_n exists and is equal to

$$\log X_n = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(X_n - I)^k}{k}.$$

We set

$$A_n := \frac{1}{h_n} \log X_n.$$

A_n is well-defined for small stepsizes since

$$\|X_n - I\| \leq \int_{t_{n-1}}^{t_n} \|A(s) X(s, t_{n-1})\| \, ds \leq M e^{h_{\max} M} h_n.$$

[9] If the linear system is derived as the linearization along a solution of a nonlinear system, integration errors from computing the solution of the nonlinear system can result in errors in the derived linear equations of order $O(1)$. In that case our perturbation results for linear theory are ineffective and nonlinear theory is necessary to study the computational error.

Similarly, we set

$$B_n := \frac{1}{h_n} \log \Phi_n$$

whenever $\|\Phi_n - I\| < 1$. Consistency of the numerical integrator and the previous estimate imply that B_n is well-defined for small stepsizes as well.

We have the following relation between A_n and B_n:

Lemma 4.2. *There is a constant*[10] $c > 0$ *independent of the stepsizes such that*

$$\|A_n - B_n\| \leq c h_n^p$$

for small h_n.

Proof. Assume h_n is small enough such that

$$\max(\|\Phi_n - I\|, \|X_n - \Phi_n\|) \leq \frac{1}{4}. \tag{4.1}$$

The logarithms of X_n and Φ_n exist. Moreover, it holds

$$\|\log(X_n) - \log(\Phi_n)\| \leq \sum_{k=1}^{\infty} \frac{\|(X_n - I)^k - (\Phi_n - I)^k\|}{k}.$$

Using

$$\|(M+E)^k - M^k)\| \leq \|E\| 2^k \max(\|M\|, \|E\|)^{k-1}$$

with $M := \Phi_n - I$ and $E := X_n - \Phi_n$, we estimate

$$\|A_n - B_n\| = \frac{1}{h_n} \|\log(X_n) - \log(\Phi_n)\| \leq \frac{1}{h_n} \left(\sum_{k=1}^{\infty} \frac{4}{2^k k} \right) \|X_n - \Phi_n\|.$$

The claim follows from consistency of the numerical integrator. □

We may regard A_n as a perturbation of the original system on $[t_{n-1}, t_n)$. The size of the perturbation depends on the continuity property of A.

Lemma 4.3. *If* A *is globally Lipschitz continuous, then there is a constant* $c > 0$ *independent of the stepsizes such that*

$$\sup_{t \in [t_{n-1}, t_n)} \|A(t) - A_n\| \leq c h_n$$

for small h_n.

Proof. If h_n is small enough such that equation (4.1) holds, then

$$\|X_n - I\| \leq \frac{1}{2}$$

and

$$\|\log(X_n) - (X_n - I)\| \leq \left(\sum_{k=2}^{\infty} \frac{\|X_n - I\|^{k-2}}{k} \right) \|X_n - I\|^2 \leq \underbrace{\left(\sum_{k=2}^{\infty} \frac{4}{2^k k} \right) M^2 e^{2h_{\max} M}}_{=:C} h_n^2.$$

[10] A more detailed analysis of the constant for systems with stable Lyapunov exponents can be found in [10].

Thus, it holds

$$\begin{aligned}\|A(t) - A_n\| &= \left\|A(t) - \frac{1}{h_n}\log X_n\right\| \leq \left\|A(t) - \frac{1}{h_n}(X_n - I)\right\| + Ch_n \\ &\leq \frac{1}{h_n}\int_{t_{n-1}}^{t_n} \|A(t) - A(s)X(s, t_{n-1})\|\, ds + Ch_n.\end{aligned}$$

for $t \in [t_{n-1}, t_n)$. Now, the claim follows from

$$A(t) - A(s)X(s, t_{n-1}) = A(t) - A(s) + A(s)(X(t_{n-1}, t_{n-1}) - X(s, t_{n-1}))$$

via Lipschitz continuity of A. □

In the following, we only consider stepsizes such that $h_n \to 0$. While we cannot guarantee that A_n and B_n are well-defined for small n, there is $N \geq 0$ such that A_n and B_n are well-defined for $n \geq N$. We define

$$A_{pc}(t) := \begin{cases} 0, & t \in [0, t_{N-1}) \\ A_n, & t \in [t_{n-1}, t_n) \text{ for } n \geq N \end{cases}$$

and

$$B_{pc}(t) := \begin{cases} 0, & t \in [0, t_{N-1}) \\ B_n. & t \in [t_{n-1}, t_n) \text{ for } n \geq N \end{cases}$$

By possibly increasing N, we ensure that A_{pc} and B_{pc} are bounded and can be estimated using the previous lemmata.[11]

Let us denote the Cauchy matrix corresponding to A_{pc} by $X_{pc}(t, s)$. If $n \geq m \geq N - 1$, then

$$X_{pc}(t_n, t_m) = e^{h_n A_n} \dots e^{h_{m+1} A_{m+1}} = X(t_n, t_m).$$

Hence, A_{pc} can be seen as a piecewise constant approximation of our original system (for large t).

Lemma 4.4. *The Lyapunov spectra of the original system and the piecewise constant approximation coincide. The same holds for their induced systems. Moreover, if the original system is regular, then the piecewise constant approximation is regular.*

Proof. Since

$$X_{pc}(t, t_{N-1}) = e^{(t - t_{n-1})A_n} X(t, t_{n-1})^{-1} X(t, t_{N-1})$$

for $t \in [t_{n-1}, t_n)$ and $n \geq N$, the growth rates of solutions with the same initial condition differ by at most a constant:

$$\left(\frac{\|X_{pc}(t, t_{N-1})v\|}{\|X(t, t_{N-1})v\|}\right)^{\pm 1} \leq e^{h_{\max}(\|A_{pc}\|_\infty + M)}$$

for any $v \neq 0$. Hence, the Lyapunov spectra of the original system and the piecewise constant approximation coincide. The remaining statements follow similarly. □

[11] This is the case if equation (4.1) is satisfied for $n \geq N$.

We write $\tilde{X}_{pc}(t,s)$ for the Cauchy matrix corresponding to B_{pc} and note that

$$\tilde{X}_{pc}(t_n,t_m) = \Phi_n \dots \Phi_{m+1}$$

for $n \geq m \geq N-1$. In particular, fixing the fundamental matrix $\tilde{X}_{pc}(t)$ satisfying $\tilde{X}_{pc}(t_{N-1}) = \Phi^{N-1}$, we have

$$\tilde{X}_{pc}(t_n) = \tilde{X}_{pc}(t_n,t_{N-1})\tilde{X}_{pc}(t_{N-1}) = \Phi_n \dots \Phi_N \Phi^{N-1} = \Phi^n$$

for $n \geq N-1$. Thus, B_{pc} can be understood as representing the numerical integration. It follows that

$$\begin{aligned}\mu_1(n) + \dots + \mu_L(n) &= \sum_{i=1}^{L} \frac{1}{t_n} \sum_{k=1}^{n} \log(R_k)_{ii} = \frac{1}{t_n} \log \left(\prod_{i=1}^{L} (R_n \dots R_1)_{ii} \right) \\ &= \frac{1}{t_n} \log \|(\wedge^L \Phi_n)(v_1 \wedge \dots \wedge v_L)\| = \frac{1}{t_n} \log \|(\wedge^L \tilde{X}_{pc}(t_n))(v_1 \wedge \dots \wedge v_L)\|\end{aligned}$$

for $n \geq N-1$. Since the stepsizes and B_{pc} are bounded, we have

$$\begin{aligned}\limsup_{n\to\infty} \mu_1(n) + \dots + \mu_L(n) &= \limsup_{n\to\infty} \frac{1}{t_n} \log \|(\wedge^L \tilde{X}_{pc}(t_n))(v_1 \wedge \dots \wedge v_L)\| \\ &= \limsup_{t\to\infty} \frac{1}{t} \log \|(\wedge^L \tilde{X}_{pc}(t))(v_1 \wedge \dots \wedge v_L)\|.\end{aligned}$$

Thus, Benettin's algorithm computes $\tilde{\lambda}_{1,L}$ of B_{pc} for Lebesgue-almost every tuple of initial vectors:

$$\limsup_{n\to\infty} \mu_1(n) + \dots + \mu_L(n) = \tilde{\lambda}_{1,L}. \tag{4.2}$$

In particular, if B_{pc} is regular, then

$$\lim_{n\to\infty} \mu_1(n) + \dots + \mu_L(n) = \tilde{\Lambda}_1 + \dots + \tilde{\Lambda}_L. \tag{4.3}$$

Theorem 4.5. *Assume equation (3.1) is globally Lipschitz continuous and has stable Lyapunov exponents. If $h_n \to 0$, then*

$$\limsup_{n\to\infty} \mu_1(n) + \dots + \mu_i(n) \leq \Lambda_1 + \dots + \Lambda_i$$

for all i and Lebesgue-almost every tuple of initial vectors.[12]

Proof. Since there are constant $c_1, c_2 > 0$ independent of the stepsizes such that

$$\sup_{t\in[t_{n-1},t_n)} \|A(t) - B_{pc}(t)\| \leq \sup_{t\in[t_{n-1},t_n)} \|A(t) - A_n\| + \|A_n - B_n\| \leq c_1 h_n + c_2 h_n^p$$

for $n \geq N$, B_{pc} is a perturbation of the original system such that $\|Q(t)\| \to 0$. In particular, Theorem 3.26 implies that B_{pc} has the same Lyapunov spectrum as the original system. Now, equation (4.2) and Proposition 3.20 imply the theorem. □

[12] A version of the estimate for finite time can be found in [11]. The authors relate the error to the departure from normality of the R-matrices obtained during Benettin's algorithm.

Theorem 4.6. *Assume equation (3.1) is globally Lipschitz continuous, regular and has stable Lyapunov exponents. If $h_n \to 0$, then*

$$\lim_{n\to\infty} \mu_i(n) = \Lambda_i$$

for all i and Lebesgue-almost every tuple of initial vectors.

Proof. The claim follows similarly to the last theorem using Proposition 3.27 and equation (4.3) instead of equation (4.2). □

Remark 4.7. If the stability of Lyapunov exponents transfers from A to A_{pc}, then the assumption that A is globally Lipschitz continuous can be dropped in Theorems 4.5 and 4.6.

Next, we derive convergence results for strongly fast invertible systems. For this, we need the following lemma:

Lemma 4.8. *The original system is L-dim. strongly fast invertible if and only if its piecewise constant approximation is L-dim. strongly fast invertible.*

Proof. The claim follows from

$$e^{(t_n-t)A_n} X_{pc}(t,s) e^{(s-t_{m-1})A_m} = X_{pc}(t_n, t_{m-1}) = X(t_n, t_{m-1}) = X(t_n, t) X(t, s) X(s, t_{m-1})$$

for $s \in [t_{m-1}, t_m)$ and $t \in [t_{n-1}, t_n)$, $n \geq m \geq N$, and the fact that strong fast invertibility can be tested on $[t_{N-1}, \infty)$ (see Lemma 3.29). □

Theorem 4.9. *Assume equation (3.1) is L-dim. strongly fast invertible. If $h_n \to 0$, then*

$$\limsup_{n\to\infty} \mu_1(n) + \cdots + \mu_L(n) \leq \Lambda_1 + \cdots + \Lambda_L$$

for Lebesgue-almost every tuple of initial vectors.

Proof. Since A_{pc} is L-dim. strongly fast invertible and its induced systems have the same Lyapunov spectra as the induced systems of the original system, Theorem 3.46 and Proposition 3.20 imply

$$\tilde{\lambda}_{1,L} \leq \lambda_{1,L} \leq \Lambda_1 + \cdots + \Lambda_L.$$

The claim follows from equation (4.2). □

Theorem 4.10. *Assume equation (3.1) is regular and L-dim. strongly fast invertible. If*

$$\sum_{n=1}^{\infty} h_n^{p+1} < \infty,$$

then

$$\limsup_{n\to\infty} \mu_1(n) + \cdots + \mu_L(n) = \Lambda_1 + \cdots + \Lambda_L$$

for Lebesgue-almost every tuple of initial vectors.

Proof. Lemma 4.2 and the stepsize condition ensure that

$$\int_0^{\infty} \|A_{pc}(t) - B_{pc}(t)\| \, dt < \infty.$$

Now, the proof is as in Theorem 4.9 using Theorem 3.47 and regularity:

$$\tilde{\lambda}_{1,L} = \lambda_{1,L} = \Lambda_1 + \cdots + \Lambda_L.$$

□

Since we do not know if regularity transfers form the original to the numerical system, Theorem 4.10 only ensures convergence to the Lyapunov exponents as a limes superior. However, when fixing a tuple of initial vectors, we can ensure convergence as a limit if the stepsizes decay fast enough.

Theorem 4.11. *Assume equation (3.1) is regular and L-dim. strongly fast invertible. For Lebesgue-almost every tuple of initial vectors, we have the following: If $\sum_{n=1}^{\infty} h_n^{p+1}$ is small enough, then*

$$\lim_{n\to\infty} \mu_1(n) + \cdots + \mu_L(n) = \Lambda_1 + \cdots + \Lambda_L.$$

Proof. The theorem follows from Corollary 3.49. □

Remark 4.12. The condition "$\sum_{n=1}^{\infty} h_n^{p+1}$ small enough" in Theorem 4.11 depends on the chosen initial vectors. Indeed, following the associated proofs, we require smaller stepsizes the smaller the first principle angle between $\operatorname{span}(v_1, \dots, v_L)$ and V'_{l+1}.

Finally, when applied to the right dimensions, Theorems 4.10 and 4.11 ensure that we may approximate the Lyapunov exponents of strongly fast invertible systems using Benettin's algorithm.

Corollary 4.13. *Assume equation (3.1) is regular and strongly fast invertible at dim. $d_1 + \cdots + d_l$ for $l = 1, \dots, p$. If*

$$\sum_{n=1}^{\infty} h_n^{p+1} < \infty,$$

then

$$d_i\lambda_i = \limsup_{n\to\infty} \mu_1(n) + \cdots + \mu_{d_1+\cdots+d_i}(n) - \limsup_{n\to\infty} \mu_1(n) + \cdots + \mu_{d_1+\cdots+d_{i-1}}(n)$$

for all i and for Lebesgue-almost every tuple of initial vectors. In particular, if the Lyapunov spectrum is simple, then

$$\lambda_i = \limsup_{n\to\infty} \mu_1(n) + \cdots + \mu_i(n) - \limsup_{n\to\infty} \mu_1(n) + \cdots + \mu_{i-1}(n)$$

for all i and for Lebesgue-almost every tuple of initial vectors.

Corollary 4.14. *Assume equation (3.1) is regular and strongly fast invertible at dim. $d_1 + \cdots + d_l$ for $l = 1, \dots, p$. For Lebesgue-almost every tuple of initial vectors, we have the following: If $\sum_{n=1}^{\infty} h_n^{p+1}$ is small enough, then*

$$d_i\lambda_i = \lim_{n\to\infty} \mu_{d_1+\cdots+d_{i-1}+1}(n) + \cdots + \mu_{d_1+\cdots+d_i}(n)$$

for all i. In particular, if the Lyapunov spectrum is simple, then

$$\lambda_i = \lim_{n\to\infty} \mu_i(n)$$

for all i.

5 Numerical examples

To showcase Benettin's algorithm, we present three numerical examples: a linear system, the Lorenz-63 and the Lorenz-96 model. While the linear system satisfies the assumptions required in our convergence analysis, the other two systems are nonlinear and exhibit additional errors that are not accounted for in our analysis. Moreover, not all Lyapunov exponents of the Lorenz models are known analytically.

All systems will be integrated using a fifth-order Runge–Kutta method (RK5) with

1. adaptive stepsizes[13],
2. constant stepsizes, and
3. two types of varying stepsizes[14] ($h_n = \frac{h_0}{\sqrt[2]{n}}$ and $h_n = \frac{h_0}{\sqrt[3]{n}}$).

In case of adaptive stepsizes, we apply the Dormand–Prince method (DP54) which uses a fourth-order Runge–Kutta method (RK4) in addition to RK5 to estimate the integration error. Since the computational costs per integration step for each approach are close[15], we can roughly associate the total computational costs to the number of integration steps to compare the performance of Benettin's algorithm for the different stepsizes.

The corresponding MATLAB code can be found in [20].

5.1 Linear system

Let

$$B(t) = \begin{pmatrix} \lambda_1 & f(t) & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix}$$

be a block-triangular matrix with $\lambda_1 > \lambda_2$ and a bounded continuous function f. One may check that the system $\dot{y} = B(t)y$ is regular and has stable Lyapunov exponents λ_1 and λ_2 due to Theorem 3.24[16]. In particular, the system is strongly fast invertible at dimensions $d_1 = 2$ and $d_1 + d_2 = 3$. These properties remain valid under Lyapunov transformations.

To get a specific example, we set $\lambda_1 = 1$, $\lambda_2 = 0$ and $f(t) = \sin(t) + 1/(t^2+1)$ and transform $\dot{y} = B(t)y$ to

$$\dot{x} = A(t)x := (LBL^{-1} + \dot{L}L^{-1})(t)x$$

[13] We use adaptive stepsizes in accordance with [9, Section 3.1] for linear systems and with [8, pp. 13–14] for nonlinear systems.

[14] We chose varying stepsizes such that

$$\sum_{n=1}^{\infty} h_n^{p+1} < \infty.$$

[15] DP54 uses the same function evaluations as RK5 and hence only increases the computational costs per step by a small amount. The possibly largest increase in costs for the adaptive method comes from stepsize corrections that require to repeat the integration step. However, at least in our examples, the increase was relatively low (a factor of ∼1.3 for the linear system and ∼1.0 for the Lorenz systems).

[16] Condition (iii) follows from

$$Y_1(t,s) = e^{\lambda_1(t-s)} \begin{pmatrix} 1 & \int_s^t f(\tau)\,d\tau \\ 0 & 1 \end{pmatrix}$$

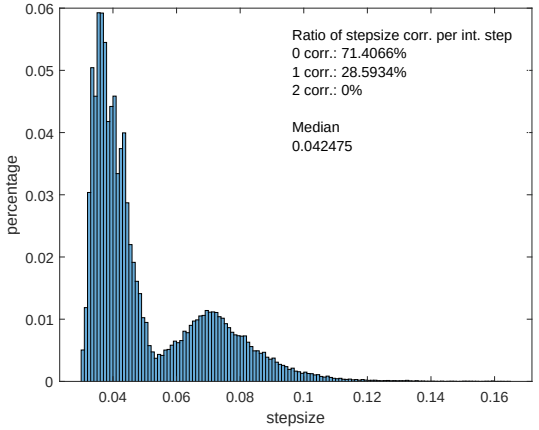
and $Y_2(t,s) = e^{\lambda_2(t-s)}$. Stability of the subsystem for λ_1 can be checked by showing $\overline{\omega}_1 = \lambda_1 = \Omega_1$ using that the constant function equal to λ_1 is a lower and upper function for the first subsystem.

using the composition $L(t)$ of two Lyapunov transformations

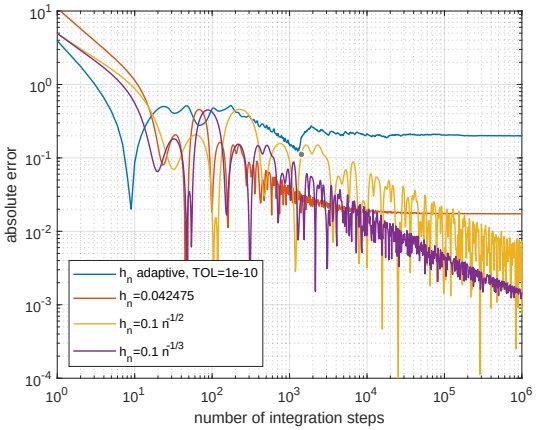
$$\begin{pmatrix} 1 & \sin(t) & \cos(t) \\ 0 & 1 & \sin(t) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(\sqrt{2}t) & 0 & \sin(\sqrt{2}t) \\ 0 & 1 & 0 \\ -\sin(\sqrt{2}t) & 0 & \cos(\sqrt{2}t) \end{pmatrix}.$$

While our theory only ensures convergence of Benettin's algorithm for $2\lambda_1$ and λ_2, we apply the algorithm to compute each individual exponent without regarding the degeneracy. The Lyapunov exponents are approximated using the different stepsizes mentioned in the beginning of Section 5. We start with the adaptive approach and take the median of the accepted stepsizes (Figure 5.1a) as the constant stepsizes for the second approach. The approximated Lyapunov exponents of all approaches are compared to their exact values ($\Lambda_1 = 1$, $\Lambda_2 = 1$ and $\Lambda_3 = 0$) and plotted against the number of integration steps (Figures 5.1b, 5.1c and 5.1d).

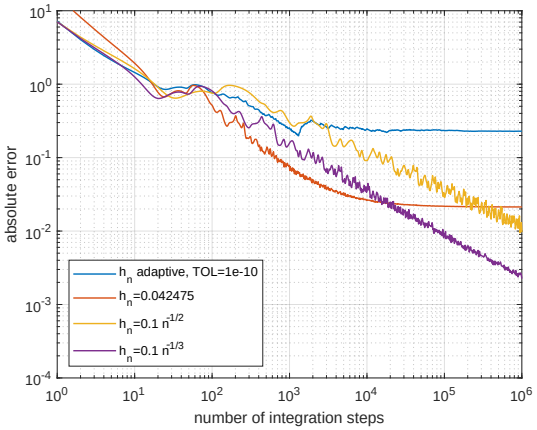
For all three Lyapunov exponents, we see that adaptive and constant stepsizes lead to an accumulation of integration errors that persists in the limit. On the other hand, the errors obtained with varying stepsizes show a steady decay as we would expect from our theory.



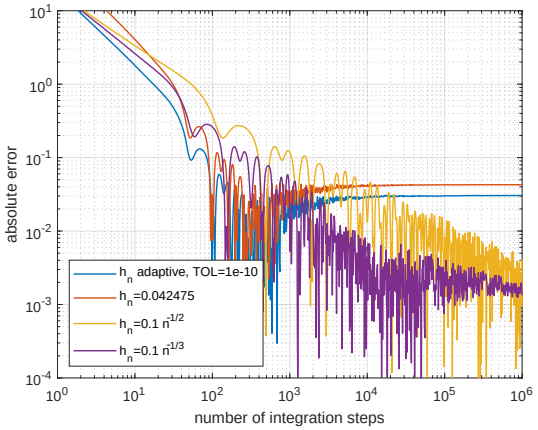
(a) PDF of accepted stepsizes



(b) LE1 (error)



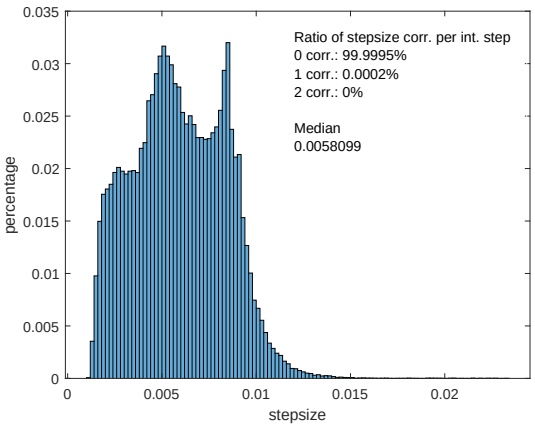
(c) LE2 (error)



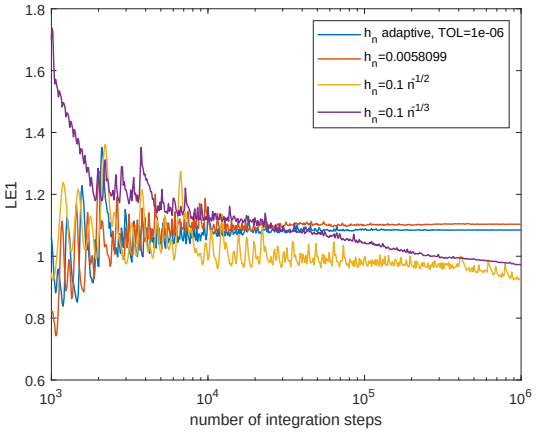
(d) LE3 (error)

Figure 5.1: The Lyapunov exponents of the system $\dot{x} = A(t)x$ from Subsection 5.1 are approximated using Benettin's algorithm with different stepsizes. The first plot shows the PDF of the accepted stepsizes of the adaptive approach, while the other plots show the errors between computed Lyapunov exponents and their exact values. To save memory, the PDF was formed over every 10th integration step.

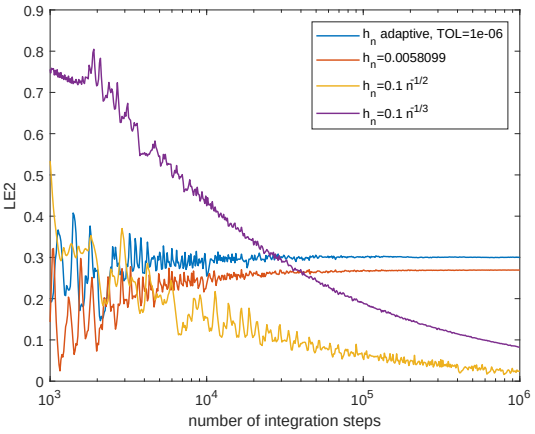
5.2 Lorenz-63 model



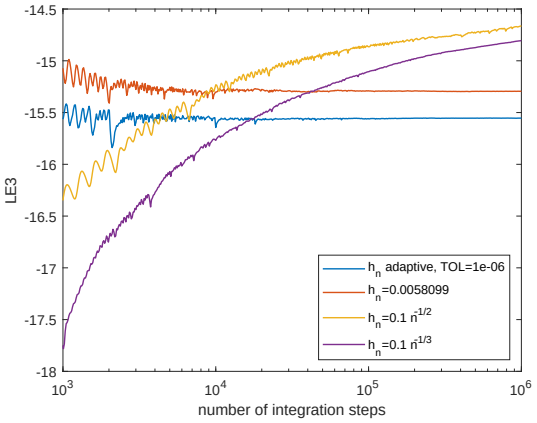
(a) PDF of accepted stepsizes



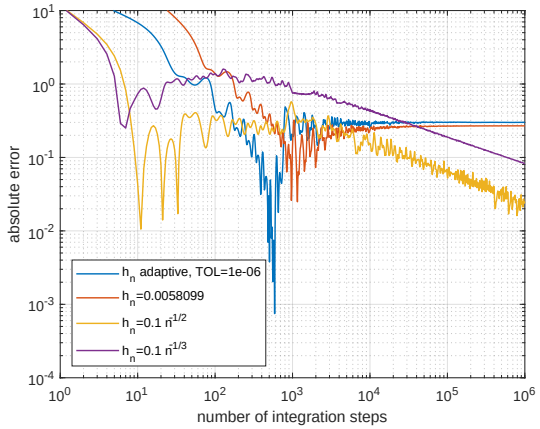
(b) LE1



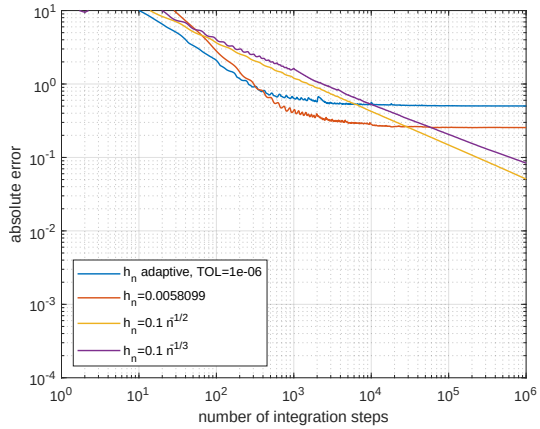
(c) LE2



(d) LE3



(e) LE2 (error)



(f) LE sum (error)

Figure 5.2: The Lyapunov exponents of the Lorenz-63 model from Subsection 5.2 are approximated using Benettin's algorithm with different stepsizes. The first plot shows the PDF of the accepted stepsizes of the adaptive approach, while the other plots show the computed Lyapunov exponents and their errors. To save memory, the PDF was formed over every 10th integration step.

Our second example is the Lorenz-63 model [15] (a reduced model for thermal convection)

given by the equations

$$\left.\begin{aligned} \dot{x}_1 &= \sigma(x_2 - x_1) \\ \dot{x}_2 &= x_1(\rho - x_3) - x_2 \\ \dot{x}_3 &= x_1 x_2 - \beta x_3 \end{aligned}\right\} := f(x)$$

with $\sigma = 10$, $\beta = 8/3$ and $\rho = 28$ as the classical parameters. The system is famous for its unpredictable behavior despite being deterministic. It exhibits a so-called *strange attractor* (the *Lorenz attractor*), which is a minimal attracting flow-invariant set that is neither a steady state nor a periodic orbit and which has a positive largest Lyapunov exponent[17]. So far, the existence of the attractor and the existence of a physical ergodic invariant measure that admits a positive largest Lyapunov exponent is only known as a consequence of the computer-aided proof by Tucker [23].

To compute the Lyapunov exponents, we require a background trajectory lying on the attractor. This is problematic, since the Lorenz attractor has Lebesgue measure zero. In particular, the initial state of our background trajectory will almost surely lie outside of the attractor. While the trajectory may be close to the attractor after a transient, a rigorous analysis would require continuity properties of the Lyapunov exponents that extend to a neighborhood of the attractor. However, in general, the Lyapunov exponents vary only measurably with the base point. Moreover, integration errors can lead to an error of order $O(1)$ in the background trajectory as well as in the derived linear system. This may drastically change the computed Lyapunov exponents.

Despite the open challenges, we try to approximate the Lyapunov exponents and compare our methods for the known cases, that is, the second Lyapunov exponent and the sum of all Lyapunov exponents. The second Lyapunov exponent corresponds to the tangent vector of the background trajectory, which is a solution of the linearized equations that is neither exponentially growing nor decaying, and thus must be zero. The sum of all three Lyapunov exponents must be equal to the trace of the Jacobian as it is constant. Indeed, it holds

$$\begin{aligned} \Lambda_1 + \Lambda_2 + \Lambda_3 &= \lim_{t\to\infty} \frac{1}{t} \log |\det(X(t))| \\ &= \lim_{t\to\infty} \frac{1}{t} \int_0^t \operatorname{tr}(Df(x(s)))ds = -(\sigma + 1 + \beta) \end{aligned}$$

for any solution $x(t)$ along which the linearized system $X(t)$ is regular. Exact values of the first and the third Lyapunov exponent are unknown.

The Lyapunov exponents are computed in a similar fashion to Subsection 5.1. This time, however, we require a background trajectory along which to propagate linear perturbations. In order to start close to the attractor, we first compute a transient starting from a randomly chosen state. After the transient, we couple the nonlinear system to its linearization. This way, we can compute the background trajectory and propagate linear perturbations simultaneously.

In this example, the adaptive approach requires almost no stepsize corrections (Figure 5.2a) and is thus well comparable to the other approaches in terms of computational costs per step. As before, we observe that Lyapunov exponents computed with adaptive and constant stepsizes seem to converge to different values depending on the given tolerance or stepsize (Figures 5.2b, 5.2c and 5.2d). A meaningful comparison between the different approaches is

[17] For nonlinear systems, the Lyapunov exponents are obtained from the linearized flow along nonlinear trajectories. If the nonlinear system has an ergodic invariant measure, the Lyapunov exponents coincide for almost every trajectory. Hence, we may associate Lyapunov exponents with the attractor (or measure) instead of the trajectory.

only possible for the second and the sum of all Lyapunov exponents (Figures 5.2e and 5.2f). We note that, similar to the linear example, the accuracy for adaptive and constant stepsizes are limited, while varying stepsizes seem to admit convergence properties that lead to the true Lyapunov exponents.

5.3 Lorenz-96 model

Our third example is the Lorenz-96 model [16], which was created to test numerical weather prediction. The model simulates the effects of external forcing, damping and advection on a scalar physical quantity, which lives on a periodic lattice representing a circle of latitude. Different from the Lorenz-63 model, the Lorenz-96 model treats each variable on the lattice the same. By changing the number of variables, one may study the effects of increased spacial resolution on the predictability (or chaoticity) of the system. There already are studies of spatiotemporal chaos in the Lorenz-96 model in which spectra of Lyapunov exponents have been computed [14].

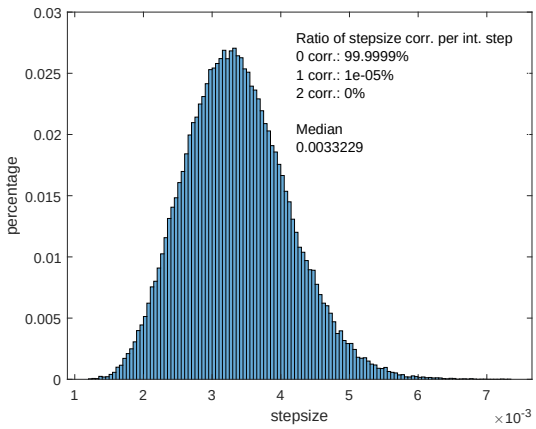
The Lorenz-96 model with $d \geq 4$ variables is given by the equations

$$\dot{x_i} = (x_{i+1} - x_{i-2})x_{i-1} - x_i + F$$

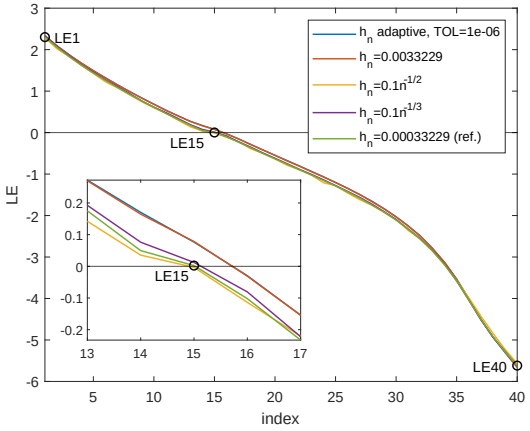
with periodic indices, i.e., $x_{i+kd} := x_i$ for $k \in \mathbb{Z}$. We choose $d = 40$ and a forcing parameter of $F = 10$ to allow for chaotic dynamics. Under the same conditions, [14] computed the Lyapunov spectrum and noticed that the shape of the spectrum does not change while increasing the number of variables from 40 to 50. They used constant stepsizes of $h_n = 1/64$ and integrated the trajectory for $T = 5 \times 10^5$ units of time or $N = 64T$ timesteps. The first $2/3$ of the trajectory were used as a transient to approach the attractor before computing the Lyapunov exponents.

Here, we explore the effects of different stepsizes on the Lyapunov spectrum in terms of the approaches mentioned in the beginning of Section 5. Similar to the Lorenz-63 model, we know that at least one Lyapunov exponent must vanish and that the sum of all Lyapunov exponents must be equal to the trace of the Jacobian, which is $-d$. However, it is not clear which index corresponds to the vanishing Lyapunov exponent. Instead of looking at each individual exponent, we compare the Lyapunov spectra of the different approaches to a reference spectrum computed with a high-resolution run. The latter is performed with ten times the number of integration steps than the other runs and with constant stepsizes that are ten times smaller than the constant stepsizes from the low-resolution run. While a higher resolution heuristically provides better estimates of the Lyapunov exponents, we still have to expect persistent errors.

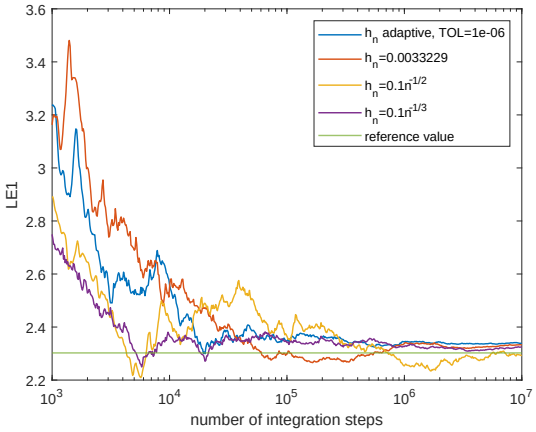
Our computed Lyapunov spectra (Figure 5.3b) take on similar shapes to the spectrum in [14, Figure 5]. We see that the spectra computed with varying stepsizes are closer to the reference spectrum than the ones computes with adaptive or constant stepsizes. The differences between the spectra vary with the specific Lyapunov exponent. A more detailed picture showing the computed extremal Lyapunov exponents and the near-zero exponent of the high-resolution run can be seen in (Figures 5.3c, 5.3d and 5.3e). Concerning the sum of all Lyapunov exponents, we observe a similar behavior as in the other two examples (Figure 5.3f).



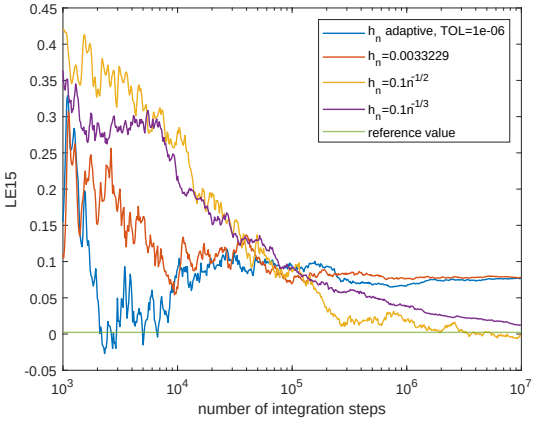
(a) PDF of accepted stepsizes



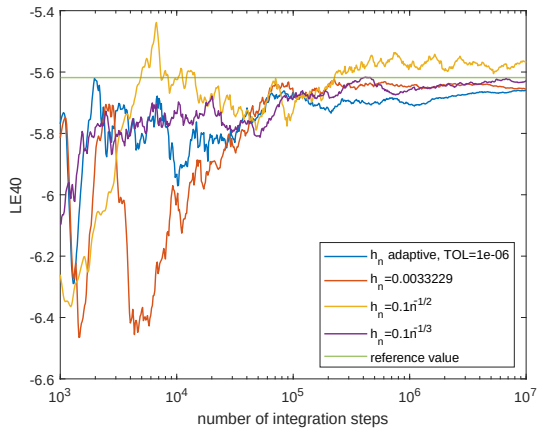
(b) LE spectrum



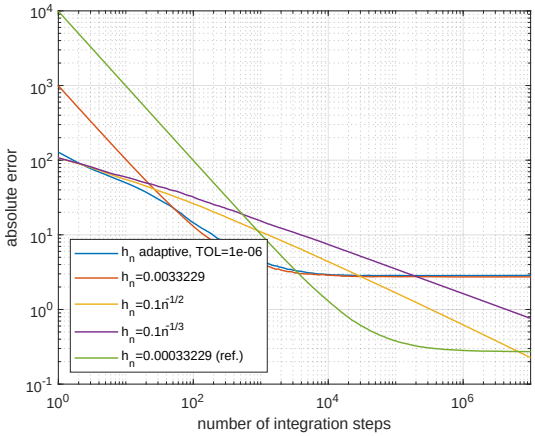
(c) LE1



(d) LE15



(e) LE40



(f) LE sum (error)

Figure 5.3: The Lyapunov exponents of the Lorenz-96 model from Subsection 5.3 are approximated using Benettin's algorithm with different stepsizes. The first plot shows the PDF of the accepted stepsizes of the adaptive approach, the second plot shows the computed spectra after 10^7 integration steps (10^8 steps for the reference), and the other plots compare the computed Lyapunov exponents to the reference (final value of the exponent after 10^8 steps). To save memory, the PDF was formed over every 100th integration step.

Acknowledgments

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Uniqueness of positive radial solutions for a class of (p,q)-Laplacian problems in a ball

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Abstract. We prove the uniqueness of positive radial solution to the (p,q)-Laplacian problem

$$\begin{cases} -\Delta_p u - \Delta_q u = \lambda f(u) \ \text{ in } \Omega, \\ u = 0 \ \text{ on } \partial\Omega, \end{cases}$$

where $p > q > 1$, $\Delta_r u = \operatorname{div}(|\nabla u|^{r-2}\nabla u)$, $\Omega = B(0,1)$ is the open unit ball in $\mathbb{R}^N$, $f : (0,\infty) \to \mathbb{R}$ is q-sublinear at ∞ with possible semipositone structure at 0, and $\lambda > 0$ is a large parameter.

Keywords: (p,q)-Laplacian, positive solutions, uniqueness.

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1 Introduction

In this paper, we study the uniqueness of positive radial solutions for the (p,q)-Laplacian boundary value problem

$$\begin{cases} -\Delta_p u - \Delta_q u = \lambda f(u) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is the open unit ball in $\mathbb{R}^N$, $p > q > 1$, $\Delta_r u = \operatorname{div}(|\nabla u|^{r-2}\nabla u(|\nabla u|^{r-2}\nabla u)$, $f : (0,\infty) \to \mathbb{R}$ is p-sublinear at ∞ i.e. $\lim_{u\to\infty} \frac{f(u)}{u^{p-1}} = 0$ and λ is a positive parameter. The (p,q)-Laplacian problems occurred in a variety of applied areas such as quantum physics, plasma physics, and reaction diffusion to name a few, see e.g. [2, 4, 6, 13, 14, 23]. When f is p-sublinear at ∞, the existence of positive solutions to (1.1) for λ large was recently established in [1] under the mere additional natural condition that $\liminf_{u\to\infty} u^\beta f(u) > 0$ and $\limsup_{u\to 0^+} u^\beta |f(u)| < \infty$ for some $\beta \in [0,1)$, which extended previous results in [10, 20]. When $p = q$, the uniqueness of positive solutions to (1.1) for all $\lambda > 0$ was established in the pioneering work [3] for $p = 2$ when $\frac{f(u)}{u}$ is strictly decreasing on $(0,\infty)$, and subsequently extended to the

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case $p > 1$ in [11]. When this monotonicity condition is assumed only for u large, uniqueness results for λ large were obtained under additional hypotheses in [5,7–9,12,15–19,21,22,24–26], where the semipositone case i.e. $-\infty < f(0) < 0$ is allowed in [18]. Note that previous uniqueness proofs depend on the homogeneity of the p-Laplace and cannot be extended to the (p,q)-Laplace operator. In this paper, we shall establish the uniqueness of positive solutions to (1.1) for λ large when f is q-sublinear at ∞ with possible semipositone structure at 0, which has not been obtained in the literature to the best of our knowledge. Related uniqueness results for (1.1) in a bounded domain of $\mathbb{R}^N$ can be found in [10,13]. In [13, Theorem 2.2], the uniqueness of positive solutions for all $\lambda > 0$ was established when $z^{1-q}f(z)$ is decreasing for $z > 0$ and the assumption that any two positive solutions u_i, $i = 1,2$, of (1.1) satisfy $\Delta_p u_i \in L^\infty(\Omega)$, and $u_i/u_j \in L^\infty(\Omega), i \neq j$. The uniqueness for λ large in the semipositone case was observed in [10] for the special Dirichlet two-point boundary problem

$$\begin{cases} -((u')^3)' - u'' = \lambda f(u) \ \text{ in } (0,1), \\ u(0) = u(1) = 0, \end{cases}$$

where $f(u) = (u+1)^\gamma - 2$ for some $\gamma \in (0,3)$. Note that this observation is generated by *Mathematica* and does not constitute an analytical proof.

Since we are looking for radial solutions for (1.1), it reduces to finding solutions of the ODE problem

$$\begin{cases} -(r^{N-1}\phi(u'))' = \lambda r^{N-1} f(u), \ 0 < r < 1, \\ u'(0) = 0, \ u(1) = 0, \end{cases} \tag{1.2}$$

where $\phi(x) = \phi_p(x) + \phi_q(x)$, $\phi_r(x) = |x|^{r-2}x$.

We make the following assumptions.

(A1) $f : (0,\infty) \to \mathbb{R}$ is continuous and f is increasing on $[L,\infty)$ for some $L > 0$.

(A2) $\limsup_{z\to\infty} \frac{zf'(z)}{f(z)} < q - 1$.

(A3) There exists $\gamma \in [0,1)$ such that $\limsup_{z\to 0^+} z^{\gamma+1}|f'(z)| < \infty$ and $f(0^+) < \infty$ if $\gamma = 0$.

By a positive solution of (1.2), we mean a function $u \in C^1[0,1]$ with $u > 0$ on $[0,1)$ and satisfying (1.2). Our main result is

Theorem 1.1. *Let (A1)–(A3) hold and suppose either*

(A4) $f(z) > 0$ *for* $z > 0$ *with* $\liminf_{z\to 0^+} \frac{f(z)}{z^{q-1}} > 0$.

or

(A5) $-\infty < f(0^+) < 0$ *and there exists* $\beta > 0$ *such that* $(z-\beta)f(z) > 0$ *for* $z \neq \beta$.

holds. Then there exists a constant $\bar{\lambda} > 0$ *such that* (1.2) *has a unique positive solution for* $\lambda > \bar{\lambda}$.

Remark 1.2.

(i) Note that (A2) implies $\frac{f(z)}{z^{q-1}}$ *is decreasing for* z *large and* $\lim_{z\to\infty} \frac{f(z)}{z^{q-1}} = 0$.

(ii) It is easily seen from (A3) that $\limsup_{z\to 0^+} z^\gamma |f(z)| < \infty$. *Since* $\liminf_{z\to\infty} z^\gamma f(z) > 0$ *and* $\limsup_{z\to 0^+} z^\gamma |f(z)| < \infty$ *, it follows from [1, Theorem 1.1] that* (1.2) *has a positive solution for* λ *large under the assumptions of Theorem 1.1.*

2 Preliminary results

Let $H(z) = \frac{\phi_p(z)}{f(z)}$. Then H is increasing for z large and $\lim_{z\to\infty} H(z) = \infty$ in view of Remark 1.2 (i).

Lemma 2.1. *Let $K > 0$. Then for $\lambda \gg 1$,*

(i) $H^{-1}(\lambda K) \le K^{\frac{1}{p-q}} H^{-1}(\lambda)$ *if* $K > 1$.

(ii) $H^{-1}(\lambda K) \ge K^{\frac{1}{p-q}} H^{-1}(\lambda)$ *if* $K < 1$.

Proof. Let $z_\lambda = H^{-1}(\lambda)$ and $\tilde{K} = K^{\frac{1}{p-q}}$. Then $z_\lambda \to \infty$ as $\lambda \to \infty$.

Suppose $K > 1$. Then $f(\tilde{K} z_\lambda) \le \tilde{K}^{q-1} f(z_\lambda)$ for λ large in view of Remark 1.2 (i), which implies

$$\frac{\phi_p(\tilde{K} z_\lambda)}{f(\tilde{K} z_\lambda)} \ge \tilde{K}^{p-q} \frac{\phi_p(z_\lambda)}{f(z_\lambda)} = \lambda \tilde{K}^{p-q} = \lambda K,$$

i.e. $\tilde{K} H^{-1}(\lambda) = \tilde{K} z_\lambda \ge H^{-1}(\lambda K)$ and (i) holds. If $K < 1$ then by replacing λ by λK and K by K^{-1} in (i), we obtain (ii). □

Lemma 2.2. *Let $\lambda, x, c > 0$. Then*

$$\phi^{-1}(\lambda x) \ge a \phi_p^{-1}(\lambda x)$$

for $\lambda x > c$, provided that $a^{p-1} + a^{q-1} c^{\frac{q-p}{p-1}} \le 1$.

Proof. Let $y = \phi_p^{-1}(\lambda x)$. Then

$$\phi(ay) = (ay)^{p-1} + (ay)^{q-1} = a^{p-1} \lambda x + a^{q-1} (\lambda x)^{\frac{q-1}{p-1}}. \tag{2.1}$$

Since $\lambda x > c$,

$$a^{q-1} (\lambda x)^{\frac{q-1}{p-1}} = a^{q-1} c^{\frac{q-1}{p-1}} \left(\frac{\lambda x}{c}\right)^{\frac{q-1}{p-1}} < a^{q-1} c^{\frac{q-1}{p-1}} \left(\frac{\lambda x}{c}\right) = a^{q-1} c^{\frac{q-p}{p-1}} \lambda x,$$

which together with (2.1) imply

$$\phi(ay) \le \left(a^{p-1} + a^{q-1} c^{\frac{q-p}{p-1}}\right) \lambda x \le \lambda x$$

if $a^{p-1} + a^{q-1} c^{\frac{q-p}{p-1}} \le 1$ i.e. $ay \le \phi^{-1}(\lambda x)$, which completes the proof. □

If (A4) holds then clearly positive solutions to (1.2) are decreasing on (0,1). The next result shows this is also true in the semipositone case under assumption (A5).

Lemma 2.3. *Let (A1) and (A5) hold. Then any positive solution u of (1.2) is decreasing on $(0,1)$ with $u(0) \ge \theta$, where $\theta > \beta$ is such that $(z-\theta)F(z) > 0$ for $z > 0$, $z \ne \theta$. Here $F(z) = \int_0^z f$.*

Proof. Let u be a positive solution of (1.2). Since $f(0^+) < 0$, it follows that $(r^{N-1}\phi(u'))' > 0$ for r near 1 and therefore $u'(r) < u'(1) \le 0$ for such r. Let $r_0 \in [0,1)$ be the smallest number such that $u' < 0$ on $(r_0, 1)$ and suppose $r_0 > 0$. Multiplying the equation in (1.2) by u' gives

$$G'(r) = -(N-1) r^{N-2} \Phi(u') + \lambda (n-1) r^{N-2} F(u) \quad \text{on } (0,1),$$

where $\Phi(z) = \int_0^z \phi$ and $G(r) = r^{N-1}(\phi(u')u' - \Phi(u') + \lambda F(u))$. Note that $G(1) = (1 - 1/p)|u'(1)|^p + (1 - 1/q)|u'(1)|^q \geq 0$.

We claim that $u(r_0) > \theta$. Indeed, if $u(r_0) \leq \theta$ then $u < \theta$ on $(r_0, 1)$, which implies G is decreasing on $(r_0, 1)$. Since $u'(r_0) = 0$, it follows that

$$0 \leq G(1) \leq G(r) < G(r_0) \leq 0 \quad \text{on } (r_0, 1),$$

a contradiction. Hence $u(r_0) > \theta$ and so $(r^{N-1}(\phi(u'))' < 0$ near r_0. Consequently, $u'(r) > 0$ for r near $r_0, r < r_0$. Let $r_1 \in [0, r_0)$ such that $u' > 0$ on (r_1, r_0) and $u'(r_1) = 0$. Since $r_0^{N-1}\phi(u'(r_0)) = 0 = r_1^{N-1}\phi(u'(r_1))$, there exists $r_2 \in (r_1, r_0)$ such that

$$-(r^{N-1}\phi(u'(r))'(r_2) = \lambda r_2^{N-1} f(u(r_2)) = 0$$

i.e. $u(r_2) = \beta$. Thus $\theta \in u((r_2, r_0])$ and there exists $r_3 \in (r_2, r_0)$ such that $u < \theta$ on $[r_2, r_3)$ and $u(r_3) = \theta$. Since $u' > 0$ on (r_1, r_0), $u < \theta$ on $[r_1, r_3)$ i.e. and so $G' < 0$ on $[r_1, r_3)$, which implies

$$0 \leq G(r_3) < G(r) < G(r_1) < 0,$$

a contradiction. Thus $r_0 = 0$, which completes the proof. □

Lemma 2.4. *Let $c > 0$ and suppose (A1), (A4) hold. Then*

$$\limsup_{x \to 0^+} \frac{\phi(cx)}{\check{f}(x)} < \infty,$$

where $\check{f}(x) = \inf_{y \geq x} f(y)$.

Proof. Since $\liminf_{y \to 0^+} \frac{f(y)}{y^{q-1}} > 0$, there exist constants $k, \delta > 0$ such that

$$f(y) \geq k\phi(cy) \quad \text{for } y \in (0, \delta). \tag{2.2}$$

Let $x < \delta$ and $y \geq x$. If $y < \delta$ then (2.2) gives

$$f(y) \geq k\phi(cx), \tag{2.3}$$

while if $y > \delta$ then

$$f(y) \geq k_0 = k_1\phi(c\delta) \geq k_1\phi(cx), \tag{2.4}$$

where $k_0 = \inf_{[\delta, \infty)} f > 0$ and $k_1 = k_0/\phi(c\delta)$.

Combining (2.3) and (2.4), we obtain

$$\check{f}(x) \geq k_2\phi(cx) \quad \text{for } x \in (0, \delta),$$

where $k_2 = \min(k, k_1)$, which completes the proof. □

The next result gives sharp lower and upper bound estimates for positive solution of (1.2) when λ is large.

Lemma 2.5. *Let the assumptions of Theorem 1.1 hold. Then there exist positive constants A_1, A_2, and λ_0 such that for $\lambda > \lambda_0$, any positive solution of (1.2) satisfies*

$$A_1 B_\lambda (1 - r) \leq u(r) \leq A_2 B_\lambda (1 - r) \tag{2.5}$$

for $r \in (0, 1)$, where $B_\lambda = \phi_p^{-1}\left(\lambda f(H^{-1}(\lambda))\right)$.

Proof. In what follows, $\lambda \gg 1$ or λ large means $\lambda > \lambda^*$ for some $\lambda^* > 0$ independent of u and λ.

Case 1. Suppose (A4) holds.

Let $\lambda > 0$ and u be a positive solution of (1.2). Then u is decreasing on $(0,1)$. By integrating, we get

$$-u'(r) = \phi^{-1}\left(\frac{\lambda}{r^{N-1}}\int_0^r s^{N-1} f(u)ds\right), \qquad 0 < r < 1.$$

Recalling that $\check{f}(x) = \inf_{y \geq x} f(y)$ and since $\check{f}$ is nondecreasing, we have

$$-u'(r) \geq \phi^{-1}\left(\frac{\lambda}{r^{N-1}}\int_0^{1/2} s^{N-1}\check{f}(u)ds\right) \geq \phi^{-1}\left(\lambda c_1 \check{f}\left(u\left(1/2\right)\right)\right) \quad \text{for } r > 1/2, \tag{2.6}$$

where $c_1 = \frac{1}{N2^N}$, which implies upon integrating on $(1/2,1)$ that

$$2u(1/2) \geq \phi^{-1}\left(\lambda c_1 \check{f}\left(u\left(1/2\right)\right)\right)$$

i.e.

$$\frac{\phi(2u(1/2))}{\check{f}\left(u\left(1/2\right)\right)} \geq \lambda c_1. \tag{2.7}$$

Since $\limsup_{x\to 0^+} \frac{\phi(2x)}{\check{f}(x)} < \infty$ in view of Lemma 2.4, we deduce from (2.7) that $u(1/2) \to \infty$ as $\lambda \to \infty$.

Hence $u(1/2) > 1$ for $\lambda \gg 1$, and so

$$\phi(2u(1/2)) \leq 2\phi_p(2u(1/2)) = 2^p \phi_p(u(1/2)),$$

which together with (2.7) and the fact that $f(z) = \check{f}(z)$ for $z > L$, gives

$$H(u(1/2)) = \frac{\phi_p(u(1/2))}{f\left(u\left(1/2\right)\right)} \geq \lambda c_2$$

where $c_2 = \min(c_1/2^p, 1)$. Hence by Lemma 2.1 (ii),

$$u(1/2) \geq H^{-1}(\lambda c_2) \geq c_3 H^{-1}(\lambda), \tag{2.8}$$

where $c_3 = c_2^{\frac{1}{p-q}}$. From (2.6) and (2.8), we get

$$-u'(r) \geq \phi^{-1}\left(\lambda c_1 f\left(c_3 H^{-1}(\lambda)\right)\right) \geq \phi^{-1}\left(\lambda c_1 c_3^{q-1} f\left(H^{-1}(\lambda)\right)\right) \quad \text{for } r > 1/2, \tag{2.9}$$

For $\lambda \gg 1, \lambda c_1 c_3^{q-1} f\left(H^{-1}(\lambda)\right) > 1$, from which (2.9) and Lemma 2.2 with $c = 1$ and $x = c_1 c_3^{q-1} f\left(H^{-1}(\lambda)\right)$ give

$$-u'(r) \geq a\phi_p^{-1}\left(\lambda c_1 c_3^{q-1} f\left(H^{-1}(\lambda)\right)\right) \equiv a_1 \phi_p^{-1}(\lambda f\left(H^{-1}(\lambda)\right) = a_1 B_\lambda \quad \text{for } r > 1/2, \tag{2.10}$$

where $B_\lambda = \phi_p^{-1}\left(\lambda f(H^{-1}(\lambda)\right)$ and $a > 0$ is such that $a^{p-1} + a^{q-1} \leq 1$ and $a_1 = a\phi_p^{-1}(c_1 c_3^{q-1})$.

Note that $B_\lambda \to \infty$ as $\lambda \to \infty$. Integrating (2.10) yields

$$u(r) \geq a_1 B_\lambda (1 - r) \tag{2.11}$$

for $r > 1/2$, while for $r < 1/2$,

$$u(r) > u(1/2) \geq \frac{a_1}{2} B_\lambda \geq \frac{a_1}{2} B_\lambda (1-r). \tag{2.12}$$

Combining (2.11) and (2.12), we obtain

$$u(r) \geq A_1 B_\lambda (1-r) \quad \text{for } r \in (0,1), \tag{2.13}$$

where $A_1 = a_1/2$ i.e. the lower bound estimate in (2.5) holds.

Let $C = \sup_{z\in(0,L)} z^\gamma f(z)$ and note that $0 < C < \infty$. Since $f > 0$ on $(0,\infty)$,

$$f(z) \leq \frac{C}{z^\gamma} + f(\max(z, L)) \tag{2.14}$$

for $z > 0$. Since $\|u\|_\infty \to \infty$ as $\lambda \to \infty$, $\|u\|_\infty > L$ for $\lambda \gg 1$ and hence (2.13)–(2.14) yield

$$f(u(\tau)) \leq \frac{C}{u^\gamma(\tau)} + f(\max(u(\tau), L)) \ \leq \frac{C_1}{B_\lambda^\gamma (1-\tau)^\gamma} + f(\|u\|_\infty),$$

for $\tau \in (0,1)$, where $C_1 = C/A_1^\gamma$. Thus

$$\begin{aligned}
u(r) &= \int_r^1 \phi^{-1}\left(\frac{\lambda}{s^{N-1}} \int_0^s \tau^{N-1} f(u) d\tau\right) ds \\
&\leq \int_r^1 \phi^{-1}\left(\frac{\lambda}{s^{N-1}} \int_0^s \tau^{N-1} \left(\frac{C_1}{B_\lambda^\gamma (1-\tau)^\gamma} + f(\|u\|_\infty)\right) d\tau\right) ds \\
&\leq \int_r^1 \phi^{-1}\left(\lambda \left(C_1 \int_0^s \frac{d\tau}{(1-\tau)^\gamma} + f(\|u\|_\infty)\right)\right) \leq \phi^{-1}\left(\lambda(C_2 + f(\|u\|_\infty))\right)(1-r) \\
&\leq \phi^{-1}(\lambda C_3 f(\|u\|_\infty))(1-r) \leq \phi_p^{-1}(\lambda C_3 f(\|u\|_\infty))(1-r)
\end{aligned} \tag{2.15}$$

for $r \in (0,1)$, where $C_2 = C_1 \int_0^1 \frac{d\tau}{(1-\tau)^\gamma}$ and $C_3 > 1$ is such that $(C_3 - 1) f(L) > C_2$.

In particular,

$$\|u\|_\infty \leq \phi_p^{-1}(\lambda C_3 f(\|u\|_\infty)),$$

which implies

$$H(\|u\|_\infty) = \frac{\phi_p(\|u\|_\infty)}{f(\|u\|_\infty)} \leq \lambda C_3,$$

and therefore

$$\|u\|_\infty \leq H^{-1}(\lambda C_3) \leq C_4 H^{-1}(\lambda), \tag{2.16}$$

in view of Lemma 2.1 (i), where $C_4 = C_3^{\frac{1}{p-q}}$.

Combining (2.15)–(2.16) and Remark 1.2 (i), we infer that

$$\begin{aligned}
u(r) &\leq \phi_p^{-1}(\lambda C_3 f\left(C_4 H^{-1}(\lambda)\right))(1-r) \leq \phi_p^{-1}(\lambda C_3 C_4^{q-1} f(H^{-1}(\lambda))(1-r) \\
&= A_2 \phi_p^{-1}(\lambda f(H^{-1}(\lambda))(1-r) \equiv A_2 B_\lambda (1-r)
\end{aligned}$$

for $r \in (0,1)$, where $A_2 = \phi_p^{-1}(C_3 C_4^{q-1})$ i.e. the upper bound estimate in (2.5) holds, which completes the proof.

Case 2. Suppose (A5) holds.

By Lemma 2.3, u is decreasing on $(0,1)$. Recall that $\beta, \theta > 0$ are such that $(z-\beta)f(z) > 0$ for $z \neq \beta$ and $(z-\theta)F(z) > 0$ for for $z \neq \theta$. Let $\rho = \frac{\beta+\theta}{2}$ and $\delta \in (0,1)$ be such that

$$\frac{\delta^N}{2N} \inf_{[\rho,\infty)} f - (1-\delta)K > 0, \tag{2.17}$$

where $K > 0$ is such that $f(z) \geq -K$ for all $z \in (0,\infty)$.

We will show that $u(\delta) \to \infty$ as $\lambda \to \infty$. To this end, we need to verify the following claims. Let $\beta_1 \in (0,\beta)$ and δ_0, δ_1 satisfy $\delta < \delta_0 < \delta_1 < 1$.

Claim 1. $u(\delta_1) > \beta_1$ *for* $\lambda \gg 1$.

Suppose to the contrary that $u(\delta_1) \leq \beta_1$. Then $u < \beta_1$ on $(\delta_1, 1)$, which implies

$$(r^{N-1}\phi(u'))' = -\lambda r^{N-1} f(u) \geq \lambda m r^{N-1} \quad \text{on } (\delta_1, 1), \tag{2.18}$$

where $m = -\sup_{(0,\beta_1)} f > 0$.

Let $\delta_2 \in (\delta_1, 1)$. By the Mean Value Theorem, there exists $\sigma \in (\delta_1, \delta_2)$ such that

$$|u'(\sigma)| = \frac{u(\delta_1) - u(\delta_2)}{\delta_2 - \delta_1} \leq \frac{\beta_1}{\delta_2 - \delta_1} \equiv \bar{\beta}.$$

Hence by integrating (2.18) on $(\sigma, 1)$, we obtain

$$0 \geq \phi(u'(1)) \geq \sigma^{N-1}\phi(u'(\sigma)) + \lambda m \int_\sigma^1 r^{N-1} dr \geq \lambda m \int_{\delta_2}^1 r^{N-1} dr - \phi(\bar{\beta}) > 0,$$

for λ large, a contradiction which proves the claim.

Claim 2. $u(\delta_0) > \rho$ *for* $\lambda \gg 1$.

Suppose to the contrary that $u(\delta_0) \leq \rho$. Then $u < \rho$ on $(\delta_0, 1)$ and therefore $G' < 0$ on $(\delta_0, 1)$, where G is defined in the proof of Lemma 2.3. Hence $G(r) \geq G(1) \geq 0$ on $(\tau_0, 1)$ i.e.

$$\phi(u')u' - \Phi(u') + \lambda F(u)) \geq 0 \quad \text{on } (\delta_0, 1),$$

or, equivalently,

$$(1 - 1/p)|u'|^p + (1 - 1/q)|u'|^q \geq -\lambda F(u) \quad \text{on } (\delta_0, 1). \tag{2.19}$$

By Claim 1 and the monotonicity of u,

$$\beta_1 \leq u \leq \rho \quad \text{on } (\delta_0, \delta_1),$$

and hence $\kappa \equiv \inf_{[\beta_1,\rho]}(-F(z)) > 0$. Consequently, (2.19) gives

$$|u'|^p + |u'|^q \geq \frac{\lambda q \kappa}{q-1} \equiv \lambda \kappa_0 \quad \text{on } (\delta_0, \delta_1).$$

For $\lambda\kappa_0 > 2$, this implies $|u'| > 1$ and hence $2|u'|^p \geq \lambda\kappa_0$ on (δ_0, δ_1) follows i.e.

$$-u' = |u'| \geq (\lambda\kappa_0/2)^{1/p} \quad \text{on } (\delta_0, \delta_1). \tag{2.20}$$

Integrating (2.20) on (δ_0, δ_1) gives

$$u(\delta_0) \geq (\lambda\kappa_0/2)^{1/p}(\delta_1 - \delta_0) > \rho$$

for $\lambda \gg 1$, a contradiction which proves Claim 2.

By Claim 2, $u > \rho$ on $(0, \delta_0)$ and hence

$$f(u) \geq \kappa_1 \quad \text{on } (0, \delta_0),$$

where $\kappa_1 = \inf_{[\rho,\infty)} f > 0$. This implies

$$u(\delta) = u(\delta_0) + \int_\delta^{\delta_0} \phi^{-1}\left(\frac{\lambda}{s^{N-1}}\int_0^s \tau^{N-1} f(u)d\tau\right) ds \geq (\delta_0 - \delta)\phi^{-1}\left(\lambda\kappa_1 \int_0^\delta \tau^{N-1}d\tau\right) ds,$$

and thus $u(\delta) \to \infty$ as $\lambda \to \infty$.

Hence for λ large, $u(\delta) > \rho$ and for $r > \delta$, we have

$$\int_0^r s^{N-1} f(u)ds = \int_0^\delta s^{N-1} f(u)ds + \int_\delta^r s^{N-1} f(u)ds \geq \frac{\delta^N}{N} f(u(\delta)) - (1-\delta)K \geq \frac{\delta^N}{2N} f(u(\delta))$$

in view of (2.17). Hence

$$-u'(r) = \phi^{-1}\left(\frac{\lambda}{r^{N-1}}\int_0^r s^{N-1} f(u)ds\right) \geq \phi^{-1}\left(\frac{\lambda\delta^N f(u(\delta)}{2N}\right), \tag{2.21}$$

for $r > \delta$, and upon integrating on $(\delta, 1)$, we get

$$u(\delta) \geq (1-\delta)\phi^{-1}\left(\frac{\lambda\delta^N f(u(\delta))}{2N}\right)$$

i.e.

$$\frac{\phi\left(cu(\delta)\right)}{f(u(\delta))} \geq \lambda D_1, \tag{2.22}$$

where $c = (1-\delta)^{-1}$ and $D_1 = \frac{\delta^N}{2N}$.

For $\lambda \gg 1$, $cu(\delta > 1$ and $\phi\left(cu(\delta)\right) \leq 2\phi_p(cu(\delta)) = 2\phi_p(c)\phi_p(\delta)$ and (2.22) becomes

$$H(u(\delta)) = \frac{\phi_p(u(\delta))}{f(u(\delta))} \geq \lambda D_2,$$

where $D_2 = \frac{D_1}{2\phi_p(c)} < 1$, which implies

$$u(\delta) \geq H^{-1}(\lambda D_2) \geq D_3 H^{-1}(\lambda) \tag{2.23}$$

follows, where $D_3 = D_2^{\frac{1}{p-q}}$.

Combining (2.21), (2.23), Remark 1.2 (i), and Lemma 2.2, we obtain for $\lambda \gg 1$,

$$\begin{aligned} -u'(r) &\geq \phi^{-1}\left(\frac{\lambda\delta^N f(D_3 H^{-1}(\lambda))}{2N}\right) \geq \phi^{-1}\left(\frac{\lambda\delta^N D_3^{q-1} f(H^{-1}(\lambda))}{2N}\right), \\ &\geq a\phi_p^{-1}\left(\frac{\lambda\delta^N D_3^{q-1} f(H^{-1}(\lambda))}{2N}\right) = D_4\phi_p^{-1}\left(\lambda f(H^{-1}(\lambda))\right), \end{aligned}$$

for $r \in (\delta, 1)$, where $D_4 = a\phi_p^{-1}(\frac{\delta^N D_3^{q-1}}{2N})$. Integrating this inequality gives

$$u(r) \geq D_4\phi_p^{-1}(\lambda f(H^{-1}(\lambda))(1-r) \quad \text{for } r \in (\delta, 1),$$

which, together with the monotonicity of u, gives the lower bound estimate in (2.5).

For the upper bound estimate, by increasing L if necessary, we can assume $L > \beta$ with $f(L) > 0$. Hence

$$f(z) \le C_1 + f(\max(z, L))$$

for $z > 0$, where $C_1 = \max_{[\beta,L]} f$.

Suppose $\|u\|_\infty > L$ and $C_2 > 1$ be such that

$$C_1 < (C_2 - 1)f(L) < (C_2 - 1)f(\|u\|_\infty).$$

Then

$$\begin{aligned} u(r) &= \int_r^1 \phi^{-1}\left(\frac{\lambda}{s^{N-1}}\int_0^s \tau^{N-1} f(u)d\tau\right) ds \le \int_r^1 \phi^{-1}\left(\frac{\lambda}{s^{N-1}}\int_0^s \tau^{N-1}\left(C_1 + f(\|u\|_\infty)\right) d\tau\right) ds \\ &\le \int_r^1 \phi^{-1}\left(\frac{\lambda}{s^{N-1}}\int_0^s \tau^{N-1}\left(C_2 f(\|u\|_\infty)\right) d\tau\right) ds \le \phi^{-1}\left(\lambda(C_2 f(\|u\|_\infty)\right)(1-r) \\ &\le \phi_p^{-1}(\lambda C_2 f(\|u\|_\infty))(1-r) \end{aligned}$$

for $r \in (0,1)$. The rest of the proof uses the same arguments as in the upper bound estimate for Case 1, which completes the proof. □

Lemma 2.6. *Let $\alpha \in (0,1)$ and $\tau > 0$. Then*

$$\min(\tau, 1)(1-\alpha) \le 1 - \alpha^\tau \le \max(\tau, 1)(1-\alpha). \tag{2.24}$$

Proof. Suppose $\tau < 1$. Then $1 - \alpha^\tau \le 1 - \alpha$. By the Mean Value Theorem with $g(\alpha) = (1-\alpha)^\tau$,

$$(1-\alpha)^\tau = |g(\alpha) - g(1)| = (1-\alpha)\frac{\tau}{(1-\xi)^{1-\tau}} \ge \tau(1-\alpha),$$

for some $\xi \in (\alpha, 1)$ i.e. (2.24) holds. The case $\tau > 1$ is proved in the same way. □

The existence of a positive solution to (1.1) in a general bounded domain is based on the following result:

Lemma 2.7 ([1, Theorem 1.1])**.** *Suppose*

(A1) $f : (0,\infty) \to \mathbb{R}$ *is continuous and* $\lim_{u\to\infty} \frac{f(u)}{u^{p-1}} = 0$.

(A2) There exist constants $A, L > 0$ and $0 \le \beta < 1$ such that

$$f(u) \ge \frac{A}{u^\beta} \quad \text{for } u > L$$

and

$$\limsup_{u\to 0^+} u^\beta |f(u)| < \infty.$$

Then there exists a constant $\lambda_0 > 0$ such that (1.1) has a positive solution u_λ for $\lambda > \lambda_0$ with $\inf_\Omega \frac{u_\lambda}{d} \to \infty$ as $\lambda \to \infty$, where $d(x)$ denotes the distance from x to $\partial\Omega$.

3 Proof of Theorem 1.1

By Lemma 2.7, (1.2) has a positive solution for λ large. Let u, v be such solutions. In view of Lemma 2.5, there exists a maximum constant $\alpha \in (0,1]$ such that $\alpha v \le u \le \alpha^{-1} v$ on $(0,1)$. Then $\alpha \ge A_1/A_2 \equiv \alpha_0$ by (2.5). Recalling that $L > 0$ is such that f is increasing on (L, ∞). By using (A2) and increasing L if necessary, we can assume that $f(L) > 0$ and $z^{-q_0} f(z)$ is decreasing on $[L, \infty)$ for some $q_0 \in (0, q-1)$. We will show that $\alpha \ge 1$. Suppose to the contrary that $\alpha < 1$. Suppose $\lambda > \lambda_0$, where λ_0 is defined in Lemma 2.5.

Let $L_1 = \alpha_0^{-1} L$. Then

$$v(1/2) > L_1$$

for $\lambda > \lambda^*$ for some $\lambda^* > 0$ independent of u, v.

For $r \le 1/2$,

$$u(r) \ge \alpha v(r) \ge \alpha_0 v(1/2) > \alpha_0 L_1 = L,$$

which implies

$$f(u(r)) \ge f(\alpha v(r)) \ge \alpha^{q_0} f(v(r)). \tag{3.1}$$

Suppose $r > 1/2$.

Case 1. $v(r) \ge L_1$

Then $u(r) > L$ and therefore (3.1) holds.

Case 2. $v(r) < L_1$.

Then

$$u(r) \le \alpha_0^{-1} v(r) < \alpha_0^{-1} L_1 \equiv L_2.$$

Let $K_0 = \sup_{z \in (0, L_2)} z^\gamma |f(z)|$ and $K_1 = \sup_{z \in (0, L_2)} z^{\gamma+1} |f'(z)|$. Then $K_0, K_1 < \infty$ by (A3) and (A5). Then the lower bound estimate in Lemma 2.5 gives,

$$|f(v(r))| \le \frac{K_0}{v^\gamma(r)} \le \frac{K_0}{A_1^\gamma B_\lambda^\gamma (1-r)^\gamma}, \tag{3.2}$$

which implies in view of Lemma 2.6,

$$f(v(r)) \ge \alpha^{q_0} f(v(r)) - (1 - \alpha^{q_0}) |f(v(r))| \ge \alpha^{q_0} f(v(r)) - \frac{K_0 c_{q_0} (1-\alpha)}{A_1^\gamma B_\lambda^\gamma (1-r)^\gamma}, \tag{3.3}$$

where $c_{q_0} = \max(1, q_0)$, and

$$\begin{aligned} |f(u(r)) - f(v(r))| = |u(r) - v(r)| |f'(\xi)| &\le \frac{K_1 (1-\alpha) v(r)}{\alpha \xi^{\gamma+1}} \\ &\le \frac{K_1 (1-\alpha)}{\alpha_0^{2+\gamma} v^\gamma(r)} \le \frac{K_1 (1-\alpha)}{\alpha_0^{2+\gamma} A_1^\gamma B_\lambda^\gamma (1-r)^\gamma} \end{aligned} \tag{3.4}$$

for some ξ between $u(r)$ and $v(r)$. From (3.3) and (3.4), we deduce that

$$f(u(r)) \ge f(v(r)) - |f(u(r)) - f(v(r))| \ge \alpha^{q_0} f(v(r)) - \frac{K_2 (1-\alpha)}{B_\lambda^\gamma (1-r)^\gamma}, \tag{3.5}$$

where $K_2 = (K_0 c_{q_0} + K_1 \alpha_0^{-2-\gamma}) / A_1^\gamma$. Hence (3.5) holds for $r > 1/2$ in both cases.

Let $q_1 \in (q_0, q-1)$. Then for $r > 1/2$, it follows from (3.2), (3.5) and Lemmas 2.5–2.6 that

$$\begin{aligned} f(u(r)) - \alpha^{q_1} f(v(r)) &\geq (\alpha^{q_0} - \alpha^{q_1}) f(v(r)) - \frac{K_2(1-\alpha)}{B_\lambda^\gamma (1-r)^\gamma} \\ &\geq -\frac{(1-\alpha^{q_1-q_0})K_0}{A_1^\gamma B_\lambda^\gamma (1-r)^\gamma} - \frac{K_2(1-\alpha)}{B_\lambda^\gamma (1-r)^\gamma} \geq -\frac{K_3(1-\alpha)}{B_\lambda^\gamma (1-r)^\gamma}, \end{aligned} \tag{3.6}$$

where $K_3 = K_0 A_1^{-\gamma} \max(1, q_1 - q_0) + K_2$.

We claim next that

$$\phi(u') \leq \alpha^{q_1} \phi(v') \tag{3.7}$$

on $(0,1)$. Using the formula

$$-r^{N-1}[\phi(u'(r))) - \alpha^{q_1}\phi(v'(r)] = \lambda \int_0^r s^{N-1}(f(u) - \alpha^{q_1} f(v))ds, \tag{3.8}$$

for $r \in (0,1)$, we see from (3.1) that for $r \leq 1/2$,

$$f(u(s)) - \alpha^{q_1} f(v(s)) \geq (\alpha^{q_0} - \alpha^{q_1}) f(v(s)) > 0 \tag{3.9}$$

for $s \leq 1/2$ i.e. (3.7) holds on $(0, 1/2]$. For $r > 1/2$, it follows from (3.6), (3.8), (3.9), and Lemma 2.6 that

$$\begin{aligned} -r^{N-1}(\phi(u') - \alpha^{q_1}\phi(v')) &= \lambda \left(\int_0^{1/2} s^{N-1}(f(u) - \alpha^{q_1} f(v))ds + \int_{1/2}^r s^{N-1}(f(u) - \alpha^{q_1} f(v))ds \right) \\ &\geq \lambda \left[(\alpha^{q_0} - \alpha^{q_1}) \int_0^{1/2} s^{N-1} f(v) ds - \frac{K_3(1-\alpha)}{B_\lambda^\gamma} \int_{1/2}^1 \frac{1}{(1-s)^\gamma} ds \right] \\ &\geq \lambda(1-\alpha) \left[K_4 f(v(1/2)) - \frac{K_3}{B_\lambda^\gamma} \int_\delta^1 \frac{1}{(1-s)^\gamma} ds \right] > 0, \end{aligned}$$

where $K_4 = \frac{\alpha_0^{q_0} \min(1, q_1 - q_0)}{N 2^N}$, provided that $\lambda > \bar{\lambda}$, where $\bar{\lambda} > \lambda_0$ is such that

$$K_4 f(L_1) - \frac{K_3}{B_\lambda^\gamma} \int_{1/2}^1 \frac{1}{(1-s)^\gamma} ds > 0.$$

Note that this is possible since $f(L_1) > 0$ and $B_\lambda \to \infty$ as $\lambda \to \infty$. Hence (3.7) holds on $(0,1)$, which implies

$$\phi(u') + \alpha^{q_1}\phi(|v'|) \leq 0. \tag{3.10}$$

Since

$$\phi\left(\alpha^{\frac{q_1}{q-1}}|v'|\right) = \alpha^{q_1}|v'|^{q-1} + \alpha^{\frac{q_1(p-1)}{q-1}}|v'|^{p-1} \leq \alpha^{q_1}(|v'|^{q-1} + |v'|^{p-1}) = \alpha^{q_1}\phi(|v'|),$$

it follows from (3.10) that

$$\phi(u') + \phi\left(\alpha^{\frac{q_1}{q-1}}|v'|\right) \leq 0,$$

i.e.

$$\phi(u') \leq -\phi\left(\alpha^{\frac{q_1}{q-1}}|v'|\right) = \phi\left(\alpha^{\frac{q_1}{q-1}} v'\right).$$

Hence $\left(u - \alpha^{\frac{q_1}{q-1}} v\right)' \leq 0$ on $(0,1)$ and since $u(1) = v(1) = 0$, it follows that $u \geq \alpha^{\frac{q_1}{q-1}} v$ on $(0,1)$, a contradiction with the maximality of α since $q_1 < q-1$. This completes the proof of Theorem 1.1. □

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A note on the stability of periodic orbits of planar Lotka–Volterra systems

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Abstract. In this paper we revisit previous results in the literature dealing with the stability of periodic solutions of periodic predator–prey Lotka–Volterra systems. These results provide stability criteria for the periodic orbits based on boundaries for the average of the coexistence states. In the way it was presented, these boundaries are independent of each other and thus provide a very practical sufficient condition for asymptotic stability. In this note we prove that this independent boundaries can be refined into an intertwined boundary, providing a more sharp sufficient condition.

Keywords: Lotka–Volterra systems, population dynamics, predator–prey model.

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1 Introduction

Consider the planar non-autonomous periodic Lotka-–Volterra system given by

$$\dot{u} = u(a(t) - b(t)u - c(t)v), \quad \dot{v} = v(d(t) + e(t)u - f(t)v), \tag{1.1}$$

with continuous and T-periodic coefficients, $T > 0$. Suppose that b, c, e and f are strictly positive functions. Under these conditions (1.1) models in the positive quadrant $\mathbb{R}^2_+ = \{(u,v) \in \mathbb{R}^2 \colon u > 0,\ v > 0\}$ a predator-prey interaction between two species in a T-periodic environment. In the last decades, there is a great effort of R. Ortega and coauthors for the understanding of the T-periodic orbits of (1.1). In what follows, we present a brief survey on the obtained results. System (1.1) can have three types T-periodic non-negative solutions $(u(t), v(t))$.

(a) The *trivial state*, given by $u(t) = v(t) = 0$ for every $t \in [0,T]$;

(b) The *semi-trivial state*, given by:

(i) $u(t) > 0$, $v(t) = 0$ for every $t \in [0,T]$, or

(ii) $u(t) = 0$, $v(t) > 0$ for every $t \in [0,T]$;

(c) The *coexistence state*, given by $u(t) > 0$ and $v(t) > 0$ for every $t \in [0,T]$.

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Observe that the trivial and semi-trivial states represents the extinction of at least one of the species. The question about the *existence* of coexistence states was completely solved by the authors in [4]. More precisely, let

$$\lambda = \frac{1}{T}\int_0^T a, \quad \mu = \frac{1}{T}\int_0^T d.$$

It follows from [4] that (1.1) has a semi-trivial state of type (i) (resp. (ii)) if, and only if, $\lambda > 0$ (resp, $\mu > 0$). Moreover, in this case the semi-trivial state $(\theta_\lambda(t), 0)$ (resp. $(0, \theta_\mu(t))$ is given by the unique positive T-periodic solution of

$$\dot{u} = u(a(t) - b(t)u) \quad \big(\text{resp. } \dot{v} = v(d(t) - f(t)v)\big).$$

In particular, if it exists then it is unique (of its type). From [4, Section 4] we have that the following statements hold (for the definition of *linear* and *asymptotic* stability, see Appendix A).

(a) The trivial state $(0, 0)$ is linearly stable if, and only if, $\lambda \leqslant 0$ and $\mu \leqslant 0$. Moreover, in this case it is also asymptotically stable in $\mathbb{R}^2_+$.

(b) The semi-trivial state $(\theta_\lambda, 0)$ is linearly stable if, and only if, $\lambda > 0$ and

$$\mu \leqslant -\frac{1}{T}\int_0^T e\theta_\lambda.$$

Moreover, in this case it is also asymptotically stable in $\mathbb{R}^2_+$.

(c) The semi-trivial state $(0, \theta_\mu)$ is linearly stable if, and only if, $\mu > 0$ and

$$\lambda \leqslant \frac{1}{T}\int_0^T c\theta_\mu.$$

Moreover, in this case it is also asymptotically stable in $\mathbb{R}^2_+$.

It can be proved [4, Theorem 4.1] that for a coexistence state to exist it is necessary to the trivial and semi-trivial states (when they exist) to be linearly unstable. That is, it is necessary

$$\mu > -\frac{1}{T}\int_0^T e\theta_\lambda, \quad \lambda > \frac{1}{T}\int_0^T c\theta_\mu. \tag{1.2}$$

On the other hand, it also follows from [4, Section 5] that (1.2) is as well a sufficient condition and thus we have a complete characterization on the existence of coexistence states for system (1.1). Moreover, it follows from [4, Theorem 5.1] that system (1.1) always has at most a finite number of coexistence states. With the existence completely characterized, the question turned to the finiteness, uniqueness and stability of the coexistence states. For this end, before we enunciate the obtained results, we introduce some technical notations.

Let $\mathbb{T}_T = \mathbb{R}/T\mathbb{Z}$ be endowed with the push-forward measure of the Lebesgue measure on $\mathbb{R}$ (see Bogachev [2, Section 3.6]). That is, the measure σ such that $\sigma(a, b) = b - a$, for $0 \leqslant a \leqslant b \leqslant T$. Given $p \in [1, \infty]$, let $L^p = L^p(\mathbb{T}_T)$ denote the usual Lebesgue L^p-space associated with $\mathbb{T}_T$. Since $\mathbb{T}_T$ has finite measure, we observe that

$$L^\infty \subset L^{p_2} \subset L^{p_1} \subset L^1,$$

where $1 \leqslant p_1 \leqslant p_2 \leqslant \infty$.

In particular, since the coefficients of (1.1) are continuous, it follows that they lie in L^∞ and thus also in L^p, for every $p \in [1,\infty]$. Let $\varphi \in L^p$. For simplicity we denote $\|\varphi\|_p = \|\varphi\|_{L^p}$, where $\|\cdot\|_{L^p}$ is the usual norm of the Banach space L^p. Let also

$$\varphi_L = \min_{t\in[0,T]}\{\varphi(t)\}, \quad \varphi_M = \max_{t\in[0,T]}\{\varphi(t)\}, \quad \overline{\varphi} = \frac{1}{T}\int_0^T \varphi,$$

be the minimum, maximum and average of φ. About the uniqueness of the coexistence states of (1.1), consider the statements

$$\overline{a} > 0, \quad -\left(\frac{e}{b}\right)_L < \frac{\overline{d}}{\overline{a}} < \left(\frac{f}{c}\right)_L, \tag{1.3}$$

and

$$\left(\frac{b}{e}\right)_L > \left(\frac{c}{f}\right)_M. \tag{1.4}$$

The authors in [1] proved that if (1.4) holds, then (1.1) has at most one coexistence state [1, Proposition 3.3]. As observed by the authors in [1, Remark 2, p. 11], if both (1.3) and (1.4) holds, then it follows from Tineo [8, Theorem 1.5] that (1.1) has exactly one coexistence state and it is *globally* asymptotically stable.

More sufficient conditions for asymptotically stability of coexistence states were obtained by R. Ortega and coauthors [1,5]. Such conditions can somewhat be called L_1*-condition* and L_∞*-condition* (see Remarks 2.3 and 2.4). In recent years V. Ortega and Rebelo [6] constructed a bridge between these two conditions, obtaining a L_p*-condition*, $p \in [1,\infty]$. Such condition requires that the L_p-norm of all possible coexistence states (u_0, v_0) to satisfy a given inequality. To this end, the authors in [6] also provide upper bounds for $\|u_0\|_p$ and $\|v_0\|_p$, independent from each other. This independence provides a *unified* sufficient condition for uniqueness and asymptotically stability of (u_0, v_0).

In this paper we obtain new *intertwined* upper bounds for $\|u_0\|_p$ and $\|v_0\|_p$, which in turn imply on new sufficient conditions for uniqueness and asymptotically stability of (u_0, v_0), that can be applied when the unified test is inconclusive. The paper is organized as follows. In Section 2 we state our main Theorem. At Section 3 we have some preliminary results to prove the main Theorem at Section 4. In Section 5 we provide an example where previous results in the literature are inconclusive, while ours is not. We also provide some further thoughts. Finally, we have an Appendix with some technicalities and illustrations.

2 Statement of the main result

Given $p \in [1,\infty)$ and $\varphi \in L^p$ non-negative, the L^p*-average* $\overline{\varphi}_p \in \mathbb{R}_{\geqslant 0}$ is given by,

$$\overline{\varphi}_p = \frac{1}{T^{\frac{1}{p}}}\left(\int_0^T \varphi^p\right)^{\frac{1}{p}} = \frac{1}{T^{\frac{1}{p}}}\|\varphi\|_p.$$

If $p = \infty$, then we define $\overline{\varphi}_\infty = \|\varphi\|_\infty$. Observe that $\overline{\varphi}_p \to \overline{\varphi}_\infty$ as $p \to \infty$.

Given a system of the form (1.1) and $p \in [1,\infty)$, let $C_p \subset \mathbb{R}^2$ be the set given by the points $(x,y) \in \mathbb{R}^2$ such that $x > 0$, $y > 0$ and

$$\begin{aligned} b_L U^{1-p}x^p + c_L V^{1-p}y^p \leqslant \overline{a} \leqslant b_M x + c_M y \\ -e_M x + f_L V^{1-p}y^p \leqslant \overline{d} \leqslant -e_L U^{1-p}x^p + f_M y, \end{aligned} \tag{2.1}$$

where,

$$U = \left(\frac{a}{b}\right)_M, \quad V = \left(\frac{d}{f}\right)_M + \left(\frac{e}{f}\right)_M U.$$

Let also $C_\infty \subset \mathbb{R}^2$ be the set given by

$$0 < x \leqslant U, \quad 0 < y \leqslant V.$$

For an illustration of C_p and more details about the definition of C_∞, see Appendix B. We observe that C_p is bounded and that it may be empty. Let also $\mathcal{J}\colon [1,\infty) \to \mathbb{R}$ be given by,

$$\mathcal{J}(q) = \int_0^{2\pi} \frac{1}{\left(|\cos\theta|^{2q} + |\sin\theta|^{2q}\right)^{\frac{1}{q}}}\, d\theta.$$

It follows from [6, Proposition 2.2] that,

$$\lim_{q\to\infty} \mathcal{J}(q) = 8.$$

Hence, we can continuously extend $\mathcal{J}$ to $[1,\infty]$ by defining $\mathcal{J}(\infty) = 8$. For more details about $\mathcal{J}(q)$, see Appendix C. Given $p \in [1,\infty]$, we recall that its *conjugate* $q \in [1,\infty]$ is given by the unique solution of $1/p + 1/q = 1$. In what follows q always denote the conjugate of p. Our main result is the following.

Theorem 2.1. *Consider a system of the form (1.1) and its respective set C_p, $p \in [1,\infty]$. Then the following statements hold.*

(a) If $(u(t), v(t))$ is a coexistence state, then $(\overline{u}_p, \overline{v}_p) \in C_p$. In particular, $C_p \neq \emptyset$ for every $p \in [1,\infty]$.

(b) Suppose we have at least one coexistence state. If

$$T\left(\sqrt{c_M e_M x_p y_p} + \frac{1}{2}(b_M x_1 + f_M y_1)\right) \leqslant \frac{\mathcal{J}(q)}{2^{2-\frac{1}{q}}}$$

for every $(x_p, y_p) \in C_p$ and $(x_1, y_1) \in C_1$, then such coexistence state is unique and asymptotically stable.

Remark 2.2. As presented in the introduction, we recall that the existence of a coexistence state is completely characterized. Therefore, the hypothesis of having at least one is not a loss of generality.

It follows from Hölder's inequality that if $\varphi \geqslant 0$, then $\overline{\varphi}_{p_1} \leqslant \overline{\varphi}_{p_2}$ for $1 \leqslant p_1 \leqslant p_2 \leqslant \infty$. Hence, we can replace statement (b) of Theorem 2.1 for the following weak version.

(b') Suppose we have at least one coexistence state. If

$$T\left(\sqrt{c_M e_M x_p y_p} + \frac{1}{2}(b_M x_p + f_M y_p)\right) \leqslant \frac{\mathcal{J}(q)}{2^{2-\frac{1}{q}}}$$

for every $(x_p, y_p) \in C_p$, then such coexistence state is unique and asymptotically stable.

Remark 2.3 (The L_1-condition)**.** Observe that if we replace $p = 1$ at Theorem 2.1(b), we obtain

$$T\left(\sqrt{c_M e_M xy} + \frac{1}{2}(b_M x + f_M y)\right) \leqslant 2$$

for every $(x, y) \in C_1$, where $C_1 \subset \mathbb{R}^2$ is the set bounded by $x > 0$, $y > 0$ and

$$\begin{aligned} b_L x + c_L y \leqslant \ \ \overline{a} \ \ &\leqslant b_M x + c_M y \\ -e_M x + f_L y \leqslant \ \ \overline{d} \ \ &\leqslant -e_L x + f_M y. \end{aligned}$$

This is precisely the L_1-condition obtained by R. Ortega [5, Theorem 5.2].

Remark 2.4 (The L_∞-condition)**.** Observe that if we replace $p = \infty$, $x = U$ and $y = V$ at Theorem 2.1(b'), we obtain

$$T\left(\sqrt{c_M e_M UV} + \frac{1}{2}(b_M U + f_M V)\right) \leqslant \pi. \tag{2.2}$$

This is the L_∞-condition obtained by R. Ortega and Amine [1, Proposition 4.5].

3 Preliminary results

In this section we recall the *L_p-condition* obtained by V. Ortega and Rebelo [6, Theorem 3.1] and also a technical lemma.

Theorem 3.1 (The L_p-condition)**.** *Suppose that all possible coexistence states (u, v) of system (1.1) satisfy*[1]

$$T^{\frac{1}{q}}\sqrt{\|eu\|_p \|cv\|_p} + \frac{1}{2}\|bu - fv\|_1 \leqslant \frac{\mathcal{J}(q)}{2^{2-\frac{1}{q}}}, \tag{3.1}$$

where p and q are conjugated indices and $p, q \in [1, \infty]$. Then the coexistence state is unique and asymptotically stable. Moreover, any coexistence state (u, v) satisfies

$$\|u\|_p \leqslant \frac{\|a\|_p}{b_L}, \quad \|v\|_p \leqslant \frac{\|d\|_p}{f_L} + \frac{e_M}{f_L}\frac{\|a\|_p}{b_L}. \tag{3.2}$$

Lemma 3.2. *Let $(u(t), v(t))$ be a coexistence state of (1.1). Then $\|u\|_\infty \leqslant U$ and $\|v\|_\infty \leqslant V$.*

Proof. Let $\tau \in [0, T]$ be such that $u(\tau) = \max_{[0,T]} u(t)$. Since $u(\tau)$ is a local maximum, it follows that $\dot{u}(\tau) = 0$ and thus it follows from the first equation of (1.1) that

$$a(\tau) = b(\tau)u(\tau) + c(\tau)v(\tau) \geqslant b(\tau)u(\tau) \Rightarrow u(\tau) \leqslant \frac{a(\tau)}{b(\tau)} \leqslant U.$$

Similarly, if we let $\tau \in [0, T]$ be such that $v(\tau) = \max_{[0,T]} v(t)$, then it follows from the second equation of (1.1) that,

$$f(\tau)v(\tau) = d(\tau) + e(\tau)u(\tau) \Rightarrow v(\tau) \leqslant \frac{d(\tau)}{f(\tau)} + \frac{e(\tau)}{f(\tau)}u(\tau) \leqslant V.$$

This finishes the proof. □

For lower bound of the coexistence states of (1.1), we refer to [4, Lemma 5.5].

[1] Actually in their paper instead of the fraction $1/2$ in the expression, it appears the fraction $T/2$. But from equation (7) in that paper one can see that it is a typo.

4 Proof of Theorem 2.1

Before we prove the theorem, we observe that given $p \in [1,\infty]$ and $\varphi \in L^P$, it follows from Hölder's inequality that,

$$\|\varphi\|_1 \leqslant T^{1-\frac{1}{p}}\|\varphi\|_p. \tag{4.1}$$

Moreover, it follows from Littlewood's inequality that

$$\|\varphi\|_p \leqslant \|\varphi\|_1^{\frac{1}{p}}\|\varphi\|_\infty^{1-\frac{1}{p}},$$

and thus

$$\|\varphi\|_1 \geqslant \frac{1}{\|\varphi\|_\infty^{p-1}}\|\varphi\|_p^p. \tag{4.2}$$

Proof of Theorem 2.1. Let $(u(t),v(t))$ be a coexistence state of (1.1). Dividing the first equation of (1.1) by u we obtain,

$$\frac{\dot{u}}{u} = a(t) - b(t)u - c(t)v. \tag{4.3}$$

Integrating (4.3) in t, from 0 to T, we obtain

$$\int_0^T a = \int_0^T bu + \int_0^T cv,$$

and thus,

$$b_L\int_0^T u + c_L\int_0^T v \leqslant \int_0^T a \leqslant b_M\int_0^T u + c_M\int_0^T v. \tag{4.4}$$

Since $u \geqslant 0$ and $v \geqslant 0$, it follows from (4.4) that,

$$b_L\|u\|_1 + c_L\|v\|_1 \leqslant \int_0^T a \leqslant b_M\|u\|_1 + c_M\|v\|_1. \tag{4.5}$$

Applying (4.1) on the right-hand side of (4.5) we obtain

$$\int_0^T a \leqslant b_M T^{1-\frac{1}{p}}\|u\|_p + c_M T^{1-\frac{1}{p}}\|v\|_p. \tag{4.6}$$

Dividing (4.6) by T and knowing that $T^{-\frac{1}{p}}\|u\|_p = \overline{u}_p$ and $T^{-\frac{1}{p}}\|v\|_p = \overline{v}_p$ we obtain,

$$\overline{a} \leqslant b_M\overline{u}_p + c_M\overline{v}_p. \tag{4.7}$$

Similarly, applying (4.2) on the left-hand side of (4.5) we obtain,

$$\int_0^T a \geqslant b_L\frac{1}{\|u\|_\infty^{p-1}}\|u\|_p^p + c_L\frac{1}{\|v\|_\infty^{p-1}}\|v\|_p^p.$$

Hence, it follows from Lemma 3.2 that,

$$\int_0^T a \geqslant b_L\frac{1}{U^{p-1}}\|u\|_p^p + c_L\frac{1}{V^{p-1}}\|v\|_p^p. \tag{4.8}$$

Multiplying (4.8) by T^{-1} we obtain,

$$\overline{a} \geqslant b_L\frac{1}{U^{p-1}}\overline{u}_p^p + c_L\frac{1}{V^{p-1}}\overline{v}_p^p. \tag{4.9}$$

Let us now look to the second equation of (1.1). Dividing it by v and integrating it in t, from 0 to T, we obtain

$$\int_0^T d = -\int_0^T eu + \int_0^T fv,$$

and thus,

$$-e_M\|u\|_1 + f_L\|v\|_1 \leqslant \int_0^T d \leqslant -e_L\|u\|_1 + f_M\|v\|_1. \tag{4.10}$$

Applying (4.1) (resp. (4.2) and Lemma 3.2) on the positive (resp. negative) term of the right-hand side of (4.10) we obtain,

$$\int_0^T d \leqslant -e_L\frac{1}{U^{p-1}}\|u\|_p^p + f_M T^{1-\frac{1}{p}}\|v\|_p. \tag{4.11}$$

Multiplying (4.11) by T^{-1} we obtain,

$$\overline{d} \leqslant -e_L\frac{1}{U^{p-1}}\overline{u}_p^p + f_M\overline{v}_p. \tag{4.12}$$

Similarly, it follows from the left-hand side of (4.10) that,

$$\overline{d} \geqslant -e_M\overline{u}_p + f_L\frac{1}{V^{p-1}}\overline{v}_p^p. \tag{4.13}$$

Now observe that equations (4.7), (4.9), (4.12) and (4.13) are the four equations given at the definition (2.1) of the set C_p. Hence, we obtained statement (a) of Theorem 2.1.

Statement (b) follows directly from Theorem 3.1. More precisely, knowing that

$$\|eu\|_p \leqslant e_M\|u\|_p, \quad \|cv\|_p \leqslant c_M\|v\|_p, \quad \|bu - fv\|_1 \leqslant b_M\|u\|_1 + f_M\|v\|_1,$$

$\|u\|_1 = T\overline{u}_1$, $\|v\|_1 = T\overline{v}_1$, and $T^{\frac{1}{q}} = T/T^{\frac{1}{p}}$, one can see that the left-hand side of (3.1) can be majored by,

$$T\left(\sqrt{c_M e_M \overline{u}_p\overline{v}_p} + \frac{1}{2}(b_M\overline{u}_1 + f_M\overline{v}_1)\right).$$

Since every possible coexistence state (u, v) satisfies $(\overline{u}_p, \overline{v}_p) \in C_p$, we have from Theorem 3.1 that if

$$T\left(\sqrt{c_M e_M x_p y_p} + \frac{1}{2}(b_M x_1 + f_M y_1)\right) \leqslant \frac{\mathcal{J}(q)}{2^{2-\frac{1}{q}}}$$

for every $(x_p, y_p) \in C_p$ and $(x_1, y_1) \in C_1$, then the coexistence state is unique and asymptotically stable. □

5 An example and further thoughts

At first glance, one could look at the proof of Theorem 2.1 and conclude that since the boundaries of C_p are obtained by majoration and minoration of the boundaries of C_1 (which in turn was obtained by R. Ortega [5, p. 11]), then no new information could be drawn from such theorem. However, as the next example will show, this is not the case.

Example 5.1. Consider a Lotka–Volterra system

$$\dot{u} = u(a - bu - cv), \quad \dot{v} = v(d + eu - fv), \tag{5.1}$$

with the coefficients being positive constant. We recall the sufficient condition

$$T\left(\sqrt{c_M e_M x_p y_p}+\frac{1}{2}(b_M x_1+f_M y_1)\right)\leqslant \frac{\mathcal{J}(q)}{2^{2-\frac{1}{q}}} \tag{5.2}$$

of Theorem 2.1(b). Let $F(q)=\mathcal{J}(q)/2^{2-\frac{1}{q}}$, where $q=p/(p-1)$ is the conjugate of p, and let also $\mathcal{F}(p)=F(p/(p-1))$ (see Appendix C). Since the coefficients of (5.1) are constant, it is not hard to see that $C_1=\{(x_1,y_1)\}$ is a single point, given by the unique solution of

$$\begin{pmatrix} b & c \\ -e & f \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}=\begin{pmatrix} a \\ d \end{pmatrix}.$$

Let $k=(b_M x_1+f_M y_1)/2$ and observe that we can write (5.2) as $x_p^p y_p^p\leqslant h(p)$, where

$$h(p)=\left[\frac{1}{\sqrt{ce}}\left(\frac{\mathcal{F}(p)}{T}-k\right)\right]^{2p}.$$

We recall that $C_p\subset\mathbb{R}^2$ is the set bounded by $x>0$, $y>0$ and

$$\begin{aligned} b_L U^{1-p}x^p+c_L V^{1-p}y^p\leqslant \overline{a}\leqslant b_M x+c_M y \\ -e_M x+f_L V^{1-p}y^p\leqslant \overline{d}\leqslant -e_L U^{1-p}x^p+f_M y, \end{aligned} \tag{5.3}$$

if $p\in[1,\infty)$, and by

$$0<x\leqslant U,\quad 0<y\leqslant V,$$

if $p=\infty$ (for an illustration, see Appendix B). Therefore, since the left-hand side of (5.2) is an increasing function of x_p and y_p (recall that x_1 and y_1 are constant), we have from (5.3) that its maximum happens somewhere on the curve

$$bU^{1-p}x_p^p+cV^{1-p}y_p^p=a.$$

Hence, the maximum occurs at some point (x_p,y_p) such that

$$y_p^p=c^{-1}V^{p-1}(a-bU^{1-p}x_p^p). \tag{5.4}$$

Replacing (5.4) at $x_p^p y_p^p\leqslant h(p)$, we obtain

$$-\left[\frac{b}{c}\left(\frac{V}{U}\right)^{p-1}\right]w^2+\left[\frac{a}{c}V^{p-1}\right]w-h(p)\leqslant 0, \tag{5.5}$$

where $w=x_p^p$. The discriminant of (5.5) in relation to w is given by,

$$\Delta(p)=V^{p-1}\left[\frac{a^2}{c^2}V^{p-1}-4\frac{b}{c}\frac{1}{U^{p-1}}h(p)\right].$$

Therefore, to study its sign it is sufficient to study the sign of

$$G(p)=\frac{a^2}{c^2}V^{p-1}-4\frac{b}{c}\frac{1}{U^{p-1}}h(p).$$

From this reasoning we have that (5.2) is equivalent to (5.5). Since the leading coefficient of the left-hand side of (5.5) is negative, it follows that the inequality holds at a given $p\in[1,\infty)$ if, and only if, $G(p)\leqslant 0$.

Hence, we conclude that we can apply Theorem 2.1 at (5.1) at a given $p \in [1,\infty)$ if, and only if, $G(p) \leqslant 0$. Therefore, if for some choice of the coefficients we have

$$G(1) > 0, \quad G(p^*) < 0, \quad \lim_{p\to\infty} G(p) > 0, \tag{5.6}$$

for some $p^* \in (1,\infty)$, then the L_1 and L_∞-conditions (recall Remarks 2.3 and 2.4) will be inconclusive. Moreover, since the coefficients are constant we have

$$\frac{\|a\|_p}{b_L} = U, \quad \frac{\|d\|_p}{f_L} + \frac{e_M}{f_L}\frac{\|a\|_p}{b_L} = V,$$

and thus the upper bound (3.2) is reduced to $\overline{u}_p \leqslant U$ and $\overline{v}_p \leqslant V$. As a consequence, it follows that when applying the L_p-condition in the way it was stated at Theorem 3.1, the only known upper bounds for $\|u\|_p$ and $\|v\|_p$, $p \in [1,\infty]$, are given by U and V, respectively. Therefore, equation (3.1) is reduced to

$$T\left(\sqrt{c_M e_M UV} + \frac{1}{2}(b_M U + f_M V)\right) \leqslant \frac{\mathcal{J}(q)}{2^{2-\frac{1}{q}}} \leqslant \pi, \tag{5.7}$$

with the last inequality following from Proposition C.1 (see also Figure C.1). Since the L_∞-condition is inconclusive, it follows that (2.2) does not hold and thus (5.7) also does not hold.

Therefore we conclude that if (5.6) holds, then the L_1 and L_∞-conditions are inconclusive, while the upper bounds (3.2) provided by the L_p-condition, in the way it was stated at Theorem 3.1, are too thick and also inconclusive. Nevertheless, Theorem 2.1 can still be applied at $p = p^*$, ensuring the uniqueness and asymptotically stability of the coexistence state.

In other words, Theorem 3.1 still holds, since it state that (3.1) must hold for all the *coexistence states* of the system. However, the practical upper bounds provided by (3.2) are too thick in the case of this example, while the upper bounds provided by Theorem 2.1 are not.

For an example of such situation, one can consider $T = 1$,

$$a = 2.0102, \quad b = 1, \quad c = 0.0051, \quad d = 2.0203, \quad e = 0.9898, \quad f = 2, \tag{5.8}$$

and see that (5.6) holds with $p^* = 2$.

Remark 5.2. We have assumed constant coefficients at Example 5.1 for the sake of simplicity. However, it follows from the continuous dependence of the boundaries of C_p, $p \in [1,\infty]$, in relation to the coefficient functions $a(t),\dots,f(t)$ that if the amplitude of such coefficients (i.e. $a_M - a_L,\dots,f_M - a_L$) is small enough and if their averages are close enough to (5.8), then the same conclusion holds. Moreover, the assumption $T = 1$ is no loss of generality since it can be obtained by time rescaling.

We think that one reason for Theorem 2.1 to be conclusive in some situations while the previous one are not is the fact that although the boundaries of C_p are obtained by inequalities on the boundaries of C_1, the function $\mathcal{F}(p)$ *also increases* (see Appendix C). Therefore, for a given $p \in (1,\infty)$, the function $\mathcal{F}(p)$ may have increased in such a way that it compensates the boundaries of C_p.

Moreover in comparison with the boundaries provided by (3.2), as anticipated in the introduction, the fact that the two inequalities are independent from each other provides a *unified* test for uniqueness and asymptotically stability of the coexistence states. More precisely, if

$$T^{\frac{1}{q}}\sqrt{c_M e_M \alpha_p \beta_p} + \frac{1}{2}(b_M\alpha_1 + f_M\beta_1) \leqslant \frac{\mathcal{J}(q)}{2^{2-\frac{1}{q}}}$$

for some $p \in [1, \infty]$, where

$$\alpha_p = \frac{\|a\|_p}{b_L}, \quad \beta_p = \frac{\|d\|_p}{f_L} + \frac{e_M}{f_L}\frac{\|a\|_p}{b_L},$$

then we are done. However, if the unified test is inconclusive then the intertwined boundaries (2.1) of C_p can reduce the lack of possibilities of the averages of the coexistence states in such a way that for all *possible* coexistence states, inequality (3.1) still holds.

A Linearly stability, asymptotic stability and Floquet theory

Consider a system T-periodic system of differential equations

$$\dot{u} = H(t, u), \tag{A.1}$$

with $u \in \mathbb{R}^n$ and $H\colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ sufficiently smooth. Let $u_0(t)$ be a T-periodic solution of (A.1). The *linearization* of (A.1) at $u_0(t)$ is the T-periodic linear system given by

$$\dot{u} = D_u H(t, u_0(t))u, \tag{A.2}$$

where $D_u H$ denotes the Jacobian matrix of H with respect to the variable u. We say that $u_0(t)$ is *linearly stable* if all *Floquet exponents* (also known as *characteristics exponents*) associated to (A.2) have non-positive real part. Fore more details about Floquet Theory, we refer to [3, Section 3.5] and [7].

We say that u_0 is *stable* if for every neighborhood $V \subset \mathbb{R}^n$ of $u_0(0)$ there is a neighborhood $W \subset V$ of $u_0(0)$ such that if $u(t)$ is a solution of (A.1) with $u(0) \in W$, then $u(t) \in V$ for $t \geqslant 0$. Moreover, given an open set $U \subset \mathbb{R}^n$, we say that $u_0(t)$ is *asymptotically stable* in U if u_0 is stable and

$$\lim_{t\to\infty} |u_0(t) - u(t)| = 0,$$

for every solution $u(t)$ of (A.1) with $u(0) \in U$.

B Illustrations of the set C_p and construction of C_∞

We recall that $C_p \subset \mathbb{R}^2$ is the set bounded by $x > 0$, $y > 0$ and

$$\begin{aligned} b_L U^{1-p} x^p + c_L V^{1-p} y^p &\leqslant \overline{a} \leqslant b_M x + c_M y \\ -e_M x + f_L V^{1-p} y^p &\leqslant \overline{d} \leqslant -e_L U^{1-p} x^p + f_M y, \end{aligned}$$

for $p \in [1, \infty)$, and by

$$0 < x \leqslant U, \quad 0 < y \leqslant V,$$

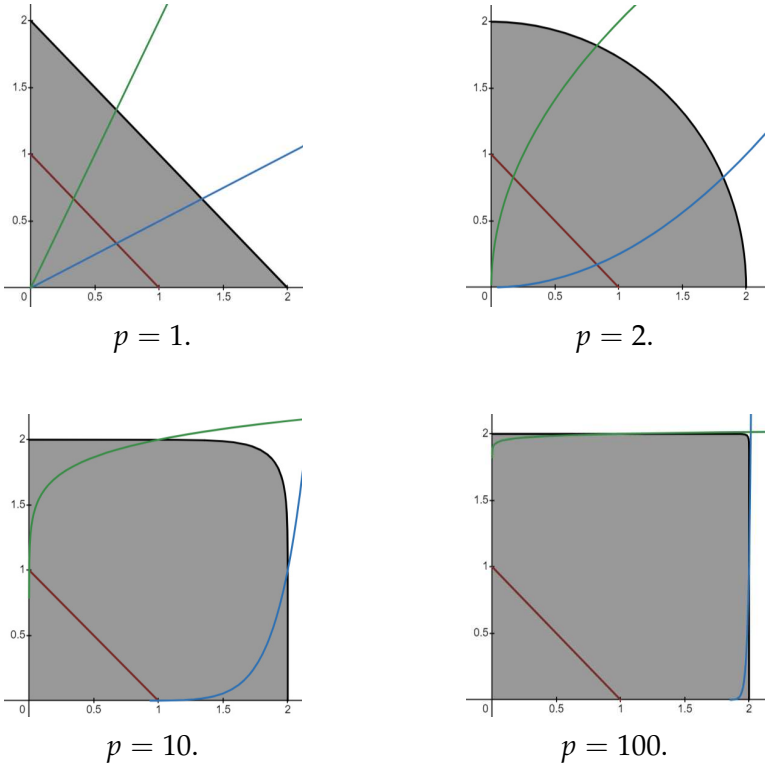
for $p = \infty$. See Figure B.1. To obtain the definition of C^∞, consider the inequality

$$b_L U^{1-p} x^p + c_L V^{1-p} y^p \leqslant \overline{a}. \tag{B.1}$$

Since the left-hand side is strictly increasing in x and y it is clear that the maximum value of x occurs when $y = 0$. Hence, replacing $y = 0$ at (B.1) we obtain,

$$x \leqslant \left(\frac{\overline{a}}{b_L}\right)^{\frac{1}{p}} U^{\frac{1}{q}}.$$

Taking the limit $p \to \infty$ we obtain $x \leqslant U$. Similarly, by replacing $x = 0$ at (B.1) and taking the limit $p \to \infty$ one obtains $y \leqslant V$.



$p = 1.$ $p = 2.$

$p = 10.$ $p = 100.$

Figure B.1: Illustrations of C_p, given by the region bounded by the four curves, with parameters $\overline{a} = 2$, $\overline{d} = 2$, $b_M = 2$, $b_L = 1$, $c_M = 2$, $c_L = 1$, $e_M = 2$, $e_L = 1$, $f_M = 2$, $f_L = 1$, $U = 2$ and $V = 2$. The gray region (resp. black curve) corresponds to the inequality (resp. equality) $b_L U^{1-p} x^p + c_L V^{1-p} y^p \leqslant \overline{a}$. The red curve correspond to $b_M x + c_M y = \overline{a}$, while the green the blue ones correspond to $-e_M x + f_L V^{1-p} y^p = \overline{d}$ and $-e_L U^{1-p} x^p + f_M y = \overline{d}$, respectively. Together, they form the boundaries of C_p. Colors available in the online version.

C Properties of the maps $\mathcal{J}$, F and $\mathcal{F}$

We recall that $\mathcal{J}\colon [1,\infty) \to \mathbb{R}$ is the function given by,

$$\mathcal{J}(q) = \int_0^{2\pi} \frac{1}{\left(|\cos\theta|^{2q} + |\sin\theta|^{2q}\right)^{\frac{1}{q}}}\, d\theta. \tag{C.1}$$

Moreover, it follows from [6, Proposition 2.2] that

$$\lim_{q\to\infty} \mathcal{J}(q) = 8,$$

and thus we can continuously extend $\mathcal{J}$ to $[1,\infty]$ by defining $\mathcal{J}(\infty) = 8$. We also recall that $F, \mathcal{F}\colon [1,\infty] \to \mathbb{R}$ are the functions given by

$$F(q) = \frac{\mathcal{J}(q)}{2^{2-\frac{1}{q}}}, \quad \mathcal{F}(p) = F\left(\frac{p}{p-1}\right) = \frac{\mathcal{J}\left(\frac{p}{p-1}\right)}{2^{1+\frac{1}{p}}}, \tag{C.2}$$

where $q = p/(p-1)$ is the conjugate of p. For a numerical plot of the graph of $\mathcal{F}(p)$, see Figure C.1.

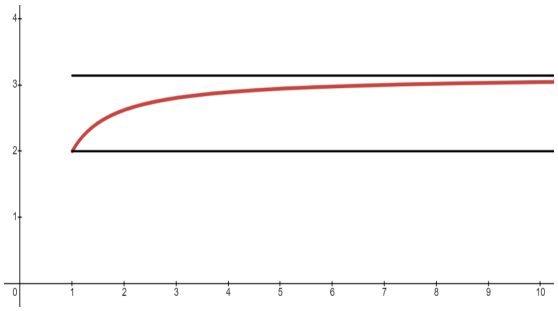


Figure C.1: Graph of the function $\mathcal{F}(p)$, in red. The black straight lines are the lower and upper bounds $y = 2$ and $y = \pi$, respectively. Colors available in the online version.

Proposition C.1. *The functions F and $\mathcal{F}$ are strictly decreasing and increasing, respectively.*

Proof. First we consider the function F. That is, we consider

$$F(q) = \frac{\mathcal{J}(q)}{2^{2-\frac{1}{q}}} = \int_0^{2\pi} 2^{\frac{1}{q}-2}\left(|\cos\theta|^{2q} + |\sin\theta|^{2q}\right)^{-\frac{1}{q}}\, d\theta. \tag{C.3}$$

Consider the integrand of the outermost right-hand side of (C.3),

$$g_1(\theta, q) = 2^{\frac{1}{q}-2}\left(|\cos\theta|^{2q} + |\sin\theta|^{2q}\right)^{-\frac{1}{q}}.$$

Differentiating it in relation to q we obtain

$$\frac{\partial g_1}{\partial q}(\theta, q) = g_1(\theta, q)\left[\frac{1}{q^2(\cos^{2q}\theta + \sin^{2q}\theta)} g_2(\theta, q) + \ln\left(2^{-1/q^2}\right)\right], \tag{C.4}$$

where

$$\begin{aligned} g_2(\theta, q) = {} & \cos^{2q}\theta\left[\ln(\cos^{2q}\theta + \sin^{2q}\theta) - \ln\cos^{2q}\theta\right] \\ & + \sin^{2q}\theta\left[\ln(\cos^{2q}\theta + \sin^{2q}\theta) - \ln\sin^{2q}\theta\right]. \end{aligned} \tag{C.5}$$

Since g_1 is strictly positive, it follows that the sign of (C.4) is given by the sign of,

$$\frac{1}{q^2(\cos^{2q}\theta + \sin^{2q}\theta)} g_2(\theta, q) + \ln\left(2^{-1/q^2}\right). \tag{C.6}$$

We claim that (C.6) is negative. Indeed, observe that (C.6) is negative if and only if,

$$g_2(\theta, q) \leqslant -q^2(\cos^{2q}\theta + \sin^{2q}\theta)\ln\left(2^{-1/q^2}\right).$$

Which by properties of the logarithm is equivalent to,

$$g_2(\theta, q) \leqslant (\cos^{2q}\theta + \sin^{2q}\theta)\ln 2.$$

Which in turn is equivalent to,

$$\begin{aligned} g_3(\theta, q) := {} & \cos^{2q}\left[\ln(\cos^{2q}\theta + \sin^{2q}\theta) - \ln(2\cos^{2q}\theta)\right] \\ & + \sin^{2q}\left[\ln(\cos^{2q}\theta + \sin^{2q}\theta) - \ln(2\sin^{2q}\theta)\right] \leqslant 0. \end{aligned} \tag{C.7}$$

Differentiating (C.7) in relation to θ we obtain,

$$\frac{\partial g_3}{\partial \theta}(\theta, q) = 2q \sin\theta \cos\theta \Big[\cos^{2(q-1)}\theta \big[\ln(2\cos^{2q}\theta) - \ln(\cos^{2q}\theta + \sin^{2q}\theta)\big] - \sin^{2(q-1)}\theta \big[\ln(2\sin^{2q}\theta) - \ln(\cos^{2q}\theta + \sin^{2q}\theta)\big]\Big]. \quad \text{(C.8)}$$

Clearly, the zeros of (C.8) in θ are given by $\theta \in \{0, \pi/2, \pi, 3\pi/2\}$ and by the zeros of

$$g_4(\theta, q) := \cos^{2(q-1)}\theta \big[\ln(2\cos^{2q}\theta) - \ln(\cos^{2q}\theta + \sin^{2q}\theta)\big] - \sin^{2(q-1)}\theta \big[\ln(2\sin^{2q}\theta) - \ln(\cos^{2q}\theta + \sin^{2q}\theta)\big]. \quad \text{(C.9)}$$

Observe that if $\theta_0 \in [0, 2\pi]$ is such that $\cos^2\theta_0 = \sin^2\theta_0$, then $g_4(\theta_0, q) = 0$. Reciprocally, we claim that if $\cos^2\theta_0 \neq 0 \sin^2\theta_0$, then $g_4(\theta_0, q) \neq 0$. Indeed, if $\cos^2\theta_0 > \sin^2\theta_0$, then it follows from the fact that the logarithm is a strictly increasing function that,

$$\ln(2\cos^{2q}\theta_0) > \ln(\cos^{2q}\theta_0 + \sin^{2q}\theta_0), \quad \ln(2\sin^{2q}\theta_0) < \ln(\cos^{2q}\theta_0 + \sin^{2q}\theta_0).$$

Hence, $g_4(\theta_0, q) > 0$. Similarly if $\cos^2\theta_0 < \sin^2\theta_0$, then $g_4(\theta_0, q) < 0$.

Therefore, we conclude that the extreme points $\theta_0 \in [0, 2\pi]$ of (C.7) are such that $\cos\theta_0 \sin\theta_0 = 0$ or $\cos^2\theta_0 = \sin^2\theta_0$. In any case, it is easy to see that (C.7) holds (with the equality holding only if $\cos^2\theta_0 = \sin^2\theta_0$), which in turn implies that the same holds for (C.4). Therefore, it follows from Leibniz integral rule that

$$F'(q) = \int_0^{2\pi} \frac{\partial g_1}{\partial q}(\theta, q)\, d\theta < 0.$$

That is, $F'(q)$ is strictly decreasing. Since $\mathcal{F}(p) = F(p/(p-1))$, it follows that

$$\mathcal{F}'(p) = -\frac{1}{(p-1)^2} F'\left(\frac{p}{p-1}\right) > 0,$$

and thus $\mathcal{F}$ is strictly increasing. □

Remark C.2. With the same reasoning of Proposition C.1, one can prove that $\mathcal{J}(q)$ is also strictly increasing. One just need to consider the integrand

$$h_1(\theta, q) = \frac{1}{\big(|\cos\theta|^{2q} + |\sin\theta|^{2q}\big)^{\frac{1}{q}}},$$

observe that

$$\frac{\partial h_1}{\partial q}(\theta, q) = \frac{h_1(\theta, q)}{q^2(\cos^{2q}\theta + \sin^{2q}\theta)} g_2(\theta, q),$$

with g_2 given by (C.5), and observe that since the logarithm is strictly increasing, it follows that

$$\ln(\cos^{2q}\theta + \sin^{2q}\theta) \geqslant \ln\cos^{2q}\theta, \quad \ln(\cos^{2q}\theta + \sin^{2q}\theta) \geqslant \ln\sin^{2q}\theta,$$

with equality holding only if $\cos\theta \sin\theta = 0$. The proof now follows from Leibniz integral rule.

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On the kernel of the Stieltjes derivative and the space of bounded Stieltjes-differentiable functions

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Abstract. We investigate the existence and uniqueness of solutions to first-order Stieltjes differential problems, focusing on the role of the Stieltjes derivative and its kernel. Unlike the classical case, the kernel of the Stieltjes derivative operator is nontrivial, leading to non-uniqueness issues in Cauchy problems. We characterize this kernel by providing necessary and sufficient conditions for a function to have a zero Stieltjes derivative. To address the implications of this nontrivial kernel, we introduce a function space which serves as a suitable framework for studying Stieltjes differential problems. We explore its topological structure and propose a metric that facilitates the formulation of existence and uniqueness results. Our findings demonstrate that solutions to first-order Stieltjes differential equations are, in general, not unique, underscoring the need for a refined analytical approach to such problems.

Keywords: Stieltjes derivative, kernel of the derivative, Stieltjes differential equations, uniqueness, Cauchy problem.

2020 Mathematics Subject Classification: 34A12, 34A30, 34A36, 26A24, 46E30.

1 Introduction

The study of existence and uniqueness of solutions to Stieltjes differential problems has gained significant attention in recent years (see, for instance, [5, 9]). Regarding uniqueness, many results rely on classical techniques—such as Lipschitz conditions—and on an appropriate definition of the concept of solution. In the case of first-order problems, solutions are typically assumed to belong either to the space of Stieltjes absolutely continuous functions [9] or to the space of continuously Stieltjes differentiable functions [5], mirroring the classical setting.

These choices, however, are not merely natural consequences of the problem's structure. A key observation is that there exist Stieltjes differentiable functions with an everywhere zero

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Stieltjes derivative that are not constant. In other words, the kernel of the Stieltjes derivative operator is larger than expected. As a direct consequence, the first-order Cauchy problem does not generally admit a unique solution if we consider solutions that are everywhere Stieltjes differentiable, possibly with a Stieltjes continuous derivative, but not necessarily Stieltjes continuous. This distinction between differentiable functions that are continuous and those that are not has been highlighted in the literature—see, for example, [6]—and stems from the absence of a mean value theorem for Stieltjes derivatives with the same strength as in the classical case. Nevertheless, certain versions of this result do exist, and we will explore some of them in this article.

Given this setting, a fundamental step is to determine the kernel of the Stieltjes derivative operator, as it parameterizes the solutions of the Cauchy problem. This is a nontrivial task, but we will provide several characterizations of the kernel along with necessary and sufficient conditions for a function to have a zero Stieltjes derivative.

The fact that we are working with differentiable functions that are not necessarily continuous raises a crucial question: *in which space are we operating?* We will introduce a function space, denoted $\mathcal{BD}$, which contains the kernel of the derivative and serves as a suitable framework for studying first-order Cauchy problems. Equipping this space with an appropriate topology is challenging. Even in the classical case, it is uncommon to work with the space of everywhere differentiable functions; instead, continuity of the derivative is usually imposed to obtain a Banach space structure. We will investigate under which conditions this is possible in our setting and propose a metric topology for the general case.

This work is structured to build a coherent development of the theoretical framework and main results. We begin in Section 2 by introducing the fundamental notions that will serve as the foundation for our analysis. A key concept in our formulation is that of derivators, which provide the necessary structure for defining and studying Stieltjes differential problems. We outline their role and significance before turning to the Stieltjes derivative itself. This derivative, which generalizes the classical notion of differentiation, possesses distinctive properties that set it apart from its conventional counterpart. To further clarify its behavior, we also discuss the set of points where a given derivator remains constant, as these play a crucial role in describing the kernel of the Stieltjes derivative operator.

Building on these preliminaries, Section 3 is devoted to a detailed study of the kernel of the Stieltjes derivative. We establish precise conditions that characterize when a function has a zero Stieltjes derivative and explore the implications of these results for uniqueness in first-order differential problems. To provide further insight, we complement the theoretical discussion with explicit examples that illustrate how the structure of the kernel depends on the choice of derivator.

In Section 4, we introduce the function space $\mathcal{BD}$, which is specifically designed to accommodate solutions to Stieltjes differential problems. Our objective is to capture the essential features of differentiability in this setting while ensuring a well-defined analytical framework. To this end, we examine the topological properties of $\mathcal{BD}$ and introduce a suitable metric structure that facilitates the development of an existence and uniqueness theory.

Finally, in Section 5, we present our main results on the existence and uniqueness of solutions to first-order Stieltjes differential equations. We show that first order problems have in general several solutions, thereby highlighting the subtleties introduced by the Stieltjes setting.

2 Preliminaries

Let $[a,b] \subset \mathbb{R}$ be an interval, $\mathbb{F}$ the field $\mathbb{R}$ or $\mathbb{C}$ and $g : \mathbb{R} \to \mathbb{R}$ a left-continuous non-decreasing function. We will refer to such functions as *derivators*. For these functions, we define the set $D_g = \{d_n\}_{n\in\Lambda}$ (where $\Lambda \subset \mathbb{N}$) as the set of all discontinuity points of g, namely, $D_g = \{t \in \mathbb{R} : \Delta g(t) > 0\}$ where $\Delta g(t) := g(t^+) - g(t)$, $t \in \mathbb{R}$, and $g(t^+)$ denotes the right hand side limit of g at t. We also define

$$C_g := \{t \in \mathbb{R} \,:\, g \text{ is constant on } (t-\varepsilon, t+\varepsilon) \text{ for some } \varepsilon > 0\}.$$

Observe that C_g is open in the usual topology of $\mathbb{R}$, so we can write

$$C_g = \bigcup_{n\in\widetilde{\Lambda}} (a_n, b_n), \tag{2.1}$$

where $\widetilde{\Lambda} \subset \mathbb{N}$ and $(a_k, b_k) \cap (a_j, b_j) = \emptyset$ for $k \neq j$. With this notation, we denote $N_g^- := \{a_n\}_{n\in\widetilde{\Lambda}} \backslash D_g$, $N_g^+ := \{b_n\}_{n\in\widetilde{\Lambda}} \backslash D_g$ and $N_g := N_g^- \cup N_g^+$.

Definition 2.1 ([5, Definition 3.7])**.** We define the *Stieltjes derivative*, or *g-derivative*, of a map $f : [a,b] \to \mathbb{C}$ at a point $t \in [0,T]$ as

$$f_g'(t) = \begin{cases} \lim\limits_{s\to t} \dfrac{f(s)-f(t)}{g(s)-g(t)}, & t \notin D_g \cup C_g, \\ \lim\limits_{s\to t^+} \dfrac{f(s)-f(t)}{g(s)-g(t)}, & t \in D_g, \\ \lim\limits_{s\to b_n^+} \dfrac{f(s)-f(b_n)}{g(s)-g(b_n)}, & t \in C_g,\ t \in (a_n, b_n), \end{cases}$$

where a_n, b_n are as in (2.1), provided the corresponding limits exist. In that case, we say that f is *g-differentiable at t*.

Remark 2.2. It is possible to further simplify the definition of the Stieltjes derivative at a point $t \in [a,b]$ by defining

$$t^* = \begin{cases} t, & t \notin C_g, \\ b_n, & t \in (a_n, b_n) \subset C_g, \end{cases}$$

with a_n, b_n as in (2.1). With this notation, we have that

$$f_g'(t) = \begin{cases} \lim\limits_{s\to t} \dfrac{f(s)-f(t)}{g(s)-g(t)}, & t \notin D_g \cup C_g, \\ \lim\limits_{s\to t^{*+}} \dfrac{f(s)-f(t^*)}{g(s)-g(t^*)}, & t \in D_g \cup C_g, \end{cases}$$

provided the corresponding limit exists. We will define the function f^* as $f^*(t) = f(t^*)$ for $t \in [a,b]$.

The following result includes some basic properties of this derivative.

Proposition 2.3 ([5, Proposition 3.9])**.** *Let $t \in [a,b]$. If $f_1, f_2 : [a,b] \to \mathbb{F}$ are g-differentiable at t, then:*

- *The function $\lambda_1 f_1 + \lambda_2 f_2$ is g-differentiable at t for any $\lambda_1, \lambda_2 \in \mathbb{F}$ and*

$$(\lambda_1 f_1 + \lambda_2 f_2)'_g(t) = \lambda_1 (f_1)'_g(t) + \lambda_2 (f_2)'_g(t).$$

- (Product rule). *The product $f_1 f_2$ is g-differentiable at t and*

$$(f_1 f_2)'_g(t) = (f_1)'_g(t) f_2(t^*) + (f_2)'_g(t) f_1(t^*) + (f_1)'_g(t)\,(f_2)'_g(t) \Delta g(t^*).$$

- (Quotient rule). *If $f_2(t^*)\,(f_2(t^*) + (f_2)'_g(t)\,\Delta g(t^*)) \neq 0$, the quotient f_1/f_2 is g-differentiable at t and*

$$\left(\frac{f_1}{f_2}\right)'_g(t) = \frac{(f_1)'_g(t) f_2(t^*) - (f_2)'_g(t) f_1(t^*)}{f_2(t^*)\,(f_2(t^*) + (f_2)'_g(t)\,\Delta g(t^*))}.$$

Furthermore, we have the following chain rule.

Proposition 2.4 ([5, Proposition 4.1]). *Let $x_0 \in [a,b]$, $f : [a,b] \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{F}$. Then, the following hold:*

1. *If $x_0 \in [a,b] \backslash (C_g \cup D_g)$ and there exist $h'(f(x_0))$ and $f'_g(x_0)$, then $h \circ f$ is g-differentiable at x_0 and*
$$(h \circ f)'_g(x_0) = h'(f(x_0)) f'_g(x_0). \tag{2.2}$$
2. *If $x_0 \in C_g$ and there exist $h'(f(x_0))$ and $f'_g(x_0)$, then $h \circ f$ is g-differentiable at x_0 and*
$$(h \circ f)'_g(x_0) = h'(f(x_0^*)) f'_g(x_0),$$
Observe that if f is g-continuous, $f(x_0) = f(x_0^*)$ and we recover the formula (2.2).
3. *If $x_0 \in D_g$ and*
$$f(s) = f(x_0), \quad s \in (x_0, x_0 + \delta) \text{ for some } \delta > 0, \tag{2.3}$$
then $f'_g(x_0) = (h \circ f)'_g(x_0) = 0$. In particular, (2.2) holds provided $h'(f(x_0))$ exists.
4. *Suppose that $x_0 \in D_g$ and condition (2.3) does not hold. If $f(x_0^+)$ exists, h is continuous at $f(x_0^+)$ and the limit*
$$\lim_{s \to x_0^+} \frac{h(f(s)) - h(f(x_0))}{f(s) - f(x_0)}$$
exists, then there exist $f'_g(x_0)$ and $(h \circ f)'_g(x_0)$ and
$$(h \circ f)'_g(x_0) = \frac{h(f(x_0^+)) - h(f(x_0))}{f(x_0^+) - f(x_0)} f'_g(x_0).$$

Remark 2.5. It is clear from the proof of [5, Proposition 4.1], which is itself a continuation of the proof of [14, Proposition 2.56 and Proposition 3.15], that it is not necessary for h to be defined on $\mathbb{R}$. It is enough that $f([a,b])$ is contained in the domain of h and that the point at which h' is evaluated in each of the cases is an accumulation point of the domain.

We shall write as μ_g the Lebesgue–Stieltjes measure associated to g given by

$$\mu_g([c,d)) = g(d) - g(c), \quad c, d \in \mathbb{R},\ c < d.$$

We will use the term "g-measurable" for a set or function to refer to μ_g-measurability in the corresponding sense, and we denote by $\mathcal{L}^1_g(X,\mathbb{F})$ the set of Lebesgue–Stieltjes μ_g-integrable functions on a g-measurable set X with values in $\mathbb{F}$, whose integral we write as

$$\int_X f(s)\ \mathrm{d}\,\mu_g(s), \quad f \in \mathcal{L}^1_g(X,\mathbb{F}).$$

Similarly, we will talk about properties holding *g-almost everywhere* in a set X (shortened to *g-a.e.* in X), or holding for *g-almost all* (or, simply, *g-a.a.*) $x \in X$, as a simplified way to express that they hold μ_g-almost everywhere in X or for μ_g-almost all $x \in X$, respectively. $L^1_g(X,\mathbb{F})$ will be the Banach space associated to $\mathcal{L}^1_g(X,\mathbb{F})$ by taking equivalence classes of functions that are equal μ_g-a.e. The spaces $L^p_g(X,\mathbb{F})$ are defined as usual (they are L^p spaces with respect to the measure μ_g) and we will denote their respective norms by $\|\cdot\|_{L^p_g}$.

Definition 2.6 ([9, Definition 3.1])**.** A function $f:[a,b]\to\mathbb{F}$ is *g-continuous* at a point $t\in[a,b]$, or *continuous with respect to g* at t, if for every $\varepsilon>0$, there exists $\delta>0$ such that

$$|f(t)-f(s)|<\varepsilon, \quad \text{for all } s\in[0,T],\ |g(t)-g(s)|<\delta.$$

If f is g-continuous at every point $t\in[a,b]$, we say that f is g-continuous on $[a,b]$. We denote by $\mathcal{C}_g([a,b];\mathbb{F})$ the *set of g-continuous functions* on $[a,b]$; and by $\mathcal{BC}_g([a,b];\mathbb{F})$ the *set of bounded g-continuous functions* on $[a,b]$.

Definition 2.7. Given $k\in\mathbb{N}$, we define $\mathcal{C}^0_g([a,b];\mathbb{F}) := \mathcal{C}_g([a,b];\mathbb{F})$ and $\mathcal{C}^k_g([a,b];\mathbb{F})$ recursively as

$$\mathcal{C}^k_g([a,b]) := \{f\in\mathcal{C}^{k-1}([a,b];\mathbb{F}):\ (f_g^{(k-1)})'_g\in\mathcal{C}_g([a,b];\mathbb{F})\},$$

where $f_g^{(0)} := f$ and $f_g^{(k)} := (f_g^{(k-1)})'_g$, $k\in\mathbb{N}$. Similarly, given $k\in\mathbb{N}$, we define $\mathcal{BC}^0_g([a,b];\mathbb{F}) := \mathcal{BC}_g([a,b];\mathbb{F})$ and $\mathcal{BC}^k_g([a,b];\mathbb{F})$ recursively as

$$\mathcal{BC}^k_g([a,b];\mathbb{F}) := \{f\in\mathcal{C}^k_g([a,b];\mathbb{F}):\ f_g^{(n)}\in\mathcal{BC}_g([a,b];\mathbb{F}),\ \forall n=0,\dots,k\}.$$

We also define $\mathcal{C}^\infty_g([a,b];\mathbb{F}) := \bigcap_{n\in\mathbb{N}}\mathcal{C}^k_g([a,b];\mathbb{F})$ and $\mathcal{BC}^\infty_g([a,b];\mathbb{F}) := \bigcap_{n\in\mathbb{N}}\mathcal{BC}^k_g([a,b];\mathbb{F})$.

Definition 2.8 ([12, Definition 5.1])**.** A function $F:[a,b]\to\mathbb{R}$ is *absolutely continuous with respect to g* (or *g-absolutely continuous*) if to each $\varepsilon>0$ there is some $\delta>0$ such that for any family $\{(a_n,b_n)\}_{n=1}^m$ of pairwise disjoint open subintervals of $[a,b]$ the inequality

$$\sum_{n=1}^m (g(b_n)-g(a_n))<\delta$$

implies

$$\sum_{n=1}^m |F(b_n)-F(a_n)|<\varepsilon.$$

Theorem 2.9 ([12, Theorem 2.4])**.** *Assume that $f:[a,b)\longrightarrow\overline{\mathbb{R}}$ is integrable on $[a,b)$ with respect to μ_g and consider its indefinite Lebesgue–Stieltjes integral*

$$F(x)=\int_{[a,x)} f\,\mathrm{d}\,\mu_g \quad \textit{for all } x\in[a,b].$$

Then there is a g-measurable set $N\subset[a,b)$ such that $\mu_g(N)=0$ and

$$F'_g(x)=f(x) \quad \textit{for all } x\in[a,b]\backslash N.$$

Theorem 2.10 (Fundamental Theorem of Calculus for the Lebesgue–Stieltjes Integral [12, Theorem 5.4]). *A function $F : [a,b] \longrightarrow \mathbb{R}$ is g-absolutely continuous on $[a,b]$ if and only if the following three conditions are fulfilled:*

1. *There exists $F_g'(t)$ for g-almost all $t \in [a,b]$;*
2. *$F_g' \in \mathcal{L}_g^1([a,b))$; and*
3. *For each $t \in [a,b]$ we have*
$$F(t) = F(a) + \int_{[a,t)} F_g' \,\mathrm{d}\,\mu_g.$$

Definition 2.11. Given a function $p : [a,b] \to \mathbb{F}$, we say that it is *g-regressive* if
$$1 + p(t)\,\Delta g(t) \neq 0, \quad t \in [a,b] \cap D_g.$$

Given a g-regressive function $p \in \mathcal{L}_g^1([0,T);\mathbb{C})$, we define the *$g$-exponential map* associated to the map p as
$$\exp_g(p;t) := \exp\left(\int_{[0,t)} \widetilde{p}(s)\ \mathrm{d}\,\mu_g(s)\right), \quad t \in [a,b],$$
where, denoting by $\ln(z) := \ln|z| + i\,\mathrm{Arg}(z)$ the principal branch of the complex logarithm,
$$\widetilde{p}(s) := \begin{cases} p(s), & s \in [0,T) \setminus D_g, \\ \dfrac{\ln\left(1 + p(s)\,\Delta g(s)\right)}{\Delta g(s)}, & s \in [0,T) \cap D_g. \end{cases}$$

Observe that $\exp_g(p;t)$ can be written in the following way:
$$\exp_g(p;t) = \prod_{s \in [0,t) \cap D_g} (1 + p(s)\,\Delta g(s)) \exp\left(\int_{[0,t)\setminus D_g} p(s)\,\mathrm{d}\,\mu_g\right).$$

Remark 2.12. The g-exponential map belongs to $\mathcal{AC}_g([a,b];\mathbb{C})$, see [5, Theorem 4.2], and, furthermore, it is the only function in that space satisfying
$$\begin{cases} v_g'(t) = p(t)\,v(t), & g\text{-a.a. } t \in [a,b], \\ v(a) = 1. \end{cases}$$

In particular, if $p \in \mathcal{BC}_g([a,b];\mathbb{C})$, then $\exp_g(p;\cdot) \in \mathcal{BC}_g^1([a,b];\mathbb{C})$. Furthermore, if $p(t) = \lambda \in \mathbb{C}$, $t \in [a,b]$, then $\exp_g(p;\cdot) \in \mathcal{BC}_g^\infty([a,b];\mathbb{C})$.

3 The Mean Value Theorem for real-valued functions and the kernel of the Stieltjes derivative

Inspired by the ideas in [1, Theorem 1.67], in this section we propose a series of results that can be described as different versions of the Mean Value Theorem for the Stieltjes derivative as a result of imposing different levels of regularity on the function. These results will be then used to study the kernel of the Stieltjes derivative under the corresponding hypotheses. It is clear that a constant function will have zero g-derivative but, as we will see, not every function with zero g-derivative has to be constant.

3.1 g-absolutely continuous functions

We first start for a theorem in the case of functions with weak derivatives, that is, the space of absolutely continuous functions $\mathcal{AC}_g([a,b],\mathbb{F})$. We will be assuming that g is not constant throughout this work.

Theorem 3.1. *Let be $f \in \mathcal{AC}_g([a,b],\mathbb{R})$. Then*

$$\mu_g\left(\left\{t \in [a,b) \ : \ f'_g(t) \geq \frac{f(b)-f(a)}{g(b)-g(a)}\right\}\right) > 0,$$
$$\mu_g\left(\left\{t \in [a,b) \ : \ f'_g(t) \leq \frac{f(b)-f(a)}{g(b)-g(a)}\right\}\right) > 0.$$

Proof. We divide the proof in different cases.

1. If $\operatorname{ess\,inf} f'_g,\ \operatorname{ess\,sup} f'_g \in \mathbb{R}$, we consider the following cases:

(a) If $\operatorname{ess\,inf} f'_g = \operatorname{ess\,sup} f'_g$, then f'_g takes the same value c μ_g-a.e. and, by Theorem 3.1, $f(b)-f(a) = \int_{[a,b)} f'_g \,\mathrm{d}\,\mu_g = c(g(b)-g(a))$, so $c = \frac{f(b)-f(a)}{g(b)-g(a)}$ and, therefore,

$$\mu_g\left(\left\{t \in [a,b) \ : \ f'_g(t) = \frac{f(b)-f(a)}{g(b)-g(a)}\right\}\right) > 0.$$

(b) In the case where

$$\operatorname{ess\,inf} f'_g \leqslant \frac{f(b)-f(a)}{g(b)-g(a)} < \operatorname{ess\,sup} f'_g,$$

we have that

$$\mu_g\left(\left\{t \in [a,b) \ : \ f'_g(t) \geqslant \frac{f(b)-f(a)}{g(b)-g(a)}\right\}\right) > 0.$$

Now, given

$$\varepsilon \in \left(0, \operatorname{ess\,sup} f'_g - \frac{f(b)-f(a)}{g(b)-g(a)}\right),$$

we define the following sets:

$$A = \left\{x \in [a,b) : \ f'_g(x) \in \left[\frac{f(b)-f(a)}{g(b)-g(a)} + \varepsilon, \operatorname{ess\,sup} f'_g\right]\right\}, \quad B = [a,b) \setminus A.$$

We have $\mu_g(A) > 0$. Indeed, assume, on the contrary, that $\mu_g(A) = 0$,

$$\operatorname{ess\,sup} f'_g \leq \frac{f(b)-f(a)}{g(b)-g(a)} + \varepsilon < \frac{f(b)-f(a)}{g(b)-g(a)} + \operatorname{ess\,sup} f'_g - \frac{f(b)-f(a)}{g(b)-g(a)} = \operatorname{ess\,sup} f'_g,$$

which leads to a contradiction. Therefore, we have that $\mu_g(A) > 0$. Now, assuming that

$$\operatorname{ess\,inf} f'_g = \frac{f(b)-f(a)}{g(b)-g(a)},$$

we have that

$$\begin{aligned} f(b)-f(a) &= \int_{[a,b]} f'_g \,\mathrm{d}\,\mu_g = \int_A f'_g \,\mathrm{d}\,\mu_g + \int_B f'_g \,\mathrm{d}\,\mu_g \\ &\geq \mu_g(A)\left(\frac{f(b)-f(a)}{g(b)-g(a)}+\varepsilon\right) + \mu_g(B)\,\operatorname{ess\,inf} f'_g \\ &= \mu_g(A)\left(\frac{f(b)-f(a)}{g(b)-g(a)}+\varepsilon\right) + \mu_g(B)\frac{f(b)-f(a)}{g(b)-g(a)} \\ &= (\mu_g(A)+\mu_g(B))\frac{f(b)-f(a)}{g(b)-g(a)} + \mu_g(A)\varepsilon \\ &= f(b)-f(a)+\mu_g(A)\varepsilon > f(b)-f(a), \end{aligned}$$

which leads to a contradiction. Therefore:

$$\operatorname{ess\,inf} f'_g < \frac{f(b)-f(a)}{g(b)-g(a)},$$

and thus:

$$\mu_g\left(\left\{t\in[a,b)\ :\ f'_g(t)\leqslant\frac{f(b)-f(a)}{g(b)-g(a)}\right\}\right)>0.$$

(c) In the case where

$$\operatorname{ess\,inf} f'_g < \frac{f(b)-f(a)}{g(b)-g(a)} \leqslant \operatorname{ess\,sup} f'_g,$$

the procedure is analogous, and we will complete it for the sake of thoroughness. As a consequence of the preceding chain of inequalities,

$$\mu_g\left(\left\{t\in[a,b)\ :\ f'_g(t)\leqslant\frac{f(b)-f(a)}{g(b)-g(a)}\right\}\right)>0.$$

Now, let us take

$$\varepsilon\in\left(0,\frac{f(b)-f(a)}{g(b)-g(a)}-\operatorname{ess\,inf} f'_g\right)$$

and define the following sets:

$$A=\left\{x\in[a,b)\colon\ f'_g(x)\in\left[\operatorname{ess\,inf} f'_g,\frac{f(b)-f(a)}{g(b)-g(a)}-\varepsilon\right]\right\},\quad B=[a,b)\setminus A.$$

We have $\mu_g(A)>0$. Indeed, if $\mu_g(A)=0$,

$$\operatorname{ess\,inf} f'_g \geq \frac{f(b)-f(a)}{g(b)-g(a)}-\varepsilon > \frac{f(b)-f(a)}{g(b)-g(a)}-\frac{f(b)-f(a)}{g(b)-g(a)}+\operatorname{ess\,inf} f'_g,$$

which leads to a contradiction. Therefore, we have that $\mu_g(A)>0$. Now, suppose that

$$\operatorname{ess\,sup} f'_g = \frac{f(b)-f(a)}{g(b)-g(a)},$$

then,

$$\begin{aligned} f(b)-f(a) &= \int_{[a,b)} f'_g \,\mathrm{d}\,\mu_g = \int_A f'_g \,\mathrm{d}\,\mu_g + \int_B f'_g \,\mathrm{d}\,\mu_g \\ &\leq \mu_g(A)\left(\frac{f(b)-f(a)}{g(b)-g(a)} - \varepsilon\right) + \mu_g(B)\operatorname{ess\,sup} f'_g \\ &= \mu_g(A)\left(\frac{f(b)-f(a)}{g(b)-g(a)} - \varepsilon\right) + \mu_g(B)\frac{f(b)-f(a)}{g(b)-g(a)} \\ &= (\mu_g(A)+\mu_g(B))\frac{f(b)-f(a)}{g(b)-g(a)} - \mu_g(A)\varepsilon \\ &= f(b)-f(a)-\mu_g(A)\varepsilon < f(b)-f(a), \end{aligned}$$

which leads to a contradiction. Therefore,

$$\operatorname{ess\,sup} f'_g > \frac{f(b)-f(a)}{g(b)-g(a)},$$

and, thus,

$$\mu_g\left(\left\{t\in[a,b) \;:\; f'_g(t) \geqslant \frac{f(b)-f(a)}{g(b)-g(a)}\right\}\right) > 0.$$

2. If $\operatorname{ess\,inf} f'_g = -\infty$ and $\operatorname{ess\,sup} f'_g \in \mathbb{R}$, then:

$$\mu_g\left(\left\{t\in[a,b) \;:\; f'_g(t) \leqslant \frac{f(b)-f(a)}{g(b)-g(a)}\right\}\right) > 0.$$

Let us see that the following also holds:

$$\mu_g\left(\left\{t\in[a,b) \;:\; f'_g(t) \geqslant \frac{f(b)-f(a)}{g(b)-g(a)}\right\}\right) > 0.$$

We are in the following situation:

$$-\infty = \operatorname{ess\,inf} f'_g < \frac{f(b)-f(a)}{g(b)-g(a)} \leq \operatorname{ess\,sup} f'_g.$$

Given $\varepsilon > 0$, we define the following sets:

$$A = \left\{x\in[a,b) :\; f'_g(x) \leq \frac{f(b)-f(a)}{g(b)-g(a)} - \varepsilon\right\}, \quad B = [a,b)\setminus A.$$

We have, since $\operatorname{ess\,inf} f'_g = -\infty$, that $\mu_g(A) > 0$. Therefore, following the reasoning applied in section (c) of the previous point,

$$\mu_g\left(\left\{t\in[a,b) \;:\; f'_g(t) \geqslant \frac{f(b)-f(a)}{g(b)-g(a)}\right\}\right) > 0.$$

3. If $\operatorname{ess\,sup} f'_g = +\infty$ and $\operatorname{ess\,inf} f'_g \in \mathbb{R}$, then:

$$\mu_g\left(\left\{t\in[a,b) \;:\; f'_g(t) \geqslant \frac{f(b)-f(a)}{g(b)-g(a)}\right\}\right) > 0.$$

Let us see that the following also holds:

$$\mu_g\left(\left\{t \in [a,b) \ : \ f_g'(t) \leqslant \frac{f(b)-f(a)}{g(b)-g(a)}\right\}\right) > 0.$$

We are in the following situation:

$$\operatorname{ess\,inf} f_g' \leqslant \frac{f(b)-f(a)}{g(b)-g(a)} < \operatorname{ess\,sup} f_g' = +\infty.$$

Given $\varepsilon > 0$, we define the following sets:

$$A = \left\{x \in [a,b) : \ f_g'(x) \geq \frac{f(b)-f(a)}{g(b)-g(a)} + \varepsilon\right\}, \quad B = [a,b) \setminus A.$$

We have, since $\operatorname{ess\,sup} f_g' = +\infty$, that $\mu_g(A) > 0$. Therefore, following the reasoning applied in section (b) of the previous point,

$$\mu_g\left(\left\{t \in [a,b) \ : \ f_g'(t) \leqslant \frac{f(b)-f(a)}{g(b)-g(a)}\right\}\right) > 0. \qquad \square$$

Corollary 3.2. *Let $f \in \mathcal{AC}_g([a,b],\mathbb{R})$. Then there exist $c,d \in [a,b]$ such that*

$$f_g'(c) \leqslant \frac{f(b)-f(a)}{g(b)-g(a)} \leqslant f_g'(d).$$

Proof. Taking into account that, by Theorem 2.10, the derivative of an g-absolutely continuous function exist g-a.e., and Theorem 3.1, we get the result. $\square$

Remark 3.3. It is clear that, in the conditions of the previous corollary, given $c \in \mathbb{R}$,

$$\mu_g\left(\left\{t \in [a,b] \ : \ f_g'(t) \geqslant c\right\}\right) + \mu_g\left(\left\{t \in [a,b] \ : \ f_g'(t) \leqslant c\right\}\right) \geqslant g(b)-g(a) > 0,$$

and, in particular, for the value $c = \frac{f(b)-f(a)}{g(b)-g(a)}$. Nevertheless, $\frac{f(b)-f(a)}{g(b)-g(a)}$ is the only value for which we can guarantee, a priori, that both sets have positive measure for every function f. Indeed, just take a $\mathcal{AC}_g$ function with constant derivative x. If $x = c$ both sets have measure $g(b)-g(a)$; otherwise, one has measure $g(b)-g(a)$ and the other measure zero.

Lemma 3.4. *Let $f \in \mathcal{AC}_g([a,b],\mathbb{R})$. Then f is constant if and only if f is g-differentiable on $[a,b]$ and $f_g' = 0$ μ_g-a.e.*

Proof. If f is constant, by the definition of g-derivative, f_g' exists everywhere in $[a,b]$ and $f_g' = 0$. Now assume $f_g' = 0$ μ_g-a.e. Since $f \in \mathcal{AC}_g([a,b],\mathbb{R})$, by Theorem 2.10, $f(t) = \int_{[a,t)} f_g'(s)\,\mathrm{d}\,s + f(a) = f(a)$, so f is constant. $\square$

3.2 Functions which are continuous with respect to g

One of the main tools in the proof of [1, Theorem 1.67] is the generalization of the Induction Principle for a given time scale, see [1, Theorem 1.7]. In our setting, we will use the following version of the Induction Principle on the real line that can be directly deduced from [2, Theorem 1].

Theorem 3.5 (Principle of real induction). *Let $c, d \in \mathbb{R}$ be such that $c < d$ and $S \subset [c, d]$. Then, $S = [c, d]$ if and only if the following conditions are satisfied:*

1. $c \in S$.
2. *If $x \in [c, d)$ is such that $x \in S$, then there exists $\delta > 0$ such that $[x, x + \delta] \subset S$.*
3. *If $x \in (c, d]$ is such that $[c, x) \subset S$, then $x \in S$.*

In this first step, we prove a version of the Mean Value Theorem for functions which are g-differentiable and g-continuous on an interval. This is, to some extent, the generalization of the set of hypotheses required for the Mean Value Theorem for the usual derivative. Nevertheless, instead of establishing this result under the mentioned conditions, we propose a formulation based on some of the properties that such functions present. We do so as we believe that this makes it easy to understand the improvements in the subsequent sections.

Theorem 3.6. *Let $f, h : [a, b] \to \mathbb{R}$ be such that the following conditions hold:*

(i) *The maps f and h are left-continuous on $(a, b]$.*

(ii) *If g is constant on some $[\alpha, \beta] \subset [a, b]$, then so are f and h.*

(iii) *For all $t \in [a, b]$, f and h are g-differentiable at t and $|f'_g(t)| \leqslant h'_g(t)$.*

Then for any $s, t \in [a, b]$,

$$|f(s) - f(t)| \leqslant |h(s) - h(t)|. \tag{3.1}$$

Proof. Let $t \in [a, b]$. We shall prove that

$$|f(s) - f(t)| \leqslant h(s) - h(t), \quad \text{for all } s \in [a, b],\ s \geqslant t. \tag{3.2}$$

Observe that, if $t = b$, this is trivial, so we shall assume that $t < b$.

Let $\varepsilon > 0$ and define

$$S_\varepsilon = \{s \in [t, b] : |f(s) - f(t)| \leqslant h(s) - h(t) + \varepsilon(g(s) - g(t))\}.$$

Note that, in order to prove (3.2), it is enough to show that $S_\varepsilon = [t, b]$ as $\varepsilon > 0$ has been arbitrarily chosen. We do this by means of Theorem 3.5. In particular, we only need to check that 2 and 3 in Theorem 3.5 are satisfied as, by definition, $t \in S_\varepsilon$, which shows 1.

In order to check 2 in Theorem 3.5, let $s \in [t, b)$ be such that $s \in S_\varepsilon$. We study two cases separately: $s \in N_g^- \cup C_g$ and $s \notin N_g^- \cup C_g$.

First, if $s \in N_g^- \cup C_g$, then we find $\delta > 0$ such that g is constant on $[s, s + \delta]$. Hence, (ii) guarantees that f and h are also constant on $[s, s + \delta]$ and thus, since $s \in S_\varepsilon$, it follows that $[s, s + \delta] \subset S_\varepsilon$.

Otherwise, we have that $s \notin N_g^- \cup C_g$ in which case, since f and h are g-differentiable at s, we know that

$$f'_g(s) = \lim_{r \to s^+} \frac{f(r) - f(s)}{g(r) - g(s)}, \quad h'_g(s) = \lim_{r \to s^+} \frac{h(r) - h(s)}{g(r) - g(s)}.$$

Hence, there exists $\rho > 0$ such that if $r \in (s, s + \rho)$, then

$$\left| \frac{f(r) - f(s)}{g(r) - g(s)} - f'_g(s) \right| < \frac{\varepsilon}{2}, \quad \left| \frac{h(r) - h(s)}{g(r) - g(s)} - h'_g(s) \right| < \frac{\varepsilon}{2},$$

or, equivalently,

$$|f(r) - f(s) - f'_g(s)(g(r) - g(s))| < \frac{\varepsilon}{2}(g(r) - g(s)),$$
$$|h(r) - h(s) - h'_g(s)(g(r) - g(s))| < \frac{\varepsilon}{2}(g(r) - g(s)).$$

In particular, for any $r \in (s, s+\rho)$, we have that

$$|f(r) - f(s)| \leqslant (g(r) - g(s)) \left(|f'_g(s)| + \frac{\varepsilon}{2}\right),$$
$$h'_g(s)(g(r) - g(s)) < h(r) - h(s) + \frac{\varepsilon}{2}(g(r) - g(s)).$$

Hence, taking $\delta \in (0, \rho)$, it follows that for any $r \in [s, s+\delta]$,

$$\begin{aligned}
|f(r) - f(t)| &\leqslant |f(r) - f(s)| + |f(s) - f(t)| \\
&\leqslant (g(r) - g(s)) \left(|f'_g(s)| + \frac{\varepsilon}{2}\right) + h(s) - h(t) + \varepsilon(g(s) - g(t)) \\
&\leqslant (g(r) - g(s)) \left(h'_g(s) + \frac{\varepsilon}{2}\right) + h(s) - h(t) + \varepsilon(g(s) - g(t)) \\
&= h'_g(s)(g(r) - g(s)) + \frac{\varepsilon}{2}(g(r) - g(s)) + h(s) - h(t) + \varepsilon(g(s) - g(t)) \\
&\leqslant h(r) - h(s) + \frac{\varepsilon}{2}(g(r) - g(s)) + \frac{\varepsilon}{2}(g(r) - g(s)) + h(s) - h(t) + \varepsilon(g(s) - g(t)) \\
&= h(r) - h(t) + \varepsilon(g(r) - g(t)),
\end{aligned}$$

which shows that $[s, s+\delta] \subset S_\varepsilon$.

Finally, for 3 in Theorem 3.5, let $s \in (t, b]$ be such that $[t, s) \subset S_\varepsilon$. In that case, since f, g and h are left-continuous at s, we see that

$$\begin{aligned}
|f(s) - f(t)| &= \left|\lim_{r \to s^-} f(r) - f(t)\right| = \lim_{r \to s^-} |f(r) - f(t)| \\
&\leqslant \lim_{r \to s^-} (h(r) - h(t) + \varepsilon(g(r) - g(t))) = h(s) - h(t) + \varepsilon(g(s) - g(t)),
\end{aligned}$$

as we needed to show.

Hence, we have proven that (3.2) holds from which (3.1) follows. □

As a direct consequence of Theorem 3.6 and [9, Proposition 3.2] we have the anticipated version of the Mean Value Theorem for functions which are continuous with respect to g.

Corollary 3.7 (Mean Value Theorem for g-continuous functions)**.** *Let $f, h : [a, b] \to \mathbb{R}$ be g-continuous on $[a, b]$. If f and h are g-differentiable on $[a, b]$ and $|f'_g(t)| \leqslant h'_g(t)$ for all $t \in [a, b]$, then,*

$$|f(s) - f(t)| \leqslant |h(s) - h(t)|, \quad s, t \in [a, b].$$

Finally, we can use Corollary 3.7 to fully describe the kernel of the Stieltjes derivative operator on the set of g-continuous functions.

Theorem 3.8 (Kernel of the Stieltjes derivative for g-continuous functions)**.** *Let $f : [a, b] \to \mathbb{R}$ be g-continuous on $[a, b]$. Then, $f'_g(t) = 0$ for all $t \in [a, b]$ if and only if f is constant on $[a, b]$.*

Proof. It is clear from the definition of g-derivative that if f is constant, then $f'_g(t) = 0$ so we shall focus on the converse implication.

Assume $f'_g(t) = 0$ for all $t \in [a,b]$. Then, the map $h(t) = 0$, $t \in [a,b]$, is g-continuous and g-differentiable on $[a,b]$ with $h'_g(t) = 0$, $t \in [a,b]$. Furthermore, we have that $|f'_g(t)| \leqslant h'_g(t)$ for all $t \in [a,b]$, so Corollary 3.7 guarantees that $|f(s) - f(t)| = 0$ for all $s,t \in [a,b]$, which finishes the proof. □

To end this section we recall some interesting and already known results in this direction.

Theorem 3.9 ([13, Theorem 2.6]). *Let $g : \mathbb{R} \to \mathbb{R}$ be a left-continuous and nondecreasing function, continuous and increasing on an interval $[a,b] \subset \mathbb{R}$. Let $f : [a,b] \to \mathbb{R}$ be g-continuous on $[a,b]$ and g-differentiable on (a,b) satisfying $f(a) = f(b)$. Then, there exists $c \in (a,b)$ such that $f'_g(c) = 0$.*

As a consequence of this theorem, the authors prove the following corollary.

Corollary 3.10 ([13, Corollary 2.7]). *Let $g : \mathbb{R} \to \mathbb{R}$ be a left-continuous and nondecreasing function, continuous and increasing on an interval $[a,b] \subset \mathbb{R}$. Let $f : [a,b] \to \mathbb{R}$ be g-continuous on $[a,b]$ and g-differentiable on (a,b). Then, there exists $c \in (a,b)$ such that $f'_g(c) = \frac{f(b)-f(a)}{g(b)-g(a)}$.*

We note that in Theorem 3.9 it is required that g be continuous and increasing on the interval $[a,b]$, a hypothesis that, in our case, is not necessary. In any case, if the derivator satisfies the hypothesis of continuity and monotonicity in the interval $[a,b]$, the proof of Theorem 3.8 is a consequence of the Corollary 3.10.

3.3 Left-continuous functions

In this next step, we aim to prove a version of the Mean Value Theorem for functions that are not necessarily g-continuous but share one basic properties with them: left-continuity.

To that end, we shall denote by $\mathcal{I}_1$ the family of connected components of $[a,b]\backslash C_g$. We have the following result conveying some properties of this family.

Lemma 3.11. *The family $\mathcal{I}_1$ is nonempty consisting of singletons or closed subintervals of $[a,b]$. Furthermore, if $I \in \mathcal{I}_1$ is an interval, then $\min I + \varepsilon \notin C_g \cup N_g^-$, for all $\varepsilon \in [0, \max I - \min I)$.*

Proof. The hypotheses under which we define the Stieltjes derivative guarantee that $b \notin C_g$, which guarantees that $[a,b]\backslash C_g \neq \emptyset$ and, thus, $\mathcal{I}_1 \neq \emptyset$. Now, by definition, the elements of $\mathcal{I}_1$ are closed and connected subsets of the real line. Hence, the first part of the result follows.

Now, let $I \in \mathcal{I}_1$ be an interval and consider $t_* = \min I$. Observe that t_* is well-defined as I is closed. Furthermore, by definition, we have that $t_* \notin C_g$. Now, reasoning by contradiction, suppose $t_* + \varepsilon \in N_g^-$. Then, by definition, there exists $r > 0$ such that g is constant on $[t_* + \varepsilon, t_* + \varepsilon + r]$, which means that $(t_* + \varepsilon, t_* + \varepsilon + r) \subset C_g$. This is a contradiction since $(t_* + \varepsilon, t_* + \varepsilon + r) \cap I \neq \emptyset$. Hence, $t_* + \varepsilon \notin N_g^-$. □

We are now in position to prove a version of the Mean Value Theorem for left-continuous functions, in which the family $\mathcal{I}_1$ plays an important role.

Theorem 3.12 (Mean Value Theorem for left-continuous functions). *Let $f, h : [a,b] \to \mathbb{R}$ be left-continuous on $(a,b]$. If f and h are g-differentiable on $[a,b]$ and $|f'_g(t)| \leqslant h'_g(t)$ for all $t \in [a,b]$, then for each $I \in \mathcal{I}_1$,*

$$|f(s) - f(t)| \leqslant |h(s) - h(t)|, \quad s,t \in I. \tag{3.3}$$

Proof. Let $I \in \mathcal{I}_1$. If I is a singleton, then (3.3) is trivially satisfied so we shall assume that $I = [c,d]$ for some $c,d \in [a,b]$, $c < d$.

Let $t \in [c,d]$. Following the ideas in the proof of Theorem 3.6, we shall prove that

$$|f(s) - f(t)| \leqslant h(s) - h(t), \quad \text{for all } s \in [c,d],\ s \geqslant t, \tag{3.4}$$

from which the result follows. Once again, if $t = d$, (3.4) is trivially satisfied, so we shall assume that $t < d$.

Let $\varepsilon > 0$ and define

$$S_\varepsilon = \{s \in [t,d] : |f(s) - f(t)| \leqslant h(s) - h(t) + \varepsilon(g(s) - g(t))\}.$$

As in the proof of Theorem 3.6, (3.4) will be proved if we show that $S_\varepsilon = [t,d]$ using Theorem 3.5. Note that 1 is trivially satisfied and 3 can be checked in an analogous manner to the proof of Theorem 3.6, so we shall focus on 2.

Let $s \in [t,d)$ be such that $s \in S_\varepsilon$. Observe that the definition of $\mathcal{I}_1$ and Lemma 3.11 ensure that $s \notin C_g \cup N_g^-$, so, since f and h are g-differentiable at s, we know that

$$f'_g(s) = \lim_{r \to s^+} \frac{f(r) - f(s)}{g(r) - g(s)}, \quad h'_g(s) = \lim_{r \to s^+} \frac{h(r) - h(s)}{g(r) - g(s)}.$$

From here, checking 2 in Theorem 3.5 is analogous to reasoning used in the proof of Theorem 3.6 and we omit it. □

We can use this new version of the Mean Value Theorem to obtain a characterization of the left-continuous functions which have null Stieltjes derivative. The proof of the following result is analogous to that of Theorem 3.8 and we omit it.

Corollary 3.13. *Let $f : [a,b] \to \mathbb{R}$ be a left-continuous function on $(a,b]$ such that $f'_g(t) = 0$ for all $t \in [a,b]$. Then,*

$$f(t) = f(s), \quad \textit{for all } s,t \in I, \quad I \in \mathcal{I}_1.$$

The reciprocal implication in Corollary 3.13 does not hold, as the following example shows.

Example 3.14. Consider g to be the Cantor function, that is,

$$g(x) = \begin{cases} \displaystyle\sum_{n=1}^{\infty} \frac{c_n}{2^{n+1}}, & x = \displaystyle\sum_{n=1}^{\infty} \frac{c_n}{3^n} \in C \text{ for } c_n \in \{0,2\}; \\ \displaystyle\sup_{y \leq x,\, y \in C} g(y), & x \in [0,1] \backslash C, \end{cases}$$

where C is Cantor's set, that is,

$$C = \bigcap_{n=0}^{\infty} E_n,$$

where

$$
\begin{aligned}
E_0 &= [0,1]\,,\\
E_1 &= \left[0,\frac{1}{3}\right]\cup\left[\frac{2}{3},1\right],\\
E_2 &= \left[0,\frac{1}{9}\right]\cup\left[\frac{2}{9},\frac{1}{3}\right]\cup\left[\frac{2}{3},\frac{7}{9}\right]\cup\left[\frac{8}{9},1\right],\\
&\vdots\\
E_m &= \left[0,\frac{1}{3^m}\right]\cup\left[\frac{2}{3^m},\frac{3}{3^m}\right]\cup\left[\frac{6}{3^m},\frac{7}{3^m}\right]\cup\left[\frac{8}{3^m},\frac{9}{3^m}\right]\cup\cdots\\
&\quad\cdots\cup\left[\frac{3^m-3}{3^m},\frac{3^m-2}{3^m}\right]\cup\left[\frac{3^m-1}{3^m},1\right],\ m\in\mathbb{N}.
\end{aligned}
$$

Observe that g is continuous and increasing and, therefore, a derivator. Furthermore, $C_g = [0,1]\backslash C$. Since C is totally disconnected, $\mathcal{I}_1 = \{\{x\} \ : \ x \in C\}$ and the conclusion of Corollary 3.13 is vacuous. Still, we have that g is g-differentiable and $g'_g = 1$, not zero.

As a final note for this case, observe that, effectively, neither Theorem 3.12 nor Corollary 3.13 describe what happens to the functions involved on the set C_g. This is because the definition of Stieltjes derivative on the set C_g does not take into account the behavior of the functions in C_g. Without any extra hypotheses, it is not possible to describe the functions on a neighborhood of such points.

3.4 Stieltjes differentiable functions

In this final section, we aim to obtain a version of the Mean Value Theorem for g-differentiable functions, without imposing any other conditions. To that end, we need to refine the family $\mathcal{I}_1$. This refined family will be $\mathcal{I}_2$, the family of connected subsets of $[a,b]\backslash(C_g \cup N_g^+ \cup D_g)$, for which we have the following result.

Lemma 3.15. *The family $\mathcal{I}_2$ is nonempty and consists of either singletons or subintervals of $[a,b]$. Moreover, letting $t_* = \inf(I)$ for any $I \in \mathcal{I}_2$, we have that, for every $\varepsilon > 0$ such that $t_* + \varepsilon < \sup(I)$, it holds that $t_* + \varepsilon \notin C_g \cup N_g \cup D_g$. Furthermore, if $t_* \in I$, then $t_* \notin C_g \cup N_g \cup D_g$.*

Proof. Let $\varepsilon > 0$ be such that $t_* + \varepsilon < \sup(I)$. If $t_* + \varepsilon \in N_g^-$, there exists $r > 0$ such that g is constant on $[t_* + \varepsilon, t_* + \varepsilon + r]$. In particular, $(t_* + \varepsilon, t_* + \varepsilon + r) \subset C_g$. This is a contradiction since $(t_* + \varepsilon, t_* + \varepsilon + r) \cap I \neq \emptyset$. Hence, $t_* + \varepsilon \notin N_g^-$. The proof for the case $t_* \in I$ is analogous to the previous one. □

This new family of sets is enough to obtain a version of the Mean Value Theorem for g-differentiable functions in a similar fashion to Theorem 3.12.

Theorem 3.16 (Mean Value Theorem for Stieltjes differentiable functions)**.** *Let $f, h : [a,b] \to \mathbb{R}$ be g-differentiable functions on $[a,b]$. If $|f'_g(t)| \leqslant h'_g(t)$ for all $t \in [a,b]$ then, for each $I \in \mathcal{I}_2$,*

$$|f(s) - f(t)| \leqslant |h(s) - h(t)|, \quad s,t \in I. \tag{3.5}$$

Proof. Let $I \in \mathcal{I}_2$. If I is a singleton, then (3.5) is trivially satisfied, so we shall assume that $\overline{I} = [c,d]$ for some $c < d$. Let us examine each case individually.

- $I = (c, d]$. If we can prove that, given any $\varepsilon \in (0, d-c)$, it holds that

$$|f(s) - f(t)| \leqslant |h(s) - h(t)|, \quad s, t \in [c+\varepsilon, d],$$

we will be done, since for any two elements $t, s \in (c, d]$, it is always possible to find a value of $\varepsilon \in (0, d-c)$ such that $t, s \in [c+\varepsilon, d]$. Thus, (3.5) is satisfied.

Now take an arbitrary $\varepsilon \in (0, d-c)$, let $t \in [c+\varepsilon, d]$ and we shall show that

$$|f(s) - f(t)| \leqslant h(s) - h(t), \quad \text{for all } s \in [c+\varepsilon, d],\ s \geqslant t. \tag{3.6}$$

We will assume that $t < d$, since, in the case $t = d$, inequality (3.6) holds trivially. Now take $\widehat{\varepsilon} > 0$ and define:

$$S_{\widehat{\varepsilon}} = \{s \in [t, d] : |f(s) - f(t)| \leqslant h(s) - h(t) + \widehat{\varepsilon}(g(s) - g(t))\}.$$

As in the previous proofs, if we show that $S_{\widehat{\varepsilon}} = [t, d]$ for any $\widehat{\varepsilon} > 0$, (3.6) holds. Let us verify that the hypotheses of Theorem 3.5 are satisfied:

1. $t \in S_{\widehat{\varepsilon}}$ trivially (equality holds).
2. Let $s \in [t, d)$ such that $s \in S_{\widehat{\varepsilon}}$. Thanks to Lemma 3.15, $s \notin C_g \cup N_g \cup D_g$, so, since f and h are g-differentiable at s, we have

$$f'_g(s) = \lim_{r \to s} \frac{f(r) - f(s)}{g(r) - g(s)}, \quad h'_g(s) = \lim_{r \to s} \frac{h(r) - h(s)}{g(r) - g(s)}.$$

In particular,

$$f'_g(s) = \lim_{r \to s^+} \frac{f(r) - f(s)}{g(r) - g(s)}, \quad h'_g(s) = \lim_{r \to s^+} \frac{h(r) - h(s)}{g(r) - g(s)}$$

and we can proceed as in the proof of Theorem 3.6.

3. Let $s \in (t, d]$ such that $[t, s) \subset S_{\widehat{\varepsilon}}$. Since $s \in [c+\varepsilon, d] \subset (c, d] \in \mathcal{I}_2$, we have $s \notin C_g \cup N_g^+ \cup D_g$. In particular, we have that $g(r) < g(s)$ for all $r < s$. Hence, since f and h are g-differentiable at s, [12, Proposition 2.1] ensures that they are left-continuous at s, so it is enough to follow the arguments in Theorem 3.6 to finish the proof of the result.

- $I = [c, d)$. This case is similar to the previous one, and we must prove that, given any $\varepsilon \in (0, d-c)$, it holds that

$$|f(s) - f(t)| \leqslant |h(s) - h(t)|, \quad s, t \in [c, d-\varepsilon].$$

Thus, (3.5) is satisfied. The reasoning is analogous to the previous case: we take $t \in [c, d-\varepsilon]$, assuming that $t < d - \varepsilon$, since the case $t = d - \varepsilon$ is trivial. Given an arbitrary $\widehat{\varepsilon} > 0$, define

$$S_{\widehat{\varepsilon}} = \{s \in [t, d-\varepsilon] : |f(s) - f(t)| \leqslant h(s) - h(t) + \widehat{\varepsilon}(g(s) - g(t))\}.$$

and verify, using Theorem 3.5, that $S_{\widehat{\varepsilon}} = [t, d-\varepsilon]$. As a consequence, we obtain that

$$|f(s) - f(t)| \leqslant h(s) - h(t), \quad \text{for all } s \in [c, d-\varepsilon],\ s \geqslant t,$$

from which it follows that

$$|f(s) - f(t)| \leqslant |h(s) - h(t)|, \quad s, t \in [c, d-\varepsilon].$$

- $I = (c, d)$. Analogous to the previous cases, if we prove that, given $\varepsilon \in (0, (d-c)/3)$, it holds that
$$|f(s) - f(t)| \leqslant |h(s) - h(t)|, \quad s, t \in [c + \varepsilon, d - \varepsilon],$$
we will be done. The proof is a combination of the arguments used in the previous two points.
- $I = [c, d]$. In this case, we directly prove that
$$|f(s) - f(t)| \leqslant |h(s) - h(t)|, \quad s, t \in [c, d],$$
with the proof being analogous to the previous cases. □

Remark 3.17. Notice that in the interior of an interval such as I in Theorem 3.16, the restriction of g satisfies the hypothesis of Theorem 3.9 and Corollary 3.10 (that is, continuous and increasing).

As a direct consequence of Theorem 3.16, we can obtain the following result which can be described as a different version of the Mean Value Theorem under a boundedness assumption. Note that this formulation of the result is closer to the corresponding result for the usual derivative.

Corollary 3.18 (Mean Value Theorem for Stieltjes differentiable functions)**.** *Let $f : [a, b] \to \mathbb{R}$ be a g-differentiable function on $[a, b]$. Then, for each $I \in \mathcal{I}_2$,*
$$|f(\sup I^-) - f(\inf I^+)| \leqslant \sup_{u \in I} |f'_g(u)|(g(\sup I^-) - g(\inf I^+)). \tag{3.7}$$

Proof. Let $I \in \mathcal{I}_2$. By the definition of $\mathcal{I}_2$, $g(\sup I^-) - g(\inf I^+) > 0$. If $\sup_{u \in I} |f'_g(u)| = \infty$, there is nothing to prove, so we will assume that $\sup_{u \in I} |f'_g(u)| < \infty$. We shall only prove (3.7) whenever $\overline{I} = [c, d]$ with $c < d$, as the case when I is a singleton is trivial.

Given $\varepsilon \in (0, (d-c)/3)$, we define the following map:
$$h_\varepsilon : t \in [c + \varepsilon, d - \varepsilon] \to h_\varepsilon(t) = \sup_{u \in I} |f'_g(u)|(g(t) - g(c + \varepsilon)).$$
Clearly h_ε is g-differentiable on $[c + \varepsilon, d - \varepsilon]$ with
$$(h_\varepsilon)'_g(t) = \sup_{u \in I} |f'_g(u)|, \ \forall\, t \in [c + \varepsilon, d - \varepsilon].$$
So, it follows that $|f'_g(t)| \leqslant (h_\varepsilon)'_g(t)$, for all $t \in [c + \varepsilon, d - \varepsilon]$. Hence, applying Theorem 3.16 on the interval $[c + \varepsilon, d - \varepsilon]$ and noting that $[c + \varepsilon, d - \varepsilon]$ is the only connected component of $[c + \varepsilon, d - \varepsilon] \setminus (C_g \cup N_g^+ \cup D_g)$, we have that
$$|f(s) - f(t)| \leqslant |h_\varepsilon(s) - h_\varepsilon(t)| = \sup_{u \in I} |f'_g(u)||g(t) - g(s)|, \quad s, t \in [c + \varepsilon, d - \varepsilon].$$
In particular,
$$|f(d - \varepsilon) - f(c + \varepsilon)| \leqslant \sup_{u \in I} |f'_g(u)|(g(d - \varepsilon) - g(c + \varepsilon)),$$
so we can conclude the proof of the result by taking $\varepsilon \to 0^+$ if we show first that the limits $f(c^+)$ and $f(d^-)$ exist. Observe that $f(c^+)$ exists by the g-differentiability of f at $c \in D_g \cup N_g^+$. On the other hand, $f(d^-)$ exists because, given $\varepsilon \in \mathbb{R}^+$, and defining
$$M = \frac{\varepsilon}{2 \max\{1, \sup_{u \in I} |f'_g(u)|\}},$$

since g is left continuous at d, there exists $\delta \in \mathbb{R}^+$ such that $|g(t) - g(d)|, |g(s) - g(d)| < M$ if $t, s \in (c, d)$, $d - t, d - s < \delta$. Thus, given $t, s \in (c, d)$, $d - t, d - s < \delta$,

$$|f(s) - f(t)| \leqslant \sup_{u \in I} |f'_g(u)||g(t) - g(s)| \leq \sup_{u \in I} |f'_g(u)|(|g(t) - g(d)| + |g(d) - g(s)|) < \varepsilon,$$

which is enough to guarantee that $f(d^-)$ exists, as it implies that, for any sequence $\{x_n\}_{n \in \mathbb{N}} \subset (c, d)$ converging to d, $\{f(x_n)\}_{n \in \mathbb{N}}$ is a Cauchy sequence—cf. [11, p. 245]. □

Theorem 3.16 can also be used to describe some properties of functions in the kernel of the Stieltjes derivative as presented in the next result. The proof of the following result is in the style of the one for Theorem 3.8 and we omit it.

Corollary 3.19. *Let $f : [a, b] \to \mathbb{R}$ be such that $f'_g(t) = 0$ for all $t \in [a, b]$. Then,*

$$f(t) = f(s), \quad \text{for all } s, t \in I, \quad I \in \mathcal{I}_2.$$

Remark 3.20. The converse of Corollary 3.19 is not generally true. For example, if we consider g as the Cantor function that we defined in Example 3.14, we have that $C_g \cup N_g^+ \cup D_g = [0, 1] \setminus \widehat{C}$, where

$$\widehat{C} = \bigcap_{n=0}^{\infty} \widehat{E}_n,$$

with

$$\begin{aligned}
\widehat{E}_0 &= [0, 1], \\
\widehat{E}_1 &= \left[0, \frac{1}{3}\right] \cup \left(\frac{2}{3}, 1\right], \\
\widehat{E}_2 &= \left[0, \frac{1}{9}\right] \cup \left(\frac{2}{9}, \frac{1}{3}\right] \cup \left(\frac{2}{3}, \frac{7}{9}\right] \cup \left(\frac{8}{9}, 1\right], \\
&\vdots \\
\widehat{E}_m &= \left[0, \frac{1}{3^m}\right] \cup \left(\frac{2}{3^m}, \frac{3}{3^m}\right] \cup \left(\frac{6}{3^m}, \frac{7}{3^m}\right] \cup \left(\frac{8}{3^m}, \frac{9}{3^m}\right] \cup \cdots \\
&\quad \cdots \cup \left(\frac{3^m - 3}{3^m}, \frac{3^m - 2}{3^m}\right] \cup \left(\frac{3^m - 1}{3^m}, 1\right], \ m \in \mathbb{N}.
\end{aligned}$$

The set $\widehat{C}$ is totally disconnected, and therefore $\mathcal{I}_2 = \{\{x\} : x \in \widehat{C}\}$. Hence, $g(t) = g(s)$, for all $s, t \in I$, with $I \in \mathcal{I}_2$. However, $g'_g(t) = 1$ for all $t \in [0, 1]$.

Now we present some characterizations of those g-differentiable functions with g-derivative 0 everywhere.

Lemma 3.21. *$f : [a, b] \to \mathbb{R}$ satisfies $f'_g(t) = 0$ for all $t \in [a, b)$ if and only if $(fh)'_g = f^* h'_g$ for every g-differentiable function $h : [a, b) \to \mathbb{R}$.*

Proof. If $f'_g(t) = 0$ for all $t \in [a, b)$, given g-differentiable function $h : [a, b) \to \mathbb{R}$, by the product rule,

$$(fh)'_g(t) = f'_g(t)h(t^*) + h'_g(t)f(t^*) + f'_g(t)h'_g(t)\Delta g(t^*) = f(t^*)h'_g(t).$$

Now assume $(fh)'_g = f^* h'_g$ for every g-differentiable function $h : [a, b] \to \mathbb{R}$. By taking $h = 1$, this equality implies that f is g-differentiable and $f'_g = 0$. □

We now aim to characterize the functions whose g-derivative is zero, using the following observation. Clearly, if φ is constant, then any function of the form $f = \varphi \circ g$ has g-derivative zero. Conversely, suppose φ is differentiable, g is continuous, and $f = \varphi \circ g$ has vanishing g-derivative. Then, by Proposition 2.4, for any t in the domain of g, we have

$$0 = f'_g(t) = \varphi'_g(g(t^*))g'_g(t) = \varphi'_g(g(t)),$$

so φ'_g has to vanish on the image of g. A similar characterization should hold for any g-differentiable function f; it suffices to express f as a composition $f = \varphi \circ g$ for a suitable function φ. This is precisely the content of Lemma 3.22.

For the next result we will denote by A'_+ the set of accumulation points of A from the right and by f'_+ the derivative of f from the right.

Lemma 3.22. *Let $f : \mathbb{R} \to \mathbb{R}$ be a g-differentiable function. f satisfies $f'_g(t) = 0$ for all $t \in [a, b)$ if and only if there exists a function $\varphi : I \to \mathbb{R}$, where I is the smallest interval containing $g(\mathbb{R})$, such that*

1. *φ is constant on $[g(t), g(t^+)]$ for $t \in \mathbb{R}$,*
2. *$\varphi(g(t^*)) = f(t^*)$ for $t \in \mathbb{R}$,*
3. *$\varphi|_{g(\mathbb{R})}$ is differentiable on $[g(\mathbb{R})\backslash g(D_g \cup N_g)] \cap g(\mathbb{R})'$, differentiable from the right on the set $[g(\mathbb{R})\backslash g(D_g)] \cap g(\mathbb{R})' \cap g(N_g^+)$ and differentiable from the left on $[g(\mathbb{R})\backslash g(D_g)] \cap g(\mathbb{R})' \cap g(N_g^-)$ and the derivative on those sets is zero, and*
4. *$\varphi|_{g(\mathbb{R})}$ is differentiable from the right on $g(D_g) \cap g(\mathbb{R})'_+$ and $(\varphi|_{g(\mathbb{R})})'_+ = 0$ on that set.*

Proof. Assume first that there exists a differentiable function $\varphi : I \to \mathbb{R}$ satisfying conditions 1-4. Let $h = \varphi \circ g = \varphi|_{g(\mathbb{R})} \circ g$. First, g is g-differentiable and $g'_g = 1$. Now we consider different cases for $t \in [a, b)$.

We start by making the observation that the points in $\varphi(\mathbb{R})\backslash\varphi(\mathbb{R})'$ are points $x = \varphi(t)$ where $t \in D_g$ and g is constant on $(t - \varepsilon, t)$ and $(t, t + \varepsilon)$ for some ε.

If $g(t) \notin \varphi(\mathbb{R})'$, $g(t) \in D_g$ and φ is constant on $[g(t), g(t^+)]$, so, given that φ is continuous,

$$h'_g(t) := \lim_{s \to t^+} \frac{\varphi(g(s)) - \varphi(g(t))}{g(s) - g(t)} = \frac{\varphi(g(t^+)) - \varphi(g(t))}{g(t^+) - g(t)} = 0.$$

If $g(t) \in \varphi(\mathbb{R})'$, taking into account Remark 2.5, we can apply points 1–3 of Lemma 2.4 to deduce that h is g-differentiable and $h'_g(t) = 0$ if $t \notin D_g$ or $t \in D_g$ and condition (2.3) holds. Suppose now that $t \in D_g$ and condition (2.3) does not hold. $g(t^+)$ exists, φ is continuous from the right at $g(t^+)$, as it is differentiable form the right, and we have that

$$\lim_{s \to t^+} \frac{\varphi(g(s)) - \varphi(g(t))}{g(s) - g(t)} = \frac{\varphi(g(t^+)) - \varphi(g(t))}{g(t^+) - g(t)} = 0.$$

Thus, by point 4 of Lemma 2.4, $h'_g = 0$.

Since f is g-differentiable, by [6, Proposition 4.1], f^* is g differentiable as well and $(f^*)'_g = f'_g$. Thus, $f'_g(t) = (f^*)_g(t)' = h'_g(t) = 0$ for $t \in [a, b)$.

To show the converse, we modify the proof of [8, Theorem 3.13]. Let $c = \sup g(\mathbb{R}) \in (-\infty, \infty]$. We start by defining the function $\sigma(t) = \sup g^{-1}(x)$ for every $x \in (-\infty, c) \cap g(\mathbb{R})$. If $c \in g(\mathbb{R})$, we define $\sigma(c) = t$ for some $t \in \mathbb{R}$ such that $g(t) = g(c)$. This way, we have defined σ on $g(\mathbb{R})$. Since g is increasing and left continuous, $\sigma(g(t)) = t^*$ for every $t \in \mathbb{R}$.

Let

$$\varphi(x) := \begin{cases} f(\sigma(x)), & x \in g(\mathbb{R}), \\ \varphi(g(t)), & x \in (g(t), g(t^+)] \backslash g(\mathbb{R}),\ t \in \mathbb{R}. \end{cases}$$

Observe that φ is well defined and, as g is nondecreasing, we have defined φ on all of I. Furthermore, φ is constant on $[g(t), g(t^+)]$ and $\varphi(g(t^*)) = f(\sigma(g(t^*))) = f(t^*)$ for $t \in \mathbb{R}$.

Now we check that $\varphi|_{g(\mathbb{R})}$ is differentiable on $[g(\mathbb{R}) \backslash g(D_g \cup N_g)] \cap g(\mathbb{R})'$ (the other cases are similar) and $\varphi|'_{g(\mathbb{R})} = 0$. Given $t \in \mathbb{R}$, $g(t)$, is an accumulation point from the left in the set $g(\mathbb{R})$ because g is left continuous, so, given that $t^* \notin N_g^+$, we can consider

$$\lim_{\substack{y \to g(t)^- \\ y \in g(\mathbb{R})}} \frac{\varphi(y) - \varphi(g(t))}{y - g(t)} = \lim_{s \to t} \frac{\varphi(g(s)) - \varphi(g(t))}{g(s) - g(t)} = \lim_{s \to t^-} \frac{f(\sigma(g(s^*))) - f(\sigma(g(t^*)))}{g(s) - g(t)}$$

$$= \lim_{s \to t^-} \frac{f(s^*) - f(t^*)}{g(s) - g(t)} = (f^*)'_g(t) = f'_g(t) = 0.$$

If $g(t)$, is an accumulation point from the right in the set $g(\mathbb{R})$, and given that $t \notin N_g^-$ we conclude, in a similar manner, that

$$\lim_{\substack{y \to g(t)^+ \\ y \in g(\mathbb{R})}} \frac{\varphi(y) - \varphi(g(t))}{y - g(t)} = (f^*)'_g(t) = f'_g(t) = 0.$$

In any case, we have proven that the derivative is zero. Condition 4 is proven in a similar way. □

Remark 3.23. While Lemma 3.21 does not require f to be g-differentiable, Lemma 3.22 imposes this condition. The reason for this is that, although g-differentiability of f implies the same for f^*, with identical g-derivatives [6, Proposition 4.1], the reverse implication does not hold in general [6, Remark 4.3].

4 $\mathcal{BD}$-spaces

We now aim to define suitable function spaces, denoted by $\mathcal{BD}_g^k([a,b];\mathbb{F})$, for g-differentiable functions. To this end, we begin by establishing some basic properties of such functions, as these must be satisfied by any element of the proposed spaces. Moreover, we seek to ensure that these spaces provide a natural framework for the study of differential problems involving g-derivatives.

Remark 4.1. Taking into account Definition 2.1, given a derivator $g : [a,b] \to \mathbb{R}$ such that $a \notin N_g^-$ and $b \notin N_g^+ \cup D_g \cup C_g$, the following conditions will be necessary for the existence of the g-derivative in all of the points of $[a,b] \setminus (C_g \cup D_g)$ (see [5, Remark 3.3]):

- Given $t \in [a,b] \setminus (C_g \cup D_g \cup N_g)$, f is g-continuous at t. In particular, f is continuous at t since g-continuous functions are, in particular, continuous at the points where g is continuous.

- Given $t \in N_g^-$, f is left g-continuous at t, in the sense that, given $\varepsilon \in \mathbb{R}^+$, there exists $\delta \in \mathbb{R}^+$ such that, if $s < t$ with $g(t) - g(s) < \delta$ then, $|f(s) - f(t)| < \varepsilon$.

Observe that the function f might not be g-continuous at such points. Indeed, take for instance

$$g : x \in [0,3] \to g(s) = \begin{cases} x, & x \in [0,1], \\ 1, & x \in (1,2), \\ x-1, & x \in [2,3]. \end{cases} \tag{4.1}$$

Then,

$$f : x \in [0,3] \to f(x) = \begin{cases} x, & x \in [0,1], \\ x+1, & x \in (1,3], \end{cases}$$

is g-differentiable at $x = 1$ since

$$\lim_{s \to 1^-} \frac{f(s) - f(1)}{g(s) - g(1)} = \lim_{s \to 1^-} \frac{s-1}{s-1} = 1.$$

However, the function f is not g-continuous at $x = 1$ since g is continuous at that point.

- Given $t \in N_g^+$, f is right g-continuous at t, in the sense that, given $\varepsilon \in \mathbb{R}^+$, there exists $\delta \in \mathbb{R}^+$ such that, if $t < s$ with $g(s) - g(t) < \delta$ then, $|f(s) - f(t)| < \varepsilon$.

  As in the previous case, the function does not have to be g-continuous at such points. Indeed, take for instance g as before and

$$f : x \in [0,3] \to f(x) = \begin{cases} x, & x \in [0,2), \\ 2x+1, & x \in [2,3], \end{cases} \tag{4.2}$$

  f is g-differentiable at $x = 2$, but f is not g-continuous at such point.

- Given $t \in D_g$, the right limit $f(t^+) = \lim_{s \to t^+} f(s)$ exists.

We conclude that, interestingly enough, the g-differentiability of a function at a point of N_g does not imply the g-continuity of the function at the point. The g-differentiability of a function only guarantees the g-continuity at the points of $[a,b] \backslash (C_g \cup D_g \cup N_g)$.

Remark 4.2. It is important to highlight that it may happen that the g-derivative of a function has more points at which it is g-continuous than the function itself. For example, let us consider the derivator (4.1) and the function (4.2). The function f is g-continuous on $[0,1) \cup (2,3]$, however, its g-derivative,

$$f'_g : x \in [0,3] \to f'_g(x) = \begin{cases} 1, & x \in [0,1], \\ 2, & x \in (1,3], \end{cases}$$

is g-continuous on $[0,1) \cup (1,3]$. That is, f'_g is g-continuous on $(1,2]$, however, f is not g-continuous on $(1,2)$ (in order for f to be continuous on the interval $(1,2)$, it would have to be constant on that interval, which is false).

Lemma 4.3. *Let $[a,b] \subset \mathbb{R}$ be a closed interval of the real line. Let us assume that $a \notin N_g^-$ and $b \notin N_g^+ \cup D_g \cup C_g$. Given a function $f : [a,b] \to \mathbb{F}$ such that $f'_g(b_n)$ exists for some $n \in \widetilde{\Lambda}$, then there exists $f'_g(x)$ for all $x \in (a_n, b_n]$, and $f'_{g|_{(a_n,b_n]}}$ is g-continuous on $(a_n, b_n]$.*

Proof. The proof is straightforward considering that f'_g is constant on $(a_n, b_n]$ since $f'_g(x) = f'_g(b_n)$ for all $x \in (a_n, b_n]$. □

Building on the observations in Remark 4.1 regarding the properties of g-differentiable functions, we now introduce the space $\mathcal{BD}_g([a,b];\mathbb{F})$, consisting of functions that satisfy those same properties.

Definition 4.4. **($\mathcal{BD}_g([a,b];\mathbb{F})$ space)** Let $[a,b] \subset \mathbb{R}$ be a closed interval of the real line. Let us assume that $a \notin N_g^-$ and $b \notin N_g^+ \cup D_g \cup C_g$. We say that $f : [a,b] \to \mathbb{F}$ belongs to $\mathcal{BD}_g([a,b];\mathbb{F})$ if it is bounded in $[a,b]$, g-continuous on $[a,b] \setminus (C_g \cup N_g \cup D_g)$, left g-continuous on N_g^- and right g-continuous on N_g^+.

In order to define the spaces $\mathcal{BD}_g^k([a,b];\mathbb{F})$, with $k \in \mathbb{N}$, we will denote by $\mathcal{BD}_g^0([a,b];\mathbb{F})$ the space $\mathcal{BD}_g([a,b];\mathbb{F})$ and by $f_g^{(0)}$ the function f.

Definition 4.5 ($\mathcal{BD}_g^k([a,b];\mathbb{F})$ space, $k \in \mathbb{N}$)**.** Let $[a,b] \subset \mathbb{R}$ be a closed interval of the real line. Let us assume that $a \notin N_g^-$ and $b \notin N_g^+ \cup D_g \cup C_g$, we say that $f : [a,b] \to \mathbb{F}$ belongs to $\mathcal{BD}_g^k([a,b];\mathbb{F})$ if $f_g^{(k-1)} \in \mathcal{BD}_g^{k-1}([a,b];\mathbb{F})$ and there exists $(f^{(k-1)})_g'(t)$, for all $t \in [a,b]$ in terms of Definition 2.1, such that $(f^{(k-1)})_g' \in \mathcal{BD}_g^0([a,b];\mathbb{F})$. We write $\mathcal{BD}_g^\infty([a,b];\mathbb{F}) = \bigcap_{k\in\mathbb{N}} \mathcal{BD}_g^k([a,b];\mathbb{F})$.

In the next result we show that, in contrast with the case of the spaces $\mathcal{BC}_g^k([a,b];\mathbb{F})$, the spaces $\mathcal{BD}_g^k([a,b];\mathbb{F})$ are closed under the product of functions, showing that they have an algebra structure.

Lemma 4.6. *Given f_1, $f_2 \in \mathcal{BD}_g^k([a,b];\mathbb{F})$, with $k \in \mathbb{N}$, we have that $f_1 f_2 \in \mathcal{BD}_g^k([a,b];\mathbb{F})$.*

Proof. Let us see the case $k = 1$ (analogous for $k \in \mathbb{N}$, $k \geq 2$). Given f_1, $f_2 \in \mathcal{BD}_g^1([a,b];\mathbb{F})$, we have, by Proposition 2.3, that, the product $f_1 f_2$ is g-differentiable at $t \in [a,b]$ and

$$(f_1 f_2)_g'(t) = (f_1)_g'(t) f_2(t^*) + (f_2)_g'(t) f_1(t^*) + (f_1)_g'(t)\,(f_2)_g'(t)\Delta g(t^*).$$

In particular,

$$(f_1 f_2)_g'(t) = (f_1)_g'(t) f_2(t) + (f_2)_g'(t) f_1(t),\ \forall t \in [a,b]\setminus(D_g \cup C_g).$$

Hence, by the definition of space $\mathcal{BD}_g^1([a,b];\mathbb{F})$ we have that $(f_1 f_2)_g'$ is bounded in $[a,b]$, g-continuous on $[a,b] \setminus (C_g \cup N_g \cup D_g)$, left g-continuous on N_g^- and right g-continuous on N_g^+. □

Remark 4.7. In general we cannot ensure in general that $\mathcal{BD}_g^1([a,b];\mathbb{F}) \subset \mathcal{AC}_g([a,b];\mathbb{F})$ since the functions of $\mathcal{BD}_g^1([a,b];\mathbb{F})$ need not be left-continuous in D_g.

Remark 4.8. Under the hypotheses of Lemma 4.3, given $f \in \mathcal{BD}_g^k([a,b];\mathbb{F})$, we have that $f_g^{(l)}|_{(a_n,b_n]}$ is g-continuous on $(a_n,b_n]$, for all $n \in \widetilde{\Lambda}$ and for all $l = 1,\ldots,k$.

Observe that given $l \in \{1,\ldots,k\}$, we only have guaranteed left g-continuity of $f_g^{(l)}$ at $b_n \in D_g$, whereas if $b_n \in N_g^+$, the function $f_g^{(l)}$ is g-continuous at b_n. In both cases the function $f_g^{(l)}|_{(a_n,b_n]}$ is g-continuous on $(a_n,b_n]$.

Example 4.9. Let us consider the example we analyzed in Remark 4.2, in this case, $D_g = \emptyset$, $C_g = (1,2)$, $N_g^- = \{1\}$, and $N_g^+ = \{2\}$. Moreover,

- f is bounded, g-continuous on $[0,3] \setminus [1,2]$, g-continuous from the left at $x = 1$, and g-continuous from the right at $x = 2$.
- f_g' is bounded, g-continuous on $[0,3] \setminus [1,2]$, g-continuous from the left at $x = 1$, and g-continuous from the right at $x = 2$. In fact, f_g' is g-continuous on $[0,3] \setminus \{1\}$.
- $f_g^{(k)} = 0$ for all $k \geq 2$, in particular, it trivially holds that it is bounded, g-continuous on $[0,3] \setminus [1,2]$, g-continuous from the left at $x = 1$, and g-continuous from the right at $x = 2$.

Thus, $f \in \mathcal{BD}_g^k([a,b];\mathbb{F})$ for all $k \in \mathbb{N}$.

Remark 4.10. In the previous example, a very important property of the spaces $\mathcal{BD}_g^k([a,b];\mathbb{F})$ is revealed, namely that non-constant functions can exist whose g-derivative is zero.

We have that $\mathcal{BD}_g^k([a,b];\mathbb{F})$ is a normed vector space with the norm

$$\begin{aligned} \mathcal{BC}_g^k([a,b]) &\xrightarrow{\quad \|\cdot\|_k \quad} \mathbb{R} \\ f &\longmapsto \|f\|_k := \sum_{0 \leq i \leq k} \|f_g^{(i)}\|_\infty \end{aligned}$$

where $\|f\|_\infty := \sup_{x \in [a,b]} |f(x)|$ is the supremum norm, and $k \in \{0\} \cup \mathbb{N}$.

4.1 Sufficient conditions for a Banach space structure

In the following results we will assume that $[a,b] \subset \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ is a derivator such that $a \notin N_g^-$ and $b \notin N_g^+ \cup D_g \cup C_g$.

Remark 4.11. We have that $(\mathcal{BD}_g([a,b];\mathbb{F}), \|\cdot\|_\infty)$ is a Banach space. Indeed: the properties in the definition of $\mathcal{BD}_g([a,b];\mathbb{F})$, g-continuity on $[a,b] \setminus (C_g \cup N_g \cup D_g)$, left g-continuity on N_g^- and right g-continuity on N_g^+ are preserved by the convergence in the supremum norm. Let us see that under some hypotheses concerning the derivator g we have that $(\mathcal{BD}_g^k([a,b];\mathbb{F}), \|\cdot\|_k)$ is also a Banach space for all $k \in \mathbb{N}$.

Lemma 4.12. *The following families of statements are equivalent:*

1. $N_g' \setminus N_g \subset D_g$ *and* $D_g' \subset D_g$.
2. (a) *For all* $x \in [a,b] \setminus (D_g \cup N_g \cup C_g)$ *there exists* $\delta > 0$ *such that* $[x-\delta, x+\delta] \subset [a,b] \setminus (D_g \cup N_g \cup C_g)$,
   (b) *for all* $x \in N_g^+$ *there exists* $\delta > 0$ *such that* $(x, x+\delta] \subset [a,b] \setminus (D_g \cup N_g \cup C_g)$ *and*
   (c) *for all* $x \in N_g^-$ *there exists* $\delta > 0$ *such that* $[x-\delta, x) \subset [a,b] \setminus (D_g \cup N_g \cup C_g)$.

Proof. Let us prove each of the implications separately.

1⇒2. On the one hand, $\overline{D_g \cup N_g \cup C_g} = C_g \cup N_g \cup D_g \cup N_g' \cup D_g' = C_g \cup N_g \cup D_g$, from which we deduce that $C_g \cup N_g \cup D_g$ is closed. Therefore, given an element $x \in [a,b] \setminus (C_g \cup N_g \cup D_g)$, there exists $\delta > 0$ such that $[x-\delta, x+\delta] \subset [a,b] \setminus (C_g \cup N_g \cup D_g)$.

Now let $x \in N_g^+$ (the case N_g^- is analogous), since D_g is a closed set, there exists an element $\delta_1 > 0$ such that $[x-\delta_1, x+\delta_1] \subset [a,b] \setminus D_g$. Since $x \notin D_g$, in particular, $x \notin N_g'$,

therefore, there exists an element $\delta_2 > 0$ such that $[x-\delta_2, x+\delta_2] \setminus \{x\} \subset [a,b] \setminus N_g$. Taking $\delta_3 = \min\{\delta_1, \delta_2\}$, we have $[x-\delta_3, x+\delta_3] \setminus \{x\} \subset [a,b] \setminus (N_g \cup D_g)$. Hence, $(x, x+\delta_3] \subset [a,b] \setminus (N_g \cup D_g)$ and, since the points of the set $N_g \cup D_g$ include the endpoints of the intervals that form the connected components of C_g, it follows that $(x, x+\delta_3] \subset [a,b] \setminus (C_g \cup N_g \cup D_g)$.

2⇒1. Let $x \in N_g$, and assume $x \in N_g^+$ (the case of N_g^- is analogous). By point 2. (b), we know that there exists an element $\delta > 0$ such that $(x, x+\delta] \subset [a,b] \setminus (C_g \cup N_g \cup D_g)$, in particular, $(x, x+\delta] \subset [a,b] \setminus (N_g \cup D_g)$. Additionally, there exists $\delta_2 > 0$ such that $[x-\delta_2, x) \subset C_g \subset [a,b] \setminus (N_g \cup D_g)$. Taking $\delta_3 = \min\{\delta_1, \delta_2\}$, we have $[x-\delta_3, x+\delta_3] \setminus \{x\} \subset [a,b] \setminus (N_g \cup D_g)$. Hence, $x \notin N_g' \cup D_g'$. Therefore, we have that $N_g \cap N_g' = N_g \cap D_g' = \emptyset$. Now, by point 2. (a), the set $C_g \cup N_g \cup D_g$ is closed in $[a,b]$, and since C_g is an open set, it follows that $N_g \cup D_g$ must be closed. Thus,

$$N_g \cup D_g = \overline{N_g \cup D_g} = N_g' \cup N_g \cup D_g \cup D_g'.$$

Finally, taking into account that $D_g \cap N_g = N_g \cap N_g' = N_g \cap D_g' = \emptyset$, we deduce that $N_g' \setminus N_g \subset D_g$ and $D_g' \subset D_g$. □

We recall the following result.

Lemma 4.13 ([5, Lemma 3.13]). *We have the continuous embedding $\mathcal{BC}_g^1([a,b]) \hookrightarrow \mathcal{AC}_g([a,b])$. Furthermore, for every $f \in \mathcal{BC}_g^1([a,b])$,*

$$f(x) = f(a) + \int_{[a,x)} f_g'(s)\, \mathrm{d}\,\mu_g(s), \quad \forall x \in [a,b].$$

The following result is a generalization of [5, Lemma 3.14]. The proof is essentially the same, but we include it here for completeness.

Lemma 4.14. *Let $h \in \mathcal{BD}_g^k([a,b];\mathbb{F})$ and consider the function*

$$H : x \in [a,b] \to H(x) = \int_{[a,x)} h(s)\, \mathrm{d}\,\mu_g(s).$$

We have that $H_g'(x) = h(x^*)$, for every $x \in [a,b]$, $h^* \in \mathcal{BD}_g^k([a,b];\mathbb{F})$, and, therefore, $H \in \mathcal{BD}_g^{k+1}([a,b];\mathbb{F})$.

Proof. On the one hand, since h and h^* have the same g-derivatives and $h \in \mathcal{BD}_g^k([a,b];\mathbb{F})$, we have that $h^* \in \mathcal{BD}_g^k([a,b];\mathbb{F})$. Furthermore, $H \in \mathcal{AC}_g([a,b];\mathbb{F})$ given that $h \in \mathcal{BD}_g([a,b];\mathbb{F}) \subset \mathcal{L}_g^1([a,b);\mathbb{F})$, so it is enough ot prove that $H_g'(x) = h(x^*)$ for every $x \in [a,b]$ to get the result. We study three different cases:

- If $x \in D_g$ it is clear that

$$\begin{aligned} H_g'(x) &= \lim_{s \to x^+} \frac{H(s) - H(x)}{g(s) - g(x)} \\ &= \lim_{s \to x^+} \frac{1}{g(s) - g(x)} \int_{[x,s)} h(s)\, \mathrm{d}\,\mu_g(s) \\ &= \lim_{s \to x^+} \frac{1}{g(s) - g(x)} \left(\int_{\{x\}} h(s)\, \mathrm{d}\,\mu_g(s) + \int_{(x,s)} h(s)\, \mathrm{d}\,\mu_g(s) \right) \\ &= \lim_{s \to x^+} \frac{h(x)\Delta^+ g(x)}{g(s) - g(x)} = h(x). \end{aligned}$$

- If $x \in [a,b]\backslash(C_g \cup D_g \cup N_g)$, let us compute the limit

$$\lim_{s\to x} \frac{H(s)-H(x)}{g(s)-g(x)},$$

on the domain where $g(s) \neq g(x)$. Fix $\varepsilon > 0$. Since h is g-continuous and g is continuous at x, there exists $\delta > 0$ such that $|h(u)-h(x)| < \varepsilon$ if $|u-x| < \delta$. Define $[\![x,s]\!] := [\min\{x,s\}, \max\{x,s\}]$. Now, for $s \in [a,b]$, $|u-s| < \delta$, we have that

$$\begin{aligned}
\left|\frac{H(s)-H(x)}{g(s)-g(x)} - h(x)\right| &= \left|\frac{\operatorname{sgn}(s-x)}{g(s)-g(x)} \int_{[\![x,s]\!]} h(u)\, \mathrm{d}\,\mu_g(u) - h(x)\right| \\
&= \frac{1}{|g(s)-g(x)|} \left|\int_{[\![x,s]\!]} (h(u)-h(x))\, \mathrm{d}\,\mu_g(u)\right| \\
&\leq \frac{1}{|g(s)-g(x)|} \int_{[\![x,s]\!]} |h(u)-h(x)|\, \mathrm{d}\,\mu_g(u) \\
&\leq \frac{1}{|g(s)-g(x)|} \int_{[\![x,s]\!]} \varepsilon\, \mathrm{d}\,\mu_g(u) = \varepsilon.
\end{aligned} \tag{4.3}$$

Thus,

$$\lim_{s\to x} \frac{H(s)-H(x)}{g(s)-g(x)} = h(x).$$

- If $x \in N_g^+\backslash D_g$, let us compute the limit

$$\lim_{s\to x^+} \frac{H(s)-H(x)}{g(s)-g(x)},$$

on the domain where $g(s) \neq g(x)$. Fix $\varepsilon > 0$. Since h is g-continuous from the right and g is continuous at x, there exists $\delta > 0$ such that $|h(u)-h(x)| < \varepsilon$ if $0 < |u-s| < \delta$. Repeating the calculations at (4.3), we conclude that

$$H_g'(x) = \lim_{s\to x^+} \frac{H(s)-H(x)}{g(s)-g(x)} = h(x).$$

- In the case $x \in N_g^-\backslash D_g$, the reasoning is analogous.
- Finally, if $x \in (a_n, b_n) \subset C_g$, it holds that

$$H_g'(x) = H_g'(b_n) = h(b_n) = h(x^*),$$

where the first equality comes from the definition of the g-derivative at the points of C_g and the last is a consequence of the definition of x^*. □

Theorem 4.15. *Assume $N_g' \setminus N_g \subset D_g$ and $D_g' \subset D_g$. Then $(\mathcal{BD}_g^k([a,b];\mathbb{F}), \|\cdot\|_k)$ is a Banach space.*

Proof. We present a similar reasoning in the one in the proof of [5, Theorem 3.13]. We will check the case $k = 1$ (the case $k \geq 2$ is analogous). Let $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{BD}_g^1([a,b])$ be a Cauchy sequence. Then, $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{BD}_g([a,b])$ and $\{(f_n)_g'\}_{n\in\mathbb{N}} \subset \mathcal{BD}_g([a,b])$ are Cauchy sequences in the Banach space $\mathcal{BD}_g([a,b])$, so there exist $f, h \in \mathcal{BD}_g([a,b])$ such that $f_n \to f$ and $(f_n)_g' \to h$ in $\mathcal{BD}_g([a,b])$. Let us check that $f_g'(x)$ exists for every $x \in [a,b]$ for all possible cases and that, furthermore, $f_g' = h$.

• Given $x \in D_g$, we have that

$$(f_n)'_g(x) = \frac{f_n(x^+) - f_n(x)}{\Delta g(x)} \to \frac{f(x^+) - f(x)}{\Delta g(x)} = f'_g(x),$$

therefore $h(x) = f'_g(x)$.

• Thanks to Lemma 4.12, given $x \in [a,b]\backslash(D_g \cup C_g \cup N_g)$, there exists $\delta > 0$ such that $[x-\delta, x+\delta] \subset [a,b]\backslash(D_g \cup N_g \cup C_g)$. We have that f_n is bounded and g-continuous in $[x-\delta, x+\delta]$ and so is $(f_n)'_g$. Therefore, by Lemma 4.13, $f_n|_{[x-\delta,x+\delta]} \in \mathcal{AC}_g([x-\delta, x+\delta]; \mathbb{F})$ and

$$f_n(t) - f_n(x-\delta) = \int_{[x-\delta,t)} (f_n)'_g(s)\ \mathrm{d}\,\mu_g(s),\ \forall t \in [x-\delta, x+\delta].$$

On the other hand,

$$\begin{aligned}\left|\int_{[x-\delta,t)} (f_n)'_g(s)\ \mathrm{d}\,\mu_g(s) - \int_{[x-\delta,t)} h(s)\ \mathrm{d}\,\mu_g(s)\right| &\\ \leq \int_{[x-\delta,t)} \left|(f_n)'_g(s) - h(s)\right|\ \mathrm{d}\,\mu_g(s) &\\ \leq \varepsilon(g(x+\delta) - g(x-\delta)),&\end{aligned}$$

where the last inequality is valid for every $n \geq N$, where $N \in \mathbb{N}$ is such that $\|(f_n)'_g - h\|_\infty \leq \varepsilon$, for every $n \geq N$. Then, we have that

$$\lim_{n\to\infty} \int_{[x-\delta,t)} (f_n)'_g(s)\ \mathrm{d}\,\mu_g(s) = \int_{[x-\delta,t)} h(s)\ \mathrm{d}\,\mu_g(s)$$

uniformly on $[x-\delta, x+\delta]$. Thus,

$$\lim_{n\to\infty} (f_n(t) - f_n(x-\delta)) = \lim_{n\to\infty} \int_{[x-\delta,t)} (f_n)'_g(s)\ \mathrm{d}\,\mu_g(s) = \int_{[x-\delta,t)} h(s)\ \mathrm{d}\,\mu_g(s)$$

uniformly on $[x-\delta, x+\delta]$. Hence,

$$f(t) = f(x-\delta) + \int_{[x-\delta,t)} h(s)\ \mathrm{d}\,\mu_g(s).$$

Since $h|_{[x-\delta,x+\delta]} \in \mathcal{BC}_g([x-\delta, x+\delta])$, by Lemma 4.14, we get that

$$f'_g(t) = h(t),\ \forall x \in [x-\delta, x+\delta],$$

as we wanted to show. Thus, we have that $f'_g(t) = h(t)$, for all $t \in [x-\delta, x+\delta]$. In particular, $f'_g(x) = h(x)$.

• For $x \in N_g^-$ (analogous for N_g^+) the reasoning is similar. In this case, thanks to Lemma 4.12, given $x \in N_g^-$, we have that there exists $\delta > 0$ such that $[x-\delta, x) \subset [a,b]\backslash(D_g \cup N_g \cup C_g)$. Now since f_n and $(f_n)'_g$ are left g-continuous at x, we have that $f_n|_{[x-\delta,x]} \in \mathcal{AC}_g([x-\delta, x]; \mathbb{F})$ and we can proceed in a way resembling that of the previous point.

• For $x \in (a_n, b_n) \subset C_g$, we have, thanks to the previous results, that $f'_g(x) = f'_g(b_n) = h(b_n)$, where $b_n \in N_g^+ \cup D_g$. Now, thanks to Remark 4.8, $(f_n)'_g|_{(a_n,b_n]}$ is g-continuous on $(a_n, b_n]$, in particular, $(f_n)'_g(x) = (f_n)'_g(b_n)$ for all $x \in (a_n, b_n]$. Therefore, by taking the limit as n tends to infinity, $h(x) = h(b_n)$ for all $x \in (a_n, b_n]$, thus $f'_g(x) = h(x)$ for all $x \in (a_n, b_n]$. □

Remark 4.16. The previous Theorem is in fact a characterization. Indeed, suppose $N_g' \setminus N_g \not\subset D_g$ (the case $D_g' \not\subset D_g$ being similar) and let us see that $\mathcal{BD}_g^1([a,b];\mathbb{F})$ is not a Banach space. There exists $t \in N_g' \setminus (C_g \cup D_g \cup N_g)$, which we may assume to be an accumulation point of N_g from the right in the interior of $[a,b]$ (when $t \in \{a,b\}$ or t is an accumulation point from the left we can argue analogously). In that case there exist a family of connected components of C_g, $\{(c_n,d_n)\}_{n\in\mathbb{N}}$, such that $d_1 > c_1 > \cdots > d_n > c_n > \cdots > t$ and $c_n, d_n \to t$. We may take $g(t) = 0$ without loss of generality and define

$$f_n(x) = \begin{cases} 0, & x < d_n, \\ \frac{1}{2}g(c_{k-1}), & d_k \le x < d_{k-1},\ k = 2,\dots,n, \\ g(b), & x \ge d_1. \end{cases}$$

Notice that whenever $n \le m$ and $x \ge d_n$, $f_n(x) = f_m(x)$. It is clear that $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{BD}_g^1([a,b];\mathbb{F})$ and $(f_n)_g' = 0$ for every n. Furthermore, by continuity of g at t, $\{f_n\}_{n\in\mathbb{N}}$ is Cauchy in $\mathcal{BD}_g^1([a,b];\mathbb{F})$; however, it cannot possibly converge in that space, since for any potential limit f we would have, for every n,

$$\frac{f(c_{n-1}) - f(t)}{g(c_{n-1}) - g(t)} = \frac{f_n(c_{n-1})}{g(c_{n-1})} = \frac{1}{2}\frac{g(c_{n-1})}{g(c_{n-1})} = \frac{1}{2} > 0 = (f_n)_g'(t).$$

Therefore $\mathcal{BD}_g^1([a,b];\mathbb{F})$ is not a Banach space.

The following example, in which $D_g = \emptyset$ and $N_g' \setminus N_g \not\subset \emptyset$, reinforces the previous remark.

Example 4.17. Let us take as the generator g the Cantor function defined in Example 3.14, and consider the sequence of functions $\{F_n\}_{n=0}^{\infty}$ defined by $F_0(x) = 1$ and, for all $n \ge 0$,

$$F_{n+1} : x \in [0,1] \to F_{n+1}(x) = \begin{cases} \frac{1}{2}F_n(3x), & 0 \le x < \frac{1}{3}, \\ \frac{1}{2}, & \frac{1}{3} \le x < \frac{2}{3}, \\ \frac{1}{2} + \frac{1}{2}F_n(3x-2), & \frac{2}{3} \le x \le 1. \end{cases} \tag{4.4}$$

As proved in [4], the sequence of functions defined by the recurrence formula (4.4) converges uniformly to the Cantor function on $[0,1]$. Additionally, $\{F_n\}_{n=0}^{\infty} \subset \mathcal{BD}_g([0,1];\mathbb{R})$ and it holds that $(F_n)_g'(x) = 0$, for all $x \in [0,1]$ and for all $n \ge 0$.

For instance, in Figure 4.1 we can observe both the function F_3 (in blue) and an approximation of the Cantor function g (in black), which corresponds to the derivator. The aim of this representation is to provide a visual intuition of the convergence of the sequence $\{F_n\}_{n\in\mathbb{N}}$ towards the Cantor function, while also highlighting the fact that each function F_n has zero g-derivative. We note that this function F_3 is g-continuous on N_g^- (in particular, left g-continuous), right g-continuous on N_g^+, and g-continuous on $[0,1] \setminus N_g$. Since the function is piecewise constant and satisfies the g-continuity conditions described above, we have that $(F_3)_g' = 0$.

Taking this into account, $\{F_n\}_{n=0}^{\infty} \subset \mathcal{BD}_g^1([0,1];\mathbb{R})$ is a Cauchy sequence in $\mathcal{BD}_g^1([0,1];\mathbb{R})$. Finally, it is clear that the function $g \in \mathcal{BD}_g^1([0,1];\mathbb{R})$, moreover, $g_g'(x) = 1$ for all $x \in [0,1]$. Therefore, the sequence $\{(F_n)_g'\}_{n=0}^{\infty}$ does not converge uniformly to g_g' on $[0,1]$, and thus the sequence $\{F_n\}_{n=0}^{\infty}$ is not convergent in $\mathcal{BD}_g^1([0,1];\mathbb{R})$.

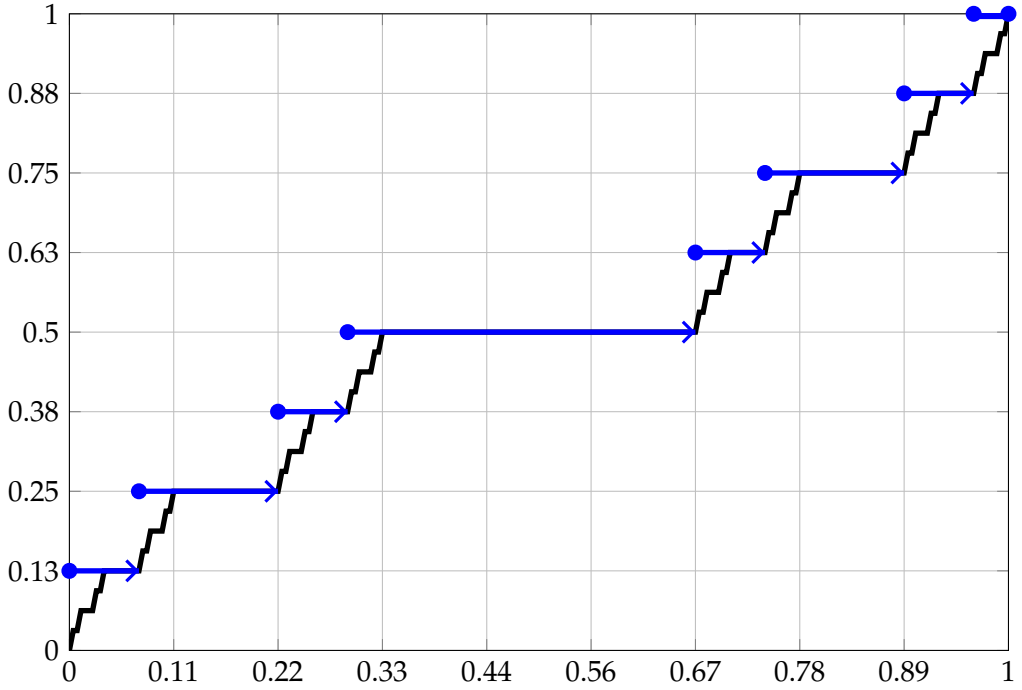


Figure 4.1: Graph of the function F_3 (in blue), along with an approximation of the Cantor function g (in black).

Proposition 4.18. *Assume $N_g' \setminus N_g \subset D_g$ and $D_g' \subset D_g$. Then $\mathcal{BC}_g^k([a,b];\mathbb{F})$ is a closed subspace of $\mathcal{BD}_g^k([a,b];\mathbb{F})$.*

Proof. It is clear that $\mathcal{BC}_g^k([a,b];\mathbb{F}) \subset \mathcal{BD}_g^k([a,b];\mathbb{F})$. Since the norm of $\mathcal{BC}_g^k([a,b];\mathbb{F})$ and that of $\mathcal{BD}_g^k([a,b];\mathbb{F})$ coincide on $\mathcal{BC}_g^k([a,b];\mathbb{F})$ and $\mathcal{BC}_g^k([a,b];\mathbb{F})$ is a Banach space, it is a closed subspace of $\mathcal{BD}_g^k([a,b];\mathbb{F})$. □

4.2 Complete metric space structure

Throughout this section, we will assume that $[a,b] \subset \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ is a derivator such that $a \notin N_g^-$ and $b \notin N_g^+ \cup D_g \cup C_g$. We will also denote by $\mathcal{BD}_g^k([a,b])$ the space $\mathcal{BD}_g^k([a,b];\mathbb{R})$, for $k \geq 0$.

Cases such as the one shown in Example 4.17 give rise to the following question: can the space $\mathcal{BD}_g^1([a,b];\mathbb{R})$ be given, without any further assumptions, a structure that makes differentiation continuous? This would imply that with such structure, the kernel of the g-derivative is a closed subspace of $\mathcal{BD}_g^1$.

A Banach or even Fréchet space structure would be desirable, but this has proven to be a rather difficult problem; one could point out that if, in Example 4.17, the supremum norm is strengthened, for instance to the total variation norm, the sequence is no longer Cauchy. If the new norm is still not strong enough, as is the case for the total variation, continuity of the g-derivative may fail, and therefore the Banach space structure; if it is too strong, however (e.g. a g-Lipschitz type norm), it may be necessary to exclude functions that we would like to have in the space (such as the ones in Example 4.17, none of which are g-Lipschitz).

In this section we provide a less ambitious (although still interesting) approach. A key tool is introduced in the following definition.

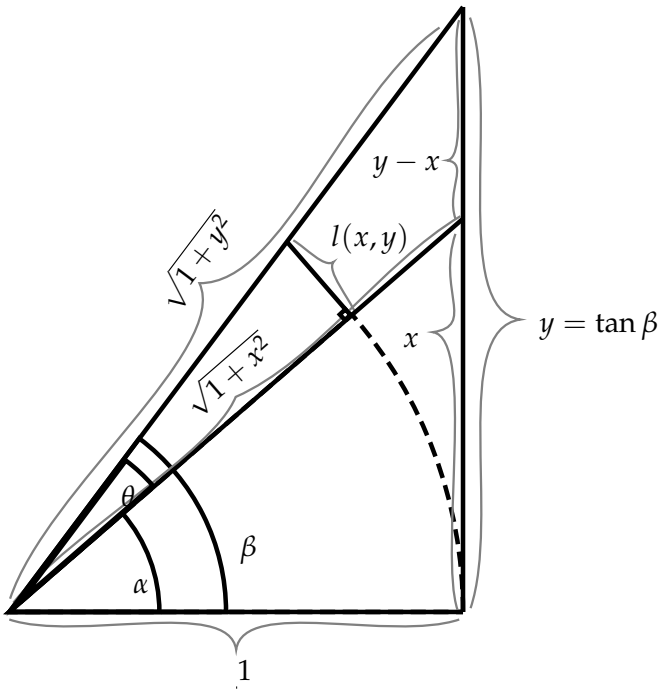


Figure 4.2: Geometric interpretation of the function l in two dimensions (without the cross product). In this figure we are assuming $y > x > 0$. Observe that $\alpha = \tan x$ and $y = \tan\beta$.

Definition 4.19. Given $x, y \in \mathbb{C}$, let

$$l(x,y) := \frac{|x-y|}{\sqrt{1+x^2}\sqrt{1+y^2}},$$

which is the *chordal distance* –see [3, 10].

Remark 4.20. The chordal distance between two complex numbers is the euclidean distance between their stereographic projections on the unit sphere. $(\mathbb{C}, l)$ is a metric space.

Remark 4.21. Another useful way of interpreting l in the case of $x, y \in \mathbb{R}$ is as $l(x,y) = \sin\theta$, where $\theta \in [0,\pi]$ is the angle between the vectors $(1,x)$ and $(1,y)$. To see this it is enough to note that

$$l(x,y) = \frac{\|(1,x,0)\times(1,y,0)\|_2}{\|(1,x,0)\|_2\|(1,y,0)\|_2},$$

and remember that $\|\vec{a}\times\vec{b}\|_2 = \|\vec{a}\|_2\|\vec{b}\|_2\sin\theta$, with $\theta \in [0,\pi]$ the (small) angle between the vectors $\vec{a}$ and $\vec{b}$.

In Figure 4.2 we show a geometric interpretation that does not rely on the cross product.

Definition 4.22. For $f, h \in \mathcal{BD}^1_g([a,b])$, we define

$$\Gamma(f,h) := \sup_{\substack{s,t\in[a,b]\\ g(s)\neq g(t)}} l\left(\frac{f(s)-f(t)}{g(s)-g(t)}, \frac{h(s)-h(t)}{g(s)-g(t)}\right),$$

and from this,

$$d(f,h) := \|f-h\|_\infty + \|f'_g - h'_g\|_\infty + \Gamma(f,h).$$

Since l is a distance on $\mathbb{R}$, Γ is symmetric and satisfies the triangle inequality (on $\mathcal{BD}^1_g([a,b])$). Therefore, $(\mathcal{BD}^1_g([a,b]), d)$ is a metric space.

Theorem 4.23. *$(\mathcal{BD}^1_g([a,b]), d)$ is a complete metric space.*

Proof. It is clear that if $\{f_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $(\mathcal{BD}^1_g([a,b]), d)$, there exist f and h such that $f_n \to f$ and $(f_n)'_g \to h$ in $(\mathcal{BD}([a,b]), \|\cdot\|_\infty)$. Let us see that $f'_g(t) = h(t)$ for every $t \in [a,b]$.

Let $t \in [a,b]$, and suppose that the g-derivative at that point is obtained by taking the right-hand limit (the remaining cases are analogous). Our goal is producing a bound in the following fashion

$$\left|\frac{f(s)-f(t)}{g(s)-g(t)} - h(t)\right| \leq \left|\frac{f(s)-f(t)}{g(s)-g(t)} - \frac{f_N(s)-f_N(t)}{g(s)-g(t)}\right| + \left|\frac{f_N(s)-f_N(t)}{g(s)-g(t)} - (f_N)'_g(t)\right| + \left|(f_N)'_g(t) - h(t)\right|,$$

where N is big enough and $s \in [t, s_N)$. Recalling its sine interpretation, for the metric l to approach zero one of the following need to happen:

- x and y both tend to infinity (or minus infinity),
- x tends to infinity, and y to minus infinity,
- x and y are close to each other (and do not approach plus or minus infinity).

That said, for a big N and s_N near t, we can bound the value $\frac{f_N(s)-f_N(t)}{g(s)-g(t)}$ (close to $(f_N)'_g(t)$, which is uniformly bounded). Since $\Gamma(f_n, f_N)$ is small if $n \geq N$, the only possibility in $[t, s_N)$ (if indices are chosen accordingly) is that $\left|\frac{f_n(s)-f_n(t)}{g(s)-g(t)} - \frac{f_N(s)-f_N(t)}{g(s)-g(t)}\right|$ is close to zero for $n \geq N$, and consequently also

$$\left|\frac{f(s)-f(t)}{g(s)-g(t)} - \frac{f_N(s)-f_N(t)}{g(s)-g(t)}\right|.$$

Thanks to all this, and the definition of d, $f'_g(t) = h(t)$.

Since $f_n \xrightarrow{\|\cdot\|_\infty} f$ (in particular there is pointwise convergence), the continuity of l guarantees that $f_n \xrightarrow{d} f$. □

Remark 4.24. Observe that the sequence $\{F_n\}_{n\in\mathbb{N}}$ defined in Example 4.17 is not a Cauchy sequence. Indeed, given $n, m \in n \in \mathbb{N}$, $n > m$, there exists a point $t \in [a,b]$ such that F_n has a jump at t whereas F_m is constant on an open interval containing that point. This implies that

$$\lim_{s\to t} \frac{F_n(s)-F_n(t)}{g(s)-g(t)} = \infty, \quad \lim_{s\to t} \frac{F_n(s)-F_n(t)}{g(s)-g(t)} = 0$$

so, taking into account Remark 4.21,

$$\Gamma(F_n, F_m) := \sup_{\substack{s,t\in[a,b]\\ g(s)\neq g(t)}} l\left(\frac{F_n(s)-F_n(t)}{g(s)-g(t)}, \frac{F_m(s)-F_m(t)}{g(s)-g(t)}\right) = 1,$$

So $\{F_n\}_{n\in\mathbb{N}}$ cannot be a Cauchy sequence. Observe that this is a general behavior occurring when we have jumps appearing in constancy intervals.

As a consequence of the previous theorem we have the following corollary.

Corollary 4.25. *The differential operator* $\partial_g : (\mathcal{BD}^1_g([a,b],\mathbb{R}),d) \to (\mathcal{BD}([a,b],\mathbb{R}),\|\cdot\|_\infty)$, $\partial_g f = f'_g$ *is continuous.*

This structure has another desirable property.

Theorem 4.26. $(\mathcal{BC}^1_g([a,b]),d)$ *has the same topology* $(\mathcal{BC}^1_g([a,b]),\|\cdot\|_{\mathcal{BC}^1_g})$.

Proof. It is equivalent to check that for any $\{f_n\}_{n\in\mathbb{N}} \cup \{f\} \subset \mathcal{BC}^1_g([a,b])$ it holds that

$$f_n \xrightarrow{d} f \iff f_n \xrightarrow{\mathcal{BC}^1_g} f.$$

By definition of d, the first implication is immediate. For the second one, it is enough to note that for $f_n \xrightarrow{\mathcal{BC}^1_g} f$, we have the following bound for the difference quotients:

$$\begin{aligned}\sup_{\substack{s,t\in[a,b]\\ g(s)\neq g(t)}} \left|\frac{f(s)-f(t)}{g(s)-g(t)} - \frac{f_n(s)-f_n(t)}{g(s)-g(t)}\right| &= \sup_{\substack{s,t\in[a,b]\\ g(s)\neq g(t)}} \left|\frac{1}{g(s)-g(t)}\int_{[t,s)} (f'_g(u)-(f_n)'_g(u))\mathrm{d}g(u)\right| \\ &\leq \|f'_g-(f_n)'_g\|_\infty.\end{aligned}$$

Hence, $\Gamma(f_n,f)\to 0$, and $d(f_n,f)\to 0$. □

Remark 4.27. Thanks to Theorem 4.26, the inclusion $(\mathcal{BC}^1_g([a,b]),\|\cdot\|_{\mathcal{BC}^1_g}) \hookrightarrow (\mathcal{BD}^1_g([a,b]),d)$ is an embedding.

The next example dispels the possibility of $(\mathcal{BD}^1_g([a,b]),d)$ being a topological vector space.

Example 4.28. Consider the functions

$$g(x)=\begin{cases} x, & x\in[-1,0],\\ x+1, & x\in(0,1].\end{cases}, \qquad f(x)=\begin{cases}0, & x\in[-1,0),\\ 1, & x\in[0,1],\end{cases}$$

and also $h=-f$, $f_k=(1-\frac{1}{k})f$, $h_k=-(1+\frac{1}{k})f$. We will check that $d(f,f_k)\to 0$, $d(h,h_k)\to 0$, but $d(f+h,f_k+h_k)\nrightarrow 0$ (which implies that addition is not a continuous operation for the product topology of $\mathcal{BD}_g([-1,1])\times\mathcal{BD}_g([-1,1])$).

On one hand, $\|f-f_k\|_\infty=\frac{1}{k}\|f\|_\infty$ and $\|f'_g-(f_k)'_g\|_\infty=\|0-0\|_\infty=0$. On the other,

$$\begin{aligned}\Gamma(f,f_k) &:= \sup_{\substack{s,t\in[a,b]\\ g(s)\neq g(t)}} l\left(\frac{f(s)-f(t)}{g(s)-g(t)}, \frac{f_k(s)-f_k(t)}{g(s)-g(t)}\right)\\ &= \sup_{\substack{s,t\in[a,b]\\ g(s)\neq g(t)}} \frac{\left|-\frac{1}{k}\frac{f(s)-f(t)}{g(s)-g(t)}\right|}{\sqrt{1+\left(\frac{f(s)-f(t)}{g(s)-g(t)}\right)^2}\sqrt{1+\left(1+\frac{1}{k}\right)\left(\frac{f(s)-f(t)}{g(s)-g(t)}\right)^2}}\\ &\leq \frac{1}{k}\sup_{\substack{s,t\in[a,b]\\ g(s)\neq g(t)}} \frac{\left|\frac{f(s)-f(t)}{g(s)-g(t)}\right|}{\sqrt{1+\left(\frac{f(s)-f(t)}{g(s)-g(t)}\right)^2}} \leq \frac{1}{k}\to 0,\end{aligned}$$

so we have that $f_k \xrightarrow{d} f$. The same reasoning is valid for h and h_k. Now,

$$\Gamma(f+h, f_k+h_k) = \Gamma\left(0, -\frac{2}{k}f\right) = \sup_{\substack{s,t\in[a,b]\\ g(s)\neq g(t)}} \frac{\left|-\frac{2}{k}\frac{f(s)-f(t)}{g(s)-g(t)}\right|}{\sqrt{1+\left(\frac{2}{k}\frac{f(s)-f(t)}{g(s)-g(t)}\right)^2}}.$$

Since f is not g-Lipschitz (take $t=0$ and $s=-1/n$) and $\lim_{x\to\infty}\frac{|x|}{\sqrt{1+x^2}}=1$, we have that $\Gamma(f+h, f_k+h_k)=1$ for each k. Consequently, $f_k+h_k \overset{d}{\nrightarrow} f+h$.

Remark 4.29. In the previous example, any f with zero g-derivative that is not g-Lipschitz could have been chosen.

4.3 Relation of $\mathcal{BD}$ spaces to other spaces

In this section, we will assume that $[a,b]\subset\mathbb{R}$ and $g:\mathbb{R}\to\mathbb{R}$ is a derivator such that $a\notin N_g^-$ and $b\notin N_g^+\cup D_g\cup C_g$. Let us examine the relationship between the space $\mathcal{BC}_g^k([a,b];\mathbb{F})$ and other more regular spaces.

We now define the Stieltjes–Sobolev spaces—see [8, Definition 5.1] and [15, Definition 3.2].

Definition 4.30. Let $p\in[1,\infty]$. We define the *Stieltjes–Sobolev spaces* as follows. $W_g^{0,1}([a,b),\mathbb{F}):=L_g^1([a,b),\mathbb{F})$, and, for $n\in\mathbb{N}$, $W_g^{n,1}([a,b),\mathbb{F}):=$

$$\left\{u\in L_g^1([a,b),\mathbb{F}) : \exists\,\widetilde{u}\in W_g^{n-1,1}([a,b),\mathbb{F}) \text{ s.t. } u(y)-u(x)=\int_x^y \widetilde{u}\,\mathrm{d}\,\mu_g,\ x,y\in[a,b)\right\}.$$

Observe that, due to Theorem 2.10, $W_g^{1,1}([a,b),\mathbb{F})=\mathcal{AC}([a,b),\mathbb{F})$.

Lemma 4.31. *Let $f\in\mathcal{BD}_g([a,b];\mathbb{F})$ be such that $f(t)=0$ μ_g-a.e. Then $f(t)=0$ for every $t\in[a,b]$.*

Proof. Let $A=\{t\in[a,b] : f(t)=0\}$. Since $f(t)=0$ μ_g-a.e., we have that $\mu_g(A)=\mu_g([a,b])$ and $\mu_g(X)=\mu_g(X\cap A)$ for any μ_g-measurable set $X\subset[a,b]$. Since $\mu_g(\{t\})>0$ for every $t\in D_g$, we conclude that $D_g\cap[a,b]\subset A$. Given $t\in(a,b)$, if there exists $\delta\in\mathbb{R}^+$, $\delta<\min\{t-a,b-t\}$, such that $[t-\delta,t+\delta)\cap A=\emptyset$, then

$$g(t+\delta)-g(t-\delta)=\mu_g([t-\delta,t+\delta))=\mu_g([t-\delta,t+\delta)\cap A)=0,$$

so $t\in C_g$. This implies that $A\cap(a,b)$ is dense in $(a,b)\backslash C_g$. Since $f\in\mathcal{BD}_g^0([a,b];\mathbb{F})$, f is g-continuous on $(a,b)\backslash(C_g\cup N_g\cup D_g)$ and, thus, continuous in that set. Given that $f|_A=0$ and $A\cap[(a,b)\backslash(C_g\cup N_g\cup D_g)]$ is dense in $(a,b)\backslash(C_g\cup N_g\cup D_g)$, we conclude that $f=0$ on $(a,b)\backslash(C_g\cup N_g\cup D_g)$.

We have proved that $D_g\cap[a,b]$, $(a,b)\backslash(C_g\cup N_g)\subset A$. It is left to see what happens on $N_g^+\backslash D_g$ and on $N_g^-\backslash D_g$. Given that $a\notin N_g^-$ and $b\notin N_g^+\cup D_g\cup C_g$, we do not have to consider a or b.

If $t\in N_g^+\backslash D_g$, since $t<b$, if there exists $\delta\in\mathbb{R}^+$, $\delta<\min\{b-t\}$, such that $[t,t+\delta)\cap A=\emptyset$, then

$$g(t+\delta)-g(t)=\mu_g([t,t+\delta))=\mu_g([t,t+\delta)\cap A=0,$$

so $t\notin N_g^+$, which is a contradiction. Therefore, we conclude that t is an accumulation point of $A\cap[a,b]$ from the right. Since f is right g-continuous and, thus, continuous from the right at $t\in N_g^+$, $f(t)=0$.

The argument is analogous for the case $t\in N_g^-\backslash D_g$, And we conclude that $f(t)=0$ for every $t\in[a,b]$. □

Remark 4.32. The condition $f \in \mathcal{BD}_g^1([a,b];\mathbb{F})$ in Lemma 4.31 is necessary as, in general, a function $f : \mathbb{R} \to \mathbb{R}$ can have g-derivative everywhere with $f_g' = 0$ μ_g-a.e., but $f_g' \neq 0$.

Indeed, let $g : \mathbb{R} \to \mathbb{R}$ be defined as

$$g(t) = t + \sum_{\substack{0<s<t \\ \frac{1}{s}\in\mathbb{Z}}} 2^{-s} + \sum_{\substack{t\leqslant s<0 \\ \frac{1}{s}\in\mathbb{Z}}} 2^{-s}.$$

Observe that $C_g = 0$, $D_g = \{\frac{1}{n}\}_{n\in\mathbb{Z}}$ and $\Delta g\left(\frac{1}{n}\right) = 2^{-|n|}$ for $n \in \mathbb{Z}$.

Define

$$f(t) := \begin{cases} \left\lfloor \frac{1}{x} \right\rfloor^{-1}, & t \in [-1,1]\backslash\{0\}, \\ 0 & t = 0. \end{cases}$$

f is right continuous. Furthermore, it is clear that if $t \in [-1,1]\backslash\{0\}$, $f_g' = 0$, so $f_g' = 0$ μ_g-a.e.

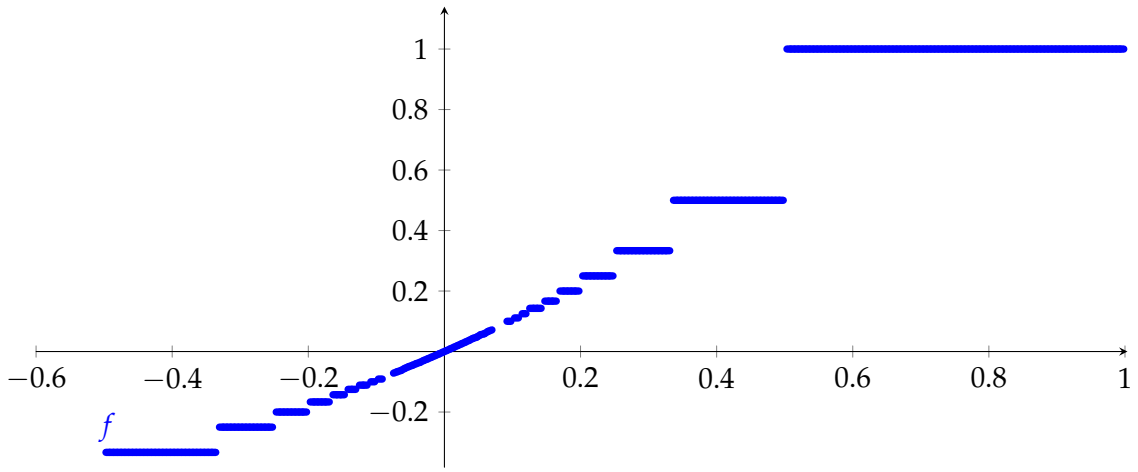


Figure 4.3: Representation of the function f.

—see Figure 4.3. Taking into account that, for $t \in [-1,1)\backslash\{0\}$,

$$t \leqslant \left\lfloor \frac{1}{t} \right\rfloor^{-1} < \frac{t}{1-t},$$

for $t \in (0,1)$,

$$\frac{f(t)-f(0)}{g(t)-g(0)} - 1 \leqslant \frac{\frac{t}{1-t}}{t + \sum\limits_{\substack{0<s<t \\ \frac{1}{s}\in\mathbb{Z}}} 2^{-s}} - 1 \leqslant \frac{\frac{t}{1-t}}{t} - 1 = \frac{1}{1-t} - 1 = \frac{t}{1-t},$$

and

$$\frac{f(t)-f(0)}{g(t)-g(0)} - 1 \geqslant \frac{t}{t + \sum\limits_{\substack{0<s<t \\ \frac{1}{s}\in\mathbb{Z}}} 2^{-s}} - 1 = \frac{t}{t} - 1 = 0,$$

whereas, for $t \in (-1/2, 0)$, given that $f(t) - f(0)$ and $g(t) - g(0)$ are negative,

$$\frac{f(t)-f(0)}{g(t)-g(0)} - 1 \leqslant \frac{t}{t + \sum_{\substack{t \leqslant s < 0 \\ \frac{1}{s} \in \mathbb{Z}}} 2^{-s} - 1} - 1 = \frac{-t}{1 - t - \sum_{\substack{t \leqslant s < 0 \\ \frac{1}{s} \in \mathbb{Z}}} 2^{-s}} - 1 \leqslant \frac{-t}{1-t},$$

and

$$\begin{aligned}\frac{f(t)-f(0)}{g(t)-g(0)} - 1 &\geqslant \frac{\frac{t}{1-t}}{t + \sum_{\substack{t \leqslant s < 0 \\ \frac{1}{s} \in \mathbb{Z}}} 2^{-s} - 1} - 1 = \frac{-\frac{t}{1-t}}{1 - t - \sum_{\substack{t \leqslant s < 0 \\ \frac{1}{s} \in \mathbb{Z}}} 2^{-s}} - 1 \geqslant \frac{\frac{-t}{1-t}}{-t} - 1 = \frac{t}{t} - 1 \\ &= \frac{t}{1-t}.\end{aligned}$$

In any case, we have that

$$f_g'(0) = \lim_{t \to 0} \frac{f(t)-f(0)}{g(t)-g(0)} = 1.$$

Thus, f is g-differentiable, $f_g' = 0$ μ_g-a.e. and

$$f_g'(x) := \begin{cases} 0, & x \neq 0, \\ 1, & x = 0. \end{cases}$$

Theorem 4.33. *Let $n \in \mathbb{N}$. $f \in \mathcal{BD}_g^n([a,b],\mathbb{F})$ if and only if $f = h + \sum_{k=1}^n \rho_k$, where $h \in W_g^{n,1}([a,b],\mathbb{F}) \cap \mathcal{BD}_g^n([a,b];\mathbb{F})$ and $\rho_k \in W^{k-1,1}([a,b];\mathbb{F}) \cap \mathcal{BD}_g^n([a,b],\mathbb{F})$ is such that $\rho_g^{(k)} = 0$, for all $k = 1, \ldots, n$.*

Proof. Given that $W_g^{n,1}([a,b],\mathbb{F}) \cap \mathcal{BD}_g^n([a,b];\mathbb{F}) \subset \mathcal{BD}_g^n([a,b];\mathbb{F})$, the sufficient condition is immediate. Let us now examine the necessary condition.

For $n = 1$, given an element $f \in \mathcal{BD}_g^1([a,b];\mathbb{F})$, we have that $f_g' \in \mathcal{BD}([a,b];\mathbb{F}) \subset \mathcal{L}_g^1([a,b];\mathbb{F})$. We can then consider $h(t) := \int_{[a,t)} f_g' \,\mathrm{d}\,\mu_g$, which is clearly $h \in W^{1,1}([a,b];\mathbb{F}) \cap \mathcal{BD}_g^1([a,b];\mathbb{F})$. Moreover, thanks to Lemma 4.14, we have $h_g' = (f_g')^* = f_g' \in \mathcal{BD}_g([a,b];\mathbb{F})$. Now, let us define $\rho_1 := f - h \in \mathcal{BD}_g^1([a,b],\mathbb{F})$. By Lemmas 4.14 and 4.31, we obtain $\rho_g' = f_g' - h_g' = 0$.

Now, let $n \in \mathbb{N}$, $n \geq 2$, and assume the result holds for $n-1$. Since $f \in \mathcal{BD}_g^n([a,b];\mathbb{F})$, we have that $f_g' \in \mathcal{BD}_g^{n-1}([a,b];\mathbb{F})$. Therefore, there exist elements $\widetilde{h} \in W^{n-1,1}([a,b];\mathbb{F}) \cap \mathcal{BD}_g^{n-1}([a,b];\mathbb{F})$ and $\widetilde{\rho}_k \in W^{k-1,1}([a,b];\mathbb{F}) \cap \mathcal{BD}_g^{n-1}([a,b];\mathbb{F})$, with $(\widetilde{\rho}_k)_g^{k)} = 0$ for all $k = 1, \ldots, n-1$, such that $f_g' = \widetilde{h} + \sum_{k=1}^{n-1} \widetilde{\rho}_k$. Define:

$$\begin{aligned} h(t) &:= \int_{[a,t)} \widetilde{h} \,\mathrm{d}\,\mu_g, \ t \in [a,b], \\ \rho_k(t) &:= \int_{[a,t)} \widetilde{\rho}_{k-1} \,\mathrm{d}\,\mu_g, \ t \in [a,b], \ k = 2, \ldots, n, \\ \rho_1 &:= f - h - \sum_{k=2}^n \rho_k. \end{aligned}$$

It follows that $h \in W^{n,1}([a,b];\mathbb{F}) \cap \mathcal{BD}_g^n([a,b];\mathbb{F})$, $\rho_k \in W^{k-1,1}([a,b];\mathbb{F}) \cap \mathcal{BD}_g^n([a,b];\mathbb{F})$ for $k = 1, \ldots, n$. Furthermore, thanks to Lemmas 4.14 and 4.31 and [6, Proposition 4.1 and Remark 4.3]

(that if φ is g-differentiable, φ^* is g-differentiable and the derivatives coincide), we obtain:

$$(\rho_k)_g^{k)} = (\widetilde{\rho}_{k-1}^*)_g^{k-1)} = (\widetilde{\rho}_{k-1})^{k-1)} = 0, \quad \forall k = 2, \dots, n,$$
$$(\rho_1)_g' = f_g' - h_g' - \sum_{k=2}^{n} (\rho_k)_g' = f_g' - \widetilde{h}^* - \sum_{k=1}^{n-1} (\widetilde{\rho}_k^*)_g = f_g' - (f_g')^* = 0. \qquad \square$$

Remark 4.34. The decomposition is unique if we impose $\rho_k(a) = 0$ for all $k = 1, \dots, n$.

Theorem 4.35. *Let $f : [a,b] \to \mathbb{F}$ a be such that $f_g'/f^* \in \mathcal{L}_g^1([a,b);\mathbb{F})$ is g-regressive. Then $f \in \mathcal{BD}_g^1([a,b],\mathbb{F})$ if and only if $f = \rho \cdot u$, where $u \in W_g^{1,1}([a,b],\mathbb{F}) \cap \mathcal{BD}_g^1([a,b];\mathbb{F})$ and $\rho \in \mathcal{BD}_g^1([a,b],\mathbb{F})$ is such that $\rho_g' = 0$.*

Furthermore, in that case, $u \in W_g^{1,1}([a,b],\mathbb{F}) \cap \mathcal{BD}_g^1([a,b];\mathbb{F})$ is a solution of equation

$$u_g'(t) = \frac{f_g'(t)}{f(t^*)} u(t), \quad \forall t \in [a,b]. \tag{4.5}$$

Proof. Since $W_g^{1,1}([a,b],\mathbb{F}) \cap \mathcal{BD}_g^1([a,b];\mathbb{F}) \subset \mathcal{BD}_g^1([a,b];\mathbb{F})$ and the product of two elements of $\mathcal{BD}_g^1([a,b];\mathbb{F})$ is in $\mathcal{BD}_g^1([a,b];\mathbb{F})$ (see Lemma 4.6), the sufficient condition is immediate. Let us now examine the necessary condition.

Let us consider the following initial value problem:

$$\begin{cases} u_g'(t) = \dfrac{f_g'(t)}{f(t^*)} u(t), \ g\text{-a.a. } t \in [a,b), \\ u(a) = 1. \end{cases}$$

Thanks to [5, Theorem 4.2 and Remark 4.4], we know that the above problem admits a unique solution in the space $\mathcal{AC}_g([a,b];\mathbb{F})$, which is also nonzero in $[a,b]$. Now, since $\frac{f_g'(t)}{f(t^*)} u(t) \in \mathcal{BD}_g([a,b];\mathbb{F})$, thanks to Lemma 4.14 and the fact that $(f_g')^* = f_g'$ and $u = u^*$ (since u is g-continuous), we can ensure that (4.5) holds at every point of the interval $[a,b]$. Consequently, we have $u \in \mathcal{AC}_g([a,b];\mathbb{F}) \cap \mathcal{BD}_g^1([a,b];\mathbb{F})$.

Now, let us define $\rho = f/u \in \mathcal{BD}_g^1([a,b];\mathbb{F})$. Clearly, we have $f = \rho \cdot u$. Moreover, given any element $t \in [a,b]$, we obtain:

$$\begin{aligned} \rho_g'(t) &= \left(\frac{f}{u}\right)_g'(t) \\ &= \frac{f_g'(t)u(t) - u_g'(t)f(t^*)}{u(t^*)\left(u(t^*) + u_g'(t)\,\Delta g(t^*)\right)} = 0, \end{aligned}$$

since (4.5) holds for all elements in the interval $[a,b]$. $\square$

5 Applications to Stieltjes differential equations

Although we have so far considered an arbitrary interval $[a,b]$, in this section, we will focus on the interval $[0,T]$ as it is a more natural choice for the study of initial value problems. In any case, the generalization to generic intervals of the form $[a,b]$ is possible. Thus, throughout this section we will assume that $[0,T] \subset \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ is a derivator such that $0 \notin N_g^-$ and $T \notin N_g^+ \cup D_g \cup C_g$.

Definition 5.1 (Kernel of the operator ∂_g)**.** We define $\ker(\partial_g) \subset \mathcal{BD}^1_g([0,T];\mathbb{F})$ as the kernel of the following operator:

$$\begin{aligned} \partial_g : \mathcal{BD}^1_g([0,T];\mathbb{F}) &\to \mathcal{BD}_g([0,T];\mathbb{F}) \\ h &\to \partial_g(h) = h'_g. \end{aligned}$$

In the following proposition, we will show that $\ker(\partial_g)$ is a non-empty set.

Proposition 5.2. *Assume $D'_g \subset D_g$ and let $h : [0,T] \to \mathbb{F}$ be bounded, right continuous at $[0,T] \cap D_g$ and constant on each connected component of $[0,T]\backslash D_g$. Then $h \in \ker(\partial_g)$.*

Proof. Given an element $t \in D_g$, we have that $h(t^+) = h(t)$, so $h'_g(t) = 0$. Now, given $t \in [0,T]\backslash D_g$, since D_g is closed, there exists $\delta > 0$ such that h is constant at $(t-\delta, t+\delta)$, therefore $h'_g(t) = 0$. □

Let us now examine the relationship between $\ker(\partial_g) \subset \mathcal{BD}^1_g([0,T];\mathbb{F})$ and the solutions of the homogeneous Stieltjes linear differential equation in the space $\mathcal{BD}^1_g([0,T];\mathbb{F})$.

Proposition 5.3. *Let $\beta \in \mathcal{BD}_g([0,T];\mathbb{F})$ be g-regressive. The solutions in $\mathcal{BD}^1_g([0,T];\mathbb{F})$ of problem*

$$\begin{cases} v'_g(t) - \beta(t)\, v(t) = 0, \ t \in [0,T), \\ v(0) = 1, \end{cases} \tag{5.1}$$

are the functions of the form $h \cdot v$ where $h \in \ker(\partial_g)$ with $h(0) = 1$ and $v \in \mathcal{BD}^1_g([0,T];\mathbb{F})$ is the unique g-absolutely continuous solution of problem (5.1).

Proof. We start by observing that the uniqueness of the g-absolutely continuous solution v of problem (5.1) was proven in [5, Theorem 4.2 and Remark 4.4]. Moreover, since v is g-absolutely continuous, it is bounded and g-continuous by [9, Proposition 5.5] and, therefore, by definition of $\mathcal{BD}_g([0,T];\mathbb{F})$, we have that $v \in \mathcal{BD}_g([0,T];\mathbb{F})$. Given that $\beta \in \mathcal{BD}_g([0,T];\mathbb{F})$, we have, by Lemma 4.6, that $\beta \cdot v \in \mathcal{BD}_g([0,T];\mathbb{F})$. Furthermore, since $v'_g = \beta \cdot v \ \mu_g$-a.e. and v is g-absolutely continuous, by Theorem 2.10 we have that $v(t) = v(0) + \int_{[0,t)} \beta \cdot v \, \mathrm{d}\, \mu_g$ for every $t \in [0,T]$. Thanks to Lemma 4.14, we have that $v'_g = \beta^* v \in \mathcal{BD}_g([0,T];\mathbb{F})$ and $v'_g(t) - \beta(t^*)\, v(t) = 0$ for all $t \in [0,T)$. Hence, $v \in \mathcal{AC}_g([0,T];\mathbb{F}) \cap \mathcal{BD}^1_g([0,T];\mathbb{F})$.

We now show that a function of the form $\widetilde{v} = h\, v$ is a solution of problem (5.1) in $\mathcal{BD}^1_g([0,T];\mathbb{F})$. Indeed, observe that, thanks to Lemma 4.6 and the fact that $h \in \mathcal{BD}^1_g([0,T];\mathbb{F})$, we have that $\widetilde{v} = h\, v \in \mathcal{BD}^1_g([0,T];\mathbb{F})$. Moreover, $\widetilde{v}(0) = h(0)\, v(0) = 1$.

Furthermore, for a given element $t \in [0,T)\backslash C_g$, $t^* = t$. If t is such that $v'_g(t) = \beta(t)\, v(t)$, it follows, thanks to Lemma 3.21, that $\widetilde{v}'_g(t) = h(t)\, v'_g(t) = h(t)\, \beta(t)\, v(t) = \beta(t)\, \widetilde{v}(t)$. We conclude that $\widetilde{v}'_g(t) = \beta(t)\, \widetilde{v}(t)$, g-a.e. $t \in [0,T)$ and, thus, $\widetilde{v}$ is a solution of problem (5.1) in $\mathcal{BD}^1_g([0,T];\mathbb{F})$.

Finally we show that if $w \in \mathcal{BD}^1_g([0,T];\mathbb{F})$ is a solution of problem (5.1), it has to be of the form $h \cdot v$ where $h \in \ker(\partial_g)$ with $h(0) = 1$. Indeed, given that $\beta = w'/w^* \in \mathcal{L}^1_g([0,T);\mathbb{F})$ is g-regressive, we can apply the construction in Theorem 4.35 to conclude that $w = h \cdot u$, for some $h \in \mathcal{BD}^1_g([0,T],\mathbb{F})$ such that $h'_g = 0$ and $u \in \mathcal{AC}_g([0,T];\mathbb{F}) \cap \mathcal{BC}^1_g([0,T];\mathbb{F})$ satisfying

$$\begin{cases} u'_g(t) = \dfrac{w'(t)}{w(t^*)}\, u(t) = \beta(t)\, u(t), \forall t \in [0,T], \\ u(0) = 1. \end{cases}$$

Hence, u is a g-absolutely continuous function that is a solution of problem (5.1). Since the g-absolutely continuous solution is unique, $u = v$, so $w = h \cdot v$, as we wanted to show. □

Proposition 5.4. *Let $\beta \in \mathcal{BD}_g([0,T];\mathbb{F})$ be g-regressive. Given $v \in \mathcal{AC}_g([0,T];\mathbb{F}) \cap \mathcal{BC}^1_g([0,T];\mathbb{F})$ the solution of (5.1) we have that a function $h \in \mathcal{BD}^1_g([0,T];\mathbb{F})$, with $h(0) = 1$, belongs to $\ker(\partial_g)$ if and only if $\widetilde{v} = h\,v \in \mathcal{BD}^1_g([0,T];\mathbb{F})$ is a solution of problem (5.1) and it satisfies that $\widetilde{v}'_g(t) - \beta(t)\,\widetilde{v}(t) = 0,\ \forall t \in [0,T] \setminus C_g$.*

Proof. Before starting the proof, it is worth remembering that the unique solution v of (5.1) in the space $\mathcal{AC}_g([0,T];\mathbb{F}) \cap \mathcal{BD}^1_g([0,T];\mathbb{F})$ is such that $v(t) \neq 0$, for all $t \in [0,T]$ —see [5, Theorem 4.2 and Remark 4.4]. Moreover, $v'_g(t) - \beta(t)\,v(t) = 0$, for all $t \in [0,T] \setminus C_g$.

Now, on the one hand, given an element $h \in \ker(\partial_g)$, by Proposition 5.3, it follows that $\widetilde{v} = h\,v \in \mathcal{BD}^1_g([0,T];\mathbb{F})$ is also a solution of (5.1). Moreover, $\widetilde{v}'_g(t) = h(t)\,v'_g(t) = h(t)\,\beta(t)\,v(t) = \beta(t)\,\widetilde{v}(t)$, for all $t \in [0,T] \setminus C_g$.

On the other hand, assume that $\widetilde{v} \in \mathcal{BD}^1_g([0,T];\mathbb{F})$ is a solution of (5.1) which also satisfies $\widetilde{v}'_g(t) - \beta(t)\,\widetilde{v}(t) = 0$, for all $t \in [0,T] \setminus C_g$. We have that:

$$\begin{aligned}\beta(t)\,\widetilde{v}(t) &= \widetilde{v}'_g(t)\\ &= (h\,v)'_g(t)\\ &= h'_g(t)\,v(t) + h(t)\,v'_g(t) + h'_g(t)\,v'_g(t)\,\Delta g(t),\ \forall t \in [0,T] \setminus C_g.\end{aligned}$$

Taking into account that $\widetilde{v} = hv$ and that $v'_g = \beta v$,

$$\beta(t)\,h(t)\,v(t) = h'_g(t)\,v(t) + \beta(t)\,h(t)\,v(t) + \beta(t)\,h'_g(t)\,v(t)\,\Delta g(t),\ \forall t \in [0,T] \setminus C_g.$$

Therefore,

$$h'_g(t)\,v(t)\,(1 + \beta(t)\,\Delta g(t)) = 0,\ \forall t \in [0,T] \setminus C_g.$$

Since $1 + \beta(t)\,\Delta g(t) \neq 0$ and $v(t) \neq 0$ for all $t \in [0,T]$, we conclude that $h'_g(t) = 0$ for all $t \in [0,T] \setminus C_g$, and thus, $h'_g(t) = 0$ for all $t \in [0,T]$. □

Remark 5.5. From Proposition 5.4 it follows that, given an element $\beta \in \mathcal{BD}_g([0,T];\mathbb{F})$ such that $1 + \beta(t)\Delta g(t) \neq 0$ for all $t \in [0,T] \cap D_g$, the dimension of the space of solutions in $\mathcal{BD}^1_g([0,T];\mathbb{F})$ of problem (5.1) is equal to the dimension of $\ker(\partial_g)$.

Example 5.6. Let be $\beta \in \mathcal{BD}_g([0,T];\mathbb{F})$ be g-regressive. Let $v = \exp_g(\beta,\cdot)$, see Definition 2.11.

Now, let us define the function $h : t \in [0,T] \to h(t) \in \mathbb{F}$, where

$$\begin{aligned}h(t) &= \exp\left(-\int_{[0,t]\cap D_g} \widetilde{\beta}(s)\ \mathrm{d}\,\mu_g(s)\right)\\ &= \exp\left(-\sum_{s\in[0,t]\cap D_g} \ln\left(1+\beta(s)\Delta g(s)\right)\right)\\ &= \left[\prod_{s\in[0,t]\cap D_g} (1+\beta(s)\,\Delta g(s))\right]^{-1}.\end{aligned}$$

The above function is well-defined since $\beta \in \mathcal{L}^1_g([0,T);\mathbb{F})$ and $T \notin D_g$. Furthermore, h is bounded, right-continuous in $[0,T] \cap D_g$, constant in the connected components of $[0,T] \setminus D_g$,

and satisfies $h(0) = 1$. Therefore, thanks to Proposition 5.2, $h \in \ker(\partial_g)$, and, by Proposition 5.3, the function $\widetilde{v} : t \in [0,T] \to \mathbb{F}$, defined by

$$\begin{aligned}\widetilde{v}(t) &= h(t)\, v(t)\\ &= \frac{\left[\prod_{s\in[0,t)\cap D_g}(1+\beta(s)\,\Delta g(s))\right]}{\left[\prod_{s\in[0,t]\cap D_g}(1+\beta(s)\,\Delta g(s))\right]}\exp\left(\int_{[0,t)\setminus D_g}\beta(s)\ \mathrm{d}\,\mu_g(s)\right)\\ &= \frac{1}{(1+\beta(t)\,\Delta g(t))}\exp\left(\int_{[0,t)\setminus D_g}\beta(s)\ \mathrm{d}\,\mu_g(s)\right)\end{aligned}$$

is another solution in the space $\mathcal{BD}^1_g([0,T];\mathbb{F})$ of the initial value problem (5.1).

For example, let $T > 2$ and consider the following generator:

$$g : t \in \mathbb{R} \to g(t) := \begin{cases} t, & t \le 1,\\ t+\delta_1, & t \in (1,2],\\ t+\delta_1+\delta_2, & t > 2,\end{cases}$$

where $\delta_k > 0$, for $k = 1,2$. In this case, $D_g = \{1,2\}$ and $N_g = \emptyset$, thus it is clear that $N_g' \setminus N_g \subset D_g$ and $D_g' \subset D_g$. Now, let $\beta : t \in [0,T] \to \beta(t) = 1$. It follows that β is g-regressive on $[0,T]$ for all $\delta_k > 0$, with $k = 1,2$. Therefore,

$$\begin{aligned}v(t) &= \exp_g(\beta,t)\\ &= \left[\prod_{s\in[0,t)\cap D_g}(1+\Delta g(s))\right]\exp\left(\mu_g([0,t)\setminus D_g\right) = \begin{cases}\exp(t), & t \le 1,\\ (1+\delta_1)\exp(t), & t \in (1,2],\\ (1+\delta_1)(1+\delta_2)\exp(t), & t > 2\end{cases}\end{aligned}$$

and

$$\begin{aligned}\widetilde{v}(t) &= h(t)\, v(t)\\ &= \frac{\prod_{s\in[0,t)\cap D_g}(1+\Delta g(s))}{\prod_{s\in[0,t]\cap D_g}(1+\Delta g(s))}\exp\left(\mu_g([0,t)\setminus D_g\right) = \begin{cases}\exp(t), & t < 1,\\ (1+\delta_1)^{-1}\exp(t), & t = 1,\\ \exp(t), & t \in (1,2),\\ (1+\delta_2)^{-1}\exp(t), & t = 2,\\ \exp(t), & t > 2.\end{cases}\end{aligned}$$

are two solutions in the space $\mathcal{BD}^1_g([0,T];\mathbb{R})$ of the initial value problem (5.1). Observe that both solutions converge to $\exp(t)$ as δ_1 and δ_2 tends to zero.

In Figure 5.1, we can observe the behavior of $v(t) = \exp_g(\beta,t)$, while in Figure 5.2, we see that of $\widetilde{v}(t) = h(t)\, v(t)$. Observe that, in both cases, the solutions exhibit discontinuities at the points of D_g. However, while v is left-continuous, $\widetilde{v}$ is not.

The significance of this example is crucial in the sense that the g-exponential function associated to β under the given conditions is unique in the space $\mathcal{AC}_g([0,T];\mathbb{F})$, while if we consider the space $\mathcal{BD}^1_g([0,T];\mathbb{F})$, the g-exponential function is not uniquely determined.

We end this work with the following corollaries of Theorem 5.3.

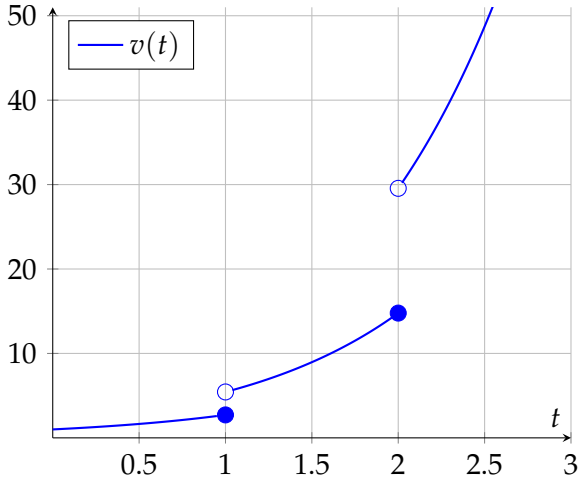


Figure 5.1: $v(t) = \exp_g(\beta, t)$.

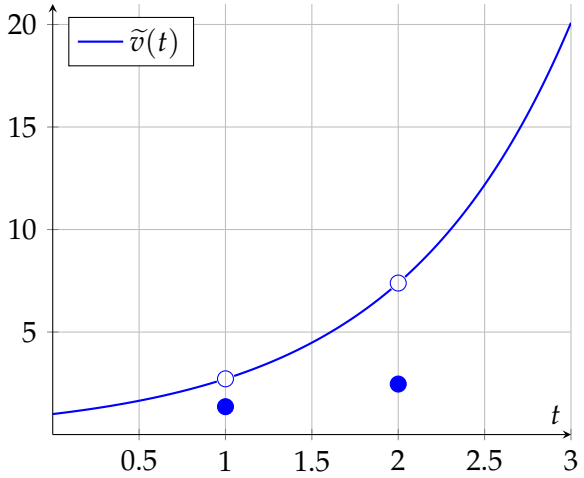


Figure 5.2: $\widetilde{v}(t) = h(t)\, v(t)$.

Corollary 5.7. *Let $n \in \mathbb{N}$, $\beta \in \mathcal{BD}_g^n([0,T];\mathbb{F})$ g-regressive. The solutions in $\mathcal{BD}_g^{n+1}([0,T];\mathbb{F})$ of problem*

$$\begin{cases} v_g'(t) - \beta(t)\, v(t) = 0, \ t \in [0,T), \\ v(0) = 1, \end{cases} \tag{5.2}$$

are the functions of the form $h \cdot v$ where $h \in \ker(\partial_g)$ with $h(0) = 1$ and $v \in \mathcal{BD}_g^{n+1}([0,T];\mathbb{F})$ is the unique g-absolutely continuous solution of problem (5.2).

Proof. We will assume $n \geqslant 2$, as the case $n = 1$ is covered by Proposition 5.3. We start by observing that $v \in \mathcal{BD}_g^n([0,T];\mathbb{F})$, where v is the unique g-absolutely continuous solution of problem (5.2). Indeed, as we reasoned in Proposition 5.3, $v \in \mathcal{BD}_g^1([0,T];\mathbb{F})$ and, given that $\beta \in \mathcal{BD}_g^n([0,T];\mathbb{F})$, we have, by Lemma 4.6, that $\beta \cdot v \in \mathcal{BD}_g^1([0,T];\mathbb{F})$. Furthermore, since $v_g' = \beta \cdot v$ μ_g-a.e. and v is g-absolutely continuous, by Theorem 2.10 we have that $v(t) = v(0) + \int_{[0,t)} \beta\, v \,\mathrm{d}\,\mu_g$ for every $t \in [0,T]$. Thanks to Lemma 4.14, we have that $v_g' = \beta\, v \in \mathcal{BD}_g^2([0,T];\mathbb{F})$ and $v_g'(t) - \beta(t^*)\, v(t) = 0$ for all $t \in [0,T]$. Hence, $v \in \mathcal{BD}_g^2([0,T];\mathbb{F})$. Repeating this argument inductively, we conclude that $v \in \mathcal{BD}_g^{n+1}([0,T];\mathbb{F})$.

Now take $\widetilde{v} = h \cdot v$ where $h \in \ker(\partial_g)$ with $h(0) = 1$ and we will show that $\widetilde{v}$ is a solution in $\mathcal{BD}_g^n([0,T];\mathbb{F})$ of problem (5.2). We already know from Proposition 5.3 that $\widetilde{v}$ is a solution in $\mathcal{BD}_g^1([0,T];\mathbb{F})$ of problem (5.2), so it is enough to show that $\widetilde{v} \in \mathcal{BD}_g^n([0,T];\mathbb{F})$, but this is evident given that, since $h_g' = 0$, and $v \in \mathcal{BD}_g^n([0,T];\mathbb{F})$, we have that $\widetilde{v}_g^{k)} = hv^{k)}$ for $k = 1, \dots, n$ and, thus, $\widetilde{v} \in \mathcal{BD}_g^n([0,T];\mathbb{F})$.

On the other hand, if $u \in \mathcal{BD}_g^n([0,T];\mathbb{F})$ is a solution of problem (5.2), by Proposition 5.3, $u = h\,v$ for some $h \in \ker \partial_g$ with $h(0) = 1$. □

The next corollary is proven following the same steps as in Proposition 5.3 and Corollary 5.7.

Corollary 5.8. *Let $n \in \mathbb{N}$, $\beta, f \in \mathcal{BD}_g^n([0,T];\mathbb{F})$ g-regressive. The solutions in $\mathcal{BD}_g^{n+1}([0,T];\mathbb{F})$ of problem*

$$\begin{cases} v_g'(t) - \beta(t)\, v(t) = f(t),\ t \in [0,T), \\ v(0) = 1, \end{cases} \tag{5.3}$$

are the functions of the form $h \cdot v$ where $h \in \ker(\partial_g)$ with $h(0) = 1$ and $v \in \mathcal{BD}_g^{n+1}([0,T];\mathbb{F})$ is the unique g-absolutely continuous solution of problem (5.3).

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Dynamics of a state-dependent delay-differential equation

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Abstract. We present a detailed study of a scalar differential equation with threshold state-dependent delayed feedback. This equation arises as a simplification of a gene regulatory model. There are two monotone nonlinearities in the model: one describes the dependence of delay on state, and the other is the feedback nonlinearity. Both increasing and decreasing nonlinearities are considered. Our analysis is exhaustive both analytically and numerically as we examine the bifurcations of the system for various combinations of increasing and decreasing nonlinearities. We identify rich bifurcation patterns including Bautin, Bogdanov–Takens, cusp, fold, homoclinic, and Hopf bifurcations whose existence depend on the derivative signs of nonlinearities. Our analysis confirms many of these patterns in the limit where the nonlinearities are switch-like and change their value abruptly at a threshold. Perhaps one of the most surprising findings is the existence of a Hopf bifurcation to a periodic solution when the nonlinearity is monotone increasing and the time delay is a decreasing function of the state variable.

Keywords: state dependent delay, threshold delay, negative feedback, positive feedback, bifurcation (Bautin, Bogdanov–Takens, cusp, fold, homoclinic, Hopf).

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1 Introduction

In considering mathematical models for the dynamics of biological feedback systems, the occurrence of delays is almost ubiquitous and this leads to very interesting biological, mathematical, and modeling problems. These delays arise because of the time required to propagate feedback signals in biological systems. For physiological examples one need only think of the

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conduction time of action potentials in a neural feedback circuit [2,38], the time required to complete DNA synthesis, mitosis and cytokinesis in cell replication [12], the significant time required to produce mature cells in the hematopoietic system [10,16], as well as a myriad of others [19]. Examples in engineering and physical settings are rich and abundant ranging from control of ships [44], vibration control [45], and the delays due to the relativistic propagation of signals at the speed of light [11,66,67].

In this paper we study the dynamics of the positive solutions of the scalar state-dependent delay differential equation (DDE)

$$x'(t) = \beta e^{-\mu\tau(t)} \frac{v(x(t)))}{v(x(t-\tau(t)))} g(x(t-\tau(t))) - \gamma x(t) \tag{1.1}$$

where for $t \geq 0$ the delay $\tau(t)$ is defined by the threshold condition

$$a = \int_{-\tau(t)}^{0} v(x(t+s))ds = \int_{t-\tau(t)}^{t} v(x(s))ds. \tag{1.2}$$

We assume that the constants β, μ, γ and a are all positive.

We will consider the system (1.1),(1.2) with both smooth and piecewise constant functions $v(x)$ and $g(x)$. In the smooth case, we take the functions to be Hill functions

$$g(x) = \frac{g^- \theta_g^n + g^+ x^n}{\theta_g^n + x^n}, \qquad v(x) = \frac{v^- \theta_v^m + v^+ x^m}{\theta_v^m + x^m}, \tag{1.3}$$

where the exponents m and n are strictly positive real numbers. We also assume that $v(x)$ is strictly positive and bounded away from zero:

$$0 < v_0 \leq v(x) \leq v_U, \qquad \text{where} \quad v_0 = \min\{v^-, v^+\}, \quad v_U = \max\{v^-, v^+\}. \tag{1.4}$$

This ensures that the delay defined by (1.2) satisfies $\tau(t) \in [a/v_U, a/v_0]$, and is thus both bounded and bounded away from zero. Moreover, applying Leibnitz's rule to (1.2) with $v(x) \geq v_0 > 0$ shows that $\frac{d}{dt}(t-\tau(t)) \geq v_0/v_U > 0$ so $t - \tau(t)$ is a strictly monotonically increasing function of t. Consequently to pose (1.1)–(1.2) as an initial value problem (IVP) it is sufficient to provide an initial function $x(t) = \phi(t)$ for $t \in [-\tau(0), 0]$ where $\tau(0)$ is defined by (1.2). We will take the function g to be non-negative and bounded with

$$0 < g_0 \leq g(x) \leq g_U, \qquad \text{where} \quad g_0 = \min\{g^-, g^+\}, \quad g_U = \max\{g^-, g^+\}. \tag{1.5}$$

We are interested in the dynamics when the Hill coefficients m, n are small or large, and also consider the case of piecewise constant functions obtained by taking the limits of $v(x)$ and $g(x)$ as m and $n \to \infty$. In that case we take the limiting functions to be set-valued at the threshold points with

$$g(x) = \begin{cases} g^-, & x < \theta_g \\ [g_0, g_U], & x = \theta_g \\ g^+, & x > \theta_g \end{cases} \tag{1.6}$$

$$v(x) = \begin{cases} v^-, & x < \theta_v \\ [v_0, v_U], & x = \theta_v \\ v^+, & x > \theta_v \end{cases} \tag{1.7}$$

Equation (1.6) is derived from (1.3) by regarding the curve $\{x, g(x)\}$ as a subset of $\mathbb{R}^2$ and taking the limit as $n \to \infty$ using the Hausdorff metric, and (1.7) is derived similarly. There is a long history of incorporating set-valued functions into differential equations, resulting in differential inclusions for ordinary differential equations [3,4], and in DDEs. Mallet-Paret and Nussbaum ([40,41]) consider set-valued limiting solutions in their study of slowly oscillating periodic solutions.

As shown in Appendix A, the system (1.1)–(1.2) arises as a quasi-steady state reduction of the state-dependent delayed extension of the Goodwin [21] operon model studied in [18,63, 65]. Thus the model is taken to describe the regulation of a bacterial operon in which the cells are growing exponentially at a rate μ and have finite transcriptional and translational velocities that are potentially dependent on the state of the system. Then the function g denotes the production flux of messenger RNA (mRNA) while v is the velocity of translation/transcription of the mRNA strand to produce the end product (protein). x represents the dimensionless effector concentration.

The function $g(x)$ in (1.3) is monotone increasing when $g^- < g^+$, which we refer to as a positive feedback case. In the operon context ([21]) this corresponds to an *inducible* operon. The classical example would be the *lac* operon regulating bacterial utilization of lactose as an energy source. Conversely, when $g^- > g^+$ the function $g(x)$ is monotone decreasing, which we refer to as a negative feedback case, and in the operon setting it would correspond to a *repressible* operon. The immediate example that comes to mind is the *tryp* operon regulating the production of the amino acid tryptophan.

In [18] it is argued that the transcription velocity for an operon should be an increasing function when g is a decreasing function, and vice versa, while the translation velocity should always be monotonically decreasing. In the current study, we do not make this assumption and instead consider the different possible combinations of increasing/decreasing/constant g and v to explore potential dynamics. To avoid confusion we use $g \uparrow$ to indicate that the function g is monotonically increasing or equivalently that $g^- < g^+$. Then $g \downarrow$ denotes that g is monotonically decreasing or equivalently that $g^- > g^+$, while we use $g \leftrightarrow$ to denote that $g^- = g^+$ and so g is a constant function. With analogous definitions for the function v we denote different cases of these pairs of functions by $(g \uparrow, v \uparrow)$, $(g \uparrow, v \downarrow)$, $(g \uparrow, v \leftrightarrow)$, etc.

Smith [55, 56] showed that a DDE with a threshold delay can be converted through a time transformation to a distributed delay DDE with a constant delay, so the theory of those equations is applicable to this model. Some authors have used this transformation to study threshold models as distributed constant delay DDEs (see for example [33,58]), but other times the existence of the time transformation to constant delay DDE has been used as an excuse to just ignore the threshold delay and treat the delay as if constant. We will demonstrate that the model (1.1) with the threshold delay (1.2) can display very different dynamics than the same model (1.1) with a constant discrete delay.

In this work we will tackle (1.1) with the threshold delay (1.2) directly without transforming to a constant delay DDE. We do this because we are also interested in problems with multiple delays. For a problem with multiple threshold delays defined by different integrands, or even for a problem with a single threshold delay and a single constant delay, it is not possible to use a time transformation to convert it to a constant delay DDE. We also point out that when $\mu \neq 0$, the term $e^{-\mu\tau}$ in (1.1) would cause the time transformed equation to be a distributed DDE with constant delay, and not a discrete delay DDE (see [64]). Finally, we believe it is interesting that the analysis and numerics of these problems can be tackled directly for the problem as originally formulated.

The outline of this paper is as follows. Section 2 discusses the semiflow generated by (1.1)–(1.2). The following Section 3 considers issues related to the positivity of solutions and the existence of a global attractor, and then Section 4 examines the linearization of the system at one of the steady states and the stability of the steady state.

Sections 5 and 6 contain a detailed investigation of the rich dynamics generated by the system (1.1)–(1.2). We combine the verification of essential necessary conditions for local bifurcations from stationary points with numerical studies of one- and two-parameter bifurcations. Our numerical techniques are extensively documented in [18,63,65], and summarised in Appendix C. Concerning Theorems 5.3, 5.4, 5.6 and 5.9 about fold- and Hopf-bifurcations the reader should be aware that for most statements we limit ourselves to proving only those parts which make the result plausible and serve as a basis for numerical investigation. In particular, we do not verify non-degeneracy conditions for the bifurcations. Theorems on Hopf bifurcation for differential equations with state-dependent delay can be found in [14,30,53].

We begin in Section 5 by examining the simpler situation in which there is only one nonlinearity, with one of g or v being constant. In Sections 5.1 and 5.2 we consider the constant delay cases $(g\downarrow, v\leftrightarrow)$ and $(g\uparrow, v\leftrightarrow)$ respectively. Then in Sections 5.3 and 5.4 we consider $(g\leftrightarrow, v\downarrow)$ and $(g\leftrightarrow, v\uparrow)$. In these two cases v and thus τ are varying, resulting in state-dependent delays. We find that the dynamics are considerably richer and more surprising in the state-dependent delay cases, with qualitative differences in the dynamics depending on whether the growth rate, μ, or the decay rate, γ, is larger.

We continue in this vein in Section 6 but considering both nonlinearities, g and v. In Section 6.1 we begin by considering the case where $\theta_g \neq \theta_v$. In principle there should be four cases to consider: $(g\uparrow, v\uparrow)$, $(g\uparrow, v\downarrow)$,$(g\downarrow, v\uparrow)$ and $(g\downarrow, v\downarrow)$, but in practice we find that the dynamics is determined by the cases studied in Section 5. For example with $(g\uparrow, v\uparrow, \theta_g \neq \theta_v)$ we find that the dynamics and bifurcations from the steady state ξ are given by the previously studied cases $(g\uparrow, v\leftrightarrow)$ for $\xi \approx \theta_g$ and by $(g\leftrightarrow, v\uparrow)$ for $\xi \approx \theta_v$. Consequently, we illustrate just two of these cases, $(g\downarrow, v\uparrow, \theta_g \neq \theta_v)$ and $(g\uparrow, v\uparrow, \theta_g \neq \theta_v)$ to show how the dynamics relates to the previous examples of Section 5. The case $(g\uparrow, v\uparrow, \theta_g \neq \theta_v)$ is interesting as it can result in up to five co-existing steady states, three of which are stable.

If $|\theta_g - \theta_v| \ll 1$ then both functions g and v influence the dynamics, so in Section 6.2 we consider the dynamics when $\theta_g = \theta_v$. In the cases $(g\uparrow, v\uparrow)$ and $(g\downarrow, v\downarrow)$ both functions are increasing or decreasing, and no new dynamics arise, beyond what was already seen in Section 5. However, the cases $(g\uparrow, v\downarrow, \theta_g = \theta_v)$ and $(g\downarrow, v\uparrow, \theta_g = \theta_v)$ reveal surprising dynamics in limiting cases.

Section 7 contains summary remarks as well as comments on possible extensions of this work.

Finally, as noted above the appendices contain the reduction of the model of [18] to the situation we consider here (Appendix A) as well as a brief elaboration in Appendix B of five previously published models that fall within the context of this paper, along with a summary and references for our numerical techniques (Appendix C).

2 The semiflow of differentiable solution operators generated by the system (1.1)–(1.2)

For delay differential equations a familiar state space is given by the space of continuous functions on a compact interval, see e.g. [13,28]. In case of variable, state-dependent delays,

however, there is a specific lack of smoothness which means that in general the initial value problem is not well-posed for only continuous initial data, not to speak of, say, smoothness of solutions with respect to initial data and linearization ([29, 59]).

Recall the definition of the segment x_t of a map $x : I \to \mathbb{R}$ for which the domain I contains the interval $[t-r, t]$ with $t \in \mathbb{R}$ and $r > 0$: x_t is the map $[-r, 0] \to \mathbb{R}$ given by $x_t(s) = x(t+s)$ for $-r \le s \le 0$. In other words, the restriction of x to $[t-r, t]$ is shifted to the interval $[-r, 0]$. In this section we reformulate the system (1.1), (1.2) as a delay differential equation

$$x'(t) = G(x_t) \tag{2.1}$$

with a functional $G : C^1 \to \mathbb{R}$ on the Banach space $C^1 = C^1([-r, 0], \mathbb{R})$ of continuously differentiable maps $[-r, 0] \to \mathbb{R}$, for some $r > 0$ which is to be determined. The norm on C^1 is given by

$$|\phi|_1 = \max_{-r \le t \le 0} |\phi(t)| + \max_{-r \le t \le 0} |\phi'(t)|,$$

We will also need the Banach space $C \supset C^1$ of continuous maps $[-r, 0] \to \mathbb{R}$, with the norm given by

$$|\phi| = \max_{-r \le t \le 0} |\phi(t)|.$$

We shall verify the hypotheses from [29, 59, 60] which guarantee the existence, uniqueness, and differentiability, with respect to initial data, of solutions to an initial value problem which is associated with (2.1) in a submanifold of the space C^1.

We make the following assumptions. The function $g : \mathbb{R} \to (0, \infty)$ is continuously differentiable with

$$0 < \inf g(\mathbb{R}) \le \sup g(\mathbb{R}) = g_U.$$

The function $v : \mathbb{R} \to [v_0, \infty)$, with $v_0 > 0$, is continuously differentiable. We fix a number $r > a/v_0$ and notice that r is an a priori bound for $\tau(t)$ in Eq. (1.2).

Next we rewrite the system Eq. (1.1), (1.2) in a form which is more convenient for our purpose. Using segment notation, Eq. (1.2) becomes

$$a = \int_{-\tau(t)}^{0} v(x_t(s))ds \tag{2.2}$$

with the segment $x_t \in C$. More generally, we consider the equation

$$a = \int_{-u}^{0} v(\phi(s))ds \tag{2.3}$$

for $u \in [0, r]$ and $\phi \in C$. Using positivity of the function v and the Intermediate Value Theorem we infer that for every $\phi \in C$ there is a uniquely determined solution $u = \delta(\phi) \in (0, r)$ of Eq. (2.3). This yields a map $\delta : C \to (0, r)$.

Proposition 2.1. *The map $\delta : C \to (0, r)$ is continuously differentiable with*

$$D\delta(\phi)\chi = -\frac{\int_{-\delta(\phi)}^{0} v'(\phi(s))\chi(s)ds}{v(\phi(-\delta(\phi)))}.$$

In case $\phi(s) = \xi$ for all $s \in [-r, 0]$,

$$\delta(\phi) = \frac{a}{v(\xi)}$$

and

$$D\delta(\phi)\chi = -\frac{v'(\xi)}{v(\xi)} \int_{-a/v(\xi)}^{0} \chi(s)ds.$$

Before giving the proof recall from [59, page 47] or [29, page 466] that the evaluation map

$$ev_C : C \times [-r, 0] \ni (\chi, u) \mapsto \chi(u) \in \mathbb{R}.$$

is continuous (but not locally Lipschitz continuous, let alone differentiable), and that the restricted evaluation map

$$ev : C^1 \times (-r, 0) \ni (\phi, u) \mapsto \phi(u) \in \mathbb{R}$$

is continuously differentiable with

$$D\,ev(\phi, u)(\hat{\phi}, \hat{u}) = D_1 ev(\phi, u)\hat{\phi} + D_2 ev(\phi, u)\hat{u} = \hat{\phi}(u) + \phi'(u)\hat{u}, \tag{2.4}$$

where D_1 and D_2 denote partial derivatives with respect to the argument in C^1 and in $(-r, 0)$, respectively. The substitution operator

$$V : C \ni \phi \mapsto v \circ \phi \in C$$

is continuously differentiable with

$$(DV(\phi)\hat{\phi})(s) = v'(\phi(s))\hat{\phi}(s) \quad \text{for all} \quad \hat{\phi} \in C, \quad s \in [-r, 0],$$

see for example [13, Appendix IV, Lemma 1.5].

Proof of Proposition 2.1. For every $\phi \in C$ the value $u = \delta(\phi)$ is the unique solution of the equation $h(u, \phi) = 0$, where $h : (0, r) \times C \to \mathbb{R}$ is given by

$$h(u, \phi) = a - \int_{-u}^{0} v(\phi(s))ds = a - ev(I(V(\phi))), -u)$$

with the continuous linear integration operator $I : C \to C^1$ defined by

$$(I\psi)(t) = \int_t^0 \psi(s)ds.$$

The map h is continuously differentiable with

$$D_1 h(u, \phi)1 = -v(\phi(-u)) < 0$$

and

$$\begin{aligned} D_2 h(u, \phi)\chi = -D_1 ev(I(V(\phi)), -u)DI(V(\phi))DV(\phi)\chi = -(DI(V(\phi))DV(\phi)\chi)(-u) \\ = -(I(DV(\phi)\chi))(-u) = -\int_{-u}^{0} v'(\phi(s))\chi(s)ds. \end{aligned}$$

The Implicit Function Theorem applies at every $(\delta(\phi), \phi) \in (-r, 0) \times C$ and yields that locally, δ is given by a continuously differentiable map. Differentiation of the equation $h(\delta(\phi), \phi) = 0$ gives

$$D\delta(\phi)\chi = -\frac{D_2 h(\delta(\phi), \phi)\chi}{D_1 h(\delta(\phi), \phi)1} = -\frac{\int_{-\delta(\phi)}^{0} v'(\phi(s))\chi(s)ds}{v(\phi(-\delta(\phi)))}.$$

In the case ϕ is constant with value $\xi \in \mathbb{R}$, Eq. (2.3) gives $a = \delta(\phi)v(\xi)$, and by the previous formula,

$$D\delta(\phi)\chi = -\frac{v'(\xi)}{v(\xi)} \int_{-a/v(\xi)}^{0} \chi(s)ds.$$

□

Using the delay functional $\delta : C \to (0, r)$ the system (1.1),(2.2) is reduced to the equation

$$x'(t) = \beta\, e^{-\mu\,\delta(x_t)}\frac{v(x(t))}{v(x(t-\delta(x_t)))}g(x(t-\delta(x_t))) - \gamma\, x(t) \tag{2.5}$$

with segments $x_t \in C$. Eq. (2.5) is of the form $x'(t) = G_C(x_t) \in \mathbb{R}$ with the map $G_C : C \to \mathbb{R}$ given by

$$G_C(\phi) = \beta\, e^{-\mu\,\delta(\phi)}\frac{v(\phi(0))}{v(\phi(-\delta(\phi)))}g(\phi(-\delta(\phi))) - \gamma\,\phi(0) \tag{2.6}$$

We observe that G_C is continuous. The restriction G of G_C to C^1 is continuously differentiable because, for $\phi \in C^1 \subset C$, we have $ev_C(\phi, -\delta(\phi)) = ev(\phi, -\delta(\phi))$. Here $ev : C^1 \times (0, r) \to \mathbb{R}$ is continuously differentiable, the map δ is continuously differentiable, and the evaluation map $ev_0 : C \ni \chi \mapsto \chi(0) \in \mathbb{R}$ is linear and continuous.

To simplify the calculations below we now introduce the continuous map

$$E_C : C \to \mathbb{R}, \quad E_C(\phi) = ev_C(\phi, -\delta(\phi)),$$

and the continuously differentiable map

$$E : C^1 \to \mathbb{R}, \quad E(\phi) = ev(\phi, -\delta(\phi)),$$

with the derivative at $\phi \in C^1$ given by

$$\begin{aligned} DE(\phi)\hat{\phi} &= D_1 ev(\phi, -\delta(\phi))\hat{\phi} + D_2 ev(\phi, -\delta(\phi))D(-\delta)(\phi)\hat{\phi} \\ &= \hat{\phi}(-\delta(\phi)) - \phi'(-\delta(\phi))D\delta(\phi)\hat{\phi} \end{aligned}$$

for all $\hat{\phi} \in C^1$. Notice that the right hand side of the previous equation makes sense also for arguments $\chi \in C$ instead of $\hat{\phi} \in C^1$. Thus they define linear extensions $D_e E(\phi) : C \to \mathbb{R}$ of $DE(\phi) : C^1 \to \mathbb{R}$. Using the continuity of the map ev_C, and the fact that differentiation $C^1 \ni \phi \mapsto \phi' \in C$ is linear and continuous we obtain the next result.

Proposition 2.2. *The map $C^1 \times C \ni (\phi, \chi) \mapsto D_e E(\phi)\chi \in \mathbb{R}$ is continuous.*

Incidentally, in the case $\phi \in C^1$ is constant with value $\xi \in \mathbb{R}$ we have

$$E(\phi) = \xi, \quad DE(\phi)\hat{\phi} = \hat{\phi}(-a/v(\xi)).$$

With the linear continuous evaluation map $ev_0 : C \ni \phi \mapsto \phi(0) \in \mathbb{R}$ we obtain for the restriction G of G_C

$$G(\phi) = \beta\, e^{-\mu\delta(\phi)}\frac{v(ev_0\phi)}{v(E(\phi))}g(E(\phi)) - \gamma\, ev_0\phi. \tag{2.7}$$

In the sequel we will show that the initial value problem

$$x'(t) = G(x_t) \quad \text{for} \quad t > 0, \quad x_0 = \phi \tag{2.8}$$

is well-posed on the set

$$X_G = \{\phi \in C^1 : \phi'(0) = G(\phi)\},$$

which is a continuously differentiable submanifold of codimension 1 in the space C^1. This result follows from results in [29,59,60] provided that the following two assertions are verified:

1. $X_G \neq \emptyset$; and
2. G has the property that

   each derivative $DG(\phi) : C^1 \to \mathbb{R}$, $\phi \in C^1$, *has a linear extension* $D_eG(\phi) : C \to \mathbb{R}$, *and the map*

   $$C^1 \times C \ni (\phi, \hat{\phi}) \mapsto D_eG(\phi)\hat{\phi} \in \mathbb{R}$$

   is continuous.

Property (2) is a version of being *almost Fréchet differentiable* from [42]. In case the delay functional $\delta : C \to (0, r)$ is bounded away from zero, which happens to be true for v bounded also from above as in the subsequent sections, the manifold X_G is simply a graph in the space C^1, given by a continuously differentiable map from an open subset of the closed hyperplane $\{\phi \in C^1 : \phi'(0) = 0\}$ into a complementary line in C^1, see [62, Theorem 2.4].

We now proceed to the proof of properties (1) and (2). We first show that $X_G \neq \emptyset$. The continuous map

$$[0, \infty) \ni \xi \mapsto \beta\, e^{-\mu\, a/v(\xi)} - \gamma\, \xi \in \mathbb{R}$$

is positive at $\xi = 0$ and tends to $-\infty$ for $\xi \to \infty$. Therefore the Intermediate Value Theorem yields a zero $\zeta > 0$ of this map. The constant function $\phi \in C^1$ with value ζ satisfies $\phi'(0) = 0 = G(\phi)$, so it belongs to the set X_G. This finishes the proof of property (1).

To prepare for the proof of the extension property (2) we compute the derivatives $DG(\phi)$, $\phi \in C^1$. For $\delta : C \to \mathbb{R}$ we use the fact that restrictions of differentiable maps $m : C \to \mathbb{R}$ to C^1 remain differentiable, with derivatives $D(m|C^1)(\phi) : C^1 \to \mathbb{R}$ being restrictions of the derivatives $Dm(\phi) : C \to \mathbb{R}$, $\phi \in C^1 \subset C$. It follows that

$$\begin{aligned} DG(\phi)\hat{\phi} = {} & -\gamma\, \hat{\phi}(0) + \\ & \beta \Bigg\{ \frac{1}{[v(E(\phi))]^2} \left[v'(\phi(0))\hat{\phi}(0) \cdot v(E(\phi)) - v(\phi(0))v'(E(\phi))DE(\phi)\hat{\phi} \right] e^{-\mu\delta(\phi)} g(E(\phi)) \\ & \quad + \frac{v(\phi(0))}{v(E(\phi))} \left[-\mu\, e^{-\mu\delta(\phi)} D\delta(\phi)\hat{\phi} \cdot g(E(\phi)) + e^{-\mu\delta(\phi)} g'(E(\phi))DE(\phi)\hat{\phi} \right] \Bigg\}. \end{aligned} \tag{2.9}$$

Now we are ready to verify property (2). In the formula for $DG(\phi)\hat{\phi}$, replace the real number $DE(\phi)\hat{\phi}$ by $D_eE(\phi)\chi$ with $\chi \in C$, replace the function $\hat{\phi}$ by χ, and replace $\hat{\phi}(0)$ by $\chi(0)$. This defines $D_eG(\phi)\chi \in \mathbb{R}$ for $\phi \in C^1$ and $\chi \in C$ so that the maps $D_eG(\phi) : C \to \mathbb{R}$, $\phi \in C^1$, are linear. Using the continuous differentiability of $\delta : C \to \mathbb{R}$ and Proposition 2.1 one shows that the map $C^1 \times C \ni (\phi, \chi) \mapsto D_eG(\phi)\chi \in \mathbb{R}$ is continuous. This finishes the proof of property (2).

With (1) and (2) verified, results from [29, 59, 60] apply and yield the following.

The set X_G is a continuously differentiable submanifold of the Banach space C^1, with codimension 1. Each $\phi \in X_G$ uniquely determines a maximal continuously differentiable solution $x : [-r, t_x) \to \mathbb{R}$, $0 < t_x \leq \infty$, of the initial value problem (2.8). That is, x is continuously differentiable and satisfies $x_0 = \phi$ and $x'(t) = G(x_t)$ for all $t \in (0, t_x)$, and any other continuously differentiable function $y : [-r, t_y) \to \mathbb{R}$, $0 < t_y \leq \infty$, which satisfies $y_0 = \phi$ and $y'(t) = G(y_t)$ for all $t \in (0, t_y)$ is a restriction of x. All segments x_t, $0 \leq t < \infty$, belong to X_G (because of Eq. (2.8)). Write $x^\phi = x$ and $t_\phi = t_x$. Let

$$\Omega_G = \{(t, \phi) \in [0, \infty) \times X_G : 0 \leq t < t_\phi\}.$$

The equation

$$S_G(t,\phi) = x_t^\phi,$$

defines a continuous semiflow $S_G : \Omega_G \to X_G$. For each $t \geq 0$ the set

$$\Omega_{G,t} = \{\phi \in X_G : t < t_\phi\}$$

is an open subset of X_G (possibly empty), $\Omega_{G,0} = X_G$, and each map

$$S_{G,t} : \Omega_{G,t} \ni \phi \mapsto S_G(t,\phi) \in X_G, \quad t \geq 0,$$

on a non-empty domain is continuously differentiable.

Moreover, the restriction of the semiflow S_G to the open subset $\{(t,\phi) \in \Omega_G : r < t\}$ of the manifold $\mathbb{R} \times X_G$ is continuously differentiable [60].

3 Positivity, dissipativity, global attractor

In addition to the assumptions made in Section 2, we assume in this section that the function v is also bounded from above by a real number $v_U \geq v_0$. Using Eq. (2.3) we infer

$$\frac{a}{v_U} \leq \delta(\phi) \leq \frac{a}{v_0}$$

for all $\phi \in C$.

Proposition 3.1. *For every $c > 0$ there exists $c' > 0$ such that for all $\phi \in X_G$ with $|\phi(t)| \leq c$ on $[-r,0]$ and for all $t \in [-r, t_\phi)$ we have $|x^\phi(t)| \leq c'$.*

Proof. Let $\phi \in X_G$ with $|\phi(t)| \leq c$ on $[-r,0]$ be given, and set $x = x^\phi$. The first term on the right hand side of Eq. (2.5) is positive and bounded by the constant

$$d^U = \beta\, g_U\, \frac{v_U}{v_0}.$$

The variation-of-constants formula yields

$$\begin{aligned} |x(t)| &= \left| x(0)e^{-\gamma t} + \int_0^t e^{-\gamma(t-s)} \left[\frac{\beta v(x(s))}{v(x(s-\delta(x_s)))} e^{-\mu\,\delta(x_s)} g(x(s-\delta(x_s))) \right] ds \right| \\ &\leq c + e^{-\gamma t} \frac{d^U}{\gamma} \left(e^{\gamma t} - 1\right) \leq c + \frac{d^U}{\gamma} \quad \text{for} \quad 0 \leq t < t_\phi. \end{aligned}$$

Set $c' = c + \frac{d^U}{\gamma}$. With $|x(t)| = |\phi(t)| \leq c$ on $[-r,0]$ we obtain $|x(t)| \leq c'$ for all $t \in [-r, t_\phi)$. □

Observe that for any (continuously differentiable) solution $x : [-r, t_x) \to \mathbb{R}$ of Eq. (2.5) and for any $t \in [0, t_x)$ the first term on the right hand side of Eq. (2.5) belongs to the interval

$$[d^L, d^U] = \left[\beta \frac{v_0}{v_U} e^{-\mu\, a/v_0} g_0, \beta\, g_U\, \frac{v_U}{v_0} \right],$$

and in the case

$$x(t) > \frac{d^U}{\gamma} \quad \text{we have} \quad x'(t) \leq d^U - \gamma\, x(t) < 0,$$

while for

$$x(t) < \frac{d^L}{\gamma} \quad \text{we have} \quad x'(t) \geq d^L - \gamma\, x(t) > 0.$$

Set

$$Q = \left[\frac{d^L}{\gamma}, \frac{d^U}{\gamma}\right] \subset \mathbb{R}$$

and

$$R = \{\phi \in C^1 : \phi([-r, 0]) \subset Q\}.$$

Proposition 3.2 (Global existence, absorption and positive invariance, positivity)**.**

(i) For all $\phi \in X_G$, $t_\phi = \infty$.

(ii) For every neighbourhood N of Q in $\mathbb{R}$ and for each $\phi \in X_G$ there exists $t(\phi, N) \in [0, \infty)$ with $x^\phi(t) \in N$ for all $t \geq t(\phi, N)$.

(iii) If $\phi \in X_G \cap R$ then $x^\phi(t) \in Q$ for all $t \geq 0$.

(iv) If $\phi \in X_G$ is strictly positive then $x^\phi(t) > 0$ for all $t \geq -r$.

Proof. 1. On (i). Let $\phi \in X_G$ be given. From Proposition 3.1 the solution $x = x^\phi$ is bounded. Using this and Eq. (2.5) we infer that x' is bounded. It follows that x is Lipschitz continuous. Assume now $t_\phi < \infty$. Then Lipschitz continuity yields that x has a limit $\xi \in \mathbb{R}$ at $t = t_\phi$ and x extends to a continuous map $\hat{x} : [-r, t_\phi] \to \mathbb{R}$. From uniform continuity on the compact interval $[-r, t_\phi]$ it follows that the curve $[0, t_\phi] \ni t \mapsto \hat{x}_t \in C$ is continuous. Using this and the equation

$$x'(t) = G(x_t) = G_C(\hat{x}_t) \quad \text{for} \quad 0 \leq t < t_\phi$$

with the continuous map $G_C : C \to \mathbb{R}$, we also conclude that x' has a limit $\eta \in \mathbb{R}$ at $t = t_\phi$. It follows that $\hat{x}$ is continuously differentiable (with $\hat{x}'(t_\phi) = \eta$), and $\hat{x}'(t_\phi) = G_C(\hat{x}_{t_\phi}) = G(\hat{x}_{t_\phi})$. In particular, $\psi = \hat{x}_{t_\phi}$ belongs to X_G, and defines a maximal solution $x^\psi : [0, t_\psi) \to \mathbb{R}$ of Eq. (2.1), with $0 < t_\psi \leq \infty$. From the semiflow properties, it follows that when $t_\psi = \infty$ we have $t_\phi = \infty$, in contradiction to the assumption above, while in the case $t_\psi < \infty$ we get $t_\phi \geq t_\phi + t_\psi$, which contradicts $t_\psi > 0$.

2. On (ii). Let a neighbourhood N of Q in $\mathbb{R}$ and $\phi \in X_G$ be given. Set $x = x^\phi$. There exists $\epsilon > 0$ so that for

$$d_{-\epsilon} = \frac{d^L}{\gamma} - \epsilon \quad \text{and} \quad d_{+\epsilon} = \frac{d^U}{\gamma} + \epsilon$$

we have $d_{-\epsilon} > 0$ and $N \supset [d_{-\epsilon}, d_{+\epsilon}]$.

2.1. Proof that when $x(t) \leq d_{+\epsilon}$ for some $t \geq 0$ we have

$$x(s) \leq d_{+\epsilon} \quad \text{for all} \quad s \geq t.$$

Otherwise $x(s) > d_{+\epsilon} = \frac{d^U}{\gamma} + \epsilon$ for some $s > t$. For the smallest $u \in [t, s]$ with $x(u) = x(s)$ we have $0 \leq x'(u)$ and, on the other hand,

$$x'(u) < d^U - \gamma\, x(u) = d^U - \gamma\, x(s) < -\epsilon\, \gamma < 0.$$

2.2. Proof that when $x(s) > d_{+\epsilon}$ for some $s \geq 0$ there exists $t > s$ with

$$x(t) \leq d_{+\epsilon}.$$

Otherwise $x(t) > d_{+\epsilon} = \frac{d^U}{\gamma} + \epsilon$ on $[s, \infty)$. Hence

$$x'(t) < d^U - \gamma\, x(t) < -\epsilon\, \gamma < 0 \quad \text{on} \quad [s, \infty),$$

and consequently $x(t) \to -\infty$ as $t \to \infty$, in contradiction to the assumption.

2.3. It follows that there exists $t^* \geq 0$ with

$$x(t) \leq d_{+\epsilon} \quad \text{for all} \quad t \geq t^*.$$

Similarly one finds $t(\phi, N) \geq t^*$ with

$$x(t) \geq d_{-\epsilon} \quad \text{for all} \quad t \geq t(\phi, N).$$

Hence

$$x(t) \in [d_{-\epsilon}, d_{+\epsilon}] \subset N \quad \text{for all} \quad t \geq t(\phi, N).$$

3. The proof of assertion (iii) begins with the assumption that for a given $\phi \in X_G \cap R$ there exists $s > 0$ with $x^\phi(s) > \frac{d^U}{\gamma}$, and is then accomplished by a simplified version of arguments as in Part 2.1.

4. On (iv). Let $\phi \in X_G$ be given with $\phi(t) > 0$ for all $t \in [-r, 0]$. Set $x = x^\phi$. The assumption $x(t) \leq 0$ for some $t > 0$ leads to a smallest $t > 0$ with $x(t) = 0$. Necessarily, $x'(t) \leq 0$ while Eq. (2.5) yields $x'(t) > 0$. It follows that $x(t) > 0$ for all $t \geq -r$. □

The solution manifold X_G is a closed subset of the space C^1, and thereby a complete metric space with respect to the metric given by the norm on C^1. The next result implies that the semiflow S_G on the complete metric space X_G is point dissipative as defined in [27].

Corollary 3.3. *There is a bounded open subset B_G of the submanifold $X_G \subset C^1$, with*

$$\phi(t) > 0 \quad \textit{for all} \quad \phi \in B_G, \quad t \in [-r, 0],$$

such that for every $\phi \in X_G$ there exists $t(\phi) \geq 0$ with

$$S_G(t, \phi) \in B_G \quad \textit{for all} \quad t \geq t(\phi).$$

Proof. Choose $c > 0$ so that $N = (0, c)$ is a neighbourhood of Q. Set $\tilde{R} = \{\phi \in C^1 : \phi([-r, 0]) \subset N\}$. Let $\phi \in X_G$ be given. Choose $t(\phi, N)$ according to Proposition 3.2 (ii). From Eq. (2.5) we see that the map G sends the set $\tilde{R}$ (which is not a bounded subset of C^1) into a bounded subset of $\mathbb{R}$, say, into $(-b, b)$ for some $b > 0$. It follows that for all $t \geq t(\phi, N) + r$ we have $|(x^\phi)'(t)| < b$. For $t \geq t(\phi, N) + 2r$ we obtain $x_t^\phi \in \{\psi \in X_G \cap \tilde{R} : |\psi'| < b\} = B_G$. The set B_G is an open and bounded subset of X_G, with $0 < \phi(t)$ for all $\phi \in B_G$, $t \in [-r, 0]$. Set $t(\phi) = t(\phi, N) + 2r$. □

Recall from [27] the definition of a global attractor of a semiflow, which in the case of our semiflow S_G is equivalent to saying that a subset $A_G \subset X_G$ is a global attractor if it is

- compact,
- invariant in the sense that for every $\phi \in A_G$ there exists an entire flowline[1] $\xi : \mathbb{R} \to X_G$ with $\xi(0) = \phi$ and $\xi(\mathbb{R}) \subset A_G$, and

[1] An entire flowline is a curve $\xi : \mathbb{R} \to X_G$ with $\xi(t + s) = S_G(t, \xi(s))$ for all $t \geq 0$ and $s \in \mathbb{R}$

- if A_G attracts every bounded set $B \subset X_G$ in the sense that given an open neighbourhood $U \supset A_G$ of A_G in X_G there exists $t_{B,U} \geq 0$ such that

$$S_G([t_{B,U}, \infty) \times B) \subset U.$$

Theorem 3.4.8 of [27] guarantees the existence of such a global attractor provided the semiflow is point-dissipative and there exists $t_1 \geq 0$ so that the semiflow S_G is *completely continuous for* $t \geq t_1$. The property of being completely continuous (for $t \geq t_1$) is explained after Lemma 3.2.1 in [27]. It means that for every $t \geq t_1$ and for every bounded set B with $S_G([0,t] \times B)$ bounded the set $S_G(\{t\} \times B)$ is precompact.

To prove the latter property, it is sufficient to verify the slightly stronger property that for every bounded set $B \subset X_G$

(i) there exists $t_1 \geq 0$ so that for every $t \geq t_1$ the set $S_G(\{t\} \times B)$ is precompact.

Theorem 3.4. *The semiflow S_G has a global attractor $A_G \subset X_G$, with $\phi(t) > 0$ for all $\phi \in A_G$, $t \in [-r, 0]$.*

Proof. 1. We first show that for every bounded subset $B \subset X_G$ there exists $c_B > 0$ with

$$|x^\phi(t)| \leq c_B \quad \text{and} \quad |(x^\phi)'(t)| \leq c_B \quad \text{for all} \quad \phi \in B \quad \text{and} \quad t \geq -r.$$

Let $B \subset X_G$ be bounded (with respect to the norm of the space C^1). Proposition 3.1 guarantees the existence of a constant $c_{B,0}$ with $|x^\phi(t)| \leq c_{B,0}$ for all $\phi \in B$, $t \geq -r$. Then (2.7) shows that the set $\{G(x_t^\phi) \in \mathbb{R} : \phi \in B, t \geq 0\}$ is bounded, and Eq. (2.1) gives that the set $\{(x^\phi)'(t) \in \mathbb{R} : \phi \in B, t \geq 0\}$ is bounded. Also the set $\{\phi'(t) \in \mathbb{R} : \phi \in B, -r \leq t \leq 0\}$ is bounded.

2. Claim: For every bounded subset $B \subset X_G$ the set $S_G(\{r\} \times B) \subset X_G$ has compact closure in C^1.

Proof: (a) Let $B \subset X_G$ be a bounded subset of C^1. Due to Part 1 the sets $\{x^\phi(t) \in \mathbb{R} : \phi \in B, -r \leq t \leq r\}$ and $\{(x^\phi)'(t) \in \mathbb{R} : \phi \in B, -r \leq t \leq r\}$ are bounded. Using the Mean Value Theorem we see that in particular the set $S_G(\{r\} \times B)$ is equicontinuous. As it also is bounded in C, the Ascoli–Arzelà Theorem implies that its closure in C is compact.

(b) We turn to the set $\{S_G(r, \phi)' \in C : \phi \in B\}$ of derivatives, which is bounded in C, and proceed to show that it is also equicontinuous. As in Part (a) one sees that the closure K of the set

$$\{S_G(t, \phi) \in C : \phi \in B, 0 \leq t \leq r\}$$

in the space C is compact. The map $G : C^1 \to \mathbb{R}$ is the restriction of the continuous map $G_C : C \to \mathbb{R}$ which is uniformly continuous on the compact set $K \subset C$. Using the boundedness of the set $\{(x^\phi)'(t) \in \mathbb{R} : \phi \in B, -r \leq t \leq r\}$ and the Mean Value Theorem one finds that the curves

$$[0, r] \ni t \mapsto S_G(t, \phi) \in C, \quad \phi \in B,$$

are uniformly Lipschitz continuous, hence equicontinuous. Now let $t_0 \in [0, r]$ and $\epsilon > 0$ be given. There exists $\delta' > 0$ with

$$|G_C(\phi) - G_C(\psi)| \leq \epsilon \quad \text{for all} \quad \phi, \psi \quad \text{in} \quad S_G([0, r] \times B) \quad \text{with} \quad |\phi - \psi| \leq \delta',$$

due to uniform continuity of G_C on K. Due to equicontinuity there exists $\eta > 0$ with

$$|S_G(t, \phi) - S_G(t_0, \phi)| \leq \delta' \quad \text{for all } \phi \in B \text{ and } t \in [0, r] \text{ with } |t - t_0| < \eta.$$

Hence

$$\begin{aligned}|(x^\phi)'(t) - (x^\phi)'(t_0)| &= |G(S_G(t,\phi)) - G(S_G(t_0,\phi))| \\ &= |G_C(S_G(t,\phi)) - G_C(S_G(t_0,\phi))| < \epsilon\end{aligned}$$

for all $\phi \in B$ and $t \in [0,r]$ with $|t - t_0| < \eta$.

(c) The Ascoli–Arzelà Theorem implies that the closure of $\{S_G(r,\phi)' \in C : \phi \in B\}$ in C is compact. For the closure of $S_G(\{r\} \times B)$ in C^1 to be compact it is sufficient to show that every sequence of points $\phi_j \in S_G(\{r\} \times B)$, $j \in \mathbb{N}$, has a subsequence which converges in C^1. Let a sequence $(\phi_j)_1^\infty$ in $S_G(\{r\} \times B)$ be given. Part a) implies that there is a subsequence which converges in C to some $\phi \in C$. Part b) shows that the subsequence has a secondary subsequence so that the derivatives of the latter subsequence converge in C to some $\psi \in C$. It follows that $\phi \in C^1$ with $\phi' = \psi$, which in turn yields convergence of the secondary subsequence to $\phi \in C^1$ as $k \to \infty$.

3. We now show that for every bounded subset $B \subset X_G$ and for every $t \geq r$ the set $S_G(\{t\} \times B) \subset X_G$ has compact closure in C^1. Let $B \subset X_G$ be bounded and let $t \geq r$. For every $\phi \in B$,

$$S_G(t,\phi) = S_G(t-r, S_G(r,\phi)),$$

hence

$$S_G(\{t\} \times B) = S_G(t-r,\cdot)(S_G(\{r\} \times B)).$$

Use that the closure of $S_G(\{r\} \times B)$ in C^1 is compact and belongs to X_G (since X_G is a closed subset of C^1), and that the map $S_G(t-r,\cdot)$ is continuous, and conclude that the closure of $S_G(\{t\} \times B)$ in C^1 is contained in a compact subset of $X_G \subset C^1$.

4. Point dissipativity from Corollary 3.3 in combination with condition (i) from the previous Part 3 of the proof yield existence of a global attractor, see the remarks preceding Theorem 3.4.

5. Finally we show that for all $\phi \in A_G$ and $t \in [-r,0]$ we have $\phi(t) > 0$. Let $\phi \in A_G$ be given. There exists a solution $x : \mathbb{R} \to \mathbb{R}$ of Eq. (2.5) with $x_0 = \phi$ and all segments x_s, $s \in \mathbb{R}$, in the compact set A_G. It suffices to deduce $x(t) > 0$ for all $t \in \mathbb{R}$. Proof of this: First, observe that x is bounded. Assume $x(t) \leq 0$ for some $t \in \mathbb{R}$. In case $x(t) = 0$ Eq. (2.5) yields $x'(t) > 0 - \gamma\, x(t) = 0$. It follows that $x(u) < 0$ for some $u \in (-\infty, t)$. In case $x(t) < 0$ set $u = t$. For every $s \leq u$ the variation-of-constants formula yields

$$x(u) \geq x(s)e^{-\gamma(u-s)} + 0,$$

hence

$$x(s) \leq x(u)e^{\gamma(u-s)} \qquad (\to -\infty \quad \text{as} \quad s \to -\infty),$$

and we arrive at a contradiction to the boundedness of x. □

4 Linearization

We now turn to linearization. At a point $\phi \in X_G$ the tangent space of the manifold X_G is given by

$$T_\phi X_G = \{\chi \in C^1 : \chi'(0) = DG(\phi)\chi\}.$$

For $\phi \in \Omega_{G,t}$ the derivative

$$DS_{G,t}(\phi) : T_\phi X_G \to T_{S_{G,t}(\phi)} X_G$$

is given by

$$DS_{G,t}(\phi)\chi = w_t^{\phi,\chi}$$

where $w^{\phi,\chi} = w$ is the unique continuously differentiable solution $[-r, t_\phi) \to \mathbb{R}$ of the IVP

$$w'(t) = DG(S_G(t,\phi))w_t \quad \text{for} \quad t > 0, \tag{4.1}$$

$$w_0 = \chi \in T_\phi X_G. \tag{4.2}$$

Equation (4.1) is called the linear variational equation along the solution x^ϕ or along the flowline

$$S_G(\cdot,\phi) : [0, t_\phi) \ni t \mapsto S_G(t,\phi) \in X_G.$$

Suppose that $\phi \in X_G$ is a stationary point of the semiflow S_G, that is, $t_\phi = \infty$ and $S_G(t,\phi) = \phi$ for all $t \geq 0$. Then ϕ is constant since for every $t \geq 0$, $x^\phi(t) = x_t^\phi(0) = S_G(t,\phi)(0) = \phi(0)$, hence

$$\phi(s) = S_G(r,\phi)(s) = x_r^\phi(s) = x^\phi(r+s) = \phi(0) \quad \text{for each } s \in [-r, 0].$$

Let $\xi \in \mathbb{R}$ denote the value of ϕ. Then $x^\phi(t) = \xi$ for all $t \geq -r$. To obtain the linear variational equation along this constant solution in terms of $g, v, a, \beta, \mu, \gamma$ we compute the values $DG(\phi)\hat{\phi}$, $\hat{\phi} \in C^1$ from the formula (2.9). Using Proposition 2.1 (for the values and for the derivatives of the map δ in case of constant arguments) and the calculation of $DE(\phi)\hat{\phi}$ right after Proposition 2.1 (in case of constant arguments), we find

$$\begin{aligned} DG(\phi)\hat{\phi} &= -\gamma\hat{\phi}(0) + \beta\Bigg\{ \frac{1}{[v(\xi)]^2}\left[v'(\xi)\hat{\phi}(0)v(\xi) - v(\xi)v'(\xi)\hat{\phi}(-a/v(\xi))\right] e^{-\mu a/v(\xi)} g(\xi) \\ &\quad + \left[-\mu e^{-\mu a/v(\xi)}\left(-\frac{v'(\xi)}{v(\xi)}\int_{-a/v(\xi)}^0 \hat{\phi}(s)ds\right) g(\xi) + e^{-\mu a/v(\xi)} g'(\xi)\hat{\phi}(-a/v(\xi))\right]\Bigg\} \\ &= -\gamma\hat{\phi}(0) + A\hat{\phi}(0) + \mu A\int_{-a/v(\xi)}^0 \hat{\phi}(s)ds + \left(\beta e^{-\mu a/v(\xi)} g'(\xi) - A\right)\hat{\phi}(-a/v(\xi)) \end{aligned} \tag{4.3}$$

with

$$A = \beta\frac{v'(\xi)}{v(\xi)} e^{-\mu a/v(\xi)} g(\xi). \tag{4.4}$$

The variational equation (4.1) along the constant solution $x^\phi : [-r, \infty) \ni t \mapsto \xi \in \mathbb{R}$ becomes

$$\begin{aligned} w'(t) = &-\gamma w(t) + Aw(t) + \left(\beta e^{-\mu a/v(\xi)} g'(\xi) - A\right) w(t - a/v(\xi)) \\ &+ \mu A\int_{-a/v(\xi)}^0 w(t+s)ds \end{aligned} \tag{4.5}$$

The derivatives $T_{G,t} = DS_{G,t}(\phi)$, $t \geq 0$, form a strongly continuous semigroup on the closed hyperplane $T_\phi X_G$ of the space C^1. This semigroup is given by $T_{G,t}\chi = T_{G,e,t}\chi$, where $T_{G,e,t} : C \to C$ is the solution operator associated with the *classical* initial value problem

$$w'(t) = D_e G(\phi)w_t \quad \text{for} \quad t > 0, \tag{4.6}$$

$$w_0 = \chi \in C, \tag{4.7}$$

with the continuous linear functional $L : C \to \mathbb{R}$, $L = D_e G(\phi)$, as in the monographs, e.g., [13,28]. Recall that by definition the solution $w : [-r, \infty) \to \mathbb{R}$ of the initial value problem (4.6), (4.7) is only continuous, with the restriction $w|_{[0,\infty)}$ continuously differentiable and satisfying Eq. (4.6).

The extended derivative $D_e G(\phi) : C \to \mathbb{R}$ in the case just considered (where $\phi \in X_G$ is a stationary point with value ξ) is given by (4.3), now for $\hat{\phi} \in C$. Therefore the equation (4.6) coincides with Eq. (4.5), considered for continuous maps $[-r, \infty) \to \mathbb{R}$ whose restrictions to $[0, \infty)$ are differentiable and satisfy Eq. (4.5) for all $t \geq 0$.

The stability of the zero solution of the linear variational equation (4.5) is determined by the spectrum $\sigma \subset \mathbb{C}$ of the generator of the semigroup $(T_{G,t})_{t\geq 0}$ on $T_\phi X_G \subset C^1$, which coincides with the spectrum $\sigma_e \subset \mathbb{C}$ of the generator of the semigroup on C.

The spectrum σ_e consists of the solutions $\lambda \in \mathbb{C}$ of the *characteristic equation*, which is obtained from the ansatz $\mathbb{R} \ni t \mapsto e^{\lambda t} \in \mathbb{C}$ for a complex-valued solution of Eq. (4.5) as follows. We write down Eq. (4.5) for $w : t \mapsto e^{\lambda t}$, multiply by $e^{-\lambda t}$, and obtain the equation

$$\lambda = -\gamma + A + \left(\beta\, e^{-\mu\, a/v(\xi)} g'(\xi) - A\right) e^{-\lambda\, a/v(\xi)} + \mu\, A \int_{-a/v(\xi)}^{0} e^{\lambda\, s} ds,$$

or equivalently,

$$0 = \lambda + \gamma - A - \left(\beta\, e^{-\mu\, a/v(\xi)} g'(\xi) - A\right) e^{-\lambda\, a/v(\xi)} - \mu\, A \int_{-a/v(\xi)}^{0} e^{\lambda\, s} ds. \tag{4.8}$$

To investigate the stability of steady states in different special cases, we make explicit various forms of (4.8).

If we study the case with a constant delay, i.e. v is a constant function with $v(\xi) = v^- = v^+$ in (1.3) and (1.7), then $v' = 0$ and hence $A = 0$. The characteristic equation reduces to

$$\lambda = -\gamma + \beta\, e^{-\mu\, a/v} g'(\xi) e^{-\lambda\, a/v}. \tag{4.9}$$

Similarly, if we study the case with g constant by setting $g^- = g^+$ in (1.3) and (1.6), then $g'(\xi) = 0$ and the characteristic equation is of the form

$$\lambda = -\gamma + A - A e^{-\lambda a/v(\xi)} + \mu A \int_{-a/v(\xi)}^{0} e^{\lambda s} ds = -\gamma + A(1 - e^{-\lambda a/v(\xi)}) \left(1 + \frac{\mu}{\lambda}\right). \tag{4.10}$$

We remark that when $\mu A \neq 0$ in (4.8), the integration of the solution $e^{\lambda t}$ over the delay interval results in a term μ/λ term as appears in (4.10). Division by λ does not arise in the characteristic functions of discrete delay DDEs, and so even if (1.1) looks at first glance like a discrete delay DDE, when combined with the threshold delay condition (1.2) it is a distributed delay DDE. This accords with the results of Smith [55,56] who showed that a DDE with a single threshold delay can be converted through a time transformation to a distributed delay DDE with a constant delay.

When we study the limiting case where both g and v are piecewise constant as defined in (1.6) and (1.7), then there are intervals where $A = v'(\xi) = 0$. For a steady state ξ in such an interval, the characteristic equation simplifies to

$$\lambda = -\gamma, \tag{4.11}$$

where there is a unique negative real characteristic root and the steady state is stable.

5 Dynamics with one Hill function

At a steady state $x(t) \equiv \xi \in \mathbb{R}$, equation (1.2) reduces to

$$\tau(\xi) = \frac{a}{v(\xi)}, \tag{5.1}$$

so the delay is still state-dependent. Equation (1.1) becomes

$$0 = h(\xi) := \beta e^{-\mu\tau(\xi)} g(\xi) - \gamma\xi \tag{5.2}$$

at a steady state. With $g(x)$ and $v(x)$ defined by (1.3) we have that

$$\beta g^- \geq h(0) = \beta e^{-\mu a/v^-} g^- \geq 0, \qquad h(x) \leq \beta e^{-\mu a/v_U} g_U - \gamma x \leq \beta g_U - \gamma x.$$

Consequently any steady state satisfies $\xi \in [0, \beta g_U/\gamma]$, and there is always at least one such steady state.

The steady states occur at the zeros of $h(\xi)$, which from (5.2) occur at the intersections of $\beta e^{-\mu\tau(\xi)} g(\xi)$ and $\gamma\xi$. Thus the number of steady states depends on the behavior of the term $e^{-\mu\tau(\xi)} g(\xi)$ in (5.2).

We begin by considering the simplified setting where either g or v is a constant function, while the other one is either a Hill function defined in (1.3), or a piecewise constant function (1.6) or (1.7). This leads to the four cases discussed below.

With $v^\pm := v^- = v^+$ in (1.3) and (1.7) we obtain $v(\xi) = v^\pm$ independent of the value of ξ and consequently a constant delay $\tau = a/v^\pm$, and equation (1.1) reduces to

$$x'(t) = \beta e^{-\mu\tau} g(x(t-\tau)) - \gamma x(t), \tag{5.3}$$

which is a constant delay DDE with a monotone feedback nonlinearity. We consider this case first for decreasing and increasing g in Sections 5.1 and 5.2, respectively. The results which we obtain correspond to semi-local properties of equilibria which are familiar for the constant delay equation (5.3) with a sufficiently smooth nonlinearity g. If g is strictly decreasing then there is a single equilibrium solution, from which periodic solutions bifurcate off in a sequence of Hopf bifurcations when a parameter multiplying g grows to infinity. These periodic solutions can be distinguished by their oscillation frequencies. Stable periodic orbits occur only at the lowest possible frequency, for so-called *slowly oscillating periodic solutions* [31]. If g is increasing then multiple equilibria are possible. Hopf bifurcations from these equilibria yield periodic orbits which are all unstable. For more detailed information about the numerous local and global results on solutions of autonomous delay differential equations which were achieved during the past decades see, for example, the survey [61].

Sections 5.1 and 5.2 contain results which are not covered by the existing theory. We consider the limiting cases as the smooth function g approaches a piecewise constant function, as these help to understand better the changes which occur in the dynamics when certain parameters grow to infinity. This also affords us the opportunity to present, in the simpler setting of constant delays, the methods we will subsequently use when the delays are state-dependent in Sections 5.3 and 5.4.

With $g^- = g^+$ in (1.3) and (1.6), g is a constant function, while $v^- \neq v^+$ results in a state-dependent delay. We consider the two cases of v decreasing or increasing in Sections 5.3 and 5.4, respectively. The general case where g and v are both non-constant is studied in Section 6.

5.1 Constant delay with decreasing g $(g\downarrow, v\leftrightarrow)$

In this section we study (1.1)–(1.2) with a constant delay τ and decreasing g. The DDE reduces to (5.3) with $\tau = a/v^{\pm}$ on setting $v^{\pm} = v^{-} = v^{+}$ in (1.3) or (1.7). We require $g^{-} > g^{+}$ in (1.3) or (1.6) to ensure that g is decreasing.

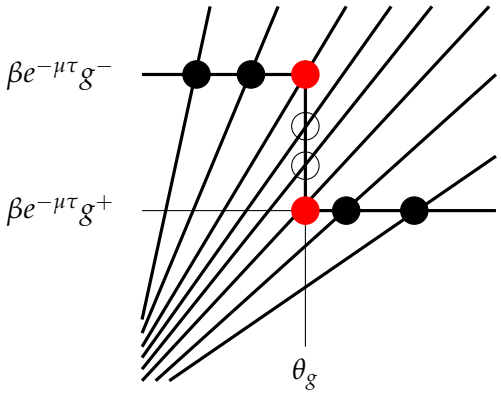


Figure 5.1: Steady states of (1.1), given by (5.2), occur at the intersections of $\xi \mapsto \beta e^{-\mu\tau} g(\xi)$ and $\xi \mapsto \gamma\xi$. These are illustrated for various γ in the limiting case of (1.6) and (1.7) with $v^{\pm} = v^{-} = v^{+}$, so the delay $\tau = a/v^{\pm}$ is constant, and $g^{-} > g^{+}$, so g is monotonically decreasing: $(g\downarrow, v\leftrightarrow)$.

As noted above, if $h(0) \geq 0$ with g decreasing we have $h'(\xi) = \beta e^{-\mu\tau} g'(\xi) - \gamma < 0$. Therefore, with smooth g defined in (1.3), for any fixed values of the parameters there is always exactly one steady state ξ. We note for later use that rearranging (5.2), for any fixed value of $\xi > 0$, there is also a unique value of γ for which ξ is a steady state.

Figure 5.1 illustrates the uniqueness of the steady state with a piecewise constant g defined by (1.6) as γ varies. At the 'corners', the steady state $\xi = \theta_g$ satisfies (5.2) with $g = g^{+}$ and $g = g^{-}$ respectively, which gives rise to

$$\gamma_1 = \frac{\beta e^{-\mu\tau} g^{+}}{\theta_g} \quad \text{and} \quad \gamma_2 = \frac{\beta e^{-\mu\tau} g^{-}}{\theta_g}. \tag{5.4}$$

We refer to the steady state where $e^{-\mu\tau} g(\xi)$ and $\gamma\xi$ intersect on the vertical line segment of the curve as a *singular steady state*. The singular steady state exists for $\gamma \in (\gamma_1, \gamma_2)$, where the bounds on γ are given by (5.4).

Next, we consider the stability of the steady state for the piecewise constant function g defined in (1.6). If $\gamma \in (0, \gamma_1) \cup (\gamma_2, +\infty)$, the intersection occurs on the horizontal line segments. Then $A = g'(\xi) = 0$ and the characteristic equation (4.8) reduces to (4.11), and hence the steady state must be stable. This implies that in the limiting case only a singular steady state may be unstable. However, the characteristic equation is not defined in this situation and we therefore consider the smooth nonlinearity g defined in (1.3) for large n. As a consequence of the choices made in this section, $A = 0$ and $g'(\xi) < 0$ with $n > 0$.

It follows immediately from (4.9) that there are no real non-negative characteristic values when $g'(\xi) < 0$. So, suppose $\lambda = \alpha \pm i\omega$, $\omega > 0$ is a solution of the characteristic equation (4.9). Then for $\lambda = \alpha + i\omega$ we have

$$\alpha + i\omega = -\gamma + \beta e^{-\mu\tau} g'(\xi) e^{-(\alpha+i\omega)\tau} = -\gamma + \beta e^{-\mu\tau} g'(\xi) e^{-\alpha\tau} \left[\cos(\omega\tau) - i\sin(\omega\tau)\right].$$

Equating real and imaginary parts respectively, we obtain

$$\alpha + \gamma = \beta e^{-\mu\tau} g'(\xi) e^{-\alpha\tau} \cos(\omega\tau), \tag{5.5}$$

$$\omega = -\beta e^{-\mu\tau} g'(\xi) e^{-\alpha\tau} \sin(\omega\tau). \tag{5.6}$$

Now suppose that $\lambda = \alpha + i\omega$ is a root of the characteristic equation with $\alpha \geq 0$. Then

$$\gamma \leq \alpha + \gamma \leq \beta e^{-\mu\tau} e^{-\alpha\tau} |g'(\xi)\cos(\omega\tau)| \leq \beta e^{-\mu\tau} |g'(\xi)|.$$

Consequently, if

$$\gamma > \beta e^{-\mu\tau} |g'(\xi)| \tag{5.7}$$

then $\alpha < 0$ for all the characteristic values and the steady state is asymptotically stable. Notice, that since (5.2) is also satisfied at a steady state, equation (5.7) is equivalent to

$$\left| \frac{\xi g'(\xi)}{g(\xi)} \right| < 1. \tag{5.8}$$

We arrive at a sufficient condition for stability of the unique steady state: when (5.8) is satisfied at a steady state, the steady state is asymptotically stable.

The function $\xi \mapsto \xi g'(\xi)/g(\xi)$ plays a central role in the analysis, so we study its properties here. Differentiating g in (1.3) gives

$$g'(\xi) = \frac{n\theta_g^n (g^+ - g^-)\xi^{n-1}}{(\theta_g^n + \xi^n)^2} \tag{5.9}$$

and hence

$$\frac{\xi g'(\xi)}{g(\xi)} = \frac{n(g^+ - g^-)(\xi/\theta_g)^n}{(1 + (\xi/\theta_g)^n)(g^- + g^+(\xi/\theta_g)^n)}. \tag{5.10}$$

Now, let

$$f(x, p, r) = \frac{p(1-r)x^p}{(1+x^p)(r+x^p)} \tag{5.11}$$

and note from (5.10) that $f(\xi/\theta_g, n, g^-/g^+) = \xi g'(\xi)/g(\xi)$. The following proposition will be essential in this and following sections.

Proposition 5.1. *Let $f(x,p,r) : \mathbb{R}^3_{>0} \to \mathbb{R}$ be defined by (5.11) then*

1. *For fixed $p > 0$, $r > 0$,*
$$\lim_{x\to 0} f(x,p,r) = \lim_{x\to\infty} f(x,p,r) = 0.$$
2. *For fixed $r > 0$ with $r \neq 1$ we have $\lim_{p\to\infty} |f(1,p,r)| = \infty$, with*
$$f(1,p,r) = \frac{p(1-r)}{2(1+r)},$$
and $|f(1,p,r)| \leq p/2$.
3. *For fixed $r > 0$ and fixed $x > 0$ with $x \neq 1$,*
$$\lim_{p\to\infty} f(x,p,r) = 0.$$
Moreover, for fixed $r > 0$ and any fixed $x^- \in (0,1)$ and $x^+ > 1$
$$\lim_{p\to\infty} \left(\max_{x\in[0,x^-]} |f(x,p,r)| \right) = \lim_{p\to\infty} \left(\max_{x\in[x^+,\infty)} |f(x,p,r)| \right) = 0.$$

4. *For fixed $r > 0$ with $r \neq 1$ and fixed $p > 0$, regarding $|f(x,p,r)|$ as a function of x only, $|f(x,p,r)|$ has a unique global maximum at $x = r^{1/2p}$ with*

$$f(r^{1/2p},p,r) = \frac{p(1-r^{1/2})}{1+r^{1/2}},$$

and $|f(r^{1/2p},p,r)| \leq p$.

Proof. Follows using elementary differentiation and algebra. □

Proposition 5.1 shows that $\xi \mapsto \xi g'(\xi)/g(\xi)$ is a unimodal function which approaches zero uniformly as $n \to 0$, but which resembles a delta-function with its peak at $\xi = \theta_g(g^-/g^+)^{1/2n} \to \theta_g$ as $n \to \infty$.

Now (5.8) and Proposition 5.1(1) show that if $\xi \ll \theta_g$ or for $\xi \gg \theta_g$ the steady state must be asymptotically stable. From (5.8), a necessary condition for the steady state to be unstable is that $|\xi g'(\xi)/g(\xi)| \geq 1$, but from Proposition 5.1(4) we have

$$\left|\frac{\xi g'(\xi)}{g(\xi)}\right| \leq |f(r^{1/2n},n,r)| = \frac{n|1-r^{1/2}|}{1+r^{1/2}}$$

where $r = r_g = g^-/g^+$. Thus a necessary condition for the steady state to be unstable is that $|f(r^{1/2n},n,r)| \geq 1$. For g decreasing, the value $r_g = g^-/g^+ > 1$, and this necessary condition for instability becomes $n > 1$ and

$$r_g = \frac{g^-}{g^+} \geq \left(1+\frac{2}{n-1}\right)^2. \tag{5.12}$$

Since the steady state is unique, a steady state bifurcation cannot occur, and we therefore investigate the existence of Hopf bifurcations. At a Hopf bifurcation, $\lambda = \pm i\omega$, and (5.5) and (5.6) reduce to

$$\gamma = \beta e^{-\mu\tau} g'(\xi)\cos(\omega\tau), \tag{5.13}$$
$$\omega = -\beta e^{-\mu\tau} g'(\xi)\sin(\omega\tau). \tag{5.14}$$

As $\beta, \gamma, \omega, e^{-\mu\tau} > 0$ and $g'(\xi) < 0$, we must have $\cos(\omega\tau) < 0 < \sin(\omega\tau)$, and hence $\omega\tau \in (\pi/2 + 2k\pi, \pi + 2k\pi)$ for $k \in \mathbb{N}$. We denote by ω_k any solution of (5.13) and (5.14) with the property that $\omega_k\tau \in (\pi/2 + 2k\pi, \pi + 2k\pi)$ for $k \in \mathbb{N}$.

Note that at a Hopf bifurcation, in addition to (5.13) and (5.14), equation (5.2) must also be satisfied. These three equations can be rearranged as

$$\gamma = \frac{\beta}{\xi} e^{-\mu\tau} g(\xi), \tag{5.15}$$
$$\omega\cot(\omega\tau) = -\gamma, \tag{5.16}$$
$$|g'(\xi)| = \frac{\sqrt{\gamma^2+\omega^2}}{\beta e^{-\mu\tau}}. \tag{5.17}$$

We will consider these equations sequentially to show that γ and ω can be regarded as functions of ξ, resulting in a single equation to solve for ξ. First note that for a smooth nonlinearity $g(\xi)$, and for arbitrary $\xi > 0$, equation (5.15) gives a unique value of $\gamma = \gamma(\xi) > 0$. Moreover, since g is monotonically decreasing, we also obtain that $\xi \mapsto \gamma(\xi)$ is monotonically decreasing. Because of the remarks following Proposition 5.1 we will be particularly interested

in the cases where $\xi = \theta_g$ and $\xi = \theta_g(g^-/g^+)^{1/2n}$. Using (5.15), (5.4) and (1.3) it is easy to see that $\gamma(\theta_g) = (\gamma_1 + \gamma_2)/2$ and $\gamma(\theta_g(g^-/g^+)^{1/2n}) = \sqrt{\gamma_1\gamma_2}/(g^-/g^+)^{1/2n}$. This shows that the γ value corresponding to the extremum of $\xi g'(\xi)/g(\xi)$ converges to the geometric mean of γ_1 and γ_2 as $n \to \infty$, while the γ value corresponding to $\xi = \theta_g$ is equal to the arithmetic mean of γ_1 and γ_2 independent of the value of n.

Next, for a given $\gamma > 0$, it follows from the properties of the cotangent function that the equation (5.16) gives a sequence of solutions $\{\omega_k\}_{k\geq 0}$ with $\omega_k\tau \in (\pi/2 + 2k\pi, \pi + 2k\pi)$ for $k = 0, 1, 2, \ldots$, with each ω_k uniquely defined as a function of γ. Recall that τ is constant because we are considering $v^- = v^+ = v$. Thus the values $\omega_k \to \infty$ as $k \to \infty$. It follows that solutions ω_k of the equation (5.16) for fixed γ must satisfy $\cot(\omega_k\tau) \to 0$ and hence $\omega_k\tau \to \pi/2 + 2k\pi$ as $k \to \infty$.

Since g is decreasing, the last equation (5.17) then becomes

$$g'(\xi) = -\frac{\sqrt{\gamma^2 + \omega_k^2}}{\beta e^{-\mu\tau}}. \tag{5.18}$$

Notice here that through (5.15) we have γ as a function of ξ, while from (5.16) we can regard each ω_k as an implicitly defined function of γ and hence of ξ. Thus it remains only to solve (5.18) for ξ, or more precisely, we need to solve for a ξ_k for every ω_k.

Combining (5.13) and (5.15) we see that at a Hopf bifurcation

$$\frac{\xi g'(\xi)}{g(\xi)}\cos(\omega\tau) = 1. \tag{5.19}$$

Thus, the function $\xi \mapsto \xi g'(\xi)/g(\xi)$ also plays a role for Hopf bifurcations. Consequently, instead of solving (5.18) directly for ξ, we proceed by combining (5.15) and (5.18) which leads to

$$\beta e^{-\mu\tau} = \frac{\gamma\xi}{g(\xi)} = -\frac{\sqrt{\gamma^2 + \omega_k^2}}{g'(\xi)},$$

and hence

$$\frac{\xi g'(\xi)}{g(\xi)} = -\sqrt{1 + (\omega_k/\gamma)^2}. \tag{5.20}$$

Combining (5.20) and (5.10), it remains to find ξ that solves

$$\frac{n(g^+ - g^-)(\xi/\theta_g)^n}{(1 + (\xi/\theta_g)^n)(g^- + g^+(\xi/\theta_g)^n)} = -\sqrt{1 + (\omega_k/\gamma)^2}. \tag{5.21}$$

We already considered the behaviour of the left-hand side of (5.21) in Proposition 5.1. The behaviour of the right-hand side is considered in the following proposition.

Proposition 5.2. *Let $s_k(\xi) = \sqrt{1 + (\omega_k/\gamma)^2}$ where $\gamma = \gamma(\xi)$ is defined by (5.15) and ω_k satisfies $\omega_k\tau \in (\pi/2 + 2k\pi, \pi + 2k\pi)$ and is a function of γ and hence of ξ through (5.16). Then*

1. $\left(1 + \left(\frac{\pi(2k+1/2)}{\beta\tau e^{-\mu\tau} g_U}\right)^2 \xi^2\right)^{\frac{1}{2}} = m_k(\xi) \leq s_k(\xi) \leq M_k(\xi) = \left(1 + \left(\frac{\pi(2k+1)}{\beta\tau e^{-\mu\tau} g_0}\right)^2 \xi^2\right)^{\frac{1}{2}}$.
2. $1 < s_k(\xi) < 1 + \frac{\pi(2k+1)}{\beta\tau e^{-\mu\tau} g_0}\xi$ *for all* $\xi > 0$.

Proof. Using (5.15)

$$s_k(\xi) = \left(1 + \left(\frac{\omega_k}{\beta e^{-\mu\tau} g(\xi)}\right)^2 \xi^2\right)^{\frac{1}{2}}.$$

Recalling (1.5) and also using the bounds on $\omega_k\tau$, the inequalities in Proposition 5.2(1), and the expressions for the bounds $m_k(\xi)$ and $M_k(\xi)$ follow easily, while Proposition 5.2(2) is weaker than Proposition 5.2(1). □

Recall that for a Hopf bifurcation, we need to solve (5.20) for ξ, but this is the same as solving $f(\xi/\theta_g, n, g^-/g^+) = -s_k(\xi)$, where the relevant properties of f and s_k are stated in Propositions 5.1 and 5.2. Propositions 5.2(1) defines bands $[m_k(\xi), M_k(\xi)]$ within which each $s_k(\xi)$ lies. Since $|s_k(\xi)| > 1$ and $\lim_{\xi\to 0} f(\xi/\theta_g, n, g^-/g^+) = \lim_{\xi\to\infty} f(\xi/\theta_g, n, g^-/g^+) = 0$, for all ξ sufficiently large or small we have $f(\xi/\theta_g, n, g^-/g^+) > -s_k(\xi)$. On the other hand, considering $\xi = \theta_g$, by Proposition 5.1(2), $f(1, n, g^-/g^+) \to -\infty$ as $n \to \infty$ while $s_k(\theta_g) \le M_k(\theta_g)$ is bounded. Consequently, for n sufficiently large $f(1, n, g^-/g^+) < -s_k(\xi)$. It follows that there are at least two points $\xi_k^- < \theta_g < \xi_k^+$ for which $f(\xi_k^\pm/\theta_g, n, g^-/g^+) = -s_k(\xi_k^\pm)$, and hence which solve (5.21). With the corresponding values of γ and ω_k defined by (5.15) and (5.16) this defines two solutions of (5.15)–(5.17).

We already noted that the steady state must be stable, and hence cannot undergo a Hopf bifurcation for ξ sufficiently small or large. On the other hand, Hopf bifurcations must occur for n sufficiently large, as for large enough n the function f will pierce through the band $s_k(\xi) \in [m_k(\xi), M_k(\xi)]$ for $\xi \approx 1$. In particular, a sufficient (but not necessary) condition for this to occur is that $M_k(\theta_g) < -f(1, n, g^-/g^+)$, or equivalently that

$$n > 2\frac{(g^-/g^+)+1}{(g^-/g^+)-1}\left(1 + \left(\frac{\pi\theta_g(2k+1)}{\beta\tau e^{-\mu\tau} g_0}\right)^2\right)^{\frac{1}{2}}. \tag{5.22}$$

This condition follows from evaluation of the function f at $x = 1$. A more complicated but tighter bound can be derived using the maximum of f. Therefore another sufficient condition to ensure that the k-th Hopf bifurcation occurs is that

$$M_k(\theta_g(g^-/g^+)^{1/(2n)}) < -f((g^-/g^+)^{1/(2n)}, n, g^-/g^+).$$

While there is a unique steady state ξ for the case of decreasing g with constant τ, the location and properties of this steady state will depend on the values of the other parameters. The following theorem collects together our results for this case.

Theorem 5.3. *Let ξ be the steady state of the DDE (1.1), (1.2) with constant delay τ and nonlinearity g defined by (1.3) with g monotonically decreasing (so $g^- > g^+$). Then:*

1. *If $|\frac{\xi g'(\xi)}{g(\xi)}| < 1$, then the steady state ξ is asymptotically stable.*
2. *If $n \le 1$, or $n > 1$ and $r_g = \frac{g^-}{g^+} < \left(1 + \frac{2}{n-1}\right)^2$, then the steady state ξ is asymptotically stable.*
3. *For any fixed $n > 1$, and for $0 < \xi \ll \theta_g$ or $\xi \gg \theta_g$, or equivalently for $\gamma \gg \gamma_2$ or $0 < \gamma \ll \gamma_1$, the steady state is asymptotically stable.*
4. *For any fixed $\xi \neq \theta_g$, let $\gamma = \gamma(n, \xi)$ be the value of γ such that (5.2) is satisfied and hence ξ is a steady state. Or, for any fixed γ with $0 < \gamma < \gamma_1$ or $\gamma > \gamma_2$ let $\xi = \xi(n, \gamma)$ satisfy (5.2) and hence be a steady state. Then ξ is asymptotically stable for all n sufficiently large.*

5. *Let (5.22) be satisfied for fixed $n \geq 2$ and fixed $k \geq 0$. Then as γ is varied*

   (a) *There are two families of (at least) $k+1$ Hopf bifurcations. One exists for $\gamma < \sqrt{\gamma_1\gamma_2}/(g^-/g^+)^{1/2n}$ and the other for $\gamma > (\gamma_1+\gamma_2)/2$. In the first family, the characteristic values $\lambda_j = i\omega_j$ satisfy $\omega_j\tau \in (\pi/2+2j\pi, \pi+2j\pi)$ for $j = 0,1,\ldots,k$ and cross the imaginary axis from left to right as γ is increased, while in the second family they cross the imaginary axis from right to left.*

   (b) *For $\xi \in [\theta_g, \theta_g(g^-/g^+)^{1/(2n)}]$ or equivalently for $\gamma \in [\sqrt{\gamma_1\gamma_2}/(g^-/g^+)^{1/2n}, (\gamma_1+\gamma_2)/2]$ the steady state is unstable with at least $k+1$ pairs of complex conjugate characteristic values $\lambda_j = \alpha_j \pm i\omega_j$ with $\alpha_j > 0$ and $\omega_j\tau \in (\pi/2+2j\pi, \pi+2j\pi)$ for $j = 0,1,\ldots,k$.*

6. *Let $\gamma \in (\gamma_1,\gamma_2)$ be fixed. Then as n is increased there is an infinite sequence of Hopf bifurcations where the real part of $\lambda_k = \alpha_k \pm i\omega_k$ becomes positive with $\omega_k\tau \in (\pi/2+2k\pi, \pi+2k\pi)$ for $k = 0,1,2,\ldots$*

Proof. (1) and (2) were already shown immediately after the proof of Proposition 5.1.

The first part of (3) then follows from Proposition 5.1(1), since $\lim_{\xi\to 0} f(\xi/\theta_g, n, g^-/g^+) = \lim_{\xi\to\infty} f(\xi/\theta_g, n, g^-/g^+) = 0$ implies $|\xi g'(\xi)/g(\xi)| < 1 < s_k(\xi)$ and $\xi g'(\xi)/g(\xi) = -s_k(\xi)$ cannot hold for ξ sufficiently small ($0 < \xi \ll \theta_g$) or large ($\xi \gg \theta_g$). The second part follows from (5.15) on noting that g monotonically decreasing implies that $\gamma(\xi)$ is monotonically decreasing with $\lim_{\xi\to 0}\gamma(\xi) = +\infty$ and $\lim_{\xi\to\infty}\gamma(\xi) = 0$.

For the first part of (4), consider a steady state at a fixed value of ξ as n is varied, with the other parameters fixed except for $\gamma = \gamma(n,\xi)$ which is determined by (5.2). The result then follows directly from the first part of Proposition 5.1(3). To prove the second part of (4) note that when $g^- > g^+$ from (1.3) we have $g^- > g(\xi) > g^+$ which using (5.4) is equivalent to

$$\gamma_2 > \frac{\beta e^{-\mu\tau} g(\xi)}{\theta_g} > \gamma_1.$$

Fix $\gamma > 0$ and using (5.15), we further rewrite this as

$$\frac{\gamma_2}{\gamma} > \frac{\xi}{\theta_g} > \frac{\gamma_1}{\gamma}. \tag{5.23}$$

Consequently, for $\xi = \xi(n,\gamma)$ satisfying (5.2), if $\gamma < \gamma_1$ then we have $\xi > \theta_g\gamma_1/\gamma > \theta_g$ while $\gamma > \gamma_2$ implies $\xi < \theta_g\gamma_2/\gamma < \theta_g$. The result then follows from the second part of Proposition 5.1(3).

For (5), equation (5.22) implies that

$$-f(1,n,g^-/g^+) > M_k(\theta_g) > M_{k-1}(\theta_g) > \ldots > M_0(\theta_g).$$

However, $s_j(\xi) \geq 1$ for all $\xi \in \mathbb{R}$ and $\lim_{x\to 0} f(x,n,g^-/g^+) = \lim_{x\to\infty} f(x,n,g^-/g^+) = 0$. Consequently for each $j \in \{0,1,\ldots,k\}$ there are at least two values of ξ which solve $-f(\xi/\theta_g, n, g^-/g^+) = s_j(\xi)$. The largest such ξ with $\xi > \theta_g(g^-/g^+)^{1/(2n)}$ and the smallest with $\xi < \theta_g$ define the required Hopf bifurcations. Since, as already noted, $\gamma(\xi)$ defined by (5.15) is a monotonically decreasing function of ξ with $\gamma(\theta_g) = (\gamma_1+\gamma_2)/2$ and $\gamma(\theta_g(g^-/g^+)^{1/(2n)}) = \sqrt{\gamma_1\gamma_2}/(g^-/g^+)^{1/2n}$, the result follows.

For (6) we consider the behaviour of the Hopf bifurcation points as $n \to \infty$. Applying Proposition 5.1 (and recalling that $g_- > g_+$ because g is decreasing), the function $|\xi g'(\xi)/g(\xi)|$ takes its maximum at $\xi = \theta_g(g^-/g^+)^{1/2n} > \theta_g$, while also $|\theta_g g'(\theta_g)/g(\theta_g)| =$

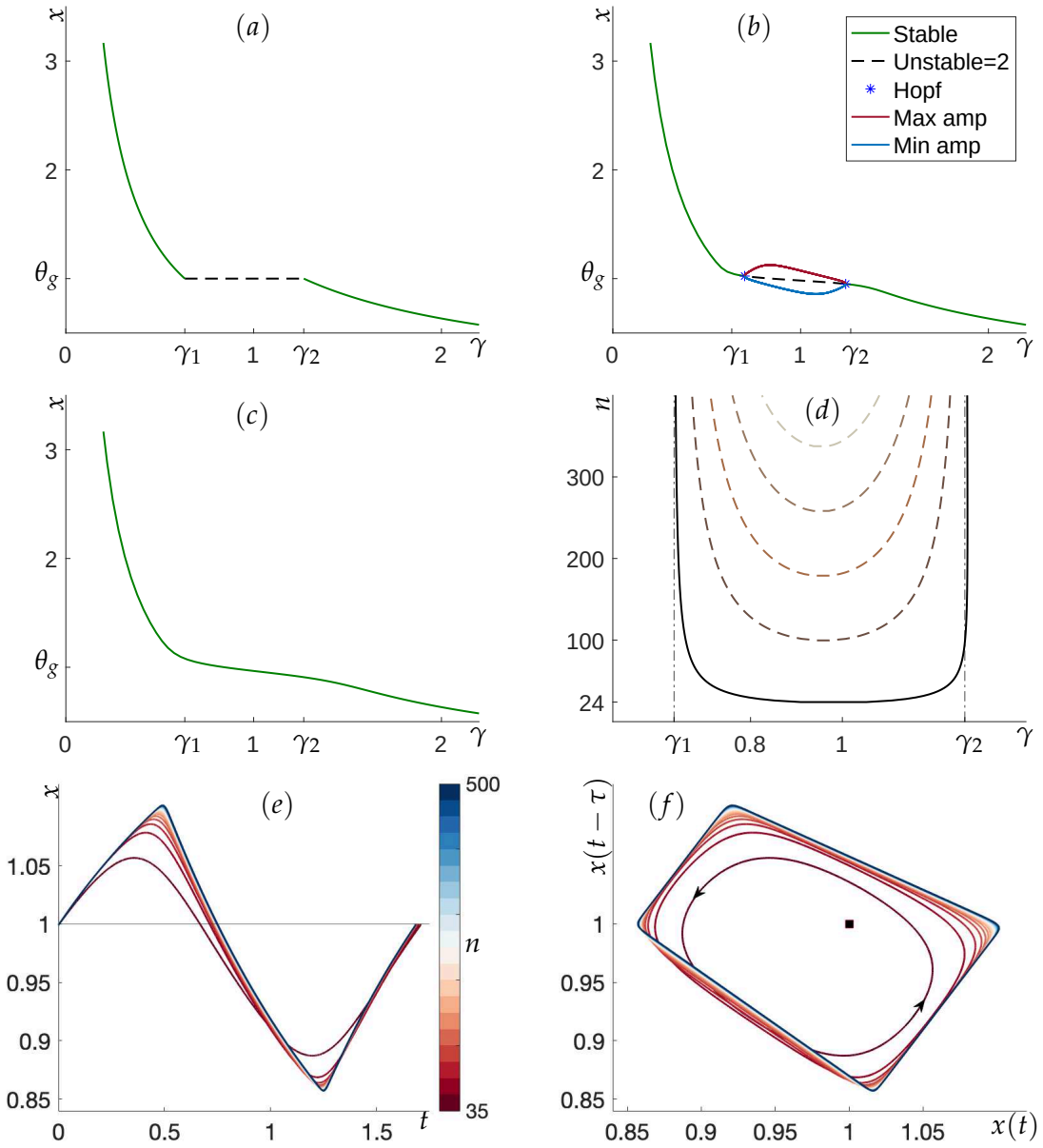


Figure 5.2: Bifurcations of (1.1)–(1.2) for $(g\downarrow, v\leftrightarrow)$ with parameters $\beta = 1.4$, $\mu = 0.2$, $g^- = 1$, $g^+ = 1/2$, $\theta_g = 1$, $\gamma = a = 1$, and $v = v^- = v^+ = 2$. (a) The limiting case with g defined by (1.6) showing the stable (green solid line) and singular (black dashed line) steady states. (b) With smooth nonlinearity g defined by (1.3) and $n = 50$. Solid lines denote stable objects including the stable steady state (green) and a stable limit cycle (represented by maximum and minimum of $x(t)$ on the periodic solution). Dashed lines represent unstable steady states which have two eigenvalues with positive real part (in black). (c) As in (b) but with $n = 23$. (d) Two-parameter continuations in n and γ of the Hopf bifurcations defined by (5.15)–(5.17) with the other parameters as above. Solid curves indicate the parts of the branch where there are no characteristic values with positive real part (and hence a stability change at the bifurcation), and dashed lines indicate the parts of the branch where there are already unstable characteristic values. The outermost curve of Hopf bifurcations is associated with the stability change seen in (b). The dash-dotted vertical black lines denote $\gamma = \gamma_1$ and $\gamma = \gamma_2$, the locations of the Hopf bifurcations in the limiting case as $n \to \infty$. (e) Profiles of the stable periodic orbits from the outermost curve of Hopf bifurcations in (d) at $\gamma = 1$ for different values of the continuation parameter n. (f) The same periodic orbits as in (e), but now shown as a projection onto the plane $(x(t), x(t-\tau))$ where $\tau = 0.5$. The arrow indicates the direction of the flow. The square denotes the singular steady state in the limiting case.

$|f(1, n, g^-/g^+)| \gg 1$. Consequently for fixed k the two Hopf bifurcation points, ξ_k^- and ξ_k^+ satisfy $\xi_k^- < \theta_g < \theta_g(g^-/g^+)^{1/2n} < \xi_k^+$. Using Proposition 5.1(3) we conclude that $\xi_k^\pm \to \theta_g$ as $n \to \infty$.

Also for fixed k, as $n \to \infty$, the value of $\omega_k \leq (2k+1)\pi/\tau$ remains bounded, as does γ by (4). Consequently for (5.18) to be satisfied it follows that $g'(\xi_k)$ must also remain bounded as $n \to \infty$. Then in the limit as $n \to \infty$ the Hopf bifurcations must converge to the "corners" of $g(\xi)$ where $\xi = \theta_g$ and $\gamma = \gamma_1$ or $\gamma = \gamma_2$. Consequently for any fixed $\gamma \in (\gamma_1, \gamma_2)$ as $n \to \infty$ there is an infinite sequence of Hopf bifurcations. □

Figure 5.2 illustrates the behaviour of (1.1)–(1.2) for $(g\downarrow, v\leftrightarrow)$. Panels (b)–(f) were computed numerically using `ddebiftool` as described in Appendix C.

Figure 5.2(a) and (b) show the similarities between the dynamics with the piecewise continuous nonlinearity (1.6) and the smooth Hill function g defined in (1.3) with $n = 50$. The singular steady state in (a) becomes an unstable steady state in the smooth case, with a bubble of stable periodic orbits existing between the pair of Hopf bifurcations where the steady state changes stability. The stable periodic orbits at $\gamma = 1$ for increasing values of n are shown as profiles in Figure 5.2(e) and projected onto the $(x(t), x(t-\tau))$-plane in Figure 5.2(f). The apparent limiting behaviour that is revealed is the topic of [31].

When the value of n is decreased, the interval of γ values between the Hopf bifurcations shrinks, until for n sufficiently small the steady state is always stable, as seen in Figure 5.2(c). Interestingly, even though there is no bifurcation in this case, the graph in Figure 5.2(c) still has a plateau around where the singular steady states exist in Figure 5.2(a).

Figure 5.2(d) shows two-parameter continuations in the (γ, n) plane of the Hopf bifurcations. This reveals the Hopf bifurcations associated with ω_k for $k = 0, 1, \ldots, 4$, with each successive Hopf bifurcation only existing for progressively larger values of n, as implied by (5.22). In particular there is no Hopf bifurcation for $n < 24$ and a second Hopf bifurcation is only seen if $n > 100$. This is why no Hopf bifurcation is seen in Figure 5.2(c) with $n = 23$, and only one pair of Hopf bifurcations is seen in Figure 5.2(b) with $n = 50$.

Figure 5.2(d) also illustrates Theorem 5.3 (points 5 and 6) where the additional Hopf bifurcations occurring as n increases approach the vertical asymptotes $\gamma = \gamma_1$ and $\gamma = \gamma_2$ in the limit as $n \to \infty$. Notice also the existence of Hopf bifurcations with $\gamma > \gamma_2$ in Figure 5.2(d); so it *is* possible for the steady state to be unstable outside the interval $\gamma \in [\gamma_1, \gamma_2]$, albeit only for a finite range of values n by Theorem 5.3 (point 3).

5.2 Constant delay with g increasing $(g\uparrow, v\leftrightarrow)$

In this section we study (1.1)–(1.2) with constant delay again, but in contrast to the previous section we assume that g is increasing. We thus assume $g^- < g^+$ in (1.3) and (1.6), with $v^- = v^+ = v^\pm$ so the delay $\tau = a/v^\pm$ is constant, independent of ξ.

When g is increasing, it is possible for multiple steady states to coexist. For example, considering the limiting case where g is given by (1.6), as shown in Figure 5.3, there are up to three coexisting steady states, including a singular steady state, as the slope of the line $\gamma\xi$ changes. The corners defined by (5.4) give rise to fold bifurcations due to a change in the number of steady states. Since these bifurcations involve a singular steady state they are not truly fold bifurcations but we use this term since, as we show below, they reflect the presence of true fold bifurcations for the smooth nonlinearity g defined by (1.3) with $g'(\xi) \gg 0$. This is illustrated in Figure 5.4(a) where there are two fold bifurcations between stable and singular steady states, with the outer steady states stable, and the middle steady state is singular.

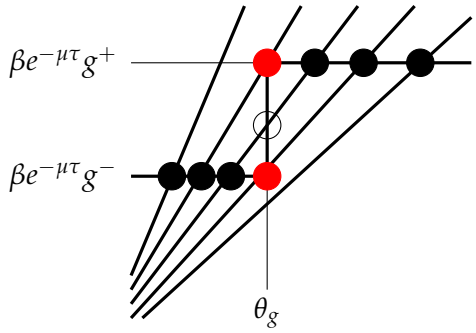


Figure 5.3: Illustration of how the number of steady states of (1.1) given by (5.2) changes with the intersections of $\xi \mapsto \beta e^{-\mu\tau} g(\xi)$ and $\xi \mapsto \gamma\xi$. These are shown in the limiting case with $v^\pm = v^- = v^+$ so $\tau = a/v^\pm$ is constant, and $g^- < g^+$ in (1.6) so g is piecewise constant and monotonically increasing: $(g \uparrow, v \leftrightarrow)$.

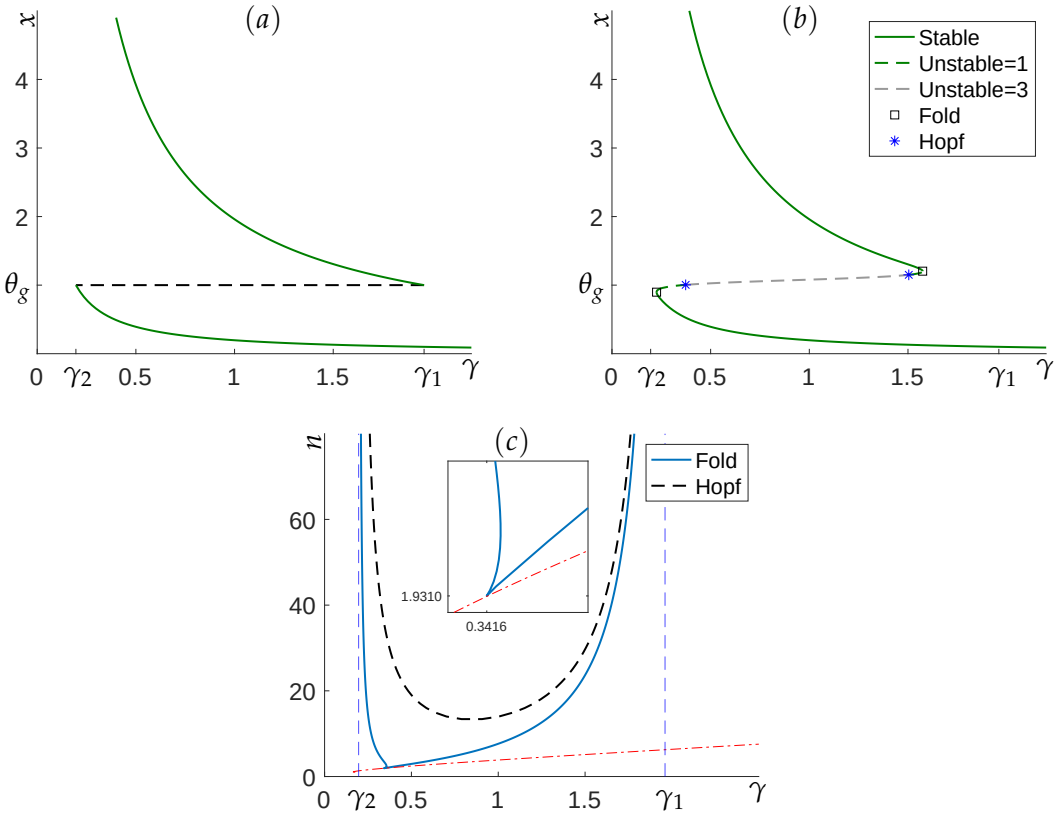


Figure 5.4: Bifurcations of (1.1)–(1.2) with $(g \uparrow, v \leftrightarrow)$ and parameters $\beta = 2$, $\mu = 0.02$, $g^- = 0.1$, $g^+ = 1$, $\theta_g = 1$, $\gamma = 1$, $a = 2$ and $v = v^- = v^+ = 2$. (a) The limiting case with g defined by (1.6). Stable steady states are shown as green solid lines, and the singular steady state as a black dashed line. (b) With a smooth nonlinearity g defined by (1.3) with $n = 30$. (c) Two-parameter continuations in n and γ of the fold (blue) and the Hopf (black) bifurcations with the other parameters as above. The dashed vertical lines denote $\gamma = \gamma_1$ and $\gamma = \gamma_2$, the location of the fold bifurcations in the limiting case as $n \to \infty$. The red dash-dotted curve denotes the bound on the fold bifurcations given by (5.28).

With g defined by the piecewise constant function (1.6), as in Section 5.1, for the non-singular steady states we have $A = g'(\xi) = 0$. The characteristic equation is again of the form (4.11), so these steady states must be stable. Only the singular steady state may be unstable, and since it coexists with two stable steady states, it is natural to regard it as unstable even though the dynamical system is infinite dimensional and the characteristic equation is not defined at the singular steady state. The bifurcation diagram for this case is illustrated in Figure 5.4(a).

For the smooth nonlinearity $g(\xi)$ defined by (1.3), from (4.9) the characteristic function is given by

$$\Delta(\lambda) = \lambda + \gamma - \beta e^{-\mu\tau} g'(\xi) e^{-\lambda\tau}. \tag{5.24}$$

Considering $\lambda \in \mathbb{R}$ first, note that $\Delta(\lambda)$ is asymptotic to $\lambda + \gamma$ for $\lambda \gg 0$. On the other hand, using (5.2) we find

$$\Delta(0) = \gamma \left(1 - \frac{\xi g'(\xi)}{g(\xi)}\right). \tag{5.25}$$

Consequently when

$$\frac{\xi g'(\xi)}{g(\xi)} > 1 \tag{5.26}$$

there is always a real characteristic value $\lambda > 0$ and the steady state is unstable. Moreover there is a characteristic value $\lambda = 0$ if and only if

$$\frac{\xi g'(\xi)}{g(\xi)} = 1, \tag{5.27}$$

and the results developed in Section 5.1, and in particular Proposition 5.1, can be applied to the fold bifurcations. Thus for $n \gg 0$ there will be two fold bifurcations $\xi^{\pm} \approx \theta_g$: one with $\gamma \approx \gamma_1$ and the other with $\gamma \approx \gamma_2$, where γ_1 and γ_2 are defined by (5.4) with $\gamma_2 < \gamma_1$ since g is increasing.

To find the fold bifurcations, equations (5.27) and (5.15) must be solved together. We first do this numerically. Figure 5.4(b) shows the resulting bifurcation diagram for the smooth nonlinearity $g(\xi)$ defined by (1.3) with $n = 30$, revealing, as expected, two smooth fold bifurcations, with an intermediate branch of unstable steady states between the stable steady states.

To further investigate when multiple steady states arise for smoothly increasing g, note that a necessary condition for multiple coexisting steady states is that $\max_{\xi \geq 0}\{h'(\xi)\} \geq 0$. This imposes a constraint on the parameters as follows. The function g has a single point of inflection $\bar{x}$ with $g''(\bar{x}) = 0$, which can be computed by differentiating g twice to find

$$\bar{x} = \theta_g \left(\frac{n-1}{n+1}\right)^{1/n}.$$

Note that $\bar{x} > 0$ requires the restriction $n > 1$. With $\bar{x}$, we can compute the maximal value of the derivative

$$\max_{\xi \geq 0}\{h'(\xi)\} = h'(\bar{x}) = \beta e^{-\mu\tau} g'(\bar{x}) - \gamma = \beta e^{-\mu\tau}(g^+ - g^-)\frac{(n+1)^{1+1/n}(1-n)^{1-1/n}}{4n\theta_g} - \gamma.$$

Therefore, if there are multiple steady states, the parameters must satisfy

$$\gamma \leq \beta e^{-\mu\tau}(g^+ - g^-)\frac{(n+1)^{1+1/n}(n-1)^{1-1/n}}{4n\theta_g}. \tag{5.28}$$

Figure 5.4(c) shows a two-parameter continuation of the fold bifurcations for increasing g. For large n the folds approach the asymptotes $\gamma = \gamma_1$ and $\gamma = \gamma_2$ defined by the limiting case. As n decreases the two fold bifurcations move closer together until they collide in a cusp bifurcation. At the cusp, the three steady states coincide, and at this point $h(\xi)$ is a monotonically decreasing function of ξ with a zero of multiplicity three at the steady state $\bar{\xi}$. Consequently not only is $h(\bar{\xi}) = h'(\bar{\xi}) = 0$ at this point, but also the function $h'(\xi)$ attains its maximum at $\bar{\xi}$ and the cusp point lies on the bounding curve defined by (5.28), as seen in Figure 5.4(c).

The analysis of Hopf bifurcations from Section 5.1 can be repeated with only minor changes for the case of increasing g. Characteristic values again satisfy (5.5) and (5.6), and at a Hopf bifurcation (5.13) and (5.14). Since $g'(\xi) > 0$, we require $\sin(\omega\tau) < 0 < \cos(\omega\tau)$ and hence

$$\omega\tau \in (2k + 3/2)\pi, (2k + 2)\pi), \quad k \in \mathbb{N}.$$

Equations (5.15), (5.16) and (5.17) can again be considered sequentially, and reduced to a single equation to solve for ξ, where for increasing g, equation (5.17) becomes

$$g'(\xi) = \frac{\sqrt{\gamma^2 + \omega_k^2}}{\beta e^{-\mu\tau}}. \tag{5.29}$$

This last equation is most easily considered by combining it with (5.15) to obtain

$$\frac{\xi g'(\xi)}{g(\xi)} = \sqrt{1 + (\omega_k/\gamma)^2}, \tag{5.30}$$

where $\xi g'(\xi)/g(\xi)$ is still given by (5.10) and Propositions 5.1 and 5.2 both apply.

We conclude that as γ is varied there is a sequence of Hopf bifurcations parameterized by a frequency $\omega_k \to \infty$ satisfying $\omega_k\tau \in ((2k+3/2)\pi, (2k+2)\pi)$ for $k \in \mathbb{N}$, where for each fixed k there is a minimal n at which this bifurcation exists. Furthermore, this minimal n grows with k. Additionally, for any fixed k as $n \to \infty$ the bifurcation points satisfy $\gamma \to \gamma_1$ and $\gamma \to \gamma_2$. Figure 5.4(b) shows that the steady states lose stability at the fold bifurcation and the resulting unstable steady state undergoes a Hopf bifurcation. This indicates that the resulting periodic orbits are unstable and thus not consequential for the asymptotic dynamics in contrast to the case of decreasing g. The two-parameter continuation of the first Hopf bifurcation is shown in Figure 5.4(c).

We collect our results from this section together in the following theorem. While there are many similarities between Theorems 5.3 and 5.4, the presence of fold bifurcations introduces some important differences. Note also that since g is increasing, in this section we have $r_g = g^-/g^+ < 1$ and also $\gamma_1 > \gamma_2$.

Theorem 5.4. *Let ξ be a steady state of the DDE (1.1), (1.2) with constant delay τ and nonlinearity g defined by (1.3) with g monotonically increasing (so $g^- < g^+$). Then*

1. *The steady state ξ is asymptotically stable if $\frac{\xi g'(\xi)}{g(\xi)} < 1$, and unstable if $\frac{\xi g'(\xi)}{g(\xi)} > 1$.*
2. *For any fixed $\xi \neq \theta_g$ let $\gamma = \gamma(n, \xi)$ be the value of γ such that (5.2) is satisfied and hence ξ is a steady state. Or, for any fixed γ with $0 < \gamma < \gamma_2$ or $\gamma > \gamma_1$ let $\xi = \xi(n, \gamma)$ satisfy (5.2) and hence be a steady state. Then ξ is asymptotically stable for all n sufficiently large.*
3. *If $n \leq 1$, or $n > 1$ and $r_g = \frac{g^-}{g^+} > \left(1 - \frac{2}{n+1}\right)^2$, then the steady state ξ is asymptotically stable.*

4. *If* $n > 1$ *and* $r = \frac{g^-}{g^+} < \left(1 - \frac{2}{n+1}\right)^2$, *then there exists* $\xi^- < \theta_g (g^-/g^+)^{1/2n} < \xi^+$ *and* $\gamma(\xi^-) < \sqrt{\gamma_1\gamma_2}/(g^-/g^+)^{1/2n} < \gamma(\xi^+)$ *such that as* γ *is varied, there is one branch of stable steady states with* $\xi < \xi^-$ *and* $\gamma > \gamma(\xi^-)$ *and another stable branch with* $\xi > \xi^+$ *and* $\gamma < \gamma(\xi^+)$. *For* $\gamma \in (\gamma(\xi^-), \gamma(\xi^+))$ *the two stable branches of steady states co-exist with a branch of unstable steady states which exists between fold bifurcations at* $(\xi, \gamma) = (\xi^-, \gamma(\xi^-))$ *and* $(\xi, \gamma) = (\xi^+, \gamma(\xi^+))$.

5. *Let fixed* $n \geq 2$ *and fixed* $k \geq 0$ *satisfy*

$$n > 2\frac{1 + (g^-/g^+)}{1 - (g^-/g^+)} \left(1 + \left(\frac{2\pi\theta_g(k+1)}{\beta\tau e^{-\mu\tau} g_0}\right)^2\right)^{\frac{1}{2}}. \tag{5.31}$$

Then as γ *is varied*

   (a) *There are two families of (at least)* $k+1$ *Hopf bifurcations from the unstable steady-state. One exists for* $\gamma < \sqrt{\gamma_1\gamma_2}/(g^-/g^+)^{1/2n}$ *and the other for* $\gamma > (\gamma_1 + \gamma_2)/2$. *In the first family, the characteristic values* $\lambda_j = \alpha_j \pm i\omega_j$ *with* $\omega_j\tau \in (3\pi/2 + 2j\pi, 2\pi + 2j\pi)$ *for* $j = 0, 1, \ldots, k$ *cross the imaginary axis from left to right as* γ *is increased, while in the second family they cross the imaginary axis from right to left.*

   (b) *For* $\xi \in [\theta_g(g^-/g^+)^{1/(2n)}, \theta_g]$, *or equivalently for* $\gamma \in (\sqrt{\gamma_1\gamma_2}/(g^-/g^+)^{1/2n}, (\gamma_1 + \gamma_2)/2)$, *the unstable steady state has one positive real characteristic value and at least* $k+1$ *pairs of complex conjugate characteristic values* $\lambda_j = \alpha_j \pm i\omega_j$ *with* $\alpha_j > 0$ *and* $\omega_j\tau \in (3\pi/2 + 2j\pi, 2\pi + 2j\pi)$ *for* $j = 0, 1, \ldots, k$.

6. *Let* $\gamma \in (\gamma_2, \gamma_1)$ *be fixed. Then as* n *is increased there is an infinite sequence of Hopf bifurcations on the branch of unstable equilibria where the real part of* $\lambda_k = \alpha_k \pm i\omega_k$ *becomes positive with* $\omega_k\tau \in (3\pi/2 + 2k\pi, 2\pi + 2k\pi)$.

Proof. The first part of (1) follows from (5.8), while the second part follows from (5.24)–(5.26).

The proof of the first part of (2) is identical to the proof of the first part of (4) of Theorem 5.3. To show the second part of (2), note that when $g^+ > g^-$ from (1.3) we have $g^+ > g(\xi) > g^-$ which, using (5.4), is equivalent to

$$\gamma_1 > \frac{\beta e^{-\mu\tau} g(\xi)}{\theta_g} > \gamma_2.$$

Fix $\gamma > 0$ and using (5.15), we further rewrite this as

$$\frac{\gamma_1}{\gamma} > \frac{\xi}{\theta_g} > \frac{\gamma_2}{\gamma}.$$

Hence, for $\xi = \xi(n, \gamma)$ satisfying (5.2) if $\gamma > \gamma_1$ then $\xi < \theta_g\gamma_1/\gamma < \theta_g$. On the other hand, if $\gamma < \gamma_2$ then $\xi > \theta_g\gamma_2/\gamma > \theta_g$. The result follows from (1) and the second part of Proposition 5.1(3)

Statement (3) is shown similarly to the corresponding result in Theorem 5.3, noting that since g is increasing $r_g = g^-/g^+ < 1$, which results in a different inequality than the one found in (5.12).

To show (4) consider the curve of steady states $(\xi, \gamma(\xi))$ for $\xi > 0$ where $\gamma(\xi)$ is defined by (5.15). Using Proposition 5.1, the conditions of (4) imply that $\xi g'(\xi)/g(\xi)$ has a maximum value larger than 1 at $\xi = \theta_g(g^-/g^+)^{1/2n}$. Let $\xi^- < \theta_g(g^-/g^+)^{1/2n} < \xi^+$ be the points where

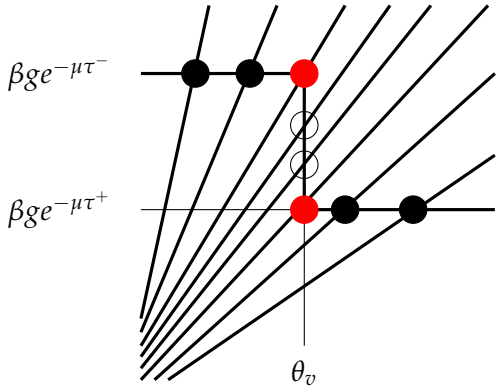


Figure 5.5: Steady states of (1.1) are given by (5.2), and hence occur at the intersections of $\xi \mapsto \beta e^{-\mu\tau(\xi)} g$ and $\xi \mapsto \gamma\xi$. These are illustrated for various $\gamma > 0$ in the limiting case of (1.6) and (1.7) with $g^\pm = g^- = g^+$, so $g(\xi) = g^\pm$ is a constant function, and $v^- > v^+$, so v is piecewise constant and monotonically decreasing : $(g \leftrightarrow, v \downarrow)$. Then $\tau(\xi) = a/v(\xi)$ is state-dependent; $\tau(\xi) = \tau^- = a/v^-$ for $\xi < \theta_v$, $\tau(\xi) = \tau^+ = a/v^+$ for $\xi > \theta_v$ and $\tau(\xi)$ is set-valued when $\xi = \theta_v$.

$\xi^- g'(\xi^-)/g(\xi^-) = \xi^+ g'(\xi^+)/g(\xi^+) = 1$. Then the steady state is stable for $\xi < \xi^-$ and $\xi > \xi^+$, and unstable for $\xi \in (\xi^-, \xi^+)$. Now differentiating (5.15) we find that

$$\gamma'(\xi) = -\frac{\beta}{\xi^2} e^{-\mu\tau} g(\xi) + \frac{\beta}{\xi} e^{-\mu\tau} g'(\xi) = \frac{\beta}{\xi^2} e^{-\mu\tau} g(\xi) \left[\frac{\xi g'(\xi)}{g(\xi)} - 1\right] = \frac{\gamma(\xi)}{\xi} \left[\frac{\xi g'(\xi)}{g(\xi)} - 1\right]. \quad (5.32)$$

Thus $\gamma(\xi)$ is a decreasing function of ξ when the steady state is stable, and an increasing function when it is unstable. From this (4) follows.

The proofs of (5) and (6) are similar to the proof of Theorem 5.3, with the main difference being that for a Hopf bifurcation from the unstable steady state we require $\omega_k \tau \in (3\pi/2 + 2k\pi, 2\pi + 2k\pi)$, while for g increasing a sufficient condition to obtain a solution of (5.30) is that $M_k(\theta_g) < f(1, n, g^-/g^+)$. □

It would be interesting if the Hopf and fold bifurcations could exchange positions on the branch, so that the steady state lost stability in a Hopf bifurcation instead of a fold bifurcation. However, this cannot happen with a constant delay because, as the theorem shows, stability is always lost when $\xi g'(\xi)/g(\xi) = 1$ at which point there is a zero characteristic value giving rise to a fold bifurcation.

5.3 State-dependent delay with v decreasing and g constant $(g \leftrightarrow, v \downarrow)$

To understand the dynamics of (1.1)–(1.2) with a state-dependent delay, we impose $g^\pm := g^- = g^+$ in (1.3) and (1.6), so $g(\xi) = g^\pm$ is a constant function. We first consider a decreasing function v with $v^- > v^+$ in (1.3) and (1.7).

It is convenient to let $\tau^+ := a/v^+$ and $\tau^- := a/v^-$. Then at a steady state ξ with $v^- > v^+$, equation (5.1) implies $\tau(\xi)$ is an increasing function of ξ with $\tau^- < \tau(\xi) < \tau^+$, so $e^{-\mu\tau(\xi)}$ is a decreasing function of ξ. Under these circumstances, $h(\xi)$ is monotonically decreasing and equation (5.2) has exactly one solution, and so there is always a unique steady state.

Figure 5.5 illustrates the uniqueness of the steady state in the limiting case when v defined

by (1.7) is piecewise constant. The steady states at the corners are associated with

$$\gamma_3 = \frac{\beta g^{\pm} e^{-\mu\tau^+}}{\theta_v} \quad \text{and} \quad \gamma_4 = \frac{\beta g^{\pm} e^{-\mu\tau^-}}{\theta_v}. \tag{5.33}$$

As discussed in Section 5.1, the steady state is stable if $\gamma \in (0,\gamma_3) \cup (\gamma_4,\infty)$ as then $A = g'(\xi) = 0$ and the characteristic equation (4.8) reduces to (4.11) with exactly one negative real characteristic value. Consequently, the steady state may only be unstable in the singular case for $\gamma \in (\gamma_3,\gamma_4)$.

Since A, which is given by (4.4), is undefined for singular steady states, the characteristic equation (4.10) cannot be used to study the stability of the singular steady states. So, instead we consider the stability of the steady states for the smooth velocity nonlinearity v defined in (1.3). There will be some similarities to the analysis in Section 5.1, but the problem studied in this section with state-dependent delay is significantly more complicated than the constant delay problem considered before.

With g constant and v defined by (1.3) the characteristic equation is of the form (4.10). Recalling the definition of A from (4.4) and using (5.2), we obtain at a steady state ξ

$$A = \beta \frac{v'(\xi)}{v(\xi)} e^{-\mu\tau(\xi)} g^{\pm} = \gamma \frac{\xi v'(\xi)}{v(\xi)} < 0, \tag{5.34}$$

since $v^- > v^+$ implies $v'(\xi) < 0$. It follows that the right-hand side of (4.10) is negative when $\lambda \geq 0$, and so there are no non-negative real characteristic values. This is not surprising since we already know that the steady state is unique, and therefore there are no steady-state bifurcations.

Furthermore, since $v(\xi)$ is a Hill function, it follows that

$$\frac{\xi v'(\xi)}{v(\xi)} = f(\xi/\theta_v, m, v^-/v^+), \tag{5.35}$$

where f is defined by (5.11). Thus Proposition 5.1 will be relevant in what follows.

To investigate the stability of the steady state we consider complex characteristic values. Let $\lambda = \alpha + i\omega$, $\omega > 0$ then (4.10) implies

$$\begin{aligned}
\alpha + i\omega &= -\gamma + A(1 - e^{-(\alpha+i\omega)\tau(\xi)})\left(1 + \frac{\mu}{\alpha + i\omega}\right), \\
&= -\gamma + \frac{A}{\alpha^2+\omega^2}\left[(1 - e^{-\alpha\tau(\xi)}\cos\omega\tau(\xi))(\alpha^2+\omega^2+\mu\alpha) + e^{-\alpha\tau(\xi)}\sin\omega\tau(\xi)\mu\omega\right] \\
&\quad + i\frac{A}{\alpha^2+\omega^2}\left[e^{-\alpha\tau(\xi)}\sin\omega\tau(\xi)(\alpha^2+\omega^2+\mu\alpha) - (1 - e^{-\alpha\tau(\xi)}\cos\omega\tau(\xi))\mu\omega\right].
\end{aligned}$$

Equating the real and imaginary parts yields

$$\begin{aligned}
\alpha + \gamma &= \frac{A}{\alpha^2+\omega^2}\left[(1 - e^{-\alpha\tau(\xi)}\cos\omega\tau(\xi))(\alpha^2+\omega^2+\mu\alpha) + e^{-\alpha\tau(\xi)}\sin\omega\tau(\xi)\mu\omega\right], \\
\omega &= \frac{A}{\alpha^2+\omega^2}\left[e^{-\alpha\tau(\xi)}\sin\omega\tau(\xi)(\alpha^2+\omega^2+\mu\alpha) - (1 - e^{-\alpha\tau(\xi)}\cos\omega\tau(\xi))\mu\omega\right].
\end{aligned}$$

Isolating $e^{-\alpha\tau(\xi)}\sin\omega\tau(\xi)$ and $1 - e^{-\alpha\tau(\xi)}\cos\omega\tau(\xi)$ respectively gives

$$A\left(1 - e^{-\alpha\tau(\xi)}\cos\omega\tau(\xi)\right)\left[(\alpha+\mu)^2+\omega^2\right] = (\alpha+\gamma)(\alpha^2+\omega^2+\mu\alpha) - \mu\omega^2, \tag{5.36}$$

$$Ae^{-\alpha\tau(\xi)}\sin\omega\tau(\xi)\left[(\alpha+\mu)^2+\omega^2\right] = \omega\Big((2\alpha+\gamma)\mu + \alpha^2 + \omega^2\Big). \tag{5.37}$$

Notice that since $A < 0$, for $\alpha \geq 0$ the left-hand side of (5.36) is non-positive. On the other hand, when

$$\gamma > \mu \tag{5.38}$$

and $\alpha \geq 0$ then the right-hand side of (5.36) is strictly positive. Consequently if (5.38) holds, then all the characteristic values must have $\alpha < 0$ and the steady state is asymptotically stable.

Next we show that the steady state is also stable if $\gamma \leq \mu$ and

$$\left|\frac{\xi v'(\xi)}{v(\xi)}\right| < \frac{1}{\mu\tau(\xi)}. \tag{5.39}$$

To do so, for contradiction suppose that $\lambda = \alpha + i\omega$ is a characteristic value with $\alpha \geq 0$ and $\omega > 0$ where $\gamma \leq \mu$ and (5.39) holds. Then $|\sin\omega\tau(\xi)| \leq |\omega\tau(\xi)|$ and (5.39) implies

$$\left|\frac{\xi v'(\xi)}{v(\xi)}\sin\omega\tau(\xi)\right| < \frac{\omega}{\mu}. \tag{5.40}$$

Consequently, using (5.34),

$$\begin{aligned}\left|Ae^{-\alpha\tau(\xi)}\sin\omega\tau(\xi)\left[(\alpha+\mu)^2+\omega^2\right]\right| &< \frac{\gamma\omega}{\mu}\left((\alpha+\mu)^2+\omega^2\right)\\ &= \omega\Big((2\alpha+\mu)\gamma + \frac{\gamma}{\mu}(\alpha^2+\omega^2)\Big)\\ &\leq \omega\big((2\alpha+\gamma)\mu + (\alpha^2+\omega^2)\big).\end{aligned}$$

However, this contradicts the assumption that $\lambda = \alpha + i\omega$ satisfies (5.37). We thus conclude that the steady state is asymptotically stable whenever (5.38) or (5.39) is satisfied.

Since the right-hand side of (5.39) is bounded below by $1/\mu\tau^+$, it follows from Proposition 5.1(1) that the steady state is stable for $0 < \xi \ll \theta_v$ and for $\xi \gg \theta_v$.

To determine when the steady state may be unstable we investigate the basic spectral condition for Hopf bifurcation, namely, the existence of a pair of complex conjugate eigenvalues on the imaginary axis. A proof that Hopf bifurcations actually occur for our equations would, of course, require in addition that a pair of eigenvalues crosses the imaginary axis at nonzero speed, that a nonresonance condition is satisfied, and furthermore that the right hand side of the delay differential equation has certain higher order smoothness properties, see [14, 30, 53] for the case of state-dependent delays.

Assume that $\lambda = \pm i\omega$, $\omega > 0$ solves equation (4.10). Then with $\alpha = 0$ equations (5.36) and (5.37) reduce to

$$A(1-\cos(\omega\tau))(\omega^2+\mu^2) = \omega^2(\gamma-\mu), \tag{5.41}$$

$$A\sin(\omega\tau)(\omega^2+\mu^2) = \omega(\omega^2+\gamma\mu). \tag{5.42}$$

Since $A < 0$ and the right-hand side of (5.42) is positive, at any Hopf bifurcation we must have $\sin(\omega\tau) < 0$ to satisfy (5.42). Moreover, the left-hand side of (5.41) is negative, and so a Hopf bifurcation is only possible if the right-hand side is also negative, that is if

$$\gamma < \mu. \tag{5.43}$$

In Section 5.1 it was so simple to rewrite (5.5) and (5.6) as (5.16) and (5.17) that we did so without comment. Equation (5.17) involves the derivative $g'(\xi)$; here the analogous term is $v'(\xi)$ which is part of A. We want to rewrite (5.41) and (5.42) as one equation for ω which is

independent of $v'(\xi)$ and one equation for $v'(\xi)$ which contains no trigonometric functions. To accomplish this we make use of half-angle formulae.

Let $U = \omega\tau/2$ then

$$\tan\frac{\omega\tau}{2} = \tan U = \frac{2\sin^2 U}{2\sin U\cos U} = \frac{1-\cos(\omega\tau)}{\sin(\omega\tau)},$$

and hence using (5.42) and (5.41)

$$\tan\frac{\omega\tau}{2} = \frac{\omega(\gamma-\mu)}{\omega^2+\gamma\mu}. \tag{5.44}$$

We next simplify (5.42) using another half-angle formula. Still with $U = \omega\tau/2$, from the standard formula

$$\sin\omega\tau = \frac{2\tan U}{1+\tan^2 U},$$

on substituting for $\tan U$ from (5.44) we obtain

$$\sin\omega\tau = \frac{2\omega(\gamma-\mu)(\omega^2+\gamma\mu)}{(\omega^2+\gamma^2)(\omega^2+\mu^2)}.$$

Substituting this into (5.42), rearranging and using (5.34) gives

$$\frac{\xi v'(\xi)}{v(\xi)} = \frac{\omega^2+\gamma^2}{2\gamma(\gamma-\mu)}. \tag{5.45}$$

At a Hopf bifurcation equations (5.2), (5.41) and (5.42) must all be satisfied. This is equivalent to solving

$$\gamma = \frac{\beta g^{\pm}}{\xi}e^{-\mu\tau(\xi)}, \tag{5.46}$$

along with (5.44) and (5.45).

We will follow similar steps as in Section 5.1, and consider (5.46), (5.44) and (5.45) sequentially, using the first two equations to define γ and ω_k as functions of ξ, so that it only remains to solve (5.45) for ξ. But because of the state-dependent delay and the constraint (5.43) the situation is not as simple as in the constant delay case considered in the previous two sections.

Note first that for any $\xi > 0$ equation (5.46) gives a unique value of $\gamma = \gamma(\xi)$. Moreover, since as already noted, τ is monotonically increasing it follows that γ is a monotonically decreasing function of ξ.

For γ satisfying (5.43), the right-hand side of (5.44) is negative. Then because of the properties of the tan function in (5.44) there will be at least one solution $\omega\tau$ to (5.44) satisfying $\omega\tau \in ((2k+1)\pi, (2k+2)\pi)$ for $k = 0, 1, 2, \ldots$. We denote by ω_k any solution of (5.44) for which $\omega_k\tau(\xi) \in ((2k+1)\pi, (2k+2)\pi)$.

At this point, we have defined γ and ω_k as functions of ξ. We still need to solve for ξ, or ξ_k from (5.45). We already considered the behaviour of the left-hand side of (5.45) in Proposition 5.1, so we now consider the behaviour of the right-hand side. Let

$$r_k(\gamma) = \frac{\omega_k^2+\gamma^2}{2\gamma(\gamma-\mu)}. \tag{5.47}$$

Usually we will take $\gamma = \gamma(\xi)$, defined by (5.46), but in the proposition below we consider $r_k(\gamma)$ for general γ. Note that a solution of (5.45) corresponds to a solution of $f(\xi/\theta_v, m, v^-/v^+) = r_k(\gamma(\xi))$, and since v is decreasing both f and r_k will be negative at such a solution, thus from (5.47) we require $\gamma < \mu$.

Proposition 5.5. *Let $r_k(\gamma)$ be defined by (5.47) and ω_k satisfy $\omega_k\tau \in ((2k+1)\pi, (2k+2)\pi)$ for integer $k \geq 0$. Then for $\gamma \in (0, \mu)$ it holds that*

1. $$|r_k(\gamma)| \geq 2\left(\frac{\omega_k}{\mu}\right)^2.$$
2. *$r_k(\gamma) \to -\infty$ as $\gamma \to 0$ or $\gamma \to \mu$,*
3. $$|r_k(\gamma)| \leq \frac{(2k+2)^2(\pi/\tau^-)^2 + \gamma^2}{2\gamma(\mu - \gamma)}.$$

Proof. The proof is elementary. □

While there is a unique steady state ξ for the case of decreasing v with constant g, just as in Section 5.1 the location and properties of this steady state will depend on the values of the other parameters. The following theorem collects together our results for this case.

Theorem 5.6. *Let ξ be the steady state of the DDE (1.1), (1.2) with $g(\xi) = g^{\pm}$ constant, and v monotonically decreasing, so $v^- > v^+$ and the state-dependent delay τ defined by (1.3) evaluated at the steady state is $\tau(\xi) = a/v(\xi)$. Then*

1. *If $\left|\frac{\xi v'(\xi)}{v(\xi)}\right| < \frac{1}{\mu\tau(\xi)}$ then the steady state ξ is asymptotically stable.*
2. *The steady state ξ is asymptotically stable if $m \leq 1/(\mu\tau(\xi))$, or if both $m > 1/(\mu\tau(\xi))$ and $r = r_v = \frac{v^-}{v^+} < \left(1 + \frac{2}{m\mu\tau(\xi) - 1}\right)^2$.*
3. *For any fixed $m > 0$, and for $0 < \xi \ll \theta_v$ or $\xi \gg \theta_v$, or equivalently for $\gamma \gg \gamma_4$ or $0 < \gamma \ll \gamma_3$, the steady state is asymptotically stable.*
4. *If $\gamma > \mu$ the steady state is asymptotically stable.*
5. *For any fixed $\xi \neq \theta_v$ let $\gamma = \gamma(m, \xi)$ be the value of γ such that (5.2) is satisfied and hence ξ is a steady state. Or, for any fixed γ with $0 < \gamma < \gamma_3$ or $\gamma > \gamma_4$ let $\xi = \xi(m, \gamma)$ satisfy (5.2) and hence be a steady state. Then ξ is asymptotically stable for all m sufficiently large.*
6. *Let $\mu > \gamma_3$.*
   (a) *For any $k > 0$, for all $m = m(k)$ sufficiently large there are two families of (at least) $k+1$ Hopf bifurcations as γ is varied. In the first family, the characteristic values $\lambda_j = \alpha_j \pm i\omega_j$ with $\omega_j\tau \in ((2j+1)\pi, (2j+2)\pi)$ for $j = 0, 1, \ldots, k$ cross the imaginary axis from left to right as γ increases, while in the second family they cross the imaginary axis from right to left.*
   (b) *Let $\gamma \in (\gamma_3, \min\{\gamma_4, \mu\})$ be fixed. Then as m is increased there is an infinite sequence of Hopf bifurcations where the real part of $\lambda_k = \alpha_k \pm i\omega_k$ becomes positive with $\omega_k\tau \in ((2k+1)\pi, (2k+2)\pi)$.*

Proof. Statements (1) and (4) were already shown; see equations (5.34), (5.39) and (5.38).

Claim (2) follows from (1) using Proposition 5.1, since

$$\left|\frac{\xi v'(\xi)}{v(\xi)}\right| \leq |f(r_v^{1/2m}, m, r_v)| = \frac{m|1 - r_v^{1/2}|}{1 + r_v^{1/2}}$$

where $r_v = v^-/v^+ > 1$.

Claim (3) also follows from (1), similarly to the proof of Theorem 5.3(3).

Statement (5) is derived similarly to Theorem 5.3 (4), by using (5.46) and (5.33) to show that $e^{-\mu\tau^+} < e^{-\mu\tau(\xi)} < e^{-\mu\tau^-}$ implies that $\gamma_3/\gamma < \xi/\theta_v < \gamma_4/\gamma$.

Finally, (6) is more delicate to prove. As noted before Proposition 5.5, to find a Hopf bifurcation we need to solve $f(\xi/\theta_v, m, v^-/v^+) = r_k(\gamma(\xi))$. Since for $\xi \neq \theta_v$ the function $f(\xi/\theta_v, m, v^-/v^+) \to 0$ as $m \to \infty$, we begin by considering $\xi = \theta_v$. Recall that $\gamma(\xi)$ defined by (5.46) is a monotonically decreasing function of ξ. Moreover, using (1.3) we see that $\tau(\theta_v) = 2a/(v^- + v^+) \in (\tau^-, \tau^+)$, which is independent of the value of m. Then

$$\gamma(\theta_v) = \frac{\beta g^{\pm}}{\theta_v} e^{-\mu\tau(\theta_v)} \in (\gamma_3, \gamma_4), \tag{5.48}$$

and $\gamma(\theta_v)$ is also independent of the value of m.

Now there are two cases to consider. First consider the case where $\gamma(\theta_v) < \mu$. If

$$m > 2\frac{(v^-/v^+)+1}{(v^-/v^+)-1}\left(\frac{(2k+2)^2(\pi/\tau(\theta_v))^2 + \gamma(\theta_v)^2}{2\gamma(\theta_v)(\mu - \gamma(\theta_v))}\right), \tag{5.49}$$

then

$$\begin{aligned} f(1, m, v^-/v^+) = -\frac{m(v^-/v^+ - 1)}{2(v^-/v^+ + 1)} &< \left(\frac{(2k+2)^2(\pi/\tau(\theta_v))^2 + \gamma(\theta_v)^2}{2\gamma(\theta_v)(\gamma(\theta_v) - \mu)}\right) \\ &\leq r_j(\gamma(\theta_v)), \quad j = 0, 1, \ldots, k. \end{aligned}$$

Here the equality comes from definition of f, the strict inequality from (5.49) and the last inequality follows from a similar argument that proves Proposition 5.5(3), the only difference being that here we use the actual value of $\tau(\theta_v)$ in the inequality, rather than the bound τ^-.

With this inequality as the starting point, we examine what happens when we increase ξ away from θ_v. If ξ is increased then $\gamma(\xi)$ decreases with $\lim_{\xi\to\infty}\gamma(\xi) = 0$. But $r_j(\gamma(\xi))$ and $f(\xi/\theta_v, m, v^-/v^+)$ are both continuous functions of ξ with $\lim_{\xi\to\infty} r_j(\gamma(\xi)) = -\infty$ and $\lim_{\xi\to\infty} f(\xi/\theta_v, m, v^-/v^+) = 0$. Consequently for each $j = 0, 1, \ldots, k$ there exists a ξ such that $f(\xi/\theta_v, m, v^-/v^+) = r_j(\gamma(\xi))$.

If instead ξ is decreased from θ_v then $\lim_{\xi\to 0}\gamma(\xi) = +\infty$, so $\gamma(\xi) > \mu$ for ξ sufficiently small. However, for $\gamma(\xi) < \mu$ we have $\lim_{\gamma\nearrow\mu} r_j(\gamma) = -\infty$, while $f(\xi/\theta_v, m, v^-/v^+)$ is bounded, so again for each $j = 0, 1, \ldots, k$ there exists a ξ such that $f(\xi/\theta_v, m, v^-/v^+) = r_j(\gamma(\xi))$.

Solutions of this equation define the Hopf bifurcation points, which gives the required Hopf bifurcations when $\gamma(\theta_v) < \mu$. To summarize the argument up to this point, for a fixed k and any large enough $m = m(k)$ satisfying (5.49), we found two families of $k+1$ Hopf bifurcations, one for $\xi_j < \theta_v$ and one for $\xi_j > \theta_v$, by finding appropriate $\gamma(\xi_j)$ that satisfy $f(\xi_j/\theta_v, m, v^-/v^+) = r_j(\gamma(\xi_j))$ for each $j = 0, \ldots, k$. See Figure 5.7(d) for illustration of these families as functions of the parameters m and γ.

Now consider the more delicate case where $\gamma_3 < \mu \leq \gamma(\theta_v) < \gamma_4$. The above argument fails in that case as $|f(\xi/\theta_v, m, v^-/v^+)| \gg 0$ for $\xi = \theta_v$ but the corresponding γ is $\gamma(\theta_v)$, with $\gamma(\theta_v) > \mu$, and by (4) the steady state would be asymptotically stable. Instead, noting that $\gamma(\xi)$ defined by (5.46) is monotonically decreasing, this function is invertible and we can instead consider $\xi = \xi(\gamma)$ as a function of γ. Fix $\gamma \in (\gamma_3, \mu)$, and consider the behaviour as $m \to \infty$. In this case the function $v(\xi)$ defined by (1.3) approaches the piecewise constant function (1.7), and the steady-state function $h(\xi)$ (recall (5.2)) approaches the case illustrated in Figure 5.5.

Since γ is fixed with $\gamma \in (\gamma_3, \mu) \subset (\gamma_3, \gamma_4)$, we find $\xi \to \theta_v$ and $v'(\xi) \to -\infty$ while $v(\xi)$ and ξ remain bounded and bounded away from zero. Consequently, for m sufficiently large $\frac{\xi v'(\xi)}{v(\xi)} < r_j(\gamma)$ for $j = 0, 1, \ldots, k$. From here the argument proceeds as in the case $\gamma(\theta_v) < \mu$. Statement (6)b also follows trivially in the case that $\gamma_3 < \mu \leq \gamma(\theta_v) < \gamma_4$. □

Although Theorem 5.6(6) is stated for one-parameter continuation in m or γ, we can also draw conclusions for two-parameter continuation of the Hopf bifurcations in the (γ, m)-parameter plane.

The argument used in the proof of Theorem 5.6(6) in the case when $\mu \in (\gamma_3, \gamma_4)$ shows that for any $\gamma_- > \gamma_3$ the k-th Hopf bifurcation occurs for $\gamma \in (\gamma_3, \gamma_-)$ for all m sufficiently large. Then because of Theorem 5.6(5) the k-th Hopf bifurcation approaches $\gamma = \gamma_3$ as $m \to \infty$, and in the (γ, m)-parameter plane the left side of the Hopf bifurcation curves asymptote to $\gamma = \gamma_3$ as $m \to \infty$. Similarly for any $\gamma_+ < \mu$ the other instance of the k-th Hopf bifurcation occurs for $\gamma \in (\gamma_+, \mu)$ for all m sufficiently large, and the right side of the Hopf bifurcation curves asymptote to $\gamma = \mu$ as $m \to \infty$.

In the case that $\mu > \gamma_4$, a similar argument can be applied to show that the k-th Hopf bifurcation curve asymptotes to $\gamma = \gamma_3$ and $\gamma = \gamma_4$ in the (γ, m)-parameter plane.

There are nevertheless differences between the cases where γ converges to γ_3 or γ_4 as $m \to \infty$ and the case where $\gamma \to \mu$ as $m \to \infty$. To see this consider for fixed k the limit as $m \to \infty$ when v approaches the piecewise constant function (1.7). Then from Theorem 5.6, we have $\xi \to \theta_v$, and $\gamma \in [\gamma_3, \min\{\gamma_4, \mu\}]$. Since $\tau(\xi) \in [a/v_U, a/v_L]$ and $\omega_k \tau(\xi) \leq (2k+2)\pi$ it also follows that ω_k is bounded. Thus the numerator of the right-hand side of (5.45) also remains bounded. Now there are two cases to consider.

First suppose that as $m \to \infty$ and $\xi \to \theta_v$ that $v'(\xi)$ becomes unbounded, that is $v'(\xi) \to -\infty$, or equivalently that $A \to -\infty$. Then the left-hand side of (5.45) becomes unbounded in the limit as $m \to \infty$. Since the numerator of the right-hand side is bounded, we must have $\gamma - \mu = \mathcal{O}(1/A) \to 0$ to satisfy equation (5.45). To summarize, if $v'(\xi) \to -\infty$ as $m \to \infty$ we must have that $\gamma \to \mu$ in this limit.

On the other hand, if $A < 0$ remains finite as $m \to \infty$, because γ, ξ and $v(\xi)$ are bounded and bounded away from zero in the limit, the only possibility in (5.34) is that $v'(\xi)$ also remains finite. But as $m \to \infty$ the function $v(\xi)$ approaches a step function, and the only places where $v'(\xi)$ is non-zero and finite are near the corners of the limiting velocity nonlinearity. Consequently, the only possibility for a Hopf bifurcation to exist for arbitrary m is that the steady state at which this Hopf bifurcation happens converges to the corners of the limiting velocity nonlinearity. That is, in the limit as $m \to \infty$ with $A < 0$ finite, we must have that $\gamma \to \gamma_3$ or $\gamma \to \gamma_4$ with $\xi \to \theta_v$, where γ_3 and γ_4 are defined by (5.33).

Below we illustrate the different possible behaviours allowed by Theorem 5.6 in the three cases: $\mu < \gamma_3$, $\mu \in (\gamma_3, \gamma_4)$ and $\mu > \gamma_4$.

Case 1: We begin with the case $\mu < \gamma_3$. By Theorem 5.6 (point 4) the steady state must be stable whenever $\gamma > \mu$, while for $\gamma \leq \mu$ we have $\gamma \leq \mu < \gamma_3$ so by Theorem 5.6(point 5) the steady state is stable for all m sufficiently large, or by Theorem 5.6 (point 3) it is stable for all γ sufficiently small.

Figure 5.6 shows an example of the behaviour of (1.1)–(1.2) for $(g \leftrightarrow, v \downarrow)$ with $\mu < \gamma_3 < \gamma_4$. Panel (b) shows the smooth case for several different values of m, which reveals that the steady state is always stable. Panel (a) shows the behavior in the limiting case with (1.7). In this case the singular steady state can only become a stable steady state for large finite m.

In the model (1.1)–(1.2) we consider the parameters γ and μ to be independent, but de-

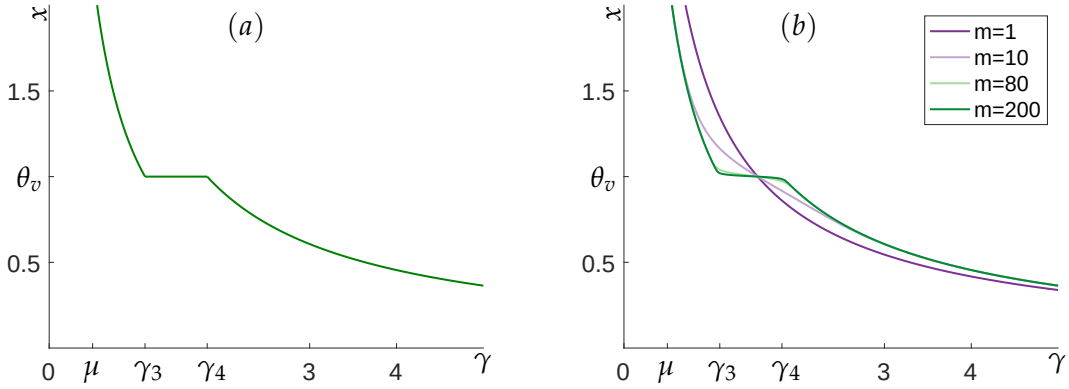


Figure 5.6: Bifurcation diagram of (1.1)–(1.2) with $(g \leftrightarrow, v \downarrow)$ and parameters $\beta = 3, \mu = 0.5, g^- = g^+ = 1, \gamma = 1, \theta_v = 1, a = 2, v^- = 2$ and $v^+ = 1$. (a) The limiting case with v defined by (1.7). The stable steady state is shown as a green solid line. (b) With smooth velocity nonlinearity v defined by (1.3) with $m = 1$, 10, 80, 200 as indicated by color. The steady state is always stable.

pending on how the model is derived, that may not be the case. For example, the current model (1.1)–(1.2) can be derived as a reduction of the operon model in [18]. In the model in [18] the parameter equivalent to γ is an effective removal rate which is the sum of the actual degradation rate with the dilution because of the growth rate μ. Consequently, in the model of [18], and in similar systems modelling gene regulatory dynamics in a growing cell, we obtain the natural parameter constraint $\gamma > \mu$, and expect dynamics corresponding to Figure 5.6.

Case 2: Next, in Figure 5.7, we illustrate the behavior of (1.1)–(1.2) for $(g \leftrightarrow, v \downarrow)$ with $\mu \in (\gamma_3, \gamma_4)$. Panel (a) depicts the behavior of the limiting case with piecewise constant v defined by (1.7). Panel (b) shows the case of the smooth velocity nonlinearity (1.3) with $m = 100$. The steady state close to $x = \theta_v$ undergoes a pair of Hopf bifurcations creating a bubble of stable periodic orbits which coexist with the unstable steady state for an interval of values of γ which is a subinterval of (γ_3, μ). As required by Theorem 5.6(4) the steady state is asymptotically stable for $\gamma > \mu$. Thus, in contrast to the previous case, the singular steady state may become either a stable or unstable steady state for very large finite values of m. Panel (c) is similar to panel (b) but for $m = 32$. In this case the steady state is always stable, and no Hopf bifurcations are seen.

Figure 5.7(d) shows a two-parameter continuation of the first three Hopf bifurcations illustrating Theorem 5.6(4-6). The steady state is stable below and outside the outermost Hopf curve and unstable otherwise, and Hopf bifurcations appear sequentially for increasing values of m. The bifurcation curve of first Hopf bifurcation (corresponding to $k = 0$ in the analysis above) is clearly seen to asymptote to $\gamma = \gamma_3$ and $\gamma = \mu$ as $m \to \infty$. The subsequent Hopf bifurcations also approach these limits as m becomes larger, but do so more slowly.

For the parameter values shown in Figure 5.7 the inequalities $\gamma_3 < \gamma(\theta_v) < \mu < \gamma_4$ hold. Consequently, (5.49) applies and gives a sufficient condition on m to ensure that the k-th Hopf bifurcation arises. For the given parameters (see the caption of Figure 5.7) these sufficient conditions are approximately $m > 84$, 316 and 703 for $k = 0, 1, 2$ respectively, whereas in Figure 5.7 the corresponding Hopf bifurcation curves have minimal m values of approximately $m = 34$, 147, and 336. Therefore, at least in this case, the sufficient condition is not tight. No Hopf bifurcations are seen for $m < 34$ in panel (d), which explains why no Hopf bifurcations

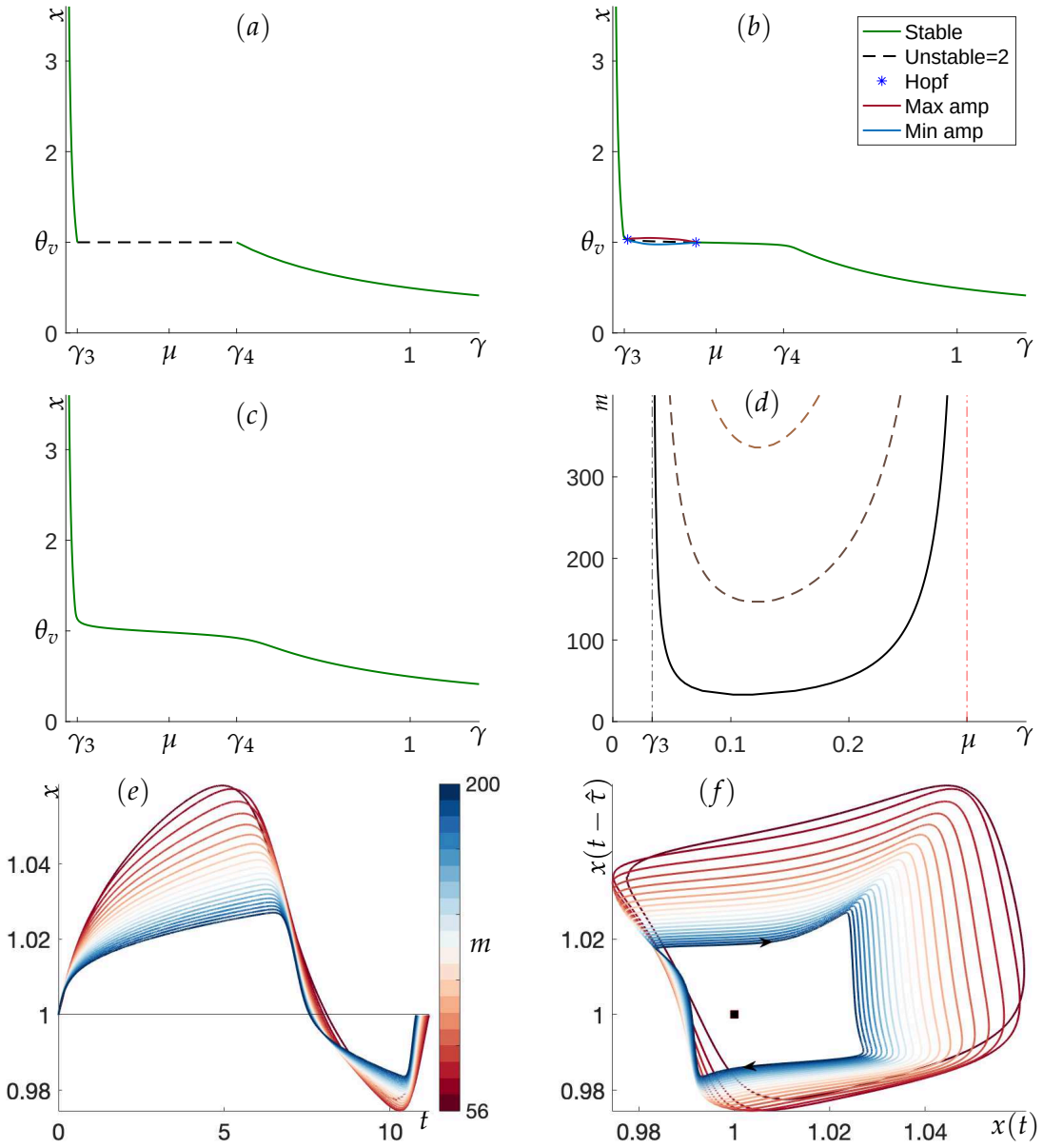


Figure 5.7: Bifurcation diagram of (1.1)–(1.2) with $(g \leftrightarrow, v \downarrow)$ and parameters $\beta = 3, \mu = 0.3, g^- = g^+ = 1, \gamma = 1, \theta_v = 1, a = 3, v^- = 0.5$ and $v^+ = 0.2$. (a) The limiting case with v defined by (1.7). The stable steady state is shown as a green solid line, and the singular steady state as a black dashed line. (b) With a smooth velocity nonlinearity v defined by (1.3) with $m = 100$. Solid lines represent stable objects including the stable steady state (in green), and envelope of the periodic orbit (in red and blue). Dashed lines represent unstable steady states which have two eigenvalues with positive real part (in black). (c) As in (b) but with $m = 32$. (d) Two-parameter continuations in m and γ of the Hopf bifurcations (shown as solid curves) with the other parameters as above. The outermost curve of Hopf bifurcations is associated with the stability change seen in (b). The black dash-dotted line denotes $\gamma = \gamma_3 = 0.0333$ and the red dash-dotted line denotes $\gamma = \mu = 0.3$, the location of the Hopf bifurcations in the limiting case as $m \to \infty$. Note that $\gamma(\theta_v) = 0.2293 < \mu < \gamma_4 = 0.4959$. (e) Profiles of the stable periodic orbits from the Hopf bifurcations in (b) at $\gamma = 0.13$ and $m \in [56, 200]$ as indicated by the color map. (f) Projection of the phase space dynamics into the $(x(t), x(t-\hat{\tau}))$ plane at $\gamma = 0.13$ where $\hat{\tau} = \tau(\theta_v) = 60/7$. The arrow indicates the direction of the flow. The square marks the unstable steady state in the limiting case at the threshold.

were detected for $m = 32$ in (c).

As m increases, the stable periodic orbits created in the Hopf bifurcation at $\gamma = 0.13 \in (\gamma_3, \mu)$ are shown in Figure 5.7(e) and (f). Panel (e) shows the profiles of these orbits, while in (f) we show their projection into the $(x(t), x(t-\hat{\tau}))$ plane. Note that since the delay τ is state-dependent we do not use the actual delay for this projection, but we took $\hat{\tau} = \tau(\theta_v)$. The limiting behaviour as $m \to \infty$ is quite different from the constant delay case considered in Section 5.1. Not only is the shape of the profile different, but comparing Figure 5.7(f) and Figure 5.2(f) we see that the direction of rotation of the orbits, as indicated by the arrows, is reversed in the two examples.

Case 3: The final case to consider for (1.1)–(1.2) with $(g \leftrightarrow, v \downarrow)$ is when $\gamma_3 < \gamma_4 < \mu$, and this is illustrated in Figure 5.8. Panels (a)–(c) show the dynamics for the limiting case with the piecewise constant velocity nonlinearity (1.7) and for the smooth case with the Hill function from (1.3) with $m = 250$ and $m = 100$ respectively. For the limiting case in (a) the singular steady state with $x = \theta_v$ appears as a straight line segment for $\gamma \in (\gamma_3, \gamma_4)$. The singular steady states become unstable steady states in the smooth case as shown in (b), and there is a bubble of stable periodic orbits between the pair of Hopf bifurcations.

Figure 5.8(d) shows the locus of the principal Hopf bifurcation for a two-parameter continuation in m and γ. Below and outside the curve of Hopf bifurcations the steady state is stable, while between the Hopf bifurcations it is unstable. We see that the Hopf bifurcation curves approach γ_3 and γ_4 as $m \to \infty$, as follows from Theorem 5.6. Since $\gamma_4 < \mu$ and necessarily $\gamma(\theta_v) \in (\gamma_3, \gamma_4)$ we are guaranteed that $\gamma(\theta_v) < \mu$, so for all such examples Theorem 5.6 guarantees that there are infinitely many Hopf bifurcations. However, they may occur for very large values of m. For the example depicted in Figure 5.8, only one Hopf bifurcation is observed with $m \geq 117$. For the parameter values of this example the sufficient condition (5.49) (with $k = 0, 1$) ensures a first Hopf bifurcation for $m > 152$, while a second one is ensured for $m > 602$, which is outside the parameter range considered in Figure 5.8.

If the value of the Hill coefficient m is reduced sufficiently then there are no longer any Hopf bifurcations and the steady state remains stable, as seen in Figure 5.8(c). Interestingly, even though there are no longer any Hopf bifurcations in this case, the dependence of the steady state x on γ is still very similar to (a) and (b), with an obvious plateau visible in the graph of x in Figure 5.8(c) for $\gamma \in (\gamma_3, \gamma_4)$.

Figure 5.8(e) and (f) respectively show the profiles and the $(x(t), x(t-\hat{\tau}))$-space projections of the stable periodic orbits at $\gamma = 0.4$ as m increases, where we choose $\hat{\tau} = \tau(\theta_v)$. The periodic orbits approach a certain structure as $m \to \infty$. Note that the direction of rotation in both Figure 5.8(f) and Figure 5.7(f) is clockwise, while in Figure 5.2(f) it is counterclockwise.

5.4 State-dependent delay with v increasing and g constant $(g \leftrightarrow, v \uparrow)$

Here we again consider the model (1.1)–(1.2) with a constant function $g(\xi) = g^{\pm}$ and state-dependent delay, but now we consider the case of increasing v, so $v^- < v^+$ in (1.3) and (1.7): $(g \leftrightarrow, v \uparrow)$.

As illustrated in Figure 5.9, it is possible for up to three steady states to coexist due to the fold bifurcations at the corners associated with $\gamma = \gamma_3$ and $\gamma = \gamma_4$. Note that in Section 5.3 we had $\gamma_3 < \gamma_4$. Since the function v is increasing in this section, it follows from the definition (5.33) that now $\gamma_4 < \gamma_3$. In the limiting case where v is defined in (1.7), at the non-singular steady states the characteristic equation is of the form (4.11), and hence these steady states are stable. For $\gamma \in (\gamma_4, \gamma_3)$ the two stable steady states co-exist with a singular steady state, at

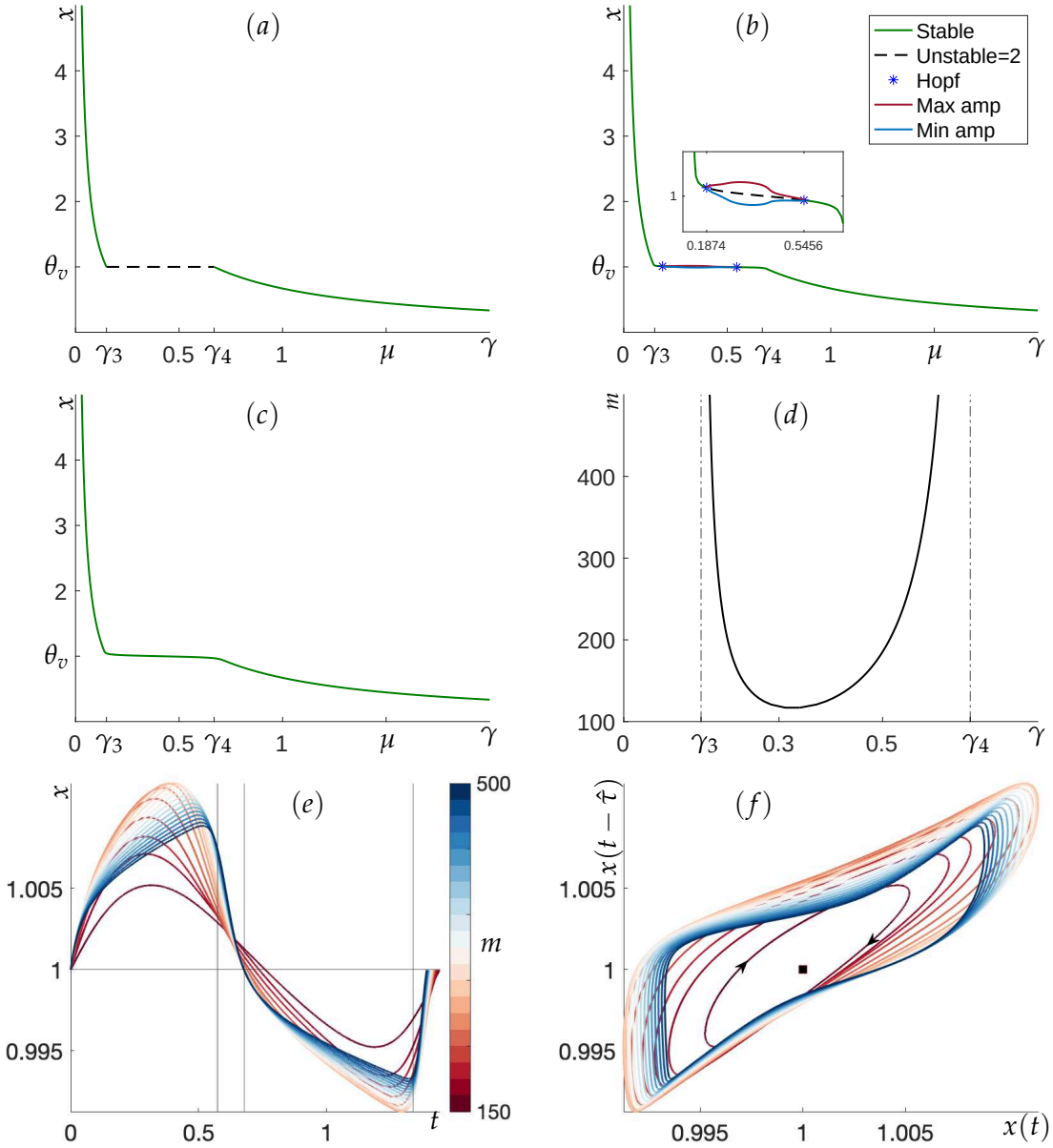


Figure 5.8: Bifurcation diagram of (1.1)–(1.2) with $(g \leftrightarrow, v \downarrow)$ and parameters $\beta = 3, \mu = 1.5, g^- = g^+ = 1, \gamma = 1, \theta_v = 1, a = 2, v^- = 2$ and $v^+ = 1$. (a) The limiting case with v defined by (1.7). The stable steady state is shown as a green solid line, and the singular steady state as a black dashed line. (b) With a smooth velocity nonlinearity v defined by (1.3) with $m = 250$. Solid lines represent stable objects including the stable steady state (in green) and a stable limit cycle (represented by maximum and minimum of $x(t)$ on the periodic solution). Dashed lines represent unstable steady states which have two eigenvalues with positive real part (in black). (c) As in (b) but with $m = 100$. (d) Two-parameter continuations in m and γ of the Hopf bifurcations seen in (b). The dash-dotted lines denote $\gamma = \gamma_3 = 0.1494$ and $\gamma = \gamma_4 = 0.6694$, the location of the Hopf bifurcations in the limiting case as $m \to \infty$. (e) Profile of the stable periodic orbits from the Hopf bifurcations in (b) at $\gamma = 0.4$. The color map indicates values of the continuation parameter m. (f) Projection of the phase space dynamics into the $(x(t), x(t-\hat{\tau}))$ plane at $\gamma = 0.4$ where $\hat{\tau} = \tau(\theta_v) = 4/3$. The arrow indicates the direction of the flow. The square marks the unstable steady state in the limiting case at the threshold.

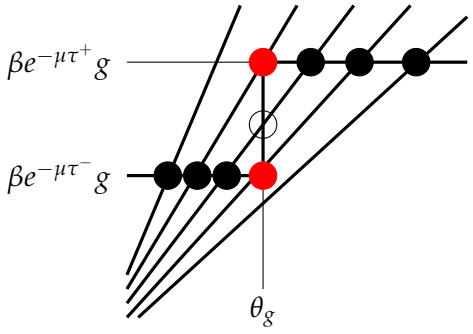


Figure 5.9: Illustration of how the number of steady states of (1.1), which are given by (5.2), changes with the intersections of $\xi \mapsto \beta e^{-\mu\tau(\xi)} g$ and $\xi \mapsto \gamma\xi$. These are shown in the limiting case of (1.6) and (1.7) with $g^{\pm} = g^- = g^+$, so $g(\xi) = g^{\pm}$ is a constant function, and $v^- < v^+$ so v is piecewise constant and monotonically increasing: $(g \leftrightarrow , v \uparrow)$. Then $\tau(\xi) = a/v(\xi)$ is state-dependent; $\tau(\xi) = \tau^- = a/v^-$ for $\xi < \theta_v$, $\tau(\xi) = \tau^+ = a/v^+$ for $\xi > \theta_v$ and $\tau(\xi)$ is set-valued when $\xi = \theta_v$.

which the characteristic function is not defined. This leads us to consider the smooth velocity nonlinearity (1.3).

Because the smooth velocity nonlinearity approaches the step function shown in Figure 5.9 as $m \to \infty$, it follows that for all m sufficiently large there will be a pair of fold bifurcations at $(\xi, \gamma) = (\xi^-, \gamma(\xi^-))$ and $(\xi, \gamma) = (\xi^+, \gamma(\xi^+))$, with $(\xi^-, \gamma(\xi^-)) \to (\theta_v, \gamma_4)$ and $(\xi^+, \gamma(\xi^+)) \to (\theta_v, \gamma_3)$ as $m \to \infty$. To study the associated bifurcations consider the characteristic equation which has the form (4.10). Writing the characteristic function $\Delta(\lambda)$ as

$$\Delta(\lambda) = \lambda + \gamma - A + Ae^{-\lambda\tau(\xi)} - \mu A \int_{-a/v(\xi)}^{0} e^{\lambda s} ds, \tag{5.50}$$

it follows from (5.2) that at a steady state

$$\begin{aligned} \Delta(0) &= \gamma - \mu A \int_{-a/v(\xi)}^{0} 1 ds = \gamma - \mu\tau(\xi)A = \gamma - \mu\tau(\xi)\gamma\frac{\xi v'(\xi)}{v(\xi)} \\ &= \gamma\mu\tau(\xi)\Big(\frac{1}{\mu\tau(\xi)} - \frac{\xi v'(\xi)}{v(\xi)}\Big). \end{aligned} \tag{5.51}$$

Hence, $\lambda = 0$ is a characteristic value if and only if

$$\frac{\xi v'(\xi)}{v(\xi)} - \frac{1}{\mu\tau(\xi)} = 0. \tag{5.52}$$

In addition, note that

$$\frac{\xi v'(\xi)}{v(\xi)} > \frac{1}{\mu\tau(\xi)} \tag{5.53}$$

implies that $\Delta(0) < 0$. At the same time, it is easily seen that for real λ the characteristic function satisfies $\Delta(\lambda) \to +\infty$ as $\lambda \to +\infty$. We conclude that when (5.53) is satisfied there is a real positive characteristic value and hence the steady state is unstable.

We now show that fold bifurcations occur when (5.52) is satisfied. To that end, consider the curve of solutions $(\gamma(\xi), \xi)$ where $\gamma(\xi)$ is defined by (5.46), which by (5.2) is the locus of steady states in (γ, ξ)-plane. Differentiating this relationship with respect to ξ we find

$$\gamma'(\xi) = -\frac{\gamma(\xi)}{\xi}\Big(1 + \mu\xi\tau'(\xi)\Big).$$

However, $\tau(\xi) = a/v(\xi)$ implies that

$$\tau'(\xi) = -\frac{av'(\xi)}{v(\xi)^2} = -\frac{v'(\xi)\tau(\xi)}{v(\xi)}. \tag{5.54}$$

Thus

$$\gamma'(\xi) = -\frac{\gamma(\xi)}{\xi}\Big(1 - \mu\xi\frac{v'(\xi)\tau(\xi)}{v(\xi)}\Big) = \frac{\mu\tau(\xi)\gamma(\xi)}{\xi}\Big(\frac{\xi v'(\xi)}{v(\xi)} - \frac{1}{\mu\tau(\xi)}\Big). \tag{5.55}$$

Hence $\operatorname{sgn}(\gamma'(\xi)) = \operatorname{sgn}\left(\frac{\xi v'(\xi)}{v(\xi)} - \frac{1}{\mu\tau(\xi)}\right)$, and $\gamma'(\xi)$ changes sign when the left-hand side of (5.52) changes sign, which ensures that fold bifurcations occur at these points in the (γ, ξ)-plane.

While the above analysis appears similar to the constant delay case considered in Section 5.2, the state-dependency of the delay introduces two significant differences.

First, with a state-dependent delay and increasing v, fold bifurcations occur when (5.52) is satisfied, whereas in Section 5.2 with increasing g we found fold bifurcations when (5.27) is satisfied. Thus for the constant delay case the fold bifurcations occur when the unimodal function $\xi \mapsto f(\xi/\theta_g, n, g^-/g^+) = \xi g'(\xi)/g(\xi)$ is equal to 1, and so there is at most one pair of fold bifurcations. In contrast, for the state-dependent delay case, fold bifurcations occur when the unimodal function (5.35) is equal to $1/\mu\tau(\xi)$. For increasing v, the function $\xi \mapsto 1/\mu\tau(\xi)$ is also an increasing function, thus the fold bifurcation occurs at the intersections of a unimodal function with an increasing function. Clearly, such functions may have multiple pairs of intersections and hence multiple pairs of fold bifurcations. From Proposition 5.1, for the Hill function $v(\xi)$ defined by (1.3) we have that $\xi \mapsto \xi v'(\xi)/v(\xi)$ is monotonically decreasing for $\xi > \theta_v(v^-/v^+)^{1/2m}$, so there will be at most one intersection with the increasing function $\xi \mapsto 1/\mu\tau(\xi)$ on this interval. Thus if there are additional fold bifurcations they must occur for $\xi < \theta_v(v^-/v^+)^{1/2m} < \theta_v$. In the current work, we will not look for these additional fold bifurcations, nor will we show that they cannot exist. Even if it were possible to show that they cannot exist when v is defined by (1.3), we would expect that additional fold bifurcations could be induced by suitable modifications to the velocity nonlinearity.

A second difference between the state-dependent and constant delay cases concerns the stability of the steady state near the fold bifurcation. Note that the argument after (5.39) in Section 5.3 still applies to show that the steady state is stable if $\gamma \leq \mu$ and (5.39) holds. Consequently, a fold bifurcation which occurs for $\gamma < \mu$ must involve a stable steady state which loses stability at the fold bifurcation. However, this argument does not apply when $\gamma > \mu$, and we will see examples where stable steady states lose stability in a Hopf bifurcation with $\gamma > \mu$, and also fold bifurcations where the steady state is unstable on both sides of the bifurcation (with different numbers of unstable characteristic values). This contrasts with the constant delay case in Section 5.2 where we showed that a stable steady state can only lose stability at a fold bifurcation.

As was the case when v was decreasing, Hopf bifurcations are governed by equations (5.41) and (5.42). However, since v is increasing we now have $A > 0$. At a Hopf bifurcation the right-hand side of (5.42) is strictly positive, and hence $\sin(\omega\tau) > 0$. Since $\sin(\omega\tau) > 0$ implies $\cos(\omega\tau) < 1$, it then follows that the left-hand side of (5.41) is also strictly positive, and a Hopf bifurcation is only possible if the right-hand side is also strictly positive, that is if

$$\gamma > \mu. \tag{5.56}$$

As in Section 5.3, we find Hopf bifurcations by sequentially solving (5.46), (5.44) and (5.45). Equation (5.44) has infinitely many solutions $\omega_k(\gamma(\xi))$ for $k = 0, 1, \ldots$ with $\omega_k\tau \in$

$(2k\pi, (2k+1)\pi)$. Finally, to solve (5.45) with $v'(\xi) > 0$, we need to determine ξ that satisfies $f(\xi/\theta_v, m, v^-/v^+) = r_k(\gamma(\xi)) > 0$ (defined in (5.47)). Usually we will take $\gamma = \gamma(\xi)$, defined by (5.46), but in the proposition below we consider $r_k(\gamma)$ for general γ.

Proposition 5.7. *Let $r_k(\gamma)$ be defined by (5.47) and ω_k satisfy $\omega_k\tau \in (2k\pi, (2k+1)\pi)$ for integer $k \geq 0$. Then for $\gamma > \mu$ it holds that $r_k(\gamma) > 0$ and*

1. *$r_k(\gamma)$ is monotonically decreasing,*
2. *$r_k(\gamma) \to +\infty$ as $\gamma \to \mu$ and $r_k(\gamma) \to 1/2$ as $\gamma \to +\infty$,*
3. $$r_k(\gamma) \leq \frac{(2k+1)^2(\pi/\tau^+)^2 + \gamma^2}{2\gamma(\gamma - \mu)}.$$

Proof. The proof is elementary. □

The following lemma will be needed in the proof of Theorem 5.9.

Lemma 5.8. *For $x > 0$ let $f(x) = \frac{1}{x}(1 - e^{-x})$ then $f(x)$ is strictly monotonically decreasing for $x > 0$ with*

$$1 = \lim_{x\to 0} f(x) > \lim_{x\to+\infty} f(x) = 0. \tag{5.57}$$

Proof. The two limits in (5.57) are easily verified. To show the monotonicity of $f(x)$ first differentiate to obtain

$$f'(x) = -\frac{1}{x^2}\left[1 - (x+1)e^{-x}\right].$$

The assumption $f'(x) = 0$ for some $x > 0$ yields $e^x = 1 + x$ which is impossible for $x \neq 0$. Using continuity we infer that on $(0, \infty)$ there is no sign change of $f'(x)$, and that $f'(x)$ is strictly monotonic for $x > 0$. Because of (5.57) f is decreasing. □

The following theorem collects our results for the case of increasing v with constant g.

Theorem 5.9. *Let ξ be a steady state of the DDE (1.1), (1.2) with $g(\xi) = g^{\pm}$ constant, and v monotonically increasing, so $v^- < v^+$, and the state-dependent delay τ defined by (1.3) evaluated at the steady state is $\tau(\xi) = a/v(\xi)$. Then*

1. *If $\frac{\xi v'(\xi)}{v(\xi)} < \frac{1}{\tau(\xi)\max\{\mu,\gamma\}}$ then the steady state ξ is asymptotically stable, while if $\frac{\xi v'(\xi)}{v(\xi)} > \frac{1}{\mu\tau(\xi)}$ it is unstable.*
2. *For any fixed $\xi \neq \theta_v$ let $\gamma = \gamma(m, \xi)$ be the value of γ such that (5.2) is satisfied and hence ξ is a steady state. Or, for any fixed γ with $0 < \gamma < \gamma_4$ or $\gamma > \gamma_3$ let $\xi = \xi(m, \gamma)$ satisfy (5.2) and hence be a steady state. Then ξ is asymptotically stable for all m sufficiently large.*
3. *The steady state ξ is asymptotically stable if $m \leq 1/(\tau(\xi)\max\{\mu, \gamma\})$, or both $m > 1/(\tau(\xi)\max\{\mu, \gamma\})$ and $1 \geq \frac{v^-}{v^+} > \left(1 - \frac{2}{m\tau(\xi)\max\{\mu,\gamma\}+1}\right)^2$.*
4. *For any fixed $m > 0$, and for $\xi \gg \theta_v$, or equivalently for $0 < \gamma \ll \gamma_4$, the steady state is asymptotically stable. For any fixed $m > 1$, and for $0 < \xi \ll \theta_v$ or equivalently for $\gamma \gg \gamma_3$ the steady state is asymptotically stable.*

5. *If* $m > 1/(\mu\tau^+)$ *and* $\frac{v^-}{v^+} < \left(1 - \frac{2}{1+m\mu\tau^+}\right)^2$ *then there exist* $\xi^- < \theta_v(v^-/v^+)^{1/2m} < \xi^+$ *with* $\gamma(\xi^-) < \gamma(\xi^+)$ *such that as* γ *is varied there is a branch of steady states with* $\xi < \xi^-$ *and* $\gamma > \gamma(\xi^-)$ *and another branch with* $\xi > \xi^+$ *and* $\gamma < \gamma(\xi^+)$*. For* $\gamma \in (\gamma(\xi^-), \gamma(\xi^+))$ *these two branches of steady states co-exist with a branch of unstable steady states which exists between fold bifurcations at* $(\xi, \gamma) = (\xi^-, \gamma(\xi^-))$ *and* $(\xi, \gamma) = (\xi^+, \gamma(\xi^+))$*.*

6. *Let* $\mu < \gamma_3$*.*

   (a) *For any* $k > 0$*, for all* $m = m(k)$ *sufficiently large there are two families of (at least)* $k+1$ *Hopf bifurcations as* γ *is varied. In the first family, the characteristic values* $\lambda_j = \alpha_j \pm i\omega_j$ *with* $\omega_j\tau \in (2j\pi, (2j+1)\pi)$ *for* $j = 0, 1, \ldots, k$ *cross the imaginary axis from left to right as* γ *is increased, while in the second family they cross the imaginary axis from right to left. For all* j *sufficiently large these bifurcations occur on the branch of unstable steady states between the fold bifurcations identified in (5).*

   (b) *Let* $\gamma \in (\max\{\gamma_4, \mu\}, \gamma_3)$ *be fixed. Then as* m *is increased there is an infinite sequence of Hopf bifurcations where the real part of* $\lambda_k = \alpha_k \pm i\omega_k$ *becomes positive with* $\omega_k\tau \in ((2k\pi, (2k+1)\pi)$*. All but finitely many of these bifurcations are located on the branch of unstable steady states between the fold bifurcations.*

Proof. The last part of (1) is shown after (5.53). To establish asymptotic stability we show that $Re(\lambda) < 0$ for all of the characteristic values that solve (4.10). From $\xi v'(\xi)/v(\xi) < 1/\mu\tau(\xi)$ and (5.51) it follows immediately that $\lambda = 0$ is not a characteristic value. To show that $\lambda > 0$ is not a characteristic value, evaluating the integral in (5.50) we obtain

$$\Delta(\lambda) = \lambda + \gamma - (1 - e^{-\lambda\tau(\xi)})\frac{A}{\lambda}(\lambda + \mu).$$

But

$$A = \gamma\frac{\xi v'(\xi)}{v(\xi)} < \frac{\gamma}{\tau(\xi)\max\{\mu, \gamma\}},$$

hence

$$\Delta(\lambda) > \lambda + \gamma - \frac{1}{\lambda\tau(\xi)}(1 - e^{-\lambda\tau(\xi)})\frac{\gamma}{\max\{\mu, \gamma\}}(\lambda + \mu) > \lambda + \gamma - \frac{\gamma}{\max\{\mu, \gamma\}}(\lambda + \mu),$$

where the last inequality follows from Lemma 5.8. Now there are two cases to consider. If $\gamma > \mu$ then

$$\Delta(\lambda) > \lambda + \gamma - (\lambda + \mu) = \gamma - \mu > 0.$$

On the other hand, if $\gamma \le \mu$ then

$$\Delta(\lambda) > \lambda + \gamma - \frac{\gamma}{\mu}(\lambda + \mu) = \lambda\left(1 - \frac{\gamma}{\mu}\right) \ge 0.$$

In both cases the characteristic function satisfies $\Delta(\lambda) > 0$ for all $\lambda > 0$ so there are no real positive characteristic values.

To complete the proof of (1) it remains to show that there are no complex characteristic values $\lambda = \alpha + i\omega$ with $\alpha \ge 0$ and $\omega > 0$. For the case $\gamma \le \mu$ the proof is the same as in Section 5.3 (where (5.39) holds independent of the sign of $v'(\xi)$). For the remaining case, if

$\gamma > \mu$ the assumption of the theorem reads $\frac{\xi v'(\xi)}{v(\xi)} < \frac{1}{\gamma\tau(\xi)}$. Let $\lambda = \alpha + i\omega$ be a characteristic value with $\alpha \geq 0$ and $\omega > 0$ then

$$\begin{aligned}|Ae^{-\alpha\tau(\xi)}\sin\omega\tau(\xi)\left[(\alpha+\mu)^2+\omega^2\right]| &\leq A\omega\tau(\xi)\left[(\alpha+\mu)^2+\omega^2\right]\\ &= \gamma\frac{\xi v'(\xi)}{v(\xi)}\omega\tau(\xi)\left[(\alpha+\mu)^2+\omega^2\right]\\ &< \omega\left[(\alpha+\mu)^2+\omega^2\right]\\ &< \omega\Big((2\alpha+\gamma)\mu+\alpha^2+\omega^2\Big).\end{aligned}$$

Consequently (5.37) is violated, and so there is no such characteristic value. Thus $\alpha < 0$ for all characteristic values and the steady state is asymptotically stable.

Statement (2) is derived similarly to Theorem 5.3(4), by first using (5.46) and (5.33) to show that $e^{-\mu\tau^-} < e^{-\mu\tau(\xi)} < e^{-\mu\tau^+}$ implies that $\gamma_4/\gamma < \xi/\theta_v < \gamma_3/\gamma$, then applying Proposition 5.1 and (1).

Statement (3) follows from (1) using Proposition 5.1 (point 4), since

$$\frac{\xi v'(\xi)}{v(\xi)} \leq f(r_v^{1/2m}, m, r_v) = \frac{m(1-r_v^{1/2})}{1+r_v^{1/2}},$$

where $r_v = v^-/v^+ \in (0,1)$. Statement (4) also follows from (1), using Proposition 5.1 (point 1), similar to Theorem 5.3 and Theorem 5.6, where for the second part of (4), $m > 1$ ensures that

$$f(\xi/\theta_v, m, v^-/v^+)\gamma(\xi) = \frac{m(1-v^-/v^+)(\xi/\theta_v)^m}{\left(1+(\xi/\theta_v)^m\right)\left(v^-/v^+ + (\xi/\theta_v)^m\right)}\frac{\beta g^{\pm}}{\xi}e^{-\mu\tau(\xi)} \to 0 \quad \text{as } \xi \to 0.$$

To show (5), note that the parameter constraints ensure that $f(\xi/\theta_v, m, v^-/v^+) = \frac{\xi v'(\xi)}{v(\xi)} > \frac{1}{\mu\tau(\xi)}$ when $\xi = \theta_v(v^-/v^+)^{1/2m}$. Then since $f(\xi/\theta_v, m, v^-/v^+) \to 0$ as $\xi \to 0$ and as $\xi \to \infty$ it follows that there exists $\xi^+ > \theta_v(v^-/v^+)^{1/2m}$ and $\xi^- < \theta_v(v^-/v^+)^{1/2m}$ which both solve (5.52). We take ξ^- to be the largest ξ which solves (5.52) with $\xi^- < \theta_v(v^-/v^+)^{1/2m}$, while as discussed after (5.55), ξ^+ is unique. Then it follows from (5.52), (5.55) and the adjacent arguments that there is a pair of fold bifurcations at $(\xi,\gamma) = (\xi^-, \gamma(\xi^-))$ and $(\xi,\gamma) = (\xi^+, \gamma(\xi^+))$, which are connected by a branch of unstable steady states, and no other fold bifurcation (besides $\xi = \xi^+$) with $\xi > \xi^-$. For $\xi > \xi^+$, it follows from (5.55) that $\gamma'(\xi) < 0$ and hence this branch exists for $\xi \in (\xi^+, \infty)$ and $\gamma \in (0, \gamma(\xi^+))$. If $\gamma(\xi^+) < \mu$ then the whole of this branch of steady states is stable, otherwise by (1) it is stable for all ξ sufficiently large, given by condition $\gamma(\xi) < \mu$.

For the branch which exists for $\xi \in (0, \xi^-)$, it follows from (5.46) that $\gamma(\xi) \to +\infty$ as $\xi \to 0$. Hence $\gamma(\xi)$ takes all values in $[\gamma(\xi^-), +\infty)$ for $\xi \in (0, \xi^-)$. However, as discussed after (5.55), it is possible for this branch to have additional fold bifurcations. If there are no additional fold bifurcations and $\gamma(\xi^-) < \mu$ then the branch of steady states is stable for $\gamma \in (\gamma(\xi^-), \mu)$. Also, if $m > 1$ by (4) it is stable for all ξ sufficiently small, or, equivalently, for all $\gamma(\xi)$ sufficiently large.

To show (6), first consider the case when $\gamma(\theta_v)$ defined by (5.48) satisfies $\gamma(\theta_v) > \mu$. If

$$m > 2\frac{1+(v^-/v^+)}{1-(v^-/v^+)}\left(\frac{(2k+1)^2(\pi/\tau(\theta_v))^2+\gamma(\theta_v)^2}{2\gamma(\theta_v)(\gamma(\theta_v)-\mu)}\right), \tag{5.58}$$

then

$$f(1, m, v^-/v^+) = \frac{m(1 - v^-/v^+)}{2(1 + v^-/v^+)} > \left(\frac{(2k+1)^2(\pi/\tau(\theta_v))^2 + \gamma(\theta_v)^2}{2\gamma(\theta_v)(\gamma(\theta_v) - \mu)} \right)$$
$$\geq r_j(\gamma(\theta_v)), \quad j = 0, 1, \ldots, k.$$

Here the equality comes from the definition of f, the strict inequality from (5.58) and the last inequality follows from a similar argument as in the proof of Proposition 5.7 (point 3), but using the actual value of $\tau(\theta_v)$ in the inequality, rather than the bound τ^+.

With this inequality as the starting point, increasing ξ away from θ_v we have $\lim_{\xi\to\infty} \gamma(\xi) = 0$. But $r_j(\gamma(\xi))$ and $f(\xi/\theta_v, m, v^-/v^+)$ are both continuous functions of ξ with $\lim_{\xi\to\infty} r_j(\gamma(\xi)) = \infty$ and $\lim_{\xi\to\infty} f(\xi/\theta_v, m, v^-/v^+) = 0$. Consequently for each $j = 0, 1, \ldots, k$ there exists a ξ such that $f(\xi/\theta_v, m, v^-/v^+) = r_j(\gamma(\xi))$.

Similarly, for ξ decreasing from θ_v (see also the proof of Theorem 5.6) it follows for each $j = 0, 1, \ldots, k$ that there exists a ξ such that $f(\xi/\theta_v, m, v^-/v^+) = r_j(\gamma(\xi))$. As in the proof of Theorem 5.6 this defines the required Hopf points for the two families of $k+1$ Hopf bifurcations.

This argument fails in the more delicate case where $\gamma_4 < \gamma(\theta_v) \leq \mu < \gamma_3$. In this case fix $\gamma \in (\mu, \gamma_3)$, and consider the behaviour as $m \to \infty$. Since $\gamma(\xi^+) \to \gamma_4$ and $\gamma(\xi^-) \to \gamma_3$, it follows that $\gamma \in (\gamma(\xi^+), \gamma(\xi^-))$ for all m sufficiently large. Then since $\gamma(\xi)$ defined by (5.46) is monotonically increasing for $\xi \in (\xi^-, \xi^+)$, this function is locally invertible and we can instead consider $\xi = \xi(\gamma)$ as a function of γ. From here the argument proceeds as in the proof of Theorem 5.6(6). □

We now present several examples to illustrate the complex dynamics allowed by Theorem 5.9. There are essentially three main cases to consider depending on whether μ falls above, below or within the interval (γ_4, γ_3). As in Section 5.3 we start with the simplest case where there are no Hopf bifurcations.

Case 1: $\gamma_4 < \gamma_3 < \mu$. This is illustrated in Figure 5.10. Panel (a) shows one-parameter continuations of the steady-states in γ for several fixed values of m, along with the limiting case (1.7), while panel (b) shows the two-parameter continuation of fold bifurcations of steady-states in the (γ, m) plane. We see that for all m sufficiently large there is a pair of fold bifurcations at $(\xi, \gamma) = (\xi^-, \gamma(\xi^-))$ and $(\xi, \gamma) = (\xi^+, \gamma(\xi^+))$, with $(\xi^-, \gamma(\xi^-)) \to (\theta_v, \gamma_4)$ and $(\xi^+, \gamma(\xi^+)) \to (\theta_v, \gamma_3)$ as $m \to \infty$. As required by Theorem 5.9, the steady state is always unstable between the fold bifurcations, while we observe it to always be stable otherwise.

Similar to the analysis in Section 5.2 that led to (5.28), imposing $\max\{h'(x)\} \geq 0$ leads to the necessary condition

$$\gamma \leq -\beta\mu\tau'(\xi)e^{-\mu\tau(\xi)}g^{\pm} \tag{5.59}$$

for the coexistence of three steady states. However, the algebra to turn this into an explicit condition (compare (5.59) with (5.28)) is challenging with a state-dependent delay (5.1), so instead we apply this condition numerically. The red dash-dotted curve in Figure 5.10(b) depicts the condition (5.59). For any fixed m, this curve provides an upper bound on γ for the existence of fold bifurcations and hence multiple coexisting steady states. By the same argument as in Section 5.2 the cusp bifurcation must lie on this curve. This can be seen in Figure 5.10(b) where the cusp point $(\gamma, m) = (0.4844, 5.0002)$ lies on this bounding curve.

Because of (5.56), there can be no Hopf bifurcations for $\gamma < \mu$, and since the fold bifurcations all occur for $\gamma < \mu$, there can be no Hopf bifurcations between the folds, as seen

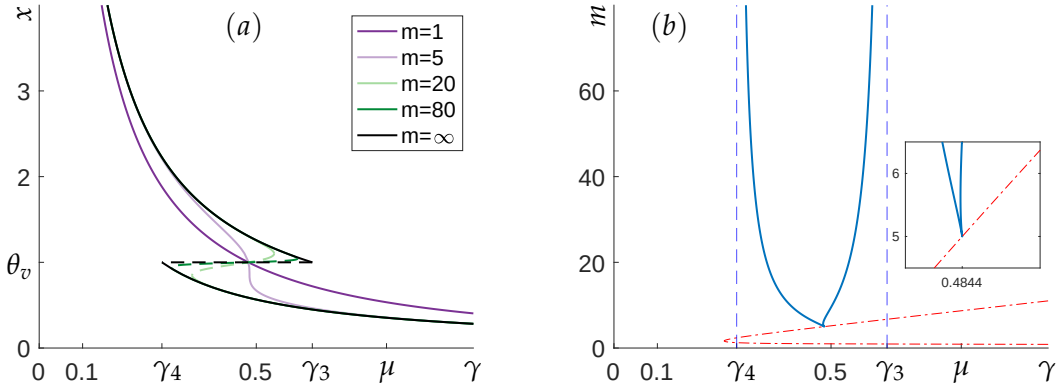


Figure 5.10: Bifurcation diagram of (1.1)–(1.2) with $(g \leftrightarrow, v \uparrow)$ and parameters $\beta = 1.4$, $\mu = 0.8$, $g^- = g^+ = 1$, $\gamma = \theta_v = a = 1$, $v^- = 0.5$ and $v^+ = 1$. By (5.33) this implies $\gamma_4 = 0.2827 < \gamma_3 = 0.6291 < \mu$ (Case 1). (a) One parameter continuation of the steady state in γ for the velocity functions in (1.7) and in (1.3) with different fixed values of m. For $m < 5$ the steady state is always stable. For $m > 5$ it is unstable between two fold bifurcations, and for large m approaches the limiting case (1.7) (denoted by $m = \infty$ in the figure). (b) Two-parameter continuation of the fold bifurcations in the (γ, m)-plane, with a cusp point at $(\gamma, m) = (0.4844, 5.0002)$. The red dash-dotted curve denotes the bound on the fold bifurcations given by (5.59).

in Figure 5.10. For $\gamma > \mu$, Theorem 5.9 ensures that the steady state is stable for γ sufficiently large or for m sufficiently large. In Figure 5.10 we see that actually there are no Hopf bifurcations at all and the steady state is stable for all values of $\gamma > \mu$ and for all $m > 0$.

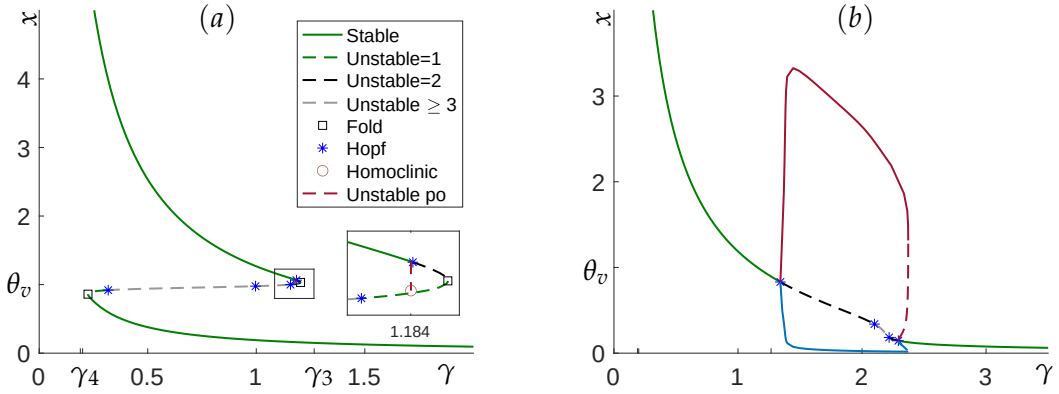


Figure 5.11: One parameter bifurcation diagrams of (1.1)–(1.2) with $(g \leftrightarrow, v \uparrow)$ for varying γ and smooth increasing v defined by (1.3), with the exponent m in the velocity nonlinearity defined by (a) $m = 50$ and (b) $m = 2$. The other parameters are taken to be $\beta = 1.4$, $\mu = 0.2$, $g^- = g^+ = 1$, $\theta_v = 1$, $a = 1$, $v^- = 0.1$ and $v^+ = 2$. By (5.33) this implies $\gamma_4 = 0.1895 < \mu < \gamma_3 = 1.2668$ (Case 2). Solid lines represent stable objects (with stable steady states shown in green), and dashed lines represent unstable objects (with unstable steady states shown in green, black or grey depending on the number of unstable characteristic values). In (a) a branch of unstable periodic orbits is represented by the 2-norm (5.60) of the periodic solution (red dashed line in inset), while in (b) red and blue lines represent the maximum and minimum values of $x(t)$ on the periodic orbit.

Case 2. The case $\mu \in (\gamma_4, \gamma_3)$ is illustrated in Figures 5.11–5.14. Figure 5.11 shows two one-parameter continuations in γ for different values of m. At first glance, Figure 5.11(a) for which $m \gg 0$, looks much like the corresponding constant delay case with increasing g illustrated in Figure 5.4(b) in Section 5.2. In both of these examples there are two fold bifurcations, leading to an interval of bi-stability of steady states, and Hopf bifurcations between the fold bifurcations leading to unstable periodic orbits. However, as shown in the inset of Figure 5.11(a), for the state-dependent delay with $\mu < \gamma_3$ the upper branch of stable steady states, which exists for $x > \theta_v$, loses stability in a (subcritical) Hopf bifurcation, and not at the fold bifurcation. Recall that for the constant delay case considered in Section 5.2 we showed that the steady-state could only lose stability in a fold bifurcation.

Figure 5.11(b) illustrates the dynamics for a much smaller value of m, but with the other parameters unchanged. In this case m is too small for fold bifurcations to occur, but we still find a pair of Hopf bifurcations, one supercritical and one subcritical, leading to a stable periodic orbit, and also a very narrow interval of bistability of this periodic orbit and a stable steady state between the subcritical Hopf bifurcation and a saddle-node of periodic orbits bifurcation. This is quite different from the dynamics seen before where, for decreasing g or v respectively in Sections 5.1 and 5.3, only supercritical Hopf bifurcations were observed, and for increasing g in Section 5.3 where Hopf bifurcations only occur between the pair of fold bifurcations.

The branches of periodic orbits in Figure 5.11(a) and (b) are represented differently. In Figure 5.11(a) we plot the L_2 norm of the periodic solution of period T, defined as

$$\|x\|_2 = \left(\frac{1}{T} \int_{t=0}^{T} |x(t)|^2 dt \right)^{1/2}. \tag{5.60}$$

In contrast, in Figure 5.11(b) we display the branch of periodic orbits by plotting both $\max x(t)$ and $\min x(t)$ over the periodic orbit, which clearly shows the amplitude of the periodic solution. Both representations can be useful on bifurcation diagrams since at a Hopf bifurcation all three expressions are equal to the steady state value and therefore the periodic orbits emanate from the steady states at Hopf bifurcations. However, the representation (5.60) has the additional property that $\|x\|_2 \to x_s$ as the solution approaches a homoclinic orbit to a saddle steady state x_s. An example of this is seen in Figure 5.11(a) where the branch of unstable periodic orbits emanating from the subcritical Hopf bifurcation appears to terminate in a homoclinic bifurcation at the middle steady state. We will investigate homoclinic bifurcations below, and so we will mainly use the L_2 norm (5.60) to represent periodic orbits.

To investigate this example further in Figure 5.12 we present two-parameter continuations of the bifurcations in γ and m, with all the other parameter values the same as in Figure 5.11.

Figure 5.12(a) shows how the γ-m parameter plane is divided into regions where there are 0, 1 or 2 stable steady states. These regions are partly bounded by a curve of fold bifurcations, but mainly by a curve of Hopf bifurcations, with the two bifurcation curves meeting at a Bogdanov–Takens (BT) bifurcation point, at which the characteristic equation has a double zero characteristic value.

Figure 5.12(b) shows the locus of the fold bifurcations, along with codimension-two bifurcations which occur on this branch. Different line types/colours are used to indicate the number of unstable characteristic values (not counting the zero characteristic value associated with the fold bifurcation itself). Since $(\xi^+, \gamma(\xi^+)) \to (\theta_v, \gamma_4)$ as $m \to \infty$, and $\mu > \gamma_4$, it follows that $\gamma(\xi^+) < \mu$ for all m sufficiently large. As explained before Proposition 5.7, these fold bifurcations which occur for $\gamma < \mu$ always involve a stable steady state. The fold bifurcations

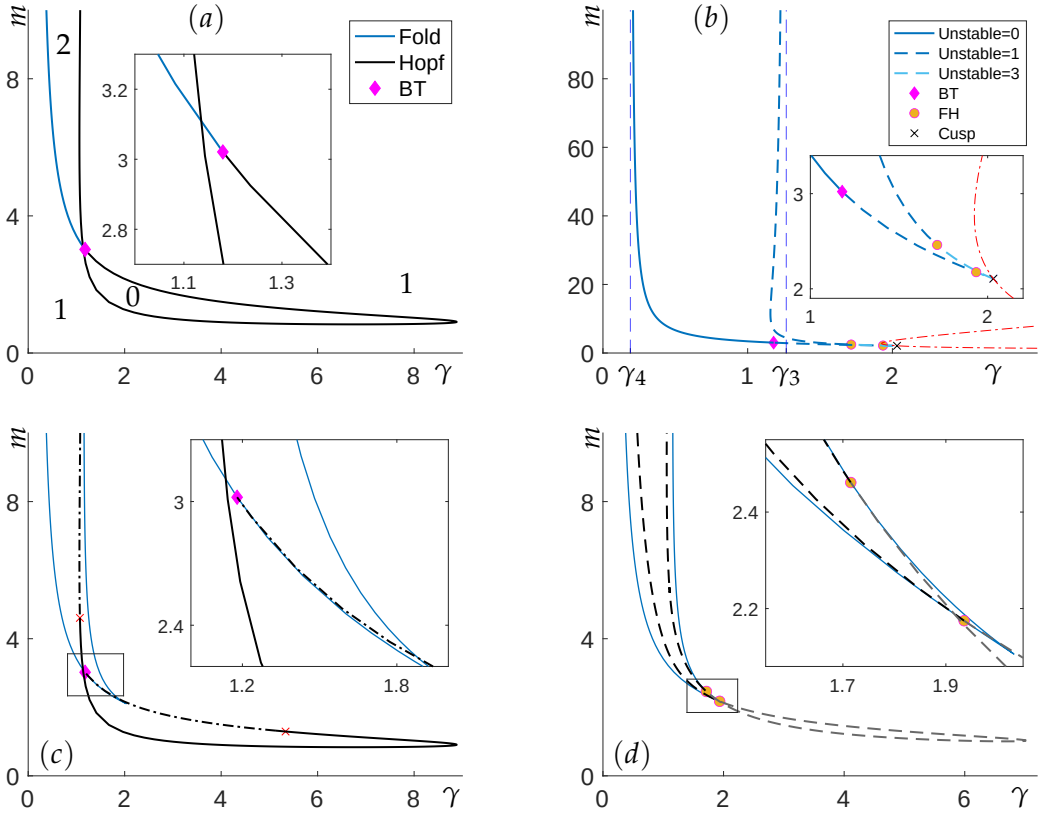


Figure 5.12: Two parameter (Case 2) bifurcation diagrams of (1.1)–(1.2) with $(g \leftrightarrow, v \uparrow)$ and smooth increasing v defined by (1.3). The bifurcation parameters are γ and m, with the other parameters for all the panels taken as stated in the caption to Figure 5.11. (a) The parameter space is divided into regions where there are 0, 1 or 2 stable steady states as indicated. These regions are partly bounded by a curve of Hopf bifurcations (black), and partly by a curve of fold bifurcations (blue). These two curves meet in a Bogdanov–Takens (BT) bifurcation point at $(\gamma, m) = (1.1795, 3.0209)$. (b) Two-parameter continuation of the fold bifurcations only. The fold curve is drawn according to the number of characteristic values with positive real part; solid blue for zero, dashed blue for one, and dashed light blue for three. The dashed vertical blue lines denote $\gamma = \gamma_3$ and $\gamma = \gamma_4$, given by (5.33), the location of the fold bifurcations in the limiting case as $m \to \infty$. The red dash-dotted curve denotes the bound on the fold bifurcations given by (5.59). There is a cusp point at $(\gamma, m) = (2.0321, 2.1058)$. The same BT point is detected again as well as two fold-Hopf points at $(\gamma, m) = (1.7153, 2.4612)$ and $(\gamma, m) = (1.9354, 2.1748)$. (c) Continuation of the branch of Hopf bifurcations emanating from the BT point (in black), with the branch of fold bifurcations shown in blue for scale. These Hopf bifurcations always involve a stable steady state that loses stability, either in a supercritical Hopf bifurcation to a stable periodic orbit (the solid black curve), or in a subcritical Hopf bifurcation to an unstable periodic orbit (the dotted-dashed black curve). Two Bautin bifurcations at $(\gamma, m) = (1.0721, 4.6069)$ and $(5.3352, 1.2929)$ where the criticality of the Hopf bifurcation changes are denoted by red stars. (d) Continuation of the branch of Hopf bifurcations which passes through both fold-Hopf points. This bifurcation curve only involves unstable steady states. The dashed black curve represents Hopf bifurcations with one unstable eigenvalue, and the dashed gray curve with two unstable eigenvalues. The change in number of eigenvalues with positive real part occurs at the fold-Hopf points.

where a stable steady state is created/destroyed are shown as the solid blue segment of the curve of fold bifurcations which is asymptotic to $\gamma = \gamma_4$ in Figure 5.12(b). In contrast, the part of the branch asymptotic to $\gamma = \gamma_3$ as $m \to \infty$ consists of fold bifurcations of two unstable steady states (with different numbers of unstable characteristic values). The BT point, already seen in Figure 5.12(a), separates these two parts of the branch of fold bifurcations. Consequently, only part of the branch of fold bifurcations delineates the boundary between regions of parameter space with different numbers of stable steady states. This contrasts starkly with the constant delay case considered in Section 5.2 and the state-dependent case with $\mu > \gamma_3$ (shown in Figure 5.10), for both of which steady states only lose stability in a fold bifurcation.

Figure 5.12(c) shows the Hopf curve which terminates at the BT point. We verified numerically that at all points on this curve the Hopf bifurcation is from a stable steady state that loses stability. Close to the BT point, and at the other end of this branch as $m \to \infty$, the Hopf bifurcation is subcritical leading to an unstable periodic orbit. But on a large segment of this curve (and in particular for the smallest values of m on the curve) the Hopf bifurcation is supercritical leading to a stable periodic orbit. Two Bautin bifurcation points, where the Hopf bifurcation changes criticality, separate the sub and supercritical segments of the curve. Figure 5.12(a) shows the number of stable steady states in the γ-m parameter plane, and just the parts of the bifurcation curves that delineate these regions. The region with no stable steady states contains a stable periodic orbit (because of the supercritical Hopf bifurcation on the left-side of this region).

Figure 5.12(d) shows another curve of Hopf bifurcations for the same problem. This curve passes through the two fold-Hopf bifurcation points seen on the curve of fold bifurcations in Figure 5.12(b). We notice that while these points are close together on the branch of fold bifurcations (they are both close to, but on different sides of, the cusp point), as the inset shows, they are on different legs of the branch of Hopf points, and hence far from each other on this branch of Hopf bifurcations. The codimension-two fold-Hopf bifurcation points are of interest, as such bifurcations are impossible for the constant delay model of Section 5.2. However the Hopf bifurcations seen in this example all generate unstable periodic orbits bifurcating from unstable steady states, and we will not study them further.

In Figure 5.13 we continue our study of the dynamics seen in Figures 5.11 and 5.12 by showing one-parameter continuations of the dynamics and bifurcations as γ is varied for different fixed values of the steepness parameter m in the Hill function (1.3). For m small there is a unique stable steady state for each value of γ and no bifurcations occur. For $m > 0.8355$ there is still a unique steady state, but it is unstable between a pair of super-critical Hopf bifurcations, where a branch of stable periodic orbits exists. This is illustrated in Figure 5.13(a) for $m = 1.2$, where we see that the period of the periodic orbits is monotonically increasing from the left Hopf point to the right Hopf point.

Increasing m further for $m > 1.2929$ we pass the lower Bautin bifurcation seen in Figure 5.12(c) and the right Hopf point becomes subcritical. The branch of periodic orbits is still contiguous, but a fold bifurcation of periodic orbits born in the Bautin bifurcation divides the branch of periodic orbits into an upper stable branch and a lower unstable branch. The period is no longer monotonically increasing on the branch of periodic orbits, but attains a maximum period near to the fold bifurcation of periodic orbits, but on the unstable branch of periodic orbits (as illustrated for $m = 2$ in Figure 5.13(b)).

For $m > 2.1058$ the cusp bifurcation seen in Figure 5.12(b) introduces two fold bifurcations of steady states (seen for $m = 2.15$ in Figure 5.13(c)). As m is increased further the maximum period seen on the branch of periodic orbits increases dramatically and the point where the

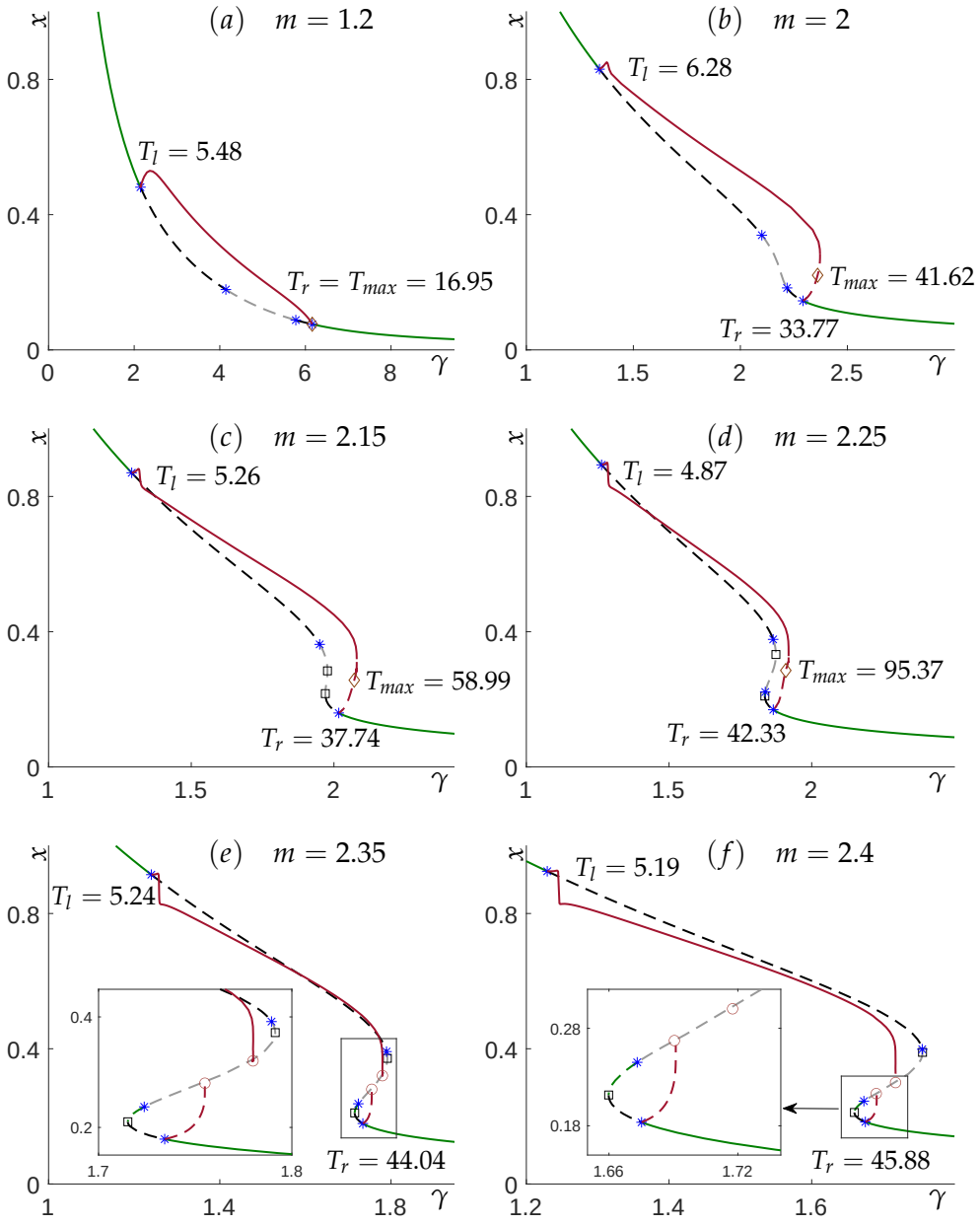


Figure 5.13: Case 2 one-parameter bifurcation diagrams for $(g \leftrightarrow, v \uparrow)$ with varying γ and different fixed values of m, with the other parameters the same as in Figures 5.11 and 5.12. Solid lines represent stable objects including stable steady states (in green) and stable periodic orbits (represented by the 2-norm (5.60)). Dashed lines represent unstable objects including unstable steady states (depending on the number of eigenvalues with positive real part, green for one, black for two and gray for three and more) and unstable periodic orbits (represented by 2-norm (5.60)). Hopf bifurcations are marked by blue stars, fold bifurcations of steady states by black squares, and homoclinic bifurcations by red circles. For the branch of periodic orbits, T_l is the period at the left Hopf bifurcation, and T_r at the right Hopf bifurcation. The point on the branch where the largest period, T_{max}, occurs is marked by a brown diamond.

maximum period occurs approaches the intermediate steady state (see Figure 5.13(d)).

Comparing panels (d) and (e) of Figure 5.13 suggests that (1.1) undergoes a codimension-two global bifurcation between these panels. The branches of periodic orbits emanating from the Hopf bifurcations where the steady states lose stability no longer join up in panels (e) and (f). Instead, each branch ends at a homoclinic bifurcation to the intermediate steady state (at different values of the parameter γ). The panels shown suggest, but do not prove, that as m is increased the branch of periodic orbits approaches the fold point with $T_{max} \to \infty$ at the first point where a homoclinic bifurcation occurs. Further evidence is presented in the two-parameter continuation of Figure 5.14(a) where the homoclinic curves seem to approach the curve of folds. For larger values of $m > 2.3$, codimension-one homoclinic bifurcations to the intermediate saddle steady state are found. Examples of dynamical systems with a single branch of homoclinic bifurcations terminating at a fold bifurcation of steady states can be found in [1, 35]. Our example is somewhat different from those, as we have two branches of homoclinic bifurcations.

The homoclinic bifurcation on the branch emanating from the lower Hopf point in Figures 5.13(e) and (f) does not persist for large m, but instead the homoclinic bifurcation and the Hopf bifurcation itself move towards the fold bifurcation and terminate at the BT-point found previously. This is illustrated in a two-parameter continuation in Figure 5.14(a), which shows the branch of fold bifurcations of steady states passing through the BT point, next to a branch of Hopf bifurcations, then a branch of homoclinic bifurcations, the latter two branches both terminating at the BT point tangential to the branch of fold bifurcations at that point. This is well-known behaviour for Bogdanov–Takens bifurcations, and can be seen in the normal form diagram for this bifurcation in Section 8.4 of [35]. Bogdanov–Takens bifurcations have recently been analysed for constant delay DDEs in [6] (where Figures 5 and S8 resemble the part of Figure 5.14(a) close to the BT point), but we are not aware of any systematic study of them in state-dependent DDE problems.

While in the classical unfolding the bifurcation curves extend arbitrarily far from the BT point, in our example in Figure 5.14(a) both the branch of homoclinic orbits and the branch of fold bifurcations terminate. The branch of fold bifurcations terminates at a cusp bifurcation with the other branch of fold bifurcations. The proximity of the cusp point to the BT point suggests that our system may be close to a codimension-three Bogdanov–Takens–cusp (BTC) point. While we are not aware of a systematic study of this bifurcation, they have been observed in a neuron model in [1], and some of the bifurcation structures that we find resemble those in [1].

The homoclinic bifurcation on the branch emanating from the upper Hopf point in Figures 5.13(e) and (f) is also shown in Figure 5.14(a) and persists for arbitrary large values of m (it was already seen with $m = 50$ in Figure 5.11(a)), but there is a change in stability on this branch for $m > 4.6069$ due to the Bautin bifurcation seen in Figure 5.12(c).

The maximum period of orbits from the one-parameter continuations described in Figure 5.13 are also shown as a curve in two-parameter space in Figure 5.12(a). This curve approaches the right most curve of fold bifurcations with the period becoming unbounded as it does so. Our computations of the two branches of homoclinic bifurcations also terminate close to this point. We conjecture that the co-dimension two of the homoclinic orbits already described in Figure 5.13 that occurs where the branch of maximum period orbits terminates will occur on the branch of fold bifurcations. This would be consistent with the previously mentioned examples of curves of homoclinic bifurcations which terminate at fold bifurcations [1, 35].

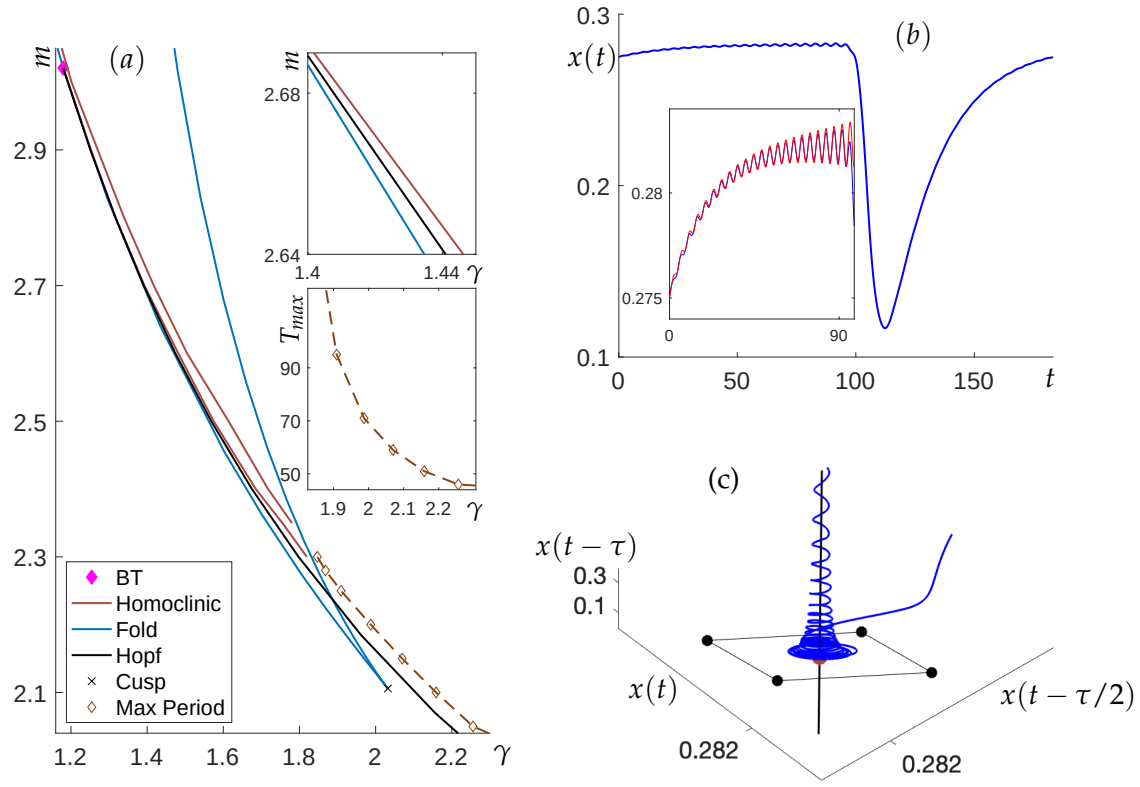


Figure 5.14: (a) Bifurcation diagram near the Bogdanov–Takens point shown in Figure 5.12 for $(g \leftrightarrow, v \uparrow)$ shows a branch of Hopf bifurcations and a branch of homoclinic bifurcations emerging from the BT point. The dashed brown line denotes the point in parameter space where the largest period was encountered on the branches of periodic orbits shown in Figure 5.13, with the period shown in the lower inset. The upper inset shows the ordering of the curves emerging from the BT point. (b) Profile of the unstable periodic orbit with parameters $\beta = 1.4$, $\mu = 0.2$, $g^- = g^+ = 1$, $m = 2.35$, $\gamma = 1.7550$, $\theta_v = 1$, $a = 1$, $v^- = 0.1$, and $v^+ = 2$. The blue curve shows the periodic orbit with period $T = 183$. In the inset the part of the periodic orbit (in blue) near to the intermediate steady state is overlayed by a linear approximation to the dynamics in red (see text for details). (c) Projection of the periodic orbit from (b) onto the space $(x(t), x(t - \tau/2), x(t - \tau))$ where $\tau = 5.1926$ is chosen to be the delay at the intermediate steady state represented by the orange dot. The solid black line represents the dominant one-dimensional linear stable manifold associated with the characteristic value -0.045. The parallelogram denotes the plane associated with the dominant unstable characteristic values $\lambda = 0.0184 \pm 1.4738i$.

In Figure 5.14(b) and (c) we display one of the periodic orbits for $m = 2.35$ from the lower branch of periodic orbits shown in Figure 5.13(e). The orbit is shown close to the homoclinic bifurcation at the end of the branch, for which the period is large. Figure 5.14(c) shows a phase space projection of the part of the orbit close to the intermediate steady state. In this projection the orbit approaches the saddle steady state from above close to the dominant stable direction (associated with the characteristic value with negative real part closest to zero which is $\lambda = -0.045$), but with a growing oscillation about this point in the plane defined by the dominant unstable direction (associated with the characteristic value with positive real part closest to zero which is $\lambda = 0.0184 \pm 1.4738i$). To further confirm that this linear

behaviour is determining the dynamics near to the intermediate steady state x^*, in the inset of Figure 5.14(b) we plot in red the curve

$$x(t) = x^* + k_1 e^{-0.045t} + k_2 e^{0.0184t} \cos(1.4738t),$$

for suitably chosen coefficients k_j, and observe that it overlays the periodic solution over most of the time interval shown. This behavior is similar to that associated with Shilnikov type complex dynamics, but in contrast with the construction of a chaotic attractor of Shilnikov type, the periodic orbit pictured in Figure 5.14(b) and (c) is unstable.

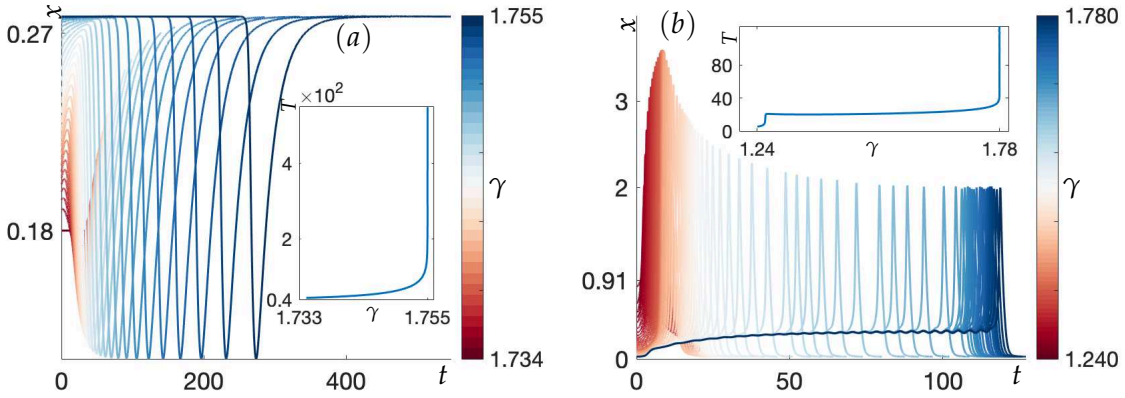


Figure 5.15: Profiles and periods of periodic orbits on the branches that approach homoclinic bifurcations in Figure 5.13(e) with $m = 2.35$ for (a) the unstable periodic orbits on the branch emanating from the subcritical Hopf bifurcation at $\gamma = 1.734$, and (b) the stable periodic orbits on the branch emanating from the supercritical Hopf bifurcation at $\gamma = 1.240$.

Figure 5.15(a) and (b) shows the profiles and periods on the two branches of periodic solutions seen in Figure 5.13(e). Comparing these two panels we see that the periodic orbits on the two branches are very different, with the unstable orbit in Figure 5.15(a) having a small amplitude and with $x(t)$ below the steady state value along the whole orbit on the entire branch. In contrast stable periodic orbits shown in panel (b) have much larger amplitude.

Case 3: It remains to consider the case where $\mu < \gamma_4 < \gamma_3$. Making a small change to the parameters considered in the example in Figures 5.11 to 5.15 by changing the value of β from $\beta = 1.4$ to $\beta = 2$ but leaving the values of the other parameters unchanged, results in $\mu = 0.2 < \gamma_4 = 0.2707 < \gamma_3 = 1.8097$. However, although $\mu < \gamma_4$ in this case μ is close to γ_4 and the dynamics and bifurcations are very similar to those shown in Figure 5.11, so we do not show them here. Instead in Figure 5.16 we consider an example with $\mu \ll \gamma_4 < \gamma_3$.

Theorem 5.9 suggests that stability may be lost in either a Hopf or a fold bifurcation, but for the example in Figure 5.16 with $\mu \ll \gamma_4$ we see that the steady state always loses stability at a Hopf bifurcation. Unlike the previous example, there is no longer a BT point, and for all m sufficiently large the steady state loses stability in a subcritical Hopf bifurcation close to the fold bifurcation, resulting in a short branch of periodic orbits that terminates in a homoclinic bifurcation to the intermediate steady state created at the fold bifurcation, as illustrated for $m = 50$ in Figure 5.16(b)–(d). For m small, below the cusp bifurcation at $m = 5.0007, \gamma = 0.7572$, the Hopf bifurcations are supercritical leading to a branch of stable periodic orbits between the Hopf bifurcations, similar to Figure 5.13(a). As seen in Figure 5.16(b) the Hopf bifurcations and homoclinic points are very close to the fold bifurcations. This makes

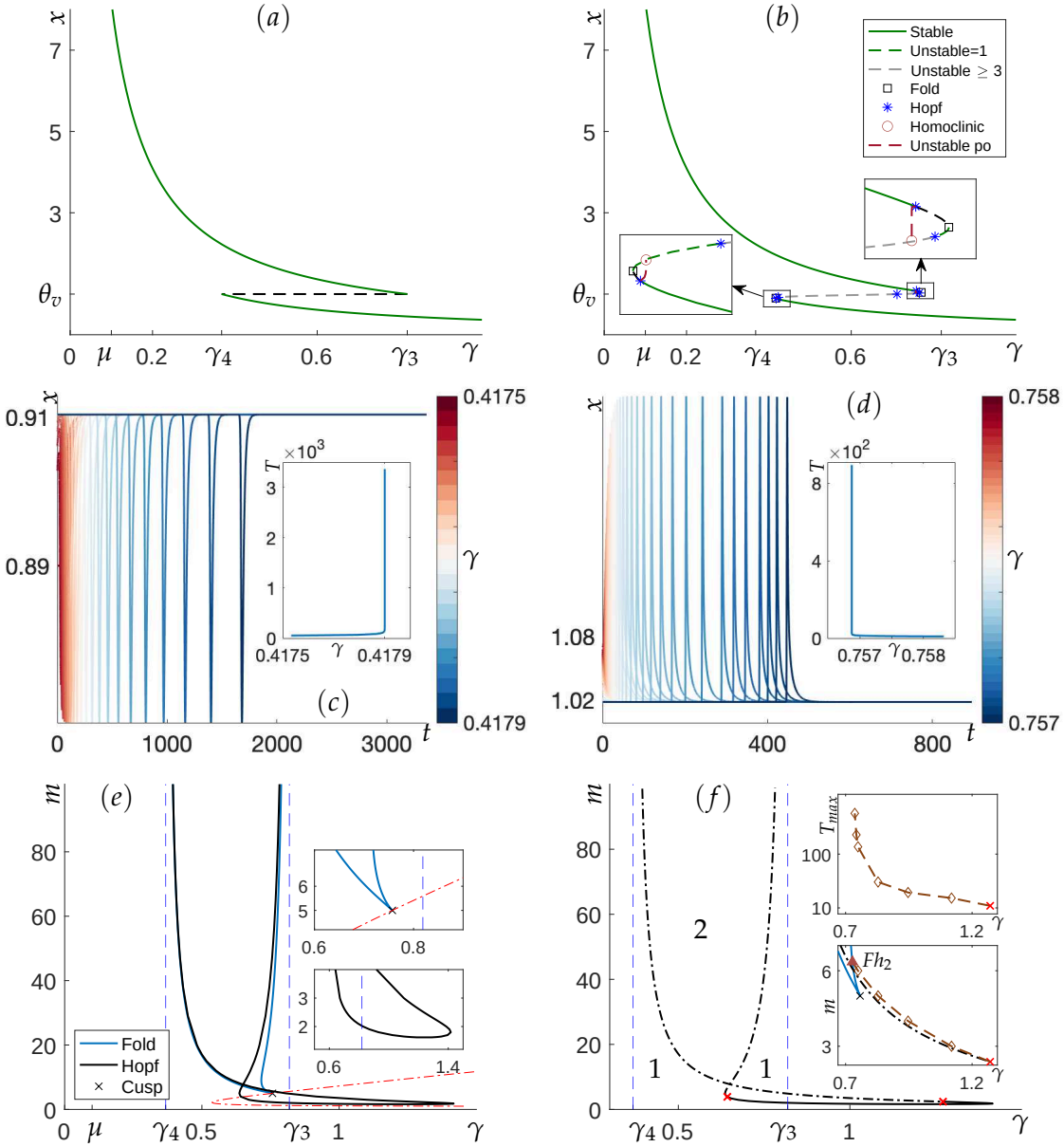


Figure 5.16: Bifurcation diagrams of (1.1)–(1.2) with $(g \leftrightarrow, v \uparrow)$ and parameters $\beta = 1$, $\mu = 0.1$, $g^- = g^+ = 1$, $\theta_v = 1$, $a = 2$, $v^- = 0.2$ and $v^+ = 1$. By (5.33) this implies $\mu < \gamma_4 = 0.3679 < \gamma_3 = 0.8187$ (Case 3). (a) The limiting case with v defined by (1.7). The stable steady state is shown as a green solid line, and the singular steady state as a black dashed line. (b) With smooth velocity nonlinearity v defined by (1.3) with $m = 50$. Solid lines represent stable steady states (in green). Dashed lines represent unstable objects including unstable steady states (depending on the number of eigenvalues with positive real part, green for one, and gray for three and more) and unstable periodic orbits (represented by 2-norm (5.60)). (c) and (d) Profiles and periods of the periodic orbits on the branches shown in (b) that terminate at homoclinic bifurcations and emanate from the subcritical Hopf bifurcations at (c) $\gamma = 0.4175$ and (d) $\gamma = 0.758$. (e) Two-parameter continuations in m and γ of the fold and the Hopf bifurcations with other parameters as above. The dashed vertical blue lines denote $\gamma = \gamma_3$ and $\gamma = \gamma_4$ the location of the fold bifurcations in the limiting case as $m \to \infty$. The red dash-dotted curve denotes the bound on the fold bifurcations. (f) Two parameter continuation of the curve of Hopf bifurcations, with the two Bautin bifurcation points and the criticality of the Hopf bifurcation indicated. The Hopf bifurcations delineate the parameter space into regions where there is 0, 1 or two stable steady states, as indicated. Also shown in the insets is the maximum period curve (similar to Figures 5.13 and 5.14), which terminates at the point marked Fh_2 with infinite period.

numerical continuation of the branches of homoclinic bifurcations very delicate, and so we do not show the curves of homoclinic bifurcations in the two parameter continuations in Figure 5.16(e) and (f). Instead in the insets in Figure 5.16(f) we show the curve formed by the periodic orbits of maximum periods from the one-parameter continuations in γ; as was done in case (b) in Figures 5.13 and 5.14. This curve is born from the Bautin bifurcation point and terminates at the point labelled Fh_2 on the curve of fold bifurcations, where the period becomes infinite. We expect two curves of homoclinic bifurcations to be born at this point, and to persist for arbitrarily large m (as seen in (b) for $m = 50$).

5.5 Summary

We briefly summarize the results of Section 5. We have separately studied cases with constant delay and decreasing (Section 5.1) and increasing (Section 5.2) function g, and then the cases with constant g but with state dependent delay with velocity v decreasing (Section 5.3) or increasing (Section 5.4). These results can be compared between constant and state-dependent cases but with the same type of nonlinearity (Section 5.1 and 5.3, and Section 5.2 and 5.4), or between the two types of nonlinearity with the same type of delay (Section 5.1 and 5.2, and Section 5.3 and 5.4), as we will do below.

In general, the dynamics with state-dependent delay is significantly richer than that of the corresponding constant delay case. For decreasing non-linearity (Section 5.1 vs Section 5.3) in the constant delay case there is a range (γ_1, γ_2) of the parameter γ where the unique equilibrium undergoes a series of Hopf bifurcations as the Hill coefficient $n \to \infty$ (Figure 5.2). In contrast, in the state dependent case a new constraint becomes important as we show that Hopf bifurcations can only occur for $\gamma < \mu$. Therefore, if the μ is below the range (γ_3, γ_4) the unique equilibrium is always stable (Figure 5.6), when it is above (γ_3, γ_4) we recover the result from Section 5.1 (Figure 5.8), and when $\mu \in (\gamma_3, \gamma_4)$, the result of Section 5.1 is confined to values of $\gamma \in (\gamma_3, \gamma_4) \cap \{\gamma < \mu\} = (\gamma_3, \mu)$; see Figure 5.7.

When the feedback nonlinearity is increasing with constant delay (Section 5.2) the system has multiple equilibria for γ in an interval which approaches (γ_2, γ_1) in the limit as the feedback nonlinearity approaches a piecewise constant function. The stable equilibria always lose stability at fold bifurcations at the ends of the interval, leading to bistability of steady states for this interval of γ values. Hopf bifurcations can only occur on the middle branch of unstable equilibria.

As documented in Section 5.4, this scenario changes dramatically for state dependent delays. For increasing v there is again a constraint and we show that Hopf bifurcations can only occur for $\gamma > \mu$. Consequently if $\mu > \gamma_3$ there are no Hopf bifurcations. On the other hand, when $\mu < \gamma_3$ new dynamics are observed. In this case stable equilibria can now lose stability in a Hopf bifurcation and the interaction of the Hopf and fold bifurcation curves in two-parameter continuation in γ and the Hill coefficient leads to existence of Bogdanov–Takens points, fold-Hopf points and a possible double homoclinic bifurcation. This last scenario, documented in Figure 5.13, suggests that a curve of periodic orbits connecting subcritical and supercritical Hopf bifurcations of the stable equilibria, collides with a fold bifurcation where the equilibrium also admits a homoclinic orbit. Continuation of the curve of periodic orbits in two-parameter space suggests that the curve splits into two curves of periodic orbits each terminating at a homoclinic bifurcation at the middle equilibrium branch, but at different values of parameter γ. A less dramatic result, but perhaps more important for applications, is the observation of bistability between a stable fixed point and a stable periodic orbit in Fig-

ure 5.13(b), a scenario that is impossible in the constant delay case of the Goodwin operon model considered in Sections 5.1 and 5.2.

6 Dynamics with two Hill functions

We now consider the model (1.1), (1.2) with both the functions g and v non-constant at the same time. In the limiting case these functions are defined by (1.6) and (1.7), and when $\theta_v \neq \theta_g$, the steps in the functions occur for different values of x, so they reduce to the cases considered in Section 5 with v constant for x in a neighbourhood of θ_g, and g constant for x in a neighbourhood of θ_v. With smooth nonlinearities defined by (1.3) and large exponents $m \gg 0$, $n \gg 0$, when $\theta_v \neq \theta_g$ we will see dynamics and bifurcations near $x = \theta_g$ similar to those observed in Sections 5.1 and 5.2, and similar to Sections 5.3 and 5.4 near $x = \theta_v$. On the other hand, when $\theta_v = \theta_g$ new dynamics are observed.

Recall the notation $(g \downarrow, v \downarrow)$ that indicates the monotonicity type of functions g and v, introduced in Section 1. Note that if $v^- > v^+$ then $v(x)$ is a decreasing function of x, then $\tau(x)$ given by (5.1) is an increasing function of x, and therefore $e^{-\mu\tau(x)}$ is a decreasing function of x. It follows immediately that for $(g \downarrow, v \downarrow)$ the function $h(x)$ (see (5.2)) is strictly monotonically decreasing and hence for g and v both monotonically decreasing there is exactly one steady state.

If either or both of g or v is increasing then it is possible to have fold bifurcations and additional steady states. To see this, consider a steady state ξ with both v and g given by (1.3), in which case the characteristic equation has the form (4.8), which can be written as $\Delta(\lambda) = 0$ with

$$\Delta(\lambda) = \lambda + \gamma(\xi) - A(\xi) - (Q(\xi) - A(\xi))e^{-\lambda\tau(\xi)} - \mu A(\xi)\int_{-\tau(\xi)}^{0} e^{\lambda s} ds \tag{6.1}$$

where (recalling (4.4) and (5.15))

$$Q(\xi) := \beta e^{-\mu\tau(\xi)} g'(\xi) = \gamma \frac{\xi g'(\xi)}{g(\xi)}, \qquad A(\xi) = \beta \frac{v'(\xi)}{v(\xi)} e^{-\mu\tau(\xi)} g(\xi) = \gamma \frac{\xi v'(\xi)}{v(\xi)}.$$

The properties of $A(\xi)$ and $Q(\xi)$ were already described in Proposition 5.1. Now

$$\begin{aligned}\Delta(0) &= \gamma(\xi) - Q(\xi) - \mu A(\xi)\int_{-\tau(\xi)}^{0} 1 ds = \gamma(\xi) - Q(\xi) - \mu\tau(\xi)A(\xi) \\ &= \gamma(\xi)\Big(1 - \frac{\xi g'(\xi)}{g(\xi)} - \mu\tau(\xi)\frac{\xi v'(\xi)}{v(\xi)}\Big).\end{aligned}$$

Hence, $\lambda = 0$ is a characteristic value if and only if

$$M(\xi) := \frac{\xi g'(\xi)}{g(\xi)} + \mu\tau(\xi)\frac{\xi v'(\xi)}{v(\xi)} - 1 = 0. \tag{6.2}$$

In addition, we note that $M(\xi) > 0$ implies that $\Delta(0) < 0$. At the same time, for real λ the characteristic function satisfies $\Delta(\lambda) \to +\infty$ as $\lambda \to +\infty$. We conclude that when $M(\xi) > 0$ there is a real positive characteristic value and hence the steady state is unstable.

We now show that fold bifurcations occur when (6.2) is satisfied. To that end, consider the curve of solutions $(\gamma(\xi), \xi)$ where $\gamma(\xi)$ is defined by

$$\gamma(\xi) = \frac{\beta g(\xi)}{\xi} e^{-\mu\tau(\xi)}, \tag{6.3}$$

which by (5.2) is the locus of steady states in the (γ, ξ)-plane. Differentiating this relationship with respect to ξ we find

$$\gamma'(\xi) = -\frac{\gamma(\xi)}{\xi} + \frac{\gamma(\xi)g'(\xi)}{g(\xi)} - \gamma(\xi)\mu\tau'(\xi).$$

Using (5.54) we obtain

$$\xi\gamma'(\xi) = \gamma(\xi)\left(\frac{\xi g'(\xi)}{g(\xi)} + \mu\tau(\xi)\frac{\xi v'(\xi)}{v(\xi)} - 1\right) = \gamma(\xi)M(\xi). \tag{6.4}$$

Hence $\operatorname{sgn}(\gamma'(\xi)) = \operatorname{sgn}(M(\xi))$, and $\gamma'(\xi)$ changes sign when $M(\xi)$ does, which ensures that fold bifurcations occur at these points in the (γ, ξ)-plane.

Note that this is a direct generalisation of the fold bifurcations found in Section 5 as (6.2) reduces to (5.27) when v is a constant function, and to (5.52) when g is constant.

Next we determine when Hopf bifurcations may arise for the general case where both g and v are non-constant. For $\lambda \neq 0$, using (6.1) we can rewrite the characteristic equation $\Delta(\lambda) = 0$ as

$$\lambda = -\gamma + A(\xi)(1 - e^{-\lambda\tau(\xi)})\left(1 + \frac{\mu}{\lambda}\right) + Q(\xi)e^{-\lambda\tau(\xi)}. \tag{6.5}$$

At a Hopf bifurcation we set $\lambda = i\omega$, with $\omega \neq 0$, and from (6.5) obtain

$$\begin{aligned} i\omega &= -\gamma + A(\xi)(1 - e^{-i\omega\tau})\left(1 + \frac{\mu}{i\omega}\right) + Q(\xi)e^{-i\omega\tau} \\ &= -\gamma + A(\xi)(1 - \cos\omega\tau + i\sin\omega\tau)\left(\frac{\omega^2 - i\omega\mu}{\omega^2}\right) + Q(\xi)(\cos\omega\tau - i\sin\omega\tau). \end{aligned}$$

Taking real and imaginary parts, this reduces to

$$\gamma = A(\xi)\left(1 - \cos\omega\tau + \frac{\mu}{\omega}\sin\omega\tau\right) + Q(\xi)\cos\omega\tau, \tag{6.6}$$

$$\omega = A(\xi)\left(\sin\omega\tau - \frac{\mu}{\omega}(1 - \cos\omega\tau)\right) - Q(\xi)\sin\omega\tau. \tag{6.7}$$

Note that when $\omega\tau = 2k\pi$ for $k \in \mathbb{Z}$ the last equation reads $\omega = 0$, which contradicts the requirement that $\omega \neq 0$ for a Hopf bifurcation. It follows that at a Hopf bifurcation

$$|\cos(\omega\tau)| < 1. \tag{6.8}$$

The equations (6.6) and (6.7) can each be rearranged in two ways: first, as $A(\xi)(1 - \cos\omega\tau) = \textit{other terms}$, or, second, as $A(\xi)\sin\omega\tau = \textit{other terms}$. Doing so, then equating the same expressions from the two equations gives

$$A(\xi)\sin\omega\tau(\omega^2 + \mu^2) = \omega(\gamma\mu + \omega^2) + Q(\xi)(\omega\sin\omega\tau - \mu\cos\omega\tau), \tag{6.9}$$

$$A(\xi)(1 - \cos\omega\tau)(\omega^2 + \mu^2) = \omega^2(\gamma - \mu) - Q(\xi)(\omega^2\cos\omega\tau + \mu\omega\sin\omega\tau). \tag{6.10}$$

Note that when $g^+ = g^-$ so g is constant and $Q(\xi) = 0$ these equations reduce to (5.41) and (5.42). On the other hand when $v^+ = v^-$ so $v'(\xi) = 0$ and hence $A(\xi) = 0$, equations (6.9) and (6.10) can be shown to reduce to (5.13) and (5.14).

Using (6.8), the left-hand side of (6.10) has the sign of $A(\xi)$ (recall also (5.41) which has an identical left-hand side).

When $Q(\xi) = 0$, the right-hand side of (6.10) has the sign of $\gamma - \mu$. For the general case with $Q(\xi) \neq 0$, since $|\cos \omega\tau| \leq 1$ and $|\sin \omega\tau| \leq \omega\tau$, we obtain

$$|Q(\xi)(\omega^2 \cos \omega\tau + \mu\omega \sin \omega\tau)| \leq |Q(\xi)|(\omega^2|\cos \omega\tau| + \mu\omega|\sin \omega\tau|) \leq |Q(\xi)|\omega^2(1 + \mu\tau).$$

Then, when $A(\xi) < 0$, and hence v is a decreasing function, the left-hand side of (6.10) is negative, while the right-hand side satisfies

$$\begin{aligned} \omega^2(\gamma - \mu) - Q(\xi)(\omega^2 \cos \omega\tau + \mu\omega \sin \omega\tau) &\geq \omega^2(\gamma - \mu) - |Q(\xi)|\omega^2(1 + \mu\tau) \\ &= \omega^2(\gamma - \mu - |Q(\xi)|(1 + \mu\tau)). \end{aligned} \tag{6.11}$$

Consequently, there can be no Hopf bifurcations if $\gamma > \mu + |Q(\xi)|(1 + \mu\tau)$.

Similarly for the case that $A(\xi) > 0$ (and hence v is an increasing function), the left-hand side of (6.10) is positive, while the right-hand side satisfies

$$\begin{aligned} \omega^2(\gamma - \mu) - Q(\xi)(\omega^2 \cos \omega\tau + \mu\omega \sin \omega\tau) &\leq \omega^2(\gamma - \mu) + |Q(\xi)|\omega^2(1 + \mu\tau) \\ &= \omega^2(\gamma - \mu + |Q(\xi)|(1 + \mu\tau)). \end{aligned} \tag{6.12}$$

Thus, when $A(\xi) > 0$ there can be no Hopf bifurcations if $\gamma < \mu - |Q(\xi)|(1 + \mu\tau)$.

Note that both the conditions here reduce to the previous ones in the limit as $Q(\xi) \to 0$, or equivalently as $g'(\xi) \to 0$. As in Section 5, we will deal with this by considering the different combinations of increasing and decreasing functions v and g separately.

6.1 Two Hill functions with $\theta_g \neq \theta_v$

With g and v both varying and $\theta_g \neq \theta_v$, the dynamics will be comparable to a combination of the four simplified cases as discussed in Section 5.1–5.4. In particular, if both g and v are decreasing functions, there is always exactly one steady state. The dynamics in the case $(g \downarrow, v \downarrow)$ will be similar to the case in Section 5.1 near to $x = \theta_g$ and similar to the case in Section 5.3 near to $x = \theta_v$. The singular steady states in the limiting case give rise to steady states in the smooth case which may undergo Hopf bifurcations where the stability of steady state changes, as illustrated in Sections 5.1 and 5.3.

For the cases when g and v have opposite monotonicity, there are up to three coexisting steady states. The bifurcations occurring near the two respective thresholds agree with the examples in Section 5. Since the dynamics can be explained using the results of Section 5 we only briefly consider the dynamics with $\theta_g \neq \theta_v$ and present two examples, one with $(g \downarrow, v \uparrow)$ in Section 6.1.1, and another for $(g \uparrow, v \uparrow)$ in Section 6.1.2. In this last example, it is possible for up to five steady states to coexist with sufficiently steep nonlinearity.

The constants γ_j for $j = 1, 2, 3, 4$ introduced in (5.4) and (5.33) in Section 5 will also play a role here. When $\theta_v \neq \theta_g$ we again define these using (5.4) and (5.33). However, since both the functions g and v are non-constant we need to define a value for τ in (5.4) and for g in (5.33). We proceed as follows. For γ_1 and γ_2 given by (5.4), the required value of τ is $\tau(\theta_g)$ and hence we take $\tau = \tau^-$ if $\theta_v > \theta_g$ and $\tau = \tau^+$ if $\theta_v < \theta_g$. Similarly in (5.33) we take $g = g^-$ if $\theta_g > \theta_v$ and $g = g^+$ if $\theta_g < \theta_v$.

6.1.1 Decreasing g and increasing v $(g \downarrow, v \uparrow, \theta_g \neq \theta_v)$

When g and v have opposite monotonicity, either $(g \downarrow, v \uparrow)$ or $(g \uparrow, v \downarrow)$, and $\theta_g \neq \theta_v$, there are up to three coexisting steady states in the limiting case. Figure 6.1 shows such an example in the case $(g \downarrow, v \uparrow, \theta_v < \theta_g)$.

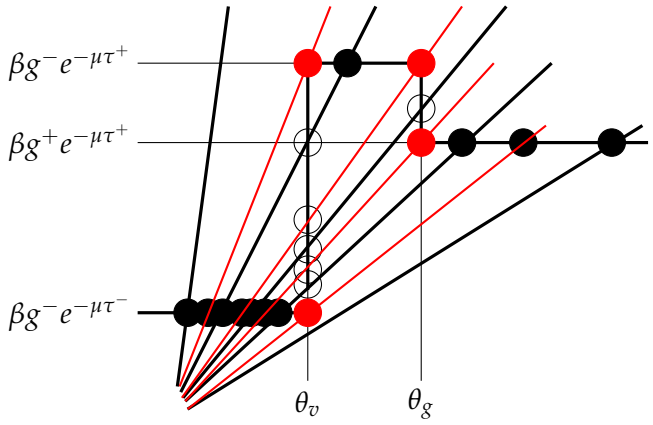


Figure 6.1: Steady states of (1.1), given by (5.2), occur at the intersections of $\xi \mapsto \beta e^{-\mu\tau(\xi)} g(\xi)$ and $\xi \mapsto \gamma\xi$. These are illustrated for various γ in the limiting case of (1.6) and (1.7) for $(g\downarrow, v\uparrow, \theta_v < \theta_g)$. The red lines denote the slopes γ_j, illustrated here with $\gamma_4 < \gamma_1 < \gamma_2 < \gamma_3$. There is a singular steady state at $\xi = \theta_v$ for $\gamma \in (\gamma_4, \gamma_3)$ and at $\xi = \theta_g$ for $\gamma \in (\gamma_1, \gamma_2)$.

Figure 6.1 shows that if $\gamma \in (\gamma_4, \gamma_3)$ then there are three steady states in the limiting case. At most two of these steady states will be stable, with the intermediate steady state at $x = \theta_v$ singular. If the intervals (γ_1, γ_2) and (γ_4, γ_3) overlap then it is possible to have one stable steady state coexisting with two singular steady states. Such a scenario is illustrated in Figure 6.2(a) which has two singular steady states and one stable steady state for $\gamma \in (\gamma_1, \gamma_2)$. The number of steady states changes when $\gamma\xi$ intersects $\beta e^{-\mu\tau(\xi)} g(\xi)$ at the corners along θ_v at parameter values γ_3 and γ_4. The singular 'fold' bifurcations give rise to classical fold bifurcations in the case of smooth nonlinearities defined by (1.3), as seen in Figure 6.2(b) for the case $m = n = 40$. We note from (6.2) that since g' and v' have opposite signs, we require $\xi v'(\xi)/v(\xi) > 1/(\mu\tau(\xi))$ for a fold bifurcation to occur, and so for $m = n$ a somewhat larger value of the steepness parameter m is required to obtain a fold bifurcation than would have been the case with constant g (for which, by (5.52), fold bifurcations occur when $\xi v'(\xi)/v(\xi) = 1/(\mu\tau(\xi))$ is satisfied).

In the example presented in Figure 6.2 the function v is increasing with $\mu < \gamma_4$, and so we expect to see similar behaviour for $x \approx \theta_v$ as in Case 3 of Section 5.4, while since g is decreasing the behaviour for $x \approx \theta_g$ should be similar to that seen in Section 5.1. This is indeed what is observed.

Since v is increasing, from (6.10) and (6.12) it is necessary that $\gamma > \mu - |Q(\xi)|(1+\mu\tau)$ for Hopf bifurcations to occur. However, this does not impose any additional constraints as in this example $\mu - |Q(\xi)|(1+\mu\tau) < \mu < \min_{j=\{1,2,3,4\}} \gamma_j$, and as we already saw in Section 5, Hopf bifurcations typically occur for $\gamma > \min_{j=\{1,2,3,4\}} \gamma_j$. So the steady state can lose stability in a Hopf bifurcation close to the fold point. This is seen in Figure 6.2(b) close to $(\gamma, x) = (\gamma_3, \theta_v)$ and is similar to the behaviour observed in Cases 2 and 3 in Section 5.4. Figure 6.2(c) shows two-parameter continuations of these bifurcations. Apart from the Hopf bifurcations giving rise to a stable periodic orbit for $\gamma \in (\gamma_1, \gamma_2)$ the rest of the dynamics observed are remarkably similar to those seen in Figure 5.12 with v increasing and constant g. In particular, as the steepness of the nonlinearities given by $m = n$ increases, the fold bifurcations approach $\gamma = \gamma_4$ and $\gamma = \gamma_3$ with the steady states at which they happen approaching θ_v, the fold bifurcations disappear at a cusp point (at $\gamma = 2.1535$), while the Hopf

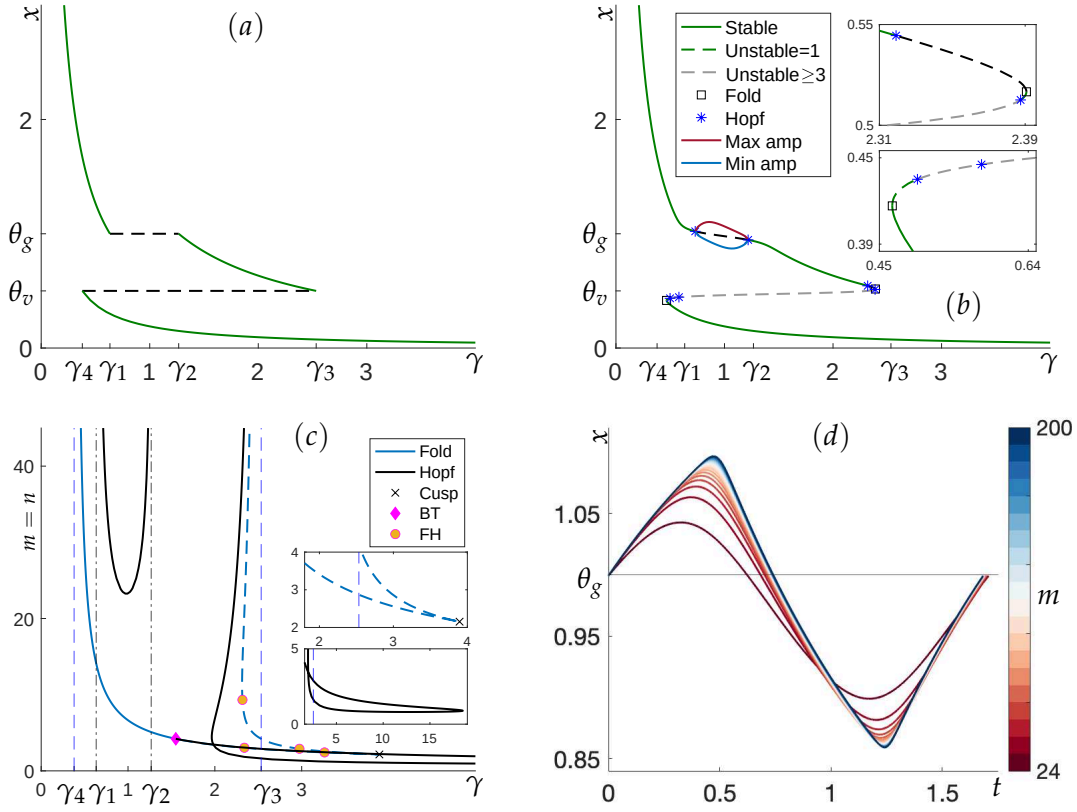


Figure 6.2: Bifurcations of (1.1)–(1.2) in the case $(g\downarrow, v\uparrow, \theta_v < \theta_g)$ with parameters $\beta = 1.4$, $\mu = 0.2$, $g^- = 1 > g^+ = 0.54$, $\theta_v = 0.5 < \theta_g = 1$, $a = 1$, and $v^- = 0.1 < v^+ = 2$. With these parameters $\mu < \gamma_4 = 0.3789 < \gamma_1 = 0.6334 < \gamma_2 = 1.2668 < \gamma_3 = 2.5335$. (a) The limiting case with g defined by (1.6) and v defined by (1.7). The stable steady state is shown as a green solid line, and the singular steady state as a black dashed line. (b) With smooth nonlinearity g and v defined by (1.3) with $m = n = 40$. Periodic solutions are represented by maximum (in red) and minimum (in blue) of x(t) on the solution. (c) Two-parameter continuations in $m = n$ and γ of the fold and the Hopf bifurcations with the other parameters as above. There is a cusp point at $(\gamma, m) = (3.8948, 2.1535)$, a BT point at $(\gamma, m) = (1.5502, 4.2073)$ and fold-Hopf points at $(\gamma, m) = (2.3399, 3.0346), (3.2632, 2.4198), (2.9741, 2.8976)$ and $(2.3178, 9.3191)$. (d) Profile of the stable periodic orbits at $\gamma = 1$ emanating from the Hopf bifurcations in (c). The color map indicates values of the continuation parameter $m = n$.

bifurcations and stable periodic orbits can occur for smaller values of $m > 0.8079$. There is also a Bogdanov–Takens bifurcation, fold-Hopf bifurcations at which Hopf bifurcations cross the fold points, and for the fold close to $\gamma = \gamma_3$ the steady state loses stability in a subcritical Hopf bifurcation for arbitrary large $m = n$, and not at the fold bifurcation.

The most significant difference between the example shown in Figure 6.2 and Cases 2 and 3 from Section 5.4 is the pair of Hopf bifurcations seen for $x \approx \theta_g$ and $\gamma \in (\gamma_1, \gamma_2)$. In Figure 6.2(c) the steady state is seen to lose stability between a pair of Hopf bifurcations for $m > 23$. Since for $m = n \gg 0$ the function v is essentially constant except for $\theta \approx \theta_v$, it follows that v is close to constant for $\theta \approx \theta_g$ and so these Hopf bifurcations and the resulting periodic orbits follow the dynamics explored in Section 5.1. Indeed, the dynamics associated with the Hopf bifurcations and periodic orbits for $x \approx \theta_g$ seen in Figure 6.2 is remarkably similar to

that seen for constant v and decreasing g in Figure 5.2 of Section 5.1. The Hopf bifurcations are supercritical, generating a stable periodic orbit which co-exists with the unstable steady state between the pair of Hopf bifurcations. The amplitude of the periodic orbit is shown for $m = n = 40$ in Figure 6.2(b), from which we see that it is always small and shrinks to zero at the Hopf bifurcations at each end of the interval.

Figure 6.2(d) depicts the profile of such stable periodic orbits for a fixed value of γ in between the two Hopf bifurcations. As $m = n \to \infty$, the periodic orbits appear to approach a limiting profile. Notice that this stable periodic orbit oscillates around θ_g but does not cross the other threshold at $\theta_v = 0.5$. Thus in the limiting case as $m = n \to \infty$ the delay will be constant and equal to τ^+ on this periodic orbit. In [31] we construct slowly oscillating periodic solutions for both constant and threshold delays, and using those constructions it can be seen that the limiting profile in Figure 6.2(d) is a slowly oscillating periodic solution.

6.1.2 Increasing g and increasing v $(g \uparrow, v \uparrow, \theta_g \neq \theta_v)$

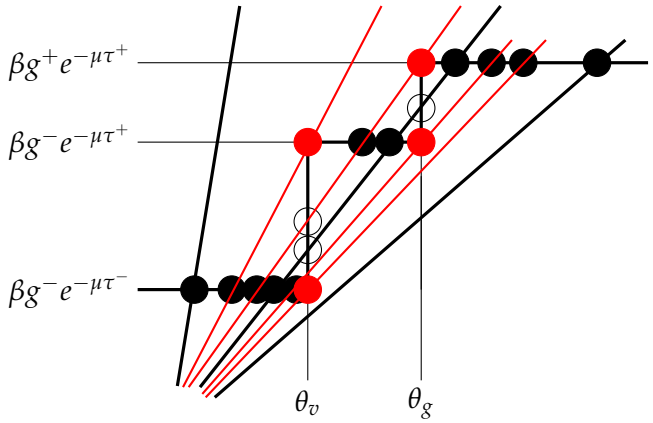


Figure 6.3: Steady states of (1.1), given by (5.2), occur at the intersections of $\xi \mapsto \beta e^{-\mu\tau(\xi)} g(\xi)$ and $\xi \mapsto \gamma\xi$. These are illustrated for various γ in the limiting case of (1.6) and (1.7) for $(g \uparrow, v \uparrow, \theta_v < \theta_g)$. The red lines denote $\gamma = \gamma_4, \gamma_2, \gamma_1, \gamma_3$ in ascending order.

When v and g are both increasing with $\theta_g \neq \theta_v$ there are up to five coexisting steady states in the limiting case as shown in Figure 6.3. The four corners at the respective thresholds θ_g and θ_v are associated with the emergence of fold bifurcations. A necessary and sufficient condition to obtain five coexisting steady states is that $(\gamma_2, \gamma_1) \cap (\gamma_4, \gamma_3) \neq \emptyset$. Note that since g is now increasing the values $\gamma_2 < \gamma_1$ have the opposite order than in previous example, see (5.4).

Figure 6.4(a) shows such an example, with five co-existing steady states in the limiting case when $\gamma \in (\gamma_2, \gamma_1)$. Referring to the discussion in Sections 5.2 and 5.4, in the limiting case the non-singular steady states are always stable, leading to tristability of steady states in this parameter interval. This then leads to tristability of steady states for smooth nonlinearities with $m = n$ sufficiently large, as depicted in Figure 6.4(b).

Following the discussion in Sections 5.2 and 5.4, the intermediate singular steady states become unstable steady states between the fold bifurcations in the smooth case. However, the behaviour is not the same at each of the folds. In the example presented in Figure 6.4 we have $\mu \in (\gamma_4, \gamma_3)$, and so we expect to see similar behaviour for $x \approx \theta_v$ as in Case 2 of Section 5.4,

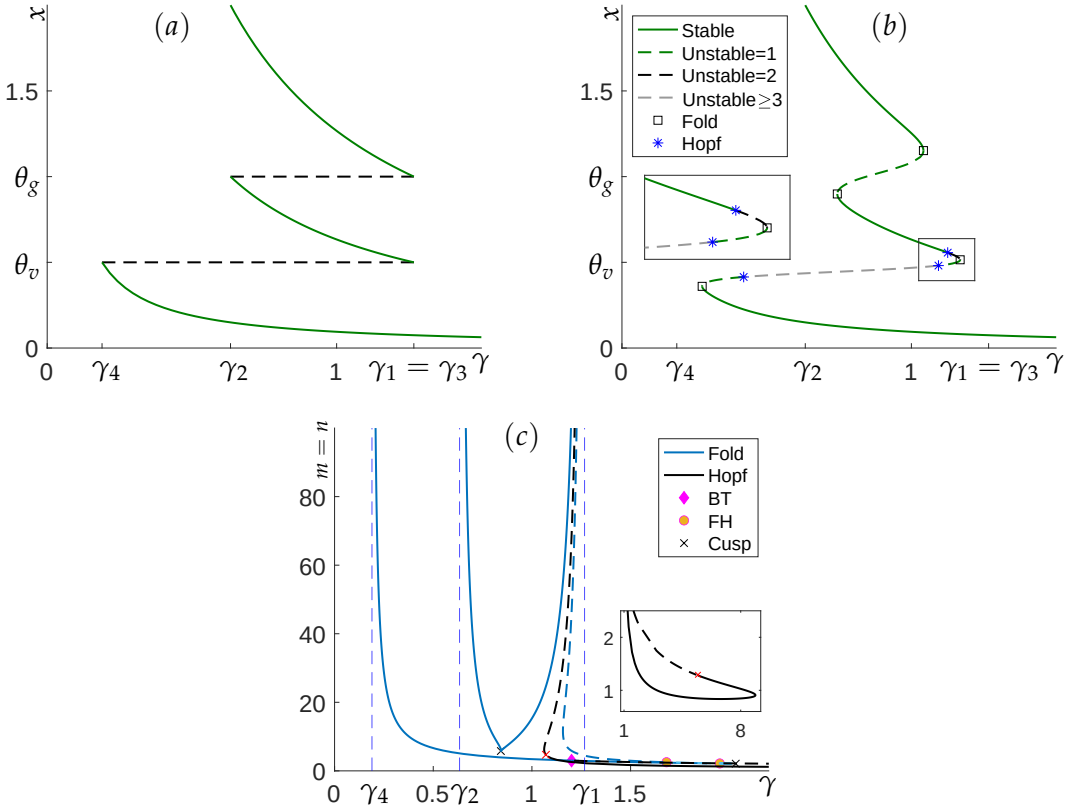


Figure 6.4: Bifurcation diagram of (1.1)–(1.2) for $(g\uparrow, v\uparrow, \theta_v < \theta_g)$ with parameters $\beta = 1.4, \mu = 0.2, g^- = 0.5, g^+ = 1, \gamma = 0.8, \theta_g = 1, \theta_v = 0.5, a = 1, v^- = 0.1$ and $v^+ = 2$. With these parameters $\gamma_4 = 0.1895 < \gamma_2 = 0.6334 < \gamma_1 = \gamma_3 = 1.2668$. (a) The limiting case with g defined by (1.6) and v defined by (1.7). The stable steady state is shown as a green solid line, and the singular steady state as a black dashed line. (b) With smooth nonlinearity g and v defined by (1.3) with $m = n = 20$. (c) Two-parameter continuations in $m = n$ and γ of the fold and the Hopf bifurcations with the other parameters as above. There are two cusp points at $(\gamma, m) = (2.0329, 2.1053)$ and $(0.84265, 5.7871)$, a BT point at $(\gamma, m) = (1.2009, 2.9825)$, two fold-Hopf points at $(\gamma, m) = (1.6820, 2.5172)$ and $(1.9519, 2.1629)$, and two Bautin bifurcations at $(\gamma, m) = (5.4365, 1.2928)$ and $(1.0709, 4.6890)$.

while since g is increasing the behaviour for $x \approx \theta_g$ should be similar to what was observed in Section 5.2, and this is what we observe.

For the fold bifurcation near $(\gamma, \xi) = (\gamma_4, \theta_v)$, the steady state loses stability at the fold for all m sufficiently large (seen for $m = n = 20$ in Figure 6.4(b)). Since v is increasing, from (6.12) there can be no Hopf bifurcations for $\gamma < \mu - |Q(\xi)|(1 + \mu\tau)$. However, $\lim_{m\to\infty} Q(\xi) = 0$ for $\xi \neq \theta_g$, and since $\gamma_4 < \mu$ there can be no Hopf bifurcations near to the fold for m sufficiently large. The same is not true for the fold close to $(\gamma, \xi) = (\gamma_3, \theta_v)$, where since $\gamma_3 > \mu$ the steady state may lose stability in a Hopf bifurcation near to the fold point, which is indeed what is seen for all m sufficiently large. This is similar to the behaviour seen in the previous example in Figure 6.2 and to Cases 2 and 3 in Section 5.4.

Following the theory of Sections 5.2 and 5.4, as $m = n$ is increased there will also be an infinite sequence of Hopf bifurcations on the unstable branches that cross θ_g and θ_v. For the branch that crosses $x = \theta_g$ these Hopf bifurcations will be confined between the fold

bifurcations as $m = n$ is increased. No Hopf bifurcations are seen on this branch in Figure 6.4(b) because the value of m is too small, with the first Hopf bifurcation only occurring for $m = n \approx 61$.

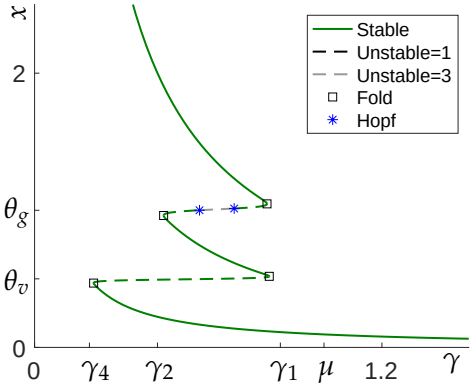


Figure 6.5: Bifurcations of (1.1)–(1.2) in the case $(g\uparrow, v\uparrow, \theta_v < \theta_g)$ with smooth nonlinearities g and v defined by (1.3) with $m = n = 100$, and other parameters $\beta = 1.4$, $\mu = 1$, $g^- = 0.5 > g^+ = 1$, $\theta_v = 0.5 < \theta_g = 1$, $a = 1$, and $v^- = 0.5 < v^+ = 2$. With these parameters $\gamma_4 = 0.1895$, $\gamma_2 = 0.4246$, and $\gamma_1 = \gamma_3 = 0.8491$.

Figure 6.5 shows another example for $(g\uparrow, v\uparrow, \theta_g \neq \theta_v)$, but this time with $\mu > \gamma_3$ which corresponds to Case 1 in Section 5.4. In this example no Hopf bifurcations are observed on the unstable branch that crosses $\xi = \theta_v$. This follows from (6.12) because there can be no Hopf bifurcations for $\gamma < \mu - |Q(\xi)|(1 + \mu\tau)$, and $|Q(\xi)| \to 0$ as $n \to \infty$ for $\xi \neq \theta_g$. On the other hand, a pair of Hopf bifurcations is observed on the branch of unstable steady states that crosses $\xi = \theta_g$. In this case (6.12) does not preclude Hopf bifurcations because Proposition 5.1(2), with $f(x, p, r)$ given by $Q(\xi)$ with parameter $p = n$ from g, shows that $|Q(\xi)|$ can be arbitrarily large for $\xi \approx \theta_g$ as $n \to \infty$.

6.2 Two Hill functions with $\theta_g = \theta_v$

The cases of $(g\uparrow, v\uparrow, \theta_v = \theta_g)$ and $(g\downarrow, v\downarrow, \theta_v = \theta_g)$ where both functions are either increasing or decreasing are reasonably straightforward. However, the two cases for which one function is increasing and the other decreasing, $(g\downarrow, v\uparrow, \theta_v = \theta_g)$ and $(g\downarrow, v\uparrow, \theta_v = \theta_g)$ are altogether more delicate and surprising.

6.2.1 g and v both increasing or decreasing $(g\uparrow, v\uparrow, \theta_g = \theta_v)$ and $(g\downarrow, v\downarrow, \theta_g = \theta_v)$

In both these cases $\xi \mapsto \beta e^{-\mu\tau(\xi)} g(\xi)$ is monotonic, so the existence of steady states is straightforward following the theory in Section 5. Writing θ_{gv} for the value of θ when $\theta_g = \theta_v$, and defining

$$\gamma_{13} = \frac{\beta e^{-\mu\tau^+} g^+}{\theta_{gv}} \quad \text{and} \quad \gamma_{24} = \frac{\beta e^{-\mu\tau^-} g^-}{\theta_{gv}}, \tag{6.13}$$

with g and v both decreasing, there is always a unique steady state. In the limiting case of piecewise constant nonlinearities (1.6)–(1.7), this steady state is stable for $\gamma < \gamma_{13}$ and $\gamma > \gamma_{24}$, and is singular for $\gamma \in (\gamma_{13}, \gamma_{24})$. In the smooth case, this gives rise to a unique stable steady state for very small or very large γ. The question then arises as to when the steady state may

lose stability in a Hopf bifurcation for γ in or near the interval $(\gamma_{13}, \gamma_{24})$. From (6.11) when g and v are both decreasing Hopf bifurcations cannot occur if

$$\gamma > \mu + |Q(\xi)|(1+\mu\tau) = \mu + \gamma(1+\mu\tau)\left|\frac{\xi g'(\xi)}{g(\xi)}\right|. \tag{6.14}$$

Thus when g is a constant function, which implies that $Q = 0$, there can only be Hopf bifurcations when $\gamma < \mu$ as seen in Section 5.3. In contrast, by Proposition 5.1, we have $|\theta_{gv} g'(\theta_{gv})/g(\theta_{gv})| \to \infty$ as $n \to \infty$ while (6.14) cannot be satisfied when $|\xi g'(\xi)/g(\xi)| > 1$. Consequently for n large there is no constraint preventing Hopf bifurcations, and we expect to see Hopf bifurcations for $\xi \approx \theta_{gv}$ for n sufficiently large, just as we did in Section 5.1.

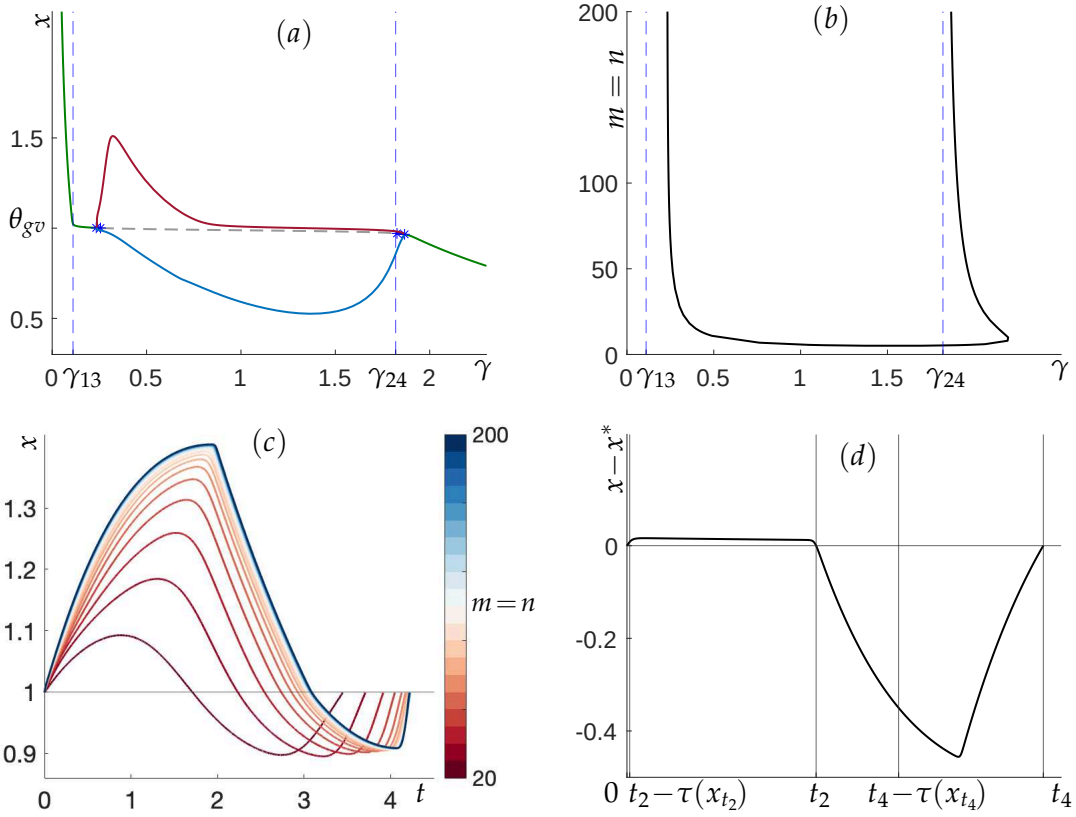


Figure 6.6: Bifurcation diagram of (1.1)–(1.2) for $(g\downarrow, v\downarrow, \theta_g = \theta_v)$ with parameters $\beta = 3$, $\mu = 0.5$, $\gamma = 1$, $\theta_g = \theta_v = \theta_{vg} = 1$, $a = 1$, $g^- = 1 > g^+ = 0.1$, and $v^- = 1 > v^+ = 0.5$. With these parameters $\gamma_{13} = 0.1104$ and $\gamma_{24} = 1.8196$. (a) With nonlinearities g and v defined by (1.3) with $m = n = 200$. The Hopf bifurcations associated with stability change occur at $(\gamma, x) = (0.2333, 1.0018)$ with $\omega = 2.3128$ and $(\gamma, x) = (1.8669, 0.9659)$ with $\omega = 2.3155$. Stable periodic orbits are represented by maximum (in red) and minimum (in blue) of $x(t)$ on the periodic solution. (b) Two-parameter continuations in $m = n$ and γ of the Hopf bifurcations with the other parameters as above. (c) Profile of the stable periodic orbits with $\gamma = 0.4$. The color map indicates values of the continuation parameter $m = n$. (d) Profile of the stable periodic orbit with $m = 200$ and $\gamma = 1.3$ and the other parameters as above. The associated steady state is at $x^* = 0.9848$. The period of the periodic orbit $T = t_4 = 2.8772$.

Figure 6.6 illustrates the case of g and v both decreasing with $\theta_v = \theta_g = \theta_{gv}$. Panel (a) shows the dynamics for smooth nonlinearities g defined by (1.3) with $m = n = 200$. In this

case the steady state is unstable between two supercritical Hopf bifurcations. These Hopf bifurcations occur close to but slightly to the right of the limiting values γ_{13} and γ_{24}. The resulting periodic orbit is stable on the whole of the interval between the Hopf bifurcations.

Figure 6.6(b) shows a two-parameter continuation of the curve of Hopf bifurcations at which the stable periodic orbit is created. The amplitude plot of the periodic orbits in panel (a) of Figure 6.6 suggests that different behavior is seen near the two Hopf points, and this is illustrated in the periodic solution profiles shown in panels (c) and (d). In (c) with $\gamma = 0.4$, as $m = n$ is increased the profile of the periodic orbit converges to a limiting profile (which we construct in [31]). The situation is different with $\gamma > 1$ as shown in panel (d) with $\gamma = 1.3$ and $m = n = 200$. While there is again a large period periodic orbit, the profile is quite different, with $x(t)$ never much greater than x^* even though there is a large segment $t \in (t_0, t_2)$ of the periodic orbit for which $x(t) > x^*$. This large subinterval of the periodic orbit on which the solution is nearly constant looks quite different to the periodic orbits seen in (c).

The case of $(g \uparrow, v \uparrow, \theta_v = \theta_g)$ is analogous to the scenarios considered in Sections 5.2 and 5.4 which lead to a pair of fold bifurcations when (6.2) is satisfied, and an interval of γ values for which two stable steady states coexist with an unstable steady state. Because v and g are both increasing, the functions $\xi \mapsto \xi g'(\xi)/g(\xi)$ and $\xi \mapsto \mu\tau(\xi)\xi v'(\xi)/v(\xi)$ are both positive, and moreover $\theta_{gv} g'(\theta_{gv})/g(\theta_{gv})$ and $\mu\tau(\theta_{gv})\theta_{gv} v'(\theta_{gv})/v(\theta_{gv})$ are increasing functions of n and m, respectively. Since the condition (6.2) is more easily satisfied in this case than when one of these functions is zero or decreasing, fold bifurcations can occur for smaller values of m or n for $(g \uparrow, v \uparrow, \theta_v = \theta_g)$ than was seen for $(g \uparrow, v \leftrightarrow)$ or $(g \leftrightarrow, v \uparrow)$. As $m = n \to \infty$ the interval of bistability converges to $(\gamma_{24}, \gamma_{13})$. Equation (6.12) imposes a constraint on Hopf bifurcations to occur when g and v are both increasing. However, since $Q(\theta_g) \to \infty$ as $n \to \infty$, Hopf bifurcations can occur for all n sufficiently large.

6.2.2 g and v with opposing monotonicity $(g \uparrow, v \downarrow, \theta_g = \theta_v)$ and $(g \downarrow, v \uparrow, \theta_g = \theta_v)$

These cases are altogether more delicate and surprising, and need explaining in some detail. For the smooth nonlinearities when the power m or n is large, and the function $g(x)$ or $v(x)$ is close to a step function, we will informally refer to the part of the function with large gradient as the interface. When $m \gg n$ the $v(x)$ function will have a much narrower interface than the function $g(x)$, while $g(x)$ will have a narrower interface when $n \gg m$. These interfaces are centred at $x = \theta_v$ and $x = \theta_g$, so when $\theta_g = \theta_v$ they will overlap or if $m \gg n$ or $m \ll n$ the interval on which one interface occurs will be inside the interval on which the other interface occurs. When g and v have opposing monotonicity the dynamics of (1.1)–(1.2) will be very different depending on which interface is narrower.

From (6.2) there will be a fold bifurcation when

$$0 = M(\xi) = \frac{\xi g'(\xi)}{g(\xi)} + \mu\tau(\xi)\frac{\xi v'(\xi)}{v(\xi)} - 1.$$

When v is increasing by Proposition 5.1

$$\lim_{m\to\infty} \frac{\theta_v v'(\theta_v)}{v(\theta_v)} = +\infty, \qquad \text{and} \qquad \lim_{\xi\to 0} \frac{\xi v'(\xi)}{v(\xi)} = \lim_{\xi\to+\infty} \frac{\xi v'(\xi)}{v(\xi)} = 0, \tag{6.15}$$

which ensures that a fold bifurcation must occur for m sufficiently large when n is held constant. If g is decreasing then $\xi g'(\xi)/g(\xi) < 0$ and the value of m required for a fold bifurcation to occur will be larger than when this term is non-negative, that is, when g is constant or increasing. Those cases were considered in Sections 5.4 and 6.2.1. An analogous argument to

(6.15) shows that when g is increasing a fold bifurcation will occur for n sufficiently large when m is held constant.

A more delicate question is, what happens when $m \to \infty$ and $n \to \infty$ with g and v having opposite monotonicities. In that case one could expect that whichever of the terms $\xi g'(\xi)/g(\xi)$ or $\xi v'(\xi)/v(\xi)$ grows faster as $m, n \to \infty$ will determine where and whether fold bifurcations occur, however, as we will see, the behaviour is more nuanced than that.

We will now consider the case $(g \downarrow, v \uparrow, \theta_g = \theta_v = \theta_{gv})$ for which $0 < r_v < 1 < r_g$, where $r_g = g^-/g^+$ and $r_v = v^-/v^+$. Evaluating $M(\theta_{gv})$ using Proposition 5.1(2) we obtain

$$M(\theta_{gv}) = \frac{\theta_{gv} g'(\theta_{gv})}{g(\theta_{gv})} + \mu\tau(\theta_{gv}) \frac{\theta_{gv} v'(\theta_{gv})}{v(\theta_{gv})} - 1 = \frac{n(1-r_g)}{2(1+r_g)} + \mu\tau(\theta_{gv}) \frac{m(1-r_v)}{2(1+r_v)} - 1. \tag{6.16}$$

Then if

$$\mu\tau(\theta_{gv}) \frac{(1-r_v)}{(1+r_v)} > \frac{(r_g-1)}{(1+r_g)}, \tag{6.17}$$

it follows that $M(\theta_{gv}) \to +\infty$ as $m = n \to \infty$, and in particular $M(\theta_{gv}) > 0$ for all $m = n$ sufficiently large. This ensures that there is a pair of fold bifurcations as γ is varied with $m = n$ sufficiently large.

Of course, since m and n appear in the different nonlinearities v and g modelling different processes, there is no reason beyond mathematical convenience to assume that $m = n$. So, it is interesting to consider whether fold bifurcations still occur as $m \to \infty$ and $n \to \infty$ independently.

The condition (6.17) was arrived at by evaluating $M(\theta_{gv})$, which has a convenient form because of Proposition 5.1(2). However, we will show below that it is not a necessary condition for existence of fold bifurcations. To see this, note that for the case we consider we have $0 < r_v < 1 < r_g$, and hence

$$r_v^{1/2m} \theta_{gv} < \theta_{gv} < r_g^{1/2n} \theta_{gv}. \tag{6.18}$$

From Proposition 5.1(4) the positive function $\xi v'(\xi)/v(\xi)$ achieves its maximum at $\xi = r_v^{1/2m}\theta_{gv} < \theta_{gv}$ and the negative function $\xi g'(\xi)/g(\xi)$ achieves its minimum at $\xi = r_g^{1/2n}\theta_{gv} > \theta_{gv}$. Thus, for $\xi \in (r_v^{1/2m}\theta_{gv}, r_g^{1/2n}\theta_{gv})$ we have $\xi v'(\xi)/v(\xi)$ and $\xi g'(\xi)/g(\xi)$ as well as $\tau(\xi)$ all decreasing functions of ξ and it follows that $M(\xi)$ is decreasing for $\xi \in (r_v^{1/2m}\theta_{gv}, r_g^{1/2n}\theta_{gv})$ as well as satisfying $\lim_{\xi \to \pm\infty} M(\xi) = -1$. Consequently, we expect $M(\xi)$ to obtain its maximum value for $\xi \lessapprox r_v^{1/2m}\theta_{gv}$. If this maximum value is positive, then (6.2) will have at least two solutions, indicating two fold bifurcations for different ξ values, one on each side of this maximum. While we will not consider the other case of $(g \uparrow, v \downarrow, \theta_g = \theta_v = \theta_{gv})$ in detail, we note that in that case g would be the increasing function, with $r_g = g^-/g^+ < 1 < r_v$, and for $\xi \in (r_g^{1/2n}\theta_{gv}, r_v^{1/2m}\theta_{gv})$ we would have $\xi g'(\xi)/g(\xi)$ positive and decreasing, $\xi v'(\xi)/v(\xi)$ negative and decreasing, and $\tau(\xi)$ increasing, from which it follows that $M(\xi)$ is decreasing for $\xi \in (r_g^{1/2n}\theta_{gv}, r_v^{1/2m}\theta_{gv})$, so $M(\xi)$ would obtain its maximum value for $\xi < \theta_{gv}$.

To study the steady states and fold bifurcations of (1.1)–(1.2) for $(g \downarrow, v \uparrow, \theta_g = \theta_v = \theta_{gv})$ we will work directly with the function $h(\xi)$ defined in (5.2). Suppose the parameters are chosen so that $\beta e^{-\mu\tau^-} g^- > \beta e^{-\mu\tau^+} g^+$. Then

$$\beta e^{-\mu\tau^-} g^- = \lim_{\xi \to 0} \beta e^{-\mu\tau(\xi)} g(\xi) > \lim_{\xi \to +\infty} \beta e^{-\mu\tau(\xi)} g(\xi) = \beta e^{-\mu\tau^+} g^+. \tag{6.19}$$

Figure 6.7 depicts such an example for which $\beta e^{-\mu\tau(\xi)} g(\xi)$ is a monotonically decreasing function of ξ in both the smooth case and the limiting case as $m = n \to \infty$, so that there is

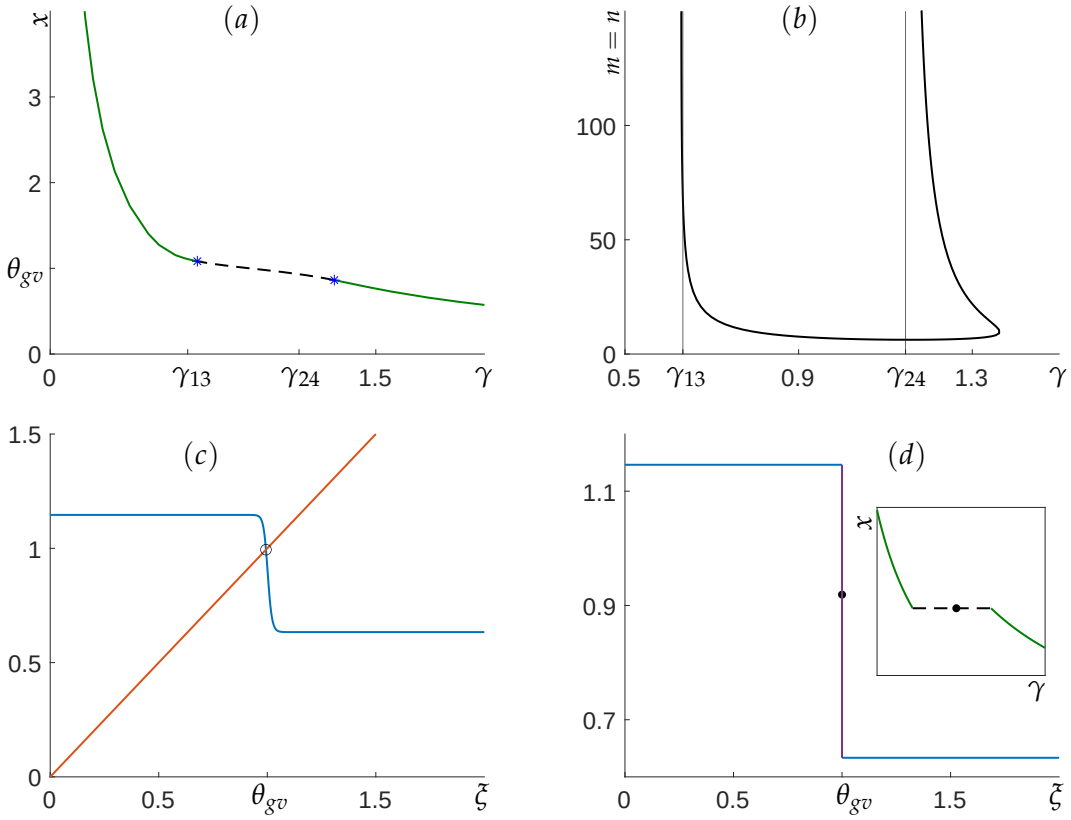


Figure 6.7: Bifurcations of (1.1)–(1.2) for $(g\downarrow, v\uparrow, \theta_g = \theta_v = \theta_{gv})$ with $\beta = 1.4$, $\mu = 0.2$, $g^- = 1$, $g^+ = 0.5$, $\theta_g = \theta_v = \theta_{gv} = 1$, $a = 1$, $v^- = 1$ and $v^+ = 2$. With these parameters $\beta e^{-\mu\tau^-} g^- > \beta e^{-\mu\tau(\theta_{gv})} g(\theta_{gv}) > \beta e^{-\mu\tau^+} g^+$. (a) With smooth nonlinearity g and v defined by (1.3) with $m = n = 100$. (b) Two-parameter continuations in $m = n$ and γ of the Hopf bifurcation at which the steady state loses stability with the other parameters as above. The vertical lines at $\gamma = \gamma_{13} = 0.6334$ and $\gamma = \gamma_{24} = 1.1462$ denote the location of the Hopf bifurcations in the limiting case as $m = n \to \infty$. (c) Graphs of $\xi \mapsto \beta e^{-\mu\tau(\xi)} g(\xi)$ and $\xi \mapsto \gamma\xi$ with $m = n = 100$ and $\gamma = 1$. Since $\xi \mapsto \beta e^{-\mu\tau(\xi)} g(\xi)$ is monotonically decreasing there is a unique intersection and hence a unique steady state for and $\gamma > 0$. (d) Graph of $\xi \mapsto \beta e^{-\mu\tau(\xi)} g(\xi)$ in the limiting case, with the black dot denoting the value of $\beta e^{-\mu\tau(\theta_{gv})} g(\theta_{gv})$. The inset depicts the continuation of the steady states in the limiting case with g and v defined by (1.6) and (1.7). Stable steady states are shown as green solid lines, and the singular steady state as a black dashed line.

always a unique steady state. One could easily and incorrectly assume that $\xi \mapsto \beta e^{-\mu\tau(\xi)} g(\xi)$ will be monotonically decreasing whenever (6.19) holds. However, this is not always true, and indeed is never true when (6.17) holds. To show this, recalling the definitions of g and v given by (1.3) in the smooth case, notice that at the threshold, the function values are independent of the nonlinearity m or n, with

$$g(\theta_g) = \frac{g^- + g^+}{2} \quad \text{and} \quad v(\theta_v) = \frac{v^- + v^+}{2}. \tag{6.20}$$

Thus for $(g\downarrow, v\uparrow, \theta_g = \theta_v = \theta_{gv})$ and $0 < r_v < 1 < r_g$ it holds that

$$g^- > g(\theta_{gv}) > g^+, \tag{6.21}$$

while $v^+ > v(\theta_{gv}) > v^-$ implies

$$\tau^- > \tau(\theta_{gv}) > \tau^+. \tag{6.22}$$

Then (6.17) implies that

$$\mu\tau^-\frac{(1-r_v)}{(1+r_v)} > \mu\tau(\theta_{gv})\frac{(1-r_v)}{(1+r_v)} > \frac{r_g-1}{1+r_g} > \ln\left(1+\frac{r_g-1}{1+r_g}\right) = \ln\left(\frac{2r_g}{1+r_g}\right). \tag{6.23}$$

But

$$\frac{2r_g}{1+r_g} = \frac{2g^-/g^+}{1+g^-/g^+} = \frac{2g^-}{g^++g^-} = \frac{g^-}{g(\theta_{gv})},$$

and

$$\tau^-\frac{(1-r_v)}{(1+r_v)} = \frac{a}{v^-}\left(\frac{v^+-v^-}{v^-+v^+}\right) = \frac{a}{v^-} - \frac{2a}{v^-+v^+} = \tau^- - \tau(\theta_{gv}).$$

Thus

$$\begin{aligned}
\mu\tau^-\frac{(1-r_v)}{(1+r_v)} > \ln\left(\frac{2r_g}{1+r_g}\right) \quad &\iff \quad \mu(\tau^- - \tau(\theta_{gv})) > \ln(g^-/g(\theta_{gv})) \\
&\iff \quad e^{-\mu(\tau(\theta_{gv})-\tau^-)} > g^-/g(\theta_{gv}) \\
&\iff \quad \beta e^{-\mu\tau(\theta_{gv})}g(\theta_{gv}) > \beta e^{-\mu\tau^-}g^-.
\end{aligned}$$

Thus if the parameters are chosen so that (6.17) and (6.19) both hold then

$$\beta e^{-\mu\tau(\theta_{gv})}g(\theta_{gv}) > \beta e^{-\mu\tau^-}g^- > \beta e^{-\mu\tau^+}g^+, \tag{6.24}$$

and the function $\xi \mapsto \beta e^{-\mu\tau(\xi)}g(\xi)$ is not monotonic. This was not the case for the example in Figure 6.7, but below we will demonstrate examples where both (6.17) and (6.24) hold, and more interesting dynamics arise. Moreover, the inequalities in (6.23) are not all tight, and we will also show that it is possible to obtain (6.24) for parameters for which (6.17) does not hold.

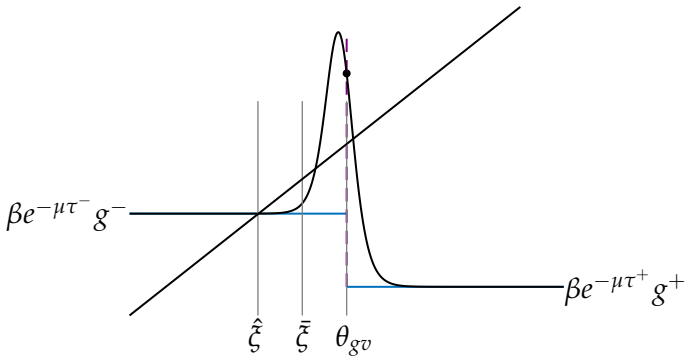


Figure 6.8: An illustration of $\beta e^{-\mu\tau(\xi)}g(\xi)$ and $\hat{\gamma}\xi$ in the case $(g\downarrow, v\uparrow, \theta_g = \theta_v = \theta_{gv})$ for both smooth nonlinearities and in the limiting case $m = n \to \infty$ when (6.24) holds.

Now suppose that (6.24) holds and consider $h(\xi)$ defined in (5.2). This situation is illustrated in Figure 6.8. Recall (6.13) and also let

$$\gamma_{gv} := \frac{\beta e^{-\mu\tau(\theta_{gv})}g(\theta_{gv})}{\theta_{gv}}. \tag{6.25}$$

We will now show that when $\gamma \in (\gamma_{24}, \gamma_{gv})$ for all m sufficiently large there are three steady states. From (5.2), we have $h(0) = \beta e^{-\mu\tau^-} g^- > 0$ and $h(\xi) \to -\infty$ as $\xi \to +\infty$. Moreover, (6.25) and $\gamma < \gamma_{gv}$ implies

$$h(\theta_{gv}) = \beta e^{-\mu\tau(\theta_{gv})} g(\theta_{gv}) - \gamma\theta_{gv} = \gamma_{gv}\theta_{gv} - \gamma\theta_{gv} = (\gamma_{gv} - \gamma)\theta_{gv} > 0.$$

From $h(\theta_{gv}) > 0$ the intermediate value theorem immediately yields the existence of one steady state $\xi > \theta_{gv}$; to show that there are three steady states it suffices to show that there exists $\bar{\xi} \in (0, \theta_{gv})$ with $h(\bar{\xi}) < 0$.

For any $\hat{\gamma} \in (\gamma_{24}, \gamma_{gv})$ we have $\hat{\gamma}\theta_{gv} \in (\beta e^{-\mu\tau^-} g^-, \beta e^{-\mu\tau(\theta_{gv})} g(\theta_{gv}))$ so for some $\varepsilon > 0$ it holds that $\hat{\gamma}\theta_{gv} = \beta e^{-\mu\tau^-} g^- + 2\hat{\gamma}\varepsilon$. This implies that the point $\hat{\xi}$ (shown in Figure 6.8) where the line $\hat{\gamma}\xi$ intersects $\beta e^{-\mu\tau^-} g^-$ satisfies $\hat{\xi} = \beta e^{-\mu\tau^-} g^- / \hat{\gamma} = \theta_{gv} - 2\varepsilon$. Let

$$\bar{\xi} = \theta_{gv} - \varepsilon = \frac{1}{2}(\hat{\xi} + \theta_{gv}) \in (\hat{\xi}, \theta_{gv}).$$

Then

$$\hat{\gamma}\bar{\xi} = \hat{\gamma}(\theta_{gv} - \varepsilon) = \beta e^{-\mu\tau^-} g^- + 2\hat{\gamma}\varepsilon - \hat{\gamma}\varepsilon = \beta e^{-\mu\tau^-} g^- + \hat{\gamma}\varepsilon.$$

We have $g(\bar{\xi}) < g^-$ since g is strictly decreasing, and then

$$h(\bar{\xi}) = \beta e^{-\mu\tau(\bar{\xi})} g(\bar{\xi}) - \hat{\gamma}\bar{\xi} < \beta e^{-\mu\tau(\bar{\xi})} g^- - (\beta e^{-\mu\tau^-} g^- + \hat{\gamma}\varepsilon) = \beta(e^{-\mu\tau(\bar{\xi})} - e^{-\mu\tau^-}) g^- - \hat{\gamma}\varepsilon.$$

Since $\bar{\xi} < \theta_{gv}$, using the pointwise limit of the Hill function to its piecewise constant limit we have $\lim_{m\to\infty} \tau(\bar{\xi}) = \tau^-$. It then follows from the previous equation that there exists $m > 0$ such that $h(\bar{\xi}) < 0$, which shows that $h(\xi)$ has at least three sign changes, and so there are three co-existing steady states.

When $m > 1$ and $n > 1$ it is easy to verify that

$$\lim_{\xi\to 0} \frac{d}{d\xi}\left(\beta e^{-\mu\tau(\xi)} g(\xi)\right) = \lim_{\xi\to+\infty} \frac{d}{d\xi}\left(\beta e^{-\mu\tau(\xi)} g(\xi)\right) = 0,$$

from which it follows that there is a unique steady state for all γ sufficiently small or large. Bifurcations from this steady state will depend on the nonlinearities g and v.

In the case of smooth nonlinearities when one of m or n is held fixed while the other one is increased to infinity the arguments of Sections 5 can be adapted to show that in the limiting case $m \to \infty$ the fold bifurcations occur at

$$\gamma_4 = \frac{\beta e^{-\mu\tau^-} g(\theta_{gv})}{\theta_{gv}} < \gamma_3 = \frac{\beta e^{-\mu\tau^+} g(\theta_{gv})}{\theta_{gv}}. \tag{6.26}$$

with singular solutions for $\gamma \in (\gamma_4, \gamma_3)$. Similarly, with m fixed in the limiting case $n \to \infty$, singular solutions occur for $\gamma \in (\gamma_1, \gamma_2)$ where

$$\gamma_1 = \frac{\beta e^{-\mu\tau(\theta_{gv})} g^+}{\theta_{gv}} < \gamma_2 = \frac{\beta e^{-\mu\tau(\theta_{gv})} g^-}{\theta_{gv}}. \tag{6.27}$$

With g decreasing, whether fold bifurcations occur at the ends of this interval will depend also on the steepness of v. These limiting cases correspond to solutions of $h(\xi) = 0$ where the line $\gamma\xi$ intersects the curve $\beta e^{-\mu\tau(\xi)} g(\xi)$ at its maxima and minima which are illustrated in panels (a) and (c) of Figure 6.9 when one of the values m, n is fixed and the other tends to ∞.

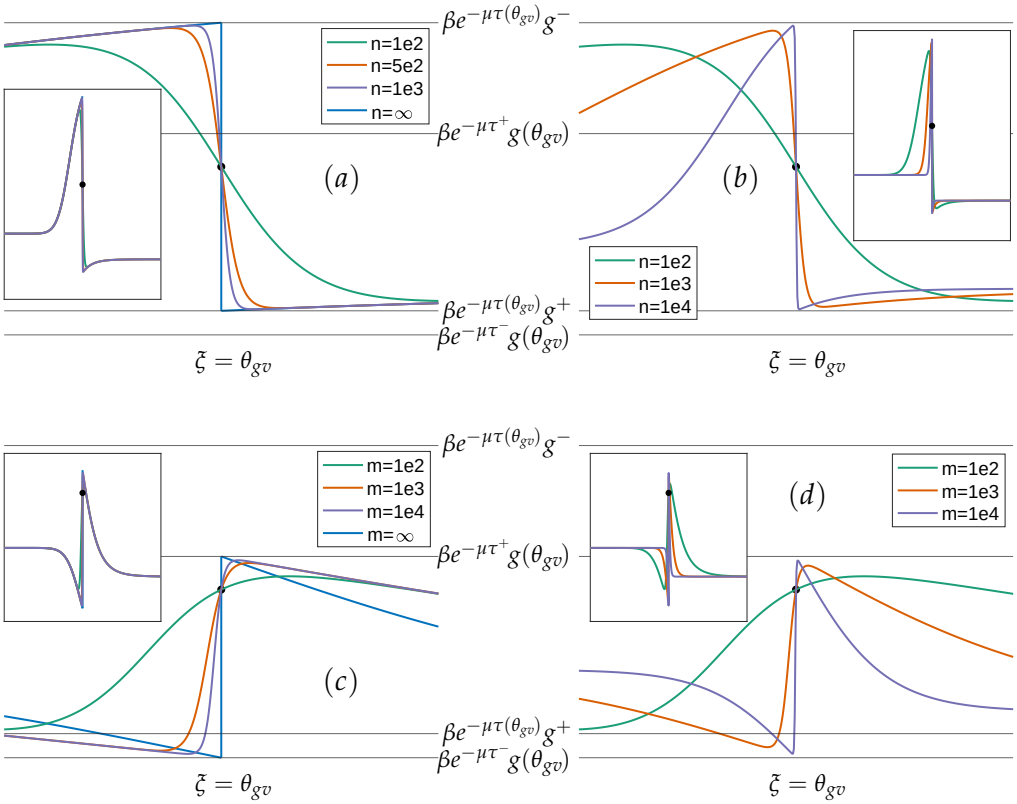


Figure 6.9: Illustrations of the behaviour of the function $\xi \mapsto \beta e^{-\mu\tau(\xi)}g(\xi)$ in the case $(g\downarrow, v\uparrow, \theta_g = \theta_v = \theta_{gv})$ with $\beta e^{-\mu\tau(\theta_{gv})}g(\theta_{gv}) > \beta e^{-\mu\tau^-}g^- > \beta e^{-\mu\tau^+}g^+$ as m and/or n tend to infinity, showing that different maxima and minima are possible depending on the limits taken. The insets show the function on the interval $\xi \in [0, 2\theta_{gv}]$ while the main panels show the behaviour for $\xi \approx \theta_{gv}$. The values of m and n are taken to be (a) $m = 10$ fixed and $n \to \infty$, (b) $m^2 = n \to \infty$, (c) $n = 10$ fixed and $m \to \infty$, and, (d) $n = \sqrt{m} \to \infty$. The values of the other parameters are $\beta = 1.4$, $\mu = 0.2$, $g^- = 1$, $g^+ = 0.5$, $\theta_g = \theta_v = \theta_{gv} = 1$, $a = 1$, $v^- = 0.3$ and $v^+ = 2$.

Panels (b) and (d) of Figure 6.9 illustrate what happens when m and n both tend to infinity, but one is increased much faster than the other, showing that the same extrema are observed as when the more slowly increasing exponent is held fixed. Since the maxima and minima of the nonlinearities converge to different values depending on whether m or n is increasing faster, and the limiting locations of the fold bifurcations are determined by the extrema, the location of the fold bifurcations as $m \to \infty$ and $n \to \infty$ will depend on how these limits are approached. We will investigate the dynamics for different relationships between m and n below.

Since the value of $\beta e^{-\mu\tau(\theta_{gv})}g(\theta_{gv})$ is independent of m and n, the maximum of $\xi \mapsto \beta e^{-\mu\tau(\xi)}g(\xi)$ cannot be smaller than this value, which imposes the bound that the rightmost fold bifurcation must occur for $\gamma \geq \gamma_{gv}$. Using (5.2) and (6.2) it is easy to show that $h'(\xi) = \gamma M(\xi)$ when $h(\xi) = 0$, that is at a steady state. Consequently, if $M(\theta_{gv}) = 0$ the right fold bifurcation will occur at $\xi = \theta_{gv}$ with $\gamma = \gamma_{gv}$, while if $M(\theta_{gv}) < 0$ then $h'(\theta_{gv}) < 0$ so the fold bifurcation occurs for $\xi < \theta_{gv}$ and $\gamma > \gamma_{gv}$. Similarly, if $M(\theta_{gv}) > 0$ the fold bifurcation will occur for $\xi > \theta_{gv}$ and $\gamma > \gamma_{gv}$. These situations are illustrated in Figures 6.10 and 6.11

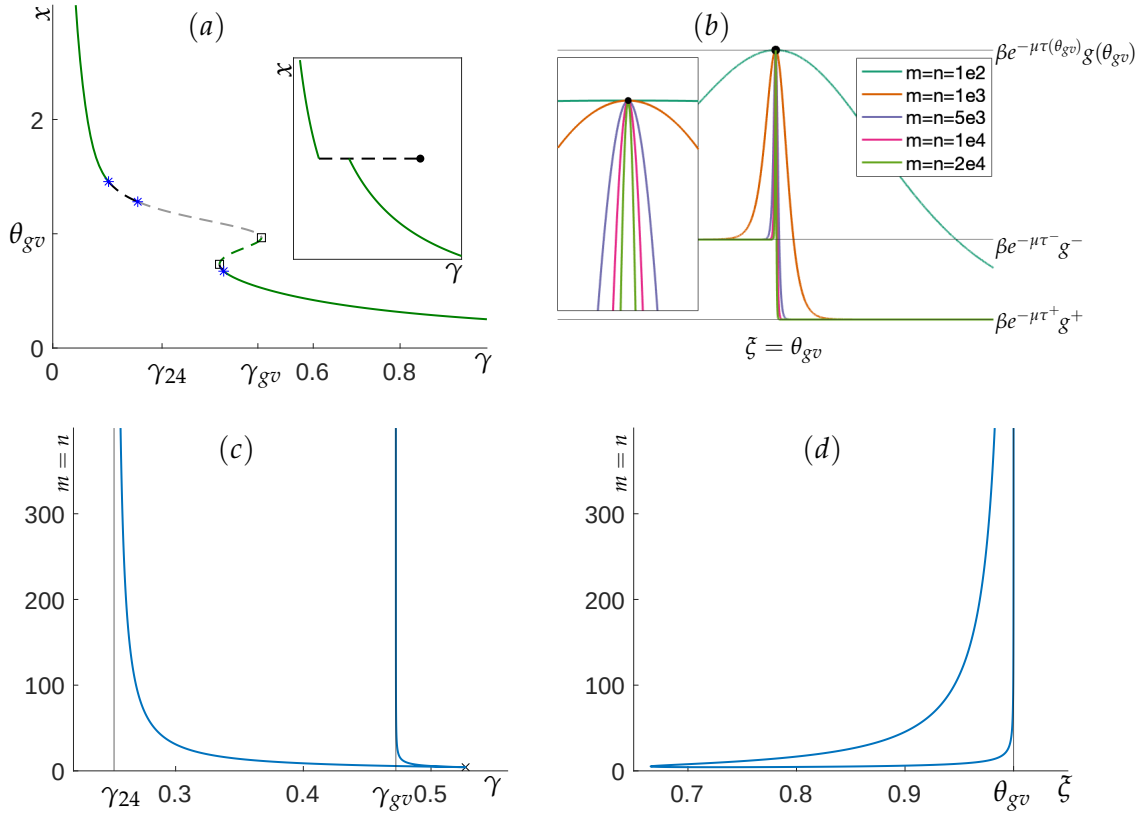


Figure 6.10: Bifurcations of (1.1)–(1.2) in the case $(g\downarrow, v\uparrow, \theta_g = \theta_v = \theta_{gv})$ with $\beta e^{-\mu\tau(\theta_{gv})}g(\theta_{gv}) > \beta e^{-\mu\tau^-}g^- > \beta e^{-\mu\tau^+}g^+$ and $m = n$. Parameter values are $\beta = 10$, $\mu = 0.2$, $g^- = 1$, $g^+ = 1/10$, $\theta_{gv} = 1$, $v^- = 1$, $v^+ = 2$ and $a = 18.4091$, for which $M(\theta_{gv}) = -1$ when $m = n$. (a) With smooth nonlinearity g and v defined by (1.3) with $m = n = 10$, and inset showing the limiting case as $m = n \to \infty$. (b) The behaviour of $\xi \mapsto \beta e^{-\mu\tau(\xi)}g(\xi)$ as $m = n \to \infty$ with the inset showing that the maximum of the function is $\beta e^{-\mu\tau(\theta_{gv})}g(\theta_{gv})$ in the limiting case. (c) Two-parameter continuation in $m = n$ and γ of the fold bifurcation with the other parameters as above. The fold bifurcations are always associated with unstable steady states, and there is a cusp point at $(\gamma, m) = (0.5271, 4.2830)$. In the limiting case as $m = n \to \infty$ the folds occur at $\gamma = \gamma_{24} = 0.2518$ (recall (6.13)) and $\gamma = \gamma_{gv} = 0.4725$. (d) Two-parameter continuation in $m = n$ and ξ of the fold bifurcations with the other parameters as above.

which we now describe in more detail.

From (6.16), for any parameter set with $0 < r_v < 1 < r_g$ taking

$$m = \frac{1}{\mu\tau(\theta_{gv})}\frac{1+r_v}{1-r_v}\left[2 + n\frac{r_g-1}{r_g+1}\right],$$

ensures that $M(\theta_{gv}) = 0$. However, for the example in Figure 6.10, we prefer to set $m = n$ and choose the other parameters to enforce equality in (6.17), which ensures that $M(\theta_{gv}) = -1$ for all $m = n > 0$. This results in a fold bifurcation at $\gamma = \gamma_{gv}$ in the limiting case as $m = n \to \infty$, since as already noted $M(\xi)$ is decreasing at $\xi = \theta_{gv}$ and also both $\xi v'(\xi)/v(\xi)$ and $\xi g'(\xi)/g(\xi)$ become increasingly steep as $m = n \to \infty$ so the point ξ at which $M(\xi) = 0$ approaches θ_{gv} as $m = n \to \infty$. This is illustrated in Figure 6.10 where panel (b) shows that the maximum of $\xi \mapsto \beta e^{-\mu\tau(\xi)}g(\xi)$ tends to $\beta e^{-\mu\tau(\theta_{gv})}g(\theta_{gv})$ as $m = n \to \infty$ while panels (c)

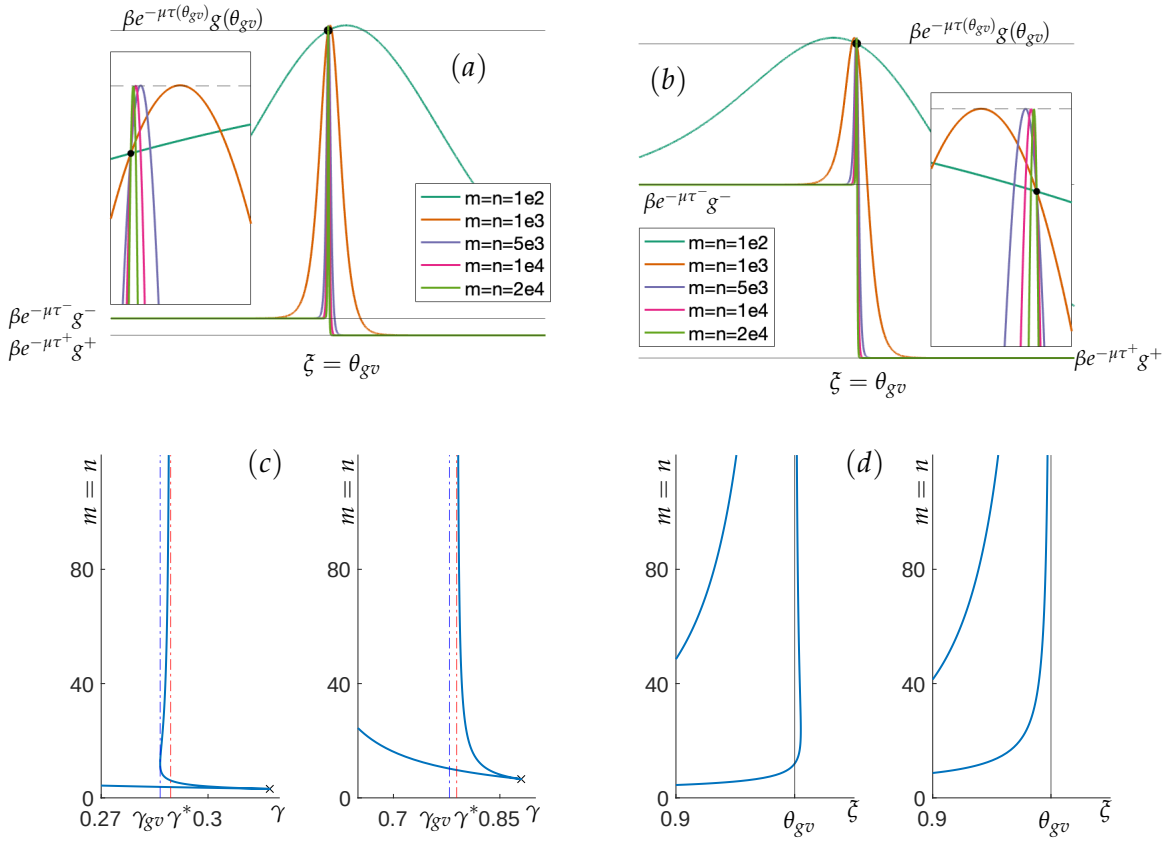


Figure 6.11: (a) and (b) The behaviour of $\xi \mapsto \beta e^{-\mu\tau(\xi)} g(\xi)$ as $m = n \to \infty$ in the case $(g\downarrow, v\uparrow, \theta_g = \theta_v = \theta_{gv})$ with $\beta e^{-\mu\tau(\theta_{gv})} g(\theta_{gv}) > \beta e^{-\mu\tau^-} g^- > \beta e^{-\mu\tau^+} g^+$ with the same parameter values as in Figure 6.10 except for the value of a. In (a) $a = 22.1591$ and in (b) $a = 14.6591$. The insets show that $\lim_{m=n\to\infty} \max_\xi \{\beta e^{-\mu\tau(\xi)} g(\xi)\} > \beta e^{-\mu\tau(\theta_{gv})} g(\theta_{gv})$, with the maximum indicated by the dashed line in the inset, and the value at θ_{gv} by the black dot. (c) and (d) show the corresponding two-parameter continuations in $m = n$ and either (c) γ or (d) ξ of the fold bifurcations with the left panel showing the case of $a = 22.1591$ and the right panel for $a = 14.6591$. The fold bifurcations are always associated with unstable steady states. In (c) the asymptote denoted γ^* for the fold bifurcation is obtained dividing the maximum value of $\beta e^{-\mu\tau(\xi)} g(\xi)$ from (a) and (b) by θ_{gv} and is seen to be strictly larger than γ_{gv} in both cases.

and (d) show that the corresponding fold bifurcation asymptotes to $\gamma = \gamma_{gv}$ and $\xi = \theta_{gv}$ as $m = n \to \infty$.

Figure 6.11 illustrates two examples with the same parameters as in Figure 6.10, except for the length parameter a. Changing just the length parameter a leaves the ratio r_v unchanged, but changes the value of $\tau(\theta_{gv})$ as well as τ^- and τ^+. With $a = 22.1591$ and $m = n$ it follows that $M(\theta_{gv}) = n/12 - 1$ and $\lim_{m=n\to\infty} M(\theta_{gv}) = +\infty$, while for $a = 14.6591$ we have $M(\theta_{gv}) = -n/12 - 1$ with $\lim_{m=n\to\infty} M(\theta_{gv}) = -\infty$. As Figure 6.11(a) and (b) show, for both of these parameter sets, (6.24) is satisfied and $\xi \mapsto \beta e^{-\mu\tau(\xi)} g(\xi)$ is non-monotone. It is also clear from the insets in these panels that in contrast to Figure 6.10, the maximum of $\xi \mapsto \beta e^{-\mu\tau(\xi)} g(\xi)$ is now strictly larger than $\beta e^{-\mu\tau(\theta_{gv})} g(\theta_{gv})$ for all $m = n \gg 0$. Consequently, as seen in Figure 6.11(c), the rightmost curve of fold bifurcations asymptotes to a value of γ which is strictly larger than γ_{gv} as $m = n \to \infty$. As Figure 6.11(d) shows in both cases the location of this fold bifurcation converges to θ_{gv} as $m = n \to \infty$. When $M(\theta_{gv}) > 0$

the maximum of $\beta e^{-\mu\tau(\xi)}g(\xi)$ occurs for $\xi > \theta_{gv}$ and consequently for the fold bifurcation as $m = n \to \infty$, $\xi \to \theta_{gv}$ from above and $\gamma \to \gamma^*$ from below for some γ^* strictly larger than γ_{gv}. For $M(\theta_{gv}) < 0$ the convergence is from the other side, but again $\gamma \to \gamma^*$ for some γ^* strictly larger than γ_{gv}.

We remark that for the example with $a = 14.6591$ in Figure 6.11, because $M(\theta_{gv}) < -1$ the condition (6.17) is violated, even though the condition (6.24) is satisfied, demonstrating as we claimed earlier that (6.17) is not a necessary condition to obtain (6.24). This also shows that it is insufficient to just evaluate the value of $M(\theta_{gv})$ to determine whether fold bifurcations occur.

In Figures 6.12 and 6.13 we return to the parameter set first considered in Figure 6.9 for which $\beta e^{-\mu\tau(\theta)}g(\theta) > \beta e^{-\mu\tau^-}g^- > \beta e^{-\mu\tau^+}g^+$. Now, we also consider the Hopf bifurcations and resulting periodic orbits. With these parameters $M(\theta_{gv}) < 0$ when $m = n > 0$ with $M(\theta_{gv}) \to -\infty$ as $m = n \to \infty$ and so the upper fold bifurcation approaches γ^* for some $\gamma^* > \gamma_{gv}$ from above as $m = n \to \infty$, as seen in Figure 6.12(c), similar to the left panel of Figure 6.11(c). The other fold bifurcation is seen in Figure 6.12(c) to approach $\gamma = \gamma_{24}$ as $m = n \to \infty$, as $\xi \mapsto \beta e^{-\mu\tau(\xi)}g(\xi)$ evolves from the smooth function shown in Figure 6.12(b) to its limiting form as in Figure 6.12(d). This agrees with the arguments under Figure 6.8 above.

For the one parameter continuation shown in Figure 6.12(a) the steady state is stable for all γ sufficiently large or sufficiently small, and loses stability in a Hopf bifurcation (and not at the fold bifurcations). A two parameter continuation of these Hopf bifurcations in Figure 6.12(c) shows that they approach γ_{13} and γ_{24} as $m = n \to \infty$, similarly to how we saw the Hopf bifurcations were forced to the ends of the interval of singular steady state solutions in Section 5.1.

In Figure 6.12(a) there are no stable steady states for $\gamma \in (0.624, 0.837)$, so we explored the stable invariant objects in this case using both numerical bifurcation detection and continuation and numerical simulation of the DDE. Using `ddebiftool` we find that the Hopf bifurcation at $\gamma = 0.624$ is supercritical and generates a branch of stable periodic orbits, which is illustrated in Figure 6.13. There is an apparent canard explosion [34] on this branch at $\gamma \approx 0.629$ where the amplitude of the periodic orbit increases dramatically. The continuation of the branch fails in `ddebiftool` at this point, but since the periodic orbits are stable we are able to continue them through the canard explosion using simulation with `ddesd`, and then continue again with `ddebiftool` after the canard explosion, to obtain the continuous branch of periodic solutions for $m = n = 40$ shown in Figure 6.12(a) and Figure 6.13. After the canard explosion for $\gamma \in (0.629, 0.84)$ the periodic orbit is stable with large amplitude and moderate period ($T \approx 13$) and evolves slowly as γ is varied. Then at $\gamma \approx 0.84$ the period of the orbit grows dramatically while the amplitude is roughly constant. In Figure 6.12(a) this branch is represented by its 2-norm as the red curve. This curve appears to terminate on the middle unstable branch of steady states when $\gamma = 0.84$. However, as seen in Figure 6.13(c) and (d) the amplitude of these orbits remains large, so there is not a Hopf bifurcation at this end of the branch.

The large period orbits shown in Figure 6.12(c) and (d) spend a lot of time close to the unstable steady state on the middle branch between the folds, which is itself very close to the threshold $x = \theta_{gv}$, and resemble relaxation oscillations. This branch of periodic orbits most likely ends in a homoclinic orbit to the unstable steady state with an infinite period. There is also bistability between the stable periodic orbit and a stable steady state for a very small range of parameters $\gamma \in [0.837, 0.84]$.

In Figure 6.12 there are two fold bifurcations, as we have seen in many examples when

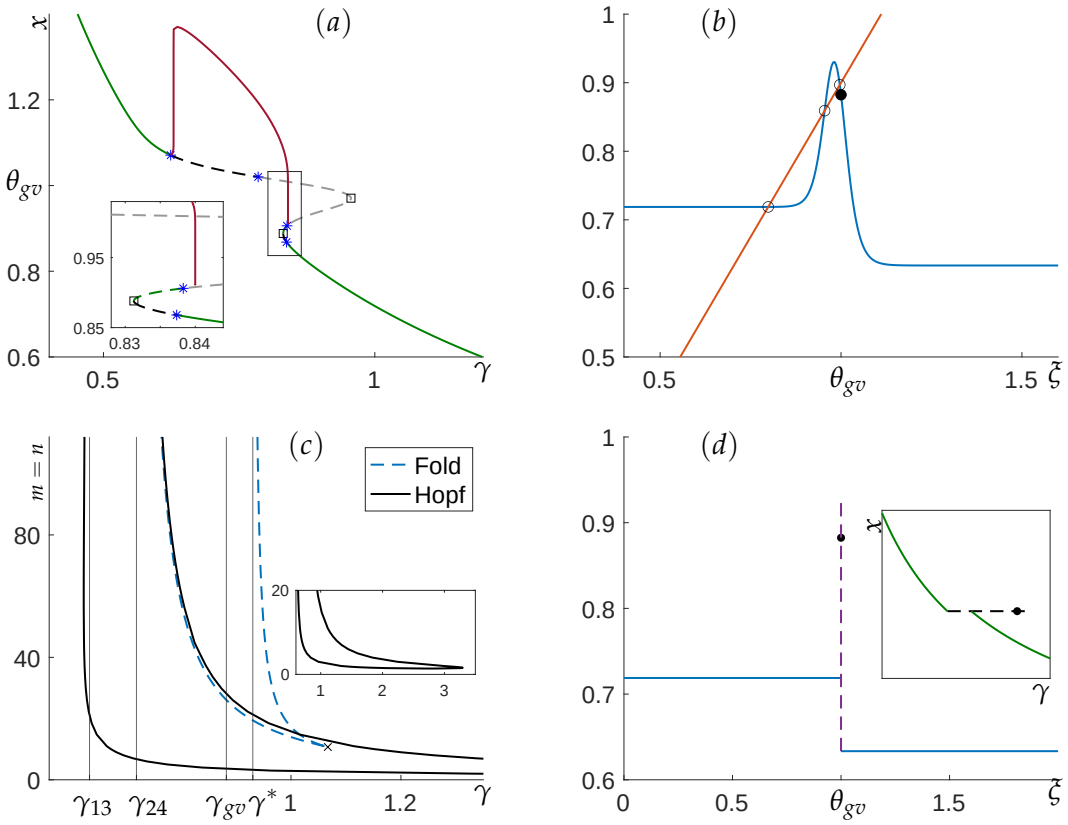


Figure 6.12: Bifurcations of (1.1)–(1.2) when $(g\downarrow, v\uparrow, \theta_g = \theta_v = \theta_{gv})$ with the same parameters as in Figure 6.9. (a) With smooth nonlinearity g and v defined by (1.3) with $m = n = 40$. The stable periodic orbits are represented by the 2-norm. (b) Steady states occur at the intersections of $\xi \mapsto \beta e^{-\mu\tau(\xi)} g(\xi)$ and $\xi \mapsto \gamma\xi$, with $\gamma\xi$ illustrated here with $\gamma = 0.9$. The black dot marks the value of $\beta e^{-\mu\tau(\theta_{gv})} g(\theta_{gv})$. (c) Two-parameter continuation in $m = n$ and γ of the fold bifurcations (in blue) and Hopf bifurcations (black) with the other parameters as above. Solid curves indicate the parts of the bifurcation branch where there are no characteristic values with positive real part (and hence a stability change at the bifurcation), and dashed lines indicate where the parts of the branch where there is at least one unstable characteristic value. In this example the steady state always loses stability in a Hopf bifurcation, and not at the fold. The four vertical lines denote γ_{13} and γ_{24} (recall (6.13)) and γ_{gv} (recall (6.25)), along with $\gamma^* = 0.9305$. (d) Plot of $\xi \mapsto \beta e^{-\mu\tau(\xi)} g(\xi)$ in the limiting case with g and v defined by (1.6) and (1.7). The black dot marks the value of $\beta e^{-\mu\tau(\theta_{gv})} g(\theta_{gv})$. The dashed line is obtained by taking the Hausdorff limit of the smooth function, and extends above the point-wise limit at θ_{gv} because $\lim_{m=n\to\infty} \max_\xi \beta e^{-\mu\tau(\xi)} g(\xi) > \beta e^{-\mu\tau(\theta_{gv})} g(\theta_{gv})$. The inset panel depicts the continuation of steady states in the limiting case. Stable steady states are shown as green solid lines, and the singular steady state as a black dashed line.

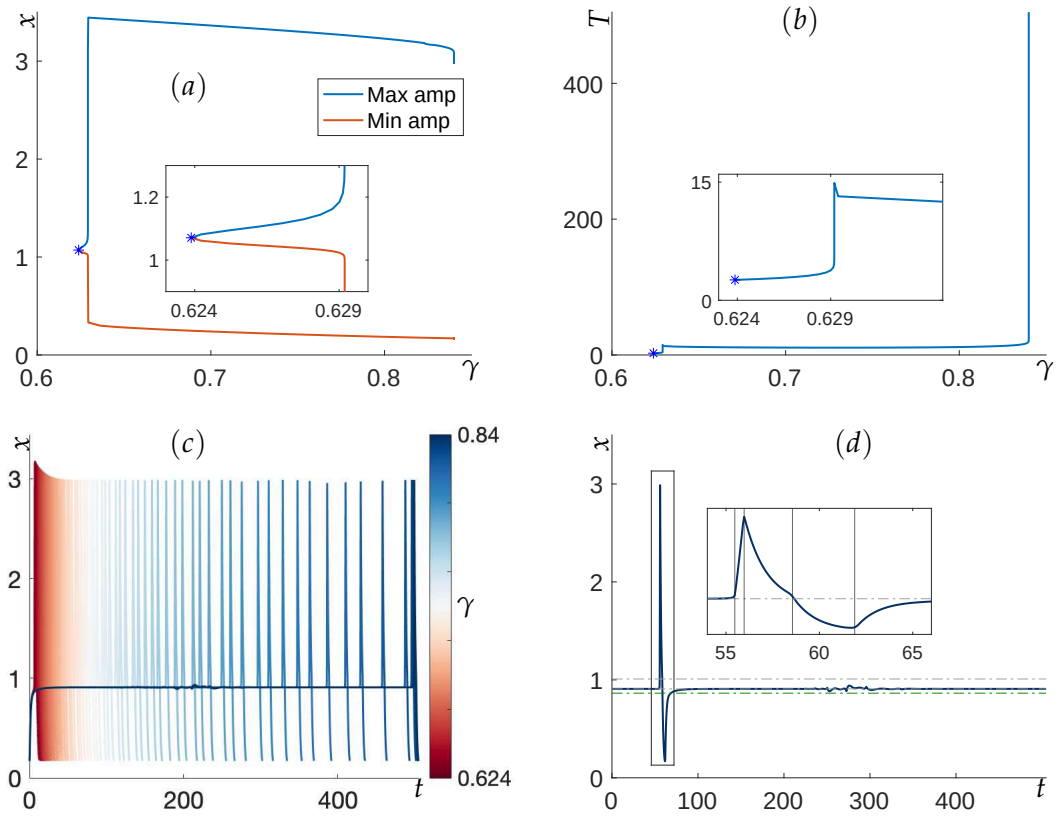


Figure 6.13: The branch of stable periodic orbits for $(g\downarrow, v\uparrow, \theta_g = \theta_v = \theta_{gv})$ when $\gamma \in [0.624, 0.84]$ from Figure 6.12(a). Panel (a) shows the amplitude of the periodic orbits along the branch. (b) The period of the orbits. (c) Profiles of the periodic solutions show that the large period orbits at the end of the branch are of relaxation oscillator type. Panel (d) depicts the last periodic orbit found by `ddebiftool`, with the inset panel showing a zoom of the dynamics near the spike.

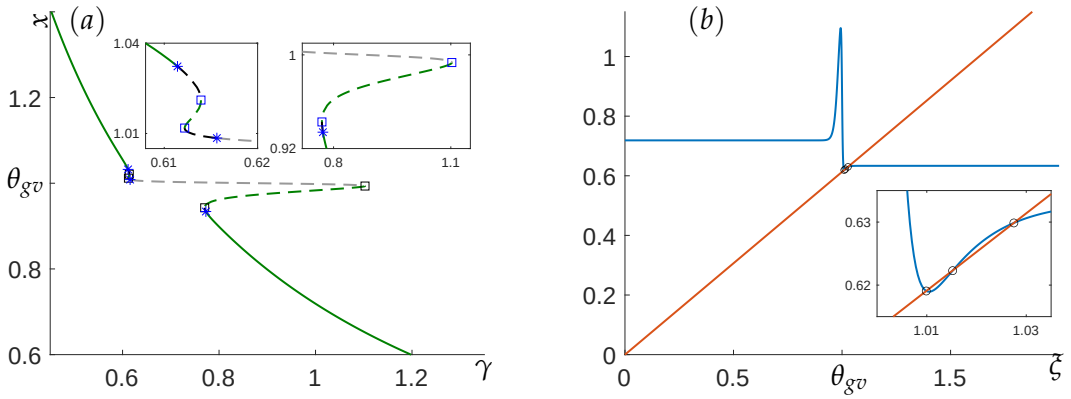


Figure 6.14: (a) One parameter continuation of the steady state when $(g\downarrow, v\uparrow, \theta_g = \theta_v = \theta_{gv})$ for varying γ with the other parameters the same as in Figure 6.12 except $m = 100$ and $n = 500$ reveals four fold bifurcations. (b) Three steady states are indicated for $\gamma = 0.6130$ at the intersections of $\xi \mapsto \beta e^{-\mu\tau(\xi)} g(\xi)$ and $\gamma\xi$.

one of g and v is increasing. However, as Figure 6.14 shows, it is possible to obtain four fold bifurcations by just changing the steepness of the nonlinearities and resulting width of the interfaces of the nonlinearities. In Figure 6.14 we do this with $n \gg m \gg 0$. Since n is very large the function g has a very sharp interface, and drops from very close to its maximum value g^+ to its minimum value g^- while v is essentially unchanged and equal to $v(\theta_{gv})$. This results in the function $\xi \mapsto \beta e^{-\mu\tau(\xi)} g(\xi)$ having extrema close to $\beta e^{-\mu\tau(\theta_{gv})} g^-$ and $\beta e^{-\mu\tau(\theta_{gv})} g^+$, as shown previously in Figure 6.9(a) and (b). Outside of this narrow interface, g is essentially constant, and therefore $\beta e^{-\mu\tau(\xi)} g(\xi)$ is increasing, since $v(\xi)$ is an increasing function. Thus the function $\xi \mapsto \beta e^{-\mu\tau(\xi)} g(\xi)$ is increasing, then decreasing, then increasing again. For m sufficiently large (so that the derivative of $\beta e^{-\mu\tau(\xi)} g(\xi)$ is larger than γ) this creates two fold bifurcations close to the ends of the inner interface for g in addition to the other two fold bifurcations close to the edges of the outer interface of v.

Figure 6.15(a) shows that with $n = 5m$ the second fold bifurcation exists for all $m \geq 66.39$. As shown in Figure 6.15(b) one pair of fold bifurcations exists for $\xi < \theta_{gv}$ and the other for $\xi > \theta_{gv}$. This happens because θ_{gv} is in the middle of the sharp interface of g, the only (very short) interval on which $\beta e^{-\mu\tau(\xi)} g(\xi)$ is decreasing, with two of the corners giving rise to the fold bifurcations on each side of θ_{gv}. Figure 6.15(b) shows that as $n = 5m \to \infty$ all of the fold bifurcations are squeezed into $\xi = \theta_{gv}$, but in Figure 6.15(a) we see that the limiting γ values of the folds all appear to be different. Because $\beta e^{-\mu\tau(\xi)} g(\xi) \to \beta e^{-\mu\tau^-} g^-$ as $m, n \to \infty$ for all $\xi < \theta_{gv}$ (and to $\beta e^{-\mu\tau^+} g^+$ for $\xi > \theta_{gv}$), two of the fold bifurcations asymptote to $\gamma = \gamma_{24}$ and $\gamma = \gamma_{13}$ as $m, n \to \infty$, as shown in Figure 6.15(a). These are the outer two fold bifurcations for the smallest and largest ξ values in Figure 6.15(b). The asymptotic γ values for the inner pair of fold bifurcations depends on the relative widths of the interfaces of the g and v functions. With $n = 5m$ it appears that the limiting γ values are strictly inside the interval (γ_1, γ_2). Different behaviour is observed if m, n approach infinity with different relationships. For example, if $n = m^2$ (as considered first in Figure 6.9(b)) then the g interface is much narrower than for v and the inner folds asymptote to $\gamma = \gamma_2$ and $\gamma = \gamma_1$ while the outer folds still asymptote to $\gamma = \gamma_{24}$ and $\gamma = \gamma_{13}$ as seen in Figure 6.15(d).

If $n = m^r \to \infty$ for $r \in (0, 1)$ then the behaviour of $\xi \mapsto \beta e^{-\mu\tau(\xi)} g(\xi)$ and the bifurcations are completely different, as shown in Figure 6.9(d) and Figure 6.15(c). In this case $m \gg n$ so the v interface is much narrower than the g interface. As g is decreasing this results in the function $\xi \mapsto \beta e^{-\mu\tau(\xi)} g(\xi)$ being decreasing for ξ small and large, and increasing only on a small interval about $\xi = \theta_{gv}$ corresponding to the narrow interface of v. In this scenario the limiting form of the function $\beta e^{-\mu\tau(\xi)} g(\xi)$ is still piecewise constant for $\xi < \theta_{gv}$ and $\xi > \theta_{gv}$, but now the function is decreasing rather than increasing at the corners corresponding to γ_{24} and γ_{13}, so these corners no longer result in fold bifurcations. Instead the folds occur close to the local extrema of $\beta e^{-\mu\tau(\xi)} g(\xi)$ seen in Figure 6.9(d), so that, for instance, when $n = m^{2/3} \to \infty$ the fold bifurcations in Figure 6.15(c) approach γ_4 and γ_3 defined by (6.26).

Figure 6.15(b) shows that the locations of the fold bifurcations are squeezed into $\xi = \theta_{gv}$ as $n = 5m \to \infty$ and the nonlinearities in (1.1)–(1.2) become piecewise constant for $\xi \neq \theta_{gv}$. Similar behaviour was observed when $n = m^r \to \infty$ for $r > 0$ but we omit the figures. Notice also that as already observed in the previous example, the outer Hopf bifurcations at which the steady state changes stability approach γ_{24} and γ_{13} as in the limiting case as $n = m^r \to \infty$ in all of these examples.

Although there are four fold bifurcations in the example in Figures 6.14 and 6.15, because of the location of the folds it is not possible to obtain five co-existing steady states.

As a final example we change just the parameters $v^\pm$ in the v function from the previous

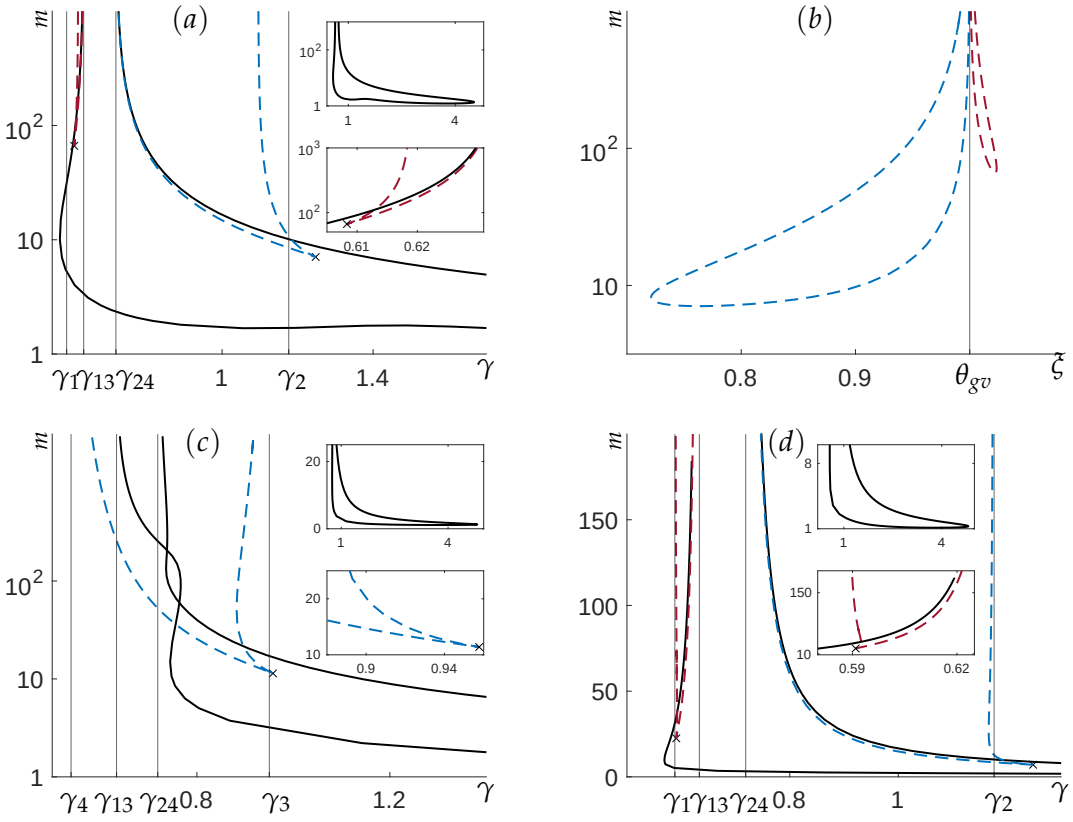


Figure 6.15: Two parameter continuations of the bifurcations of (1.1)–(1.2) when $(g \downarrow, v \uparrow, \theta_g = \theta_v = \theta_{gv})$ with the same parameters as in Figures 6.9, 6.12 and 6.14, except for m and n. The branches of fold bifurcations are shown as blue and red curves and the Hopf bifurcations in black. Solid curves indicate the parts of the bifurcation branch where there are no characteristic values with positive real part (and hence a stability change at the bifurcation), and dashed lines indicate where the parts of the branch where there is at least one unstable characteristic value. In all four examples the steady state always loses stability at a Hopf bifurcation, and not at the fold bifurcation. (a) and (b) Continuation of the fold bifurcations in γ and m with $n = 5m$, showing in (a) the γ values of the fold bifurcations and (b) the value of the steady state ξ at the fold bifurcation. Cusp bifurcations occur at $(\gamma, m) = (0.6083, 66.3853)$ and $(\gamma, m) = (1.2480, 7.0681)$. (c) and (d) are similar to (a), except in (c) $n = m^{2/3}$ and in (d) $n = m^2$. Cusp bifurcations occur in (c) at $(\gamma, m) = (0.9577, 11.3879)$, and in (d) at $(\gamma, m) = (0.5909, 22.4752)$ and $(\gamma, m) = (1.2484, 7.0591)$.

example, to obtain a new example with $\beta e^{-\mu\tau(\theta)} g(\theta) > \beta e^{-\mu\tau^+} g^+ > \beta e^{-\mu\tau^-} g^-$, whose dynamics are explored in Figure 6.16. With $m = n$ the dynamics with one and two parameter continuations is very similar to the earlier case shown in Figure 6.12 with a single pair of fold bifurcations and so we omit the figures here. But different choices of m and n lead to more interesting dynamics shown in Figure 6.16. One-parameter continuation in γ with n and m fixed with $n \gg m \gg 0$, as shown in Figure 6.16(a) reveals four fold bifurcations, a large interval of γ values on which there are three coexisting steady states with two stable, and a small interval for $\gamma \in (0.5352, 0.5513)$ for which there are five co-existing steady states of which one or two are stable. Figure 6.16(b) illustrates how these five steady states arise because for $n \gg m$ the downward g interface is much narrower than the interface for the increasing v

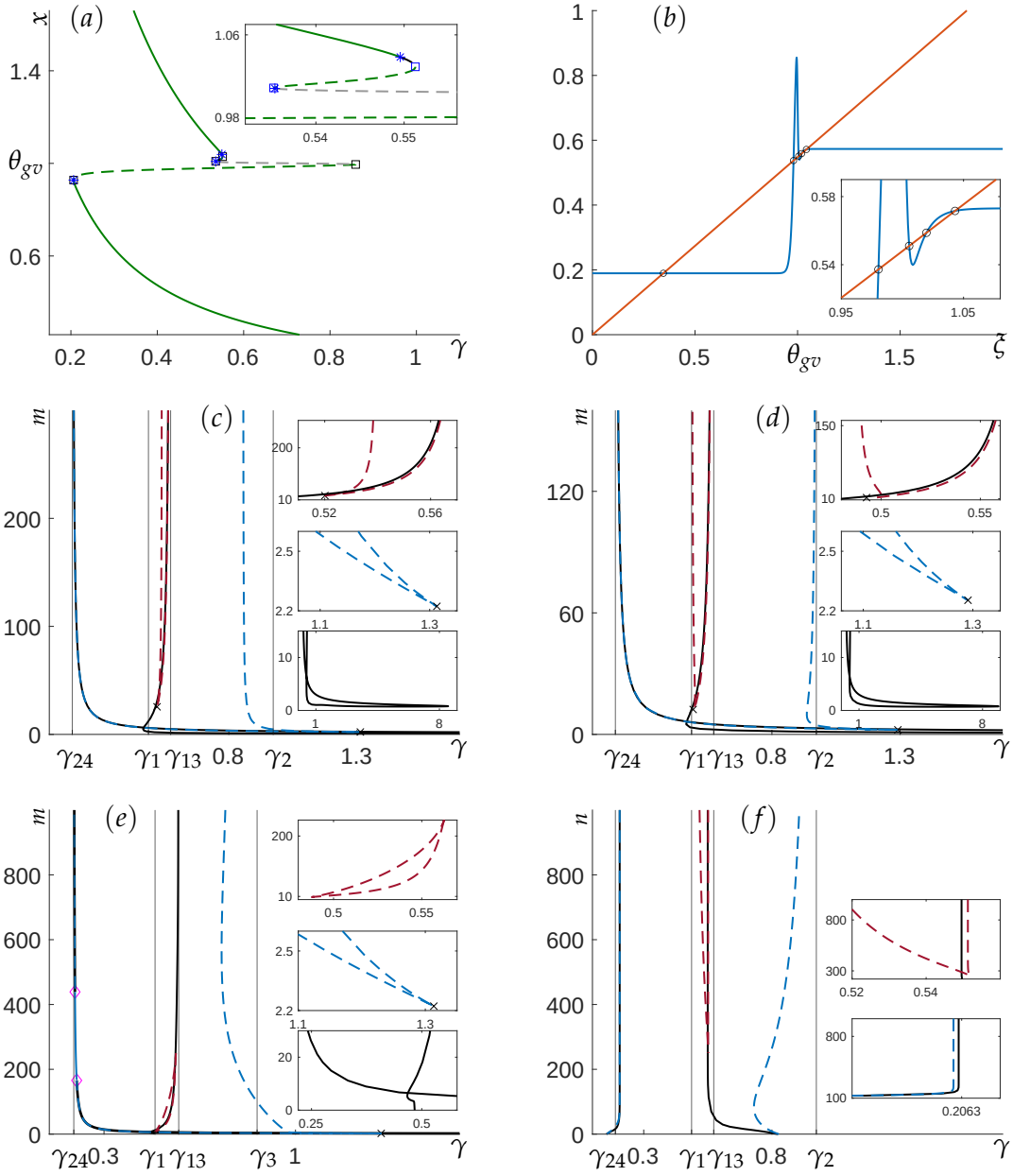


Figure 6.16: Bifurcations of (1.1)–(1.2) when $(g\downarrow, v\uparrow, \theta_g = \theta_v = \theta_{gv})$ with $\beta = 1.4$, $\mu = 0.2$, $g^- = 1$, $g^+ = 0.5$, $\theta_g = \theta_v = \theta_{gv} = 1$, $a = 1$, $v^- = 0.1$ and $v^+ = 1$. Apart from m, n, v^- and v^+, these are the same parameters as in Figures 6.9 and 6.12–6.15. (a) One parameter continuation of steady states in γ with smooth nonlinearities g and v defined by (1.3) with $m = 100$ and $n = 500$. This reveals four fold bifurcations and up to five coexisting steady states (at most two of which are stable). (b) Illustration of the five steady states occurring in (a) at the intersections of $\xi \mapsto \beta e^{-\mu\tau(\xi)} g(\xi)$ and $\xi \mapsto \gamma\xi$ with $\gamma = 0.5480$. (c) to (f) Two parameter continuations of the fold bifurcations (red and blue) and Hopf bifurcations (black) with (c) $n = 5m$ and (d) $n = m^2$ (e) m varying with $n = 500$ fixed and (f) n varying with $m = 100$ fixed. The bifurcation curves are drawn according to the number of characteristic values with positive real part; solid for zero and dashed for one or more. Cusp points are denoted by crosses. In (e) there is a change of stability at the fold bifurcation between two Bogdanov–Takens bifurcations (denoted by pink diamonds) at $(\gamma, m) = (0.2, 165.62)$ and $(\gamma, m) = (0.1938, 438.82)$, otherwise in these examples the stability change is always at the Hopf bifurcation (as was the case in Figure 6.15).

function (compare with Figures 5.3 and 6.14). Panels (c) and (d) of Figure 6.16 show two-parameter continuations with $n = 5m$ and $n = m^2$ which reveal that with these parameter constraints all four fold bifurcations persist to arbitrary large values of m with one pair of fold bifurcations contained in the interval (γ_1, γ_{13}) and the other in the interval (γ_{24}, γ_2) for all m sufficiently large, just as was seen in Figure 6.15. But for the parameters used in Figure 6.16 we have $(\gamma_1, \gamma_{13}) \subset (\gamma_{24}, \gamma_2)$ so there are five coexisting steady states between the red curves in Figure 6.16(c) and (d) for arbitrary large m.

In the examples shown in Figure 6.16(c) and (d) the stable steady states loses stability in a Hopf bifurcation just before the fold bifurcation. As in the previous examples in this section these Hopf bifurcations asymptote to γ_{24} and γ_{13} as $m, n \to \infty$. In Figure 6.16(e) and (f) we explore a different scenario, where we keep one of m or n fixed but much larger than zero, and allow the other one to vary on the positive real line.

The value $m = 0$ in Figure 6.16(e) corresponds to the v function being constant, and since g is decreasing, this corresponds to the scenario considered in Section 5.1. This is why the curves of Hopf bifurcations, but not the folds, extend all the way down to $m = 0$. For $0 < m \ll n$ the g function has a much narrower interface than the v function, so $\beta e^{-\mu\tau(\xi)} g(\xi)$ will be first increasing then decreasing and finally increasing again, similar to many of the examples in this section. However for $m \gg n \gg 0$ the v function will have a narrower interface, so $\beta e^{-\mu\tau(\xi)} g(\xi)$ will be first decreasing, like in Figures 6.9(d) and 6.15(c). Interesting dynamics is observed with $n \gg m > 0$. In particular, in Figure 6.16(e) there is a pair of Bogdanov–Takens bifurcations, and on the curve of fold bifurcations between these points a steady state loses stability at the fold bifurcation, unlike the previous examples in this section, but similar to examples in Section 5.4. In Figure 6.16(f) we explore varying n with $m \gg 0$ fixed. Thus when $n = 0$ we are in the scenario of g fixed and v increasing explored in Section 5.4 which is why fold bifurcations persist down to $n = 0$ in Figure 6.16(f). As n increases, once the interface of the g function becomes narrow enough there is a cusp bifurcation which leads to a second pair of fold bifurcations and five coexisting steady states for all n sufficiently large. For n large the folds are contained in the intervals (γ_{24}, γ_2) and (γ_1, γ_{13}), but do not asymptote to the ends of these intervals as $n \to \infty$ because m is held fixed and finite.

6.3 Summary

In Section 6 we have considered the situation where both functions g and v are non-constant. We first derived formulae for the existence of both fold bifurcations (6.2) and Hopf bifurcations (6.9)–(6.10), which reduce to formulae derived in Section 5 if either v or g is a constant function.

In Section 6.1 we study the case when the thresholds are different; $\theta_g \neq \theta_v$. Not surprisingly, we recover the same dynamics and bifurcation structures as in Section 5 near each threshold when $m \gg 0$, or $n \gg 0$ since in these limits the behavior localizes in the area where the other function is essentially constant. In particular, in the case $(g\downarrow, v\downarrow)$ the dynamics is similar to that observed in Section 5.1 near $x = \theta_g$ and to that in Section 5.3 near $x = \theta_v$. In Section 6.1.1 we consider the example $(g\downarrow, v\uparrow)$ which supports bistability between an equilibrium and periodic orbit and in Section 6.1.2 we examine $(g\uparrow, v\uparrow)$ which can exhibit up to five equilibria and tristability.

In Section 6.2 we look at the situation when $\theta_v = \theta_g$. First in Section 6.2.1 we consider a relatively straightforward situation where both functions g and v are either increasing $(g\uparrow, v\uparrow)$ or both decreasing $(g\downarrow, v\downarrow)$. Both of these cases are similar to their counterparts discussed in

Section 5.

The most interesting case is when $\theta_v = \theta_g$, but the monotonicity of g and v do not agree i.e. $(g\uparrow, v\downarrow)$ and $(g\downarrow, v\uparrow)$. This is discussed in Section 6.2.2. The main result is that which behavior dominates (i.e. the behavior observed for increasing g or v or behavior for decreasing g or v) depends sensitively on the way in which m and n approach infinity. We capture this informally in a concept of "interface width"; if $m \gg n$ then the interface of v is much narrower than the interface of g and vice versa.

Figure 6.9 illustrates that the function $\xi \mapsto \beta e^{-\mu\tau(\xi)} g(\xi)$ converges to different limits as m and n go to infinity at different rates, indicating that the fold bifurcations will occur at different places in these limits. We also observe that additional equilibria may appear when m and n approach infinity at different rates (Figures 6.15 and 6.16). Finally, we observe a canard-like explosion in amplitude of a periodic orbit that emanates from a Hopf bifurcation, see Figure 6.13.

7 Conclusions

This paper is long and complicated, and hence difficult to summarize succinctly. Suffice to say that we have introduced a generalization of a prokaryotic gene regulatory model, in (1.1)–(1.2), originally developed in [18]. The generalization has both nonlinearities in terms of feedback but also nonlinearities appearing in state dependent delays. Moreover the delays are generated by a threshold condition. It is thus somewhat novel for both the mathematical and modeling literature. The first three sections of the paper are concerned with establishing important and relevant mathematical properties of the basic system (1.1)–(1.2). Section 2 establishes properties of the semiflow generated by (1.1)–(1.2), while Section 3 establishes conditions for the existence of a global attractor of the semiflow. Section 4 deals with the linearization of (1.1)–(1.2) which is of significant utility in the following Sections 5 and 6 where we examine the behaviour of (1.1)–(1.2) both numerically and analytically.

At the ends of both Sections 5 and 6 we have offered general synopses of the results of each, which we will not repeat here. As an aid to the reader, in Table 7.1 we have listed all of the figures in this paper illustrating dynamics for various combinations of g and v, along with the types of bifurcations that we observed numerically in each case.

We have studied the threshold model (1.1), (1.2) directly without applying the time transformation of Smith [55,56] to convert the equations to a distributed delay DDE with constant delay. We showed in Sections 5.3–6.2.2 that the dynamics of the model (1.1) with a threshold delay is altogether richer and more surprising than the dynamics of the corresponding constant delay model considered in Sections 5.1–5.1. In so doing we demonstrated that problems with threshold delays can be analysed and studied numerically without time transforming the problems to constant delay problems. Our methods are applicable to problems with multiple delays including one or more threshold delays.

Much of the algebra in this paper is rooted in the use of monotone Hill functions for $g(\xi)$ and $v(\xi)$ which is both relevant and quite justifiable for modeling in the biological realm. Our algebra is helped by the use of the function $\xi \mapsto \xi f'(\xi)/f(\xi)$ which has a tractable form (5.11) when f is a Hill function. Notice that if $f(\xi) = C\xi^p$ then

$$\frac{\xi f'(\xi)}{f(\xi)} = p,$$

so we can think of $\xi f'(\xi)/f(\xi)$ as a description of a local power of a non-polynomial function.

(g, v)	Corresponding figure number and bifurcations
$(g\downarrow, v\leftrightarrow)$	5.1($\beta\gamma$), 5.2(1,2,H,S)
$(g\uparrow, v\leftrightarrow)$	5.3($\beta\gamma$), 5.4(1,2,F,H,Cu)
$(g\leftrightarrow, v\downarrow)$	5.5($\beta\gamma$), 5.6(1,S), 5.7(1,2,H,S), 5.8(1,2,H,S)
$(g\leftrightarrow, v\uparrow)$	5.9($\beta\gamma$), 5.10(1,2,F,S,Cu), 5.11(1,F,h,H,F_P), 5.12(1,2,Ba,BT,FH,Cu), 5.13(1,F,F_P,H,h), 5.14(2,F,H,h,BT,Cu), 5.15(1,H,h), 5.16(1,2,F,H,h,Cu,Ba,Fh_2)
$(g\downarrow, v\uparrow)$	6.1($\beta\gamma$), 6.2(1,2,F,H,Cu,BT,FH), 6.7($\beta\gamma$,1,2,H), 6.8($\beta\gamma$), 6.9(β), 6.10(β,1,2,F,Cu), 6.11(β,2,F,Cu), 6.12($\beta\gamma$,1,2,F,H,Cu), 6.13(1,H,h,Ca), 6.14($\beta\gamma$,1,4F,H), 6.15(2,4F,H,Cu), 6.16($\beta\gamma$,1,2,4F,5SS,H,Cu,BT)
$(g\uparrow, v\uparrow)$	6.3($\beta\gamma$), 6.4(1,2,4F,5SS,H,Ba,BT,FH), 6.5(1,4F,5SS,H)
$(g\downarrow, v\downarrow)$	6.6(1,2,H)
$(g\uparrow, v\downarrow)$	

Table 7.1: Summary of various dynamic behaviours seen in Sections 5 and 6. The notation of 'n.m' refers to figure 'm' of Section 'n' and the symbols after indicate the type of continuation explored and the bifurcations observed. $\beta\gamma$ and β refers to a figure in which $\beta e^{-\mu\tau} g(\xi)$ (with or without $\gamma\xi$) is plotted to show qualitatively the number of steady states of the system (1.1)–(1.2). 1 and 2 indicate that 1 and/or 2 parameter numerical continuation was performed. The bifurcations observed are denoted by F (Fold), F_P (Fold of Periodic Orbits), H (Hopf), S (Stable-no bifurcation), and h (homoclinic) and for higher co-dimension bifurcations: Ba (Bautin *or* generalized Hopf), BT (Bogdanov-Takens), Cu (cusp), FH (fold-Hopf), Fh_2 (fold-homoclinic). 4F denotes examples with 4 fold bifurcations, 5SS denotes 5 co-existing steady states, and Ca a canard. Note that 'left clicking' on a number will take you to that figure, while 'Alt ←' will bring you back to this table.

While a similar approach could be adopted for other nonlinearities, the results would likely be quantitatively different, but we expect that for monotone functions they would not show qualitative alterations. However, if non-monotone nonlinearities were considered then we expect that there would be substantial qualitative as well as quantitative differences.

Future directions. In [20] the authors examine the various dynamic patterns (what they term 'multi-synchronization' or 'multi-rhythmicity') that emerge from models of biochemical and cellular oscillators coupled to each other, and illustrate their considerations with examples of the circadian oscillator and the cell cycle. While certainly of interest biologically, perhaps the more interesting question is whether or not one could construct a catalog, or dictionary, of possible dynamical behaviours arising from the coupling of N different dynamics generated by DDEs as we have considered here.

In particular, consider

$$x_i'(t) = \beta_i e^{-\mu_i \tau_i(t)} \frac{v_i(x_i(t)))}{v_i(x_i(t-\tau_i(t)))} g_i(x_i(t-\tau_i(t))) - \gamma_i x_i(t) \tag{7.1}$$

where for $t \geq 0$ the delay $\tau_i(t)$ is defined by the threshold condition

$$a_i = \int_{-\tau_i(t)}^{0} v_i(x_i(t+s))ds = \int_{t-\tau_i(t)}^{t} v_i(x_i(s))ds \tag{7.2}$$

for $i = 1, \ldots, N$. Assuming that these systems are coupled through one or more of the parameters $\beta_i, \mu_i, \gamma_i, a_i$ or through the functions τ_i, v_i, g_i, then what can be said about the resulting

dynamics? The paper [20] considered both 'series' and 'parallel' coupling between their two dynamics, but one would have to precisely define what was meant by this within the context of (7.1)–(7.2).

For $N = 2$ one might consider a system

$$x_1'(t) = \beta_1 x_2 e^{-\mu_1 \tau_1(t)} \frac{v_1(x_1(t)))}{v_1(x_1(t-\tau_1(t)))} g_1(x_1(t-\tau_1(t))) - \gamma_1 x_1(t),$$
$$x_2'(t) = \beta_2 x_1 e^{-\mu_i \tau_2(t)} \frac{v_2(x_2(t)))}{v_2(x_2(t-\tau_2(t)))} g_2(x_2(t-\tau_2(t))) - \gamma_2 x_2(t).$$

One could regulate the strength of the coupling by an additional constant i.e. by considering the term $\beta_1 a_1 x_2$ rather than just $\beta_1 x_2$. This would allow an examination of the strength of coupling while keeping β's the same for comparison with a single DDE oscillator. This seems to be the simplest linear coupling; an alternative might be considering $\beta_1(x_2), \beta_2(x_1)$ where these are Hill functions. This coupling results in a positive mutual feedback in the sense that more x_1 makes $g(x_2)$ bigger and vice versa.

For stable oscillation in a single DDE we need at least one of the functions v or g to be decreasing. Are there differences if we couple same kind oscillators $(-,+)$ with $(-,+)$ or $(+,-)$ with $(+,-)$ on one hand and two different oscillators $(+,-)$ with $(-,+)$? If we mutually couple two DDEs in a way that they form a negative feedback loop, can we identify oscillations that arise due to local negative feedback (i.e. the oscillator type $(-,+)$ and $(+,-)$) and the global negative feedback, that arises from the mutual coupling of the oscillators? In particular, consider two types of mutually coupled equations. First consider the coupling through the nonlinearities g

$$x_1'(t) = \beta_1 e^{-\mu_1 \tau_1(t)} \frac{v_1(x_1(t)))}{v_1(x_1(t-\tau_1(t)))} g_1(x_2(t-\tau_1(t))) - \gamma_1 x_1(t),$$
$$x_2'(t) = \beta_2 e^{-\mu_2 \tau_2(t)} \frac{v_2(x_2(t)))}{v_2(x_2(t-\tau_2(t)))} g_2(x_1(t-\tau_2 t))) - \gamma_2 x_2(t),$$

where both v_1, v_2 are decreasing, so that each equation can support oscillations with constant input from the other oscillator (see Section 5.3). If both g_1, g_2 are also decreasing this represents mutual inhibition between two oscillators and suggests the possible existence of bistability.

Alternatively, one can consider coupling of the equations through the v functions rather than nonlinearities g, with all functions v_1, v_2, g_1, g_2 still decreasing:

$$x_1'(t) = \beta_1 e^{-\mu_1 \tau_1(t)} \frac{v_1(x_2(t)))}{v_1(x_2(t-\tau_1(t)))} g_1(x_1(t-\tau_1(t))) - \gamma_1 x_1(t),$$
$$x_2'(t) = \beta_2 e^{-\mu_2 \tau_2(t)} \frac{v_2(x_1(t)))}{v_2(x_1(t-\tau_2(t)))} g_2(x_2(t-\tau_2 t))) - \gamma_2 x_2(t).$$

Does either system support bistability between periodic solutions? If so, does the shape of the periodic solutions reflect the fact that in mutually repressible coupled systems one component is high while the other is low?

One can also consider larger system of equations that are cyclically coupled

$$x_1'(t) = \beta_1 e^{-\mu_1 \tau_1(t)} \frac{v_1(x_1(t)))}{v_1(x_1(t-\tau_1(t)))} g_1(x_N(t-\tau_1(t))) - \gamma_1 x_1(t), \tag{7.3}$$
$$x_j'(t) = \beta_j e^{-\mu_j \tau_j(t)} \frac{v_j(x_j(t)))}{v_j(x_j(t-\tau_j(t)))} g_j(x_{j-1}(t-\tau_j(t))) - \gamma_j x_j(t), \quad j = 2, \ldots, N. \tag{7.4}$$

We have already tackled a particular problem of this form in [18] where we considered

$$\begin{aligned}
\frac{dM}{dt}(t) &= \beta_M e^{-\mu\tau_M(t)} \frac{v_M(E(t))}{v_M(E(t-\tau_M(t)))} f(E(t-\tau_M(t))) - \bar{\gamma}_M M(t), \\
\frac{dI}{dt}(t) &= \beta_I e^{-\mu\tau_I(t)} \frac{v_I(M(t))}{v_I(M(t-\tau_I(t)))} M(t-\tau_I(t)) - \bar{\gamma}_I I(t), \\
\frac{dE}{dt}(t) &= \beta_E I(t) - \bar{\gamma}_E E(t).
\end{aligned}$$

which is exactly of the form (7.3),(7.4) with $N = 3$ and $g_2(x) = g_3(x) = x$ and $\mu_3 = \tau_3 = 0$.

To conclude, the work presented here opens many interesting questions on how the interplay between the character of the nonlinearity, state dependent delay and coupling affects the local and global dynamics of DDEs.

Appendices

A Relation of (1.1)–(1.2) to the model of Gedeon et al. [18]

The Goodwin model [22] for operon dynamics considers a large population of cells, each of which contains one copy of a particular operon. (M, I, E) respectively denote the mRNA, intermediate protein, and effector protein concentrations. For a generic operon [21, 22, 24, 25, 46, 51] the dynamics are assumed to be given by

$$\frac{dM}{dt} = \beta_M f(E) - \gamma_M M, \tag{A.1}$$

$$\frac{dI}{dt} = \beta_I M - \gamma_I I, \tag{A.2}$$

$$\frac{dE}{dt} = \beta_E I - \gamma_E E. \tag{A.3}$$

The production flux f of mRNA is assumed to be a function of the effector level E. Furthermore, the model assumes that the flux of protein and metabolite production are proportional (at rates β_I, β_E respectively) to the amount of mRNA and intermediate protein respectively. All three of the components (M, I, E) are subject to degradation at rates $\gamma_M, \gamma_I, \gamma_E$. The parameters $\beta_I, \beta_E, \gamma_M, \gamma_I$ and γ_E have dimensions [time^{-1}].

In [18] we extended the classic Goodwin model for the regulation of the bacterial operon to a situation in which the cells are growing exponentially at a rate μ and have finite transcriptional and translational velocities that are potentially dependent on the state of the system. For the full model of [18], retain the notation of the original Goodwin operon model, denote the transcriptional velocity by $v_M(E(t))$ and the translational velocity by $v_I(M(t))$, and let $\bar{\gamma}_i = \gamma_i + \mu$. Then the extension in [18] is

$$\frac{dM}{dt} = \beta_M e^{-\mu\tau_M(t)} \frac{v_M(E)}{v_M(E(t-\tau_M(t)))} f(E(t-\tau_M(t))) - \bar{\gamma}_M M, \tag{A.4}$$

$$\frac{dI}{dt} = \beta_I e^{-\mu\tau_I(t)} \frac{v_I(M)}{v_I(M(t-\tau_I(t)))} M(t-\tau_I(t)) - \bar{\gamma}_I I, \tag{A.5}$$

$$\frac{dE}{dt} = \beta_E I - \bar{\gamma}_E E. \tag{A.6}$$

These equations are supplemented by the two additional equations which implicitly define the delays τ_M and τ_I by threshold conditions, namely

$$a_M = \int_{t-\tau_M(t)}^{t} v_M(E(s))ds = \int_{-\tau_M(t)}^{0} v_M(E(t+s))ds, \tag{A.7}$$

$$a_I = \int_{t-\tau_I(t)}^{t} v_I(M(s))ds = \int_{-\tau_I(t)}^{0} v_I(M(t+s))ds. \tag{A.8}$$

In our extended model [18], as in the original Goodwin [21] formulation, the function f is a monotone increasing function for an inducible operon, and for a repressible operon f is a monotone decreasing function.

To make the transition from the full model (A.4)–(A.6) presented in [18] to the apparently simpler situation of (1.1)–(1.2) considered here is relatively straightforward. We simply need two assumptions:

1. We first assume that the translational velocity v_I is not regulated but is large with respect to the other characteristic times in the system so the translational delay $\tau_I \simeq 0$.
2. Our second assumption involves the existence of fast and slow variables [36, Section 2.3] so there is a clear separation of time scales.

With these assumptions and the substitutions: $E \to x$, $\beta \to \bar{\gamma}_I \frac{\beta_M \beta_I \beta_E}{\bar{\gamma}_M \bar{\gamma}_I \bar{\gamma}_E}$ (or $\beta \to \bar{\gamma}_E \frac{\beta_M \beta_I \beta_E}{\bar{\gamma}_M \bar{\gamma}_I \bar{\gamma}_E}$), $f \to g$, and $\tau_M(t) \to \tau(t)$ we immediately obtain the system (1.1)–(1.2) that we study here.

One might think that (1.1) is somewhat novel but in fact similar formulations are available in different situations. Below, in Appendix B we give several other examples of models in the spirit of (1.1), but which differ in details that may or may not offer significant differences in behaviour.

B Other examples

In this appendix, we mention four different examples of previously published studies which can be thought of as extensions of the considerations in this paper.

A bistable genetic switch

The two types of operon dynamics originally considered were classified as either repressible (in which the production flux function f in (A.1) is a decreasing function of its argument), or inducible (in which f is an increasing function of its argument). In the language of dynamical systems these are negative or positive feedback systems. However, there is a third type of fundamental dynamical entity found in prokaryotes, namely the so called bistable switch in which the effector produced by operon X inhibits the transcription of DNA from operon Y and vice versa (which we might denote as a $(-,-)$ system). The paradigmatic molecular biology example of a bistable switch due to reciprocal negative feedback is the bacteriophage (or phage) λ, which is a virus capable of infecting E. coli bacteria. Originally described in [32] and very nicely treated in [48], it is but one of many examples of mutually inhibitory bistable switches that have been found since. Models of this process are to be found in [26], [50], and [9] among others.

Consider a simple model for this process in which the dynamics of the intermediate of both the X and Y operons is fast relative to the dynamics of the corresponding mRNA M_X

and M_Y, and effectors E_X and E_Y. Then we can write down a reduced version of the model (A.4)–(A.6) in the form

$$\frac{dM_X}{dt} = \beta_{M_X} e^{-\mu\tau_{M_X}(t)} \frac{v_{M_X}(E_Y)}{v_{M_X}(E_Y(t-\tau_{M_X}(t)))} f_X(E_Y(t-\tau_{M_X}(t))) - \bar{\gamma}_{M_X} M_X, \tag{B.1}$$

$$\frac{dE_X}{dt} = \beta_{E_X} M_X - \bar{\gamma}_{E_X} E_X, \tag{B.2}$$

$$\frac{dM_Y}{dt} = \beta_{M_Y} e^{-\mu\tau_{M_Y}(t)} \frac{v_{M_Y}(E_X)}{v_{M_Y}(E_X(t-\tau_{M_Y}(t)))} f(E_X(t-\tau_{M_Y}(t))) - \bar{\gamma}_{M_Y} M_Y, \tag{B.3}$$

$$\frac{dE_Y}{dt} = \beta_{E_Y} M_Y - \bar{\gamma}_{E_Y} E_Y. \tag{B.4}$$

Analysis of the network dynamics of ordinary differential equation systems [23, 47] suggests that the system

$$\begin{aligned}
\frac{dM_X}{dt} &= \beta_{M_X} f_X(E_Y) - \bar{\gamma}_{M_X} M_X,\\
\frac{dE_X}{dt} &= \beta_{E_X} M_X - \bar{\gamma}_{E_X} E_X,\\
\frac{dM_Y}{dt} &= \beta_{M_Y} f(E_X) - \bar{\gamma}_{M_Y},\\
\frac{dE_Y}{dt} &= \beta_{E_Y} M_Y - \bar{\gamma}_{E_Y} E_Y,
\end{aligned}$$

corresponding to (B.1)–(B.4) in both $(++)$ and $(--)$ cases is a positive cyclic feedback system [17] and will display bistability [23, 47]. The recent preprint [49] carefully analyzes the boundary of bistability regions in both $(++)$ and $(--)$ cases.

Based on the results from [18], inclusion of distributed delays could add additional equilibria, resulting in multistability. Adding delays also can change the range of bistability in the parameter space.

System $(+-)$ is a negative feedback system where we expect that the trivial equilibrium will lose stability through Hopf bifurcation if the slope(s) of nonlinearities are sufficiently steep at the equilibrium. Adding delays may result in additional stable equilibria and secondary Hopf bifurcations, as was observed in [18].

A forest growth model

In [39], the authors considered an age structured model for the growth of a single tree species forest. They showed that the dynamics of the number of adult trees A is governed by

$$\frac{dA(t)}{dt} = e^{-\mu\tau(t)} \frac{f(A(t))}{f(A(t-\tau(t)))} rb(A(t-\tau(t))) - \gamma A(t) \tag{B.5}$$

in conjunction with the condition

$$\int_{t-\tau(t)}^{t} f(A(s))ds = \bar{s}. \tag{B.6}$$

In (B.5) μ is the mortality rate of saplings, f is the velocity of maturation of saplings, r is the birth rate, b is the reproduction (birth) function and γ is the mortality rate of adults.

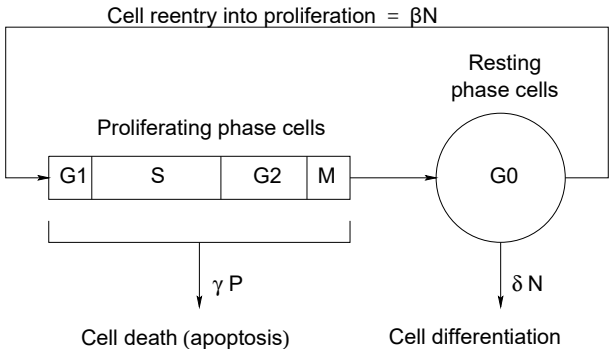


Figure B.1: The Burns and Tannock [7] model for the cell cycle consisting of a resting phase (G_0) and the proliferating phase P with the sub-phases G_1, S (DNA synthesis), G_2, and M (mitosis and cytokinesis). It is assumed that cells can die from the proliferating phase at a rate γ and exit into the differentiation pathway from G_0 at a rate δ. Redrawn from [37].

The G_0 cell cycle model

The original G_0 cell cycle model proposed by Burns and Tannock in [7] is illustrated in Figure B.1, and captures the essence of what is known of the cellular replication process at a intermediate (cellular or non-molecular) level of sophistication and knowledge. The proliferating phase cells are denoted by P while the resting G_0 phase cells are denoted by N.

In the elaboration of [37], it is presumed that cells die from the proliferative phase at a rate γ so the flux of cells to death is γP, and differentiate from G_0 at a rate δ so the differentiation flux is δN. Cells in the G_0 phase can, in addition to differentiating, re-enter the proliferative phase at a rate β so the flux of cells into proliferation is βN. Cells that enter the proliferative phase are assumed to proceed through the stages G_1, S, G_2, and M in a lock step fashion that takes a time τ to complete if death does not intervene. Once mitosis is completed, cytokinesis produces two daughter cells that then enter G_0.

Based on the assumption that the rate β of cell entry from G_0 into P, is a function of the size N of G_0, then it is straightforward to show that the dynamics of the augmented Burns/Tannock cell cycle model are governed by the differential delay equation

$$\frac{dN}{dt} = -[\delta + \beta(N)]N + 2e^{-\gamma\tau}N_\tau\beta(N_\tau), \tag{B.7}$$

where we have used the notation $N \equiv N(t)$ and $N_\tau \equiv N(t-\tau)$. Equation (B.7) has been the subject of an extensive bifurcation analysis in the work of [12].

Now we consider a slightly modified cell cycle model identical to the Burns/Tannock model with the additional wrinkle that the velocity v with which cells move through the proliferative phase of the cell cycle is under the control of the number N of non-proliferative cells.

Remark B.1. One would think that the velocity of movement through the proliferative phase P would be an increasing function of decreased N which would lead to a delay τ that decreases as N decreases. This would lead to cell density dependent inter-division times.

If we take the number of proliferative phase cells at time t and age a to be $p(t,m)$, the maximum age at cytokinesis to be a_m, and the velocity with which they move through the cell

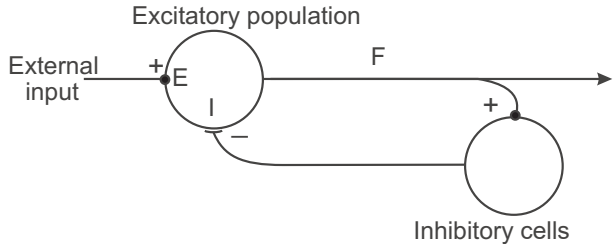


Figure B.2: A schematic representation of the recurrent inhibition circuit considered in [15]. This figure is modified from the original publication.

cycle to be $v(N(t))$, then the evolution equation for $p(t,m)$ is given by

$$\frac{\partial p(t,m)}{\partial t} + v(N(t))\frac{\partial p(t,m)}{\partial m} = -\gamma p(t,m) \tag{B.8}$$

and we have the initial condition

$$p(t,m=0) = N(t)\beta(N(t)). \tag{B.9}$$

Following the same derivation procedure as in [18], the final equation for the dynamics of the non-proliferative cellular population N is given by

$$\frac{dN(t)}{dt} = 2e^{-\gamma\tau(t)}\frac{v(N(t))}{v(N(t-\tau(t)))}N(t-\tau(t))\beta(N(t-\tau(t)) - (\delta+\beta(N(t)))N(t). \tag{B.10}$$

We also have the additional condition that

$$a_m = \int_{t-\tau(t)}^{t} v(N(s))ds. \tag{B.11}$$

Remark B.2. Note that, not unsurprisingly, the augmented cell cycle model is identical in formulation with the forest growth model of [39].

A model for recurrent inhibition

A model for the dynamics of a recurrent inhibitory neural feedback circuit is considered in [15], however the model was incomplete omitting the velocity ratio term which appears in all of the other models here. The term $v(x(t))/v(x(t-\tau(t)))$ in (1.1) is essential as shown in [5, 10, 18] from conservation and flux arguments. Here, we complete the model of [15].

To capture the spirit of the recurrent inhibitory neural circuitry and offer a complete formulation incorporating the ratio term, we consider the situation with dynamics described by

$$\frac{dV(t)}{dt} = \gamma(E - V(t)) - \frac{\Sigma(V(t))}{\Sigma(V(t-\tau(t)))}G(F(V(t-\tau(t)), \tag{B.12}$$

where $V = E - I$ is the net potential in the neuron, $E(I)$ is the excitatory (inhibitory) potential, Σ is the velocity of propagation of the action potentials around the inhibitory loop back to the soma, and G is the response of the neuron to the recurrent inhibitory drive. $F \approx \alpha V$ ($\alpha > 0$) is the firing frequency of the cell, an increasing function of the postsynaptic potential V. In conjunction with (B.12), if the length of the recurrent inhibitory pathway is L we have the ancillary condition

$$L = \int_{t-\tau(t)}^{t} \Sigma(V(s))ds. \tag{B.13}$$

In [15] it was argued that Σ is an increasing function of the postsynaptic potential V so, as a consequence the delay time τ is a decreasing function of V. It was further noted that, on physiological grounds, G is an increasing function of the firing frequency F for small values of F and decreasing with increasing F past a certain maximum.

This model can be further generalized to a state dependent DDE with distributed state dependent delays. This generalization relaxes the assumption that the length L is constant. A more realistic assumption is that the lengths are distributed in population of neurons described by a distribution K_L. Note that this is equivalent to having variable firing threshold at the soma that receives the inhibitory input. The model takes the form

$$\frac{dV(t)}{dt} = \gamma(E - V(t)) - \int_0^\infty \frac{\Sigma(V(t))}{\Sigma(V(t-\phi(t)))} G(F(V(t-\phi(t))) K_L \left(\int_{t-\phi}^{t} \Sigma(V(s)) ds \right) d\phi, \quad \text{(B.14)}$$

where $V = E - I$ is the net potential in the neuron, $E(I)$ is the excitatory (inhibitory) potential, Σ is the velocity of propagation of the action potentials around the inhibitory loop back to the soma, and G is the response of the neuron to the recurrent inhibitory drive. $F \approx \alpha V$ ($\alpha > 0$) is the firing frequency of the cell, an increasing function of the postsynaptic potential V. To derive this equation we note that the distribution of the recurrent loop lengths L is analogous to the distributed maturation times in Kendrick–McCormack age structured models analyzed in [8]. The analog to the Kendrick–McCormack model of the age distributed population is

$$\begin{aligned} &\partial_t E(t,s) + \partial_s(\Sigma(V(t))E(s,t)) = -h(s)E(s,t), \\ &\Sigma(V(t))E(t,0) = G(F(V(t))), \qquad E(t_0,s) = f(s), s \in [0,\infty) \end{aligned}$$

where s is a position along a loop, $E(t,s)$ is the action potential at position s of the loop and the propagation velocity Σ depends on the voltage at time t. The term $h(s)$ is the discharge of the action potential in the soma when the circuit has length s. The function $h(s)$ can be expressed (see [8]) using the distribution K_L of circuit lengths as

$$h(s(t)) = \frac{K_L(s(t))}{1 - \int_0^\sigma K_L(s(\sigma))d\sigma} \frac{d}{dt} s(t).$$

Solving the partial differential equation along the characteristics

$$\frac{d}{d\phi} t(\phi) = 1, \qquad \frac{d}{dt} s(t) = \Sigma(V(t))$$

gives the equation (B.14). For details of the derivation see [8].

C Numerical techniques

Our numerical techniques are documented in detail in [18,63–65], so here we will only briefly summarise them, and note some extensions of the previous methods.

All of our computations are performed using MATLAB [43]. Mainly we perform numerical continuation and bifurcation analyses using the MATLAB package `ddebiftool` [54]. Existing versions of `ddebiftool` are only implemented for discrete delays, and cannot be applied directly to problems with threshold delay. Since the largest possible delay is a/v_0, to implement threshold delay problems in `ddebiftool` we introduce up to 100 dummy constant delays to discretize the time interval $[t - a/v_0, t]$ on a equally spaced mesh. The threshold

delay $\tau(t)$ that satisfies (1.2) can then be computed be applying numerical quadrature. The implementation is explained in detail in [64, 65], and is a refinement of our earlier methods [18, 63].

One of the features that makes `ddebiftool`, and numerical continuation in general, so powerful is its ability to follow stable and unstable branches of solutions equally well. Occasionally we also use numerical simulation of the DDE to find stable solutions. The standard MATLAB solver for state-dependent DDEs is `ddesd` [52], but this is also only implemented for discrete delays. To use it to solve (1.1)–(1.2) as an initial value problem we first differentiate the threshold condition (1.2) to obtain

$$\frac{d\tau(t)}{dt} = 1 - \frac{v(x(t))}{v(x(t-\tau(t)))}, \tag{C.1}$$

and then solve (1.1), (C.1) using `ddesd`. See [18, 63, 65] for implementation details, and also [64] for an investigation of the differences between (1.1), (1.2) and (1.1), (C.1).

The numerical collocation underpinning the `ddebiftool` differential equation solver, is best suited to smooth nonlinearities and solutions. When m or n is large and one of the Hill functions in (1.3) approaches a step function issues may arise in the computations. Here we mention some tricks we applied to compute the figures in those cases.

To circumvent underflow, overflow and division by zero errors in MATLAB when evaluating $\beta e^{-\mu\tau(\xi)} g(\xi)$ near θ_g or θ_v (such as in Figure 6.9), we rewrite the Hill function as

$$g(x) = \begin{cases} \dfrac{g^- + g^+(x/\theta_g)^n}{1+(x/\theta_g)^n}, & x \le \theta_g, \\ \dfrac{g^-(\theta_g/x)^n + g^+}{(\theta_g/x)^n + 1}, & x > \theta_g, \end{cases}$$

and similarly for $v(x)$. It can still be delicate to continue branches of fold bifurcations when m or n are large. In practice, when needed, we apply two different techniques for extending branches of fold bifurcations to very large values of m and/or n. Firstly, the MATLAB nonlinear system solver `fsolve` [57] can be used to solve (6.2) for each fixed value of m or n, to extend the branch to large values of these nonlinearity parameters. Alternatively, since $M(\xi)$ changes sign at the fold bifurcation, these can be revealed by performing a contour plot of $M(\xi)$ in the (ξ, m) or (ξ, n) plane and displaying only the $M(\xi) = 0$ curve. Furthermore, to perform two-parameter continuation of the fold bifurcations in (γ, m) or (γ, n), we evaluate the right hand side of (6.3) to get the corresponding γ for the fold bifurcation at ξ. We apply such tricks extensively in Section 6.2.2.

We were unable to use `ddebiftool` subroutines for detecting codimension-two bifurcations along the fold and the Hopf curves. This was likely due to the complexity of the stability computations with the augmented system `ddebiftool` uses for two-parameter continuation combined with the large number of delays in our implementation. To find the codimension-two bifurcation points we compute the stability of the points on the codimension-one bifurcation branches and track the number of characteristic values with positive real part along the branch. On the fold branch, a fold-Hopf bifurcation occurs when a complex conjugate pair of characteristic values cross the imaginary axis. Fold-Hopf points are also found on a branch of Hopf bifurcations at points where a single real characteristic value changes sign. A Bogdanov–Takens point is characterised by a single real characteristic value changing sign on a fold branch, at a point in parameter space where a branch of Hopf bifurcations also terminates. Note that we do not compute the normal form coefficients of these bifurcations.

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Coexisting steady-state solutions of a class of reaction-diffusion systems with different boundary conditions

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Abstract. In this work, we investigate a class of reaction-diffusion system in which both species are influenced by self-diffusion. By introducing two particular functions, we provide a complete characterization of the parameter ranges such that coexisting steady-state solutions of the system do not exist under three boundary conditions. Then based on the maximum principle, a sufficient condition for the existence of constant coexisting solutions of the system under Neumann boundary conditions was derived.

Keywords: reaction-diffusion system, steady-state, existence, boundary condition.

2020 Mathematics Subject Classification: 35B50, 35J57.

1 Introduction

In [14], Shigesada, Kawasaki and Teramoto introduced the following system with cross-diffusions and self-diffusions, when they took a nonlinear dispersive force and an environmental potential function into consideration,

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta[(d_1 + a_{11}u + a_{12}v)u] + u(1 - u - a_1 v), & x \in \Omega, t > 0, \\ \dfrac{\partial v}{\partial t} = \Delta[(d_2 + a_{21}u + a_{22}v)v] + v(1 - a_2 u - v), & x \in \Omega, t > 0, \\ \alpha_1 u + \beta_1 \dfrac{\partial u}{\partial \nu} = \alpha_2 v + \beta_2 \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x) \geq 0,\ v(x,0) = v_0(x) \geq 0, & x \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n (n \geq 1)$ is a bounded domain with smooth boundary, and satisfies the interior ball condition at any $x \in \partial\Omega$, ν is the outward unit normal vector on $\partial\Omega$. u and v are the densities of two competing species, α_i, β_i and a_{ij} $(i, j = 1, 2)$ are nonnegative constants, $\alpha_i + \beta_i > 0$ $(i = 1, 2)$, a_i and d_i $(i = 1, 2)$ are all positive constants, a_{11} and a_{22} stand for the

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self-diffusion pressures, while a_{12} and a_{21} are the cross-diffusion pressures, a_1, a_2 describe the inter-specific competitions, and d_1, d_2 are their diffusion rates [19].

In order to analyze and describe the above reaction-diffusion model, it is necessary to clarify the boundary conditions of the region. When $\alpha_i = 0$ $(i = 1, 2)$, we have $\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0$, which is called Neumann boundary condition. At this point, individuals who reach the boundary will be reflected back into the region without leaving, meaning that the species is in an isolated environment. When $\beta_i = 0$ $(i = 1, 2)$, we have $u = v = 0$. These indicate that individuals who encounter the boundary cross it immediately and thereby maintain the density on the boundary at zero [1]. This means that the boundary $\partial\Omega$ can effectively absorb all individuals encountering it. Therefore, this boundary is called absorbed, also known as Dirichlet boundary condition. When $\alpha_i, \beta_i > 0$ $(i = 1, 2)$, it is called Robin boundary condition in the mathematical literature.

Since the model, which was abbreviated as SKT model, was proposed, numerous experts and scholars have conducted extensive and in-depth research on it. These results mainly include the existence, boundedness and global convergence of classical solutions, the local and global existence of weak solutions, the existence, nonexistence and stability of steady-state solutions, the existence of traveling wave solutions, and so on.

In fact, the study of standard SKT models is quite difficult. More experts and scholars are turning to the study of certain special forms of SKT model; see [2, 4–8, 12, 15, 16]. Among them, when the spatial dimension is 1 and $d_1 = d_2$, [6] proved the global existence of smooth solutions. Under the same conditions, [15] obtained the uniform boundedness and convergence of smooth solutions. In 2015, Lou and Winkler [12] used the comparison principle and Sobolev regularity theory to obtain the global existence and uniform boundedness of smooth solutions on bounded convex domain when the spatial dimension is less than 3 and $d_1 = d_2$. Taking self-diffusion into consideration, [5] proved the global existence of the unique smooth solution in any spatial dimension; and [8] obtained the global existence of the unique classical solution by using Sobolev embedding theory under the condition $d_1 = d_2$.

For research on steady-state solutions, one can refer to [9–11]. In 1996, Lou and Ni [9] used the maximum principle and Lyapunov functional theory to prove that in the weak competition case, if self-diffusion and/or cross-diffusion are relatively weaker than diffusion, then there is still no nonconstant steady-state solution. At the same time, they proved that in the weak competition case, with one of the cross-diffusion pressures arbitrarily given but fixed, it is expect to find non-constant steady-state solutions if the other cross-diffusion pressure is large enough [9]. Without considering the influence of self-diffusion, [10] obtained a sufficient condition such that the SKT model has no nonconstant steady-state solutions. When $a_{21} = a_{22} = 0$, Lou et al. [11] provided the parameter ranges such that the system has no nonconstant positive solutions for $a_{11} = 0$ and $a_{11} \neq 0$, respectively.

It is obvious to see that the above studies were conducted under Neumann boundary conditions, which is also the most extensively studied scenario. In addition, some scholars have also studied the SKT model under Dirichlet boundary conditions, which can be found in [3, 13, 17, 20] and references therein. For example, the sufficient conditions for the existence of positive steady-state solutions of the system are given in [13] using the fixed point theory in the case of fixed or sufficiently large cross-diffusion coefficients, respectively. The existence of steady-state solutions for a one-dimensional system was studied using the singular perturbation method [17].

Based on the previous work [19], where one species has cross-diffusion and another species has self-diffusion, now we consider the following steady-state model, which indicates that

there are self-diffusions in both competing species and there is no cross-diffusion in either,

$$\begin{cases} \Delta[(d_1 + a_{11}u)u] + u(1 - u - a_1v) = 0, & x \in \Omega, \\ \Delta[(d_2 + a_{22}v)v] + v(1 - a_2u - v) = 0, & x \in \Omega, \\ \alpha_1 u + \beta_1 \dfrac{\partial u}{\partial \nu} = \alpha_2 v + \beta_2 \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases} \tag{1.2}$$

We aim to obtain sufficient conditions such that the system (1.2) has no coexisting solutions under three different boundary conditions, and establish the parameter ranges for the existence of constant coexisting solutions under Neumann boundary conditions. Considering that u and v represent species densities, we focus on the nonnegative classical solution (u, v) of (1.2), which means that $(u, v) \in (C^1(\overline{\Omega}) \cap C^2(\Omega))^2$, $u, v \geq 0$ in $\overline{\Omega}$, and satisfies (1.2) in the pointwise sense [19].

The remainder of this work is organized as follows. Section 2 gives some basic preliminaries which imply the strict positivity of the nontrivial solutions of system (1.2). In Section 3, based on the boundedness of solutions, we obtain two different parameter ranges for nonexistence of coexisting solutions under three boundary conditions. In Section 4, we establish the sufficient conditions for the existence of constant coexisting solutions under Neumann boundary conditions.

2 Preliminaries

First of all, we can obtain the positivity of the classical solutions of (1.2) in Ω, which is crucial in subsequent section.

Proposition 2.1. *Let (u, v) be a nonnegative classical solution of (1.2). Then if $u \not\equiv 0$, we have $u > 0$ in Ω, and if $v \not\equiv 0$, we have $v > 0$ in Ω.*

Proof. We only prove $u > 0$ in Ω whenever $u \not\equiv 0$, since the positivity of v in Ω can be proved in a similar way. Otherwise, there is $x_0 \in \Omega$ such that $u(x_0) = \min_{x\in\overline{\Omega}} u(x) = 0$.

It follows from the first equation of (1.2) that

$$(d_1 + 2a_{11}u)\Delta u + 2a_{11}|\nabla u|^2 + u(1 - u - a_1v) = 0.$$

Let

$$Lu = -(d_1 + 2a_{11}u)\Delta u - 2a_{11}|\nabla u|^2 + cu \quad \text{with } c = u + a_1v.$$

Then

$$c \geq 0 \quad \text{and} \quad Lu = u \geq 0 \quad \text{in } \Omega.$$

So, an application of the strong maximum principle shows that u is a constant in Ω, and thus $u = 0$, a contradiction to $u \not\equiv 0$. This completes the proof. □

Remark 2.2. In the case of Neumann or Robin boundary conditions, we can get further that $u, v > 0$ in $\overline{\Omega}$ by Hopf's boundary lemma. In fact, for example, considering the case of Robin boundary conditions, suppose that there is $x_0 \in \overline{\Omega}$ such that $u(x_0) = \min_{x\in\overline{\Omega}} u(x) = 0$. If $x_0 \in \Omega$, we can directly derive a contradiction by Proposition 2.1. If $x_0 \in \partial\Omega$, then $u(x_0) < u(x)$ for all $x \in \Omega$. Since $Lu \geq 0$ in Ω and Ω satisfies the interior ball condition at $x_0 \in \partial\Omega$, it follows from Hopf's boundary lemma that $\frac{\partial u}{\partial \nu}(x_0) < 0$. Hence

$$\alpha_1 u(x_0) + \beta_1 \frac{\partial u}{\partial \nu}(x_0) < 0,$$

a condiction. Therefore, $u > 0$ in $\overline{\Omega}$.

3 Nonexistence of coexisting steady-state solutions

In this section, we will discuss the nonexistence of coexisting solutions for system (1.2) under three different boundary conditions.

3.1 Neumann boundary condition

In this subsection, assume that $\alpha_1 = \alpha_2 = 0$, that is, we consider the following system,

$$\begin{cases} \Delta[(d_1 + a_{11}u)u] + u(1 - u - a_1 v) = 0, & x \in \Omega, \\ \Delta[(d_2 + a_{22}v)v] + v(1 - a_2 u - v) = 0, & x \in \Omega, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases} \tag{3.1}$$

Firstly, we give the following lemma, which indicates that u and v are both bounded.

Lemma 3.1. *Suppose that (u, v) is a nonnegative classical solution of (3.1). If $u \not\equiv 0$, then $u \leq 1$ in $\overline{\Omega}$. Similarly, if $v \not\equiv 0$, then $v \leq 1$ in $\overline{\Omega}$.*

Proof. We only prove the boundedness of u, while the boundedness of v can be similarly obtained, leaving it for interested readers. Rewrite the first equation of the system (3.1) as follows

$$\begin{cases} (d_1 + 2a_{11}u)\Delta u + 2a_{11}|\nabla u|^2 + u(1 - u - a_1 v) = 0, & x \in \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases} \tag{3.2}$$

Let $f(x) = u(x)(1 - u(x) - a_1 v(x))$, $x \in \Omega$. Proposition 2.2 in [9] implies that there exist $x_1, x_2 \in \overline{\Omega}$ such that

$$u(x_1) = \max_{x \in \overline{\Omega}} u(x) \triangleq M, \qquad u(x_2) = \min_{x \in \overline{\Omega}} u(x) \triangleq m,$$

and

$$f(x_1) \geq 0, \qquad f(x_2) \leq 0,$$

that is

$$M(1 - M - a_1 v(x_1)) \geq 0, \qquad m(1 - m - a_1 v(x_2)) \leq 0.$$

We thus have

$$1 - a_1 v(x_2) \leq m \leq u(x) \leq M \leq 1 - a_1 v(x_1) \quad \text{for all } x \in \overline{\Omega}.$$

Combining this with the nonnegativity of v, we obtain $u \leq 1$ in $\overline{\Omega}$. □

Next, we provide a sufficient condition for the nonexistence of coexisting solutions under Neumann boundary conditions.

Theorem 3.2. *Suppose that (u, v) is a nonnegative classical solution of (3.1). If*

(i) $$a_1 > 1 > a_2, \qquad d_1 > d_2 \quad \text{and} \quad d_1 \geq d_2 + 2a_{22}$$

or

(ii) $$a_1 < 1 < a_2, \qquad d_1 < d_2 \quad \text{and} \quad d_2 \geq d_1 + 2a_{11},$$

then system (3.1) *has no coexisting solutions, that is, at least one species is extinct.*

Proof. (i) We argue by contradiction. Suppose that $u \not\equiv 0$ and $v \not\equiv 0$. It follows from Proposition 2.1 that $u, v > 0$ in Ω, which allows us to rewrite system (3.1) as follows,

$$\begin{cases} \dfrac{\Delta[(d_1 + a_{11}u)u]}{u} = -1 + u + a_1 v, & x \in \Omega, \\ \dfrac{\Delta[(d_2 + a_{22}v)v]}{v} = -1 + a_2 u + v, & x \in \Omega, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases} \tag{3.3}$$

Let

$$w_1 = d_1 + a_{11}u \quad \text{and} \quad w_2 = d_2 + a_{22}v. \tag{3.4}$$

Since $a_1 > 1 > a_2$, we can obtain that $\frac{\Delta(w_1 u)}{u} > \frac{\Delta(w_2 v)}{v}$, that is,

$$\frac{u\Delta w_1 + 2\nabla u \cdot \nabla w_1 + w_1 \Delta u}{u} > \frac{v\Delta w_2 + 2\nabla v \cdot \nabla w_2 + w_2 \Delta v}{v} \quad \text{in } \Omega. \tag{3.5}$$

We procedure the following calculation based on the inequality above,

$$\begin{aligned} &\operatorname{div}\Big[(uv\nabla w_1 - uv\nabla w_2 + vw_1\nabla u - uw_2\nabla v)\frac{u}{v}\Big] \\ &\quad = \big(uv\Delta w_1 + v\nabla u\cdot\nabla w_1 + u\nabla v\cdot\nabla w_1 - uv\Delta w_2 - v\nabla u\cdot\nabla w_2 - u\nabla v\cdot\nabla w_2 \\ &\qquad + vw_1\Delta u + v\nabla u\cdot\nabla w_1 + w_1\nabla u\cdot\nabla v - uw_2\Delta v - u\nabla v\cdot\nabla w_2 - w_2\nabla u\cdot\nabla v\big)\frac{u}{v} \\ &\qquad + \big(uv\nabla w_1 - uv\nabla w_2 + vw_1\nabla u - uw_2\nabla v\big)\cdot\Big(\frac{1}{v}\nabla u - \frac{u}{v^2}\nabla v\Big) \\ &\quad > \big(u\nabla v\cdot\nabla w_1 - v\nabla u\cdot\nabla w_2 + w_1\nabla u\cdot\nabla v - w_2\nabla u\cdot\nabla v\big)\frac{u}{v} \\ &\qquad + \big(uv\nabla w_1 - uv\nabla w_2 + vw_1\nabla u - uw_2\nabla v\big)\cdot\Big(\frac{1}{v}\nabla u - \frac{u}{v^2}\nabla v\Big) \\ &\quad = |\nabla u|^2\big(a_{11}u + w_1\big) + |\nabla v|^2\big(a_{22}v + w_2\big)\frac{u^2}{v^2} \\ &\qquad + \nabla u\cdot\nabla v\Big[(a_{11}u + w_1)\frac{u}{v} + (-a_{22}v - w_2)\frac{u}{v} + (-a_{11}u - w_1 - a_{22}v - w_2)\frac{u}{v}\Big] \\ &\quad = |\nabla u|^2\big(d_1 + 2a_{11}u\big) + |\nabla v|^2\big(d_2 + 2a_{22}v\big)\frac{u^2}{v^2} - 2(d_2 + 2a_{22}v)\frac{u}{v}\nabla u\cdot\nabla v \\ &\quad = |\nabla u|^2\big(d_1 + 2a_{11}u - d_2 - 2a_{22}v\big) + \Big(\sqrt{d_2 + 2a_{22}v}\nabla u - \frac{u}{v}\sqrt{d_2 + 2a_{22}v}\nabla v\Big)^2. \end{aligned}$$

As $d_1 > d_2$, $d_1 \geq d_2 + 2a_{22}$ and $v \leq 1$, we have

$$d_1 + 2a_{11}u - d_2 - 2a_{22}v \geq d_1 + 2a_{11}u - d_2 - 2a_{22} \geq 0 \quad \text{in } \Omega.$$

Thus,

$$\operatorname{div}\Big[(uv\nabla w_1 - uv\nabla w_2 + vw_1\nabla u - uw_2\nabla v)\frac{u}{v}\Big] > 0. \tag{3.6}$$

On the other hand, we can see from Neumann boundary conditions that

$$\begin{aligned}
&\int_\Omega \operatorname{div}\left[\left(uv\nabla w_1 - uv\nabla w_2 + vw_1\nabla u - uw_2\nabla v\right)\frac{u}{v}\right]\mathrm{d}x\\
&\quad= \int_{\partial\Omega}\left(uv\frac{\partial w_1}{\partial \nu} - uv\frac{\partial w_2}{\partial \nu} + vw_1\frac{\partial u}{\partial \nu} - uw_2\frac{\partial v}{\partial \nu}\right)\frac{u}{v}\mathrm{d}S\\
&\quad= \int_{\partial\Omega}\left(a_{11}uv\frac{\partial u}{\partial \nu} - a_{22}uv\frac{\partial v}{\partial \nu} + vw_1\frac{\partial u}{\partial \nu} - uw_2\frac{\partial v}{\partial \nu}\right)\frac{u}{v}\mathrm{d}S\\
&\quad= 0,
\end{aligned}$$

which is contradict to (3.6). Therefore, $u \equiv 0$ or $v \equiv 0$.

(ii) In this case, due to $a_1 < 1 < a_2$, we can see that $\frac{\Delta(w_1 u)}{u} < \frac{\Delta(w_2 v)}{v}$, which can be expanded into the following form

$$\frac{u\Delta w_1 + 2\nabla u\cdot\nabla w_1 + w_1\Delta u}{u} < \frac{v\Delta w_2 + 2\nabla v\cdot\nabla w_2 + w_2\Delta v}{v} \quad \text{in } \Omega, \tag{3.7}$$

where w_1, w_2 are defined in (3.4). Similar to the calculations of (i), we consider the process as follows

$$\begin{aligned}
&\operatorname{div}\left[\left(uv\nabla w_1 - uv\nabla w_2 + vw_1\nabla u - uw_2\nabla v\right)\frac{v}{u}\right]\\
&\quad< \left(u\nabla v\cdot\nabla w_1 - v\nabla u\cdot\nabla w_2 + w_1\nabla u\cdot\nabla v - w_2\nabla u\cdot\nabla v\right)\frac{v}{u}\\
&\qquad+ \left(uv\nabla w_1 - uv\nabla w_2 + vw_1\nabla u - uw_2\nabla v\right)\cdot\left(-\frac{v}{u^2}\nabla u + \frac{1}{u}\nabla v\right)\\
&\quad= |\nabla u|^2\left(-a_{11}u - w_1\right)\frac{v^2}{u^2} + |\nabla v|^2\left(-a_{22}v - w_2\right)\\
&\qquad+ \nabla u\cdot\nabla v\left[\left(a_{11}u + w_1\right)\frac{v}{u} + \left(-a_{22}v - w_2\right)\frac{v}{u} + \left(a_{11}u + w_1 + a_{22}v + w_2\right)\frac{v}{u}\right]\\
&\quad= -|\nabla u|^2\left(d_1 + 2a_{11}u\right)\frac{v^2}{u^2} - |\nabla v|^2\left(d_2 + 2a_{22}v\right) + 2\left(d_1 + 2a_{11}u\right)\frac{v}{u}\nabla u\cdot\nabla v\\
&\quad= |\nabla v|^2\left(-d_2 - 2a_{22}v + d_1 + 2a_{11}u\right) - \left(\frac{v}{u}\sqrt{d_1 + 2a_{11}u}\nabla u - \sqrt{d_1 + 2a_{11}u}\nabla v\right)^2.
\end{aligned}$$

Since $d_1 < d_2$, $d_2 \geq d_1 + 2a_{11}$ and $u \leq 1$, we see

$$-d_2 - 2a_{22}v + d_1 + 2a_{11}u \leq -d_2 - 2a_{22}v + d_1 + 2a_{11} \leq 0 \quad \text{in } \Omega.$$

It is shown that

$$\operatorname{div}\left[\left(uv\nabla w_1 - uv\nabla w_2 + vw_1\nabla u - uw_2\nabla v\right)\frac{v}{u}\right] < 0. \tag{3.8}$$

Now, based on the boundary conditions, we can still obtain the following result

$$\int_\Omega \operatorname{div}\left[\left(uv\nabla w_1 - uv\nabla w_2 + vw_1\nabla u - uw_2\nabla v\right)\frac{v}{u}\right]\mathrm{d}x = 0,$$

which leads to a contradiction. This completes the proof. □

Remark 3.3. Based on Proposition 2.1, we constructed two subtle auxiliary functions and proved the conclusion by contradiction. From the conclusion, it can be seen that the theorem provides a specific characterization that when the inter-specific competition rate and diffusion rate of some species are relatively high, it will lead to some species extinction.

Furthermore, under the conditions (i) and (ii) of Theorem 3.2, we selected two sets of values and conducted numerical simulations for the following system respectively,

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta[(d_1 + a_{11}u)u] + u(1 - u - a_1 v), & x \in (0,1), t > 0, \\ \dfrac{\partial v}{\partial t} = \Delta[(d_2 + a_{22}v)v] + v(1 - a_2 u - v), & x \in (0,1), t > 0, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x = 0, 1, t > 0, \\ u(x,0) = u_0(x) \geq 0,\ v(x,0) = v_0(x) \geq 0, & x \in (0,1), \end{cases} \tag{3.9}$$

as shown in Figure 3.1 and 3.2. The simulation results show that, species u becomes extinct under condition (i) while species v becomes extinct under condition (ii). Therefore, it is nature to ask whether species u necessarily extinct under condition (i) and species v necessarily extinct under condition (ii). Our results have not provided an answer, and this is our future work.

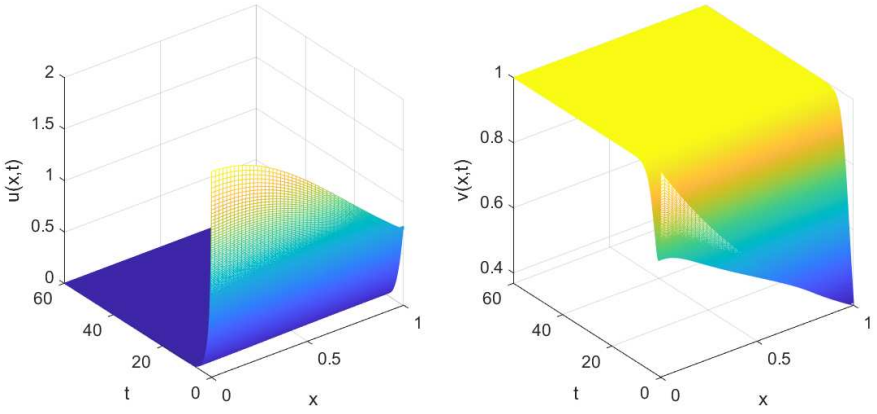


Figure 3.1: Numerical simulations for system (3.9) with $u_0(x) = 2e^{-x}$, $v_0(x) = e^{-x}$, $d_1 = 0.7, d_2 = 0.2, a_{11} = 0.1, a_{22} = 0.2$, $a_1 = 1.8$ and $a_2 = 0.5$.

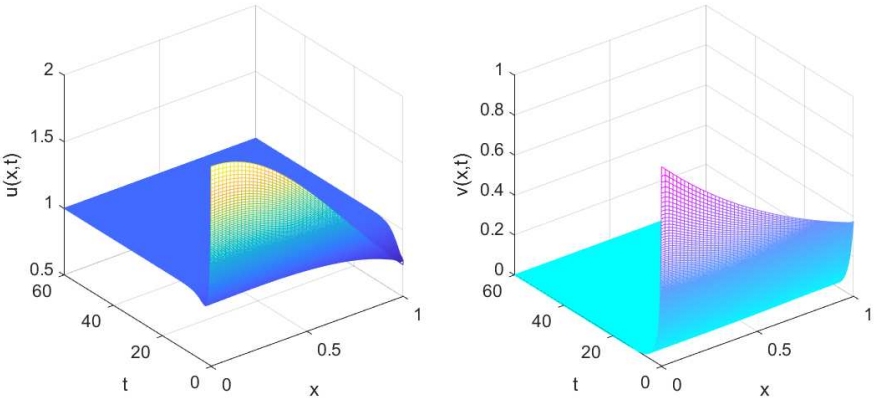


Figure 3.2: Numerical simulations for system (3.9) with $u_0(x) = 2e^{-x}$, $v_0(x) = e^{-x}$, $d_1 = 0.1, d_2 = 0.4, a_{11} = 0.1, a_{22} = 0.2$, $a_1 = 0.8$ and $a_2 = 1.5$.

3.2 Dirichlet boundary condition

In this subsection, we consider the case of $\beta_1 = \beta_2 = 0$, which leads to the following system,

$$\begin{cases} \Delta[(d_1 + a_{11}u)u] + u(1 - u - a_1 v) = 0, & x \in \Omega, \\ \Delta[(d_2 + a_{22}v)v] + v(1 - a_2 u - v) = 0, & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega. \end{cases} \tag{3.10}$$

Now, we can also deduce that u and v are both bounded.

Lemma 3.4. *Suppose that (u, v) is a nonnegative classical solution of (3.10). If $u \not\equiv 0$, then $u \leq 1$ in $\overline{\Omega}$. Similarly, if $v \not\equiv 0$, then $v \leq 1$ in $\overline{\Omega}$.*

Proof. We only need to prove the boundedness of v. Proposition 2.1 implies that $v > 0$ in Ω if $v \not\equiv 0$. The second equation of system (3.10) can be transformed into the following form,

$$\begin{cases} (d_2 + 2a_{22}v)\Delta v + 2a_{22}|\nabla v|^2 + v(1 - a_2 u - v) = 0, & x \in \Omega, \\ v = 0, & x \in \partial\Omega. \end{cases} \tag{3.11}$$

Suppose on the contrary that there exists a point x_0 such that $v(x_0) = \max_{\overline{\Omega}} v(x) > 1$. Obviously, $x_0 \in \Omega$. So we have

$$\Delta v(x_0) \leq 0,\ \nabla v(x_0) = 0 \text{ and } v(x_0)(1 - a_2 u(x_0) - v(x_0)) \geq 0.$$

Since $v > 0$ in Ω, we obtain that $1 - a_2 u(x_0) - v(x_0) \geq 0$, that is, $1 \geq a_2 u(x_0) + v(x_0)$, a contradiction. □

If we restrict attention to the Dirichlet boundary conditions, we can establish the parameter ranges such that the system (3.10) has no coexisting solutions.

Theorem 3.5. *Suppose that (u, v) is a nonnegative classical solution of (3.10). If*

(i) $$a_1 > 1 > a_2, \qquad d_1 > d_2 \quad \text{and} \quad d_1 \geq d_2 + 2a_{22}$$

or

(ii) $$a_1 < 1 < a_2, \qquad d_1 < d_2 \quad \text{and} \quad d_2 \geq d_1 + 2a_{11},$$

then system (3.10) has no coexisting solutions.

Proof. (i) According to the proof of Theorem 3.2-(i), the following inequality still holds

$$\operatorname{div}\left[(uv\nabla w_1 - uv\nabla w_2 + vw_1\nabla u - uw_2\nabla v)\frac{u}{v}\right] > 0,$$

where $w_1 = d_1 + a_{11}u$ and $w_2 = d_2 + a_{22}v$.

Now, we consider the following integral

$$\begin{aligned} &\int_\Omega \operatorname{div}\left[\left(uv\nabla w_1 - uv\nabla w_2 + vw_1\nabla u - uw_2\nabla v\right)\frac{u}{v}\right] \mathrm{d}x \\ &= \int_{\partial\Omega}\left(uv\frac{\partial w_1}{\partial \nu} - uv\frac{\partial w_2}{\partial \nu} + vw_1\frac{\partial u}{\partial \nu} - uw_2\frac{\partial v}{\partial \nu}\right)\frac{u}{v}\mathrm{d}S \\ &= \int_{\partial\Omega}\left(a_{11}u^2\frac{\partial u}{\partial \nu} - a_{22}u^2\frac{\partial v}{\partial \nu} + uw_1\frac{\partial u}{\partial \nu} - \left(d_2\frac{u^2}{v} + a_{22}u^2\right)\frac{\partial v}{\partial \nu}\right)\mathrm{d}S. \end{aligned}$$

It is easy to see that the function $\frac{u^2}{v}$ in the last term of the integrand does not make sense on $\partial\Omega$. So in such a case we cannot make calculations directly. Let

$$\Omega_\varepsilon = \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) > \varepsilon\} \quad \text{for any small } \varepsilon > 0.$$

Since $(u, v) \in (C^1(\overline{\Omega}) \cap C^2(\Omega))^2$, we observe that

$$\big(uv\nabla w_1 - uv\nabla w_2 + vw_1\nabla u - uw_2\nabla v\big)\frac{u}{v} \in C^1(\overline{\Omega_\varepsilon}).$$

Then divergence theorem implies that

$$\begin{aligned}
&\int_\Omega \operatorname{div}\Big[\big(uv\nabla w_1 - uv\nabla w_2 + vw_1\nabla u - uw_2\nabla v\big)\frac{u}{v}\Big]\,\mathrm{d}x \\
&\quad= \lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} \operatorname{div}\Big[\big(uv\nabla w_1 - uv\nabla w_2 + vw_1\nabla u - uw_2\nabla v\big)\frac{u}{v}\Big]\,\mathrm{d}x \\
&\quad= \lim_{\varepsilon\to 0}\int_{\partial\Omega_\varepsilon}\left(a_{11}u^2\frac{\partial u}{\partial \nu} - a_{22}u^2\frac{\partial v}{\partial \nu} + uw_1\frac{\partial u}{\partial \nu} - \left(d_2\frac{u^2}{v} + a_{22}u^2\right)\frac{\partial v}{\partial \nu}\right)\mathrm{d}S.
\end{aligned}$$

Let

$$\begin{aligned}
I_1(\varepsilon) &= \int_{\partial\Omega_\varepsilon}\left(a_{11}u^2\frac{\partial u}{\partial \nu} - a_{22}u^2\frac{\partial v}{\partial \nu} + uw_1\frac{\partial u}{\partial \nu} - a_{22}u^2\frac{\partial v}{\partial \nu}\right)\mathrm{d}S, \\
I_2(\varepsilon) &= \int_{\partial\Omega_\varepsilon} d_2\frac{u^2}{v}\frac{\partial v}{\partial \nu}\mathrm{d}S.
\end{aligned}$$

Obviously, $I_1(\varepsilon)$ approaches zero as $\varepsilon \to 0$ in terms of Dirichlet boundary conditions. In order to deal with the term $I_2(\varepsilon)$, we write

$$V = \left\{\varphi(x) \in C^1(\overline{\Omega}) \mid \varphi(x) > 0, x \in \Omega;\ \varphi|_{\partial\Omega} = 0;\ \frac{\partial\varphi}{\partial\nu}\Big|_{\partial\Omega} < 0\right\}.$$

By Hopf's Lemma, we have $\frac{\partial u(x_0)}{\partial \nu} < 0$ and $\frac{\partial v(x_0)}{\partial \nu} < 0$ for any $x_0 \in \partial\Omega$, and thus $u \in V$ and $v \in V$. We now define

$$g(x) := \begin{cases} \dfrac{u(x)}{v(x)}, & x \in \Omega, \\ \dfrac{\partial u(x)}{\partial \nu}\Big/\dfrac{\partial v(x)}{\partial \nu}, & x \in \partial\Omega. \end{cases}$$

Then Lemma 2.4 in [18] shows that $g(x) \in C\big(\overline{\Omega}, (0, +\infty)\big)$. Therefore, we conclude that $I_2(\varepsilon)$ also approaches zero as $\varepsilon \to 0$. These considerations motivate that

$$\int_\Omega \operatorname{div}\Big[\big(uv\nabla w_1 - uv\nabla w_2 + vw_1\nabla u - uw_2\nabla v\big)\frac{u}{v}\Big]\,\mathrm{d}x = 0 \tag{3.12}$$

thanks to Lebesgue dominated convergence theorem and Dirichlet boundary conditions, a contradiction.

(ii) On one hand, it is obvious to see from Theorem 3.2-(ii) that

$$\operatorname{div}\Big[\big(uv\nabla w_1 - uv\nabla w_2 + vw_1\nabla u - uw_2\nabla v\big)\frac{v}{u}\Big] < 0, \tag{3.13}$$

due to $d_1 < d_2$, $d_2 \geq d_1 + 2a_{11}$ and Lemma 3.4.

On the other hand, we can deduce from the discussion on boundary integrals similar to (i) that

$$\int_\Omega \operatorname{div}\Big[\big(uv\nabla w_1 - uv\nabla w_2 + vw_1\nabla u - uw_2\nabla v\big)\frac{v}{u}\Big]\,\mathrm{d}x = 0. \tag{3.14}$$

Therefore, a contradiction can also be obtained, and the conclusion is valid. □

Remark 3.6. It is easy to see that the conditions of this theorem are the same as those of Theorem 3.2. In fact, since the equations in system (3.1) and (3.10) are the same, the conditions of Theorem 3.2 and 3.5 are given to ensure that the constructed auxiliary function is strictly positive or strictly negative within the region, which is independent of boundary conditions.

3.3 Robin boundary condition

In this subsection, we consider the following system

$$\begin{cases} \Delta[(d_1 + a_{11}u)u] + u(1 - u - a_1v) = 0, & x \in \Omega, \\ \Delta[(d_2 + a_{22}v)v] + v(1 - a_2u - v) = 0, & x \in \Omega, \\ \alpha_1 u + \beta_1 \dfrac{\partial u}{\partial \nu} = \alpha_2 v + \beta_2 \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, \end{cases} \tag{3.15}$$

where $\alpha_i > 0$, $\beta_i > 0$, $i = 1, 2$. Under this boundary conditions, the boundedness of u and v is also not difficult to obtain.

Lemma 3.7. *Suppose that (u, v) is a nonnegative classical solution of (3.15). If $u \not\equiv 0$, then $u \leq 1$ in $\overline{\Omega}$. Similarly, if $v \not\equiv 0$, then $v \leq 1$ in $\overline{\Omega}$.*

Proof. We only prove the previous statement. First, we can obtain that

$$\begin{cases} (d_1 + 2a_{11}u)\Delta u + 2a_{11}|\nabla u|^2 + u(1 - u - a_1v) = 0, & x \in \Omega, \\ \alpha_1 u + \beta_1 \dfrac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases} \tag{3.16}$$

Suppose that there exists a point $x_0 \in \overline{\Omega}$, such that $u(x_0) = \max_{x\in\overline{\Omega}} u(x) > 1$.

(i) If $x_0 \in \Omega$, we have $\Delta u(x_0) \leq 0$, $\nabla v(x_0) = 0$ and $u(x_0)(1 - u(x_0) - a_1v(x_0)) \geq 0$. Since $u > 0$ in Ω, we obtain that $1 - u(x_0) - a_1v(x_0) \geq 0$, that is, $1 \geq u(x_0) + a_1v(x_0)$, a contradiction.

(ii) If $x_0 \in \partial\Omega$, then $u(x_0) > u(x)$ for all $x \in \Omega$. Thus we can see that $\frac{\partial u}{\partial \nu}(x_0) \geq 0$. Since $u > 0$ in $\overline{\Omega}$ by Remark 2.2, we have $\alpha_1 u(x_0) + \beta_1 \frac{\partial u}{\partial \nu}(x_0) > 0$, a contradiction. □

Theorem 3.8. *Suppose that (u, v) is a nonnegative classical solution of (3.15). If*

(i) $a_1 > 1 > a_2$, $d_1 > d_2$, $d_1 \geq d_2 + 2a_{22}$, $\alpha_1 \geq \beta_1$ *and* $\alpha_2 \leq \beta_2$

or

(ii) $a_1 < 1 < a_2$, $d_1 < d_2$, $d_2 \geq d_1 + 2a_{11}$, $\alpha_1 \leq \beta_1$ *and* $\alpha_2 \geq \beta_2$,

then system (3.15) has no coexisting solutions.

Proof. We only prove the first case. Now, we can still obtain

$$\operatorname{div}\left[(uv\nabla w_1 - uv\nabla w_2 + vw_1\nabla u - uw_2\nabla v)\frac{u}{v}\right] > 0,$$

where $w_1 = d_1 + a_{11}u$ and $w_2 = d_2 + a_{22}v$.

Notice that $\frac{\partial u}{\partial \nu} = -\frac{\alpha_1}{\beta_1}u$, $\frac{\partial v}{\partial \nu} = -\frac{\alpha_2}{\beta_2}v$ in $\partial\Omega$. It consequently seems advisable to consider the following integral

$$\begin{aligned}
&\int_\Omega \operatorname{div}\left[\left(uv\nabla w_1 - uv\nabla w_2 + vw_1\nabla u - uw_2\nabla v\right)\frac{u}{v}\right]\mathrm{d}x \\
&= \int_{\partial\Omega}\left(a_{11}uv\frac{\partial u}{\partial \nu} - a_{22}uv\frac{\partial v}{\partial \nu} + vw_1\frac{\partial u}{\partial \nu} - uw_2\frac{\partial v}{\partial \nu}\right)\frac{u}{v}\mathrm{d}S \\
&= \int_{\partial\Omega}\left(-a_{11}\frac{\alpha_1}{\beta_1}u^3 + a_{22}\frac{\alpha_2}{\beta_2}u^2v - \frac{\alpha_1}{\beta_1}u^2w_1 + \frac{\alpha_2}{\beta_2}u^2w_2\right)\mathrm{d}S \\
&= \int_{\partial\Omega} u^2\left(-2a_{11}\frac{\alpha_1}{\beta_1}u + 2a_{22}\frac{\alpha_2}{\beta_2}v - d_1\frac{\alpha_1}{\beta_1} + d_2\frac{\alpha_2}{\beta_2}\right)\mathrm{d}S \\
&\le \int_{\partial\Omega} u^2\left(-2a_{11}\frac{\alpha_1}{\beta_1}u + 2a_{22}\frac{\alpha_2}{\beta_2} - d_1\frac{\alpha_1}{\beta_1} + d_2\frac{\alpha_2}{\beta_2}\right)\mathrm{d}S \\
&\le \int_{\partial\Omega} u^2\left(-2a_{11}\frac{\alpha_1}{\beta_1}u + 2a_{22} - d_1 + d_2\right)\mathrm{d}S \\
&\le 0,
\end{aligned}$$

due to Lemma 3.7, $\alpha_1 \ge \beta_1$, $\alpha_2 \le \beta_2$ and $d_1 \ge d_2 + 2a_{22}$, a contradiction. □

4 Existence of coexisting solutions

In this section, we will investigate the existence of coexisting solutions for system (3.1).

Theorem 4.1. *Let $a_1 < 1$ and $a_2 < 1$. Suppose that (u, v) is a nonnegative classical solution of (3.1). If $u \not\equiv 0$ and $v \not\equiv 0$, then*

$$u \equiv \frac{1-a_1}{1-a_1a_2} \quad \text{and} \quad v \equiv \frac{1-a_2}{1-a_1a_2}. \tag{4.1}$$

Proof. Recall from Lemma 3.1 that there exist $x_1, x_2 \in \overline{\Omega}$ such that

$$u(x_1) = \max_{x\in\overline{\Omega}} u(x) \triangleq M, \qquad u(x_2) = \min_{x\in\overline{\Omega}} u(x) \triangleq m,$$

and

$$1 - a_1v(x_2) \le m \le u(x) \le M \le 1 - a_1v(x_1) \quad \text{for all } x \in \overline{\Omega}. \tag{4.2}$$

Similarly, there exist $x_3, x_4 \in \overline{\Omega}$ such that

$$v(x_3) = \max_{x\in\overline{\Omega}} v(x) \triangleq M', \qquad v(x_4) = \min_{x\in\overline{\Omega}} v(x) \triangleq m',$$

and

$$1 - a_2u(x_4) \le m' \le v(x) \le M' \le 1 - a_2u(x_3) \quad \text{for all } x \in \overline{\Omega}. \tag{4.3}$$

We first prove $u \equiv \frac{1-a_1}{1-a_1a_2}$. Combining (4.2) with (4.3), we obtain

$$m \ge 1 - a_1v(x_2) \ge 1 - a_1(1 - a_2u(x_3)) \ge 1 - a_1 + a_1a_2m, \tag{4.4}$$

and

$$M \leq 1 - a_1 v(x_1) \leq 1 - a_1(1 - a_2 u(x_4)) \leq 1 - a_1 + a_1 a_2 M. \tag{4.5}$$

Notice that since $a_1 < 1$ and $a_2 < 1$, we have

$$\frac{1 - a_1}{1 - a_1 a_2} \leq m \leq u(x) \leq M \leq \frac{1 - a_1}{1 - a_1 a_2}. \tag{4.6}$$

Hence

$$u \equiv \frac{1 - a_1}{1 - a_1 a_2}.$$

Similarly, it can be proved that

$$\frac{1 - a_2}{1 - a_1 a_2} \leq m' \leq v(x) \leq M' \leq \frac{1 - a_2}{1 - a_1 a_2}. \tag{4.7}$$

Consequently,

$$v \equiv \frac{1 - a_2}{1 - a_1 a_2}.$$

This completes the proof. □

Remark 4.2. Here we only give sufficient conditions for the existence of nonzero constant solutions for the system under Neumann boundary conditions. In fact, for systems under the other two boundary conditions, it is easy to see that if there is a constant solution, it can only be the zero solution.

In addition, under the conditions of Theorem 4.1, we also selected a set of values and conducted numerical simulations for system (3.9), as shown in Figure 4.1. The simulation results show that species u and v tend to constant coexisting solutions $\left(\frac{1-a_1}{1-a_1a_2}, \frac{1-a_2}{1-a_1a_2}\right) \approx (0.33, 0.83)$.

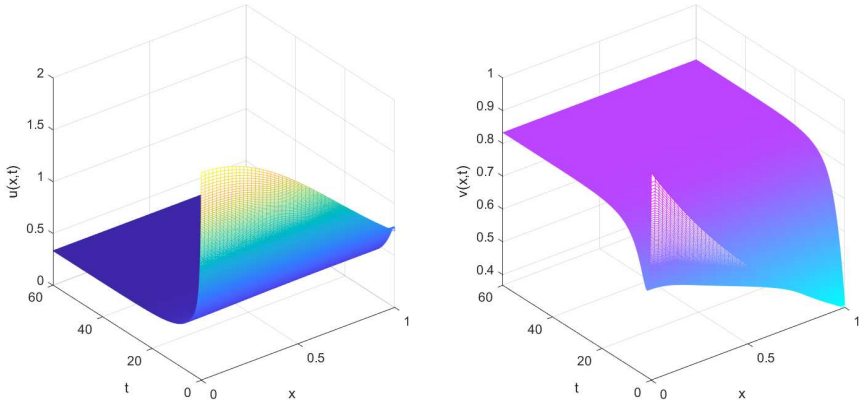


Figure 4.1: Numerical simulations for system (3.9) with $u_0(x) = 2e^{-x}$, $v_0(x) = e^{-x}$, $d_1 = 0.1, d_2 = 0.2, a_{11} = 0.1, a_{22} = 0.2$, $a_1 = 0.8$ and $a_2 = 0.5$.

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Combined effects of singular attractive and asymptotically linear terms in a Kirchhoff Dirichlet problem

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Abstract. In this paper, we consider the following Kirchhoff Dirichlet problem with singular attractive term

$$\begin{cases} -\left(a+b\int_\Omega |\nabla u|^2 dx\right)\Delta u = u^{-\gamma}+\lambda f(u) & \text{in } \Omega, \\ u=0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ $(N \geq 3)$, the parameters $a, b, \lambda, \gamma > 0$ and $f(u)$ is asymptotically linear reaction. In particular, we investigate both the strong singular case $(\gamma \geq 1)$ and the weak singular case $(0 < \gamma < 1)$ employing different techniques to reflect the distinct nature of each scenario. In the first case, ground state solutions are obtained via a direct minimizing methods, while in the latter case we combine variational theory with perturbation methods to prove the existence of ground state solutions for above Kirchhoff problem.

Keywords: positive solution, Kirchhoff Dirichlet problem, singularity, asymptotically linear reaction.

2020 Mathematics Subject Classification: 35B25, 35J60.

1 Introduction

Considering the following Kirchhoff Dirichlet problem

$$\begin{cases} -\left(a+b\int_\Omega |\nabla u|^2 dx\right)\Delta u = f(x,u) & \text{in } \Omega, \\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ $(N \geq 3)$, the parameters $a, b \geq 0$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a continuous function. The equation (1.1) was first introduced by Kirchhoff [13] as

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an extension of the classical D'Alembert's wave equations for free vibration of elastic strings. It is worth mentioning that fundamental analysis approach was introduced by Lions in the pioneer work [21]. After then, the existence results for above equations have been widely investigated, we refer the reader to [5,6,9–12,14,18,19,23–26,30,31] and the references therein.

Very recently, there are some papers on the Kirchhoff Dirichlet problem involving singularity, see [15,16,20,22,27,28] and the references therein. For example, in [28], Wang and Yan consider a class of Kirchhoff Dirichlet problem with singularity and nonlinearity in a bounded domain:

$$\begin{cases} -\left(a+b\int_\Omega |\nabla u|^2 dx\right)\Delta u = \dfrac{f(x)}{u} - \mu u^p & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $1 < p < 2^* - 1$, $2^* = \frac{2N}{N-2}$, Ω is a smooth bounded domain in $\mathbb{R}^N (N \geq 3)$, the function $f(x) \in L^2(\Omega)$ is positive, and a, b, μ are real numbers. They using the approximation method, a unique positive solution is obtained.

In particular, in [20,22], the following Kirchhoff Dirichlet problem with singularity and Sobolev subcritical perturbation is considered:

$$\begin{cases} -\left(a+b\int_\Omega |\nabla u|^2 dx\right)\Delta u = u^p + \dfrac{\lambda}{u^\gamma} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $0 < p < 5$, $0 < \gamma < 1$, and existence of solutions was obtained via using variational methods. Later, Lei, Liao and Tang [16] extended main results in [20,22] from Sobolev subcritical case to Sobolev critical case $(p = 5)$:

$$\begin{cases} -\left(a+b\int_\Omega |\nabla u|^2 dx\right)\Delta u = u^5 + \dfrac{\lambda}{u^\gamma} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^3$, and λ is a real parameter, $\gamma \in (0,1)$ is a constant. Such problem may occur in various branches of mathematical physics. The authors obtain two positive solutions via the variational and perturbation methods. Moreover, in [15], Lei and Liao studied the multiplicity of solutions for above Kirchhoff Dirichlet problem with more general nonlinearity $f(x,u)$, instead of power term u^p. In addition, many scholars are also interested in the p-Laplacian type Kirchhoff equation with critical reactions [3,17].

To the best of our knowledge, for strong singular case $\gamma \geq 1$, the existence results of ground state solution has not been studied up to now. The study of ground state solutions was started in the seminal work of Berestycki and Lions [4] and Coleman, Glazer and Martin [8]. In the calculus of variations, the ground state solution is the function that minimizes a given energy functional. Thus in this paper we study the ground state solutions of Kirchhoff Dirichlet problem involving singular and asymptotically linear terms:

$$\begin{cases} -\left(a+b\int_\Omega |\nabla u|^2 dx\right)\Delta u = u^{-\gamma} + \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N (N \geq 3)$, the parameters $a, b, \lambda, \gamma > 0$. The continuous function $f : \mathbb{R}^+ \mapsto \mathbb{R}^+, \mathbb{R}^+ = [0, +\infty)$ satisfies

(f_1) $\lim_{s\to\infty}\frac{f(s)}{s}=\theta$, for some $\theta\in(0,+\infty)$;

(f_2) the function $s\mapsto\frac{f(s)}{s}$ is non-increasing in $(0,+\infty)$.

The function $u(x)$ is called a weak solution of (1.2) if $0<u(x)\in H_0^1(\Omega)$ and for all $\varphi\in H_0^1(\Omega)$ it holds

$$(a+b\|u\|^2)\int_\Omega\nabla u\nabla\varphi dx-\lambda\int_\Omega f(u)\varphi dx-\int_\Omega u^{-\gamma}\varphi dx=0.$$

Define the function $G(t)$ as follows:

- for $0<\gamma<1$, $G(t)=\frac{(t^+)^{1-\gamma}}{1-\gamma}$, $t\in\mathbb{R}$;
- for $\gamma=1$, $G(t)=\begin{cases}\ln t, & t>0,\\ +\infty, & t=0;\end{cases}$ and for $\gamma>1$, $G(t)=\begin{cases}\frac{t^{1-\gamma}}{1-\gamma}, & t>0,\\ +\infty, & t=0.\end{cases}$

Let

$$f_0(t)=\begin{cases}f(t), & t\geq 0,\\ f(0), & t<0,\end{cases}\quad\text{and}\quad F(t)=\int_0^t f_0(s)ds.$$

Denote by δ_1 the first eigenvalue of $\left(-\Delta,H_0^1(\Omega)\right)$ [29] and set

$$\lambda_*=\frac{a\delta_1}{\theta}. \tag{1.3}$$

Now the main results can be described as follows.

Theorem 1.1. *Assume that $\gamma\geq 1$, the conditions (f_1)-(f_2) hold and*

$$D=\left\{u\in H_0^1(\Omega):\int_\Omega G(|u|)dx\in\mathbb{R}\right\}\neq\emptyset.$$

Then for every $\lambda\in(0,\lambda_*)$, there exists a positive ground state solution u_λ of (1.2). Moreover, u_λ is a global minimizer of the action functional I_λ, where

$$I_\lambda(u)=\frac{a}{2}\|u\|^2+\frac{b}{4}\|u\|^4-\lambda\int_\Omega F(u)dx-\int_\Omega G(|u|)dx.$$

Theorem 1.2. *Assume that $0<\gamma<1$ and the conditions (f_1)–(f_2) hold. Then for every $\lambda\in(0,\lambda_*)$, there exists a positive ground state solution $\tilde{u}_\lambda$ of (1.2). Moreover, $\tilde{u}_\lambda$ is a global minimizer of I_λ.*

Remark 1.3. (i) The set D in Theorem 1.1 is not closed as usual, see [1]. (ii) Theorems 1.1 extend main results in [15] from weak singular case $0<\gamma<1$ to strong singular case $\gamma\geq 1$. Moreover, following the ideas in [1,2], the methods in Theorem 1.2 to obtain global minimizer are more direct.

Notations: Throughout this paper, we make use of the following notations:

- The space $H_0^1(\Omega)$ is equipped with the norm $\|u\|^2=\int_\Omega|\nabla u|^2dx$, the norm in $L^p(\Omega)$ is denoted by $|\cdot|_p$;
- If A is a measurable set in $\mathbb{R}^{\mathbb{N}}$, we denote by $|A|$ the Lebesgue measure of A;
- $C,C_1,C_2,C_3,\ldots$, denote different positive constants ;

- Denote by d the distance function to the boundary;
- $u^+ = \max\{0, u\}$, $u_- = \max\{0, -u\}$.

The remainder of this paper is organized as follows. In Section 2, we consider the strong singular case and give the proof of Theorem 1.1. We consider the weak singular case and give the proof of Theorem 1.2 in Section 3.

2 The strong singular case: $\gamma \geq 1$

Since we are interested in positive solutions, we introduce the set

$$D^+ = \{u \in D : u \geq 0 \text{ a.e.} \quad \text{in } \Omega\}.$$

Now we show that $\inf_{u\in D} I_\lambda(u) = \inf_{u\in D^+} I_\lambda(u)$. In fact, it is easy to see that $\inf_{u\in D} I_\lambda(u) \leq \inf_{u\in D^+} I_\lambda(u)$. On the other hand, we have

$$\begin{aligned} I_\lambda(u) &= \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \lambda\int_\Omega F(u)dx - \int_\Omega G(|u|)dx \\ &= \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \lambda\int_{u<0} f(0)udx - \lambda\int_{u\geq 0} F(u)dx - \int_\Omega G(|u|)dx \\ &\geq \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \lambda\int_\Omega F(|u|)dx - \int_\Omega G(|u|)dx \\ &= I_\lambda(|u|) \geq \inf_{u\in D^+} I_\lambda(u), \quad \forall\, u \in D. \end{aligned}$$

Hence, $\inf_{u\in D} I_\lambda(u) = \inf_{u\in D^+} I_\lambda(u)$. This implies that a global minimum for $I_\lambda(u)$ exists if and only if there exists $v \in D^+$, such that $I_\lambda(v) = \inf_{u\in D^+} I_\lambda(u)$.

Now, we are in position to state the following lemma, which provides the existence of a global minimizer of $I_\lambda(u)$ for every $\lambda \in (0, \lambda_*)$.

Lemma 2.1. *Assume $\lambda \in (0, \lambda_*)$. Then, there exists $u_\lambda \in D^+$, such that*

$$I_\lambda(u_\lambda) = \inf_{u\in D^+} I_\lambda(u).$$

Proof. Take $\varepsilon > 0$, satisfies $\frac{\lambda(\theta+\varepsilon)}{a\delta_1} < 1$. From (f_1)–(f_2), for $\forall\, \varepsilon > 0$, there exist C_1, C_2, such that

$$\theta s \leq f(s) \leq (\theta+\varepsilon)s + C_1, \quad \forall\, s \geq 0, \tag{2.1}$$

$$\theta\frac{s^2}{2} \leq F(s) \leq \frac{\theta+\varepsilon}{2}s^2 + C_2 s, \quad \forall\, s \geq 0. \tag{2.2}$$

By (2.2), Sobolev embedding and Poincaré inequality, it follows that

$$\begin{aligned} I_\lambda(u) &\geq \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \lambda\int_\Omega \left(\frac{\theta+\varepsilon}{2}u^2 + C_2 u\right)dx - \int_\Omega G(|u|)dx \\ &\geq \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{\lambda(\theta+\varepsilon)}{2}\frac{\|u\|^2}{\delta_1} - C_3\|u\| - \int_\Omega G(|u|)dx \\ &= \frac{a}{2}\left(1 - \frac{\lambda(\theta+\varepsilon)}{a\delta_1}\right)\|u\|^2 + \frac{b}{4}\|u\|^4 - C_3\|u\| - \int_\Omega G(|u|)dx, \quad \forall\, u \in D^+. \end{aligned} \tag{2.3}$$

Now, we consider the following two cases:

Case i: $\gamma = 1$. By the inequality $\ln|u| \leq |u|$, (2.3) and Sobolev embedding, we deduce that

$$\begin{aligned} I_\lambda(u) &\geq \frac{a}{2}\left(1 - \frac{\lambda(\theta+\varepsilon)}{a\delta_1}\right)\|u\|^2 + \frac{b}{4}\|u\|^4 - C_3\|u\| - \int_\Omega \ln|u|dx \\ &\geq \frac{a}{2}\left(1 - \frac{\lambda(\theta+\varepsilon)}{a\delta_1}\right)\|u\|^2 + \frac{b}{4}\|u\|^4 - C_3\|u\| - \int_\Omega |u|dx \\ &\geq \frac{a}{2}\left(1 - \frac{\lambda(\theta+\varepsilon)}{a\delta_1}\right)\|u\|^2 + \frac{b}{4}\|u\|^4 - C_4\|u\|, \end{aligned}$$

which implies that $I_\lambda(u)$ is coercive in D^+. Let $\{u_n\} \subset D^+$ be a minimizing sequence, such that $I_\lambda(u_n) \to \inf_{u\in D^+} I_\lambda(u)$ as $n \to \infty$. Therefore, we may assume that $u_\lambda \in H_0^1(\Omega)$, such that

$$\begin{cases} u_n \rightharpoonup u_\lambda & \text{in } H_0^1(\Omega), \\ u_n \to u_\lambda & \text{in } L^p(\Omega), \quad p \in (0,6), \\ u_n \to u_\lambda & \text{a.e. in } \Omega. \end{cases} \tag{2.4}$$

Since $\{u_n\} \subset D^+$ one has $u_n \geq 0$ a.e. in Ω, thus $u_\lambda \geq 0$ a.e. in Ω. Moreover, since $u_n \to u_\lambda$ in $L^1(\Omega)$ there exists $h \in L^1(\Omega)$ such that $|u_n| \leq h$ a.e. in Ω. So, $\ln|u_n| \leq |u_n| \leq h$ a.e. in Ω, applying the Reverse Fatou's lemma, we have

$$-\infty < \limsup_{n\to\infty} \int_\Omega \ln|u_n|dx \leq \int_\Omega \limsup_{n\to\infty} \ln|u_n|dx = \int_\Omega \ln|u_\lambda|dx. \tag{2.5}$$

Note that if $u_\lambda = 0$ in a set of positive measure A, then $\int_\Omega \ln|u_\lambda|dx = \int_{\Omega\setminus A} \ln|u_\lambda|dx + \int_A \ln|u_\lambda|dx \to -\infty$, because $\ln|u_\lambda| = -\infty$ in A, it contradicts (2.5). Therefore, we obtain $u_\lambda > 0$ in Ω. Next, we show that the sequence $\{\int_\Omega G(|u_n|)dx\} = \{\int_\Omega \ln|u_n|dx\}$ is bounded, if not, we should have $-\int_\Omega \ln|u_n|dx \to +\infty$, then $\liminf_{n\to\infty} I_\lambda(u_n) = +\infty$, which is a contradiction with $\inf_{u\in D^+} I_\lambda(u) = \liminf_{n\to\infty} I_\lambda(u_n) \in \mathbb{R}$. This shows that $\{\int_\Omega G(|u_n|)dx\} = \{\int_\Omega \ln|u_n|dx\}$ is bounded. Then, we have $u_\lambda \in D^+$ and

$$\begin{aligned} \inf_{u\in D^+} I_\lambda(u) &= \liminf_{n\to\infty} I_\lambda(u_n) \\ &\geq \frac{a}{2}\|u_\lambda\|^2 + \frac{b}{4}\|u_\lambda\|^4 - \lambda\int_\Omega F(u_\lambda)dx - \limsup_{n\to\infty}\int \ln|u_n|dx \\ &\geq \frac{a}{2}\|u_\lambda\|^2 + \frac{b}{4}\|u_\lambda\|^4 - \lambda\int_\Omega F(u_\lambda)dx - \int_\Omega \ln|u_\lambda|dx = I_\lambda(u_\lambda) \geq \inf_{u\in D^+} I_\lambda(u). \end{aligned}$$

Thus, $I_\lambda(u_\lambda) = \inf_{u\in D^+} I_\lambda(u)$.

Case ii: $\gamma > 1$. From (2.3), we have

$$\begin{aligned} I_\lambda(u) &\geq \frac{a}{2}\left(1 - \frac{\lambda(\theta+\varepsilon)}{a\delta_1}\right)\|u\|^2 + \frac{b}{4}\|u\|^4 - C_3\|u\| - \frac{1}{1-\gamma}\int_\Omega |u|^{1-\gamma}dx \\ &> \frac{a}{2}\left(1 - \frac{\lambda(\theta+\varepsilon)}{a\delta_1}\right)\|u\|^2 + \frac{b}{4}\|u\|^4 - C_3\|u\|, \quad \forall u \in D^+, \end{aligned}$$

which implies that $I_\lambda(u)$ is coercive in D^+. Let $\{u_n\} \subset D^+$ be a minimizing sequence, such that $I_\lambda(u_n) \to \inf_{u\in D^+} I_\lambda(u)$ as $n \to \infty$. Therefore, we may assume that $0 \leq u_\lambda \in H_0^1(\Omega)$, such that (2.4) hold. Similar with Case i, we have $u_\lambda > 0$ and the sequence $\{\int_\Omega G(|u_n|)dx\} = \{\int_\Omega |u_n|^{1-\gamma}dx\}$ is bounded. By Fatou's lemma, one has

$$-\infty < \limsup_{n\to\infty} \int_\Omega \frac{1}{1-\gamma}|u_n|^{1-\gamma}dx \leq \int_\Omega \limsup_{n\to\infty} \frac{1}{1-\gamma}|u_n|^{1-\gamma}dx = \frac{1}{1-\gamma}\int_\Omega |u_\lambda|^{1-\gamma}dx,$$

which implies that $u_\lambda \in D^+$ and

$$\begin{aligned}\inf_{u\in D^+} I_\lambda(u) &= \liminf_{n\to\infty} I_\lambda(u_n)\\ &\geq \frac{a}{2}\|u_\lambda\|^2 + \frac{b}{4}\|u_\lambda\|^4 - \lambda\int_\Omega F(u_\lambda)dx - \limsup_{n\to\infty}\frac{1}{1-\gamma}\int_\Omega |u_n|^{1-\gamma}dx\\ &\geq \frac{a}{2}\|u_\lambda\|^2 + \frac{b}{4}\|u_\lambda\|^4 - \lambda\int_\Omega F(u_\lambda)dx - \int_\Omega \frac{1}{1-\gamma}|u_\lambda|^{1-\gamma}dx\\ &= I_\lambda(u_\lambda) \geq \inf_{u\in D^+} I_\lambda(u).\end{aligned}$$

Thus, $I_\lambda(u_\lambda) = \inf_{u\in D^+} I_\lambda(u)$. □

Proof of Theorem 1.1. Let u_λ be as in Lemma 2.1, we prove that u_λ is a solution of Eq. (1.2). For arbitrary $\varepsilon > 0$ and $\phi \in H_0^1(\Omega)$, $\phi \geq 0$ *in* Ω, to show that $u_\lambda + \varepsilon\phi \in D^+$, we divide the proof into two cases.

Case 1. $\gamma = 1$. Since $\ln(u) \leq u$ for all $u > 0$, and $u_\lambda \in D^+ \subset D$, then

$$-\infty < \int_\Omega \ln(u_\lambda) \leq \int_\Omega \ln(u_\lambda + \varepsilon\phi) \leq \int_\Omega (u_\lambda + \varepsilon\phi) < +\infty,$$

that is, $u_\lambda + \varepsilon\phi \in D^+$.

Case 2. $\gamma > 1$. Since $\gamma > 1, u_\lambda + \varepsilon\phi \geq u_\lambda > 0$, then

$$\int_\Omega |u_\lambda + \varepsilon\phi|^{1-\gamma}dx \leq \int_\Omega |u_\lambda|^{1-\gamma}dx < +\infty,$$

that is, $u_\lambda + \varepsilon\phi \in D^+$.

Thus, for $\gamma \geq 1$, we have

$$I_\lambda(u_\lambda + \varepsilon\phi) \geq I_\lambda(u_\lambda). \tag{2.6}$$

It follows from (2.6) that

$$\begin{aligned}&\frac{a}{2}(\|u_\lambda + \varepsilon\phi\|^2 - \|u_\lambda\|^2) - \lambda\int_\Omega (F(u_\lambda + \varepsilon\phi - F(u_\lambda))dx)\\ &+ \frac{b}{4}(\|u_\lambda + \varepsilon\phi\|^4 - \|u_\lambda\|^4) - \int_\Omega (G(u_\lambda + \varepsilon\phi) - G(u_\lambda))dx \geq 0.\end{aligned} \tag{2.7}$$

Dividing by $\varepsilon > 0$ and passing to the limit as $\varepsilon \to 0^+$ in (2.7), it gives

$$a\int_\Omega \nabla u_\lambda \nabla\phi dx + b\|u_\lambda\|^2\int_\Omega \nabla u_\lambda\nabla\phi dx - \lambda\int_\Omega f(u_\lambda)\phi dx - \int_\Omega u_\lambda^{-\gamma}\phi dx \geq 0. \tag{2.8}$$

Suppose for any $t > 0$, $tu_\lambda \in D^+$. Define $\psi \in C^1((0,+\infty),\mathbb{R})$ by $\psi(t) = I_\lambda(tu_\lambda)$. Obviously, ψ has a global minimum at $t = 1$, and so

$$0 = \psi'(1) = a\|u_\lambda\|^2 + b\|u_\lambda\|^4 - \lambda\int_\Omega f(u_\lambda)u_\lambda dx - \int_\Omega u_\lambda^{1-\gamma}dx. \tag{2.9}$$

Set $\Psi(x) = (u_\lambda(x) + \varepsilon\phi(x))^+$, $\forall\, \phi \in H_0^1(\Omega)$, $\varepsilon > 0$. From (2.8) and (2.9), it follows that

$$\begin{aligned}
0 &\le a\int_\Omega \nabla u_\lambda \nabla \Psi dx + b\|u_\lambda\|^2 \int_\Omega \nabla u_\lambda \nabla \Psi dx - \lambda \int_\Omega f(u_\lambda)\Psi dx - \int_\Omega u_\lambda^{-\gamma}\Psi dx \\
&= \int_{u_\lambda+\varepsilon\phi\ge 0} (a + b\|u_\lambda\|^2)\nabla u_\lambda \nabla (u_\lambda + \varepsilon\phi)dx \\
&\quad - \lambda \int_{u_\lambda+\varepsilon\phi\ge 0} f(u_\lambda)(u_\lambda+\varepsilon\phi)dx - \int_{u_\lambda+\varepsilon\phi\ge 0} u_\lambda^{-\gamma}(u_\lambda+\varepsilon\phi)dx \\
&= \int_\Omega [(a + b\|u_\lambda\|^2)\nabla u_\lambda \nabla (u_\lambda + \varepsilon\phi) - \lambda f(u_\lambda)(u_\lambda+\varepsilon\phi) - u_\lambda^{-\gamma}(u_\lambda+\varepsilon\phi)]dx \\
&\quad - \int_{u_\lambda+\varepsilon\phi<0} [(a + b\|u_\lambda\|^2)\nabla u_\lambda \nabla (u_\lambda + \varepsilon\phi) - \lambda f(u_\lambda)(u_\lambda+\varepsilon\phi) - u_\lambda^{-\gamma}(u_\lambda+\varepsilon\phi)]dx \qquad (2.10) \\
&\le \psi'(1) + \varepsilon \int_\Omega [(a + b\|u_\lambda\|^2)\nabla u_\lambda \nabla \phi - \lambda f(u_\lambda)\phi - u_\lambda^{-\gamma}\phi]dx \\
&\quad - \varepsilon \int_{u_\lambda+\varepsilon\phi<0} (a + b\|u_\lambda\|^2)\nabla u_\lambda \nabla \phi dx \\
&= \varepsilon \int_\Omega [(a + b\|u_\lambda\|^2)\nabla u_\lambda \nabla \phi - \lambda f(u_\lambda)\phi - \mu u_\lambda^{-\gamma}\phi]dx - \varepsilon(a + b\|u_\lambda\|^2)\int_{u_\lambda+\varepsilon\phi<0} \nabla u_\lambda \nabla \phi dx.
\end{aligned}$$

Since the measure of the domain of integration $|\{u_\lambda + \varepsilon\phi < 0\}| \to 0$ as $\varepsilon \to 0^+$, it follows that

$$\lim_{\varepsilon\to 0^+} \int_{u_\lambda+\varepsilon\phi<0} \nabla u_\lambda \nabla \phi dx = 0.$$

Therefore, dividing by $\varepsilon > 0$ and passing to the limit as $\varepsilon \to 0^+$ in (2.10), one sees that

$$0 \le (a + b\|u_\lambda\|^2)\int_\Omega \nabla u_\lambda \nabla \phi dx - \lambda \int_\Omega f(u_\lambda)\phi dx - \int_\Omega u_\lambda^{-\gamma}\phi dx.$$

By the arbitrariness of ϕ, this inequality also holds for $-\phi$, i.e.

$$(a + b\|u_\lambda\|^2)\int_\Omega \nabla u_\lambda \nabla \phi dx - \lambda \int_\Omega f(u_\lambda)\phi dx - \int_\Omega u_\lambda^{-\gamma}\phi dx = 0,$$

for every $\phi \in H_0^1(\Omega)$, which implies that u_λ is a solution of Eq. (1.2). □

3 The weak singular case: $0 < \gamma < 1$

Lemma 3.1. *If $u \in H_0^1(\Omega)$ is a global minimum of I_λ for some positive λ,then*

(a_1) *$u > 0$ a.e. in Ω;*

(a_2) *$u^{-\gamma}\varphi \in L^1$, $\forall\, \varphi \in H_0^1(\Omega)$.*

Proof. (a_1) Since u is a global minimum of I_λ, for $t \in [0,1]$, one has

$$\begin{aligned}
0 &\le \frac{I_\lambda(u + tu_-) - I_\lambda(u)}{t} \\
&= a\int_\Omega \nabla u \nabla u_- dx + b\|u\|^2 \int_\Omega \nabla u \nabla u_- dx + \frac{a}{2}t\|u_-\|^2 + bt\|u_-\|^4 \\
&\quad + \frac{b}{2}t\|u\|^2\|u_-\|^2 + bt^2\|u_-\|^2 \int_\Omega \nabla u \nabla u_- dx + \frac{b}{4}t^3\|u_-\|^3 \\
&\quad - \lambda \int_\Omega \frac{F(u + tu_-) - F(u)}{t}dx - \frac{1}{1-\gamma}\int_\Omega \frac{((u + tu_-)^+)^{1-\gamma} - (u^+)^{1-\gamma}}{t}dx.
\end{aligned} \qquad (3.1)$$

Notice that

$$\int_\Omega \frac{F(u+tu_-)-F(u)}{t}dx = \int_{u<0} \frac{F((1-t)u)-F(u)}{t}dx = -f(0)\int_{u<0} udx \geq 0, \tag{3.2}$$

$$\int_\Omega \frac{((u+tu_-)^+)^{1-\gamma}-(u^+)^{1-\gamma}}{t}dx = 0. \tag{3.3}$$

Combine (3.1), (3.2) and (3.3), we have

$$0 \leq -(a+b\|u\|^2)\|u_-\|^2 + \frac{a}{2}t\|u_-\|^2 + bt\|u_-\|^4 + \frac{b}{2}t\|u\|^2\|u_-\|^2 - bt^2\|u_-\|^4 + \frac{b}{4}t^3\|u_-\|^3.$$

As $t \to 0^+$, we obtain $\|u_-\|^2 \leq 0$, which implies that $u \geq 0$ a.e. in Ω.

Now we assume that there exists a set $A \subset \Omega$ of positive measure such that $u = 0$ for every $x \in A$. Thus we can choose a positive function $\varphi \in C_0^1(\bar{\Omega})$ such that for $t \in [0,1]$, we have $(u(x)+t\varphi(x))^{1-\gamma} \geq u(x)^{1-\gamma}$ a.e. in Ω. Notice that

$$\begin{aligned}\int_\Omega \frac{(u+t\varphi)^{1-\gamma}-u^{1-\gamma}}{t}dx &= \left(\int_{\Omega\setminus A}+\int_A\right)\frac{(u+t\varphi)^{1-\gamma}-u^{1-\gamma}}{t}dx \\ &\geq \int_A \frac{(u+t\varphi)^{1-\gamma}-u^{1-\gamma}}{t}dx = t^{-\gamma}\int_A \varphi^{1-\gamma}dx.\end{aligned}$$

From (3.1), one has

$$\begin{aligned}0 &\leq \frac{I_\lambda(u+t\varphi)-I_\lambda(u)}{t} \\ &\leq a\int_\Omega \nabla u\nabla\varphi dx + b\|u\|^2\int_\Omega \nabla u\nabla\varphi dx + \frac{a}{2}t\|\varphi\|^2 + bt\|\varphi\|^4 \\ &\quad + \frac{b}{2}t\|u\|^2\|\varphi\|^2 + bt^2\|\varphi\|^2\int_\Omega \nabla u\nabla\varphi dx + \frac{b}{4}t^3\|\varphi\|^3 \\ &\quad - \lambda\int_\Omega \frac{F(u+t\varphi)-F(u)}{t}dx - \frac{1}{1-\gamma}t^{-\gamma}\int_A \varphi^{1-\gamma}dx.\end{aligned} \tag{3.4}$$

As $t \to 0^+$, we obtain

$$\begin{aligned}&\frac{a}{2}t\|\varphi\|^2 + bt\|\varphi\|^4 + \frac{b}{2}t\|u\|^2\|\varphi\|^2 + bt^2\|\varphi\|^2\int_\Omega \nabla u\nabla\varphi dx \\ &\qquad + \frac{b}{4}t^3\|\varphi\|^3 - \lambda\int_\Omega \frac{F(u+t\varphi)-F(u)}{t}dx \to -\lambda\int_\Omega f(u)\varphi,\end{aligned}$$

and

$$-\frac{1}{1-\gamma}t^{-\gamma}\int_A \varphi^{1-\gamma}dx \to -\infty,$$

we know the right-hand of (3.4) goes to $-\infty$, which is a contradiction. Thus, $u > 0$ a.e. in Ω.
(a_2) If $\varphi \in H_0^1(\Omega)$ and t_n is a sequence in $[0,1]$, define

$$h_n(x) = \frac{[(u_n(x)+t_n\varphi(x))^+]^{1-\gamma}-u(x)^{1-\gamma}}{t_n}, \quad \text{a.e. } x \in \Omega,\ n \in \mathbb{N}.$$

In order to prove (a_2), it is enough to take $\varphi \in H_0^1(\Omega)$ with $\varphi \geq 0$. Fix a decreasing sequence $t_n \subseteq [0,1]$ with $\lim_{n\to+\infty} t_n = 0$, it follows that

$$h_n(x) = \frac{(u_n(x)+t_n\varphi(x))^{1-\gamma}-u(x)^{1-\gamma}}{t_n}, \tag{3.5}$$

are measurable non-negative functions and $t_n \subseteq [0,1]$ with $\lim_{n\to+\infty} h_n(x) = (1-\gamma)u(x)^{-\gamma}\varphi(x)$ for almost $x \in \Omega$. By Fatou's lemma, one has

$$(1-\gamma)\int_\Omega u^{-\gamma}\varphi dx \leq \liminf_{n\to+\infty}\int_\Omega h_n(x)dx. \tag{3.6}$$

From (3.5) and (3.6), we complete the proof of (a_2). □

As above, we are going to construct a minimum of I_λ under our assumptions. Set

$$C_+ = \{u \in C_0^1(\bar{\Omega}) : u(x) \geq 0 \quad \forall\, x \in \Omega\},$$

has a non-empty interior given by

$$\text{int}(C^+) = \left\{u \in C^+ : u(x) > 0, \forall\, x \in \Omega, \frac{\partial u}{\partial n}(x) < 0, \forall\, x \in \partial\Omega\right\}.$$

Lemma 3.2. *Assume that (f_1)–(f_2) hold and let λ_* be as in (1.3). Then, for any $\lambda \in (0,\lambda_*)$, there exists $\tilde{u}_\lambda \in H_0^1(\Omega)$ with the following properties*

(b_1) *$\tilde{u}_\lambda \geq c_\mu d(x)$ in Ω for some $c_\mu > 0$ (independent of λ);*

(b_2) *$I_\lambda(\tilde{u}_\lambda) = \inf_{u\in H_0^1(\Omega)} I_\lambda(u)$;*

(b_3) *$\tilde{u}_\lambda^{-\gamma}\varphi \in L^1, \forall\, \varphi \in H_0^1(\Omega)$.*

Proof. Choose a sequence $\{\varepsilon_n\} \subset [0,1]$ with $\lim_{n\to+\infty}\varepsilon_n = 0$. We consider the perturbed problem

$$\begin{cases} -\left(a + b\displaystyle\int_\Omega |\nabla u|^2 dx\right)\Delta u = \lambda f(u) + (u+\varepsilon_n)^{-\gamma} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.7}$$

which have variational structure. Indeed, if $G_n : \mathbb{R} \to \mathbb{R}$ is defined as

$$G_n(t) = \begin{cases} \frac{(t+\varepsilon_n)^{1-\gamma}}{1-\gamma}, & t \geq 0, \\ \varepsilon_n^{-\gamma}t + \frac{\varepsilon_n^{1-\gamma}}{1-\gamma}, & t < 0, \end{cases}$$

then $G_n'(t) = (t^+ + \varepsilon_n)^{-\gamma}$ for every $t \in \mathbb{R}$. The functional $I_n : H_0^1(\Omega) \to \mathbb{R}$

$$I_n(u) = \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \lambda\int_\Omega F(u)dx - \int_\Omega G_n(u)dx$$

is well defined, sequentially weakly lower semi-continuous, coercive and Gâteaux differentiable in $H_0^1(\Omega)$. In particular,

$$I_n'(u)(\varphi) = (a + b\|u\|^2)\int_\Omega \nabla u\nabla\varphi dx - \lambda\int_\Omega f(u)\varphi dx - \int_\Omega (u^+ + \varepsilon_n)^{-\gamma}\varphi dx, \quad \forall\, u \in H_0^1(\Omega).$$

Therefore critical point of I_n turn out to be weak solution of (3.7). Let $u_n \in H_0^1(\Omega)$ such that $I_n(u_n) = \inf_{u\in H_0^1(\Omega)} I_n(u)$. From Lemma 3.1, $u_n > 0$ a.e. in Ω and is not zero, by the maximum principle, $u_n \in \text{int}(C_+)$. First, we prove that there exists a positive constant c_μ (independent n and λ), such that $u_n(x) \geq c_\mu d(x)$ for every $x \in \bar{\Omega}$. Consider the semilinear problem

$$\begin{cases} -\Delta u = 1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

and denote by $\hat{u}$ its unique solution. By [7], we know that there exists a constant C such that $\hat{u}(x) \geq Cd(x)$, for every $x \in \bar{\Omega}$. Fix some positive number β, such that $\frac{\beta(a+b\beta^2\|\hat{u}\|^2)}{(\beta\|\hat{u}\|_\infty+\varepsilon_n)^{-\gamma}} < 1$. Then $\beta\hat{u}$ is a subsolution of (3.7). Indeed, if $\varphi \in H_0^1(\Omega)$, $\varphi \geq 0$, then

$$\begin{aligned}
&(a+b\|\beta\hat{u}\|^2)\int_\Omega \nabla(\beta\hat{u})\nabla\varphi dx - \lambda\int_\Omega f(\beta\hat{u})\varphi dx - \int_\Omega(\beta\hat{u}+\varepsilon_n)^{-\gamma}\varphi dx \\
&\leq (a+b\beta^2\|\hat{u}\|^2)\int_\Omega \beta\varphi dx - \int_\Omega(\beta\|\hat{u}\|_\infty+\varepsilon_n)^{-\gamma}\varphi dx \\
&= \int_\Omega[\beta(a+b\beta^2\|\hat{u}\|^2) - (\beta\|\hat{u}\|_\infty+\varepsilon_n)^{-\gamma}]\varphi dx \leq 0.
\end{aligned}$$

We have u_n is a supersolution of (3.7). Since the map $t \to \lambda\frac{f(t)}{t} + \frac{(t+\varepsilon_n)^{-\gamma}}{t}$ is decreasing in $(0,+\infty)$, then, $u_n \geq \beta\hat{u}$, hence, we obtain

$$u_n \geq c_\mu d(x). \tag{3.8}$$

Since u_n is a critical point of I_n, one has

$$(a+b\|u_n\|^2)\int_\Omega \nabla u_n\nabla\varphi dx = \lambda\int_\Omega f(u_n)\varphi dx + \int_\Omega(u_n+\varepsilon_n)^{-\gamma}\varphi dx, \tag{3.9}$$

for every $\varphi \in H_0^1(\Omega)$. Taking the test function $\varphi = u_n$, by (f_1), we obtain that

$$\begin{aligned}
(a+b\|u_n\|^2)\|u_n\|^2 &= \lambda\int_\Omega f(u_n)u_n dx + \int_\Omega(u_n+\varepsilon_n)^{-\gamma}u_n dx \\
&\leq \lambda\int_\Omega(\theta u_n^2 + C_5 u_n)dx + \int_\Omega u_n^{1-\gamma}dx \\
&\leq \frac{\lambda\theta}{\delta_1}\|u_n\|^2 + \lambda C_6\|u_n\| + C_7\|u_n\|^{1-\gamma}.
\end{aligned}$$

If $\|u_n\| \to +\infty$, as $n \to +\infty$, from above inequality, we have $0 < b \leq 0$, which is a contradiction. Thus, u_n is bounded in $H_0^1(\Omega)$. Therefore, there exist a subsequence (still denoted by u_n) and $u_0 \in H_0^1$ such that

$$\begin{cases}
u_n \rightharpoonup u_0 & \text{in } H_0^1(\Omega), \\
u_n \to u_0 & \text{in } L^p(\Omega), \quad p \in (0,6), \\
u_n \to u_0 & \text{a.e. in } \Omega.
\end{cases} \tag{3.10}$$

Now we prove the result with $\tilde{u}_\lambda = u_0$.

(b_1) It follows from (3.8). In particular, $u_0 > 0$ a.e. in Ω

(b_2) From (3.10), we have

$$\|u_0\|^2 \leq \liminf_{n\to\infty}\|u_n\|^2,$$

$$\int_\Omega F(u_n)dx \to \int_\Omega F(u_0)dx,$$

$$\int_\Omega G_n(u_n)dx \to \frac{1}{1-\gamma}\int_\Omega u_0^{1-\gamma}dx.$$

Thus we can deduce that $I_\lambda(u_0) \leq \liminf_{n\to\infty} I_n(u_n)$. Since u_n is a global minimum of I_n, we have

$$I_n(u_n) \leq I_n(|u|), \quad \forall\, u \in H_0^1(\Omega). \tag{3.11}$$

By passing to the limit as $n \to +\infty$, it gives

$$\begin{aligned} I_n(|u|) &= \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \lambda \int_\Omega F(|u|)dx - \frac{1}{1-\gamma}\int_\Omega (|u|+\varepsilon_n)^{1-\gamma}dx \\ &\leq \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \lambda \int_\Omega F(|u|)dx - \frac{1}{1-\gamma}\int_\Omega |u|^{1-\gamma}dx = I_\lambda(|u|), \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} I_\lambda(|u|) &= \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \lambda \int_{\Omega_+} F(u)dx - \lambda \int_{\Omega_-} F(-u)dx \\ &\quad - \frac{1}{1-\gamma}\int_\Omega (u^+)^{1-\gamma}dx - \frac{1}{1-\gamma}\int_\Omega (u_-)^{1-\gamma}dx \\ &\leq \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \lambda \int_{\Omega_+} F(u)dx - \frac{1}{1-\gamma}\int_\Omega (u^+)^{1-\gamma}dx \\ &\leq \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \lambda \int_{\Omega_+} F(u)dx - \lambda f(0)\int_{\Omega_-} udx - \frac{\int_\Omega (u^+)^{1-\gamma}dx}{1-\gamma} = I_\lambda(u). \end{aligned} \tag{3.13}$$

By putting together (3.11), (3.12) and (3.13), we obtain that

$$I_\lambda(u_0) \leq \liminf_{n\to\infty} I_n(u_n) \leq \liminf_{n\to\infty} I_n(|u_n|) \leq I_\lambda(|u|) \leq I_\lambda(u).$$

(b_3) Since u_0 is a global minimum of I_λ, from Lemma 3.1-(a_2), we obtain that $\tilde{u}_0^{-\gamma}\varphi \in L^1, \forall\ \varphi \in H_0^1(\Omega)$. □

Proof of the Theorem 1.2. Choose $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$, from (3.11) we get

$$\int_\Omega \nabla u_n \nabla \varphi dx \to \int_\Omega \nabla u_0 \nabla \varphi dx,$$

$$\int_\Omega f(u_n)\varphi dx \to \int_\Omega f(u_0)\varphi dx,$$

$$(u_n+\varepsilon_n)^{-\gamma}\varphi \to u_0^{-\gamma}\varphi \quad \text{a.e. in } \Omega.$$

By (3.8), one has

$$|(u_n+\varepsilon_n)^{-\gamma}\varphi| \leq u_n^{-\gamma}|\varphi| \leq c_\mu^{-\gamma} d(x)^{-\gamma}\|\varphi\|_\infty \in L^1(\Omega).$$

By the Lebesgue dominated convergence theorem, we have

$$\int_\Omega (u_n+\varepsilon_n)^{-\gamma}\varphi dx \to \int_\Omega u_0^{-\gamma}\varphi dx.$$

Thus, according to (3.9), we deduce that

$$(a+b\|u_0\|^2)\int_\Omega \nabla u_0 \nabla \varphi dx = \lambda \int_\Omega f(u_0)\varphi dx + \int_\Omega u_0^{-\gamma}\varphi dx. \tag{3.14}$$

Let $0 \leq \varphi \in H_0^1(\Omega)$ and fix $\varepsilon > 0$, taking the test function $\varphi_\varepsilon = \frac{\varphi}{1+\varepsilon\varphi} \in H_0^1(\Omega) \cap L^\infty(\Omega)$, we have

$$(a+b\|u_0\|^2)\int_\Omega \nabla u_0 \frac{\nabla \varphi}{(1+\varepsilon\varphi)^2}dx = \lambda \int_\Omega f(u_0)\frac{\varphi}{1+\varepsilon\varphi}dx + \int_\Omega u_0^{-\gamma}\frac{\varphi}{1+\varepsilon\varphi}dx.$$

By the Lebesgue dominated convergence theorem and Lemma 3.2, as $\varepsilon \to 0^+$, we obtain

$$\int_\Omega \nabla u_0 \frac{\nabla \varphi}{(1+\varepsilon\varphi)^2} dx \to \int_\Omega \nabla u_0 \nabla \varphi dx,$$
$$\int_\Omega u_0^{-\gamma} \frac{\varphi}{1+\varepsilon\varphi} dx \to \int_\Omega u_0^{-\gamma} \varphi dx,$$
$$\int_\Omega f(u_0) \frac{\varphi}{1+\varepsilon\varphi} dx \to \int_\Omega f(u_0) \varphi dx,$$

which imply equality (3.14). Therefore, the result of theorem 1.2 follows at once, by recalling that for any $\varphi \in H_0^1(\Omega)$, $\varphi = \varphi^+ - \varphi_-$. □

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Theodore A. Burton (1935–2025)

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Outstanding researcher, author, and educator of differential equations, Theodore A. Burton passed away on January 5, 2025, in his 90th year.

"Ted", as he was known to many, was a renowned personage of the community of differential equation people. He worked in the theory of ordinary, functional, partial differential equations, integral equations, and fractional differential equations. He enriched several branches of these fields, especially those of stability, periodicity, and oscillations, with fundamental achievements. He published more than 220 scientific papers in mathematical journals devoted to research in the theory of ordinary, integral, functional and fractional differential equations. He is the author of five monographs and also the editor of two books on mathematical biology. According to Mathematical Reviews, his works have been cited more than 2000 times by other leading authors and researchers, Google Scholar gives more than 9000 citations.

Born on September 7, 1935 in Longton, Kansas, USA, Theodore Allen Burton completed his elementary and high school in the Cascade Mountains of the state of Washington. In 1959, he graduated from Washington State College with a Bachelor of Science with Honors. On August 5, 1961, he married Fredda Jean Anderson.

At Washington State University, Ted earned both a Master's degree (1962) and his Ph.D. (1964), the latter under the direction of Professor Donald W. Bushaw. After completing his Ph.D., Ted accepted an academic position at the University of Alberta in Edmonton, Canada, where he worked for two years. In 1966, he joined the faculty of Southern Illinois University at Carbondale (SIUC) as an Associate Professor of Mathematics, and quickly contributed to the development of the doctoral program in Mathematics as led by J. M. H. Olmsted. Often referred to as the "Bear", perhaps because his wife, Fredda, called him "Mister Bear" and because of his features at the time (a bit rotund and a bushy beard). Ted attracted students to his research interests in differential equations early on. In fact, his first three Ph.D. students, all named John, were among the first to receive Ph.D.s in Mathematics from SIUC; there were only two earlier graduates of the program.

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According to Ted's own assertion, one of his primary achievements in academia that gave him the most pleasure was his success at mentoring doctoral students. In all, he supervised thirteen young mathematicians who earned Ph.D.s in mathematics, each of whom enjoyed a successful career of their own as university professors. In addition to these thirteen students, the Mathematics Genealogy Project lists nine other Mathematical descendants of Ted. Speaking of descendants, Bushaw was a student of Solomon Lefschetz, not a bad lineage for Ted's students.

In 1998, Ted retired from teaching at SIUC and was immediately named Emeritus Professor. He relocated to the state of Washington and founded the Northwest Research Institute, where he continued doing research. He remained active as a plenary speaker at multiple international conferences.

Ted's activity in teaching and research was international in scope. Twice he was awarded a Fulbright senior scholarship to Eastern Europe, once at the University of Szeged and later at the Technical University of Budapest. He also held research appointments at the University of Florence and the University of Madrid. He lectured at more than eighty conferences around the world. He served on the editorial boards of multiple international mathematics journals.

He was one of the founding Editors-in-Chief (along with László Hatvani of the University of Szeged, Hungary) of the *Electronic Journal of Qualitative Theory of Differential Equations (EJQTDE)*, the first volume of which appeared in 1998. Ted was of the opinion that mathematical journals were too expensive, and as a consequence, some mathematical institutes were not able to ensure that new scientific information was readily available to their researchers. He wanted to start a new mathematical journal that would be free to anyone wishing to see its contents. Thus, the EJQTDE was born.

Ted loved the EJQTDE. He once said that, as one of the Editors-in-Chief, he read every paper before it appeared to ensure that there was nothing written that might make the Journal liable in any way. The editorial leadership of EJQTDE has since transitioned to Jeff Webb in 2012 and Tibor Krisztin in 2015. True to Ted's own standards with respect to scholarly publication, EJQTDE is a diamond open-access journal, meaning that no charges and fees to either the reader or the author. Today, according to several academic journal rankings, EJQTDE is a high quality journal in the field of differential equations.

Ted's own research embraced several different aspects of differential equations. During his doctoral years he studied the Liénard equation, a natural nonlinear generalization of the damped linear oscillator equation. Of primary interest, he showed that solutions of the nonlinear equation share properties similar to those of solutions to the linear oscillator equation.

Ted had a lifelong interest in stability theory and in Lyapunov's second (direct) method in particular. He felt that his most interesting contributions were in the construction of some explicit Lyapunov functions and functionals. It is clear that he contributed significantly to the theoretical foundations of Lyapunov theory and furthered the development of the direct method as well. Some of his early work was on the boundedness and stability of solutions of second-order nonlinear equations. This included his work on the Liénard equation and other results on the oscillation of nonlinear equations. He was an advocate of applying the Krasnosel'skii fixed point theorem for a mapping that is a sum of a compact map and a contraction map. In the last few years of his life, Ted became especially interested in various aspects of fractional differential equations.

The Bear loved mathematics and passed that love on to his students. His influence on them was profound; he was their mentor, their friend, and their colleague all at once.

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Global existence of strong solutions to nonhomogeneous MHD system in thin three-dimensional domains

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Abstract. We establish the global existence of strong solutions to the nonhomogeneous incompressible magnetohydrodynamics (MHD) equations in a thin three-dimensional domain $\Omega = \mathbb{R}^2 \times (0,\epsilon)$, with $\epsilon \in (0,1]$, subject to Dirichlet boundary conditions on the top and bottom boundaries. Global well-posedness may hold for large initial data, provided the vertical thickness ϵ is sufficiently small. Moreover, when $\epsilon \to 0^+$, both the velocity and magnetic field tend to vanish away from the initial time. The analysis is based on *a priori* H^2 estimates of the solutions, with particular attention to the dependence on the vertical parameter ϵ.

Keywords: MHD system, thin domains, global existence.

2020 Mathematics Subject Classification: 35Q35, 76D03, 76W05.

1 Introduction

This work investigates nonhomogeneous incompressible magnetohydrodynamic (MHD) flows in thin 3D domains. The equations involve a coupling between the nonhomogeneous Navier–Stokes system and the Maxwell equations in $Q := \Omega \times (0,\infty)$, where $\Omega := \mathbb{R}^2 \times (0,\epsilon)$, with $0 < \epsilon \leq 1$, is a thin domain of $\mathbb{R}^3$. The governing equations, described extensively in textbooks such as [8,13,20], are as follows:

$$\begin{cases} \rho \boldsymbol{u}_t + \rho(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \mu\Delta\boldsymbol{u} + \nabla\left(P + \frac{1}{2}|\boldsymbol{b}|^2\right) = (\boldsymbol{b}\cdot\nabla)\boldsymbol{b}, \\ \boldsymbol{b}_t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{b} - \eta\Delta\boldsymbol{b} = (\boldsymbol{b}\cdot\nabla)\boldsymbol{u}, \\ \rho_t + \boldsymbol{u}\cdot\nabla\rho = 0, \\ \operatorname{div}\boldsymbol{u} = 0, \quad \operatorname{div}\boldsymbol{b} = 0. \end{cases} \tag{1.1}$$

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Here, $\boldsymbol{u}(\boldsymbol{x},t) \in \mathbb{R}^3$ and $\boldsymbol{b}(\boldsymbol{x},t) \in \mathbb{R}^3$ denote the velocity and magnetic field of the fluid at each point $(\boldsymbol{x},t) \in Q$, respectively. The real-valued functions $\rho(\boldsymbol{x},t) \in \mathbb{R}^+$ and $P(\boldsymbol{x},t) \in \mathbb{R}$ represent the fluid density and hydrostatic pressure, respectively. The quantity $|\boldsymbol{b}|^2/2$ corresponds to the *magnetic pressure*. Accordingly, we define the total pressure of the fluid as $p := P + \frac{1}{2}|\boldsymbol{b}|^2$. The positive constants μ and η denote the viscosity and the resistivity coefficient, respectively. The latter is inversely proportional to the electrical conductivity and serves as the magnetic diffusivity of the magnetic field.

We complete the system (1.1) with the following initial and boundary conditions:

$$\begin{cases} (\rho, \boldsymbol{u}, \boldsymbol{b})\,\big|_{t=0} = (\rho_0, \boldsymbol{u}_0, \boldsymbol{b}_0) & \text{in } \Omega, \\ (\boldsymbol{u}(\boldsymbol{x},t), \boldsymbol{b}(\boldsymbol{x},t)) = (\boldsymbol{0}, \boldsymbol{0}) & \text{for all } (\boldsymbol{x},t) \in \Gamma \times (0,\infty), \end{cases} \tag{1.2}$$

where $\Gamma := \{(x_1, x_2, x_3) \,/\, (x_1, x_2) \in \mathbb{R}^2, x_3 = 0 \text{ or } x_3 = \epsilon\}$.

The magnetohydrodynamic (MHD) equations with variable density have been studied extensively in recent years; see [1,9,14,19,25] and the references therein (see also [3,4,6,10–12] for further results concerning the MHD equations). These results are closely related to those obtained for the variable-density Navier-Stokes equations. The study of the classical Navier–Stokes system in thin domains was initiated by Raugel and Sell in [22] and subsequently generalized in several directions; see, for example, [16] and the references therein. The first work addressing the variable-density Navier-Stokes equations in thin domains was carried out by Liao in [18].

In this work, we show that results analogous to those of Liao [18] are valid for the system (1.1)–(1.2). Specifically, we prove that global well-posedness may be achieved for large initial data when the vertical thickness ϵ is sufficiently small. For additional related work on fluid mechanics equations in thin domains, see, for instance, [5,7,17,23,24].

The paper is organized as follows. In Section 2, we introduce the basic notation and state the main result. In Section 3, we provide the proof of our main theorem (see Theorem 2.6 below).

2 Preliminaries and main result

In this section, we introduce the function spaces, definitions, and the main result that will be used throughout this work.

2.1 Functional framework

Throughout this paper, we employ the standard Lebesgue space $L^p(\Omega)$, for $1 \le p \le \infty$, and the Sobolev space

$$W^{m,p}(\Omega) := \left\{ f \in L^p(\Omega) \,:\, \|\partial^k f\|_p < \infty \text{ for all } |k| \le m \right\}, \quad m \in \mathbb{N} \cup \{0\},$$

with norms denoted by $\|\cdot\|_p$ and $\|\cdot\|_{W^{m,p}}$, respectively. In particular, we denote the L^2 norm and the L^2 inner product by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, respectively.

Corresponding Sobolev spaces of vector-valued functions are denoted in boldface. For example, we write $\boldsymbol{L}^2(\Omega) := (L^2(\Omega))^n$ and $\boldsymbol{W}^{m,p}(\Omega) := (W^{m,p}(\Omega))^n$, where $n \in \mathbb{N}$.

Furthermore, we denote by $\boldsymbol{W}_0^{m,p}(\Omega)$ the closure of $\boldsymbol{C}_0^\infty(\Omega)$ in $\boldsymbol{W}^{m,p}(\Omega)$. When $p = 2$, we set

$$\boldsymbol{H}^m(\Omega) := \boldsymbol{W}^{m,2}(\Omega) \quad \text{and} \quad \boldsymbol{H}_0^m(\Omega) := \boldsymbol{W}_0^{m,2}(\Omega).$$

We define

$$\boldsymbol{\mathcal{V}}(\Omega) := \{\boldsymbol{v} \in \boldsymbol{C}_0^\infty(\Omega) \,:\, \operatorname{div} \boldsymbol{v} = 0 \text{ in } \Omega\},$$

and denote by $\boldsymbol{H}$ and $\boldsymbol{V}$ the closures of $\boldsymbol{\mathcal{V}}(\Omega)$ in $\boldsymbol{L}^2(\Omega)$ and $\boldsymbol{H}_0^1(\Omega)$, respectively.

Let X be a Banach space and $0 < T \leq \infty$. We denote by $L^p(0,T;X)$, or alternatively $L^p_T(X)$, the Banach space of X-valued (equivalence classes of) functions defined almost everywhere on $[0,T]$ that are L^p-integrable in the sense of Bochner. The associated norm is denoted by $\|\cdot\|_{L^p_T(X)}$.

We will frequently work with spaces of the form $L^p(0,T;X)$ where $X = \boldsymbol{W}^{m,q}(\Omega)$. In such cases, for any $\boldsymbol{v} \in L^p(0,T;\boldsymbol{W}^{m,q}(\Omega))$, the notation $\boldsymbol{v}(t)$ refers to the spatial function $\boldsymbol{v}(\cdot,t)$. We also use the notation $C^k([0,T);X)$ for the space of k times continuously differentiable functions on $[0,T)$ with values in X. Moreover, as usual, the letters $C, K, C_1, K_1, \ldots$ are positive constants, independents of $(\rho, \boldsymbol{u}, \boldsymbol{b})$, but their values may change from line to line.

Notation

As usual, we set

$$\|(\varphi,\psi)\|_X^2 := \|\varphi\|_X^2 + \|\psi\|_X^2.$$

In addition, we introduce the pair $\boldsymbol{z}(t) := (\boldsymbol{u}(t), \boldsymbol{b}(t))$ with initial data $\boldsymbol{z}_0 := \boldsymbol{z}(0) = (\boldsymbol{u}_0, \boldsymbol{b}_0)$, and the following norms:

$$\begin{aligned}
\|\boldsymbol{z}(t)\|^2 &:= \|\boldsymbol{u}(t)\|^2 + \|\boldsymbol{b}(t)\|^2, & \|\boldsymbol{z}_t(t)\|^2 &:= \|\boldsymbol{u}_t(t)\|^2 + \|\boldsymbol{b}_t(t)\|^2,\\
\|\nabla\boldsymbol{z}(t)\|^2 &:= \|\nabla\boldsymbol{u}(t)\|^2 + \|\nabla\boldsymbol{b}(t)\|^2, & \|\Delta\boldsymbol{z}(t)\|^2 &:= \|\Delta\boldsymbol{u}(t)\|^2 + \|\Delta\boldsymbol{b}(t)\|^2,\\
\|\nabla^2\boldsymbol{z}(t)\|^2 &:= \|\nabla^2\boldsymbol{u}(t)\|^2 + \|\nabla^2\boldsymbol{b}(t)\|^2, & \|\nabla\boldsymbol{z}_t(t)\|^2 &:= \|\nabla\boldsymbol{u}_t(t)\|^2 + \|\nabla\boldsymbol{b}_t(t)\|^2.
\end{aligned}$$

2.2 Auxiliary results

Following the approach in [18, Theorem 1.1], we obtain the following result concerning strong solutions to the system (1.1)–(1.2).

Theorem 2.1. *Assume that the initial data $(\rho_0, \boldsymbol{u}_0, \boldsymbol{b}_0)$ satisfy*

$$0 < \alpha \leq \rho_0(\boldsymbol{x}) \leq \beta < \infty \quad \text{in } \Omega, \text{ with } \alpha, \beta \in \mathbb{R}^+, \tag{2.1}$$

$$\boldsymbol{u}_0,\, \boldsymbol{b}_0 \in \boldsymbol{V}. \tag{2.2}$$

Then, there exists a time $T^* > 0$ and a unique strong solution $(\rho, \boldsymbol{u}, p, \boldsymbol{b})$ in $\Omega \times [0,T^*]$ for the problem (1.1)–(1.2) satisfying

$$\begin{gathered}
\alpha \leq \rho(x,t) \leq \beta,\\
\boldsymbol{u},\, \boldsymbol{b} \in L^\infty(0,T^*;\boldsymbol{V}) \cap L^2(0,T^*;\boldsymbol{V} \cap \boldsymbol{H}^2(\Omega)),\\
\nabla\boldsymbol{u},\, \nabla\boldsymbol{b} \in L^2(0,T^*;\boldsymbol{L}^\infty(\Omega) \cap \boldsymbol{W}^{1,6}(\Omega)),\\
\boldsymbol{u}_t,\, \boldsymbol{b}_t \in L^2(0,T^*;\boldsymbol{H}),\\
\nabla p \in L^\infty(0,T^*;\boldsymbol{L}^2(\Omega)) \cap L^2(0,T^*;\boldsymbol{L}^6(\Omega)).
\end{gathered}$$

Moreover, if $\boldsymbol{u}_0,\, \boldsymbol{b}_0 \in \boldsymbol{V} \cap \boldsymbol{H}^2(\Omega)$, then the unique solution satisfies

$$\boldsymbol{u},\, \boldsymbol{b} \in L^\infty(0,T^*;\boldsymbol{V} \cap \boldsymbol{H}^2(\Omega)) \quad \text{and} \quad \boldsymbol{u}_t,\, \boldsymbol{b}_t \in L^\infty(0,T^*;\boldsymbol{H}) \cap L^2(0,T^*;\boldsymbol{V}).$$

We will show that the unique solution to the problem (1.1)–(1.2) exists globally in time, provided that the initial data $\boldsymbol{u}_0$ and $\boldsymbol{b}_0$ satisfy the "smallness" condition

$$\epsilon^{\frac{1}{2}}\left(\|\nabla \boldsymbol{u}_0\|^2+\|\nabla \boldsymbol{b}_0\|^2\right)^{\frac{1}{2}} \leq c_0, \tag{2.3}$$

for some constant $c_0>0$ sufficiently small, depending only on α and β.

To this end, the following estimates will be instrumental.

Lemma 2.2 (See [18], Lemma 1.6, p. 174)**.** *If $\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega)$, then the following Poincaré and Sobolev's inequalities hold:*

$$\|\boldsymbol{v}\| \leq \epsilon\|\partial_3 \boldsymbol{v}\|, \qquad \|\boldsymbol{v}\|_6 \leq C\|\nabla \boldsymbol{v}\|.$$

Remark 2.3. Observe that if $\boldsymbol{u}_0, \boldsymbol{b}_0$ satisfy (2.2)–(2.3), then

$$\begin{aligned}
&\|\boldsymbol{u}_0\|\|\nabla \boldsymbol{u}_0\| \leq \epsilon\|\partial_3 \boldsymbol{u}_0\|\|\nabla \boldsymbol{u}_0\| \leq c_0^2, \qquad \|\boldsymbol{u}_0\|\|\nabla \boldsymbol{b}_0\| \leq \epsilon\|\partial_3 \boldsymbol{u}_0\|\|\nabla \boldsymbol{b}_0\| \leq c_0^2,\\
&\|\boldsymbol{b}_0\|\|\nabla \boldsymbol{u}_0\| \leq \epsilon\|\partial_3 \boldsymbol{b}_0\|\|\nabla \boldsymbol{u}_0\| \leq c_0^2, \qquad \|\boldsymbol{b}_0\|\|\nabla \boldsymbol{b}_0\| \leq \epsilon\|\partial_3 \boldsymbol{b}_0\|\|\nabla \boldsymbol{b}_0\| \leq c_0^2.
\end{aligned}$$

In particular, we have

$$\left(\|\boldsymbol{u}_0\|^2+\|\boldsymbol{b}_0\|^2\right)\left(\|\nabla \boldsymbol{u}_0\|^2+\|\nabla \boldsymbol{b}_0\|^2\right) \leq 4c_0^4.$$

Remark 2.4. From smallness condition (2.3), applying the Poincaré's inequality in the vertical direction on $\boldsymbol{u}_0$ and $\boldsymbol{b}_0$, we get the following smallness assumption:

$$\|\boldsymbol{u}_0\| \leq \epsilon\|\partial_3 \boldsymbol{u}_0\| \leq \epsilon\|\nabla \boldsymbol{u}_0\| \leq c_0\epsilon^{\frac{1}{2}} \quad \text{and} \quad \|\boldsymbol{b}_0\| \leq \epsilon\|\partial_3 \boldsymbol{b}_0\| \leq \epsilon\|\nabla \boldsymbol{b}_0\| \leq c_0\epsilon^{\frac{1}{2}}.$$

Remark 2.5. The "smallness" assumption (2.3) closely resembles the one proposed in [21]:

$$\|\boldsymbol{u}_0\|\|\nabla \boldsymbol{u}_0\| \leq \vartheta, \quad \text{with } \vartheta>0 \text{ sufficiently small.}$$

In fact, from Remark 2.2, we have

$$\|\boldsymbol{u}_0\|\|\nabla \boldsymbol{u}_0\| \leq c_0^2 \quad \text{and} \quad \|\boldsymbol{b}_0\|\|\nabla \boldsymbol{b}_0\| \leq c_0^2.$$

Therefore, under the smallness condition

$$\|\boldsymbol{u}_0\|\|\nabla \boldsymbol{u}_0\|+\|\boldsymbol{b}_0\|\|\nabla \boldsymbol{b}_0\| \leq 2c_0^2,$$

and following the approach in [21], we conclude that the strong solution given in Theorem 2.1 is global in time (i.e., $T^*=T$, and T could be $+\infty$).

2.3 Statement of the main result

Initially, we define

$$\begin{aligned}
\mathcal{U}(t) &:= \left\|(\boldsymbol{u},\boldsymbol{b})\right\|^2_{L^\infty_t(\boldsymbol{L}^2(\Omega))}+\left\|(\sqrt{\mu}\,\nabla \boldsymbol{u},\sqrt{\eta}\,\nabla \boldsymbol{b})\right\|^2_{L^2_t(\boldsymbol{L}^2(\Omega))},\\
\mathcal{V}(t) &:= \left\|(\nabla \boldsymbol{u},\nabla \boldsymbol{b})\right\|^2_{L^\infty_t(\boldsymbol{L}^2(\Omega))}+\left\|(\boldsymbol{u}_t,\boldsymbol{b}_t,\Delta \boldsymbol{u},\Delta \boldsymbol{b})\right\|^2_{L^2_t(\boldsymbol{L}^2(\Omega))},\\
\mathcal{W}(t) &:= \left\|(\boldsymbol{u}_t,\boldsymbol{b}_t,\nabla^2 \boldsymbol{u},\nabla^2 \boldsymbol{b},\nabla p)\right\|^2_{L^\infty_t(\boldsymbol{L}^2(\Omega))}+\left\|(\nabla \boldsymbol{u}_t,\nabla \boldsymbol{b}_t)\right\|^2_{L^2_t(\boldsymbol{L}^2(\Omega))},
\end{aligned} \tag{2.4}$$

and

$$\mathcal{U}_0 := \|\boldsymbol{u}_0\|^2+\|\boldsymbol{b}_0\|^2, \quad \mathcal{V}_0 := \|\nabla \boldsymbol{u}_0\|^2+\|\nabla \boldsymbol{b}_0\|^2, \quad \mathcal{W}_0 := \|\nabla^2 \boldsymbol{u}_0\|^2+\|\nabla^2 \boldsymbol{b}_0\|^2. \tag{2.5}$$

The main result of this paper is the following.

Theorem 2.6. *Suppose that the initial data $(\rho_0, \boldsymbol{u}_0, \boldsymbol{b}_0)$ satisfy (2.1)–(2.3). Then the problem (1.1)–(1.2) has a unique global in time strong solution $(\rho, \boldsymbol{u}, p, \boldsymbol{b})$ such that, for all $T \in (0, \infty)$, the following inequalities hold*

$$\alpha \le \rho(\boldsymbol{x}, t) \le \beta, \qquad \mathcal{U}(T)C \le C\,\mathcal{U}_0, \qquad \mathcal{V}(T) \le \widetilde{C}\,\mathcal{V}_0,$$
$$\big\|(\boldsymbol{u}, \boldsymbol{b})\big\|^2_{L^\infty_T(\boldsymbol{L}^2(\Omega))} \le C\,\mathcal{U}_0\, e^{-\gamma\, T/\epsilon^2},$$
$$\big\|(\nabla \boldsymbol{u}, \nabla \boldsymbol{b})\big\|^2_{L^\infty_T(\boldsymbol{L}^2(\Omega))} \le \widetilde{C}\,\mathcal{V}_0\, e^{-\overline{\gamma}\, T/\epsilon^2},$$

where $C = C(\alpha, \beta) > 0$, $\widetilde{C} = \widetilde{C}(\alpha, \beta, \mu, \eta, c_0) > 0$, $\gamma := \min\big\{\frac{2\mu}{\beta}, 2\eta\big\}$ and $\overline{\gamma} := \min\big\{\frac{\widetilde{\gamma}}{\mu}, \frac{\widetilde{\gamma}}{\eta}\big\}$, with $\widetilde{\gamma} := \min\big\{\frac{\mu^2}{16\beta}, \frac{\eta^2}{16}\big\}$. Moreover, if $\boldsymbol{u}_0, \boldsymbol{b}_0 \in \boldsymbol{V} \cap \boldsymbol{H}^2(\Omega)$, then

$$\mathcal{W}(T) \le \widetilde{C}\,\mathcal{W}_0 \quad \text{and} \quad \big\|(\nabla^2 \boldsymbol{u}, \nabla^2 \boldsymbol{b})\big\|^2_{L^\infty_T(\boldsymbol{L}^2(\Omega))} \le \widetilde{C}\,\mathcal{W}_0\, e^{-\sigma\, T/\epsilon^2},$$

where $\sigma := \min\{\gamma, \overline{\gamma}\}$. In particular, for all $t_\star > 0$, we conclude that

$$\lim_{\epsilon \to 0^+} (\boldsymbol{u}, \boldsymbol{b}) = (\boldsymbol{0}, \boldsymbol{0}) \quad \text{uniformly in } C\big([t_\star, \infty); \boldsymbol{H}^2(\Omega)\big).$$

3 Proof of the main result

Theorem 2.6 follows as a consequence of Theorem 2.1 above, together with Propositions 3.1 and 3.3 established below.

A priori bounds

From now on, we denote by C, K, and $\widetilde{C}$ generic positive constants, which may vary from line to line. More specifically, C denotes a constant depending only on α and β; K depends solely on α, β, μ, and η; and $\widetilde{C}$ depends on α, β, μ, η, and c_0.

In the following, we derive *a priori* estimates for the strong solution $(\rho, \boldsymbol{u}, p, \boldsymbol{b})$ to the problem (1.1)–(1.2), defined on $\Omega \times [0, T]$, for $T \in (0, \infty)$.

Proposition 3.1 (Estimates in the $\boldsymbol{H}^1(\Omega)$ norm)**.** *Assume that the initial data $(\rho_0, \boldsymbol{u}_0, \boldsymbol{b}_0)$ satisfy (2.1)–(2.3). Let $(\rho, \boldsymbol{u}, p, \boldsymbol{b})$ be the strong solution furnished by Theorem 2.1 defined on $\Omega \times [0, T]$. There exist constants $C = C(\alpha, \beta) > 0$ and $\widetilde{C} = \widetilde{C}(\alpha, \beta, \mu, \eta, c_0) > 0$, such that*

$$\alpha \le \rho(\boldsymbol{x}, t) \le \beta, \qquad \forall\, (\boldsymbol{x}, t) \in \Omega \times [0, T], \quad (3.1)$$
$$\|\boldsymbol{u}(t)\|^2 + \|\boldsymbol{b}(t)\|^2 \le C\left(\|\boldsymbol{u}_0\|^2 + \|\boldsymbol{b}_0\|^2\right), \qquad \forall\, t \in [0, T], \quad (3.2)$$
$$\int_0^t \left(\mu\|\nabla \boldsymbol{u}(s)\|^2 + \eta\|\nabla \boldsymbol{b}(s)\|^2\right) ds \le C\left(\|\boldsymbol{u}_0\|^2 + \|\boldsymbol{b}_0\|^2\right), \qquad \forall\, t \in [0, T], \quad (3.3)$$
$$\|\boldsymbol{u}(t)\|^2 + \|\boldsymbol{b}(t)\|^2 \le C\left(\|\boldsymbol{u}_0\|^2 + \|\boldsymbol{b}_0\|^2\right) e^{-\gamma\, t/\epsilon^2}, \qquad \forall\, t \in [0, T], \quad (3.4)$$
$$\|\nabla \boldsymbol{u}(t)\|^2 + \|\nabla \boldsymbol{b}(t)\|^2 \le \widetilde{C}\left(\|\nabla \boldsymbol{u}_0\|^2 + \|\nabla \boldsymbol{b}_0\|^2\right), \qquad \forall\, t \in [0, T], \quad (3.5)$$
$$\int_0^t \left(\|\boldsymbol{u}_t(s)\|^2 + \|\boldsymbol{b}_t(s)\|^2\right) ds \le \widetilde{C}\left(\|\nabla \boldsymbol{u}_0\|^2 + \|\nabla \boldsymbol{b}_0\|^2\right), \qquad \forall\, t \in [0, T], \quad (3.6)$$
$$\int_0^t \left(\|\Delta \boldsymbol{u}(s)\|^2 + \|\Delta \boldsymbol{b}(s)\|^2\right) ds \le \widetilde{C}\left(\|\nabla \boldsymbol{u}_0\|^2 + \|\nabla \boldsymbol{b}_0\|^2\right), \qquad \forall\, t \in [0, T], \quad (3.7)$$
$$\|\nabla \boldsymbol{u}(t)\|^2 + \|\nabla \boldsymbol{b}(t)\|^2 \le \widetilde{C}\left(\|\nabla \boldsymbol{u}_0\|^2 + \|\nabla \boldsymbol{b}_0\|^2\right) e^{-\overline{\gamma}\, t/\epsilon^2}, \qquad \forall\, t \in [0, T], \quad (3.8)$$

with $\gamma := \min\big\{\frac{2\mu}{\beta}, 2\eta\big\}$ and $\overline{\gamma} := \min\big\{\frac{\widetilde{\gamma}}{\mu}, \frac{\widetilde{\gamma}}{\eta}\big\}$, where $\widetilde{\gamma} := \min\big\{\frac{\mu^2}{16\beta}, \frac{\eta^2}{16}\big\}$.

Proof. For a given $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}, t)$, one gets ρ, solution of $(1.1)_3$, through the classical method of characteristics. Indeed, if

$$\Pi(s) := \rho(\boldsymbol{x} + s\boldsymbol{u},\, t + s), \quad \forall\, s \in [-t, \infty) \qquad (\text{with } t \geq 0),$$

then

$$\frac{d}{ds}\Pi(s) = \nabla\rho(\boldsymbol{x} + s\boldsymbol{u},\, t + s) \cdot \boldsymbol{u}(\boldsymbol{x} + s\boldsymbol{u},\, t + s) + \rho_t(\boldsymbol{x} + s\boldsymbol{u},\, t + s) = 0, \quad \forall\, s \in (-t, \infty).$$

Thus, $\Pi(\cdot)$ is a constant function of s. Hence,

$$\rho(\boldsymbol{x}, t) = \Pi(0) = \Pi(-t) = \rho(\boldsymbol{x} - t\boldsymbol{u},\, 0) = \rho_0(\boldsymbol{x} - t\boldsymbol{u}).$$

Therefore, from the hypothesis (2.1), it follows that ρ satisfies bound (3.1). Testing equation $(1.1)_1$ by $\boldsymbol{u}$ and equation $(1.1)_2$ by $\boldsymbol{b}$, one obtains, respectively

$$\frac{1}{2}\frac{d}{dt}\left(\|\sqrt{\rho}\boldsymbol{u}\|^2\right) + \mu\|\nabla\boldsymbol{u}\|^2 = \langle(\boldsymbol{b}\cdot\nabla)\boldsymbol{b}, \boldsymbol{u}\rangle$$

and

$$\frac{1}{2}\frac{d}{dt}\left(\|\boldsymbol{b}\|^2\right) + \eta\|\nabla\boldsymbol{b}\|^2 = \langle(\boldsymbol{b}\cdot\nabla)\boldsymbol{u}, \boldsymbol{b}\rangle,$$

since $\rho_t = -\boldsymbol{u}\cdot\nabla\rho$ and $\operatorname{div}\boldsymbol{u} = 0$. Adding the above identities and noting that $\langle(\boldsymbol{b}\cdot\nabla)\boldsymbol{b}, \boldsymbol{u}\rangle + \langle(\boldsymbol{b}\cdot\nabla)\boldsymbol{u}, \boldsymbol{b}\rangle = 0$, because $\operatorname{div}\boldsymbol{b} = 0$, we get the energy equation

$$\frac{d}{dt}\left(\|\sqrt{\rho}\boldsymbol{u}\|^2 + \|\boldsymbol{b}\|^2\right) + 2\left(\mu\|\nabla\boldsymbol{u}\|^2 + \eta\|\nabla\boldsymbol{b}\|^2\right) = 0. \tag{3.9}$$

Integrating (3.9) in time and using (3.1), we find

$$\alpha\|\boldsymbol{u}(t)\|^2 + \|\boldsymbol{b}(t)\|^2 + 2\int_0^t \left(\mu\|\nabla\boldsymbol{u}(s)\|^2 + \eta\|\nabla\boldsymbol{b}(s)\|^2\right) ds \leq \beta\|\boldsymbol{u}_0\|^2 + \|\boldsymbol{b}_0\|^2, \tag{3.10}$$

which proves the estimates (3.2) and (3.3).

Furthermore, applying Poincaré's inequality (see Lemma 2.2) to identity (3.9), we obtain

$$\frac{d}{dt}\left(\|\sqrt{\rho}\boldsymbol{u}\|^2 + \|\boldsymbol{b}\|^2\right) + \frac{\gamma}{\epsilon^2}\left(\|\sqrt{\rho}\boldsymbol{u}\|^2 + \|\boldsymbol{b}\|^2\right) \leq 0, \tag{3.11}$$

where $\gamma := \min\{2\mu/\beta, 2\eta\}$. Multiplying the above inequality by the integrating factor $e^{\gamma\epsilon^{-2}t}$, we get

$$\frac{d}{dt}\left(e^{\gamma\epsilon^{-2}t}\left[\|\sqrt{\rho}\boldsymbol{u}\|^2 + \|\boldsymbol{b}\|^2\right]\right) \leq 0.$$

Integrating from 0 to t, one obtains

$$\|\boldsymbol{u}(t)\|^2 + \|\boldsymbol{b}(t)\|^2 \leq C\left(\|\boldsymbol{u}_0\|^2 + \|\boldsymbol{b}_0\|^2\right)e^{-\gamma t/\epsilon^2}. \tag{3.12}$$

This proves bound (3.4).

On the other hand, taking the $\boldsymbol{L}^2$ scalar product of first and second equations in (1.1) by $\boldsymbol{u}_t$ and $\boldsymbol{b}_t$ respectively, and adding the resulting identities, one gets

$$\begin{aligned}\|\sqrt{\rho}\boldsymbol{u}_t\|^2 + \|\boldsymbol{b}_t\|^2 + \frac{1}{2}\frac{d}{dt}\left(\mu\|\nabla\boldsymbol{u}\|^2 + \eta\|\nabla\boldsymbol{b}\|^2\right)&\\ = \langle(\boldsymbol{b}\cdot\nabla)\boldsymbol{b}, \boldsymbol{u}_t\rangle + \langle\rho(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}, -\boldsymbol{u}_t\rangle + \langle(\boldsymbol{b}\cdot\nabla)\boldsymbol{u}, \boldsymbol{b}_t\rangle + \langle(\boldsymbol{u}\cdot\nabla)\boldsymbol{b}, -\boldsymbol{b}_t\rangle.&\end{aligned} \tag{3.13}$$

Analogously, making the $\boldsymbol{L}^2$ inner product of first and second equations in (1.1) by $-\kappa\Delta\boldsymbol{u}$ and $-\widetilde{\kappa}\Delta\boldsymbol{b}$ respectively, with $\kappa, \widetilde{\kappa} > 0$ to be chosen later on, we obtain, after adding the resulting equations, that

$$\begin{aligned}\kappa\mu\|\Delta\boldsymbol{u}\|^2 + \widetilde{\kappa}\eta\|\Delta\boldsymbol{b}\|^2 = \kappa\langle\rho\boldsymbol{u}_t, \Delta\boldsymbol{u}\rangle + \kappa\langle\rho(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}, \Delta\boldsymbol{u}\rangle + \kappa\langle(\boldsymbol{b}\cdot\nabla)\boldsymbol{b}, -\Delta\boldsymbol{u}\rangle \\ + \widetilde{\kappa}\langle\boldsymbol{b}_t, \Delta\boldsymbol{b}\rangle + \widetilde{\kappa}\langle(\boldsymbol{u}\cdot\nabla)\boldsymbol{b}, \Delta\boldsymbol{b}\rangle + \widetilde{\kappa}\langle(\boldsymbol{b}\cdot\nabla)\boldsymbol{u}, -\Delta\boldsymbol{b}\rangle. \qquad (3.14)\end{aligned}$$

Then applying Cauchy–Schwarz and Young's inequalities, we bound each term on the right-hand side of equalities (3.13) and (3.14) as follows:

$$\begin{aligned}|\langle(\boldsymbol{b}\cdot\nabla)\boldsymbol{b}, \boldsymbol{u}_t\rangle| &\le \frac{1}{6}\|\sqrt{\rho}\boldsymbol{u}_t\|^2 + \frac{3}{2\alpha}\|(\boldsymbol{b}\cdot\nabla)\boldsymbol{b}\|^2,\\
|\langle\rho(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}, -\boldsymbol{u}_t\rangle| &\le \frac{1}{6}\|\sqrt{\rho}\boldsymbol{u}_t\|^2 + \frac{3\beta}{2}\|(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\|^2,\\
|\langle(\boldsymbol{b}\cdot\nabla)\boldsymbol{u}, \boldsymbol{b}_t\rangle| &\le \frac{1}{6}\|\boldsymbol{b}_t\|^2 + \frac{3}{2}\|(\boldsymbol{b}\cdot\nabla)\boldsymbol{u}\|^2,\\
|\langle(\boldsymbol{u}\cdot\nabla)\boldsymbol{b}, -\boldsymbol{b}_t\rangle| &\le \frac{1}{6}\|\boldsymbol{b}_t\|^2 + \frac{3}{2}\|(\boldsymbol{u}\cdot\nabla)\boldsymbol{b}\|^2,\\
\kappa|\langle\rho\boldsymbol{u}_t, \Delta\boldsymbol{u}\rangle| &\le \frac{1}{6}\|\sqrt{\rho}\boldsymbol{u}_t\|^2 + \frac{3}{2}\beta\kappa^2\|\Delta\boldsymbol{u}\|^2,\\
\kappa|\langle\rho(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}, \Delta\boldsymbol{u}\rangle| &\le \frac{\beta}{4}\|(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\|^2 + \frac{3}{2}\beta\kappa^2\|\Delta\boldsymbol{u}\|^2,\\
\kappa|\langle(\boldsymbol{b}\cdot\nabla)\boldsymbol{b}, -\Delta\boldsymbol{u}\rangle| &\le \frac{1}{4\alpha}\|(\boldsymbol{b}\cdot\nabla)\boldsymbol{b}\|^2 + \beta\kappa^2\|\Delta\boldsymbol{u}\|^2,\\
\widetilde{\kappa}|\langle\boldsymbol{b}_t, \Delta\boldsymbol{b}\rangle| &\le \frac{1}{6}\|\boldsymbol{b}_t\|^2 + \frac{3}{2}\widetilde{\kappa}^2\|\Delta\boldsymbol{b}\|^2,\\
\widetilde{\kappa}|\langle(\boldsymbol{u}\cdot\nabla)\boldsymbol{b}, \Delta\boldsymbol{b}\rangle| &\le \frac{1}{6}\|(\boldsymbol{u}\cdot\nabla)\boldsymbol{b}\|^2 + \frac{3}{2}\widetilde{\kappa}^2\|\Delta\boldsymbol{b}\|^2,\\
\widetilde{\kappa}|\langle(\boldsymbol{b}\cdot\nabla)\boldsymbol{u}, -\Delta\boldsymbol{b}\rangle| &\le \frac{1}{4}\|(\boldsymbol{b}\cdot\nabla)\boldsymbol{u}\|^2 + \widetilde{\kappa}^2\|\Delta\boldsymbol{b}\|^2.\end{aligned}$$

Consequently, applying the estimates above to resulting equation of the sum of the identities (3.13) and (3.14) and choosing $\kappa = \mu/8\beta$ and $\widetilde{\kappa} = \eta/8$, one finds

$$\begin{aligned}\|\sqrt{\rho}\boldsymbol{u}_t\|^2 + \|\boldsymbol{b}_t\|^2 + \frac{d}{dt}\left(\mu\|\nabla\boldsymbol{u}\|^2 + \eta\|\nabla\boldsymbol{b}\|^2\right) + \frac{1}{8}\left(\frac{\mu^2}{\beta}\|\Delta\boldsymbol{u}\|^2 + \eta^2\|\Delta\boldsymbol{b}\|^2\right) \\ \le C\left(\|(\boldsymbol{b}\cdot\nabla)\boldsymbol{b}\|^2 + \|(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\|^2 + \|(\boldsymbol{b}\cdot\nabla)\boldsymbol{u}\|^2 + \|(\boldsymbol{u}\cdot\nabla)\boldsymbol{b}\|^2\right). \qquad (3.15)\end{aligned}$$

Now, using Hölder, standard interpolation, Sobolev and Young's inequalities (see [2]), we have

$$\begin{aligned}C\|(\boldsymbol{b}\cdot\nabla)\boldsymbol{b}\|^2 \le C\|\boldsymbol{b}\|_6^2\|\nabla\boldsymbol{b}\|_3^2 \le C\|\boldsymbol{b}\|_6^2\|\nabla\boldsymbol{b}\|\,\|\Delta\boldsymbol{b}\| \le C\|\nabla\boldsymbol{b}\|^3\|\Delta\boldsymbol{b}\| \\ \le \frac{\eta^2}{32}\|\Delta\boldsymbol{b}\|^2 + K\|\nabla\boldsymbol{b}\|^6, \qquad (3.16)\\
C\|(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\|^2 \le \frac{\mu^2}{32\beta}\|\Delta\boldsymbol{u}\|^2 + K\|\nabla\boldsymbol{u}\|^6, \qquad (3.17)\\
C\|(\boldsymbol{b}\cdot\nabla)\boldsymbol{u}\|^2 \le \frac{\mu^2}{32\beta}\|\Delta\boldsymbol{u}\|^2 + K\|\nabla\boldsymbol{b}\|^4\|\nabla\boldsymbol{u}\|^2, \qquad (3.18)\\
C\|(\boldsymbol{u}\cdot\nabla)\boldsymbol{b}\|^2 \le \frac{\eta^2}{32}\|\Delta\boldsymbol{b}\|^2 + K\|\nabla\boldsymbol{u}\|^4\|\nabla\boldsymbol{b}\|^2. \qquad (3.19)\end{aligned}$$

Thus, applying the estimates (3.16)–(3.19) to inequality (3.15), we find

$$\begin{aligned}\|\sqrt{\rho}\boldsymbol{u}_t\|^2+\|\boldsymbol{b}_t\|^2+\frac{d}{dt}\left(\mu\|\nabla\boldsymbol{u}\|^2+\eta\|\nabla\boldsymbol{b}\|^2\right)+\widetilde{\gamma}\left(\|\Delta\boldsymbol{u}\|^2+\|\Delta\boldsymbol{b}\|^2\right)\\ \leq K\left(\|\nabla\boldsymbol{u}\|^4+\|\nabla\boldsymbol{b}\|^4\right)\left(\|\nabla\boldsymbol{u}\|^2+\|\nabla\boldsymbol{b}\|^2\right),\end{aligned}\tag{3.20}$$

where $\widetilde{\gamma}:=\min\left\{\frac{\mu^2}{16\beta},\frac{\eta^2}{16}\right\}$. By integration of the above inequality over $[0,t]$, we obtain

$$\|\nabla\boldsymbol{z}(t)\|^2+\int_0^t\left(\|\boldsymbol{z}_t(s)\|^2+\|\Delta\boldsymbol{z}(s)\|^2\right)ds\leq K\|\nabla\boldsymbol{z}_0\|^2+K\int_0^t\|\nabla\boldsymbol{z}(s)\|^4\|\nabla\boldsymbol{z}(s)\|^2ds.$$

So we deduce from estimate (3.10) that

$$\begin{aligned}\|\nabla\boldsymbol{z}(t)\|^2+\int_0^t\left(\|\boldsymbol{z}_t(s)\|^2+\|\Delta\boldsymbol{z}(s)\|^2\right)ds&\leq K\left(\|\nabla\boldsymbol{z}_0\|^2+\|\nabla\boldsymbol{z}\|^4_{L^\infty_t(\boldsymbol{L}^2(\Omega))}\|\nabla\boldsymbol{z}\|^2_{L^2_t(\boldsymbol{L}^2(\Omega))}\right)\\&\leq K\left(\|\nabla\boldsymbol{z}_0\|^2+\|\boldsymbol{z}_0\|^2\|\nabla\boldsymbol{z}\|^4_{L^\infty_t(\boldsymbol{L}^2(\Omega))}\right).\end{aligned}$$

Having in mind the identities (2.4) and (2.5), one concludes that

$$\mathcal{V}(t)\leq K\left(\mathcal{V}_0+\mathcal{U}_0\,\mathcal{V}(t)^2\right).$$

Hence, if c_0 is sufficiently small, we conclude that

$$\mathcal{V}(t)\leq\widetilde{C}\mathcal{V}_0,$$

for some positive constant $\widetilde{C}$ depending of α, β, μ, η and c_0. This proves inequalities (3.5)–(3.7).

Lastly, we will focus on the estimate (3.8). Notice that, by bound (3.20), one has

$$\frac{d}{dt}(\mu\|\nabla\boldsymbol{u}\|^2+\eta\|\nabla\boldsymbol{b}\|^2)+\widetilde{\gamma}(\|\Delta\boldsymbol{u}\|^2+\|\Delta\boldsymbol{b}\|^2)\leq K\Big(\|\nabla\boldsymbol{u}\|^4+\|\nabla\boldsymbol{b}\|^4\Big)(\|\nabla\boldsymbol{u}\|^2+\|\nabla\boldsymbol{b}\|^2).\tag{3.21}$$

Then similarly as we got estimate (3.11), we deduce, from bound (3.21) and Poincaré's inequality (see Lemma 2.2), that

$$\frac{d}{dt}(\mu\|\nabla\boldsymbol{u}\|^2+\eta\|\nabla\boldsymbol{b}\|^2)+\frac{\overline{\gamma}}{\epsilon^2}\Big(\mu\|\nabla\boldsymbol{u}\|^2+\eta\|\nabla\boldsymbol{b}\|^2\Big)\leq K\Big(\|\nabla\boldsymbol{u}\|^4+\|\nabla\boldsymbol{b}\|^4\Big)(\|\nabla\boldsymbol{u}\|^2+\|\nabla\boldsymbol{b}\|^2),$$

where $\overline{\gamma}:=\min\left\{\frac{\widetilde{\gamma}}{\mu},\frac{\widetilde{\gamma}}{\eta}\right\}$. Multiplying by the integrating factor $e^{\overline{\gamma}\epsilon^{-2}t}$ gives

$$\frac{d}{dt}\left\{e^{\overline{\gamma}\epsilon^{-2}t}\left(\mu\|\nabla\boldsymbol{u}\|^2+\eta\|\nabla\boldsymbol{b}\|^2\right)\right\}\leq K\,e^{\overline{\gamma}\epsilon^{-2}t}\left(\|\nabla\boldsymbol{u}\|^4+\|\nabla\boldsymbol{b}\|^4\right)\left(\|\nabla\boldsymbol{u}\|^2+\|\nabla\boldsymbol{b}\|^2\right).$$

Thus, integrating in time and using estimate (3.3), one finds

$$\begin{aligned}\|\nabla\boldsymbol{z}(t)\|^2&\leq K\|\nabla\boldsymbol{z}_0\|^2e^{-\overline{\gamma}\epsilon^{-2}t}+K\int_0^t e^{-\overline{\gamma}\epsilon^{-2}(t-s)}\|\nabla\boldsymbol{z}(s)\|^4\|\nabla\boldsymbol{z}(s)\|^2ds\\&\leq K\|\nabla\boldsymbol{z}_0\|^2e^{-\overline{\gamma}\epsilon^{-2}t}+K\|\nabla\boldsymbol{z}\|^4_{L^\infty_t(\boldsymbol{L}^2(\Omega))}\|\nabla\boldsymbol{z}\|^2_{L^2_t(\boldsymbol{L}^2(\Omega))}\\&\leq K\|\nabla\boldsymbol{z}_0\|^2e^{-\overline{\gamma}\epsilon^{-2}t}+K\|\boldsymbol{z}_0\|^2\|\nabla\boldsymbol{z}\|^4_{L^\infty_t(\boldsymbol{L}^2(\Omega))}.\end{aligned}$$

Consequently, if c_0 is small enough, then we deduce that

$$\|\nabla\boldsymbol{z}(t)\|^2\leq\widetilde{C}\|\nabla\boldsymbol{z}_0\|^2e^{-\overline{\gamma}t/\epsilon^2},$$

where $\widetilde{C}=\widetilde{C}(\alpha,\beta,\mu,\eta,c_0)>0$. The proposition follows. □

Remark 3.2 (Estimates for the "initial data" $\boldsymbol{u}_t(0)$ and $\boldsymbol{b}_t(0)$)**.** Multiplying the first equation in (1.1) by $\boldsymbol{u}_t$ in $\boldsymbol{L}^2(\Omega)$ and applying Cauchy–Schwarz and Young's inequalities, we get

$$\|\sqrt{\rho}\boldsymbol{u}_t\|^2 \leq 3\beta\|(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\|^2 + \frac{3\mu^2}{\alpha}\|\Delta\boldsymbol{u}\|^2 + \frac{3}{\alpha}\|(\boldsymbol{b}\cdot\nabla)\boldsymbol{b}\|^2,$$

since $\operatorname{div}\boldsymbol{u}_t = 0$. It follows from Gagliardo–Nirenberg inequalities that

$$\|\boldsymbol{u}_t\|^2 \leq K_1\left(\|\boldsymbol{u}\|\|\nabla\boldsymbol{u}\|\|\nabla^2\boldsymbol{u}\|^2 + \|\Delta\boldsymbol{u}\|^2 + \|\boldsymbol{b}\|\|\nabla\boldsymbol{b}\|\|\nabla^2\boldsymbol{b}\|^2\right),$$

where $K_1 > 0$ depends only on the constants α, β and μ. Due to Remark 2.3, we have

$$\begin{aligned}\|\boldsymbol{u}_t(0)\|^2 &\leq K_1\left(\|\boldsymbol{u}_0\|\|\nabla\boldsymbol{u}_0\|\|\nabla^2\boldsymbol{u}_0\|^2 + \|\Delta\boldsymbol{u}_0\|^2 + \|\boldsymbol{b}_0\|\|\nabla\boldsymbol{b}_0\|\|\nabla^2\boldsymbol{b}_0\|^2\right)\\ &\leq K_1\left(c_0^2\|\nabla^2\boldsymbol{u}_0\|^2 + \|\Delta\boldsymbol{u}_0\|^2 + c_0^2\|\nabla^2\boldsymbol{b}_0\|^2\right)\\ &\leq \widetilde{C}_1\|\nabla^2\boldsymbol{z}_0\|^2,\end{aligned} \tag{3.22}$$

where $\widetilde{C}_1 = \widetilde{C}_1(\alpha, \beta, \mu, c_0) > 0$. With similar arguments, we can find

$$\|\boldsymbol{b}_t(0)\|^2 \leq \widetilde{C}_2\|\nabla^2\boldsymbol{z}_0\|^2, \tag{3.23}$$

where $\widetilde{C}_2 = \widetilde{C}_2(\eta, c_0) > 0$. Hence, by bounds (3.22) and (3.23), one concludes that

$$\|\boldsymbol{u}_t(0)\|^2 + \|\boldsymbol{b}_t(0)\|^2 \leq \widetilde{C}\left(\|\nabla^2\boldsymbol{u}_0\|^2 + \|\nabla^2\boldsymbol{b}_0\|^2\right), \tag{3.24}$$

where $\widetilde{C} > 0$ is a constant depending only on α, β, μ, η and c_0.

Proposition 3.3 (Bounds in the $\boldsymbol{H}^2(\Omega)$ norm)**.** *Suppose that the initial data $(\rho_0, \boldsymbol{u}_0, \boldsymbol{b}_0)$ satisfy (2.1) and (2.3), with $\boldsymbol{u}_0, \boldsymbol{b}_0 \in \boldsymbol{V}\cap\boldsymbol{H}^2(\Omega)$. Let $(\rho, \boldsymbol{u}, p, \boldsymbol{b})$ be the strong solution furnished by Theorem 2.1 defined on $\Omega\times[0,T]$. There exists a constant $\widetilde{C} = \widetilde{C}(\alpha, \beta, \mu, \eta, c_0) > 0$ such that*

$$\|\boldsymbol{u}_t(t)\|^2 + \|\boldsymbol{b}_t(t)\|^2 \leq \widetilde{C}\left(\|\nabla^2\boldsymbol{u}_0\|^2 + \|\nabla^2\boldsymbol{b}_0\|^2\right), \tag{3.25}$$

$$\int_0^t \left(\|\nabla\boldsymbol{u}_t(s)\|^2 + \|\nabla\boldsymbol{b}_t(s)\|^2\right)ds \leq \widetilde{C}\left(\|\nabla^2\boldsymbol{u}_0\|^2 + \|\nabla^2\boldsymbol{b}_0\|^2\right), \tag{3.26}$$

$$\left\|(\nabla^2\boldsymbol{u}, \nabla^2\boldsymbol{b}, \nabla p)\right\|^2_{L^\infty_t(\boldsymbol{L}^2(\Omega))} \leq \widetilde{C}\left(\|\nabla^2\boldsymbol{u}_0\|^2 + \|\nabla^2\boldsymbol{b}_0\|^2\right), \tag{3.27}$$

$$\|\boldsymbol{u}_t(t)\|^2 + \|\boldsymbol{b}_t(t)\|^2 \leq \widetilde{C}\left(\|\nabla^2\boldsymbol{u}_0\|^2 + \|\nabla^2\boldsymbol{b}_0\|^2\right)e^{-\sigma t/\epsilon^2}, \tag{3.28}$$

$$\left\|(\nabla^2\boldsymbol{u}, \nabla^2\boldsymbol{b}, \nabla p)\right\|^2_{L^\infty_t(\boldsymbol{L}^2(\Omega))} \leq \widetilde{C}\left(\|\nabla^2\boldsymbol{u}_0\|^2 + \|\nabla^2\boldsymbol{b}_0\|^2\right)e^{-\sigma t/\epsilon^2}, \tag{3.29}$$

for all $t\in[0,T]$, with $\sigma := \min\{\widehat{\gamma}, \overline{\gamma}\}$, where $\widehat{\gamma} := \min\{\frac{\mu}{\beta}, \eta\}$, $\overline{\gamma} := \min\{\frac{\widetilde{\gamma}}{\mu}, \frac{\widetilde{\gamma}}{\eta}\}$ and $\widetilde{\gamma} := \min\{\frac{\mu^2}{16\beta}, \frac{\eta^2}{16}\}$.

Proof. Differentiating the equations $(1.1)_1$ and $(1.1)_2$ with respect to t, taking the dot product with $\boldsymbol{u}_t$ and $\boldsymbol{b}_t$, respectively, integrating the resulting identities on Ω and adding both results up, we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|\sqrt{\rho}\boldsymbol{u}_t\|^2 + \|\boldsymbol{b}_t\|^2\right) + \mu\|\nabla\boldsymbol{u}_t\|^2 + \eta\|\nabla\boldsymbol{b}_t\|^2\\ &\quad= \langle(\boldsymbol{b}_t\cdot\nabla)\boldsymbol{b}, \boldsymbol{u}_t\rangle + 2\langle\rho(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}_t, -\boldsymbol{u}_t\rangle + \langle\rho_t(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}, -\boldsymbol{u}_t\rangle\\ &\qquad+ \langle\rho(\boldsymbol{u}_t\cdot\nabla)\boldsymbol{u}, -\boldsymbol{u}_t\rangle + \langle(\boldsymbol{b}_t\cdot\nabla)\boldsymbol{u}, \boldsymbol{b}_t\rangle + \langle(\boldsymbol{u}_t\cdot\nabla)\boldsymbol{b}, -\boldsymbol{b}_t\rangle,\end{aligned} \tag{3.30}$$

since $\rho_t = -\boldsymbol{u}\cdot\nabla\rho$ and $\operatorname{div}\boldsymbol{u} = \operatorname{div}\boldsymbol{u}_t = \operatorname{div}\boldsymbol{b} = 0$.

Now, we estimate each term on the right-hand side of identity (3.30). Using Hölder, Sobolev, interpolation and Young's inequalities, we find

$$\begin{aligned}
|\langle(\boldsymbol{b}_t\cdot\nabla)\boldsymbol{b},\boldsymbol{u}_t\rangle| &\le \|\boldsymbol{b}_t\|_6\|\nabla\boldsymbol{b}\|\|\boldsymbol{b}_t\|_3\\
&\le \delta_1\|\nabla\boldsymbol{b}_t\|^2 + C_{\delta_1}\|\nabla\boldsymbol{b}\|^4\|\boldsymbol{b}_t\|^2,\\
2|\langle\rho(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}_t,-\boldsymbol{u}_t\rangle| &\le 2\|\rho\|_\infty\|\boldsymbol{u}\|_6\|\nabla\boldsymbol{u}_t\|\|\boldsymbol{u}_t\|_3\\
&\le \delta_2\|\nabla\boldsymbol{u}_t\|^2 + C_{\delta_2}\|\nabla\boldsymbol{u}\|^4\|\boldsymbol{u}_t\|^2,\\
|\langle\rho_t(\boldsymbol{u}\cdot\nabla)\boldsymbol{u},-\boldsymbol{u}_t\rangle| &= |\langle\operatorname{div}(\rho\boldsymbol{u})(\boldsymbol{u}\cdot\nabla)\boldsymbol{u},\boldsymbol{u}_t\rangle|\\
&\le C\|\nabla\boldsymbol{u}\|^2\|\nabla^2\boldsymbol{u}\|\|\nabla\boldsymbol{u}_t\|\\
&\le \delta_2\|\nabla\boldsymbol{u}_t\|^2 + C_{\delta_2}\|\nabla\boldsymbol{u}\|^4\|\nabla^2\boldsymbol{u}\|^2,\\
|\langle\rho(\boldsymbol{u}_t\cdot\nabla)\boldsymbol{u},-\boldsymbol{u}_t\rangle| &\le \|\rho\|_\infty\|\boldsymbol{u}_t\|_6\|\nabla\boldsymbol{u}\|\|\boldsymbol{u}_t\|_3\\
&\le \delta_2\|\nabla\boldsymbol{u}_t\|^2 + C_{\delta_2}\|\nabla\boldsymbol{u}\|^4\|\boldsymbol{u}_t\|^2,\\
|\langle(\boldsymbol{b}_t\cdot\nabla)\boldsymbol{u},\boldsymbol{b}_t\rangle| &\le \|\boldsymbol{b}_t\|_6\|\nabla\boldsymbol{u}\|\|\boldsymbol{b}_t\|_3\\
&\le \delta_1\|\nabla\boldsymbol{b}_t\|^2 + C_{\delta_1}\|\nabla\boldsymbol{u}\|^4\|\boldsymbol{b}_t\|^2,\\
|\langle(\boldsymbol{u}_t\cdot\nabla)\boldsymbol{b},-\boldsymbol{b}_t\rangle| &\le \|\boldsymbol{u}_t\|_6\|\nabla\boldsymbol{b}\|\|\boldsymbol{b}_t\|_3\\
&\le C\|\nabla\boldsymbol{u}_t\|\|\nabla\boldsymbol{b}\|\|\boldsymbol{b}_t\|^{1/2}\|\nabla\boldsymbol{b}_t\|^{1/2}\\
&\le \delta_1\|\nabla\boldsymbol{b}_t\|^2 + \delta_2\|\nabla\boldsymbol{u}_t\|^2 + C_{\delta_1,\delta_2}\|\nabla\boldsymbol{b}\|^4\|\boldsymbol{b}_t\|^2.
\end{aligned}$$

So, applying the estimates above to equality (3.30) and taking $\delta_1 = \eta/6$ and $\delta_2 = \mu/8$, we obtain

$$\begin{aligned}
&\frac{d}{dt}\left(\|\sqrt{\rho}\boldsymbol{u}_t\|^2 + \|\boldsymbol{b}_t\|^2\right) + \mu\|\nabla\boldsymbol{u}_t\|^2 + \eta\|\nabla\boldsymbol{b}_t\|^2\\
&\quad\le K\left(\|\nabla\boldsymbol{u}\|^4\|\boldsymbol{u}_t\|^2 + \|\nabla\boldsymbol{u}\|^4\|\boldsymbol{b}_t\|^2 + \|\nabla\boldsymbol{b}\|^4\|\boldsymbol{b}_t\|^2 + \|\nabla\boldsymbol{u}\|^4\|\nabla^2\boldsymbol{u}\|^2\right)\\
&\quad\le K\left[\left(\|\nabla\boldsymbol{u}\|^4 + \|\nabla\boldsymbol{b}\|^4\right)\left(\|\boldsymbol{u}_t\|^2 + \|\boldsymbol{b}_t\|^2\right) + \|\nabla\boldsymbol{u}\|^4\|\nabla^2\boldsymbol{u}\|^2\right].
\end{aligned}\tag{3.31}$$

Moreover, since $\alpha \le \rho(\boldsymbol{x},t) \le \beta$ for all $(\boldsymbol{x},t)\in\Omega\times[0,T]$, we deduce that

$$\begin{aligned}
&\frac{d}{dt}\left(\|\sqrt{\rho}\boldsymbol{u}_t\|^2 + \|\boldsymbol{b}_t\|^2\right) + \mu\|\nabla\boldsymbol{u}_t\|^2 + \eta\|\nabla\boldsymbol{b}_t\|^2\\
&\quad\le K\left(\|\nabla\boldsymbol{u}\|^4 + \|\nabla\boldsymbol{b}\|^4\right)\left(\frac{1}{\alpha}\|\sqrt{\rho}\boldsymbol{u}_t\|^2 + \|\boldsymbol{b}_t\|^2\right) + K\|\nabla\boldsymbol{u}\|^4\|\nabla^2\boldsymbol{u}\|^2\\
&\quad\le C\left(\|\nabla\boldsymbol{u}\|^4 + \|\nabla\boldsymbol{b}\|^4\right)\left(\|\sqrt{\rho}\boldsymbol{u}_t\|^2 + \|\boldsymbol{b}_t\|^2\right) + C\|\nabla\boldsymbol{u}\|^4\|\nabla^2\boldsymbol{u}\|^2.
\end{aligned}\tag{3.32}$$

Then, applying a generalized Gronwall inequality (see, for instance, [15, Lemma 4, p. 656]) and taking into account (3.24), we have

$$\begin{aligned}
\|\boldsymbol{z}_t(t)\|^2 &= \|\boldsymbol{u}_t(t)\|^2 + \|\boldsymbol{b}_t(t)\|^2\\
&\le \exp\left(C\int_0^t\left(\|\nabla\boldsymbol{u}(s)\|^4 + \|\nabla\boldsymbol{b}(s)\|^4\right)ds\right)\left(\widetilde{C}\|\nabla^2\boldsymbol{z}_0\|^2 + C\int_0^t\|\nabla\boldsymbol{u}(s)\|^4\|\nabla^2\boldsymbol{u}(s)\|^2\,ds\right)\\
&\le \exp\left(C\|\nabla\boldsymbol{u}\|^2_{L^2_t(\boldsymbol{L}^2(\Omega))}\|\nabla\boldsymbol{u}\|^2_{L^\infty_t(\boldsymbol{L}^2(\Omega))} + C\|\nabla\boldsymbol{b}\|^2_{L^2_t(\boldsymbol{L}^2(\Omega))}\|\nabla\boldsymbol{b}\|^2_{L^\infty_t(\boldsymbol{L}^2(\Omega))}\right)\\
&\quad\times\left(\widetilde{C}\|\nabla^2\boldsymbol{z}_0\|^2 + C\|\nabla\boldsymbol{u}\|^2_{L^2_t(\boldsymbol{L}^2(\Omega))}\|\nabla\boldsymbol{u}\|^2_{L^\infty_t(\boldsymbol{L}^2(\Omega))}\|\nabla^2\boldsymbol{z}\|^2_{L^\infty_t(\boldsymbol{L}^2(\Omega))}\right).
\end{aligned}$$

This estimate, together with (3.3), (3.5), and Remark 2.3, implies that

$$\begin{aligned}\|z_t(t)\|^2 &\le \exp\Big(\widetilde{C}\|z_0\|^2\|\nabla z_0\|^2\Big)\Big(\widetilde{C}\|\nabla^2 z_0\|^2+\widetilde{C}\|z_0\|^2\|\nabla z_0\|^2\|\nabla^2 z\|^2_{L^\infty_t(\boldsymbol{L}^2(\Omega))}\Big)\\ &\le \widetilde{C}\exp\Big(\widetilde{C}c_0^4\Big)\Big(\|\nabla^2 z_0\|^2+c_0^4\|\nabla^2 z\|^2_{L^\infty_t(\boldsymbol{L}^2(\Omega))}\Big)\\ &\le \widetilde{C}\Big(\|\nabla^2 z_0\|^2+c_0^4\|\nabla^2 z\|^2_{L^\infty_t(\boldsymbol{L}^2(\Omega))}\Big). \end{aligned} \tag{3.33}$$

Now, integrating (3.32) in time, and using again (3.3), (3.5), Remark 2.3, as well as (3.33), we obtain

$$\begin{aligned}&\|z_t(t)\|^2+\int_0^t\|\nabla z_t(s)\|^2\,ds\\ &\le \widetilde{C}\|\nabla^2 z_0\|^2+C\int_0^t\Big(\|\nabla \boldsymbol{u}(s)\|^4+\|\nabla \boldsymbol{b}(s)\|^4\Big)\|z_t(s)\|^2\,ds+C\int_0^t\|\nabla \boldsymbol{u}(s)\|^4\|\nabla^2\boldsymbol{u}(s)\|^2\,ds\\ &\le \widetilde{C}\|\nabla^2 z_0\|^2+C\|\nabla z\|^2_{L^2_t(\boldsymbol{L}^2(\Omega))}\|\nabla z\|^2_{L^\infty_t(\boldsymbol{L}^2(\Omega))}\|z_t\|^2_{L^\infty_t(\boldsymbol{L}^2(\Omega))}\\ &\quad+C\|\nabla z\|^2_{L^2_t(\boldsymbol{L}^2(\Omega))}\|\nabla z\|^2_{L^\infty_t(\boldsymbol{L}^2(\Omega))}\|\nabla^2 z\|^2_{L^\infty_t(\boldsymbol{L}^2(\Omega))}\\ &\le \widetilde{C}\|\nabla^2 z_0\|^2+\widetilde{C}\|z_0\|^2\|\nabla z_0\|^2\|z_t\|^2_{L^\infty_t(\boldsymbol{L}^2(\Omega))}+\widetilde{C}\|z_0\|^2\|\nabla z_0\|^2\|\nabla^2 z\|^2_{L^\infty_t(\boldsymbol{L}^2(\Omega))}\\ &\le \widetilde{C}\|\nabla^2 z_0\|^2+\widetilde{C}c_0^4\|z_t\|^2_{L^\infty_t(\boldsymbol{L}^2(\Omega))}+\widetilde{C}c_0^4\|\nabla^2 z\|^2_{L^\infty_t(\boldsymbol{L}^2(\Omega))}\\ &\le \widetilde{C}\|\nabla^2 z_0\|^2+\widetilde{C}c_0^4\|\nabla^2 z_0\|^2+\widetilde{C}c_0^8\|\nabla^2 z\|^2_{L^\infty_t(\boldsymbol{L}^2(\Omega))}+\widetilde{C}c_0^4\|\nabla^2 z\|^2_{L^\infty_t(\boldsymbol{L}^2(\Omega))}\\ &\le \widetilde{C}\Big(\|\nabla^2 z_0\|^2+c_0^4\|\nabla^2 z\|^2_{L^\infty_t(\boldsymbol{L}^2(\Omega))}\Big), \end{aligned} \tag{3.34}$$

since $c_0>0$ is sufficiently small.

On the other side, by identity (3.14), one concludes that

$$\|\nabla^2 z(t)\|^2\le K\big(\|z_t\|^2+\|(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\|^2+\|(\boldsymbol{b}\cdot\nabla)\boldsymbol{b}\|^2+\|(\boldsymbol{u}\cdot\nabla)\boldsymbol{b}\|^2+\|(\boldsymbol{b}\cdot\nabla)\boldsymbol{u}\|^2\big). \tag{3.35}$$

In addition, since $\operatorname{div}\boldsymbol{u}=0$, applying the divergence operator to the first in (1.1), we obtain

$$\Delta p=-\operatorname{div}\big(\rho\boldsymbol{u}_t+\rho(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}-(\boldsymbol{b}\cdot\nabla)\boldsymbol{b}\big),$$

and therefore the total pressure p may be recovered by

$$\nabla p=-\nabla\Delta^{-1}\operatorname{div}\big(\rho\boldsymbol{u}_t+\rho(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}-(\boldsymbol{b}\cdot\nabla)\boldsymbol{b}\big). \tag{3.36}$$

Hence, we have

$$\|\nabla p(t)\|^2\le C_\beta\left(\|\boldsymbol{u}_t\|^2+\|(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\|^2+\|(\boldsymbol{b}\cdot\nabla)\boldsymbol{b}\|^2\right).$$

So, from bounds (3.35) and (3.36), we get

$$\|(\nabla^2\boldsymbol{u},\nabla^2\boldsymbol{b},\nabla p)\|^2\le K\left(\|\boldsymbol{u}_t\|^2+\|\boldsymbol{b}_t\|^2+F(t)\right), \tag{3.37}$$

where $F(t):=\|(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\|^2+\|(\boldsymbol{u}\cdot\nabla)\boldsymbol{b}\|^2+\|(\boldsymbol{b}\cdot\nabla)\boldsymbol{b}\|^2+\|(\boldsymbol{b}\cdot\nabla)\boldsymbol{u}\|^2$. To obtain the estimate of $\big\|(\nabla^2\boldsymbol{u},\nabla^2\boldsymbol{b},\nabla p)\big\|^2_{L^\infty_t(\boldsymbol{L}^2(\Omega))}$, it remains to control $F(t)$. Using Hölder, standard interpolation

and Sobolev's inequalities (see [2]), we find, from estimates (3.2) and (3.5), and Remark 2.3, that

$$\begin{aligned}\|(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\|^2 &\le \|\boldsymbol{u}\|_3^2\,\|\nabla\boldsymbol{u}\|_6^2 \le C\|\boldsymbol{u}\|\|\nabla\boldsymbol{u}\|\|\nabla^2\boldsymbol{u}\|^2\\ &\le \widetilde{C}\|z_0\|\|\nabla z_0\|\|\nabla^2\boldsymbol{u}\|^2\\ &\le \widetilde{C}c_0^2\|\nabla^2\boldsymbol{u}\|^2, \qquad (3.38)\\ \|(\boldsymbol{u}\cdot\nabla)\boldsymbol{b}\|^2 &\le \|\boldsymbol{u}\|_3^2\,\|\nabla\boldsymbol{b}\|_6^2 \le C\|\boldsymbol{u}\|\|\nabla\boldsymbol{u}\|\|\nabla^2\boldsymbol{b}\|^2\\ &\le \widetilde{C}\|z_0\|\|\nabla z_0\|\|\nabla^2\boldsymbol{b}\|^2\\ &\le \widetilde{C}c_0^2\|\nabla^2\boldsymbol{b}\|^2. \qquad (3.39)\end{aligned}$$

Analogously, one has

$$\|(\boldsymbol{b}\cdot\nabla)\boldsymbol{b}\|^2 \le \widetilde{C}c_0^2\|\nabla^2\boldsymbol{b}\|^2, \qquad (3.40)$$

$$\|(\boldsymbol{b}\cdot\nabla)\boldsymbol{u}\|^2 \le \widetilde{C}c_0^2\|\nabla^2\boldsymbol{u}\|^2. \qquad (3.41)$$

Consequently, by inequality (3.37), one obtains

$$\|(\nabla^2 z,\nabla p)\|^2 \le \widetilde{C}\big(\|z_t\|^2 + c_0^2\|\nabla^2 z\|^2\big). \qquad (3.42)$$

Since $c_0 > 0$ is sufficiently small, one gets

$$\|(\nabla^2 z,\nabla p)\|^2 \le \widetilde{C}\|z_t\|^2. \qquad (3.43)$$

Furthermore, having in mind the identities (2.4)–(2.5) and summing the bounds (3.34) and (3.43), one concludes that

$$\mathcal{W}(t) \le \widetilde{C}\Big(\mathcal{W}_0 + c_0^4\mathcal{W}(t)\Big).$$

Hence, the smallness of c_0 ensures that

$$\mathcal{W}(t) \le \widetilde{C}\mathcal{W}_0.$$

This proves estimates (3.25)–(3.27). Finally, using Poincaré's inequality (see Lemma 2.2), we get

$$\|\nabla\boldsymbol{u}_t\| \ge \epsilon^{-1}\|\boldsymbol{u}_t\| \quad \text{and} \quad \|\nabla\boldsymbol{b}_t\| \ge \epsilon^{-1}\|\boldsymbol{b}_t\|.$$

Thus, by estimate (3.31), we have

$$\begin{aligned}&\frac{d}{dt}\big(\|\sqrt{\rho}\boldsymbol{u}_t\|^2 + \|\boldsymbol{b}_t\|^2\big) + \frac{\widehat{\gamma}}{\epsilon^2}\big(\|\sqrt{\rho}\boldsymbol{u}_t\|^2 + \|\boldsymbol{b}_t\|^2\big)\\ &\quad\le K\Big[\Big(\|\nabla\boldsymbol{u}\|^4 + \|\nabla\boldsymbol{b}\|^4\Big)\big(\|\boldsymbol{u}_t\|^2 + \|\boldsymbol{b}_t\|^2\big) + \|\nabla\boldsymbol{u}\|^4\|\nabla^2\boldsymbol{u}\|^2\Big], \qquad (3.44)\end{aligned}$$

where $\widehat{\gamma} := \min\{\mu/\beta, \eta\}$. Multiplying inequality (3.44) by integrating factor $e^{\widehat{\gamma}\epsilon^{-2}t}$ yields

$$\begin{aligned}&\frac{d}{dt}\Big\{e^{\widehat{\gamma}\epsilon^{-2}t}\big(\|\sqrt{\rho}\boldsymbol{u}_t\|^2 + \|\boldsymbol{b}_t\|^2\big)\Big\}\\ &\quad\le K\,e^{\widehat{\gamma}\epsilon^{-2}t}\Big[\Big(\|\nabla\boldsymbol{u}\|^4 + \|\nabla\boldsymbol{b}\|^4\Big)\big(\|\boldsymbol{u}_t\|^2 + \|\boldsymbol{b}_t\|^2\big) + \|\nabla\boldsymbol{u}\|^4\|\nabla^2\boldsymbol{u}\|^2\Big].\end{aligned}$$

Then, we deduce from estimate (3.24) that

$$\|\sqrt{\rho}\boldsymbol{u}_t\|^2 + \|\boldsymbol{b}_t\|^2 \leq \widetilde{C}\|\nabla^2 \boldsymbol{z}_0\|^2\, e^{-\widehat{\gamma}\epsilon^{-2}t} + K\int_0^t e^{-\widehat{\gamma}\epsilon^{-2}(t-s)}\left[\left(\|\nabla\boldsymbol{u}(s)\|^4 + \|\nabla\boldsymbol{b}(s)\|^4\right)\|\boldsymbol{z}_t(s)\|^2 + \|\nabla\boldsymbol{u}(s)\|^4\|\nabla^2\boldsymbol{u}(s)\|^2\right] ds.$$

Consequently, we have

$$\begin{aligned}\|\boldsymbol{z}_t(t)\|^2 \leq \widetilde{C}\|\nabla^2\boldsymbol{z}_0\|^2\, e^{-\widehat{\gamma}\epsilon^{-2}t} &+ K\|\nabla\boldsymbol{u}\|^2_{L^2_t(\boldsymbol{L}^2(\Omega))}\|\nabla\boldsymbol{u}\|^2_{L^\infty_t(\boldsymbol{L}^2(\Omega))}\|\boldsymbol{z}_t\|^2_{L^\infty_t(\boldsymbol{L}^2(\Omega))}\\ &+ K\|\nabla\boldsymbol{b}\|^2_{L^2_t(\boldsymbol{L}^2(\Omega))}\|\nabla\boldsymbol{b}\|^2_{L^\infty_t(\boldsymbol{L}^2(\Omega))}\|\boldsymbol{z}_t\|^2_{L^\infty_t(\boldsymbol{L}^2(\Omega))}\\ &+ K\|\nabla\boldsymbol{u}\|^2_{L^2_t(\boldsymbol{L}^2(\Omega))}\|\nabla\boldsymbol{u}\|^2_{L^\infty_t(\boldsymbol{L}^2(\Omega))}\|\nabla^2\boldsymbol{u}\|^2_{L^\infty_t(\boldsymbol{L}^2(\Omega))}.\end{aligned}$$

Let us recall that estimates (3.3), (3.8), (3.25), and (3.27) yield the following bounds:

$$\|(\sqrt{\mu}\,\nabla\boldsymbol{u}, \sqrt{\eta}\,\nabla\boldsymbol{b})\|^2_{L^2_t(\boldsymbol{L}^2(\Omega))} \leq C\,\|\boldsymbol{z}_0\|^2,$$

$$\|(\nabla\boldsymbol{u}, \nabla\boldsymbol{b})\|^2_{L^\infty_t(\boldsymbol{L}^2(\Omega))} \leq \widetilde{C}\,\|\nabla\boldsymbol{z}_0\|^2\, e^{-\overline{\gamma}\,t/\epsilon^2},$$

$$\|(\boldsymbol{u}_t, \boldsymbol{b}_t)\|^2_{L^\infty_t(\boldsymbol{L}^2(\Omega))} \leq \widetilde{C}\,\|\nabla^2\boldsymbol{z}_0\|^2$$

and

$$\|(\nabla^2\boldsymbol{u}, \nabla^2\boldsymbol{b})\|^2_{L^\infty_t(\boldsymbol{L}^2(\Omega))} \leq \widetilde{C}\,\|\nabla^2\boldsymbol{z}_0\|^2,$$

respectively. These imply that

$$\|\boldsymbol{z}_t(t)\|^2 \leq \widetilde{C}\|\nabla^2\boldsymbol{z}_0\|^2\, e^{-\widehat{\gamma}\epsilon^{-2}t} + \widetilde{C}\|\boldsymbol{z}_0\|^2\|\nabla\boldsymbol{z}_0\|^2\|\nabla^2\boldsymbol{z}_0\|^2\, e^{-\overline{\gamma}\epsilon^{-2}t},$$

where $\overline{\gamma} := \min\left\{\frac{\widetilde{\gamma}}{\mu}, \frac{\widetilde{\gamma}}{\eta}\right\}$ and $\widetilde{\gamma} := \min\left\{\frac{\mu^2}{16\beta}, \frac{\eta^2}{16}\right\}$. Moreover, by Remark 2.3, we have

$$\|\boldsymbol{z}_0\|^2\|\nabla\boldsymbol{z}_0\|^2 \leq 4c_0^4.$$

Hence,

$$\|\boldsymbol{z}_t(t)\|^2 \leq \widetilde{C}\|\nabla^2\boldsymbol{z}_0\|^2\, e^{-\widehat{\gamma}\epsilon^{-2}t} + 4\widetilde{C}c_0^4\|\nabla^2\boldsymbol{z}_0\|^2\, e^{-\overline{\gamma}\epsilon^{-2}t}.$$

Therefore, choosing $\sigma := \min\{\widehat{\gamma}, \overline{\gamma}\}$, we infer

$$\|\boldsymbol{z}_t(t)\|^2 \leq \widetilde{C}\|\nabla^2\boldsymbol{z}_0\|^2\, e^{-\sigma t/\epsilon^2},$$

which together with inequality (3.43) implies that

$$\|(\nabla^2\boldsymbol{z}, \nabla p)\|^2 \leq \widetilde{C}\|\nabla^2\boldsymbol{z}_0\|^2\, e^{-\sigma t/\epsilon^2}.$$

This proves estimates (3.28)–(3.29). The proof of Proposition 3.3 is finished. □

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A bifurcation analysis on a nonlocal overdetermined problem

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Abstract. In this paper, we study an overdetermined problem with Kirchhoff type nonlocal terms related to the celebrated work by Serrin. We obtain the precise number of solutions according to the value of the bifurcation parameter and study asymptotics of bifurcation curves of solutions when the bifurcation parameter is large in some cases.

Keywords: nonlocal elliptic equations, overdetermined problems.

2020 Mathematics Subject Classification: 34C23, 37G99.

1 Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with C^2 boundary. In this paper, we consider the following overdetermined elliptic problem with Kirchhoff type nonlocal terms:

$$\begin{cases} -A\left(\|u\|_{p_1}, \|\nabla u\|_{q_1}\right) \Delta u = \lambda B\left(\|u\|_{p_2}, \|\nabla u\|_{q_2}\right) & \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \\ \frac{\partial u}{\partial \nu} = c \quad \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $p_1, p_2, q_1, q_2 > 0$, $\lambda > 0$ are given constants, $\|\cdot\|_t$ stands for $\|\cdot\|_{L^t(\Omega)}$ for any $t > 0$, c is an unknown constant, ν is an outer unit normal to $\partial\Omega$, and $A(s,t)$, $B(s,t)$ are positive continuous functions in $(s,t) \in \mathbb{R}_+ \times \mathbb{R}_+$. The positive constant λ plays a role as a bifurcation parameter.

In a celebrated paper by Serrin [5], see also Weinberger [10], it is proven that if the overdetermined elliptic problem

$$\begin{cases} -\Delta u = 1 \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \\ \frac{\partial u}{\partial \nu} = c \quad \text{on } \partial\Omega \end{cases} \tag{1.2}$$

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admits a solution $u \in C^2(\overline{\Omega})$, then Ω must be a ball and u is radially symmetric. Thus if $\Omega = B_R(x_0)$, a ball of radius R with center x_0, then the unique solution of (1.2) must be of the form

$$U_{R,x_0}(x) = \frac{R^2 - |x - x_0|^2}{2N}. \tag{1.3}$$

By a simple computation, we have

$$\|U_{R,x_0}\|^p_{L^p(B_R(x_0))} = \left(\frac{1}{2N}\right)^p \left(\frac{\omega_{N-1}}{2}\right) R^{2p+N} B\left(\frac{N}{2}, p+1\right),$$
$$\|\nabla U_{R,x_0}\|^q_{L^q(B_R(x_0))} = \left(\frac{1}{N}\right)^q \left(\frac{\omega_{N-1}}{N+q}\right) R^{N+q}$$

for any $p, q > 0$, where ω_{N-1} is the area of the unit sphere in $\mathbb{R}^N$ and $B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt$ denotes the Beta function.

After [5, 10], many researches have been done around this topic and various generalizations (differential operators, spaces, and so on) have been obtained up to now. We refer to a survey paper [3] and the references there in. However, to the best of authors' knowledge, overdetermined problems with Kirchhoff type nonlocal terms have not been studied so far. Thus we initiate to study it in this paper.

In this paper, following an argument in [1], we prove

Theorem 1.1. *Let $A, B \in C(\mathbb{R}_+ \times \mathbb{R}_+; \mathbb{R}_+)$. Then if (1.1) admits a solution $u \in C^2(\overline{\Omega})$, then Ω must be a ball and u must be radially symmetric.*

On the other hand, consider the system of equations with respect to $(s_1, s_2, t_1, t_2) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+$:

$$\begin{cases} s_1 = \lambda \dfrac{B(s_2,t_2)}{A(s_1,t_1)} \|U_{R,x_0}\|_{L^{p_1}(B_R(x_0))} \\ s_2 = \lambda \dfrac{B(s_2,t_2)}{A(s_1,t_1)} \|U_{R,x_0}\|_{L^{p_2}(B_R(x_0))} \\ t_1 = \lambda \dfrac{B(s_2,t_2)}{A(s_1,t_1)} \|\nabla U_{R,x_0}\|_{L^{q_1}(B_R(x_0))} \\ t_2 = \lambda \dfrac{B(s_2,t_2)}{A(s_1,t_1)} \|\nabla U_{R,x_0}\|_{L^{q_2}(B_R(x_0))} \end{cases} \tag{1.4}$$

where U_{R,x_0} defined in (1.3) is the unique solution of (1.2) for $\Omega = B_R(x_0)$. Then for $\lambda > 0$, the problem (1.1) for $\Omega = B_R(x_0)$ has the same number of solutions of the system of equations (1.4). Also the number of solutions of (1.4) is the same as the number of solutions to the equation

$$g(s) = \lambda \|U_{R,x_0}\|_{L^{p_1}(B_R(x_0))} \tag{1.5}$$

with respect to $s > 0$, where

$$g(s) = \frac{sA\left(s, \dfrac{\|\nabla U_{R,x_0}\|_{L^{q_1}(B_R(x_0))}}{\|U_{R,x_0}\|_{L^{p_1}(B_R(x_0))}} s\right)}{B\left(\dfrac{\|U_{R,x_0}\|_{L^{p_2}(B_R(x_0))}}{\|U_{R,x_0}\|_{L^{p_1}(B_R(x_0))}} s, \dfrac{\|\nabla U_{R,x_0}\|_{L^{q_2}(B_R(x_0))}}{\|U_{R,x_0}\|_{L^{p_1}(B_R(x_0))}} s\right)} \tag{1.6}$$

Moreover, any solution u_λ of (1.1) is of the form

$$u_\lambda(x) = s_1 \frac{U_{R,x_0}(x)}{\|U_{R,x_0}\|_{L^{p_1}(B_R(x_0))}} \tag{1.7}$$

where s_1 is a solution of (1.5).

As a result of Theorem 1.1, many bifurcation diagrams can be obtained for specific A and B in (1.1). Next are typical examples of the result.

Corollary 1.2. *Let $\Omega = B_R(x_0)$ and let $A(s,t) = (s-a)^2 + b$, $B(s,t) = s$ in* (1.1) *for $a, b > 0$. Then*

(i) if $0 < \lambda < \frac{b}{\|U_{R,x_0}\|_{L^{p_2}(B_R(x_0))}}$, there is no solution to (1.1),

(ii) if $\lambda = \frac{b}{\|U_{R,x_0}\|_{L^{p_2}(B_R(x_0))}}$, there is a unique solution to (1.1),

(iii) if $\frac{b}{\|U_{R,x_0}\|_{L^{p_2}(B_R(x_0))}} < \lambda < \frac{(a^2+b)}{\|U_{R,x_0}\|_{L^{p_2}(B_R(x_0))}}$, there are just two solutions to (1.1),

(iv) if $\lambda \geq \frac{(a^2+b)}{\|U_{R,x_0}\|_{L^{p_2}(B_R(x_0))}}$, there is a unique solution to (1.1).

Also any solution u_λ is of the form

$$u_\lambda(x) = s_\lambda \frac{U_{R,x_0}(x)}{\|U_{R,x_0}\|_{L^{p_1}(B_R(x_0))}}, \quad x \in B_R(x_0)$$

where s_λ is a positive solution of the quadratic equation

$$(s-a)^2 + b = \lambda \|U_{R,x_0}\|_{L^{p_2}(B_R(x_0))}.$$

Corollary 1.3. *Let $\Omega = B_R(x_0)$ and let $A(s,t) = e^s$, $B(s,t) = t^r$ in* (1.1) *where $r > 1$. Then*

(i) if $0 < \lambda \|U_{R,x_0}\|_{L^{p_1}(B_R(x_0))} < \left(\frac{e}{r-1}\right)^{r-1} \left(\frac{\|U_{R,x_0}\|_{L^{p_1}(B_R(x_0))}}{\|\nabla U_{R,x_0}\|_{L^{q_2}(B_R(x_0))}}\right)^r$, there is no solution to (1.1),

(ii) if $\lambda \|U_{R,x_0}\|_{L^{p_1}(B_R(x_0))} = \left(\frac{e}{r-1}\right)^{r-1} \left(\frac{\|U_{R,x_0}\|_{L^{p_1}(B_R(x_0))}}{\|\nabla U_{R,x_0}\|_{L^{q_2}(B_R(x_0))}}\right)^r$, there is a unique solution to (1.1),

(iii) if $\lambda \|U_{R,x_0}\|_{L^{p_1}(B_R(x_0))} > \left(\frac{e}{r-1}\right)^{r-1} \left(\frac{\|U_{R,x_0}\|_{L^{p_1}(B_R(x_0))}}{\|\nabla U_{R,x_0}\|_{L^{q_2}(B_R(x_0))}}\right)^r$, there are just two solutions $u_{1,\lambda}$, $u_{2,\lambda}$ to (1.1).

Also we have

$$u_{1,\lambda}(x) = \frac{\lambda^{-\frac{1}{r-1}}}{\|\nabla U_{R,x_0}\|^{\frac{r}{r-1}}_{L^{q_2}(B_R(x_0))}} \left(1 + \frac{1}{r-1}(1+o(1))\lambda^{-\frac{1}{r-1}} \frac{\|U_{R,x_0}\|_{L^{p_1}(B_R(x_0))}}{\|\nabla U_{R,x_0}\|^{\frac{r}{r-1}}_{L^{q_2}(B_R(x_0))}}\right) U_{R,x_0}(x)$$

for $x \in B_R(x_0)$ as $\lambda \to \infty$, and

$$u_{2,\lambda}(x) = \{\log\lambda + (r-1)(\log\log\lambda)(1+o(1))\} \frac{U_{R,x_0}(x)}{\|U_{R,x_0}\|_{L^{p_1}(B_R(x_0))}}$$

for $x \in B_R(x_0)$ as $\lambda \to \infty$.

Here we mention some related works. Let $I = (-1,1) \subset \mathbb{R}$ be an interval. T. Shibata performed a bifurcation analysis on the one-dimensional Dirichlet problems with Kirchhoff type nonlocal terms of the form

$$\begin{cases} -A\left(\|u\|_{q_1}, \|u'\|_{r_1}\right) u''(x) = \lambda B\left(\|u\|_{q_2}, \|u'\|_{r_2}\right) u^p(x), & x \in I, \\ u(x) > 0, & x \in I, \\ u(\pm 1) = 0, \end{cases}$$

where $p > 1$ and $\lambda > 0$, and $\|\cdot\|_t$ stands for $\|\cdot\|_{L^t(I)}$ for any $t > 0$. We refer the readers to a series of works by T. Shibata [6–9]. Also, in the same spirit, we performed a bifurcation analysis on the nonlocal boundary blowup problem in one space dimension

$$\begin{cases} A\left(\|u\|_{q_1}, \|u'\|_{r_1}\right) u''(x) = \lambda B\left(\|u\|_{q_2}, \|u'\|_{r_2}\right) u^p(x), & x \in I, \\ u(x) > 0, & x \in I, \\ \lim_{x \to \pm 1} u(x) = +\infty, \end{cases}$$

where $p > 1$, $\lambda > 0$, $q_1, q_2 \in \left(0, \frac{p-1}{2}\right)$, and $r_1, r_2 \in \left(0, \frac{p-1}{p+1}\right)$, see [2,4]. We consider the problem (1.1) is a higher dimensional analogue of the problems above.

2 Proof of Theorem 1.1.

In this section, we prove Theorem 1.1.

Proof. First assume that there exists a solution $u \in C^2(\overline{\Omega})$ of (1.1) and put

$$v = \gamma u$$

where $\gamma > 0$ is chosen so that

$$\gamma^{-1} = \lambda \frac{B(\|u\|_{L^{p_2}(\Omega)}, \|\nabla u\|_{L^{q_2}(\Omega)})}{A(\|u\|_{L^{p_1}(\Omega)}, \|\nabla u\|_{L^{q_1}(\Omega)})}.$$

Then

$$-\Delta v(x) = \gamma(-\Delta u(x)) \overset{(1.2)}{=} \gamma\lambda \frac{B(\|u\|_{L^{p_2}(\Omega)}, \|\nabla u\|_{L^{q_2}(\Omega)})}{A(\|u\|_{L^{p_1}(\Omega)}, \|\nabla u\|_{L^{q_1}(\Omega)})} = 1$$

in Ω. Also we see $v = 0$ on $\partial\Omega$ and $\frac{\partial v}{\partial \nu} = \gamma c$ is a constant on $\partial\Omega$. Thus by the result of Serrin, we see that Ω must be a ball, say $\Omega = B_R(x_0)$ for some $R > 0$ and $x_0 \in \mathbb{R}^N$ and $v \equiv U_{R,x_0}(x)$. This implies that

$$u(x) = \gamma^{-1} U_{R,x_0}(x), \quad \nabla u(x) = \gamma^{-1} \nabla U_{R,x_0}(x). \tag{2.1}$$

Define

$$s_1 = \|u\|_{L^{p_1}(B_R(x_0))}, \quad s_2 = \|u\|_{L^{p_2}(B_R(x_0))}, \quad t_1 = \|\nabla u\|_{L^{q_1}(B_R(x_0))}, \quad t_2 = \|\nabla u\|_{L^{q_2}(B_R(x_0))}.$$

Then by (2.1), we have

$$\begin{cases} s_1 = \gamma^{-1} \|U_{R,x_0}\|_{L^{p_1}(B_R(x_0))} \\ s_2 = \gamma^{-1} \|U_{R,x_0}\|_{L^{p_2}(B_R(x_0))} \\ t_1 = \gamma^{-1} \|\nabla U_{R,x_0}\|_{L^{q_1}(B_R(x_0))} \\ t_2 = \gamma^{-1} \|\nabla U_{R,x_0}\|_{L^{q_2}(B_R(x_0))} \end{cases}$$

which is equivalent to (1.4). This shows that

$$(s_1, s_2, t_1, t_2) = (\|u\|_{L^{p_1}(B_R(x_0))}, \|u\|_{L^{p_2}(B_R(x_0))}, \|\nabla u\|_{L^{q_1}(B_R(x_0))}, \|\nabla u\|_{L^{q_2}(B_R(x_0))})$$

is a solution to (1.4) and thus

$$\sharp\{u : \text{solutions of (1.1)}\} \le \sharp\{(s_1, s_2, t_1, t_2) \in (\mathbb{R}_+)^4 : \text{solutions of (1.4)}\},$$

where $\sharp A$ denotes the cardinality of the set A.

On the other hand, let $\Omega = B_R(x_0)$ for some $R > 0$ and $x_0 \in \mathbb{R}^N$ and let $(s_1, s_2, t_1, t_2) \in (\mathbb{R}_+)^4$ be any solution to (1.4). Note that by (1.4), we see

$$\begin{aligned} \frac{s_1}{\|U_{R,x_0}\|_{L^{p_1}(B_R(x_0))}} &= \frac{s_2}{\|U_{R,x_0}\|_{L^{p_2}(B_R(x_0))}} \\ &= \frac{t_1}{\|\nabla U_{R,x_0}\|_{L^{q_1}(B_R(x_0))}} = \frac{t_2}{\|\nabla U_{R,x_0}\|_{L^{q_2}(B_R(x_0))}} \\ &= \lambda \frac{B(s_2, t_2)}{A(s_1, t_1)}. \end{aligned} \tag{2.2}$$

Thus if we define

$$u(x) = s_1 \frac{U_{R,x_0}(x)}{\|U_{R,x_0}\|_{L^{p_1}(B_R(x_0))}} \left(= s_2 \frac{U_{R,x_0}(x)}{\|U_{R,x_0}\|_{L^{p_2}(B_R(x_0))}} = t_1 \frac{U_{R,x_0}(x)}{\|\nabla U_{R,x_0}\|_{L^{q_1}(B_R(x_0))}} = t_2 \frac{U_{R,x_0}(x)}{\|\nabla U_{R,x_0}\|_{L^{q_2}(B_R(x_0))}} \right),$$

then we have $u(x) > 0$, $u = 0$, $\frac{\partial u}{\partial \nu} = const.$ on $\partial B_R(x_0)$ and

$$s_1 = \|u\|_{L^{p_1}(B_R(x_0))},\ s_2 = \|u\|_{L^{p_2}(B_R(x_0))},\ t_1 = \|\nabla u\|_{L^{q_1}(B_R(x_0))},\ t_2 = \|\nabla u\|_{L^{q_2}(B_R(x_0))}.$$

Moreover, by the definition of u, we have

$$\begin{aligned} -A\left(\|u\|_{L^{p_1}(B_R(x_0))}, \|\nabla u\|_{L^{q_1}(B_R(x_0))}\right) \Delta u(x) &= -A(s_1, t_1)\Delta u(x) \\ &= -A(s_1, t_1) \frac{s_1}{\|U_{R,x_0}\|_{L^{p_1}(B_R(x_0))}} \underbrace{\Delta U_{R,x_0}(x)}_{=-1} \\ &= A(s_1, t_1) \frac{s_1}{\|U_{R,x_0}\|_{L^{p_1}(B_R(x_0))}} \\ &\overset{(2.2)}{=} \lambda B(s_2, t_2). \end{aligned}$$

This shows that

$$\sharp\{u : \text{solutions of (1.1)}\} \ge \sharp\{(s_1, s_2, t_1, t_2) \in (\mathbb{R}_+)^4 : \text{solutions of (1.4)}\}.$$

Thus the number of solutions of (1.1) for $\Omega = B_R(x_0)$ and that of (1.4) are the same.

Also by (2.2), we can rewrite the system of equations (1.4) into a single equation for $s = s_1$

$$s_1 = \lambda \frac{B\left(\frac{\|U_{R,x_0}\|_{L^{p_2}(B_R(x_0))}}{\|U_{R,x_0}\|_{L^{p_1}(B_R(x_0))}} s_1, \frac{\|\nabla U_{R,x_0}\|_{L^{q_2}(B_R(x_0))}}{\|U_{R,x_0}\|_{L^{p_1}(B_R(x_0))}} s_1\right)}{A\left(s_1, \frac{\|\nabla U_{R,x_0}\|_{L^{q_1}(B_R(x_0))}}{\|U_{R,x_0}\|_{L^{p_1}(B_R(x_0))}} s_1\right)} \|U_{R,x_0}\|_{L^{p_1}(B_R(x_0))}$$

which is equivalent to (1.5) with $g(s)$ in (1.6). Thus the number of solutions of (1.4) for $\Omega = B_R(x_0)$ and that of (1.5) are also the same. □

Remark 2.1. Theorem 1.1 can be generalized to the overdetermined problem of the form

$$\begin{cases} -A\left(\|u\|_{\vec{p}}, \|\nabla u\|_{\vec{q}}\right)\Delta u = \lambda B\left(\|u\|_{\vec{r}}, \|\nabla u\|_{\vec{s}}\right), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \\ \frac{\partial u}{\partial \nu} = c, & \text{on } \partial\Omega \end{cases}$$

where

$$A\left(\|u\|_{\vec{p}}, \|\nabla u\|_{\vec{q}}\right) = A\left(\|u\|_{L^{p_1}(\Omega)}, \dots, \|u\|_{L^{p_I}(\Omega)}, \|\nabla u\|_{L^{q_1}(\Omega)}, \dots, \|\nabla u\|_{L^{q_J}(\Omega)}\right),$$
$$B\left(\|u\|_{\vec{r}}, \|\nabla u\|_{\vec{s}}\right) = B\left(\|u\|_{L^{r_1}(\Omega)}, \dots, \|u\|_{L^{r_K}(\Omega)}, \|\nabla u\|_{L^{s_1}(\Omega)}, \dots, \|\nabla u\|_{L^{s_L}(\Omega)}\right)$$

for $\{p_i\}_{i=1}^I, \{q_j\}_{j=1}^J, \{r_k\}_{k=1}^K, \{s_l\}_{l=1}^L \subset \mathbb{R}_+$. We treat A, B of the forms in Theorem 1.1 just for a simple presentation.

3 Proof of corollaries

In this section, we prove corollaries in §1. By Theorem 1.1, the number of solutions of (1.1) is the same as the number of solutions of (1.5) when $\Omega = B_R(x_0)$, and we have the expression of solution $u(x) = s\frac{U_{R,x_0}(x)}{\|U_{R,x_0}\|_{L^{p_1}}}$ to (1.1) where $s > 0$ is any solution of (1.5). Here and in what follows, $\|\cdot\|_p$ stands for $\|\cdot\|_{L^p(B_R(x_0))}$ for any $p > 0$. Thus we just need to solve the equation (1.5) with $\lambda > 0$ for specific A and B.

Proof of Corollary 1.2. Since $A(s,t) = \{(s-a)^2 + b\}$ and $B(s,t) = s$, the function $g(s)$ defined in (1.6) becomes

$$g(s) = \frac{\|U_{R,x_0}\|_{p_1}}{\|U_{R,x_0}\|_{p_2}}\left\{(s-a)^2 + b\right\}.$$

Thus $g(s)$ is a quadratic function with respect to $s > 0$ and

$$\min_{s>0} g(s) = g(a) = b\frac{\|U_{R,x_0}\|_{p_1}}{\|U_{R,x_0}\|_{p_2}}, \qquad g(0) = (a^2+b)\frac{\|U_{R,x_0}\|_{p_1}}{\|U_{R,x_0}\|_{p_2}}.$$

Now, the equation (1.5) reads

$$\frac{\|U_{R,x_0}\|_{p_1}}{\|U_{R,x_0}\|_{p_2}}\left\{(s-a)^2 + b\right\} = \lambda\|U_{R,x_0}\|_{p_1}.$$

By a consideration of the graph of $g(s)$, we see that the number of positive solutions s of (1.5) according to the value of $\lambda > 0$ is:

(i) 0, if $0 < \lambda\|U_{R,x_0}\|_{p_1} < g(a) = \min_{s>0} g(s)$,

(ii) 1, if $\lambda\|U_{R,x_0}\|_{p_1} = g(a)$,

(iii) 2, if $g(a) < \lambda\|U_{R,x_0}\|_{p_1} < g(0)$,

(iv) 1, if $\lambda\|U_{R,x_0}\|_{p_1} \geq g(0)$.

This completes the proof of Corollary 1.2. □

Proof of Corollary 1.3. Let $A(s,t) = e^s$ and $B(s,t) = t^r$ for $r > 1$. Then the function $g(s)$ defined in (1.6) becomes

$$g(s) = \left(\frac{\|U_{R,x_0}\|_{p_1}}{\|\nabla U_{R,x_0}\|_{q_2}} \right)^r e^s s^{1-r}.$$

A computation yields that

$$\min_{s>0} g(s) = g(r-1) = \left(\frac{\|U_{R,x_0}\|_{p_1}}{\|\nabla U_{R,x_0}\|_{q_2}} \right)^r \left(\frac{e}{r-1} \right)^{r-1}$$

and $\lim_{s\to+0} g(s) = \lim_{s\to\infty} g(s) = +\infty$. Thus the number of solutions to (1.5) is

(i) 0, if $0 < \lambda\|U_{R,x_0}\|_{p_1} < \left(\frac{\|U_{R,x_0}\|_{p_1}}{\|\nabla U_{R,x_0}\|_{q_2}} \right)^r \left(\frac{e}{r-1} \right)^{r-1}$,

(ii) 1, if $\lambda\|U_{R,x_0}\|_{p_1} = \left(\frac{\|U_{R,x_0}\|_{p_1}}{\|\nabla U_{R,x_0}\|_{q_2}} \right)^r \left(\frac{e}{r-1} \right)^{r-1}$,

(iii) 2, if $\lambda\|U_{R,x_0}\|_{p_1} > \left(\frac{\|U_{R,x_0}\|_{p_1}}{\|\nabla U_{R,x_0}\|_{q_2}} \right)^r \left(\frac{e}{r-1} \right)^{r-1}$.

Let $0 < s_{1,\lambda} < s_{2,\lambda}$ be two solutions of $g(s) = \lambda\|U_{R,x_0}\|_{p_1}$ in the case (iii). Then we see $s_{1,\lambda} \to 0$ and $s_{2,\lambda} \to \infty$ as $\lambda \to \infty$. Since

$$e^{s_{i,\lambda}} = s_{i,\lambda}^{r-1} \lambda K$$

for $i = 1, 2$, where

$$K = \frac{\|\nabla U_{R,x_0}\|_{q_2}^r}{\|U_{R,x_0}\|_{p_1}^{r-1}},$$

by taking log of the above equation, we obtain

$$s_{i,\lambda} = (r-1)\log s_{i,\lambda} + \log(\lambda K). \tag{3.1}$$

Since $s_{1,\lambda} = o(1)$ as $\lambda \to \infty$, we have

$$o(1) = (r-1)\log s_{1,\lambda} + \log(\lambda K)$$

as $\lambda \to \infty$. By solving this with respect to $s_{1,\lambda}$, we see

$$s_{1,\lambda} = (\lambda K)^{-\frac{1}{r-1}} e^{o(1)} = (\lambda K)^{-\frac{1}{r-1}} (1+\delta) \tag{3.2}$$

where $\delta \to 0$ as $\lambda \to \infty$. Inserting (3.2) into the both sides of (3.1) and computing, we have

$$(\lambda K)^{-\frac{1}{r-1}} (1+\delta) = (r-1)\log(1+\delta) = (r-1)\delta(1+o(1)).$$

From this, we have

$$(\lambda K)^{-\frac{1}{r-1}} = (r-1)\frac{\delta}{1+\delta}(1+o(1)) = (r-1)\delta(1+o(1)).$$

Thus

$$\delta = \frac{1}{r-1}(1+o(1))(\lambda K)^{-\frac{1}{r-1}} \quad (\lambda \to \infty).$$

Inserting this into (3.2), we have

$$s_{1,\lambda} = (\lambda K)^{-\frac{1}{r-1}} \left(1 + \frac{1}{r-1}(1+o(1))(\lambda K)^{-\frac{1}{r-1}} \right)$$

and recalling $u_{1,\lambda} = s_{1,\lambda}\frac{U_{R,x_0}(x)}{\|U_{R,x_0}\|_{p_1}}$, we obtain the formula for $u_{1,\lambda}$:

$$\begin{aligned} u_{1,\lambda}(x) &= (\lambda K)^{-\frac{1}{r-1}}\left(1+\frac{1}{r-1}(1+o(1))(\lambda K)^{-\frac{1}{r-1}}\right)\frac{U_{R,x_0}(x)}{\|U_{R,x_0}\|_{p_1}} \\ &= \frac{\lambda^{-\frac{1}{r-1}}}{\|\nabla U_{R,x_0}\|_{q_2}^{\frac{r}{r-1}}}\left(1+\frac{1}{r-1}(1+o(1))\lambda^{-\frac{1}{r-1}}\frac{\|U_{R,x_0}\|_{p_1}}{\|\nabla U_{R,x_0}\|_{q_2}^{\frac{r}{r-1}}}\right)U_{R,x_0}(x) \end{aligned}$$

for $x \in B_R(x_0)$.

Similarly, from (3.1), we have

$$s_{2,\lambda} = (r-1)\log s_{2,\lambda} + \log(\lambda K).$$

Now, since $s_{2,\lambda} \to \infty$ as $\lambda \to \infty$, we have

$$s_{2,\lambda}\left(1-(r-1)\underbrace{\frac{\log s_{2,\lambda}}{s_{2,\lambda}}}_{=o(1)}\right) = \log(\lambda K) = (\log\lambda)\left(1+\underbrace{\frac{\log K}{\log\lambda}}_{=o(1)}\right)$$

as $\lambda \to \infty$. From this, we have

$$s_{2,\lambda} = \frac{1}{1-o(1)}(\log\lambda)(1+o(1)) = (1+\varepsilon)\log\lambda \tag{3.3}$$

as $\lambda \to \infty$, where $\varepsilon \to 0$ as $\lambda \to \infty$. Again, inserting this into (3.1), we have

$$(1+\varepsilon)\log\lambda = (r-1)\{\log(1+\varepsilon) + \log\log\lambda\} + \log\lambda + \log K,$$

which leads to

$$\begin{aligned} \varepsilon &= (r-1)\frac{\log\log\lambda}{\log\lambda}\left\{1+\underbrace{\frac{\log(1+\varepsilon)}{\log\log\lambda}+\frac{1}{r-1}\frac{\log K}{\log\log\lambda}}_{=o(1)}\right\} \\ &= (r-1)\frac{\log\log\lambda}{\log\lambda}(1+o(1)) \end{aligned}$$

as $\lambda \to \infty$. Then coming back to (3.3) and recalling that $u_{2,\lambda} = s_{2,\lambda}\frac{U_{R,x_0}(x)}{\|U_{R,x_0}\|_{p_1}}$, we obtain the asymptotic formula for $u_{2,\lambda}$:

$$\begin{aligned} s_{2,\lambda} &= (1+\varepsilon)\log\lambda = \left(1+(r-1)\frac{\log\log\lambda}{\log\lambda}(1+o(1))\right)\log\lambda, \\ u_{2,\lambda}(x) &= \{\log\lambda + (r-1)(\log\log\lambda)(1+o(1))\}\frac{U_{R,x_0}(x)}{\|U_{R,x_0}\|_{p_1}} \end{aligned}$$

as $\lambda \to \infty$. □

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A study on the oscillation of second-order neutral differential equations in noncanonical form

This article is dedicated to the memory of my father, Müslim Tunç (1932–1978).

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Abstract. This paper analyzes the oscillation of the second-order neutral differential equation in noncanonical form

$$\left[r(t)\left((x(t)+p(t)x(\pi(t)))'\right)^{\alpha}\right]'+k(t)x^{\beta}(\eta(t))=0.$$

Using a combination of the linearization technique and the monotonicity properties of the neutral term, we derive new conditions for the studied equation to be oscillatory. Our findings provide new results applicable to linear, sublinear, superlinear, half-linear and other nonlinear forms of the studied equation, and moreover, for all these cases, the oscillation of the studied equation is attained via only one condition. Further, we illustrate the significance of our results with various examples, highlighting their superiority over known criteria in the existing literature.

Keywords: noncanonical, second-order, oscillation, neutral.

2020 Mathematics Subject Classification: 34C10, 34K11, 34K40.

1 Introduction

The purpose of the present paper is to investigate the oscillatory behavior of solutions to the second-order noncanonical differential equation with bounded and unbounded neutral coefficients

$$\left[r(t)\left((x(t)+p(t)x(\pi(t)))'\right)^{\alpha}\right]'+k(t)x^{\beta}(\eta(t))=0,\quad t\in\mathcal{I}_0:=[t_0,\infty),\tag{1.1}$$

where $t_0\in\mathbb{R}_+:=(0,\infty)$, α and β are the ratios of odd positive integers with $\alpha\geq 1$. We also assume that:

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(C_1) $r \in C(\mathcal{I}_0, \mathbb{R}_+)$ satisfies

$$\int_{t_0}^{\infty} \frac{1}{r^{1/\alpha}(t)} dt < \infty; \tag{1.2}$$

(C_2) $p, k \in C(\mathcal{I}_0, \mathbb{R}_0)$, $p(t) \geq \ell > 1$, and k does not vanish eventually;

(C_3) $\pi, \eta \in C(\mathcal{I}_0, \mathbb{R})$, $\pi(t) \leq t$, π is strictly increasing, η is nondecreasing, $\eta(t) \leq \pi(t)$, and $\lim_{t\to\infty} \pi(t) = \lim_{t\to\infty} \eta(t) = \infty$.

For simplicity, we put $z(t) := x(t) + p(t)x(\pi(t))$. Under a *solution* of (1.1), we mean a function $x \in C(\mathcal{I}_x, \mathbb{R})$, $\mathcal{I}_x := [t_x, \infty)$ for some $t_x \geq t_0$ such that $z \in C^1(\mathcal{I}_x, \mathbb{R})$, $r(z')^{\alpha} \in C^1(\mathcal{I}_x, \mathbb{R})$ and x satisfies (1.1) on $\mathcal{I}_x$. Since we are only interested in the oscillatory behavior of solutions, every solution $x(t)$ of (1.1) considered here is assumed to be continuable and nontrivial, i.e., $x(t)$ exists on some half-line $\mathcal{I}_x$ and $\sup_{t\geq T}\{|x(t)|\} > 0$ for any $T \geq t_x$. We say that a solution of (1.1) is *oscillatory* if it has an unbounded set of zeros on $\mathcal{I}_x$; it is called *nonoscillatory*, otherwise. The equation itself is called oscillatory if all its solutions oscillate.

Neutral differential equations are a type of functional differential equations in which the highest-order derivative of the unknown function appears both with and without deviating arguments. Equations of this type have attracted the interest of researchers not only from a theoretical point of view, but also because of their numerous applications in different fields. Those interested in the applications of such equations can refer to [17, 18] for some classical applications and to [7, 12] for more recent applications.

It is clearly observed from the literature review that the oscillation theory is one of the continuously developing important branches of the qualitative theory of differential equations. Its foundations are based on the pioneering work of Sturm [23] on well-known results concerning the zeros of solutions of second-order self-adjoint differential equations. Since then, oscillation criteria have been established for various classes of differential equations by researchers using different techniques and/or methods and the interesting results obtained have contributed significantly to the growth and development of this theory. There are different techniques and/or methods to establish oscillation criteria; among them, Riccati technique, integral averaging technique, comparison theorems and linearization techniques are some of the most influential; see, e.g., the books by Agarwal et al. [2,3], the papers [1,5,6,9–11,13,15,16,20–22,24] and the references quoted there.

We would like to point out that Eq. (1.1) is said to be in *noncanonical* form if (1.2) is satisfied, and Eq. (1.1) is said to be in *canonical* form if

$$\int_{t_0}^{\infty} \frac{1}{r^{1/\alpha}(t)} dt = \infty. \tag{1.3}$$

In particular, for the delay differential equation in canonical form

$$x''(t) + k(t)x(\eta(t)) = 0, \tag{1.4}$$

Koplatadze et al. [19] presented the following interesting oscillation criterion of lim sup-type

$$\limsup_{t\to\infty} \left\{ \eta(t) \int_t^{\infty} k(s)ds + \int_{\eta(t)}^{t} \eta(s)k(s)ds + \frac{1}{\eta(t)} \int_{t_0}^{\eta(t)} s\eta(s)k(s)ds \right\} > 1. \tag{1.5}$$

Later, Baculikova [4] extended the technique developed by Koplatadze et al.[19] to the second-order differential equation with deviating argument in noncanonical form

$$(r(t)x'(t))' + k(t)f(x(\eta(t))) = 0, \tag{1.6}$$

where f is nondecreasing and covers sublinear and linear cases ($f(u) = u^\beta$ with $\beta \in (0,1]$) and provided new oscillation criteria for equation (1.6).

Grace et al. [14] applied the same technique to the second-order neutral differential equation in noncanonical form

$$\left[r(t)\left(x(t)+p(t)x(\pi(t))\right)'\right]' + k(t)x^\beta(\eta(t)) = 0, \tag{1.7}$$

where $0 \le p(t) \le d < 1$ and $\beta \in (0,1]$, and derived new oscillation results for (1.7).

Tunç et al. [25] implemented the same technique to the equation with distributed deviating arguments in noncanonical form

$$\left[r(t)\left(x(t)+p(t)x(\pi(t))\right)'\right]' + \int_{c_1}^{c_2} k(t,\varrho)x^\beta(\phi(t,\varrho))d\varrho = 0, \tag{1.8}$$

where $0 < c_1 < c_2 < \infty$, $p(t) \ge \ell > 1$ and $\beta \in (0,1]$, and established novel criteria for the oscillation of (1.8).

From the above observations, it is clear that the results in [4, 14, 25] only cover the linear and sublinear cases, i.e., the results in [4, 14, 25] are only applicable to the cases ($\alpha = \beta = 1$) and ($\alpha = 1$ and $\beta < 1$); and so they provide no information for the superlinear case ($\alpha = 1$ and $\beta > 1$), half-linear case ($\beta = \alpha \ne 1$) and the other nonlinear cases such as ($1 \ne \alpha \ne \beta$) and ($\alpha > 1$ and $\beta = 1$).

Motivated by the above mentioned researches; using a combination of the linearization technique and the monotonicity properties of the neutral term, our goal here is to establish new oscillation criteria for Equation (1.1) that are applicable not only to the linear and sublinear cases, but also to the superlinear, half-linear and the other nonlinear cases the mentioned above. Furthermore, in contrast to the results reported in [14, 25], a key feature of our results is that the oscillation of the equation considered here is ensured through only one condition. It should also be noted that, as can be seen from the details in the proofs, the proofs for all these cases of α and β are not straightforward and require considerable effort. On the other hand, since we focus on the cases $1 < p(t) < \infty$ and $p(t) \to \infty$ as $t \to \infty$, our current results cannot be applied to the case $0 \le p(t) < 1$. However, with some standard calculations, the criteria here can be adjusted for the case $0 \le p(t) < 1$ as well. Additionally, a series of carefully selected examples are provided to illustrate the significance of our results and their superiority over known criteria in the existing literature.

We wish also point out that if x is a solution of (1.1), under our assumptions, $-x$ is also a solution of (1.1). Therefore, when considering nonoscillatory solutions, it is sufficient to consider only positive ones since the proofs for negative solutions are similar due to the form of (1.1).

2 Oscillation results for (1.1) in the case where $\beta \le \alpha$

In this section, we present the oscillation results for (1.1) in the case where $\beta \le \alpha$. For notational purposes, it would be convenient to set:

$$\kappa = (\ell-1)^\beta \ell^{-\beta},\ b(t) = \pi^{-1}(\eta(t)),\ A(t) = \int_{t_*}^{t} \frac{1}{r^{1/\alpha}(s)}ds,\ t_* \in \mathcal{I}_0,\ \lambda(t) = \int_t^\infty \frac{1}{r^{1/\alpha}(s)}ds.$$

Lemma 2.1. *Let x be an eventually positive solution of (1.1) on $\mathcal{I}_0$. If*

$$\int_{t_0}^\infty \lambda^\alpha(t)k(t)p^{-\beta}(b(t))dt = \infty, \tag{2.1}$$

then the corresponding function $z(t) = x(t) + p(t)x(\pi(t))$ *belongs to the class* $\mathcal{N}_0$ *eventually, where*

$$z(t) \in \mathcal{N}_0 \Leftrightarrow z(t) > 0, \quad r^{1/\alpha}(t)z'(t) < 0, \quad \left(r^{1/\alpha}z'\right)'(t) \leq 0.$$

Proof. Since x is an eventually positive solution of (1.1) on $\mathcal{I}_0$, there exists a $t_1 \in \mathcal{I}_0$ such that $x(t) > 0$ for $t \geq t_1$. In view of $\lim_{t\to\infty} \pi(t) = \lim_{t\to\infty} \eta(t) = \infty$, we can choose $t_2 \geq t_1$ such that $x(\pi(t)) > 0$ and $x(\eta(t)) > 0$ for $t \geq t_2$. Now, for $t \geq t_2$, $z(t) > 0$ and

$$\left(r(z')^{\alpha}\right)'(t) = -k(t)x^{\beta}(\eta(t)) \leq 0,$$

which implies that z' is of one sign eventually, i.e., either $z'(t) > 0$ or $z'(t) < 0$ eventually. We claim that $z'(t) < 0$ eventually. If this is not the case, then there exists a $t_3 \geq t_2$ such that $z'(t) > 0$ for $t \geq t_3$. Applying the chain-rule to $\left(r(z')^{\alpha}\right)'(t)$ yields

$$\left(r(z')^{\alpha}\right)'(t) = \alpha\left(r^{1/\alpha}(t)z'(t)\right)^{\alpha-1}\left(r^{1/\alpha}(t)z'(t)\right)'. \tag{2.2}$$

From (2.2) and the fact that α is the ratio of odd positive integers, we deduce that

$$\operatorname{sgn}\left(r^{1/\alpha}(t)z'(t)\right)' = \operatorname{sgn}\left(r(t)(z'(t))^{\alpha}\right)'.$$

Hence

$$z(t) > 0, \quad r^{1/\alpha}(t)z'(t) > 0, \quad \left(r^{1/\alpha}z'\right)'(t) \leq 0 \quad \text{for } t \geq t_3.$$

Now

$$z(t) \geq \int_{t_3}^{t} \frac{r^{1/\alpha}(s)z'(s)}{r^{1/\alpha}(s)}ds = A(t)r^{1/\alpha}(t)z'(t),$$

which implies that

$$\left(\frac{z(t)}{A(t)}\right)' \leq 0,$$

for $t \geq t_4$ for some $t_4 > t_3$, i.e., $z(t)/A(t)$ is nonincreasing for $t \geq t_4$. The definition of z leads to

$$\begin{aligned} x(t) &= \frac{1}{p(\pi^{-1}(t))}\left[z(\pi^{-1}(t)) - x(\pi^{-1}(t))\right] \\ &\geq \frac{z(\pi^{-1}(t))}{p(\pi^{-1}(t))} - \frac{z(\pi^{-1}(\pi^{-1}(t)))}{p(\pi^{-1}(t))p(\pi^{-1}(\pi^{-1}(t)))}. \end{aligned} \tag{2.3}$$

By (C_3), we see that π^{-1} is increasing, $t \leq \pi^{-1}(t)$, and

$$\pi^{-1}(t) \leq \pi^{-1}(\pi^{-1}(t)). \tag{2.4}$$

(2.4) together with the noincreasing nature of z/A yields

$$z\left(\pi^{-1}(\pi^{-1}(t))\right) \leq \frac{A\left(\pi^{-1}(\pi^{-1}(t))\right)z(\pi^{-1}(t))}{A\left(\pi^{-1}(t)\right)}. \tag{2.5}$$

Using (2.5) in (2.3) gives

$$x(t) \geq \frac{z(\pi^{-1}(t))}{p(\pi^{-1}(t))}\left[1 - \frac{A(\pi^{-1}(\pi^{-1}(t)))}{A(\pi^{-1}(t))}\frac{1}{p(\pi^{-1}(\pi^{-1}(t)))}\right] \quad \text{for } t \geq t_4. \tag{2.6}$$

By (C_2) and that

$$\lim_{t\to\infty}\frac{A(\pi^{-1}(t))}{A(t)}=1,$$

we see that there exists $t_5\geq t_4$ such that for any $\epsilon\in(0,\ell-1)$ and $t\geq t_5$,

$$\frac{A(\pi^{-1}(\pi^{-1}(t)))}{A(\pi^{-1}(t))}\frac{1}{p(\pi^{-1}(\pi^{-1}(t)))}\leq\frac{1+\epsilon}{\ell}.$$

Using this in (2.6) gives

$$x(t)\geq d\frac{z(\pi^{-1}(t))}{p(\pi^{-1}(t))}\quad\text{for } t\geq t_5, \tag{2.7}$$

where $d=1-(1+\epsilon)/\ell>0$. Substituting (2.7) into (1.1) yields

$$\left(r(z')^{\alpha}\right)'(t)+d^{\beta}k(t)p^{-\beta}(\pi^{-1}(\eta(t)))z^{\beta}\left(\pi^{-1}(\eta(t))\right)\leq 0, \tag{2.8}$$

for $t\geq t_6$. For $t\geq t_6$, we have $z(t)\geq z(t_6):=c>0$, and so inequality (2.8) leads to

$$\left(r(z')^{\alpha}\right)'(t)+c^{\beta}d^{\beta}k(t)p^{-\beta}(b(t))\leq 0\quad\text{for } t\geq t_7\geq t_6. \tag{2.9}$$

Since λ is positive, decreasing and $\lim_{t\to\infty}\lambda(t)=0$, we can choose a sufficiently large $t_*\in\mathcal{I}_0$ such that

$$0<\lambda(t)<1\quad\text{for } t\geq t_*. \tag{2.10}$$

Let $t_8=\max\{t_7,t_*\}$. Integrating (2.9) from t_7 to ∞ and taking (2.10) into account, we obtain

$$\begin{aligned}r(t_7)(z'(t_7))^{\alpha}\geq c^{\beta}d^{\beta}\int_{t_7}^{\infty}k(t)p^{-\beta}(b(t))dt&\geq c^{\beta}d^{\beta}\int_{t_8}^{\infty}k(t)p^{-\beta}(b(t))dt\\&\geq c^{\beta}d^{\beta}\int_{t_8}^{\infty}\lambda^{\alpha}(t)k(t)p^{-\beta}(b(t))dt,\end{aligned}$$

which is impossible in view of (2.1) and so $z'(t)<0$ eventually and thus the proof ends. □

Lemma 2.2. *Suppose that x is an eventually positive solution of (1.1) on $\mathcal{I}_0$. If (2.1) holds, then the following are satisfied eventually:*

(i) $$z(t)+\lambda(t)r^{1/\alpha}(t)z'(t)\geq 0;$$

(ii) $$\frac{z(t)}{\lambda(t)}\quad\textit{is nondecreasing;}$$

(iii) $$\left(\frac{z(b(t))}{r^{1/\alpha}(t)z'(t)}\right)^{\alpha-1}\geq\lambda^{\alpha-1}(t);$$

and

(iv) $$\lim_{t\to\infty}z(t)=0.$$

Proof. Pick a $t_1\in\mathcal{I}_0$ such that $x(t)>0$, $x(\pi(t))>0$ and $x(\eta(t))>0$ for $t\geq t_1$. Proceeding similarly as in the proof of Lemma 2.1, we again see that the corresponding function $z(t)$ belongs to the class $\mathcal{N}_0$ eventually, say for $t\geq t_2$, for some $t_2\geq t_1$. Since $r^{1/\alpha}z'$ is negative and decreasing,

$$z(t)\geq-\int_t^{\infty}\frac{r^{1/\alpha}(s)z'(s)}{r^{1/\alpha}(s)}ds\geq-\lambda(t)r^{1/\alpha}(t)z'(t), \tag{2.11}$$

which means that part (i) is fulfilled.

From (2.11),

$$\left(\frac{z(t)}{\lambda(t)}\right)' = \frac{\lambda(t)r^{1/\alpha}(t)z'(t) + z(t)}{\lambda^2(t)r^{1/\alpha}(t)} \geq 0,$$

i.e., part (ii) holds.

Since $b(t) \leq t$, we get

$$z(b(t)) \geq z(t). \tag{2.12}$$

By (2.11) and (2.12), we observe that

$$\left(\frac{z(b(t))}{r^{1/\alpha}(t)z'(t)}\right)^{\alpha-1} \geq \lambda^{\alpha-1}(t), \tag{2.13}$$

which proves part (iii).

Finally, we will show that part (iv) is true. From the monotonic properties of z on $[t_2, \infty)$, we can choose a constant $L_1 \geq 0$ such that

$$\lim_{t\to\infty} z(t) = L_1.$$

We will show that $L_1 = 0$. If $L_1 > 0$, then there exists a $T_1 \geq t_2$ such that for any $\epsilon > 0$,

$$L_1 < z(t) < L_1 + \epsilon \quad \text{for } t \geq T_1. \tag{2.14}$$

By (2.4) and $z'(t) < 0$,

$$z(\pi^{-1}(t)) \geq z(\pi^{-1}(\pi^{-1}(t))). \tag{2.15}$$

Using (2.15) in (2.3) leads to

$$x(t) \geq \frac{1}{p(\pi^{-1}(t))}\left[1 - \frac{1}{p(\pi^{-1}(\pi^{-1}(t)))}\right] z(\pi^{-1}(t)) \quad \text{for } t \geq t_2, \tag{2.16}$$

which together with (C_2) yields

$$x(t) \geq \kappa^{1/\beta}\frac{z(\pi^{-1}(t))}{p(\pi^{-1}(t))} \quad \text{for } t \geq t_2, \tag{2.17}$$

Substituting (2.17) into (1.1) gives

$$\left(r(z')^{\alpha}\right)'(t) + \kappa k(t)p^{-\beta}(b(t))z^{\beta}(b(t)) \leq 0, \tag{2.18}$$

for $t \geq t_3$, for some $t_3 \geq t_2$. In view of (2.2), we can rewrite (2.18) in the form

$$\left(r^{1/\alpha}z'\right)'(t) + \frac{\kappa}{\alpha}\frac{z^{\alpha-1}(b(t))}{\left(r^{1/\alpha}(t)z'(t)\right)^{\alpha-1}}k(t)p^{-\beta}(b(t))z^{\beta-\alpha+1}(b(t)) \leq 0. \tag{2.19}$$

By virtue of (2.13) and (2.19), we obtain

$$\left(r^{1/\alpha}z'\right)'(t) + \frac{\kappa}{\alpha}\lambda^{\alpha-1}(t)k(t)p^{-\beta}(b(t))z^{\beta-\alpha+1}(b(t)) \leq 0 \quad \text{for } t \geq t_3. \tag{2.20}$$

Let $t_4 = \max\{t_3, T_1\}$. Then, from (2.14) and (2.20), we observe that

$$\left(r^{1/\alpha}z'\right)'(t) + d\lambda^{\alpha-1}(t)k(t)p^{-\beta}(b(t)) \leq 0 \quad \text{for } t \geq t_4, \tag{2.21}$$

where $d = \kappa L_1^{1+\beta}/\alpha(L_1+\epsilon)^{\alpha} > 0$. Integrating (2.21) two times gives

$$\begin{aligned} z(t_4) &\geq d\int_{t_4}^{\infty} \frac{1}{r^{1/\alpha}(u)} \int_{t_4}^{u} \lambda^{\alpha-1}(s)k(s)p^{-\beta}(b(s))dsdu \\ &= d\int_{t_4}^{\infty} \lambda^{\alpha}(s)k(s)p^{-\beta}(b(s))ds = \infty, \end{aligned}$$

which is a contradiction. The contradiction obtained proves that $L_1 = 0$. The proof is over. □

Lemma 2.3. *Let x be an eventually positive solution of (1.1) on $\mathcal{I}_0$. If (2.1) holds, then the corresponding function z satisfies the linear inequalities*

$$\left(r^{1/\alpha}z'\right)'(t) + \frac{\kappa}{\alpha}\lambda^{\alpha-1}(t)k(t)p^{-\beta}(b(t))z(b(t)) \leq 0 \tag{2.22}$$

and

$$\left(z + \lambda r^{1/\alpha}z'\right)'(t) + \frac{\kappa}{\alpha}\lambda^{\alpha}(t)k(t)p^{-\beta}(b(t))z(b(t)) \leq 0, \tag{2.23}$$

eventually.

Proof. Following again the same arguments in the proof of Lemma 2.2, we arrive at (2.20) for $t \geq t_3$. Since z is positive, decreasing and $\lim_{t\to\infty} z(t) = 0$, there exists a sufficiently large $t_4 \geq t_3$ such that

$$0 < z(t) < 1 \quad \text{for } t \geq t_4. \tag{2.24}$$

By (2.24) and $\beta \leq \alpha$, we have

$$z^{\beta/\alpha}(t) \geq z(t) \quad \text{for } t \geq t_4. \tag{2.25}$$

From (2.24) and the fact that $\beta - \alpha \leq 0$, we obtain

$$z^{\beta-\alpha}(t) \geq z^{(\beta-\alpha)/\alpha}(t) \quad \text{for } t \geq t_4. \tag{2.26}$$

Using (2.26) in (2.20) gives

$$\left(r^{1/\alpha}z'\right)'(t) + \frac{\kappa}{\alpha}\lambda^{\alpha-1}(t)k(t)p^{-\beta}(b(t))z^{\beta/\alpha}(b(t)) \leq 0 \tag{2.27}$$

for $t \geq t_5$, for some $t_5 \geq t_4$. Using (2.25) in (2.27) leads to

$$\left(r^{1/\alpha}z'\right)'(t) + \frac{\kappa}{\alpha}\lambda^{\alpha-1}(t)k(t)p^{-\beta}(b(t))z(b(t)) \leq 0 \quad \text{for } t \geq t_5, \tag{2.28}$$

which proves (2.22).

From (2.28) and the fact that

$$\left(z + \lambda r^{1/\alpha}z'\right)'(t) = \lambda(t)\left(r^{1/\alpha}(t)z'(t)\right)', \tag{2.29}$$

we observe that

$$\left(z + \lambda r^{1/\alpha}z'\right)'(t) + \frac{\kappa}{\alpha}\lambda^{\alpha}(t)k(t)p^{-\beta}(b(t))z(b(t)) \leq 0 \quad \text{for } t \geq t_5, \tag{2.30}$$

i.e., (2.23) holds. The proof is complete. □

We are now ready to give the oscillation results for (1.1) in the case where $\beta \leq \alpha$ based on Lemmas 2.1–2.3.

Theorem 2.4. *If*

$$\limsup_{t\to\infty} \mathrm{W}(t) > \frac{\alpha}{\kappa}, \tag{2.31}$$

where

$$\mathrm{W}(t) := \lambda(t)\int_{t_0}^{t} \lambda^{\alpha-1}(s)k(s)p^{-\beta}(b(s))ds + \frac{1}{\lambda(b(t))}\int_{t}^{\infty} \lambda^{\alpha}(s)k(s)p^{-\beta}(b(s))\lambda(b(s))ds,$$

then equation (1.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of (1.1), say $x(t) > 0$, $x(\pi(t)) > 0$ and $x(\eta(t)) > 0$ for $t \geq t_1$, for some $t_1 \geq t_0$. It follows from (2.31) that there exists a positive constant k such that

$$\limsup_{t\to\infty} \mathrm{W}(t) \geq \mathrm{k}. \tag{2.32}$$

We now assert that (2.32) implies (2.1). Indeed, if not, then

$$\int^{\infty} \lambda^{\alpha}(t)k(t)p^{-\beta}(b(t))dt < \infty.$$

Thus, there exists a sufficiently large $t_2 \in [t_1, \infty)$ such that

$$\int_{t_2}^{\infty} \lambda^{\alpha}(t)k(t)p^{-\beta}(b(t))dt < \frac{\mathrm{k}}{4}. \tag{2.33}$$

Recalling again that λ is positive, decreasing and $\lim_{t\to\infty}\lambda(t) = 0$, we again see that there exists a sufficiently large $t_* \in [t_0, \infty)$ such that (2.10) holds. Let $t_3 = \max\{t_2, t_*\}$. Then, for $t \geq t_3$, it follows from (2.10) and (2.33) that

$$\begin{aligned}
\lambda(t)\int_{t_1}^{t} \lambda^{\alpha-1}(s)k(s)p^{-\beta}(b(s))ds &= \lambda(t)\int_{t_1}^{t_3} \lambda^{\alpha-1}(s)k(s)p^{-\beta}(b(s))ds \\
&\quad + \lambda(t)\int_{t_3}^{t} \lambda^{\alpha-1}(s)k(s)p^{-\beta}(b(s))ds \\
&\leq \lambda(t)\int_{t_1}^{t_3} \lambda^{\alpha-1}(s)k(s)p^{-\beta}(b(s))ds \\
&\quad + \int_{t_3}^{t} \lambda^{\alpha}(s)k(s)p^{-\beta}(b(s))ds \\
&\leq \lambda(t)\int_{t_1}^{t_3} \lambda^{\alpha-1}(s)k(s)p^{-\beta}(b(s))ds + \frac{\mathrm{k}}{4}.
\end{aligned} \tag{2.34}$$

Also, for $t \geq t_3$,

$$\begin{aligned}
\frac{1}{\lambda(b(t))}\int_{t}^{\infty} \lambda^{\alpha}(s)k(s)p^{-\beta}(b(s))\lambda(b(s))ds &\leq \int_{t}^{\infty} \lambda^{\alpha}(s)k(s)p^{-\beta}(b(s))ds \\
&\leq \int_{t_3}^{\infty} \lambda^{\alpha}(s)k(s)p^{-\beta}(b(s))ds < \frac{\mathrm{k}}{4}.
\end{aligned} \tag{2.35}$$

From (2.34) and (2.35), we deduce that

$$\limsup_{t\to\infty} \mathrm{W}(t) \leq \frac{\mathrm{k}}{2},$$

which contradicts (2.32), and the contradiction obtained shows that (2.32) implies (2.1). Thereby, all results of Lemmas 2.1–2.3 are fulfilled. Proceeding similarly as in proof in Lemma 2.3, we again arrive at (2.28) and (2.30) for $t \geq t_5$. Integrating (2.30) from $t(\geq t_5)$ to ∞ leads to

$$(z + \lambda r^{1/\alpha} z')(t) \geq \frac{\kappa}{\alpha} \int_t^\infty \lambda^\alpha(s) k(s) p^{-\beta}(b(s)) z(b(s)) ds. \tag{2.36}$$

An integration of (2.28) from t_5 to t gives

$$-(\lambda r^{1/\alpha} z')(t) \geq \frac{\kappa}{\alpha} \lambda(t) \int_{t_5}^t \lambda^{\alpha-1}(s) k(s) p^{-\beta}(b(s)) z(b(s)) ds. \tag{2.37}$$

From (2.36) and (2.37), we obtain

$$\begin{aligned} z(t) \geq \frac{\kappa}{\alpha} \lambda(t) \int_{t_5}^t \lambda^{\alpha-1}(s) k(s) p^{-\beta}(b(s)) z(b(s)) ds \\ + \frac{\kappa}{\alpha} \int_t^\infty \lambda^\alpha(s) k(s) p^{-\beta}(b(s)) z(b(s)) ds. \end{aligned} \tag{2.38}$$

In view of the monotonicity properties of z, b and z/λ, we obtain

$$\int_{t_5}^t \lambda^{\alpha-1}(s) k(s) p^{-\beta}(b(s)) z(b(s)) ds \geq \left(\int_{t_5}^t \lambda^{\alpha-1}(s) k(s) p^{-\beta}(b(s)) ds \right) z(t), \tag{2.39}$$

and

$$\begin{aligned} \int_t^\infty \lambda^\alpha(s) k(s) p^{-\beta}(b(s)) z(b(s)) ds &= \int_t^\infty \lambda^\alpha(s) k(s) p^{-\beta}(b(s)) \lambda(b(s)) \frac{z(b(s))}{\lambda(b(s))} ds \\ &\geq \left(\frac{1}{\lambda(b(t))} \int_t^\infty \lambda^\alpha(s) k(s) p^{-\beta}(b(s)) \lambda(b(s)) ds \right) z(t). \end{aligned} \tag{2.40}$$

Using (2.39) and (2.40) in (2.38), we arrive at

$$\frac{\alpha}{\kappa} \geq \lambda(t) \int_{t_5}^t \lambda^{\alpha-1}(s) k(s) p^{-\beta}(b(s)) ds + \frac{1}{\lambda(b(t))} \int_t^\infty \lambda^\alpha(s) k(s) p^{-\beta}(b(s)) \lambda(b(s)) ds.$$

Taking lim sup as $t \to \infty$ in the latter inequality, we obtain a contradiction with (2.31). This completes the proof. □

Theorem 2.5. *If*

$$\limsup_{t \to \infty} \mathrm{H}(t) > \frac{\alpha}{\kappa}, \tag{2.41}$$

where

$$\begin{aligned} \mathrm{H}(t) := \lambda(b(t)) \int_{t_0}^{b(t)} \lambda^{\alpha-1}(s) k(s) p^{-\beta}(b(s)) ds + \int_{b(t)}^t \lambda^\alpha(s) k(s) p^{-\beta}(b(s)) ds \\ + \frac{1}{\lambda(b(t))} \int_t^\infty \lambda^\alpha(s) k(s) p^{-\beta}(b(s)) \lambda(b(s)) ds, \end{aligned}$$

then (1.1) *oscillates.*

Proof. Suppose that $x(t)$ is a nonoscillatory solution of (1.1) such that $x(t) > 0$, $x(\pi(t)) > 0$ and $x(\eta(t)) > 0$ for $t \geq t_1$, for some $t_1 \geq t_0$. As in the proof of Theorem 2.4, it is not difficult to see that (2.1) holds. Thereby, all results of Lemmas 2.1–2.3 are fulfilled. Using exactly the same arguments as in the proof of Theorem 2.4, we see that (2.38) holds and leads to

$$z(b(t)) \geq \frac{\kappa}{\alpha}\lambda(b(t)) \int_{t_5}^{b(t)} \lambda^{\alpha-1}(s)k(s)p^{-\beta}(b(s))z(b(s))ds + \frac{\kappa}{\alpha}\int_{b(t)}^{t} \lambda^{\alpha}(s)k(s)p^{-\beta}(b(s))z(b(s))ds + \frac{\kappa}{\alpha}\int_{t}^{\infty} \lambda^{\alpha}(s)k(s)p^{-\beta}(b(s))z(b(s))ds.$$

In view of the monotonicity properties of z and z/λ, we conclude from the latter inequality that

$$\frac{\alpha}{\kappa} \geq \lambda(b(t)) \int_{t_5}^{b(t)} \lambda^{\alpha-1}(s)k(s)p^{-\beta}(b(s))ds + \int_{b(t)}^{t} \lambda^{\alpha}(s)k(s)p^{-\beta}(b(s))ds + \frac{1}{\lambda(b(t))}\int_{t}^{\infty} \lambda^{\alpha}(s)k(s)p^{-\beta}(b(s))\lambda(b(s))ds.$$

The remainder of the proof follows from that of Theorem 2.4. □

Theorem 2.6. *If*

$$\limsup_{t\to\infty} \int_{b(t)}^{t} \lambda^{\alpha}(s)k(s)p^{-\beta}(b(s))ds > \frac{\alpha}{\kappa}, \tag{2.42}$$

then equation (1.1) *is oscillatory.*

Proof. Suppose that $x(t)$ is a nonoscillatory solution of (1.1) such that $x(t) > 0$, $x(\pi(t)) > 0$ and $x(\eta(t)) > 0$ for $t \geq t_1$, for some $t_1 \geq t_0$. It is obvious that (2.42) implies (2.1). Thereby, all results of Lemmas 2.1–2.3 are fulfilled. By Lemma 2.3, we again have (2.30). Setting

$$y(t) := z(t) + \lambda(t)r^{1/\alpha}(t)z'(t),$$

and taking $0 \leq y(t) \leq z(t)$ into account, we observe from (2.30) that

$$y'(t) + \frac{\kappa}{\alpha}\lambda^{\alpha}(t)k(t)p^{-\beta}(b(t))y(b(t)) \leq 0.$$

Integrating the latter inequality from $b(t)$ to t gives

$$y(b(t)) \geq \frac{\kappa}{\alpha}\int_{b(t)}^{t} \lambda^{\alpha}(s)k(s)p^{-\beta}(b(s))y(b(s))ds \geq \left(\frac{\kappa}{\alpha}\int_{b(t)}^{t} \lambda^{\alpha}(s)k(s)p^{-\beta}(b(s))ds\right) y(b(t)),$$

from which

$$\frac{\alpha}{\kappa} \geq \int_{b(t)}^{t} \lambda^{\alpha}(s)k(s)p^{-\beta}(b(s))ds.$$

Taking the lim sup as $t \to \infty$, we find a contradiction to (2.42). The proof is complete. □

3 Oscillation results for (1.1) in the case where $\beta > \alpha$

In this section, we will establish the oscillation criteria for the case when $\beta > \alpha$. While in the previous section we have established the oscillation criteria via $z(t) \to 0$ as $t \to \infty$; in this section we will establish the oscillation criteria via $z(t)/\lambda(t) \to \infty$ as $t \to \infty$.

Lemma 3.1. *Let x be an eventually positive solution of (1.1) on $\mathcal{I}_0$. If*

$$\int_{t_0}^{\infty} \lambda^{\beta}(t)k(t)p^{-\beta}(b(t))dt = \infty, \tag{3.1}$$

then $z(t) \in \mathcal{N}_0$ and

$$\lim_{t\to\infty} \frac{z(t)}{\lambda(t)} = \infty.$$

Proof. Recalling again the monotonic properties of λ, we again see that there exists a sufficiently large $t_* \in [t_0, \infty)$ such that (2.10) holds. By (2.10) and $\beta > \alpha$, we have

$$\lambda^{\beta}(t) \leq \lambda^{\alpha}(t) \quad \text{for } t \geq t_*, \tag{3.2}$$

from which,

$$\lambda^{\beta}(t)k(t)p^{-\beta}(b(t)) \leq \lambda^{\alpha}(t)k(t)p^{-\beta}(b(t)). \tag{3.3}$$

Now, in view of (3.3), it is clear that (3.1) implies (2.1). So, all results of Lemma 2.1 and lemma 2.2 are fulfilled. Following similar arguments as in the proof of Lemma 2.2, we see that (2.18) holds. Since $\lambda(t) \to 0$ as $t \to \infty$ and $z(t) \to 0$ as $t \to \infty$, by L'Hôpital's rule,

$$\lim_{t\to\infty} \frac{z(t)}{\lambda(t)} = \lim_{t\to\infty} -r^{1/\alpha}(t)z'(t).$$

We now will show that $\lim_{t\to\infty} -r^{1/\alpha}(t)z'(t) = \infty$. Suppose to the contrary that positive and increasing function $-r^{1/\alpha}(t)z'(t)$ has finite limit. This assumption implies that there is a constant $b_1 > 0$ such that

$$-r^{1/\alpha}(t)z'(t) \leq b_1 < \infty.$$

This together with an integration of (2.18) from t_3 to t yields

$$\begin{aligned} b_1^{\alpha} \geq -r(t)(z'(t))^{\alpha} &\geq \kappa \int_{t_3}^{t} k(s)p^{-\beta}(b(s))z^{\beta}(b(s))ds \\ &\geq \kappa \int_{t_3}^{t} \lambda^{\beta}(s)k(s)p^{-\beta}(b(s)) \left(\frac{z(s)}{\lambda(s)}\right)^{\beta} ds \\ &\geq \kappa \left(\frac{z(t_3)}{\lambda(t_3)}\right)^{\beta} \int_{t_3}^{t} \lambda^{\beta}(s)k(s)p^{-\beta}(b(s))ds, \end{aligned}$$

which contradicts (3.1) and we deduce that

$$\lim_{t\to\infty} \frac{z(t)}{\lambda(t)} = \lim_{t\to\infty} -r^{1/\alpha}(t)z'(t) = \infty. \qquad \square$$

Lemma 3.2. *Let x be an eventually positive solution of (1.1) on $\mathcal{I}_0$. If (3.1) holds, then*

$$\left(r^{1/\alpha}z'\right)'(t) + \frac{\kappa}{\alpha}\lambda^{\beta}(t)k(t)p^{-\beta}(b(t))\frac{z(b(t))}{\lambda(b(t))} \leq 0 \tag{3.4}$$

and

$$\left(z + \lambda r^{1/\alpha}z'\right)'(t) + \frac{\kappa}{\alpha}\lambda^{\beta+1}(t)k(t)p^{-\beta}(b(t))\frac{z(b(t))}{\lambda(b(t))} \leq 0, \tag{3.5}$$

eventually.

Proof. Since (3.1) implies (2.1), we again see that all the results of Lemma 2.1, Lemma 2.2 and Lemma 3.1 are fulfilled. Following similar arguments as in the proof of Lemma 2.2, we see that (2.20) for $t \geq t_3$ is satisfied. Since $z(t)/\lambda(t) \to \infty$ as $t \to \infty$, there exists a sufficiently large $T_1 \in [t_0, \infty)$ such that

$$\frac{z(t)}{\lambda(t)} \geq 1 \quad \text{for } t \geq T_1. \tag{3.6}$$

By (3.6) and $\beta > \alpha$,

$$\left(\frac{z(t)}{\lambda(t)}\right)^{\beta/\alpha} \geq \frac{z(t)}{\lambda(t)} \quad \text{for } t \geq T_1, \tag{3.7}$$

and

$$\beta - \alpha \geq \frac{\beta - \alpha}{\alpha}. \tag{3.8}$$

Now (3.6) and (3.8) leads to

$$\left(\frac{z(t)}{\lambda(t)}\right)^{\beta-\alpha} \geq \left(\frac{z(t)}{\lambda(t)}\right)^{(\beta-\alpha)/\alpha} \quad \text{for } t \geq T_1. \tag{3.9}$$

Since $\lim_{t\to\infty} b(t) = \infty$, we can choose $T_2 \geq T_1$ such that $b(t) \geq T_1$ for $t \geq T_2$ and so from (3.9)

$$\left(\frac{z(b(t))}{\lambda(b(t))}\right)^{\beta-\alpha} \geq \left(\frac{z(b(t))}{\lambda(b(t))}\right)^{(\beta-\alpha)/\alpha} \quad \text{for } t \geq T_2. \tag{3.10}$$

Let $t_4 = \max\{t_3, T_2\}$. Using (3.10) in (2.20) gives

$$\left(r^{1/\alpha} z'\right)'(t) + \frac{\kappa}{\alpha}\lambda^{\alpha-1}(t)k(t)p^{-\beta}(b(t))\lambda^{\beta-\alpha+1}(b(t))\left(\frac{z(b(t))}{\lambda(b(t))}\right)^{\beta/\alpha} \leq 0 \quad \text{for } t \geq t_4. \tag{3.11}$$

Using that λ is decreasing, $b(t) \leq t$ and $\beta > \alpha$, it follows from (3.11) that

$$\left(r^{1/\alpha} z'\right)'(t) + \frac{\kappa}{\alpha}\lambda^{\beta}(t)k(t)p^{-\beta}(b(t))\left(\frac{z(b(t))}{\lambda(b(t))}\right)^{\beta/\alpha} \leq 0 \quad \text{for } t \geq t_4. \tag{3.12}$$

Using (3.7) in (3.12) yields

$$\left(r^{1/\alpha} z'\right)'(t) + \frac{\kappa}{\alpha}\lambda^{\beta}(t)k(t)p^{-\beta}(b(t))\frac{z(b(t))}{\lambda(b(t))} \leq 0 \quad \text{for } t \geq t_4, \tag{3.13}$$

which proves (3.4). Recalling (2.29) again, we can rewrite (3.13) in the form

$$\left(z + \lambda r^{1/\alpha} z'\right)'(t) + \frac{\kappa}{\alpha}\lambda^{\beta+1}(t)k(t)p^{-\beta}(b(t))\frac{z(b(t))}{\lambda(b(t))} \leq 0 \quad \text{for } t \geq t_4, \tag{3.14}$$

which proves (3.5). The proof is complete. □

Theorem 3.3. *If*

$$\limsup_{t\to\infty} \mathrm{K}(t) > \frac{\alpha}{\kappa}, \tag{3.15}$$

where

$$\mathrm{K}(t) := \lambda(t)\int_{t_0}^{t} \lambda^{\beta}(s)k(s)p^{-\beta}(b(s))\lambda^{-1}(b(s))ds + \frac{1}{\lambda(b(t))}\int_{t}^{\infty} \lambda^{\beta+1}(s)k(s)p^{-\beta}(b(s))ds,$$

then (1.1) *oscillates.*

Proof. Pick $t_1 \in \mathcal{I}_0$ such that $x(t) > 0$, $x(\pi(t)) > 0$ and $x(\eta(t)) > 0$ for $t \geq t_1$. From (3.15), we can choose a $\delta > 0$ such that

$$\limsup_{t\to\infty} \mathrm{K}(t) \geq \delta. \tag{3.16}$$

We now claim that (3.16) implies (3.1). Indeed, if not, then

$$\int^{\infty} \lambda^{\beta}(t)k(t)p^{-\beta}(b(t))dt < \infty.$$

Thus, for a sufficiently large $t_2 \in [t_1, \infty)$,

$$\int_{t_2}^{\infty} \lambda^{\beta}(t)k(t)p^{-\beta}(b(t))dt < \frac{\delta}{4}. \tag{3.17}$$

Recalling again the monotonicity properties of λ, we see that there exists a sufficiently large $t_* \in [t_0, \infty)$ such that (2.10) holds. Let $t_3 = \max\{t_2, t_*\}$. Then, for $t \geq t_3$, (2.10) and (3.17) yields

$$\begin{aligned}
\lambda(t)\int_{t_1}^{t} \lambda^{\beta}(s)k(s)p^{-\beta}(b(s))\lambda^{-1}(b(s))ds &= \lambda(t)\int_{t_1}^{t_3} \lambda^{\beta}(s)k(s)p^{-\beta}(b(s))\lambda^{-1}(b(s))ds \\
&\quad + \lambda(t)\int_{t_3}^{t} \lambda^{\beta}(s)k(s)p^{-\beta}(b(s))\lambda^{-1}(b(s))ds \\
&\leq \lambda(t)\int_{t_1}^{t_3} \lambda^{\beta}(s)k(s)p^{-\beta}(b(s))\lambda^{-1}(b(s))ds \\
&\quad + \int_{t_3}^{t} \lambda^{\beta}(s)k(s)p^{-\beta}(b(s))ds \\
&\leq \lambda(t)\int_{t_1}^{t_3} \lambda^{\beta}(s)k(s)p^{-\beta}(b(s))\lambda^{-1}(b(s))ds + \frac{\delta}{4}.
\end{aligned} \tag{3.18}$$

Also, for $t \geq t_3$,

$$\begin{aligned}
\frac{1}{\lambda(b(t))}\int_{t}^{\infty} \lambda^{\beta+1}(s)k(s)p^{-\beta}(b(s))ds &\leq \int_{t}^{\infty} \lambda^{\beta}(s)k(s)p^{-\beta}(b(s))ds \\
&\leq \int_{t_3}^{\infty} \lambda^{\beta}(s)k(s)p^{-\beta}(b(s))ds < \frac{\delta}{4}.
\end{aligned} \tag{3.19}$$

From (3.18) and (3.19), we conclude that

$$\limsup_{t\to\infty} \mathrm{K}(t) \leq \frac{\delta}{2},$$

which contradicts (3.16). The contradiction obtained shows that (3.1) holds, and hence all results of Lemma 3.1 and Lemma 3.2 are fulfilled. Next, proceeding similarly as in proof in Lemma 3.2, we arrive at (3.13) and (3.14) for $t \geq t_4$. Integrating (3.14) from t to ∞ yields

$$\left(z + \lambda r^{1/\alpha} z'\right)(t) \geq \frac{\kappa}{\alpha}\int_{t}^{\infty} \lambda^{\beta+1}(s)k(s)p^{-\beta}(b(s))\frac{z(b(s))}{\lambda(b(s))}ds. \tag{3.20}$$

Integrating (3.13) from t_4 to t and then multiplying by $\lambda(t)$, we get

$$-\left(\lambda r^{1/\alpha} z'\right)(t) \geq \frac{\kappa}{\alpha}\lambda(t)\int_{t_4}^{t} \lambda^{\beta}(s)k(s)p^{-\beta}(b(s))\frac{z(b(s))}{\lambda(b(s))}ds. \tag{3.21}$$

It follows from (3.20) and (3.21) that

$$z(t) \geq \frac{\kappa}{\alpha}\lambda(t)\int_{t_4}^{t}\lambda^{\beta}(s)k(s)p^{-\beta}(b(s))\frac{z(b(s))}{\lambda(b(s))}ds + \frac{\kappa}{\alpha}\int_{t}^{\infty}\lambda^{\beta+1}(s)k(s)p^{-\beta}(b(s))\frac{z(b(s))}{\lambda(b(s))}ds. \tag{3.22}$$

In view of the monotonicity properties of z, b and z/λ, we obtain

$$\int_{t_4}^{t}\lambda^{\beta}(s)k(s)p^{-\beta}(b(s))\frac{z(b(s))}{\lambda(b(s))}ds \geq \left(\int_{t_4}^{t}\lambda^{\beta}(s)k(s)p^{-\beta}(b(s))\lambda^{-1}(b(s))ds\right)z(b(t)), \tag{3.23}$$

and

$$\int_{t}^{\infty}\lambda^{\beta+1}(s)k(s)p^{-\beta}(b(s))\frac{z(b(s))}{\lambda(b(s))}ds \geq \left(\frac{1}{\lambda(b(t))}\int_{t}^{\infty}\lambda^{\beta+1}(s)k(s)p^{-\beta}(b(s))ds\right)z(b(t)). \tag{3.24}$$

Using (3.23) and (3.24) in (3.22) yields

$$\frac{\alpha}{\kappa}z(t) \geq \left(\lambda(t)\int_{t_4}^{t}\lambda^{\beta}(s)k(s)p^{-\beta}(b(s))\lambda^{-1}(b(s))ds\right)z(b(t)) + \left(\frac{1}{\lambda(b(t))}\int_{t}^{\infty}\lambda^{\beta+1}(s)k(s)p^{-\beta}(b(s))ds\right)z(b(t)). \tag{3.25}$$

Using that z is decreasing and $b(t) \leq t$, we deduce from (3.25) that

$$\frac{\alpha}{\kappa} \geq \lambda(t)\int_{t_4}^{t}\lambda^{\beta}(s)k(s)p^{-\beta}(b(s))\lambda^{-1}(b(s))ds + \frac{1}{\lambda(b(t))}\int_{t}^{\infty}\lambda^{\beta+1}(s)k(s)p^{-\beta}(b(s))ds.$$

Now take the lim sup as $t \to \infty$, we get a contradiction to (3.15) and complete the proof. □

Theorem 3.4. *If*

$$\limsup_{t\to\infty} \mathrm{V}(t) > \frac{\alpha}{\kappa}, \tag{3.26}$$

where

$$\mathrm{V}(t) := \lambda(b(t))\int_{t_0}^{b(t)}\lambda^{\beta}(s)k(s)p^{-\beta}(b(s))\lambda^{-1}(b(s))ds + \int_{b(t)}^{t}\lambda^{\beta+1}(s)k(s)p^{-\beta}(b(s))\lambda^{-1}(b(s))ds + \frac{1}{\lambda(b(t))}\int_{t}^{\infty}\lambda^{\beta+1}(s)k(s)p^{-\beta}(b(s))ds,$$

then equation (1.1) *is oscillatory.*

Proof. Pick $t_1 \in \mathcal{I}_0$ such that $x(t) > 0$, $x(\pi(t)) > 0$ and $x(\eta(t)) > 0$ for $t \geq t_1$. As in the proof of Theorem 3.3, it is not difficult to see that (3.1). Thus, as in the proof of Theorem 3.3, we see that (3.22) holds and leads to

$$z(b(t)) \geq \frac{\kappa}{\alpha}\lambda(b(t))\int_{t_4}^{b(t)}\lambda^{\beta}(s)k(s)p^{-\beta}(b(s))\frac{z(b(s))}{\lambda(b(s))}ds + \frac{\kappa}{\alpha}\int_{b(t)}^{t}\lambda^{\beta+1}(s)k(s)p^{-\beta}(b(s))\frac{z(b(s))}{\lambda(b(s))}ds + \frac{\kappa}{\alpha}\int_{t}^{\infty}\lambda^{\beta+1}(s)k(s)p^{-\beta}(b(s))\frac{z(b(s))}{\lambda(b(s))}ds. \tag{3.27}$$

Using the monotonic properties of z and z/λ, we deduce from (3.27) that

$$z(b(t)) \geq \left(\frac{\kappa}{\alpha}\lambda(b(t))\int_{t_4}^{b(t)}\lambda^{\beta}(s)k(s)p^{-\beta}(b(s))\lambda^{-1}(b(s))ds\right)z(b(t)) + \left(\frac{\kappa}{\alpha}\int_{b(t)}^{t}\lambda^{\beta+1}(s)k(s)p^{-\beta}(b(s))\lambda^{-1}(b(s))ds\right)z(b(t)) + \left(\frac{\kappa}{\alpha}\int_{t}^{\infty}\lambda^{\beta+1}(s)k(s)p^{-\beta}(b(s))ds\right)\frac{z(b(t))}{\lambda(b(t))},$$

from which

$$\frac{\alpha}{\kappa} \geq \lambda(b(t))\int_{t_4}^{b(t)}\lambda^{\beta}(s)k(s)p^{-\beta}(b(s))\lambda^{-1}(b(s))ds + \int_{b(t)}^{t}\lambda^{\beta+1}(s)k(s)p^{-\beta}(b(s))\lambda^{-1}(b(s))ds + \frac{1}{\lambda(b(t))}\int_{t}^{\infty}\lambda^{\beta+1}(s)k(s)p^{-\beta}(b(s))ds.$$

The rest of the proof is as that of Theorem 3.3. □

Theorem 3.5. *If*

$$\limsup_{t\to\infty}\int_{b(t)}^{t}\lambda^{\beta+1}(s)k(s)p^{-\beta}(b(s))\lambda^{-1}(b(s))ds > \frac{\alpha}{\kappa}, \tag{3.28}$$

then (1.1) *is oscillatory.*

Proof. Pick $t_1 \in \mathcal{I}_0$ such that $x(t) > 0$, $x(\pi(t)) > 0$ and $x(\eta(t)) > 0$ for $t \geq t_1$. It is clear that (3.28) implies that

$$\int_{t_0}^{\infty}\lambda^{\beta+1}(s)k(s)p^{-\beta}(b(s))\lambda^{-1}(b(s))ds = \infty. \tag{3.29}$$

Using the monotonicity properties of λ and b, we see that

$$\int_{b(t)}^{t}\lambda^{\beta+1}(s)k(s)p^{-\beta}(b(s))\lambda^{-1}(b(s))ds \leq \int_{b(t)}^{t}\lambda^{\beta}(s)k(s)p^{-\beta}(b(s))ds. \tag{3.30}$$

In view of (3.29) and (3.30), we observe that (3.28) implies (3.1). Thereby, all results of Lemmas 3.1–3.2 are fulfilled. Proceeding as in the proof of Lemma 3.2, we again arrive at (3.14). Integrating (3.14) from $b(t)$ to t yields

$$z(b(t)) + \lambda(b(t))r^{1/\alpha}(b(t))z'(b(t)) \geq \frac{\kappa}{\alpha}\int_{b(t)}^{t}\lambda^{\beta+1}(s)k(s)p^{-\beta}(b(s))\frac{z(b(s))}{\lambda(b(s))}ds \geq \left(\frac{\kappa}{\alpha}\int_{b(t)}^{t}\lambda^{\beta+1}(s)k(s)p^{-\beta}(b(s))\lambda^{-1}(b(s))ds\right)z(b(t)),$$

which, together with $z'(t) < 0$, gives

$$z(b(t)) \geq \left(\frac{\kappa}{\alpha}\int_{b(t)}^{t}\lambda^{\beta+1}(s)k(s)p^{-\beta}(b(s))\lambda^{-1}(b(s))ds\right)z(b(t)). \tag{3.31}$$

We now observe from (3.31) that

$$\frac{\alpha}{\kappa} \geq \int_{b(t)}^{t}\lambda^{\beta+1}(s)k(s)p^{-\beta}(b(s))\lambda^{-1}(b(s))ds.$$

Taking the lim sup as $t \to \infty$, we get a contradiction to (3.28) and ends the proof. □

4 Examples

Now the applicability of all theorems presented in this article will be demonstrated with examples for all cases of α and β, i.e., for $\alpha = \beta$, $\beta > \alpha$ and $\beta < \alpha$, respectively.

Example 4.1. Let us consider the half-linear equation

$$\left(t^{10}\left(\left(x(t)+2x\left(\frac{t}{2}\right)\right)'\right)^5\right)' + at^4x^5\left(\frac{t}{4}\right) = 0, \quad t \geq 1. \tag{4.1}$$

Here, $r(t) = t^{10}$, $p(t) = 2$, $\pi(t) = t/2$, $\eta(t) = t/4$, $\alpha = \beta = 5$, and $k(t) = at^4$ with $a > 0$. Then

$$\lambda(t) = 1/t, \quad \pi^{-1}(t) = 2t, \quad b(t) = t/2, \text{ and } \quad \lambda(b(t)) = 2/t.$$

Now, condition (2.31) becomes

$$\limsup_{t\to\infty} \mathrm{W}(t) = \frac{a}{16},$$

condition (2.41) becomes

$$\limsup_{t\to\infty} \mathrm{H}(t) = a2^{-5}\left(2+\ln 2\right),$$

condition (2.42) becomes

$$\limsup_{t\to\infty} \int_{t/2}^{t} \frac{a2^{-5}}{s} ds = a2^{-5}\ln 2.$$

Next, by applying Theorems 2.4–2.6 to equation (4.1), we deduce that (4.1) is oscillatory by Theorem 2.4 if $a > 2560$, Eq. (4.1) oscillates by Theorem 2.5 if $a > 1901.4$, and (4.1) is oscillatory by Theorem 2.6 if $a > 7386$.

Note that none of the results in [4,14,25] can be applied to equation (4.1) since the equation is half-linear and ($\alpha = \beta = 5$). Also, our findings required only one condition to obtain the result, but the approach in [14,25] requires two conditions.

Example 4.2. Consider the superlinear equation

$$\left(t^2\left(x(t)+4tx\left(\frac{t}{6}\right)\right)'\right)' + at^5x^3\left(\frac{t}{8}\right) = 0, \quad t \geq 1. \tag{4.2}$$

Here $r(t) = t^2$, $p(t) = 4t$, $\pi(t) = t/6$, $\eta(t) = t/8$, $\alpha = 1$, $\beta = 3$, and $k(t) = at^5$ with $a > 0$. Then

$$\lambda(t) = 1/t, \quad \pi^{-1}(t) = 6t, \quad b(t) = 3t/4, \quad \text{and} \quad \lambda(b(t)) = 4/3t.$$

Now, condition (3.15) becomes

$$\limsup_{t\to\infty} \mathrm{K}(t) = \frac{a}{18},$$

condition (3.26) becomes

$$\limsup_{t\to\infty} \mathrm{V}(t) = \frac{a}{36}(2+\ln\frac{4}{3}),$$

condition (3.28) becomes

$$\limsup_{t\to\infty} \int_{3t/4}^{t} \frac{a}{36s} ds = \frac{a}{36}\ln\frac{4}{3}.$$

Next, by applying Theorems 3.3–3.5 to equation (4.2), we deduce that (4.2) is oscillatory by Theorem 3.3 if $a > 128/3 = 42.667$, Eq. (4.2) oscillates by Theorem 3.4 if $a > 256/3(2 + \ln 4/3) = 37.301$, and (4.2) is oscillatory by Theorem 3.5 if $a > 296.62$.

Note that none of the results in [4,14,25] can be applied to equations (4.2) since the equation is superlinear ($\beta = 3 > \alpha = 1$).

Example 4.3. Consider the nonlinear equation

$$\left(t^{10}\left(\left(x(t)+10x\left(\frac{t}{2}\right)\right)'\right)^5\right)' + at^6x^7\left(\frac{t}{4}\right) = 0, \quad t \geq 1. \tag{4.3}$$

Here, $r(t) = t^{10}$, $p(t) = 10$, $\pi(t) = t/2$, $\eta(t) = t/4$, $\alpha = 5$, $\beta = 7$, and $k(t) = at^6$. Then

$$\lambda(t) = 1/t, \quad \pi^{-1}(t) = 2t, \quad b(t) = t/2, \quad \text{and} \quad \lambda(b(t)) = 2/t.$$

Now, condition (3.15) becomes

$$\limsup_{t\to\infty} \mathrm{K}(t) = a10^{-7},$$

condition (3.26) becomes

$$\limsup_{t\to\infty} \mathrm{V}(t) = a10^{-7}2^{-1}(2+\ln 2),$$

condition (3.28) becomes

$$\limsup_{t\to\infty} \int_{t/2}^{t} \frac{a10^{-7}}{2s} ds = a10^{-7}2^{-1}\ln 2.$$

Next, by applying Theorems 3.3–3.5 to equation (4.3), we deduce that (4.3) is oscillatory by Theorem 3.3 if $a > 10^{15}/2 \times 9^7$, Eq. (4.3) oscillates by Theorem 3.4 if $a > 10^{15}/9^7(2+\ln 2)$, and (4.3) is oscillatory by Theorem 3.5 if $a > 10^{15}/9^7 \ln 2$.

Example 4.4. Consider the nonlinear equation

$$\left(t^{6}\left(\left(x(t)+2tx\left(\frac{t}{3}\right)\right)'\right)^3\right)' + at^3x\left(\frac{t}{6}\right) = 0, \quad t \geq 1. \tag{4.4}$$

Then

$$\lambda(t) = 1/t, \quad \pi^{-1}(t) = 3t, \quad b(t) = t/2, \text{ and } \quad \lambda(b(t)) = 2/t.$$

Now, condition (2.31) becomes

$$\limsup_{t\to\infty} \mathrm{W}(t) = 2a,$$

condition (2.41) becomes

$$\limsup_{t\to\infty} \mathrm{H}(t) = a(2+\ln 2),$$

(2.42) becomes

$$\limsup_{t\to\infty} \int_{t/2}^{t} \frac{a}{s} ds = a\ln 2.$$

Next, by applying Theorems 2.4–2.6 to equation (4.4), we deduce that (4.4) is oscillatory by Theorem 2.4 if $a > 3$, Eq. (4.4) oscillates by Theorem 2.5 if $a > 2.2279$, and (4.4) is oscillatory by Theorem 2.6 if $a > 8.6562$.

Note that none of the results in [4, 14, 25] can be applied to equations (4.3) and (4.4) since ($\alpha = 5$ and $\beta = 7$) and ($\alpha = 3$ and $\beta = 1$), respectively.

Remark 4.5. It would be of interest to extend the results in [8] to equations of type (1.1) using the technique in [8].

5 Conclusion

This study analyzed the oscillation of solutions to a class of second-order differential equations with bounded and unbounded neutral coefficients in noncanonical form. The oscillation results were obtained by a combination of the linearization technique and the monotonicity properties of the neutral term. Our findings advance the current understanding in this area by providing new results that can be applied to linear ($\alpha = \beta = 1$), superlinear ($\alpha = 1$ and $\beta > 1$), sublinear ($\alpha = 1$ and $\beta < 1$), half-linear ($\alpha = \beta \neq 1$) and the other nonlinear [$(1 \neq \alpha \neq \beta)$ and $(\alpha > 1$ and $\beta = 1)$] forms of the equation under study. Moreover, the oscillation results for all these cases are established by means of only one condition. Hence, results designed specifically for different classes are important in their own right and have both weak and strong points. In some cases, they improve and extend some existing results in the literature; and in some cases, they are interesting in their own right, i.e., they neither include nor are included by the existing results, and these differences are explained to some extent by examples.

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Global boundedness in a quasilinear predator–prey chemotaxis system with nonlinear indirect signal production

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Abstract. This paper deals with the quasilinear predator–prey chemotaxis system with nonlinear indirect signal production under homogeneous Neumann boundary conditions. Based on the maximal Sobolev regularity and a priori estimates, the global boundedness of solution is investigated. Meanwhile, this work partially improves or extends the results of Wang and Ke [*Acta Appl. Math.* **190**(2024), Paper No. 9, 14 pp.].

Keywords: chemotaxis, predator–prey, nonlinear production, boundedness.

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1 Introduction

The mathematical modeling of predator–prey has a long and distinguished history. The research on the predator–prey model was pioneered by Lotka and Volterra [26]. Predators naturally move towards areas with many prey, while prey move towards areas with fewer predators, effectively avoiding theirs pursuers. Thus, their movements are not accidental but are directional, forming what we call a chemotaxis phenomenon. Chemotaxis is the widespread phenomena in nature, for instance, the directional movement of biological cells, bacteria or organisms in response to some chemical substance, including the positive (chemo-attraction) chemotaxis and negative (chemo-repulsion) chemotaxis. The research on the famous classical Keller–Segel model described the chemotaxis was initiated by Keller and Segel [18]. Chemotaxis is the ability of certain living organisms to orient their movement along a chemical concentration gradient of chemical signal substance in their environment, plays an important role in a wide range of biological processes. From then on, the studies on modeling and analysis of predator–prey chemotaxis system have received more attention from theoretical and mathematical ecologists due to it has various interesting issues, such as, global solvability, finite time blow-up, time asymptotic behavior.

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In this paper, we study the following quasilinear predator–prey chemotaxis with nonlinear indirect signal production

$$\begin{cases} u_t = \nabla\cdot(D_1(u)\nabla u) + \chi\nabla\cdot(u(u+1)^{\alpha-1}\nabla\omega) + \lambda_1 u(1-u^{r_1-1}-\mu_1 v), & x\in\Omega,\ t>0,\\ v_t = \nabla\cdot(D_2(v)\nabla v) - \xi\nabla\cdot(v(v+1)^{\beta-1}\nabla\omega) + \lambda_2 v(1-v^{r_2-1}+\mu_2 u), & x\in\Omega,\ t>0,\\ \omega_t = \Delta\omega - \omega + z^{\gamma}, & x\in\Omega,\ t>0,\\ 0 = \Delta z - z + f(u) + g(v), & x\in\Omega,\ t>0,\\ \frac{\partial u}{\partial\nu} = \frac{\partial v}{\partial\nu} = \frac{\partial\omega}{\partial\nu} = \frac{\partial z}{\partial\nu} = 0, & x\in\partial\Omega,\ t>0,\\ u(x,0)=u_0(x), v(x,0)=v_0(x), \omega(x,0)=\omega_0(x) & x\in\Omega, \end{cases} \tag{1.1}$$

where $\Omega\subset\mathbb{R}^n (n\geq 1)$ is a bounded domain with smooth boundary, ν is the unit outer normal vector on the boundary, the parameters $\chi,\xi,\lambda_1,\lambda_2,\mu_1,\mu_2,\gamma>0, r_1,r_2>1$ and $\alpha,\beta\in\mathbb{R}$. The initial date u_0, v_0 and ω_0 satisfy

$$u_0, v_0\in C^0(\overline{\Omega}) \quad\text{and}\quad \omega_0\in W^{1,\infty}(\Omega), \quad u_0, v_0, \omega_0\geq 0(\not\equiv 0) \text{ in } \overline{\Omega}. \tag{1.2}$$

Here $u(x,t)$ and $v(x,t)$ represent the population density of the prey and the predator, respectively. $\omega(x,t)$, $z(x,t)$ denote the concentration of chemical attractants, $z(x,t)$ is produced by $u(x,t)$ and $v(x,t)$, and $\omega(x,t)$ is secreted by $z(x,t)$. The nonlinear functions $D_1, D_2\in C^2([0,\infty))$ satisfy

$$D_1(u)\geq (u+1)^{m_1-1}, \qquad D_2(v)\geq (v+1)^{m_2-1} \tag{1.3}$$

with $m_1, m_2\in\mathbb{R}$. The functions $f(u), g(v)$ are smooth and satisfy

$$0<f(u)\leq (u+1)^{\gamma_1}, \qquad 0<g(v)\leq (v+1)^{\gamma_2} \tag{1.4}$$

with $\gamma_1,\gamma_2>0$.

Before entering our motivation and main results of this paper, we first retrace some important developments on the (1.1) and its variants. We begin with the following well-known chemotaxis system. The Keller–Segel system was first proposed in [17] to describe the chemotaxis of cellular slime system, which reads as

$$\begin{cases} u_t = \Delta u - \chi\nabla\cdot(u\nabla v), & x\in\Omega,\ t>0,\\ v_t = \Delta v - v + u, & x\in\Omega,\ t>0, \end{cases} \tag{1.5}$$

where $u(x,t)$ and $v(x,t)$ denote the cell density and chemical concentration, respectively. Osaki et al. [30] obtained that the system (1.5) possesses a unique global classical solution which is uniformly bounded if $n=1$. The solution of the system (1.5) may blow up in finite or infinite time for $n=2$ or $n\geq 3$ in [11,13,32,45,46].

The main focus of the above Keller–Segel system is chemoattraction, however, in practical application, chemorepulsion is also involved in many biological processes. In fact, there is a repulsive Keller–Segel system

$$\begin{cases} u_t = \Delta u + \xi\nabla\cdot(u\nabla v), & x\in\Omega,\ t>0,\\ v_t = \Delta v - v + u, & x\in\Omega,\ t>0. \end{cases} \tag{1.6}$$

Cieślak et al. [5] obtained the global existence of the smooth solutions to system (1.6) in the two dimensions and weak solutions in dimension $n=3,4$. for results on the repulsive Keller–Segel model were given in [35,57].

In order to better understand the system (1.1), we first mention the following chemotaxis system

$$\begin{cases} u_t = \nabla \cdot (D(u)\nabla u) - \nabla \cdot (S(u)\nabla v) + f(u), & x \in \Omega,\ t > 0, \\ v_t = \Delta v - v + g(u), & x \in \Omega,\ t > 0, \end{cases} \tag{1.7}$$

where $\Omega \subset \mathbb{R}^n (n \geq 1)$, $D(u)$ and $S(u)$ represent the self-diffusion of the cells and the density dependent cross-diffusion mechanism, respectively. Meanwhile, The function $f(u)$ and $g(u)$ denote the logistic source and the production of chemical substances, respectively.

• At first, we introduce the case where $f(u) \equiv 0, g(u) = u$. For the case $D(u)$ varies in algebraic form, the known results indicate that the asymptotic of $\frac{S(u)}{D(u)} \simeq u^{\frac{2}{n}} (u > 1)$ is the critical condition distinguishing between blow-up and global boundedness to (1.7). Winkler [43] proved that if $\frac{S(u)}{D(u)} \geq cu^{\frac{2}{n}+\varepsilon}$ for all $u > 1$, the model (1.7) with Ω being a ball admits smooth solutions that blow-up either in finite or infinite time, while Tao and Winkler [36] showed that if Ω is convex and $\frac{S(u)}{D(u)} \leq cu^{\frac{2}{n}-\varepsilon}$ for all $u > 1$, then the solutions are globally bounded. Later, if $D(u) = (u+1)^{\alpha-1}, S(u) = u(u+1)^{\beta-1}$, there existed a global bounded classical solution under $\alpha > \beta - \frac{2}{n} + 1$ [16]. and if $\alpha < \beta - \frac{2}{n} + 1$ and when $\Omega \subset \mathbb{R}^n$ is a ball, then for any $\alpha > 0$, the solutions for system (1.7) might blow up in finite or infinite time when $n \geq 2$ [6,48].

• Turn to the case that more general $g(u)$ satisfies $0 < g(u) \leq u^\gamma$. When $D(u) \equiv 1$ and $S(u) = u$, Liu [24] showed the solutions of (1.7) are bounded in time if $\gamma \in (0, \frac{2}{n})$ with $n \geq 2$; whereas when $\gamma > \frac{2}{n}$, Winkler obtained radial blow-up solutions to (1.7) with replacing the second equation by $0 = \Delta v + u^\gamma - \frac{1}{|\Omega|}\int_\Omega u^\gamma$ [47]. Moreover, Other more results on global boundedness or blow-up of solutions related to the system (1.7), we refer to [4,23,49] for more details.

• Now, we will introduce the situation with logistic source $f(u) = ru - \mu u^{1+\sigma}$. If $D(u) \equiv 1, S(u) = u, g(u) = u$, it has been proven that (1.7) admits a unique globally bounded solution with $\sigma = 1, n \geq 3$ and $\mu > \mu_0$, where $\mu_0 > 0$ [44]. Furthermore, Xiang [51] proved the boundedness and convergence of solutions to this model by giving the explicit formula of $\mu_0 > 0$. If there is no bigness restriction to $\mu > 0$, Lankeit [20] showed that the model has a global weak solution when $r > 0$ and $n \geq 3$, and then established the eventual smoothness of the weak solution under the condition that $n = 3$ and r is sufficiently small. If $D(u) \equiv 1, S(u) = u, 0 \leq g(u) \leq u(u+1)^{\gamma-1}$ with $\gamma > 0$, Zhuang et al. [59] established that (1.7) admits a globally bounded solution under $\beta + \gamma = \sigma + 1$ with sufficiently large μ, or under $\beta + \gamma < \sigma + 1$ for all $\mu > 0$; Next, we assume that $D(u) \simeq (u+1)^{-\alpha}$, $S(u) \simeq u(u+1)^{\beta-1}, g(u) = u$ with $\alpha, \beta \in \mathbb{R}, r, \mu, \sigma > 0$, it is shown that in [55] that the model admits globally bounded for $0 < \alpha + \beta < \max\{\sigma + \alpha, \frac{2}{n}\}$ or $\beta = \sigma$ with μ large enough. Whereas if $g(u) = u^\gamma$ with $\gamma > 0$, Ding et al. [9] proved that the model (1.7) possesses a globally bounded and classical solution if one of the following holds:

(i) $\alpha + \beta + \gamma < 1 + \dfrac{2}{n}$,

(ii) $\beta + \gamma < \sigma + 1$, or $\beta + \gamma = \sigma + 1$ with μ large enough.

One of the well-known variants form of (1.7) is the two-species and one-stimuli chemotaxis

system. To describe the movement of two species, the following chemotaxis model

$$\begin{cases} u_t = \nabla \cdot (D_1(u)\nabla u - S_1(u)\nabla \omega) + f(u,v), & x \in \Omega,\ t > 0, \\ v_t = \nabla \cdot (D_2(v)\nabla v - S_2(v)\nabla \omega) + g(u,v), & x \in \Omega,\ t > 0, \\ \tau \omega_t = \Delta \omega - \omega + h(u,v), & x \in \Omega,\ t > 0, \end{cases} \tag{1.8}$$

was firstly proposed by Tello and Winkler in [38], where $\tau \in \{0,1\}$. For system (1.8), $u(x,t)$ and $v(x,t)$ denote the density of different species respectively, and $\omega(x,t)$ represents the concentrations of chemical substances. The functions $f(u,v), g(u,v)$ are functional response functions which are used to describe the relationship between two species, for instance, the predator–prey effect or competition mechanisms, and $h(u,v)$ is the production of chemical substances.

Before stating our motivation and main results of this paper, we mention some previous contributions of model (1.8). On the one hand, if $D_1(u) \equiv 1, D_2(v) \equiv 1, S_1(u) = \chi u, S_2(v) = \xi v$ and $f(u,v) = \mu_1 u(1-u-a_1 v), g(u,v) = \mu_2 v(1-v-a_2 u), h(u,v) = u+v$, Tello and Winkler [38] showed that the system (1.8) with $\tau = 0$ admits a global existence and asymptotic behavior of solutions when $0 \le a_1, a_2 < 1$. For $a_1 > 1 > a_2 \ge 0$, Stinner et al. [34] proved that the global existence and convergence of classical solutions. Later on, for $\tau = 1$, Bai and Winkler [2] showed that the global boundedness and asymptotical behavior of solution with sufficiently large $\mu_1, \mu_2 > 0$ for the two cases $a_1 \ge 1, 0 < a_2 < 1$ and $0 < a_1, a_2 < 1$ when $n \le 2$. In corresponding of strong competition $a_1, a_2 \ge 1$, Pan et al. [31] have developed a method to solve the asymptotic behavior. In addition, When $K_{1i}(s+1)^{\alpha_i - 1} \le D_i(s) \le K_{2i}(s+1)^{\beta_i - 1}, \frac{S_i(s)}{D_i(s)} \le K_i(s+1)^{\gamma_i}, (i = 1,2)$, and $f(u,v), g(u,v) \equiv 0, h(u,v) = u+v$, it is proved that system (1.8) admits a global bounded solution for $0 < \gamma_i < \frac{2}{n}$ in [39].

On the other hand, if $f(u,v) = \mu_1 u(1-u-a_1 v)$ and $g(u,v) = \mu_2 v(1-v+a_2 u)$, the system (1.8) can be seen as a two-species predator–prey model with chemotaxis mechanism. When $D_1(u) \equiv 1, D_2(v) \equiv 1, S_1(u) = -\chi u, S_2(v) = \xi v$, in the case of $\tau = 1$, Fu and Miao [10] proved that the system (1.8) admits a globally bounded classical solution under the parameters μ_1, μ_2, a_1, a_2 satisfy some suitable conditions and $n \le 2$. Furthermore, the unique positive equilibrium point and the semi-trivial equilibrium point were established, respectively. For $n = 3$, Miao et al.[29] proved that the system (1.8) possesses a global and bounded classical solution. In the case $\tau = 0$, Ma et al. [28] derived that the system (1.8) admits a boundedness and convergence of solutions for $n = 2, 3$. Recently, Yang-Xu-Pan [54] extended the work in [28] to arbitrary dimensional $n \ge 1$. When $D_1(u) \equiv 1, D_2(v) \equiv 1, S_1(u) = -\chi u^m, S_2(v) = \xi v^m$ and $f(u,v) = \mu_1 u(1-u-a_1 v), g(u,v) = \mu_2 v(1-v+a_2 u), h(u,v) \le \alpha u^l + \beta v^l$, it is shown that system (1.8) with $\tau = 1$ possesses a global bounded solution if $\frac{\mu_2 a_2^{p+1}}{\mu_1} \le C$ with $p > \frac{n}{2}, n \ge 2$ and $l + m < 1 + \frac{2}{n}$ with $m \ge 1$ in [33]. Moreover, for more results on the predator–prey chemotaxis system with two chemicals, which can refer to the details in [22,25].

In the system mentioned above, the chemical stimulus are produced by population, directly. However, in some realistic biological processes, there are very few research results in multi-species and multi-stimuli chemotaxis systems, where the production mechanism of stimulus is indirect. This process can be described as

$$\begin{cases} u_t = d_1 \Delta u - \chi_1 \nabla \cdot (u \nabla \omega) + \mu_1 u(1-u-a_1 v), & x \in \Omega,\ t > 0, \\ v_t = d_2 \Delta v - \chi_2 \nabla \cdot (v \nabla \omega) + \mu_2 v(1-v-a_2 u), & x \in \Omega,\ t > 0, \\ \tau \omega_t = \Delta \omega - \omega + z, & x \in \Omega,\ t > 0, \\ \tau z_t = \Delta z - z + u + v, & x \in \Omega,\ t > 0, \end{cases} \tag{1.9}$$

where $\tau \in \{0,1\}$, $u(x,t)$ and $v(x,t)$ denote the density of two species, $\omega(x,t)$ and $z(x,t)$ represent the concentration of chemical substance, $\omega(x,t)$ is secreted by $z(x,t)$, $z(x,t)$ is secreted by $u(x,t), v(x,t)$. If $\tau = 1, n \leq 2$, Xiang et al. [53] proved that the system (1.9) has a unique global classical solution, which is bounded in $\Omega \times (0,\infty)$. And if $\tau = 0$, it is shown that regardless of the size of u_0, v_0, the system admits a global bounded classical solution under two-dimension. Furthermore, asymptotic stabilization of global bounded solution to system (1.9) is researched under some suitable conditions of a_i, μ_i. Based on system (1.9), Xiang et al. [52] also discussed a two-competing-species chemotaxis model with indirect signal consumption, which admits a globally bounded and classical solution and obtains asymptotic stabilization under certain conditions. Besides, there are also other results of relevant systems in [41, 58]. More results involving the indirect signal production mechanism can refer to [3, 21, 27, 42, 50, 56].

Based on the above analysis, it is an interesting topic to study the impact of nonlinear indirect signal mechanisms on the dynamic behavior of predator–prey systems. Therefore, we will consider the following quasilinear predator–prey system with nonlinear indirect signal production

$$\begin{cases} u_t = \nabla \cdot (D_1(u)\nabla u) + \chi \nabla \cdot (u(u+1)^{\alpha-1}\nabla \omega) + \lambda_1 u(1 - u^{r_1 - 1} - \mu_1 v), & x \in \Omega,\ t > 0, \\ v_t = \nabla \cdot (D_2(v)\nabla v) - \xi \nabla \cdot (v(v+1)^{\beta-1}\nabla \omega) + \lambda_2 v(1 - v^{r_2 - 1} + \mu_2 u), & x \in \Omega,\ t > 0, \\ \omega_t = \Delta \omega - \omega + z^{\gamma}, & x \in \Omega,\ t > 0, \\ 0 = \Delta z - z + f(u) + g(v), & x \in \Omega,\ t > 0, \end{cases} \tag{1.10}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary. When $D_1(u) = D_2(v) \equiv 1, \alpha = \beta = 1$ and $f(u) = u^{\gamma_1}, g(v) = v^{\gamma_2}$, Wang and Ke [40] proved that if $n \geq 1$ and $\gamma, \gamma_1, \gamma_2, n$ and r_1, r_2 satisfy suitable conditions, then the system (1.10) admitted a global bounded solution.

For the best of our knowledge, few rigorous result seem to be known about the quasilinear predator–prey system with nonlinear indirect signal production of (1.1). Thus, we will study the boundedness of the system (1.1), now we show the main result of the present paper.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n (n \geq 1)$ be a bounded domain with smooth boundary and the parameters fulfill $\chi, \xi, \lambda_1, \lambda_2, \mu_1, \mu_2, \gamma, \gamma_1, \gamma_2 > 0$ and $m_1, m_2, \alpha, \beta \in \mathbb{R}$. If $r_1 > 1, r_2 > 2$, $m_1 + \frac{2}{n} > \max\{\alpha + \gamma(\gamma_1 + \gamma_2), \beta + \gamma(\gamma_1 + \gamma_2), 1\}$ and $m_2 + \frac{2}{n} > \max\{\alpha + \gamma(\gamma_1 + \gamma_2), \beta + \gamma(\gamma_1 + \gamma_2), 1\}$. Assume that the initial date $u_0, v_0 \in C^0(\overline{\Omega})$ and $\omega_0 \in W^{1,\infty}(\Omega)$ are nonnegative, then the system (1.1) admits a nonnegative global classical solution*

$$(u, v, \omega, z) \in (C^0(\overline{\Omega} \times [0,\infty)) \cap C^{2,1}(\overline{\Omega} \times (0,\infty)))^3 \times (C^{2,0}(\overline{\Omega} \times (0,\infty))).$$

Moreover, the solution is bounded in $\Omega \times (0,\infty)$, in other words, there exists $C > 0$ such that

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{L^\infty(\Omega)} + \|\omega(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|z(\cdot,t)\|_{W^{1,\infty}(\Omega)} \leq C \tag{1.11}$$

for all $t > 0$.

2 Preliminaries

In this section, we first state the local existence of classical solutions to (1.1), which can be achieved by the fixed point theory and parabolic regularity. Refer to [8, 14, 19, 37] for more details.

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^n (n \geq 1)$ be a bounded domain with smooth boundary and the initial date $u_0, v_0 \in C^0(\overline{\Omega})$ and $\omega_0 \in W^{1,\infty}(\Omega)$ are nonnegative. Then there exists $T_{\max} \in (0, \infty]$ and a uniquely determined trip*

$$(u, v, \omega, z) \in \left(C^0(\overline{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\overline{\Omega} \times [0, T_{\max}))\right)^3 \times (C^{2,0}(\overline{\Omega} \times (0, T_{\max})))$$

solving (1.1) in the classical sense in $\Omega \times (0, T_{\max})$.

With

$$u, v, \omega, z \geq 0 \quad \text{in } \overline{\Omega} \times (0, T_{\max}).$$

Furthermore, if $T_{\max} < \infty$, then

$$\lim_{t \to T_{\max}} \|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{L^\infty(\Omega)} + \|\omega(\cdot, t)\|_{W^{1,\infty}(\Omega)} + \|z(\cdot, t)\|_{W^{1,\infty}(\Omega)} = \infty.$$

Lemma 2.2. *Assume that (u, v, ω, z) be a solution to (1.1). Then for any $\tilde{\eta} > 0$ and $\theta > 1$, there exists $\eta = \eta(\theta, \tilde{\eta}, |\Omega|)$ such that*

$$\int_\Omega z^\theta \leq \eta \int_\Omega (f(u)^\theta + g(v)^\theta) \quad \text{for all } t \in (0, T_{\max}), \tag{2.1}$$

with $f(u), g(v)$ as given in system (1.1).

Proof. The proof can be obtained via the same method as in [12, Lemma 2.2], but we will provide a detailed proof process here. Integrating the fourth of (1.1) to obtain

$$\int_\Omega z \leq \int_\Omega \big(f(u) + g(v)\big). \tag{2.2}$$

Multiplying the fourth equation of (1.1) by $z^{\theta-1}$, and integrating on Ω, we get

$$\begin{aligned} \frac{4(\theta-1)}{\theta^2} \int_\Omega |\nabla z^{\frac{\theta}{2}}|^2 + \int_\Omega z^\theta &= \int_\Omega \big(f(u) + g(v)\big) z^{\theta-1} \\ &\leq \frac{\theta-1}{\theta} \int_\Omega z^\theta + \frac{1}{\theta} \int_\Omega \big(f(u) + f(v)\big)^\theta \end{aligned} \tag{2.3}$$

for all $t \in (0, T_{\max})$. By Young's inequality, and thus

$$\|z\|_{L^\theta(\Omega)} \leq \|f(u) + g(v)\|_{L^\theta(\Omega)}, \qquad t \in (0, T_{\max}), \tag{2.4}$$

$$\frac{4(\theta-1)}{\theta} \int_\Omega |\nabla z^{\frac{\theta}{2}}|^2 \leq \int_\Omega \big(f(u) + g(v)\big)^\theta, \qquad t \in (0, T_{\max}). \tag{2.5}$$

Using Ehrling's lemma, for any $\tilde{\eta} > 0$, $\theta > 1$, there exists $\tilde{C} = \tilde{C}(\tilde{\eta}, \theta) > 0$ such that

$$\|\phi\|^2_{L^2(\Omega)} \leq \tilde{\eta} \|\phi\|^2_{W^{1,2}(\Omega)} + \tilde{C} \|\phi\|^2_{L^{\frac{2}{\theta}}(\Omega)}, \ \phi \in W^{1,2}(\Omega). \tag{2.6}$$

Take $\phi = z^{\frac{\theta}{2}}$. we know from (2.2), (2.4) and (2.5) that

$$\int_\Omega z^\theta \leq \tilde{\eta} \int_\Omega \big(f(u) + g(v)\big)^\theta + c_0 \|f(u) + g(v)\|^\theta_{L^1} \tag{2.7}$$

with $c_0 = c_0(\tilde{\eta}, \theta) > 0$. We have by the Hölder inequality with (2.7) that

$$\begin{aligned} \int_\Omega z^\theta &\leq \tilde{\eta} \int_\Omega \big(f(u) + g(v)\big)^\theta + c_0 |\Omega|^{\theta-1} \int_\Omega \big(f(u) + g(v)\big)^\theta \\ &= (\tilde{\eta} + c_0 |\Omega|^{\theta-1}) \int_\Omega \big(f(u) + g(v)\big)^\theta \\ &\leq \eta \int_\Omega \big(f(u)^\theta + g(v)^\theta\big) \end{aligned} \tag{2.8}$$

with $\eta := 2^{\theta-1}(\tilde{\eta} + c_0 |\Omega|^{\theta-1})$. □

Lemma 2.3. *Let $\Omega \subset \mathbb{R}^n (n \geq 1)$ be a bounded domain with smooth boundary and the parameters satisfy $\chi, \xi, \lambda_1, \lambda_2, \mu_1, \mu_2, \gamma, \gamma_1, \gamma_2 > 0, r_1, r_2 > 1$ and $\alpha, \beta, m_1, m_2 \in \mathbb{R}$. Assume that (u, v, ω, z) be a solution to (1.1). Then there exists $C > 0$ such that*

$$\int_\Omega u + \int_\Omega v \leq C \tag{2.9}$$

for all $t \in (0, T_{\max})$.

Proof. Integrating the first two equation in (1.1) with respect to $x \in \Omega$, we get

$$\frac{d}{dt}\int_\Omega u + \int_\Omega u = \int_\Omega \lambda_1 u(1 - u^{r_1 - 1} - \mu_1 v) + \int_\Omega u \tag{2.10}$$

and

$$\frac{d}{dt}\int_\Omega v + \int_\Omega v = \int_\Omega \lambda_2 v(1 - v^{r_2 - 1} + \mu_2 u) + \int_\Omega v \tag{2.11}$$

for all $t \in (0, T_{\max})$. Multiplying the equation (2.10) and the equation (2.11) with $\lambda_2\mu_2$ and $\lambda_1\mu_1$, respectively, then we add them up and invoke Young's inequality to derive that there exists $c_1 > 0$ such that

$$\begin{aligned}
&\lambda_2\mu_2 \frac{d}{dt}\int_\Omega u + \lambda_2\mu_2 \int_\Omega u + \lambda_1\mu_1 \frac{d}{dt}\int_\Omega v + \lambda_1\mu_1 \int_\Omega v \\
&= \lambda_2\mu_2(\lambda_1 + 1)\int_\Omega u - \lambda_1\lambda_2\mu_2 \int_\Omega u^{r_1} + \lambda_1\mu_1(\lambda_2 + 1)\int_\Omega v - \lambda_1\lambda_2\mu_1 \int_\Omega v^{r_2} \\
&\leq -\frac{\lambda_1\lambda_2\mu_2}{2}\int_\Omega u^{r_1} - \frac{\lambda_1\lambda_2\mu_1}{2}\int_\Omega v^{r_2} + c_1 \\
&\leq c_1
\end{aligned} \tag{2.12}$$

for all $t \in (0, T_{\max})$. Thus, applying the ODE comparison principle to (2.12), there exists $c_2 > 0$ such that

$$\int_\Omega u + \int_\Omega v \leq c_2 \tag{2.13}$$

for all $t \in (0, T_{\max})$. □

Lemma 2.4 ([15, Lemma 3.6] and [7, Lemma 3.2])**.** *Let $q \in (1, \infty)$ and $T \in (0, \infty]$, Then, for $g \in L^q([0, T); L^q(\Omega))$ and $z_0 \in W^{2,q}(\Omega)$, with $\frac{\partial z_0}{\partial \nu} = 0$ on $\partial\Omega$, there exists $C_q = C_q(\Omega, q, \|z_0\|_{W^{2,q}(\Omega)}) > 0$ such that every solution $z \in W^{1,q}_{loc}([0, T); L^q(\Omega)) \cap L^q_{loc}([0, T); W^{2,q}(\Omega))$ of*

$$\begin{cases} z_t = \Delta z - z + g, & x \in \Omega,\ t > 0, \\ \frac{\partial z}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ z(x, 0) = z_0, & x \in \Omega, \end{cases} \tag{2.14}$$

satisfies

$$\int_0^t e^s \int_\Omega |\Delta z(\cdot, t)|^q ds \leq C_q \left(1 + \int_0^t e^s \int_\Omega |g(\cdot, t)|^q ds\right) \quad \text{for all } t \in (0, T). \tag{2.15}$$

3 Global existence and boundedness

In this section, in order to prove the global existence and uniform boundedness of solutions to the system (1.1), we first establish the L^p-estimate for u, v.

Lemma 3.1. *Let $\Omega \subset \mathbb{R}^n (n \geq 1)$ be a bounded domain with smooth boundary and the parameters satisfy $\chi, \xi, \lambda_1, \lambda_2, \mu_1, \mu_2, \gamma, \gamma_1, \gamma_2 > 0$ and $m_1, m_2, \alpha, \beta \in \mathbb{R}$. If $r_1 > 1, r_2 > 2$, $m_1 + \frac{2}{n} > \max\{\alpha + \gamma_(\gamma_1 + \gamma_2), \beta + \gamma(\gamma_1 + \gamma_2), 1\}$ and $m_2 + \frac{2}{n} > \max\{\alpha + \gamma(\gamma_1 + \gamma_2), \beta + \gamma(\gamma_1 + \gamma_2), 1\}$, then for any $p > \max\{1 - m_1, 1 - \alpha, 1 - \beta, 1 - m_2, 1 - \alpha + (1 - \gamma)(\gamma_1 + \gamma_2), 1 - \beta + (1 - \gamma)(\gamma_1 + \gamma_2), \frac{(2-r_1)(r_2-1)}{r_2-2}, 1\}$, there exists $C > 0$ such that*

$$\int_\Omega u^p + \int_\Omega v^p \leq C \quad \text{for all } t \in (0, T_{\max}). \tag{3.1}$$

Proof. For any $p > 1$, testing the first equation in system (1.1) by $(u+1)^{p-1}$, we have

$$\begin{aligned}
\frac{1}{p}\frac{d}{dt}\int_\Omega (u+1)^p =& -(p-1)\int_\Omega (u+1)^{p-2} D_1(u)|\nabla u|^2 - \chi(p-1)\int_\Omega u(u+1)^{p+\alpha-3}\nabla u \cdot \nabla \omega \\
&+ \lambda_1 \int_\Omega u(u+1)^{p-1} - \lambda_1 \int_\Omega u^{r_1}(u+1)^{p-1} - \lambda_1\mu_1 \int_\Omega u(u+1)^{p-1} v \\
\leq& -(p-1)\int_\Omega (u+1)^{p+m_1-3}|\nabla u|^2 - \chi(p-1)\int_\Omega u(u+1)^{p+\alpha-3}\nabla u \cdot \nabla \omega \\
&+ \lambda_1 \int_\Omega u(u+1)^{p-1} - \lambda_1 \int_\Omega u^{r_1}(u+1)^{p-1}
\end{aligned} \tag{3.2}$$

for all $t \in (0, T_{\max})$. Due to

$$(a+b)^s \leq 2^{s-1}(a^s + b^s)$$

for all $a, b > 0, s > 1$. We get

$$\frac{1}{2^{r_1-1}}\int_\Omega (u+1)^{p+r_1-1} \leq \int_\Omega (u+1)^{p-1}u^{r_1} + \int_\Omega (u+1)^{p-1} \tag{3.3}$$

with $t \in (0, T_{\max})$. Let $\varphi(t) := \frac{1}{p}\int_\Omega (u+1)^p$, $h_\alpha(u(\cdot,t)) := \int_0^u s(s+1)^{p+\alpha-3}ds$. Thus, by a simple calculation with (3.2) and (3.3), we obtain

$$\begin{aligned}
\varphi'(t) \leq& -\frac{4(p-1)}{(p+m_1-1)^2}\int_\Omega \left|\nabla (u+1)^{\frac{p+m_1-1}{2}}\right|^2 - \chi(p-1)\int_\Omega \nabla h_\alpha(u)\cdot\nabla\omega + \lambda_1\int_\Omega u(u+1)^{p-1} \\
&- \lambda_1 \int_\Omega u^{r_1}(u+1)^{p-1} \\
\leq& -\frac{4(p-1)}{(p+m_1-1)^2}\int_\Omega \left|\nabla (u+1)^{\frac{p+m_1-1}{2}}\right|^2 + \chi(p-1)\int_\Omega h_\alpha(u)\Delta\omega + \lambda_1\int_\Omega (u+1)^p \\
&+ \lambda_1 \int_\Omega (u+1)^{p-1} - \frac{\lambda_1}{2^{r_1-1}}\int_\Omega (u+1)^{p+r_1-1}
\end{aligned} \tag{3.4}$$

for all $t \in (0, T_{\max})$. For the term $h_\alpha(u(\cdot t))$, we achieve

$$\begin{aligned}
h_\alpha(u(\cdot,t)) &= \int_0^{u(\cdot,t)} s(s+1)^{p+\alpha-3}ds \\
&\leq \frac{1}{p+\alpha-1}(u(\cdot,t)+1)^{p+\alpha-1}
\end{aligned} \tag{3.5}$$

for all $t \in (0, T_{\max})$. Thus, by the simple calculation with (3.5) and Young's inequality, we get

$$\begin{aligned}
\chi(p-1)\int_\Omega h_\alpha(u)\Delta\omega &\leq \frac{\chi(p-1)}{p+\alpha-1}\int_\Omega (u+1)^{p+\alpha-1}|\Delta\omega| \\
&\leq \int_\Omega (u+1)^{p+\alpha-1+\gamma(\gamma_1+\gamma_2)} + c_3\int_\Omega |\Delta\omega|^{\frac{p+\alpha-1+\gamma(\gamma_1+\gamma_2)}{\gamma(\gamma_1+\gamma_2)}}
\end{aligned} \tag{3.6}$$

for all $t \in (0, T_{\max})$, where $c_3 = \left(\frac{\chi(p-1)}{p+\alpha-1}\right)^{\frac{p+\alpha-1+\gamma(\gamma_1+\gamma_2)}{\gamma(\gamma_1+\gamma_2)}}$. Using Young's inequality once more, for $r_1 > 1$ we have

$$\begin{aligned}
&\lambda_1 \int_\Omega (u+1)^p + \lambda_1 \int_\Omega (u+1)^{p-1} - \frac{\lambda_1}{2^{r_1-1}} \int_\Omega (u+1)^{p+r_1-1} \\
&\quad \le \frac{\lambda_1}{2^{r_1}} \int_\Omega (u+1)^{p+r_1-1} - \frac{\lambda_1}{2^{r_1-1}} \int_\Omega (u+1)^{p+r_1-1} + c_4 \\
&\quad = -\frac{\lambda_1}{2^{r_1}} \int_\Omega (u+1)^{p+r_1-1} + c_4
\end{aligned} \tag{3.7}$$

for all $t \in (0, T_{\max})$. Substituting (3.6)–(3.7) into (3.4), we obtain

$$\begin{aligned}
\varphi'(t) \le & -\frac{4(p-1)}{(p+m_1-1)^2} \int_\Omega |\nabla (u+1)^{\frac{p+m_1-1}{2}}|^2 + \int_\Omega (u+1)^{p+\alpha-1+\gamma(\gamma_1+\gamma_2)} \\
& + c_3 \int_\Omega |\Delta \omega|^{\frac{p+\alpha-1+\gamma(\gamma_1+\gamma_2)}{\gamma(\gamma_1+\gamma_2)}} - \frac{\lambda_1}{2^{r_1}} \int_\Omega (u+1)^{p+r_1-1} + c_4
\end{aligned} \tag{3.8}$$

for all $t \in (0, T_{\max})$.

Now, we add to both sides of (3.8) the term $\frac{1}{p}\int_\Omega (u+1)^p$, and we multiply by e^t. Since $e^t\varphi'(t) + e^t\varphi(t) = \frac{d}{dt}\left(e^t\varphi(t)\right)$, an integration over $(0,t)$ provides for all $t \in (0, T_{\max})$

$$\begin{aligned}
e^t\varphi(t) \le \varphi(0) & - \frac{4(p-1)}{(p+m_1-1)^2} \int_0^t e^s \int_\Omega |\nabla (u+1)^{\frac{p+m_1-1}{2}}|^2 + \int_0^t e^s \int_\Omega (u+1)^{p+\alpha-1+\gamma(\gamma_1+\gamma_2)} \\
& + c_3 \int_0^t e^s \int_\Omega |\Delta \omega|^{\frac{p+\alpha-1+\gamma(\gamma_1+\gamma_2)}{\gamma(\gamma_1+\gamma_2)}} - \frac{\lambda_1}{2^{r_1}} \int_0^t e^s \int_\Omega (u+1)^{p+r_1-1} + \frac{1}{p} \int_0^t e^s \int_\Omega (u+1)^p \\
& + c_4 \int_0^t e^s ds.
\end{aligned} \tag{3.9}$$

Taking $q = \frac{p+\alpha-1+\gamma(\gamma_1+\gamma_2)}{\gamma(\gamma_1+\gamma_2)} > 1$ in the Lemma 2.4, according to (2.15) and the third equation in system (1.1), we know

$$\int_0^t e^s \int_\Omega |\Delta \omega|^{\frac{p+\alpha-1+\gamma(\gamma_1+\gamma_2)}{\gamma(\gamma_1+\gamma_2)}} \le C_q \left(1 + \int_0^t e^s \int_\Omega z^{\frac{p+\alpha-1+\gamma(\gamma_1+\gamma_2)}{\gamma_1+\gamma_2}} ds\right) \tag{3.10}$$

for all $t \in (0, T_{\max})$. Applying Lemma 2.2 with $\theta = \frac{p+\alpha-1+\gamma(\gamma_1+\gamma_2)}{\gamma_1+\gamma_2} > 1$, and by (1.4), this implies

$$\int_\Omega z^{\frac{p+\alpha-1+\gamma(\gamma_1+\gamma_2)}{\gamma_1+\gamma_2}} \le \eta \int_\Omega \left((1+u)^{\frac{\gamma_1(p+\alpha-1+\gamma(\gamma_1+\gamma_2))}{\gamma_1+\gamma_2}} + (1+v)^{\frac{\gamma_2(p+\alpha-1+\gamma(\gamma_1+\gamma_2))}{\gamma_1+\gamma_2}}\right) \tag{3.11}$$

for all $t \in (0, T_{\max})$. By Young's inequality, we have

$$\int_\Omega z^{\frac{p+\alpha-1+\gamma(\gamma_1+\gamma_2)}{\gamma_1+\gamma_2}} \le \int_\Omega \left((1+u)^{p+\alpha-1+\gamma(\gamma_1+\gamma_2)} + (1+v)^{p+\alpha-1+\gamma(\gamma_1+\gamma_2)}\right) + c_5 \tag{3.12}$$

for all $t \in (0, T_{\max})$. Collecting (3.10)–(3.12), we know

$$\begin{aligned}
e^t\varphi(t) \le \varphi(0) & - \frac{4(p-1)}{(p+m_1-1)^2} \int_0^t e^s \int_\Omega |\nabla (u+1)^{\frac{p+m_1-1}{2}}|^2 + c_6 \int_0^t e^s \int_\Omega (u+1)^{p+\alpha-1+\gamma(\gamma_1+\gamma_2)} \\
& + c_7 \int_0^t e^s \int_\Omega (v+1)^{p+\alpha-1+\gamma(\gamma_1+\gamma_2)} + \frac{1}{p} \int_0^t e^s \int_\Omega (u+1)^p - \frac{\lambda_1}{2^{r_1}} \int_0^t e^s \int_\Omega (u+1)^{p+r_1-1} \\
& + c_8 \int_0^t e^s ds
\end{aligned} \tag{3.13}$$

for all $t \in (0, T_{\max})$. We get from (2.9) with the Gagliardo–Nirenberg inequality that

$$\begin{aligned}
&\hat{c}\int_\Omega (u+1)^{p+\alpha+\gamma(\gamma_1+\gamma_2)-1} \\
&\quad = \hat{c}\left\|(u+1)^{\frac{p+m_1-1}{2}}\right\|^{\frac{2(p+\alpha+\gamma(\gamma_1+\gamma_2)-1)}{p+m_1-1}}_{L^{\frac{2(p+\alpha+\gamma(\gamma_1+\gamma_2)-1)}{p+m_1-1}}(\Omega)} \\
&\quad \le c_9\left\|\nabla(u+1)^{\frac{p+m_1-1}{2}}\right\|^{\theta_1\frac{2(p+\alpha+\gamma(\gamma_1+\gamma_2)-1)}{p+m_1-1}}_{L^2(\Omega)} \left\|(u+1)^{\frac{p+m_1-1}{2}}\right\|^{(1-\theta_1)\frac{2(p+\alpha+\gamma(\gamma_1+\gamma_2)-1)}{p+m_1-1}}_{L^{\frac{2}{p+m_1-1}}(\Omega)} \\
&\qquad + c_9\left\|(u+1)^{\frac{p+m_1-1}{2}}\right\|^{\frac{2(p+\alpha+\gamma(\gamma_1+\gamma_2)-1)}{p+m_1-1}}_{L^{\frac{2}{p+m_1-1}}(\Omega)} \\
&\quad \le c_{10}\left(\int_\Omega \left|\nabla(u+1)^{\frac{p+m_1-1}{2}}\right|^2\right)^{\theta_1\frac{p+\alpha+\gamma(\gamma_1+\gamma_2)-1}{p+m_1-1}} + c_{11}
\end{aligned} \tag{3.14}$$

for all $t \in (0, T_{\max})$ and $\theta_1 = \frac{\frac{p+m_1-1}{2} - \frac{p+m_1-1}{2(p+\alpha+\gamma(\gamma_1+\gamma_2)-1)}}{\frac{p+m_1-1}{2} - \frac{1}{2} + \frac{1}{n}} \in (0,1)$.

Since $m_1 + \frac{2}{n} > \alpha + \gamma(\gamma_1+\gamma_2)$ implies $\theta_1\frac{p+\alpha+\gamma(\gamma_1+\gamma_2)-1}{p+m_1-1} < 1$, we have by Young's inequality that

$$\hat{c}\int_\Omega (u+1)^{p+\alpha+\gamma(\gamma_1+\gamma_2)-1} \le \frac{(p-1)}{(p+m_1-1)^2}\int_\Omega \left|\nabla(u+1)^{\frac{p+m_1-1}{2}}\right|^2 + c_{12} \tag{3.15}$$

for all $t \in (0, T_{\max})$. By (2.9) with the Gagliardo–Nirenberg inequality once more, we get

$$\begin{aligned}
\hat{c}\int_\Omega (u+1)^p &= \hat{c}\left\|(u+1)^{\frac{p+m_1-1}{2}}\right\|^{\frac{2p}{p+m_1-1}}_{L^{\frac{2p}{p+m_1-1}}(\Omega)} \\
&\le c_{13}\left\|\nabla(u+1)^{\frac{p+m_1-1}{2}}\right\|^{\theta_2\frac{2p}{p+m_1-1}}_{L^2(\Omega)} \left\|(u+1)^{\frac{p+m_1-1}{2}}\right\|^{(1-\theta_2)\frac{2p}{p+m_1-1}}_{L^{\frac{2}{p+m_1-1}}(\Omega)} \\
&\quad + c_{13}\left\|(u+1)^{\frac{p+m_1-1}{2}}\right\|^{\frac{2p}{p+m_1-1}}_{L^{\frac{2}{p+m_1-1}}(\Omega)} \\
&\le c_{14}\left(\int_\Omega \left|\nabla(u+1)^{\frac{p+m_1-1}{2}}\right|^2\right)^{\theta_2\frac{p}{p+m_1-1}} + c_{15}
\end{aligned} \tag{3.16}$$

for all $t \in (0, T_{\max})$ and $\theta_2 = \frac{\frac{p+m_1-1}{2} - \frac{p+m_1-1}{2p}}{\frac{p+m_1-1}{2} - \frac{1}{2} + \frac{1}{n}} \in (0,1)$. Due to $m_1 + \frac{2}{n} > 1$ implies $\theta_2\frac{p}{p+m_1-1} < 1$, we get by Young's inequality that

$$\hat{c}\int_\Omega (u+1)^p \le \frac{(p-1)}{(p+m_1-1)^2}\int_\Omega \left|\nabla(u+1)^{\frac{p+m_1-1}{2}}\right|^2 + c_{15} \tag{3.17}$$

for all $t \in (0, T_{\max})$. Combining (3.15) and (3.17), we obtain

$$\begin{aligned}
e^t\varphi(t) \le \varphi(0) &- \frac{2(p-1)}{(p+m_1-1)^2}\int_0^t e^s\int_\Omega \left|\nabla(u+1)^{\frac{p+m_1-1}{2}}\right|^2 + c_7\int_0^t e^s\int_\Omega (v+1)^{p+\alpha-1+\gamma(\gamma_1+\gamma_2)} \\
&- \frac{\lambda_1}{2^{r_1}}\int_0^t e^s\int_\Omega (u+1)^{p+r_1-1} + c_{16}\int_0^t e^s ds
\end{aligned} \tag{3.18}$$

for all $t \in (0, T_{\max})$. Multiplying the second equation in system (1.1) with $(1+v)^{p-1}$, for any

$p > 1$, we derive from integration by parts that

$$\begin{aligned}\tilde{\varphi}'(t) \leq & -\frac{4(p-1)}{(p+m_2-1)^2}\int_\Omega \left|\nabla(v+1)^{\frac{p+m_2-1}{2}}\right|^2 - \frac{\xi(p-1)}{p+\beta-1}\int_\Omega (v+1)^{p+\beta-1}|\Delta\omega| \\ & + \lambda_2\left(\int_\Omega (v+1)^p + \int_\Omega (v+1)^{p-1}\right) \\ & - \frac{\lambda_2}{2^{r_2-1}}\int_\Omega (v+1)^{p+r_2-1} + \lambda_2\mu_2\int_\Omega (v+1)^p u\end{aligned} \tag{3.19}$$

for all $t \in (0, T_{\max})$, where $\tilde{\varphi}(t) = \frac{1}{p}\int_\Omega (v+1)^p$. Due to $r_2 > 1$, applications of Young's inequality imply that

$$\lambda_2\int_\Omega (v+1)^p + \lambda_2\int_\Omega (v+1)^{p-1} \leq \frac{\lambda_2}{2^{r_2-2}}\int_\Omega (v+1)^{p+r_2-1} + c_{17} \tag{3.20}$$

and

$$\begin{aligned}\lambda_2\mu_2\int_\Omega (1+v)^p u &\leq \frac{\lambda_2}{2^{r_2-2}}\int_\Omega (v+1)^{p+r_2-1} + c_{18}\int_\Omega u^{\frac{p+r_2-1}{r_2-1}} \\ &\leq \frac{\lambda_2}{2^{r_2-2}}\int_\Omega (v+1)^{p+r_2-1} + c_{18}\int_\Omega (u+1)^{\frac{p+r_2-1}{r_2-1}}\end{aligned} \tag{3.21}$$

for all $t \in (0, T_{\max})$. Substituting (3.20)–(3.21) into (3.19), we conclude that

$$\begin{aligned}\tilde{\varphi}'(t) \leq & -\frac{4(p-1)}{(p+m_2-1)^2}\int_\Omega \left|\nabla(v+1)^{\frac{p+m_2-1}{2}}\right|^2 - \frac{\xi(p-1)}{p+\beta-1}\int_\Omega (v+1)^{p+\beta-1}|\Delta\omega| \\ & + c_{18}\int_\Omega (u+1)^{\frac{p+r_2-1}{r_2-1}} + c_{19}\end{aligned} \tag{3.22}$$

for all $t \in (0, T_{\max})$. Invoking the same procedure as dealing with u, we have

$$\begin{aligned}e^t\tilde{\varphi}(t) \leq \tilde{\varphi}(0) & - \frac{2(p-1)}{(p+m_2-1)^2}\int_0^t e^s\int_\Omega \left|\nabla(v+1)^{\frac{p+m_2-1}{2}}\right|^2 + c_{20}\int_0^t e^s\int_\Omega (u+1)^{p+\beta-1+\gamma(\gamma_1+\gamma_2)} \\ & + c_{18}\int_0^t e^s\int_\Omega (u+1)^{\frac{p+r_2-1}{r_2-1}} + c_{21}\int_0^t e^s ds\end{aligned} \tag{3.23}$$

for all $t \in (0, T_{\max})$. By (2.9) with the Gagliardo–Nirenberg inequality once more, we obtain

$$\begin{aligned}&\hat{c}\int_\Omega (u+1)^{p+\beta+\gamma(\gamma_1+\gamma_2)-1} \\ &\quad = \hat{c}\left\|(u+1)^{\frac{p+m_1-1}{2}}\right\|^{\frac{2(p+\beta+\gamma(\gamma_1+\gamma_2)-1)}{p+m_1-1}}_{L^{\frac{2(p+\beta+\gamma(\gamma_1+\gamma_2)-1)}{p+m_1-1}}(\Omega)} \\ &\quad \leq c_{22}\left\|\nabla(u+1)^{\frac{p+m_1-1}{2}}\right\|^{\theta_3\frac{2(p+\beta+\gamma(\gamma_1+\gamma_2)-1)}{p+m_1-1}}_{L^2(\Omega)} \left\|(u+1)^{\frac{p+m_1-1}{2}}\right\|^{(1-\theta_1)\frac{2(p+\beta+\gamma(\gamma_1+\gamma_2)-1)}{p+m_1-1}}_{L^{\frac{2}{p+m_1-1}}(\Omega)} \\ &\qquad + c_{22}\left\|(u+1)^{\frac{p+m_1-1}{2}}\right\|^{\frac{2(p+\beta+\gamma(\gamma_1+\gamma_2)-1)}{p+m_1-1}}_{L^{\frac{2}{p+m_1-1}}(\Omega)} \\ &\quad \leq c_{23}\left(\int_\Omega \left|\nabla(u+1)^{\frac{p+m_1-1}{2}}\right|^2\right)^{\theta_3\frac{p+\beta+\gamma(\gamma_1+\gamma_2)-1}{p+m_1-1}} + c_{24}\end{aligned} \tag{3.24}$$

for all $t \in (0, T_{\max})$ and $\theta_3 = \frac{\frac{p+m_1-1}{2} - \frac{p+m_1-1}{2(p+\beta+\gamma(\gamma_1+\gamma_2)-1)}}{\frac{p+m_1-1}{2} - \frac{1}{2} + \frac{1}{n}} \in (0,1)$.

Since $m_1 + \frac{2}{n} > \beta + \gamma(\gamma_1+\gamma_2)$ implies $\theta_3\frac{p+\beta+\gamma(\gamma_1+\gamma_2)-1}{p+m_1-1} < 1$, which in conjunction with

Young's inequality deduces that

$$\hat{c}\int_\Omega (u+1)^{p+\beta+\gamma(\gamma_1+\gamma_2)-1} \le \frac{2(p-1)}{(p+m_1-1)^2}\int_\Omega |\nabla (u+1)^{\frac{p+m_1-1}{2}}|^2 + c_{25} \tag{3.25}$$

for all $t \in (0, T_{\max})$. In view of the Gagliardo–Nirenberg inequality, there exists $c_{26} > 0$ such that

$$\begin{aligned}
&\hat{c}\int_\Omega (v+1)^{p+\alpha+\gamma(\gamma_1+\gamma_2)-1} \\
&\quad = \hat{c}\left\|(v+1)^{\frac{p+m_2-1}{2}}\right\|^{\frac{2(p+\alpha+\gamma(\gamma_2+\gamma_3)-1)}{p+m-1}}_{L^{\frac{2(p+\alpha+\gamma(\gamma_1+\gamma_2)-1)}{p+m_2-1}}(\Omega)} \\
&\quad \le c_{26}\left\|\nabla(v+1)^{\frac{p+m_2-1}{2}}\right\|^{\theta_4\frac{2(p+\alpha+\gamma(\gamma_1+\gamma_2)-1)}{p+m_2-1}}_{L^2(\Omega)} \left\|(u+1)^{\frac{p+m_2-1}{2}}\right\|^{(1-\theta_4)\frac{2(p+\alpha+\gamma(\gamma_1+\gamma_2)-1)}{p+m_2-1}}_{L^{\frac{2}{p+m_2-1}}(\Omega)} \\
&\qquad + c_{26}\left\|(v+1)^{\frac{p+m_2-1}{2}}\right\|^{\frac{2(p+\alpha+\gamma(\gamma_1+\gamma_2)-1)}{p+m_2-1}}_{L^{\frac{2}{p+m_2-1}}(\Omega)} \\
&\quad \le c_{27}\left(\int_\Omega \left|\nabla(v+1)^{\frac{p+m_2-1}{2}}\right|^2\right)^{\theta_4\frac{p+\alpha+\gamma(\gamma_1+\gamma_2)-1}{p+m_2-1}} + c_{28}
\end{aligned} \tag{3.26}$$

for all $t \in (0, T_{\max})$ and $\theta_4 = \frac{\frac{p+m_2-1}{2} - \frac{p+m_2-1}{2(p+\alpha+\gamma(\gamma_1+\gamma_2)-1)}}{\frac{p+m_2-1}{2}-\frac{1}{2}+\frac{1}{n}} \in (0,1)$.

Due to $m_2 + \frac{2}{n} > \alpha + \gamma(\gamma_1+\gamma_2)$ implies $\theta_4\frac{p+\alpha+\gamma(\gamma_1+\gamma_2)-1}{p+m_2-1} < 1$, we get by Young's inequality that

$$\hat{c}\int_\Omega (v+1)^{p+\alpha+\gamma(\gamma_1+\gamma_2)-1} \le \frac{2(p-1)}{(p+m_2-1)^2}\int_\Omega \left|\nabla(v+1)^{\frac{p+m_2-1}{2}}\right|^2 + c_{30} \tag{3.27}$$

for all $t \in (0, T_{\max})$. Combing (3.18), (3.23), (3.25), and (3.27), we obtain

$$\begin{aligned}
\frac{1}{p}e^t\int_\Omega u^p + \frac{1}{p}e^t\int_\Omega v^p &\le \frac{1}{p}e^t\int_\Omega (u+1)^p + \frac{1}{p}e^t\int_\Omega (v+1)^p \\
&= e^t\varphi(t) + e^t\tilde{\varphi}(t) \\
&\le \varphi(0) + \tilde{\varphi}(0) - \frac{\lambda_1}{2^{r_1}}\int_0^t e^s\int_\Omega (u+1)^{p+r_1-1} \\
&\quad + c_{18}\int_0^t e^s\int_\Omega (u+1)^{\frac{p+r_2-1}{r_2-1}} + c_{31}\int_0^t e^s ds
\end{aligned} \tag{3.28}$$

for all $t \in (0, T_{\max})$. Since $r_2 > 2$, $p > \frac{(2-r_1)(r_2-1)}{r_2-2}$, it is not difficult to compute that

$$\begin{aligned}
p + r_1 - 1 - \frac{p+r_2-1}{r_2-1} &= \frac{(p+r_1-1)(r_2-1) - (p+r_2-1)}{r_2-1} \\
&= \frac{p(r_2-2) + (r_1-1)(r_2-1) - (r_2-1)}{r_2-1} \\
&= \frac{p(r_2-2)}{r_2-1} + r_1 - 2 > 0.
\end{aligned} \tag{3.29}$$

Thus, we can conclude from Young's inequality that

$$c_{18}\int_0^t e^s\int_\Omega (u+1)^{\frac{p+r_2-1}{r_2-1}} \le \frac{\lambda_1}{2^{r_1}}\int_0^t e^s\int_\Omega (u+1)^{p+r_1-1} + c_{32} \tag{3.30}$$

for all $t \in (0, T_{\max})$. Combining (3.28) and (3.30), we have

$$e^t \left(\int_\Omega u^p + \int_\Omega v^p \right) \leq c_{33} + c_{34}(e^t - 1) \tag{3.31}$$

for all $t \in (0, T_{\max})$. Thus, we conclude that

$$\int_\Omega u^p + \int_\Omega v^p \leq c_{35} \tag{3.32}$$

for all $t \in (0, T_{\max})$. Therefore, the proof of Lemma 3.1 is complete. □

Now, we are in a position to complete the Theorem 1.1.

Proof of Theorem 1.1. For any $p > \max\{1 - m_1, 1 - m_2, 1 - \alpha, 1 - \beta, 1 - \alpha + (1 - \gamma)(\gamma_1 + \gamma_2), 1 - \beta + (1 - \gamma)(\gamma_1 + \gamma_2), \frac{(2-r_1)(r_2-1)}{r_2-2}, n\gamma_1, n\gamma_2, 1\}$ with $\gamma, \gamma_1, \gamma_2 > 0, m_1, m_2, \alpha, \beta \in \mathbb{R}$ and $n \geq 1$, by the elliptic L^p-estimate to the fourth equation in system (1.1) ensures the existence of $c_{36} > 0$ fulfilling

$$\|z(\cdot, t)\|_{W^{2, \frac{p}{\max\{\gamma_1, \gamma_2\}}}(\Omega)} \leq c_{36}, \qquad t \in (0, T_{\max}).$$

Thus

$$\|z(\cdot, t)\|_{W^{1,\infty}(\Omega)} \leq c_{37}, \quad t \in (0, T_{\max}), \quad \text{with } c_{37} > 0,$$

by the Sobolev imbedding theorem. Applying the parabolic regularity to the third equation in system (1.1), we have $\|\omega(\cdot, t)\|_{W^{1,\infty}(\Omega)} \leq c_{38}$ for all $t \in (0, T_{\max})$, with $c_{38} > 0$. From Lemma 3.1, we apply Moser iteration technique [1, 36] ensures

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq c_{39},$$

$$\|v(\cdot, t)\|_{L^\infty(\Omega)} \leq c_{40}$$

for all $t \in (0, T_{\max})$, with $c_{39}, c_{40} > 0$. Recalling Lemma 2.1, we infer that $T_{\max} = \infty$. Thus, we complete the proof of Theorem 1.1. □

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Second order neutral delay differential equations and their oscillatory criteria derived by linearization

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Abstract. In this paper, we investigate the oscillatory behavior of second-order neutral delay differential equations with a canonical operator of the form

$$\left(a(x)\left(v'(x)\right)^m\right)' + C(x, u(\varphi(x))) = 0.$$

We introduce new monotonicity properties of the non-oscillatory solutions of these equations, which are then used to linearize the equations and derive new oscillatory criteria. The presented results significantly improve upon existing criteria.

Keywords: second order neutral delay differential equations, asymptotic properties, linearization, oscillation theory.

2020 Mathematics Subject Classification: 34K40, 34K25, 34K06, 34K11.

1 Introduction

The aim of our paper is to investigate the asymptotic and oscillatory behavior of solutions for second-order neutral delay differential equations

$$\left(a(x)\left(v'(x)\right)^m\right)' + C(x, u(\varphi(x))) = 0, \tag{1.1}$$

where $v(x) = u(x) + b(x)u(\psi(x))$.

We make the following assumptions throughout the paper:

(H1) $a(x) \in C([x_0, \infty), (0, \infty))$ and m is a ratio of two positive odd integers,

(H2) $c(x), b(x) \in C([x_0, \infty), (0, \infty))$ and $\frac{C(x,u(x))}{u^m(x)} \geq c(x)$, $b(x) \leq 1$,

(H3) $\psi(x) \in C([x_0, \infty))$ and $\psi(x) \leq x$, $\lim_{x \to \infty} \psi(x) = \infty$,

(H4) $\varphi(x) \in C([x_0, \infty))$ and $\varphi(x) \leq x$, $\lim_{x \to \infty} \varphi(x) = \infty$.

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Additionally, we require the canonical condition

$$A(x) = \int_{x_0}^{x} \frac{dy}{a^{1/m}(y)} \to \infty \quad \text{as } x \to \infty \tag{1.2}$$

which ensures proper asymptotic behavior of solutions.

A solution of (1.1) is defined to be a function $u(x) \in C^1([T_0, \infty))$ with $T_0 \geq x_0$, satisfying $a(x)(v'(x))^m \in C^1([T_0, \infty))$ and equation (1.1) on $[T_0, \infty)$. We only consider solutions $u(x)$ of (1.1) that satisfy $\sup\{|u(x)| : x \geq T\} > 0$ for all $T \geq T_0$, and assume that such solutions exist. A solution of (1.1) is classified as oscillatory if it has arbitrarily large zeros on $[T_0, \infty)$, and non-oscillatory otherwise. The equation is said to be oscillatory if all of its solutions are oscillatory.

Second-order delay differential equations have been the focus of significant research for several decades. Numerous papers have been dedicated to the problem of establishing oscillatory criteria for various types of differential equations. Notable monographs in this area include those by Blanka Baculíková and Josef Džurina [4], Agarwal et al. [1], Došly and Řehák [5], Erbe et al. [8], Kiguradze and Chanturia [10], Ladde et al. [13], and Győri and Ladas [9], as well as numerous works [2,3,6,7,11,14,15].

Such a criterion for the second-order delay differential equations, as presented by Koplatadze et al. [11], is based on the monotonic properties $u(x) \uparrow$ and $\frac{u(x)}{x} \downarrow$ of positive solutions of $u''(x) + c(x)u(\varphi(x)) = 0$. These properties have been widely used in the literature to establish oscillatory behavior in various types of differential equations.

In this paper, we aim to establish new comparison theorems and oscillatory criteria for the investigation of second-order neutral delay differential equations by utilizing the monotonic properties of non-oscillatory solutions. Building upon the previous work of Blanka Baculíková and Josef Džurina [4], we present novel monotonic properties that can be utilized to linearize the equation, which in turn allows us to deduce the oscillation of the original equation from its linear forms. Our first objective is to establish new comparison theorems for equation (1.1). The second objective is to provide oscillatory criteria that consider the linear forms of (1.1). Finally, we seek to test the strength of the general criteria with an example. The presented results in this paper significantly enhance the existing oscillatory criteria.

Our work builds upon previous research in the field and provides a novel and useful contribution to the theory of second-order neutral delay differential equations.

2 Preliminary results

We start with some useful lemmas concerning monotonic properties of nonoscillatory solutions for studied equation.

Lemma 2.1. *Let $u(x)$ be a positive solution of (1.1). Then*

(i) $a(x)(v'(x))^m > 0$,

(ii) $\frac{v(x)}{A(x)}$ *is decreasing*

for $x \geq x_1 \geq x_0$. Moreover if

$$\int_{x_0}^{\infty} A^m(\varphi(y))(1 - b(\varphi(y)))^m c(y)dy = \infty \tag{2.1}$$

holds, then

(iii) $\lim_{x\to\infty} \frac{v(x)}{A(x)} = 0$.

Proof. Suppose that $u(x)$ is a positive solution of (1.1). Then

$$\Big(a(x)\big(v'(x)\big)^m\Big)' < 0,$$

and there exists $x_1 \geq x_0$ such that $a(x)(v'(x))^m$ has a constant sign for $x \geq x_1$. Now assume the opposite, i.e., $a(x)(v'(x))^m < 0$. This implies the existence of a constant $k > 0$ such that $a(x)(v'(x))^m \leq -k < 0$. Integrating the previous inequality from x_1 to x and using (1.2), we obtain

$$v(x) \leq v(x_1) - k^{1/m}A(x) \to -\infty \quad \text{as } x \to \infty.$$

Hence, we have shown that assuming $a(x)(v'(x))^m < 0$ leads to a contradiction, and therefore we can conclude that $a(x)(v'(x))^m > 0$.

Using the monotonicity of $a^{1/m}(x)v'(x)$, we get

$$v(x) \geq \int_{x_1}^{x} \frac{a^{1/m}(y)v'(y)}{a^{1/m}(y)}dy \geq a^{1/m}(x)v'(x)A(x), \tag{2.2}$$

which implies $\big(\frac{v(x)}{A(x)}\big)' < 0$.

On the other hand, as $\frac{v(x)}{A(x)}$ is a positive and decreasing function, there exists a positive constant δ such that

$$\lim_{x\to\infty} \frac{v(x)}{A(x)} = \delta \geq 0.$$

Assume on the contrary that $\delta > 0$. Then $\frac{v(x)}{A(x)} \geq \delta$ for $x \geq x_1$. From the definition of $v(x)$ we have

$$\begin{aligned} u(x) &= v(x) - b(x)u(\psi(x)) \\ &\geq v(x) - b(x)v(\psi(x)) \\ &\geq v(x)(1 - b(x)). \end{aligned} \tag{2.3}$$

Integrating (1.1) from x_1 to x, we obtain

$$a(x_1)\big(v'(x_1)\big)^m \geq \delta^m \int_{x_1}^{x} c(y)A^m(\varphi(y))(1 - b(\varphi(y)))^m dy$$

which for $x \to \infty$ contradicts with (2.1). So that $\lim_{x\to\infty} \frac{v(x)}{A(x)} = 0$. The proof is completed. □

Since $A(x)$ is increasing, there exists $\lambda \geq 1$ such that

$$\frac{A(x)}{A(\varphi(x))} \geq \lambda. \tag{2.4}$$

Theorem 2.2. *Let* (2.1) *hold and there exist a positive constant* n *such that*

$$\frac{1}{m}A^m(\varphi(x))a^{1/m}(x)A(x)c(x)(1 - b(\varphi(x)))^m \geq n \quad \text{for } x \geq x_0. \tag{2.5}$$

If $u(x)$ *is a positive solution of* (1.1)*, then*

(i) $\frac{v(x)}{A^{1-n}(x)}$ *is decreasing for* $x \geq x_1$,

(ii) $\frac{v(x)}{A^{n_0}(x)}$ *is increasing for* $x \geq x_1$, *where* $n_0 = n^{1/m}\lambda^n$.

Proof. Assume that $u(x)$ is a positive solution of (1.1). It is important to note that condition (iii) of Lemma 2.1 implies

$$\lim_{x\to\infty} a^{1/m}(x)v'(x) = 0. \tag{2.6}$$

Therefore, an integration of (1.1) yields

$$a^{1/m}(x)v'(x) = \left(\int_x^\infty C(y, u(\varphi(y)))dy\right)^{1/m}. \tag{2.7}$$

It is easy to see that

$$\left[\left(a^{1/m}(x)v'(x)\right)^m\right]' = m\left(a^{1/m}(x)v'(x)\right)^{m-1}\left(a^{1/m}(x)v'(x)\right)'. \tag{2.8}$$

Setting into (1.1), we have

$$\left(a^{1/m}(x)v'(x)\right)' + \frac{1}{m}\left(a^{1/m}(x)v'(x)\right)^{1-m} C(x, u(\varphi(x))) = 0. \tag{2.9}$$

Then $w(x) = a^{1/m}(x)v'(x)$ is positive decreasing and satisfies

$$w'(x) + \frac{1}{m}w^{1-m}(x)C(x, u(\varphi(x))) = 0,$$

and from (H2) we get

$$w'(x) + \frac{1}{m}w^{1-m}(x)c(x)u^m(\varphi(x)) \le 0. \tag{2.10}$$

On the other hand, (2.2) and (2.3) implies

$$\begin{aligned} v(x) &\ge a^{1/m}(x)v'(x)A(x) \\ v(x) &\ge w(x)A(x) \\ \frac{u(x)}{(1-b(x))} &\ge w(x)A(x) \end{aligned}$$

and so

$$u(\varphi(x)) \ge w(x)A(\varphi(x))(1 - b(\varphi(x))). \tag{2.11}$$

Substituting the last inequality into (2.10), we get

$$w'(x) + \frac{1}{m}c(x)A^m(\varphi(x))(1 - b(\varphi(x)))^m w(x) \le 0$$

and

$$w'(x) + \frac{n}{A(x)a^{1/m}(x)}w(x) \le 0$$

which implies

$$-w'(x)A(x) \ge \frac{n}{a^{1/m}(x)}w(x) = nv'(x).$$

We can skip the proof of the theorem as it is similar to [4, Theorem 2.3]. □

The previous results did not distinguish between the cases $m < 1$ and $m \ge 1$. However, in order to provide oscillatory criteria for (1.1), we need to consider these cases separately.

3 Oscillatory results for $m \geq 1$

To simplify our notation let us denote

$$\kappa = \frac{(1-n)^{1-m}\lambda^{n(m-1)}}{m}.$$

Now we will provide new comparison principles that significantly simplify the examination of neutral delay differential equations.

Theorem 3.1. *Let $m \geq 1$, and (2.1), (2.5) hold. Then (1.1) is oscillatory provided that*

$$\Big(a^{1/m}(x)v'(x)\Big)' + \kappa A^{m-1}(\varphi(x))c(x)(1-b(\varphi(x)))^m v(\varphi(x)) = 0 \tag{3.1}$$

is oscillatory.

Proof. Suppose the opposite of the desired result, namely that $u(x)$ is a positive solution of (1.1). Since $\frac{v(x)}{A^{1-n}(x)}$ is decreasing, we have the inequality

$$v(x) \geq \frac{a^{1/m}(x)v'(x)}{(1-n)}A(x), \tag{3.2}$$

which for $m \geq 1$ yields

$$v^{1-m}(x) \leq \frac{\big(a^{1/m}(x)v'(x)\big)^{1-m}}{(1-n)^{1-m}}A^{1-m}(x).$$

Hence

$$\Big(a^{1/m}(x)v'(x)\Big)^{1-m} \geq (1-n)^{1-m}\frac{v^{1-m}(x)}{A^{1-m}(x)}. \tag{3.3}$$

Using again the monotonic property of $\frac{v(x)}{A^{1-n}(x)}$, we get

$$v^{1-m}(x) \geq \frac{v^{1-m}(\varphi(x))}{A^{(1-m)(1-n)}(\varphi(x))}A^{(1-m)(1-n)}(x). \tag{3.4}$$

Substituting (3.4) from (3.3), we have in view of (2.4) that

$$\begin{aligned}\Big(a^{1/m}(x)v'(x)\Big)^{1-m} &\geq (1-n)^{1-m}\frac{A^{(1-m)(1-n)}(x)}{A^{(1-m)(1-n)}(\varphi(x))}\frac{v^{1-m}(\varphi(x))}{A^{1-m}(x)} \\ &\geq (1-n)^{1-m}\lambda^{n(m-1)}\frac{v^{1-m}(\varphi(x))}{A^{1-m}(\varphi(x))}.\end{aligned} \tag{3.5}$$

By combining (2.10) and (3.5), we can derive that $v(x)$ satisfies the linear differential inequality

$$\Big(a^{1/m}(x)v'(x)\Big)' + \kappa c(x)A^{m-1}(\varphi(x))(1-b(\varphi(x)))^m v(\varphi(x)) \leq 0. \tag{3.6}$$

Conversely, by [12, Corollary 1], it is ensured that the corresponding differential equation (3.1) possesses a positive solution, which contradicts the assumption made earlier, thus completing the proof. □

We will utilize the outcomes of the previous theorem to establish novel oscillatory criteria.

Theorem 3.2. *Let $m \geq 1$, and (2.1), (2.5) hold. If*

$$\limsup_{x\to\infty}\Bigg\{A^{n-1}(\varphi(x))\int_{x_1}^{\varphi(x)} c(y)A(y)A^{m-n}(\varphi(y))(1-b(\varphi(y)))^m dy + A^{n}(\varphi(x))\int_{\varphi(x)}^{x} c(y)A^{m-n}(\varphi(y))(1-b(\varphi(y)))^m dy + A^{1-n_0}(\varphi(x))\int_{x}^{\infty} c(y)A^{m+n_0-1}(\varphi(y))(1-b(\varphi(y)))^m dy\Bigg\} > \frac{1}{\kappa}$$

then (1.1) is oscillatory.

Proof. The proof of the theorem closely follows the approach taken in [4, Theorem 4.1], making it unnecessary to provide a full proof here. □

For practical applications, we often need criteria that are easier to verify. The next theorem provides such conditions:

Theorem 3.3. *Let $m \geq 1$, and (2.1), (2.5) hold. If*

$$\liminf_{x\to\infty}\int_{\varphi(x)}^{x} c(y)A^m(\varphi(y))(1-b(\varphi(y)))^m dy > \frac{1}{\kappa e}$$

then (1.1) is oscillatory.

Proof. Suppose, for the sake of contradiction, that (1.1) has a positive solution $u(x)$. By Theorem 3.2, equation (3.1) is nonoscillatory, and we can assume that it has an eventually positive solution $v(x)$. Then, we can define $w(x) = a^{1/m}(x)v'(x)$, which is positive and decreasing. Therefore, we have

$$\begin{aligned} v(x) &= v(x_1) + \int_{x_1}^{x} v'(y)dy \geq \int_{x_1}^{x} \frac{a^{1/m}(y)v'(y)}{a^{1/m}(y)}dy \\ &\geq a^{1/m}(x)v'(x)\int_{x_1}^{x}\frac{dy}{a^{1/m}(y)} = w(x)A(x). \end{aligned}$$

By substituting the expression for $v(x)$ into (3.1), we obtain a first-order differential inequality of the form

$$w'(x) + \kappa A^m(\varphi(x))c(x)(1-b(\varphi(x)))^m w(\varphi(x)) \leq 0 \tag{3.7}$$

with positive solution $w(x)$. According to [13, Theorem 2.1.1], (3.7) has no positive solution if

$$\liminf_{x\to\infty}\int_{\varphi(x)}^{x} \kappa c(y)A^m(\varphi(y))(1-b(\varphi(y)))^m dy > \frac{1}{e}.$$

This contradicts our assumption that (1.1) has a positive solution. Hence, the proof is complete. □

4 Oscillatory results for $m \in (0,1)$

We now turn our attention to the case where $0 < m < 1$. This case requires different techniques due to the different nature of the nonlinearity. The following results parallel those of the previous section, but with important modifications to account for the changed parameter range.

To simplify the notation, we define

$$\omega = \frac{n^{\frac{1-m}{m}}\lambda^{1-m}}{m(1-n_0)^{\frac{1-m}{m}}}.$$

Theorem 4.1. *Let $0 < m < 1$, and (2.1), (2.5) hold. Then (1.1) is oscillatory provided that*

$$\left(a^{1/m}(x)v'(x)\right)' + \omega A^{m-1}(x)c(x)(1-b(\varphi(x)))^m v(\varphi(x)) = 0 \tag{4.1}$$

is oscillatory.

Proof. Assume the contrary that $u(x)$ is a positive solution of (1.1). Differentiating (2.7) with respect to x leads to the equation

$$\left(a^{1/m}(x)v'(x)\right)' + \frac{1}{m}\left(\int_x^\infty C(y,u(\varphi(y)))dy\right)^{\frac{1-m}{m}} C(x,u(\varphi(x))) = 0.$$

Employing the fact that $\frac{v(x)}{A^{n_0}(x)}$ is an increasing function, we can rewrite the previous inequality as follows:

$$\left(a^{1/m}(x)v'(x)\right)' + \frac{1}{m}\frac{v^{1-m}(\varphi(x))}{A^{(1-m)n_0}(\varphi(x))}\left(\int_x^\infty \frac{c(y)b_0^m(x)dy}{A^{-mn_0}(\varphi(y))}\right)^{\frac{1-m}{m}} \frac{c(x)v^m(\varphi(x))}{b_0^{-m}(x)} \le 0,$$

where $b_0(x) = 1 - b(\varphi(x))$. Consequently, we can conclude that $u(x)$ must satisfy the linear differential inequality

$$\left(a^{1/m}(x)v'(x)\right)' + \frac{1}{m}\left(\int_x^\infty \frac{c(y)b_0^m(y)dy}{A^{-mn_0}(\varphi(y))}\right)^{\frac{1-m}{m}} \frac{b_0^m(x)c(x)v(\varphi(x))}{A^{(1-m)n_0}(\varphi(x))} \le 0. \tag{4.2}$$

Additionally, by utilizing (2.4) and (2.5), we can derive

$$\begin{aligned}\int_x^\infty \frac{c(y)(1-b(\varphi(y)))^m}{A^{-mn_0}(\varphi(y))}dy &\ge \int_x^\infty mn\frac{A^{mn_0}(\varphi(y))}{A^m(\varphi(y))a^{1/m}(y)A(y)}dy \\ &\ge mn\lambda^{m(n_0-1)}\int_x^\infty \frac{A^{m(n_0-1)-1}(y)}{a^{1/m}(y)}dy \\ &\ge \frac{n\lambda^{m(n_0-1)}}{1-n_0}A^{m(n_0-1)}(x).\end{aligned}$$

Substituting (4.2), we get

$$\left(a^{1/m}(x)v'(x)\right)' + \frac{n^{\frac{1-m}{m}}\lambda^{1-m}}{m(1-n_0)^{\frac{1-m}{m}}}\frac{A^{(1-m)(n_0-1)}(x)}{A^{(1-m)n_0}(\varphi(x))}(1-b(\varphi(x)))^m c(x)v(\varphi(x)) \le 0.$$

By applying [12, Corollary 1], we can conclude that the corresponding differential equation (4.1) also has a positive solution. This contradicts the assumption that (4.1) has no positive solutions, and therefore the proof is complete. □

Theorem 4.2. *Let* $0 < m < 1$, *and* (2.1), (2.5) *hold. If*

$$\limsup_{x\to\infty}\Bigg\{A^{n-1}(\varphi(x))\int_{x_1}^{\varphi(x)}A^{1-n}(\varphi(y))A^m(y)c(y)(1-b(\varphi(y)))^m dy + A^n(\varphi(y))\int_{\varphi(x)}^{x}A^{1-n}(\varphi(y))A^{m-1}(y)c(y)(1-b(\varphi(y)))^m dy + A^{1-n_0}(\varphi(y))\int_{x}^{\infty}A^{n_0}(\varphi(y))A^{m-1}(y)c(y)(1-b(\varphi(y)))^m dy\Bigg\} > \frac{1}{\omega},$$

then (1.1) *is oscillatory.*

Proof. The proof of the theorem closely follows the approach used in [4, Theorem 4.2]. □

Theorem 4.3. *Let* $0 < m < 1$, *and* (2.1), (2.5) *hold. If*

$$\liminf_{x\to\infty}\int_{\varphi(x)}^{x}A(\varphi(y))A^{m-1}(y)c(y)(1-b(\varphi(y)))^m dy > \frac{1}{\omega e}$$

then (1.1) *is oscillatory.*

Proof. Since the proof of the theorem closely resembles that of [4, Theorem 4.3], it may be omitted. □

5 Example

Example 5.1. Consider a second-order neutral delay differential equation in the form of a general Euler differential equation

$$\left(x^k\left(v'(x)\right)^m\right)' + \frac{c_0}{x^{(m-k+1)}}u^m(\varphi_0 x) = 0, \tag{5.1}$$

where $v(x) = u(x) + b_0u(\psi(x))$ with $c_0 > 0$, $\varphi_0 \in (0,1)$ and $0 \le b_0 < 1$.

Then

$$A(x) = \frac{mx^{\left(1-\frac{k}{m}\right)}}{m-k},\quad A(\varphi_0 x) = \frac{m(\varphi_0 x)^{\left(1-\frac{k}{m}\right)}}{m-k},\quad \lambda = \frac{A(x)}{A(\varphi_0 x)} = \varphi_0^{\left(\frac{k}{m}-1\right)},$$

$$n = \frac{c_0\varphi_0^{(m-k)}}{m}\left(\frac{m}{m-k}\right)^{(m+1)}(1-b_0)^m,\quad n_0 = \frac{c_0^{\left(\frac{1}{m}\right)}m\varphi_0^{\left(1-\frac{k}{m}\right)(1-n)}}{(m-k)^{\left(1+\frac{1}{m}\right)}}(1-b_0)$$

$$\kappa = \frac{(1-n)^{(1-m)}\varphi_0^{n(1-m)\left(1-\frac{k}{m}\right)}}{m}$$

and

$$\omega = \frac{n^{\frac{1}{m}-1}\varphi_0^{(1-m)\left(\frac{k}{m}-1\right)}}{m(1-n_0)^{\frac{1}{m}-1}}.$$

Equation (5.1) with $m \geq 1$ is oscillatory provided that,

$$mn\left\{\frac{\varphi_0^{n\left(\frac{k}{m}-1\right)}}{1-n}+\frac{\varphi_0^{n\left(\frac{k}{m}-1\right)}-1}{n}+\frac{1}{1-n_0}\right\}>\frac{1}{\kappa}$$

by Theorem 3.2, or

$$n(m-k)\ln\frac{1}{\varphi_0}>\frac{1}{e\kappa}$$

by Theorem 3.3.

Equation (5.1) with $0 < m < 1$ is oscillatory provided that,

$$mn\varphi_0^{(1-m)\left(1-\frac{k}{m}\right)}\left\{\frac{\varphi_0^{n\left(\frac{k}{m}-1\right)}}{1-n}+\frac{\varphi_0^{n\left(\frac{k}{m}-1\right)}-1}{n}+\frac{1}{1-n_0}\right\}>\frac{1}{\omega}$$

by Theorem 4.2 or

$$n(m-k)\varphi_0^{(1-m)\left(1-\frac{k}{m}\right)}\ln\frac{1}{\varphi_0}>\frac{1}{e\omega}$$

by Theorem 4.3.

6 Conclusion

In this paper, we have established new oscillation criteria for second-order neutral delay differential equations by utilizing the monotonic properties of non-oscillatory solutions. Our approach generalizes previous results by considering a neutral term and a more general nonlinearity. The presented theorems provide effective tools for determining the oscillatory behavior of these equations.

It is worth noting that when we set $b(x) = 0$ (eliminating the neutral term) and $C(x, u(x)) = c(x)u^m(x)$, our results reduce to those obtained by Baculíková and Džurina [4]. This confirms the consistency of our approach while demonstrating its broader applicability to more complex equations.

Declaration of generative AI and AI-assisted technologies in the writing process

During the preparation of this work, the authors used ChatGPT in order to improve the readability and language of the article. After using this tool, the authors reviewed and edited the content as needed and take full responsibility for the content of the published article.

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Decreasing solutions of cyclic second-order difference systems

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Abstract. The existence and asymptotic behavior of positive decreasing solutions to the cyclic second-order nonlinear difference system

$$\Delta(p_i(n)|\Delta x_i(n)|^{\alpha_i-1}\Delta x_i(n)) = q_i(n)|x_{i+1}(n+1)|^{\beta_i-1}x_{i+1}(n+1), \quad i=\overline{1,N},$$

are studied, where $x_{N+1}=x_1$, $p_i=\{p_i(n)\}$ and $q_i=\{q_i(n)\}$ are positive real sequences, and the constants α_i and β_i, $i=\overline{1,N}$ are positive and satisfy the sublinear condition $\alpha_1\alpha_2\cdot\ldots\cdot\alpha_N>\beta_1\beta_2\cdot\ldots\cdot\beta_N$. Two distinct types of positive decreasing solutions are considered, depending on whether the series $\sum_{n=1}^{\infty}p_i(n)^{-1/\alpha_i}$ is divergent or convergent. In the first case, necessary and sufficient conditions for the existence of solutions tending to a positive constant as well as solutions tending to zero, while their associated quasi-differences approach a nonzero limit, are rigorously derived using fixed point techniques. In the second case, the analysis is focused on solutions whose components and quasi-differences both tend to zero. Under the additional assumption that the coefficient sequences are regularly varying, necessary and sufficient conditions for the existence of such solutions are obtained, and their precise asymptotic behavior is determined using the theory of discrete regular variation.

Keywords: cyclic system of difference equations, Emden–Fowler type difference equation, nonlinear difference equations, decreasing solutions, asymptotic behavior, regularly varying sequence.

2020 Mathematics Subject Classification: 39A22, 39A12, 26A12.

1 Introduction

The cyclic second-order nonlinear system of difference equations examined in this paper is of the following form:

$$\Delta(p_i(n)|\Delta x_i(n)|^{\alpha_i-1}\Delta x_i(n)) = q_i(n)|x_{i+1}(n+1)|^{\beta_i-1}x_{i+1}(n+1), \tag{SE}$$

where $i=\overline{1,N}$, $x_{N+1}=x_1$, $n\in\mathbb{N}$, and following conditions hold:

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(a) α_i and β_i, $i = \overline{1,N}$ are positive constants such that

$$\alpha_1\alpha_2 \cdot \ldots \cdot \alpha_N > \beta_1\beta_2 \cdot \ldots \cdot \beta_N;$$

(b) $p_i = \{p_i(n)\}$ and $q_i = \{q_i(n)\}$ are positive real sequences;

(c) All p_i, $i = \overline{1,N}$ simultaneously satisfy either

(I)
$$S_i = \sum_{n=1}^{\infty} \frac{1}{p_i(n)^{1/\alpha_i}} = \infty,$$

or

(II)
$$S_i = \sum_{n=1}^{\infty} \frac{1}{p_i(n)^{1/\alpha_i}} < \infty.$$

If (I) holds, the following notation will be used

$$P_i(n) = \sum_{k=1}^{n-1} \frac{1}{p_i(k)^{1/\alpha_i}}, \quad i = \overline{1,N} \tag{1.1}$$

while if (II) holds, the following notation will be used

$$\pi_i(n) = \sum_{k=n}^{\infty} \frac{1}{p_i(k)^{1/\alpha_i}}, \quad i = \overline{1,N} \tag{1.2}$$

System (SE) is referred to as sublinear if the condition (a) holds. When the opposite inequality is satisfied, the system is termed superlinear. If $\alpha_1\alpha_2 \cdot \ldots \cdot \alpha_N = \beta_1\beta_2 \cdot \ldots \cdot \beta_N$, then system is called half-linear.

Over the past fifty years, the application of difference equations in solving problems throughout statistics, engineering, and various scientific disciplines has significantly expanded. The development of high-speed digital computer technology has motivated the application of difference equations to ordinary and partial differential equations. Apart from this, difference equations are very useful for analyzing electrical, mechanical, thermal and other systems, the behavior of electric-wave filters and other filters, insulator strings, crankshafts of multi-cylinder engines among others.

As the generalization of the most studied second-order nonlinear differential equation, known as Emden–Fowler type equation, many authors have studied the differential equation

$$\big(p(t)|x'(t)|^{\alpha-1}x'(t)\big)' \pm q(t)|x(t)|^{\beta-1}x(t) = 0,$$

whose properties such as existence, uniqueness and continuity of the solution, oscillatory and nonoscillatory behavior of solutions have been examined in monographs [15, 27] as well as in papers [9, 10, 22, 35, 49]. The discrete counterpart of this equation

$$\Delta(p(n)|\Delta x(n)|^{\alpha-1}\Delta x(n)) \pm q(n)|x(n+1)|^{\beta-1}x(n+1) = 0, \tag{1.3}$$

has also attracted many researchers, see e.g. [11–14] and monographs [1, 3].

The qualitative analysis of second-order nonlinear difference equations was further extended to the study of two-dimensional first-order and second-order nonlinear difference systems [4, 21, 28, 30, 31], as well as to symmetric and close-to-symmetric systems of difference equations [6, 41, 42, 44–46].

As a further development of the research on second-order difference equations of Emden–Fowler type, the consideration of cyclic second-order systems of difference equations was naturally imposed. The study of such systems was first suggested in [17], and further attention has been given to them and to the close-to-cyclic difference equations in [29,38–40,43,47], in addition to the already quoted papers. A system of the form (SE) could be observed in the analysis of numerical methods for heat and fluid transfer in layered materials (e.g., cylindrical thermal insulation or heat transfer through geological layers).

In the continuous case, this type of system has been studied by Jaroš and Kusano in [18–20]. However, in the existing literature, there are no results concerning asymptotic analysis of solutions of a cyclic system of second-order difference equations, except in the work of Kapešić [23]. In this regard, the first task will be to classify positive decreasing solutions based on their behavior at infinity. Finding the necessary and sufficient conditions for the existence of all possible types of positive decreasing solutions is the second task. The last and most challenging task is obtaining the precise asymptotic formulas of these solutions.

It should be mentioned that the obtained results can be applied to cyclic systems of N first-order difference equations, in the case N is even.

2 Classification of positive decreasing solutions

By a solution of (SE) we mean a vector sequence

$$\mathbf{x} = (x_1, x_2, \dots, x_N) \in {}^{\mathbb{N}}\mathbb{R} \times \dots \times {}^{\mathbb{N}}\mathbb{R}, \qquad x_i = \{x_i(n)\}_{n\in\mathbb{N}}$$

where ${}^{\mathbb{N}}\mathbb{R} = \{f \mid f : \mathbb{N} \to \mathbb{R}\}$, whose components $x_i = \{x_i(n)\}_{n\in\mathbb{N}}$, $i = \overline{1,N}$ satisfy (SE). In what following, we will observe sequences x_i for sufficiently large n, i.e. $n \geq n_0$, for some $n_0 \in \mathbb{N}$. Therefore, we introduced notation ${}^{\mathbb{N}_{n_0}}\mathbb{R} = \{f \mid f : \mathbb{N}_{n_0} \to \mathbb{R}\}$, where $\mathbb{N}_{n_0} = \{n \in \mathbb{N} \mid n \geq n_0\}$.

A solution $\mathbf{x}$ is called nonoscillatory if all its components are eventually of one sign. Because of sign condition on the coefficients, if one component is nonoscillatory, then all components are nonoscillatory and eventually monotone, and therefore they have limit. A nonoscillatory solution is called positive if all its components are eventually positive. Our aim is to study the existence and asymptotic behavior of positive decreasing solutions of (SE), that is, solutions whose components are both eventually positive and decreasing, i.e. satisfying

$$x_i(n) > 0, \quad \Delta x_i(n) < 0, \quad \text{for } n \geq n_0, \quad i = \overline{1,N}. \tag{2.1}$$

We denote by $\mathcal{DS}$ the set of all the solutions of (SE) whose components are all eventually positive decreasing. From (2.1), for x_i, $i = \overline{1,N}$, one of the following two cases holds:

$$(i) \quad \lim_{n\to\infty} x_i(n) = k_i,\ k_i > 0 \quad \text{or} \quad (ii) \quad \lim_{n\to\infty} x_i(n) = 0.$$

For every component of any solution $\mathbf{x}$ of (SE), let we denote by $x_i^{[1]} = \{x_i^{[1]}(n)\}$ its quasi-difference $x_i^{[1]}(n) = p_i(n)|\Delta x_i(n)|^{\alpha_i - 1}\Delta x_i(n)$, $i = \overline{1,N}$. Then, for $x_i^{[1]}$, $i = \overline{1,N}$, one of the following two cases holds:

$$(iii) \quad \lim_{n\to\infty} x_i^{[1]}(n) = -\omega_i,\ \omega_i > 0 \quad \text{or} \quad (iv) \quad \lim_{n\to\infty} x_i^{[1]}(n) = 0.$$

However, if (I) holds, then only (iv) may hold. Indeed, if (iii) holds, then $-p_i(n)(-\Delta x_i(n))^{\alpha_i} \leq -\omega_i$, $n \geq n_0$ i.e. $x_i(n) \leq x_i(n_0) - \omega_i^{1/\alpha_i} \sum_{k=n_0}^{n-1} p_i(k)^{-1/\alpha_i}$. As the right-hand side tends to $-\infty$ contradicts positivity of x_i, we have the desired conclusion.

These leads to the following classification of positive decreasing solution: if (I) holds, each component x_i of positive decreasing solution $\mathbf{x}$ satisfies:

$$(SD1)\ \lim_{n\to\infty} x_i(n) = \lim_{n\to\infty} x_i^{[1]}(n) = 0,$$

$$(AC)\ \lim_{n\to\infty} x_i(n) = k_i > 0 \quad \Leftrightarrow \quad x_i(n) \sim k_i,\ n \to \infty,$$

while if (II) holds each component x_i of positive decreasing solution $\mathbf{x}$ satisfies

$$(SD2)\ \lim_{n\to\infty} x_i(n) = \lim_{n\to\infty} x_i^{[1]}(n) = 0,$$

$$(P1)\ \lim_{n\to\infty} x_i(n) = 0,\ \lim_{n\to\infty} x_i^{[1]}(n) = -\omega_i < 0,$$
$$\Leftrightarrow \quad x_i(n) \sim W_i\, \pi_i(n),\ n \to \infty, \quad W_i = \omega_i^{1/\alpha_i}$$

$$(AC)\ \lim_{n\to\infty} x_i(n) = k_i > 0 \quad \Leftrightarrow \quad x_i(n) \sim k_i,\ n \to \infty,$$

where the following asymptotic relation has been used

$$f(n) \sim g(n),\ n \to \infty \ \Leftrightarrow\ \lim_{n\to\infty} \frac{f(n)}{g(n)} = 1.$$

Using the Stolz–Cesàro Theorem (see Theorem 4.5), solutions satisfying $(SD1)$ and $(SD2)$ can be characterized as

$$(SD1) \quad x_i(n) \prec P_i(n),\ n \to \infty, \qquad (SD2) \quad x_i(n) \prec \pi_i(n),\ n \to \infty,$$

where the following asymptotic relation has been used

$$f(n) \prec g(n),\ n \to \infty \ \Leftrightarrow\ \lim_{n\to\infty} \frac{f(n)}{g(n)} = 0.$$

Solutions of type $(P1)$ and (AC) are called *primitive solutions*, while solutions $(SD1)$, $(SD2)$ are called *strongly decreasing*. Necessary and sufficient conditions for the existence of primitive solutions will be established in the Section 5. On the other hand, the existence and precise asymptotic formulas of strongly decreasing solutions are not easy to determine in the general case. Therefore, in Section 6, we will assume that the coefficients of the system are regularly varying sequences and thus, restricting consideration to the class of regularly varying solutions, we provide necessary and sufficient conditions for the existence of such solutions.

3 Regularly varying sequences

The theory of regularly varying sequences, often called Karamata sequences (see [26]), was developed during the seventies by Galambos, Seneta and Bojanić in [8, 16]. However, until the appearance of the paper of Matucci and Řehák [32], the connection between regularly varying sequences and difference equations was not considered. In this paper, as well as in the following ones [33, 34, 36, 37], the theory of regularly varying sequences is further developed and applied in the asymptotic analysis of linear and half-linear difference equations of the second-order, giving necessary and sufficient conditions for the existence of regularly varying solutions of these equations. After this, further development of the discrete theory of regular variation, as well as its application to nonlinear difference equations of type Emden–Fowler type, can be found in [5], as well as in [24].

In this section, we present basic definitions and properties of regularly varying sequences that will be used in the main results. For a thorough discussion of regular variation, the reader is referred to Bingham et al. [7].

There are two main approaches in the basic theory of regularly varying sequences: the approach due to Karamata [26], based on a definition that can be understood as a direct discrete counterpart of simple and elegant continuous definition (Definition 3.1), and the approach due to Galambos and Seneta [16], based on purely sequential definition (Definition 3.2). Bojanić and Seneta have shown in [8] the equivalence of these two definitions.

Definition 3.1 (Karamata [26])**.** A positive sequence $y = \{y(k)\}$, $k \in \mathbb{N}$ is said to be *regularly varying of index* $\rho \in \mathbb{R}$ if

$$\lim_{k\to\infty} \frac{y([\lambda k])}{y(k)} = \lambda^\rho \quad \text{for } \forall \lambda > 0,$$

where $[n]$ denotes the integer part of n.

Definition 3.2 (Galambos and Seneta [16])**.** A positive sequence $y = \{y(k)\}, n \in \mathbb{N}$ is said to be *regularly varying of index* $\rho \in \mathbb{R}$ if there exists a positive sequence $\{\alpha(k)\}$ satisfying

$$\lim_{k\to\infty} \frac{y(k)}{\alpha(k)} = C, \quad 0 < C < \infty, \qquad \lim_{k\to\infty} k\,\frac{\Delta\alpha(k-1)}{\alpha(k)} = \rho\,.$$

If $\rho = 0$, then y is said to be *slowly varying*. The sets of regularly varying sequences with index ρ and slowly varying sequences are denoted $\mathcal{RV}(\rho)$ and $\mathcal{SV}$, respectively.

The concept of normalized regularly varying sequences was introduced by Matucci and Rehak in [33], where they also offered a modification of Definition 3.2, i.e. they proved that the second limit in Definition 3.2 can be replaced with

$$\lim_{k\to\infty} k\,\frac{\Delta\alpha(k)}{\alpha(k)} = \rho\,.$$

Definition 3.3. A positive sequence $y = \{y(k)\}, k \in \mathbb{N}$ is said to be *normalized regularly varying of index* $\rho \in \mathbb{R}$ if it satisfies

$$\lim_{k\to\infty} \frac{k\Delta y(k)}{y(k)} = \rho.$$

If $\rho = 0$, then y is called a *normalized slowly varying sequence.*

The notations $\mathcal{NRV}(\rho)$ and $\mathcal{NSV}$ are most commonly used to denote the set of all normalized regularly varying sequences of index ρ and the set of all normalized slowly varying sequences, respectively.

Typical examples are:

$$\{\log k\} \in \mathcal{NSV}, \quad \{k^\rho \log k\} \in \mathcal{NRV}(\rho), \quad \{1 + (-1)^k/k\} \in \mathcal{SV} \setminus \mathcal{NSV}\,.$$

In order to present results for a system of difference equations, we need to define a regularly varying vector $\mathbf{x} \in {}^{\mathbb{N}}\mathbb{R} \times \ldots \times {}^{\mathbb{N}}\mathbb{R}$, where ${}^{\mathbb{N}}\mathbb{R} = \{f \mid f : \mathbb{N} \to \mathbb{R}\}$.

Definition 3.4. A vector $\mathbf{x} \in {}^{\mathbb{N}}\mathbb{R} \times \ldots \times {}^{\mathbb{N}}\mathbb{R}$, $\mathbf{x} = (\{x_1(n)\}, \ldots, \{x_N(n)\})$ is said to be regularly varying of index $(\rho_1, \rho_2, \ldots, \rho_N)$ if $x_i = \{x_i(n)\} \in \mathcal{RV}(\rho_i)$ for $i = \overline{1, N}$. If all ρ_i are positive (or negative), then $\mathbf{x}$ is called regularly varying vector sequence of positive (or negative) index $(\rho_1, \rho_2, \ldots, \rho_N)$. The set of all regularly varying vectors of index $(\rho_1, \rho_2, \ldots, \rho_N)$ is denoted by $\mathcal{RV}(\rho_1, \rho_2, \ldots, \rho_N)$.

Various necessary and sufficient conditions for a sequence of positive numbers to be regularly varying have been established (see [8,16,32,33]) and consequently, any of these can be used to define regularly varying sequence. The one that is the most important is the following Representation theorem (see [8, Theorem 3]), while some other representation formula for regularly varying sequences was established in [33, Lemma 1].

Theorem 3.5 (Representation theorem). *A positive sequence $\{y(k)\}, k \in \mathbb{N}$ is said to be regularly varying of index $\rho \in \mathbb{R}$ if and only if there exists sequences $\{c(k)\}$ and $\{\delta(k)\}$ such that*

$$\lim_{k\to\infty} c(k) = c_0 \in (0,\infty) \quad \textit{and} \quad \lim_{k\to\infty} \delta(k) = 0,$$

and

$$y(k) = c(k)\, k^{\rho} \exp\left(\sum_{i=1}^{k} \frac{\delta(i)}{i}\right).$$

In [8], a very useful embedding theorem was proved, which gives the possibility of using the continuous theory in developing a theory of regularly varying sequences. However, as noted in [8], such development is not generally straightforward and sometimes far from a simple imitation of arguments for regularly varying functions.

Theorem 3.6 (Embedding Theorem). *If $y = \{y(n)\}$ is regularly varying sequence of index $\rho \in \mathbb{R}$, then function $Y(t)$ defined on $[0,\infty)$ by $Y(t) = y([t])$ is a regularly varying function of index ρ. Conversely, if $Y(t)$ is a regularly varying function on $[0,\infty)$ of index ρ, then a sequence $\{y(k)\}$, $y(k) = Y(k)$, $k \in \mathbb{N}$ is regularly varying of index ρ.*

Next, we state some important properties of $\mathcal{RV}$ sequences useful for the development of the asymptotic behavior of solutions of (SE) in the subsequent sections (for more properties and proofs see [8,32]).

Theorem 3.7. *The following properties hold:*

(i) $y \in \mathcal{RV}(\rho)$ *if and only if* $y(k) = k^{\rho}\, l(k)$, *where* $l = \{l(k)\} \in \mathcal{SV}$.

(ii) Let $x \in \mathcal{RV}(\rho_1)$ *and* $y \in \mathcal{RV}(\rho_2)$. *Then,* $xy \in \mathcal{RV}(\rho_1+\rho_2)$, $x+y \in \mathcal{RV}(\rho)$, $\rho = \max\{\rho_1,\rho_2\}$ *and* $1/x \in \mathcal{RV}(-\rho_1)$.

(iii) If $y \in \mathcal{RV}(\rho)$, *then* $\lim_{k\to\infty} \frac{y(k+1)}{y(k)} = 1$.

(iv) If $l \in \mathcal{SV}$ *and* $l(k) \sim L(k)$, $k \to \infty$, *then,* $L \in \mathcal{SV}$.

(v) If $l \in \mathcal{SV}$, *then for any* $\varepsilon > 0$,

$$\lim_{k\to\infty} k^{\varepsilon} l(k) = \infty, \qquad \lim_{k\to\infty} k^{-\varepsilon} l(k) = 0,$$

(vi) If $y \in \mathcal{NRV}(\rho)$, *then* $\{k^{-\sigma}y(k)\}$ *is eventually increasing for each* $\sigma < \rho$ *and* $\{k^{-\mu}y(k)\}$ *is eventually decreasing for each* $\mu > \rho$.

The following theorem can be seen as *the discrete analog of the Karamata's integration theorem* and plays a central role in proving this paper main results. Proof of this theorem can be found in [24]. Also, some parts of this theorem's proof can be found in [8] and [36].

Theorem 3.8. *Let $l = \{l(n)\} \in \mathcal{SV}$.*

(i) If $\alpha > -1$, then $\lim_{n\to\infty} \frac{1}{n^{\alpha+1}l(n)} \sum_{k=1}^{n} k^{\alpha} l(k) = \frac{1}{1+\alpha}$;

(ii) If $\alpha < -1$, then $\lim_{n\to\infty} \frac{1}{n^{\alpha+1}l(n)} \sum_{k=n}^{\infty} k^{\alpha} l(k) = -\frac{1}{1+\alpha}$;

(iii) If $\sum_{k=1}^{\infty} \frac{l(k)}{k} < \infty$, then $S_{\star}(n) = \sum_{k=n}^{\infty} \frac{l(k)}{k}$, $S_{\star} \in \mathcal{SV}$ and $\lim_{n\to\infty} \frac{S_{\star}(n)}{l(n)} = \infty$;

(iv) If $\sum_{k=1}^{\infty} \frac{l(k)}{k} = \infty$, then $S^{\star}(n) = \sum_{k=1}^{n} \frac{l(k)}{k}$, $S^{\star} \in \mathcal{SV}$ and $\lim_{n\to\infty} \frac{S^{\star}(n)}{l(n)} = \infty$.

Remark 3.9. It is easy to see, in view of Theorem 3.7-(iii) and Theorem 3.8-(i), that for $l \in \mathcal{SV}$, if $\alpha > -1$, we have

$$\sum_{k=1}^{n-1} k^{\alpha} l(k) \sim \frac{(n-1)^{\alpha+1} l(n-1)}{\alpha+1} \sim \frac{n^{\alpha+1} l(n)}{\alpha+1} \sim \sum_{k=1}^{n} k^{\alpha} l(k), \qquad n \to \infty,$$

and since $\lim_{n\to\infty} \sum_{k=1}^{n-1} k^{\alpha} l(k) = \infty$, we also get

$$\sum_{k=n_0}^{n} k^{\alpha} l(k) \sim \sum_{k=1}^{n} k^{\alpha} l(k), \qquad n \to \infty.$$

If $\lim_{n\to\infty} \sum_{k=1}^{n} k^{-1} l(k) = \infty$, we have

$$\sum_{k=n_0}^{n} k^{-1} l(k) \sim \sum_{k=1}^{n} k^{-1} l(k), \qquad n \to \infty.$$

4 Basic concepts

In this section, we will state the basic notation and assertions necessary for the proofs of the main results in the following sections.

The existence of solutions will be demonstrated using fixed point techniques. Actually, the following two fixed point theorems will be applied throughout the paper.

Theorem 4.1 (Knaster–Tarski fixed point theorem [2])**.** *Let X be a partially ordered Banach space with ordering $\leq$. Let M be a subset of X with the following properties: The infimum of M belongs to M and every nonempty subset of M has a supremum which belongs to M. Let $\mathcal{F} : M \to M$ be an increasing mapping, i.e. $x \geq y$ implies $\mathcal{F}x \geq \mathcal{F}y$. Then $\mathcal{F}$ has a fixed point in M.*

Theorem 4.2 (Schauder–Tychonoff fixed point theorem [2])**.** *Let S be closed, convex, nonempty subset of a locally convex topological vector space X. Let T be a continuous mapping from S to itself, such that TS is relatively compact. Then T has a fixed point.*

To prove that appropriately constructed operator T is continuous, we will apply the following theorem.

Theorem 4.3 (Discrete Lebesgue's dominated convergence theorem [1])**.** *Let $\{a^{(m)}(k)\}$ be a double real sequence, $a^{(m)}(k) \geq 0$ for $m, k \in \mathbb{N}$ such that $\lim_{m\to\infty} a^{(m)}(k) = A(k)$, for every $k \in \mathbb{N}$. Assume that the series $\sum_{k=1}^{\infty} a^{(m)}(k)$ is totally convergent, that is, there exists a sequence $\{\alpha(k)\}$ such that $a^{(m)}(k) \leq \alpha(k)$ for all $m, k \in \mathbb{N}$, with $\sum_{k=1}^{\infty} \alpha(k) < \infty$. Then, the series $\sum_{k=1}^{\infty} A(k)$ converges and*

$$\lim_{m\to\infty} \sum_{k=1}^{\infty} a^{(m)}(k) = \sum_{k=1}^{\infty} A(k).$$

To apply the Schauder–Tychonoff fixed point theorem, relatively compactness of the set TS must be verified and for that purpose the following statement will be used. This theorem represent a discrete version of the Arzelà–Ascoli theorem and it is known as Cheng–Patula theorem (see [14]).

Theorem 4.4. *A bounded, uniformly Cauchy subset Ω of l^∞ is relatively compact.*

For the proof of regularity of solutions, Stolz–Cesàro Theorem will be used. Therefore, we recall two variants of the Stolz–Cesàro theorem (see [48]).

Theorem 4.5. *If $f = \{f(n)\}$ is a strictly increasing sequence of positive real numbers, such that $\lim_{n\to\infty} f(n) = \infty$, then for any sequence $g = \{g(n)\}$ of positive real numbers one has the inequalities:*

$$\liminf_{n\to\infty} \frac{\Delta f(n)}{\Delta g(n)} \leq \liminf_{n\to\infty} \frac{f(n)}{g(n)} \leq \limsup_{n\to\infty} \frac{f(n)}{g(n)} \leq \limsup_{n\to\infty} \frac{\Delta f(n)}{\Delta g(n)}.$$

In particular, if the sequence $\{\Delta f(n)/\Delta g(n)\}$ has a limit, then

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{\Delta f(n)}{\Delta g(n)}. \tag{4.1}$$

Theorem 4.6. *Let $f = \{f(n)\}$, $g = \{g(n)\}$ be sequences of positive real numbers, such that*

(i) $\lim_{n\to\infty} f(n) = \lim_{n\to\infty} g(n) = 0$;

(ii) the sequence g is strictly monotone;

(iii) the sequence $\{\Delta f(n)/\Delta g(n)\}$ has a limit.

Then, a sequence $\{f(n)/g(n)\}$ is convergent and (4.1) holds.

In Section 6 analyzing the regularly varying solutions of the system with regularly varying coefficients, we assume $p_i \in \mathcal{RV}(\lambda_i)$, $q_i \in \mathcal{RV}(\mu_i)$, $i = \overline{1,N}$ and express them as follows:

$$p_i(n) = n^{\lambda_i} l_i(n), \quad q_i(n) = n^{\mu_i} m_i(n), \quad l_i, m_i \in \mathcal{SV}, \quad i = \overline{1,N}, \tag{4.2}$$

while components of the regularly varying solutions $\mathbf{x} \in \mathcal{RV}(\rho_1, \rho_2, \ldots, \rho_N)$ of the observed system are expressed in the form

$$x_i(n) = n^{\rho_i} \xi_i(n), \quad \xi_i \in \mathcal{SV}, \quad i = \overline{1,N}. \tag{4.3}$$

We also assume that all sequences p_i, $i = \overline{1,N}$ satisfy either (I) or (II). Condition (I) is satisfied if and only if index of regularity λ_i satisfies either

$$\lambda_i < \alpha_i, \tag{4.4}$$

or

$$\lambda_i = \alpha_i \quad \text{and} \quad S_i = \sum_{n=1}^{\infty} n^{-1} l_i(n)^{-\frac{1}{\alpha_i}} = \infty. \tag{4.5}$$

Using Theorem 3.8, if (4.4) holds, the following asymptotic relation is obtained for the sequence $P_i = \{P_i(n)\}$ defined by (1.1):

$$P_i(n) \sim \frac{\alpha_i}{\alpha_i - \lambda_i} n^{\frac{\alpha_i - \lambda_i}{\alpha_i}} l_i(n)^{-\frac{1}{\alpha_i}}, \quad n \to \infty, \tag{4.6}$$

implying that $P_i \in \mathcal{RV}\left(\frac{\alpha_i-\lambda_i}{\alpha_i}\right)$. Clearly, if (4.5) holds, $P_i \in \mathcal{SV}$. Condition (II) is satisfied if and only if either

$$\lambda_i > \alpha_i, \tag{4.7}$$

or

$$\lambda_i = \alpha_i \quad \text{and} \quad S_i = \sum_{n=1}^{\infty} n^{-1} l_i(n)^{-\frac{1}{\alpha_i}} < \infty. \tag{4.8}$$

If (4.7) holds, using Theorem 3.8, the following asymptotic relation is obtained for the sequence $\pi_i = \{\pi_i(n)\}$ defined by (1.2):

$$\pi_i(n) \sim \frac{\alpha_i}{\lambda_i - \alpha_i} n^{\frac{\alpha_i-\lambda_i}{\alpha_i}} l_i(n)^{-\frac{1}{\alpha_i}}, \quad n \to \infty, \tag{4.9}$$

implying that $\pi_i \in \mathcal{RV}\left(\frac{\alpha_i-\lambda_i}{\alpha_i}\right)$. If (4.8) holds, $\pi_i \in \mathcal{SV}$.

Also, to simplify notation we denote $A_N = \alpha_1\alpha_2 \cdot \ldots \cdot \alpha_N$, $B_N = \beta_1\beta_2 \cdot \ldots \cdot \beta_N$ and use matrix

$$M = \begin{pmatrix} 1 & \frac{\beta_1}{\alpha_1} & \frac{\beta_1\beta_2}{\alpha_1\alpha_2} & \ldots & \frac{\beta_1\beta_2\cdot\ldots\cdot\beta_{N-2}}{\alpha_1\alpha_2\cdot\ldots\cdot\alpha_{N-2}} & \frac{\beta_1\beta_2\cdot\ldots\cdot\beta_{N-1}}{\alpha_1\alpha_2\cdot\ldots\cdot\alpha_{N-1}} \\ \frac{\beta_2\beta_3\cdot\ldots\cdot\beta_N}{\alpha_2\alpha_3\cdot\ldots\cdot\alpha_N} & 1 & \frac{\beta_2}{\alpha_2} & \ldots & \frac{\beta_2\beta_3\cdot\ldots\cdot\beta_{N-2}}{\alpha_2\alpha_3\cdot\ldots\cdot\alpha_{N-2}} & \frac{\beta_2\beta_3\cdot\ldots\cdot\beta_{N-1}}{\alpha_2\alpha_3\cdot\ldots\cdot\alpha_{N-1}} \\ \frac{\beta_3\beta_4\cdot\ldots\cdot\beta_N}{\alpha_3\alpha_4\cdot\ldots\cdot\alpha_N} & \frac{\beta_3\cdot\ldots\cdot\beta_N\beta_1}{\alpha_3\cdot\ldots\cdot\alpha_N\alpha_1} & 1 & \ldots & \frac{\beta_3\beta_4\cdot\ldots\cdot\beta_{N-2}}{\alpha_3\alpha_4\cdot\ldots\cdot\alpha_{N-2}} & \frac{\beta_3\beta_4\cdot\ldots\cdot\beta_{N-1}}{\alpha_3\alpha_4\cdot\ldots\cdot\alpha_{N-1}} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{\beta_{N-1}\beta_N}{\alpha_{N-1}\alpha_N} & \frac{\beta_{N-1}\beta_N\beta_1}{\alpha_{N-1}\alpha_N\alpha_1} & \frac{\beta_{N-1}\beta_N\beta_1\beta_2}{\alpha_{N-1}\alpha_N\alpha_1\alpha_2} & \ldots & 1 & \frac{\beta_{N-1}}{\alpha_{N-1}} \\ \frac{\beta_N}{\alpha_N} & \frac{\beta_N\beta_1}{\alpha_N\alpha_1} & \frac{\beta_N\beta_1\beta_2}{\alpha_N\alpha_1\alpha_2} & \ldots & \frac{\beta_N\beta_1\cdot\ldots\cdot\beta_{N-2}}{\alpha_N\alpha_1\cdot\ldots\cdot\alpha_{N-2}} & 1 \end{pmatrix}, \tag{4.10}$$

whose elements will be denoted by $M = (M_{ij})$. The i-th row of (M_{ij}) is obtained by shifting the vector

$$\left(1, \frac{\beta_i}{\alpha_i}, \frac{\beta_i\beta_{i+1}}{\alpha_i\alpha_{i+1}}, \ldots, \frac{\beta_i\beta_{i+1}\cdot\ldots\beta_{i+(N-2)}}{\alpha_i\alpha_{i+1}\cdot\ldots\alpha_{i+(N-2)}}\right), \quad \alpha_{N+j} = \alpha_j,\ \beta_{N+j} = \beta_j,\ j = \overline{1, N-2}$$

$(i-1)$-times to the right cyclically, so that elements $M_{ij}, M_{ji}, i > j$, satisfy the relation

$$M_{ij}M_{ji} = \frac{\beta_1\beta_2 \cdot \ldots \cdot \beta_N}{\alpha_1\alpha_2 \cdot \ldots \cdot \alpha_N}, \quad i > j, \quad i = \overline{2, N}.$$

It is easy to see that elements of matrix M satisfy for $i = \overline{1, N}$, $j = \overline{1, N}$

$$M_{i+1,i}\frac{\beta_i}{\alpha_i} = \frac{B_N}{A_N}, \quad M_{i+1,j}\frac{\beta_i}{\alpha_i} = M_{ij} \quad \text{for } j \neq i,\ M_{N+1,j} = M_{1,j}. \tag{4.11}$$

The next matrix also plays important note in the proof of main results:

$$A = \begin{pmatrix} 1 & -\frac{\beta_1}{\alpha_1} & 0 & \ldots & 0 & 0 \\ 0 & 1 & -\frac{\beta_2}{\alpha_2} & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & & \vdots & \vdots \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & -\frac{\beta_{N-1}}{\alpha_{N-1}} \\ -\frac{\beta_N}{\alpha_N} & 0 & 0 & \ldots & 0 & 1 \end{pmatrix}. \tag{4.12}$$

Since,

$$\det(A) = 1 - \frac{\beta_1\beta_2 \cdot \ldots \cdot \beta_N}{\alpha_1\alpha_2 \cdot \ldots \cdot \alpha_N} > 0\,,$$

the matrix A is invertible and its inverse matrix is given by

$$A^{-1} = \frac{A_N}{A_N - B_N} M. \tag{4.13}$$

Throughout the text, $n \geq n_0$ means that n is sufficiently large so that n_0 need not to be the same at each occurrence.

5 Existence of primitive decreasing solutions

The asymptotic behavior of primitive solutions is evident from the classification itself. The following theorems provide necessary and sufficient conditions for the existence of these solutions, using Fixed point theory and without the assumption that coefficients are regularly varying sequences.

Theorem 5.1. *The system (SE) has a solution $\mathbf{x} \in \mathcal{DS}$ with each component satisfying (AC) if and only if*

$$\sum_{n=1}^{\infty} \left(\frac{1}{p_i(n)} \sum_{k=n}^{\infty} q_i(k) \right)^{\frac{1}{\alpha_i}} < \infty, \quad i = \overline{1, N}. \tag{5.1}$$

Proof. The "only if" part: Let $\mathbf{x} = (x_1, x_2, \ldots, x_N) \in \mathcal{DS}$ be a solution of (SE) with each component satisfying $\lim_{n\to\infty} x_i(n) = c_i$. Then, there exist $n_0 \in \mathbb{N}$ such that $x_i(n) \geq c_i$, $n \geq n_0$, $i = \overline{1, N}$.

If (I) holds, as previously shown $\lim_{n\to\infty} x_i^{[1]}(n) = 0$, $i = \overline{1, N}$. By summing equations of (SE) twice, first from n to ∞, and then from n_0 to ∞, we derive that

$$x_i(n_0) = c_i + \sum_{n=n_0}^{\infty} \left(\frac{1}{p_i(n)} \sum_{k=n}^{\infty} q_i(k) x_{i+1}(k+1)^{\beta_i} \right)^{\frac{1}{\alpha_i}} \geq c_{i+1}^{\frac{\beta_i}{\alpha_i}} \sum_{n=n_0}^{\infty} \left(\frac{1}{p_i(n)} \sum_{k=n}^{\infty} q_i(k) \right)^{\frac{1}{\alpha_i}},$$

for $i = \overline{1, N}$. From the previous equation, we see that condition (5.1) is satisfied.

If (II) holds, then, since $x_i^{[1]}$, $i = \overline{1, N}$ are eventually negative and increasing, we may assume that $x_i^{[1]}(n) \leq -\omega_i \leq 0$ for $n \geq n_0$, $i = \overline{1, N}$. By summing equations of (SE) from n_0 to n, we obtain for $i = \overline{1, N}$

$$-\omega_i - x_i^{[1]}(n_0) \geq x_i^{[1]}(n) - x_i^{[1]}(n_0) = \sum_{k=n_0}^{n} q_i(k) x_{i+1}(k+1)^{\beta_i} \geq c_{i+1}^{\beta_i} \sum_{k=n_0}^{n} q_i(k), \quad n \geq n_0.$$

Letting $n \to \infty$, we find

$$\sum_{n=n_0}^{\infty} q_i(n) < \infty, \quad i = \overline{1, N},$$

which together with (II) yields that the condition (5.1) is satisfied.

The "if" part: Suppose that (5.1) holds. Then, there exists $n_0 \in \mathbb{N}$ such that

$$\sum_{n=n_0}^{\infty} \left(\frac{1}{p_i(n)} \sum_{k=n}^{\infty} q_i(k) \right)^{\frac{1}{\alpha_i}} < 2^{1-\frac{\beta_i}{\alpha_i}}, \quad i = \overline{1, N}. \tag{5.2}$$

Denote by $\mathcal{L}_{n_0}$ the space of all vectors $\mathbf{x} = (x_1, x_2, \ldots, x_N)$, such that $x_i = \{x_i(n)\} \in {}^{\mathbb{N}_{n_0}}\mathbb{R}$, $i = \overline{1, N}$ are bounded. Then, $\mathcal{L}_{n_0}$ is a Banach space endowed with the norm

$$\|\mathbf{x}\| = \max_{1\le i\le N}\Big\{\sup_{n\ge n_0}|x_i(n)|\Big\}. \tag{5.3}$$

Set

$$\Lambda_1 = \Big\{\mathbf{x}\in\mathcal{L}_{n_0}\;\Big|\; c_i\le x_i(n)\le 2c_i,\quad n\ge n_0,\quad i=\overline{1,N}\Big\}, \tag{5.4}$$

where c_i, $i=\overline{1,N}$ are positive constants such that

$$c_i \ge 2c_{i+1}^{\frac{\beta_i}{\alpha_i}},\quad i=\overline{1,N},\quad c_{N+1}=c_1. \tag{5.5}$$

Define operators $\mathcal{F}_i : {}^{\mathbb{N}_{n_0}}\mathbb{R}\to{}^{\mathbb{N}_{n_0}}\mathbb{R}$ by

$$\mathcal{F}_i x(n) = c_i + \sum_{k=n}^{\infty}\left(\frac{1}{p_i(k)}\sum_{s=k}^{\infty}q_i(s)x(s+1)^{\beta_i}\right)^{\frac{1}{\alpha_i}},\quad n\ge n_0,\quad i=\overline{1,N}, \tag{5.6}$$

and the mapping $\Theta:\Lambda_1\to\mathcal{L}_{n_0}$ by

$$\Theta(x_1,x_2,\dots,x_N) = \big(\mathcal{F}_1x_2,\mathcal{F}_2x_3,\dots,\mathcal{F}_Nx_{N+1}\big),\quad x_{N+1}=x_1. \tag{5.7}$$

We will show that Θ has a fixed point by using Schauder–Tychonoff fixed point theorem. Namely, the operator Θ has the following properties:

(i) Θ *maps* Λ_1 *into itself*: Let $\mathbf{x}\in\Lambda_1$. Then, using (5.2), (5.4), (5.5) and (5.6), we see that

$$c_i \le \mathcal{F}_i x_{i+1}(n) \le c_i + (2c_{i+1})^{\frac{\beta_i}{\alpha_i}}\sum_{k=n}^{\infty}\left(\frac{1}{p_i(k)}\sum_{s=k}^{\infty}q_i(s)\right)^{\frac{1}{\alpha_i}} \le c_i + 2^{\frac{\beta_i}{\alpha_i}}\cdot\frac{c_i}{2}\cdot 2^{1-\frac{\beta_i}{\alpha_i}} = 2c_i,$$

for $i=\overline{1,N}$ and $n\ge n_0$.

(ii) Θ *is continuous*: Let $\varepsilon_i>0, i=\overline{1,N}$ and $\{\mathbf{x}^{(m)}\}_{m\in\mathbb{N}} = \big\{(x_1^{(m)},x_2^{(m)},\dots,x_N^{(m)})\big\}_{m\in\mathbb{N}}$, be a sequence in Λ_1 which converges to $\mathbf{x}=(x_1,x_2,\dots,x_N)$ as $m\to\infty$. Since, Λ_1 is closed, $\mathbf{x}\in\Lambda_1$. The rest of the proof does not depend on i, so let $i\in\{1,2,\dots,N\}$ be arbitrary fixed. For every $n\ge n_0$, we have

$$\begin{aligned}&\Big|\mathcal{F}_i x_{i+1}^{(m)}(n) - \mathcal{F}_i x_{i+1}(n)\Big|\\ &\qquad\le \sum_{k=n}^{\infty}\frac{1}{p_i(k)^{\frac{1}{\alpha_i}}}\left|\left(\sum_{s=k}^{\infty}q_i(s)x_{i+1}^{(m)}(s+1)^{\beta_i}\right)^{\frac{1}{\alpha_i}} - \left(\sum_{s=k}^{\infty}q_i(s)x_{i+1}(s+1)^{\beta_i}\right)^{\frac{1}{\alpha_i}}\right|.\end{aligned}$$

Since (5.1) implies that $\sum_{n=n_0}^{\infty}q_i(n)$ is convergent, we conclude that $\sum_{n=n_0}^{\infty}q_i(n)\,x_{i+1}^{(m)}(n+1)^{\beta_i}$ is totally convergent, because $q_i(n)\,x_{i+1}^{(m)}(n+1)^{\beta_i}\le(2c_{i+1})^{\beta_i}q_i(n)$, for every $n\ge n_0, m\in\mathbb{N}$. Then, by a discrete analogue of Lebesgue dominated convergence theorem (Theorem 4.3), it holds for every $k\ge n_0$

$$\lim_{m\to\infty}\left|\left(\sum_{s=k}^{\infty}q_i(s)x_{i+1}^{(m)}(s+1)^{\beta_i}\right)^{\frac{1}{\alpha_i}} - \left(\sum_{s=k}^{\infty}q_i(s)x_{i+1}(s+1)^{\beta_i}\right)^{\frac{1}{\alpha_i}}\right| = 0,$$

which shows that

$$\lim_{m\to\infty}\sup_{n\ge n_0}\Big|\mathcal{F}_i x_{i+1}^{(m)}(n) - \mathcal{F}_i x_{i+1}(n)\Big| = 0.$$

Therefore, $\|\Theta\mathbf{x}^{(m)} - \Theta\mathbf{x}\| \to 0$ as $m \to \infty$, i.e. Θ is continuous.

(iii) $\Theta(\Lambda_1)$ *is relatively compact*: To show this, by Theorem 4.4, it is sufficient to show that $\Theta(\Lambda_1)$ is uniformly Cauchy in the topology of $\mathcal{L}_{n_0}$. For $\mathbf{x} \in \Lambda_1$ and $m > n \geq n_0$ we have

$$\begin{aligned}|\mathcal{F}_i x_{i+1}(m) - \mathcal{F}_i x_{i+1}(n)| &= \left|\sum_{k=n}^{m-1}\left(\frac{1}{p_i(k)}\sum_{s=k}^{\infty} q_i(s)x_{i+1}(s+1)^{\beta_i}\right)^{\frac{1}{\alpha_i}}\right| \\ &\leq \sum_{k=n}^{m-1}\frac{1}{p_i(k)^{\frac{1}{\alpha_i}}}\left|\sum_{s=k}^{\infty} q_i(s)x_{i+1}(s+1)^{\beta_i}\right|^{\frac{1}{\alpha_i}} \\ &\leq (2c_{i+1})^{\frac{\beta_i}{\alpha_i}}\sum_{k=n}^{m-1}\left(\frac{1}{p_i(k)}\sum_{s=k}^{\infty} q_i(s)\right)^{\frac{1}{\alpha_i}}.\end{aligned}$$

According to (5.1) it follows that $\Theta(\Lambda_1)$ is uniformly Cauchy.

Therefore, all the hypotheses of Schauder–Tychonoff fixed point theorem are fulfilled implying the existence of a fixed point $\mathbf{x} \in \Lambda_1$ of the mapping Θ, which satisfies

$$x_i(n) = c_i + \sum_{k=n}^{\infty}\left(\frac{1}{p_i(k)}\sum_{s=k}^{\infty} q_i(s)x_{i+1}(s+1)^{\beta_i}\right)^{\frac{1}{\alpha_i}}, \quad n \geq n_0, \quad i = \overline{1,N}.$$

It is clear that $\mathbf{x}$ is a positive decreasing solution of (SE) whose all components tend to constants. □

Theorem 5.2. *Let (II) hold. The system (SE) has a solution $\mathbf{x} \in \mathcal{DS}$ with each component satisfying $(P1)$ if and only if*

$$\sum_{n=1}^{\infty} q_i(n)\pi_{i+1}(n+1)^{\beta_i} < \infty, \quad i = \overline{1,N}. \tag{5.8}$$

Proof. The "only if" part: Let $\mathbf{x} = (x_1, x_2, \ldots, x_N) \in \mathcal{DS}$ be a solution with each component satisfying $(P1)$. Since $x_i^{[1]}$, $i = \overline{1,N}$ are eventually negative and increasing, we may assume that $x_i^{[1]}(n) \leq -\omega_i < 0$ for $n \geq n_0$, $i = \overline{1,N}$. Then, for $n \geq n_0$, we have

$$x_i(n) = \sum_{k=n}^{\infty}\frac{\left(-x_i^{[1]}(k)\right)^{1/\alpha_i}}{p_i(k)^{1/\alpha_i}} \geq \omega_i^{\frac{1}{\alpha_i}}\pi_i(n), \quad i = \overline{1,N}.$$

Summing equations of the system (SE) from n_0 to n we get for $n \geq n_0$,

$$-\omega_i - x_i^{[1]}(n_0) \geq x_i^{[1]}(n+1) - x_i^{[1]}(n_0) = \sum_{k=n_0}^{n} q_i(k)x_{i+1}(k+1)^{\beta_i} \geq \omega_{i+1}^{\frac{\beta_i}{\alpha_{i+1}}}\sum_{k=n_0}^{n} q_i(k)\pi_{i+1}(k+1)^{\beta_i} \tag{5.9}$$

Letting that $n \to \infty$ in (5.9), we conclude that (5.8) holds.

The "if" part: Suppose that (5.8) holds. Then, there exists $n_0 \in \mathbb{N}$ such that

$$\sum_{n=n_0}^{\infty} q_i(n)\pi_{i+1}(n+1)^{\beta_i} < 2^{\alpha_i-\beta_i}\left(2^{\alpha_i}-1\right), \quad i = \overline{1,N}. \tag{5.10}$$

Denote by $\mathcal{X}_{n_0}$ the space of all vectors $\mathbf{x} = (x_1, x_2, \ldots, x_N)$, $x_i \in {}^{\mathbb{N}_{n_0}}\mathbb{R}$, $i = \overline{1,N}$, such that $\{x_i(n)/\pi_i(n)\}$, $i = \overline{1,N}$ are bounded. Then, $\mathcal{X}_{n_0}$ is a Banach space endowed with the norm

$$\|\mathbf{x}\| = \max_{1\le i\le N}\left\{\sup_{n\ge n_0}\left|\frac{x_i(n)}{\pi_i(n)}\right|\right\}. \tag{5.11}$$

Further, $\mathcal{X}_{n_0}$ is partially ordered, with usual pointwise ordering $\le$: for $\mathbf{x}, \mathbf{y} \in \mathcal{X}_{n_0}$, $\mathbf{x} \le \mathbf{y}$ means $x_i(n) \le y_i(n)$ for all $n \ge n_0$ and $i = \overline{1,N}$. Set

$$\Lambda_2 = \left\{\mathbf{x} \in \mathcal{X}_{n_0} \;\middle|\; c_i\pi_i(n) \le x_i(n) \le 2c_i\pi_i(n), \quad n \ge n_0, \quad i = \overline{1,N}\right\}, \tag{5.12}$$

where c_i, $i = \overline{1,N}$ are positive constants which satisfy (5.5). For any subset B of Λ_2, it is obvious that $\sup B \in \Lambda_2$ and $\inf B \in \Lambda_2$.

Let $\Theta : \Lambda_2 \to \mathcal{X}_{n_0}$ be the mapping given by (5.7), where $\mathcal{F}_i : {}^{\mathbb{N}_{n_0}}\mathbb{R} \to {}^{\mathbb{N}_{n_0}}\mathbb{R}$ is operator defined by

$$\mathcal{F}_i x(n) = \sum_{k=n}^{\infty}\left(\frac{1}{p_i(k)}\left(c_i^{\alpha_i} + \sum_{s=k}^{\infty} q_i(s)x(s+1)^{\beta_i}\right)\right)^{\frac{1}{\alpha_i}}, \quad n \ge n_0, \quad i = \overline{1,N}, \tag{5.13}$$

We will show that Θ has a fixed point by using Theorem 4.1. Namely, the operator Θ has the following properties:

(i) Θ *maps* Λ_2 *into itself*: Let $\mathbf{x} \in \Lambda_2$. Then, using (5.5), (5.10), (5.12) and (5.13), we see that for $n \ge n_0$

$$\begin{aligned} c_i\pi_i(n) \le \mathcal{F}_i x_{i+1}(n) &\le \sum_{k=n}^{\infty}\left(\frac{1}{p_i(k)}\left(c_i^{\alpha_i} + (2c_{i+1})^{\beta_i}\sum_{s=k}^{\infty} q_i(s)\pi_{i+1}(s+1)^{\beta_i}\right)\right)^{\frac{1}{\alpha_i}} \\ &\le \sum_{k=n}^{\infty}\left(\frac{1}{p_i(k)}\left(c_i^{\alpha_i} + 2^{\alpha_i}\left(2^{\alpha_i}-1\right)c_{i+1}^{\beta_i}\right)\right)^{\frac{1}{\alpha_i}} \\ &\le \sum_{k=n}^{\infty}\left(\frac{1}{p_i(k)}\left(c_i^{\alpha_i} + \left(2^{\alpha_i}-1\right)c_i^{\alpha_i}\right)\right)^{\frac{1}{\alpha_i}} = 2c_i\pi_i(n), \end{aligned}$$

(ii) Θ *is increasing*, i.e. for any $\mathbf{x}, \mathbf{y} \in \Lambda_2$, $\mathbf{x} \le \mathbf{y}$ implies $\Theta\mathbf{x} \le \Theta\mathbf{y}$.

Thus, all the hypotheses of Theorem 4.1 are fulfilled, implying the existence of a fixed point $\mathbf{x} \in \Lambda_2$ of the mapping Θ. It is easy to see that $\mathbf{x}$ is a positive decreasing solution of the system (SE) and that its components satisfy $(P1)$. □

6 Existence and asymptotic behavior of strongly decreasing $\mathcal{RV}$-solutions

This section is devoted to the problem of existence and asymptotic behavior of strongly decreasing solutions for system (SE). Every strongly decreasing solution of (SE) is a solution of the system

$$x_i(n) = \sum_{k=n}^{\infty}\left(\frac{1}{p_i(k)}\sum_{j=k}^{\infty} q_i(j)x_{i+1}(j+1)^{\beta_i}\right)^{\frac{1}{\alpha_i}}, \quad i = \overline{1,N}, \quad n \ge n_0, \tag{6.1}$$

for some $n_0 \in \mathbb{N}$. We consider existence and asymptotic behavior of strongly decreasing $\mathcal{RV}$-solutions with a negative index of regularity. However, due to computing difficulties, the existence problem and asymptotic behavior of strongly decreasing slowly varying solutions will be excluded for the time being.

Assuming that condition (I) is satisfied and that coefficients are regularly varying, the following theorem gives the necessary and sufficient conditions for a system (SE) to posses a regularly varying solution $\mathbf{x}$ of a negative index $(\rho_1, \rho_2, \dots, \rho_N)$ of regularity and moreover determine their asymptotic behavior at infinity precisely.

Theorem 6.1. *Let $p_i \in \mathcal{RV}(\lambda_i)$, $q_i \in \mathcal{RV}(\mu_i)$, $i = \overline{1,N}$. Suppose that (I) holds. The system (SE) has a regularly varying solution $\mathbf{x} \in \mathcal{RV}(\rho_1, \rho_2, \dots, \rho_N)$ with $\rho_i < 0$, $i = \overline{1,N}$ if and only if*

$$\sum_{j=1}^{N} M_{ij} \frac{\alpha_j - \lambda_j + \mu_j + 1}{\alpha_j} < 0, \quad i = \overline{1,N} \tag{6.2}$$

holds, in which case ρ_i are uniquely determined by

$$\rho_i = \frac{A_N}{A_N - B_N} \sum_{j=1}^{N} M_{ij} \frac{\alpha_j - \lambda_j + \mu_j + 1}{\alpha_j}, \quad i = \overline{1,N} \tag{6.3}$$

and the asymptotic behavior of each component of any such solution is governed by the unique formula

$$x_i(n) \sim \left[\prod_{j=1}^{N} \left(\frac{n^{\frac{\alpha_j+1}{\alpha_j}} p_j(n)^{-\frac{1}{\alpha_j}} q_j(n)^{\frac{1}{\alpha_j}}}{D_j} \right)^{M_{ij}} \right]^{\frac{A_N}{A_N - B_N}}, \quad n \to \infty, \quad i = \overline{1,N}, \tag{6.4}$$

where

$$D_j = \left(\alpha_j - \lambda_j - \alpha_j \rho_j\right)^{\frac{1}{\alpha_j}} (-\rho_j), \quad j = \overline{1,N}. \tag{6.5}$$

Proof. The "only if" part: Let $\mathbf{x} \in \mathcal{RV}(\rho_1, \rho_2, \dots, \rho_N)$ with all $\rho_i < 0$ be a solution of (SE). Then, by Theorem 3.7-(vi) and Theorem 3.7-(v) each x_i satisfies (2.1) and $x_i(n) \to 0, n \to \infty$. Also, since (I) holds, as shown in Section 2 we have that $x_i^{[1]}(n) \to 0$, as $n \to \infty$.

Using (4.2) and (4.3), and applying Theorem 3.7-(iii), we obtain

$$\begin{aligned} -x_i^{[1]}(n) &= \sum_{k=n}^{\infty} q_i(k) x_{i+1}(k+1)^{\beta_i} \sim \sum_{k=n}^{\infty} q_i(k) x_{i+1}(k)^{\beta_i} \\ &= \sum_{k=n}^{\infty} k^{\mu_i + \beta_i \rho_{i+1}} m_i(k) \xi_{i+1}(k)^{\beta_i}, \quad n \to \infty, \ i = \overline{1,N}. \end{aligned} \tag{6.6}$$

and

$$-\Delta x_i(n) = \left(\frac{1}{p_i(n)} \sum_{k=n}^{\infty} q_i(k) x_{i+1}(k+1)^{\beta_i} \right)^{\frac{1}{\alpha_i}} \sim n^{-\frac{\lambda_i}{\alpha_i}} l_i(n)^{-\frac{1}{\alpha_i}} H_i(n)^{\frac{1}{\alpha_i}}, \quad n \to \infty, \tag{6.7}$$

where $H_i(n) = \sum_{k=n}^{\infty} k^{\mu_i + \beta_i \rho_{i+1}} m_i(k) \xi_{i+1}(k)^{\beta_i}$. As the left-hand side of (6.6) tends to zero as $n \to \infty$, it must be $\mu_i + \beta_i \rho_{i+1} \leq -1$, $i = \overline{1,N}$. If $\mu_i + \beta_i \rho_{i+1} = -1$, for some i, then $H_i \in \mathcal{SV}$, so that summing (6.7) from n to ∞, we obtain

$$x_i(n) \sim \sum_{k=n}^{\infty} k^{-\frac{\lambda_i}{\alpha_i}} l_i(k)^{-\frac{1}{\alpha_i}} H_i(k)^{\frac{1}{\alpha_i}}, \quad n \to \infty. \tag{6.8}$$

Since $x_i(n) \to 0, n \to \infty$, using that either (4.4) or (4.5) is satisfied, it is only possible that $\lambda_i = \alpha_i$, in which case $x_i \in \mathcal{SV}$, contradicting the assumption that $\rho_i < 0$. Therefore, it follows that $\mu_i + \beta_i \rho_{i+1} < -1$ for all $i = \overline{1,N}$. Application of Theorem 3.8 to (6.7) gives for $i = \overline{1,N}$

$$-\Delta x_i(n) \sim \frac{n^{\frac{-\lambda_i+\mu_i+\beta_i\rho_{i+1}+1}{\alpha_i}} l_i(n)^{-\frac{1}{\alpha_i}} m_i(n)^{\frac{1}{\alpha_i}} \xi_{i+1}(n)^{\frac{\beta_i}{\alpha_i}}}{(-(\mu_i + \beta_i\rho_{i+1} + 1))^{\frac{1}{\alpha_i}}}, \tag{6.9}$$

as $n \to \infty$. Summing (6.9) from n to ∞, we obtain

$$x_i(n) \sim \sum_{k=n}^{\infty} \frac{k^{\frac{-\lambda_i+\mu_i+\beta_i\rho_{i+1}+1}{\alpha_i}} l_i(k)^{-\frac{1}{\alpha_i}} m_i(k)^{\frac{1}{\alpha_i}} \xi_{i+1}(k)^{\frac{\beta_i}{\alpha_i}}}{(-(\mu_i + \beta_i\rho_{i+1} + 1))^{\frac{1}{\alpha_i}}}, \quad n \to \infty. \tag{6.10}$$

Using that $x_i(n) \to 0, n \to \infty$, we conclude that $(-\lambda_i + \mu_i + \beta_i\rho_{i+1} + 1)/\alpha_i \le -1$, $i = \overline{1,N}$. All inequalities should be strict, because if the equality holds for some i, then (6.10) leads to the contradiction that $\rho_i = 0$. Therefore, $(-\lambda_i + \mu_i + \beta_i\rho_{i+1} + 1)/\alpha_i < -1$, $i = \overline{1,N}$. Applying Theorem 3.8 at (6.10), we conclude that

$$x_i(n) \sim \frac{n^{\frac{-\lambda_i+\mu_i+\beta_i\rho_{i+1}+1}{\alpha_i}+1} l_i(n)^{-\frac{1}{\alpha_i}} m_i(n)^{\frac{1}{\alpha_i}} \xi_{i+1}(n)^{\frac{\beta_i}{\alpha_i}}}{-\left(\frac{-\lambda_i+\mu_i+\beta_i\rho_{i+1}+1}{\alpha_i} + 1\right)(-(\mu_i + \beta_i\rho_{i+1} + 1))^{\frac{1}{\alpha_i}}}, \quad n \to \infty, \quad i = \overline{1,N}. \tag{6.11}$$

From the previous relation, we see that

$$\rho_i = \frac{-\lambda_i + \mu_i + \beta_i\rho_{i+1} + 1}{\alpha_i} + 1, \quad i = \overline{1,N}, \quad \rho_{N+1} = \rho_1 \tag{6.12}$$

which is equivalent to a linear cyclic system of equations

$$\rho_i - \frac{\beta_i}{\alpha_i}\rho_{i+1} = \frac{\alpha_i - \lambda_i + \mu_i + 1}{\alpha_i}, \quad i = \overline{1,N}, \quad \rho_{N+1} = \rho_1. \tag{6.13}$$

The matrix of the system (6.13) is given by (4.12). As shown in Section 4 the matrix A is invertible, implying that the system (6.13) has the unique solution $(\rho_1, \dots, \rho_N)$. Using (4.13), we derive that these ρ_i are given explicitly by (6.3). It is obvious that $\rho_i < 0$, $i = \overline{1,N}$ if and only if (6.2) holds.

Using (4.2) and (4.3), we can transform (6.11) into the following system of asymptotic relations

$$x_i(n) \sim \frac{n^{\frac{\alpha_i+1}{\alpha_i}} p_i(n)^{-\frac{1}{\alpha_i}} q_i(n)^{\frac{1}{\alpha_i}} x_{i+1}(n)^{\frac{\beta_i}{\alpha_i}}}{D_i}, \quad n \to \infty, \quad i = \overline{1,N}, \tag{6.14}$$

where D_i are given by (6.5). Without difficulty, we can obtain that each component x_i of regularly varying solution $\mathbf{x}$ satisfies the explicit asymptotic formula

$$\begin{aligned} x_i(n) &\sim \left[\prod_{j=1}^{N} \left(\frac{n^{\frac{\alpha_j+1}{\alpha_j}} p_j(n)^{-\frac{1}{\alpha_j}} q_j(n)^{\frac{1}{\alpha_j}}}{D_j}\right)^{M_{ij}}\right]^{\frac{A_N}{A_N - B_N}} \\ &= n^{\rho_i}\left[\prod_{j=1}^{N} \left(\frac{l_j(n)^{-\frac{1}{\alpha_j}} m_j(n)^{\frac{1}{\alpha_j}}}{D_j}\right)^{M_{ij}}\right]^{\frac{A_N}{A_N - B_N}}, \quad n \to \infty, \quad i = \overline{1,N}. \end{aligned} \tag{6.15}$$

It is clear from (6.15) that the regularity index of each x_i is exactly ρ_i.

The "if" part: Suppose now that (6.2) holds. Define ρ_i with (6.3) and sequences X_i, $i=\overline{1,N}$, by

$$X_i(n)=\left[\prod_{j=1}^{N}\left(\frac{n^{\frac{\alpha_j+1}{\alpha_j}}p_j(n)^{-\frac{1}{\alpha_j}}q_j(n)^{\frac{1}{\alpha_j}}}{D_j}\right)^{M_{ij}}\right]^{\frac{A_N}{A_N-B_N}}=n^{\rho_i}\chi_i(n),\quad i=\overline{1,N},\tag{6.16}$$

where D_j, $j=\overline{1,N}$ are given by (6.5) and

$$\chi_i(n)=\left[\prod_{j=1}^{N}\left(\frac{l_j(n)^{-\frac{1}{\alpha_j}}m_j(n)^{\frac{1}{\alpha_j}}}{D_j}\right)^{M_{ij}}\right]^{\frac{A_N}{A_N-B_N}},\quad \chi_i\in\mathcal{SV}.$$

From, (6.2) we have that $\rho_i<0$, $i=\overline{1,N}$. First, we show that sequences $X_i\in\mathcal{RV}(\rho_i)$ satisfy the system of asymptotic relations

$$\sum_{k=n}^{\infty}\left(\frac{1}{p_i(k)}\sum_{s=k}^{\infty}q_i(s)X_{i+1}(s+1)^{\beta_i}\right)^{\frac{1}{\alpha_i}}\sim X_i(n),\quad n\to\infty,\quad i=\overline{1,N},\tag{6.17}$$

where $X_{N+1}=X_1$. To that end, let us note that from "the only if" part we conclude that $(\rho_1,\dots,\rho_N)$, with ρ_i given by (6.3) is in fact the unique solution of the linear cyclic system of equations (6.13). In order to apply Theorem 3.8, it must hold that $\mu_i+\rho_{i+1}\beta_i<-1$, $i=\overline{1,N}$. From the fact $\rho_i<0$ and (6.12) we have

$$\mu_i+\rho_{i+1}\beta_i=(\rho_i-1)\alpha_i+\lambda_i-1<-\alpha_i+\lambda_i-1\le-1.$$

Therefore, using (6.12) and applying Theorem 3.8, we obtain

$$\left(\frac{1}{p_i(n)}\sum_{k=n}^{\infty}q_i(k)X_{i+1}(k+1)^{\beta_i}\right)^{\frac{1}{\alpha_i}}\sim\frac{n^{\rho_i-1}l_i(n)^{-\frac{1}{\alpha_i}}m_i(n)^{\frac{1}{\alpha_i}}\chi_{i+1}(n)^{\frac{\beta_i}{\alpha_i}}}{(\alpha_i-\lambda_i-\alpha_i\rho_i)^{\frac{1}{\alpha_i}}},\quad n\to\infty,$$

and

$$\sum_{k=n}^{\infty}\left(\frac{1}{p_i(k)}\sum_{s=k}^{\infty}q_i(s)X_{i+1}(s+1)^{\beta_i}\right)^{\frac{1}{\alpha_i}}\sim\frac{n^{\rho_i}l_i(n)^{-\frac{1}{\alpha_i}}m_i(n)^{\frac{1}{\alpha_i}}\chi_{i+1}(n)^{\frac{\beta_i}{\alpha_i}}}{D_i},\quad n\to\infty.\tag{6.18}$$

Using relation (4.11) for matrix elements M_{ij}, the right-hand side of the relation (6.18) can be transformed as follows

$$\begin{aligned}\frac{l_i(n)^{-\frac{1}{\alpha_i}}m_i(n)^{\frac{1}{\alpha_i}}}{D_i}\chi_{i+1}(n)^{\frac{\beta_i}{\alpha_i}}&=\frac{l_i(n)^{-\frac{1}{\alpha_i}}m_i(n)^{\frac{1}{\alpha_i}}}{D_i}\left[\prod_{j=1}^{N}\left(\frac{l_j(n)^{-\frac{1}{\alpha_j}}m_j(n)^{\frac{1}{\alpha_j}}}{D_j}\right)^{M_{i+1,j}\frac{\beta_i}{\alpha_i}}\right]^{\frac{A_N}{A_N-B_N}}\\&=\left[\prod_{j=1}^{N}\left(\frac{l_j(n)^{-\frac{1}{\alpha_j}}m_j(n)^{\frac{1}{\alpha_j}}}{D_j}\right)^{M_{ij}}\right]^{\frac{A_N}{A_N-B_N}}=\chi_i(n),\quad i=\overline{1,N},\quad \chi_{N+1}=\chi_N,\end{aligned}$$

so from (6.18), we obtain that X_i, $i = \overline{1,N}$ satisfy (6.17).

It follows from (6.17) that there exists $n_0 > 1$ such that for $n > n_0$ holds

$$\frac{1}{2}X_i(n) \le \sum_{k=n}^{\infty}\left(\frac{1}{p_i(k)}\sum_{s=k}^{\infty}q_i(s)X_{i+1}(s+1)^{\beta_i}\right)^{\frac{1}{\alpha_i}} \le 2X_i(n), \quad i = \overline{1,N}. \tag{6.19}$$

Let we choose positive constants ω_i and W_i so that

$$\omega_i \le \frac{1}{2}\omega_{i+1}^{\frac{\beta_i}{\alpha_i}}, \quad W_i \ge 2W_{i+1}^{\frac{\beta_i}{\alpha_i}}, \quad i = \overline{1,N}, \quad \omega_{N+1} = \omega_1, \quad W_{N+1} = W_1. \tag{6.20}$$

Consider the space Y_{n_0} of all vectors $\mathbf{x} = (x_1, x_2, \dots, x_N)$, $x_i \in {}^{\mathbb{N}_{n_0}}\mathbb{R}$, $i = \overline{1,N}$, such that $\{x_i(n)/X_i(n)\}$, $i = \overline{1,N}$ are bounded. Then, Y_{n_0} is a Banach space endowed with the norm

$$\|\mathbf{x}\| = \max_{1\le i\le N}\left\{\sup_{n\ge n_0}\left|\frac{x_i(n)}{X_i(n)}\right|\right\}$$

Further, Y_{n_0} is partially ordered, with the usual pointwise ordering $\le$: For $\mathbf{x}, \mathbf{y} \in \mathrm{Y}_{n_0}$, $\mathbf{x} \le \mathbf{y}$ means $x_i(n) \le y_i(n)$ for all $n \ge n_0$ and $i = \overline{1,N}$. Define the subset $\mathcal{X} \subset \mathrm{Y}_{n_0}$ with

$$\mathcal{X} = \left\{\mathbf{x} \in \mathrm{Y}_{n_0} \;\middle|\; \omega_i X_i(n) \le x_i(n) \le W_i X_i(n), \quad n \ge n_0, \quad i = \overline{1,N}\right\}. \tag{6.21}$$

It is easy to see that for any $\mathbf{x} \in \mathcal{X}$, the norm of $\mathbf{x}$ is finite. Also, for any subset $B \subset \mathcal{X}$, it is obvious that $\inf B \in \mathcal{X}$ and $\sup B \in \mathcal{X}$. We will define the operators $\mathcal{F}_i : {}^{\mathbb{N}_{n_0}}\mathbb{R} \to {}^{\mathbb{N}_{n_0}}\mathbb{R}$ by

$$\mathcal{F}_i x(n) = \sum_{k=n}^{\infty}\left(\frac{1}{p_i(k)}\sum_{s=k}^{\infty}q_i(s)x(s+1)^{\beta_i}\right)^{\frac{1}{\alpha_i}}, \quad n \ge n_0, \quad i = \overline{1,N}, \tag{6.22}$$

and define the mapping $\Phi : \mathcal{X} \to \mathrm{Y}_{n_0}$ by

$$\Phi(x_1, x_2, \dots, x_N) = \left(\mathcal{F}_1 x_2, \mathcal{F}_2 x_3, \dots, \mathcal{F}_N x_{N+1}\right), \quad x_{N+1} = x_1. \tag{6.23}$$

We will show that Φ has a fixed point by using Theorem 4.1. Namely, the operator Φ has the following properties:

(i) Φ *maps* $\mathcal{X}$ *into itself*: Let $\mathbf{x} \in \mathcal{X}$. Then, using (6.19)–(6.22), we see that for $n \ge n_0$,

$$\mathcal{F}_i x_{i+1}(n) \le W_{i+1}^{\frac{\beta_i}{\alpha_i}}\sum_{k=n}^{\infty}\left(\frac{1}{p_i(k)}\sum_{s=k}^{\infty}q_i(s)X_{i+1}(s+1)^{\beta_i}\right)^{\frac{1}{\alpha_i}} \le 2W_{i+1}^{\frac{\beta_i}{\alpha_i}}X_i(n) \le W_i X_i(n),$$

and

$$\mathcal{F}_i x_{i+1}(n) \ge \omega_{i+1}^{\frac{\beta_i}{\alpha_i}}\sum_{k=n}^{\infty}\left(\frac{1}{p_i(k)}\sum_{s=k}^{\infty}q_i(s)X_{i+1}(s+1)^{\beta_i}\right)^{\frac{1}{\alpha_i}} \ge \frac{1}{2}\omega_{i+1}^{\frac{\beta_i}{\alpha_i}}X_i(n) \ge \omega_i X_i(n).$$

This shows that $\Phi\mathbf{x} \in \mathcal{X}$, that is, Φ is a self-map on $\mathcal{X}$.

(ii) Φ *is increasing*, i.e. for any $\mathbf{x}, \mathbf{y} \in \mathcal{X}$, $\mathbf{x} \le \mathbf{y}$ implies $\Phi\mathbf{x} \le \Phi\mathbf{y}$.

Thus all the hypotheses of Theorem 4.1 are fulfilled implying the existence of a fixed point $\mathbf{x} \in \mathcal{X}$ of Φ, which satisfies

$$x_i(n) = \mathcal{F}_i x_{i+1}(n) = \sum_{k=n}^{\infty}\left(\frac{1}{p_i(k)}\sum_{s=k}^{\infty}q_i(s)x_{i+1}(s+1)^{\beta_i}\right)^{\frac{1}{\alpha_i}}, \quad n \ge n_0, \quad i = \overline{1,N}.$$

It is easy to see that this solution is positive and decreasing. Since $\mathbf{x} \in \mathcal{X}$, it is clear that $x_i(n) \to 0$, $n \to \infty$. Furthermore, using (6.12), (6.16) and applying Theorem 3.8 we have

$$\begin{aligned} p_i(n)(-\Delta x_i(n))^{\alpha_i} &= \sum_{k=n}^{\infty} q_i(k) x_{i+1}(k+1)^{\beta_i} \le W_{i+1}^{\beta_i} \sum_{k=n}^{\infty} q_i(k) X_{i+1}(k+1)^{\beta_i} \\ &\sim W_{i+1}^{\beta_i} \frac{n^{(\rho_i-1)\alpha_i+\lambda_i} m_i(n) \chi_{i+1}(n)^{\beta_i}}{(1-\rho_i)\alpha_i - \lambda_i}, \quad n \to \infty, \quad i = \overline{1,N}. \end{aligned}$$

Since $(\rho_i - 1)\alpha_i + \lambda_i < \lambda_i - \alpha_i \le 0$, we conclude that $x_i^{[1]}(n) \to 0$, $n \to \infty$. This shows that $\mathbf{x} \in \mathcal{X}$ is a strongly decreasing solution of the system (SE).

It remains to verify that $\mathbf{x} \in \mathcal{RV}(\rho_1, \rho_2, \dots, \rho_N)$. We define

$$u_i(n) = \sum_{k=n}^{\infty} \left(\frac{1}{p_i(k)} \sum_{s=k}^{\infty} q_i(s) X_{i+1}(s+1)^{\beta_i} \right)^{\frac{1}{\alpha_i}}, \quad i = \overline{1,N},$$

and put

$$r_i = \liminf_{n\to\infty} \frac{x_i(n)}{u_i(n)}, \quad R_i = \limsup_{n\to\infty} \frac{x_i(n)}{u_i(n)}.$$

Since $\omega_i X_i(n) \le x_i(n) \le W_i X_i(n)$, $n \ge n_0$, $i = \overline{1,N}$ and

$$u_i(n) \sim X_i(n), \quad n \to \infty, \quad i = \overline{1,N}, \tag{6.24}$$

it follows that $0 < r_i \le R_i < \infty$, $i = \overline{1,N}$. Using Lemma 4.5 we obtain

$$\begin{aligned} r_i &\ge \liminf_{n\to\infty} \frac{\Delta x_i(n)}{\Delta u_i(n)} = \liminf_{n\to\infty} \frac{-\left(\frac{1}{p_i(n)} \sum_{k=n}^{\infty} q_i(k) x_{i+1}(k+1)^{\beta_i}\right)^{\frac{1}{\alpha_i}}}{-\left(\frac{1}{p_i(n)} \sum_{k=n}^{\infty} q_i(k) X_{i+1}(k+1)^{\beta_i}\right)^{\frac{1}{\alpha_i}}} \\ &= \liminf_{n\to\infty} \left(\frac{\sum_{k=n}^{\infty} q_i(k) x_{i+1}(k+1)^{\beta_i}}{\sum_{k=n}^{\infty} q_i(k) X_{i+1}(k+1)^{\beta_i}} \right)^{\frac{1}{\alpha_i}} = \left(\liminf_{n\to\infty} \frac{\sum_{k=n}^{\infty} q_i(k) x_{i+1}(k+1)^{\beta_i}}{\sum_{k=n}^{\infty} q_i(k) X_{i+1}(k+1)^{\beta_i}} \right)^{\frac{1}{\alpha_i}} \\ &\ge \left(\liminf_{n\to\infty} \frac{-q_i(n) x_{i+1}(n+1)^{\beta_i}}{-q_i(n) X_{i+1}(n+1)^{\beta_i}} \right)^{\frac{1}{\alpha_i}} = \liminf_{n\to\infty} \left(\frac{x_{i+1}(n+1)}{X_{i+1}(n+1)} \right)^{\frac{\beta_i}{\alpha_i}} = r_{i+1}^{\frac{\beta_i}{\alpha_i}} \end{aligned}$$

where (6.24) has been used in the last step. Thus, r_i satisfy the cyclic system of inequalities

$$r_i \ge r_{i+1}^{\frac{\beta_i}{\alpha_i}}, \quad i = \overline{1,N}, \quad r_{N+1} = r_1. \tag{6.25}$$

Similarly, by taking the upper limits instead of the lower limits, we get that R_i satisfy the cyclic system of inequalities

$$R_i \le R_{i+1}^{\frac{\beta_i}{\alpha_i}}, \quad i = \overline{1,N}, \quad R_{N+1} = R_1. \tag{6.26}$$

From (6.25) and (6.26) we easily see that

$$r_i \ge r_i^{\frac{\beta_1\beta_2\dots\beta_N}{\alpha_1\alpha_2\dots\alpha_N}}, \quad R_i \le R_i^{\frac{\beta_1\beta_2\dots\beta_N}{\alpha_1\alpha_2\dots\alpha_N}},$$

whence, because of the hypothesis $\beta_1\beta_2\dots\beta_N/\alpha_1\alpha_2\dots\alpha_N < 1$, we find that $r_i \ge 1$ and $R_i \le 1$, $i = \overline{1,N}$. It follows therefore that $r_i = R_i = 1$ i.e. $\lim_{n\to\infty} x_i(n)/u_i(n) = 1$ for $i = \overline{1,N}$. Combined this with (6.24) implies that $x_i(n) \sim u_i(n) \sim X_i(n)$ as $n \to \infty$, which shows that each x_i is a regularly varying sequence of index ρ_i. Thus, the proof of the "if" part of Theorem 6.1. is completed. □

Assuming that condition (II) is satisfied, taking into account that (II) is satisfied if and only if (4.7) or (4.8) holds, the next two theorems provide the necessary and sufficient conditions for a system (SE) to posses a regularly varying solution $\mathbf{x}$ with the index of regularity $(\rho_1, \rho_2, \dots, \rho_N)$ such that $\rho_i < \frac{\alpha_i - \lambda_i}{\alpha_i}$, $i = \overline{1,N}$ if (4.7) holds and such that $\rho_i < 0$, $i = \overline{1,N}$ if (4.8) holds, and moreover determine their asymptotic behavior at infinity precisely.

Theorem 6.2. *Let $p_i \in \mathcal{RV}(\lambda_i)$, $q_i \in \mathcal{RV}(\mu_i)$. Suppose that (4.7) holds. The system (SE) has a regularly varying solution $\mathbf{x} \in \mathcal{RV}(\rho_1, \rho_2, \dots, \rho_N)$ with $\rho_i < \frac{\alpha_i - \lambda_i}{\alpha_i}$, $i = \overline{1,N}$ if and only if*

$$\sum_{j=1}^{N} M_{ij} \left(\frac{\mu_j + 1}{\alpha_j} + \frac{\beta_j(\alpha_{j+1} - \lambda_{j+1})}{\alpha_j \alpha_{j+1}} \right) < 0, \quad i = \overline{1,N}, \ \ \alpha_{N+1} = \alpha_1, \lambda_{N+1} = \lambda_1, \tag{6.27}$$

holds, in which case ρ_i are uniquely determined by (6.3) and the asymptotic behavior of each component of any such solution is governed by the unique formula (6.4) with D_j, $j = \overline{1,N}$ given by (6.5).

Proof. The "only if" part: Let $\mathbf{x} \in \mathcal{RV}(\rho_1, \rho_2, \dots, \rho_N)$ with all $\rho_i < (\alpha_i - \lambda_i)/\alpha_i$ be a solution of (SE). Since from (4.7), ρ_i, $i = \overline{1,N}$ are negative, from Theorem 3.7-(v) we have that $x_i(n) \to 0$, $n \to \infty$. Also, we have that $\pi_i \in \mathcal{RV}\left(\frac{\alpha_i - \lambda_i}{\alpha_i}\right)$, from the asymptotic relation (4.9) for π_i holding if (4.7) is satisfied. Therefore, we conclude that index of regularity of each x_i/π_i, $i = \overline{1,N}$ is less then zero, implying by Theorem 3.7-(v) that $\lim_{n\to\infty} x_i(n)/\pi_i(n) = 0$. Therefore, $\lim_{n\to\infty} x_i^{[1]}(n) = 0$. Using (4.2) and (4.3), and applying Theorem 3.7-(iii), we obtain (6.6). Then, $\lim_{n\to\infty} x_i^{[1]}(n) = 0$ implies that $\mu_i + \beta_i \rho_{i+1} \le -1$, $i = \overline{1,N}$. If $\mu_i + \beta_i \rho_{i+1} = -1$ for some i, then as in the proof of Theorem 6.1 we obtain that (6.8) holds. Since (4.7) holds, application of Theorem 3.8 gives

$$x_i(n) \sim \sum_{k=n}^{\infty} k^{-\frac{\lambda_i}{\alpha_i}} l_i(k)^{-\frac{1}{\alpha_i}} H_i(k)^{\frac{1}{\alpha_i}} \sim \frac{\alpha_i}{\lambda_i - \alpha_i} n^{\frac{\alpha_i - \lambda_i}{\alpha_i}} l_i(n)^{-\frac{1}{\alpha_i}} H_i(n)^{\frac{1}{\alpha_i}}, \quad n \to \infty,$$

where $H_i(n) = \sum_{k=n}^{\infty} k^{-1} m_i(k) \xi_{i+1}(k)^{\beta_i}$, $H_i \in \mathcal{SV}$. Thus, $x_i \in \mathcal{RV}\left(\frac{\alpha_i - \lambda_i}{\alpha_i}\right)$, contradicting the hypothesis $\rho_i < \frac{\alpha_i - \lambda_i}{\alpha_i}$. Therefore, $\mu_i + \beta_i \rho_{i+1} < -1$ for all $i = \overline{1,N}$. Proceeding exactly as in the proof of Theorem 6.1, we get that (6.9) holds and conclude that $(-\lambda_i + \mu_i + \beta_i \rho_{i+1} + 1)/\alpha_i \le -1$ for all i. All inequalities should be strict, because if the equality holds for some i, then $\lambda_i - \alpha_i = \mu_i + \beta_i \rho_{i+1} + 1 < 0$, contradicting the assumption (4.7). Thus, (6.11) holds, which yields that $(\rho_1, \dots, \rho_N)$ is the unique solution of the linear cyclic system of equations (6.13), so that ρ_i, $i = \overline{1,N}$ are uniquely determined by (6.3). That the asymptotic behavior of each component of solution $\mathbf{x} \in \mathcal{RV}(\rho_1, \rho_2, \dots, \rho_N)$ with all $\rho_i < (\alpha_i - \lambda_i)/\alpha_i$ is governed by the unique formula (6.4) with D_j, $j = \overline{1,N}$ given by (6.5) is obtain as in the proof of Theorem 6.1.

To verify the condition (6.27), let we denote $d_i = \rho_i + \lambda_i/\alpha_i - 1$, $i = \overline{1,N}$. Then, the linear system of equations (6.13) is transformed into the system

$$d_i - \frac{\beta_i}{\alpha_i} d_{i+1} = \frac{\mu_i + 1}{\alpha_i} + \frac{\beta_i(\alpha_{i+1} - \lambda_{i+1})}{\alpha_i \alpha_{i+1}}, \quad i = \overline{1,N}, \quad d_{N+1} = d_1. \tag{6.28}$$

Matrix of the system (6.28) is given by (4.12). Since A is nonsingular, the system (6.28) has the unique solution $d_i, i = \overline{1,N}$, where

$$d_i = \sum_{j=1}^{N} M_{ij} \left(\frac{\mu_j + 1}{\alpha_j} + \frac{\beta_j(\alpha_{j+1} - \lambda_{j+1})}{\alpha_j \alpha_{j+1}} \right), \quad i = \overline{1,N}. \tag{6.29}$$

Since $\rho_i < \frac{\alpha_i - \lambda_i}{\alpha_i}$ if and only if $d_i < 0$, we conclude that the condition (6.27) holds.

The "if" part: Suppose that (6.27) holds. Define ρ_i with (6.3) and sequences $X_i \in \mathcal{RV}(\rho_i)$, $i = \overline{1,N}$, by (6.16). The indices of regularity of X_i, are the unique solution of the linear cyclic system (6.13). According to the assumption (6.27), we conclude that $\rho_i < (\alpha_i - \lambda_i)/\alpha_i < 0$, $i = \overline{1,N}$. Then, in a manner similar to that of the previous theorem proof, it can be demonstrated that X_i, $i = \overline{1,N}$, satisfy the asymptotic relation (6.17). Therefore, the "if" part of the theorem as well as asymptotic formulas for each component x_i of $\mathcal{RV}$-solution $\mathbf{x}$ can be obtained as in the "if" part proof of the previous theorem. □

Theorem 6.3. *Let $p_i \in \mathcal{RV}(\lambda_i)$, $q_i \in \mathcal{RV}(\mu_i)$. Suppose that (4.8) holds. The system (SE) has a regularly varying solution $\mathbf{x} \in \mathcal{RV}(\rho_1, \rho_2, \ldots, \rho_N)$ with $\rho_i < 0$, $i = \overline{1,N}$ if and only if*

$$\sum_{j=1}^{N} M_{ij} \frac{\mu_j + 1}{\alpha_j} < 0, \quad i = \overline{1,N} \tag{6.30}$$

in which case ρ_i are uniquely determined by

$$\rho_i = \frac{A_N}{A_N - B_N} \sum_{j=1}^{N} M_{ij} \frac{\mu_j + 1}{\alpha_j}, \quad i = \overline{1,N} \tag{6.31}$$

and the asymptotic behavior of any such solution is governed by the unique formulas (6.4) with $D_j = \left(\alpha_j(-\rho_j)^{\alpha_j+1}\right)^{1/\alpha_j}$, $j = \overline{1,N}$.

Proof. The "only if" part: Suppose that the system (SE) has a solution $\mathbf{x} \in \mathcal{RV}(\rho_1, \rho_2, \ldots, \rho_N)$, $\rho_i < 0$, $i = \overline{1,N}$. The assumption (4.8) implies that

$$\pi_i(n) = \sum_{k=n}^{\infty} k^{-1} l_i(k)^{-\frac{1}{\alpha_i}},$$

so that $\pi_i \in \mathcal{SV}$, implying that index of regularity of each x_i/π_i is $\rho_i < 0$ for $i = \overline{1,N}$. Therefore, by Theorem 3.7-(v), we have that $\lim_{n\to\infty} x_i(n) = 0$ and $\lim_{n\to\infty} x_i(n)/\pi_i(n) = 0$, implying that $\lim_{n\to\infty} x_i^{[1]}(n) = 0$. From (6.6), since the left-hand side tends to zero, we conclude that it must be $\mu_i + \beta_i\rho_{i+1} \leq -1$. If $\mu_i + \beta_i\rho_{i+1} = -1$ holds for some i, then as in the proof of Theorem 6.1 we obtain from (6.8) with $H_i \in \mathcal{SV}$, which due to the assumption (4.8) has the form

$$x_i(n) \sim \sum_{k=n}^{\infty} k^{-1} l_i(k)^{-\frac{1}{\alpha_i}} H_i(k)^{\frac{1}{\alpha_i}}, \quad n \to \infty, \quad i = \overline{1,N},$$

implying the contradiction $x_i \in \mathcal{SV}$. Therefore, $\mu_i + \beta_i\rho_{i+1} < -1$ for all $i = \overline{1,N}$. Then as in the proof of Theorem 6.1 we obtain (6.10) and since $x_i(n) \to 0$, $n \to \infty$ it must be

$$\frac{-\lambda_i + \mu_i + \beta_i\rho_{i+1} + 1}{\alpha_i} = -1 + \frac{\mu_i + \beta_i\rho_{i+1} + 1}{\alpha_i} \leq -1, \quad i = \overline{1,N}.$$

If equality holds for some i, then (6.10) yields the contradiction $x_i \in \mathcal{SV}$. Thus, $\mu_i + \beta_i\rho_{i+1} < -1$, $i = \overline{1,N}$, and applying Theorem 3.8 at (6.10), we conclude that

$$x_i(n) \sim \frac{n^{\frac{\mu_i+\beta_i\rho_{i+1}+1}{\alpha_i}} l_i(n)^{-\frac{1}{\alpha_i}} m_i(n)^{\frac{1}{\alpha_i}} \xi_{i+1}(n)^{\frac{\beta_i}{\alpha_i}}}{-\frac{\mu_i+\beta_i\rho_{i+1}+1}{\alpha_i} \left(-(\mu_i + \beta_i\rho_{i+1} + 1)\right)^{\frac{1}{\alpha_i}}}, \quad n \to \infty, \quad i = \overline{1,N}. \tag{6.32}$$

Therefore, (6.32) implies that $(\rho_1, \dots, \rho_N)$ is the unique solution of the linear cyclic system of equations

$$\rho_i - \frac{\beta_i}{\alpha_i}\rho_{i+1} = \frac{\mu_i + 1}{\alpha_i}, \quad i = \overline{1, N}, \quad \rho_{N+1} = \rho_1, \tag{6.33}$$

since the matrix of the system (6.33), given by (4.12), is invertible. Using (4.13), we obtain that $\rho_i, i = \overline{1, N}$ is given by (6.31). Proceeding exactly as in the proof of Theorem 6.1, we obtain that the unique asymptotic formula for each x_i is given by (6.4), where constants D_j are reduced to $D_j = \left(\alpha_j/(-\rho_i)^{\alpha_j+1}\right)^{1/\alpha_j}, j = \overline{1, N}$. Since all ρ_i are negative if and only if (6.30) holds, the "only if" part of the theorem is proved.

The "if" part of the theorem is proved and the explicit asymptotic formula for each component x_i of $\mathcal{RV}$-solution $\mathbf{x}$ can be obtained as in the proof of Theorem 6.1. □

Application. The one-dimensional system with positive coefficients $\{p(n)\}, \{q(n)\}$ is in fact the second-order difference equation of Emden–Fowler type (see (1.3)):

$$\Delta(p(n)|\Delta x(n)|^{\alpha-1}\Delta x(n)) = q(n)|x(n+1)|^{\beta-1}x(n+1), \tag{6.34}$$

This equation has been studied in [24, 25]. In this case, Theorems 6.1 and 6.2 reduce to Theorem 3.8 from [25]. However, since the case where the regularity index of the coefficient p is equal to α, was not considered neither in [25] nor in the existing literature so far, the new result is obtained as the corollary of Theorem 6.1 and Theorem 6.3. This result provides necessary and sufficient condition for the existence of $\mathcal{RV}$ strongly decreasing solutions of type $(SD1)$ if $S = \infty$ as well as of $\mathcal{RV}$ strongly decreasing solutions of type $(SD2)$ if $S < \infty$, where $S = \sum_{n=1}^{\infty} \frac{1}{p(n)^{\alpha}}$.

Corollary 6.4. *Let $\{p(n)\} \in \mathcal{RV}(\alpha)$ and $\{q(n)\} \in \mathcal{RV}(\mu)$. The equation (6.34) possesses a regularly varying solution of index $\rho < 0$ if and only if $\mu < -1$, in which case ρ is given by*

$$\rho = \frac{\mu + 1}{\alpha - \beta}$$

and the asymptotic behavior of any such solution x is governed by the unique formula

$$x(n) \sim \left[\frac{n^{\alpha+1}p(n)^{-1}q(n)}{\alpha(-\rho)^{\alpha+1}}\right]^{\frac{1}{\alpha-\beta}}, \quad n \to \infty.$$

Following example illustrate results obtained in the last corollary.

Example 6.5. Consider the difference equation

$$\Delta\left(-n^2\sqrt{\log n}(\Delta x(n))^2\right) = \frac{\gamma(n)}{n^6 \log^{17/6} n}\sqrt[3]{x(n+1)}, \quad n \geq 2, \tag{6.35}$$

where $\gamma(n)$ is positive real-valued sequence such that $\lim_{n\to\infty}\gamma(n) = \delta$. In this equation, $\alpha = 3$, $\beta = \frac{1}{3}$, $\{p(n)\} \in \mathcal{RV}(2)$, and $\{q(n)\} \in \mathcal{RV}(-6)$.

Since $\mu = -6 < -1$, by Corollary 6.4 we conclude that equation (6.35) has a strongly decreasing $\mathcal{RV}-$solution of index $\rho = -3$. The asymptotic behavior of such a solution is

$$x(n) \sim \left(\frac{\delta}{54}\right)^{\frac{3}{5}} n^{-3}(\log n)^{-2}, \quad n \to \infty.$$

If

$$\gamma(n) = n^2(n+1)(\log n)^{\frac{9}{2}}(\log(n+1))^{\frac{2}{3}}\left(\psi(n) - \left(\frac{n}{n+1}\right)^4 \frac{\sqrt{\log(n+1)}}{\log n}\psi(n+1)\right),$$

where

$$\psi(n) = \left(\left(\frac{n}{n+1}\right)^3 \frac{1}{(\log(n+1))^2} - \frac{1}{(\log n)^2}\right)^2,$$

then $\delta = 54$ and equation (6.35) has the exact solution $x(n) = n^{-3}(\log n)^{-2}$.

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Systems of second order Stieltjes differential equations with three-point boundary conditions

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Abstract. In this paper, existence results are obtained for a system of second order boundary value problems

$$\begin{cases} u''_{\overline{g}}(t) = f(t, u(t), u'_{\overline{g}}(t)), \ \mu_{\overline{g}}\text{-a.e. } t \in [0,1) \\ u(0) = \overline{\alpha} \\ u(1) + ku(\theta) = \overline{\beta}, \end{cases}$$

with Stieltjes derivatives with respect to different derivators on different coordinates.

Using an appropriate Green function and Schauder's fixed point theorem, a very general existence result is obtained, thus overcoming several outcomes in literature.

Keywords: Stieltjes derivative, three-point boundary value problem, second order equation.

2020 Mathematics Subject Classification: 34A06, 34B10, 26A24, 26A42.

1 Introduction

The interest for second order differential equations with multi-point boundary conditions has robust roots in engineering (e.g. [17]) and it was, therefore, intensively studied. We refer the reader to [17, 31, 33] and the references therein and to [7] or [1, 24] for the framework of impulsive problems, respectively of dynamic equations on time scales.

At the same time, problems involving the Stieltjes derivative with respect to nondecreasing functions have recently found significant applications in studying real processes where stationary intervals and abrupt changes are equally present (see [11–13, 18] or [19]); this kind of behaviour was investigated through the theory of measure differential equations (e.g. [6]) and inclusions (for instance, [4] or [5]). Starting with the paper [16] (following an idea in [32]), a more convenient, equivalent writing using the Stieltjes derivative became increasingly popular (see [8, 9, 11, 12, 20, 23, 27, 28] for the single-valued case or [26, 29] for the set-valued setting).

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Second order equations with Stieltjes derivative were considered, to the best of the author's knowledge, only in the linear case with initial value boundary conditions in [8,9] and in the nonlinear case with periodic conditions in [22].

When speaking about systems of differential problems, as described in [12,14,19] or [21], it is important to allow the variation speed of different unknown variables to depend on different nondecreasing functions and thus, to have different impulsive moments or stationary intervals of time; this means to consider a vector-valued derivator.

With all these in mind, we hereby develop a study of systems of nonlinear second order Stieltjes differential equations with three-point boundary conditions and several derivators:

$$\begin{cases} u''_{\overline{g}}(t) = f(t, u(t), u'_{\overline{g}}(t)), \ \mu_{\overline{g}}\text{-a.e. } t \in [0,1) \\ u(0) = \overline{\alpha} \\ u(1) + ku(\theta) = \overline{\beta}. \end{cases}$$

Inspired by [33], a Green's function is adapted to the present framework, allowing one to get existence and uniqueness for single-valued linear second order equations on the real line. A fixed point theorem is then applied to get an existence result in the nonlinear case for systems of second order Stieltjes differential problems with three-point boundary conditions and several derivators.

It is well-known that the theory of Stieltjes differential equations is strongly related to other types of problems: generalized differential equations ([30]), impulsive differential equations ([16]) or dynamic problems on time scales ([6]); consequently, the outcomes presented here can be used to get existence and uniqueness results for systems of second-order equations with three-point boundary conditions in the mentioned settings.

2 Notations and auxiliary results.

Let $g : [0,1] \to \mathbb{R}$ be a nondecreasing left-continuous function. Without any loss of generality, one may suppose that $g(0) = 0$. The g-measurability means the measurability with respect to (shortly, w.r.t.) the σ-algebra defined by g, μ_g is the Stieltjes measure generated by g (see [10]) and the Lebesgue–Stieltjes (shortly, LS-) integrability w.r.t. g is the abstract Lebesgue integrability w.r.t. the measure μ_g. Let $L^1_g([0,1])$ be the space of real LS-integrable functions w.r.t. g with its natural topological structure given by the norm

$$\|f\|_1 = \int_{[0,1)} |f(t)| dg(t).$$

Consider the following sets:

$$D_g = \{t \in [0,1] : g(t+) - g(t) > 0\}$$

and

$$C_g = \{t \in [0,1] : g \text{ is constant on } (t-\varepsilon, t+\varepsilon) \text{ for some } \varepsilon > 0\}$$

along with

$$N_g = \{u_n, v_n : n \in \mathbb{N}\} \setminus D_g,$$

where $C_g = \bigcup_{n\in\mathbb{N}}(u_n, v_n)$ with $(u_n, v_n)_n$ pairwise disjoint. Let $N_g^- = \{u_n : n \in \mathbb{N}\} \setminus D_g$ and $N_g^+ = \{v_n : n \in \mathbb{N}\} \setminus D_g$. As $\mu_g(C_g) = \mu_g(N_g) = 0$ (proved in [16]), these two sets are irrelevant in the study of differential equations. Note also that $D_g \cap C_g = \emptyset$.

Let us now recall the notion of differentiability related to Stieltjes integrals introduced in [8] (which extends that in [16] in such a way that the points in C_g are also covered).

Definition 2.1. Let $g : [0,1] \to \mathbb{R}$ be a nondecreasing left-continuous function such that $0 \notin N_g^-$ and $1 \notin C_g \cup N_g^+$. The derivative with respect to g (or the g-derivative) of $f : [0,1] \to \mathbb{R}$ at $\bar{t} \in [0,1]$ is defined by

$$f'_g(\bar{t}) = \lim_{t \to \bar{t}} \frac{f(t) - f(\bar{t})}{g(t) - g(\bar{t})} \quad \text{if } t \notin D_g \cup C_g,$$

$$f'_g(\bar{t}) = \lim_{t \to \bar{t}+} \frac{f(t) - f(\bar{t})}{g(t) - g(\bar{t})} \quad \text{if } t \in D_g,$$

$$f'_g(\bar{t}) = \lim_{t \to v_n+} \frac{f(t) - f(v_n)}{g(t) - g(v_n)} \quad \text{if } t \in (u_n, v_n) \subseteq C_g,$$

if the limits exist. In this case f is said to be g-differentiable at t.

The points of N_g must be approached in the following manner:

$$f'_g(\bar{t}) = \lim_{t \to \bar{t}+} \frac{f(t) - f(\bar{t})}{g(t) - g(\bar{t})} \quad \text{if } t \in N_g^+,$$

$$f'_g(\bar{t}) = \lim_{t \to \bar{t}-} \frac{f(t) - f(\bar{t})}{g(t) - g(\bar{t})} \quad \text{if } t \in N_g^-.$$

Note that if $t \in D_g$, the g-derivative $f'_g(t)$ exists if and only if the right limit $f(t+)$ exists, and in this case

$$f'_g(t) = \frac{f(t+) - f(t)}{g(t+) - g(t)},$$

while if $t \in (u_n, v_n) \subseteq C_g$ the g-derivative $f'_g(t)$ exists if and only if there exists the right g-derivative at v_n.

The g-derivative is very useful when trying to solve many interesting problems where abrupt modifications (corresponding to discontinuity points of g) and stationary times (corresponding to intervals where g is constant) are both part of the state behaviour, such as in [11, 12] or [13].

Connecting Stieltjes integrals and the Stieltjes derivative is one of the main technical issues of the theory; such connections are provided by Fundamental Theorems of Calculus ([16, Theorems 5.4, 6.2, 6.5]).

In order to state them, let us remind the reader (e.g. [16]) that f is called g-absolutely continuous ($f \in \mathcal{AC}_g([0,1])$) if for every $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that

$$\sum_{j=1}^{m} |f(b_j) - f(a_j)| < \varepsilon$$

for any set $\{(a_j, b_j); j = 1, \dots, m\}$ of disjoint subintervals of $[0,1]$ satisfying

$$\sum_{j=1}^{m} (g(b_j) - g(a_j)) < \delta_\varepsilon.$$

Theorem 2.2.

a) ([16, Theorem 2.4]) Let $f : [0,1] \to \mathbb{R}$ be LS-integrable with respect to g. Then

$$F(t) = \int_{[0,t)} f(s)\, dg(s), \quad t \in [0,1]$$

defines a map g-absolutely continuous and μ_g-a.e. g-differentiable on $[0,1)$ with the property that $F_g'(t) = f(t)$, μ_g-a.e.

b) ([16, Theorem 5.4], see also [11, Theorem 5.1]) If $F : [0,1] \to \mathbb{R}$ is g-absolutely continuous, then F_g' exists μ_g-a.e. and

$$F(t) = F(0) + \int_{[0,t)} F_g'(s)\, dg(s) \quad \text{for every } t \in [0,1].$$

Remark 2.3. It easily follows that if $f \in L_g^1([0,1])$,

$$\tilde{F}(t) = \int_{[t,1)} f(s)\, dg(s), \quad t \in [0,1)$$

defines a map μ_g-a.e. g-differentiable with the property that $\tilde{F}_g'(t) = -f(t)$, μ_g-a.e.

Let us also recall ([11]) that a map $f : [0,1] \to \mathbb{R}$ is g-continuous at a point $t \in [0,1]$ if for every $\varepsilon > 0$ one can find $\delta_{t,\varepsilon} > 0$ such that

$$s \in [0,1],\ |g(t) - g(s)| < \delta_{t,\varepsilon} \ \Rightarrow\ |f(t) - f(s)| < \varepsilon$$

and that g-continuity on $[0,1]$ means g-continuity at every $t \in [0,1]$.

Any g-absolutely continuous function is g-continuous and it was proved in [16, Proposition 5.3] that g-absolutely continuous functions are left-continuous and constant on the intervals where g is constant.

Note that g-continuous functions are not necessarily bounded, this is the reason to consider the space $\mathcal{BC}_g([0,1])$ of functions which are bounded and g-continuous ([11]). It is a Banach space when endowed with the norm

$$\|u\|_C = \sup_{t \in [0,1]} |u(t)|$$

and $\mathcal{AC}_g([0,1]) \subset \mathcal{BC}_g([0,1])$.

We add to Theorem 2.2 the result below.

Lemma 2.4. *Let $F : [0,1] \to \mathbb{R}$ be g-absolutely continuous such that there exists $f \in \mathcal{BC}_g([0,1])$ satisfying*

$$F_g'(t) = f(t), \quad \mu_g\text{-a.e.}$$

Then F is g-differentiable everywhere and $F_g't) = f(t)$ for every $t \in [0,1]$.

Proof. By Theorem 2.2.b), for every $t \in [0,1]$,

$$F(t) = F(0) + \int_{[0,t)} F_g'(t) dg(t)$$

whence, by hypothesis,

$$F(t) = F(0) + \int_{[0,t)} f(t) dg(t), \quad \text{for every } t \in [0,1].$$

Applying [8, Lemma 3.14] we obtain that F is g-differentiable everywhere and $F_g't) = f(t)$ for every $t \in [0,1]$. □

The following consequence of [8, Proposition 3.9] (see [9, Remark 2.11]) provides a useful rule for the g-derivative of a product.

Lemma 2.5. *Let $f_1, f_2 : [0,1] \to \mathbb{R}$ be g-continuous and g-differentiable at the point $t \in [0,1]$. Then*

$$(f_1 f_2)'_g(t) = (f_1)'_g(t) f_2(t) + f_1(t)(f_2)'_g(t) + (f_1)'_g(t)(f_2)'_g(t)\Delta g(t^*),$$

where $\Delta g(t^*) = g(t^*+) - g(t^*)$ is the jump of g at the point

$$t^* = \begin{cases} t, & \text{if } t \notin C_g \\ v_n, & \text{if } t \in (u_n, v_n) \subset C_g. \end{cases}$$

The recent works [8, 9] opened the way to study higher order differential equations with Stieltjes derivative. The space $\mathcal{BC}^1_g([0,1])$ was defined as the space of functions $f : [0,1] \to \mathbb{R}$ such that f is g-differentiable everywhere on $[0,1]$ and its g-derivative f'_g is bounded and g-continuous. Endowed with the norm

$$\|f\|_{\mathcal{BC}^1_g([0,1])} = \|f\|_C + \|f'_g\|_C$$

it is, by [8, Theorem 3.15], a Banach space.

The matter of compactness in $\mathcal{BC}_g([0,1])$ was addressed in [11].

Theorem 2.6 ([11, Proposition 5.6])**.** *Let $\mathcal{S} \subset \mathcal{AC}_g([0,1])$ be such that $\{u(0) : u \in \mathcal{S}\}$ is bounded and there exists $\phi \in L^1_g([0,1])$satisfying*

$$|u'_g(t)| \le \phi(t), \quad \mu_g\text{-a.e. in } (0,1] \text{ and for all } u \in \mathcal{S}.$$

Then $\mathcal{S}$ is relatively compact in $\mathcal{BC}_g([0,1])$.

Besides,

Lemma 2.7 ([15, Lemma 3.8])**.** *Let $(u_k)_k \subset \mathcal{AC}_g([0,1])$ be pointwise convergent to $u : [0,1] \to \mathbb{R}$. If there exists $\phi \in L^1_g([0,1])$ such that for all $k \in \mathbb{N}$,*

$$|(u_k)'_g(t)| \le \phi(t), \quad \mu_g\text{-a.e. in } (0,1],$$

then u is also g-absolutely continuous.

3 Main results

3.1 Second order linear Stieltjes differential equations with a single derivator

The aim of this subsection is to study the linear single-valued setting, namely the problem

$$\begin{cases} u''_g(t) = f(t), \ \mu_g\text{-a.e. } t \in [0,1) \\ u(0) = \alpha \\ u(1) + ku(\theta) = \beta \end{cases} \tag{3.1}$$

where $g : [0,1] \to \mathbb{R}$ is a nondecreasing left-continuous function, $k \in \mathbb{R}$ and $\theta \in (0,1)$ satisfy $g(1) + kg(\theta) \neq 0$ and $\alpha, \beta \in \mathbb{R}$.

Definition 3.1. A function $u \in \mathcal{BC}_g^1([0,1))$ is a solution of (3.1) if it verifies the boundary conditions $u(0) = \alpha$ and $u(1) + ku(\theta) = \beta$, its first g-derivative u_g' is g-absolutely continuous on $[0,1)$ and its second g-derivative (which is well defined μ_g-a.e. by Theorem 2.2) satisfies

$$u_g''(t) = f(t), \quad \mu_g\text{-a.e. in } [0,1).$$

The existence result given below involves a Green-type function appropriate for the present framework, inspired by [33].

Theorem 3.2. *Let $f : [0,1] \to \mathbb{R}$ be LS-integrable with respect to the nondecreasing left-continuous function $g : [0,1] \to \mathbb{R}$. Then $u_f : [0,1] \to \mathbb{R}$,*

$$\begin{aligned} u_f(t) = &\int_{[0,1)} G(t,s) f(s) dg(s) - \chi_{(\theta,1]}(t) \cdot A(g(t) - g(\theta)) \\ &+ \frac{g(1) - g(t) + k(g(\theta) - g(t))}{g(1) + kg(\theta)} \alpha + \frac{g(t)}{g(1) + kg(\theta)} \tilde{\beta} \end{aligned}$$

is a solution of (3.1).

Here $G : [0,1] \times [0,1] \to \mathbb{R}$ is defined as follows: if $t \leq \theta$,

$$G(t,s) = \begin{cases} \frac{-g(s)[g(1)-g(t)+k(g(\theta)-g(t))]}{g(1)+kg(\theta)} - \Delta g(s^*), & \text{if } 0 \leq s < t \\ \frac{-g(t)[g(1)-g(s)+k(g(\theta)-g(s))]}{g(1)+kg(\theta)}, & \text{if } t \leq s < \theta \\ \frac{-g(t)(g(1)-g(s))}{g(1)+kg(\theta)}, & \text{if } \theta \leq s \leq 1, \end{cases}$$

while if $t > \theta$,

$$G(t,s) = \begin{cases} \frac{-g(s)[g(1)-g(t)+k(g(\theta)-g(t))]}{g(1)+kg(\theta)} - \Delta g(s^*), & \text{if } 0 \leq s < \theta \\ \frac{-g(s)(g(1)-g(t))+kg(\theta)(g(t)-g(s))}{g(1)+kg(\theta)} - \Delta g(s^*) & \text{if } \theta \leq s < t \\ \frac{-g(t)(g(1)-g(s))}{g(1)+kg(\theta)}, & \text{if } t \leq s \leq 1. \end{cases}$$

Besides,

$$\chi_{(\theta,1]}(t) = \begin{cases} 0, & \text{if } 0 \leq t \leq \theta \\ 1, & \text{if } \theta < t \leq 1, \end{cases}$$

$$A = \frac{-k}{g(1) + kg(\theta)} \int_{[0,\theta)} f(s)(g(s) + 1) dg(s)$$

and

$$\tilde{\beta} = \beta + \left(\int_{[0,1)} \Delta g(s^*) f(s) dg(s) + k \int_{[0,\theta)} \Delta g(s^*) f(s) dg(s) + A(g(1) - g(\theta)) \right).$$

Proof. Let us first note that if $t \leq \theta$,

$$\begin{aligned} u_f(t) = &\int_{[0,t)} -g(s) f(s) dg(s) + \frac{g(t)}{g(1) + kg(\theta)} \int_{[0,t)} g(s)(1+k) f(s) dg(s) - \int_{[0,t)} \Delta g(s^*) f(s) dg(s) \\ &- \frac{g(t)}{g(1) + kg(\theta)} \int_{[t,\theta)} [g(1) - g(s) + k(g(\theta) - g(s))] f(s) dg(s) \\ &- \frac{g(t)}{g(1) + kg(\theta)} \int_{[\theta,1)} (g(1) - g(s)) f(s) dg(s) \\ &+ \frac{g(1) - g(t) + k(g(\theta) - g(t))}{g(1) + kg(\theta)} \alpha + \frac{g(t)}{g(1) + kg(\theta)} \tilde{\beta}. \end{aligned}$$

Then, using Theorem 2.2.a) and Lemma 2.5, u_f is g-absolutely continuous on $[0,\theta]$ and μ_g-a.e. on $[0,\theta)$,

$$\begin{aligned}(u_f)'_g(t) = & -g(t)f(t) + \frac{1+k}{g(1)+kg(\theta)}\int_{[0,t)} g(s)f(s)dg(s) + \frac{g^2(t)}{g(1)+kg(\theta)}(1+k)f(t)\\ & + \frac{1}{g(1)+kg(\theta)}g(t)(1+k)f(t)\Delta g(t^*) - \Delta g(t^*)f(t)\\ & - \frac{1}{g(1)+kg(\theta)}\int_{[t,\theta)}[g(1)-g(s)+k(g(\theta)-g(s))]f(s)dg(s)\\ & + \frac{g(t)}{g(1)+kg(\theta)}[g(1)-g(t)+k(g(\theta)-g(t))]f(t)\\ & + \frac{g(1)-g(t)+k(g(\theta)-g(t))}{g(1)+kg(\theta)}f(t)\Delta g(t^*)\\ & - \frac{1}{g(1)+kg(\theta)}\int_{[\theta,1)}(g(1)-g(s))f(s)dg(s) - \frac{(1+k)\alpha-\tilde{\beta}}{g(1)+kg(\theta)}.\end{aligned}$$

Using Lemma 2.4 we get that for every $t\in[0,\theta)$,

$$\begin{aligned}(u_f)'_g(t) = & \frac{1+k}{g(1)+kg(\theta)}\int_{[0,t)} g(s)f(s)dg(s)\\ & - \frac{1}{g(1)+kg(\theta)}\int_{[t,\theta)}[g(1)-g(s)+k(g(\theta)-g(s)]f(s)dg(s)\\ & - \frac{1}{g(1)+kg(\theta)}\int_{[\theta,1)}(g(1)-g(s))f(s)dg(s)\\ & - \frac{(1+k)\alpha-\tilde{\beta}}{g(1)+kg(\theta)}.\end{aligned} \tag{3.2}$$

It is g-absolutely continuous on $[0,\theta]$ and for μ_g-almost every $t\in[0,\theta)$,

$$(u_f)''_g(t) = \frac{1+k}{g(1)+kg(\theta)}g(t)f(t) + \frac{g(1)-g(t)+kg(\theta)-kg(t)}{g(1)+kg(\theta)}f(t) = f(t).$$

Also, for $t>\theta$,

$$\begin{aligned}u_f(t) = & \int_{[0,\theta)} -g(s)f(s)dg(s) + \frac{g(t)}{g(1)+kg(\theta)}\int_{[0,\theta)}(g(s)-k)f(s)dg(s)\\ & - \int_{[0,\theta)}\Delta g(s^*)f(s)dg(s) + \int_{[\theta,t)} -g(s)f(s)dg(s)\\ & + \frac{g(t)}{g(1)+kg(\theta)}\int_{[\theta,t)}(g(s)+kg(\theta))f(s)dg(s) - \int_{[\theta,t)}\Delta g(s^*)f(s)dg(s)\\ & - \frac{g(t)}{g(1)+kg(\theta)}\int_{[t,1)}(g(1)-g(s))f(s)dg(s)\\ & - A(g(t)-g(\theta)) + \frac{g(1)-g(t)+k(g(\theta)-g(t))}{g(1)+kg(\theta)}\alpha + \frac{g(t)}{g(1)+kg(\theta)}\tilde{\beta}.\end{aligned}$$

Then u_f is g-absolutely continuous on $(\theta,1)$ and μ_g-a.e. on $(\theta,1)$,

$$\begin{aligned}(u_f)'_g(t) = &\frac{1}{g(1)+kg(\theta)}\int_{[0,\theta)}(g(s)-k)f(s)dg(s) - g(t)f(t)\\ &+\frac{1}{g(1)+kg(\theta)}\int_{[\theta,t)}(g(s)+kg(\theta))f(s)dg(s)\\ &+\frac{g(t)}{g(1)+kg(\theta)}(g(t)+kg(\theta))f(t)+\frac{(g(t)+kg(\theta))f(t)}{g(1)+kg(\theta)}\Delta g(t^*)-\Delta g(t^*)f(t)\\ &-\frac{1}{g(1)+kg(\theta)}\int_{[t,1)}(g(1)-g(s))f(s)dg(s)+\frac{g(t)}{g(1)+kg(\theta)}(g(1)-g(t))f(t)\\ &+\frac{(g(1)-g(t))f(t)}{g(1)+kg(\theta)}\Delta g(t^*)-A-\frac{(1+k)\alpha-\tilde{\beta}}{g(1)+kg(\theta)}.\end{aligned}$$

Again by Lemma 2.4 we infer that u_f is g-differentiable on $(\theta,1)$ and that

$$\begin{aligned}(u_f)'_g(t) = &\frac{1}{g(1)+kg(\theta)}\int_{[0,\theta)}(g(s)-k)f(s)dg(s)\\ &+\frac{1}{g(1)+kg(\theta)}\int_{[\theta,t)}(g(s)+kg(\theta))f(s)dg(s)\\ &-\frac{1}{g(1)+kg(\theta)}\int_{[t,1)}(g(1)-g(s))f(s)dg(s)-A-\frac{(1+k)\alpha-\tilde{\beta}}{g(1)+kg(\theta)}.\end{aligned}\tag{3.3}$$

Consequently, $(u_f)'_g$ is g-absolutely continuous on $(\theta,1)$ and for μ_g-a.e. $t\in(\theta,1)$,

$$(u_f)''_g(t)=\frac{1}{g(1)+kg(\theta)}(g(t)+kg(\theta))f(t)+\frac{1}{g(1)+kg(\theta)}(g(1)-g(t))f(t)=f(t).$$

On the other hand, if $t=\theta\in D_g$, then using (3.2), (3.3) we can see that

$$\begin{aligned}(u_f)''_g(\theta) &= \frac{(u_f)'_g(\theta+)-(u_f)'_g(\theta-)}{\Delta g(\theta)}\\ &= \frac{1}{\Delta g(\theta)}\left(f(\theta)\Delta g(\theta)-\frac{k}{g(1)+kg(\theta)}\cdot\int_{[0,\theta)}f(s)(1+g(s))dg(s)-A\right)\\ &= f(\theta).\end{aligned}$$

Since in the case where $\theta\notin D_g$, $\mu_g(\{\theta\})=0$, we can conclude that $(u_f)''_g(t)=f(t)$ for μ_g-a.e. $t\in[0,1)$.

Besides, as u_f and $(u_f)'_g$ are both continuous at θ if $\theta\notin D_g$, it follows that $u\in\mathcal{BC}^1_g([0,1])$ and $(u_f)'_g$ is g-absolutely continuous on $[0,1]$.

As for the boundary conditions, it can be easily checked that

$$u_f(0)=\int_{[0,1)}G(0,s)f(s)dg(s)+\alpha=\alpha$$

and

$$\begin{aligned}
u_f(1)+ku_f(\theta) = &\int_{[0,1)} G(1,s)f(s)dg(s) - A(g(1)-g(\theta)) + \frac{k(g(\theta)-g(1))}{g(1)+kg(\theta)}\alpha + \frac{g(1)\tilde{\beta}}{g(1)+kg(\theta)}\\
&+k\int_{[0,1)} G(\theta,s)f(s)dg(s) + k\frac{g(1)-g(\theta)}{g(1)+kg(\theta)}\alpha + k\frac{g(\theta)\tilde{\beta}}{g(1)+kg(\theta)}\\
=\tilde{\beta} &+ \int_{[0,\theta)} \frac{-kg(s)(g(\theta)-g(1)))}{g(1)+kg(\theta)}f(s)dg(s) - \int_{[0,\theta)} \Delta g(s^*)f(s)dg(s)\\
&+\int_{[\theta,1)} \frac{kg(\theta)(g(1)-g(s))}{g(1)+kg(\theta)}f(s)dg(s)\\
&-\int_{[\theta,1)} \Delta g(s^*)f(s)dg(s) - A(g(1)-g(\theta))\\
&+k\int_{[0,\theta)} \frac{-g(s)(g(1)-g(\theta))}{g(1)+kg(\theta)}f(s)dg(s)\\
&-k\int_{[0,\theta)} \Delta g(s^*)f(s)dg(s) + k\int_{[\theta,1)} \frac{-g(\theta)(g(1)-g(s))}{g(1)+kg(\theta)}f(s)dg(s) = \beta. \qquad \square
\end{aligned}$$

Corollary 3.3. *A map $u:[0,1]\to\mathbb{R}$ is a solution of (3.1) if and only if*

$$\begin{aligned}
u(t) = &\int_{[0,1)} G(t,s)f(s)dg(s) - \chi_{(\theta,1]}(t)\cdot A(g(t)-g(\theta))\\
&+\frac{g(1)-g(t)+k(g(\theta)-g(t))}{g(1)+kg(\theta)}\alpha + \frac{g(t)}{g(1)+kg(\theta)}\tilde{\beta}
\end{aligned}$$

Proof. One implication is proved in Theorem 3.2; as for the other one, suppose u_1, u_2 are two solutions of the considered problem. Then

$$\begin{cases}
(u_1-u_2)''_g(t) = 0,\ \mu_g\text{-a.e. } t\in[0,1)\\
(u_1-u_2)(0) = 0\\
(u_1-u_2)(1)+k(u_1-u_2)(\theta) = 0.
\end{cases} \tag{3.4}$$

Using the g-absolute continuity of $(u_1-u_2)'_g$ one infers that for every $t\in[0,1]$,

$$(u_1-u_2)'_g(t) = (u_1-u_2)'_g(0) + \int_{[0,t)} (u_1-u_2)''_g(s)dg(s) = (u_1-u_2)'_g(0)$$

and so, by the g-absolute continuity of u_1-u_2,

$$(u_1-u_2)(t) = (u_1-u_2)(0) + \int_{[0,t)} (u_1-u_2)'_g(s)dg(s) = (u_1-u_2)(0) + (u_1-u_2)'_g(0)\cdot g(t).$$

Now the boundary conditions lead us to

$$(u_1-u_2)(0) = 0$$

respectively

$$(u_1-u_2)'_g(0)\cdot(g(1)+kg(\theta)) = 0.$$

It follows that $(u_1-u_2)'_g(0) = 0$ and, consequently, $u_1(t) = u_2(t)$ for every $t\in[0,1]$. $\square$

Remark 3.4. Taking into account that, by Lemma 2.5, $s = s^*$ μ_g-almost everywhere, in Theorem 3.2 the following expression could have been used instead for $G(t,s)$:
if $t \leq \theta$,

$$G(t,s) = \begin{cases} \frac{-g(s)[g(1)-g(t)+k(g(\theta)-g(t))]}{g(1)+kg(\theta)} - \Delta g(s), & \text{if } 0 \leq s < t \\ \frac{-g(t)[g(1)-g(s)+k(g(\theta)-g(s))]}{g(1)+kg(\theta)}, & \text{if } t \leq s < \theta \\ \frac{-g(t)(g(1)-g(s))}{g(1)+kg(\theta)}, & \text{if } \theta \leq s \leq 1, \end{cases}$$

while if $t > \theta$,

$$G(t,s) = \begin{cases} \frac{-g(s)[g(1)-g(t)+k(g(\theta)-g(t))]}{g(1)+kg(\theta)} - \Delta g(s), & \text{if } 0 \leq s < \theta \\ \frac{-g(s)(g(1)-g(t))+kg(\theta)(g(t)-g(s))}{g(1)+kg(\theta)} - \Delta g(s), & \text{if } \theta \leq s < t \\ \frac{-g(t)(g(1)-g(s))}{g(1)+kg(\theta)}, & \text{if } t \leq s \leq 1. \end{cases}$$

3.2 Second order linear Stieltjes differential systems with several derivators

As described in [12, 19] or [21], differential systems with several derivators naturally occur when investigating physical processes; more precisely, it leads to studying the existence (and uniqueness) of $u : [0,1] \to \mathbb{R}^d$ such that

$$\begin{cases} u''_{\overline{g}}(t) = \overline{f}(t), \ \mu_{\overline{g}}\text{-a.e. } t \in [0,1) \\ u(0) = \overline{\alpha} \\ u(1) + ku(\theta) = \overline{\beta} \end{cases} \tag{3.5}$$

where $\overline{g} = (g_1, \ldots, g_d) : [0,1] \to \mathbb{R}^d$ with g_i left-continuous and nondecreasing for each $i \in \{1, \ldots, d\}$, $\overline{f} : [0,1] \to \mathbb{R}^d$, $\overline{\alpha} = (\alpha_1, \ldots, \alpha_d), \overline{\beta} = (\beta_1, \ldots, \beta_d) \in \mathbb{R}^d$ and $\theta \in (0,1)$ satisfy the assumption: $g_i(1) + kg_i(\theta) \neq 0$ for each $i \in \{1, \ldots, d\}$.

The $\overline{g}$-derivative is to be understood as the vector $((u_1)'_{g_1}, \ldots, (u_d)'_{g_d})$, thus the system (3.5) must be understood as:

$$\begin{cases} (u_i)''_{g_i}(t) = f_i(t), \ \mu_{g_i}\text{-a.e. } t \in [0,1) \\ u_i(0) = \alpha_i \\ u_i(1) + ku_i(\theta) = \beta_i \end{cases} \tag{3.6}$$

for every $i = 1, \ldots, d$.

Denote (as in [19, page 6]) by

$$\mathcal{BC}_{\overline{g}}([0,1], \mathbb{R}^d) = \prod_{i=1}^{d} \mathcal{BC}_{g_i}([0,1]),$$

by

$$\mathcal{BC}^1_{\overline{g}}([0,1], \mathbb{R}^d) = \prod_{i=1}^{d} \mathcal{BC}^1_{g_i}([0,1]),$$

respectively by

$$\mathcal{AC}_{\overline{g}}([0,1], \mathbb{R}^d) = \prod_{i=1}^{d} \mathcal{AC}_{g_i}([0,1]).$$

Definition 3.5. A function $u \in \mathcal{BC}^1_{\overline{g}}([0,1],\mathbb{R}^d)$ is a solution of (3.5) if for each i it verifies the boundary conditions $u_i(0) = \alpha_i$ and $u_i(1) + ku_i(\theta) = \beta_i$, the first g_i-derivative $(u_i)'_{g_i}$ is g_i-absolutely continuous and its g_i-derivative (which is well defined μ_{g_i}-a.e.) satisfies

$$(u_i)''_{g_i}(t) = f_i(t), \quad \mu_{g_i}\text{-a.e. in } [0,1).$$

Reasoning componentwise, Theorem 3.2 and Corollary 3.3 involve the following result.

Theorem 3.6. *Let $\overline{g} = (g_1,\dots,g_d) : [0,1] \to \mathbb{R}^d$ where g_i is left-continuous and nondecreasing for each $i \in \{1,\dots,d\}$ and let $\overline{f} \in L^1_{\overline{g}}([0,1],\mathbb{R}^d) = \prod_{i=1}^d L^1_{g_i}([0,1])$. Then $u : [0,1] \to \mathbb{R}^d$ is a solution of (3.5) if and only if for every $i \in \{1,\dots,d\}$:*

$$\begin{aligned} u_i(t) = &\int_{[0,1)} G_i(t,s) f_i(s) dg(s) - \chi_{(\theta,1]}(t) \cdot A_i(g_i(t) - g_i(\theta)) \\ &+ \frac{g_i(1) - g_i(t) + k(g_i(\theta) - g_i(t))}{g_i(1) + kg_i(\theta)} \alpha_i + \frac{g_i(t)}{g_i(1) + kg_i(\theta)} \tilde{\beta}_i, \quad \forall\, t \in [0,1], \end{aligned}$$

where $G_i : [0,1] \times [0,1] \to \mathbb{R}$ is defined as follows: if $t \le \theta$,

$$G_i(t,s) = \begin{cases} \frac{-g_i(s)[g_i(1)-g_i(t)+k(g_i(\theta)-g_i(t))]}{g_i(1)+kg_i(\theta)} - \Delta g_i(s_i^*), & \text{if } 0 \le s < t \\ \frac{-g_i(t)[g_i(1)-g_i(s)+k(g_i(\theta)-g_i(s))]}{g_i(1)+kg_i(\theta)}, & \text{if } t \le s < \theta \\ \frac{-g_i(t)(g_i(1)-g_i(s))}{g_i(1)+kg_i(\theta)}, & \text{if } \theta \le s \le 1, \end{cases}$$

while if $t > \theta$,

$$G_i(t,s) = \begin{cases} \frac{-g_i(s)[g_i(1)-g_i(t)+k(g_i(\theta)-g_i(t))]}{g_i(1)+kg_i(\theta)} - \Delta g_i(s_i^*), & \text{if } 0 \le s < \theta \\ \frac{-g_i(s)(g_i(1)-g_i(t))+kg_i(\theta)(g_i(t)-g_i(s))}{g_i(1)+kg_i(\theta)} - \Delta g_i(s_i^*), & \text{if } \theta \le s < t \\ \frac{-g_i(t)(g_i(1)-g_i(s))}{g_i(1)+kg_i(\theta)}, & \text{if } t \le s \le 1, \end{cases}$$

$$A_i = \frac{-k}{g_i(1) + kg_i(\theta)} \int_{[0,\theta)} f_i(s)(g_i(s) + 1) dg_i(s)$$

and

$$\tilde{\beta}_i = \beta_i + \left(\int_{[0,1)} \Delta g_i(s_i^*) f_i(s) dg_i(s) + k \int_{[0,\theta)} \Delta g_i(s_i^*) f_i(s) dg_i(s) + A_i(g_i(1) - g_i(\theta)) \right).$$

3.3 Existence result for systems of nonlinear second order boundary value equations

In this subsection, we focus on nonlinear differential problems of second order with boundary value conditions

$$\begin{cases} u''_{\overline{g}}(t) = f(t, u(t), u'_{\overline{g}}(t)), \ \mu_{\overline{g}}\text{-a.e. } t \in [0,1) \\ u(0) = \overline{\alpha} \\ u(1) + ku(\theta) = \overline{\beta}, \end{cases} \tag{3.7}$$

with $\overline{g} = (g_1,\dots,g_d) : [0,1] \to \mathbb{R}^d$ such that g_i is left-continuous and nondecreasing for each $i \in \{1,\dots,d\}$, $\overline{\alpha} = (\alpha_1,\dots,\alpha_d), \overline{\beta} = (\beta_1,\dots,\beta_d) \in \mathbb{R}^d$ and $k \in \mathbb{R}$, $\theta \in (0,1)$ satisfy the assumption: $g_i(1) + kg_i(\theta) \neq 0$ for each $i \in \{1,\dots,d\}$.

Definition 3.7. A function $u \in \mathcal{BC}^1_{\overline{g}}([0,1],\mathbb{R}^d)$ is a solution of (3.7) if $u(0)=\overline{\alpha}$, $u(1)+ku(\theta)=\overline{\beta}$, its first $\overline{g}$-derivative $u'_{\overline{g}} \in \mathcal{AC}_{\overline{g}}([0,1],\mathbb{R}^d)$ and its second $\overline{g}$-derivative satisfies

$$u''_{\overline{g}}(t)=f(t,u(t),u'_{\overline{g}}(t)), \quad \mu_{\overline{g}}\text{-a.e. in } [0,1),$$

i.e. for every $i \in \{1,\dots,d\}$:

$$(u_i)''_{g_i}(t)=f_i(t,u(t),u'_{\overline{g}}(t)), \quad \mu_{g_i}\text{-a.e. in } [0,1).$$

Theorem 3.8. *Let $f:[0,1]\times\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}^d$ satisfy, for each $i\in\{1,\dots,d\}$, the following conditions:*

(1) *there exists $M_i \in L^1_{g_i}([0,1])$ such that*

$$|f_i(t,x,y)|\le M_i(t), \quad \mu_{g_i}\text{-a.e. } t\in[0,1],\ \forall\, x,y\in\mathbb{R}^d;$$

(2) *$f_i(\cdot,x,y)$ is g_i-measurable, for every $x,y\in\mathbb{R}^d$;*

(3) *$f_i(t,\cdot,\cdot)$ is continuous on $\mathbb{R}^d\times\mathbb{R}^d$, for μ_{g_i}-almost every $t\in[0,1]$.*

Then the problem (3.7) has solutions.

Proof. Let us prove that the set

$$\mathcal{K}=\left\{u\in\mathcal{BC}^1_{\overline{g}}([0,1],\mathbb{R}^d);|(u_i)''_{g_i}(t)|\le M_i(t),\mu_{g_i}\text{-a.e., }\forall\, i \text{ and } u(0)=\overline{\alpha},\ u(1)+ku(\theta)=\overline{\beta}\right\}$$

is nonempty, convex and compact in $\mathcal{BC}^1_{\overline{g}}([0,1],\mathbb{R}^d)$.

First, note that any map $f\in L^1_{\overline{g}}([0,1],\mathbb{R}^d)$ such that $|f_i(t)|\le M_i(t)$ for all $i\in\{1,\dots,d\}$ provides, thanks to Theorem 3.2, an element of $\mathcal{K}$ (which is, therefore, nonempty) and that the convexity can be easily checked.

Then, let us see that $\mathcal{K}$ is compact in $\mathcal{BC}^1_{\overline{g}}([0,1],\mathbb{R}^d)$.

To this aim, consider a sequence $(u_n)_n\subset\mathcal{K}$. It can be seen from the proof of Theorem 3.2 that for each $i\in\{1,\dots,n\}$, if $t<\theta$,

$$\begin{aligned}((u_n)_i)'_{g_i}(t)=&\frac{1+k}{g_i(1)+kg_i(\theta)}\int_{[0,t)}g_i(s)((u_n)_i)''_{g_i}(s)dg_i(s)\\&-\frac{1}{g_i(1)+kg_i(\theta)}\int_{[t,\theta)}[g_i(1)-g_i(s)+k(g_i(\theta)-g_i(s)]((u_n)_i)''_{g_i}(s)dg_i(s)\\&-\frac{1}{g_i(1)+kg_i(\theta)}\int_{[\theta,1)}(g_i(1)-g_i(s))((u_n)_i)''_{g_i}(s)dg_i(s)-\frac{(1+k)\alpha_i-\tilde{\beta}_i}{g_i(1)+kg_i(\theta)},\end{aligned}$$

while if $t>\theta$,

$$\begin{aligned}((u_n)_i)'_{g_i}(t)=&\frac{1}{g_i(1)+kg_i(\theta)}\int_{[0,\theta)}(g_i(s)-k)((u_n)_i)''_{g_i}(s)dg(s)-g_i(t)((u_n)_i)''_{g_i}(t)\\&+\frac{1}{g_i(1)+kg_i(\theta)}\int_{[\theta,t)}(g_i(s)+kg_i(\theta))((u_n)_i)''_{g_i}(s)dg_i(s)\\&+\frac{g_i(t)}{g_i(1)+kg_i(\theta)}(g_i(t)+kg_i(\theta))((u_n)_i)''_{g_i}(t)\\&-\frac{1}{g_i(1)+kg_i(\theta)}\int_{[t,1)}(g_i(1)-g_i(s))((u_n)_i)''_{g_i}(s)dg_i(s)\\&+\frac{g_i(t)}{g_i(1)+kg_i(\theta)}(g_i(1)-g_i(t))((u_n)_i)''_{g_i}(t)-A_i-\frac{(1+k)\alpha_i-\tilde{\beta}_i}{g_i(1)+kg_i(\theta)}.\end{aligned}$$

Using Theorem 2.6 on each coordinate, $((u_n)'_{\overline{g}})_n$ is relatively compact in the space $\mathcal{BC}_{\overline{g}}([0,1],\mathbb{R}^d)$. One can choose a subsequence, denoted again by $(u_n)_n$ such that $\left((u_n)'_{\overline{g}}\right)_n$ is uniformly convergent to a function $v \in \mathcal{AC}_{\overline{g}}([0,1],\mathbb{R}^d)$.

In both cases, as $u_n(0) = \overline{\alpha}, \forall\, n$, $(u_n)_n$ satisfies as well the hypotheses of Theorem 2.6, therefore one can choose a further subsequence (not re-labelled) such that $(u_n)_n$ uniformly converges to a function $u \in \mathcal{AC}_{\overline{g}}([0,1],\mathbb{R}^d)$.

Since for every $i \in \{1,\dots,d\}$, the sequence $\left(((u_n)_i)'_{g_i}\right)_n$ uniformly converges to v, we infer that for each $t \in [0,1]$:

$$\int_{[0,t)} ((u_n)_i)'_{g_i}(s)dg_i(s) \to \int_{[0,t)} v(s)dg_i(s)$$

and remark that, by Theorem 2.2,

$$\int_{[0,t)} ((u_n)_i)'_{g_i}(s)dg_i(s) = (u_n)_i(t) - \alpha_i \to u_i(t) - \alpha_i;$$

it follows that

$$v(t) = u'_{\overline{g}}(t).$$

Moreover, note that $((u_n)''_{\overline{g}})_n$ is, by hypothesis, relatively weakly compact in $L^1_{\overline{g}}([0,1],\mathbb{R}^d)$, so on convex combinations we may suppose that $((u_n)''_{\overline{g}})_n$ converges pointwise to a function $w \in L^1_{\overline{g}}([0,1],\mathbb{R}^d)$. Applying a dominated convergence theorem one gets for each $i \in \{1,\dots,d\}$, if $t < \theta$,

$$\begin{aligned}((u_n)_i)'_{g_i}(t) \to\ & \frac{1+k}{g_i(1)+kg_i(\theta)} \int_{[0,t)} g_i(s)w_i(s)dg_i(s) \\ & - \frac{1}{g_i(1)+kg_i(\theta)} \int_{[t,\theta)} [g_i(1) - g_i(s) + k(g_i(\theta) - g_i(s)]w_i(s)dg_i(s) \\ & - \frac{1}{g_i(1)+kg_i(\theta)} \int_{[\theta,1)} (g_i(1) - g_i(s))w_i(s)dg_i(s) - \frac{(1+k)\alpha_i - \tilde{\beta}_i}{g_i(1)+kg_i(\theta)}.\end{aligned}$$

while if $t > \theta$,

$$\begin{aligned}((u_n)_i)'_{g_i}(t) \to\ & \frac{1}{g_i(1)+kg_i(\theta)} \int_{[0,\theta)} (g_i(s) - k)w_i(s)dg(s) - g_i(t)w_i(t) \\ & + \frac{1}{g_i(1)+kg_i(\theta)} \int_{[\theta,t)} (g_i(s) + kg_i(\theta))w_i(s)dg_i(s) \\ & + \frac{g_i(t)}{g_i(1)+kg_i(\theta)} (g_i(t) + kg_i(\theta))w_i(t) \\ & - \frac{1}{g_i(1)+kg_i(\theta)} \int_{[t,1)} (g_i(1) - g_i(s))w_i(s)dg_i(s) \\ & + \frac{g_i(t)}{g_i(1)+kg_i(\theta)} (g_i(1) - g_i(t))w_i(t) - A_i - \frac{(1+k)\alpha_i - \tilde{\beta}_i}{g_i(1)+kg_i(\theta)}.\end{aligned}$$

On the other hand, it was seen that for each $i \in \{1,\dots,d\}$, $\left(((u_n)_i)'_{g_i}\right)_n$ uniformly converges to $(u_i)'_{g_i}$ which is equal to

$$\begin{aligned}& \frac{1+k}{g_i(1)+kg_i(\theta)} \int_{[0,t)} g_i(s)(u_i)''_{g_i}(s)dg_i(s) \\ & - \frac{1}{g_i(1)+kg_i(\theta)} \int_{[t,\theta)} [g_i(1) - g_i(s) + k(g_i(\theta) - g_i(s)](u_i)''_{g_i}(s)dg_i(s) \\ & - \frac{1}{g_i(1)+kg_i(\theta)} \int_{[\theta,1)} (g_i(1) - g_i(s))(u_i)''_{g_i}(s)dg_i(s) - \frac{(1+k)\alpha_i - \tilde{\beta}_i}{g_i(1)+kg_i(\theta)}.\end{aligned}$$

if $t < \theta$, respectively to

$$\begin{aligned}
&\frac{1}{g_i(1)+kg_i(\theta)}\int_{[0,\theta)}(g_i(s)-k)(u_i)''_{g_i}(s)dg(s) - g_i(t)(u_i)''_{g_i}(t)\\
&\quad+\frac{1}{g_i(1)+kg_i(\theta)}\int_{[\theta,t)}(g_i(s)+kg_i(\theta))(u_i)''_{g_i}(s)dg_i(s)\\
&\quad+\frac{g_i(t)}{g_i(1)+kg_i(\theta)}(g_i(t)+kg_i(\theta))(u_i)''_{g_i}(t)\\
&\quad-\frac{1}{g_i(1)+kg_i(\theta)}\int_{[t,1)}(g_i(1)-g_i(s))(u_i)''_{g_i}(s)dg_i(s)\\
&\quad+\frac{g_i(t)}{g_i(1)+kg_i(\theta)}(g_i(1)-g_i(t))(u_i)''_{g_i}(t) - A_i - \frac{(1+k)\alpha_i-\tilde{\beta}_i}{g_i(1)+kg_i(\theta)}
\end{aligned}$$

if $t > \theta$.

Consequently,

$$w(t) = u''_{\overline{g}}(t), \quad \mu_{\overline{g}}\text{-almost everywhere on } [0,1).$$

Therefore, $|(u_i)''_{g_i}(t)| \leq M_i(t), \mu_{g_i}$-a.e., $\forall\, i$ and so, $u \in \mathcal{K}$. The compactness of $\mathcal{K}$ is thus proved.

Let us now check that the operator $\Xi : \mathcal{K} \to \mathcal{K}$,

$$\Xi(u) = v \in \mathcal{K} \quad \text{such that } v''_{\overline{g}}(t) = f(t, u(t), u'_{\overline{g}}(t)),\ \mu_{\overline{g}}\text{-a.e.}$$

satisfies the conditions of Schauder's fixed point theorem.

At the beginning, note that Ξ is well-defined by Theorem 3.2 and Corollary 3.3, since as in [11, Lemma 7.2] it can be proved that for any $u \in \mathcal{K}$, $f(\cdot, u(\cdot), u'_{\overline{g}}(\cdot)) \in L^1_{\overline{g}}([0,1], \mathbb{R}^d)$.

All we have to check now is that Ξ is continuous.

Let $(u_n)_n \subset \mathcal{K}$ converge to $u \in \mathcal{K}$ and we aim to prove that $(\Xi(u_n))_n = (v_n)_n$ converges to $\Xi(u) = v$.

As $(u_n)_n$ uniformly converges to u and $\left((u_n)'_{\overline{g}}\right)_n$ uniformly converges to $u'_{\overline{g}}$, hypothesis (3) involves that for each $i \in \{1, \dots, d\}$,

$$\left(f_i(\cdot, u_n(\cdot), (u_n)'_{\overline{g}}(\cdot))\right)_n \quad \text{converges to } f_i(\cdot, u(\cdot), u'_{\overline{g}}(\cdot)),\ \mu_{g_i}\text{-a.e.}$$

By Theorem 3.2, for every $n \in \mathbb{N}$ and $i \in \{1, \dots, d\}$:

$$\begin{aligned}
(v_n)_i(t) = &\int_{[0,1)} G_i(t,s) f_i(s, u_n(s), (u_n)'_{\overline{g}}(s)) dg_i(s) - \chi_{(\theta,1]}(t)\cdot A_i(g_i(t)-g_i(\theta))\\
&+\frac{g_i(1)-g_i(t)+k(g_i(\theta)-g_i(t))}{g_i(1)+kg_i(\theta)}\alpha_i + \frac{g_i(t)}{g_i(1)+kg_i(\theta)}\tilde{\beta}_i
\end{aligned}$$

and

$$\begin{aligned}
v_i(t) = &\int_{[0,1)} G_i(t,s) f_i(s, u(s), u'_{\overline{g}}(s)) dg_i(s) - \chi_{(\theta,1]}(t)\cdot A_i(g_i(t)-g_i(\theta))\\
&+\frac{g_i(1)-g_i(t)+k(g_i(\theta)-g_i(t))}{g_i(1)+kg_i(\theta)}\alpha_i + \frac{g_i(t)}{g_i(1)+kg_i(\theta)}\tilde{\beta}_i
\end{aligned}$$

for all $t \in [0,1]$.

The hypothesis (1) ensures that the dominated convergence theorem can be applied on each coordinate and so,

$$(v_n)_n \to v \quad \text{uniformly on } [0,1].$$

Likewise, from the proof of Theorem 3.2 it can be seen that for each $i \in \{1, \dots, n\}$, if $t < \theta$,

$$\begin{aligned}((v_n)_i)'_{g_i}(t) = {} & \frac{1+k}{g_i(1)+kg_i(\theta)} \int_{[0,t)} g_i(s) f_i(s, u_n(s), (u_n)'_{\overline{g}}(s)) dg_i(s) \\ & - \frac{1}{g_i(1)+kg_i(\theta)} \int_{[t,\theta)} [g_i(1) - g_i(s) + k(g_i(\theta) - g_i(s)] f_i(s, u_n(s), (u_n)'_{\overline{g}}(s)) dg_i(s) \\ & - \frac{1}{g_i(1)+kg_i(\theta)} \int_{[\theta,1)} (g_i(1) - g_i(s)) f_i(s, u_n(s), (u_n)'_{\overline{g}}(s)) dg_i(s) - \frac{(1+k)\alpha_i - \tilde{\beta}_i}{g_i(1)+kg_i(\theta)}\end{aligned}$$

while if $t > \theta$,

$$\begin{aligned}& ((v_n)_i)'_{g_i}(t) \\ & = \frac{1}{g_i(1)+kg_i(\theta)} \int_{[0,\theta)} (g_i(s) - k) f_i(s, u_n(s), (u_n)'_{\overline{g}}(s)) dg(s) - g_i(t) f_i(t, u_n(t), (u_n)'_{\overline{g}}(t)) \\ & \quad + \frac{1}{g_i(1)+kg_i(\theta)} \int_{[\theta,t)} (g_i(s) + kg_i(\theta)) f_i(s, u_n(s), (u_n)'_{\overline{g}}(s)) dg_i(s) \\ & \quad + \frac{g_i(t)}{g_i(1)+kg_i(\theta)} (g_i(t) + kg_i(\theta)) f_i(t, u_n(t), (u_n)'_{\overline{g}}(t)) \\ & \quad - \frac{1}{g_i(1)+kg_i(\theta)} \int_{[t,1)} (g_i(1) - g_i(s)) f_i(s, u_n(s), (u_n)'_{\overline{g}}(s)) dg_i(s) \\ & \quad + \frac{g_i(t)}{g_i(1)+kg_i(\theta)} (g_i(1) - g_i(t)) f_i(t, u_n(t), (u_n)'_{\overline{g}}(t)) - A_i - \frac{(1+k)\alpha_i - \tilde{\beta}_i}{g_i(1)+kg_i(\theta)}.\end{aligned}$$

Also, if $t < \theta$,

$$\begin{aligned}(v_i)'_{g_i}(t) = {} & \frac{1+k}{g_i(1)+kg_i(\theta)} \int_{[0,t)} g_i(s) f_i(s, u(s), u'_{\overline{g}}(s)) dg_i(s) \\ & - \frac{1}{g_i(1)+kg_i(\theta)} \int_{[t,\theta)} [g_i(1) - g_i(s) + k(g_i(\theta) - g_i(s)] f_i(s, u(s), u'_{\overline{g}}(s)) dg_i(s) \\ & - \frac{1}{g_i(1)+kg_i(\theta)} \int_{[\theta,1)} (g_i(1) - g_i(s)) f_i(s, u(s), u'_{\overline{g}}(s)) dg_i(s) - \frac{(1+k)\alpha_i - \tilde{\beta}_i}{g_i(1)+kg_i(\theta)}\end{aligned}$$

while if $t > \theta$,

$$\begin{aligned}(v_i)'_{g_i}(t) = {} & \frac{1}{g_i(1)+kg_i(\theta)} \int_{[0,\theta)} (g_i(s) - k) f_i(s, u(s), u'_{\overline{g}}(s)) dg(s) - g_i(t) f_i(t, u(t), u'_{\overline{g}}(t)) \\ & + \frac{1}{g_i(1)+kg_i(\theta)} \int_{[\theta,t)} (g_i(s) + kg_i(\theta)) f_i(s, u(s), u'_{\overline{g}}(s)) dg_i(s) \\ & + \frac{g_i(t)}{g_i(1)+kg_i(\theta)} (g_i(t) + kg_i(\theta)) f_i(t, u(t), u'_{\overline{g}}(t)) \\ & - \frac{1}{g_i(1)+kg_i(\theta)} \int_{[t,1)} (g_i(1) - g_i(s)) f_i(s, u(s), u'_{\overline{g}}(s)) dg_i(s) \\ & + \frac{g_i(t)}{g_i(1)+kg_i(\theta)} (g_i(1) - g_i(t)) f_i(t, u(t), u'_{\overline{g}}(t)) - A_i - \frac{(1+k)\alpha_i - \tilde{\beta}_i}{g_i(1)+kg_i(\theta)}.\end{aligned}$$

In both cases, again by dominated convergence theorem it follows that $((v_n)'_{\overline{g}})_n$ uniformly converges to $v'_{\overline{g}}$.

So, $\Xi(u_n) = v_n \to \Xi(u) = v$ in $\mathcal{BC}^1_{\overline{g}}([0,1], \mathbb{R}^d)$ and the continuity of Ξ is proved.

In conclusion, Ξ has fixed points which are solutions of the considered problem. □

Remark 3.9. To the best of the author's knowledge, this is the first study of systems of second order equations involving Stieltjes derivatives (besides, with several derivators) and three-point conditions on the boundary.

When $g(t) = t$ on $[0,1]$ we cover or complement already known results with various boundary value conditions (e.g. [17,31,33] and their references, see also [2,3,25]). Note finally that the strong connection between the theory of Stieltjes differential equations and other types of differential problems implies that the outcomes presented in this work lead to new results for systems of second-order generalized differential equations, impulsive differential equations (e.g. [7]) and also dynamic problems on time scales (see [1, 24]) with three-point boundary conditions.

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Unilateral global bifurcation for an overdetermined problem in $\mathbb{S}^N \times \mathbb{R}$ and $\mathbb{H}^N \times \mathbb{R}$

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Abstract. We establish the Dancer-type unilateral global bifurcation theorem for nonlinear operator equation of $u + f(\lambda, u) = F(\lambda, u) = 0$, where X is a real Banach space and $f : \mathbb{R} \times X \to X$ is completely continuous with $f(\lambda, 0) = 0$ and C^1 with respect to u at $u = 0$. We shall show that, if $\dim \mathrm{Ker}(D_u F(\mu, 0)) = 1$ for some $\mu \in \mathbb{R}$ and $D_u F(\lambda, 0)$ has an odd crossing number at $\lambda = \mu$, there exist two branches $\mathscr{C}_\mu^\nu$ ($\nu \in \{+, -\}$) emanating from $(\mu, 0)$, such that either $\mathscr{C}_\mu^+$ and $\mathscr{C}_\mu^-$ are both unbounded or $\mathscr{C}_\mu^+ \cap \mathscr{C}_\mu^- \neq \{(\mu, 0)\}$. As one of applications, we obtain the unilateral global bifurcation result for an overdetermined problem in $\mathbb{S}^N \times \mathbb{R}$ and $\mathbb{H}^N \times \mathbb{R}$.

Keywords: unilateral global bifurcation, odd crossing number, completely continuous, overdetermined problem.

2020 Mathematics Subject Classification: 35B32, 35N05, 47A75, 47H14.

1 Introduction

Let X be real Banach space. We investigate the structure of the set of nontrivial solutions for the following nonlinear parameter-dependent problem

$$u + f(\lambda, u) = F(\lambda, u) = 0, \ (\lambda, u) \in \mathbb{R} \times X, \tag{1.1}$$

where $f : \mathbb{R} \times X \to X$ is completely continuous with $f(\lambda, 0) = 0$ for $\lambda \in \mathbb{R}$ and C^1 with respect to u at $u = 0$.

For $f(\lambda, u) = -\lambda L u - H(\lambda, u)$ where $L : X \to X$ is a linear compact operator and $H : \mathbb{R} \times X \to X$ is completely continuous with $H = o(\|u\|)$ near $u = 0$ uniformly on bounded λ sets, Krasnosel'skii [12] has shown that all characteristic values of L which are of odd multiplicity are bifurcation points. Rabinowitz [20] has extended this result by showing that bifurcation has global consequences.

Rabinowitz [20] also established two unilateral global bifurcation theorems from simple eigenvalues, i.e., Theorem 1.27 and Theorem 1.40 of [20]. As pointed out by Dancer [6], the

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proofs of these two theorems both contain gaps. Dancer [6,7] constructed a counterexample to Theorem 1.40 of [20]. López-Gómez [15] also pointed out that the proof of Theorem 1.27 is insufficient. Moreover, Dancer [6] established the so-called Dancer-type unilateral global bifurcation theorem from simple eigenvalues. Concretely, if μ^{-1} is an eigenvalue of L of algebraic multiplicity 1, then there exist two sub-continua, $\mathscr{C}_\mu^+$ and $\mathscr{C}_\mu^-$, of $\mathscr{C}_\mu$ bifurcating from $(\mu,0)$, such that either $\mathscr{C}_\mu^+$ and $\mathscr{C}_\mu^-$ are both unbounded or $\mathscr{C}_\mu^+ \cap \mathscr{C}_\mu^- \neq \{(\mu,0)\}$. Further, Dancer [7] also proved that the above beautiful unilateral global bifurcation result is also valid for the case of μ^{-1} being an eigenvalue of L with geometric multiplicity 1 and odd algebraic multiplicity. López-Gómez [15] also established a unilateral global bifurcation theorem. López-Gómez's result indicates that $\mathscr{C}_\mu^\nu$ with each $\nu \in \{+,-\}$ either satisfies Rabinowitz-type global alternative or contains a nontrivial element of complement of kernel space.

Following the Rabinowitz's reflection argument in the proof of [20, Theorem 1.27], Kielhofer [11, Theorem II.5.9] established a unilateral global bifurcation theorem via the conception of odd crossing number. To present the Kielhofer's unilateral global bifurcation theorem, we recall the conception of odd crossing number. Let 0 be an isolated eigenvalue of algebraic multiplicity m of $D_uF(\mu,0)$ for some $\mu \in \mathbb{R}$. It is well known that the number m is an invariant, i.e., the dimension of eigenspace is invariant under perturbation near μ. The set of all perturbed eigenvalues near 0 is called 0-*group*. Further, define $\sigma(\lambda) = 1$ if there are no negative real eigenvalues in the 0-group of $D_uF(\lambda,0)$ and

$$\sigma(\lambda) = (-1)^{m_1+m_2+\cdots+m_k}$$

if $\mu_1,\mu_2,\ldots,\mu_k$ are all negative real eigenvalues in the 0-group having algebraic multiplicities $m_1,m_2,\ldots,m_k$, respectively. From now on, for simplicity, $\sigma(\lambda)$ is called 0-*group index* of λ. If $D_uF(\lambda,0)$ is regular for $\lambda \in (\mu-\delta,\mu)\cup(\mu,\mu+\delta)$ and if $\sigma(\lambda)$ changes at $\lambda=\mu$, then $D_uF(\lambda,0)$ has an *odd crossing number* at $\lambda=\mu$.

If $\dim \mathrm{Ker}(D_uF(\mu,0)) = 1$ and $D_uF(\lambda,0)$ has an odd crossing number at $\lambda=\mu$, Kielhofer [11, Theorem II.5.9] proved that there exist two sub-continua, $\mathscr{C}_\mu^+$ and $\mathscr{C}_\mu^-$, of $\mathscr{C}_\mu$ bifurcating from $(\mu,0)$, such that $\mathscr{C}_\mu^\nu$ with each $\nu \in \{+,-\}$ either satisfies Rabinowitz-type global alternative or contains a pair of points (λ,u) and $(\lambda,-u)$ with $u \neq 0$. If $f(\lambda,u) = \lambda f(u)$ and 0 is the simple eigenvalue of $D_uF(\mu,0)$, Kielhofer's unilateral global bifurcation theorem is just the Rabinowitz's unilateral global bifurcation theorem of [20, Theorem 1.27]. So, Kielhofer's result extends Rabinowitz's unilateral global bifurcation theorem of [20, Theorem 1.27]. As the mentioned above, since the proof of Theorem 1.27 of [20] is insufficient, Kielhofer's argument is also insufficient because he also adopted the Rabinowitz's reflection argument.

The first aim of this work is to establish the Dancer-type unilateral global bifurcation theorem under the assumptions of Kielhofer, which fills the above gap by providing a corrected unilateral global bifurcation theorem. Let $\mathcal{S}$ be the closure of the set of nontrivial solutions of equation (1.1). The following Dancer-type unilateral global bifurcation theorem is our first main result.

Theorem 1.1. *Assume that* $\dim \mathrm{Ker}(D_uF(\mu,0)) = 1$ *for some* $\mu \in \mathbb{R}$ *and* $D_uF(\lambda,0)$ *has an odd crossing number at* $\lambda=\mu$. *Then* $\mathcal{S}$ *possesses two maximal sub-continua* $\mathscr{C}_\mu^\pm$ *emanating from* $(\mu,0)$, *such that either* $\mathscr{C}_\mu^+$ *and* $\mathscr{C}_\mu^-$ *are both unbounded or* $\mathscr{C}_\mu^+ \cap \mathscr{C}_\mu^- \neq \{(\mu,0)\}$.

If $f(\lambda,u)$ has the form of $\lambda f(u)$ and 0 is the simple eigenvalue of $D_uF(\mu,0)$, the conclusion of Theorem 1.1 would degenerate to the famous Dancer-type unilateral global bifurcation theorem from simple eigenvalues [6]. While, $f(\lambda,u)$ has the form of $\lambda f(u)$ and 0 is odd algebraic multiplicity eigenvalue of $D_uF(\mu,0)$, Theorem 1.1 degenerates to the Dancer-type

unilateral global bifurcation theorem from odd algebraic multiplicity eigenvalues [7]. As pointed out by Dancer [7], although the conditions are the same, Dancer-type unilateral global bifurcation conclusion is better than López-Gómez's [15, Theorem 6.4.3].

As one of applications for our unilateral global bifurcation theorem, we next investigate the unilateral global bifurcation phenomenon for an overdetermined elliptic problem in $\mathbb{S}^N \times \mathbb{R}$ and $\mathbb{H}^N \times \mathbb{R}$, where $\mathbb{S}^N$ is the N-dimensional sphere, and $\mathbb{H}^N$ is the N-dimensional hyperbolic space. We consider the following overdetermined elliptic problem

$$\begin{cases} \Delta_g u + \lambda u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ g(\nabla u, \nu) = \text{const.} & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbb{M}^N \times \mathbb{R}$ with $\mathbb{M}^N = \mathbb{S}^N$ or $\mathbb{H}^N$, $N \geq 2$, g denotes the standard metric of $\mathbb{M}^N \times \mathbb{R}$ and Δ_g is the Laplace–Beltrami operator.

When $\mathbb{M}^N = \mathbb{R}^N$, Sicbaldi [24] constructed periodic solutions of (1.2) that are perturbations of a cylinder, which can be seen the first unbounded case counterexample to the following BCN Conjecture.

BCN Conjecture: If Ω is a smooth domain and $\mathbb{R}^N \setminus \overline{\Omega}$ is connected such that problem (1.2) exists a bounded solution, then Ω is either a ball, a half-space, a generalized cylinder $B^k \times \mathbb{R}^{N-k}$ where B^k is a ball in $\mathbb{R}^k$, or the complement of one of them.

In [23], it is shown that such new solutions belong in fact to a smooth local 1-parameter family. Generalizations of such results have been done in the Riemannian manifolds $\mathbb{S}^N \times \mathbb{R}$ and $\mathbb{H}^N \times \mathbb{R}$ [5,16], and in the Euclidean case for some functions f [22]. The boundaries of the domains Ω constructed in [24] are related to the Delaunay surfaces in $\mathbb{R}^N$. The analogy between constant mean curvatures surfaces and overdetermined elliptic problems has inspired a lot of works [9, 14, 21, 25]. Such analogy motivated also the study of overdetermined elliptic problems in the Riemannian manifolds $\mathbb{S}^N \times \mathbb{R}$ and $\mathbb{H}^N \times \mathbb{R}$. It is well known that the Thurston's Geometrization Conjecture proved by G. Perelman in 2003 [17–19]. Since these two manifolds with $N = 2$ represent two of the eight Thurston's 3-dimensional geometries [1], the theory of constant mean curvature surfaces in such ambient spaces are extremely important.

While, we note that bifurcation results in [5,16] are all local. A natural question is whether these local bifurcation results can be extended to the global? Our second main result provides a positive answer to this question.

Theorem 1.2. *Let* $\mathcal{C}^{2,\alpha}_{\text{even},0}(\mathbb{R}/2\pi\mathbb{Z})$ *be the space of even* 2π*-periodic* $\mathcal{C}^{2,\alpha}$ *functions of mean zero. There exist two nontrivial branches* $\mathscr{C}^+$ *and* $\mathscr{C}^-$ *in* $\mathcal{V} \times (0, T_0)$ *emanating from* $(0, T_*(N))$*, such that* $\mathscr{C}^\nu \cap (\{0\} \times (0, +\infty)) = \{(0, T_*(N))\}$ *and for any* $(v^\nu, T) \in \mathscr{C}^\nu \setminus \{(0, T_*(N))\}$ *with* $\nu \in \{+, -\}$*, the overdetermined problem* (1.2) *has a positive* T*-periodic solution* $u \in \mathcal{C}^{2,\alpha}(\Omega^\nu)$ *on the modified cylinder*

$$\Omega^\nu = \left\{ (x, t) \in \mathbb{M}^N \times \mathbb{R} : |x| < 1 + v^\nu\left(\frac{2\pi}{T} t\right) \right\},$$

where $\mathcal{V}$ *is an open neighborhoods of the* 0 *in* $\left\{ v \in \mathcal{C}^{2,\alpha}_{\text{even},0}(\mathbb{R}/2\pi\mathbb{Z}) : v > -1 \right\}$ *for some positive constant* T_0. Moreover, $\mathscr{C}^\nu$ satisfies at least one of the following three properties:

(i) $\mathscr{C}^\nu \cap \partial\mathcal{O} \neq \emptyset$ with $\mathcal{O} = \mathcal{V} \times (0, T_0)$,

(ii) $\mathscr{C}^\nu$ *is unbounded,*

(iii) $\mathscr{C}^\nu$ *contains a point* $(T^*, 0) \in \mathcal{O}$ *with some* $T^* \neq T_*$.

The outline of the rest of this article is as follows. In Section 2, we mainly establish a new Dancer-type unilateral global bifurcation theorem via the so-called 0-group, which ends the proof of Theorem 1.1. In Section 3, we finish the proof of Theorem 1.2 by using the new Dancer-type unilateral global bifurcation Theorem 1.1.

2 Proof of Theorem 1.1

Under the assumptions of Theorem 1.1, the operator F can be rewritten as

$$F(\lambda, u) = u + D_u f(\lambda, 0)u + H(\lambda, u),$$

where $H(\lambda, u)$ is $o(\|u\|)$ for $u \in X$ near 0 uniformly on the bounded λ interval. Since $f : \mathbb{R} \times X \to X$ is a completely continuous operator, $D_u f(\lambda, 0)$ is also a completely continuous operator (see [8] or [13]). It further follows that $H : \mathbb{R} \times X \to X$ is a completely continuous operator.

Let $\mathbb{X} = \mathbb{R} \times X$. Given any $\iota \in \mathbb{R}$ and $0 < s < +\infty$, we consider an open neighborhood of $(\iota, 0)$ in $\mathbb{X}$ defined by

$$\mathbb{B}_s(\iota, 0) = \{(\lambda, u) \in \mathbb{X} : \|u\| + |\lambda - \iota| < s\}.$$

Let X_0 be a closed subspace of X such that

$$X = \operatorname{span}\{w_0\} \oplus X_0,$$

where w_0 is a nonzero element in $\operatorname{Ker}(D_u F(\mu, 0))$. Without loss of generality, we assume that $\|w_0\| = 1$. According to the Hahn–Banach theorem, there exists a linear functional $l \in X^*$ such that

$$l(w_0) = 1 \quad \text{and} \quad X_0 = \{u \in X : l(u) = 0\},$$

where X^* denotes the dual space of X. For any $0 < \eta < 1$, define

$$K_\eta = \{(\lambda, u) \in \mathbb{X} : |l(u)| > \eta\|u\|\}.$$

Obviously, K_η is an open subset of $\mathbb{X}$ consisting of two disjoint components K_η^+ and K_η^- with

$$K_\eta^+ = \{(\lambda, u) \in \mathbb{X} : l(u) > \eta\|u\|\},$$
$$K_\eta^- = \{(\lambda, u) \in \mathbb{X} : l(u) < -\eta\|u\|\}.$$

Clearly, both K_η^+ and K_η^- are convex cones, $K_\eta^- = -K_\eta^+$, and $\nu t w_0 \in K_\eta^\nu$ for every $t > 0$ and each $\nu \in \{+, -\}$.

Applying [15, Lemma 6.4.1] or [3, Lemma 2.2], we have the the following lemma, which localizes the possible solutions of problem (1.1) bifurcating from $(\mu, 0)$.

Lemma 2.1. *For every* $\eta \in (0, 1)$ *there exists a number* $\delta_0 > 0$ *such that for each* $0 < \delta < \delta_0$, *there holds*

$$(\mathcal{S} \backslash \{(\mu, 0)\}) \cap \bar{\mathbb{B}}_\delta(\mu, 0) \subseteq K_\eta,$$

and when

$$(\lambda, u) \in (\mathcal{S} \setminus \{(\mu, 0)\}) \cap \bar{\mathbb{B}}_\delta(\mu, 0)$$

there are $s \in \mathbb{R}$ *and a unique* $y \in X_0$ *such that* $u = sw_0 + y$ *and* $|s| > \eta \|u\|$. *Furthermore, for each* (λ, u), *there holds* $\lambda = \mu + o(1)$ *and* $y = o(s)$ *as* $s \to 0$.

It follows from [11, Theorem II.3.3] that $(\mu, 0)$ is a bifurcation point for equation (1.1), and $\mathcal{S}$ possesses a maximal continuum $\mathscr{C}_\mu$ such that $(\mu, 0) \in \mathscr{C}_\mu$ and $\mathscr{C}_\mu$ either meets at infinity in $\mathbb{R} \times X$, or meets at $(\hat{\mu}, 0)$ with some $\hat{\mu} \neq \mu$. Furthermore, by [15, Lemma 6.4.2] or [3, Lemma 2.3], we have that $\mathscr{C}_\mu$ possesses a subcontinuum in each of the cones $K_\eta^+ \cup \{(\mu, 0)\}$ and $K_\eta^- \cup \{(\mu, 0)\}$ each of which meets $(\mu, 0)$ and $\partial \bar{\mathbb{B}}_\varrho(\mu, 0)$ for all $\varrho > 0$ sufficiently small, which is the local unilateral bifurcation structure of $\mathscr{C}_\mu$.

Proof of Theorem 1.1. For any $\varepsilon > 0$ small enough, let $a = \mu - \varepsilon$ and $b = \mu + \varepsilon$. Since $D_u F(a, 0)$ and $D_u F(b, 0)$ are isomorphism, the isolated zero index formula is well-defined for $I + D_u f(a, 0)$ and $I + D_u f(b, 0)$, which are denoted by $i(I + D_u f(a, 0), 0)$ and $i(I + D_u f(b, 0), 0)$. From the definition of 0-group index we see that

$$i(I + D_u f(a, 0), 0) = \sigma(a)$$

and

$$i(I + D_u f(b, 0), 0) = \sigma(b).$$

Thus, we have that

$$i(I + D_u f(a, 0), 0) \neq i(I + D_u f(b, 0), 0).$$

That is to say

$$\deg(I + D_u f(a, 0), \mathfrak{B}_r(0), 0) \neq \deg(I + D_u f(a, 0), \mathfrak{B}_r(0), 0),$$

where $\mathfrak{B}_r(0) = \{u \in X : \|u\| < r\}$ is an isolating neighborhood of the trivial solution. Applying [4, Theorem 3.1], we obtain that $\mathcal{S}$ possesses two maximal sub-continua $\mathscr{C}_\mu^\pm$ emanating from $(\mu, 0)$, such that either $\mathscr{C}_\mu^+$ and $\mathscr{C}_\mu^-$ are both unbounded or $\mathscr{C}_\mu^+ \cap \mathscr{C}_\mu^- \neq \{(\mu, 0)\}$. □

Note that the unilateral global bifurcation result of [4, Theorem 3.1] is for multiparameter problem. Here we use its special case of single parameter. If f is not globally defined, it is not difficult to get the following result.

Corollary 2.2. *Assume that* $\mathcal{O}$ *is an open subset of* $\mathbb{R} \times X$ *and* F *is defined on* $\mathcal{O}$. *Under the assumptions of Theorem 1.1, either* $\mathscr{C}_\mu^+$ *and* $\mathscr{C}_\mu^-$ *satisfy the alternatives of Theorem 1.1 or at least one of them meets* $\partial\mathcal{O}$.

3 Proof of Theorem 1.2

Let k be the sectional curvature of the manifold $\mathbb{M}^N$ (i.e. $k = 1$ if $\mathbb{M}^N = \mathbb{S}^N$ and $k = -1$ if $\mathbb{M}^N = \mathbb{H}^N$). If we choose spherical coordinates (r, θ), with $\theta \in \mathbb{S}^{N-1}$ and $r \in [0, +\infty)$ if $k < 0$ and $r \in [0, \pi]$ if $k > 0$, the usual metric in $\mathbb{M}^N$ [2, Section II.5, Theorem 1] can be written as

$$g_{\mathbb{M}^N} = \mathrm{d}r^2 + S_k^2(r)\, \mathrm{d}\theta^2$$

where

$$S_k(r) = \begin{cases} \sinh r & \text{if } k = -1, \\ \sin r & \text{if } k = 1. \end{cases}$$

Consider the eigenvalue problem

$$\begin{cases} \Delta_{g_{\mathbb{M}^N}} u + \lambda u = 0 & \text{in } B_1, \\ u = 0 & \text{on } \partial B_1, \end{cases} \tag{3.1}$$

where B_1 is the unit geodesic ball of $\mathbb{M}^N$. It is well known that (3.1) possesses a unique principal eigenvalue λ_1. Let $\widetilde{\phi}_1$ be the positive eigenfunction associated to λ_1 normalized so that $\int_{B_1} \widetilde{\phi}_1^2 \, \mathrm{dvol}_{g_{\mathbb{M}^N}} = 1/2\pi$. Then, if g denotes the standard product metric of $\mathbb{M}^N \times \mathbb{R}$ and $r(x)$ denotes the geodesic distance of $x \in \mathbb{M}^N$ from a fixed point $0 \in \mathbb{M}^N$ (the origin), the function $\phi_1(x,t) = \widetilde{\phi}_1(x)$ is a solution of

$$\begin{cases} \Delta_g \phi_1 + \lambda_1 \phi_1 = 0 & \text{in } C_1^T, \\ \phi_1 = 0 & \text{on } \partial C_1^T, \end{cases}$$

where

$$C_1^T = \Big\{ (x,t) \in \mathbb{M}^N \times \mathbb{R}/T\mathbb{Z} : r(x) < 1 \Big\}.$$

It is easy to see that

$$\int_{C_1^{2\pi}} \phi_1^2 \, \mathrm{dvol}_g = 1. \tag{3.2}$$

For each $v \in \mathcal{C}^{2,\alpha}_{\mathrm{even},0}(\mathbb{R}/2\pi\mathbb{Z})$ with $v(t) > -1$, we set

$$C_{1+v}^T = \left\{ (x,t) \in \mathbb{M}^N \times \mathbb{R}/T\mathbb{Z} : r(x) < 1 + v\left(\frac{2\pi t}{T}\right) \right\}.$$

It follows from [2, 10] that there exists a unique positive function $\phi_v \in \mathcal{C}^{2,\alpha}\big(C_{1+v}^T\big)$ and a constant λ_v such that

$$\begin{cases} \Delta_g \phi_v + \lambda_v \phi_v = 0 & \text{in } C_{1+v}^T, \\ \phi_v = 0 & \text{on } \partial C_{1+v}^T \end{cases} \tag{3.3}$$

and

$$\int_{C_{1+v}^{2\pi}} \phi_v^2 \left(x, \frac{T}{2\pi} t \right) \mathrm{dvol}_g = 1.$$

Define the operator

$$N(v,T) = g(\nabla \phi_v, \omega) \mid_{\partial C_{1+v}^T} - \frac{1}{\mathrm{Vol}_g\big(\partial C_{1+v}^T\big)} \int_{\partial C_{1+v}^T} g(\nabla \phi_v, \omega) \, \mathrm{dvol}_g,$$

where ω denotes the unit normal vector field to ∂C_{1+v}^T. By the rotational symmetry of C_{1+v}^T, it is easy to show that N depends only on the variable t [16]. Set $F(v,T) = N(v,T)\big(\frac{T}{2\pi} t\big)$. Obviously, $F(0,T) = 0$ for any $T > 0$. From [5, Lemma 2.1] we know that F is a C^1 operator in a neighborhood of $(0,T)$ for any fixed $T > 0$.

Let ψ be the unique solution of

$$\begin{cases} \Delta_g \psi + \lambda_1 \psi = 0 & \text{in } C_1^T, \\ \psi = -\partial_r \phi_1 v\big(\frac{2\pi t}{T}\big) & \text{on } \partial C_1^T. \end{cases}$$

Define the function $\widetilde{H}_T(\cdot)$ as follows

$$\widetilde{H}_T(v) = \left(\partial_r \psi + \partial_r^2 \phi_1 v\left(\frac{2\pi}{T} t\right) \right) \Big|_{\partial C_1^T}$$

and set

$$H_T(v) = \widetilde{H}_T(v)\left(\frac{T}{2\pi}t\right).$$

It follows from [16, Proposition 3.2] that the linearization of F with respect to v at the point $(0, T)$ is just H_T.

If $v \in \mathcal{C}^{2,\alpha}_{\text{even},0}(\mathbb{R}/2\pi\mathbb{Z})$, its Fourier expansion is

$$v = \sum_{m\geq 1} a_m \cos(mt).$$

Let V_m be the space spanned by the function $\cos(mt)$. It follows from [16, Proposition 4.3] that H_T preserves the eigenspaces V_m. Let $\sigma_m(T)$ be the eigenvalue of H_T associated with the eigenfunction $\cos(mt)$. It is known (see [16]) that

$$\sigma_m(T) = \partial_r c_m(1) + \partial_r^2 \phi_1(1),$$

where c_m is the continuous solution on $[0, 1]$ of

$$\left(\partial_r^2 + (n-1)\frac{C_k(r)}{S_k(r)}\partial_r + \lambda_1\right)c - \left(\frac{2m\pi}{T}\right)^2 c = 0$$

with $c_m(1) = -\partial_r\phi_1(1)$, where

$$C_k(r) = \begin{cases} \cosh r & \text{if } k = -1, \\ \cos r & \text{if } k = 1. \end{cases}$$

It follows from [5, Proposition 2.1] that the function $\sigma_1(T)$ satisfies $\sigma_1'(T) < 0$ for any $T > 0$. Moreover, σ_1 has exactly one zero in $(0, +\infty)$, which is denoted by T_*. Furthermore, by [5, Proposition 2.2], we also know that the linearized operator

$$H_T = D_v F(0, T) : \mathcal{C}^{2,\alpha}_{\text{even},0}(\mathbb{R}/2\pi\mathbb{Z}) \longrightarrow \mathcal{C}^{1,\alpha}_{\text{even},0}(\mathbb{R}/2\pi\mathbb{Z})$$

is a formally self-adjoint, first order elliptic operator. It preserves the eigenspaces V_m for all m and $T > 0$. Moreover, the kernel of H_{T_m} is just V_m and the eigenvalue associated to the eigenspace V_m has a unique zero which is just T_m. Note that $\sigma_m(T) = \sigma_1(T/m)$, which indicates the property of σ_m can be deduced from the property of σ_1. So we next only consider the case of $m = 1$.

We now present the proof of Theorem 1.2.

Proof of Theorem 1.2. From the property of $\sigma_1(T)$ we know that there exists $T_0 > T_*$ such that $\sigma_1(T) > -1$ for any $T \in (0, T_0)$. We *claim* that $H_T + \mathrm{Id}$ is invertible for any $T \in (0, T_0)$. For any $v \in \mathcal{C}^{2,\alpha}_{\text{even},0}(\mathbb{R}/2\pi\mathbb{Z})$ such that $H_T v + v = 0$, it follows from the Fourier expansion $v = \sum_{m\geq 1} a_m \cos(mt)$ and $\sigma_m(T) = \sigma(T/m)$ that

$$(\sigma(T) + 1)\int_{-\pi}^{\pi} v^2 \,\mathrm{d}t \leq \int_{-\pi}^{\pi} (H_T v^2 + v^2)\,\mathrm{d}t = 0.$$

It follows that $v \equiv 0$. Clearly, $H_T + \mathrm{Id}$ is linear continuous. By Banach inverse operator theorem, $H_T + \mathrm{Id}$ is an isomorphism for any $T \in (0, T_0)$.

Define $G : (0, T_0) \times \mathcal{V} \to \mathcal{W}$ by

$$G(T, v) = F(v, T) + v,$$

where $\mathcal{V} \subset \mathcal{C}^{2,\alpha}_{\text{even},0}(\mathbb{R}/2\pi\mathbb{Z})$ and $\mathcal{W} \subset \mathcal{C}^{1,\alpha}_{\text{even},0}(\mathbb{R}/2\pi\mathbb{Z})$ are open neighborhoods of the 0 function. Since the operator $H_T + \mathrm{Id}$ is invertible for $T \in (0, T_0)$, $D_v G(T, 0)$ is an isomorphism for all $T \in (0, T_0)$. For $w \in \mathcal{W}$, there exists a unique $v \in \mathcal{V}$ such that $G(\lambda, v) = w$. Let $v = G^{-1}(w)$. Clearly, G^{-1} maps $\mathcal{W}$ into $\mathcal{V}$. Let $R(T, w) = w - G^{-1}(w)$, which maps $(0, T_0) \times \mathcal{W}$ into $\mathcal{W}$ because $\mathcal{V} \subset \mathcal{W}$. Since the embedding of $\mathcal{C}^{2,\alpha}_{\text{even},0}(\mathbb{R}/2\pi\mathbb{Z}) \hookrightarrow \mathcal{C}^{1,\alpha}_{\text{even},0}(\mathbb{R}/2\pi\mathbb{Z})$ is compact, $R : (0, T_0) \times \mathcal{W} \to \mathcal{W}$ is compact. Then $F(v, T) = 0$ is equivalent to $R(T, v) = 0$ for all $T \in (0, T_0)$. We see that $D_w R(T, 0)w = \mu w$ is equivalent to $H_\lambda(w) = \mu w/(1 - \mu)$ with $\mu < 1$. It follows that $D_w R(T, 0)$ has the same number of negative eigenvalues as H_T.

We have known that $\dim \mathrm{Ker}(H_{T_*}) = 1$. So we also have that $\dim \mathrm{Ker}(D_w R(T_*, 0)) = 1$. For any $\varepsilon > 0$ small enough, the property of $\sigma(T)$ implies that 0-group index $\sigma(T_* - \varepsilon) = (-1)^0 = 1$ and $\sigma(T_* + \varepsilon) = (-1)^1 = -1$. It further indicates that $D_w R(T, 0)$ has an odd crossing number at $T = T_*$. Applying Theorem 1.1 to $R(T, v) = 0$, we can conclude the desired unilateral global bifurcation result. □

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On uniform asymptotic *h*-stability for linear nonautonomous systems

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Abstract. A classical result in the theory of stability of nonautonomous linear systems is that the uniform exponential decay of its corresponding transition matrix is equivalent to the uniform asymptotic stability of the trivial solution. The aim of this article is to extend this equivalence for non-exponential decays $h(t)$. In this article we prove that, in some suitable cases, the function $h(t)$ allows the construction of a topological abelian group that makes possible to formulate a more general definition of uniform stability which is equivalent with a decay dominated by $h(t)$. Moreover, we can use this group to establish the necessary elements to develop a theory of h-stable systems. As a first step in this direction, we provide integral conditions on the solutions of a uniformly asymptotically h-stable system.

Keywords: linear nonautonomous systems, uniform asymptotic h-stability, group theory.

2020 Mathematics Subject Classification: 34A30, 34C11, 34D20.

1 Introduction and motivation

1.1 Preliminaries

This article provides new results about the property of uniform asymptotic h-stability of the nonautonomous linear system of ordinary differential equations:

$$\dot{x} = A(t)x \quad \text{for any } t \in J := (a, +\infty), \tag{1.1}$$

where a can be either a real number or $a = -\infty$, $A\colon J \to M_n(\mathbb{R})$ is a piecewise continuous matrix valued function. Moreover, $\Phi(t)$ and $\Phi(t,s) = \Phi(t)\Phi^{-1}(s)$ will denote respectively a fundamental matrix of (1.1) and its corresponding transition matrix. The solution of (1.1) passing through x_0 at time $t = t_0 > a$ will be denoted by $t \mapsto x(t, t_0, x_0) := \Phi(t, t_0)x_0$. For a

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given norm $\|\cdot\|$ on $\mathbb{R}^n$ and any matrix $U \in M_n(\mathbb{R})$, the induced operator norm is defined as $\|U\|_{\mathcal{L}} := \sup\{\|Ux\| : \|x\| \leq 1\}$.

The stability and asymptotic stability of the nonautonomous linear systems (1.1) are classic research topics and there exists a myriad of asymptotic stabilities whose classification and description has been carried out in authoritative monographs as [5, 9, 14]. In particular, the uniform asymptotic h-stability is defined as follows:

Definition 1.1. The nonautonomous system (1.1) is *uniformly asymptotically h-stable* if there exist constants $K \geq 1$ and $\alpha > 0$ and a function $h : J \to (0, +\infty)$ such that:

$$\|\Phi(t, t_0)\|_{\mathcal{L}} \leq K \left(\frac{h(t_0)}{h(t)}\right)^{\alpha} \quad \text{for any } t \geq t_0 > a, \tag{1.2}$$

where $h(\cdot)$ is continuous, surjective and strictly increasing.

To the best of our knowledge, the above definition has been introduced by M. Pinto in [13] in a more general context, including the nonlinear case and considering $\alpha = 1$. This definition of stability has been explored by the current research, either by considering a fixed function $h(\cdot)$ such as in the cases of polynomial stability [7] or algebraic dichotomy [11], or by considering a large family of functions, such as in [12].

The previous definition is a particular case of the *uniform stability* and encompasses a wide range of well known stabilities, for example, the classical property of *uniform asymptotic stability*, which is ubiquitous in theoretical and applied research and will be recalled to make this article self contained:

Definition 1.2 ([2, Ch. III.1])**.** The nonautonomous linear system (1.1) is *uniformly stable* if any solution $t \mapsto x(t) := x(t, t_0, x_0)$ satisfies the following property:

a) For each $\varepsilon > 0$ there is a $\delta_\varepsilon > 0$ such that if $\|x(t_0)\| < \delta_\varepsilon$ for some $t_0 \in J$ then $\|x(t)\| < \varepsilon$ for all $t \geq t_0$.

The nonautonomous system (1.1) is *uniformly asymptotically stable* if it is uniformly stable and in addition:

b) There is a $\delta_0 > 0$ such that for all $\varepsilon > 0$ there exists a constant $T_\varepsilon \in (0, +\infty)$ such that if $\|x(t_0)\| < \delta_0$ then $\|x(t)\| < \varepsilon$ for all $t \geq t_0 + T_\varepsilon$.

The proof that the uniform asymptotic stability described in Definition 1.2 is a particular case of the uniform asymptotic h-stability follows from the following results considering $h(t) = e^t$:

Proposition 1.3 ([9, Prop.3.3.1])**.** *The nonautonomous linear system* (1.1) *is uniformly stable if and only if there exists a constant $K \geq 1$ such that:*

$$\|\Phi(t, s)\|_{\mathcal{L}} \leq K \quad \textit{for all } t \geq s > a.$$

Proposition 1.4 ([10, Th.4.11])**.** *The nonautonomous linear system* (1.1) *is uniformly asymptotically stable if and only if it is uniformly exponentially stable, namely, there exist constants $K \geq 1$ and $\alpha > 0$ such that:*

$$\|\Phi(t, s)\|_{\mathcal{L}} \leq K e^{-\alpha(t-s)} \quad \textit{for all } t \geq s > a. \tag{1.3}$$

Another equivalent characterization of uniform asymptotic stability in terms of the transition matrix is given by:

Proposition 1.5. *The nonautonomous linear system* (1.1) *is uniformly asymptotically stable if and only if it is uniformly stable and there exists a constant* $T \in (0, +\infty)$ *such that*

$$\sup_{s\in J} \|\Phi(T+s,s)\|_{\mathcal{L}} < 1. \tag{1.4}$$

The step proving the necessity of the above result can be found as an intermediate step in the proof of Proposition 1.4 given by [2, Ch. III.1] and as Lemma 5 in [6], while the sufficiency step of the is an immediate consequence of Proposition 1.4.

1.2 Novelty of the article and main results

We stress that the equivalence between the ε–δ characterization of uniform asymptotic stability stated in Definition 1.2 and the estimations for the transition matrices given by Propositions 1.4 and 1.5 is a classical topic in stability theory. Nevertheless, a generalization for decays described by $h(\cdot)$ functions has not been addressed and is the aim of this article.

A possible way to study the uniform asymptotic h-stability is to introduce the change of time variables

$$t = h^{-1}(e^{\tilde{t}}) \quad \text{and} \quad \tilde{t} = \log(h(t)) \quad \text{with } \log(h(a^+)) < \tilde{t} < +\infty, \tag{1.5}$$

and the matrix $\Phi_h(\tilde{u})$:

$$\Phi_h(\tilde{u}) = \Phi(h^{-1}(e^{\tilde{u}})) \quad \text{for any } \tilde{u} > \log(h(a^+)). \tag{1.6}$$

These change of variables allow to reinterpret the estimation (1.2) as follows:

$$\|\Phi_h(\tilde{t},\tilde{t}_0)\|_{\mathcal{L}} \le Ke^{-\alpha(\tilde{t}-\tilde{t}_0)} \quad \text{for any } \tilde{t} \ge \tilde{t}_0 > \log(h(a^+)). \tag{1.7}$$

Notice that the right-hand side of the inequalities (1.3) and (1.7) describes a similar exponential decay. Nevertheless, we warn that this analogy could induce the erroneous belief that the uniform asymptotic h-stability can be trivially revisited as a uniform exponential stability via a suitable change of variables. To support this warning, we state that the columns of the transformed matrix $\Phi_h(\cdot)$ are not a basis of solutions and the left side of (1.7) as well as (1.5)–(1.6) do not have an intuitively clear meaning. These issues deserve a meticulous study and a big contribution of this article is to show that any function $h(\cdot)$ allows to construct a topological group $(J,*)$ such that the change of variables described by (1.5) can be seen as the commutative diagrams of isomorphisms of topological groups such that

$$\begin{array}{ccc} (\mathbb{R},+) & \xrightarrow{e} & (\mathbb{R}^+,\cdot) \\ & & \downarrow{\scriptstyle h^{-1}} \\ & & (J,*) \end{array} \quad \text{and} \quad \begin{array}{ccc} (J,*) & \xrightarrow{h} & (\mathbb{R}^+,\cdot) \\ & & \downarrow{\scriptstyle \log} \\ & & (\mathbb{R},+), \end{array}$$

where the isomorphism $h\colon (J,*) \to (\mathbb{R}^+,\cdot)$ verifies $h(t*s) = h(t)h(s)$, which mimics the exponential identity $e^{t+s} = e^t e^s$.

The above described approach will enable to state the main results of this article:

- To provide an ε–δ definition of uniform asymptotic h-stability generalizing the one described in Definition 1.2.
- To generalize the Proposition 1.5 for the uniformly asymptotically h-stable case.

- To deduce integral characterizations for the uniform asymptotic h-stability which generalizes those obtained for the uniform asymptotic stability.

We stress that Definition 1.2 and Proposition 1.5 are stated in term of the additive group $(\mathbb{R}, +)$ whereas its generalization will be carried out in terms of the *additive* structure of $(J, *)$. We stress the originality of this approach, its remarkable simplicity and its potential application for the study of h-dichotomies.

1.3 Structure of the article

In Section 2 we provide examples of systems which are uniformly h-stable but not uniformly asymptotically stable. Section 3 is devoted to the construction the topological group $(J, *)$, an invariant measure on J under the action of this group, and the study of their properties. In Section 4 we state and prove the main results, namely, the generalization of the equivalences between Definition 1.2 and Propositions 1.4 and 1.5 to the uniform h-stability framework. Finally, in Section 5 we generalize, to uniform h-stable systems, a pair of results characterizing the uniform exponentially stable systems in terms integral conditions.

2 Examples of uniform asymptotic h-stability

We will consider simple h-stable scalar equations and use them to show the existence of an equation that is logarithmically stable but not exponentially.

Example 2.1. Given $h : J \to (0, +\infty)$ continuously differentiable, surjective and strictly increasing, together with a constant $\alpha > 0$, let us consider the equation:

$$x' = -\alpha \left(\frac{h'(t)}{h(t)} \right) x,$$

which provides a simple example of h-stability. Indeed, the transition matrix is given by

$$\Phi(t, s) = \left(\frac{h(s)}{h(t)} \right)^{\alpha},$$

and the Definition 1.1 is verified with $K = 1$.

Example 2.2. The above example allows us to construct a nonautonomous linear equation whose solutions $t \mapsto x(t, s, x_0)$ asymptotically approach the origin but are not uniformly exponentially stable. In fact, when considering $h(t) = \log(1 + t)$, the previous example says that the corresponding transition matrix is given by

$$\Phi(t, s) = \left(\frac{\log(1 + s)}{\log(1 + t)} \right)^{\alpha},$$

this system is uniformly asymptotically log-stable and for any $s \in J := (0, +\infty)$ and $x_0 \in \mathbb{R}$ it follows that

$$\lim_{t \to +\infty} \Phi(t, s) x_0 = \lim_{t \to +\infty} \left(\frac{\log(1 + s)}{\log(1 + t)} \right)^{\alpha} x_0 = 0.$$

However, for any pair of positive numbers α and $\beta > 0$, we have the estimation

$$\lim_{t \to +\infty} \log(1 + t) e^{-t^{\frac{\beta}{\alpha}}} = 0.$$

Hence for a fixed $s \in J$ and $K > 1$ there exists t large enough such that

$$\log(1+t)e^{-t\frac{\beta}{\alpha}} < K^{-1}\log(1+s)e^{-s\frac{\beta}{\alpha}},$$

and we can rearrange this inequality to obtain

$$Ke^{\beta(s-t)} < \left(\frac{\log(1+s)}{\log(1+t)}\right)^{\alpha},$$

which shows that the inequality (1.3) is not true for all pairs $t \geq s$ and the uniform exponential stability described by (1.3) cannot be verified.

3 A topological group induced by $h(\cdot)$

The conditions we have given to the function $h(\cdot)$ on Definition 1.1 are enough to carry out a deep study of the solutions of (1.1). In fact, as any continuous bijection between two open intervals is an homeomorphism, the function $h : (a, +\infty) \to (0, +\infty)$ has a continuous inverse $h^{-1}(\cdot)$. Then, we introduce the operation

$$t * s := h^{-1}(h(t)h(s)), \tag{3.1}$$

on $J = (a, +\infty)$, namely, the domain of h. To distinguish $*$ from the usual operations on $\mathbb{R}$, we will denote the integer powers of t as t^{*n}. This operation is associative and commutative, it has an identity element defined by

$$e_* = h^{-1}(1)$$

and each $t \in J$ has an inverse

$$t^{*-1} = h^{-1}\left(\frac{1}{h(t)}\right). \tag{3.2}$$

These properties, alongside the continuity of h and h^{-1}, imply that $(J, *)$, with the standard real open interval topology, is a locally compact abelian topological group.

Example 3.1. Let $h(t) = \log(t)$ on the interval $(1, +\infty)$. The corresponding group operation is

$$t * s = e^{\log(t)\log(s)} = t^{\log(s)} = s^{\log(t)},$$

and the identity on this group is $h^{-1}(1) = e^1 = e$.

It is straightforward to deduce the following properties as a consequence of (3.1) and (3.2):

$$\begin{aligned} h(t*s) &= h(t)h(s) \\ h^{-1}(t) * h^{-1}(s) &= h^{-1}(ts) \\ h(s*t^{*-1}) &= \frac{h(s)}{h(t)}. \end{aligned} \tag{3.3}$$

The first two identities of (3.3) imply that the function $h : J \mapsto \mathbb{R}^+$ is an isomorphism of topological groups between $(J, *)$ and the multiplicative group of positive real numbers $(\mathbb{R}^+, \cdot)$. Due to this isomorphism, we may understand properties of $(J, *)$ by first considering the analogous case on $(\mathbb{R}^+, \cdot)$ and passing from one group to the other by using the functions h and h^{-1}.

Remark 3.2. In the case of uniform exponential stability, namely, when $h(t) = e^t$, this operation becomes $t * s = \log(e^t e^s) = t + s$ and, consequently, the group $(\mathbb{R}, *)$ coincides with the additive group $(\mathbb{R}, +)$.

Remark 3.3. Note that inequality (1.2), which defines uniform asymtptotic h-stability, can be seen from another perspective by using the identity (3.3) as follows:

$$\|\Phi(t,s)\|_{\mathcal{L}} \leq Kh(s * t^{*-1})^{\alpha} \quad \text{for all } t \geq s > a.$$

We also can define a measure μ_* on the interval J, which is invariant under the action of elements of $(J, *)$. This is an absolutely continuous measure whose Radon–Nikodym derivative is the logarithmic derivative of h, that is, for a Borel measurable set $A \subset J$ we define its measure μ_* as

$$\mu_*(A) = \int_A \frac{h'(u)}{h(u)} \mathrm{d}m(u), \tag{3.4}$$

where $m(\cdot)$ is the Lebesgue measure. To see that this is an invariant measure, it is enough to prove that it is invariant for compact intervals, as these sets generate the Borel σ-algebra.

Lemma 3.4. *Given a compact interval $[s,t] \subset J$, its measure is invariant under translation by elements of $(J, *)$, that is, for any $\gamma \in J$*

$$\mu_*([s,t]) = \mu_*([\gamma * s, \gamma * t])$$

Proof. The measure of the interval $[s,t] \subset J$ is

$$\mu_*([s,t]) = \int_s^t \frac{h'(u)}{h(u)} \mathrm{d}m(u) = \log\left(\frac{h(t)}{h(s)}\right). \tag{3.5}$$

Notice that, given any constant $\gamma \in J$, by (3.3) we have

$$\begin{aligned} \mu_*([\gamma * s, \gamma * t]) &= \log\left(\frac{h(\gamma * t)}{h(\gamma * s)}\right) \\ &= \log\left(\frac{h(t)}{h(s)}\right) \\ &= \mu_*([s,t]). \end{aligned}$$

□

Remark 3.5. The identity (3.5) allows an additional characterization of the uniform asymptotic h-stability in terms of the measure μ_* as follows:

$$\|\Phi(t,s)\|_{\mathcal{L}} \leq Ke^{-\alpha\mu_*([s,t])} \quad \text{for all } t \geq s \geq a.$$

Remark 3.6. It will be of interest to understand the behavior of the iterates of elements on $(J, *)$ in the sense

$$\begin{aligned} J \times J &\to J \\ (\gamma, \gamma) &\mapsto \gamma^{*2} := \gamma * \gamma. \end{aligned}$$

Firstly, by using the isomorphism between $(J, *)$ and $(\mathbb{R}^+, \cdot)$, it follows that the n-the power of γ is

$$\gamma^{*n} = h^{-1} \circ h(\gamma^{*n}) = h^{-1}(h(\gamma)^n). \tag{3.6}$$

Secondly, given an element $\gamma > e_*$ we have that $h(\gamma) > 1$, which implies that $h(\gamma)^n \to +\infty$ and $h(\gamma)^{-n} \to 0$ as $n \to +\infty$. This means that any forward orbit of the action $J \to \gamma * J$ approaches $+\infty$ while any backward orbit approaches a.
Analogously, for $\gamma < e_*$ forward orbits approach a while backward orbits approach $+\infty$.

Ref.	$h(t)$	J	$c * d$	$\frac{h'(t)}{h(t)}$
[2]	e^t	$\mathbb{R}$	$c+d$	1
[4,7]	t	$\mathbb{R}^+$	cd	$\frac{1}{t}$
	$\log(1+t)$	$\mathbb{R}^+$	$(1+c)^{\log(1+d)}-1$	$\frac{1}{(t+1)\log(1+t)}$
[11]	$t+\sqrt{t^2+1}$	$\mathbb{R}$	$c\sqrt{d^2+1}+d\sqrt{c^2+1}$	$\frac{1}{\sqrt{t^2+1}}$

Table 3.1: Some examples for functions $h(\cdot)$ with its respective interval of stability, group operation, and logarithmic derivative.

A useful construction on $\mathbb{R}$ is the partition on intervals $[k, k+1)$ of equal length, where $k \in \mathbb{Z}$. In proofs of exponential stability theory such as [8], this partition allows us to approximate the continuous system (1.1) by a discretization of $\Phi(t,s)$. However, when the decay rate of $\Phi(t,s)$ is given by a function $h(\cdot)$ rather than the exponential function, the notion of length, given by the invariant measure, changes. For this reason, we propose the following partition:

Lemma 3.7. *For any $\gamma \in (e_*, +\infty)$ the intervals $J_k = [\gamma^{*k}, \gamma^{*(k+1)})$ with $k \in \mathbb{Z}$ define a partition of J into sets of constant μ_* measure.*

Proof. The function h^{-1} is strictly increasing and $h(\gamma) > 1$, hence the sequence $\{\gamma^{*k}\}_{k\in\mathbb{Z}}$ which is defined by (3.6), that is, by the identity $\gamma^{*k} = h^{-1}(h(\gamma)^k)$, is strictly increasing, which implies that the intervals J_k are disjoint. It remains to show that the intervals J_k form a covering of J. Without loss of generality we assume that $t \in [e_*, +\infty)$. As discussed in Remark 3.6, for any $t \in [e_*, +\infty)$ there exists $n \in \mathbb{N} \cup \{0\}$ such that $\gamma^{*(n+1)} > t$. Since the inequality $\gamma^{*(n+1)} > t$ is satisfied by a set of natural numbers, there must be a least element of this set, which we call m. The number m is the minimal natural such that $\gamma^{*(m+1)} > t$, hence $\gamma^{*m} \leq t$, that is, $t \in J_m$. □

A noticeable byproduct of the above result is:

Corollary 3.8. *The measure μ_* is σ-finite.*

Example 3.9. In the case corresponding to $h(t) = t$, which defines polynomial stability in [7] and [1], the operation is defined by $t * s = ts$, the domain of h is the interval $(0, +\infty)$ and, for any $\gamma > 1$, the subintervals $\{[1,\gamma), [\gamma,\gamma^2), [\gamma^2,\gamma^3), \ldots\}$, and $\{[\gamma^{-1},1), [\gamma^{-2},\gamma^{-1}), [\gamma^{-3},\gamma^{-2}), \ldots\}$ define a uniform partition of $(0,+\infty)$. Moreover, given a constant $C > 0$ and considering the measure defined by (3.4), we have $\mu_*([C\gamma^k, C\gamma^{k+1})) = \mu_*([\gamma^k, \gamma^{k+1})) = \log(\gamma)$ for any $k \in \mathbb{Z}$.

4 Characterizations of uniform asymptotic *h*-stability

The main result of this work is a characterization of the uniform asymptotic h-stability which emulates and generalizes the equivalence between uniform asymptotic stability with Proposition 1.4 and Proposition 1.5. The group $(J, *)$ introduced in the previous section, as well as its properties, will play an essential role on the proof.

Theorem 4.1. *Given a nonautonomous system (1.1) with a transition matrix $\Phi(t,s)$, the following statements are equivalent:*

i) The system (1.1) is uniformly asymptotically h-stable.

ii) The system (1.1) is uniformly stable and there exists a constant $T \in (e_*, +\infty)$ such that

$$\sup_{s\in J} \|\Phi(T * s, s)\|_{\mathcal{L}} < 1. \tag{4.1}$$

iii) The solutions $x(t) = \Phi(t, t_0)x_0$ of the system (1.1) satisfy the following two properties:

a) For each $\varepsilon > 0$ there is a $\delta_\varepsilon > 0$ such that if $\|x(t_0)\| < \delta_\varepsilon$ for some $t_0 \in J$ then $\|x(t)\| < \varepsilon$ for all $t \geq t_0$.

b) There is a $\delta_0 > 0$ such that for all $\varepsilon > 0$ there exists a constant $T_\varepsilon \in (e_*, +\infty)$ such that if $\|x(t_0)\| < \delta_0$ then $\|x(t)\| < \varepsilon$ for all $t \geq t_0 * T_\varepsilon$.

Proof. We begin with i) $\Rightarrow$ ii): Suppose that the system (1.1) is uniformly asymptotically h-stable. By using (1.2) it follows that an uniformly asymptotically h-stable system always satisfies the bound $\|\Phi(t,s)\|_{\mathcal{L}} \leq K$ since $\frac{h(s)}{h(t)} \leq 1$ for all pairs s, t with $a < s \leq t < +\infty$, that is, for any h an uniformly asymptotically h-stable system is also uniformly stable, due to Proposition 1.3.

Now, note that for all $s \in J$ and $T \in (e_*, +\infty)$, by using (1.2) and (3.3) we have

$$\begin{aligned}\|\Phi(T * s, s)\|_{\mathcal{L}} &\leq K\left(\frac{h(s)}{h(T * s)}\right)^{\alpha} \\ &= K\left(\frac{h(s)}{h(T)h(s)}\right)^{\alpha} \\ &= K\left(\frac{1}{h(T)}\right)^{\alpha},\end{aligned}$$

thus, by choosing $T \in (e_*, +\infty)$ large enough such that $K\left(\frac{1}{h(T)}\right)^{\alpha} < 1$, we obtain (4.1) since the right hand of the above inequality is independent of s.

We proceed to prove ii) $\Rightarrow$ i): Let us consider a pair of values (t, s) such that $a < s \leq t < +\infty$, by following a construction analogue to that of Lemma 3.7, we construct a partition of the interval J into the sets $s * A_n := s * [T^{*n}, T^{*(n+1)})$ for $n \in \mathbb{Z}$. As $\{s * A_n\}_{n\in\mathbb{Z}}$ is a partition of J, there must be an unique integer k such that $t \in s * A_k$. We can then rewrite $\Phi(t, s)$ as

$$\Phi(t,s) = \Phi(t, T^{*k} * s)\Phi(T^{*k} * s, T^{*(k-1)} * s) \cdots \Phi(T * s, s). \tag{4.2}$$

On the other hand, due to inequality (4.1) and the uniform stability of the system, we know that there exists constants $\theta < 1$ and $1 \leq K$ respectively, such that

$$\|\Phi(T * s, s)\|_{\mathcal{L}} < \theta \quad \text{and} \quad \|\Phi(t, T^{*k} * s)\|_{\mathcal{L}} \leq K.$$

Using these two bounds along with the submultiplicative property of the operator norm applied to (4.2), we get

$$\|\Phi(t,s)\|_{\mathcal{L}} \leq K\theta^k = Ke^{-k|\log(\theta)|}, \tag{4.3}$$

where $\log(\theta) = -|\log(\theta)|$ because $\theta < 1$. As the integer k depends on the values of T, t and s, we can bound it by a function of these numbers, to do this we notice that $t \in s * A_k$ is equivalent to the inequalities

$$s * T^{*k} \leq t < s * T^{*(k+1)}. \tag{4.4}$$

For any $\gamma \in J$, the group action on J, $\gamma * (\cdot)$, is an strictly increasing function, as can be seen by noting that $\gamma * (\cdot) = h^{-1}(h(\gamma)h(\cdot))$, where the left hand side is a composition of increasing functions. Hence, the action of s^{*-1} on (4.4) mantains the inequalities

$$T^{*k} \leq t * s^{*-1} < T^{*(k+1)}.$$

By passing the above inequality through the monotone function $u \mapsto \log(h(u))$ combined with the identities (3.2) and (3.3), we get

$$k \log(h(T)) \leq \log\left(\frac{h(t)}{h(s)}\right) < (k+1)\log(h(T)).$$

Notice that $\log\left(\frac{h(t)}{h(s)}\right)\log(h(T))^{-1} - 1 < k$, then by (4.3) and, due to the fact that the exponent $-k|\log(\theta)| < 0$, we deduce

$$\begin{aligned}\|\Phi(t,s)\|_{\mathcal{L}} &\leq K\theta^k \\ &\leq Ke^{-[\log(\frac{h(t)}{h(s)})\log(h(T))^{-1}-1]|\log(\theta)|} \\ &= Ke^{|\log(\theta)|}\left(\frac{h(s)}{h(t)}\right)^{\frac{|\log(\theta)|}{\log(h(T))}},\end{aligned}$$

as this is true for any pair (t,s), we have shown that $\Phi(t,s)$ is uniformly asymptotically h-stable with constants $Ke^{|\log(\theta)|} \geq 1$ and $\frac{|\log(\theta)|}{\log(h(T))} > 0$.

We now prove i) $\Rightarrow$ iii): Suppose that the system (1.1) is uniformly asymptotically h-stable. The property a) of iii) is immediately verfied since, as we said in the step i) $\Rightarrow$ ii), the uniform h-stability implies the uniform stability from Definition 1.2 which is equivalent to iii).

In order to prove that the solutions $x(t)$ satisfy property $b)$, let us consider $\delta_0 = 1$, $T \in (e_*, +\infty)$ and $t_0 \in (a, +\infty)$. Now, if $x(t)$ is a solution of (1.1) such that $\|x_0\| < \delta_0$ and $t \geq T * t_0$ by using the fact that (1.1) is uniformly asymptotically h-stable combined with (3.3) we can deduce that

$$\begin{aligned}\|x(t)\| &= \|\Phi(t, T*t_0)\Phi(T*t_0, t_0)x_0\| \\ &\leq \|\Phi(t, T*t_0)\|_{\mathcal{L}} \cdot \|\Phi(T*t_0, t_0)\|_{\mathcal{L}} \cdot \|x_0\| \\ &\leq K^2\left(\frac{h(t_0)}{h(T*t_0)}\right)^{\alpha}\delta_0 \\ &= K^2\left(\frac{1}{h(T)}\right)^{\alpha}.\end{aligned}$$

As the function h is surjective onto the interval $(0, +\infty)$, for any given $\varepsilon > 0$ we can choose a T_ε such that

$$K^2\left(\frac{1}{h(T_\varepsilon)}\right)^{\alpha} < \varepsilon,$$

which proves property $b)$.

Finally, we need to prove iii) $\Rightarrow$ ii): If (1.1) satisfies iii) then it has property $a)$, which is the uniform stability described by Definition 1.2.

We now prove (4.1). Let δ_0 be as in property $b)$ of statement iii). Given any $0 < \varepsilon < \delta_0/2$ we define $\theta = \varepsilon(2/\delta_0) < 1$. There exists $T_\varepsilon > e_*$ such that

$$\|\Phi(T_\varepsilon * s, s)\|_{\mathcal{L}} = \sup_{\|x\|=1} \frac{2}{\delta_0}\|\Phi(T_\varepsilon * s, s)\frac{\delta_0}{2}x\| < \theta,$$

which proves iii). □

5 Integral conditions

The results proved in the previous sections will allow us to carry out a wider study of the properties of uniform asymptotic h-stability, we will extend two existing results regarding necessary and sufficient conditions for uniform exponential stability. One of these theorems is proved in [9, Th. 3.3.15] in a finite dimensional framework. Besides the case of uniform exponential stability, these results were proved for uniform polynomial stability in [7], within the more general framework of evolutionary processes in Banach spaces.

Theorem 5.1. *A nonautonomous linear system* (1.1) *with a transition matrix* $\Phi(t,s)$ *is uniformly asymptotically h-stable if and only if it is uniformly stable and*

$$\sup_{s\in J}\int_s^{+\infty}\|\Phi(u,s)x\|\mathrm{d}\mu_*(u)<+\infty \quad \textit{for all } x\in\mathbb{R}^n. \tag{5.1}$$

Proof. We begin with the forward direction of the equivalence. Let $\Phi(t,s)$ be a transition matrix of an uniformly asymptotically h-stable system. As a uniformly asymptotically h-stable system is also uniformly stable, we only need to prove (5.1). To do this we can use (1.2) to bound the integral as follows

$$\begin{aligned}\int_s^{+\infty}\|\Phi(u,s)x\|\mathrm{d}\mu_*(u) &\leq K\|x\|\int_s^{+\infty}\left(\frac{h(s)}{h(u)}\right)^{\alpha}\frac{h'(u)}{h(u)}\mathrm{d}m(u)\\ &= K\|x\|h(s)^{\alpha}\int_s^{+\infty}\frac{h'(u)}{h(u)^{\alpha+1}}\mathrm{d}m(u)\\ &= K\alpha^{-1}\|x\|\lim_{t\to+\infty}\left(1-\frac{h(s)^{\alpha}}{h(t)^{\alpha}}\right)\\ &= K\alpha^{-1}\|x\|,\end{aligned}$$

which gives, for each $x\in\mathbb{R}^n$, a finite bound that is independent from s and the property (5.1) is verified.

To prove the backwards direction of the equivalence we will use Theorem 4.1. Given a compact interval $[c,d]\subset J$, the fact that $\Phi(t,s)$ is the transition matrix of an uniformly stable system implies that there exists a constant $K>1$ such that

$$\begin{aligned}rcl\|\Phi(d,c)\|_{\mathcal{L}} &\leq \|\Phi(d,u)\|_{\mathcal{L}}\cdot\|\Phi(u,c)\|_{\mathcal{L}}\\ &\leq K\|\Phi(u,c)\|_{\mathcal{L}} \text{ for } c\leq u\leq d.\end{aligned}$$

Given that the integral in (5.1) has a non-negative integrand we have the inequalities

$$\begin{aligned}+\infty &> \sup_{s\in J}\int_s^{+\infty}\|\Phi(u,s)x\|\mathrm{d}\mu_*(u)\\ &\geq \int_c^d\|\Phi(u,c)x\|\mathrm{d}\mu_*(u)\\ &\geq \int_c^d K^{-1}\|\Phi(d,c)x\|\mathrm{d}\mu_*(u)\\ &= K^{-1}\|\Phi(d,c)x\|\mu_*([c,d]).\end{aligned}$$

As the above inequality is satisfied by all $x\in\mathbb{R}^n$ and $[c,d]\subset J$, the Banach–Steinhaus theorem implies the existence of a constant M such that

$$\mu_*([c,d])K^{-1}\|\Phi(d,c)\|_{\mathcal{L}}\leq M$$

for any compact interval $[c,d] \subset J$.

There exists $T \in (e_*, +\infty)$ large enough such that $\frac{MK}{\mu_*([e_*,T])} < 1$. Then, by the invariance of μ_* under $*$-translations given for intervals in Lemma 3.4 we have:

$$\|\Phi(T*s,s)\|_{\mathcal{L}} \leq \frac{MK}{\mu_*([s,T*s])} = \frac{MK}{\mu_*([e_*,T])} < 1,$$

which due to Theorem 4.1 proves that $\Phi(t,s)$ is uniformly asymptotically h-stable. □

Theorem 5.2. *A nonautonomous system (1.1) with a transition matrix $\Phi(t,s)$ is uniformly asymptotically h-stable if and only if it is uniformly stable and*

$$\sup_{t\in J} \int_a^t \|\Phi(t,u)\|_{\mathcal{L}} \mathrm{d}\mu_*(u) < \infty. \tag{5.2}$$

Proof. We begin with the forward direction of the equivalence. As uniform asymptotic h-stability implies uniform stability, we only need to prove that $\Phi(t,s)$ satisfies (5.2). For any $t \in J = (a, +\infty)$ we replace $\|\Phi(t,s)\|_{\mathcal{L}}$ with its bound (1.2) to obtain the inequality

$$\begin{aligned}
\int_a^t \|\Phi(t,u)\| \mathrm{d}\mu_*(u) &\leq K \int_a^t \left(\frac{h(u)}{h(t)}\right)^{\alpha} \mathrm{d}\mu_*(u) \\
&= K \int_a^t \left(\frac{h(u)}{h(t)}\right)^{\alpha} \frac{h'(u)}{h(u)} \mathrm{d}m(u) \\
&= \frac{K}{h(t)^{\alpha}} \lim_{s\to a^+} \int_s^t h(u)^{\alpha-1} h'(u) \mathrm{d}m(u) \\
&= K\alpha^{-1},
\end{aligned}$$

which proves the first part of the theorem.

Now we prove the backward direction of the equivalence. Let $K > 0$ be the bound of $\|\Phi(t,s)\|_{\mathcal{L}}$ obtained by the uniform stability and $M > 0$ a bound of (5.2). For all $a < s \leq t < +\infty$ it is true that

$$\begin{aligned}
KM &> K \int_s^t \|\Phi(t,u)\|_{\mathcal{L}} \mathrm{d}\mu_*(u) \\
&> \int_s^t \|\Phi(t,u)\|_{\mathcal{L}} \|\Phi(u,s)\|_{\mathcal{L}} \mathrm{d}\mu_*(u) \\
&\geq \int_s^t \|\Phi(t,s)\|_{\mathcal{L}} \mathrm{d}\mu_*(u) \\
&= \log\left(\frac{h(t)}{h(s)}\right) \|\Phi(t,s)\|_{\mathcal{L}}.
\end{aligned} \tag{5.3}$$

The function h is upper unbounded, hence there exists $T \in (e_*, +\infty)$ such that $\log(h(T)) > KM$. Given any $s \in J$, the inequality (5.3) for the pair $(T*s, s)$ is

$$\|\Phi(T*s,s)\|_{\mathcal{L}} \leq \frac{KM}{\log(h(T))} < 1.$$

It follows from Theorem 4.1 that $\Phi(t,s)$ is uniformly h-stable. □

6 Discussion

In this work, we revisited the property of uniform h-stability for nonautonomous linear systems and proved a set of properties emulating the classical results for the uniform asymptotic stability and its equivalence with the uniform exponential one. The main tool was the characterization of h as an isomorphism of topological abelian groups $h : (J, *) \to (\mathbb{R}, \cdot)$ which, in the specific case $h(t) = e^t$ with $(J, *) = (\mathbb{R}, +)$, recovers the uniform exponential stability and its characterizations.

To the best of knowledge, there are few results on stability for nonautonomous linear systems based on group theory. However, the methods we have presented in this paper can be applied to generalize other existing theorems of the theory of exponential stability to uniform asymptotic h-stability. In particular, we have obtained preliminary results in the following topics:

a) The problem of $(\mathcal{B}, \mathcal{D})$-admissibility, that is, if for each $f \in \mathcal{B}$ the system

$$\dot{x} = A(t)x + f(t), \tag{6.1}$$

has a solution $x \in \mathcal{D}$, where $\mathcal{B}$ and $\mathcal{D}$ are suitable function spaces. An affirmative answer to this problem, allows us to give a functional characterization of stability properties of the linear system associated to (6.1).

b) The roughness of uniform asymptotic h-stability, which concerns sufficient smallness conditions for a matrix valued function $B(\cdot)$ such that the system

$$\dot{x} = A(t)x + B(t)x,$$

is uniformly asymptotically h-stable, provided that (1.1) is a uniformly asymptotically h-stable system.

To further emphasize the usefulness of the ideas developed in this article, we will also mention that the function $\frac{h'(t)}{h(t)}$ has been used by other authors to prove theorems regarding h-stability and dichotomy, see for example [7] and [3]. Despite this, the way it relates to the algebraic structure of these problems, such as the fact that it defines an invariant measure, has neither been established nor employed to its further extent in previous works.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal internships that could have appeared to influence the work reported on the paper.

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Stability sets for fractional differential equations with two delays

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Abstract. The main subject of this study is a linear fractional-order differential equation with two delayed terms. By applying the D-decomposition method to the characteristic equation, we present a stability region as a necessary and sufficient condition for the asymptotic stability of the zero solution. Given the relationship between the Caputo fractional derivative and the integer-order derivative, we compare the conditions in this study with the results of the corresponding first-order delay differential equation to demonstrate the validity of the extension.

Keywords: fractional-order differential equation, delay differential equation, two delays, asymptotic stability, stability region.

2020 Mathematics Subject Classification: 34K37, 34K25, 34K20.

1 Introduction

Fractional calculus is a generalization concept of the classical calculus of integer order, including integrals and derivatives of any arbitrary real or complex order. There are several different families of fractional integrals and fractional derivatives, and the potentially useful properties of these definitions have been identified (see [7, 10, 12, 18] and references therein). As the basic theory of fractional calculus improves, its advantages in describing the memory and hereditary properties of processes has been gradually acknowledged. For example, empirical analyses of the transmission dynamics of COVID-19 (see [16, 22]) reveal that models incorporating Caputo fractional-order derivatives demonstrate superior performance in data fitting accuracy and epidemic trend prediction capabilities over their integer-order counterparts. By now, fractional differential equations have become one of the best tools in the areas of fluid flow, rheology, dynamical processes in self-similar and porous structures, electrical networks, and biophysics. Because stability is an important characteristic of the normal operation of a practical system, the stability theory of fractional differential equations have attracted the attention of mathematicians and engineers.

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In engineering and natural systems, the time delay in the process of material exchange and information transmission cannot be ignored. The existence of a time delay is usually the source of the instability of the system, which makes the discussion essential on the stability of fractional differential equations with time delay. Moreover, several factors can cause time delays in large or complex systems. In general, secondary factors are often overlooked for the convenience of processing; thus, the system is simplified into a single time-delay system with only the main time-delay parameter. However, for a complex system in which it is difficult to select the factors that cause a time delay, a model with multiple delay parameters should be considered.

The research on fractional differential equations with multiple delays has increased in the last double decades; see, e.g., [1, 6, 9, 20, 21, 23, 24] and references therein. The basic criterion of these studies are based on [6] in which an n-dimensional system of linear fractional differential equations with n^2 different delays was considered. In 2007, Deng et al. [6] established that the zero solution of the system is asymptotically stable if all the roots of the associated characteristic equation have negative real parts. It should be pointed out that the characteristic equation of a linear delay differential equation is a transcendental equation that cannot be solved using algebraic tools, and there are practical difficulties in condition verification.

This study focuses on a linear fractional differential equation with two delays

$$\begin{aligned} {}^{C}\mathrm{D}_0^{\alpha} x(t) &= a x(t-\tau_1) + b x(t-\tau_2), \quad t \geq 0, \\ x(t) &= \phi(t), \quad t \in [-\tau, 0], \quad \tau = \max\{\tau_1, \tau_2\}. \end{aligned} \tag{1.1}$$

Here, $\alpha \in (0,1)$ is a constant and ${}^{C}\mathrm{D}_0^{\alpha}$ is the Caputo fractional differentiation operator of order α defined by

$${}^{C}\mathrm{D}_0^{\alpha} x(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{x'(r)}{(t-r)^{\alpha}} dr, \quad t > 0.$$

Parameters a and b are real numbers and τ_1 and τ_2 are positive real numbers. The initial function ϕ is continuous on the interval $[-\tau, 0]$. The main purpose of this study is to investigate the asymptotic behavior of the solutions to (1.1) and present the stability set of (a, b) for the zero solution to be asymptotically stable.

The study on (1.1) with $b = 0$ can be found in [5, 13] and references therein. In 2011, Krol [13] determined the range of parameter a where the zero solution is asymptotically stable.

Theorem A. Let $b = 0$ and $\tau_1 = \tau$. Then, the zero solution to (1.1) is asymptotically stable if and only if

$$-\left(\frac{\pi - \alpha\pi/2}{\tau}\right)^{\alpha} < a < 0.$$

This is stable but not asymptotically stable if and only if

$$a = -\left(\frac{\pi - \alpha\pi/2}{\tau}\right)^{\alpha}.$$

Stability criteria for (1.1) with $\tau_1 = 0$ was studied in [2, 3, 11] and references therein. Especially, Čermák et al. [3] gave the stability set of (a, b) for (1.1) with $\tau_1 = 0$ and $0 < \alpha < 1$, which has been extended for the case $\alpha > 1$ by Čermák et al. [4]. By virtue of their work, we have the following result.

Theorem B. Let $\alpha > 0$, $\tau_1 = 0$, and $\tau_2 = \tau$.

(i) The zero solution to (1.1) is asymptotically stable if and only if $(a,b) \in D$ where D is an open region enclosed by the line $a+b=0$ from above and by the parametric curve

$$a = \frac{\omega^\alpha \sin(\tau\omega + \alpha\pi/2)}{\sin(\tau\omega)}, \quad b = -\frac{\omega^\alpha \sin(\alpha\pi/2)}{\sin(\tau\omega)} \tag{1.2}$$

for $\omega \in ((\pi - \alpha\pi)/\tau, \pi/\tau)$ from below.

(ii) The zero solution to (1.1) is stable but not asymptotically stable if and only if either

$$a+b=0, \quad a \le \frac{(\pi(1-\alpha))^\alpha}{2\tau^\alpha \cos(\alpha\pi/2)},$$

or a and b satisfy (1.2) for an admissible value ω.

Letting $\alpha \to 1^-$, (1.1) becomes the first-order delay differential equation

$$x'(t) = ax(t-\tau_1) + bx(t-\tau_2), \quad t \ge 0. \tag{1.3}$$

When $b=0$ or $\tau_1 = 0$ in (1.3), relevant classical results exist for the asymptotic stability of the zero solution. A simple calculation shows that the asymptotic stability condition in Theorems A and B with $\alpha = 1$ coincides with the result for the corresponding case in (1.3).

Stability criteria for (1.3) can be found in [14, 15, 19, 25]. In [19], Sakata discussed the stability set of (a,b) for (1.3) when $\tau_2 = n\tau_1$, $n = 2, 3$. Here, we present the result for $\tau_2 = 2\tau_1$.

Theorem C. Let $\tau_2 = 2\tau_1 = 2\tau$. Then, the zero solution to (1.3) is asymptotically stable if and only if $(a,b) \in D_1$ where D_1 is an open region enclosed by the line $a+b=0$ from the right and by the parametric curve

$$a = \frac{\omega\cos(2\tau\omega)}{\sin(\tau\omega)}, \quad b = -\frac{\omega\cos(\tau\omega)}{\sin(\tau\omega)}, \quad 0 < \omega < \frac{2\pi}{3\tau}$$

from the left.

In this study, we consider the special case of (1.1) in which $\tau_2 = 2\tau_1$. For simplicity, we denote τ_1 as τ and τ_2 as 2τ. Subsequently, (1.1) is reduced to

$$\begin{aligned} {}^C\mathrm{D}_0^\alpha x(t) &= ax(t-\tau) + bx(t-2\tau), \quad t \ge 0, \\ x(t) &= \phi(t), \quad t \in [-2\tau, 0]. \end{aligned} \tag{1.4}$$

The following theorem is the main result of this study.

Theorem 1.1. *Let $0 < \alpha < 1$, a, b and $\tau > 0$ be real numbers.*

(i) *The zero solution to (1.4) is asymptotically stable if and only if $(a,b) \in D_{0,0}$ where $D_{0,0}$ is an open region enclosed by the line $a+b=0$ from the right and by the parametric curve*

$$a = \frac{\rho^\alpha \sin(\alpha\pi/2 + 2\tau\rho)}{\sin(\tau\rho)}, \quad b = -\frac{\rho^\alpha \sin(\alpha\pi/2 + \tau\rho)}{\sin(\tau\rho)} \tag{1.5}$$

for

$$\rho \in \left(\frac{(1-\alpha)\pi}{3\tau}, \frac{(3-\alpha)\pi}{3\tau} \right)$$

from the left.

(ii) *The zero solution to* (1.4) *is stable but not asymptotically stable if and only if* $a+b=0$ *or* a *and* b *satisfy* (1.5) *for*

$$\rho = \frac{(2-\alpha)\pi}{4\tau} \quad or \quad \rho \in \left[\frac{(2-\alpha)\pi}{2\tau}, \frac{(3-\alpha)\pi}{3\tau}\right).$$

Remark 1.2. We observe that the conditions for parameter a in Theorem 1.1 are the same as those in Theorem A by letting $b=0$, that is, $\tau\rho=(2-\alpha)\pi/2$ in (1.5).

Remark 1.3. As previously mentioned, (1.4) corresponds to (1.3) with $\tau_1=\tau$ and $\tau_2=2\tau$ as $\alpha\to 1^-$. Comparing the conditions of Theorems C and 1.1 (i), the regions for the asymptotic stability of the zero solution are the same if $\alpha=1$.

Remark 1.4. As $\alpha\to 1^+$, (1.4) does not become (1.3) with $\tau_1=\tau$ and $\tau_2=2\tau$ if $x'(0)\neq 0$. Therefore, the case of $\alpha>1$ is out of the scope of the extension of Theorem C that we aim to achieve.

The set of the pair (a,b) that guarantees the asymptotic stability of the zero solution when the fractional order takes different values with a fixed delay is shown in Figure 1.1. Notably, these regions have no inclusion relationships with each other.

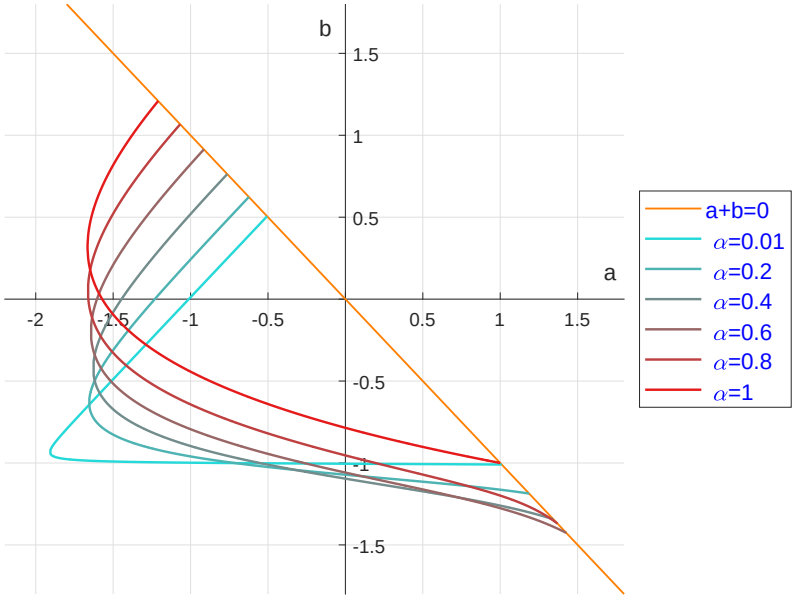


Figure 1.1: Parametric curve (1.5) for $\alpha=0.2,0.4,0.6,0.8,1$, and $\tau=1$.

Remark 1.5. According to [12,18], as $\alpha\to 0^+$, (1.4) is reduced to

$$x(t)-\phi(0)=ax(t-\tau)+bx(t-2\tau), \quad t\geq 0.$$

Figure 1.1 indicates that as $\tau=1$ and $\alpha\to 0^+$, $D_{0,0}$ given in Theorem 1.1 becomes the open triangle region D_0 enclosed by lines $a+b=0$, $b=a+1$, and $b=-1$. We note that D_0 can be regarded as the stability set for the equilibrium $\phi(0)/(1-a-b)$ of the second-order difference equation

$$x(k)-ax(k-1)-bx(k-2)=\phi(0), \quad k=0,1,2,\ldots,$$

where a, b, and $\phi(0)$ are real numbers with $a+b<0$; see, e.g., [8].

The structure of this article is arranged as follows: In the next section, we present the expression for the solutions using the Laplace transform. In Section 3, we provide a detailed analysis of the characteristic equation to present a set of (a,b) in which all the characteristic roots are located in the left half of the complex plane. The proof of the main theorem is completed in Section 4. Finally, we summarize our results in Section 5 and compare them with those of the corresponding first-order differential equation with two delays. The scope for future directions is also provided.

2 Preliminaries

In this section, we introduce some definitions and properties as preliminary for investigating the asymptotic behavior of the solutions to (1.1).

First, we introduce the Laplace transform; see, e.g. [7,10,12,18].

Definition 2.1. The Laplace transform of the function $f:\mathbb{R}_+\to\mathbb{R}$ is defined as

$$\mathcal{L}[f(t)](s)=\mathcal{L}_f(s):=\int_0^\infty e^{-st}f(t)dt,\quad s\in\mathbb{C}.$$

Suppose that $f\in L^1_{loc}(\mathbb{R}_+)$ and there exists $r\geq 0$ such that $|f(t)|\leq ce^{rt}$ for a large t. Then, $\mathcal{L}_f(s)$ exists and is analytic for $\mathrm{Re}(s)>r$. Moreover, if the integral converges at point $s_0\in\mathbb{C}$, then it converges absolutely for $s\in\mathbb{C}$ with $\mathrm{Re}(s)>\mathrm{Re}(s_0)$. The infimum R_f of the real part of s for which the Laplace integral of $f(t)$ converges, is called the abscissa of convergence. Clearly, the Laplace transform is linear. Next, we provide common rules and formulae for the Laplace transform.

Lemma 2.2.

(i) $\mathcal{L}[t^k](s)=\Gamma(k+1)/s^{k+1}$ *for* $k>-1$ *and* $s>0$.

(ii) $\mathcal{L}\big[\int_0^t f(t-u)g(u)du\big](s)=\mathcal{L}_f(s)\mathcal{L}_g(s)$.

(iii) $\mathcal{L}[f^{(n)}(t)](s)=s^n\mathcal{L}_f(s)-\sum_{k=1}^n s^{n-k}f^{(k-1)}(s)$.

(iv) *For* $c\in\mathbb{R}$, *let*

$$f_c(t)=\begin{cases}0, & t<c,\\ f(t-c), & t\geq c.\end{cases}$$

Then, $\mathcal{L}[f_c(t)](s)=e^{-cs}\mathcal{L}_f(s)$.

Definition 2.3. The inverse formula for the Laplace transform is given by

$$f(t)=\mathcal{L}^{-1}[\mathcal{L}_f(s)](t)=\frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty}e^{ts}\mathcal{L}_f(s)ds.$$

Here, the integral is along the vertical line $\mathrm{Re}(s)=r$ in $\mathbb{C}$ such that $r>R_f$, that is, it is greater than the real part of all the singularities of $\mathcal{L}_f$, and $\mathcal{L}_f$ is bounded on the line.

Next, we establish a representation formula for the solution $x(t)$ to (1.1). Let $X(s)=\mathcal{L}[x(t)](s)$. Then, the Laplace transform of (1.1) becomes

$$s^\alpha X(s)-s^{\alpha-1}\phi(0)=ae^{-\tau_1 s}X(s)+a\int_{-\tau_1}^0\phi(u)e^{-s(\tau_1+u)}du+be^{-\tau_2 s}X(s)+b\int_{-\tau_2}^0\phi(u)e^{-s(\tau_2+u)}du.$$

This implies that

$$X(s) = \frac{s^{\alpha-1}\phi(0) + a\int_{-\tau_1}^{0}\phi(u)e^{-s(\tau_1+u)}du + b\int_{-\tau_2}^{0}\phi(u)e^{-s(\tau_2+u)}du}{s^\alpha - ae^{-\tau_1 s} - be^{-\tau_2 s}}.$$

Let

$$\Phi^{\alpha,\beta}(t) = \mathcal{L}^{-1}\left(\frac{s^{\alpha-\beta}}{s^\alpha - ae^{-\tau_1 s} - be^{-\tau_2 s}}\right)(t).$$

Subsequently, from Lemma 2.2, it follows that

$$\begin{aligned} X(s) &= \phi(0)\mathcal{L}\left(\Phi^{\alpha,1}(t)\right)(s) + a\mathcal{L}\left(\Phi^{\alpha,\alpha}(t)\right)(s)\int_{-\tau_1}^{0}\phi(u)e^{-s(\tau_1+u)}du \\ &\quad + b\mathcal{L}\left(\Phi^{\alpha,\alpha}(t)\right)(s)\int_{-\tau_2}^{0}\phi(u)e^{-s(\tau_2+u)}du \\ &= \phi(0)\mathcal{L}\left(\Phi^{\alpha,1}(t)\right)(s) + a\int_{-\tau_1}^{0}\phi(u)\mathcal{L}\left({\Phi^{\alpha,\alpha}}_{\tau_1+u}(t)\right)(s)du \\ &\quad + b\int_{-\tau_2}^{0}\phi(u)\mathcal{L}\left({\Phi^{\alpha,\alpha}}_{\tau_2+u}(t)\right)(s)du. \end{aligned}$$

Using the inverse Laplace transform of $X(s)$, we obtain the following representation formula for the solution to (1.1):

$$x(t) = \phi(0)\Phi^{\alpha,1}(t) + a\int_{-\tau_1}^{0}\phi(u){\Phi^{\alpha,\alpha}}_{\tau_1+u}(t)du + b\int_{-\tau_2}^{0}\phi(u){\Phi^{\alpha,\alpha}}_{\tau_2+u}(t)du.$$

In particular, we obtain the representation formula for the solution to (1.4):

$$x(t) = \phi(0)\Phi^{\alpha,1}(t) + a\int_{-\tau}^{0}\phi(u){\Phi^{\alpha,\alpha}}_{\tau+u}(t)du + b\int_{-2\tau}^{0}\phi(u){\Phi^{\alpha,\alpha}}_{2\tau+u}(t)du, \tag{2.1}$$

where

$$\Phi^{\alpha,1}(t) = \mathcal{L}^{-1}\left[\frac{s^{\alpha-1}}{s^\alpha - ae^{-\tau s} - be^{-2\tau s}}\right](t), \tag{2.2}$$

$$\Phi^{\alpha,\alpha}{}_c(t) = \mathcal{L}^{-1}\left[\frac{e^{-cs}}{s^\alpha - ae^{-\tau s} - be^{-2\tau s}}\right](t). \tag{2.3}$$

3 Analysis of characteristic equation

The representation formula (2.1) shows that

$$P(s) := s^\alpha - ae^{-\tau s} - be^{-2\tau s} = 0 \tag{3.1}$$

is a characteristic equation associated with (1.4). Therefore, it is essential to analyze the roots of (3.1) in detail. First, we note that $P(0) = -a - b$ and $P(s) > 0$ for s real and sufficiently large. Thus, if $a + b > 0$, then (3.1) has at least one positive root. Moreover, as $\overline{e^z} = e^{\bar{z}}$ for any $z \in \mathbb{C}$, we observe that

$$\overline{P(s)} = \overline{s^\alpha} - \overline{ae^{-\tau s}} - \overline{be^{-2\tau s}} = \bar{s}^\alpha - ae^{-\tau\bar{s}} - be^{-2\tau\bar{s}} = P(\bar{s}).$$

This implies that for any $s \in \mathbb{C}$, $P(s) = 0$ if and only if $P(\bar{s}) = 0$. Based on this property, we consider (3.1) only in the upper half of the complex plane in the following discussion.

Proposition 3.1. *Let $0 < \alpha < 1$, a, b and $\tau > 0$ be real numbers.*

(i) (3.1) *has the root 0 if and only if $a + b = 0$.*

(ii) (3.1) *has a finite number of roots in the domain of $\{s \in \mathbb{C} \mid |\arg(s)| \leq \sigma\}$ for any given $\sigma \in (\pi/2, \pi)$.*

(iii) (3.1) *has a root ρi with $\rho > 0$ if and only if a and b satisfy*

$$a = \frac{\rho^\alpha \sin(\alpha\pi/2 + 2\tau\rho)}{\sin(\tau\rho)}, \quad b = -\frac{\rho^\alpha \sin(\alpha\pi/2 + \tau\rho)}{\sin(\tau\rho)}, \quad \rho \neq \frac{m\pi}{\tau}, \quad m \in \mathbb{N}. \tag{3.2}$$

(iv) (3.1) *has no quadruple roots. Moreover, if $a \neq 0$ and $b > 0$, then* (3.1) *has no double roots on the imaginary axis.*

Proof. (i) This part is obvious.

(ii) Let $s = \rho e^{\theta i}$ be a root of (3.1), where $\rho \geq 0$ and $\theta \in [0, \pi]$. The real and imaginary parts of $P(\rho e^{\theta i}) = 0$ are separated to obtain

$$\rho^\alpha \cos(\alpha\theta) = a\cos(\tau\rho\sin\theta)e^{-\tau\rho\cos\theta} + b\cos(2\tau\rho\sin\theta)e^{-2\tau\rho\cos\theta}, \tag{3.3}$$

$$\rho^\alpha \sin(\alpha\theta) = -a\sin(\tau\rho\sin\theta)e^{-\tau\rho\cos\theta} - b\sin(2\tau\rho\sin\theta)e^{-2\tau\rho\cos\theta}. \tag{3.4}$$

The sum of the squares of (3.3) and (3.4) is

$$\rho^{2\alpha} = e^{-2\tau\rho\cos\theta}\left[a^2 + 2ab\cos(\tau\rho\sin\theta)e^{-\tau\rho\cos\theta} + b^2e^{-2\tau\rho\cos\theta}\right]. \tag{3.5}$$

If $\theta \in [0, \pi/2]$, then

$$e^{-2\tau\rho\cos\theta}\left(a^2 + 2ab\cos(\tau\rho\sin\theta)e^{-\tau\rho\cos\theta} + b^2e^{-2\tau\rho\cos\theta}\right) \leq \left(|a| + |b|\right)^2.$$

This implies that (3.5) is impossible if $\rho > \left(|a| + |b|\right)^{1/\alpha}$. Thus, (3.1) has no roots in the region of

$$\{s \in \mathbb{C} \mid |s| > \left(|a| + |b|\right)^{1/\alpha}, \arg(s) \in [0, \pi/2]\}.$$

Similarly, if $\theta \in (\pi/2, \pi)$, it follows that

$$e^{-2\tau\rho\cos\theta}\left(a^2 + 2ab\cos(\tau\rho\sin\theta)e^{-\tau\rho\cos\theta} + b^2e^{-2\tau\rho\cos\theta}\right) \geq e^{-2\tau\rho\cos\theta}\left(|a| - |b|e^{-\tau\rho\cos\theta}\right)^2.$$

Because there exists a $\rho_1 > 0$ such that

$$\rho^{2\alpha} < e^{-2\tau\rho\cos\theta}\left(|a| - |b|e^{-\tau\rho\cos\theta}\right)^2$$

for all $\rho > \rho_1$, (3.1) has no root in the region of

$$\{s \in \mathbb{C} \mid |s| > \rho_1, \arg(s) \in (\pi/2, \pi)\}.$$

Therefore, there exists a $\bar{\rho}$ such that (3.1) has no root with $|s| > \bar{\rho}$ and $|\arg(s)| \in (\pi/2, \pi)$. Furthermore, if $a + b \neq 0$, then (3.3) does not hold as $\rho \to 0$. Hence, there exists $\underline{\rho} > 0$ such that (3.1) has no roots with $|s| < \underline{\rho}$. Let

$$\Omega = \{s \in \mathbb{C} \mid \underline{\rho} < |s| < \bar{\rho}, \ |\arg(s)| \in [0, \sigma]\}.$$

Because the function P is analytic on Ω, (3.1) has finite number of roots in Ω.

(iii) Let $P(\rho i) = 0$ where $\rho > 0$. Then, (3.3) and (3.4) with $\theta = \pi/2$ become

$$\begin{aligned}\rho^\alpha \cos(\alpha\pi/2) &= a\cos(\tau\rho) + b\cos(2\tau\rho),\\ \rho^\alpha \sin(\alpha\pi/2) &= -a\sin(\tau\rho) - b\sin(2\tau\rho),\end{aligned}$$

namely,

$$a\sin(\tau\rho) = \rho^\alpha \sin(\alpha\pi/2 + 2\tau\rho), \quad b\sin(\tau\rho) = -\rho^\alpha \sin(\alpha\pi/2 + \tau\rho). \tag{3.6}$$

Therefore, $P(\rho i) = 0$ is equivalent to (3.6). If $\sin(\tau\rho) = 0$, then (3.6) yields $\rho = 0$, which is a contradiction. Thus, $\sin(\tau\rho) \neq 0$, that is, $\tau\rho \neq m\pi$ for $m \in \mathbb{N}$; hence, $P(\rho i) = 0$ is equivalent to (3.2).

(iv) Finally, we consider the multiplicity of the roots of (3.1). Let $P'(s) = 0$. Then,

$$\frac{dP}{ds} = \alpha s^{\alpha-1} + a\tau e^{-\tau s} + 2b\tau e^{-2\tau s} = 0. \tag{3.7}$$

Clearly, $s = 0$ is not the root of (3.7). From Part (ii), we observe that if $s = \rho e^{\theta i}$ is a root of (3.7), then $\bar{s} = \rho e^{-\theta i}$ is also a root of (3.7). Eliminating s^α in (3.1) and (3.7) yields

$$a(\tau s + \alpha) + b(2\tau s + \alpha)e^{-\tau s} = 0.$$

Eliminating $e^{-2\tau s}$ in (3.1) and (3.7) yields

$$2\tau s^\alpha + \alpha s^{\alpha-1} - a\tau e^{-\tau s} = 0.$$

Then, we obtain

$$\frac{2\tau s + \alpha}{a\tau s^{1-\alpha}} = e^{-\tau s} = -\frac{a(\tau s + \alpha)}{b(2\tau s + \alpha)}. \tag{3.8}$$

This implies that

$$b(2\tau s + \alpha)^2 = -a^2\tau s^{1-\alpha}(\tau s + \alpha). \tag{3.9}$$

We observe that (3.9) has no positive real root s if $b > 0$. Let $s = \rho e^{\varphi i}$ where $\rho > 0$, $\varphi \in (0, 2\pi)$. Then, we have

$$\begin{aligned}b(4\tau^2\rho^2\cos 2\varphi + 4\alpha\tau\rho\cos\varphi + \alpha^2) &= -a^2\tau\rho^{1-\alpha}\left(\tau\rho\cos((2-\alpha)\varphi) + \alpha\cos((1-\alpha)\varphi)\right),\\ b(4\tau^2\rho^2\sin 2\varphi + 4\alpha\tau\rho\sin\varphi) &= -a^2\tau\rho^{1-\alpha}\left(\tau\rho\sin((2-\alpha)\varphi) + \alpha\sin((1-\alpha)\varphi)\right).\end{aligned}$$

If $\varphi = \pi$, then ρ satisfies

$$\begin{aligned}b(2\tau\rho - \alpha)^2 &= a^2\tau\rho^{1-\alpha}(\alpha - \tau\rho)\cos\alpha\pi,\\ 0 &= a^2\tau\rho^{1-\alpha}(\tau\rho - \alpha)\sin\alpha\pi.\end{aligned}$$

The second equation implies that $\rho = 0$ or $\rho = \alpha/\tau$; however, none satisfies the first equation. Therefore, (3.9) has no negative real root.

Suppose that (3.9) has a root $s = \rho i$. Then, ρ satisfies

$$\begin{aligned}b(\alpha^2 - 4\tau^2\rho^2) &= -a^2\tau\rho^{1-\alpha}\left(\alpha\sin(\alpha\pi/2) - \tau\rho\cos(\alpha\pi/2)\right),\\ 4b\alpha\tau\rho &= -a^2\tau\rho^{1-\alpha}\left(\tau\rho\sin(\alpha\pi/2) + \alpha\cos(\alpha\pi/2)\right).\end{aligned}$$

If $b > 0$, then the second equation does not hold for any $\rho > 0$; hence, (3.9) has no root on the imaginary axis. If $b < 0$, we divide the two equations to obtain

$$\frac{\alpha^2 - 4\tau^2\rho^2}{4\alpha\tau\rho} = \frac{\alpha\sin(\alpha\pi/2) - \tau\rho\cos(\alpha\pi/2)}{\tau\rho\sin(\alpha\pi/2) + \alpha\cos(\alpha\pi/2)}.$$

This implies that

$$F(\rho) := \rho^3 + \frac{3\alpha^2}{4\tau^2}\rho - \frac{\alpha^3}{4\tau^3}\cot\frac{\alpha\pi}{2} = 0.$$

As $F(0) < 0$ and $F'(\rho) > 0$ for $\rho \geq 0$, there is a unique positive root ρ^* for this equation. Therefore, (3.9) contains only one pair of roots $s = \pm\rho^* i$ if $b < 0$.

In summary, (a) if $a \neq 0$ and $b > 0$, (3.1) has no real or purely imaginary double roots; (b) if $a \neq 0$ and $b < 0$, (3.1) has a positive real double root and one pair of purely imaginary double roots, but has no negative real double roots.

Let $P''(s) = 0$. Then,

$$\frac{d^2P}{ds^2} = \alpha(1-\alpha)s^{\alpha-2} + a\tau^2e^{-\tau s} + 4b\tau^2e^{-2\tau s} = 0. \tag{3.10}$$

From (3.7) and (3.10), we obtain

$$e^{-\tau s} = -\frac{a(\tau s + \alpha - 1)}{2b(2\tau s + \alpha - 1)}. \tag{3.11}$$

From (3.8) and (3.11), we obtain

$$\alpha = (2\tau s + \alpha)(\tau s + \alpha). \tag{3.12}$$

Note that (3.12) has only one positive real root and one negative real root. Therefore, (3.1) has a positive real triple root if $b < 0$.

Let $P'''(s) = 0$. Then,

$$\frac{d^3P}{ds^3} = \alpha(1-\alpha)(2-\alpha)s^{\alpha-3} + a\tau^3e^{-\tau s} + 8b\tau^3e^{-2\tau s} = 0. \tag{3.13}$$

From (3.10) and (3.13), we obtain

$$e^{-\tau s} = -\frac{a(\tau s + \alpha - 2)}{4b(2\tau s + \alpha - 2)}. \tag{3.14}$$

From (3.11) and (3.14), we obtain

$$3\tau s + 3\alpha - 2 = (2\tau s + \alpha)(\tau s + \alpha). \tag{3.15}$$

Note that (3.12) and (3.15) have only one common root, $s^* = (2-2\alpha)/(3\tau)$. Substituting s^* into (3.12), we obtain $\alpha^2 - 7\alpha - 8 = 0$. This yields $\alpha = 1$ or $\alpha = -8$. This result is in conflict with $\alpha \in (0,1)$. Therefore, (3.1) has no quadruple root. □

For simplicity, we define the family of parametric curves as follows:

$$\begin{gathered} a = a_m(\rho) := \frac{\rho^\alpha\sin(\alpha\pi/2 + 2\tau\rho)}{\sin(\tau\rho)}, \quad b = b_m(\rho) := -\frac{\rho^\alpha\sin(\alpha\pi/2 + \tau\rho)}{\sin(\tau\rho)}, \\ \Gamma_m : (a_m(\rho), b_m(\rho)), \quad \frac{m\pi}{\tau} < \rho < \frac{(m+1)\pi}{\tau}, \quad m = 0, 1, 2, \ldots \end{gathered} \tag{3.16}$$

Proposition 3.1 (ii) indicates that conjugate complex numbers $\pm\rho i$ with $\rho > 0$ are roots of (3.1) if and only if the pair (a, b) satisfies (3.16). The parametric curve in (1.5) corresponds to a part of the Γ_0.

Lemma 3.2. *Let $m, n \in \mathbb{N}$.*

(i) *Each curve Γ_m intersects with the a-axis at $((-1)^m\rho^\alpha, 0)$ when $\tau\rho = (2-\alpha)\pi/2 + m\pi$; intersects with the b-axis at two points, which are $(0, -\rho^\alpha)$ when $\tau\rho = (2-\alpha)\pi/4 + m\pi$ and $(0, \rho^\alpha)$ when $\tau\rho = (4-\alpha)\pi/4 + m\pi$.*

(ii) *Each curve Γ_{2n} intersects with the line $a + b = 0$ when $\tau\rho = (1-\alpha)\pi/3 + 2n\pi$ and $\tau\rho = (3-\alpha)\pi/3 + 2n\pi$; intersects with the line $a = b$ when $\tau\rho = (2-\alpha)\pi/3 + 2n\pi$.*

(iii) *Each curve Γ_{2n+1} intersects with the line $a + b = 0$ when $\tau\rho = (2-\alpha)\pi/3 + (2n+1)\pi$; intersects with the line $a = b$ when $\tau\rho = (1-\alpha)\pi/3 + (2n+1)\pi$ and $\tau\rho = (3-\alpha)\pi/3 + (2n+1)\pi$.*

(iv) *If $\tau\rho$ tends to $2n\pi$, curves Γ_{2n-1} and Γ_{2n} approach their common asymptotes*

$$l_{2n} : b = -a + \left(\frac{2n\pi}{\tau}\right)^\alpha \cos\left(\frac{\alpha\pi}{2}\right).$$

(v) *If $\tau\rho$ tends to $(2n+1)\pi$, curves Γ_{2n} and Γ_{2n+1} approach their common asymptotes*

$$l_{2n+1} : b = a + \left(\frac{(2n+1)\pi}{\tau}\right)^\alpha \cos\left(\frac{\alpha\pi}{2}\right).$$

Proof. Parts (i), (ii), and (iii) are clear. A direct calculation reveals that

$$\begin{aligned} a_m(\rho) + \cos(\tau\rho) b_m(\rho) &= \frac{\rho^\alpha \sin(\alpha\pi/2 + 2\tau\rho) - \rho^\alpha \cos(\tau\rho)\sin(\alpha\pi/2 + \tau\rho)}{\sin(\tau\rho)} \\ &= \rho^\alpha \cos(\alpha\pi/2 + \tau\rho). \end{aligned}$$

Then, parts (iv) and (v) are obvious. □

Proposition 3.1 indicates that the range of $\tau\rho$ for the curve $\Gamma_{2n}(\rho)$ to lie in the region $\overline{D} := \{(a,b) \in \mathbb{R}^2 \mid a + b \le 0\}$ is

$$\tau\rho \in \left(\frac{(1-\alpha)\pi}{3} + 2n\pi, \frac{(3-\alpha)\pi}{3} + 2n\pi\right) =: J_{2n}$$

and the range of $\tau\rho$ for the curve $\Gamma_{2n+1}(\rho)$ to lie in the region $\overline{D}$ is

$$\tau\rho \in \left((2n+1)\pi, \frac{(2-\alpha)\pi}{3} + (2n+1)\pi\right)$$

for $n \in \mathbb{N}$.

Proposition 3.3. *Let $\alpha \in (0,1)$, $a + b \le 0$, and $m, n \in \mathbb{N}$. Suppose the point (a,b) lies on the curves Γ_m at $\tau = \tau^*$. Then, for $\tau^*\rho \in J_{2n} \cup J_{2n+1}$, the root ρi of (3.1) enters the right half of the complex plane as τ increases from τ^*, where*

$$J_{2n+1} := \left(\frac{(1-\alpha)\pi}{3} + (2n+1)\pi, \frac{(2-\alpha)\pi}{3} + (2n+1)\pi\right).$$

Proof. It suffices to demonstrate that

$$\mathrm{Re}\left(\frac{ds}{d\tau}\right)\Big|_{\substack{s=\rho i \\ \tau=\tau^*}} > 0 \quad \text{for} \quad \tau^*\rho \in J_{2n} \cup J_{2n+1}.$$

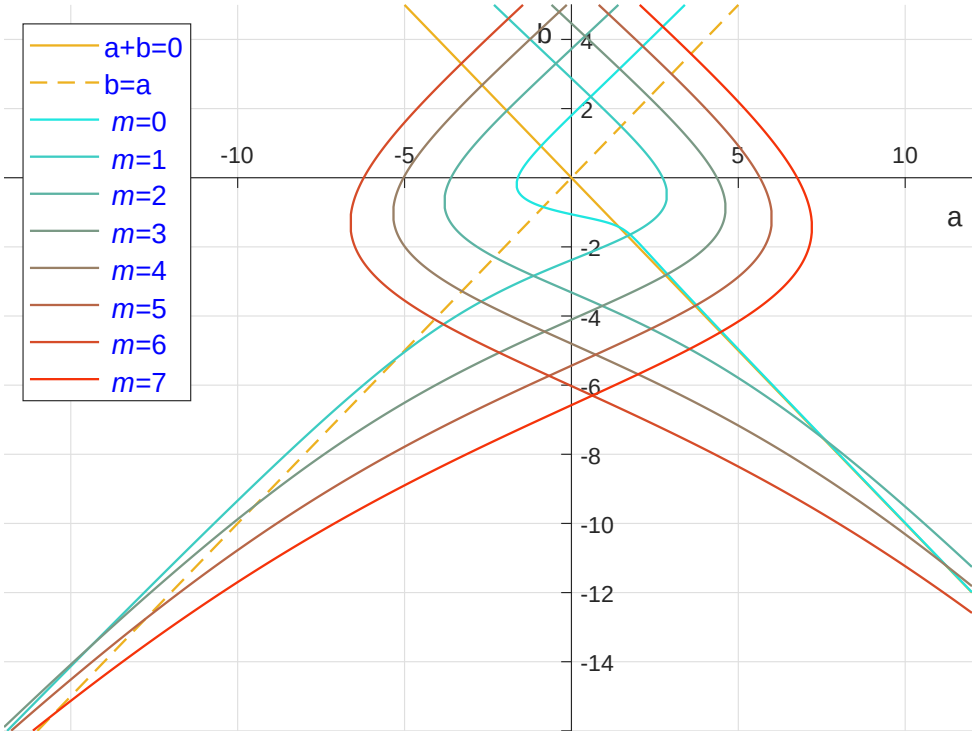


Figure 3.1: The graph of Γ_m for $m \in \{0,1,2,3,4,5,6,7\}$ with $\alpha = 0.6$ and $\tau = 1$.

From (3.1), we have

$$\frac{ds}{d\tau} = -\frac{s(ae^{-\tau s} + 2be^{-2\tau s})}{\alpha s^{\alpha-1} + a\tau e^{-\tau s} + 2b\tau e^{-2\tau s}}. \tag{3.17}$$

Substituting $s = \rho i$, $\tau = \tau^*$, $a = a_m(\rho)$, $b = b_m(\rho)$ into (3.17), we obtain

$$\left.\frac{ds}{d\tau}\right|_{\substack{s=\rho i \\ \tau=\tau^*}} = -\frac{A + Bi}{C + Di}, \tag{3.18}$$

where

$$\begin{aligned}
A &= \rho a_m(\rho)\sin(\tau^*\rho) + 2\rho b_m(\rho)\sin(2\tau^*\rho),\\
B &= \rho a_m(\rho)\cos(\tau^*\rho) + 2\rho b_m(\rho)\cos(2\tau^*\rho),\\
C &= \alpha\rho^{\alpha-1}\cos\left(\frac{(\alpha-1)\pi}{2}\right) + \tau^* a_m(\rho)\cos(\tau^*\rho) + 2\tau^* b_m(\rho)\cos(2\tau^*\rho),\\
D &= \alpha\rho^{\alpha-1}\sin\left(\frac{(\alpha-1)\pi}{2}\right) - \tau^* a_m(\rho)\sin(\tau^*\rho) - 2\tau^* b_m(\rho)\sin(2\tau^*\rho).
\end{aligned}$$

A direct calculation reveals that

$$\begin{aligned}
\mathrm{Re}\left(\frac{ds}{d\tau}\right)\Big|_{\substack{s=\rho i \\ \tau=\tau^*}} \cdot (C^2 + D^2) &= -AC - BD\\
&= \alpha\rho^{\alpha} a_m(\rho)\left(\cos(\tau^*\rho)\cos\left(\frac{\alpha\pi}{2}\right) - \sin(\tau^*\rho)\sin\left(\frac{\alpha\pi}{2}\right)\right)\\
&\quad + 2\alpha\rho^{\alpha} b_m(\rho)\left(\cos(2\tau^*\rho)\cos\left(\frac{\alpha\pi}{2}\right) - \sin(2\tau^*\rho)\sin\left(\frac{\alpha\pi}{2}\right)\right)\\
&= \frac{\alpha\rho^{2\alpha}}{2}\left(3 - \frac{\sin(\alpha\pi + 3\tau^*\rho)}{\sin(\tau^*\rho)}\right).
\end{aligned}$$

Therefore, if we show that

$$f(x) = 3 - \frac{\sin(\alpha\pi + 3x)}{\sin x} > 0 \quad \text{for } x \in J_{2n} \cup J_{2n+1}, \tag{3.19}$$

then the proof will be complete. We only need to consider the behavior of $f(x)$ on the interval $(0, \pi)$ because $f(x)$ is a periodic function with period π and satisfies $\lim_{x\to k\pi^-} f(x) = +\infty$ and $\lim_{x\to k\pi^+} f(x) = -\infty$ for any integer k. First, we obtain $f((r-\alpha)\pi/3) = 3$ for $r = 1, 2, 3$, and

$$\begin{aligned} f'(x) &= \frac{\sin(\alpha\pi + 3x)\cos(x) - 3\cos(\alpha\pi + 3x)\sin(x)}{\sin^2(x)} \\ &= \frac{2\sin(\alpha\pi + 2x) - \sin(\alpha\pi + 4x)}{\sin^2(x)} =: \frac{g(x)}{\sin^2(x)}. \end{aligned}$$

Note that $g((1-\alpha)\pi/3) = 3\sin((1-\alpha)\pi/3) > 0$, $g((2-\alpha)\pi/3) = -3\sin((2-\alpha)\pi/3) < 0$, $g((3-\alpha)\pi/3) = 3\sin((3-\alpha)\pi/3) > 0$, and $g'(x) = 8\sin(x)\sin(\alpha\pi + 3x) = 0$ only at $x = (r-\alpha)\pi/3$ for $r = 1, 2, 3$. Therefore, there only exist two numbers x_1^* and x_2^* with

$$\frac{(1-\alpha)\pi}{3} < x_1^* < \frac{(2-\alpha)\pi}{3} < x_2^* < \frac{(3-\alpha)\pi}{3}$$

such that $g(x_1^*) = g(x_2^*) = 0$. Then we see that the minimum of $f(x)$ on the interval $((1-\alpha)\pi/3, (3-\alpha)\pi/3)$ is $f(x_2^*)$. Next, we will show that $f(x_2^*) > 0$. From $f'(x_2^*) = 0$, we have

$$\sin(\alpha\pi + 3x_2^*)\cos(x_2^*) = 3\cos(\alpha\pi + 3x_2^*)\sin(x_2^*).$$

This yields

$$\sin^2(\alpha\pi + 3x_2^*)\left(1 - \sin^2(x_2^*)\right) = 9\left(1 - \sin^2(\alpha\pi + 3x_2^*)\right)\sin^2(x_2^*).$$

Thus,

$$\left|\frac{\sin(\alpha\pi + 3x_2^*)}{\sin(x_2^*)}\right| = \frac{3}{\sqrt{1 + 8\sin^2(x_2^*)}} < 3,$$

which implies that $f(x_2^*) > 0$. Therefore, we obtain

$$f(x) > 0 \quad \text{for} \quad x \in \left(\frac{(1-\alpha)\pi}{3} + k\pi, \frac{(3-\alpha)\pi}{3} + k\pi\right), \ k \in \mathbb{N}.$$

In other words, (3.19) is satisfied. □

Proposition 3.4. *Let $\alpha \in (0, 1)$, a, b and $\tau > 0$ be constants. Then, all roots of (3.1) are located in the left half of the complex plane if and only if $(a, b) \in D_{0,0}$, where the open region $D_{0,0}$ is bounded by the line $a + b = 0$ and the curve Γ_0 with $\rho \in ((1-\alpha)\pi/3\tau, (3-\alpha)\pi/3\tau)$.*

Proof. The region $\overline{D}$ is divided into an infinite number of regions by the curves Γ_m for $m \in \mathbb{N}$. As shown in Figure 3.2, the open regions that are enclosed by the line $a + b = 0$ and the curves Γ_m are denoted by $D_{h,k}$ with nonnegative integers h and k. It is well established that the number of roots with positive real part of the characteristic equation changes only if the pure imaginary root appears. Then the number of roots of (3.1) with positive real parts does not change in these regions $D_{h,k}$ if the time delay is fixed. Let $N(D_{h,k})$ denote the number of

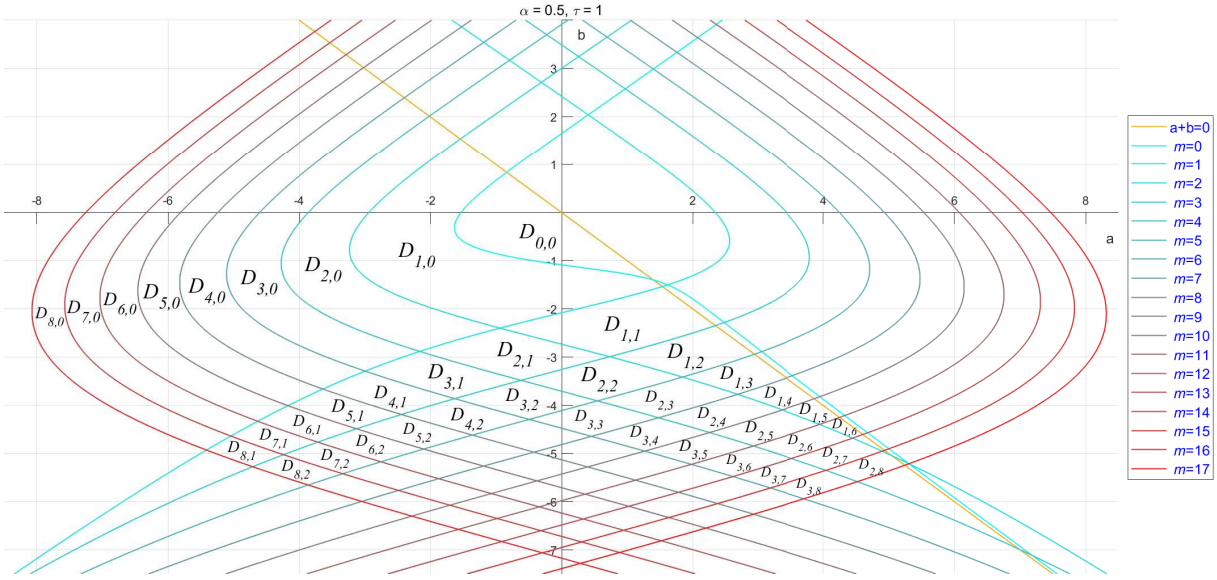


Figure 3.2: The graph of Γ_m for $\alpha = 0.5$, $\tau = 1$.

roots of (3.1) with positive real parts for $(a,b) \in D_{h,k}$. From Theorem A and Lemma 3.2 (i), we see that $N(D_{0,0}) = 0$ and $N(D_{h,k}) > 0$ for $D_{h,k}$ which intersects the a-axis or the b-axis.

Suppose there exists a point (a,b) lies on the common boundary of $D_{1,0}$ and $D_{1,1}$, which is on the curve Γ_1. Especially, we suppose the point (a,b) is close to the common point of Γ_1 and the line $a+b=0$, which guarantees that $\tau^*\rho \in J_1$. From Lemma 3.2 (i), the point (a,b) moves from $D_{1,0}$ into $D_{1,1}$ as τ grows near τ^*. Then, Proposition 3.3 shows that the number of roots of (3.1) with positive real parts increases as the point (a,b) moves from $D_{1,0}$ into $D_{1,1}$. Therefore, $N(D_{1,1}) > N(D_{1,0})$. Similarly, we can prove that $N(D_{1,k+1}) > N(D_{1,k}) > 0$ for any nonnegative integer k.

Suppose there exists a point (a,b) lies on the common boundary of $D_{1,0}$ and $D_{2,0}$, which is on the curve Γ_2. From Lemma 3.2 (i), the point (a,b) moves from $D_{1,0}$ into $D_{2,0}$ as τ grows near τ^*. Then, Proposition 3.3 shows that the number of roots of (3.1) with positive real parts increases as the point (a,b) moves from $D_{1,0}$ into $D_{2,0}$. Therefore, $N(D_{2,0}) > N(D_{1,0})$. Similarly, we can prove that $N(D_{2,k}) > N(D_{1,k})$ for any nonnegative integer k.

By repeating the same argument, we conclude that for any $D_{h,k}$ with $h = 1,2,\ldots$ and $k = 0,1,2,\ldots$, (3.1) has at least one pair of conjugate roots with positive real part. This completes the proof. □

4 Proof of main theorem

To prove that the zero solution to (1.4) is asymptotically stable or stable, we must examine the behavior of the solution to (1.4) given in (2.1) as $t \to \infty$. Because the representation formula (2.1) comprises functions (2.2) and (2.3), the problem is reduced to studying the behavior of the family of functions Φ as $t \to \infty$.

Because function Φ is defined by the inverse Laplace transform, we introduce an integral curve that is commonly used in inverse Laplace transforms. For brevity, denote by $\gamma(\varepsilon,\vartheta)$ the

contour comprising the following parts:

$$\{s \in \mathbb{C} \mid |s| = \varepsilon, \, |\arg(s)| \le \vartheta\}, \quad \{s \in \mathbb{C} \mid |s| \ge \varepsilon, \, |\arg(s)| = \vartheta\},$$

where $\varepsilon > 0$ and $0 < \vartheta \le \pi$.

The following lemma is utilized to compute the inverse Laplace transform of Φ.

Lemma 4.1. *Let $\alpha \in (0,1)$, $\beta \in (0,1]$, $\varepsilon > 0$, $\sigma \in (\pi/2, \pi)$, and $\ell = 0$ or $\ell = \alpha$. Define*

$$w(t) = \frac{1}{2\pi\alpha i} \int_{\gamma(\varepsilon^\alpha, \alpha\sigma)} \frac{z^{\frac{1-\beta}{\alpha}} \exp\left(z^{\frac{1}{\alpha}}(1 + 2\tau/t)\right)}{z \exp\left((2\tau/t) z^{\frac{1}{\alpha}}\right) - at^\alpha \exp\left((\tau/t) z^{\frac{1}{\alpha}}\right) - bt^\alpha} dz.$$

If there exists $K > 0$ such that

$$\left| z \exp\left(\frac{2\tau}{t} z^{\frac{1}{\alpha}}\right) - at^\alpha \exp\left(\frac{\tau}{t} z^{\frac{1}{\alpha}}\right) - bt^\alpha \right| \ge Kt^\ell \left| \exp\left(\frac{2\tau}{t} z^{\frac{1}{\alpha}}\right) \right| \tag{4.1}$$

for all $z \in \gamma(\varepsilon^\alpha, \alpha\sigma)$, then $w(t) = O(t^{-\ell})$ as $t \to \infty$.

Proof. From (4.1), we observe that:

$$\begin{aligned}
|w(t)| &\le \frac{1}{2\pi\alpha} \int_{\gamma(\varepsilon^\alpha, \alpha\sigma)} \frac{\left|z^{\frac{1-\beta}{\alpha}}\right| \left|\exp\left(z^{\frac{1}{\alpha}}(1 + 2\tau/t)\right)\right|}{Kt^\ell \left|\exp\left((2\tau/t) z^{\frac{1}{\alpha}}\right)\right|} |dz| \\
&= \frac{t^{-\ell}}{2\pi\alpha K} \int_{\gamma(\varepsilon^\alpha, \alpha\sigma)} \left|z^{\frac{1-\beta}{\alpha}}\right| \left|\exp\left(z^{\frac{1}{\alpha}}\right)\right| |dz| \\
&= \frac{t^{-\ell}}{2\pi\alpha K} \left(\varepsilon^{1+\alpha-\beta} \int_{-\alpha\sigma}^{\alpha\sigma} \exp\left(\varepsilon \cos\frac{\theta}{\alpha}\right) d\theta + 2 \int_{\varepsilon^\alpha}^{\infty} r^{\frac{1-\beta}{\alpha}} \exp\left(\cos\sigma r^{\frac{1}{\alpha}}\right) dr \right).
\end{aligned}$$

From $\sigma \in (\pi/2, \pi)$, it follows that

$$\begin{aligned}
\int_{\varepsilon^\alpha}^{\infty} r^{\frac{1-\beta}{\alpha}} \exp\left(\cos\sigma r^{\frac{1}{\alpha}}\right) dr &= \frac{\alpha}{(-\cos\sigma)^{1+\alpha-\beta}} \int_{\varepsilon^\alpha}^{\infty} \left(-\cos\sigma r^{\frac{1}{\alpha}}\right)^{\alpha-\beta} \exp\left(\cos\sigma r^{\frac{1}{\alpha}}\right) d\left(-\cos\sigma r^{\frac{1}{\alpha}}\right) \\
&< \frac{\alpha\Gamma(1+\alpha-\beta)}{(-\cos\sigma)^{1+\alpha-\beta}}.
\end{aligned}$$

Therefore, we obtain

$$|w(t)| < \frac{t^{-\ell}}{\pi\alpha K} \left(\alpha\sigma\varepsilon^{1+\alpha-\beta} e^\varepsilon + \frac{\alpha\Gamma(1+\alpha-\beta)}{(-\cos\sigma)^{1+\alpha-\beta}} \right) = O(t^{-\ell}) \quad \text{as } t \to \infty,$$

as desired. □

Proposition 4.2. *Let $0 < \alpha < 1$, $0 < \beta \le 1$.*

(i) *If all the roots of (3.1) are located in the left half-plane, then $\Phi^{\alpha,\beta}(t) = O(t^{\beta-\alpha-1})$ as $t \to \infty$.*

(ii) *If there exists the root 0 of (3.1) and other roots of (3.1) are located in the left half-plane, then $\Phi^{\alpha,\beta}(t) = O(t^{\beta-1})$ as $t \to \infty$.*

(iii) *If all the roots of (3.1) are located in the left half-plane or on the imaginary axis, and those of the roots on the imaginary axis are simple, then $\Phi^{\alpha,\beta}(t)$ is bounded.*

(iv) *If all the roots of* (3.1) *are located in the left half-plane or on the imaginary axis, and those of the roots on the imaginary axis are double, then*

$$\limsup_{t\to\infty}\left|\frac{\Phi^{\alpha,\beta}(t)}{t}\right| = \text{const} > 0. \tag{4.2}$$

(v) *If there are some roots* s_i *of* (3.1) *in the right half-plane, then*

$$\limsup_{t\to\infty}\left|\frac{\Phi^{\alpha,\beta}(t)}{t^2e^{Mt}}\right| = \text{const} > 0, \tag{4.3}$$

where $M = \max_{s_i}\{\mathrm{Re}(s_i)\}$.

Proof. Proposition 3.1 (i) implies that only the finite number of roots of (3.1) satisfy $|\arg(s)| < \sigma$ for $\sigma \in (\pi/2, \pi)$. Then, for $t > 1$, there exist $R > 0$ and $\varepsilon > 0$ with $R > \varepsilon$ such that all the roots of (3.1) are located to the left of $\gamma(R,\sigma)$ and the roots of (3.1) with $|\arg(s)| \le \sigma$ are located to the right of $\gamma(\varepsilon/t, \sigma)$. Subsequently, $\Phi^{\alpha,\beta}(t)$ is expressed as

$$\begin{aligned}\Phi^{\alpha,\beta}(t) &= \frac{1}{2\pi i}\int_{\gamma(R,\sigma)}\frac{s^{\alpha-\beta}e^{ts}}{P(s)}ds\\ &= \frac{1}{2\pi i}\int_{\gamma(R,\sigma)-\gamma(\varepsilon/t,\sigma)}\frac{s^{\alpha-\beta}e^{ts}}{P(s)}ds + \frac{1}{2\pi i}\int_{\gamma(\varepsilon/t,\sigma)}\frac{s^{\alpha-\beta}e^{ts}}{P(s)}ds.\end{aligned}$$

Using the residue theorem, we obtain

$$I_1(t) := \frac{1}{2\pi i}\int_{\gamma(R,\sigma)-\gamma(\varepsilon/t,\sigma)}\frac{s^{\alpha-\beta}e^{ts}}{P(s)}ds = \sum_{k=1}^{N}\mathrm{Res}\left[\frac{s^{\alpha-\beta}e^{ts}}{P(s)}; s_k\right].$$

Here, s_k, $k = 1, 2, \ldots, N$, represent the roots of (3.1) in the area enclosed by $\gamma(R,\sigma) - \gamma(\varepsilon/t, \sigma)$. Proposition 3.1 (iii) implies that

$$\begin{aligned}\frac{s^{\alpha-\beta}}{P(s)} &= a^k_{-3}(s-s_k)^{-3} + a^k_{-2}(s-s_k)^{-2} + a^k_{-1}(s-s_k)^{-1} + a^k_0 + \cdots,\\ e^{ts} &= e^{ts_k}\left(1 + t(s-s_k) + t^2(s-s_k)^2/2! + \cdots\right),\end{aligned}$$

where a^k_j, $j = -3, -2, \ldots$, are complex constants that are not all zero. Hence,

$$I_1(t) = \sum_{k=1}^{N}\left(a^k_{-1} + a^k_{-2}t + \frac{a^k_{-3}}{2!}t^2\right)e^{ts_k}.$$

By changing the variable $s = z^{1/\alpha}/t$, we obtain

$$\begin{aligned}I_2(t) &:= \frac{1}{2\pi i}\int_{\gamma(\varepsilon/t,\sigma)}\frac{s^{\alpha-\beta}e^{ts}}{P(s)}ds\\ &= \frac{1}{2\pi i}\int_{\gamma(\varepsilon/t,\sigma)}\frac{s^{\alpha-\beta}e^{ts}}{s^\alpha - ae^{-\tau s} - be^{-2\tau s}}ds\\ &= \frac{t^{\beta-1}}{2\pi\alpha i}\int_{\gamma(\varepsilon^\alpha,\alpha\sigma)}\frac{z^{\frac{1-\beta}{\alpha}}\exp\left(z^{\frac{1}{\alpha}}(1+2\tau/t)\right)}{z\exp\left((2\tau/t)z^{\frac{1}{\alpha}}\right) - at^\alpha\exp\left((\tau/t)z^{\frac{1}{\alpha}}\right) - bt^\alpha}dz\\ &= t^{\beta-1}w_{\frac{1-\beta}{\alpha},2}(t).\end{aligned}$$

If $a+b=0$, then (3.1) has the root 0. Note that the points in

$$\{s \in \mathbb{C} \mid s = (\varepsilon/t)e^{\theta i},\ \theta \in [-\sigma, \sigma]\}$$

are the closest points of $\gamma(\varepsilon/t,\ \sigma)$ with respect to the root 0. Thus, we have

$$|s^\alpha - ae^{-\tau s} - be^{-2\tau s}| \geq \left| \left(\frac{\varepsilon}{t}\right)^\alpha e^{\alpha\theta i} - a\exp\left(-\frac{\varepsilon\tau}{t}e^{\theta i}\right) - b\exp\left(-\frac{2\varepsilon\tau}{t}e^{\theta i}\right)\right|.$$

Because

$$\lim_{t\to\infty}\left| \left(\frac{\varepsilon}{t}\right)^\alpha e^{\alpha\theta i} - a\exp\left(-\frac{\varepsilon\tau}{t}e^{\theta i}\right) - b\exp\left(-\frac{2\varepsilon\tau}{t}e^{\theta i}\right)\right| t^\alpha = \varepsilon^\alpha,$$

we observe that there exist $\eta_0 > 0$ and $T > 0$ such that

$$|s^\alpha - ae^{-\tau s} - be^{-2\tau s}| \geq \eta_0 t^{-\alpha} \quad \text{for } t > T.$$

By changing the variable $s = z^{1/\alpha}/t$, we obtain

$$\left| \frac{z}{t^\alpha} - a\exp\left(-\frac{\tau}{t}z^{\frac{1}{\alpha}}\right) - b\exp\left(-\frac{2\tau}{t}z^{\frac{1}{\alpha}}\right)\right| \geq \eta_0 t^{-\alpha} \quad \text{for } t > T,$$

namely,

$$\left| z\exp\left(\frac{2\tau}{t}z^{\frac{1}{\alpha}}\right) - at^\alpha\exp\left(\frac{\tau}{t}z^{\frac{1}{\alpha}}\right) - bt^\alpha\right| \geq \eta_0 \left|\exp\left(\frac{2\tau}{t}z^{\frac{1}{\alpha}}\right)\right| \quad \text{for } t > T.$$

Then, from Lemma 4.1 with $\ell = 0$, we obtain

$$I_2(t) = t^{\beta-1}w_{\frac{1-\beta}{\alpha},2}(t) = t^{\beta-1}O(1) = O\big(t^{\beta-1}\big) \quad \text{as } t \to \infty.$$

If $a+b \neq 0$, then (3.1) contains only nonzero roots. Because contour $\gamma(\varepsilon/t,\ \sigma)$ does not contain any root of (3.1), there exists $\eta_1 > 0$ such that

$$|s^\alpha - ae^{-\tau s} - be^{-2\tau s}| \geq \eta_1$$

for all $s \in \gamma(\varepsilon/t,\ \sigma)$. By changing the variable $s = z^{1/\alpha}/t$, we obtain

$$\left| z\exp\left(\frac{2\tau}{t}z^{\frac{1}{\alpha}}\right) - at^\alpha\exp\left(\frac{\tau}{t}z^{\frac{1}{\alpha}}\right) - bt^\alpha\right| \geq \eta_1 t^\alpha \left|\exp\left(\frac{2\tau}{t}z^{\frac{1}{\alpha}}\right)\right|.$$

Then, from Lemma 4.1 with $\ell = \alpha$, it follows that

$$I_2(t) = t^{\beta-1}w_{\frac{1-\beta}{\alpha},2}(t) = t^{\beta-1}O(t^{-\alpha}) = O\big(t^{\beta-\alpha-1}\big) \quad \text{as } t \to \infty.$$

(i) If all the roots of (3.1) are located in the left half-plane, then

$$\Phi^{\alpha,\beta}(t) = I_2(t) = O\big(t^{\beta-\alpha-1}\big) \quad \text{as } t \to \infty.$$

(ii) If there exists the root 0 of (3.1) and the other roots of (3.1) are located in the left half-plane, then

$$\Phi^{\alpha,\beta}(t) = I_2(t) = O\big(t^{\beta-1}\big) \quad \text{as } t \to \infty.$$

(iii) If all the roots of (3.1) are located in the left half-plane or on the imaginary axis, and those roots on the imaginary axis are simple, then

$$\Phi^{\alpha,\beta}(t) = I_1(t) + I_2(t) = \sum_{k=1}^{N_1} a_{-1}^k e^{\pm\rho_k i t} + O\big(t^{\beta-1}\big),$$

which implies that $\Phi^{\alpha,\beta}(t)$ is bounded.

(iv) If all the roots of (3.1) are located in the left half-plane or on the imaginary axis, and those roots on the imaginary axis are double, then

$$\Phi^{\alpha,\beta}(t) = I_1(t) + I_2(t) = \sum_{k=1}^{N_1} \big(a_{-1}^k + a_{-2}^k t\big) e^{\pm\rho_k i t} + O\big(t^{\beta-1}\big) \quad \text{as } t \to \infty.$$

This yields (4.2).

(v) If there are some roots s_i of (3.1) in the right half-plane, then

$$\Phi^{\alpha,\beta}(t) = I_1(t) + I_2(t) = \sum_{k=1}^{N} \big(a_{-1}^k + a_{-2}^k t + a_{-3}^k t^2\big) e^{t s_k} + O\big(t^{\beta-1}\big) \quad \text{as } t \to \infty.$$

This implies (4.3). □

Remark 4.3. The asymptotic behaviors in Proposition 4.2 are fully characterized by Matignon's Theorem 1 in [17]: the algebraic decay $O(t^{-\alpha})$ for stable solutions corresponds to characteristic roots satisfying $|\arg(\lambda)| > \alpha\pi/2$; bounded but non-asymptotically stable solutions on the stability boundary occur when characteristic roots satisfy $|\arg(\lambda)| \geq \alpha\pi/2$ and the geometric multiplicity of a characteristic root with $|\arg(\lambda)| = \alpha\pi/2$ is one; the exponential growth rate e^{Mt} in unstable regimes directly follows from his analysis of $|\arg(\lambda)| \leq \alpha\pi/2$.

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. Recall the representation formula (2.1). There exist $A > 0$ and $B > 0$ such that

$$\begin{aligned}\left|a\int_{-\tau}^{0} \phi(u)\Phi^{\alpha,\alpha}{}_{\tau+u}(t)du\right| &\leq |a|\int_{-\tau}^{0} |\phi(u)\Phi^{\alpha,\alpha}{}_{\tau+u}(t)|\,du \\ &\leq |a| \sup_{t-\tau\leq\xi\leq t} |\Phi^{\alpha,\alpha}(\xi)| \int_{-\tau}^{0} |\phi(u)|\,du \\ &= A \sup_{t-\tau\leq\xi\leq t} |\Phi^{\alpha,\alpha}(\xi)|,\end{aligned}$$

and

$$\left|b\int_{-2\tau}^{0} \phi(u)\Phi^{\alpha,\alpha}{}_{2\tau+u}du\right| \leq B \sup_{t-2\tau\leq\xi\leq t} |\Phi^{\alpha,\alpha}(\xi)|.$$

Thus, we have:

$$|x(t)| \leq |\phi(0)\Phi^{\alpha,1}(t)| + A \sup_{t-\tau\leq\xi\leq t} |\Phi^{\alpha,\alpha}(\xi)| + B \sup_{t-2\tau\leq\xi\leq t} |\Phi^{\alpha,\alpha}(\xi)|.$$

Then, from Proposition 4.2, we conclude that:

(i) If all the roots of (3.1) are located in the left half-plane, then

$$x(t) = |\phi(0)|\,O(t^{-\alpha}) + A\cdot O(t^{-1}) + B\cdot O(t^{-1}) \quad \text{as } t \to \infty.$$

Therefore, $x(t) = O(t^{-\alpha})$ if $\phi(0) \neq 0$ and $x(t) = O(t^{-1})$ if $\phi(0) = 0$ as $t \to \infty$.

(ii) If there exists the root 0 of (3.1) and the other roots of (3.1) are located in the left half-plane, then we obtain

$$x(t) = |\phi(0)|\, O(1) + A \cdot O(t^{\alpha-1}) + B \cdot O(t^{\alpha-1}) \quad \text{as } t \to \infty.$$

Therefore, $x(t) = O(1)$ if $\phi(0) \neq 0$ and $x(t) = O(t^{\alpha-1})$ if $\phi(0) = 0$ as $t \to \infty$.

(iii) If all the roots of (3.1) are located in the left half-plane or on the imaginary axis, and those roots on the imaginary axis are simple, then $x(t)$ is bounded.

(iv) If all the roots of (3.1) are located in the left half-plane or on the imaginary axis, and those roots on the imaginary axis are double, then

$$\limsup_{t\to\infty} \left| \frac{x(t)}{t} \right| = \text{const} > 0.$$

(v) If there are some roots s_i of (3.1) in the right half-plane, then

$$\limsup_{t\to\infty} \left| \frac{x(t)}{t^2 e^{Mt}} \right| = \text{const} > 0,$$

where $M = \max_{s_i}\{\mathrm{Re}(s_i)\}$.

Hence, along with Propositions 3.1 and 3.3, Theorem 1.1 holds true. □

5 Conclusion

In this study, we discussed a Caputo-type fractional-order differential equation with two delays

$$^{C}\mathrm{D}_0^{\alpha} x(t) = ax(t-\tau) + bx(t-2\tau), \quad t \geq 0,$$

where $\alpha \in (0,1)$. Using the Laplace transform, we obtained the representation formula of the solution and the associated characteristic equation. From a detailed root analysis of the characteristic equation, we presented a region enclosed by a line and a curve in the ab-plane to guarantee that all the roots of the characteristic equation are located in the left half-plane for given α and τ. Moreover, we proved that the zero solution is asymptotically stable if and only if the pair (a,b) is an interior point of this region, and it is only stable if the pair (a,b) is on the partial boundary of this region. Because the Caputo fractional derivative coincides with the first-order derivative if $\alpha \to 1^-$, we compared Theorem 1.1 and Theorem C, which describes the asymptotic stability of the corresponding first-order delay differential equation. On the one hand, the expression of the set of (a,b) for asymptotic stability is the same if we set $\alpha = 1$ in the criterion of Theorem 1.1. On the other hand, we can judge from Figure 1.1, which shows the regions of asymptotic stability when α takes different values, that there exists no relationship of one set being a subset of another. This implies that the fractional order does not make it easier or more difficult to stabilize the zero solution than the integer order.

The asymptotic stability of two cases will be the focus of our future studies: (1.1) with $\tau_1 = \tau$, $\tau_2 = 3\tau$, and $0 < \alpha < 1$; and (1.4) with $1 < \alpha < 2$. This work aims to confirm the effect of the fractional-order α.

Statements and declarations

The authors declare that there are no conflicts of interest.

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Note on the oscillation of second-order hybrid type delay differential equations

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Abstract. This note presents some new criteria for the oscillation of all solutions of the second-order hybrid type delay differential equation

$$(a(t)x'(t))' - p_1(t)x(t) + p_2(t)x^{\alpha}(\sigma(t)) = 0,$$

by making use of a positive solution of the associated linear differential equation

$$(a(t)u'(t))' - p_1(t)u(t) = 0.$$

The results obtained are new and are illustrated by three examples.

Keywords: second-order, hybrid type, delay differential equation, oscillation.

2020 Mathematics Subject Classification: 34C10, 34K11.

1 Introduction

Consider the second-order hybrid type delay differential equation of the form

$$(a(t)x'(t))' - q_1(t)x(t) + q_2(t)x^{\alpha}(\sigma(t)) = 0, \qquad t \geq t_0, \tag{E}$$

where we assume throughout that the following conditions hold:

(H_1) $a \in C^1([t_0,\infty),(0,\infty))$, $q_1,\ q_2 \in C([t_0,\infty),(0,\infty))$, and α is a ratio of odd positive integers;

(H_2) $\sigma \in C^1([t_0,\infty),\mathbb{R})$ with $\sigma'(t) \geq 0$, $\sigma(t) \leq t$, and $\sigma(t) \to \infty$ as $t \to \infty$.

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By a *proper solution* of (E), we mean a function $x : [T_x, \infty) \to \mathbb{R}$ that satisfies (E) for all sufficiently large t and $\sup\{|x(t)| : t \geq T\} > 0$ for all $T \geq T_x$. We always assume that equation (E) possesses proper solutions. A solution of (E) is said to be *oscillatory* if it has arbitrarily large zeros, and otherwise it is called *nonoscillatory*. Equation (E) is said to be oscillatory if all its solutions are oscillatory.

Note that if $q_1 = 0$, then (E) reduces to the second order delay differential equation

$$(a(t)x'(t))' + q_2(t)x^{\alpha}(\sigma(t)) = 0, \qquad t \geq t_0, \tag{E$_1$}$$

and if $q_2 = 0$, then we obtain the linear differential equation without a delay

$$(a(t)x'(t))' - q_1(t)x(t) = 0, \qquad t \geq t_0. \tag{E$_2$}$$

Therefore, we may refer to equation (E) as a "hybrid type" differential equation.

Oscillation properties of solutions of second-order nonlinear delay differential equations were first studied in [9]. Since then, there has been a large number of papers regarding the oscillation of delay differential equations of different forms; see, for example, the monographs [1,2,10], the recent papers [3,5,7,8,13,17], and the references cited therein. This is due to the fact that oscillation and delay/advanced phenomena appear in different models arising from real world applications.

Recently in [8], the authors studied oscillation properties of the trinomial differential equation

$$x''(t) - q_1(t)x(\tau(t)) - q_2(t)x(\sigma(t)) = 0, \qquad t \geq t_0, \tag{E$_3$}$$

where $\tau(t) \geq t$ and $\sigma(t) \leq t$ for all $t \geq t_0$. Moreover, in [5,17], the authors presented oscillation criteria for equations of the form

$$(a(t)x'(t))' + q_1(t)x(t) + q_2(t)x(\sigma(t)) = 0, \qquad t \geq t_0, \tag{E$_4$}$$

in case (E$_4$) is in either canonical or noncanonical form. Note that if $a(t) > 0$, $q_1(t) > 0$, and $q_2(t) > 0$ for all $t \geq t_0$, then for any positive solution $x(t)$ of (E$_3$) or (E$_4$), the sign of the second-order derivative becomes

$$x''(t) > 0 \quad \text{or} \quad (a(t)x'(t))' < 0, \quad \text{for } t \geq t_0,$$

and from this it can easily be seen that $x'(t) > 0$ or $x'(t) < 0$ eventually. However, equation (E) includes both of the binomial equations (E$_1$) and (E$_2$) as special cases. Properties of both of these equations have been studied by several authors; see, for example, the monographs [1,2,15] and the references contained therein for details.

It is clear that the solutions spaces of equations (E$_1$) and (E$_2$) are completely different. If we denote by S the set of all nonoscillatory solutions of a considered equation, then for (E$_1$), the set S becomes

$$S = S_1 \cup S_0$$

where positive solutions satisfy

$$x(t) \in S_1 \quad \text{if and only if} \quad x'(t) > 0 \text{ and } (a(t)x'(t))' < 0,$$

and

$$x(t) \in S_0 \quad \text{if and only if} \quad x'(t) < 0 \text{ and } (a(t)x'(t))' < 0.$$

On the other hand, for (E_2), the set S has the structure

$$S = S_0^* \cup S_2^*$$

where positive solutions satisfy

$$x(t) \in S_0^* \quad \text{if and only if} \quad x'(t) < 0 \text{ and } (a(t)x'(t))' > 0,$$

and

$$x(t) \in S_2^* \quad \text{if and only if} \quad x'(t) > 0 \text{ and } (a(t)x'(t))' > 0.$$

Consequently, the structure of the set of nonoscillatory solutions of the hybrid equation (E) with positive and negative parts is unclear.

A method often employed in the study of the oscillation of solutions of trinomial differential equations is to omit one term. If, we omit the negative part of (E), then we are led to the differential inequality

$$\{(a(t)x'(t))' + q_2(t)x^\alpha(\sigma(t))\} \operatorname{sgn} x(t) \geq 0. \tag{E_5}$$

However, it is well known that properties of the corresponding differential equation (E_1) are connected to solutions of a differential inequality in the opposite direction. Similarly, omitting the positive term in (E) gives the differential inequality

$$\{(a(t)x'(t))' - q_1(t)x(t)\} \operatorname{sgn} x(t) \leq 0, \tag{E_6}$$

which is again opposite to the one that we would need. There are only a limited number of papers for equations like (E) with positive and negative parts. In this paper, we use a different approach that overcomes those difficulties caused by the presence of both negative and positive terms in (E). We then use comparison methods to obtain criteria for the oscillation of all solutions of (E). Three examples are provided to illustrate the importance and novelty of our main results.

2 Main results

In this section, we first transform the hybrid equation (E) into the form of a binomial equation, and then use a positive solution of the related auxiliary second-order linear ordinary differential equation

$$(a(t)u'(t))' = q_1(t)u(t), \qquad t \geq t_0. \tag{2.1}$$

Lemma 2.1. *The differential equation* (2.1) *has a positive nonincreasing solution on* $[t_0, \infty)$.

Proof. The proof is similar to Theorem 2.46 of [15] and so the details are omitted. □

Next, using a positive solution $u(t)$ of (2.1), we can obtain the following result.

Lemma 2.2. *Let* $u(t)$ *be a positive decreasing solution of* (2.1) *on* $[t_0, \infty)$. *Then* (E) *can be written in the form*

$$\left(a(t)u^2(t)\left(\frac{x(t)}{u(t)}\right)'\right)' + u(t)q_2(t)x^\alpha(\sigma(t)) = 0, \qquad t \geq t_0. \tag{E_7}$$

Proof. By a direct calculation and using (2.1), we see that

$$\begin{aligned}\frac{1}{u(t)}\left(a(t)u^2(t)\left(\frac{x(t)}{u(t)}\right)'\right)' &= \frac{1}{u(t)}\left(a(t)u(t)x'(t)-a(t)u'(t)x(t)\right)'\\ &= \frac{1}{u(t)}\left(u(t)(a(t)x'(t))'-(a(t)u'(t))'x(t)\right)\\ &= (a(t)x'(t))'-\frac{(a(t)u'(t))'x(t)}{u(t)}\\ &= (a(t)x'(t))'-q_1(t)x(t). \end{aligned} \tag{2.2}$$

Substituting (2.2) into (E) yields the desired result, and completes the proof. □

For the sake of convenience, we define the functions:

$$b(t)=a(t)u^2(t), \qquad B(t)=\int_{t_*}^{t}\frac{1}{a(s)u^2(s)}ds,$$

$$Q(t)=u(t)q_2(t)u^{\alpha}(\sigma(t)), \quad \text{and} \quad z(t)=\frac{x(t)}{u(t)}$$

for $t \geq t_* \geq t_0$. Employing the oscillation-preserving transformation $x(t)=u(t)z(t)$, where u is a positive solution of the differential equation (2.1), we can obtain the following consequence of Lemma 2.2.

Theorem 2.3. *Let $u(t)$ be a positive decreasing solution of (2.1). Then (E) is oscillatory if and only if the differential equation*

$$(b(t)z'(t))'+Q(t)z^{\alpha}(\sigma(t))=0, \qquad t \geq t_0, \tag{2.3}$$

is oscillatory.

For convenience in what follows, we will assume that (2.3) is in canonical form, that is,

$$B(t) \to \infty \quad \text{as } t \to \infty. \tag{2.4}$$

Remark 2.4. If $B(t) < \infty$ so that equation (2.3) is in noncanonical form, define

$$\Omega(t)=\int_{t}^{\infty}\frac{1}{b(s)}ds, \qquad b_1(t)=b(t)\Omega^2(t), \qquad \mu(t)=z(t)/\Omega(t),$$

and

$$Q_1(t)=\Omega(t)Q(t)\Omega^{\alpha}(\sigma(t)).$$

Then equation (2.3) can be written in the canonical form as

$$(b_1(t)\mu'(t))'+Q_1(t)\mu^{\alpha}(\sigma(t))=0, \qquad t \geq t_0. \tag{2.5}$$

The oscillation of (2.5) can then be deduced as for equation (2.3).

Theorem 2.5. *Let (2.4) hold and $u(t)$ be a positive decreasing solution of (2.1). Assume that $z(t)$ is an eventually positive (nonoscillatory) solution of (2.3). Then,*

$$z'(t) > 0 \quad \textit{and} \quad b(t)z'(t))' \leq 0 \tag{2.6}$$

for all $t \geq t_1 \geq t_0$.

Proof. Let $z(t)$ be an eventually positive solution of (2.3), say $z(t) > 0$ and $z(\sigma(t)) > 0$ for $t \geq t_1$ for some $t_1 \geq t_0$. Then from (2.3), we see that $(b(t)z'(t))' \leq 0$ and so $b(t)z'(t) > 0$ or $b(t)z'(t) < 0$ for all $t \geq t_1$.

If $b(t)z'(t) < 0$ for $t \geq t_1$, then

$$b(t)z'(t) \leq b(t_1)z'(t_1) < 0.$$

Dividing the last inequality by $b(t)$ and then integrating from t_1 to t gives

$$z(t) \leq z(t_1) + b(t_1)z'(t_1)\int_{t_1}^{t} \frac{1}{b(s)}ds \to -\infty$$

as $t \to \infty$ by (2.4). This contradicts the positivity of $z(t)$ and proves the theorem. □

Theorem 2.6. *Let* (2.4) *hold and* u *be a positive decreasing solution of* (2.1). *If*

$$\int_{t_1}^{\infty} Q(t)dt = \infty, \tag{2.7}$$

then equation (E) *is oscillatory.*

Proof. Assume, to the contrary, that $x(t)$ is an eventually positive solution of (E), say $x(t) > 0$ and $x(\sigma(t)) > 0$ for $t \geq t_1$ for some $t_1 \geq t_0$. Then, by our transformation, $z(t) = \frac{x(t)}{u(t)}$ is an eventually positive solution of (2.3), say $z(t) > 0$ and $z(\sigma(t)) > 0$ for all $t \geq t_1$. By Theorem 2.5, $z(t)$ is increasing, so by (H_1), we have $z(\sigma(t)) \geq z(\sigma(t_1))$ for $t \geq t_1$. Using this in (2.3) and then integrating the resulting inequality from t_1 to t yields

$$z^{\alpha}(\sigma(t_1))\int_{t_1}^{t} Q(s)ds \leq b(t_1)z'(t_1) - b(t)z'(t) \leq b(t_1)z'(t_1)$$

since $b(t)z'(t)$ is nonnegative. Letting $t \to \infty$ in the last inequality contradicts (2.7). This completes the proof. □

Remark 2.7. The above theorem is independent of α and the delay argument, so it holds for linear, super-linear, and sublinear equations as well as for delay or advanced type equations.

Next, we give two results for the oscillation of equation (E) without condition (2.7) holding.

Theorem 2.8. *Let* (2.4) *hold and* u *be a positive decreasing solution of* (2.1). *If* $\alpha = 1$ *and*

$$\liminf_{t\to\infty} B(\sigma(t))\int_{t}^{\infty} Q(s)ds > \frac{1}{4}, \tag{2.8}$$

then equation (E) *is oscillatory.*

Proof. Assume that $x(t)$ is an eventually positive solution of (E), say $x(t) > 0$ and $x(\sigma(t)) > 0$ for all $t \geq t_1$ for some $t_1 \geq t_0$. Then, proceeding as in the proof of Theorem 2.6, we have $z(t) > 0$, $z(\sigma(t)) > 0$, and condition (2.6) holds for all $t \geq t_1$. Using Corollary 2 in [7], we see that condition (2.8) implies that $z(t)$ is oscillatory. This contradiction completes the proof. □

Theorem 2.9. *Let* (2.4) *hold and* u *be a positive decreasing solution of* (2.1). *If the first order delay differential equation*

$$w'(t) + Q(t)B^{\alpha}(\sigma(t))w^{\alpha}(\sigma(t)) = 0 \tag{2.9}$$

is oscillatory, then equation (E) *is oscillatory.*

Proof. Assume that $x(t)$ is an eventually positive solution of (E), say $x(t) > 0$ and $x(\sigma(t)) > 0$ for $t \geq t_1 \geq t_0$. Proceeding as in the proof of Theorem 2.6, we see that $z(t)$ is a positive solution of (2.3) and (2.6) holds for $t \geq t_1$. Using the monotonicity of $b(t)z'(t) > 0$, we have

$$z(t) \geq \int_{t_1}^{t} \frac{b(s)z'(s)}{b(s)} ds \geq B(t)b(t)z'(t).$$

Using this in (2.3) yields

$$(b(t)z'(t))' + Q(t)B^{\alpha}(\sigma(t))(b(\sigma(t))z'(\sigma(t)))^{\alpha} \leq 0.$$

Let $w(t) = b(t)z'(t) > 0$; then $w(t)$ is a positive solution of the inequality

$$w'(t) + Q(t)B^{\alpha}(\sigma(t))w^{\alpha}(\sigma(t)) \leq 0.$$

By [14, Theorem 1], equation (2.9) also has a positive solution, which is a contradiction. This completes the proof. □

Next, we present some explicit oscillation criteria for equation (E) with the help of the results given in [10] and [16] for $\alpha = 1$, $0 < \alpha < 1$, and $\alpha > 1$.

Corollary 2.10. *Let* (2.4) *hold and* u *be a positive decreasing solution of* (2.1). *If*

$$\liminf_{t\to\infty} \int_{\sigma(t)}^{t} Q(s)B(\sigma(s))ds > \frac{1}{e} \quad \text{for } \alpha = 1, \tag{2.10}$$

or

$$\int_{t_0}^{\infty} Q(t)B^{\alpha}(\sigma(t))dt = \infty \quad \text{for } 0 < \alpha < 1, \tag{2.11}$$

then equation (E) *is oscillatory.*

Proof. Based on results in [10] (see [10, Theorem 2.1.1] and [10, Theorem 3.6.3], we see that conditions (2.10) and (2.11), respectively, imply equation (2.3) is oscillatory. The conclusion then follows from Theorem 2.3. □

Corollary 2.11. *Let* $\alpha > 1$, (2.4) *hold, and* u *be a positive decreasing solution of* (2.1). *Then every solution of* (E) *is oscillatory if any of the following conditions hold:*

(i) $\sigma(t) = t - \tau$, $\tau > 0$, *and there exists* $\lambda > \tau^{-1} \ln \alpha$ *such that*

$$\liminf_{t\to\infty} \left[Q(t)B^{\alpha}(\sigma(t)) \exp(-e^{\lambda t}) \right] > 0; \tag{2.12}$$

(ii) $\sigma(t) = \theta t$, $0 < \theta < 1$ *and there exists* $\lambda > -\ln \alpha / \ln \theta$ *such that*

$$\liminf_{t\to\infty} \left[Q(t)B^{\alpha}(\sigma(t)) \exp(-t^{\lambda}) \right] > 0; \tag{2.13}$$

(iii) $\sigma(t) = t^{\theta}$, $0 < \theta < 1$ *and there exists* $\lambda > -\ln \alpha / \ln \theta$ *such that*

$$\liminf_{t\to\infty} \left[Q(t)B^{\alpha}(\sigma(t)) \exp(-(\ln t)^{\lambda}) \right] > 0. \tag{2.14}$$

Proof. The proof follows from Theorem 3(i), Theorem 4(i), and Theorem 5(i) in [16], respectively, along with Theorem 2.9 above. This proves the corollary. □

Before presenting our final theorem, we define the constants

$$\beta_* = \liminf_{t\to\infty} b(t)B(\sigma(t))B(t)Q(t) \quad \text{and} \quad \lambda_* = \liminf_{t\to\infty} \frac{B(t)}{B(\sigma(t))}.$$

Theorem 2.12. *Let* (2.4) *hold and* u *be a positive decreasing solution of* (2.1). *If* $\alpha = 1$ *and*

$$\beta_* > \begin{cases} 0, & \text{if } \lambda_* = \infty, \\ \frac{d+2-\sqrt{(d+2)^2-4d}}{2d}, & \text{if } \lambda_* < \infty, \end{cases} \tag{2.15}$$

where $d = \ln \lambda_*$, *then equation* (E) *is oscillatory.*

Proof. From Corollary 2 and Theorem 4 in [11], we see that equation (2.3) is oscillatory. The conclusion then follows from Theorem 2.3. □

3 Examples

In this section, we present three examples to show the importance and novelty of the main results.

Example 3.1. Consider the second order hybrid differential equation

$$\left(t^{3/2}x'(t)\right)' - \frac{1}{2\sqrt{t}}x(t) + \frac{q_0}{\sqrt{t}}x(\lambda t) = 0, \qquad t \geq 1, \tag{3.1}$$

where $q_0 > 0$, and $\lambda \in (0,1)$. Here we have $a(t) = t^{3/2}$, $q_1(t) = \frac{1}{2\sqrt{t}}$, $q_2(t) = \frac{q_0}{\sqrt{t}}$, $\alpha = 1$, and $\sigma(t) = \lambda t$.

The auxiliary equation (2.1) becomes

$$(t^{3/2}u'(t))' = \frac{1}{2\sqrt{t}}u(t), \qquad t \geq 1$$

which has the positive decreasing solution $u(t) = 1/t$. Thus, $b(t) = t^{-1/2}$, $B(t) = \frac{2}{3}\left(t^{3/2} - 1\right)$, $Q(t) = q_0/(\lambda t^{5/2})$, and $z(t) = tx(t)$.

The transformed binomial form (2.3) of (3.1) is

$$\left(\frac{1}{\sqrt{t}}z'(t)\right)' + \frac{q_0}{\lambda t^{5/2}}z(\lambda t) = 0, \qquad t \geq 1. \tag{3.2}$$

Condition (2.4) clearly holds and condition (2.8) becomes

$$\liminf_{t\to\infty} \frac{2}{3}\left(\lambda^{3/2}t^{3/2} - 1\right)\int_t^\infty \frac{q_0}{\lambda s^{5/2}}ds > \frac{4}{9}\sqrt{\lambda}q_0,$$

so it holds if $q_0 > \frac{9}{16\sqrt{\lambda}}$. Hence, by Theorem 2.8, equation (3.1) is oscillatory if $q_0 > \frac{9}{16\sqrt{\lambda}}$.

In view of condition (2.10), we see that if

$$q_0 > \frac{3}{2e\sqrt{\lambda}\ln\frac{1}{\lambda}},$$

then equation (3.1) is oscillatory.

For $\lambda = 1/4$, the condition (2.8) becomes $q_0 > 9/8$ and the condition (2.11) becomes $q_0 > 0.796107$. Thus, we see that condition (2.11) is better than (2.8).

Also note that equation (3.1) is of the noncanonical type (since $\int_{t_0}^\infty \frac{1}{a(s)}ds < \infty$) whereas the transformed equation (3.2) is in canonical form. Thus, our results apply to noncanonical equations as well as those in canonical form.

Example 3.2. Consider the second order sub-linear hybrid delay differential equation

$$x''(t) - x(t) + e^{\frac{2t}{3}} x^{\frac{1}{3}}(t-\pi) = 0, \qquad t \geq \pi. \tag{3.3}$$

We have $a(t) = 1$, $q_1(t) = 1$, $q_2(t) = e^{\frac{2t}{3}}$, $\alpha = \frac{1}{3}$, and $\sigma(t) = t - \pi$, so the auxiliary equation is

$$u''(t) - u(t) = 0$$

for which $u(t) = e^{-t}$ is a positive decreasing solution. The transformed equation is

$$\left(e^{-2t} z'(t)\right)' + e^{\frac{-2t}{3} + \frac{\pi}{3}} z^{\frac{1}{3}}(t-\pi) = 0, \qquad t \geq \pi.$$

Simple calculations show that $B(t) \approx \frac{e^{2t}}{2}$, $Q(t) = e^{\frac{-2t}{3} + \frac{\pi}{3}}$ and condition (2.11) becomes

$$\int_{\pi}^{\infty} \frac{e^{\frac{-2t}{3} + \frac{\pi}{3}} e^{\frac{2t}{3}}}{2^{\frac{1}{3}}} dt = \infty,$$

so it holds. Therefore, by Corollary 2.10, equation (3.3) is oscillatory.

Example 3.3. Consider the second order super-linear hybrid delay differential equation

$$x''(t) - x(t) + e^{t^2+2t} x^3(t/2) = 0, \qquad t \geq 1. \tag{3.4}$$

Now $a(t) = 1$, $q_1(t) = 1$, $q_2(t) = e^{t^2+2t}$, $\alpha = 3$, and $\sigma(t) = t/2$. The auxiliary equation is

$$u''(t) - u(t) = 0$$

and $u(t) = e^{-t}$ is a positive decreasing solution. The transformed equation is

$$\left(e^{-2t} z'(t)\right)' + e^{t^2-2t} z^3(t/2) = 0, \qquad t \geq 1.$$

Additional calculations give $b(t) = e^{-2t}$, $B(t) \approx \frac{e^{2t}}{2}$, and $Q(t) = e^{t^2 - t/2}$. Choosing $\lambda = 2$ we see that $2 > \ln 3 / \ln 2$. The condition (2.13) becomes

$$\liminf_{t\to\infty} \left[\frac{1}{8} e^{t^2 - t/2} e^{3t} e^{-t^2}\right] = \infty > 0,$$

so it holds. Therefore, by Corollary 2.11(ii), equation (3.4) is oscillatory.

4 Conclusion

In this paper, we studied the oscillatory properties of hybrid delay differential equations of second-order. This is achieved by transforming the equation under consideration to a binomial type equation. Then by applying the comparison method and integral averaging technique, we established conditions for the oscillation of all solutions. None of the results reported in [5, 8, 17] are applicable to Examples 3.1 or 3.3 since these equations contain both positive and negative parts. Thus, the oscillation results presented here are a significant new contribution to the theory of oscillation of trinomial delay differential equations. In a future work, we plan to extend the results in this paper to equations of the form

$$(a(t)x'(t))' + p_1(t)x(t) - p_2(t)x^{\alpha}(\sigma(t)) = 0,$$

where $\sigma(t) \geq t$.

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A systematic method for complete stability problem of a class of delayed neural networks in parameter space

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Abstract. This paper presents a systematic method to address the complete stability problem of delayed neural networks with heterogeneous free parameters. First, we adopt an algebraic method to investigate the complete stability problem with respect to the free delay parameter τ. Then, the stability analysis is extended to the scenario with additional free system parameters, denoted by a vector X. We can investigate the complete root classification for the auxiliary characteristic equation in the entire (X, τ)-space. As a result, we can analytically calculate the number of stability τ-intervals and characterize all classifications of stability property over the whole (X, τ)-space. Finally, we will give a systematic method for determining the stability set in the whole (X, τ)-space. Some representative examples show the effectiveness of the approach.

Keywords: neural networks, time delays, stability, complete root classification.

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1 Introduction

During the last four decades, the interest in investigating the stability of neural networks (NNs) has steadily increased since the pioneering work of Hopfield (see [19]). Based on the Hopfield NN model, Marcus and Westervelt incorporated time delays in an NN model (see [32]). Since then, various types of delayed NN models have been proposed, and various issues of these models (including stability, periodic solution, and bifurcation, etc.) have been studied extensively by many researchers (see e.g., [1–3, 7, 12, 15, 16, 18, 20, 23, 28, 40, 42, 44, 45, 47]).

The appearance of delays causes a dynamical system to be infinitely dimensional. Hence, the stability analysis for a time delay system will usually be much more complicated than that for a delay-free system (see e.g., [14, 34, 37]).

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From a practical point of view, besides a time delay, a system usually involves a number of parameters. One can design the system parameters to render the system asymptotically stable (see e.g., [36, 41] for the controller design, [22] for the economic system, [48] for the laser system, and [4] for the biological system).

The stability problem becomes much more complex if the system contains the delay free parameter as well as additional system free parameters (i.e., heterogeneous free parameters).

For delayed NNs, the stability set of delay τ subject to the form $\tau \in [0, \overline{\tau})$ is extensively addressed. Such a $\overline{\tau}$ is termed the delay margin (see [14]), a notion widely used in the field of time-delay systems. See, for instance, [5] and [21] for delayed Hopfield NNs, [9] and [38] for delayed bidirectional associative memory (BAM) NNs, and [8] and [46] for delayed NNs with an annular topology.

Recently, the complete stability problem w.r.t. the delay parameter τ* has been studied for NNs in [24]. It is found therein that a delayed NN may have more than one stability τ-interval including or excluding $\tau = 0$.

Nevertheless, the problem considered in [24] contains only a free delay parameter τ. In this paper, for the delayed NN, we will address the complete stability problem with heterogeneous free parameters, including a delay parameter τ and additional system parameters (throughout this paper, we denote them by a vector X).

It will be interesting to see in this paper that the stability parameter space of a delayed NN may have multiple disjoint parts.

As far as we know, the complete stability problem of delayed NNs with heterogeneous free parameters has not been well investigated. In our opinion, the difficulties in studying this problem mainly come from two aspects:

(i) The free parameter space is multiple-dimensional.

(ii) The free parameters are of heterogeneous types.

To the best of the authors' knowledge, the above two technical points can not be appropriately covered by the existing results for delayed NNs. In order to solve such a complete stability problem, we will develop a systematic approach.

First, for the case where τ is the only free parameter, an algebraic approach will be employed for the complete stability analysis w.r.t. τ of the NN.

Notably, the above approach covers the general case (to be explained in Remark 3.7).

Next, in the scenario where the free parameters contain delay parameter τ plus system parameter vector X, we will employ the discrimination system, a mathematical tool for polynomial algebra. Then, we can obtain the complete root classification (CRC) for the auxiliary characteristic equation.

Consequently, we are able to characterize the stability property over the entire (X, τ)-space. To be more precise, we can calculate the possible number of stability τ-intervals and analytically calculate them.

Combining the abovementioned results, we will develop a systematic method to identify the stability set in the whole (X, τ)-space.

This systematic method can contribute to a more refined design process. It offers a possible "stabilization" way such that a pair of (X, τ) can be found under which a practical NN system can be stable.

*We call the problem of analyzing the stability property along the entire positive τ-axis as the complete stability problem w.r.t. τ (see e.g., [26]).

We will present some examples with different NN architectures and show that all classifications of stability property can be characterized in the entire (X, τ)-space by using our approach. It will be seen in the examples that a very small change of the vector X may make the $NU(\tau)$ distribution ($NU(\tau)$ stands for the number of characteristic roots in $\mathbb{C}_+$) have a structural variation.

This paper is organized as follows. In Section 2, three types of delayed NNs and the corresponding characteristic functions are reviewed. The main results are proposed in Section 3. Illustrative examples are presented in Section 4. Finally, the paper concludes in Section 5.

Notations: Throughout the paper, the following standard notations are used. $\mathbb{R}$ ($\mathbb{R}_+$) is the set of (positive) real numbers. $\mathbb{N}$ stands for the set of non-negative integers. $\mathbb{C}$ represents the set of complex numbers. We use $\mathbb{C}_-$ and $\mathbb{C}_+$ respectively to denote the left half-plane and the right half-plane in $\mathbb{C}$. $\deg(\cdot)$ is the degree of a polynomial. Finally, $\lceil \gamma \rceil$ stands for the smallest integer greater than or equal to γ, where $\gamma \in \mathbb{R}$.

2 Neural network models and characteristic functions

We will review in this section three types of delayed NN models, including the Hopfield NN model, the bidirectional associative memory (BAM) NN model, and the NN model with an annular topology. In recent years, these NN models have attracted much attention.

We will show that the local stability is determined by the corresponding characteristic function $f(\lambda, \tau)$.

2.1 Delayed Hopfield neural network model

The first model to be recalled in this paper is the delayed Hopfield NN model (see e.g., [5] and [21]):

$$\dot{y}_i(t) = -\mu_i y_i(t) + \sum_{j=1}^{n} c_{ij} f(y_j(t - \tau_{ij})), \qquad i = 1, \ldots, n, \tag{2.1}$$

where $y_i(t)$ stands for the voltage on the input of the i-th neuron at time t; $\mu_i \in \mathbb{R}_+$ is a positive constant; $c_{ij} \in \mathbb{R}$ represents the connection weight of the unit j on the unit i; f is the activation function; $\tau_{ij} \in \mathbb{R}_+ \cup 0$ denotes the signal transmission delay.

Here, we consider a simplified Hopfield NN model, which has three neurons in series (see Fig. 2.1 (a) borrowed from [29]):

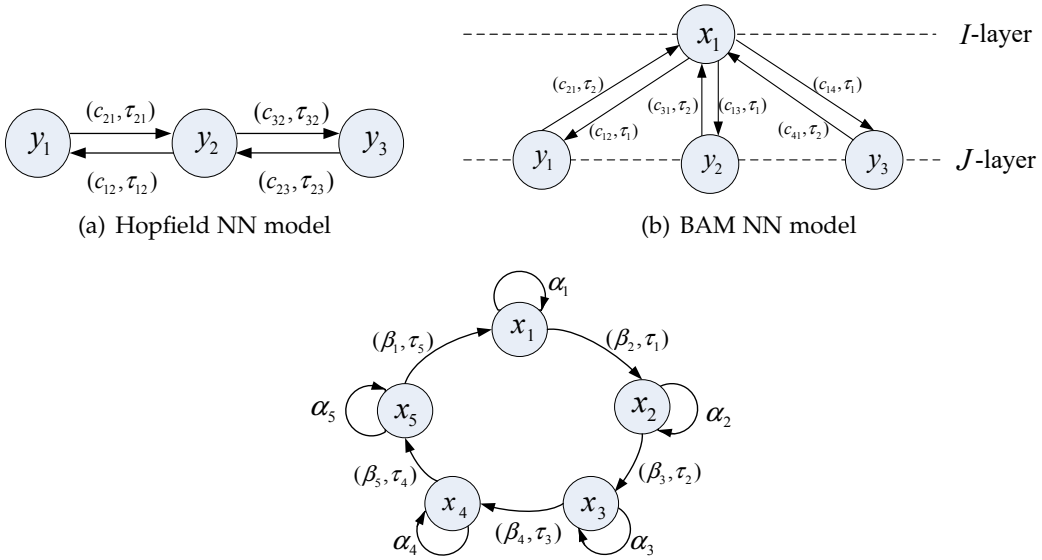
$$\begin{cases} \dot{y}_1(t) = -\mu_1 y_1(t) + c_{12} f(y_2(t - \tau_{12})), \\ \dot{y}_2(t) = -\mu_2 y_2(t) + c_{21} f(y_1(t - \tau_{21})) + c_{23} f(y_3(t - \tau_{23})), \\ \dot{y}_3(t) = -\mu_3 y_3(t) + c_{32} f(y_2(t - \tau_{32})), \end{cases} \tag{2.2}$$

where $c_{ij} = 0$ if $i = j$, $f \in C^1$, and $f(0) = 0$.

For the sake of simplicity, we assume that both the loops between neurons have the same sum of delays, i.e., $\tau_{12} + \tau_{21} = \tau_{23} + \tau_{32}$. We denote this value by τ.

It is not hard to see that for NN (2.2), the origin is the equilibrium and the linearization is

$$\begin{cases} \dot{y}_1(t) = -\mu_1 y_1(t) + h_{12} y_2(t - \tau_{12}), \\ \dot{y}_2(t) = -\mu_2 y_2(t) + h_{21} y_1(t - \tau_{21}) + h_{23} y_3(t - \tau_{23}), \\ \dot{y}_3(t) = -\mu_3 y_3(t) + h_{32} y_2(t - \tau_{32}), \end{cases}$$



(a) Hopfield NN model (b) BAM NN model

(c) Annular NN model

Figure 2.1: Architectures of three delayed NN models.

where $h_{ij} = c_{ij}f'(0)$.

The corresponding characteristic function is

$$a_0(\lambda) + a_1(\lambda)e^{-\tau\lambda}, \tag{2.3}$$

where $a_0(\lambda) = \prod_{i=1}^{3}(\lambda+\mu_i)$, $a_1(\lambda) = -(h_{21}h_{12}+h_{23}h_{32})\lambda - h_{21}h_{12}\mu_3 - h_{23}h_{32}\mu_1$. It follows that $\deg(a_0(\lambda)) = 3$ and $\deg(a_1(\lambda)) = 1$.

For such a Hopfield NN model, the coefficients μ_i and h_{ij} are allowed to have free system parameters (as mentioned previously, we denote them by a vector X). In Example 4.1 of this paper, we will choose $X = (h_{21})$ and study the whole stability set in (h_{21},τ)-plane.

2.2 Delayed bidirectional associative memory (BAM) neural network model

Next, we recall the BAM NNs (see e.g., [9] and [38]).

A delayed BAM NN is described by the model

$$\begin{cases} \dot{x}_i(t) = -\mu_i x_i(t) + \sum_{j=1}^{m} c_{ji} f_i(y_j(t-\tau_2)), \\ \dot{y}_j(t) = -\nu_j y_j(t) + \sum_{i=1}^{n} d_{ij} g_j(x_i(t-\tau_1)), \end{cases} \tag{2.4}$$

where $x_i(t)$ and $y_j(t)$ are the state of the neurons in the I-layer and the J-layer at time t, respectively ($i = 1,\dots,n$, $j = 1,\dots,m$, n and m denote the numbers of neurons); f_i and g_j are the activation functions; $c_{ji} \in \mathbb{R}$ and $d_{ij} \in \mathbb{R}$ are the connection weights; τ_1 and τ_2 are the signal transmission delays; $\mu_i \in \mathbb{R}_+$ and $\nu_j \in \mathbb{R}_+$ describe the stability of internal neuron process.

Here, we consider the case where $n = 1$ and $m = 3$ (the architecture of this model is described in Fig. 2.1 (b)).

For simplicity, we redefine $\mu_2 = \nu_1, \mu_3 = \nu_2, \mu_4 = \nu_3$, $c_{21} = c_{11}$, $c_{31} = c_{21}$, $c_{41} = c_{31}$,

$c_{12} = d_{11}$, $c_{13} = d_{12}$, $c_{14} = d_{13}$, $f_2 = g_1$, $f_3 = g_2$, $f_4 = g_3$. Then, the BAM NN model (2.4) reads:

$$\begin{cases} \dot{x}_1(t) = -\mu_1 x_1(t) + c_{21} f_1(y_1(t-\tau_2)) + c_{31} f_1(y_2(t-\tau_2)) + c_{41} f_1(y_3(t-\tau_2)), \\ \dot{y}_1(t) = -\mu_2 y_1(t) + c_{12} f_2(x_1(t-\tau_1)), \\ \dot{y}_2(t) = -\mu_3 y_2(t) + c_{13} f_3(x_1(t-\tau_1)), \\ \dot{y}_3(t) = -\mu_4 y_3(t) + c_{14} f_4(x_1(t-\tau_1)), \end{cases} \tag{2.5}$$

where $f_i \in C^1$ and $f_i(0) = 0, i = 1, 2, 3, 4$.

By letting $u_1(t) = x_1(t-\tau_1), u_2(t) = y_1(t), u_3(t) = y_2(t)$, $u_4(t) = y_3(t)$, and $\tau = \tau_1 + \tau_2$, the BAM NN model (2.5) may be rewritten as the following model

$$\begin{cases} \dot{u}_1(t) = -\mu_1 u_1(t) + c_{21} f_1(u_2(t-\tau)) + c_{31} f_1(u_3(t-\tau)) + c_{41} f_1(u_4(t-\tau)), \\ \dot{u}_2(t) = -\mu_2 u_2(t) + c_{12} f_2(u_1(t)), \\ \dot{u}_3(t) = -\mu_3 u_3(t) + c_{13} f_3(u_1(t)), \\ \dot{u}_4(t) = -\mu_4 u_4(t) + c_{14} f_4(u_1(t)). \end{cases} \tag{2.6}$$

Then, the linearization of the model (2.6) at the equilibrium $(0, 0, 0, 0)$ is

$$\begin{cases} \dot{u}_1(t) = -\mu_1 u_1(t) + h_{21} u_2(t-\tau) + h_{31} u_3(t-\tau) + h_{41} u_4(t-\tau), \\ \dot{u}_2(t) = -\mu_2 u_2(t) + h_{12} u_1(t), \\ \dot{u}_3(t) = -\mu_3 u_3(t) + h_{13} u_1(t), \\ \dot{u}_4(t) = -\mu_4 u_4(t) + h_{14} u_1(t), \end{cases} \tag{2.7}$$

where $h_{ij} = c_{ij} f_j'(0)$.

The corresponding characteristic function is

$$a_0(\lambda) + a_1(\lambda) e^{-\tau\lambda}, \tag{2.8}$$

where $a_0(\lambda) = \lambda^4 + (\mu_1 + \mu_2 + \mu_3 + \mu_4)\lambda^3 + (\mu_1\mu_2 + \mu_3\mu_4 + \mu_1\mu_3 + \mu_1\mu_4 + \mu_2\mu_3 + \mu_2\mu_4)\lambda^2 + (\mu_1\mu_2\mu_3 + \mu_1\mu_2\mu_4 + \mu_1\mu_3\mu_4 + \mu_2\mu_3\mu_4)\lambda + \mu_1\mu_2\mu_3\mu_4$ and $a_1(\lambda) = -(h_{12}h_{21} + h_{13}h_{31} + h_{14}h_{41})\lambda^2 - (h_{12}h_{21}\mu_3 + h_{12}h_{21}\mu_4 + h_{13}h_{31}\mu_2 + h_{13}h_{31}\mu_4 + h_{14}h_{41}\mu_2 + h_{14}h_{41}\mu_3)\lambda - (h_{12}h_{21}\mu_3\mu_4 + h_{13}h_{31}\mu_2\mu_4 + h_{14}h_{41}\mu_2\mu_3)$. It follows that $\deg(a_0(\lambda)) = 4$ and $\deg(a_1(\lambda)) = 2$.

For this BAM NN model, the coefficients μ_i and h_{ij} are allowed to have free system parameters. In Examples 3.12 and 4.2 of this paper, we will choose $X = (h_{12}, h_{21})$ and study the whole stability set in (h_{12}, h_{21}, τ)-space. In Example 4.2, we will also obtain the stability set in the corresponding 4D parameter space.

2.3 Delayed annular neural network model

Finally, we recall the delayed annular NN model (see e.g., [8] and [46]). Now consider the following NN model with five neurons:

$$\begin{cases} \dot{x}_1(t) = -\mu_1 x_1(t) + \alpha_1 f(x_1(t)) + \beta_1 g(x_5(t-\tau_5)), \\ \dot{x}_2(t) = -\mu_2 x_2(t) + \alpha_2 f(x_2(t)) + \beta_2 g(x_1(t-\tau_1)), \\ \dot{x}_3(t) = -\mu_3 x_3(t) + \alpha_3 f(x_3(t)) + \beta_3 g(x_2(t-\tau_2)), \\ \dot{x}_4(t) = -\mu_4 x_4(t) + \alpha_4 f(x_4(t)) + \beta_4 g(x_3(t-\tau_3)), \\ \dot{x}_5(t) = -\mu_5 x_5(t) + \alpha_5 f(x_5(t)) + \beta_5 g(x_4(t-\tau_4)), \end{cases} \tag{2.9}$$

where $x_i(t)$ stands for the voltage on the input of the ith neuron at time t; $\mu_i \in \mathbb{R}_+$ denotes the ratio of the capacitance to the resistance; τ_i represents the signal transmission delay; α_i and β_i are the nonzero connection weights; the activation functions f and g are assumed to satisfy $f, g \in C^1$ and $f(0) = g(0) = 0$.

The above model (2.9) represents the dynamics of a ring of neurons, and the architecture of this model is described in Fig. 2.1 (c).

We now calculate the associated characteristic function.

For NN model (2.9), the origin is the equilibrium and the linearization is

$$\begin{cases} \dot{x}_1(t) = (s_1 - \mu_1)x_1(t) + h_1 x_5(t - \tau_5), \\ \dot{x}_2(t) = (s_2 - \mu_2)x_2(t) + h_2 x_1(t - \tau_1), \\ \dot{x}_3(t) = (s_3 - \mu_3)x_3(t) + h_3 x_2(t - \tau_2), \\ \dot{x}_4(t) = (s_4 - \mu_4)x_4(t) + h_4 x_3(t - \tau_3), \\ \dot{x}_5(t) = (s_5 - \mu_5)x_5(t) + h_5 x_4(t - \tau_4), \end{cases} \tag{2.10}$$

where $s_i = \alpha_i f'(0), h_i = \beta_i g'(0), i = 1, \ldots, 5$.

The corresponding characteristic function is

$$a_0(\lambda) + a_1(\lambda)e^{-\tau\lambda}, \tag{2.11}$$

where $a_0(\lambda) = \prod_{i=1}^{5}(\lambda + \mu_i - s_i)$, $a_1(\lambda) = -\prod_{i=1}^{5}(h_i)$, and $\tau = \sum_{i=1}^{5}\tau_i$. It follows that $\deg(a_0(\lambda)) = 5$ and $\deg(a_1(\lambda)) = 0$.

For such an annular NN model, the coefficients μ_i, s_i, and h_i are allowed to have free system parameters. In Example 4.3 of this paper, we will choose $X = (s_1)$ and study the whole stability set in (s_1, τ)-plane.

2.4 Characteristic function

It is clear that the characteristic functions for abovementioned three types of delayed NNs are subject to the same form

$$f(\lambda, \tau) = a_0(\lambda) + a_1(\lambda)e^{-\lambda\tau}, \tag{2.12}$$

where $a_0(\lambda)$ and $a_1(\lambda)$ ($\deg(a_0(\lambda)) > \deg(a_1(\lambda))$) are polynomials with real coefficients. The coefficients may contain free parameters.

The characteristic function $f(\lambda, \tau)$ is a quasipolynomial, and, unsurprisingly, it also covers many other types of delayed NNs.

The local asymptotic stability of delayed NNs may be analytically determined by the above characteristic function (2.12). More specifically, the delayed NN is locally asymptotically stable if all the characteristic roots are located in the left half-plane $\mathbb{C}_-$.

3 Main results

The complete stability problem w.r.t. the delay parameter τ has been studied for delayed NNs by using a frequency-sweeping approach in a very recent paper (see [24]). It is pointed out that a delayed NN may have more than one stability τ-interval including or excluding $\tau = 0$.

However, the frequency-sweeping approach, as a graphical one, is difficult to apply in the case with additional free system parameters. When a delayed NN has multiple free parameters, we need to study the parameter space and the difficulties mentioned in Introduction arise.

In order to study the complete stability problem of delayed NN with heterogeneous free parameters, we will fulfill a systematic analysis in the whole (X, τ)-space.

More specifically, when τ is the only free parameter, we will adopt an algebraic approach for the complete stability analysis (to be given in Subsection 3.1), and in the case where the free parameters include delay parameter τ plus system parameter vector X, we will adopt a tool for polynomial algebra and develop a method to achieve the CRC of effective W roots (to be given in Subsection 3.2). Finally, we will present a systematic method to detect the stability set in the whole (X, τ)-space (to be given in Subsection 3.3).

3.1 Complete stability problem w.r.t. τ

When τ is the only free parameter, we use $NU(\tau) \in \mathbb{N}$ to denote the number of characteristic roots in $\mathbb{C}_+$. $NU(\tau)$ may increase or decrease only when τ is a critical delay (CD) where the system has a critical imaginary root (CIR) $\lambda = j\omega$, $\omega \in \mathbb{R}$. We call the corresponding pair (λ, τ) a critical pair.

We denote the CIRs as $\lambda_\alpha = j\omega_\alpha$, $\alpha = 0, \ldots, \mathfrak{u} - 1$. Concerning a CIR $\lambda_\alpha = j\omega_\alpha$, the CDs are $\tau_{\alpha,k} = \tau_{\alpha,0} + \frac{2k\pi}{\omega_\alpha}, k \in \mathbb{N}$, where $\tau_{\alpha,0}$ is the minimum CD satisfying

$$e^{-\tau\omega_\alpha j} = -\frac{a_0(j\omega_\alpha)}{a_1(j\omega_\alpha)}. \tag{3.1}$$

For a critical pair $(\lambda_\alpha, \tau_{\alpha,k} > 0)$, the influence of the asymptotic behavior on $NU(\tau)$ can be reflected by the notation $\Delta NU_{\lambda_\alpha}(\tau_{\alpha,k}) \in \mathbb{N}$, representing the variation of $NU(\tau)$ due to λ_α as τ is increased from $\tau_{\alpha,k} - \varepsilon$ to $\tau_{\alpha,k} + \varepsilon$.

First, along the similar idea in [13], we may obtain the following lemma straightforwardly.

Lemma 3.1. *For the characteristic function $f(\lambda, \tau)$ of a delayed NN considered in this paper, there exists a critical imaginary root $\lambda = j\omega \neq 0$ if and only if (iff) the auxiliary characteristic equation $\mathcal{F}(W) = 0$ has a positive real root $W = \omega^2$, where*

$$\begin{aligned} \mathcal{F}(W) &= |a_0(j\omega)|^2 - |a_1(j\omega)|^2 \\ &= (\mathrm{Re}(a_0(j\omega)))^2 + (\mathrm{Im}(a_0(j\omega)))^2 - (\mathrm{Re}(a_1(j\omega)))^2 - (\mathrm{Im}(a_1(j\omega)))^2. \end{aligned} \tag{3.2}$$

Remark 3.2. The application of auxiliary characteristic function can be traced back to [13], with an almost 40-year history. It has been widely used for the stability analysis of time-delay systems. However, the existing methods have the restriction of not being able to address the multiple auxiliary characteristic roots. In the sequel, we will give some results covering the restriction.

For the auxiliary characteristic function (3.2), we are only concerned with the positive real W roots since $W = \omega^2$. Such roots are termed the *effective W roots*, denoted by W_α, $\alpha = 0, \ldots, \mathfrak{u} - 1$, with $W_\alpha = \omega_\alpha^2$. Without any loss of generality, assume that among W_α, there are $q_o \in \mathbb{N}$ ones with odd multiplicities (we denote them by $W_0^o, \ldots, W_{q_o-1}^o$), and $q_e \in \mathbb{N}$ ones with even multiplicities (we denote them by $W_0^e, \ldots, W_{q_e-1}^e$). We label them as:

$$W_0^o > \cdots > W_{q_o-1}^o > 0, \; W_0^e > \cdots > W_{q_e-1}^e > 0. \tag{3.3}$$

For each W_i^o and W_i^e, we use $(\lambda_i^o = j\omega_i^o, \tau_{i,k}^o)$, where $\omega_i^o = \sqrt{W_i^o} > 0$, and $(\lambda_i^e = j\omega_i^e, \tau_{i,k}^e)$, where $\omega_i^e = \sqrt{W_i^e} > 0, k \in \mathbb{N}$, respectively to denote critical pairs. The CDs are $\tau_{i,k}^o = \tau_{i,0}^o + \frac{2k\pi}{\omega_i^o}$ and $\tau_{i,k}^e = \tau_{i,0}^e + \frac{2k\pi}{\omega_i^e}$, respectively.

Lemma 3.3. *For any $\varepsilon > 0$, when W varies slightly near the effective W root of delayed NNs considered in this paper, one of the following three cases must happen:*

Case (1): For a $W = W_i^e$, when W increases from $W_i^e - \varepsilon$ to $W_i^e + \varepsilon$, the $\mathcal{F}(W)$ does not change sign.

Case (2): For a $W = W_i^o$ (i is even), when W increases from $W_i^o - \varepsilon$ to $W_i^o + \varepsilon$, the sign of $\mathcal{F}(W)$ changes from negative to positive.

Case (3): For a $W = W_i^o$ (i is odd), when W increases from $W_i^o - \varepsilon$ to $W_i^o + \varepsilon$, the sign of $\mathcal{F}(W)$ changes from positive to negative.

Proof of Lemma 3.3. In light of Lemma 3.1, it is clear that $\mathcal{F}(W) = 0$ is a polynomial equation. If the polynomial equation has a W_i^e root, the Case (1) is true. For delayed NNs considered in this paper, the characteristic function (2.12) is satisfied with $\deg(a_0(\lambda)) > \deg(a_1(\lambda))$. It can be seen that $\mathcal{F}(W) > 0$ when $W \to \infty$. Hence, Cases (2) and (3) are true. □

Lemma 3.4. *Suppose $\widetilde{W} = (\widetilde{\omega})^2$ is the effective W root of delayed NNs considered in this paper. $(\widetilde{\lambda} = j\widetilde{\omega}, \widetilde{\tau})$ is the critical pair with $f(\widetilde{\lambda}, \widetilde{\tau}) = 0$. As W increases from $\widetilde{W} - \varepsilon$ to $\widetilde{W} + \varepsilon$, we have the following three results.*

(i) $\Delta NU_{j\widetilde{\omega}}(\widetilde{\tau}_k) = 0$ iff the $\mathcal{F}(W)$ does not change sign.

(ii) $\Delta NU_{j\widetilde{\omega}}(\widetilde{\tau}_k) = +1$ iff the sign of $\mathcal{F}(W)$ changes from negative to positive.

(iii) $\Delta NU_{j\widetilde{\omega}}(\widetilde{\tau}_k) = -1$ iff the sign of $\mathcal{F}(W)$ changes from positive to negative.

Proof of Lemma 3.4. First, letting $z = e^{-\lambda\tau}$, we rewrite the characteristic function (2.12) as $p(\lambda, z) = a_0(\lambda) + a_1(\lambda)z$. For each λ, we have a solution of z such that $p(\lambda, z) = 0$. Hence, we can obtain that $z = -a_0(\lambda)/a_1(\lambda)$, denoted by $z(\lambda)$. Here, we suppose $\widetilde{\lambda} = j\widetilde{\omega}$ is a critical imaginary root (calculated by $\mathcal{F}(W) = 0$, $\widetilde{W} = \widetilde{\omega}^2$ is the effective W root). The corresponding critical delays are $\widetilde{\tau}_k = \widetilde{\tau}_0 + \frac{2k\pi}{\widetilde{\omega}}$, where $\widetilde{\tau}_0$ is the minimum critical delay satisfying $z(j\widetilde{\omega}) = e^{-\tau\widetilde{\omega}j} = -a_0(j\widetilde{\omega})/a_1(j\widetilde{\omega})$. It follows that $|z(j\widetilde{\omega})| = 1$.

Next, for a $\widetilde{\lambda}$, it is clear that under a small perturbation $+\varepsilon j$, we hold that $|z(j(\widetilde{\omega} + \varepsilon))| = |a_0(j(\widetilde{\omega} + \varepsilon)/a_1(j(\widetilde{\omega} + \varepsilon)|$. Hence, $|z(j(\widetilde{\omega} + \varepsilon))| - |z(j\widetilde{\omega})| > 0 (< 0)$ iff $|a_0(j(\widetilde{\omega} + \varepsilon))| - |a_1(j(\widetilde{\omega} + \varepsilon))| > 0 (< 0)$, which is the same sign as $\mathcal{F}((\widetilde{\omega} + \varepsilon)^2)$. The above analysis also applies when the small perturbation is $-\varepsilon j$.

Finally, as W increases from $\widetilde{W} - \varepsilon$ to $\widetilde{W} + \varepsilon$ (ω increases from $\widetilde{\omega} - \varepsilon$ to $\widetilde{\omega} + \varepsilon$), in accordance with Section III(B) of [26] and Theorem 1 of [30], we can obtain the value of $\Delta NU_{j\widetilde{\omega}}(\widetilde{\tau}_k)$ by the sign of $|z(j(\widetilde{\omega} + \varepsilon))| - |z(j\widetilde{\omega})|$ and $|z(j(\widetilde{\omega} - \varepsilon))| - |z(j\widetilde{\omega})|$. Then the Lemma 3.4 can be proved. □

Following from Lemma 3.3 and Lemma 3.4, we can obtain the algebraic criterion for the complete stability problem of the delayed NN as follows:

Lemma 3.5. *When $\widetilde{W} = (\widetilde{\omega}_i^+)^2$ ($+$ is "o" or "e") is the effective W root and τ is the only free parameter for the delayed NN considered in this paper, we have the following three results.*

(1) For a $\lambda_i^e = j\widetilde{\omega}_i^e$ ($\widetilde{W} = W_i^e = (\widetilde{\omega}_i^e)^2$), $\Delta NU_{j\widetilde{\omega}_i^e}(\tau_{i,k}^e) = 0$ for all $\tau_{i,k}^e > 0$.

(2) For a $\lambda_i^o = j\widetilde{\omega}_i^o$ ($\widetilde{W} = W_i^o = (\widetilde{\omega}_i^o)^2$) with i is even, $\Delta NU_{j\widetilde{\omega}_i^o}(\tau_{i,k}^o) = +1$ for all $\tau_{i,k}^o > 0$.

(3) For a $\lambda_i^o = j\widetilde{\omega}_i^o$ ($\widetilde{W} = W_i^o = (\widetilde{\omega}_i^o)^2$) with i is odd, $\Delta NU_{j\widetilde{\omega}_i^o}(\tau_{i,k}^o) = -1$ for all $\tau_{i,k}^o > 0$.

(The λ_i^o is also listed in order from largest to smallest, with λ_0^o ($i = 0$) being the largest.)

Remark 3.6. The study on the stability of time-delay systems can be roughly classified into two types, namely a time domain approach and a frequency-domain approach. For the current study by the time-domain approach, the Lyapunov–Krasovskii method [14] is widely used, but it usually yield conservative results and cannot deal with the case when the delay is a free parameter. Besides the Lyapunov-based methods, the CTCR method [35], the direct method [39], and the matrix pencil method [10] can effectively analyze the stability of systems. However, it is difficult to apply these methods to deal with the asymptotic behavior of multiple CIRs.

In this paper, we address the complete stability problem of the delayed NN. Based on the perspective of characteristic root, the stability of systems is studied analytically. The conclusion of our approach is almost non-conservative, and it is very easy to obtain the sufficient and necessary conditions.

Remark 3.7. For the stability analysis of time-delay systems, the asymptotic behavior of multiple CIRs is essential (see e.g., [6, 11, 25, 26, 33]). It was recently pointed out in [25] that the appearance of a multiple and degenerate CIR may cause the system to become 'asymptotically stable' from 'unstable'. As far as we know, in the earlier references of delayed NNs, the case with multiple and/or degenerate CIRs has not been studied. Our method covers the case with multiple and/or degenerate CIRs, as such information is not a technical constraint for Lemma 3.5 (unlike for previous results).

In view of Lemma 3.5, $\Delta NU_{\lambda_\alpha}(\tau_{\alpha,k})$ is a constant belonging to the set $\{-1, 0, +1\}$. Next, we are able to derive the expression of $NU(\tau)$ in view of the root continuity argument.

Theorem 3.8. *Consider a delayed NN under consideration in this paper. For any $\tau > 0$ which is not a critical delay, $NU(\tau)$ for the characteristic equation $f(\lambda, \tau) = 0$ can be explicitly expressed as:*

$$NU(\tau) = NU(+\varepsilon) + \sum_{i=0}^{q_o-1} NU_i^o(\tau), \tag{3.4}$$

$$NU_i^o(\tau) = \begin{cases} 0, \ \tau < \tau_{i,0}^o, \\ 2(-1)^i \left\lceil \frac{\tau - \tau_{i,0}^o}{2\pi/\omega_i^o} \right\rceil, \ \tau > \tau_{i,0}^o, \end{cases} \quad \text{if } \tau_{i,0}^o \neq 0,$$

$$NU_i^o(\tau) = \begin{cases} 0, \ \tau < \tau_{i,1}^o, \\ 2(-1)^i \left\lceil \frac{\tau - \tau_{i,1}^o}{2\pi/\omega_i^o} \right\rceil, \ \tau > \tau_{i,1}^o, \end{cases} \quad \text{if } \tau_{i,0}^o = 0.$$

(In light of Theorem 5.1 in [26], one can calculate the value of $NU(+\varepsilon)$.)

The equilibrium corresponding to the characteristic equation $f(\lambda, \tau) = 0$ is locally asymptotically stable if τ is not a CD and belongs to the set with $NU(\tau) = 0$.

In view of the "$NU(\tau)$ vs. τ" plot, the equilibrium may undergo a Hopf bifurcation (see [17] for the Hopf Bifurcation Theorem) as τ is increased near a CD. The stability interval(s) and bifurcation values of τ can be exhaustively obtained in the whole positive τ-axis.

As mentioned, a delayed NN may contain multiple stability τ-intervals including or excluding $\tau = 0$. Hence, we introduce a notion from [27] that can be applied in the general case.

For the case where τ is the only free parameter, without loss of generality, we assume that the non-empty stability τ-set is subject to the form

$$\tau \in (\underline{\tau}_1, \overline{\tau}_1) \cup \cdots \cup (\underline{\tau}_s, \overline{\tau}_s), \tag{3.5}$$

plus possible $\tau = 0$, with $0 \leq \underline{\tau}_1 < \overline{\tau}_1 < \cdots < \underline{\tau}_s < \overline{\tau}_s$. $\overline{\tau}_s$ is termed the *generalized delay margin*.

Example 3.9. Consider the delayed BAM NN (2.5) with the activation functions $f_i(\cdot) = \tanh(\cdot)$. It is true that $h_{ij} = c_{ij}$. Here, we choose the coefficients as: $\mu_1 = 2.46$, $\mu_2 = 4.5769$, $\mu_3 = 0.8561$, $\mu_4 = 0.9669$, $h_{12} = 4.1$, $h_{13} = -0.3896$, $h_{14} = 2.3488$, $h_{21} = -4.57$, $h_{31} = -2.8466$, and $h_{41} = 0.7057$.

First, we calculate that $NU(0) = 0$ by solving the characteristic equation $f(\lambda, 0) = 0$.

Then, according to Lemma 3.1, we have $\mathcal{F}(W) = W^4 + 28.6674W^3 - 82.5660W^2 - 0.0347W + 69.9926$.

By calculating the roots of equation $\mathcal{F}(W) = 0$, we have two sets of CIRs: $\lambda_0^o = 1.4869j$ (associated with the CDs $\tau_{0,k}^o = 1.7802 + 4.2258k$, $k \in \mathbb{N}$) and $\lambda_1^o = 1.1136j$ (associated with the CDs $\tau_{1,k}^o = 2.6181 + 5.6420k$, $k \in \mathbb{N}$).

Furthermore, we can obtain that $\Delta NU_{\lambda_0^o}(\tau_{0,k}^o) = +1$ and $\Delta NU_{\lambda_1^o}(\tau_{1,k}^o) = -1$ in view of Lemma 3.5. Next, according to Theorem 3.8, we have the "$NU(\tau)$ vs. τ" plot, as shown in Fig. 3.1.

Consequently, we know that the equilibrium is locally asymptotically stable if $\tau \in [0, 1.7802) \cup (2.6181, 6.0059) \cup (8.2602, 10.2317) \cup (13.9022, 14.4575)$ and that a Hopf bifurcation may occur when $\tau = 1.7802$, 2.6181, 6.0059, 8.2602, 10.2317, 13.9022 and 14.4575. To verify the above analysis, we give the simulations for $\tau = 1.2$ (in the first stability τ-interval) in Fig. 3.2 (a), Fig. 3.2 (d) and Fig. 3.3 (a), $\tau = 2.3$ (between the first and the second stability τ-intervals) in Fig. 3.2 (b), Fig. 3.2 (e) and Fig. 3.3 (b), and $\tau = 5$ (in the second stability τ-interval) in Fig. 3.2 (c), Fig. 3.2 (f) and Fig. 3.3 (c).

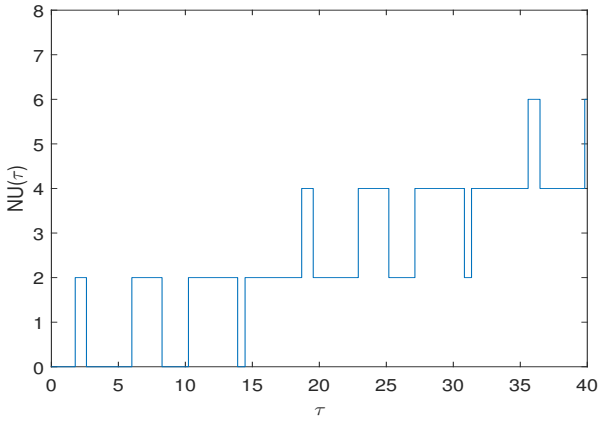
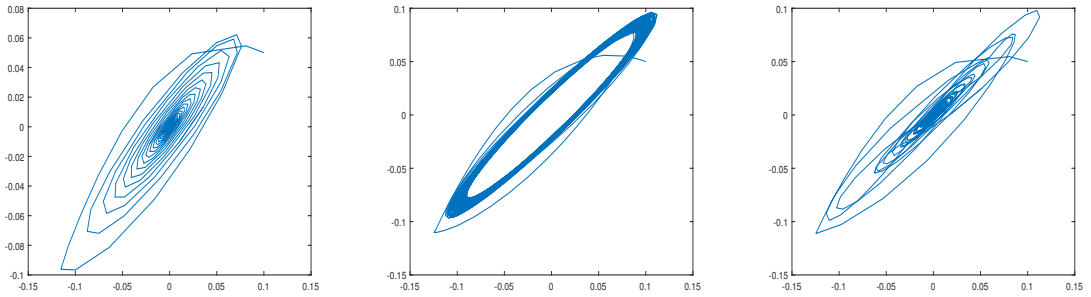


Figure 3.1: $NU(\tau)$ vs. τ for Example 3.9.

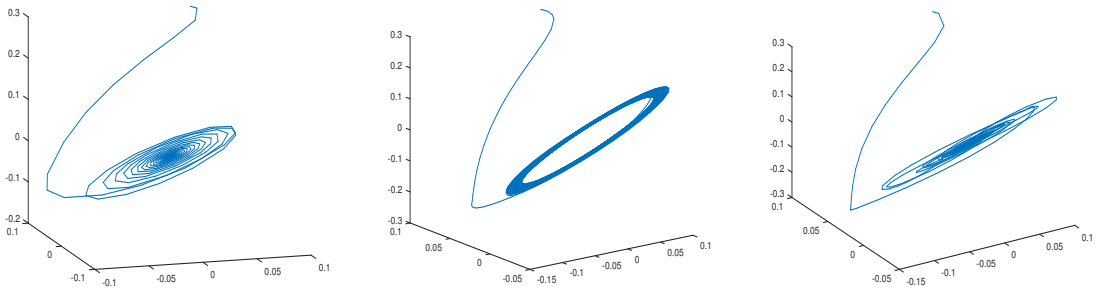
3.2 Complete root classification (CRC) of effective W roots

We proceed to address the scenario with additional free system parameters. In order to appropriately address such a case, we now investigate the real W root classification and the effective W root classification. The real (effective) W root classification refers to the information about the numbers and multiplicities of different real (effective) W roots. The complete root classification (CRC) of real (effective) W roots is the collection of all possible real (effective) W root classifications.

In the sequel, we recall a mathematical tool, termed the *discrimination system* (one may refer to [31] and [43] for details).

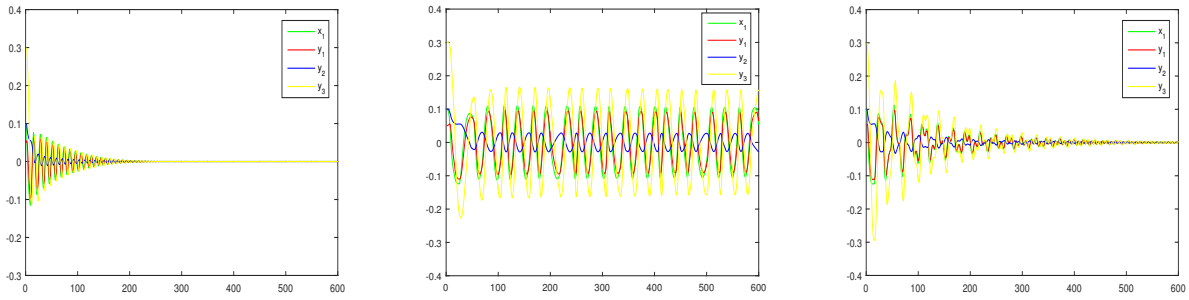


(a) $x_1 - y_1$ plane with $\tau = 1.2$ (b) $x_1 - y_1$ plane with $\tau = 2.3$ (c) $x_1 - y_1$ plane with $\tau = 5$



(d) $y_1 - y_2 - y_3$ space with $\tau = 1.2$ (e) $y_1 - y_2 - y_3$ space with $\tau = 2.3$ (f) $y_1 - y_2 - y_3$ space with $\tau = 5$

Figure 3.2: Phase portraits for Example 3.9.



(a) $\tau = 1.2$ (b) $\tau = 2.3$ (c) $\tau = 5$

Figure 3.3: Simulations for Example 3.9.

Let $\mathcal{Q}(z)$ be a polynomial with real coefficients,

$$\mathcal{Q}(z) = \zeta_q z^q + \zeta_{q-1} z^{q-1} + \cdots + \zeta_0, \zeta_q \neq 0. \tag{3.6}$$

The $2q \times 2q$ matrix $M =$

$$\begin{pmatrix}
\zeta_q & \zeta_{q-1} & \zeta_{q-2} & \cdots & \zeta_0 & & & \\
0 & q\zeta_q & (q-1)\zeta_{q-1} & \cdots & \zeta_1 & & & \\
 & \zeta_q & \zeta_{q-1} & \cdots & \zeta_1 & \zeta_0 & & \\
 & 0 & q\zeta_q & \cdots & 2\zeta_2 & \zeta_1 & & \\
 & & & \vdots & \vdots & & & \\
 & & & \zeta_q & \zeta_{q-1} & \zeta_{q-2} & \cdots & \zeta_0 \\
 & & & 0 & q\zeta_q & (q-1)\zeta_{q-1} & \cdots & \zeta_1
\end{pmatrix}$$

is the *discrimination matrix* of $\mathcal{Q}(z)$.

The determinant of the submatrix of M consisting of the first 2α rows and the first 2α columns $(\alpha = 1, \ldots, q)$ is represented by D_α. We call the *discriminant sequence* of $\mathcal{Q}(z)$ as $D = [D_1, \ldots, D_q]$.

In light of the signs of $D_1, \ldots, D_q$, we can obtain the sign list $[\text{sign}(D_1), \ldots, \text{sign}(D_q)]$ (sign(x)= 1(0)[-1] if $x > 0 (= 0)[< 0]$) and then the revised sign list for $\mathcal{Q}(z)$ (see [31] for more details).

Without loss of generality, let $\mathcal{F}(W)$ (3.2) be a qth-order polynomial

$$\mathcal{F}(W) = \psi_q W^q + \psi_{q-1} W^{q-1} + \cdots + \psi_0, \psi_q \neq 0, \tag{3.7}$$

where $\psi_q, \psi_{q-1}, \ldots, \psi_0$ are coefficients. These coefficients are allowed to be functions of the free system parameters denoted by the vector X.

As for the delayed Hopfield NN model where $\deg(a_0(\lambda)) = 3$, the sign list of $\mathcal{F}(W)$ is [sign(D_1),sign(D_2),sign(D_3)]. If sign(D_2) = 0, the revised sign list is [sign(D_1), -1, sign(D_3)]. Otherwise, the revised sign list is exactly the sign list.

As for the delayed BAM NN model where $\deg(a_0(\lambda)) = 4$, the sign list of $\mathcal{F}(W)$ is [sign(D_1),sign(D_2),sign(D_3),sign(D_4)]. If sign(D_i) = 0, $i = 2, 3$, the revised sign list is listed in Table 3.1. Otherwise, the revised sign list is exactly the sign list.

sign(D_i) = 0	The revised sign list
sign(D_2) = 0	[sign(D_1), -1, sign(D_3), sign(D_4)]
sign(D_3) = 0	[sign(D_1), sign(D_2), -1, sign(D_4)]
sign(D_2) = 0 and sign(D_3) = 0	[sign(D_1), -1, -1, sign(D_4)]

Table 3.1: The revised sign list with sign(D_i) = 0, $i = 2, 3$ for the delayed BAM NN model.

As for the delayed annular NN model where $\deg(a_0(\lambda)) = 5$, the sign list of $\mathcal{F}(W)$ is [sign(D_1),sign(D_2),sign(D_3),sign(D_4),sign(D_5)]. If sign(D_i) = 0, $i = 2, 3, 4$, the revised sign list is listed in Table 3.2. Otherwise, the revised sign list is exactly the sign list.

sign(D_i) = 0	The revised sign list
sign(D_2) = 0	[sign(D_1), -1, sign(D_3), sign(D_4), sign(D_5)]
sign(D_3) = 0	[sign(D_1), sign(D_2), -1, sign(D_4), sign(D_5)]
sign(D_4) = 0	[sign(D_1), sign(D_2), sign(D_3), -1, sign(D_5)]
sign(D_2) = 0 and sign(D_3) = 0	[sign(D_1), -1, -1, sign(D_4), sign(D_5)]
sign(D_2) = 0 and sign(D_4) = 0	[sign(D_1), -1, sign(D_3), -1, sign(D_5)]
sign(D_3) = 0 and sign(D_4) = 0	[sign(D_1), sign(D_2), -1, -1, sign(D_5)]
sign(D_2) = 0, sign(D_3) = 0, and sign(D_4) = 0	[sign(D_1), -1, -1, 1, sign(D_5)]

Table 3.2: The revised sign list with sign(D_i) = 0, $i = 2, 3$ for the delayed BAM NN model.

Proposition 3.10 ([43]). *Suppose that the revised sign list of $\mathcal{F}(W)$ has l non-vanishing members and the number of sign changes is v. It follows that the number of distinct real roots of $\mathcal{F}(W)$ is $l - 2v$.*

For example, if the revised sign list for $\mathcal{F}(W)$ is $[1,1,-1]$, then $v=1$, $l=3$, and $\mathcal{F}(W)$ has $l-2v=1$ real W root. If the revised sign list for $\mathcal{F}(W)$ is $[1,1,1,0]$, then $v=0$, $l=3$, and F(W) has $l-2v=3$ distinct real W roots. If the revised sign list for $\mathcal{F}(W)$ is $[1,-1,-1,1,0]$, then $v=2$, $l=4$, and $\mathcal{F}(W)$ has $l-2v=0$ real W root.

Applying the discrimination system to $\mathcal{F}(W)$ (3.7), one can obtain the discriminant sequence $D=[D_1,\ldots,D_q]$ and then the CRC of real W roots.

As earlier mentioned, we are only concerned with the positive real W roots (since $W=\omega^2$), i.e., the effective W roots. We will present a method to examine the CRC of effective W roots.

Theorem 3.11. *For the auxiliary characteristic function $\mathcal{F}(W)$ (3.7), consider a region partitioned in view of the CRC of real W roots. If the set $\psi_0=0$ separates this region into some subregions in X space, then the positive real W root classification keeps unchanged in each subregion. Otherwise, in the whole region, the positive real root classification remains unchanged.*

Proof of Theorem 3.11. For $\mathcal{F}(W)$, there exists a root $W=0$ iff $\psi_0=0$. If a point in X space continuously moves in a region partitioned in view of the CRC of real W roots, all the real W roots' signs remain unchanged if it does not cross the set $\psi_0=0$. □

For each region partitioned in accordance with the CRC of real W roots, we can obtain the effective W root classifications. According to Theorem 3.11, for each region not separated (or subregion separated) by the set $\psi_0=0$, we can select any point to solve $\mathcal{F}(W)=0$ and this point represents the effective W root classification for all the points in the region (subregion).

We follow the notations adopted in [43]. For example: "$\{1,1,1,1\}$" stands for four different simple real W roots; "$\{2,1,1\}$" stands for one double real W root plus two simple real W roots; "$\{\ \}$" stands for no real W root. Furthermore, the effective W root classification is represented by $\{\cdot,\ldots,\cdot\}^+$. For example, if a real root classification $\{2,1,1\}$ involves one double *negative* real W root and two different simple *positive* real W roots, it is represented by $\{1,1\}^+$ (two different simple effective W roots).

Example 3.12. Consider the delayed BAM NN of Example 3.9. Here, we choose h_{12} and h_{21} as free system parameters, i.e., $X=(h_{12},h_{21})$. For simplicity, we study the case $(h_{12},h_{21})\in[1,6]\times[-6,-1]$.

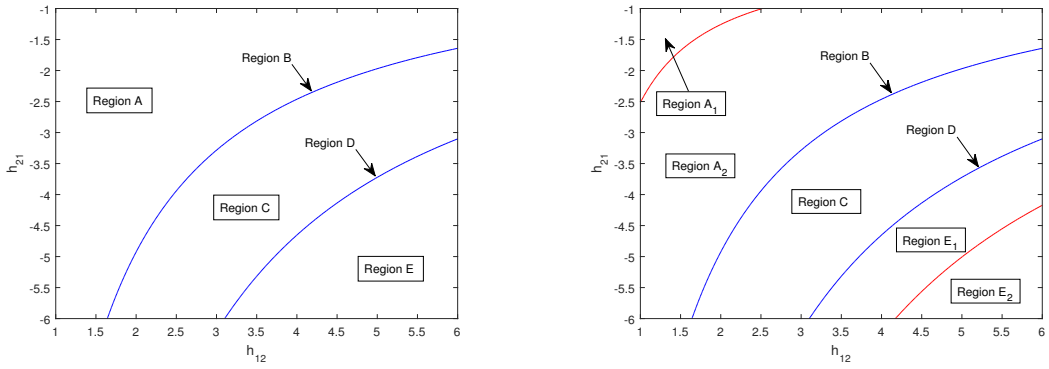
First, according to Lemma 3.1, we obtain $\mathcal{F}(W)=W^4+28.6674W^3+(172.4842-(h_{12}h_{21}+2.7666)^2)W^2+((1.6555h_{12}h_{21}+22.8053)(h_{12}h_{21}+2.7666)-(1.8230h_{12}h_{21}+15.1537)^2+229.9256)W+86.8611-(0.8278h_{12}h_{21}+11.4027)^2$.

Then, applying the discrimination system, the discriminant sequence of $\mathcal{F}(W)$ is $[D_1,D_2,D_3,D_4]$, where $D_1=4$, $D_2=8h_{12}^2h_{21}^2+44.2653h_{12}h_{21}+1146.8204$, $D_3=8h_{12}^6h_{21}^6+132.7960h_{12}^5h_{21}^5-316.9872h_{12}^4h_{21}^4+2920.3114h_{12}^3h_{21}^3+114921.2317h_{12}^2h_{21}^2+801885.8568h_{12}h_{21}+8309682.1306$, and $D_4=0.1632h_{12}^{10}h_{21}^{10}+11.7962h_{12}^9h_{21}^9+1602.5531h_{12}^8h_{21}^8+46407.3671h_{12}^7h_{21}^7+418378.3143h_{12}^6h_{21}^6+2094423.4796h_{12}^5h_{21}^5+46116871.4106h_{12}^4h_{21}^4+738951226.2778h_{12}^3h_{21}^3+6714650368.4108h_{12}^2h_{21}^2+40072389757.5584h_{12}h_{21}+111789468423.0607$.

For this example, $D_1>0$, $D_2>0$ (one may easily prove it), D_3 and D_4 may represent different signs w.r.t. $X=(h_{12},h_{21})$. We can obtain the CRC of real W roots and thereby partition the selected domain into five regions, as shown in Fig. 3.4 (a).

Region A with $\{1,1,1,1\}$: $D_3>0\cap D_4>0$; Region B with $\{2,1,1\}$: $D_3>0\cap D_4=0$; Region C with $\{1,1\}$: $(D_3>0\cap D_4<0)\cup(D_3=0\cap D_4<0)\cup(D_3<0\cap D_4<0)$; Region D with $\{2,1,1\}$: $D_3>0\cap D_4=0$; Region E with $\{1,1,1,1\}$: $D_3>0\cap D_4>0$.

Second, the set $\psi_0=0$, i.e., the set $86.8611-(0.8278h_{12}h_{21}+11.4027)^2=0$, corresponds to the red curves in Fig. 3.4 (b).



(a) Real W root classifications (b) Effective W root classifications

Figure 3.4: CRCs of real and effective W roots.

Region A is separated by the set $\psi_0 = 0$ into subregions A_1 and A_2, and Region E is separated by the set $\psi_0 = 0$ into subregions E_1 and E_2.

In accordance with Theorem 3.11, we have the following results: Region A_1: $\{1,1,1,1\} \to \{1\}^+$; Region A_2: $\{1,1,1,1\} \to \{\ \}^+$; Region B: $\{2,1,1\} \to \{\ \}^+$; Region C: $\{1,1\} \to \{\ \}^+$; Region D: $\{2,1,1\} \to \{2\}^+$; Region E_1: $\{1,1,1,1\} \to \{1,1\}^+$; Region E_2: $\{1,1,1,1\} \to \{1\}^+$.

Furthermore, we can combine Regions A_2, B, and C into one region, labeled by Region $B^{'}$. Consequently, we may partition the domain into five regions: Regions A_1, $B^{'}$, D, E_1, and E_2. Region D is the boundary between Regions $B^{'}$ and E_1, whose analytic condition is $D_3 > 0 \cap D_4 = 0$.

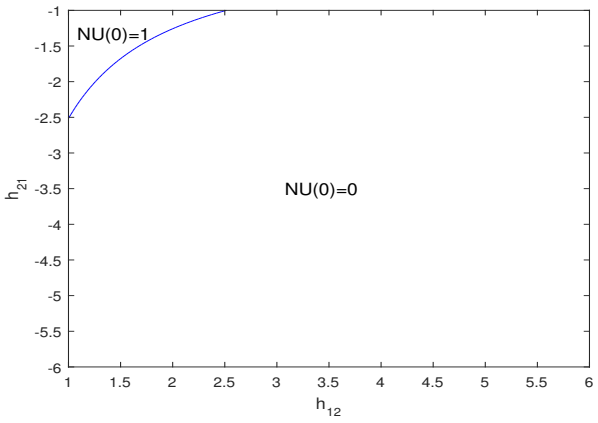


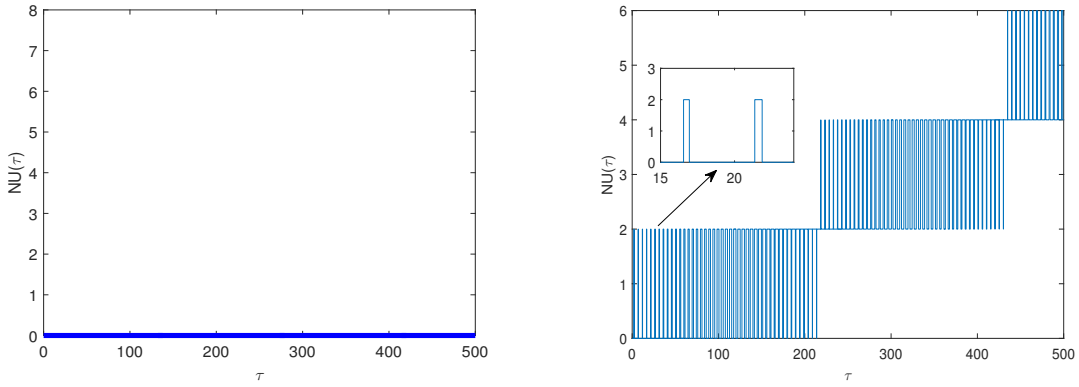
Figure 3.5: $NU(0)$ distribution w.r.t. (h_{12}, h_{21}) for Example 3.12.

Next, based on the equation $f(\lambda, 0) = 0$, $NU(0)$ distribution w.r.t. (h_{12}, h_{21}) is obtained (see Fig. 3.5).

According to the above analysis, we can obtain the following results:

(1) In Region A_1 $(NU(0) = 1)$: As there does not exist an effective W root, the system is not asymptotically stable for any $\tau \in [0, +\infty)$.

(2) In Region $B^{'}$ $(NU(0) = 0)$: As there does not exist an effective W root, the system is delay-independently stable.



(a) $h_{21} = -4.5396$ (b) $h_{21} = -4.5397$

Figure 3.6: $NU(\tau)$ vs. τ for Example 3.12.

(3) On Region D $(NU(0) = 0)$: As there exists a double effective W root, the system is asymptotically stable for all $\tau \geq 0$ except at the CDs.

(4) In Region E_1 $(NU(0) = 0)$: As there exist two simple effective W roots, the system may contain more than one stability τ-interval including $\tau = 0$.

(5) In Region E_2 $(NU(0) = 0)$: As there exists one simple effective W root, the system has only one stability τ-interval of the form $[0, \overline{\tau}_1)$.

For illustrations, we choose a point $(h_{12} = 4.1, h_{21} = -4.5396)$ in Region B' and a point $(h_{12} = 4.1, h_{21} = -4.5397)$ in Region E_1. The corresponding "$NU(\tau)$ vs. τ" plots are shown in Fig. 3.6 (a) and Fig. 3.6 (b), respectively. We can see that the $NU(\tau)$ distribution undergoes a structural variation due to a very small change of h_{21}.

Remark 3.13. The analytic condition for the boundary of different effective W root classifications is available. As the $NU(\tau)$ distribution does not have a structural variation inside a region with identical effective W root classification, we may study the $NU(\tau)$ distribution by employing the parameter-sweeping technique. It is worth noting that such a technique is not for the qualitative test, and hence it is not necessary to set a very fine grid in practice.

3.3 Systematic approach for determining stability set in (X, τ)-space

Based on Theorems 3.8 and 3.11, we now propose a systematic approach to investigate the stability of delayed NNs in the (X, τ)-space, consisting of the following steps.

Step 0: Linearize the delayed NN under consideration at the equilibrium and calculate the characteristic function $f(\lambda, \tau)$. Next, obtain the analytic expression of auxiliary characteristic function $\mathcal{F}(W)$.

Step 1: Obtain the CRC of real W roots in the entire X space with the aid of discrimination system.

Step 2: Determine all possible effective W root classifications in accordance with Theorem 3.11 in the whole X space.

Step 3: Obtain the $NU(0)$ distribution in the X space.

Step 4: Scan the X space. For each point, investigate the complete stability problem w.r.t. τ using Theorem 3.8.

With the steps above, we are able to determine the stability set in the entire (X,τ)-space. Moreover, we can characterize all classifications of stability property in the entire (X,τ)-space.

4 Illustrative examples

In this section, we will give some examples with different NN architectures to illustrate the proposed approach. We adopt the widely-used hyperbolic tangent functions as the activation functions, i.e., $f_i(\cdot)=\tanh(\cdot)$, $g_i(\cdot)=\tanh(\cdot)$, $i=1,2,\ldots,n$. Then, $f_i(0)=g_i(0)=0$ and $f_i'(0)=g_i'(0)=1$, $i=1,2,\ldots,n$.

We will show that an NN with different coefficients can exhibit different classifications of stability property. It is worth mentioning that a very small change of the vector X may make the $NU(\tau)$ distribution have a structural variation, which may alter the classification of stability property. This phenomenon demonstrates the necessity of dividing the parameter space.

Example 4.1. Consider the delayed Hopfield NN (2.2). With the activation functions adopted in this section, it is true that $h_{ij}=c_{ij}$. Here, we choose the coefficients as: $\mu_1=1.4147, \mu_2=1.0102, \mu_3=0.4610, h_{12}=1.5176, h_{23}=-1.4432, h_{32}=0.9080$. We let $X=(h_{21})$.

Step 0: The characteristic function is given by (2.3). In light of Lemma 3.1, we have $\mathcal{F}(W)=W^3+3.2344W^2+(-2.3031h_{21}^2+3.9774h_{21}+0.9674)W-0.4895h_{21}^2+2.5940h_{21}-3.0027$.

Step 1: The discriminant sequence is $[D_1,D_2,D_3]$, where $D_1=3$, $D_2=13.8187h_{21}^2-23.8644h_{21}+15.1182$, and $D_3=48.8657h_{21}^6-253.1688h_{21}^5+490.2890h_{21}^4-623.2539h_{21}^3+742.0196h_{21}^2-443.8572h_{21}+0.0105$.

For this example, $D_1>0$, $D_2>0$ (one may easily prove it), and D_3 may represent different signs w.r.t. h_{21}. We can obtain the CRC of real W roots: (1) {1,1,1} iff $h_{21}\in(-\infty,0.00002373)\cup(1.265628,1.766693)\cup(2.341787,+\infty)$ (where $D_3>0$); (2) {1} iff $h_{21}\in(0.00002373,1.265628)\cup(1.766693,2.341787)$ (where $D_3<0$); (3) {2,1} iff $h_{21}\in\{0.00002373,1.265628,1.766693,2.341787\}$ (where $D_3=0$).

Step 2: When $\psi_0=0$, i.e., $-0.4895h_{21}^2+2.5940h_{21}-3.0027=0$, we can calculate that $h_{21}=1.708129$ or $h_{21}=3.591537$. Hence, the intervals $(1.265628,1.766693)$ and $(2.341787,+\infty)$ are respectively separated into $(1.265628,1.708129)$, $(1.708129,1.766693)$, $(2.341787,3.591537)$, and $(3.591537,+\infty)$.

Furthermore, in light of Theorem 3.11, we can obtain the CRC of effective W roots:

(1) $h_{21}\in(-\infty,1.708129)\cup(3.591537,+\infty)$, $\{1\}^+$;

(2) $h_{21}\in(1.708129,2.341787)$, $\{\ \}^+$;

(3) $h_{21}=\{2.341787\}$, $\{2\}^+$;

(4) $h_{21}\in(2.341787,3.591537)$, $\{1,1\}^+$.

Step 3: We analyze the $NU(0)$ distribution w.r.t. h_{21} by solving the characteristic equation $f(\lambda,0)=0$. We have that when $h_{21}\in(-\infty,2.342233)$, $NU(0)=0$.

In view of the above analysis, we have the following results:

(1) The system contains only 1 stability τ-interval of the form $[0,\overline{\tau}_1)$ iff $h_{21}\in(-\infty,1.708129)$;

(2) The system is delay-independently stable iff $h_{21} \in (1.708129, 2.341787)$;

(3) The system is asymptotically stable for all $\tau \geq 0$ except at the CDs iff $h_{21} = 2.341787$;

(4) The system may contain more than one stability τ-interval including $\tau = 0$ iff $h_{21} \in (2.341787, 2.342233)$;

(5) The system may contain stability τ-interval(s) excluding $\tau = 0$ iff $h_{21} \in (2.342233, 3.591537)$;

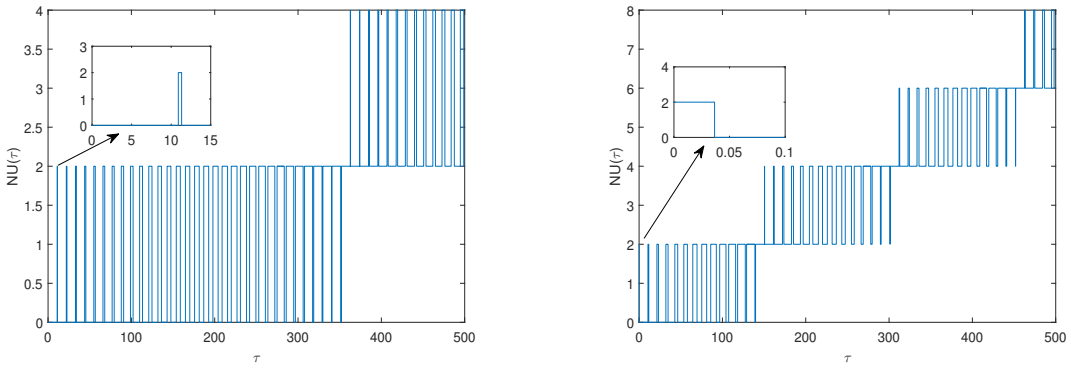
(6) The system has no stability τ-interval iff $h_{21} \in (3.591537, +\infty)$.

It can be seen that all classifications of stability property can be characterized in the whole (h_{21},τ)-plane. To illustrate the above results and to show the qualitative change of stability property, we next choose some values of h_{21} and give the detailed results.

At point $h_{21} = 1.70$, the system contains only 1 stability τ-interval $[0, 32.1647)$. At point $h_{21} = 1.71$, the system is delay-independently stable.

At point $h_{21} = 2.342$, the system has 32 stability τ-intervals including $\tau = 0$ and the generalized delay margin is 351.8083. At point $h_{21} = 2.343$, the system has 13 stability τ-intervals excluding $\tau = 0$ and the generalized delay margin is 139.7930. To intuitively show the difference, we give "$NU(\tau)$ vs. τ" plots in Fig. 4.1 (a) and Fig. 4.1 (b), respectively.

At point $h_{21} = 3.59$, the system contains only 1 stability τ-interval $(2.4170, 2.9154)$. At point $h_{21} = 3.60$, no stability τ-interval exists.



(a) $h_{21} = 2.342$ (b) $h_{21} = 2.343$

Figure 4.1: $NU(\tau)$ vs. τ for Example 4.1.

Step 4: We can address the whole (h_{21},τ)-plane and exhaustively determine the stability set. For a clear demonstration, we provide the stability set with $(h_{21}, \tau) \in [-2, 5] \times [0, 50]$ in Fig. 4.2. □

Example 4.2. For the delayed BAM NN (2.5) in Example 3.12, we here choose a domain $(h_{12}, h_{21}) \in [4.1, 4.7] \times [-4.9, -4.57]$ for a clear illustration. Scan the (h_{12}, h_{21})-plane, and for each point we calculate the stability set by using Theorem 3.8. Then we can exhaustively determine the stability set, as shown in Fig. 4.3.

It is interesting to see in Fig. 4.3 that the stability parameter space of the delayed BAM NN has multiple disjoint parts.

Using the approach proposed in Subsection 3.3, we can also obtain the stability set in the corresponding 4-D parameter space, as reflected in Fig. 4.4. We here choose a domain

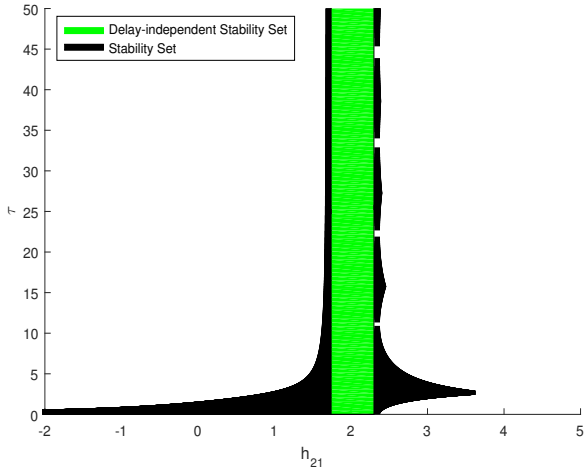


Figure 4.2: Stability set in the (h_{21}, τ)-plane for Example 4.1.

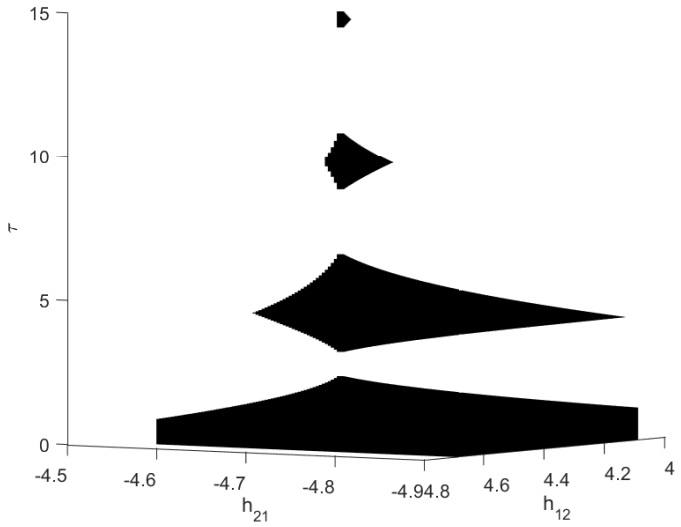


Figure 4.3: Stability set in the (h_{12}, h_{21}, τ)-space for Example 4.2.

$(h_{12}, h_{13}, h_{14}) \in [4.1, 4.2] \times [-0.38, -0.36] \times [2.2, 2.4]$ and the color information represents the generalized delay margin. Here, the maximum generalized delay margin is 14.5923. □

Example 4.3. Consider the delayed annular NN (2.9) with coefficients as in [8] and [46]: $\mu_i = 2, \alpha_i = 1, i = 1, \ldots, 5$, $\beta_1 = 2, \beta_2 = 1, \beta_3 = 1, \beta_4 = 1.3$, and $\beta_5 = -0.5$.

With the activation functions adopted in this section, it is true that $s_i = \alpha_i$ and $h_i = \beta_i$. Here, we let $X = (s_1)$.

Step 0: The characteristic function is given by (2.11). In light of Lemma 3.1, we have that $\mathcal{F}(W) = W^5 + (s_1^2 - 4s_1 + 8)W^4 + (4s_1^2 - 16s_1 + 22)W^3 + (6s_1^2 - 24s_1 + 28)W^2 + (4s_1^2 - 16s_1 + 17)W + s_1^2 - 4s_1 + 2.31$.

Step 1: The discriminant sequence is $[D_1, D_2, D_3, D_4, D_5]$, where $D_1 = 5$, $D_2 = 4s_1^4 - 32s_1^3 + 88s_1^2 - 96s_1 + 36$, $D_3 = 0$, $D_4 = -285.61s_1^4 + 2284.88s_1^3 - 6283.42s_1^2 + 6854.64s_1 - 2570.49$, and $D_5 = -1235.6631s_1^{10} + 24713.2621s_1^9 - 216241.0432s_1^8 + 1087383.5315s_1^7 - 3472213.3222s_1^6 +$

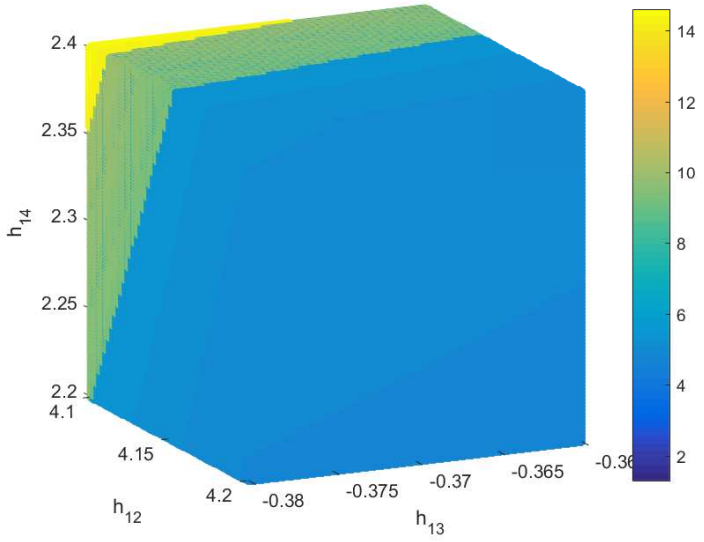


Figure 4.4: Generalized delay margin w.r.t. (h_{12}, h_{13}, h_{14}).

$7344781.4902s_1^5 - 10416639.9667s_1^4 + 9786451.7837s_1^3 - 5838508.1664s_1^2 + 2001774.2285s_1 - 274774.5492$.

For this example, $D_1 > 0$, $D_2 \geq 0$, $D_3 = 0$, $D_4 \leq 0$ (one may easily prove the properties concerning the signs of D_2 and D_4), and D_5 may represent different signs w.r.t. s_1. We can obtain the CRC of real W roots: (1) {1,1,1} iff $s_1 \in (-\infty, 0.3172) \cup (3.6828, +\infty)$ (in the case $D_1 > 0, D_2 > 0, D_3 = 0, D_4 < 0, D_5 < 0$); (2) {1} iff $s_1 \in (0.3172, 3.6828)$ (in the case $D_1 > 0, D_2 > 0, D_3 = 0, D_4 < 0, D_5 > 0$ or $D_1 > 0, D_2 = 0, D_3 = 0, D_4 = 0, D_5 > 0$); (3) {2,1} iff $s_1 \in \{0.3172, 3.6828\}$ (in the case $D_1 > 0, D_2 > 0, D_3 = 0, D_4 < 0, D_5 = 0$).

Step 2: When $\psi_0 = 0$, i.e., $s_1^2 - 4s_1 + 2.31 = 0$, we can calculate that $s_1 = 0.7$ or $s_1 = 3.3$. Hence, the interval $(0.3172, 3.6828)$ is separated into $(0.3172, 0.7)$, $(0.7, 3.3)$, and $(3.3, 3.6828)$.

Furthermore, in light of Theorem 3.11, we can obtain the CRC of effective W roots:

(1) $s_1 \in (0.7, 3.3)$, $\{1\}^+$;

(2) $s_1 \in (-\infty, 0.7) \cup (3.3, +\infty)$, $\{\}^+$.

Step 3: We analyze the $NU(\tau)$ distribution w.r.t. s_1 through solving the characteristic equation $f(\lambda, 0) = 0$. We have that when $s_1 \in (-\infty, 1.6138356)$, $NU(0) = 0$.

In view of the above analysis, we have that there are three classifications of stability property as described below:

(1) The system is delay-independently stable iff $s_1 \in (-\infty, 0.7)$;

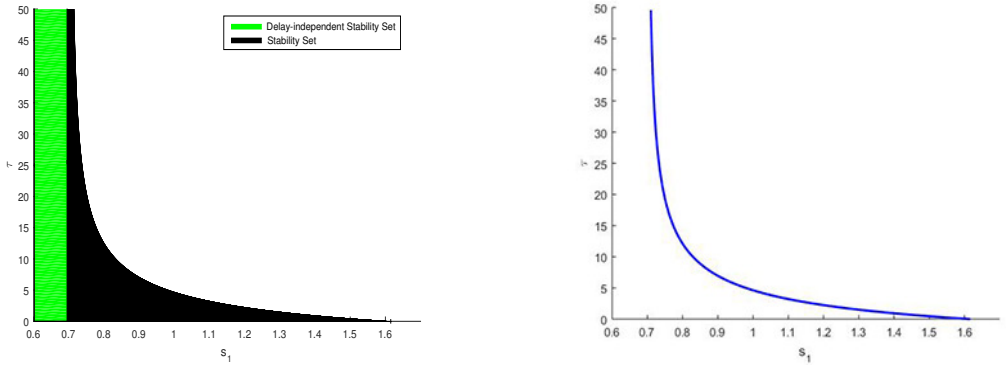
(2) The system contains only one stability τ-interval of the form $[0, \overline{\tau}_1)$ iff $s_1 \in (0.7, 1.6138356)$;

(3) The system has no stability τ-interval iff $s_1 \in (1.6138356, +\infty)$.

Step 4: We can study the whole (s_1,τ)-plane and exhaustively determine the stability set. For a clear demonstration, we provide the stability set with $(s_1, \tau) \in [0.6, 1.7] \times [0, 50]$ in Fig. 4.5 (a).

It can be seen that the system has one stability τ-interval iff $s_1 \in (0.7, 1.6138356)$. The boundary of the corresponding stability region is shown in Fig. 4.5 (b).

To illustrate our analysis, we choose some values of s_1 and analyze the stability of the system. We list the details in Table 4.1. □



(a) Stability set (b) Boundary of stability set

Figure 4.5: Stability set and the boundary curve for Example 4.3.

s_1	stability τ-set	s_1	stability τ-set	s_1	stability τ-set
0.69	$[0,+\infty)$	0.99	$[0,4.7956)$	1.61	$[0,0.0139)$
0.71	$[0,49.4106)$	1	$[0,4.6174)$	1.62	ϕ
0.72	$[0,33.4746)$	1.01	$[0,4.4477)$	1.63	ϕ

Table 4.1: Stability τ-set for Example 4.3.

5 Conclusion and future work

In this paper, we considered the stability of delayed neural networks (NNs) with a free delay parameter τ and a free system parameter vector X. As far as we know, the complete stability problem of delayed NNs involving heterogeneous free parameters has not been well investigated.

We proposed a systematic method to investigate the stability in the (X,τ)-space. As a consequence, we can exhaustively determine the stability set in the whole (X,τ)-space. The effectiveness of the approach is illustrated by some numerical examples. It is interesting to see that the stability parameter space of a delayed NN may have multiple disjoint parts.

Based on the research of this paper, we may further investigate more general delayed NNs. In this paper, we focus on the case where the characteristic functions are in the form $f(\lambda,\tau) = a_0(\lambda) + a_1(\lambda)e^{-\tau\lambda}$. A more general form is $f(\lambda,\tau) = a_0(\lambda) + a_1(\lambda)e^{-\tau\lambda} + \cdots + a_q(\lambda)e^{-q\tau\lambda}$, where $a_0(\lambda),\ldots,a_q(\lambda)$, $q \in \mathbb{N}_+$, are polynomials. The stability analysis when $q > 1$ will be much more complicated. In the future, we would extend the approach to such a case.

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Smooth linearization of contractive random dynamical systems in continuous time

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Abstract. We establish that uniformly exponentially stable random dynamical systems on the half line have equivalent dynamics through a C^m-conjugacy. This result was obtained for random differential equations as well as for random dynamical systems with a uniformly exponentially stable linear part.

Keywords: random dynamical systems, topological equivalence, smooth equivalence, Carathéodory differential equations, stochastic differential equations.

2020 Mathematics Subject Classification: 37H05, 37H15, 37C15, 37D05.

1 Introduction

The Hartman–Grobman theorem is a fundamental result in the local theory of dynamical systems. It establishes a connection between the qualitative behavior of a nonlinear system and its linearization around an equilibrium point. The classical theorem states that if a nonlinear system possesses a hyperbolic equilibrium point, then the qualitative behavior of the system near the equilibrium point is topologically equivalent to that of its linearization.

The original theorem was obtained by Philip Hartman [20, 21] and David Grobman [18] independently in the 1960s. It has since become a cornerstone of dynamical systems theory. This theorem provides a powerful tool for analyzing, e.g., stability of nonlinear systems by reducing the problem to the study of linear systems.

It should be mentioned that the standard framework in Hartman–Grobman theory relies on a fundamental assumption about the hyperbolicity of the linearized system. However, when eigenvalues lie on the imaginary axis, the dynamics can undergo significant changes. For this reason, the original paper on the linearization theorem has inspired pioneering work and extensive research, leading to numerous generalizations and extensions.

Speaking about generalizations, it is important to mention results for autonomous systems in Banach spaces [32,35], linearization for nonautonomous problems [28,33,34] and results for Carathéodory-type differential equations in Banach spaces [3].

In recent years, there has been growing interest in extending the Hartman–Grobman theorem to the case of random dynamical systems. These systems incorporate the effects of

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stochasticity or randomness into the evolution of dynamical systems (see, e.g., [1,14,17]), and are particularly relevant in modeling real-world phenomena where uncertainties or random perturbations play a significant role. The generalization of the Hartman–Grobman theorem to random dynamical systems aims to establish similar qualitative connections between the behavior of a nonlinear system and its linearization in the presence of random perturbations. The goal is to understand how the stability properties, attractors, and bifurcations of the deterministic system are affected by the stochastic components. Accordingly, extensions of the Hartman–Grobman theorem to random dynamical systems go back to the works by Arnold [1] resulting from Wanner [44]. As in the deterministic case, there are many generalizations of the random linearization theorem, for instance, in [25,26,47], and others.

In linearization theory, a compelling research question revolves around investigating the smoothness properties of the linearization map, since this allows a better understanding of the information that is carried from one system to another. Smooth linearization ensures that invariant manifolds and their geometric structures, such as stable and unstable tangent directions, are consistently preserved. This is important for accurately capturing the flow near fixed points, periodic orbits, or other invariant sets, allowing to study complex dynamics in a simplified framework. This research problem is widely explored in deterministic settings. For example, in [7,37,39] Hölder properties of linearization homeomorphisms were established. The C^1-linearization result for contractions has been obtained in [5,38,46]. Significant contributions in the higher order smoothness within the autonomous context are the results of Sternberg [42,43], who proved that under certain non-resonance conditions the local conjugation between two systems can actually be a C^m map. Later, Belickii [8,9] achieved the same result under assumptions related to the spectrum of the diffeomorphism that defines the dynamics.

The authors in [11] delved into the differentiability properties within a particular case of Palmer's linearization theorem [33], particularly when restricting the problem to the nonnegative real line. Within this framework, sufficient conditions ensuring that the conjugating map is a C^2-preserving orientation diffeomorphism were obtained. These conditions were established under the premise that the linear system was uniformly asymptotically stable. Recent generalizations and improvements of this result were presented in [12].

Like in deterministic scenarios, the smoothness properties of the conjugating map in random dynamical systems were equally intriguing. At this point, it is worth mentioning the paper [4], where conditions for Hölder continuity of the conjugacies on bounded subsets for infinite-dimensional random dynamical systems with a non-invertible linear part were established.

In the paper [6], the Hölder continuity of conjugacy for both maps and flows was established. Notably, the analysis demonstrated that such continuity could be achieved without imposing any nonresonance condition. In [27], a proof of a $C^{1,\beta}$ generalization of the random Hartman–Grobman theorem was presented. Additionally, temperedness for the conjugacy was obtained, further enhancing the understanding of the relationship between random dynamical systems and their associated linearizations.

In [36], stochastic differential equations (SDEs) with non-Gaussian Lévy processes for two-sided time were studied. Topological equivalence between these SDEs and transformed random differential equations (RDEs) with Lévy processes was established. This allowed for a stochastic Hartman–Grobman theorem for the linearization of SDEs to be proved.

The aim of the present paper is to derive sufficient conditions for C^m-linearizations of random dynamical systems on the nonnegative half-line. The advantage of considering the

positive half-line is that no information about the past is required, allowing the focus to be solely on future dynamics. Specifically, the main contribution of the paper can be considered a generalization of the approach proposed in [12] in several directions.

Firstly, in the class of Carathéodory differential equations. Carathéodory-type models include, for instance, random differential equations [1,14] where the time dependence is often irregular and non-smooth, as well as control systems [16] with control functions that are merely measurable in time and not necessarily smooth. The generality of the Carathéodory framework makes it particularly well suited for modeling real-world systems with time-irregular behavior. Within this setting, we not only derived sufficient conditions for the topological equivalence of the nonlinear equation and its linearization, but also provided additional information about the conjugating map. Specifically, we established that the conjugating map is globally Lipschitz and belongs to the class C^m. These results hold for arbitrary (possibly infinite-dimensional) Banach spaces.

Secondly, in the class of random dynamical systems generated by random differential equations, we obtained both global and local C^m-linearization results. These results were derived under the assumption that the underlying Banach space is finite-dimensional and provide the foundation to extend our analysis to the class of random dynamical systems generated by stochastic differential equations (SDEs). To achieve smooth linearization results for random dynamical systems generated by SDEs, we leveraged the connection between random differential equations (RDEs) and SDEs established in [22].

The paper is arranged as follows. In the remaining part of this section, we introduce the basic notations and definitions needed throughout the paper. Section 2 focuses on semilinear Carathéodory equations, where we establish the smooth topological equivalence results, including sufficient conditions for C^m-linearizations and detailed properties of the conjugating map. In Section 3, we apply these results to the setting of random dynamical systems. Specifically, we analyze systems generated by random differential equations (RDEs) and stochastic differential equations (SDEs), leveraging the connection between RDEs and SDEs to derive smooth C^m-linearization results for the corresponding random dynamical systems. The Appendix contains some key theorems used in proving our results.

1.1 Preliminaries

Throughout the paper, let $I \subseteq \mathbb{R}$ be an interval unbounded from above, X be a Banach space equipped with a norm $\|\cdot\|$, $L(X)$ is the space of linear operator on X, and $(\Omega, \mathfrak{F})$ be a measurable space. Moreover, let $\{\|\cdot\|_{t,\omega}\}_{t\in I,\,\omega\in\Omega}$ denote a family of norms on X such that

(i) the mapping $(t,\omega,x) \mapsto \|x\|_{t,\omega}$ is strongly measurable,

(ii) there exists a measurable mapping $\ell : I \times \Omega \to (0,\infty)$ such that $\ell(\cdot,\omega)$ is locally bounded and

$$\frac{1}{\ell(t,\omega)}\,\|x\| \le \|x\|_{t,\omega} \le \ell(t,\omega)\,\|x\| \quad \text{for all } t \in I,\ \omega \in \Omega,\ x \in X. \tag{1.1}$$

Furthermore, for a linear operator $T \in L(X)$ we define

$$\|T\|_{s,t,\omega} := \sup_{\|x\|_{s,\omega}=1} \|Tx\|_{t,\omega} \quad \text{for all } s,t \in I,\ \omega \in \Omega,$$

and obtain $\|Tx\|_{t,\omega} \le \|T\|_{s,t,\omega}\,\|x\|_{s,\omega}$ for all $t,s \in I$ and $x \in X$.

Given this, a function $\mu : I \to X$ is called *bounded* (for $\omega \in \Omega$), if

$$\|\mu\|_\omega := \sup_{t\in I} \|\mu(t)\|_{t,\omega} < \infty.$$

We write $BC_\omega(I, X)$ for the set of all bounded and continuous functions $\mu : I \to X$, which equipped with the norm $\|\cdot\|_\omega$ becomes a Banach space.

We denote $f : I \times X \to Y$ as *Carathéodory function*, Y being another Banach space, provided the following hold:

- $f(\cdot, x) : I \to Y$ is strongly measurable for each $x \in X$ (w.r.t. the Borel σ-algebras on I and X),
- $f(t, \cdot) : X \to Y$ is continuous for almost all $t \in I$.

Let $f : I \times X \times \Omega \to X$ be a mapping such that for every choice of $\omega \in \Omega$ the mapping $f(\cdot, \cdot, \omega) : I \times X \to X$ has the Carathéodory property. Then the equation

$$\dot{x} = f(t, x, \omega) \tag{1.2}$$

is called an *ω-dependent Carathéodory differential equation,* and an absolutely continuous mapping $\phi : J \to X$ is called a *solution* of (1.2) (to the parameter ω), if the mapping $f(\cdot, \phi(\cdot), \omega)$ is locally integrable and the identity

$$\phi(t) = \phi(\tau_0) + \int_{\tau_0}^{t} f(s, \phi(s), \omega)ds \tag{1.3}$$

holds for all $\tau_0, t \in J$, where $J \subseteq I$ is a subinterval. If in addition the mapping ϕ satisfies the condition $\phi(\tau) = \xi$ then we say that ϕ solves the initial value problem for $\tau, \xi \in X$.

The next lemma bridges the integral and differential formulations of solutions to the Carathéodory equation, providing a tool for analyzing these solutions within a functional-analytic framework.

Lemma 1.1. *An absolutely continuous mapping $\phi\colon J \to X$ solves (1.2) if and only if $\dot{\phi}(t) = f(t, \phi(t), \omega)$ for almost all $t \in J$.*

The following theorem presents the results on existence, uniqueness and continuous dependence of solutions on initial values and parameters.

Theorem 1.2 ([2, Thm. 2.4, p. 49], [10])**.** *Let $f : I \times X \times \Omega \to X$ be a mapping such that, for every $\omega \in \Omega$, the mapping $f(\cdot, \cdot, \omega) : I \times X \to X$ satisfies the Carathéodory property. Furthermore, let $l, l_0 : I \to \mathbb{R}_0^+$ be locally integrable functions such that the two estimates*

$$\|f(t, x, \omega) - f(t, \bar{x}, \omega)\| \le l(t)\|x - \bar{x}\|,$$
$$\|f(t, 0, \omega)\| \le l_0(t)$$

hold for almost all $t \in I$ and for all $x, \bar{x} \in X$ and $\omega \in \Omega$.

Then for any $(\tau, \xi, \omega_0) \in I \times X \times \Omega$ the differential equation (1.2) has a unique solution $\phi(\cdot; \tau, \xi, \omega_0) : I \to X$ to the parameter value ω_0, satisfying the initial condition $x(\tau) = \xi$, and the so-defined mapping $\phi(\cdot, \cdot, \cdot, \omega) : I \times I \times X \to X$ is continuous.

The mapping $\varphi : I \times I \times X \times \Omega \to X$ provided by Theorem 1.2 is called a *general solution* of differential equation (1.2). For all $\tau, t_1, t_2 \in I, \xi \in X$ and $\omega \in \Omega$, this mapping satisfies the two-parameter group property

$$\varphi(t_2; \tau, \xi, \omega) = \varphi(t_2; t_1, \varphi(t_1; \tau, \xi, \omega), \omega). \tag{1.4}$$

2 Semilinear Carathéodory equations

In this section, we focus on semilinear Carathéodory differential equations with parameter and investigate their topological equivalence to linear systems. Specifically, we aim to derive sufficient conditions for smooth linearization of class C^m.

Let $I \subseteq \mathbb{R}$ be an interval unbounded from above and $(\Omega, \mathfrak{F})$ be a measurable space, $A_\omega : I \to L(X)$ be locally integrable and $F_\omega : I \times X \to X$ with a given $\omega \in \Omega$.

Consider the system of semilinear Carathéodory differential equations with parameter $\omega \in \Omega$

$$\dot{x} = A_\omega(t)x + F_\omega(t,x). \tag{N}$$

Together with the system (N) consider a linear system

$$\dot{x} = A_\omega(t)x. \tag{L}$$

Lemma 2.1 ([2, Lemma 2.9, p. 56])**.** *Let $\phi_\omega : I \times I \times X \to X$ denote the general solution of the linear equation (L). Then for any $s,t \in I$ the mapping $\Phi_\omega : I \times I \to L(X)$,*

$$\Phi_\omega(t,s) := \phi_\omega(t;s,\cdot),$$

is called the evolution operator of (L) and satisfies the following properties:

(i) each $\Phi_\omega(t,s)$, $s,t \in I$ is an element of the group $GL(X)$ of invertible operators in $L(X)$ and the mapping $\Phi_\omega : I \times I \to GL(X)$ is continuous;

(ii) the identities

$$\begin{aligned} \Phi_\omega(t,t) &= \mathrm{Id}_X, \\ \Phi_\omega(t,s) &= \Phi_\omega(t,r)\Phi_\omega(r,s), \\ \Phi_\omega(t,s)^{-1} &= \Phi_\omega(s,t) \end{aligned}$$

hold for all $r,s,t \in I$.

2.1 Topological linearization

Assume that the following assumptions hold for all $\omega \in \Omega$

(H_1) $A_\omega : I \to L(X)$ is locally integrable and there exist reals $K(\omega) \geq 1$ and $\alpha > 0$ such that

$$\|\Phi_\omega(t,s)\|_{s,t,\omega} \leq K(\omega)e^{-\alpha(t-s)} \quad \text{for all } t,s \in I,\ s \leq t. \tag{2.1}$$

(H_2^0) $F_\omega : I \times X \to X$ is a Carathéodory function and there exist reals $L(\omega), M(\omega) \geq 0$ such that

$$F_\omega(t,0) = 0, \tag{2.2}$$

$$\|F_\omega(t,x)\|_{t,\omega} \leq M(\omega), \tag{2.3}$$

$$\|F_\omega(t,x) - F_\omega(t,\bar{x})\|_{t,\omega} \leq L(\omega)\,\|x - \bar{x}\|_{t,\omega} \tag{2.4}$$

for almost all $t \in I$ and all $x, \bar{x} \in X$.

It is easy to verify that under assumptions (H_1), (H_2^0), the semilinear equation (N) satisfies the assumptions of Theorem 1.2 with $f(t,x,\omega) = A_\omega(t)x + F_\omega(t,x)$, $l(t) := \|A_\omega(t)\| + L(\omega)$ and $l_0(t) = 0$. Then, for every initial condition $x(\tau) = \xi$ and fixed $\omega \in \Omega$ there exists exactly one solution of (N) on I. From now, we denote the general solution of (N) by φ_ω.

Before delving into the main result, let us first introduce some technical lemmas. Throughout the paper, $\tau_0 \in \mathbb{R}$ is fixed and $\tau \in I := [\tau_0, \infty)$.

Lemma 2.2 (the operator $\mathcal{L}_\omega$)**.** *If (H_1) holds, then*

$$\mathcal{L}_\omega : I \times X \to BC_\omega(I,X), \qquad \mathcal{L}_\omega(\tau,\xi) := \Phi_\omega(\cdot,\tau)\xi$$

defines an operator such that $L_\omega(\tau,\cdot)$ is bounded and linear for all $\tau \in I$, $\omega \in \Omega$. Moreover, for all $\omega \in \Omega$ and $\xi \in X$, the following holds:

(i) $(\tau,\xi) \mapsto \mathcal{L}_\omega(\tau,\xi)$ *is continuous,*

(ii) $\mathcal{L}_\omega(\tau,\xi) : I \to X$ *is absolutely continuous.*

Proof. Let $\omega \in \Omega$, $\tau \in I$, and $\xi \in X$. Then $\Phi_\omega(\cdot,\tau)\xi : I \to X$ solves the initial value problem $\dot{x} = A_\omega(t)x$, $x(\tau) = \xi$, and is therefore absolutely continuous.

Without loss of generality, let $\tilde{\tau} \le \tau \le t$. To prove the continuity of the map $(\tau,\xi) \mapsto \mathcal{L}_\omega(\tau,\xi)$, consider the following expression:

$$\begin{aligned}\|\mathcal{L}_\omega(\tau,\xi) - \mathcal{L}_{,\omega}(\tilde{\tau},\xi_0)\|_{t,\omega} &\le \|\Phi_\omega(t,\tau)\xi - \Phi_\omega(t,\tau)\xi_0\|_{t,\omega} + \|\Phi_\omega(t,\tau)\xi_0 - \Phi_\omega(t,\tilde{\tau})\xi_0\|_{t,\omega} \\ &\le K(\omega)e^{-\alpha(t-\tau)}\|\xi - \xi_0\| + \|\Phi_\omega(t,\tau) - \Phi_\omega(t,\tilde{\tau})\|_{t,\omega}\|\xi_0\|.\end{aligned}$$

From the continuous dependence of $\Phi_\omega(t,\tau)$ on the initial time τ, it follows that for any $\varepsilon > 0$, there exists $\delta > 0$ such that if $|\tau - \tilde{\tau}| + \|\xi - \xi_0\| < \delta$, then

$$\|\mathcal{L}_\omega(\tau,\xi) - \mathcal{L}_\omega(\tilde{\tau},\xi_0)\|_{t,\omega} \le K(\omega)\delta + \|\xi_0\|\,\varepsilon \le \tilde{\varepsilon},$$

where $\tilde{\varepsilon}$ can be made arbitrarily small. Therefore, the continuity of $(\tau,\xi) \mapsto \mathcal{L}_\omega(\tau,\xi)$ follows.

Moreover, one has

$$\begin{aligned}\|[\mathcal{L}_\omega(\tau,\xi)](t)\|_{t,\omega} = \|\Phi_\omega(t,\tau)\xi\|_{t,\omega} &\overset{(2.1)}{\le} K(\omega)e^{-\alpha(t-\tau)}\|\xi\|_{\tau,\omega} = K(\omega)e^{-\alpha(t-\tau_0)}e^{\alpha(\tau-\tau_0)}\|\xi\|_{\tau,\omega} \\ &\le K(\omega)e^{\alpha(\tau-\tau_0)}\|\xi\|_{\tau,\omega} \overset{(1.1)}{\le} K(\omega)e^{\alpha(\tau-\tau_0)}\ell(\tau,\omega)\|\xi\| \quad \text{for all } \tau_0 \le t \qquad (2.5)\end{aligned}$$

and passing to the supremum over $t \in I$ yields $\|\mathcal{L}_\omega(\tau,\xi)\|_\omega \le K(\omega)e^{\alpha(\tau-\tau_0)}\ell(\tau,\omega)\|\xi\|$, which is the claim. □

Lemma 2.3 (the operator $\mathcal{F}_\omega$)**.** *If (H_1) and (H_2^0) hold, then the nonlinear operator*

$$\mathcal{F}_\omega : BC_\omega(I,X) \to BC_\omega(I,X), \qquad \mathcal{F}_\omega(\phi) := \int_{\tau_0}^{\cdot} \Phi_\omega(\cdot,s)F_\omega(s,\phi(s))\,ds \qquad (2.6)$$

is well-defined, and, for all $\phi, \bar{\phi} \in BC_\omega(I,X)$, $\omega \in \Omega$ satisfies

$$\|\mathcal{F}_\omega(\phi) - \mathcal{F}_\omega(\bar{\phi})\|_\omega \le \frac{K(\omega)L(\omega)}{\alpha}\|\phi - \bar{\phi}\|_\omega, \qquad (2.7)$$

$$\|\mathcal{F}_\omega(\phi)\|_\omega \le \frac{K(\omega)M(\omega)}{\alpha}. \qquad (2.8)$$

Proof. Let $\omega \in \Omega$ and $\phi, \bar{\phi} \in BC_\omega(I,X)$. Then $\int_{\tau_0}^{\cdot} \Phi_\omega(\cdot,s)F_\omega(s,\phi(s))\,ds$ solves the linearly inhomogeneous initial value problem $\dot{x} = A_\omega(t)x + F_\omega(t,\phi(t))$, $x(\tau_0) = 0$. Moreover,

$$\begin{aligned}\|\mathcal{F}_\omega(\phi)(t)\|_{t,\omega} &\le \int_{\tau_0}^{t} \|\Phi_\omega(t,\tau)\|_{s,t,\omega}\,\|F_\omega(s,\phi(s))\|_{s,\omega}\,ds \overset{(2.1)}{\le} K(\omega)\int_{\tau_0}^{t} e^{-\alpha(t-s)}\,\|F_\omega(s,\phi(s))\|_{s,\omega}\,ds \\ &\overset{(2.3)}{\le} K(\omega)M(\omega)\int_{\tau_0}^{t} e^{-\alpha(t-s)}\,ds \le \frac{K(\omega)M(\omega)}{\alpha} \quad \text{for all } \tau_0 \le t,\end{aligned}$$

and passing to the supremum over $t \in I$ yields $\mathcal{F}_\omega(\phi) \in BC_\omega(I,X)$.

Furthermore,

$$\begin{aligned}\|\mathcal{F}_\omega(\phi)(t) - \mathcal{F}_\omega(\bar{\phi})(t)\|_{t,\omega} &\le \int_{\tau_0}^{t} \|\Phi_\omega(t,\tau)\|_{s,t,\omega}\,\|F_\omega(s,\phi(s)) - F_\omega(s,\bar{\phi}(s))\|_{s,\omega}\,ds \\ &\overset{(2.1)}{\le} K(\omega)\int_{\tau_0}^{t} e^{-\alpha(t-s)}\,\|F_\omega(s,\phi(s)) - F_\omega(s,\bar{\phi}(s))\|_{s,\omega}\,ds \\ &\overset{(2.4)}{\le} K(\omega)L(\omega)\int_{\tau_0}^{t} e^{-\alpha(t-s)}\,\|\phi(s) - \bar{\phi}(s)\|_{s,\omega}\,ds \\ &\le \frac{K(\omega)L(\omega)}{\alpha}\,\|\phi - \bar{\phi}\|_{\omega} \quad \text{for all } \tau_0 \le t.\end{aligned}$$

Passing to the supremum over $t \in I$ yields (2.7) for all $\phi, \bar{\phi} \in BC_\omega(I,X)$. Finally, $\mathcal{F}(\phi)$ is absolutely continuous. □

Theorem 2.4 (topological linearization of (N))**.** *If* (H_1) *and* (H_2^0) *hold with*

$$K(\omega)L(\omega) < \alpha,$$

then there exists a map $H_\omega : I \times X \to X$ *such that, for every* $\omega \in \Omega$, $s,t \in I$*:*

(i) $H_\omega(t,\cdot)$ *is a homeomorphism on* X *with* $H_\omega(t,0) = 0$*;*

(ii) the linear equation (L) *and the semilinear equation* (N) *are conjugated in the sense that*

$$H_\omega(t,\Phi_\omega(t,s)\cdot) = \varphi_\omega(t,s,H_\omega(s,\cdot));$$

(iii) the operators $H_\omega(t,\cdot)$ *and its inverse are near identity, i.e. for all* $\xi \in X$, $\eta \in X$*:*

$$\|H_\omega(t,\xi) - \xi\|_{t,\omega} \le \frac{K(\omega)M(\omega)}{\alpha} \quad \text{and} \quad \left\|H_\omega(t,\cdot)^{-1}(\eta) - \eta\right\|_{t,\omega} \le \frac{K(\omega)M(\omega)}{\alpha};$$

(iv) the operators $H_\omega(t,\cdot) : X \to X$ *and its inverse satisfy a global Lipschitz condition;*

(v) the operators $H_\omega : I \times X \to X$ *and* $(t,x) \mapsto H_\omega^{-1}(t,\cdot)(x)$ *are continuous.*

We denote H_ω *as a (topological) conjugacy between the semilinear equation* (N) *and its linear part* (L)*.*

Proof. For clarity of presentation, we carry out the proof in four steps.

Step 1. We fix $\tau \in I, \omega \in \Omega$. Let us define $\mathcal{T}_\omega : BC_\omega(I,X) \times X \to BC_\omega(I,X)$ as

$$\mathcal{T}_\omega(\phi;\xi,\tau) := \mathcal{F}_\omega(\phi + \mathcal{L}_\omega(\tau,\xi)), \tag{2.9}$$

where $\mathcal{F}_\omega : BC_\omega(I,X) \to BC_\omega(I,X)$ is introduced in Lemma 2.3 and $\mathcal{L}_\omega : X \to BC_\omega(I,X)$ is introduced in Lemma 2.1. We have

$$\begin{aligned}\|\mathcal{T}_\omega(\phi;\xi,\tau)-\mathcal{T}_\omega(\bar\phi;\xi,\tau)\|_\omega &\overset{(2.9)}{=}\|\mathcal{F}_\omega(\phi+\mathcal{L}_\omega(\tau,\xi))-\mathcal{F}_\omega(\bar\phi+\mathcal{L}_\omega(\tau,\xi))\|_\omega\\ &\overset{(2.7)}{\leq}\frac{K(\omega)L(\omega)}{\alpha}\|\phi-\bar\phi\|_\omega\end{aligned}$$

for all $\phi,\bar\phi\in BC_\omega(I,X)$. Hence, $\mathcal{T}_\omega$ is a contraction uniformly in $\xi\in X$ and in $\tau\in I$ and by uniform contraction principle (Theorem A.4) there is a unique fixed point $\varphi^*_{\tau,\omega}(\xi)\in BC_\omega(I,X)$ of $\mathcal{T}_\omega(\cdot;\xi,\tau)$ for all $\xi\in X$, $\tau\in I$ and the mapping $(\tau,\xi)\mapsto\varphi^*_{\tau,\omega}(\xi)$ is continuous.

Moreover,

$$\begin{aligned}\left\|\mathcal{T}_\omega(\phi;\xi,\tau)-\mathcal{T}_\omega(\phi;\bar\xi,\tau)\right\|_{\tau,\omega} &\overset{(2.9)}{=}\left\|\mathcal{F}_\omega(\phi+\mathcal{L}_\omega(\tau,\xi))-\mathcal{F}_\omega(\phi+\mathcal{L}_\omega(\tau,\bar\xi))\right\|_{\tau,\omega}\\ &\overset{(2.7)}{\leq}\frac{K(\omega)L(\omega)}{\alpha}\left\|\mathcal{L}_\omega(\tau,\xi)-\mathcal{L}_\omega(\tau,\bar\xi)\right\|_{\tau,\omega}\\ &\overset{(2.5)}{\leq}\frac{K^2(\omega)L(\omega)}{\alpha}e^{\alpha(\tau-\tau_0)}\ell(\tau,\omega)\left\|\xi-\bar\xi\right\|.\end{aligned}\tag{2.10}$$

This leads us to conclude that $\mathcal{T}_\omega(\phi,\cdot,\tau)$ is Lipschitz with Lipschitz constant $\frac{K^2(\omega)L(\omega)}{\alpha}e^{\alpha(\tau-\tau_0)}\ell(\tau,\omega)$. Therefore, the mapping $\xi\mapsto\varphi^*_{\tau,\omega}(\xi)$ is also Lipschitz with Lipschitz constant $\frac{K^2(\omega)L(\omega)e^{\alpha(\tau-\tau_0)}\ell(\tau,\omega)}{\alpha-K(\omega)L(\omega)}$.

Furthermore, we show that $\varphi^*_{\tau,\omega}(\xi)$ satisfies the identity

$$\varphi^*_{\tau,\omega}(\xi)=\varphi^*_{r,\omega}(\Phi_\omega(r,\tau)\xi)\tag{2.11}$$

for all $\xi\in X$ and almost all $\tau,r\in I$.

Indeed,

$$\varphi^*_{\tau,\omega}(\xi)=\int_{\tau_0}^t\Phi_\omega(t,s)F_\omega(s,\varphi^*_{\tau,\omega}(\xi)(s)+\Phi_\omega(t,\tau)\xi)\,ds$$

is a unique solution to the initial value problem $\dot x=A_\omega(t)x+F_\omega(t,x+\Phi_\omega(t,\tau)\xi),x(\tau_0)=0$. On the other hand,

$$\begin{aligned}\varphi^*_{r,\omega}(\Phi_\omega(r,\tau)\xi)&=\int_{\tau_0}^t\Phi_\omega(t,s)F_\omega(s,\varphi^*_{r,\omega}(\Phi_\omega(r,\tau)\xi)(s)+\Phi_\omega(t,r)\Phi_\omega(r,\tau)\xi)\,ds\\ &=\int_{\tau_0}^t\Phi_\omega(t,s)F_\omega(s,\varphi^*_{r,\omega}(\Phi_\omega(r,\tau)\xi)(s)+\Phi_\omega(t,\tau)\xi)\,ds\end{aligned}$$

is a unique solution to the same initial value problem. As consequence of uniqueness of solution we obtain the desired result.

Step 2. Let us introduce $H_\omega:I\times X\to X$ and $G_\omega:I\times X\to X$ as follows

$$\begin{aligned}H_\omega(t,\xi)&:=\xi+\int_{\tau_0}^t\Phi_\omega(t,s)F_\omega(s,\varphi^*_{t,\omega}(\xi)(s)+\Phi_\omega(t,s)\xi)\,ds\\ &=\xi+\varphi^*_{t,\omega}(\xi)(t).\\ G_\omega(t,\eta)&:=\eta-\int_{\tau_0}^t\Phi_\omega(t,s)F_\omega(s,\varphi_\omega(s,t,\eta))\,ds.\end{aligned}\tag{2.12}$$

We can show the boundedness of $\xi \mapsto H_\omega(t,\xi) - \xi$ and $\xi \mapsto G_\omega(t,\eta) - \eta$ as follows

$$\begin{aligned}\|H_\omega(t,\xi) - \xi\|_{t,\omega} &\leq \int_{\tau_0}^{t} \|\Phi_\omega(t,s)\|_{s,t,\omega} \left\|F_\omega(s, \varphi^*_{t,\omega}(\xi)(s) + \Phi_\omega(t,s)\xi)\right\|_{s,\omega} ds \\ &\overset{(2.1)}{\leq} K(\omega) \int_{\tau_0}^{t} e^{-\alpha(t-s)} \left\|F_\omega(s, \varphi^*_{t,\omega}(\xi)(s) + \Phi_\omega(t,s)\xi)\right\|_{s,\omega} ds \\ &\overset{(2.3)}{\leq} K(\omega)M(\omega) \int_{\tau_0}^{t} e^{-\alpha(t-s)}\, ds \leq \frac{K(\omega)M(\omega)}{\alpha} \quad \text{for all } \xi \in X,\end{aligned}$$

and respectively,

$$\begin{aligned}\|G_\omega(t,\eta) - \eta\|_{t,\omega} &\leq \int_{\tau_0}^{t} \|\Phi_\omega(t,s)\|_{s,t,\omega} \|F_\omega(s, \varphi_\omega(s,t,\eta)\|_{s,\omega}\, ds \\ &\leq \frac{K(\omega)M(\omega)}{\alpha} \quad \text{for all } \eta \in X.\end{aligned}$$

Now, let us demonstrate the conjugation of (*N*) and (*L*). First, we claim that $H_\omega(t, \Phi_\omega(t,\tau)\xi)$ is differentiable in t. Indeed,

$$H_\omega(t, \Phi_\omega(t,\tau)\xi) = \Phi_\omega(t,\tau)\xi + \int_{\tau_0}^{t} \Phi_\omega(t,s) F_\omega\left(s, \varphi^*_{t,\omega}(\Phi_\omega(t,\tau)\xi)(s) + \Phi_\omega(t,s)\Phi_\omega(t,\tau)\xi\right) ds.$$

The function $\Phi_\omega(t,s)$ is the solution of (*L*), and the conditions in (H_1) guarantee that Φ_ω is differentiable almost everywhere with respect to t for $t \geq s$. Specifically:

$$\frac{d}{dt}\Phi_\omega(t,s) = A_\omega(t)\Phi_\omega(t,s).$$

To analyze the second term, we note that (H_2^0) guarantees that F_ω is sufficiently regular for the integral term to be differentiable. Moreover, $\varphi^*_{t,\omega}(\xi)$ solves (*N*) and depends on t in a smooth way through $\Phi_\omega(t,s)$ and the nonlinearity F_ω. The Lipschitz continuity of F_ω ensures that the solution $\varphi^*_{t,\omega}(\xi)$ is differentiable with respect to t.

Combining all this with the differentiation under the integral Theorem A.7, we can compute the derivative in a direct form:

$$\begin{aligned}\frac{d}{dt}H_\omega(t, \Phi_\omega(t,\tau)\xi) &= \frac{d}{dt}\Phi_\omega(t,\tau)\xi + \frac{d}{dt}\varphi^*_{t,\omega}(\Phi_\omega(t,\tau)\xi)(t) \\ &= A_\omega(t)\left(\Phi_\omega(t,\tau)\xi + \varphi^*_{t,\omega}(\Phi_\omega(t,\tau)\xi)(t)\right) + F_\omega\left(t, H_\omega(t, \Phi_\omega(t,\tau)\xi)\right) \\ &= A_\omega(t)H_\omega(t, \Phi_\omega(t,\tau)\xi) + F_\omega\left(t, H_\omega(t, \Phi_\omega(t,\tau)\xi)\right).\end{aligned}$$

This means that $t \mapsto H_\omega(t, \Phi_\omega(t,\tau)\xi)$ solves the initial value problem:

$$\dot{x} = A_\omega(t)x + F_\omega(t,x), \quad x(\tau) = H_\omega(\tau,\xi).$$

By the uniqueness of the solution (due to Theorem 1.2), we obtain:

$$H_\omega(t, \Phi_\omega(t,\tau)\xi) = \varphi_\omega(t,\tau, H_\omega(\tau,\xi)) \quad \forall\, \xi \in X.$$

In the same way it can be shown that

$$G_\omega(t, \varphi_\omega(t,s,\eta)) = \Phi_\omega(t,s)G_\omega(s,\eta) \quad \text{for all } \eta \in X.$$

Step 3. In order to complete the proof and to show that $H_\omega(t,\cdot)$ is a homeomorphism, we start with the claim that $H_\omega(t,\cdot): X \to X$ is bijective. Indeed, for any $\eta \in X$, we have

$$\begin{aligned} H_\omega(t, G_\omega(t,\varphi_\omega(s,t,\eta))) &\overset{(2.12)}{=} G_\omega(t,\varphi_\omega(t,\tau,\eta)) + \int_{\tau_0}^t \Phi_\omega(t,s)F_\omega(s,\varphi^*_{t,\omega}(G_\omega(s,\varphi_\omega(t,\tau,\eta)))(s) \\ &\quad + \Phi_\omega(t,s)G_\omega(s,\varphi_\omega(t,\tau,\eta)))\,ds \\ &= \varphi_\omega(t,\tau,\eta) - \int_{\tau_0}^t \Phi_\omega(t,s)F_\omega(s,\varphi_\omega(t,\tau,\eta))ds \\ &\quad + \int_{\tau_0}^t \Phi_\omega(t,s)F_\omega(s,\varphi^*_{t,\omega}(G_\omega(s,\varphi_\omega(t,\tau,\eta)))(s) \\ &\quad + \Phi_\omega(t,s)G_\omega(s,\varphi_\omega(t,\tau,\eta)))\,ds. \end{aligned}$$

If we abbreviate by $\nu(t) = \|H_\omega(t, G_\omega(t,\varphi_\omega(s,t,\eta))) - \varphi_\omega(t,\tau,\eta)\|_{t,\omega}$, then we have

$$\begin{aligned} \nu(t) &\le \int_{\tau_0}^t \|\Phi_\omega(t,s)\|_{s,t,\omega}\,\|F_\omega(s,\varphi^*_{t,\omega}(G_\omega(s,\varphi_\omega(t,\tau,\eta)))(s) \\ &\qquad + \Phi_\omega(t,s)G_\omega(s,\varphi_\omega(t,\tau,\eta))) - F_\omega(s,\varphi_\omega(t,\tau,\eta))\|_{s,\omega}ds \\ &\le K(\omega)L(\omega)\int_{\tau_0}^t e^{-\alpha(t-s)}\|\varphi^*_{t,\omega}(G_\omega(s,\varphi_\omega(t,\tau,\eta)))(s) \\ &\qquad + \Phi_\omega(t,s)G_\omega(s,\varphi_\omega(t,\tau,\eta))) - \varphi_\omega(t,\tau,\eta)\|_{s,\omega}\,ds \\ &\le K(\omega)L(\omega)\int_{\tau_0}^t e^{-\alpha(t-s)}\,\|H_\omega(s,\Phi_\omega(t,s)G_\omega(s,\varphi_\omega(t,\tau,\eta)))) - \varphi_\omega(t,\tau,\eta)\|_{s,\omega}\,ds \\ &\le K(\omega)L(\omega)\int_{\tau_0}^t e^{-\alpha(t-s)}\nu(s)\,ds \le \frac{K(\omega)L(\omega)}{\alpha}\sup_{s\in I}\nu(s) \quad \text{for all } t\in I. \end{aligned}$$

Hence, from the above estimate and the assumption $K(\omega)L(\omega) < \alpha$, it follows that $\nu(t) = 0$ for any $t \in I$ and with $t = \tau$, $H_\omega(\tau, G_\omega(\tau,\eta)) = \eta$.

Now we prove that for all $\xi \in X$, it holds that $G_\omega(t, H_\omega(t,\xi)) = \xi$. We have

$$\begin{aligned} &G_\omega(t, H_\omega(t,\Phi_\omega(t,\tau)\xi)) \\ &\quad = H_\omega(t,\Phi_\omega(t,\tau)\xi) - \int_{\tau_0}^t \Phi_\omega(t,s)F_\omega(s,\varphi_\omega(s,t,H_\omega(t,\Phi_\omega(t,\tau)\xi)))\,ds \\ &\quad = \Phi_\omega(t,\tau)\xi + \int_{\tau_0}^t (F_\omega(s,H_\omega(s,\Phi_\omega(s,\tau)\xi)) - F_\omega(s,\varphi_\omega(s,\tau,H_\omega(\tau,\xi))))\,ds \\ &\quad = \Phi_\omega(t,\tau)\xi. \end{aligned}$$

Now, if we set $t = \tau$, then $G_\omega(\tau, H_\omega(\tau,\xi)) = \xi$ holds.

As a consequence, for any $t \in I$, $H_\omega(t,\cdot)\colon X \to X$ is a bijection, and $G_\omega(t,\cdot)\colon X \to X$ is its inverse.

Step 4. In this step, we show that the conjugacy and its inverse satisfy Lipschitz conditions with Lipschitz constants L_H and L_G respectively, where

$$L_G = 1 + \frac{K^2(\omega)L(\omega)}{2\alpha - K(\omega)L(\omega)}, \qquad L_H = 1 + \frac{K^2(\omega)L(\omega)}{2\alpha} + \frac{K^3(\omega)L^2(\omega)e^{\alpha(\tau-\tau_0)}\ell(\tau,\omega)}{\alpha(\alpha - K(\omega)L(\omega))}.$$

In order to show the claim for the operator G_ω, we start with additional estimates.

Note that the general solution $\varphi_\omega(\cdot,\tau,\eta)$ of (N) also solves the inhomogeneous equation

$$\dot{x} = A_\omega(t)x + F_\omega(t,\varphi_\omega(t,\tau,\eta)).$$

Hence, by variation of constants formula (Theorem A.2), we have

$$\varphi_\omega(s,t,\eta) = \Phi_\omega(t,s)\eta + \int_s^t \Phi_\omega(t,\tau)F_\omega(\tau,\varphi_\omega(t,\tau,\eta))\,d\tau,$$

so we obtain

$$\begin{aligned}
&\|\varphi_\omega(s,t,\eta) - \varphi_\omega(s,t,\bar{\eta})\|_{t,\omega} \\
&\quad\le \|\Phi_\omega(t,s)(\eta-\bar{\eta})\|_\omega + \int_s^t \|\Phi_\omega(t,\tau)\|_{s,t,\omega}\,\|F_\omega(\tau,\varphi_\omega(t,\tau,\eta)) - F_\omega(\tau,\varphi_\omega(t,\tau,\bar{\eta}))\|_{s,\omega}\,ds \\
&\quad\le K(\omega)e^{-\alpha(t-s)}\|\eta-\bar{\eta}\| + K(\omega)L(\omega)\int_s^t e^{-\alpha(t-\tau)}\|\varphi_\omega(t,\tau,\eta) - \varphi_\omega(t,\tau,\bar{\eta})\|_{t,\omega}\,ds.
\end{aligned}$$

Multiplying both sides by $e^{\alpha(t-s)}$ and using Gronwall's inequality (A.1), gives us

$$\|\varphi_\omega(s,t,\eta) - \varphi_\omega(s,t,\bar{\eta})\|_{t,\omega}\,e^{\alpha(t-s)} \le K(\omega)e^{K(\omega)L(\omega)(t-s)}\|\eta-\bar{\eta}\|.$$

From the estimate above we obtain

$$\|\varphi_\omega(s,t,\eta) - \varphi_\omega(s,t,\bar{\eta})\|_{t,\omega} \le K(\omega)e^{(-\alpha+K(\omega)L(\omega))(t-s)}\|\eta-\bar{\eta}\|. \tag{2.13}$$

Now, using (2.13), we have that

$$\begin{aligned}
\|G_\omega(t,\eta) - G_\omega(t,\bar{\eta})\|_{t,\omega} &\le \|\eta-\bar{\eta}\| + K(\omega)L(\omega)\int_{\tau_0}^t e^{-\alpha(t-s)}\|\varphi_\omega(t,s,\bar{\eta}) - \varphi_\omega(t,s,\eta)\|_{s,\omega}\,ds \\
&\le \|\eta-\bar{\eta}\|\left(1 + K^2(\omega)L(\omega)\int_{\tau_0}^t e^{(-2\alpha+K(\omega)L(\omega))(t-s)}ds\right) \\
&\le \left(1 + \frac{K^2(\omega)L(\omega)}{2\alpha - K(\omega)L(\omega)}\right)\|\eta-\bar{\eta}\| \\
&\le L_G\|\eta-\bar{\eta}\|.
\end{aligned} \tag{2.14}$$

Now we show that H_ω satisfy a global Lipschitz condition. We have

$$\begin{aligned}
&\left\|H_\omega(t,\xi) - H_\omega(t,\bar{\xi})\right\|_{t,\omega} \\
&\le \|\xi-\bar{\xi}\| + K(\omega)L(\omega)\int_{\tau_0}^t e^{-\alpha(t-s)}\left\{\left\|\Phi_\omega(t,s)\xi - \Phi_\omega(t,s)\widetilde{\xi}\right\|_{s,\omega} + \left\|\varphi^*_{t,\omega}(\xi) - \varphi^*_{t,\omega}(\bar{\xi})\right\|_\omega\right\}ds \\
&\le \|\xi-\bar{\xi}\|\left(1 + K^2(\omega)L(\omega)\int_{\tau_0}^t e^{-2\alpha(t-s)}ds\right) + \frac{K(\omega)L(\omega)}{\alpha}\left\|\varphi^*_{t,\omega}(\xi) - \varphi^*_{t,\omega}(\bar{\xi})\right\|_\omega \\
&\le \|\xi-\bar{\xi}\|\left(1 + K^2(\omega)L(\omega)\int_{\tau_0}^t e^{-2\alpha(t-s)}ds\right) + \frac{K^3(\omega)L^2(\omega)e^{\alpha(\tau-\tau_0)}\ell(\tau,\omega)}{\alpha(\alpha - K(\omega)L(\omega))}\|\xi-\bar{\xi}\| \\
&\le \left(1 + \frac{K^2(\omega)L(\omega)}{2\alpha} + \frac{K^3(\omega)L^2(\omega)e^{\alpha(\tau-\tau_0)}\ell(\tau,\omega)}{\alpha(\alpha - K(\omega)L(\omega))}\right)\|\xi-\bar{\xi}\| \\
&\le L_H\left\|\xi-\bar{\xi}\right\|.
\end{aligned} \tag{2.15}$$

Based on (2.14) and (2.15), we conclude that $H_\omega(t,\cdot)$ and $G_\omega(t,\cdot)$ are Lipschitz, and therefore uniformly continuous for any $t \in I$.

Moreover, it can be shown that the operator $H_\omega : I \times X \to X$ and its inverse are continuous.

Indeed, from the continuity of the solution of (L), it follows that for any $\varepsilon_1 > 0$ there exists $\delta_1(t_0, \xi_0, \varepsilon_1) > 0 : |t - t_0| + \|\xi - \xi_0\| < \delta_1 \Rightarrow \|\Phi_\omega(t,s)\xi - \Phi_\omega(t_0,s)\xi_0\|_{s,\omega} < \varepsilon_1$.

Thus, at the first step we proved the continuity of the mapping $(t, \xi) \mapsto \varphi^*_{t,\omega}(\xi)$. This means that, for any $\varepsilon_2 > 0$, there exists $\delta_2(t_0, \xi_0, \varepsilon_2) > 0 : |t - t_0| + \|\xi - \xi_0\| < \delta_2 \Rightarrow \left\|\varphi^*_{t,\omega}(\xi) - \varphi^*_{t,\omega}(\xi_0)\right\|_{t,\omega} < \varepsilon_2$.

Without loss of generality, let $\tau_0 \le t_0 \le t$. In order to prove the continuity of the operator H_ω, we consider the following expression

$$\begin{aligned}
H_\omega(t,\xi) - H_\omega(t_0,\xi_0) = \xi - \xi_0 &+ \int_{\tau_0}^{t} \Phi_\omega(t,s)F_\omega(s,\varphi^*_{t,\omega}(\xi)(s) + \Phi_\omega(t,s)\xi)\,ds \\
&- \int_{\tau_0}^{t_0} \Phi_\omega(t_0,s)F_\omega(s,\varphi^*_{t_0,\omega}(\xi_0)(s) + \Phi_\omega(t_0,s)\xi_0)\,ds = \xi - \xi_0 \\
&+ \int_{\tau_0}^{t_0} (\Phi_\omega(t,s) - \Phi_\omega(t_0,s))F_\omega(s,\varphi^*_{t,\omega}(\xi)(s) + \Phi_\omega(t,s)\xi)\,ds \\
&+ \int_{\tau_0}^{t_0} \Phi_\omega(t_0,s)\left(F_\omega(s,\varphi^*_{t,\omega}(\xi)(s) + \Phi_\omega(t,s)\xi)\right. \\
&- \left. F_\omega(s,\varphi^*_{t_0,\omega}(\xi_0)(s) + \Phi_\omega(t_0,s)\xi_0)\,ds\right) \\
&+ \int_{t_0}^{t} \Phi_\omega(t,s)F_\omega(s,\varphi^*_{t,\omega}(\xi)(s) + \Phi_\omega(t,s)\xi)\,ds.
\end{aligned}$$

Let $C = \max\{\|\Phi(t,s)\|_{s,t,\omega} : t,s \in I, t \ge s\}$, then we obtain

$$\begin{aligned}
\|H_\omega(t,\xi) - H_\omega(t_0,\xi_0)\|_{t,\omega} &\le \|\xi - \xi_0\| + M(\omega)\int_{\tau_0}^{t_0} \|\Phi_\omega(t,s) - \Phi_\omega(t_0,s)\|_{s,t,\omega}\,ds \\
&\quad + L(\omega)C\int_{\tau_0}^{t_0} \left\|\varphi^*_{t,\omega}(\xi)(s) - \varphi^*_{t_0,\omega}(\xi_0)(s)\right\|_{t,\omega}\,ds \\
&\quad + M(\omega)C(t - t_0) + L(\omega)C(t_0 - \tau_0)\varepsilon \\
&\le \|\xi - \xi_0\| + (2M(\omega) + L(\omega)\,(\varepsilon_1 + \varepsilon_2))\,C(t_0 - \tau_0) \\
&\quad + M(\omega)C(t - t_0),
\end{aligned}$$

which proves the continuity of H_ω in any $(t_0, \xi_0) \in I \times X$.

In the same way, using the continuity of general solution $\varphi_\omega(s,t,\eta)$ of (N), we know that for any $\varepsilon_3 > 0$ there exists $\delta_3(t_0, \xi_0, \varepsilon_3) > 0 : |t - t_0| + \|\xi - \xi_0\| < \delta_3 \Rightarrow \|\varphi_\omega(s,t,\eta) - \varphi_\omega(s,t_0,\eta_0)\|_{t,\omega} < \varepsilon_3$. Thus,

$$\begin{aligned}
\|G_\omega(t,\eta) - G_\omega(t_0,\eta_0)\|_{t,\omega} \le\ & \|\eta - \eta_0\| + M(\omega)\int_{\tau_0}^{t_0} \|\Phi_\omega(t,s) - \Phi_\omega(t_0,s)\|_{s,t,\omega}\,ds \\
&+ L(\omega)C\int_{\tau_0}^{t_0} \|\varphi_\omega(s,t,\eta) - \varphi_\omega(s,t_0,\eta_0)\|_{t,\omega}\,ds + M(\omega)C(t - t_0) \\
\le\ & \|\eta - \eta_0\| + (L(\omega)\varepsilon_3 + 2M(\omega))C(t_0 - \tau_0) \\
&+ M(\omega)C(t - t_0),
\end{aligned}$$

which completes the proof. □

If we set $\tilde{u} = 0$ in (2.13), we see that $\|\varphi_\omega(t,s,u)\|_{t,\omega} \le K(\omega)e^{(-\alpha + K(\omega)L(\omega))(t-s)}\|u\|$ which leads us to conclude that the trivial solution of the (N) is globally uniformly exponentially stable.

2.2 Smooth linearization

We write $D_2^j F_\omega : I \times X \to L_j(X)$ for the j-th order partial derivative of a mapping $F_\omega : I \times X \to X$ with respect to the second variable. And we substitute the assumption (H_2^0) by the following condition supposed to hold for some $m \in \mathbb{N}$ and all $\omega \in \Omega$:

(H_2^m) $F_\omega : I \times X \to X$ and its partial derivatives $D_2^j F_\omega : I \times X \to L_j(X)$, $1 \leq j \leq m$, are Carathéodory functions and there exist reals $M_j(\omega) \geq 0$ such that

$$\left\| D_2^j F_\omega(t,x) \right\|_{t,\omega} \leq M_j(\omega) \quad \text{for almost all } t \in I, \text{ all } x \in X \text{ and } 0 \leq j \leq m. \tag{2.16}$$

Lemma 2.5 (Differentiation under the integral). *Let $m \in \mathbb{N}$, $U \subseteq X, V \subseteq Y$ be open and $f : U \times V \to X$ be such that $f(\cdot, y)$ is integrable for any $y \in V$. Suppose, that for any $x \in U, y \in V$, there exist derivatives $D_2^j f(x,y)$, and*

$$\left\| D_2^j f(x,y) \right\| \leq g(x)$$

for all $0 \leq j \leq m$, where $g : U \to \mathbb{R}^+$ is integrable. Then the mapping

$$F(y) = \int_U f(x,y)dx$$

is m times differentiable and its j-th derivative is given by

$$D^j F(y) = \int_U D_2^j f(x,y)dx$$

for all $0 \leq j \leq m$.

Theorem 2.6 (C^m-linearization of (N)). *Assume (H_1) and (H_2^m) hold and $K(\omega)M_1(\omega) < \alpha$ for all $\omega \in \Omega$. Then all statements of Theorem 2.4 hold with $H_\omega(t,\cdot)$ being a C^m-diffeomorphism. In the present setting, H_ω is called a C^m-conjugacy between (N) and its linearization (L).*

Note that C^m-conjugacies guarantee that the integral manifolds of (N) are tangential to the invariant linear integral manifolds of (L).

Proof. First, we aim to show that the general solution $\varphi_\omega(s,t,\cdot) : X \to X$ of (N) is m times differentiable with continuous bounded partial derivatives $D_3^j \varphi_\omega : I \times I \times X \to L_j(X)$ for any $\eta \in X$, $s,t \in I$ and $1 \leq j \leq m$.

It is well-known (see, e.g., Theorem 1.16 in [40]) that $\varphi_\omega \in C^m$. Our goal is to show the boundedness of the partial derivatives.

From the fact that φ_ω is the general solution of (N) it follows that, for any $\eta \in X$, the mapping φ_ω satisfies the equation $D_1\varphi_\omega(t,s,\eta) = A_\omega(t)\varphi_\omega(t,s,\eta) + F_\omega(t,\varphi_\omega(t,s,\eta))$. Then,

$$D_1 D_3 \varphi_\omega(t,s,\cdot) = A_\omega(t) D_3\varphi_\omega(t,s,\eta) + D_2 F_\omega(t,\varphi_\omega(t,s,\eta)) D_3\varphi_\omega(t,s,\eta)$$

and, therefore, $D_3\varphi_\omega(\cdot,s,\eta)$ solves the initial value problem $\dot{x} = [A_\omega(t) + D_2F_\omega(t,\varphi_\omega(t,s,\eta))]x$, $x(s) = \mathrm{Id}_X$. Hence, by variation of constants formula we have

$$D_3\varphi_\omega(t,s,\eta) = \Phi_\omega(t,s) + \int_s^t \Phi_\omega(r,s) D_2F_\omega(r,\varphi_\omega(r,s,\eta)) D_3\varphi_\omega(r,s,\eta)\, dr,$$

and we estimate

$$\begin{aligned}\|D_3\varphi_\omega(t,s,\eta)\|_{t,\omega} \leq & \int_s^t \|\Phi_\omega(r,s)\|_{s,t,\omega} \|D_2F_\omega(r,\varphi_\omega(r,s,\eta))\|_{s,\omega} \|D_3\varphi_\omega(r,s,\eta)\|_{s,\omega}\, dr \\ & + \|\Phi_\omega(t,s)\|_{s,t,\omega} \leq K(\omega)e^{-\alpha(t-s)} \\ & + K(\omega)M_1(\omega)\int_s^t e^{-\alpha(r-s)} \|D_3\varphi_\omega(r,s,\eta)\|_{s,\omega}\, dr.\end{aligned}$$

Multiplying both sides by $e^{\alpha(t-s)}$ and using Gronwall's inequality (A.1), gives us

$$\|D_3\varphi_\omega(t,s,\eta)\|_{t,\omega}\, e^{\alpha(t-s)} \leq K(\omega)e^{K(\omega)M_1(\omega)(t-s)}.$$

From the estimate above, we obtain

$$\|D_3\varphi_\omega(t,s,\eta)\|_{t,\omega} \leq K(\omega)e^{(K(\omega)M_1(\omega)-\alpha)(t-s)},$$

and the boundedness of the first-order partial derivative follows. Moreover, the boundedness of higher-order partial derivatives can be demonstrated using the same arguments by induction.

Now our goal is to show that the map $\eta \mapsto G_\omega(t,\eta)$ is C^m.

This follows from the fact that $\varphi_\omega(s,t,\cdot) : X \to X$ is C^m, hypothesis (H_2^m), and from applying Lemma 2.5 to the function $f(t,\eta) = \Phi_\omega(t,s)F_\omega(s,\varphi_\omega(s,t,\eta))$. To justify the application of Lemma 2.5, we observe that

$$\|D_2f(t,\eta)\|_{t,\omega} \leq M_1K^2(\omega)e^{(K(\omega)M_1-2\alpha)(t-s)}.$$

This allows us to conclude the desired statement and to express $D_2G_\omega(t,\eta)$ in explicit form using the chain rule

$$D_2G_\omega(t,\eta) = \mathrm{Id}_X - \int_{\tau_0}^t \Phi_\omega(t,s)D_2F_\omega(s,\varphi_\omega(s,t,\eta))D_3\varphi_\omega(s,t,\eta)\, ds. \tag{2.17}$$

From (H_2^m) and the fact that φ_ω is the general solution of (N), it follows that $D_3\varphi_\omega(s,t,\cdot)$ satisfies the differential equation

$$\dot{x} = A_\omega(t)x + D_2F_\omega(t,\varphi_\omega(s,t,\eta))x, \quad x(s) = \mathrm{Id}_X. \tag{2.18}$$

Combining (2.17) and (2.18) we obtain

$$\begin{aligned}D_2G_\omega(t,\eta) &= \mathrm{Id}_X - \int_{\tau_0}^t \Phi_\omega(t,s)\left(D_1D_3\varphi_\omega(s,t,\eta) - A_\omega(t)D_3\varphi_\omega(s,t,\eta)\right) ds \\ &= \mathrm{Id}_X - \int_{\tau_0}^t D_1\left(\Phi_\omega(t,s)D_3\varphi_\omega(s,t,\eta)\right) ds \\ &= \Phi_\omega(t,\tau_0)D_3\varphi_\omega(\tau_0,t,\eta).\end{aligned} \tag{2.19}$$

From Lemma 2.1, we know that $\Phi_\omega(t,\tau_0) \in GL(X)$ is invertible. Moreover, (2.18) demonstrates that $D_3\varphi_\omega(\tau_0,t,\eta)$ is a transition matrix of $\dot{x} = [A_\omega(t) + D_2F_\omega(t,\varphi_\omega(t,s,\eta))]x$, and hence it is also invertible. Consequently, the invertibility of $D_2G_\omega(t,\eta)$ follows.

Hence, by Theorem A.5, the map $\xi \mapsto G_\omega(t,\xi)$ is a local C^m diffeomorphism for any fixed $t \in I$. As the final step, since we have shown that G_ω is a homeomorphism, therefore it preserves the compactness of sets. Thus, we can apply Theorem A.6 to G_ω, which completes the proof. □

3 Random dynamical systems

In the previous section, we presented a smooth linearization result specifically for semilinear Carathéodory differential equations in general Banach spaces. Our current objective is to apply this result to continuous-time random dynamical systems, considering random differential equations (RDEs) and stochastic differential equations (SDEs).

RDEs and SDEs are complementary approaches commonly used in the study of random dynamical systems. Random differential equations, introduced in, e.g., [1, 41], and systematically treated in [19, 23], provide a pathwise framework for modeling dynamical systems under uncertainty. Their analysis typically relies on existence and uniqueness results for ordinary differential equations with Carathéodory-type vector fields [29]. Stochastic differential equations, on the other hand, offer a probabilistic framework and are well established in both theory and applications [30, 31].

In order to mimic the constructions from [1, 44], we restrict ourselves to finite-dimensional Banach spaces X. Before delving into the details, we first introduce the research object of this section.

A random dynamical system is an object consisting of a measurable dynamical system and a cocycle over this system [1], therefore we begin with a definition of a measurable dynamical system.

Definition 3.1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. We call $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in I})$ to be a *measurable dynamical system (MDS)* , if the mapping $\theta_t : \Omega \to \Omega$ satisfies the following conditions:

(i) the mapping $(t, \omega) \mapsto \theta_t\omega$ is measurable;

(ii) the family $(\theta_t)_{t\in I}$ forms a group, i.e. $\theta_0 = \mathrm{Id}_\Omega$ and $\theta_{t+s} = \theta_t \circ \theta_s$, for arbitrary $s, t \in I$;

(iii) the mapping θ_t is $\mathbb{P}$-preserving, i.e., for arbitrary $t \in I$ and $F \in \mathcal{F}$, the identity $\mathbb{P}(\theta_t^{-1}(F)) = \mathbb{P}(F)$ holds.

An MDS $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in I})$ is called *ergodic* if every θ_t- invariant set has probability 0 or 1, i.e. if for all $F \in \mathcal{F}$ satisfying $\theta_t^{-1}(F) = F$ for every $t \in I$ we have either $\mathbb{P}(F) = 0$ or $\mathbb{P}(F) = 1$.

Definition 3.2. A *measurable random dynamical system (RDS)* on X over some given MDS $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in I})$ is a measurable mapping $\psi\colon I \times \Omega \times X \to X$, $(s, \omega, x) \to \psi(s, \omega, x)$ forming a *cocycle* over θ_t, i.e., the mapping $\psi(t, \omega) := \psi(t, \omega, \cdot)$ satisfies

$$\psi(0, \omega) = \mathrm{Id}_X,$$
$$\psi(s + t, \omega) = \psi(t, \theta_s\omega) \circ \psi(s, \omega),$$

for arbitrary $s, t \in I$, and $\omega \in \Omega$.

The random dynamical system ψ is called *linear,* if the mapping $\psi(t, \omega)$ is linear, *continuous,* if the mapping $\psi(\cdot, \omega)$ is continuous. Furthermore, it is called *smooth of class* C^m, if it is continuous, the mapping $\psi(t, \omega)$ is of class C^m and the derivatives are continuous with respect to the (t, x), for arbitrary $t \in I$, $\omega \in \Omega$, and $x \in X$.

Definition 3.3. Let $\Phi : I \times \Omega \to X$ be a linear RDS over an ergodic MDS θ_t. We define the *Lyapunov exponents* of Φ for all $x \in X, x \neq 0$ as

$$\lambda(\omega, x) = \limsup_{t\to\infty} \frac{1}{t} \|\Phi(t, \omega)x\| .$$

We denote the distinct values that $\lambda(\omega, x)$ can take for $x \neq 0$ as

$$-\infty < \lambda_{k(\omega)}(\omega) < \cdots < \lambda_1(\omega).$$

The value $\lambda_1(\omega)$ is called the *top Lyapunov exponent*.

In the setting of the Multiplicative Ergodic Theorem A.8, the sets $U_\lambda := \{x : \lambda(\omega, x) \leq \lambda(\omega)\}$ are linear subspaces of X, $U_i := U_{\lambda_i}$, and they form a filtration (flag of subspaces)

$$\{0\} \subset U_{k(\omega)} \subset \cdots \subset U_1 = X,$$

where

$$\lambda(\omega, x) = \lambda_i(\omega) \iff x \in U_i \setminus U_{i+1}, \quad i = 1, \ldots, k(\omega).$$

We say that $d_i(\omega) := \dim U_i - \dim U_{i+1}$ is the *multiplicity* of $\lambda_i(\omega)$.

In order to apply the results from the previous section to the random dynamical system, we first answer the following questions:

(1) What is the connection between the semilinear Carathéodory differential equations (N) and the corresponding random dynamical system?

(2) How should we choose the new norm depending on both time and randomness in order to achieve our goal?

The answer to the first question is given in the following lemma.

Lemma 3.4. *Let $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in I})$ be a MDS. If φ_ω denotes the general solution of the Carathéodory differential equation (N), then the following statements are equivalent:*

(i) The mapping ψ_ω defined as $\psi_\omega(t, x) := \varphi_\omega(t, 0, x)$ is a measurable RDS over $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in I})$.

(ii) The general solution of (N) satisfies

$$\varphi_\omega(t, \tau, x) = \varphi_{\theta_\tau \omega}(t - \tau, 0, x) \tag{3.1}$$

for arbitrary $\tau, t \in I$ and $x \in X$.

Proof. $(i) \Rightarrow (ii)$: From the cocycle property of RDS we know that

$$\varphi_\omega(s + t, 0, x) = \varphi_{\theta_s \omega}(t, 0, \varphi_\omega(s, 0, x)). \tag{3.2}$$

Moreover, it is well known that the general solution of (N) satisfies the property

$$\varphi_\omega(t_2, t_0, x) = \varphi_\omega(t_2, t_1, \varphi_\omega(t_1, t_0, x)). \tag{3.3}$$

Thus,

$$\varphi_\omega(t, \tau, x) = \varphi_\omega((t - \tau) + \tau, \tau, x) \overset{(3.2)}{=} \varphi_{\theta_\tau \omega}(t - \tau, \tau, \varphi_\omega(\tau, 0, x)) \overset{(3.3)}{=} \varphi_{\theta_\tau \omega}(t - \tau, 0, x).$$

$(ii) \Rightarrow (i)$: In order to prove this implication it is enough to check the cocycle property. Namely,

$$\varphi_\omega(s + t, 0, x) \overset{(3.3)}{=} \varphi_\omega(s + t, s, \varphi_\omega(s, 0, x)) \overset{(3.1)}{=} \varphi_{\theta_s \omega}(t, 0, \varphi_\omega(s, 0, x)),$$

so the lemma follows. □

To address the second question, we introduce a new norm in the following way. In the setting of the Multiplicative Ergodic Theorem A.8, suppose that Φ is a linear RDS with the Lyapunov exponents $\lambda_1(\omega), \dots, \lambda_{k(\omega)}(\omega)$. It is well known from, e.g., [1], that when θ_t is ergodic, $k(\omega)$ and $\lambda_i(\omega)$ are independent of ω. We assume ergodicity for the remainder of this paper.

Let $a > 0$ denote a fixed real constant such that the intervals $[\lambda_i - a, \lambda_i + a]$, $i = 1, \dots, k$, are disjoint. For $\omega \in \tilde{\Omega}$ and $x = x^1 + \cdots + x^{k(\omega)} \in U_1(\omega) \oplus \cdots \oplus U_{k(\omega)}(\omega)$, we define a new norm

$$|x|_\omega = \sqrt{|x^1|_\omega^2 + \cdots + |x^{k(\omega)}|_\omega^2},$$

where $|x^i|_\omega = \left(\int_0^\infty \|\Phi(t,\omega)x^i\|^2 e^{-2(\lambda_i t + at)} dt \right)^{\frac{1}{2}}$ for $x^i \in U_i(\omega)$. For $\omega \notin \tilde{\Omega}$, we set $|x|_\omega = \|x\|$. Then (according to [1,44]) the following properties holds

(i) $|\cdot|_\omega$ is a random norm on X;

(ii) for every $\varepsilon > 0$, there is a measurable mapping $B_\varepsilon : \Omega \to [1, \infty)$ such that, for every $x \in X$, and almost every $\omega \in \tilde{\Omega}$, and $t \in I$,

$$\frac{1}{B_\varepsilon(\omega)}\|x\| \le |x|_\omega \le B_\varepsilon(\omega)\|x\|,$$
$$B_\varepsilon(\omega)e^{-\varepsilon t} \le B_\varepsilon(\theta_t\omega) \le B_\varepsilon(\omega)e^{\varepsilon t};$$

(iii) for almost every $\omega \in \tilde{\Omega}$, $t \in I$, and every $i = 1, \dots, k(\omega)$, $x \in U_i(\omega)$,

$$e^{(\lambda_i - a)t} \le \left|\Phi(t,\omega)|_{U_i(\omega)}\right|_{\omega,\theta_t\omega} \le e^{(\lambda_i + a)t},$$

where $\left|\Phi(t,\omega)|_{U_i(\omega)}\right|_{\omega,\theta_t\omega} := \sup\{|\Phi(t,\omega)x|_{\theta_t\omega} : x \in U_i(\omega), |x|_\omega \le 1\}$.

It is important to note that the constant a is chosen in such a way that $\lambda_i + a$ remains a negative expression for all $i = 1, \dots, k$. In the following, we will focus on the case, when all of Lyapunov exponents of the system are negative, i.e., $\lambda_k < \cdots < \lambda_1 < 0$.

Remark 3.5. *Let Φ be a linear RDS satisfying the Multiplicative Ergodic Theorem A.8 and suppose $\Phi_\omega(t,s)$ is the evolution operator of a linear random differential equation $\dot{x} = A_\omega(t)x$. In this case $(H1)$ holds with $\|x\|_{t,\omega} := |x|_{\theta_t\omega}$, $K(\omega) = 1$ and $\alpha = -\lambda_1 - a$.*

Indeed, in [1,44] is shown that Φ and $\Phi_\omega(t,s)$ are related by

$$\Phi_\omega(t,s) = \Phi(t-s, \theta_s\omega).$$

From estimates in (iii) for $|\Phi(t,\omega)|_{\omega,\theta_t\omega}$, we have

$$\begin{aligned}
|\Phi(t,\omega)x|_{\theta_t\omega} &= \sqrt{|\Phi(t,\omega)x^1|_{\theta_t\omega} + \cdots + |\Phi(t,\omega)x^k|_{\theta_t\omega}} \\
&\le \sqrt{(e^{(\lambda_1+a)t}|x^1|_\omega)^2 + \cdots + (e^{(\lambda_k+a)t}|x^k|_\omega)^2} \\
&\le e^{(\lambda_1+a)t}\sqrt{|x^1|_\omega^2 + \cdots + |x^k|_\omega^2} \\
&\le e^{(\lambda_1+a)t}|x|_\omega,
\end{aligned}$$

and therefore for all $t \in I$

$$|\Phi(t,\omega)|_{\omega,\theta_t\omega} \le e^{(\lambda_1+a)t}. \tag{3.4}$$

Thus,

$$\|\Phi_\omega(t,s)\|_{s,t,\omega} = |\Phi(t-s,\theta_s\omega)|_{\theta_s\omega,\theta_t\omega} = |\Phi(t-s,\theta_s\omega)|_{\theta_s\omega,\theta_{t-s}\theta_s\omega},$$

which together with (3.4) gives us

$$\|\Phi_\omega(t,s)\|_{s,t,\omega} \leq e^{(\lambda_1+a)(t-s)}. \tag{3.5}$$

In the above definition of the family of norms $\|\cdot\|_{t,\omega}$, the function ℓ can be chosen as $\ell(t,\omega) := B_1(\theta_t\omega)$, since $B_1(\theta.\omega)$ is locally bounded for any $\omega \in \tilde{\Omega}$.

3.1 C^m linearization of random dynamical systems

After establishing the connection between random dynamical systems and Carathéodory differential equations, we apply the results of the second section to RDS. To this end, let Φ be a linear random dynamical system on X over the ergodic MDS $(\Omega,\mathcal{F},\mathbb{P},(\theta_t)_{t\in I})$, satisfying conditions of the MET and generated by the following linear equation

$$\dot{x} = A(\theta_t\omega)x. \tag{3.6}$$

Together with Φ we will consider a random dynamical system ψ on X over the ergodic MDS $(\Omega,\mathcal{F},\mathbb{P},(\theta_t)_{t\in I})$, generated by

$$\dot{x} = A(\theta_t\omega)x + F(\theta_t\omega,x). \tag{3.7}$$

Assume that ψ has 0 as a fixed point, i.e. $\psi(t,\omega,0) = 0$ for almost all $t \in I$, $\omega \in \Omega$. Let the "nonlinear" part of ψ be given by $\Psi(t,\omega,x) = \psi(t,\omega,x) - \Phi(t,\omega)x$.

While the validity of $(H1)$ has already been shown, to satisfy the remaining hypotheses, it is necessary to impose additional conditions on the random dynamical system.

(H_3^0) Assume that for almost all $t \in I$, $\omega \in \Omega$ and all $x, \bar{x} \in X$, there exist reals $L, M \geq 0$ such that

$$|\Psi(t,\omega,x)|_{\theta_t\omega} \leq M \tag{3.8}$$

$$|\Psi(t,\omega,x) - \Psi(t,\omega,\bar{x})|_{\theta_t\omega} \leq L|x-\bar{x}|_\omega. \tag{3.9}$$

Then the following theorem establishes the connection between a random dynamical system ψ and its linearization.

Theorem 3.6 (Topological linearization of RDS)**.** *Assume that the linear random dynamical system Φ, generated by (3.6), satisfies the assumptions of the MET A.8 and has negative Lyapunov exponents. Assume that the nonlinear part Ψ of a RDS ψ, generated by (3.7), satisfies (H_3^0). If $L \leq \alpha$, then the RDS ψ and Φ are topologically equivalent, i.e. there is a measurable mapping $h : \Omega \times X \to X$ with the following properties:*

(i) $h(\omega) = h(\omega,\cdot)$ is a homeomorphism on X with $h(\omega,0) = 0$ for almost all $\omega \in \Omega$;

(ii) $h(\omega)$ maps ω-orbits of Φ onto ω-orbits of ψ in the sense that for every $\omega \in \Omega$, arbitrary $t \in I$ and $x \in X$

$$h(\omega) \circ \Phi(t,\omega)x = \psi(t,\omega,h(\omega,x));$$

(iii) h has property of being near identity, i.e. for all $\xi, \eta \in X$

$$|h(\omega,\xi)-\xi|_{\theta_t\omega} \le \frac{M}{\alpha} \quad \textit{and} \quad |h^{-1}(\omega,\eta)-\eta|_{\theta_t\omega} \le \frac{M}{\alpha},$$

where $\alpha = -\lambda_1 - a > 0$.

Proof. Applying Theorem 2.4 ω-by-ω to the system (3.7) with Ω replaced by $\tilde{\Omega}$ will yield a mapping $h : \Omega \times I \times X \to X$, defined as follows

$$\begin{cases} h(\omega,\xi) = \xi + \int_0^t \Phi(t-s,\theta_s\omega)\Psi(s,\theta_s\omega,\varphi^*_{\theta_t\omega}(\xi)(s) + \Phi(t-s,\theta_s\omega)\xi)\,\mathrm{d}s, \quad \omega \in \tilde{\Omega}, \\ h(\omega,\xi) = 0, \quad \omega \notin \tilde{\Omega}, \end{cases} \tag{3.10}$$

with the inverse

$$\begin{cases} h^{-1}(\omega,\eta) = \eta - \int_0^t \Phi(t-s,\theta_s\omega)\Psi(s,\theta_s\omega,\psi(s,\omega,\eta))\,\mathrm{d}s, \quad \omega \in \tilde{\Omega}, \\ h(\omega,\eta) = 0, \quad \omega \notin \tilde{\Omega}, \end{cases} \tag{3.11}$$

such that statements (i) and (ii) holds.

Let us check that the homeomorphism defined in (3.10) and its inverse are near identity. Indeed, it is easy to verify that

$$\begin{aligned} |h^{-1}(\omega,\eta)-\eta|_{\theta_t\omega} &\overset{(3.11)}{\le} \int_0^t |\Phi(t-s,\theta_s\omega)|_{\theta_s\omega,\theta_t\omega}|\Psi(s,\theta_s\omega,\psi(s,\omega,x))|_{\theta_t\omega}\,ds \\ &\overset{(3.5)}{\le} \int_0^t e^{(\lambda_1+a)(t-s)}|\Psi(s,\theta_s\omega,\psi(s,\omega,x))|_{\theta_t\omega}\,ds \overset{(3.12)}{\le} M\int_0^t e^{(\lambda_1+a)(t-s)}\,ds \\ &\le \frac{M}{-\lambda_1-a}. \end{aligned}$$

Using the same arguments, we obtain

$$|h(\omega,\xi)-\xi|_{\theta_t\omega} \le \frac{M}{-\lambda_1-a}. \qquad \square$$

Theorem 3.7 (Smooth linearization of RDS)**.** *Assume that the linear random dynamical system Φ generated by (3.6) satisfies the assumptions of the MET A.8, has negative Lyapunov exponents. Assume further that all partial derivatives $D_2^j\Psi : I \times \Omega \times X \to L_j(X)$, $1 \le j \le m$ satisfy Carathéodory property, and, moreover, there exist reals $M_j \ge 0$ such that*

$$|D_2^j\Psi(\omega,x)|_{\theta_t\omega} \le M_j \quad \textit{for almost all } t \in I, \omega \in \Omega, \textit{ all } x \in X, 0 \le j \le m, \tag{3.12}$$

and $M_1 \le \alpha$, $\alpha = -\lambda_1 - a$, $\alpha > 0$, then all statements of the previous theorem hold and $h(\omega,\cdot)$ is a C^m-diffeomorphism.

Proof. It suffices to demonstrate that the homeomorphism (3.10) is C^m. To do this we express the $D_2h(\omega,\cdot)^{-1}$ in a direct form, using the same arguments as in Theorem 2.6 :

$$D_2h(\omega,\cdot)^{-1} = \Phi(t,\omega)D_3\psi(s,\omega,\cdot).$$

Applying Theorem [1, Thm. 2.2.2, p. 60], we see that the determinant $\det D\psi(t,\omega,x)$ satisfies Liouville's equation:

$$\det D\psi(t,\omega,x) = \exp\int_0^t \operatorname{trace} D\Psi(\theta_s\omega,\psi(s,\omega,x))\,\mathrm{d}s.$$

Moreover, the linear part of RDS satisfies:

$$\det \Phi(t,\omega) = \exp \int_0^t \operatorname{trace} \Phi(t-s,\theta_s\omega)\, \mathrm{d}s.$$

Therefore, $\det(\Phi(t,\omega)D_3\psi(s,\omega,\eta)) > 0$ for any $t \in I$, $\omega \in \Omega$ which gives us the invertibility of $D_2h^{-1}(\omega,\eta)$ and the theorem follows. □

The assurance of the existence and smallness of a Lipschitz constant, as required in the last theorem, is, of course, not guaranteed in general. Therefore, we will adjust the nonlinear part of the vector fields by cutting them off when they exceed a certain threshold. This will bring us to the local linearization result, which is more suitable to stochastic systems and can be applied in the next section.

For deriving local results from global ones we consider a random dynamical system ψ having a fixed point at the origin. Our objective is to construct a new random dynamical system, denoted as $\tilde{\psi}$, which faithfully reproduces the behavior of ψ within a localized random neighborhood of origin. To make this construction easier, we use the insights from the following lemmas.

Lemma 3.8 ([44, Lemma 4.2, p. 6262])**.** *Let $L_0 > 0$, $(\Omega,\mathcal{F},\mathbb{P},(\theta_t)_{t\in I})$ denote a measurable dynamical system, and consider a measurable mapping $F : \Omega \times X \to X$ such that $F(\omega,\cdot)$ is continuous with $F(\omega,0) = 0$ for all $\omega \in \Omega$. If*

$$\lim_{(x,y)\to(0,0)} \frac{\|F(\omega,x) - F(\omega,y)\|}{\|x-y\|} = 0 \tag{3.13}$$

and

$$\sup_{\|x\|\le c,\|y\|\le c, x\neq y} \frac{\|F(\omega,x) - F(\omega,y)\|}{\|x-y\|} \in L^1(\Omega,\mathcal{F},\mathbb{P}) \tag{3.14}$$

hold for some $c > 0$, then there are measurable mappings $\sigma : \Omega \to (0,c]$, $L : \Omega \to \mathbb{R}^+$ and $\tilde{F} : \Omega \times X \to X$, as well as a θ_t-invariant set $\hat{\Omega} \in \mathcal{F}$ with $\mathbb{P}(\hat{\Omega}) = 1$ such that the following holds:

(i) if we define the random neighborhood of 0 by

$$U(\omega) := \{x \in X : \|x\| < \sigma(\omega)\}, \tag{3.15}$$

then the identity $F(\omega,x) = \tilde{F}(\omega,x)$ holds for all $\omega \in \hat{\Omega}$ and $x \in U(\omega)$. Additionally, we have $\tilde{F}(\omega,x) = 0$ for $\omega \in \Omega \setminus \hat{\Omega}$ and $x \in X$;

(ii) for arbitrary $\omega \in \hat{\Omega}$ and $x,y \in X$ we have

$$|\tilde{F}(\omega,x) - \tilde{F}(\omega,y)|_\omega \le L(\omega)|x-y|_\omega \quad \text{and} \quad |\tilde{F}(\omega,x)|_\omega \le L(\omega)c,$$

as well as $\int_0^1 L(\theta_s\omega)ds \le L_0$;

(iii) for all $\omega \in \hat{\Omega}$ and $a,b \in \mathbb{R}$ with $a \le b$ the estimate $\inf_{a\le t\le b} \sigma(\theta_t\omega) > 0$ holds, i.e., the mapping $\sigma(\theta_t\omega)$ is locally bounded away from 0.

Lemma 3.9 ([44, Prop. 4.3, p. 264])**.** *Assume we are given a continuous random dynamical system ψ on X over the ergodic measurable dynamical system $(\Omega,\mathcal{F},\mathbb{P},(\theta_t)_{t\in I})$ with fixed point 0, which is generated by the random differential equation (3.7). If $\|A\| \in L^1(\Omega,\mathcal{F},\mathbb{P})$, $F(\omega,0) = 0$ and*

$$\lim_{(x,y)\to(0,0)} \frac{\|F(\omega,x) - F(\omega,y)\|}{\|x-y\|} = 0$$

for all $\omega \in \Omega$, then the following holds

(i) There is a θ_t-invariant set $\hat{\Omega} \in \mathcal{F}$ with $\mathbb{P}(\hat{\Omega}) = 1$, as well as a measurable mapping $\sigma : \Omega \to \mathbb{R}^+$ and a continuous random dynamical system $\tilde{\psi}(t,\omega,x) = \Phi(t,\omega)x + \tilde{\Psi}(t,\omega,x)$ over $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in I})$ satisfying (H_3^0), as well as

$$\psi(t,\omega,x) = \tilde{\psi}(t,\omega,x) \tag{3.16}$$

for $\omega \in \hat{\Omega}$, $x \in U(\omega)$ and $t \in I_{\max}(\omega,x) := \{0, t_{\max}(\omega,x)\}$, where $U(\omega)$ is defined in (3.15) and

$$t_{\max}(\omega,x) := \sup\{t \in I : \psi(\tau,\omega,x) \in U(\theta_\tau\omega) \quad \text{for all } 0 \le \tau \le t\} \ge 0. \tag{3.17}$$

(ii) For every $\omega \in \hat{\Omega}$ we have $\lim_{x\to 0} t_{\max}(\omega,x) = \infty$. In particular, this implies that for every choice of $\omega \in \hat{\Omega}$ and $\tau_0 \in I$ the existence of a neighborhood $U_{\tau_0}(\omega) \subset X$ of the origin such that the equality (3.16) holds for all $t \in (0,\tau_0)$ and $x \in U_{\tau_0}(\omega)$.

Lemma 3.10. *If the conditions of Lemma 3.9 holds, then the nonlinear part $\tilde{\Psi}$ of the newly defined local random dynamical system $\tilde{\psi}$ satisfies the assumptions contained in (H_3^0) with Ω replaced by $\hat{\Omega}$.*

Proof. Let $\tilde{\Phi}_\omega(t,s)$ denote the evolution operator of the linear random differential equation $\dot{x} = A(\theta_t\omega)x$. The correspondence between the linear random dynamical system $\Phi(t-s,\theta_s\omega)$ and $\tilde{\Phi}_\omega(t,s)$ shown in Remark 3.5, together with (3.4) implies

$$|\tilde{\Phi}_\omega(t,s)|_{\theta_s\omega,\theta_t\omega} = |\Phi(t-s,\omega)|_{\theta_s\omega,\theta_t\omega} \le e^{(\lambda_1+a)(t-s)} \tag{3.18}$$

for arbitrary $s \le t, \omega \in \hat{\Omega}$.

Let $\tilde{F}$ denote the mapping guaranteed by Lemma 3.8. Since $\tilde{\psi}(t,\omega,x)$ solves the initial value problem $\dot{x} = A(\theta_t\omega)x + \tilde{F}(\theta_t\omega,x)$, $x(0) = x$, from the variation of constants formula we have

$$\tilde{\psi}(t,\omega,x) = \tilde{\Phi}_\omega(t,0)x + \int_0^t \tilde{\Phi}_\omega(t,s)\tilde{F}(\theta_s\omega,\tilde{\psi}(s,\omega,x))ds,$$

and therefore

$$\tilde{\Psi}(t,\omega,x) = \int_0^t \tilde{\Phi}_\omega(t,s)\tilde{F}(\theta_s\omega,\tilde{\Phi}_\omega(s,0)x + \tilde{\Psi}(s,\omega,x))ds,$$

for arbitrary $t \in I$, $x \in X$ and $\omega \in \hat{\Omega}$. This implies the estimate

$$\begin{aligned}
|\tilde{\Psi}(t,\omega,x)|_{\theta_t\omega} &\le \int_0^t |\tilde{\Phi}_\omega(t,s)|_{\theta_s\omega,\theta_t\omega}|\tilde{F}(\theta_s\omega,\tilde{\Phi}_\omega(s,0)x + \tilde{\Psi}(s,\omega,x))|_{\theta_s\omega}\, ds \\
&\le c\int_0^t e^{(\lambda_1+a)(t-s)}L(\theta_s\omega)\, ds \le ce^{(\lambda_1+a)}\int_0^1 L(\theta_s\omega)ds \\
&\le L_0 ce^{(\lambda_1+a)},
\end{aligned}$$

for any $t \in [0,1]$, $\omega \in \hat{\Omega}$ and $x \in X$.

Now, our aim is to verify the Lipschitz condition. For any $t \in [0,1]$, $\omega \in \hat{\Omega}$ and $x, \bar{x} \in X$ we have:

$$\begin{aligned}
&|\tilde{\Psi}(t,\omega,x) - \tilde{\Psi}(t,\omega,\bar{x})|_{\theta_t\omega} \\
&\le \int_0^t |\tilde{\Phi}_\omega(t,s)|_{\theta_s\omega,\theta_t\omega}|\tilde{F}(\theta_s\omega,\tilde{\Phi}_\omega(s,0)x + \tilde{\Psi}(s,\omega,x)) - \tilde{F}(\theta_s\omega,\tilde{\Phi}_\omega(s,0)\bar{x} + \tilde{\Psi}(s,\omega,\bar{x}))|_{\theta_s\omega}\, ds \\
&\le c\int_0^t e^{(\lambda_1+a)(t-s)}L(\theta_s\omega)(|\tilde{\Phi}_\omega(s,0)(x-\bar{x})|_{\theta_s\omega}\, ds + |\tilde{\Psi}(s,\omega,x) - \tilde{\Psi}(s,\omega,\bar{x})|_{\theta_s\omega})\, ds.
\end{aligned}$$

Denote by $\vartheta(t) := e^{-(\lambda_1+a)t}|\tilde{\Psi}(s,\omega,x) - \tilde{\Psi}(s,\omega,\bar{x})|_{\theta_s\omega}$. Then the last estimation can be expressed as follows

$$\begin{aligned}\vartheta(t) &\leq \int_0^t e^{-(\lambda_1+a)s} L(\theta_s\omega)|\tilde{\Phi}_\omega(s,0)|_{\omega,\theta_s\omega}|x-\bar{x}|_\omega\, ds + \int_0^t L(\theta_s\omega)\vartheta(s)\, ds \\ &\leq |x-\bar{x}|_\omega \int_0^t L(\theta_s\omega)\, ds + \int_0^t L(\theta_s\omega)\vartheta(s)\, ds.\end{aligned}$$

By applying Gronwall's inequality, we obtain

$$\begin{aligned}\vartheta(t) &\leq |x-\bar{x}|_\omega \int_0^t L(\theta_s\omega)\, ds + \int_0^t |x-\bar{x}|_\omega L(\theta_s\omega)\left(\int_0^s L(\theta_\tau\omega)\, d\tau\right) e^{\int_s^t L(\theta_\tau\omega)\, d\tau}\, ds \\ &\leq |x-\bar{x}|_\omega (L_0 + L_0^2 e^{L_0}),\end{aligned}$$

for any $s,t \in [0,1], s \leq t$, $x, \bar{x} \in X$ and $\omega \in \hat{\Omega}$.

Finally, we obtain

$$|\tilde{\Psi}(t,\omega,x) - \tilde{\Psi}(t,\omega,\bar{x})|_{\theta_t\omega} \leq e^{\lambda_1+a}(L_0 + L_0^2 e^{L_0})|x-\bar{x}|_\omega,$$

which completes the proof. □

Now, we are able to formulate a local C^m result as well.

Theorem 3.11 (Local topological linearization of RDS)**.** *Let ψ be a given continuous random dynamical system on X over the ergodic metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in I})$ with fixed point 0, generated by the random differential equation (3.7). And and Φ be a linear random dynamical system on X, generated by the random differential equation (3.6). If Φ satisfies MET, has negative Lyapunov exponents, $F(\omega,0) = 0$ and*

$$\lim_{(x,y)\to(0,0)} \frac{\|F(\omega,x) - F(\omega,y)\|}{\|x-y\|} = 0$$

for all $\omega \in \Omega$, then the newly defined local random dynamical system $\tilde{\psi}$ and Φ are topologically equivalent on $I_{\max}(\omega,x) := (0, t_{\max})$ in the sense of Theorem 3.6 for all $x \in U(\omega)$ and $\omega \in \hat{\Omega}$, where $t_{\max}$ defined in (3.17) and $U(\omega)$ defined in (3.15).

Theorem 3.12 (Local smooth linearization of RDS)**.** *If the assumptions of Theorem 3.11 hold and additionally there exist reals $M_j \geq 0$ such that all partial derivatives $D_2^j F : \hat{\Omega} \times U(\omega) \to L_j(U(\omega))$, $1 \leq j \leq m$ satisfy*

$$|D_2^j F(\omega,x)|_{\theta_t\omega} \leq M_j \quad \text{for almost all } t \in I_{\max},\ \omega \in \hat{\Omega}, \text{ all } x \in U(\omega) \text{ and } 0 \leq j \leq m, \tag{3.19}$$

then random dynamical system $\tilde{\psi}$ and Φ are C^m-topologically equivalent on $I_{\max}$.

Remark 3.13. *The proof of both above Theorems 3.11 and 3.12 follow from the application of the Theorems 3.6 and 3.7 respectively to the new random dynamical system $\tilde{\psi}$ with X restricted to $U(\omega)$ and I restricted to $I_{\max}(\omega,x)$.*

3.2 C^m linearization of stochastic differential equations

In this section, we establish a local correspondence between a random dynamical system, generated by a stochastic differential equation, and its linearization by using conjugate transformations and applying previously established local linearization results. Assume that X is a finite dimensional Banach space. We will also need the following notations borrowed from [24]. For $k \in \mathbb{N}, \delta \in (0,1]$, denote by $C_b^{k,\delta}$ the set of functions $f : X \to X$ which have partial derivatives $D^i f$ of order up to $|\alpha| \leq k$, are linearly bounded and for which the derivatives of order k are δ-Hölder continuous. In other words,

$$\sup_{x\in X} \frac{|f(x)|}{1+|x|} + \sum_{|\alpha|=1}^{k} \sup_{x\in X} |D^\alpha f(x)| + \sum_{|\alpha|=k} \sup_{\substack{x,\bar{x}\in X \\ x\neq\bar{x}}} \frac{|D^\alpha f(x) - D^\alpha f(\bar{x})|}{|x-\bar{x}|^\delta} < \infty.$$

In order to obtain an SDE result, the approach of [22] plays a key role for us. Specifically, the next two theorems demonstrate the feasibility of a suitable random stationary coordinate transformation (cohomology) that establishes a connection between cocycles of stochastic and random differential equations.

Theorem 3.14 ([22, Thm. 3.1, p. 140])**.** *If $f_1, \ldots, f_k \in C_b^{2,\delta}$ satisfy $\sum_{i=1}^k f_i^j \frac{\partial f_i}{\partial x_j} \in C_b^{2,\delta}$, then there exists a random flow of diffeomorphisms $\Phi : I \times X \times \Omega \to X$ such that the following properties hold:*

(i) for any $(\eta, \tau) \in X \times I$ the process $\Phi(x,\tau)$ satisfies the stochastic integral equation

$$\Phi_t(x,\tau) = x + e^{-\tau} \sum_{i=1}^{k} \int_{-\infty}^{t} e^s f_i(\Phi_s(x,\tau)) \circ dW_s^i; \tag{3.20}$$

(ii) if

$$H_t = \Phi_t(\cdot, t), \tag{3.21}$$

$$\Gamma_t = \frac{\partial}{\partial \tau} \Phi_t(\cdot, t), \tag{3.22}$$

then H is a stationary cocycle of diffeomorphisms, Γ is a stationary random field, and for any $x \in X$ the processes $H(x)$ and $\Gamma(x)$ satisfy the SDE

$$dH_t(x) = \sum_{i=1}^{k} f_i(H_t(x)) \circ dW_t^i + \Gamma_t(x)dt;$$

(iii) for $x \in X$, $\Gamma(x)$ satisfies the stochastic integral equation

$$\Gamma_t(x) = -(H_t(x) - x) + e^{-t} \sum_{i=1}^{k} \int_{-\infty}^{t} e^s \frac{\partial}{\partial x} f_i(H_t(x)) \Gamma_s(x) \circ dW_s^i.$$

Theorem 3.15 ([22, Thm. 3.2, p. 142])**.** *If $f_0 \in C_b^{1,\delta}$ and $f_1, \ldots, f_k \in C_b^{2,\delta}$ satisfy $\sum_{i=1}^k f_i^j \frac{\partial f_i}{\partial x_j} \in C_b^{2,\delta}$, then the following holds for the random vector field $g : \Omega \times X \to X$,*

$$g(\cdot, y) := \frac{\partial}{\partial y} H_0^{-1}(y)[f_0(H_0(y)) + \Gamma_0(y)],$$

with H and Γ given by the previous theorem:

(i) *the random differential equation*

$$dy_t = g(\theta_t \cdot, y_t)dt$$

generates a random cocycle of diffeomorphism Ψ*;*

(ii) *the SDE*

$$dx_t = f_0(x_t)dt + \sum_{i=1}^{k} f_i(x_t) \circ dW_t^i$$

generates a random cocycle of diffeomorphism Φ*;*

(iii) Φ *and* Ψ *are conjugate with cohomology* H_0*, i.e., for* $t \in I$, $\omega \in \Omega$

$$\Phi_t(\omega) = H_0(\theta_t\omega, \cdot) \circ \Psi(\omega) \circ H_0^{-1}(\omega, \cdot).$$

Remark 3.16. *It is important to note the following. Consider Theorems 3.14 and 3.15 in the context of a higher degree of smoothness for the vector fields* $f_1, \dots, f_k$*. Specifically, suppose* $f_1, \dots, f_k \in C_b^{m+1,\delta}$*, and* $\sum_{i=1}^{m} f_i^j \frac{\partial f_i}{\partial x_j} \in C_b^{m+1,\delta}$*. Then* Φ *and, consequently,* H *are* C^m *diffeomorphisms.*

This fact is a direct consequence of applying the following Theorem 3.17 to system (3.20).

Theorem 3.17 ([1, Thm. 2.3.32, p. 93])**.** *If* $f_0 \in C_b^{m,\delta}$ *and* $f_1, \dots, f_k \in C_b^{m+1,\delta}$ *satisfy* $\sum_{i=1}^{k} f_i^j \frac{\partial f_i}{\partial x_j} \in C_b^{m,\delta}$*, then the classical Stratonovich stochastic differential equation*

$$dx_t = f_0(x_t)dt + \sum_{j=1}^{k} f_j(x_t) \circ dW_t^j$$

generates a unique C^m *random dynamical system over the filtered dynamical system describing a Brownian motion.*

Transitioning between SDEs and RDEs through coordinate changes must maintain the principles of a multiplicative ergodic theory, such as Lyapunov exponents and Oseledets spaces. The following proposition demonstrates that these properties are indeed preserved, given that the coordinate change fulfills specific integrability criteria.

Lemma 3.18 ([22, Prop. 5.1, p. 156])**.** *Let* $\Phi : I \times \Omega \to X$ *be a linear cocycle such that*

$$\sup_{0\le t\le 1} \log^+ \|\Phi(t,\omega)\| \in L^1(\Omega, \mathcal{F}, \mathbb{P}), \qquad \sup_{0\le t\le 1} \log^+ \left\|\Phi^{-1}(t,\omega)\right\| \in L^1(\Omega, \mathcal{F}, \mathbb{P}).$$

If $H : \Omega \times X \to X$ *is a random linear mapping satisfying*

(i) $t \mapsto H(\theta_t\omega)$ *is continuous for every* $\omega \in \Omega$*,*

(ii) *one has the integrability condition*

$$\sup_{0\le t\le 1} \log^+ \|H(\theta_t\omega)\| \in L^1(\Omega, \mathcal{F}, \mathbb{P}), \qquad \sup_{0\le t\le 1} \log^+ \left\|H^{-1}(\theta_t\omega)\right\| \in L^1(\Omega, \mathcal{F}, \mathbb{P}),$$

then

$$\Psi(t,\omega) = H(\theta_t\omega, \cdot) \circ \Psi(t,\omega) \circ H^{-1}(\omega, \cdot)$$

for any $t \in I$ *and almost all* $\omega \in \Omega$ *defines a linear cocycle which satisfies MET.* Ψ *possesses the same Lyapunov exponents as* Φ*. If* $U_1, \dots, U_k$ *are the Oseledets spaces of* Φ*, then the Oseletets spaces of* Ψ *are given by* $HU_1, \dots, HU_k$*.*

Following the preliminary discussions, we establish a local C^m-linearization result for random dynamical systems (RDS) generated by stochastic differential equations (SDEs).

Theorem 3.19 (Local smooth linearization of SDEs)**.** *Let the vector fields $f_0 \in C_b^{m,\delta}$ and $f_1, \dots, f_k \in C_b^{m+1,\delta}$ satisfy $f_i(0) = 0$, $i = 0, \dots, k$, and additionally assume*

$$\sum_{i=1}^{k} \sum_{j=1}^{d} f_i^j \frac{\partial f_i}{\partial x_j} \in C_b^{m+1,\delta}.$$

If the random dynamical system Φ generated by the linear stochastic differential equation

$$dx_t = \frac{\partial}{\partial x} f_0(0) x_t \, dt + \sum_{i=1}^{k} \frac{\partial}{\partial x} f_i(0) x_t \circ dW_t^i \tag{3.23}$$

satisfies MET with negative Lyapunov exponents, then it is C^{m-1}-topologically equivalent to the RDS ψ generated by the stochastic differential equation

$$dx_t = f_0(x_t) \, dt + \sum_{i=1}^{k} f_i(x_t) \circ dW_t^i \tag{3.24}$$

for all $x \in X$, $t \in I$ and almost all $\omega \in \Omega$.

Proof. To prove the theorem, first, we need to conjugate the nonlinear SDE (3.24) with the corresponding random differential equation.

In order to do this, let us construct Γ and H and for $y \in X$ define the random field $g : \Omega \times X \to X$ according to Theorems 3.14, 3.15 by

$$g(\cdot, x) = \frac{\partial}{\partial x} H_0^{-1}(x)(f_0(H_0(y)) + \Gamma_0(y)).$$

The random differential equation induced by g has 0 as a fixed point by our hypothesis and construction and is differentiable. Let

$$A = \frac{\partial}{\partial x} g.$$

Denote by ψ^0 the RDS generated by g, and by Φ^0 the RDS generated by

$$dx_t = A(\theta_t \cdot) x_t \, dt.$$

By the construction, the RDS ψ is conjugated to ψ^0 via the cohomology H, and the RDS Φ generated by (3.23) is conjugated to Φ^0 via $\frac{\partial H}{\partial x}$.

Now we turn to the application of the local linearization result (Theorem 3.11) to the newly obtained systems ψ^0 and Φ^0. In order to do this, we have to check the assumptions of Theorem 3.11.

By the construction of Γ and H, and the smoothness properties of f_i for $0 \leq i \leq k$, it follows that all required properties for g are satisfied. Moreover, from Lemma 3.18 and the construction of H, it follows that Φ^0 satisfies the integrability condition of the MET. Therefore, by applying the local linearization theorem, we conclude that Φ and ψ^0 are topologically equivalent via C^m-diffeomorphism h.

With all the groundwork laid, we now proceed to conclude that random dynamical systems Φ and ψ are topological equivalent via the C^{m-1}-diffeomorphism

$$\tilde{h} = \frac{\partial}{\partial x} H_0(\theta_t \omega, \cdot) \circ h \circ H_0^{-1}(\omega, \cdot)$$

for all $x \in X$, $t \in I$, and almost all $\omega \in \Omega$. □

A Appendix

This appendix contains all the essential results and lemmas used throughout the paper to support the proofs of our results. It is organized into two logical parts: deterministic and random.

In the deterministic part, we start with Gronwall's Lemma, which is fundamental for estimating the growth of solutions of differential equations.

Throughout, X, Y denote Banach spaces.

Lemma A.1 (Gronwall's inequality [2, Lemma 2.5, p. 53], [10]). *Suppose we are given a nonempty interval I, a point $\tau \in I$ and let $\alpha, u : I \to \mathbb{R}$ be continuous with α non-decreasing. If for fixed $\beta > 0$ the inequality*

$$u(t) \leq \alpha(t) + \int_\tau^t \beta u(s) ds$$

holds, then

$$u(t) \leq \alpha(t) e^{\beta(t-\tau)} \quad \text{for all } t \in I. \tag{A.1}$$

Theorem A.2 (Variation of Constants Formula [2, Thm. 2.10, p. 58], [15]). *Consider the system*

$$\dot{x} = A_\omega(t)x + F_\omega(t, x), \tag{A.2}$$

where $A_\omega \colon I \to L(X)$ and $F_\omega \colon I \times X \to X$ are locally integrable. Then the solution $\varphi(t, s, \xi)$ of (A.2) is given by the variation of constants formula:

$$\varphi(t, s, \xi) = \Phi_\omega(t, s)\xi + \int_s^t \Phi_\omega(t, \tau) F_\omega(\tau, \varphi(\tau, s, \xi))\, d\tau,$$

which is valid for all $(t, s, \xi) \in I \times I \times X$.

Now we present the well-known fixed point theorem of Banach and some related results.

Theorem A.3 (Banach's Fixed Point Theorem [2, Thm. B.1, p. 114], [45]). *If $\mathcal{T} \colon X \to X$ a contraction, i.e. there exists $0 \leq c < 1$ such that*

$$\|\mathcal{T}(x) - \mathcal{T}(\bar{x})\| \leq c \|x - \bar{x}\| \quad \text{for all } x, \bar{x} \in X,$$

then there exists exactly one fixed point $\varphi \in X$ of $\mathcal{T}$.

Theorem A.4 (Uniform Contraction Principle [13, Thm. 2.2, p. 25]). *Suppose $U \subseteq X$, $V \subseteq Y$ are open, let $\bar{U}$ denote the closure of U and $m \in \mathbb{N}_0$. If $\mathcal{T} \in C^m(\bar{U} \times V, X)$, is a uniform contraction, i.e. there exists a $0 \leq c < 1$ such that*

$$\|\mathcal{T}(x, y) - \mathcal{T}(\bar{x}, y)\| \leq c \|x - \bar{x}\| \quad \text{for all } x, \bar{x} \in \bar{U} \text{ and } y \in V,$$

then the unique fixed point $\varphi(y)$ of $\mathcal{T}(\cdot, y)$ in $\bar{U}$ satisfies $\varphi \in C^m(V, X)$.

Theorem A.5 (Local Inverse Mapping Theorem [45, Thm. 4.F, p. 172]). *Let $U \subseteq X, x_0 \in U$ and $f : U \to Y$ be a C^m mapping for $m \in \mathbb{N}$. Then f is a local C^m-diffeomorphism at x_0 if and only if $Df(x_0)$ is bijective.*

Theorem A.6 (Global Inverse Mapping Theorem [45, Thm. 4.G, p. 174]). *Let $f : X \to Y$ be a local C^m-diffeomorphism for $m \in \mathbb{N}$ at every point of X. Then f is a C^m-diffeomorphism if and only if f is proper.*

Theorem A.7 (Differentiation under the integral [40, Thm. A.5, p. 263]). *Let $U \subseteq X, V \subseteq Y$ be open and $f : U \times V \to X$ be such that $f(\cdot, y)$ is integrable for any $y \in V$. Suppose, that for any $x \in U, y \in V$, there exist derivative $D_2 f(x,y)$, and*

$$\|D_2 f(x,y)\| \leq g(x),$$

where $g : U \to \mathbb{R}^+$ is integrable. Then the mapping

$$F(y) = \int_U f(x,y)dx$$

is differentiable with derivative is given by

$$DF(y) = \int_U D_2 f(x,y)dx.$$

Now let us turn to a key result in random dynamical systems: the fundamental Oseledets's multiplicative ergodic theorem (MET).

In the following we will write $\log^+(x) = \max\{0, \log(x)\}$ for any $x \in X$. For a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we denote by $L^1(\Omega, \mathcal{F}, \mathbb{P})$ the space of all integrable measurable functions.

Theorem A.8 (Multiplicative Ergodic Theorem, half line, continuous time [1, Thm. 3.4.1, p. 134]). *Let Φ be a linear cocycle over an ergodic MDS $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in I})$. Assume that*

$$\sup_{0\leq t\leq 1} \log^+ \|\Phi(t,\omega)\| \in L^1(\Omega, \mathcal{F}, \mathbb{P}), \qquad \sup_{0\leq t\leq 1} \log^+ \left\|\Phi^{-1}(t,\omega)\right\| \in L^1(\Omega, \mathcal{F}, \mathbb{P}). \tag{A.3}$$

Then there exist a forward invariant set $\tilde{\Omega} \in \mathcal{F}$ of full measure such that for every $\omega \in \tilde{\Omega}$ the following statements hold:

(i) There exist $k(\omega)$ numbers $\lambda_1(\omega) > \cdots > \lambda_{k(\omega)}(\omega)$ and the invariant splitting $X = U_1(\omega) \oplus \cdots \oplus U_{k(\omega)}(\omega)$ such that

$$\begin{aligned} k(\theta_t\omega) &= k(\omega), \\ \lambda_i(\theta_t\omega) &= \lambda_i(\omega) \quad \text{for all } i \in \{1, 2, \ldots, k(\omega)\}, \\ d_i(\theta_t\omega) &= d_i(\omega) \quad \text{for all } i \in \{1, 2, \ldots, k(\omega)\}, \end{aligned}$$

where $d_i(\omega) := \dim U_i(\omega)$.

(ii) Put $V_{k(\omega)+1}(\omega) := 0$ and for $i = 1, \ldots, k(\omega)$

$$V_i(\omega) := U_i(\omega) \oplus \cdots \oplus U_{k(\omega)}(\omega)$$

so that

$$V_{k(\omega)}(\omega) \subset \cdots \subset V_i(\omega) \subset \cdots \subset V_1(\omega)$$

defines a filtration on X. Then for each $x \in X \setminus \{0\}$ the Lyapunov exponent

$$\lambda(\omega, x) := \lim_{t\to\infty} \frac{1}{t} \log \|\Phi(t,\omega)x\|$$

exists and

$$\lambda(\omega, x) = \lambda_i(\omega) \iff x \in V_i(\omega) \setminus V_{i+1}(\omega),$$

i.e.

$$V_i(\omega) = \{x \in X : \lambda(\omega, x) \leq \lambda_i(\omega)\}.$$

(iii) For all $x \in X \setminus \{0\}$

$$\lambda(\theta_t\omega, \Phi(t,\omega)x) = \lambda(\omega, x),$$

whence

$$\Phi(t,\omega)V_i(\omega) \subset V_i(\theta_t\omega) \quad \text{for all } i \in \{1, 2, \ldots, k(\omega)\}.$$

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Complex oscillations of non-definite Sturm–Liouville problems, II

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Abstract. We correct and update a result of R. G. D. Richardson [*Amer. J. Math.* **40**(1918), 283–316] dealing with the separation of zeros of the real and imaginary parts of non-real eigenfunctions of non-definite Sturm–Liouville eigenvalue problems. We then extend it to the case where the weight function is allowed to be identically zero on a subinterval that excludes the end-points and study the behavior of the zeros of the real and imaginary parts when the end-points are included. Examples are given illustrating the sharpness of the results along with open questions.

Keywords: Sturm–Liouville, eigenvalues, non-definite, complex ghosts, oscillations, zeros, separation theorem

2020 Mathematics Subject Classification: 34B24, 34B09, 34C10.

1 Introduction

We consider the Sturm–Liouville eigenvalue problem with real piecewise continuous coefficients over an interval $[a, b]$,

$$-y'' + q(x)y = \lambda r(x)y, \tag{1.1}$$

$$y(a) = y(b) = 0. \tag{1.2}$$

in the non-definite case, [11], i.e., this is essentially the case where the quadratic forms induced by the two operators on the left and on the right are sign-indefinite on their domains. The more general equation

$$-(p(x)y')' + q(x)y = \lambda r(x)y$$

where $p(x)$ is of one sign may be reduced to (1.1) by means of a Liouville transformation, and so leads to no new results. Thus, we shall restrict ourselves to the formulation (1.1) in this paper.

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Tacit assumptions and notation

In the sequel we will always assume that the eigenvalue problem (1.1)–(1.2) is non-definite and has at least one pair of non-real eigenvalues, usually denoted by $\lambda, \bar{\lambda} \in \mathbf{C}$. The problem of the actual *existence* of non-real eigenvalues for problems of this type is difficult and very few results exist in this direction, see, e.g., [1,14]. Given a non-real eigenvalue of (1.1)–(1.2), an associated non-real eigenfunction will be denoted by $y = \varphi + i\psi$, where φ, ψ are its, normally unspecified, real and imaginary parts. A non-real eigenfunction will be called a complex ghost state or a *complex ghost*, for short [11]. Hereinafter, all solutions will be assumed non-trivial unless otherwise specified. In addition, since Im $\lambda \neq 0$ we will always assume that, unless otherwise specified, Im $\lambda > 0$. This is always possible as we can replace λ by $\bar{\lambda}$ and y by $\bar{y}$ in (1.1). Insofar as tacit assumptions are concerned we will always assume that $q, r \in L^1(a,b)$, so that solutions of initial value problems associated with (1.1) over $[a,b]$ exist and are unique in the Carathéodory sense, [4], i.e., $y, y' \in AC[a,b]$ and (1.1) holds a.e. in $[a,b]$.

The following original result is essentially stated in Richardson [13], see also [9,11].

Theorem 1.1 ([13, Theorem X])**.** Let r be continuous in $[a,b]$. If $r(x)$ changes its sign precisely once in (a,b) then the roots of the real and imaginary parts of any complex ghost separate one another.

A discussion of Theorem 1.1 is necessary. First, it was understood at the time of Richardson's paper (as it is now in classical Sturmian cases, such as in Sturm's separation theorem) that the expression *separate one another* meant that between any two consecutive zeros of one solution there is always exactly one zero (of any other independent solution) that is distinct from either of the two original zeros. It is unlikely that Richardson had in mind the possibility that the weight function, $r(x)$, could be identically zero in a subinterval, however the latter possibility does change the result(s). To be clear, in this paper, the statement that two continuous functions f, g have zeros that separate one another on $[a,b]$ means that between any two consecutive zeros of one there is exactly one zero of the other that is different from either of the other two.

We will show below that Theorem 1.1 as stated is, in general, not true (see e.g., Example 3.2 and Figure 3.1 below). This was alluded to in [9] but without discussion. It is true if one excludes the endpoints as one of the two zeros in question, i.e., we are dealing only with interior zeros. However, counterexamples can be found in cases where the endpoints are included (see the examples at the end of this article). Indeed, we show in Remark 2.11 that if $r(x)$ is identically zero in a right-neighborhood of $x = a$ that includes the endpoint, then Theorem 1.1 is false. In fact, we go further and show in Lemma 2.1 below, that the statement in Theorem 1.1 cannot be true if the endpoints are included (if even the weight function never vanishes identically in any subinterval). As a result, Richardson's Theorem X in [13] must be replaced by something like Theorem 2.10. Indeed, in Theorem 2.10 we show that the single turning point or zero required in Theorem 1.1 may be replaced by an *interval of zeros* without affecting the conclusion.

In addition, we will find conditions under which given two consecutive zeros (one of which is an endpoint) of either the real (or imaginary) part of a complex ghost there is or there isn't a zero of the imaginary (or real) part. In this sense, this paper continues a study of the zeros of the real and imaginary parts of such ghosts, research that began with [9,10,12,13] and continued with [7,8] and the results therein.

Remark 1.2. We note that Theorem 1.1 appears to be exceptional in the sense that it has no apparent equivalent in the two turning point case (with a weight function that does not vanish

identically in a subinterval). For example, in [8] there are examples showing that the interior zeros of the real and imaginary parts of non-real eigenfunctions may not separate one another.

2 Main results

First, we show that Theorem 1.1 cannot be true if one or more end-points are included in its conclusion.

Lemma 2.1. There is no non-trivial complex ghost whose real and imaginary parts have zeros that separate one another on an interval that includes the endpoints of $[a, b]$.

Proof. Without loss of generality we may assume that the endpoint in question is $x = a$. Assuming the contrary, let $y = \varphi + \psi i$ be a non-trivial complex ghost where φ satisfies $\varphi(a) = 0$ and $\varphi(x_0) = 0$ where x_0 is a zero of φ larger than $x = a$. Then, by hypothesis, $\psi(x)$ has a unique zero t_0 in (a, x_0). Now $\psi(a) = \psi(t_0) = 0$, thus, by assumption $\varphi(x_1) = 0$ for some unique $x_1 \in (a, t_0)$. Since $\varphi(a) = \varphi(x_1) = 0$ there must be a unique zero $t_1 \in (a, x_1)$ of ψ. Continuing in this way we see that the now infinite sequence of zeros x_i, t_i must accumulate at $x = a$. On the other hand the theorem of the mean then implies that both $\varphi'(a) = 0$ and $\psi'(a) = 0$. Thus, $y'(a) = 0$ and so y is the trivial solution, a contradiction. □

Lemma 2.2. Let λ be a non-real eigenvalue of (1.1)–(1.2) and $y = \varphi + i\psi$ be a corresponding eigenfunction. Then φ, ψ are linearly independent over $[a, b]$.

Proof. Assume the contrary. Then $\psi = \mu\varphi$ where $\mu \in \mathbf{C} \setminus 0$. Since $y = (1 + i\mu)\varphi$ satisfies (1.1) we get that $-\varphi'' + q\varphi = \lambda r\varphi$. Since the left side of the previous equation is real and $\operatorname{Im}\lambda \neq 0$ we find that $r(x)\varphi(x) \equiv 0$ on $[a, b]$. Since, by hypothesis, (1.1)–(1.2) is non-definite, $r(x)$ cannot be identically zero and so there is an interval $J \subset [a, b]$ on which $r(x) > 0$, say. Thus, $\varphi(x) \equiv 0$ on J. Hence $\psi(x) \equiv 0$ on J so that $y \equiv 0$ on J. This, however, violates the uniqueness of solutions of an initial value problem with initial conditions on a point of J. This contradiction proves the lemma. □

Lemma 2.3. Let $c > a$, r be piecewise continuous on (a, c) and $r(x) < 0$ there. In addition, let $r(x) \equiv 0$ on $[c, d] \subset (a, b)$ and $r(x) > 0$ and continuous on (d, b). Then, for any complex ghost y of (1.1),

$$G(x) \equiv \int_a^x r|y|^2 \, dt, \qquad x \in [a, b], \tag{2.1}$$

is of one sign in (a, b), in fact, $G(x) < 0$, for all $x \in (a, b)$.

Proof. Since $r(x) < 0$ on $(a, c) \subseteq (a, d)$, by hypothesis, then $G(x) < 0$ too, as an eigenfunction cannot vanish identically on an interval unless it is identically zero there. Assume, on the contrary, that there is a smallest point $x_1 \in (a, b)$ with $G(x_1) = 0$. Since $G(x)$ must change its sign around x_1 it follows that $x_1 \in [d, b)$.

Next, observe that $G(x_1) = G(b) = 0$ and $G \in C^1(d, b)$. (Note that $G(b) = 0$ for complex ghosts, see [9, Theorem 1]. Hence there exists $\eta \in (d, b)$ such that $r(\eta)|y(\eta)|^2 = 0$, i.e., $y(\eta) = 0$. Consider the boundary problem (1.1) subject to the boundary conditions $y(\eta) = y(b) = 0$. This new eigenvalue problem admits our complex ghost y as an eigenfunction corresponding to the same non-real eigenvalue. Thus, this problem must be non-definite on $[\eta, b]$, i.e., $r(x)$ must change its sign on this interval, but this is a contradiction as $r(x) > 0$ there. Hence no such x_1 can exist. This completes the proof. □

The next result is an immediate consequence of Lemma 2.3.

Corollary 2.4. If $r(x) > 0$ on (a,c), $r(x) \equiv 0$ on $[c,d] \subset (a,b)$, and $r(x) < 0$ on (d,b), then $G(x) > 0$ on (a,b).

Proof. Replace λ by $-\lambda$ and r by $-r$ in Lemma 2.3 so that the eigenfunction does not change. □

Corollary 2.5 ([13, Theorem X])**.** If $r(x)$ is not identically zero in any right interval of a and $r(x)$ has a unique zero around which $r(x)$ changes sign in (a,b), then $G(x)$ is of one sign in (a,b).

Proof. This is the case where $c = d$ in Lemma 2.3 so either Lemma 2.3 or Corollary 2.4 applies. □

Corollary 2.6. Let $c > a$, r be piecewise continuous on (a,c) and $r(x) < 0$ there. In addition, let $r(x) \equiv 0$ on $[c,d] \subset (a,b)$ and $r(x) > 0$ and continuous on (d,b). Then, for any $x \in (a,b)$, the function $H(x) = \int_x^b r|y|^2\,dt$ is of one sign on (a,b), i.e., $H(x) > 0$ on (a,b).

Proof. This is an application of Lemma 2.3 and the fact that $\int_a^b r|y|^2\,dx = 0$ must hold for any complex ghost. □

Lemma 2.7. Let $y = \varphi + i\psi$ be a complex ghost. Then, for any two points $x_1, x_2 \in [a,b]$ we have,

$$\left[\varphi'\psi - \varphi\psi'\right]\Big|_{x_1}^{x_2} = \operatorname{Im}\lambda \int_{x_1}^{x_2} r|y|^2\ dx.$$

Proof. Let $y = \varphi + i\psi$ be a complex ghost. Then, from (1.1), we find that (suppressing the variables for simplicity of exposition),

$$-\varphi'' + (q - r\operatorname{Re}\lambda)\varphi = -(\operatorname{Im}\lambda)\ r\ \psi, \tag{2.2}$$
$$-\psi'' + (q - r\operatorname{Re}\lambda)\psi = (\operatorname{Im}\lambda)\ r\ \varphi, \tag{2.3}$$

where $\operatorname{Im}\lambda \neq 0$ by hypothesis. For any two points $x_1, x_2 \in [a,b]$ a simple Sturmian argument applied to (2.2) and (2.3) now yields,

$$\left[\varphi'\psi - \varphi\psi'\right]\Big|_{x_1}^{x_2} = \operatorname{Im}\lambda \int_{x_1}^{x_2} r|y|^2\ dx. \tag{2.4}$$

From (2.4) since $y(a) = 0$ is equivalent to $\varphi(a) = \psi(a) = 0$ we get that, for any $x \in [a,b]$,

$$\varphi'(x)\psi(x) - \varphi(x)\psi'(x) = \operatorname{Im}\lambda \int_a^x r|y|^2\ dt = (\operatorname{Im}\lambda)\ G(x). \tag{2.5}$$

By Lemma 2.3 and Corollary 2.4 $G(x)$ is of one sign only on (a,b) (depending on the sign of $r(x)$ on (a,c)). □

The next lemma is an application of classical folklore so a proof is given for the sake of completeness.

Lemma 2.8. Let $q, r \in C[a,b]$, and for some $c > a$, let $r(x) < 0$ on (a,c), $r(x) \equiv 0$ on $[c,d] \subset (a,b)$ and $r(x) > 0$ and continuous on (d,b). Let φ, ψ be two non-trivial solutions of the system (2.2) and (2.3). If the zeros of φ (or ψ) have a point of accumulation $x_0 \in (a,b)$, then the non-real eigenfunction, y, satisfies $y(x_0) = 0$. In addition, the end-points, $x = a, b$, cannot be points of accumulation of either φ or ψ.

Proof. By fundamental existence theorems we know that the continuity of the coefficients q, r implies that both solutions $\varphi, \psi \in C^2(a,b)$ (whenever they exist, or whenever there are non-real eigenvalues).

If possible let $x_0 \in (a,b)$ be a point of accumulation of φ and let $x_n \to x_0$ be an infinite sequence of zeros of φ accumulating at x_0. Then, by continuity, $\varphi(x_0) = 0$. On the other hand, $\varphi \in C^1$ so that $0 = (\varphi(x_n) - \varphi(x_0))/(x_n - x_0) = \varphi'(\xi_n)$ holds for all n for some ξ_n between x_0 and x_n. There follows that $\xi_n \to x_0$ as well and thus, since $\varphi \in C^1$, $\varphi'(x_0) = 0$. Since $\varphi \in C^2$ we get that $0 = (\varphi'(\xi_n) - \varphi'(x_0))/(\xi_n - x_0) = \varphi''(\zeta_n)$ for some ζ_n between ξ_n and x_0. Since $\zeta_n \to x_0$ there follows that $\varphi''(x_0) = 0$. From this and (2.2) we obtain, $r(x_0)\psi(x_0) = 0$ (as $\operatorname{Im}\lambda > 0$, by hypothesis.)

So, $\psi(x_0) = 0$ whenever $r(x_0) \neq 0$. On the other hand, if $r(x) \equiv 0$ on some subinterval $[c,d] \subset (a,b)$ where, in addition, $x_0 \in [c,d]$, then, using (2.2) and (2.3), we get that both φ, ψ are solutions of the same equation, namely, $-z'' + (q - r\operatorname{Re}\lambda)z = 0$, where $x \in [c,d]$. Thus either φ, ψ are linearly dependent, in which case $\varphi(x_0) = 0 = \psi(x_0)$ is immediate, or they are linearly independent, in which case Sturm's Separation Theorem implies the interlacing of the zeros of φ, ψ and thus, once again we must have $\psi(x_0) = 0$. Therefore, in any case we must have $\psi(x_0) = 0$. Finally, $\varphi(x_0) = \psi(x_0) = 0$ is equivalent to $y(x_0) = 0$ as was to be shown.

The same argument applies to ψ so that all its interior zeros are isolated as well.

If either $x_0 = a$ or $x_0 = b$ is an accumulation point of φ, we proceed as follows: Assume that $x = a$ is such a point. Since $y'(a) \neq 0$ (as y is non-trivial) at least one, maybe both, of $\varphi'(a), \psi'(a)$ are non-zero. We may assume that $\varphi'(a) \neq 0$ (or else if $\varphi'(a) = 0$, then $\psi'(a) \neq 0$ and we can apply the following argument to ψ as $\psi(a) = 0$). Applying the argument in the second paragraph of the proof of Lemma 2.8 above, we can conclude that $\varphi'(a) = 0$, which is a contradiction. Hence $x = a$ cannot be a point of accumulation of φ (similarly for ψ). If $x = b$ a similar argument applies whose proof is omitted. □

Corollary 2.9. Let q, r be continuous on $[a,b]$. For some $c > a$, let $r(x) < 0$ on $[a,c)$, $r(x) \equiv 0$ on $[c,d] \subset (a,b)$, and $r(x) > 0$ on $(d,b]$. Then the interior zeros of any solution φ of (2.2)–(2.3) are isolated.

Proof. Since we are dealing with interior zeros, by Lemma 2.8 if there is such an accumulation point x_0 of φ that satisfies $a < x_0 < b$, then it must also satisfy $y(x_0) = 0$ where $y = \varphi + i\psi$ is an eigenfunction of (1.1)–(1.2). Thus, $y(a) = y(x_0) = y(b) = 0$. Note that if $x_0 \in (a,d]$ then $r(x) \leq 0$ in which case the boundary problem (1.1) subject to the boundary condition $y(a) = y(x_0) = 0$ must have a non-real eigenvalue which isn't possible as the eigenvalue problem on $(a,x_0]$ is now right-definite. On the other hand, if $x_0 \in [d,b)$ then $r(x) > 0$ in $[d,b)$ and so in $[x_0,b)$. Hence, the problem (1.1) subject to the boundary condition $y(x_0) = y(b) = 0$ must have also have a non-real eigenvalue which isn't possible as this problem is also right-definite. Thus all interior zeros of φ are isolated. □

Theorem 2.10. Let $q, r \in C[a,b]$, and for some $c > a$, let $r(x) < 0$ on (a,c), $r(x) \equiv 0$ on $[c,d] \subset (a,b)$ and $r(x) > 0$. Then the interior zeros of the real and imaginary part of a complex ghost separate one another.

Proof. Since, by Lemma 2.8, the interior zeros of φ are isolated, we let $x_1 < x_2 \in (a,b)$ be two consecutive zeros of $\varphi(x)$ where, without loss of generality, we may take it that $\varphi(x) > 0$ in (x_1,x_2). (If φ is negative, we replace φ by $-\varphi$ and ψ by $-\psi$ in what follows.) Assume, if possible, that $\psi(x) \neq 0$ in $[x_1,x_2]$.

Since $r(x) < 0$ in (a, c), Lemma 2.3 implies that $G(x) < 0$ in (a, b). Assume, if possible, that $\psi(x) > 0$ in $[x_1, x_2]$. Using (2.5) we get, for $x_1 \leq x \leq x_2$,

$$\left(\frac{\varphi}{\psi}\right)' = \frac{\operatorname{Im}\lambda}{\psi^2(x)}\, G(x) < 0. \tag{2.6}$$

Thus, $0 = \varphi(x_1)/\psi(x_1) > \varphi(x)/\psi(x)$ which is impossible. Hence, $\psi(x) > 0$ in $[x_1, x_2]$ cannot hold, i.e., $\psi(x)$ must vanish somewhere in $[x_1, x_2]$. A similar argument holds in the case where $\psi(x) < 0$ in $[x_1, x_2]$ as from (2.6), $\varphi(x)/\psi(x) > \varphi(x_2)/\psi(x_2) = 0$, which is also impossible.

Now we show that $\psi(x_1)\psi(x_2) \neq 0$. Assuming, if possible, that $\psi(x_1) = 0$, then $y(x_1) = 0$ as well. Since $y(a) = y(x_1) = 0$ and y is a non-real eigenfunction, (1.1) must be non-definite on $[a, x_1]$, so $r(x)$ cannot have a fixed sign there. Therefore, we must have that $x_1 \in (d, b]$ (see the hypotheses on $r(x)$). Once again the problem (1.1), subject to the boundary conditions $y(x_1) = 0, y(b) = 0$, must be non-definite. But this is impossible since $r(x) < 0$ on (d, b). This contradiction implies that $\psi(x_1) \neq 0$. A similar argument applies to the case where $\psi(x_2) = 0$, and we leave this to the reader.

It follows from the preceding argument that $\psi(x) = 0$ somewhere in (x_1, x_2), say, at $x = t_1$. We take it that t_1 is the smallest such zero, so that $\psi(x) \neq 0$ in (t_1, t_2). However, since $\varphi(x) \neq 0$ in $[t_1, t_2]$, a modification of (2.6) using (2.5), gives

$$\left(\frac{\psi}{\varphi}\right)' = -\frac{\operatorname{Im}\lambda}{\varphi^2(x)}\, G(x) > 0, \tag{2.7}$$

from which we obtain a contradiction as before, as ψ/φ must be increasing in (t_1, t_2), yet ψ vanishes at the end-points. This contradiction shows that between any two consecutive interior zeros of φ, the function ψ has a unique zero. □

Remark 2.11. This extends and corrects Theorem 1.1 from a single turning point (or a zero around which $r(x)$ changes sign) to an interval of zeros. The condition that, for some $c > a$, $r(x) \neq 0$ on (a, c) is necessary. This is because if r vanishes identically in some subinterval of $[a, c]$ containing $x = a$, the separation property of the roots fails as φ, ψ must be linearly dependent on account of (2.4) applied to said interval.

Lemma 2.12. Let $G(x) < 0$ for all $x \in (a, b)$. Let $\varphi'(a) > 0$, (resp. $\varphi'(a) < 0$) and let $x_0 \in (a, b)$ be the smallest zero of $\varphi(x)$.

1. If $\psi'(a) > 0$, then $\psi(x) > 0$ in $(a, x_0]$ (resp. if $\psi'(a) < 0$, then $\psi(x) < 0$ in $(a, x_0]$)
2. If $\psi'(a) < 0$, then $\psi(x) = 0$ exactly once in (a, x_0) (resp. if $\psi'(a) > 0$, then $\psi(x) = 0$ exactly once in (a, x_0))
3. If $\psi'(a) = 0$, then one of the first two cases must occur.

Proof. By Lemma 2.8 we can let x_0, where $a < x_0 < b$, be the smallest zero of φ (assuming there is a zero at all). Since $\varphi'(a) > 0$ and x_0 is the smallest zero of $\varphi(x)$, then $\varphi(x) > 0$ on (a, x_0). Assume, for the moment that $\psi(x_0) \leq 0$. Geometrical considerations give us that $\varphi'(x_0)\psi(x_0) \geq 0$. Applying (2.5) with $x = x_0$ we obtain a contradiction. Thus, it must be the case that $\psi(x_0) > 0$.

There are now three possibilities, each of which may occur (see Example 3.2, Example 3.3 and Example 3.4, below).

Case 1. Let $\psi'(a) > 0$. Then ψ increases in a right-neighborhood of $x = a$ and we will show that $\psi(x) \neq 0$ in (a, x_0), so that $\psi(x) > 0$ there.

For assume, on the contrary, that $\psi(x_1) = 0$ at some smallest point x_1 where $a < x_1 < x_0$ and $\psi(x) > 0$ for $a < x < x_1$. From (2.5) we know that

$$\varphi'(x_1)\psi(x_1) - \varphi(x_1)\psi'(x_1) = \operatorname{Im}\lambda \int_a^{x_1} r|y|^2 \, dx < 0,$$

which gives a contradiction since $\varphi(x_1) > 0$ and $\psi'(x_1) < 0$ (note that $\psi'(x_1) = 0$ is excluded from the previous display). Hence, in this case, $\psi(x) \neq 0$ whenever $a < x < x_0$, i.e., $\psi(x) > 0$ in $(a, x_0]$.

Case 2. Let $\psi'(a) < 0$. In this case, ψ decreases in a right neighborhood of $x = a$ and so $\psi(x)$ must be negative. On the other hand $\psi(x_0) > 0$ and so $\psi(x_3) = 0$ at some smallest point $x = x_3$, where $a < x_3 < x_0$ and $\psi(x) < 0$ on (a, x_3). Note that ψ cannot vanish again for $x > x_3$ for, if $\psi(x_4) = 0$, where $x_3 < x_4 < x_0$, then this would violate the separation of internal zeros property in Theorem 2.10 as $\varphi(x) > 0$ in $[x_3, x_4]$.

Case 3. Let $\psi'(a) = 0$. This final case is shown by noting that $\psi(x)$ must be of one sign in a right-neighborhood of $x = a$ after which one of the proofs in cases 1 and 2 must apply, depending on the sign.

The alternative ("respective") result is handled from the first by replacing y by $-y$, or φ, ψ by $-\varphi, -\psi$, a transformation which keeps (2.4) invariant, and the eigenvalue the same. □

Corollary 2.13. Let $G(x) < 0$ for all $x \in (a, b)$. Let $\psi'(a) < 0$, and let $x_0 \in (a, b)$ be the smallest zero of $\psi(x)$.

1. If $\varphi'(a) > 0$, then $\varphi(x) > 0$ in (a, x_0)
2. If $\varphi'(a) < 0$, then $\varphi(x) = 0$ exactly once in (a, x_0)
3. If $\varphi'(a) = 0$, then one of the first two cases must occur.

Proof. Replace $y = \varphi + i\psi$ by $iy = -\psi + i\varphi$ which is another eigenfunction satisfying (1.1)–(1.2) and apply the lemma to this eigenfunction. Observe that this change of variable also keeps (2.4) invariant. □

Remark 2.14. Observe that the proof of Lemma 2.12 fails if $x_0 = b$ as then $G(b) = 0$ so that the right-hand side of (2.5) is zero.

Remark 2.15. Note that if $\psi'(a) = 0$, then one of Case 1, 2 above must apply, i.e., either ψ increases or decreases in a right neighborhood of $x = a$. In either case we have the same results. See Example 3.2 for an application of Corollary 2.13(1) and Example 3.3 for an application of Lemma 2.12 (2).

Results analogous to Lemma 2.12 and Corollary 2.13 may be formulated in the event that information is available at the right endpoint. Two samples follow, others may be readily formulated using the change of variables in Lemma 2.12 and Corollary 2.13. We sketch the proofs.

Lemma 2.16. Let $G(x) < 0$ for all $x \in (a, b)$. Let $\varphi'(b) < 0$, (resp. $\varphi'(b) > 0$) and let $x_0 \in (a, b)$ be the largest zero of $\varphi(x)$.

1. If $\psi'(b) > 0$, then $\psi(x) < 0$ in (x_0, b) (resp. if $\psi'(b) < 0$, then $\psi(x) > 0$ in (x_0, b))

2. If $\psi'(b) < 0$, then $\psi(x) = 0$ exactly once in (x_0, b) (resp. if $\psi'(b) > 0$, then $\psi(x) = 0$ exactly once in (x_0, b))

3. If $\psi'(b) = 0$, then one of the first two cases must occur.

Proof. Let $a < x_0 < b$ be the largest zero at which φ vanishes (again, assuming there is a zero at all). Since $\varphi'(b) < 0$ and x_0 is the largest zero of $\varphi(x)$, then $\varphi(x) > 0$ on (x_0, b). Assume, for the moment that $\psi(x_0) > 0$. Geometrical considerations give us that $\varphi'(x_0)\psi(x_0) \geq 0$. Applying (2.5) with $x = x_0$ we obtain a contradiction. Thus, it must be the case that $\psi(x_0) \leq 0$. On the other hand, if $\psi(x_0) = 0$, then the left hand side of (2.5) is zero (at $x = x_0$) while the right hand is non-zero. This contradiction proves that $\psi(x_0) < 0$.

Case 1. Let $\psi'(b) > 0$. Then ψ increases in a left-neighborhood of $x = b$ and we will show that $\psi(x) \neq 0$ in (x_0, b), so that $\psi(x) < 0$ there.

For assume, on the contrary, that $\psi(x_1) = 0$ at some largest point x_1 where $x_0 < x_1 < b$ and $\psi(x) < 0$ for $x_1 < x < b$. From (2.5) we know that

$$\varphi'(x_1)\psi(x_1) - \varphi(x_1)\psi'(x_1) = \operatorname{Im}\lambda \int_a^{x_1} r|y|^2 \, dx < 0,$$

which gives a contradiction since $\varphi(x_1) > 0$ and $\psi'(x_1) < 0$ (note that $\psi'(x_1) = 0$ is excluded from the previous display). Hence, in this case, $\psi(x) \neq 0$ whenever $x_0 < x < b$, i.e., $\psi(x) < 0$, $x \in (x_0, b)$.

Case 2. If $\psi'(b) < 0$ then ψ decreases and is positive in a left-neighborhood of $x = b$. However, since $\psi(x_0) < 0$, it follows that $\psi(x_5) = 0$ for some $x_0 < x_5 < b$, where x_5 is taken to be the largest (in fact, the only) such zero, i.e., $\psi(x) > 0$ in (x_5, b). If possible, assume that $\psi(x)$ vanishes again at $x = x_6$, where now $x_0 < x_6 < x_5$. This now violates the separation of zeros property in Theorem 2.10 as these are internal zeros and φ is not zero there.

Case 3. This is shown as in the corresponding cases in the above lemmas, so the proofs are left to the reader. □

Corollary 2.17. Let $G(x) < 0$ for all $x \in (a, b)$. Let $\psi'(b) > 0$, and let $x_0 \in (a, b)$ be the largest zero of $\psi(x)$.

1. If $\varphi'(b) > 0$, then $\varphi(x) < 0$ in (x_0, b).

2. If $\varphi'(b) < 0$, then $\varphi(x) = 0$ exactly once in (x_0, b).

3. If $\varphi'(b) = 0$, then one of the first two cases must occur.

Proof. Consider the eigenfunction $iy = -\psi + i\varphi$ as before. □

We recall that if $G(x) > 0$, analogous results may be obtained by replacing λ by $-\lambda$ and r by $-r$ in the above theorems and lemmas as the eigenfunction remains the same although the non-real eigenvalue is the negative of the original one corresponding to a different weight function, r.

3 Examples

In the examples that follow the coefficients q, r appearing in (1.1) are piecewise continuous rather than continuous, this is not a real lack of generality as one expects the same results holding in this slightly weaker case of piecewise continuity as continuous functions on $[a,b]$ are dense in $L(a,b)$. Thus, solutions of problems with piecewise continuous coefficients may be approximated arbitrarily closely by a differential equation with continuous coefficients.

In this section we shall make use of the following result,

Lemma 3.1 ([10, Theorem 4.2.1])**.** Let $p(x) > 0$ a.e. on $[a,b]$, $q \in L(a,b)$ and $r(x)$ is a real-valued function defined on $[a,b]$ and such that $r(x)$ takes both signs on some subsets of positive measure. Then the eigenvalue problem

$$-(p(x)y')' + q(x)y = \lambda r(x)y, \qquad y(a) = y(b) = 0, \tag{3.1}$$

possesses at most a finite number of non-real eigenvalues. If we let M the number of pairs of distinct non-real eigenvalues of (3.1), N the number of distinct negative eigenvalues of (3.1) with $r(x) \equiv 1$, which we know is finite by classical Sturm–Liouville theory, [6, §5.8, Theorem 2], then $M \leq N$.

Example 3.2. As an application of Lemma 2.3 and the second conclusion in Lemma 2.12, consider the eigenvalue problem for the equation

$$-y''(x) - q\,y(x) = \lambda\,\mathrm{sgn}(x)y(x), \qquad y(-1) = 0,\ y(1) = 0.$$

where $\lambda \in \mathbf{C}$, $q > 0$ is a constant, and $\mathrm{sgn}(x) = -1$ for $x \leq 0$ and $\mathrm{sgn}(x) = 1$ for $x > 0$. Its eigenfunctions are given by

$$f(x) = \begin{cases} \sin\left(\sqrt{-q-\lambda}\ (1+x)\right), & -1 \leq x \leq 0 \\ \sin\left(\sqrt{-q-\lambda}\right)\cos\left(x\sqrt{-q+\lambda}\right) + \dfrac{\sqrt{-q-\lambda}\,\cos\left(\sqrt{-q-\lambda}\right)\sin\left(x\sqrt{-q+\lambda}\right)}{\sqrt{-q+\lambda}}, & 0 \leq x \leq 1 \end{cases}$$

where the $\lambda \in \mathbf{C}$ satisfy the dispersion relation, see [3],

$$\sqrt{-z-q}\,\cos(\sqrt{-z-q})\sin(\sqrt{z-q}) + \sqrt{z-q}\,\cos(\sqrt{z-q})\sin(\sqrt{-z-q}) = 0. \tag{3.2}$$

This problem always has a doubly infinite sequence of real eigenvalues along with a finite number of complex eigenvalues, [11, 13], one of which is, in the case where $q = 40$, $\lambda \approx 26.9376 + 6.9215i$. Writing $y = \varphi + \psi i$, the graphs of φ, ψ, and $G(x)$ are reproduced above.

Example 3.3. consider the initial value problem for the equation

$$-y''(x) - (7 + \lambda x)y(x) = 0, \qquad y(-1) = 0,\ y(1) = 0,$$

where $\lambda \in \mathbf{C}$, is solvable in terms of Airy functions, so that if $y'(-1) = 1$, then the solution is

$$\frac{\left(-\mathrm{AiryAi}\left(\frac{\lambda-7}{\lambda^{\frac{2}{3}}}\right)\mathrm{AiryBi}\left(-\frac{\lambda x+7}{\lambda^{\frac{2}{3}}}\right) + \mathrm{AiryBi}\left(\frac{\lambda-7}{\lambda^{\frac{2}{3}}}\right)\mathrm{AiryAi}\left(-\frac{\lambda x+7}{\lambda^{\frac{2}{3}}}\right)\right)\pi}{\lambda^{\frac{1}{3}}}.$$

The associated Dirichlet problem on $[-1,1]$, i.e.,

$$y(-1) = 0 = y(1)$$

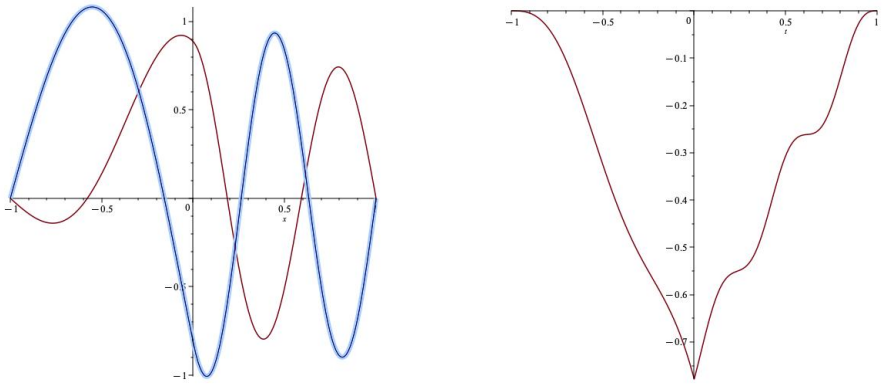


Figure 3.1: Here, φ is the left-most curve of the graph on the left, ψ being the other. The graph on the right is that of $G(x)$.

then gives a doubly infinite sequence of real eigenvalues along with a finite number of complex eigenvalues, [11,13], one of which is $\lambda \approx 12.3076i$.

Now, let $y(-1) = 0$, $y'(-1) = 1 - i$. Of course, the spectrum is the same but the eigenfunctions differ slightly. In this case, the eigenfunctions take the form,

$$y = -\frac{\mathrm{AiryBi}\left(\frac{\lambda-7}{\lambda^{\frac{2}{3}}}\right) \cdot \alpha \cdot \mathrm{AiryAi}\left(-\frac{\lambda\cdot x+7}{\lambda^{\frac{2}{3}}}\right)}{\mathrm{AiryAi}\left(\frac{\lambda-7}{\lambda^{\frac{2}{3}}}\right)} + \alpha \cdot \mathrm{AiryBi}\left(-\frac{\lambda\cdot x+7}{\lambda^{\frac{2}{3}}}\right)$$

where

$$\alpha = \frac{(i-1)\cdot \mathrm{AiryAi}\left(\frac{\lambda-7}{\lambda^{\frac{2}{3}}}\right)\pi}{\lambda^{\frac{1}{3}}}.$$

The plots of the real and imaginary parts of y are given below in Figure 3.2 and the qualitative behavior of these functions is in conformity with the conclusions obtained in Lemma 2.12 (2) and Lemma 2.16 (1).

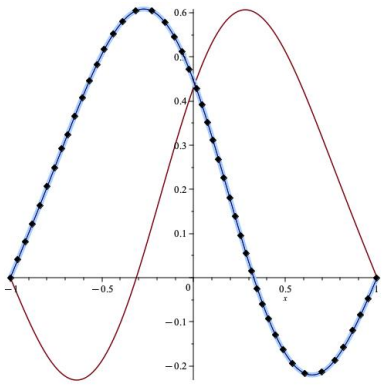


Figure 3.2: Here, φ is the upper curve on the left.

Example 3.4. In this example we exhibit the oscillations in the more general case where the weight function vanishes on an interval, rather than just a point, as in [13].

Here, the piecewise continuous weight function r is given by

$$r(x) = \begin{cases} 1, & \text{if } -2 \le x < -1, \\ 0, & \text{if } -1 \le x < 1. \\ -1, & \text{if } 1 \le x \le 2. \end{cases}$$

and $q(x) = q > 0$, is a constant function. In order to simplify the notation, the Dirichlet problem is defined by considering (1.1) (with q there replaced by $-q$ and r by $-r$) subject to $y(-2) = 0 = y(2)$. Of course, this changes the sign of $G(x)$ in general, if we wish to maintain $\operatorname{Im}\lambda > 0$.

In the following we use the following simplified notation: $\sin\sqrt{q-\lambda} = \sin(\sqrt{q-\lambda})$, $\sin 2\sqrt{q} = \sin(2\sqrt{q})$, $\sin\sqrt{q-\lambda}x = \sin(\sqrt{q-\lambda}x)$ etc.

Thus, the solution of the problem satisfying $y(-2) = 0$, $y'(-2) = 1$ is given by

$$y(x) = \frac{\sin 2\sqrt{q-\lambda}}{\sqrt{q-\lambda}} \left(\cot\ 2\sqrt{q-\lambda} \sin\sqrt{q-\lambda}\, x + \cos\sqrt{q-\lambda}\, x \right),$$

for $-2 \le x \le -1$.

On the other hand, on the interval $-2 \le x < 1$, the solution y is given by,

$$y(x) = \frac{1}{\sqrt{q-\lambda}} \left(\{A - B\} \sin\sqrt{q}x \right) + C\cos\sqrt{q}x$$

where

$$\begin{aligned} A &= \frac{(\sin\sqrt{q-\lambda}\,\sqrt{q}\cot\sqrt{q} + \sqrt{q-\lambda}\cos\sqrt{q-\lambda})\cot\sqrt{q}}{\sqrt{q}\csc\sqrt{q}}, \\ B &= \sin\sqrt{q-\lambda}\csc\sqrt{q}, \\ C &= \frac{1}{\sqrt{q}\sqrt{q-\lambda}\csc\sqrt{q}} \sin\sqrt{q-\lambda}\sqrt{q}\cot\sqrt{q} + \sqrt{q-\lambda}\cos\sqrt{q-\lambda}. \end{aligned}$$

Finally, the expression for y on $1 \le x \le 2$ is given by

$$\begin{aligned} y(x) = &\left(\frac{1}{\sqrt{q+\lambda}\sqrt{q}\sqrt{q-\lambda}} \left\{ A' + B' \right\} \right) \sin\sqrt{q+\lambda}x \\ &+ \left(\frac{1}{\sqrt{q+\lambda}\sqrt{q}\sqrt{q-\lambda}} \left\{ C' + D' \right\} \right) \cos\sqrt{q+\lambda}x \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} A' &= \sqrt{q+\lambda}\sqrt{q}\sin\sqrt{q+\lambda}\cos 2\sqrt{q}\sin\sqrt{q-\lambda} \\ &\quad + \sqrt{q-\lambda}\sqrt{q+\lambda}\sin\sqrt{q+\lambda}\sin 2\sqrt{q}\cos\sqrt{q-\lambda}, \\ B' &= \sqrt{q}\sqrt{q-\lambda}\cos\sqrt{q+\lambda}\cos\sqrt{q-\lambda}\cos 2\sqrt{q} \\ &\quad - q\ \cos\sqrt{q+\lambda}\sin\sqrt{q-\lambda}\sin 2\sqrt{q}, \\ C' &= \sqrt{q}\sqrt{q+\lambda}\cos\sqrt{q+\lambda}\cos 2\sqrt{q}\sin\sqrt{q-\lambda} \\ &\quad + \sqrt{q+\lambda}\sqrt{q-\lambda}\cos\sqrt{q+\lambda}\sin 2\sqrt{q}\cos\sqrt{q-\lambda}, \\ D' &= q\sin 2\sqrt{q}\sin\sqrt{q+\lambda}\sin\sqrt{q-\lambda} \\ &\quad - \sqrt{q}\cos 2\sqrt{q}\sqrt{q-\lambda}\cos\sqrt{q-\lambda}\sin\sqrt{q+\lambda}. \end{aligned}$$

The dispersion relation for the eigenvalues of this non-definite problem is given by setting $y(2) = 0$ in (3.3) and solving for $\lambda \in \mathbf{C}$. Using the Maple© subroutine *RootFinding* we find a non-real eigenvalue $\lambda \approx 6.29625i$ in the case where $q = 8$. It appears as if $G(x) < 0$ in this

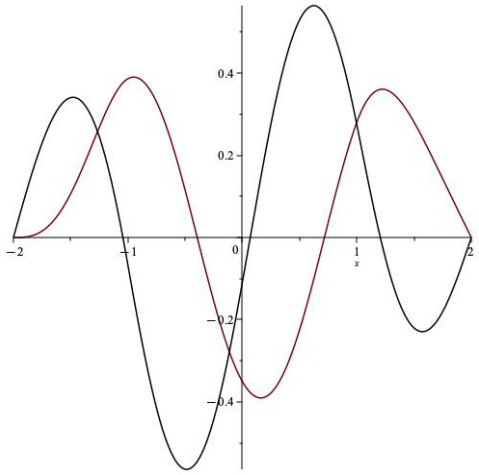


Figure 3.3: Here, the left-most curve is φ, the real part.

case in which case Lemma 2.16 (1) applies.

Example 3.5. Returning to Example 3.2 above,

$$-y''(x) - q\,y(x) = \lambda\,\mathrm{sgn}(x)y(x), \qquad y(-1) = 0,\ y(1) = 0.$$

where now $q = \pi^2/4 + 2$, we get a non-real eigenvalue pair $\lambda \approx \pm 3.8741i$.

Writing $y = \varphi + \psi i$, the graphs of φ, ψ, and $G(x)$ are reproduced below,

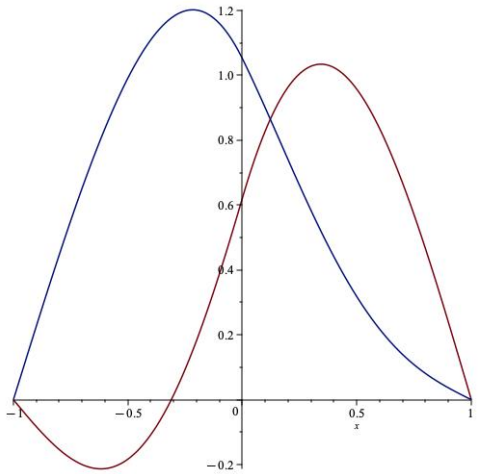


Figure 3.4: Here, φ is the left-most curve of the graph on the left, ψ being the other. Note the positivity of the real part in $(-1, 1)$. Again, $G(x) < 0$, here.

4 Sturm theory for non-real eigenfunctions

In this section the word "eigenvalue" refers to a non-real eigenvalue of (1.1)–(1.2) and all questions here, unless otherwise specified, refer to the case where the weight function has a unique turning point around which $r(x)$ changes sign. For example, we can ask:

1. Under what conditions does there necessarily exist an eigenfunction whose real or imaginary part does not vanish in (a,b)?

   Now, (1.1)–(1.2) always has a finite number of pairs of non-real eigenvalues by Lemma 3.1, so this is a sensible question. In Example 3.3 there is a complex ghost whose real and imaginary part both have a zero in (a,b). In this case, a simple calculation shows that $M = 1$ in Lemma 3.1, and thus there can only be at most one pair of non-real eigenvalues. Since this eigenvalue is one of the only possible pairs of eigenvalues of the problem, the answer is negative in this case.

   On the other hand, in Example 3.5, for $q = \pi^2/4 + 2$ we have an eigenvalue pair $\lambda \approx \pm 3.8741i$ with an eigenfunction having a real part that is seemingly positive in $(-1,1)$. Once again, $M = 1$ in Lemma 3.1 and thus this is the only non-real pair.

   In relation to 1) above, we can ask for a weaker result, that is,

2. Can a non-real eigenfunction $y(x,\lambda)$ vanish for some $x \in (a,b)$?

   Generally speaking this cannot occur in the case of one turning point such as in Richardson [13] or, more generally, as in Theorem 2.10. This is because of the separation of the interior zeros of the real and imaginary parts of the eigenfunction.

   However, the possibility of a non-real eigenfunction vanishing in the interior of an interval may be true in the case of two turning points as the results in [8] show. There, an example of a non-definite problem with two turning points was found in which a non-real eigenfunction vanished as a function of x inside (a,b). The result is therefore likely true in the case of more than two turning points, but this has not been verified yet. In [9] it was shown that if the weight function has n turning points, then any non-real eigenfunction can vanish at most $n-1$ times in (a,b).

3. Under what conditions can we infer the existence of an eigenfunction, corresponding to an eigenvalue of largest modulus, that does not vanish in (a,b)?

   In the case of one turning point or as in Theorem 2.10 this is clear as eigenfunctions cannot vanish in the interior because of the separation property. In particular, see Example 3.5 where we have only one-non-real pair. This statement is then reminiscent of the Perron–Frobenius Theorem [5] in the finite dimensional (matrix) case. (See Corollary 3.2 in [12] for a related result.) Since our boundary value problem always has a finite number of pairs of non-real eigenvalues by Lemma 3.1, this too is a ponderable question. To what extent is it true in cases where there are multiple turning points?

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Critical points approaches to fourth-order elastic beam equations with variable coefficients having combined effects of concave and convex nonlinearities

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Abstract. This study investigates the theoretical and practical aspects of fourth-order elastic beam equations characterized by variable coefficients and influenced by combined nonlinearities. The equations under consideration arise naturally in the context of high-order differential problems encountered in material science, particularly in the study of phase transitions. The research focuses on establishing the existence and multiplicity of solutions for these equations, which involve both concave and convex nonlinearities. Using advanced mathematical techniques, such as variational methods and critical point theory, the study provides rigorous proofs for the existence of solutions under specific conditions. A central result is the application of Bonanno's local minimum theorem, which ensures the existence of at least one solution. Moreover, the research shows that by imposing additional algebraic conditions, particularly the classical Ambrosetti–Rabinowitz condition, the presence of two distinct solutions can be guaranteed. Beyond this, critical point theorems from Averna and Bonanno are employed to demonstrate scenarios where three solutions are possible.

Keywords: multiple solutions, fourth-order equation, critical point, variational methods.

2020 Mathematics Subject Classification: 35J35, 47J10, 58E05, 34B15.

1 Introduction

Elastic beam equations, particularly those of fourth order, represent a cornerstone of modern theoretical and applied mechanics. These equations are invaluable in modeling the behavior of elastic materials and structures under various forces, enabling scientists and engineers to predict deformation, stability, and stress distribution with high precision. Fourth-order elastic

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equations are especially important because they incorporate effects that lower-order models neglect, such as bending and torsion, and account for variability in material properties and external forces. The significance of fourth-order differential equations extends across multiple disciplines. In structural engineering, these equations are essential for analyzing the deflection and stability of beams and plates, forming the theoretical basis for the design of buildings, bridges, and other critical infrastructures. The pioneering work of Timoshenko and Gere [21] introduced a comprehensive framework for understanding elastic stability, which remains relevant for modern engineering practices. In materials science, fourth-order equations play a central role in studying phase transitions phenomena where materials shift between states, such as from liquid to solid, under controlled conditions. This is particularly important for developing advanced materials in industries such as aerospace and automotive, where performance and reliability are critical. These equations' ability to model real-world conditions, including variable coefficients and nonlinear behaviors, makes them indispensable for tackling complex scientific and engineering challenges. Despite their utility, solving fourth-order elastic equations is a mathematically challenging task. These equations often involve nonlinearities, such as concave and convex effects, which significantly complicate the solution process. Nonlinearities arise in real-world scenarios where material responses to stress are non-proportional to the applied force. For example, when a beam experiences large deformations, the relationship between load and deflection becomes nonlinear. Variable coefficients add another layer of complexity, representing changes in material properties or external conditions over space or time. Traditional analytical methods frequently fall short in addressing these issues, necessitating the use of advanced techniques such as variational methods and critical point theory. These tools provide a rigorous framework for proving the existence and uniqueness of solutions to higher-order differential equations. A key aspect of this study is the application of variational methods to solve fourth-order elastic equations. These methods involve reformulating differential equations into problems of finding critical points of associated functionals. This approach is particularly advantageous for addressing the nonlinearities and boundary conditions inherent in these equations. Bonanno's local minimum theorem [3] and the Ambrosetti-Rabinowitz condition [1] are instrumental in this context, as they provide conditions under which solutions exist. The Palais–Smale condition, a cornerstone of variational calculus, ensures that functionals possess the compactness properties needed to identify critical points. Furthermore, Sobolev spaces offer a robust mathematical framework for handling boundary conditions and variable coefficients with precision. Research by Averna and Bonanno [2] has demonstrated the power of variational methods in proving not only the existence but also the multiplicity of solutions. These results highlight the ability of fourth-order elastic equations to model a range of physical phenomena, from stable configurations to unstable states. The practical implications of this research are vast, spanning both theoretical and applied domains. In materials science, understanding the multiplicity of solutions allows for the prediction of material behavior under different conditions. For instance, during a phase transition, multiple solutions may correspond to distinct physical states, such as stable and unstable configurations. This insight is critical for designing materials with tailored properties, such as high-performance alloys or composites. In structural engineering, these equations inform the design of beams, plates, and other structural elements to ensure safety and reliability. For example, aerospace engineers rely on these models to optimize the design of lightweight yet robust materials for aircraft and spacecraft. By predicting behaviors such as buckling, vibration, and resonance, fourth-order elastic equations provide the tools needed to address the demands of modern engineering.

In this paper, we are going to establish the existence result for the following problem

$$\begin{cases} (p(t)z'')'' - (q(t)z')' + r(t)z = \lambda h(t, z(t)) + \mu g(t, z(t)), \quad t \in [0,1], \\ z(a) = z(b) = 0, \\ z''(a) = z''(b) = 0 \end{cases} \tag{$P^{h,g}$}$$

where $\lambda > 0$, $\mu \geq 0$, h, g are continuous functions, $p \in C^2[0,1]$, $q \in C^1[0,1]$, $r \in C[0,1]$ are regular functions with $p = \operatorname{ess\,inf}_{[0,1]} p > 0$. We aim to advance the understanding of fourth-order elastic equations by addressing both their theoretical and practical challenges. Specifically, we seek to: 1. Establish the existence of solutions using variational methods and critical point theory. 2. Explore conditions that lead to the multiplicity of solutions, highlighting the influence of nonlinearities and variable coefficients.

The remainder of this paper is organized as follows. Section 2 introduces the necessary mathematical preliminaries, including definitions, theorems, and frameworks that form the foundation of the analysis. Section 3 examines the existence of a single solution, employing foundational techniques such as the Palais–Smale condition. Section 4 extends the discussion to explore conditions under which multiple solutions arise, with particular attention to nonlinearities and coefficient variations. Section 5 addresses special cases, such as when specific parameters are set to zero, providing additional insights into the behavior of these equations. Finally, the study concludes with a discussion of the implications of the findings and potential directions for future research.

For additional information, we recommend consulting sources such as [7, 9, 11, 13, 14, 22]. For example, Bonanno et al. in [9] applied multiple critical points theorems to demonstrate the existence of two non-trivial solutions for equation ($P^{h,g}$), in the case $\mu = 0$.

2 Preliminaries

For relevant notations and foundational results, we direct the reader to references [17, 19]. Let E be represent a real Banach space. A functional $I : \mathrm{E} \to \mathbb{R}$, which is continuously Gâteaux differentiable, is said to satisfy the *Palais–Smale condition* (shortened as (PS)-condition) if every sequence $\{z_n\}$ such that $\{I(z_n)\}$ is bounded and $\lim_{n\longrightarrow\infty} \|I'(z_n)\|_{\mathrm{E}^*} = 0$, has a convergent subsequence. This condition is crucial in the framework of variational methods, as it ensures that critical points of λ, correspond to solutions of the associated boundary value problem.

Now, consider two continuously Gâteaux differentiable functions $\Phi, \Psi : \mathrm{E} \longrightarrow \mathbb{R}$ and the functional

$$I = \Phi - \Psi.$$

Let $s_1, s_2 \in [-\infty, \infty]$ with $s_1 < s_2$. The functional I is said to satisfy the *Palais–Smale condition with bounds s_1 and s_2* (denoted ${}^{[s_1]}(\mathrm{PS})^{[s_2]}$-condition) if any sequence $\{z_n\}$ such that $\{I(z_n)\}$ is bounded, $\lim_{n\longrightarrow\infty} \|I'(z_n)\|_{\mathrm{E}^*} = 0$ and $s_1 < \Phi(z_n) < s_2$ for each $n \in \mathbb{N}$, has a convergent subsequence.

When $s_1 = -\infty$ and $s_2 = \infty$, this condition reduces to the classical (PS)-condition. If $s_1 = -\infty$ and $s_2 \in \mathbb{R}$, it is referred to as the $(\mathrm{PS})^{[s_2]}$-condition. Similarly, when $s_1 \in \mathbb{R}$ and $s_2 = \infty$, it is denoted as ${}^{[s_1]}(\mathrm{PS})$-condition. Indeed, if Φ and Ψ be two continuously Gâteaux differentiable functionals defined on a real Banach space E and fix $s \in \mathbb{R}$. The functional $I = \Phi - \Psi$ is said to verify the Palais–Smale condition cut off upper at r (in short $(\mathrm{PS})^{[s]}$) if any sequence $\{z_n\}_{n\in\mathbb{N}}$ in E such that $\{I(z_n)\}$ is bounded, $\lim_{n\longrightarrow\infty} \|I'(z_n)\|_{\mathrm{E}^*} = 0$ and

$\Phi(z_n) < s$ for each $n \in \mathbb{N}$, has a convergent subsequence. Furthermore, if I satisfies the ${}^{[s_1]}(\mathrm{PS})^{[s_2]}$-condition, it automatically satisfies the ${}^{[\varrho_1]}(\mathrm{PS})^{[\varrho_2]}$-condition for all $\varrho_1, \varrho_2 \in [-\infty, \infty]$ such that $s_1 \leq \varrho_1 < \varrho_2 \leq s_2$.

Notably, if I adheres to the classical (PS)-condition, it also satisfies ${}^{[\varrho_1]}(\mathrm{PS})^{[\varrho_2]}$-condition for any $\varrho_1, \varrho_2 \in [-\infty, \infty]$ with $\varrho_1 < \varrho_2$.

The following four theorems will be utilized in proving the results of this paper.

Theorem 2.1 ([5, Theorem 2.3]). *Let* E *be a real Banach space and let* $\Phi, \Psi : \mathrm{E} \longrightarrow \mathbb{R}$ *be two continuously Gâteaux differentiable functions such that* $\inf_{z \in \mathrm{E}} \Phi(z) = \Phi(0) = \Psi(0) = 0$. *Assume that there exist* $s > 0$ *and* $\bar{z} \in \mathrm{E}$, *with* $0 < \Phi(\bar{z}) < s$, *such that*

(a_1) $$\frac{\sup_{\Phi(z) \leq s} \Psi(z)}{s} < \frac{\Psi(\bar{z})}{\Phi(\bar{z})},$$

(a_2) *for each* $\lambda \in \left(\dfrac{\Phi(\bar{z})}{\Psi(\bar{z})}, \dfrac{s}{\sup_{\Phi(z) \leq s} \Psi(z)} \right)$ *the functional* $I_\lambda = \Phi - \lambda \Psi$ *satisfies* $(\mathrm{PS})^{[s]}$*-condition.*

Then, for each

$$\lambda \in \Lambda_s = \left(\frac{\Phi(\bar{z})}{\Psi(\bar{z})}, \frac{s}{\sup_{\Phi(z) \leq s} \Psi(z)} \right)$$

there exists $z_{0,\lambda} \in \Phi^{-1}(0, s)$ *such that* $I'_\lambda(z_{0,\lambda}) = 0$ *and* $I_\lambda(z_{0,\lambda}) \leq I_\lambda(z)$ *for each* $z \in \Phi^{-1}(0, s)$.

Theorem 2.2. *[5, Theorem 3.2] Let* E *be a real Banach space,* $\Phi, \Psi : \mathrm{E} \longrightarrow \mathbb{R}$ *be two continuously Gâteaux differentiable functionals such that* Φ *is bounded from below and* $\Phi(0) = \Psi(0) = 0$. *Fix* $s > 0$ *and assume that, for each*

$$\lambda \in \left(0, \frac{s}{\sup_{z \in \Phi^{-1}(-\infty, s)} \Psi(z)} \right),$$

the functional $I_\lambda = \Phi - \lambda \Psi$ *satisfies* (PS)*-condition and it is unbounded from below. Then, for each*

$$\lambda \in \left(0, \frac{s}{\sup_{z \in \Phi^{-1}(-\infty, s)} \Psi(z)} \right),$$

the functional I_λ *admits two distinct critical points.*

Theorem 2.3 ([2, Theorem A]). *Let* E *be a reflexive real Banach space,* $\Phi : \mathrm{E} \longrightarrow \mathbb{R}$ *a continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on* E^*, *and* $\Psi : \mathrm{E} \longrightarrow \mathbb{R}$ *a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Assume that*

(b_1) $\lim_{\|z\| \longrightarrow \infty} (\Phi(z) + \lambda \Psi(z)) = \infty$ *for each* $\lambda \in [0, \infty)$,

(b_2) *there is* $s \in \mathbb{R}$ *such that*

$$\inf_{\mathrm{E}} \Phi < s$$

and

$$\varphi_1(s) < \varphi_2(s)$$

where

$$\varphi_1(s) = \inf_{z \in \Phi^{-1}(-\infty, s)} \frac{\Psi(z) - \inf_{\overline{\Phi^{-1}(-\infty, s)}^{w}} \Psi}{s - \Phi(z)},$$

$$\varphi_2(s) = \inf_{z\in\Phi^{-1}(-\infty,s)} \sup_{v\in\Phi^{-1}[s,\infty)} \frac{\Psi(z)-\Psi(v)}{\Phi(v)-\Phi(z)},$$

and $\overline{\Phi^{-1}(-\infty,s)}^{w}$ *is the closure of* $\Phi^{-1}(-\infty,s)$ *in the weak topology.*

Then, for each $\lambda \in \left(\frac{1}{\varphi_2(s)}, \frac{1}{\varphi_1(s)}\right)$, *the functional* $\Phi+\lambda\Psi$ *has at least three critical points in* E.

It is important to note that $\varphi_1(s)$ in Theorem 2.3 may equal 0. In such cases, and in similar situations both here and later, we will interpret $\frac{1}{0}$ as ∞.

We will also utilize the following two theorems related to critical points.

Theorem 2.4 ([4, Theorem 1.1]). *Let* E *be a reflexive real Banach space, and let* $\Phi, \Psi : \mathrm{E} \longrightarrow \mathbb{R}$ *be two sequentially weakly lower semicontinuous and Gâteaux differentiable functions. Assume that* Φ *is (strongly) continuous and satisfies*

$$\lim_{\|z\|\longrightarrow\infty} \Phi(z) = \infty.$$

Assume also that there exist two constants s_1 *and* s_2 *such that*

(c_1) $\inf_{\mathrm{E}} \Phi < s_1 < s_2$,

(c_2) $\varphi_1(s_1) < \varphi_2^*(s_1,s_2)$,

(c_3) $\varphi_1(s_2) < \varphi_2^*(s_1,s_2)$, *where* φ_1 *is defined as in Theorem 2.3 and*

$$\varphi_2^*(s_1,s_2) = \inf_{z\in\Phi^{-1}(-\infty,s_1)} \sup_{v\in\Phi^{-1}[s_1,s_2)} \frac{\Psi(z)-\Psi(v)}{\Phi(v)-\Phi(z)}.$$

Then, for each

$$\lambda \in \left(\frac{1}{\varphi_2^*(s_1,s_2)}, \min\left\{\frac{1}{\varphi_1(s_1)}, \frac{1}{\varphi_1(s_2)}\right\}\right),$$

the functional $\Phi+\lambda\Psi$ *admits at least two critical points which lie in* $\Phi^{-1}(-\infty,s_1]$ *and* $\Phi^{-1}[s_1,s_2)$, *respectively.*

It is worth noting that Theorems 2.3 and 2.4 are based on Ricceri's variational principle [20]. For further details, readers are encouraged to consult [6], where Theorems 2.1 and 2.2 were employed to prove the existence of at least one and two solutions for elliptic Dirichlet problems with variable exponents. Similarly, in [10], Theorems 2.3 and 2.4 were utilized to establish the existence of at least two and three solutions for a boundary value problem on the half-line. Additionally, reference [12, 15, 16] demonstrates how Theorems 2.1 through 2.4 were applied to guarantee the existence of solutions for boundary value problems.

In this section, we present fundamental notations and supporting results to incorporate equation ($P^{h,g}$) into a variational framework. Let E denote the Sobolev space $W^{2,2}([0,1]) \cap W_0^{1,2}([0,1])$, equipped with the norm

$$\|z\| = \left(\|z''\|_2^2 + \|z'\|_2^2 + \|z\|_2^2\right)^{\frac{1}{2}} \tag{2.1}$$

for every $z \in \mathrm{E}$ where $\|\cdot\|_2$ is the usual norm in $L^2[a,b]$. It is well known that $\|\cdot\|$ is induced by the inner product

$$\int_a^b \left(z''(t)v''(t) + z'(t)v'(t) + z(t)v(t)\right) \mathrm{d}t$$

for every $z, v \in \mathrm{E}$.

We highlight the following Poincaré-type inequalities, which can be found in the works of [18].

Proposition 2.5. *For every $z \in \mathrm{E}$, if $k = \frac{1}{\pi^2}$, one has*

$$\|z^{(i)}\|_2^2 \leq k^{j-i}\|z^{(j)}\|_2^2, \quad i = 0,1, \quad j = 1,2 \ \text{ with } i < j. \tag{2.2}$$

Now, defining p^- as previously mentioned, and letting $q^- = \operatorname{ess\,inf}_{[0,1]} q$ and $r^- = \operatorname{ess\,inf}_{[0,1]} r$, we will examine the following set of conditions based on the signs of these quantities:

(H_1) $p^- > 0, q^- \geq 0, r^- \geq 0,$

(H_2) $p^- > 0, q^- < 0, r^- \geq 0$ and $p^- + q^- k > 0,$

(H_3) $p^- > 0, q^- \geq 0, r^- < 0$ and $p^- + r^- k > 0,$

(H_4) $p^- > 0, q^- < 0, r^- < 0$ and $p^- + q^- k + r^- k > 0.$

Additionally, take into account the following condition:

(H) $\min\{p^- + q^- k,\ p^- + r^- k,\ p^- + q^- k + r^- k\} > 0.$

Put

$$\sigma = \min\{p^-,\ p^- + q^- k,\ p^- + r^- k,\ p^- + q^- k + r^- k\}.$$

Clearly, assuming the condition (H) implies that $\sigma > 0$. Furthermore, a simple calculation reveals the following result.

Proposition 2.6. *Condition (H) is satisfied if and only if at least one of the conditions (H_1) through (H_4) is met.*

We will now introduce a useful norm, which is equivalent to $\|\cdot\|$ and still ensures that E remains a Hilbert space. Therefore, for the fixed values of p, q and r mentioned earlier, we define the function $N : \mathrm{E} \to \mathbb{R}$ as follows

$$N(z) = \int_0^1 (p(t)|z''(t)|^2 + q(t)|z'(t)|^2 + r(t)|z(t)|^2)\mathrm{d}t$$

holds for any $z \in \mathrm{E}$. We have the following proposition, which will be helpful in confirming that $\sqrt{N(\cdot)}$ is a norm equivalent to the standard one.

Proposition 2.7 ([9, Proposition 2.3])**.** *Assume (H) holds. Then, there exits $m > 0$ such that*

$$N(z) \geq m\|z\|^2 \tag{2.3}$$

for any $z \in \mathrm{E}$, with $m = \frac{\sigma}{1+k+k^2}$. Moreover, one has

$$N(z) \geq \sigma\|z''\|_2^2 \tag{2.4}$$

for any $z \in \mathrm{E}$.

Proposition 2.8 ([9, Proposition 2.4])**.** *Assume that condition (H) is satisfied and define*

$$\|\cdot\|_{\mathrm{E}} = \sqrt{N(\cdot)}$$

for any $z \in \mathrm{E}$. Then, $\|\cdot\|_{\mathrm{E}}$ is a norm equivalent to the usual one defined in (2.2) and $(\mathrm{E}, \|\cdot\|_{\mathrm{E}})$ is a Hilbert space.

Now, assuming again (H), put

$$\delta = \sqrt{\sigma} = \left(\min\{p^-,\ p^- + q^- k,\ p^- + r^- k,\ p^- + q^- k + r^- k\}\right)^{\frac{1}{2}}.$$

The constant δ is well-defined, given that $\sigma > 0$ holds under condition (H).

The following proposition will be useful in the next section.

Proposition 2.9 ([9, Proposition 2.5]). *Assume that (H) holds. One has*

$$\|z\|_\infty \leq \frac{1}{2\pi\delta}\|z\|_{\mathrm{E}}$$

for any $z \in \mathrm{E}$.

Definition 2.10. A function $z \in \mathrm{E}$ is called a weak solution of problem $(P^{h,g})$, if

$$\int_0^1 \left(p(t)z''(t)v''(t) + q(t)z'(t)v'(t) + r(t)z(t)v(t)\right)\mathrm{d}t - \lambda\int_0^1 h(t,z(t))\mathrm{d}t - \mu\int_0^1 g(t,z(t))\mathrm{d}t = 0$$

holds for any $v \in \mathrm{E}$.

Put

$$H(t,m) = \int_0^m h(t,x)\mathrm{d}x \quad \text{for any } (t,m) \in [0,1]\times\mathbb{R}$$

and

$$G(t,m) = \int_0^m g(t,x)\mathrm{d}x \quad \text{for any } (t,m) \in [0,1]\times\mathbb{R}.$$

We define the functionals Φ and Ψ for each $z \in \mathrm{E}$, as follows

$$\Phi(z) = \frac{1}{2}\|z\|_{\mathrm{E}}^2 \tag{2.5}$$

and

$$\Psi(z) = \int_0^1 H(t,z(t))\mathrm{d}t + \frac{\mu}{\lambda}\int_0^1 G(t,z(t))\mathrm{d}t \tag{2.6}$$

and we put

$$I_\lambda(z) = \Phi(z) - \lambda\Psi(z)$$

for every $z \in \mathrm{E}$.

Proposition 2.11 ([9, Proposition 2.6]). *Function $z \in \mathrm{E}$ is a generalized solution of $(P^{h,g})$ if only if $z \in \mathrm{E}$ is a critical point of I_λ.*

We need the following Proposition for existence our main results.

Proposition 2.12. *Let $S : \mathrm{E} \longrightarrow \mathrm{E}^*$ be the operator defined by*

$$S(z)(v) = \int_0^1 \left(p(t)z''(t)v''(t) + q(t)z'(t)v'(t) + r(t)z(t)v(t)\right)\mathrm{d}t$$

for every $z, v \in \mathrm{E}$. Then, S admits a continuous inverse on E^*.

Proof. It is obvious that

$$S(z)(z) = \int_0^1 \left(p(t)|z''(t)|^2 + q(t)|z'(t)|^2 + r(t)|z(t)|^2\right) \mathrm{d}t = \|z\|_{\mathrm{E}}.$$

This follows that S is coercive. Owing to our assumptions on the data, one has

$$\langle S(z) - S(v), z - v\rangle \geq C\|z - v\|_{\mathrm{E}}^2 > 0$$

for some $C > 0$, for every $z, v \in \mathrm{E}$, which means that S is strictly monotone. Moreover, since E is reflexive, for $z_n \longrightarrow z$ strongly in E as $n \to +\infty$, one has $S(z_n) \to S(z)$ weakly in E^* as $n \to \infty$. Hence, S is demicontinuous, so by [23, Theorem 26.A(d)], the inverse operator S^{-1} of S exists and it is continuous. Indeed, let e_n be a sequence of E^* such that $e_n \to e$ strongly in E^* as $n \to \infty$. Let z_n and u in E such that $S^{-1}(e_n) = z_n$ and $S^{-1}(e) = z$. Taking into account that S is coercive, one has that the sequence z_n is bounded in the reflexive space E. For a suitable subsequence, we have $z_n \to \hat{z}$ weakly in E as $n \to \infty$, which concludes

$$\langle S(z_n) - S(z), z_n - \hat{z}\rangle = \langle e_n - e, z_n - \hat{z}\rangle = 0.$$

Note that if $z_n \to \hat{z}$ weakly in E as $n \to +\infty$ and $S(z_n) \to S(\hat{z})$ strongly in E^* as $n \to +\infty$, one has $z_n \to \hat{z}$ strongly in E as $n \to +\infty$, and since S is continuous, we have $z_n \to \hat{z}$ weakly in E as $n \to +\infty$ and $S(z_n) \to S(\hat{z}) = S(z)$ strongly in E^* as $n \to +\infty$. Hence, taking into account that S is an injection, we have $z = \hat{z}$. □

3 Existence of one solution

In this section, we address the existence of a solution for the problem ($P^{h,g}$).

First, put

$$B = \frac{1}{2}\left(\frac{4096}{27}p^- + \frac{64}{9}q^- + \frac{13}{20}r^-\right)$$

and

$$D = \frac{1}{2}\left(\frac{4096}{27}p^+ + \frac{64}{9}q^+ + \frac{13}{20}r^+\right)$$

where p^+, q^+ and r^+ are the ess sup in $[0,1]$ of the functions p, q and r respectively. For ease of reference, we define

$$G^\theta = \int_0^1 \max_{|m|\leq\theta} G(t,m)\mathrm{d}t \quad \text{for each } \theta > 0$$

and

$$G_\sigma = \inf_{t\in[0,1]} G(t,\sigma) \quad \text{for each } \sigma > 0.$$

If g is sign-changing, then clearly $G^\theta \geq 0$ and $G_\sigma \leq 0$.

To achieve our objective, we will fix two positive constants θ and σ, and set

$$\underline{\delta}_{\lambda,g} = \min\left\{\frac{2\delta^2\pi^2\theta^2 - \lambda\int_0^1 \max_{|m|\leq\theta} H(t,m)\mathrm{d}t}{G^\theta}, \frac{D\sigma^2 - \lambda\int_{\frac{3}{8}}^{\frac{5}{8}} H(t,\sigma)\mathrm{d}t}{G_\sigma}\right\}$$

and

$$\overline{\delta}_{\lambda,g} = \min\left\{\underline{\delta}_{\lambda,g},\ \frac{1}{\max\left\{0, \frac{1}{2\delta^2\pi^2}\limsup_{|m|\longrightarrow\infty}\frac{G(t,m)}{|m|^2}\right\}}\right\}, \tag{3.1}$$

where we read $\epsilon/0 = +\infty$, so that, for instance, $\overline{\delta}_{\lambda,g} = +\infty$ when

$$\limsup_{|m|\longrightarrow\infty}\frac{G(t,m)}{|m|^2} \leq 0$$

and $G_\sigma = G^\theta = 0$.

Theorem 3.1. *Assume that there exist two positive constants θ and σ with the property*

$$\sqrt{\frac{D}{2}}\frac{\sigma}{\pi\delta} < \theta$$

such that

(A_1) $h(t,v) \geq 0$ *for each* $(t,v) \in \left([0,\frac{3}{8}) \cup (\frac{5}{8},1]\right) \times \mathbb{R}$,

(A_2) $\frac{\int_0^1 \max_{|m|\leq\theta} H(t,m)\mathrm{d}t}{\theta^2} < \frac{2\delta^2\pi^2}{D}\frac{\int_{\frac{3}{8}}^{\frac{5}{8}} H(t,\sigma)\mathrm{d}t}{\sigma^2}$,

(A_3) $\min_{t\in[0,1]}\limsup_{|m|\longrightarrow\infty}\frac{H(t,m)}{|m|^2} \in (-\infty,0]$.

Then, for each

$$\lambda \in \Lambda = \left(\frac{D\sigma^2}{\int_{\frac{3}{8}}^{\frac{5}{8}} H(t,\sigma)\mathrm{d}t}, \frac{2\delta^2\pi^2\theta^2}{\int_0^1 \max_{|m|\leq\theta} H(t,m)\mathrm{d}t}\right)$$

and for each function $g : [0,1]\times\mathbb{R} \longrightarrow \mathbb{R}$ satisfying the condition

$$\limsup_{|m|\longrightarrow\infty}\frac{G(t,m)}{|m|^2} \in (-\infty,0], \tag{3.2}$$

there exists $\overline{\delta}_{\lambda,g} > 0$ given by (3.1) such that for each $\mu \in [0,\overline{\delta}_{\lambda,g})$, the problem ($P^{h,g}$) admits at least one solution z_λ in E *such that*

$$|z_\lambda(t)| < \theta.$$

Proof. The objective is to apply Theorem 2.1 to address the problem ($P^{h,g}$). To do this, we consider the functionals Φ and Ψ as given in 2.5 and 2.6, respectively. We aim to show that these functionals satisfy the necessary conditions outlined in Theorem 2.1. It is well-known that both functionals are well-defined Gâteaux differentiable, and one has

$$\Psi'(z)(v) = \int_0^1 h(t,z(t))v(t)\,\mathrm{d}t + \frac{\mu}{\lambda}\int_0^1 g(t,z(t))v(t)\,\mathrm{d}t$$

and

$$\Phi'(z)(v) = \int_0^1 \left(p(t)z''(t)v''(t) + q(t)z'(t)v'(t) + r(t)z(t)v(t)\right)\mathrm{d}t$$

for every $z, v \in \mathrm{E}$. Furthermore, Φ and Ψ are C^1-functions. By utilizing the definition of Φ, it follows that

$$\lim_{\|z\|_{\mathrm{E}} \to +\infty} \Phi(z) = +\infty,$$

which implies that Φ is coercive. Furthermore, proposition 2.12 guarantees that Φ has a continuous inverse on E^*. Therefore, we can conclude that the regularity condition on Φ and Ψ, required in Theorem 2.1 are satisfied. The critical points of the functional I_λ in E correspond precisely to the generalized solutions of the considered problem ($P^{h,g}$). To establish the existence of a critical point I_λ in E, we verify that the regularity conditions on Φ and Ψ, as outlined in Theorem 2.1, are indeed fulfilled. It is crucial to note that the operator I_λ is a $C^1(\mathrm{E}, \mathbb{R})$ functional on E, and the critical points of I_λ correspond to weak solutions of the problem ($P^{h,g}$). Furthermore, for $\lambda > 0$, the functional I_λ is coercive. Indeed, since $\mu < \delta_\lambda$ we can fix t such that

$$\limsup_{|m| \longrightarrow \infty} \frac{G(t,m)}{|m|^2} \in (-\infty, 0]$$

and $\mu t < 2\pi^2\delta^2$. Consequently, there exists a positive constant ι such that

$$G(t,m) \leq tm^2 + \iota$$

for each $(t,m) \in [0,1] \times \mathbb{R}$. Now, we fix

$$0 < \varepsilon < \frac{4\pi^2\delta^2}{\lambda}\left(\frac{1}{2} - \mu t \frac{1}{4\pi^2\delta^2}\right).$$

Based on the assumption (A_3) there is a bounded function ϕ_ε such that

$$H(t,m) \leq \varepsilon m^2 + \phi$$

for each $(t,m) \in [0,1] \times \mathbb{R}$. It follows that, for each $z \in \mathrm{E}$, we have

$$\begin{aligned}
\Phi(z) - \lambda\Psi(z) &= \frac{1}{2}\|z\|_{\mathrm{E}}^2 - \lambda\left(\int_0^1 H(t, z(t))\mathrm{d}t + \frac{\mu}{\lambda}\int_0^1 G(t, z(t))\mathrm{d}t\right) \\
&\geq \frac{1}{2}\|z\|_0^2 - \lambda\left(\varepsilon\frac{1}{4\pi^2\delta^2}\|z\|_{\mathrm{E}}^2 + \phi\right) - \mu\left(t\frac{1}{4\pi^2\delta^2}\|z\|_{\mathrm{E}}^2 + \iota\right) \\
&\geq \left(\frac{1}{2} - \lambda\varepsilon\frac{1}{4\pi^2\delta^2} - \mu t\frac{1}{4\pi^2\delta^2}\right)\|z\|_{\mathrm{E}}^2 - \lambda\phi - \mu\iota
\end{aligned}$$

and thus

$$\lim_{\|z\|_{\mathrm{E}} \longrightarrow \infty} \Phi(z) - \lambda\Psi(z) = \infty,$$

which means the functional $I_\lambda = \Phi - \lambda\Psi$ is coercive. Thus, by [3, Proposition 2.1] the functional $I_\lambda = \Phi - \lambda\Psi$ verifies $(\mathrm{PS})^{[s]}$-condition for each $s > 0$ and so the condition (a_2) of Theorem 2.1 is verified. Fix $\lambda \in (0, \lambda^*)$, thus

$$\frac{\int_{\frac{3}{8}}^{\frac{5}{8}} H(t,\sigma)\mathrm{d}t + \frac{\mu}{\lambda}G_\sigma}{D\sigma^2} > \frac{1}{\lambda}.$$

Put $s = 2\delta^2\pi^2\theta^2$ and

$$w_\sigma(t) = \begin{cases} -\dfrac{64\sigma}{9}\left(t^2 - \dfrac{3}{4}t\right), & \text{if } t \in [0, \frac{3}{8}), \\ \sigma, & \text{if } t \in [\frac{3}{8}, \frac{5}{8}] \\ -\dfrac{64\sigma}{9}\left(t^2 - \dfrac{5}{4}t + \dfrac{1}{4}\right), & \text{if } t \in (\frac{5}{8}, 1]. \end{cases}$$

Clearly, $w_\sigma \in \mathrm{E}$. Obviously, one has

$$\begin{aligned}
&\left(\frac{4096}{27}p^- + \frac{64}{9}q^- + \frac{13}{20}r^-\right)\sigma^2 \\
&\quad \le \int_0^{\frac{3}{8}} p(t)\frac{16384}{81}\sigma^2 \mathrm{d}t + \int_{\frac{5}{8}}^1 p(t)\frac{16384}{81}\sigma^2 \mathrm{d}t + \int_0^{\frac{3}{8}} q(t)\left(-\frac{64\sigma}{9}\left(2t - \frac{3}{4}\right)\right)^2 \mathrm{d}t \\
&\qquad + \int_{\frac{5}{8}}^1 q(t)\left(-\frac{64\sigma}{9}\left(2t - \frac{5}{4}\right)\right)^2 \mathrm{d}t + \int_0^{\frac{3}{8}} r(t)\left(-\frac{64\sigma}{9}\left(t^2 - \frac{3}{4}t\right)\right)^2 \mathrm{d}t \\
&\qquad + \int_{\frac{5}{8}}^1 r(t)\left(-\frac{64\sigma}{9}\left(t^2 - \frac{5}{4}t + \frac{1}{4}\right)\right)^2 \mathrm{d}t + \int_{\frac{3}{8}}^{\frac{5}{8}} r(t)\sigma^2 \mathrm{d}t \\
&\quad = \int_0^1 (p(t)|z''(t)|^2 + q(t)|z'(t)|^2 + r(t)|z(t)|^2)\mathrm{d}t \\
&\quad = \|w_\sigma\|_{\mathrm{E}}^2 \le \left(\frac{4096}{27}p^+ + \frac{64}{9}q^+ + \frac{13}{20}r^+\right)\sigma^2.
\end{aligned}$$

Then, we have $\Phi(0) = \Psi(0) = 0$ and

$$\frac{1}{2}\left(\frac{4096}{27}p^- + \frac{64}{9}q^- + \frac{13}{20}r^-\right)\sigma^2 \le \Phi(w_\sigma) = \frac{1}{2}\|w_\sigma\|_{\mathrm{E}}^2 \le \frac{1}{2}\left(\frac{4096}{27}p^+ + \frac{64}{9}q^+ + \frac{13}{20}r^+\right)\sigma^2.$$

By using condition (A_1), we have

$$\begin{aligned}
\Psi(w_\sigma) &= \int_0^1 H(t, w_\sigma(t))\mathrm{d}t + \frac{\mu}{\lambda}\int_0^1 G(t, w_\sigma(t))\mathrm{d}t \\
&\ge \int_{\frac{3}{8}}^{\frac{5}{8}} H(t, \sigma)\mathrm{d}t + \frac{\mu}{\lambda}G_\sigma.
\end{aligned}$$

Thus, by the assumption

$$\sqrt{\frac{D}{2}}\frac{\sigma}{\pi\delta} < \theta,$$

we have $0 < \Phi(w_\sigma) < s$. Owing to Proposition 2.9, one has

$$\|z\|_\infty \le \frac{1}{2\pi\delta}\|z\|_{\mathrm{E}} \le \frac{1}{2\pi\delta}\sqrt{2s} = \theta,$$

hence, we have

$$\sup_{\Phi(z)<s} \Psi(z) \le \int_0^1 \max_{|m|\le\theta} H(t, m)\mathrm{d}t + \frac{\mu}{\lambda}G^\theta.$$

Therefore, we have

$$\begin{aligned}
&\frac{\sup_{z\in\Phi^{-1}(-\infty,s]}\Psi(z)}{s} \\
&\quad = \frac{\sup_{z\in\Phi^{-1}(-\infty,s]}\left(\int_0^1 H(t, z(t))\mathrm{d}t + \frac{\mu}{\lambda}\int_0^1 G(t, z(t))\mathrm{d}t\right)}{s} \\
&\quad \le \frac{\int_0^1 \max_{|m|\le\theta} H(t, m)\mathrm{d}t + \frac{\mu}{\lambda}G^\theta}{2\delta^2\pi^2\theta^2}
\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}\frac{\Psi(w_\sigma)}{\Phi(w_\sigma)} &\geq \frac{\int_0^1 H(t,w(t))\mathrm{d}t + \frac{\mu}{\lambda}\int_0^1 G(t,w(t))\mathrm{d}t}{D\sigma^2} \\ &\geq \frac{\int_{\frac{3}{8}}^{\frac{5}{8}} H(t,\sigma)\mathrm{d}t + \frac{\mu}{\lambda}G_\sigma}{D\sigma^2}.\end{aligned} \tag{3.4}$$

Since

$$\mu < \frac{2\delta^2\pi^2\theta^2 - \lambda\int_0^1 \max_{|m|\leq\theta} H(t,m)\mathrm{d}t}{G^\theta},$$

we have

$$\frac{\int_0^1 \max_{|m|\leq\theta} H(t,m)\mathrm{d}t + \frac{\mu}{\lambda}G^\theta}{2\delta^2\pi^2\theta^2} < \frac{1}{\lambda}.$$

Furthermore,

$$\mu < \frac{D\sigma^2 - \lambda\int_{\frac{3}{8}}^{\frac{5}{8}} H(t,\sigma)\mathrm{d}t}{G_\sigma},$$

this means

$$\frac{\int_{\frac{3}{8}}^{\frac{5}{8}} H(t,\sigma)\mathrm{d}t + \frac{\mu}{\lambda}G_\sigma}{D\sigma^2} > \frac{1}{\lambda}.$$

Then,

$$\frac{1}{2\delta^2\pi^2}\frac{\int_0^1 \max_{|m|\leq\theta} H(t,m)\mathrm{d}t + \frac{\mu}{\lambda}G^\theta}{\theta^2} < \frac{1}{\lambda} < \frac{\int_{\frac{3}{8}}^{\frac{5}{8}} H(t,\sigma)\mathrm{d}t + \frac{\mu}{\lambda}G_\sigma}{D\sigma^2}. \tag{3.5}$$

Hence, from (3.3)–(3.5), the condition (a_1) of Theorem 2.1 is fulfilled. Since

$$\lambda \in \left(\frac{\Phi(w_\sigma)}{\Psi(w_\sigma)}, \frac{s}{\sup_{\Phi(z)\leq s}\Psi(z)}\right),$$

Theorem 2.1 with $\bar{z} = w$ guarantees the existence of a local minimum point z_λ for the functional I_λ such that $0 < \Phi(z_\lambda) < s$ and so z_λ is a nontrivial solution of the problem ($P^{h,g}$) such that

$$|z_\lambda(t)| < \theta. \qquad \square$$

Remark 3.2. We note that the preceding theorem also holds for a Carathéodory function h alongside $p \in W^{2,1}([0,1])$, $q \in W^{1,1}([0,1])$ and $r \in L^\infty([0,1])$. It is evident that, in this context, the solutions are generalized (see [8]).

We will now illustrate Theorem 3.1 by providing the following example.

Example 3.3. We consider the following problem

$$\begin{cases} z^{(4)} - z'' + z = \lambda h(z(t)) + \mu g(z(t)), & t \in [0,1], \\ z(0) = z(1) = 0, \\ z''(0) = z''(1) = 0 \end{cases} \tag{3.6}$$

where

$$h(m) = \begin{cases} 5m^4, & \text{for any } m \in (-\infty, 1), \\ \dfrac{5}{m}, & \text{for any } m \in [1, +\infty) \end{cases}$$

and

$$g(m) = \begin{cases} 2m, & \text{for any } m \in (-\infty, 1), \\ \dfrac{2}{m}, & \text{for any } m \in [1, +\infty). \end{cases}$$

Based on the expressions for h and g, we can conclude that

$$H(m) = \begin{cases} m^5, & \text{for each } m \in (-\infty, 1), \\ 5\ln(m) + 1, & \text{for each } m \in [1, +\infty) \end{cases}$$

and

$$G(m) = \begin{cases} m^2, & \text{for each } m \in (-\infty, 1), \\ 2\ln(m) + 1, & \text{for each } m \in [1, +\infty). \end{cases}$$

Through straightforward calculations, we derive $\delta = 1$ and $D = \frac{86111}{1080}$. Hence, $\lim_{|m|\longrightarrow\infty} \frac{H(m)}{|m|^2} = 0$, thus (A_3) is holds. Choose $\theta = 10^{-2}$, and $\sigma = 1$. Since

$$\frac{\int_0^1 \max_{|m|\leq\theta} H(m)\mathrm{d}t}{\theta^2} = \frac{1}{10^6} < \frac{540\pi^2}{86111} = \frac{2\delta^2\pi^2}{D}\frac{\int_{\frac{3}{8}}^{\frac{5}{8}} H(\sigma)\mathrm{d}t}{\sigma^2},$$

therefore, if condition (A_2) is satisfied, all the requirements of Theorem 3.1 are met. Consequently, it follows that for each

$$\lambda \in \left(\frac{86111}{270}, 2\pi^2 \times 10^6\right)$$

and since

$$\limsup_{|m|\longrightarrow\infty} \frac{G(,m)}{|m|^2} \in (-\infty, 0]$$

it follows that for each

$$\mu \in \left[0, \min\left\{\frac{2\pi^2 \times 10^{-4} - \lambda 10^{-10}}{10^{-4}}, \frac{86111}{1080} - \frac{\lambda}{4}\right\}\right)$$

the problem (3.6) admits at least one solution z_λ in E such that

$$|z_\lambda(t)| < 10^{-2}.$$

4 Existence of two solutions

Applying Theorem 2.2, we obtain the following results:

Theorem 4.1. *Assume that there exist two positive constants θ and σ with the property*

$$\sqrt{\frac{D}{2}}\frac{\sigma}{\pi\delta} < \theta$$

and

(A_4) *there exist $\nu > 2$ and $R > 0$ such that*

$$0 < \nu H(t, m) \leq mh(t, m) \tag{4.1}$$

for each $|m| \geq R$ and for each $t \in [0, 1]$.

Then, for each

$$\lambda \in \left(0, \frac{2\delta^2\pi^2\theta^2}{\int_0^1 \max_{|m|\leq\theta} H(t,m)\mathrm{d}t}\right),$$

and for each function $g : [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}$ *satisfying the condition* (A_4), *there exists* $\delta_\lambda > 0$ *for each* $\mu \in [0, \delta_\lambda[$, *the problem* ($P^{h,g}$) *admits at least two solutions* z_1 *and* z_2 *in* E *such that*

$$|z_1(t)| < \theta.$$

Proof. Our objective is to utilize Theorem 2.2 in the context of the space E, and the functionals Φ and Ψ as outlined in the proof of Theorem 3.1. Additionally, the functional I_λ adheres to the (PS)-condition. Indeed, assume that $\{z_k\}_{k\in\mathbb{N}} \subset \mathrm{E}$ such that $\{I_\lambda(z_k)\}_{k\in\mathbb{N}}$ is bounded and $I'_\alpha(z_k) \longrightarrow 0$ as $k \longrightarrow +\infty$. Then, there exists a positive constant C_0 such that $|I_\lambda(z_k)| \leq C_0$ for each $k \in \mathbb{N}$. Therefore, we infer to deduce from the definition of I'_λ and the assumption (A_4), for k large enough, we get

$$\begin{aligned} C_0 + C_1\|z_n\|_{\mathrm{E}} \geq \nu I_\lambda(z_n) - I'_\lambda(z_n)(z_n) &= \left(\frac{\nu}{2} - 1\right)\|z_n\|_{\mathrm{E}}^2 \\ &\quad - \lambda\int_0^1 \left(\nu H(t,z_n(t)) - h(t,z_n(t))\, z_n(t)\mathrm{d}t\right. \\ &\quad - \mu\int_0^1 \left(\nu G(t,z_n(t)) - g(t,z_n(t))\, z_n(t)\mathrm{d}t\right. \\ &\geq \left(\frac{\nu}{2} - 1\right)\|z_n\|_{\mathrm{E}}^2 \end{aligned}$$

for some $C_1 > 0$. Since $\nu > 2$, this implies that $\{z_k\}_{k\in\mathbb{N}}$ is bounded. Consequently, since E is a reflexive Banach space, we have, up to a subsequence, $z_k \rightharpoonup z$ in E, $z_k \to z$ in $L^2([0,1])$ and $z_k \to z$ a.e. on $[0,1]$. By $I'_\lambda(z_k) \to 0$ and $z_k \rightharpoonup z$ in E, we obtain

$$\left(I'_\lambda(z_k) - I'_\lambda(z)\right)(z_k - z) \to 0.$$

From the continuity of h and g, we have

$$\int_0^1 \left(h(t,z_k(t)) - h(t,z(t))\right)(z_k(t) - z(t))\mathrm{d}t \to 0$$

and

$$\int_0^1 \left(g(t,z_k(t)) - g(t,z(t))\right)(z_k(t) - z(t))\mathrm{d}t \to 0.$$

Thus

$$0 = \lim_{k\to+\infty}\left(I'_\lambda(z_k) - I'_\lambda(z)\right)(z_k - z) \geq \lim_{k\to+\infty}\|z_k - z\|_{\mathrm{E}}^2.$$

So $\|z_k - z\|_{\mathrm{E}} \to 0$ as $k \to +\infty$, which implies that $\{z_k\}$ converges strongly to u in E. Therefore, I_λ satisfies the Palais–Smale condition. Additionally, by incorporating the condition (4.1), there are constants $a_1, a_2, a_3, a_4 > 0$ such that

$$H(t,z) \geq a_1|z|^\nu - a_2 \quad \text{and} \quad G(t,z) \geq a_3|z|^\nu - a_4$$

for each $(t,z) \in [0,1] \times \mathbb{R}$. Now, by selecting any $z \in \mathrm{E} \setminus \{0\}$, we have

$$\begin{aligned} I_\lambda(z) &= (\Phi + \lambda\Psi)(z) \\ &= \frac{1}{2}\|z\|_{\mathrm{E}}^2 - \lambda\int_0^1 H(t,z(t))\mathrm{d}t - \mu\int_0^1 G(t,z(t))\mathrm{d}t \\ &\leq \frac{1}{2}\|z\|_{\mathrm{E}}^2 - \lambda a_1\frac{1}{2^\nu\pi^\nu\delta^\nu}\|z\|_{\mathrm{E}}^\nu - \mu a_3\frac{1}{2^\nu\pi^\nu\delta^\nu}\|z\|_{\mathrm{E}}^\nu + \lambda a_2 + \mu a_4. \end{aligned}$$

Given that $\nu > 2$, this condition ensures that I_λ is unbounded from below. Consequently, all the requirements of Theorem 2.2 are satisfied. Thus, for each

$$\lambda \in \left(0, \frac{2\delta^2\pi^2\theta^2}{\int_0^1 \max_{|m|\leq\theta} H(t,m)\mathrm{d}t}\right),$$

the functional I_λ admits two distinct critical points that are solutions of the problem $(P^{h,g})$. □

Remark 4.2. In Theorem 2.1, if either $h(t,0) \neq 0$ for some $t \in [0,1]$ or $g(t,0) \neq 0$ for some $t \in [0,1]$ or if both conditions are satisfied, then Theorem 4.1 guarantees the existence of two nontrivial solutions for the problem $(P^{h,g})$. if neither $h(t,0) \neq 0$ and $g(t,0) \neq 0$ is satisfied for any $t \in [0,1]$, the second solution z_2 to the problem $(P^{h,g})$ may be trivial. However, there is still at least one nontrivial solution.

Remark 4.3. Using similar arguments as presented in the proof of [5, Theorem 3.5], the nontrivial nature of the second solution guaranteed by Theorem 4.1 can also be established in the scenario where $h(t,0) = 0$ for each $t \in [0,1]$. This requires an additional condition at zero: namely, there exists a non-empty open set $D \subseteq [0,1]$ and a subset $B \subset D$ with positive Lebesgue measure such that

$$\limsup_{m\longrightarrow 0^+} \frac{\operatorname{ess\,inf}_{t\in B} H(t,m)}{|m|^2} = \infty \quad \text{and} \quad \liminf_{m\longrightarrow 0^+} \frac{\operatorname{ess\,inf}_{t\in D} H(t,m)}{|m|^2} > -\infty.$$

5 Another multiplicity result for the case $\mu = 0$

In this section, we discuss the existence of at least two and three solutions for the problem $(P^{h,g})$ when $\mu = 0$. To do this, we define

$$F^c = \int_0^1 \max_{|m|\leq c} H(t,m)\mathrm{d}t$$

and

$$F_c = \left(\frac{1}{4}\right) \inf_{t\in[0,1]} H(t,c)$$

for each $c > 0$.

Theorem 5.1. *Assume that there exist two positive constants $\bar\theta$ and $\bar\sigma$ such that*

$$\sqrt{\frac{D}{2}}\frac{\bar\sigma}{\pi\delta} < \bar\theta \tag{5.1}$$

and suppose that the assumptions (A_1) and (A_3) in Theorem 3.1 hold. Moreover, assume that

(A_5) $\frac{F^{\bar\theta}}{\bar\theta^2} < \frac{2\delta^2\pi^2}{D}\frac{F_{\bar\sigma}-F^{\bar\theta}}{\bar\sigma^2}$.

Then, for each

$$\lambda \in \left(\frac{D\bar\sigma^2}{F_{\bar\sigma} - F^{\bar\theta}}, \frac{2\delta^2\pi^2\bar\theta^2}{F^{\bar\theta}}\right),$$

the problem $(P^{h,g})$ in the case $\mu = 0$ admits at least three solutions in E.

Proof. Put $I_\lambda = \Phi + \lambda\Psi$, where

$$\Phi(z) = \frac{1}{2}\|z\|_{\mathrm{E}}^2 \tag{5.2}$$

and

$$\Psi(z) = -\int_0^1 H(t, z(t))\mathrm{d}t$$

for each $z \in \mathrm{E}$. Standard arguments demonstrate that the functionals Φ and Ψ are Gâteaux differentiable. The Gâteaux derivatives at a point $z \in \mathrm{E}$ are expressed as follows

$$\Phi'(z)(v) = \int_0^1 z'(t)v'(t)\mathrm{d}t + \int_0^1 \delta(t)(z(t))v(t)\mathrm{d}t$$

and

$$\Psi'(z)(v) = -\int_0^1 h(t, z(t))v(t)\mathrm{d}t$$

for each $z, v \in \mathrm{E}$, respectively. Therefore, a critical point of the functional $\Phi + \lambda\Psi$ corresponds to a solution of equation ($P^{h,g}$) when $\mu = 0$. Our goal is to apply Theorem 2.3 to the functionals Φ and Ψ. The Functional Φ is sequentially weakly lower semicontinuous, and, as discussed in Section 2, Φ is continuously Gâteaux differentiable Moreover, Proposition 2.12 indicates that its Gâteaux derivative has a continuous inverse on E^*. The functional $\Psi : \mathrm{E} \longrightarrow \mathbb{R}$ is well defined, continuously Gâteaux differentiable, and its Gâteaux derivative is compact. Thus, we need to verify that Φ and Ψ satisfy conditions (c_1) and (c_2) as stated in Theorem 2.3. We will now fix $0 < \varepsilon < \frac{2\pi^2\delta^2}{\lambda}$. From the assumption (A_3) there is a function $\phi_\varepsilon : [0,1] \longrightarrow \mathbb{R}$ with $\phi_\varepsilon(t) < \infty$ for each $t \in [0,1]$ such that

$$H(t, \xi) \leq \varepsilon\xi^2 + \phi_\varepsilon(t)$$

for each $(t, \xi) \in [0,1]$. It can be concluded that for each $z \in \mathrm{E}$

$$\begin{aligned}
\Phi(z) + \lambda\Psi(z) &= \frac{1}{2}\|z\|_{\mathrm{E}}^2 - \lambda\int_0^1 H(t, z(t))\mathrm{d}t \\
&\leq \frac{1}{2}\|z\|_{\mathrm{E}}^2 - \lambda\varepsilon\frac{1}{4\pi^2\delta^2}\|z\|_{\mathrm{E}}^2 - \lambda\int_0^1 \phi_\varepsilon(t)dt \\
&= \left(\frac{1}{2} - \lambda\varepsilon\frac{1}{4\pi^2\delta^2}\right)\|z\|_{\mathrm{E}}^2 - \lambda\int_0^1 \phi_\varepsilon(t)dt
\end{aligned}$$

and thus

$$\lim_{\|z\|_{\mathrm{E}} \longrightarrow \infty}(\Phi(z) + \lambda\Psi(z)) = \infty,$$

which means the functional $I_\lambda = \Phi + \lambda\Psi$ is coercive. Now, it remains to show that (c_2) of Theorem 2.3 is fulfilled. Let $\bar{s} = 2\delta^2\pi^2\bar{\theta}^2$ and

$$\bar{w}_{\bar{\sigma}}(t) = \begin{cases} -\frac{64\bar{\sigma}}{9}\left(t^2 - \frac{3}{4}t\right), & \text{if } t \in \left[0, \frac{3}{8}\right), \\ \bar{\sigma}, & \text{if } t \in \left[\frac{3}{8}, \frac{5}{8}\right] \\ -\frac{64\bar{\sigma}}{9}\left(t^2 - \frac{5}{4}t + \frac{1}{4}\right), & \text{if } t \in \left(\frac{5}{8}, 1\right]. \end{cases}$$

Clearly $w \in \mathrm{E}$. Therefore, we can conclude that

$$B\bar{\sigma}^2 \leq \Phi(\bar{w}_{\bar{\sigma}}) \leq D\bar{\sigma}^2.$$

Thus by (5.1), $\Phi(w_\sigma) > \bar{s}$. Moreover

$$\Psi(\bar{w}_{\bar{\sigma}}) = -\int_0^1 H(t, \bar{w}_{\bar{\sigma}})\mathrm{d}t \leq -\int_{\frac{3}{8}}^{\frac{5}{8}} H(t, \bar{\sigma})\mathrm{d}t = -F_{\bar{\sigma}}.$$

For each $z \in \mathrm{E}$ such that $\Phi(z) < \bar{s}$, we obtain

$$\sup_{t\in[0,1]} |z(t)| \leq \bar{\theta}.$$

Thus,

$$\sup_{z\in\Phi^{-1}(-\infty,\bar{s})} \Psi(x) \leq \int_0^1 \max_{|m|\leq\bar{\theta}} H(t, m)\mathrm{d}t = F^{\bar{\theta}}. \tag{5.3}$$

Through straightforward calculations and based on the definition of $\varphi(\bar{s})$, we find that since $\Phi(0) = \Psi(0) = 0$ and $\overline{\Phi^{-1}(-\infty, \bar{s})}^{w} = \Phi^{-1}(-\infty, \bar{s})$, it follows that By simple calculations and from the definition of $\varphi(\bar{s})$, since $\Phi(0) = \Psi(0) = 0$ and $\overline{\Phi^{-1}(-\infty, \bar{s})}^{w} = \Phi^{-1}(-\infty, \bar{s})$, one has

$$\begin{aligned}\varphi_1(\bar{s}) &= \inf_{z\in\Phi^{-1}(]-\infty,\bar{s}[)} \frac{\Psi(z) - \inf_{\overline{\Phi^{-1}(-\infty,\bar{s})}^{w}} \Psi}{\bar{s} - \Phi(z)} \leq \frac{-\inf_{\overline{\Phi^{-1}(-\infty,\bar{s})}^{w}} \Psi}{\bar{s}} \\ &\leq \frac{\int_0^1 \max_{|m|\leq\bar{\theta}} H(t, m)\mathrm{d}t}{2\delta^2\pi^2\bar{\theta}^2} = \frac{F^{\bar{\theta}}}{2\delta^2\pi^2\bar{\theta}^2}.\end{aligned}$$

Alternatively, according to (5.3), one can conclude that

$$\begin{aligned}\varphi_2(\bar{s}) &= \inf_{z\in\Phi^{-1}(-\infty,\bar{s})} \sup_{v\in\Phi^{-1}[\bar{s},\infty)} \frac{\Psi(z) - \Psi(v)}{\Phi(z) - \Phi(v)} \geq \inf_{z\in\Phi^{-1}(-\infty,\bar{s})} \frac{\Psi(z) - \Psi(\bar{w}_{\bar{\sigma}})}{\Phi(\bar{w}_{\bar{\sigma}}) - \Phi(z)} \\ &\geq \frac{\inf_{z\in\Phi^{-1}(-\infty,\bar{s})} \Psi(z) - \Psi(\bar{w}_{\bar{\sigma}})}{\Phi(\bar{w}_{\bar{\sigma}}) - \Phi(z)} \\ &\geq \frac{-\int_0^1 \max_{|m|\leq\bar{\theta}} H(t, m)\mathrm{d}t + \int_{\frac{3}{8}}^{\frac{5}{8}} H(t, \bar{\sigma})\mathrm{d}t}{\Phi(w_\sigma) - \Phi(z)} \\ &\geq \frac{F_{\bar{\sigma}} - F^{\bar{\theta}}}{D\sigma^2}.\end{aligned}$$

Consequently, from (A_5), we derive

$$\varphi_1(\bar{s}) < \varphi_2(\bar{s}).$$

Thus, based on Theorem 2.3, while also considering that

$$\frac{1}{\varphi_2(\bar{s})} \leq \frac{D\sigma^2}{F_{\bar{\sigma}} - F^{\bar{\theta}}}$$

and

$$\frac{1}{\varphi_1(\bar{s})} \geq \frac{2\delta^2\pi^2\bar{\theta}^2}{F^{\bar{\theta}}},$$

we arrive at the intended conclusion. □

Remark 5.2. When the assumption (A_5) of Theorem 5.1 is satisfied, straightforward calculations demonstrate that the condition

(A_6) $\frac{F^{\bar{\theta}}}{2\delta^2\pi^2\bar{\theta}^2} < \frac{F_{\bar{\sigma}}}{D\bar{\sigma}^2}$

implies (A_5) of Theorem 5.1. Therefore, if the conditions (5.1) and (A_5) are satisfied, then for each

$$\lambda \in \left(\frac{D\bar{\sigma}^2}{F_{\bar{\sigma}}}, \frac{2\delta^2\pi^2\bar{\theta}^2}{F^{\bar{\theta}}}\right),$$

the problem $(P^{h,g})$ in the case $\mu = 0$ admits at least three solutions.

We now apply Theorem 2.4, which will later be used to establish the existence of multiple solutions for the problem $(P^{h,g})$ when $\mu = 0$, without relying on assumption (A_3).

Theorem 5.3. *Assume that there exist three positive constants $\bar{\theta}_1$, $\bar{\sigma}$ and $\bar{\theta}_2$ with*

$$\bar{\theta}_1 < \sqrt{\frac{B}{2}}\frac{\bar{\sigma}}{\pi\delta} \tag{5.4}$$

and

$$\sqrt{\frac{D}{2}}\frac{\bar{\sigma}}{\pi\delta} < \bar{\theta}_2 \tag{5.5}$$

in a manner that satisfies the assumption (A_4) in Theorem 2.3 and

(A_7) $\frac{1}{2\delta^2\pi^2}\max\left\{\frac{F^{\bar{\theta}_1}}{\bar{\theta}_1^2}, \frac{F^{\bar{\theta}_2}}{\bar{\theta}_2^2}\right\} < \frac{F_{\bar{\sigma}}}{D\bar{\sigma}^2}$.

Then, for each

$$\lambda \in \Lambda = \left(\frac{D\bar{\sigma}^2}{F_{\bar{\sigma}}}, \min\left\{\frac{2\delta^2\pi^2\bar{\theta}_1^2}{F^{\bar{\theta}_1}}, \frac{2\delta^2\pi^2\bar{\theta}_2^2}{F^{\bar{\theta}_2}}\right\}\right),$$

the problem $(P^{h,g})$ in the case $\mu = 0$ admits at least two solutions $z_{1,\lambda}$ and $z_{2,\lambda}$ such that $|z_{1,\lambda}(t)| < \bar{\theta}_1$ and $|z_{2,\lambda}(t)| < \bar{\theta}_2$.

Proof. Put

$$\overline{h}(t,\epsilon) = \begin{cases} h(t,-\bar{\theta}_2) & \text{if } [0,1]\times(-\infty,\bar{\theta}_2), \\ h(t,\epsilon) & \text{if } [0,1]\times[-\bar{\theta}_2,\bar{\theta}_2], \\ h(t,\bar{\theta}_2) & \text{if } [0,1]\times(\bar{\theta}_2,\infty). \end{cases}$$

Clearly, $\overline{h} : [0,1]\times\mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function. Now put $\overline{H}(t,m) = \int_0^m \overline{h}(t,\epsilon)\mathrm{d}\epsilon$ for each $(t,m) \in [0,1]\times\mathbb{R}$ and take E and Φ as given in (5.2), and

$$\Psi(z) = -\int_0^1 \overline{H}(t,z(t))\mathrm{d}t$$

for each $z \in \mathrm{E}$. Our goal is to apply Theorem 2.4 to Φ and Ψ. It is well known that $\lim_{\|z\|_{\mathrm{E}}\longrightarrow\infty}\Phi(z) = \infty$ and Ψ is a differentiable functional whose differential at the point $z \in \mathrm{E}$ is

$$\Psi'(z)(v) = -\int_0^1 \overline{h}(t,z(t))v(t)\mathrm{d}t$$

for any $v \in \mathrm{E}$ as well as it is sequentially weakly lower semicontinuous. Furthermore $\Psi' : \mathrm{E} \longrightarrow \mathrm{E}^*$ is a compact operator. Therefore, it suffices to demonstrate that Φ and Ψ meet the criteria (c_1), (c_2) and (c_3) outlined in Theorem 2.4. Let

$$\bar{s}_1 = 2\delta^2\pi^2\bar{\theta}_1^2, \quad \bar{s}_2 = 2\delta^2\pi^2\bar{\theta}_2^2$$

and $w \in \mathrm{E}$ in a manner similar to the proof of Theorem 2.4, we note that under the assumptions (5.4) and (5.5), it follows that $\bar{s}_1 < \Phi(w_\sigma) < \bar{s}_2$ and $\inf_{\mathrm{E}} \Phi < \bar{s}_1 < \bar{s}_2$. Furthermore, by applying the reasoning used in the proof of Theorem 5.1 and considering Remark 5.2, we arrive at

$$\varphi_1(\bar{s}_1) \leq \frac{\int_0^1 \max_{|m|\leq\bar{\theta}_1} \overline{H}(t,m)\mathrm{d}t}{2\delta^2\pi^2\bar{\theta}_1^2} = \frac{F^{\bar{\theta}_1}}{2\delta^2\pi^2\bar{\theta}_1^2},$$

$$\varphi_1(\bar{s}_2) \leq \frac{\int_0^1 \max_{|m|\leq\bar{\theta}_2} \overline{H}(t,m)\mathrm{d}t}{2\delta^2\pi^2\bar{\theta}_2^2} = \frac{F^{\bar{\theta}_2}}{2\delta^2\pi^2\bar{\theta}_2^2}$$

and

$$\varphi_2^*(\bar{s}_1,\bar{s}_2) \geq \frac{F_{\bar{\sigma}}}{D\bar{\sigma}^2}.$$

Therefore, based on (A_7), the conditions (c_2) and (c_3) of Theorem 2.4 are satisfied. As a result, we can conclude from Theorem 2.4 that for each $\lambda \in \Lambda$, the problem

$$\begin{cases} -z'' + \alpha(t)z' + \delta(t)z = \lambda\overline{h}(t,z(t)) + \mu\overline{g}(t,z(t)), & t \in [0,1], \\ z(a) = z(b) = 0 \end{cases}$$

admits at least two solutions $z_{1,\lambda}$ and $z_{2,\lambda}$ such that $|z_{1,\lambda}(t)| < \bar{\theta}_1$ and $|z_{2,\lambda}(t)| < \bar{\theta}_2$. Noting that these solutions also satisfy the problem $(P^{h,g})$ when $\mu = 0$, we can draw the conclusion. □

In this section, we will provide some observations regarding our results.

Remark 5.4. In Theorems 5.1 and 5.3, we examined the critical points of the functional I_λ, which is naturally linked to the problem $(P^{h,g})$ when $\mu = 0$. It is important to note that, in general, I_λ can be unbounded from below in the space E. For instance, consider the scenario where $h(t) = 1 + |t|^{v-2}t$ for each $t \in \mathbb{R}$ with $v > 2$. In this case, for any fixed $z \in \mathrm{E}\backslash\{0\}$ and $\iota \in \mathbb{R}$, we can derive

$$\begin{aligned} I_\lambda(\iota z) &= \frac{1}{2}\|\iota z\|_{\mathrm{E}}^2 - \lambda\int_0^1 H(\iota z(t))\mathrm{d}t \\ &\leq \frac{\iota^2}{2}\|z\|_{\mathrm{E}}^2 - \lambda\frac{1}{2\pi\delta}\|z\|_{\mathrm{E}} - \lambda\frac{1}{2^v\pi^v\delta^v}\|z\|_{\mathrm{E}}^v \longrightarrow -\infty \end{aligned}$$

as $\iota \longrightarrow \infty$. Thus, direct minimization cannot be used to locate the critical points of I_λ.

Remark 5.5. If h is non-negative, Theorem 5.3 provides a bifurcation result where the pair $(0,0)$ is included in the closure of the set:

$$\left\{(z_\lambda,\lambda) \in \mathrm{E}\times(0,\infty) : z_\lambda \text{ is a non-trivial solution of } (P^{h,g}),\ \mu = 0\right\} \subset \mathrm{E}\times\mathbb{R}.$$

Indeed, if λ goes to zero, by Theorem 5.3 we have that $\bar{\theta}_i \longrightarrow 0$, $i = 1, 2$ and since $|z_{i,\lambda}(t)| < \bar{\theta}_i$, $i = 1, 2$, there exist two sequences $\{t_j\}$ in E and $\{\lambda_j\}$ in $\mathbb{R}^+$ (here $t_j = z_{\lambda_j}$) such that

$$\lambda_j \longrightarrow 0^+ \quad \text{and} \quad \|t_j\|_{\mathrm{E}} \longrightarrow 0,$$

as $j \longrightarrow \infty$. Moreover, since f is nonnegative, $\Psi(z) < 0$ for each $z \in \mathbb{R}$ and thus

$$(0, \lambda^*) \ni \lambda \mapsto I_\lambda(z_\lambda)$$

is strictly decreasing. Hence, for each $\lambda_1, \lambda_2 \in (0, \lambda^*)$, with $\lambda_1 \neq \lambda_2$, solutions z_{λ_1} and z_{λ_2} ensured by Theorem 2.4 are different.

Remark 5.6. As noted in [18, Remark 3.10], if h is non-negative, the solutions guaranteed by Theorems 5.1 and 5.3 are also non-negative.

6 Ethical statement

Author contributions: All authors contributed equally to this work.

Code availability: Not applicable.

Declarations

Conflict of interest: The authors declare that they have no conflict of interest.

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Positive solutions for semipositon Φ-Laplacian involving nonlocal term in Orlicz–Sobolev space

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Abstract. This paper establishes the existence of positive weak solutions for a class of semipositone Kirchhoff-type problems involving nonlocal operators and anisotropic Φ-Laplacians. By combining variational methods with careful asymptotic analysis, we prove that for sufficiently small $a > 0$, the problem admits a positive solution. Our results extend previous work on semipositone problems to the nonlocal Kirchhoff setting, overcoming challenges arising from the interplay between the nonlocal term and the indefinite nonlinearity.

Keywords: Kirchhoff type problem, semipositone nonlinearity, Orlicz–Sobolev spaces, variational methods, positive solutions.

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1 Introduction

Semipositone problems, where the nonlinearity satisfies $\hbar(0) < 0$ (see (1.2)), arise naturally in many physical and biological contexts, including population dynamics with harvesting and chemical reaction processes. These problems present substantial mathematical challenges compared to their positone counterparts, as the standard techniques based on sub-super solutions often fail due to the lack of a natural subsolution at zero (see [2–5,7–15,19–23,26,30]).

In 2021, Figueiredo, Massa and Santos [16], proved the existence of positive solution for the semipositone problem

$$\begin{cases} -M(\int_Y |\nabla u|^2 dx)\Delta u = \hbar(u) - a & \text{in } Y, \\ u(x) > 0 & \text{in } Y, \\ u = 0 & \text{on } \partial Y, \end{cases} \tag{1.1}$$

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where $Y \subset \mathbb{R}^N$ is a smooth bounded domain, M, f are continuous nonnegative functions and $a > 0$ and small enough. They generalize the classical superlinear and sublinear problems. Recently, Razani and Figueiredo [30] studied a Dirichlet semipositone anisotropic p-Laplacian problem.

In this work, we investigate the existence of positive solutions for a class of nonlocal elliptic problems of Kirchhoff type with semipositone nonlinearities. Specifically, we consider:

$$\begin{cases} -\Gamma\left(\int_Y \Phi(u)dx\right)\Delta_\Phi u = \hbar(u) - \varepsilon & \text{in } Y, \\ u > 0 & \text{in } Y, \\ u = 0 & \text{on } \partial Y, \end{cases} \tag{1.2}$$

where Δ_Φ denotes the Φ-Laplacian operator, a generalization of the p-Laplacian that arises in the study of non-Newtonian fluids and nonlinear elasticity. The term $\Gamma(\int \Phi(u)dx)$ introduces a nonlocal dependence on the solution, modeling phenomena where the system's behavior at a point depends on its global state.

Here we assume $Y \subset \mathbb{R}^N$ is a smooth bounded domain, $\varepsilon > 0$ and $\hbar : [0; +\infty) \to \mathbb{R}$ is a continuous function where

($\hbar_1$) $0 = \hbar(0) = \min_{t\in 0,+\infty)} \hbar(t)$.

($\hbar_2$) $\lim_{t\to 0^+} \frac{\hbar(t)}{\phi(t)t} = 0$.

($\hbar_3$) There is $q \in (m, l^\star)$ such that

$$\lim_{|t|\to+\infty} \frac{|\hbar(t)|}{|t|^{q-1}} < +\infty.$$

($\hbar_4$) There are $\theta > m$ and $t_0 > 0$ such that

$$\theta H(t) \le \hbar(t)t, \quad \text{for all } t \ge t_0,$$

where $H(t) = \int_0^t \hbar(\tau)d\tau$ is the primitive of $\hbar$.

Notice that

$$\Delta_\Phi(u) = \operatorname{div}(\phi(|\nabla u|)\nabla u)$$

where $\Phi(t) := \int_0^{|t|} \phi(s)sds$ and ϕ satisfies the following conditions:

(φ_1) $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is a C^1-function.

(φ_2) $\phi(t), (\phi(t)t)' > 0,\ t > 0$

(φ_3) There exists $l, m \in (1, N)$ with $m \in [l, l^\star)$ and $l^\star = \frac{lN}{N-l}$ such that

$$l \le \frac{\Phi'(t)t}{\Phi(t)} \le m, \qquad \forall t > 0$$

(φ_4) There exists $\bar{l}, \overline{m} > 0$ such that

$$\bar{l} \le \frac{\Phi''(t)t}{\Phi'(t)} \le \overline{m}, \quad \text{for all } t > 0.$$

Assume $\Gamma : [0, +\infty) \to \mathbb{R}^+$ is an increasing and continuous function and $\Gamma(t) \ge m_0 > 0$ for all $t \in [0, +\infty)$. We set $\digamma(t) = \int_0^t \Gamma(\tau)d\tau$ and we assume

(Γ_0) $\liminf_{t\to 0} \frac{F(t)}{t} > 0$.

(Γ_∞) There exists $\theta_1 \in (1,\theta)$ such that $\limsup_{t\to+\infty} \frac{F(t)}{t^{\theta_1}} < \infty$.

(Γ_{AR}) There exists $\theta > 1,\ D, \beta > 0$ such that

$$\theta F(t) - \Gamma(t)t \geq \beta t - D \quad \text{for every } t \geq 0,$$

Definition 1.1. A function $u \in W_0^{1,\Phi}(\mathrm{Y})$ is called a weak solution of problem (1.2) if:

(i) u is continuous and positive in Y;

(ii) For every test function $\varphi \in W_0^{1,\Phi}(\mathrm{Y})$, the following identity holds:

$$\Gamma\left(\int_{\mathrm{Y}} \Phi(u)\,dx\right)\int_{\mathrm{Y}} \phi(|\nabla u|)\nabla u\cdot\nabla\varphi\,dx = \int_{\mathrm{Y}} \big(\hbar(u)-\varepsilon\big)\varphi\,dx,$$

where $\hbar$ and Γ satisfy the assumptions stated above.

The mathematical study of semipositone problems was initiated by Castro, Castro, and Shivaji [13], who revealed fundamental differences from positone problems, including the breakdown of symmetry properties and more complex solution structures. Subsequent works [4,5,12] developed various techniques to handle these challenges, particularly for local operators. For problems of Kirchhoff type, recent progress has been made in [2,16,27,29], but the semipositone case with general Φ-Laplacian operators remains largely unexplored.

Our work makes several significant contributions:

- We extend the existence theory for semipositone problems to the nonlocal Kirchhoff setting with anisotropic Φ-Laplacian operators.
- We develop new estimates to handle the interplay between the nonlocal coefficient Γ and the semipositone nonlinearity.
- We establish precise threshold values for the parameter ε that guarantee existence of positive solutions.
- Our framework accommodates both degenerate and non-degenerate cases of the Kirchhoff term.

The proof strategy combines variational methods in Orlicz–Sobolev spaces with careful asymptotic analysis. Key challenges include:

- Establishing the mountain pass geometry despite the semipositone nature of the problem.
- Deriving uniform L^∞ estimates independent of the parameter ε.
- Handling the nonlocal coefficient's effect on the compactness properties.
- Ensuring positivity of solutions through delicate comparison arguments.

Our results generalize and extend previous works in several directions, including [5] (local Φ-Laplacian case), [16] (Kirchhoff Laplacian), and [30] (anisotropic semipositone problems). The techniques developed here may be applicable to other nonlocal problems with indefinite nonlinearities.

The paper is organized as follows: Section 2 introduces the necessary framework of Orlicz–Sobolev spaces. Section 3 establishes existence for an auxiliary problem, while Section 4 completes the proof of our main result through careful asymptotic analysis and positivity arguments.

2 Preliminaries on Orlicz–Sobolev spaces

This section provides the necessary background on Orlicz and Orlicz–Sobolev spaces, which serve as the functional framework for our analysis. We recall fundamental definitions, key properties, and essential embedding results that will be crucial for our subsequent arguments. For a comprehensive treatment of these topics, we refer to the standard references [1] and [28].

We begin by recalling key properties of the Orlicz–Sobolev space $W^{1,\Phi}(Y)$, which becomes a Banach space when equipped with the standard norm

$$\|u\|_{1,\Phi} = \|u\|_{\Phi} + \|\nabla u\|_{\Phi}. \tag{2.1}$$

Its subspace $W_0^{1,\Phi}(Y)$, defined as the completion of $C_0^\infty(Y)$ under this norm, inherits these Banach space properties. Crucially, both spaces are separable and reflexive when Φ and its conjugate $\widetilde{\Phi}$ satisfy the Δ_2-condition. This condition further characterizes convergence in $L_\Phi(Y)$: a sequence $\{u_n\}$ converges to u if and only if

$$\int_Y \Phi(|u_n - u|)dx \to 0. \tag{2.2}$$

Three fundamental inequalities govern these spaces:

1. **Poincaré inequality**: There exists $\Lambda > 0$ such that for all $u \in W_0^{1,\Phi}(Y)$,

$$\int_Y \Phi(|u|)dx \leq \Lambda \int_Y \Phi(|\nabla u|)dx. \tag{2.3}$$

   This implies $\|u\| := \|\nabla u\|_\Phi$ is an equivalent norm on $W_0^{1,\Phi}(Y)$, which we adopt hereafter.

2. **Sobolev embedding**: There exists $S_N > 0$ such that

$$\|u\|_{\Phi_*} \leq S_N \|u\|, \tag{2.4}$$

   ensuring the continuous embedding $W_0^{1,\Phi}(Y) \hookrightarrow L_{\Phi_*}(Y)$. For bounded domains, this embedding becomes compact if the N-function B satisfies the growth condition

$$\limsup_{|t|\to\infty} \frac{B(t)}{\Phi_*(t)} = 0. \tag{2.5}$$

3. **Structural assumptions**: For our analysis, we assume $\phi(s) = \varphi(s)s$ and define

$$\Phi(t) = \int_0^{|t|} \varphi(s)s\,ds, \tag{2.6}$$

   extended continuously at $t = 0$. Under conditions (φ_1), (φ_2), (φ_3), both Φ and $\widetilde{\Phi}$ become N-functions satisfying the Δ_2-condition, as established in [17].

Lemma 2.1 (Growth estimates). *Under assumptions (φ_1) and (φ_2), define the comparison functions*

$$\zeta(t) = \min\{t^l, t^m\} \quad \text{and} \quad \kappa(t) = \max\{t^l, t^m\}, \quad t \geq 0.$$

Then the following estimates hold:

(i) Scaling property: For all $\rho, t \geq 0$,

$$\zeta(\rho)\Phi(t) \leq \Phi(\rho t) \leq \kappa(\rho)\Phi(t).$$

(ii) Norm control: For any $u \in L_\Phi(\mathrm{Y})$,

$$\zeta(\|u\|_\Phi) \leq \int_{\mathrm{Y}} \Phi(|u|)dx \leq \kappa(\|u\|_\Phi).$$

Lemma 2.2 (Convergence criterion). *Assume (φ_1), (φ_2), (φ_3) hold and $\mathrm{Y} \subset \mathbb{R}^N$ is a smooth bounded domain. Let $u \in W_0^{1,\Phi}(\mathrm{Y})$ and $\{u_n\}$ be a bounded sequence in $W_0^{1,\Phi}(\mathrm{Y})$ satisfying*

$$\begin{aligned}\int_{\mathrm{Y}} \Bigg(\Gamma\left(\int_{\mathrm{Y}} \Phi(u_n)dx\right) \phi(|\nabla u_n|)\nabla u_n \\ - \Gamma\left(\int_{\mathrm{Y}} \Phi(u)dx\right) \phi(|\nabla u|)\nabla u \Bigg) \cdot \nabla(u_n - u)\, dx \to 0.\end{aligned} \tag{2.7}$$

Then $u_n \to u$ strongly in $W_0^{1,\Phi}(\mathrm{Y})$.

The proof follows from [5, Lemma 2.2].

Lemma 2.3 (Regularity result). *Under assumptions (φ_1), (φ_2), (φ_3), (φ_4), let $h \in L^\infty(\mathrm{Y})$ and $u \in W_0^{1,\Phi}(\mathrm{Y})$ be a weak solution of*

$$\begin{cases} -\Gamma\left(\int_{\mathrm{Y}} \Phi(u)dx\right) div(\phi(|\nabla u|)\nabla u) = h & \text{in } \mathrm{Y}, \\ u = 0 & \text{on } \partial\mathrm{Y}. \end{cases}$$

Then:

- *$u \in C^{1,\alpha}(\overline{\mathrm{Y}})$ for some $\alpha > 0$ depending on $\bar{l}$ and $\overline{m}$*
- *There exists $K = K(\|h\|_\infty, \bar{l}, \overline{m}, \mathrm{Y}) > 0$ such that $|u|_{1,\alpha} \leq K$*
- *$K \to 0$ as $\|h\|_\infty \to 0$*

The proof follows from the regularity results in [24, 25].

Definition 2.4 (Differential inequality). For $u, v \in W_0^{1,\Phi}(\mathrm{Y})$, we say that

$$-\Gamma\left(\int_{\mathrm{Y}} \Phi(u)dx\right) \Delta_\Phi u \leq -\Gamma\left(\int_{\mathrm{Y}} \Phi(v)dx\right) \Delta_\Phi v \quad \text{in } \mathrm{Y}$$

if for all $w \in W_0^{1,\Phi}(\mathrm{Y})$ with $w \geq 0$,

$$\Gamma\left(\int_{\mathrm{Y}} \Phi(u)dx\right) \int_{\mathrm{Y}} \phi(|\nabla u|)\nabla u \cdot \nabla w\, dx \leq \Gamma\left(\int_{\mathrm{Y}} \Phi(v)dx\right) \int_{\mathrm{Y}} \phi(|\nabla v|)\nabla v \cdot \nabla w\, dx.$$

Lemma 2.5 (Comparison principle). *Let $u, v \in W_0^{1,\Phi}(\Upsilon)$ satisfy*

$$\Gamma\left(\int_\Upsilon \Phi(u)dx\right)\int_\Upsilon \phi(|\nabla u|)\nabla u\cdot\nabla w\,dx \le \Gamma\left(\int_\Upsilon \Phi(v)dx\right)\int_\Upsilon \phi(|\nabla v|)\nabla v\cdot\nabla w\,dx$$

for all nonnegative $w \in W_0^{1,\Phi}(\Upsilon)$. Then $u \le v$ in Υ.

Proof. Consider $(u-v)^+ \in W_0^{1,\Phi}(\Upsilon)$. On the set $\Upsilon_1 = \{x \in \Upsilon : u(x) - v(x) \ge 0\}$, we have:

$$\int_{\Upsilon_1}\langle\Gamma\left(\int_\Upsilon \Phi(u)dx\right)\phi(|\nabla u|)\nabla u - \Gamma\left(\int_\Upsilon \Phi(v)dx\right)\phi(|\nabla v|)\nabla v, \nabla(v-u)\rangle dx \le 0.$$

By strict convexity of Φ, this implies $\nabla(u-v) = 0$ a.e. in Υ_1. Since $(u-v)^+ = 0$ on $\partial\Upsilon$, we conclude $u \le v$ in Υ. □

The following maximum principle was established by Guedda et al. [18]:

Theorem 2.6 (Maximum principle). *Let $\Upsilon \subset \mathbb{R}^N$ be a smooth bounded domain and $u \in C^1(\Upsilon)$ a nonnegative function satisfying*

$$-\Delta_\Phi u \ge 0 \quad \text{in } \Upsilon. \tag{2.8}$$

Then

- *If $u \ne 0$, then $u > 0$ in Υ*
- *For $u \in C^1(\Upsilon \cup \{x_0\})$ with $u(x_0) = 0$ at $x_0 \in \partial\Upsilon$, the normal derivative satisfies $\frac{\partial u}{\partial \nu}(y) \ge c > 0$ for all $y \in \partial\Upsilon$.*
- *In particular, for $u \in C^1(\overline{\Upsilon})$ with $u = 0$ on $\partial\Upsilon$, there exists $c > 0$ such that $\frac{\partial u}{\partial \nu}(y) \ge c$ for all $y \in \partial\Upsilon$.*

3 Auxiliary problem and functional framework

We begin by defining the modified nonlinearity $\hbar_\varepsilon : \mathbb{R} \to \mathbb{R}$:

$$\hbar_\varepsilon(t) = \begin{cases} \hbar(t) - \varepsilon & \text{if } t \ge 0, \\ -\varepsilon(t+1) & \text{if } t \in [-1,0], \\ 0 & \text{if } t \le -1, \end{cases}$$

where $0 < \varepsilon < 1$ and $-\varepsilon = \min_{t\in\mathbb{R}} \hbar_\varepsilon(t)$. This leads us to study the auxiliary problem:

$$\begin{cases} -\Gamma\left(\int_\Upsilon \Phi(u)dx\right)\Delta_\Phi u = \hbar_\varepsilon(u) & \text{in } \Upsilon, \\ u > 0 & \text{in } \Upsilon, \\ u = 0 & \text{on } \partial\Upsilon. \end{cases} \tag{3.1}$$

The corresponding energy functional $I_\varepsilon : W_0^{1,\Phi}(\Upsilon) \to \mathbb{R}$ is defined by:

$$I_\varepsilon(u) = F\left(\int_\Upsilon \Phi(|\nabla u|)dx\right) - \int_\Upsilon H_\varepsilon(u)dx,$$

where H_ε is the primitive of $\hbar_\varepsilon$:

$$H_\varepsilon(t) = \begin{cases} H(t) - \varepsilon t & \text{if } t \geq 0, \\ -\frac{\varepsilon}{2}(1-(t+1)^2) & \text{if } t \in [-1,0], \\ \frac{\varepsilon}{2} & \text{if } t \leq -1. \end{cases}$$

The functional I_ε is Fréchet differentiable and I'_ε is given by

$$\langle I'_\varepsilon(u), v\rangle = \Gamma\left(\int_Y \Phi(|\nabla u|)dx\right)\int_Y \phi(|\nabla u|)\nabla u \nabla v dx - \int_Y \hbar_\varepsilon(u)v dx,$$

for all $v \in W_0^{1,\Phi}(Y)$.

Lemma 3.1 (Mountain Pass Geometry)**.** *There exist constants $r, \varepsilon_1 > 0$ such that for any $\rho \in (0,r)$ and $\|u\| = \rho$, we have:*

- *$I_\varepsilon(u) \geq \alpha(\rho) > 0$ for all $\varepsilon \in (0,\varepsilon_1)$.*
- *The constants r and ρ are independent of $\varepsilon \in (0,\varepsilon_1)$.*

Proof. Notice that $H_\varepsilon(t) \leq \frac{1}{\Lambda}\Phi(t) + C\frac{|t|^q}{q} + \frac{\varepsilon}{2}$ for all $t \in \mathbb{R}$ and some $C > 0$. Also by (Γ_0) one can have $\Gamma(t) \geq C_2 t$. Therefore

$$\begin{aligned} I_\varepsilon(u) &= \Gamma\left(\int_Y \Phi(|\nabla u|)dx\right) - \int_Y H_\varepsilon(u)dx \\ &\geq \Gamma\left(\int_Y \Phi(|\nabla u|)dx\right) - \frac{C}{q}\int_Y |u|^q dx - \frac{\varepsilon}{2}|Y| \\ &\geq \zeta(\|u\|) - \frac{C}{q}\int_Y |u|^q dx - \frac{\varepsilon}{2}|Y|, \end{aligned}$$

where ζ was given in Lemma 2.1. Thus, there is $C_1 > 0$ verifying

$$I_\varepsilon(u) \geq \zeta(\|u\|) - \frac{C}{q}\int_Y |u|^q dx - \frac{\varepsilon}{2}|Y|.$$

Taking $\|u\| = \rho$ with ρ small enough and using Lemma 2.1, we get

$$I_\varepsilon(u) \geq \rho^m(1 - C_1\rho^{q-m}) - \frac{\varepsilon}{2}|Y|.$$

Now, we fix $\varepsilon_1 = \varepsilon_1(\rho, q, m) > 0$ and $r > 0$ such that

$$\rho^m(1 - C_1\rho^{q-m}) - \frac{\varepsilon}{2}|Y| \geq \frac{\rho^m(1 - C_1\rho^{q-m})}{2} > 0,$$

for all $\varepsilon \in (0,\varepsilon_1)$ and $\rho \in (0,r)$.

From this, $I_\varepsilon(u) \geq \alpha > 0$ if $\|u\| = \rho$ where $\alpha := \frac{\rho^m(1-C_1\rho^{q-m})}{2}$. □

Lemma 3.2 (Mountain Pass Condition)**.** *For each $\varepsilon \in (0,\varepsilon_1)$, there exists $v \in W_0^{1,\Phi}(Y)$ with $\|v\| > \rho$ and $I_\varepsilon(v) < 0$.*

Proof. Let $\varphi \in C_0^\infty(\Upsilon)$ be a function verifying

$$\varphi > 0, \text{ in } \Upsilon \text{ and } \|\varphi\| = 1.$$

Note that for all $t > 1$,

$$\begin{aligned} I_\varepsilon(t\varphi) &= \digamma\left(\int_\Upsilon \Phi(|\nabla t\varphi|)dx\right) - \int_\Upsilon H_\varepsilon(t\varphi)dx \\ &= \digamma\left(\int_\Upsilon \Phi(|\nabla t\varphi|)dx\right) - \int_\Upsilon H_\varepsilon(t\varphi)dx + \varepsilon\int_\Upsilon t\varphi dx. \end{aligned}$$

By $(\hbar_4)$, there are $A_1, B_1 > 0$ verifying

$$H(t) \geq A_1|t|^\theta - B_1, \text{ for all } t \in \mathbb{R}. \tag{3.2}$$

and By (Γ_∞) there exists $C, D > 0$ such that $\digamma(t) \leq Ct^{\theta_1} + D$ for $t \geq 0$. The last inequality together with Lemma 2.1 leads to

$$\begin{aligned} I_\varepsilon(t\varphi) &\leq C\kappa(t)\kappa(\|\varphi\|) - \varepsilon_1 t^\theta \int_\Upsilon \varphi^\theta dx + ta\|\varphi\|_1 + (D + B_1)|\Upsilon| \\ &\leq Ct^m - \varepsilon_1 t^\theta \|\varphi\|_\theta^\theta + ta\|\varphi\|_1 + (D + B_1)|\Upsilon|. \end{aligned}$$

where κ was given in Lemma 2.1. Since $\theta > m$ and $\varepsilon \in (0, \varepsilon_1)$, we can fix $t_0 > 1$ large enough so that $I_\varepsilon(v) < 0$, where $v = t_0\varphi \in W_0^{1,\Phi}(\Upsilon)$. □

3.1 Compactness and existence results

Lemma 3.3 (Palais–Smale Condition)**.** *The functional I_ε satisfies the Palais–Smale condition for all $a > 0$.*

Proof. Let $\{u_n\}$ be a sequence in $W_0^{1,\Phi}(\Upsilon)$ such that $\{I_\varepsilon(u_n)\}$ is bounded and $I'_\varepsilon(u_n) \to 0$. Hence, there exists $n_0 \in \mathbb{N}$ such that $|\langle I'_\varepsilon(u_n), u_n\rangle| \leq \|u_n\|$ for $n > n_0$. Thus,

$$\begin{aligned} -\|u_n\| &\leq \langle I'_\varepsilon(u_n), u_n\rangle \\ &= \Gamma\left(\int_\Upsilon \Phi(|\nabla u_n|)dx\right)\int_\Upsilon \phi(|\nabla u_n|)\nabla u_n \nabla u_n dx - \int_\Upsilon \hbar_\varepsilon(u_n)u_n dx \\ &\leq \theta\digamma\left(\int_\Upsilon \Phi(|\nabla u_n|)dx\right) + \beta\left(\int_\Upsilon \Phi(|\nabla u_n|)dx\right) - D - \int_\Upsilon \hbar_\varepsilon(u_n)u_n dx, \end{aligned}$$

so

$$-\|u_n\| - \theta\digamma\left(\int_\Upsilon \Phi(|\nabla u_n|)dx\right) - \beta\left(\int_\Upsilon \Phi(|\nabla u_n|)dx\right) + D \leq -\int_\Upsilon \hbar_\varepsilon(u_n)u_n dx. \tag{3.3}$$

On the other hand, as there exists $K > 0$ such that $|I_\varepsilon(u_n)| \leq K$ for all $n = 1, 2, \ldots$, it follows that

$$\digamma\left(\int_\Upsilon \Phi(|\nabla u_n|)dx\right) - \int_\Upsilon H_\varepsilon(u_n)dx \leq K, \quad \text{for all } n \in \mathbb{N}. \tag{3.4}$$

This implies that

$$\digamma\left(\int_\Upsilon \Phi(|\nabla u|)dx\right) - \frac{1}{\theta}\int_\Upsilon \hbar_\varepsilon(u_n)u_n dx + \frac{1}{\theta}M|\Upsilon| \leq K, \quad \text{for all } n \in \mathbb{N}. \tag{3.5}$$

This shows

$$F\left(\int_Y \Phi(|\nabla u|)dx\right) - \frac{1+\beta}{\theta}\|u_n\| \leq K - \frac{1}{\theta}\left(M|Y| + D\right),$$

for n large enough, which is a contradiction. This shows that $\{u_n\}$ is bounded in $W_0^{1,\Phi}(Y)$. Thus, without loss of generality, we may assume that $u_n \rightharpoonup u$ in $W_0^{1,\Phi}(Y)$ and since that $B(t) = |t|^q$ verifies (2.5), we get $u_n \to u$ in $L^q(Y)$, $m < q < l^*$. By conditions of $\hbar$,

$$\int_Y \hbar_\varepsilon(u_n)(u_n - u)dx \to 0.$$

Since

$$\begin{aligned} o_n &= I'_\varepsilon(u_n)(u_n - u) \\ &= \Gamma\left(\int_Y \Phi(|\nabla u_n|)dx\right)\int_Y \phi(|\nabla u_n|)\nabla u_n \nabla(u_n - u)dx - \int_Y \hbar_\varepsilon(u_n)(u_n - u)dx \end{aligned}$$

we have

$$\Gamma\left(\int_Y \Phi(|\nabla u_n|)dx\right)\int_Y \phi(|\nabla u_n|)\nabla u_n \nabla(u_n - u)dx \to 0. \tag{3.6}$$

The weak convergence $u_n \rightharpoonup u$ in $W_0^{1,\Phi}(Y)$ yields

$$\Gamma\left(\int_Y \Phi(|\nabla u|)dx\right)\int_Y \phi(|\nabla u|)\nabla u \nabla(u_n - u)dx \to 0. \tag{3.7}$$

This implies that

$$\int_Y \left\{\Gamma\left(\int_Y \Phi(|\nabla u_n|)dx\right)\phi(|\nabla u_n|)\nabla u_n - \Gamma\left(\int_Y \Phi(|\nabla u|)dx\right)\phi(|\nabla u|)\nabla u\right\}\nabla(u_n - u)dx \to 0.$$

This shows $u_n \to u$ in $W_0^{1,\Phi}(Y)$ by Lemma 2.2. □

Lemma 3.4 (Existence of solutions). *For $\varepsilon \in (0, \varepsilon_1)$, problem (1.2) has a solution $u_\varepsilon \in W_0^{1,\Phi}(Y)$ with $I_\varepsilon(u_\varepsilon) \leq C(\varepsilon_1, \theta, m, |Y|)$.*

Proof. The verification of the Mountain Pass Geometry (Lemma 3.1), linking condition (Lemma 3.2), and Palais–Smale condition (Lemma 3.3) allows us to apply the Mountain Pass Theorem [6], yielding for each $a \in (0, \varepsilon_1)$ a critical point $u_\varepsilon \in W_0^{1,\Phi}(Y)$ at level $d_\varepsilon > 0$, where d_ε is the Mountain Pass level associated with I_ε.

Now, letting $\phi \in C_0^\infty(Y), \phi \geq 0$, $t > 0$ and using (3.2), we obtain

$$\begin{aligned} I_\varepsilon(t\phi) &= F\left(\int_Y \Phi(|\nabla t\phi|)dx\right) - \int_Y H_\varepsilon(t\phi)dx \\ &\leq C\kappa(t)\kappa(\|\Phi\|) - \varepsilon_1\int_Y (t\phi)^\theta dx + B_1|Y| + \int_Y \varepsilon t\phi dx \end{aligned}$$

for all $\varepsilon \in (0, \varepsilon_1)$. Then

$$I_\varepsilon(t\phi) \leq C_1\kappa(t) - C_2 t^\theta + C_3 t + C_4,$$

where $C_1 = \kappa(\|\phi\|), C_2 = A_1\|\phi\|_\theta^\theta, C_3 = \varepsilon_1\|\phi\|_1$ and $C_4 = B_1|Y|$. Setting $h(t) = C_1\kappa(t) - C_2 t^\theta + C_3 t + C_4$, we find

$$d_\varepsilon \leq \max\{I_\varepsilon(t\phi), t \geq 0\} \leq \max_{t\geq 0} h(t) = C(\varepsilon_1, m, \theta, |Y|) > 0.$$

Thus $I_\varepsilon(u_\varepsilon) \leq C(\varepsilon_1, m, \theta, |Y|)$. □

Lemma 3.5 (Uniform boundedness). *There exists $K = K(\varepsilon_1, q, m, \theta, |\mathrm{Y}|) > 0$ such that $\|u_\varepsilon\| \leq K$ for all $\varepsilon \in (0, \varepsilon_1)$.*

Proof. Notice that

$$\begin{aligned}
C(\varepsilon_1, m, \theta, |\mathrm{Y}|) &\geq I_\varepsilon(u_\varepsilon) - \frac{1}{\theta} I'_\varepsilon(u_\varepsilon)u_\varepsilon \\
&= \digamma\left(\int_{\mathrm{Y}} \Phi(|\nabla u_\varepsilon|)dx\right) - \int_{\mathrm{Y}} H_\varepsilon(u_\varepsilon)dx \\
&\quad - \frac{1}{\theta}\left(\Gamma\left(\int_{\mathrm{Y}} \Phi(|\nabla u_\varepsilon|)dx\right)\int_{\mathrm{Y}} \phi(|\nabla u_\varepsilon|)|\nabla u_\varepsilon|^2 dx - \int_{\mathrm{Y}} \hbar_\varepsilon(u_\varepsilon)u_\varepsilon dx\right) \\
&= \left(\digamma\left(\int_{\mathrm{Y}} \Phi(|\nabla u_\varepsilon|)dx\right) - \frac{1}{\theta}\Gamma\left(\int_{\mathrm{Y}} \Phi(|\nabla u_\varepsilon|)dx\right)\int_{\mathrm{Y}} \phi(|\nabla u_\varepsilon|)|\nabla u_\varepsilon|^2 dx\right) \\
&\quad - \left(\int_{\mathrm{Y}} H_\varepsilon(u_\varepsilon)dx - \frac{1}{\theta}\int_{\mathrm{Y}} \hbar_\varepsilon(u_\varepsilon)u_\varepsilon dx\right) \\
&\geq \left(\digamma\left(\int_{\mathrm{Y}} \Phi(|\nabla u_\varepsilon|)dx\right) - \frac{m}{\theta}\Gamma\left(\int_{\mathrm{Y}} \Phi(|\nabla u_\varepsilon|)dx\right)\int_{\mathrm{Y}} \Phi(|\nabla u_\varepsilon|)dx\right) \\
&\quad - \left(\int_{\mathrm{Y}} H_\varepsilon(u_\varepsilon)dx - \frac{1}{\theta}\int_{\mathrm{Y}} \hbar_\varepsilon(u_\varepsilon)u_\varepsilon dx\right) \\
&\geq \frac{1}{\theta}\int_{\mathrm{Y}} \Phi(|\nabla u_\varepsilon|)dx - \frac{1}{\theta}D - \frac{M}{\theta}|\mathrm{Y}|.
\end{aligned}$$

Now the Δ_2-condition combines with the last inequality to gives

$$\|u_\varepsilon\| \leq K \quad \text{for all } \varepsilon \in (0, \varepsilon_1),$$

for some $K = K(\varepsilon_1, q, m, \theta, |\mathrm{Y}|)$. □

3.2 Regularity and positivity

Lemma 3.6 (L^∞-regularity). *There exists $\varepsilon_2 \in (0, \varepsilon_1)$ such that:*

- *$u_\varepsilon \in L^\infty(\mathrm{Y})$ for all $\varepsilon \in (0, \varepsilon_2)$.*
- *$\|u_\varepsilon\|_\infty \leq C$ uniformly for $\varepsilon \in (0, \varepsilon_2)$.*

Proof. To establish the uniform L^∞ bound, consider any sequence $\varepsilon_j \to 0^+$ with corresponding solutions $u_j := u_{\varepsilon_j} \in W_0^{1,\Phi}(\mathrm{Y})$. Our proof proceeds in three steps:

1. Compactness: Since $\{u_j\}$ is bounded in $W_0^{1,\Phi}(\mathrm{Y})$ by Lemma 3.5, the subcritical growth of $\hbar$ and the compact embedding $W_0^{1,\Phi}(\mathrm{Y}) \hookrightarrow L^q(\mathrm{Y})$ (for $q < l^*$) yield a subsequence (still denoted $\{u_j\}$) converging strongly to some $u \in W_0^{1,\Phi}(\mathrm{Y})$.

2. Regularity transfer: Each u_j satisfies the equation

$$-\Gamma\left(\int_{\mathrm{Y}} \Phi(u_j)\right)\Delta_\Phi u_j = \hbar(u_j) - \varepsilon_j.$$

The right-hand side remains bounded in $L^s(\mathrm{Y})$ for some $s > 1$ due to the growth condition on $\hbar$ and the uniform $W_0^{1,\Phi}$ bounds.

3. Bootstrap: Applying the regularity arguments from [31, Theorem 3.1], we obtain uniform L^∞ estimates:

$$\|u_j\|_{L^\infty(\mathrm{Y})} \leq C(\|\hbar(u_j)\|_{L^s} + \|\varepsilon_j\|_\infty + 1) \leq C',$$

where C' depends only on the structural constants and is independent of j. □

Lemma 3.7 (Lower bound). *There exist $\varepsilon_3 \in (0, \varepsilon_2)$ and $\delta > 0$ such that $\|u_\varepsilon\|_\infty \geq \delta$ for all $\varepsilon \in (0, \varepsilon_3)$.*

Proof. If u_ε is a solution of (1.2), then

$$\Gamma\left(\int_{\mathrm{Y}} \Phi(|\nabla u_\varepsilon|)dx\right) \int_{\mathrm{Y}} \phi(|\nabla u_\varepsilon|)\nabla u_\varepsilon \nabla \phi dx = \int_{\mathrm{Y}} \hbar_\varepsilon(u_\varepsilon)\phi dx$$

for all $\phi \in W_0^{1,\Phi}(\mathrm{Y})$. For $\phi = u_\varepsilon$

$$\int_{\mathrm{Y}} \hbar_\varepsilon(u_\varepsilon)u_\varepsilon dx = \Gamma\left(\int_{\mathrm{Y}} \Phi(|\nabla u_\varepsilon|)dx\right) \int_{\mathrm{Y}} \phi(|\nabla u_\varepsilon|)|\nabla u_\varepsilon|^2 dx \geq ml \int_{\mathrm{Y}} \Phi(|\nabla u_\varepsilon|)dx.$$

By Δ_2-condition,

$$\int_{\mathrm{Y}} \Phi(|\nabla u_\varepsilon|)dx \to 0 \quad \text{as } \varepsilon \to 0 \quad \Longleftrightarrow \quad \|u_\varepsilon\| \to 0 \quad \text{as } \varepsilon \to 0.$$

Since $I_\varepsilon(u_\varepsilon) \geq \alpha$ for all $\varepsilon \in (0, \varepsilon_2)$, there exists $\varepsilon_3 \in (0, \varepsilon_2)$ and $\alpha_0 > 0$ such that

$$\int_{\mathrm{Y}} \Phi(|\nabla u_\varepsilon|)dx \geq \alpha_0, \quad \text{for all } \varepsilon \in (0, \varepsilon_3). \tag{3.8}$$

Thus,

$$\int_{\mathrm{Y}} \hbar_\varepsilon(u_\varepsilon)u_\varepsilon dx \geq lm\alpha =: A > 0, \quad \text{for all } \varepsilon \in (0, \varepsilon_3).$$

There exists $C > 0$ such that

$$|\hbar_\varepsilon(t)| \leq C(|t|^q + \phi(|t|)|t|) + \varepsilon, \quad \text{for all } t \in \mathbb{R} \text{ and for all } \varepsilon \in (0, \varepsilon_3).$$

Therefore

$$\begin{aligned} A &\leq \int_{\mathrm{Y}} (C(|u_\varepsilon|^q + \phi(|u_\varepsilon|)|u_\varepsilon|) + \varepsilon|u_\varepsilon|)dx \\ &\leq (C(\|u_\varepsilon\|_\infty^q + \phi(\|u_\varepsilon\|_\infty)|u_\varepsilon|) + \varepsilon\|u_\varepsilon\|_\infty). \end{aligned}$$

Thus $\|u_\varepsilon\|_\infty \geq \delta$ for some $\delta > 0$ for all $\varepsilon \in (0, \varepsilon_3)$, decreasing ε_3 if necessary. □

4 Existence of positive solution

The study of positive solutions for semipositone problems has long been challenging due to the lack of natural subsolutions at zero. When combined with nonlocal Kirchhoff-type operators and anisotropic Φ-Laplacian nonlinearities, these problems present intricate mathematical structures that require novel analytical approaches. In this section, we establish our principal existence result, which not only extends classical semipositone theory but also provides new insights into the interplay between nonlocal effects and anisotropic operators.

Theorem 4.1 (Existence of positive solutions)**.** *Under the structural conditions:*

(S1) Regularity and growth assumptions on ϕ ((φ_1)–(φ_4)),

(S2) Nonlocal term properties ((Γ_0), (Γ_∞), (Γ_{AR})),

(S3) Nonlinearity behavior ($(\hbar_1)$–$(\hbar_4)$),

there exists a critical threshold $\varepsilon^\star > 0$ such that for all $\varepsilon \in (0, \varepsilon^\star)$, the semipositone Kirchhoff-type problem (1.2) admits at least one positive weak solution $u_\varepsilon \in W_0^{1,\Phi}(Y)$. Moreover, the solution satisfies:

(i) $u_\varepsilon \in C^{1,\alpha}(\overline{Y})$ for some $\alpha > 0$.

(ii) $\inf_K u_\varepsilon > 0$ for any compact $K \subset Y$,

(iii) Uniform bounds $\delta \leq \|u_\varepsilon\|_\infty \leq C$ independent of ε,

Proof. First, we prove that the solution u_ε is positive for $\varepsilon \in (0, \varepsilon_3)$, decreasing ε_3 if necessary. Indeed, let $\{\varepsilon_g\} \in (0, \varepsilon_3)$ be a sequence with $\varepsilon_j \to 0$ as $j \to \infty$, and let u_j be a solution of (1.2) with $\varepsilon = \varepsilon_j$. Setting $\hbar_j(u_j) = \hbar_{\varepsilon_j}(u_j)$, we have

$$\begin{cases} -\Gamma(\int_Y \Phi(u_j)dx)\Delta_\Phi(u_j) = \hbar_j(u_j) & \text{in } Y, \\ u_j = 0 & \text{on } \partial Y. \end{cases}$$

By Lemma 3.6, there is $C > 0$ such that $\|u_j\|_\infty \leq C$ for all $j \in \mathbb{N}$, then $\|\hbar_j(u_j)\|_\infty \leq C_1$ for all $j\mathbb{N}$ and some $C_1 > 0$. Thereby, by Lemma 2.3, $u_j \in C^{1,\beta}(\overline{Y})$ for some $\beta \in (0,1)$, and

$$|u_j|_{1,\beta} \leq \Gamma(\|\hbar_j(u_j)\|_\infty) \leq M_2, \quad \text{for all } j \in \mathbb{N} \text{ and for some } M_2 > 0.$$

From compactness embedding $C^{1,\beta}(\overline{Y}) \hookrightarrow C^{1,\tau}(\overline{Y})$ for $\tau \in (0, \beta)$, there exists a subsequence for $\{u_j\}$, still denoted by $\{u_j\}$, and $u \in C^{1,\tau}(\overline{Y})$ such that $u_j \to u$ in $C^{1,\tau}(\overline{Y})$.

Now we estimate

$$-\Delta_\Phi u_j = \frac{\hbar_j(u_j)}{\Gamma(\int_Y \Phi(v_j)dx)} \geq \frac{-\varepsilon_j}{h_{\varepsilon_3}}$$

Assume v_j is the solution of

$$\begin{cases} -\Delta_\Phi(v_j) = k_j := \frac{-\varepsilon_j}{h_{\varepsilon_3}} & \text{in } Y, \\ v_j = 0 & \text{on } \partial Y, \end{cases}$$

where $k_j = \min\{\hbar_j(t), t \in \mathbb{R}\} = -\varepsilon_j \to 0^-$ as $j \to \infty$. Then $u_j, v_j \in W_0^{1,\Phi}(Y)$ and

$$-\Delta_\Phi v_j \leq -\Delta_\Phi u_j$$

and by comparison principle

$$v_j \leq u_j \quad \text{in } Y, \text{ for all } j \in \mathbb{N}. \tag{4.1}$$

On the other hand, as $k_j \to 0$, the Lemma 2.3 gives

$$\|v_j\|_\infty \to 0. \tag{4.2}$$

Then, (4.1) combines with (4.2) to give $u \geq 0$ in Y. Notice that

- $\nabla u_j \to \nabla u$ uniformly in $\overline{\mathrm{Y}}$,
- $\int_{\mathrm{Y}} \Phi(|\nabla u_j|)dx \to \int_{\mathrm{Y}} \Phi(|\nabla u|)dx$ and then

$$\Gamma\left(\int_{\mathrm{Y}} \Phi(|\nabla u_j|)dx\right) \to \Gamma\left(\int_{\mathrm{Y}} \Phi(|\nabla u|)dx\right),$$

- $\{\hbar_j(u_j)\}$ is bounded in $L^s(\mathrm{Y})$, $s > 1$,
- $\hbar_j(u_j) \rightharpoonup z$ in $L^s(\mathrm{Y})$,
- $\hbar_j(u_j(x)) \to \hbar_0(u(x))$ a.e. $x \in \mathrm{Y}$,

where $\hbar_0(t) = \hbar(t)$ if $t \geq 0$, and $\hbar_0(t) = 0$ if $t < 0$. Thus $z = \hbar_0(u)$, and for any $\phi \in C_0^\infty(\mathrm{Y})$,

$$\begin{aligned}\Gamma\left(\int_{\mathrm{Y}} \Phi(|\nabla u|)dx\right)\int_{\mathrm{Y}} \phi(|\nabla u|)\nabla u\nabla\phi dx &= \lim_{j\to+\infty} \Gamma\left(\int_{\mathrm{Y}} \Phi(|\nabla u_j|)dx\right)\int_{\mathrm{Y}} \phi(|\nabla u_j|)\nabla u_j\nabla\phi dx \\ &= \lim_{j\to+\infty}\int_{\mathrm{Y}} \hbar_j(u_j)\phi dx \\ &= \int_{\mathrm{Y}} z\phi dx,\end{aligned}$$

consequently

$$\begin{cases} -\Delta_\Phi(u) = \frac{z}{\Gamma\left(\int_{\mathrm{Y}} \Phi(u)dx\right)} & \text{in } \mathrm{Y}, \\ u \geq 0 & \text{in } \mathrm{Y}, \\ u = 0 & \text{on } \partial\mathrm{Y}. \end{cases}$$

As $\|u_j\|_\infty \geq C_0$ for all $j \in \mathbb{N}$, we derive that $\|u_j\|_\infty \geq C_0$, and so $u \neq 0$. As $z \geq 0$, by Theorem 2.6,

$$u > 0, \quad \text{in } \mathrm{Y} \quad \text{and} \quad \frac{\partial u}{\partial \eta} < 0 \quad \text{on } \partial\mathrm{Y}$$

and by

$$u_j \to u \quad \text{in } C^{1,\tau}(\overline{\mathrm{Y}})$$

we have $u_j(x) > 0$, $x \in \mathrm{Y}$, for j large enough. Decreasing ε_3 if necessary, the above analysis guarantee that u_ε is positive for $\varepsilon \in (0, \varepsilon_3)$. □

Conclusion

In this work, we have established the existence of positive weak solutions for a class of semipositone Kirchhoff-type problems involving nonlocal operators and anisotropic Φ-Laplacians. By employing variational methods in the framework of Orlicz–Sobolev spaces, we successfully overcame the challenges posed by three key features of the problem:

- The semipositone nature of the nonlinearity ($\hbar(0) < 0$).
- The nonlocal Kirchhoff term $\Gamma(\int \Phi(u)dx)$.
- The anisotropic behavior of the Φ-Laplacian operator.

Our main result (Theorem 4.1) demonstrates that for sufficiently small positive values of the parameter ε, the problem admits at least one positive solution. The proof combined mountain pass techniques with careful asymptotic analysis and delicate estimates to handle the interaction between the nonlocal term and the indefinite nonlinearity.

Several interesting directions for future research emerge from this work:

- Investigation of multiplicity results for positive solutions.
- Extension to more general nonlocal operators.
- Study of critical growth cases in this framework.
- Analysis of related parabolic versions of the problem.

The methods developed in this paper may be applicable to other classes of nonlocal problems with indefinite nonlinearities, particularly those arising in mathematical physics and biology where such structural features naturally occur.

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Optimal $C^{1,\alpha}$ regularity for quasilinear elliptic equations with Orlicz growth

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Abstract. In this paper we obtain the interior optimal $C^{1,\alpha}$ regularity of weak solutions for the following quasilinear elliptic equations with Orlicz growth in divergence form

$$-\operatorname{div} a(x,Du) = -\operatorname{div}\mathbf{F}+f \quad \text{in } \Omega\subset\mathbb{R}^n,$$

including the following two special models

$$-\operatorname{div}\left((ADu\cdot Du)^{\frac{p-2}{2}}ADu\right) = -\operatorname{div}\mathbf{F}+f$$

and

$$-\operatorname{div}\left((ADu\cdot Du)^{\frac{p-2}{2}}\ln^{\theta}\left(1+(ADu\cdot Du)^{\frac{1}{2}}\right)ADu\right) = -\operatorname{div}\mathbf{F}+f$$

for $\theta>0$, $n\geq 2$, $\mathbf{F}(x)\in C^{0,\sigma_0}(\Omega)$ for some $\sigma_0\in(0,1)$ and $f\in L^q_{loc}(\Omega)$ with $n<q\leq\infty$, where the symmetric matrix $A(x)$ of coefficients is a Hölder continuous function satisfying the uniformly elliptic condition. Moreover, we would like to remark that this work can be viewed as a continuation and follow-up to the works [4,33,34].

Keywords: $C^{1,\alpha}$, regularity, divergence, weak solutions, Orlicz, quasilinear, elliptic.

2020 Mathematics Subject Classification: 35B65, 35J60.

1 Introduction

In this paper we are mainly concerned with the interior optimal $C^{1,\alpha}$ regularity estimates for weak solutions of the following quasilinear elliptic equations with Orlicz growth in divergence form

$$-\operatorname{div} a(x,Du) = -\operatorname{div}\mathbf{F}+f \quad \text{in } \Omega\subset\mathbb{R}^n, \tag{1.1}$$

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where $n \geq 2$, $f \in L^q(\Omega)$ for $n < q \leq \infty$, $\mathbf{F}(x) \in C^{0,\sigma_0}(\Omega)$ for some $\sigma_0 \in (0,1)$ and the vector field $a(x,\xi) : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ is C^1-regular in the variable ξ satisfying

$$|a(x,\xi)| + \left|D_\xi a(x,\xi)\right| |\xi| \leq C\varphi(|\xi|), \tag{1.2}$$

$$D_\xi a(x,\xi)\eta \cdot \eta \geq C\varphi'(|\xi|)|\eta|^2, \tag{1.3}$$

$$|a(x_1,\xi) - a(x_2,\xi)| \leq w\left(|x_1 - x_2|\right)\varphi(|\xi|). \tag{1.4}$$

Here $w : [0,+\infty) \to [0,+\infty)$ belongs to $C^{0,\sigma}\left([0,+\infty)\right)$ for some $\sigma \in (0,1)$, $w(0) = 0$ and the function $\varphi(t) : [0,+\infty) \to [0,+\infty)$ belongs to $C^1\left([0,+\infty)\right)$ satisfying

$$0 < i_a := \inf_{t>0} \frac{t\varphi'(t)}{\varphi(t)} \leq \sup_{t>0} \frac{t\varphi'(t)}{\varphi(t)} =: s_a < \infty. \tag{1.5}$$

Actually, the two special models of (1.1) are the nonhomogeneous p-Laplacian equation with varying coefficients in divergence form

$$-\operatorname{div}\left((ADu \cdot Du)^{\frac{p-2}{2}} ADu\right) = -\operatorname{div}\mathbf{F} + f$$

and the nonhomogeneous p-Laplacian equation with varying coefficients and the logarithmic growth in divergence form

$$-\operatorname{div}\left((ADu \cdot Du)^{\frac{p-2}{2}} \ln^\theta\left(1 + (ADu \cdot Du)^{\frac{1}{2}}\right) ADu\right) = -\operatorname{div}\mathbf{F} + f$$

for $\theta > 0$.

It is well-known that the classical elliptic p-Laplacian equation

$$-\operatorname{div}\left(|Du|^{p-2} Du\right) = f$$

can be derived from the variational problem

$$\mathcal{P}(u,\Omega) := \min_{v-g \in W_0^{1,p}(\Omega)} \mathcal{P}(v,\Omega),$$

where

$$\mathcal{P}(v,\Omega) := \int_\Omega \left(\frac{1}{p}|\nabla v|^p - fv\right) dx \quad \text{for any } v \in W^{1,p}(\Omega).$$

Then $\mathcal{P}(v,\Omega)$ attains its minimum at a unique function u, which implies u is the weak solution of elliptic p-Laplacian equation with the boundary condition $u = g$ on $\partial\Omega$, satisfying

$$\int_\Omega \left[\,|Du|^{p-2} Du \cdot D\phi - f\phi\right] dx = 0 \quad \text{for any } \phi \in W_0^{1,p}(\Omega).$$

In reality, the nonlinear elliptic and parabolic PDEs can be derived from many important practical problems among the natural sciences: nonlinear elasticity mechanics and dynamic glaciology, non-Newtonian fluid mechanics, turbulent flows of a gas in porous media, thermodynamics and so on. At the same time, they can also come from some financial and economic problems and simultaneously the solutions of the nonlinear PDEs and their properties illustrate the features of these problems. Since the structure models in some real financial products and the option price can be reduced to some nonlinear PDE boundary problems, it is useful to adopt the existing theory and methods of PDEs as a fundamental approach to the study of the

financial and economic theory (see [22]). For these reasons it is very meaningful and useful for us to study various kinds of regularity estimates for the nonlinear elliptic and parabolic PDEs with different coefficients and domain conditions. There have been a wide research activities [5, 6, 8–10, 17, 23, 25] on L^p-type estimates for weak solutions of elliptic quasilinear equations of p-Laplacian type with different coefficient and domain assumptions. On the other hand, Duzaar, Kuusi and Mingione [19, 20, 24] also made a deep study of sharp local a priori estimates and regularity results for solutions to

$$-\operatorname{div} a(x, Du) = \mu,$$

whose prototype is the elliptic p-Laplacian equation with coefficients and the right-hand side measure μ

$$-\operatorname{div}\left(\gamma(x)\,|Du|^{p-2}\,Du\right) = \mu.$$

Moreover, Cianchi and Maz'ya [12–14] proved global Lipschitz regularity and sharp estimates for weak solutions of

$$\operatorname{div}\left(\varphi'\left(|\nabla u|\right)\nabla u\right) = f \quad \text{in } \Omega \tag{1.6}$$

with the condition (1.5). Meanwhile, Baroni [7] proved pointwise gradient estimates via linear Riesz potentials for solutions to the following nonlinear elliptic equations with the right-hand side measure

$$\operatorname{div}\left(\varphi'\left(|\nabla u|\right)\nabla u\right) = \mu.$$

The theory of $C^{1,\alpha}$ regularity estimates for weak solutions of the elliptic p-Laplacian equation

$$-\operatorname{div}\left(|Du|^{p-2}\,Du\right) = 0 \tag{1.7}$$

is well-known. Evans, Lewis, Tolksdorf, Uhlenbeck and Ural'ceva (see [21, 26, 29–31] and references therein) have studied the theory of $C^{1,\alpha}$ regularity estimates for weak solutions of the elliptic p-Laplace equation (1.7) and the more general cases with variable coefficients. Moreover, Wang [32] used compactness methods to give a different proof of the interior $C^{1,\alpha}$ regularity for weak solutions of (1.7). In addition, Colombo and Mingione [15, 16] obtained the interior $C^{1,\alpha}$ regularity of weak solutions for a class of variational problems whose model is given by the functional

$$w \mapsto \int \left(|Dw|^p + a(x)\,|Dw|^q\right) dx.$$

A longstanding conjecture in elliptic regularity theory inquires whether a $W^{1,p}$ function whose p-Laplacian is bounded is locally of class $C^{p'} = C^{1,\frac{1}{p-1}}$. Recently, Araújo, Teixeira and Urbano [2] proved the planar counterpart of $C^{p'}$-regularity in the plane that weak solutions of the elliptic degenerate p-Laplacian equation

$$-\operatorname{div}\left(|Du|^{p-2}\,Du\right) = f(x) \quad \text{in } \Omega \subset \mathbb{R}^2$$

with a bounded source $f \in L^\infty$ are locally of class $C^{p'} = C^{1,\frac{1}{p-1}}$, in which they gave the precise control on a new oscillation of weak solutions in terms of the magnitude of its gradient and then improved $C^{1,\alpha}$ regularity estimates by geometric iteration. Moreover, we would like to remark that this regularity in [2] is optimal. Subsequently, Araújo, Teixeira and Urbano [3] solved the $C^{p'}$-regularity conjecture for weak solutions of the degenerate elliptic p-Laplacian equation in higher dimensions $n > 2$. Very recently, Araújo and Zhang [4] established the

interior optimal sharp $C^{1,\alpha}$ estimates for weak solutions of quasilinear elliptic equations of p-Laplacian type with varying coefficients

$$-\operatorname{div} a(x, Du) = f \quad \text{in } \Omega \subset \mathbb{R}^n,$$

where $n \geq 2$, $f \in L^q(\Omega)$ for $n < q \leq \infty$ and the vector field $a(x,\xi) : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ is C^1-regular in the variable ξ satisfying the following structural assumptions

$$\begin{aligned} |a(x,\xi)| + \left|D_\xi a(x,\xi)\right| |\xi| &\leq C|\xi|^{p-1}, \\ D_\xi a(x,\xi)\eta \cdot \eta &\geq C|\xi|^{p-2}|\eta|^2, \\ |a(x_1,\xi) - a(x_2,\xi)| &\leq w\left(|x_1 - x_2|\right) |\xi|^{p-1} \end{aligned}$$

with $w \in C^{0,\sigma}(\Omega)$ for some $\sigma \in (0,1)$, $w(0) = 0$ and $f \in L^q(\Omega)$. Additionally, Ding, Zhang and Zhou [18] then studied the optimal $C^{1,\alpha}$ estimates for the elliptic $p(x)$-Laplacian equation

$$\operatorname{div}\left(a(x)\,|Du|^{p(x)-2}\,Du\right) = \operatorname{div} \mathbf{h}(x) + f(x) \quad \text{in } \Omega.$$

Just recently Teixeira [28] undertook further research of the p-degenerate elliptic equations in a heterogeneous medium

$$-\operatorname{div} a(x, Du) = f(x, u, Du) \quad \text{in } \Omega,$$

which cover the following degenerate-elliptic nonhomogeneous PDE of the general form

$$-\operatorname{div}\left(|Du|^{p-2}\,Du\right) = f(x, u, Du) \quad \text{in } \Omega.$$

For the sake of convenience, we first elaborate on some definitions and fundamental results about the general Orlicz spaces, which have been widely used in the area of analysis as one of the most natural generalizations of Sobolev spaces (see [1,27]). A function $\Phi : [0,+\infty) \to [0,+\infty)$ is said to be a Young function if it is convex and $\Phi(0) = 0$. Moreover, a Young function Φ is called an N-function if $0 < \Phi(t) < \infty$ for $t > 0$ and

$$\lim_{t\to+\infty} \frac{\Phi(t)}{t} = \lim_{t\to 0+} \frac{t}{\Phi(t)} = +\infty.$$

Additionally, we call that a Young function Φ belongs to Δ_2 if there exists a positive constant K such that

$$\Phi(2t) \leq K\Phi(t) \quad \text{for any } t > 0.$$

Furthermore, we say that a Young function Φ belongs to ∇_2 if there exists a number $\theta > 1$ such that

$$\Phi(t) \leq \frac{\Phi(\theta t)}{2\theta} \quad \text{for any } t > 0.$$

Definition 1.1 (see [1]). *The Orlicz class $K^\Phi(\Omega)$ is the set of all measurable functions $g : \Omega \to \mathbb{R}$ satisfying*

$$\int_\Omega \Phi(|g|)dx < \infty.$$

The Orlicz space $L^\Phi(\Omega)$ is the linear hull of $K^\Phi(\Omega)$. Furthermore, we define $W^{1,\Phi}(\Omega)$ as

$$W^{1,\Phi}(\Omega) := \left\{u \in L^\Phi(\Omega) \mid Du \in L^\Phi(\Omega)\right\}.$$

The space $W_0^{1,\Phi}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{1,\Phi}(\Omega)$.

In this paper we define

$$\Phi(t) := \int_0^t \varphi(\tau)d\tau \quad \text{for } t \geq 0. \tag{1.8}$$

Then from (1.5) it is easy to check that

$$\varphi(t) \text{ is strictly increasing and continuous over } [0, +\infty], \tag{1.9}$$

and

$$\Phi(t) \text{ is increasing over } [0, +\infty]. \tag{1.10}$$

As usual, the solutions of (1.1) are taken in a weak sense. We now state the definition of weak solutions.

Definition 1.2. A function $u \in W^{1,\Phi}_{loc}(\Omega)$ is a local weak solution of of (1.1) in Ω if

$$\int_\Omega \left[a(x, Du) \cdot D\phi - \mathbf{F} \cdot D\phi - f\phi\right] dx = 0$$

holds for any $\phi \in C_0^\infty(\Omega)$.

A direct example

$$v(x) = \frac{p-1}{p}|x|^{\frac{p}{p-1}} = \frac{p-1}{p}|x|^{1+\frac{1}{p-1}} \quad \text{for } p > 1,$$

which satisfies the elliptic p-Laplacian equation

$$\operatorname{div}\left(|Dv|^{p-2}\, Dv\right) = n,$$

shows that if the optimal regularity for the elliptic p-Laplacian equations is $C^{1,\alpha}$, then we can get to know that α is surely smaller than $\frac{1}{p-1}$. In particular, in [34] the authors proved that weak solutions for a class of the homogeneous quasilinear elliptic equations

$$\operatorname{div}\left(\varphi'\left(|\nabla u|\right)\nabla u\right) = 0 \quad \text{in } \Omega \tag{1.11}$$

with $f, w \equiv 0$ are locally $C^{1,\alpha}$ for some $\alpha \in (0,1)$ depending on n, i_a, s_a. Following the same approach as Definition 3 of the article [28], we define the maximal C^{1,α_m} regularity for weak solutions of (1.11) as corresponding to the maximal $\alpha_m \in (0,1]$, where α_m is given by

$$\alpha_m := \sup\left\{\alpha \in (0,1) \mid h \text{ belongs to } C^{1,\alpha}_{loc}(\Omega) \text{ for every local weak solution } h \in W^{1,\Phi}_{loc}(\Omega) \text{ of (1.11) with the condition (1.5)}\right\}.$$

Furthermore, in [33] authors have also extended the corresponding results in [34] to the non-uniformly nonlinear elliptic equations.

Now let us state the main result of this work.

Theorem 1.3. *Assume that the conditions (1.2)–(1.5) hold and let $u \in W^{1,\Phi}_{loc}(\Omega)$ be a local weak solution of (1.1). Then we have $u \in C^{1,\alpha}_{loc}(\Omega)$ for some $\alpha \in (0,1)$, where*

$$\alpha := \min\left\{\alpha_m - \epsilon, \alpha_{\sigma, i_a, s_a, q}\right\} \tag{1.12}$$

for any positive number ϵ less than α_m, and

$$\alpha_{\sigma,i_a,s_a,q} := \begin{cases} \min\left\{\sigma,\sigma_0,1-\dfrac{n}{q}\right\}\cdot\min\left\{1,\dfrac{1}{s_a}\right\} & \text{if } n<q<\infty, \\ \min\{\sigma,\sigma_0\}\cdot\min\left\{1,\dfrac{1}{s_a}\right\} & \text{if } q=\infty. \end{cases} \tag{1.13}$$

Moreover, for every $\Omega' \subset\subset \Omega$, the following estimates

$$\sup_{\substack{x,y\in\Omega' \\ x\neq y}} \frac{|u(x)-u(y)|}{|x-y|^{\alpha}} \leq C \tag{1.14}$$

and

$$\sup_{\substack{x,y\in\Omega' \\ x\neq y}} \frac{|Du(x)-Du(y)|}{|x-y|^{\alpha}} \leq C \tag{1.15}$$

hold, where C is a positive constant depending on $n, i_a, s_a, \alpha, \sigma, \sigma_0$, $\|u\|_{W^{1,\Phi}_{loc}(\Omega)}$, $\|w\|_{C^{0,\sigma}_{loc}(\Omega)}$, $[\boldsymbol{F}]_{C^{0,\sigma_0}_{loc}(\Omega)}$, $\|f\|_{L^q_{loc}(\Omega)}$ and dist $(\Omega',\partial\Omega)$.

2 Auxiliary lemmas

In the following, we will introduce several auxiliary lemmas, which will be employed in the proof of the conclusions stated in Theorem 1.3. Moreover, we stress the point that the proof is much influenced by the papers [2–4, 11]. Let us first state a crucial lemma under present assumptions on the function $\varphi(t)$.

Lemma 2.1. *If $\varphi(t)$ satisfies (1.5), then we find that*

$$\theta^{i_a}\varphi(t) \leq \varphi(\theta t) \leq \theta^{s_a}\varphi(t) \quad \text{for } \theta \geq 1 \tag{2.1}$$

and

$$C\theta^{i_a-1}\varphi'(t) \leq \varphi'(\theta t) \leq C\theta^{s_a-1}\varphi'(t) \quad \text{for } \theta \geq 1. \tag{2.2}$$

Proof. Firstly, (1.5) implies that

$$\frac{i_a}{\theta t} \leq \frac{\varphi'(\theta t)}{\varphi(\theta t)} \leq \frac{s_a}{\theta t} \quad \text{for any } t>0,$$

which implies that

$$\int_1^{\theta} \frac{i_a}{\theta t}d\theta \leq \frac{1}{t}\ln\left[\frac{\varphi(\theta t)}{\varphi(t)}\right] = \int_1^{\theta}\frac{\varphi'(\theta t)}{\varphi(\theta t)}d\theta \leq \int_1^{\theta}\frac{s_a}{\theta t}d\theta.$$

So, we can get the desired estimate (2.1)

$$\theta^{i_a}\varphi(t) \leq \varphi(\theta t) \leq \theta^{s_a}\varphi(t) \quad \text{for } \theta \geq 1.$$

On the other hand, by using (1.5) again we deduce that

$$i_a\frac{\varphi(\theta t)}{\theta t} \leq \varphi'(\theta t) \leq s_a\frac{\varphi(\theta t)}{\theta t} \quad \text{for any } t>0$$

and

$$i_a \frac{\varphi(t)}{t} \le \varphi'(t) \le s_a \frac{\varphi(t)}{t} \quad \text{for any } t > 0,$$

which imply, thanks to (2.1), that

$$\frac{i_a}{s_a}\theta^{i_a-1}\varphi'(t) \le i_a \frac{\theta^{i_a}\varphi(t)}{\theta t} \le \varphi'(\theta t) \le s_a \frac{\theta^{s_a}\varphi(t)}{\theta t} \le \frac{s_a}{i_a}\theta^{s_a-1}\varphi'(t).$$

Thus, we finish the proof of this lemma. □

In the subsequent analysis, we proceed to verify that the weak solution u of equation (1.1) can be approximated by that of a reference equation with constant coefficients, which exhibits favorable analytical properties. Before proceeding, we first define the ball $B_r(x_0)$ as the open ball with center x_0 and radius r, and specify B_r as $B_r := B_r(0)$.

Lemma 2.2. *Assume that the conditions (1.2)–(1.5) hold and let $u \in W^{1,\Phi}_{loc}(\Omega)$ be a local weak solution of (1.1) in $\Omega \supset B_1$ with $\|u\|_{L^\infty(B_1)} \le 1$ and $u(0) = 0$. For any $\epsilon > 0$, there exists a positive constant δ, depending on $n, q, i_a, s_a, \sigma, \sigma_0$ and ϵ, such that if*

$$\|f\|_{L^q(B_1)} \le \delta, \qquad [\boldsymbol{F}]_{C^{0,\sigma_0}(B_1)} \le \delta$$

and

$$\sup_{B_1} \left|a(x,\xi) - a(0,\xi)\right| \le \varphi(|\xi|)\delta,$$

then there exists a function h weak solution of

$$-\operatorname{div} \bar{a}(Dh) = 0 \quad \text{in } B_{3/4} \text{ with } h(0) = 0 \tag{2.3}$$

for a constant coefficient field $\bar{a}$ satisfying (1.2)–(1.5) and $w \equiv 0$, such that

$$\sup_{B_{1/2}} |u - h| + |Du(0) - Dh(0)| \le \epsilon.$$

Proof. We prove this by contradiction. Suppose, for the sake of contradiction, that the result is false. Then there exist $\epsilon_0 > 0$, $\{f_j\}_{j=1}^\infty$, $\{\mathbf{F}_j\}_{j=1}^\infty$, $\{a_j(x,\xi)\}_{j=1}^\infty$ and $\{u_j\}_{j=1}^\infty$ satisfying

$$\int_\Omega \left[a_j(x, Du_j)\cdot D\phi - \mathbf{F}_j \cdot D\phi - f_j\phi\right]dx = 0 \quad \text{for any } \phi \in C_0^\infty(\Omega), \tag{2.4}$$

$$\begin{aligned}
\|u_j\|_{L^\infty(B_1)} &\le 1,\\
u_j(0) &= 0,\\
\|f_j\|_{L^q(B_1)} &\le \frac{1}{j},\\
[\mathbf{F}_j]_{C^{0,\sigma_0}(B_1)} &\le \frac{1}{j},\\
\left|a_j(x,\xi) - a_j(0,\xi)\right| &\le \frac{1}{j}\varphi(|\xi|),
\end{aligned}$$

so that for any weak solution h of the homogeneous constant-coefficient equation (2.3) in $B_{3/4}$ satisfying $h(0) = 0$, we have

$$\sup_{B_{1/2}} |u_j - h| + |Du_j(0) - Dh(0)| \ge \epsilon_0. \tag{2.5}$$

In light of this, a standard regularity estimate for weak solutions of (2.4) guarantees that the sequence $\{u_j\}_{j=1}^{\infty}$ satisfies $|Du_j| \le L/2$ for some $L > 0$ and all $x \in B_{3/4}$, and forms a pre-compact sequence in the C^1-topology (see [7, Corollary 1.3]). This further implies the existence of a function $u_\infty \in C^1$ with $u_\infty(0) = 0$ and

$$\sup_{B_{1/2}} |u_j - u_\infty| + |Du_j(0) - Du_\infty(0)| \to 0 \quad \text{as } j \to +\infty. \tag{2.6}$$

Define

$$b_j(x,\xi) := a_j(x,\xi)\chi_{\{|\xi|\le L\}} + L\,\chi_{\{|\xi|\ge L\}}.$$

Since $\{b_j(0,\cdot)\}$ is equicontinuous and uniformly bounded, by the Ascoli–Arzelà theorem, we can find $b_\infty(0,\cdot)$ such that

$$b_j(0,\cdot) \to b_\infty(0,\cdot) \quad \text{uniformly in } B_L := \{|\xi| \le L\} \text{ as } j \to \infty,$$

which implies that

$$\big|a_j(x,\xi) - b_\infty(0,\xi)\big| \le \varphi(|\xi|)\frac{1}{j} + \big|a_j(0,\xi) - b_\infty(0,\xi)\big| \to 0 \quad \text{as } j \to +\infty \tag{2.7}$$

uniformly in $B_{3/4} \times B_L$. Now, applying (2.4), (2.6)–(2.7) and standard arguments, we find that u_∞ is a weak solution of the constant-coefficient equation

$$\int_{B_{3/4}} b_\infty(0, Du_\infty) \cdot D\phi \, dx = 0 \quad \text{for any } \phi \in C_0^\infty(B_{3/4}) \tag{2.8}$$

with $u_\infty(0) = 0$. However, this contradicts (2.5) by (2.6) because u_∞ can serve as the function h described in (2.5). Thus, the proof is complete. □

In addition, by making full use of the above lemma we are able to prove the following crucial results which play an essential role in the subsequent discussion.

Lemma 2.3. *Assume that the conditions (1.2)–(1.5) hold and let $u \in W^{1,\Phi}_{loc}(\Omega)$ be a local weak solution of (1.1) in $\Omega \supset B_1$ with $\|u\|_{L^\infty(B_1)} \le 1$ and $u(0) = 0$.*

1. *There exist two positive constants δ_0 and $\rho_0 \in (0,1)$, depending on $n, q, i_a, s_a, \sigma, \sigma_0$ and α, such that if*

$$\|f\|_{L^q(B_1)} \le \delta_0, \quad [\boldsymbol{F}]_{C^{0,\sigma_0}(B_1)} \le \delta_0, \quad \sup_{B_1} \big|a(x,\xi) - a(0,\xi)\big| \le \varphi(|\xi|)\delta_0 \tag{2.9}$$

and

$$|Du(0)| \le \frac{1}{4}\rho_0^\alpha,$$

then we have

$$\sup_{B_{\rho_0}} |u(x)| \le \rho_0^{1+\alpha}.$$

2. *If*

$$|Du(0)| \le \frac{1}{4}\rho_0^{i\alpha}$$

for some positive integer $i \in \mathbb{N}$ and (2.9) hold, then we have

$$\sup_{B_{\rho_0^i}} |u(x)| \le C\rho_0^{i(1+\alpha)},$$

where C depends on $n, q, i_a, s_a, \sigma, \sigma_0$ and α.

3. *If*

$$|Du(0)| \le \frac{1}{4}\rho^{\alpha}$$

for any $0 < \rho \le \rho_0 < 1$ and (2.9) hold, then we have

$$\sup_{B_\rho} |u(x)| \le C\rho^{1+\alpha},$$

where C depends on $n, q, i_a, s_a, \sigma, \sigma_0$ and α.

Proof. (1) In accordance with Lemma 2.2, for any $\epsilon > 0$, there exists a weak solution h to the constant-coefficient equation (2.3) satisfying

$$|Du(0) - Dh(0)| \le \epsilon$$

and

$$\sup_{B_\rho} |u| \le \epsilon + \sup_{B_\rho} |h|$$

for any $0 < \rho \le 1/2$. Meanwhile, from the interior C^{1,α_m}-regularity estimates for the constant coefficient equation (2.3) together with the fact that $h(0) = 0$, we conclude that

$$\sup_{B_\rho} |h| \le C\rho^{1+\alpha_m} + |Dh(0)|\rho,$$

where the positive constant $C > 1$ depends on n, i_a and s_a, which implies that

$$\begin{aligned}\sup_{B_{\rho_0}} |u| &\le \epsilon + C\rho_0^{1+\alpha_m} + (\epsilon + |Du(0)|)\,\rho_0 \\ &\le \epsilon + C\rho_0^{1+\alpha_m} + \left(\epsilon + \frac{1}{4}\rho_0^{\alpha}\right)\rho_0 \\ &\le \rho_0^{1+\alpha}\end{aligned}$$

by choosing $\rho_0 = (4C)^{\frac{1}{\alpha-\alpha_m}}$ and $\epsilon = \frac{1}{4}\rho_0^{1+\alpha}$.

(2) We prove it by induction. From (1) we know that the result is true for $i = 1$. Let us assume the conclusion is true for $1, 2, \dots i$. Suppose u satisfies

$$|Du(0)| \le \frac{1}{4}\rho_0^{(i+1)\alpha}. \tag{2.10}$$

We denote $u_i(x)$ by

$$u_i(x) := \frac{u\left(\rho_0^i x\right)}{\rho_0^{i(\alpha+1)}}.$$

Then $u_i(x)$ solves

$$-\operatorname{div} a_i(x, Du_i) = f_i \quad \text{in } B_1,$$

where

$$\begin{aligned}a_i(x,\xi) &:= \frac{a(\rho_0^i x, \rho_0^{i\alpha}\xi)}{\varphi(\rho_0^{i\alpha})}, \\ f_i &:= \frac{\rho_0^i f(\rho_0^i x)}{\varphi(\rho_0^{i\alpha})}, \\ \mathbf{F}_i &:= \frac{\mathbf{F}(\rho_0^i x)}{\varphi(\rho_0^{i\alpha})}.\end{aligned}$$

We set

$$\varphi_i(t) := \frac{\varphi(\rho_0^{i\alpha} t)}{\varphi(\rho_0^{i\alpha})}.$$

It is easy to check that $a_i(x,\xi)$ and φ_i still satisfy the assumptions (1.2)–(1.5), and $f_i(x)$ and $\mathbf{F}_i(x)$ satisfy

$$\|f_i\|_{L^q(B_1)} \le \frac{\rho_0^{i(1-\frac{n}{q})}}{\varphi(\rho_0^{i\alpha})}\|f\|_{L^q(B_1)} \le \rho_0^{i(1-\frac{n}{q}-\alpha s_a)}\|f\|_{L^q(B_1)} \le \delta_0$$

and

$$\begin{aligned}
[\mathbf{F}_i]_{C^{0,\sigma_0}(B_1)} &:= \sup_{x,y\in B_1} \frac{|\mathbf{F}_i(x)-\mathbf{F}_i(y)|}{|x-y|^{\sigma_0}} \\
&= \frac{1}{\varphi(\rho_0^{i\alpha})} \sup_{x,y\in B_1} \frac{|\mathbf{F}(\rho_0^i x)-\mathbf{F}(\rho_0^i y)|}{|x-y|^{\sigma_0}} \\
&\le \frac{\rho_0^{i\sigma_0}}{\varphi(\rho_0^{i\alpha})}[\mathbf{F}]_{C^{0,\sigma_0}(B_1)} \\
&\le \rho_0^{i(\sigma_0-\alpha s_a)}[\mathbf{F}]_{C^{0,\sigma_0}(B_1)} \\
&\le \delta_0
\end{aligned}$$

in view of (1.12), (1.13), (2.1) and the facts that

$$1-\frac{n}{q}-\alpha s_a \ge 0 \quad \text{and} \quad \sigma_0 - \alpha s_a \ge 0.$$

Moreover, using the inductive result for the case i, one can deduce that

$$|Du_i(0)| \le \frac{1}{4}\rho_0^{\alpha}$$

by the assumed condition (2.10). Then it follows from (1) that

$$\sup_{B_{\rho_0}} |u_i(x)| \le \rho_0^{1+\alpha}.$$

Finally, the definition of u_i implies that

$$\sup_{B_{\rho_0^{i+1}}} |u(x)| \le \rho_0^{(i+1)(1+\alpha)},$$

which completes the proof of (2).

(3) For $0 < \rho \le \rho_0 < 1$, there exists a positive integer $k \in \mathbb{N}$ such that $\rho_0^{k+1} < \rho \le \rho_0^k$. By the given conditions, we derive

$$|Du(0)| \le \frac{1}{4}\rho^{\alpha} \le \frac{1}{4}\rho_0^{k\alpha}$$

and then by (2),

$$\sup_{B_\rho} |u(x)| \le \sup_{B_{\rho_0^k}} |u(x)| \le \rho_0^{k(1+\alpha)} = \rho_0^{-(1+\alpha)}\rho_0^{(k+1)(1+\alpha)} \le \rho_0^{-(1+\alpha)}\rho^{1+\alpha}.$$

This completes our proof. □

Our approach is based on the following two oscillation estimates, which are similarly to the classical Taylor expansion, themselves interesting and can reveal some essential qualities or characteristics of the solutions to the problem (1.1). More precisely, they can give the precise control on the oscillation of weak solutions to the problem (1.1) in term of the magnitude of the gradients

$$\sup_{B_r} \left|u(x) - u(0)\right| \lesssim r^{1+\alpha} + |Du(0)|\, r.$$

It is noteworthy that we shall need to solve some difficulties cased by the differences of balls B_r with large radii: $|Du(0)| \lesssim r$ and with small radii: $r \lesssim |Du(0)|$. Next, we shall first derive the following sharp regularity estimates for the large radii. Moreover, we would like to remark that the positive constants C are called universal if they only depend on $n, i_a, s_a, \sigma, \sigma_0, \alpha$, $\|u\|_{W^{1,\Phi}_{loc}(\Omega)}$, $\|w\|_{C^{0,\sigma}_{loc}(\Omega)}$, $[\mathbf{F}]_{C^{0,\sigma_0}_{loc}(\Omega)}$, $\|f\|_{L^q_{loc}(\Omega)}$ and $\operatorname{dist}(\Omega', \partial\Omega)$.

Proposition 2.4. *Assume that the conditions (1.2)–(1.5) hold and let $u \in W^{1,\Phi}_{loc}(\Omega)$ be a local weak solution of (1.1). Let $\Omega' \subset\subset \Omega$ and let $x_0 \in \Omega'$, there exist universal positive constants κ, C and $\bar{\rho}$ such that if*

$$|Du(x_0)| \le \kappa \rho^{\alpha} \tag{2.11}$$

for some $0 < \rho \le \rho_0$ with $B_{\rho_0}(x_0) \subset \Omega' \subset\subset \Omega$, then we have

$$\sup_{B_\rho(x_0)} \left|u(x) - u(x_0)\right| \le C\rho^{1+\alpha} \tag{2.12}$$

and

$$\sup_{B_\rho(x_0)} \left|u(x) - u(x_0) - Du(x_0)\cdot(x - x_0)\right| \le C\rho^{1+\alpha} \tag{2.13}$$

for any $0 < \rho \le \bar{\rho} < \rho_0 < 1$.

Proof. We denote $v(x)$ by

$$v(x) := \frac{u(x_0 + A_0 x) - u(x_0)}{B_0}$$

and then $v(x)$ solves

$$-\operatorname{div} a_0(x, Dv) = -\operatorname{div} \mathbf{F}_0 + f_0 \quad \text{in } B_1,$$

where

$$\begin{aligned}
A_0 &:= \min\left\{1, \frac{d_0}{2}, \frac{w^{-1}(\delta_0)}{d_0}\right\} \quad \text{with } d_0 = \operatorname{dist}(\Omega', \partial\Omega),\\
B_0 &:= \max\left\{1, 2\|u\|_{L^\infty(\Omega')}, \left(\|f\|_{L^q(\Omega')}\delta_0^{-1}\right)^{\frac{1}{i_a}}, \left([\mathbf{F}]_{C^{0,\sigma_0}(\Omega')}\delta_0^{-1}\right)^{\frac{1}{i_a}}\right\},\\
a_0(x,\xi) &:= \frac{a(x_0 + A_0 x, \frac{B_0}{A_0}\xi)}{\varphi\left(\frac{B_0}{A_0}\right)},\\
f_0 &:= \frac{A_0 f(x_0 + A_0 x)}{\varphi\left(\frac{B_0}{A_0}\right)},\\
\mathbf{F}_0 &:= \frac{\mathbf{F}_0(x_0 + A_0 x)}{\varphi\left(\frac{B_0}{A_0}\right)}.
\end{aligned}$$

We set

$$\varphi_0(t) := \frac{\varphi(\frac{B_0}{A_0}t)}{\varphi\left(\frac{B_0}{A_0}\right)}.$$

It is easy to check that $a_0(x,\xi)$ and φ_0 still satisfy the assumptions (1.2)–(1.5), $v(0) = 0$, $\|v\|_{L^\infty(B_1)} \le 1$,

$$\begin{aligned}
|a_0(x,\xi) - a_0(0,\xi)| &= \left| \frac{a(x_0 + A_0 x, \frac{B_0}{A_0}\xi)}{\varphi\left(\frac{B_0}{A_0}\right)} - \frac{a(x_0, \frac{B_0}{A_0}\xi)}{\varphi\left(\frac{B_0}{A_0}\right)} \right| \\
&\le w\left(A_0|x|\right)\varphi_0(|\xi|) \\
&\le \delta_0 \varphi_0(|\xi|),
\end{aligned}$$

$$\begin{aligned}
\|f_0\|_{L^q(B_1)} &\le \frac{A_0^{1-\frac{n}{q}}}{\varphi\left(\frac{B_0}{A_0}\right)} \|f\|_{L^q(\Omega')} \\
&\le A_0^{1-\frac{n}{q}+i_a} B_0^{-i_a} \|f\|_{L^q(\Omega')} \\
&\le B_0^{-i_a} \|f\|_{L^q(\Omega')} \le \delta_0,
\end{aligned}$$

$$\begin{aligned}
[\mathbf{F}_0]_{C^{0,\sigma_0}(B_1)} &\le \frac{A_0^{\sigma_0}}{\varphi\left(\frac{B_0}{A_0}\right)} [\mathbf{F}]_{C^{0,\sigma_0}(\Omega')} \\
&\le A_0^{\sigma_0+i_a} B_0^{-i_a} [\mathbf{F}]_{C^{0,\sigma_0}(\Omega')} \\
&\le \delta_0
\end{aligned}$$

in view of (1.5) and (2.1), and

$$|Dv(0)| \le C\rho^\alpha$$

for $\rho \le \rho_0$, by the assumed condition (2.11). Then by Lemma 2.3, we obtain

$$\sup_{B_\rho} |v(x)| \le C\rho^{1+\alpha}$$

for some $0 < \rho \le \rho_0$, which implies that

$$\sup_{B_{A_0\rho}(x_0)} |u(x) - u(x_0)| \le C\rho^{1+\alpha} \le C\left(A_0\rho\right)^{1+\alpha}.$$

Thus, we can finish the proof of (2.12)–(2.13) by choosing $\bar{\rho} = A_0\rho_0 \le \rho_0$. □

Here we are going to prove another regularity estimates in the balls B_r with small radii $r \lesssim |Du|$ which enable the vector field $a(x,\xi)$ can be viewed as essentially a function with linear growth in the variable ξ.

Proposition 2.5. *Assume that the conditions (1.2)–(1.5) hold and let $u \in W^{1,\Phi}_{loc}(\Omega)$ be a local weak solution of (1.1). Let $\Omega' \subset\subset \Omega$ and let $x_0 \in \Omega'$, there exist universal positive constants κ, C and $\tilde{\rho}$ such that if*

$$|Du(x_0)| > \kappa\rho^\alpha \tag{2.14}$$

for $0 < \rho \le \tilde{\rho}$ with $B_{\tilde{\rho}}(x_0) \subset \Omega' \subset\subset \Omega$, then for $0 < \rho \le \tilde{\rho}$ we have

$$\sup_{B_\rho(x_0)} \big|u(x) - u(x_0)\big| \le C\rho^\alpha \tag{2.15}$$

and

$$\sup_{B_\rho(x_0)} \big|u(x) - u(x_0) - Du(x_0)\cdot(x - x_0)\big| \le C\rho^{1+\alpha}. \tag{2.16}$$

Proof. We define $v(x)$ by

$$v(x) := \frac{u(x_0 + \rho_* x) - u(x_0)}{\rho_*^{1+\alpha}}$$

and then $v(x)$ solves

$$-\operatorname{div} a_*(x, Dv) = -\operatorname{div}\mathbf{F}_*(x) + f_*(x) \quad \text{in } B_1, \tag{2.17}$$

where

$$\begin{aligned}
\rho_* &:= \Big[\kappa^{-1}|Du(x_0)|\Big]^{\frac{1}{\alpha}},\\
a_*(x,\xi) &:= \frac{a(x_0 + \rho_* x, \rho_*^\alpha \xi)}{\varphi(\rho_*^\alpha)},\\
f_* &:= \frac{\rho_* f(x_0 + \rho_* x)}{\varphi(\rho_*^\alpha)},\\
\mathbf{F}_* &:= \frac{\mathbf{F}(x_0 + \rho_* x)}{\varphi(\rho_*^\alpha)}.
\end{aligned}$$

We set

$$\varphi_*(t) := \frac{\varphi(\rho_*^\alpha t)}{\varphi(\rho_*^\alpha)}.$$

It is easy to check that $a_*(x,\xi)$ and φ_* still satisfy the assumptions (1.2)–(1.5). Now we divide into two cases.

Case 1: $\rho_* \le \bar{\rho} < 1$, where $\bar{\rho}$ is defined in Proposition 2.4. Then we find that

$$|Du(x_0)| = \kappa\rho_*^\alpha \le \kappa\bar{\rho}^\alpha.$$

From Proposition 2.4, we know that

$$\sup_{B_{\rho_*}(x_0)} \big|u(x) - u(x_0)\big| \le C\rho_*^{1+\alpha} \tag{2.18}$$

and

$$\sup_{B_{\rho_*}(x_0)} \big|u(x) - u(x_0) - Du(x_0)\cdot(x - x_0)\big| \le C\rho_*^{1+\alpha}, \tag{2.19}$$

which implies that

$$\sup_{B_1} |v(x)| \le C \sup_{B_{\rho_*}(x_0)} \frac{\big|u(x) - u(x_0)\big|}{\rho_*^{1+\alpha}} \le C.$$

Consequently, by applying the C^0-estimates to Dv, there exists $\tau_* > 0$ such that

$$\operatorname{osc}_{B_{\tau_*}} |Dv| < \frac{\kappa}{2}.$$

Since $Dv(0) = \kappa$, it follows that $|Dv(x)| > \kappa/2$ in B_{τ_*}. Thus, there is a universal constant $c_0 > 1$ satisfying

$$c_0^{-1} \le |Dv(x)| \le c_0 \quad \text{in } B_{\tau_*}.$$

At this stage, the nonlinear PDE (2.17) can be treated as essentially linear because

$$|a_*(x,\xi)| + |D_\xi a_*(x,\xi)||\xi| \le C\varphi'(c_0)|\xi|,$$
$$D_\xi a_*(x,\xi)\eta \cdot \eta \ge C\varphi'\left(\frac{1}{c_0}\right)|\eta|^2,$$
$$|a_*(x_1,\xi) - a_*(x_2,\xi)| \le Cw(|x_1 - x_2|)\varphi'(c_0)|\xi|$$

in view of (1.5) and Lemma 2.1. Hence, by the $C^{1,\beta}$ estimates for linear equations (see Theorem 3.1 in [4]), we obtain

$$\sup_{B_r} |v(x) - v(0)| \le Cr^\beta$$

and

$$\sup_{B_r} |v(x) - v(0) - Dv(0)\cdot x| \le Cr^{1+\beta}$$

for any $0 < r \le \tau_*/2$ and

$$\beta := \begin{cases} \min\left\{\sigma, \sigma_0, 1 - \frac{n}{q}\right\} & \text{if } n < q < \infty, \\ \min\{\sigma, \sigma_0\} & \text{if } q = \infty. \end{cases}$$

Consequently, we deduce that

$$\sup_{B_{\rho_* r}(x_0)} |u(x) - u(x_0)| \le C\rho_*^{1+\alpha} r^\beta \le C(\rho_* r)^\alpha$$

and

$$\sup_{B_{\rho_* r}(x_0)} |u(x) - u(x_0) - Du(x_0)\cdot(x - x_0)| \le C\rho_*^{1+\alpha} r^{1+\beta} \le C(\rho_* r)^{1+\alpha}$$

for any $0 < r \le \tau_*/2$, which implies that

$$\sup_{B_\rho(x_0)} |u(x) - u(x_0)| \le C\rho^\alpha$$

and

$$\sup_{B_\rho(x_0)} |u(x) - u(x_0) - Du(x_0)\cdot(x - x_0)| \le C\rho^{1+\alpha}$$

for any $0 < \rho \le \tau_*\rho_*/2$. On the one hand, if $\tau_*\rho_*/2 < \rho \le \rho_* \le \bar{\rho} < 1$, (2.18) implies that

$$\sup_{B_\rho(x_0)} |u(x) - u(x_0)| \le C\rho_*^{1+\alpha} \le C\rho_*^\alpha \le C\left(\frac{2}{\tau_*}\right)^\alpha \rho^\alpha \le C\rho^\alpha$$

and then

$$\sup_{B_\rho(x_0)} |u(x) - u(x_0) - Du(x_0)\cdot(x - x_0)| \le C\rho_*^{1+\alpha} \le C\left(\frac{2}{\tau_*}\right)^{1+\alpha} \rho^{1+\alpha} \le C\rho^{1+\alpha},$$

which completes the proof of Case 1.

Case 2: $\rho_* > \bar{\rho}$. Then the definition of ρ_* implies that

$$|Du(x_0)| = \kappa\rho_*^{\alpha} > \kappa\bar{\rho}^{\alpha}.$$

Since the argument applied previously in Case 1 for the function v can also be directly used to the function u for the universal constant $\kappa\bar{\rho}^{\alpha} > 0$, we can directly get the desired results (2.15)–(2.16) for any $0 < \rho < \rho'$ with some positive constant ρ' and then finish the final proof of this proposition by choosing $\tilde{\rho} := \min\{\rho', \bar{\rho}\}$. □

3 Final proof

Finally, we shall combine the previous Proposition 2.4 and Proposition 2.5 for the large radii and small radii cases respectively to finish the proof of the main result in this paper.

Proof of Theorem 1.3. From Proposition 2.4 and Proposition 2.5 we can find that for any compact subset Ω' of Ω there exist positive constants κ, C and $\tilde{\rho}$, depending on $n, i_a, s_a, \alpha, \sigma, \sigma_0$, $\|u\|_{W^{1,\Phi}_{loc}(\Omega)}$, $\|w\|_{C^{0,\sigma}_{loc}(\Omega)}$, $[\mathbf{F}]_{C^{0,\sigma_0}_{loc}(\Omega)}$, $\|f\|_{L^{q}_{loc}(\Omega)}$ and $\operatorname{dist}(\Omega', \partial\Omega)$, such that for $0 < \rho \le \tilde{\rho} < 1$ with $B_{\tilde{\rho}}(x_0) \subset \Omega' \subset\subset \Omega$ we have

$$\sup_{B_\rho(x_0)} \big|u(x) - u(x_0)\big| \le C\rho^{\alpha} \tag{3.1}$$

and

$$\sup_{B_\rho(x_0)} \big|u(x) - u(x_0) - Du(x_0)\cdot(x - x_0)\big| \le C\rho^{1+\alpha}, \tag{3.2}$$

where α is defined in (1.13). Let $x, y \in \Omega'$ and $R = |x - y| \le \tilde{\rho} < 1$. Then from (3.1) we deduce that

$$\big|u(x) - u(y)\big| \le CR^{\alpha}.$$

And then we can conclude that

$$\sup_{\substack{x,y\in\Omega'\\ x\neq y}} \frac{|u(x) - u(y)|}{|x - y|^{\alpha}} \le C$$

holds true by a standard finite covering argument, which implies that (1.14) is true. Furthermore, without loss of generality we may as well assume that $x = (x_1, x') \in \Omega'$ and $y = (x_2, x') \in \Omega'$ with $x' \in \mathbb{R}^{n-1}$. At this time we can obtain that $R = |x - y| = |x_1 - x_2| \le \tilde{\rho} < 1$,

$$u(y) = u(x) + \frac{\partial u(x)}{\partial x_1}(x_2 - x_1) + O\left(|x_1 - x_2|^{\alpha+1}\right)$$

and

$$u(x) = u(y) - \frac{\partial u(y)}{\partial x_1}(x_2 - x_1) + O\left(|x_1 - x_2|^{\alpha+1}\right),$$

which implies that

$$\left|\frac{\partial u(x)}{\partial x_1} - \frac{\partial u(y)}{\partial x_1}\right| |x_2 - x_1| = O\left(|x_1 - x_2|^{\alpha+1}\right).$$

Therefore, we can reach the conclusion that

$$\sup_{\substack{x,y\in\Omega'\\ x\neq y}} \frac{\left|\frac{\partial u(x)}{\partial x_1} - \frac{\partial u(y)}{\partial x_1}\right|}{|x - y|^{\alpha}} \le C$$

by a standard finite covering argument. Similarly, we also can prove that

$$\sup_{\substack{x,y\in\Omega'\\ x\neq y}} \frac{\left|\frac{\partial u(x)}{\partial x_i}-\frac{\partial u(y)}{\partial x_i}\right|}{|x-y|^{\alpha}} \leq C \quad \text{for } 2\leq i\leq n.$$

Thus, we can finish the final proof of the conclusion (1.15). □

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Novel estimates of the impact of hostility on the equilibrium stability in an arms race model

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Abstract. This article proposes a non-autonomous mathematical model of confrontation between two countries and examines the influence of the factor of hostility between countries on the stability of the equilibrium state. Namely, conditions for stability (asymptotic stability) at zero hostility have been established; an estimate was found for the total level of armament of the two countries, provided that the zero equilibrium state is stable in the linear approximation of the model; conditions for stability under parametric disturbances in the linear approximation of the model are established. For a non-autonomous quasi-linear model of confrontation between two countries, a new estimate of the deviation of the weapons vector from the equilibrium ray is obtained and stability conditions are indicated.

Keywords: non-autonomous mathematical models, arms race, hostility, stability, equilibrium.

2020 Mathematics Subject Classification: 70K20, 93D30, 91F10.

1 Introduction

Mathematical modeling of the arms race process goes back to the original work of Richardson [20] published in 1960. Over the past period of time, many researchers have developed the Richardson model or proposed alternative models. See, for example, [1, 3, 5, 18, 21–25, 29, 31] and the references therein.

These works are of theoretical importance, since they contain general conditions for which arms race can be controlled (stable) or can lead to real conflict between opposing countries. It is not by chance that the authors in [6] discussed the "responsibility of mathematicians and mathematics educators for achieving peace and dignity for mankind via mathematics (research and development) and mathematics education".

The question of reinterpreting and improving the mathematical models of the arms race has become very relevant recently against the background of the increasing number of conflicts between two countries. In fact, the relevance of research in the field of stability of the

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equilibrium of opposing countries and their alliances increases significantly, including analysis of the strategy of countries that own atomic weapons. This motivate numerous researchers to extend the existing models and advance the theory. For example, in [4] mathematical models of confrontation between two and n countries, including countries with nuclear weapons, have been proposed and analyzed. As a result of the analysis, stability conditions of the equilibrium state of the warring countries are established, and also the influence of hostility on the decrease (increase) in the norm of the vector of weapons n countries participating in alliances is considered. The edited by Gleditsch book [8] presents a very good overview of the recent research directed on the development of the Richardson arms race models. The paper [19] applied the fractional-order approach to extend the classical arms race model.

Most of the above papers investigated the stability behavior of the corresponding models. In fact, stability analysis is a fundamental approach in a wide range of fields in control theory and the qualitative theory of differential equations [11, 13, 26, 28]. This analysis is of a significant importance in the study of the patterns of many real world phenomena and opens the possibilities of additional knowledge about their behavior [7, 9, 10, 27].

One of the more strong and effective approaches applied in the stability analysis of systems is the second method of Lyapunov or the method of Lyapunov-like functions [2, 12, 30]. This interesting and fruitful technique is gaining more and more importance and gives a decisive impetus for the modern development of the stability theory of differential equations and their applications.

The Lyapunov-like function method has been applied in the recent paper [4] to establish conditions for the stability of the zero equilibrium state of a quasi-linear mathematical model of the confrontation between two countries.

The aim of this paper is to analyze the influence of the level of hostility between two opposing countries on the stability of the equilibrium state based on a non-autonomous quasi-linear model that generalizes the Richardson arms race model. In the article [4], such an analysis was started based on a quasi-linear model with an autonomous linear approximation.

In this case, the factor of hostility between opposing countries is filled with expanded content. In addition, it is noted that hostility between countries is formed by the media in order to control public opinion about the arms race of opposing countries. Thus, the dynamic properties of the model proposed in the article are rich and meaningful, which can motivate its future analysis, as well as the application of the obtained results in the study of arms race processes.

The analysis of the proposed mathematical model of confrontation is carried out using the Lyapunov function method and nonlinear integral inequalities [14–17].

The paper is arranged using the following agenda. Section 1 provides a generalized non-autonomous mathematical model of confrontation between two countries. Section 2 discusses the stability of the equilibrium state under zero hostility and under parametric perturbations of the weapon matrix. Section 3 discusses the mathematical model of confrontation with neutral stability of the linear approximation of the original Richardson's model of confrontation between countries. Section 4 examines the general problem of stability in a generalized Richardson's arms race model. In Section 5, for the non-autonomous quasi-linear model of confrontation between two countries conditions for the stability of weapon vector evasion from the beam balance of arms are established. The final Section 6 provides comments on the results obtained. Some future directions for research are also discussed.

2 Generalized Richardson model of arms race

The classical Richardson's model of a two country arms race is given by a system of differential equations:

$$\frac{dx}{dt} = ky - ax + g, \qquad \frac{dy}{dt} = lx - by + h, \tag{2.1}$$

where $x(t)$ and $y(t)$ represents the armament expenditures (or armament levels) of two countries (nations) at time t, $\frac{dx}{dt}$ or $\dot{x}$ is the rate of change in expenditures over time for the country N_1, and $\frac{dy}{dt}$ or $\dot{y}$ is the rate of change in expenditures over time for the country N_2, k, l, a, b are positive constants and g, h are real constants. The constants k and l describe mutual fear; the constants a and b measure proportionality factors for the "internal brakes" to further arms increases (economic burden parameters, called "fatigue and expense" initially by Richardson). Positive values for g and h correspond to underlying factors of ill will or distrust that would persist even if arms expenditures dropped to zero. Negative values for g and h represent a contribution based on good will [20].

The model (2.1) has been generalized and modified in different directions. In [4], the following generalization is studied:

$$\frac{dx}{dt} = ky - ax + g(t,x,y), \qquad \frac{dy}{dt} = lx - by + h(t,x,y), \tag{2.2}$$

where $t \in \mathbb{R}_\tau$, τ is a finite number or the symbol $+\infty$, $x, y \in \mathbb{R}_+$ represent measures of the levels of the countries' N_1 and N_2 armament, correspondingly, the parameters $k > 0$ and $l > 0$ describe the "threat" between the countries, parameters $a > 0$, $b > 0$ characterize the "expenses" of each from countries to maintain the level of their armament. The functions $g(t,x,y)$ and $h(t,x,y)$ characterize the factor of "hostility" between the countries N_1 and N_2, which depends on the level of public opinion about the confrontation between countries and their armament.

In this paper, we will extend the model (2.2) considering the non-autonomous quasi-linear system of differential equations:

$$\begin{aligned} \frac{dx}{dt} &= k(t)y - a(t)x + g(t,x,y), \qquad x(t_0) = x_0, \\ \frac{dy}{dt} &= l(t)x - b(t)y + h(t,x,y), \qquad y(t_0) = y_0, \end{aligned} \tag{2.3}$$

where $t_0 \in \mathbb{R}_\tau$, all the parameters $k(t) > 0$, $l(t) > 0$, $a(t) > 0$ and $b(t) > 0$ are supposed to be bounded for all $t \in \mathbb{R}_\tau$, and the rest of the parameters are as in [4].

Remark 2.1. The proposed model (2.3) extends and improve the classical Richardson's model [20] and numerous related models [1, 3, 5, 8, 25, 31] in which all the parameters $k(t)$, $l(t)$, $a(t)$ and $b(t)$ are positive constants and the hostility functions $g(t,x,y)$ and $h(t,x,y)$ are real constants, too. The model (2.3) extends also the recent model proposed in [4] in which the parameters $k(t)$, $l(t)$, $a(t)$ and $b(t)$ are positive constants.

Remark 2.2. If the countries N_1 and N_2 possess nuclear weapons, then the parameters $k(t)$ and $l(t)$ can take arbitrarily large values for any $t \in \mathbb{R}_\tau$ while remaining bounded.

From a practical point of view, we will also assume that hostility between countries N_1 and N_2 does not exceed some threshold level $\bar{b} = (\bar{b}_1, \bar{b}_2)^T$ on any finite time interval at any

level of armament of the opposing countries. Thus, for the components of the hostility vector function $(g(t,x,y),h(t,x,y))^T$ there exist constants $\bar{b}_1 > 0$ and $\bar{b}_2 > 0$ such that

$$|g(t,x,y)| \leq \bar{b}_1, \qquad |h(t,x,y)| \leq \bar{b}_2 \tag{2.4}$$

for all $t \in \mathbb{R}_\tau$ and $x, y \in \mathbb{R}_+$.

Remark 2.3. As noted in [4], limit values of permissible hostility $\bar{b}_1$ and $\bar{b}_2$ between the countries N_1 and N_2 should be under the control of the civil society of these countries and/or international organizations, given that modern means of influence on the human psyche have reached the level of criminality.

3 Stability conditions for time-zero hostility

In this section, together with system (2.3) we will consider the following system

$$\frac{dx}{dt} = k(t)y - a(t)x, \qquad \frac{dy}{dt} = l(t)x - b(t)y, \tag{3.1}$$

with zero hostility $g(t,x,y) = 0$ and $h(t,x,y) = 0$ for all $t \in \mathbb{R}_\tau$ and $x, y \in \mathbb{R}_+$.

It is clear that $(x,y)^T = (0,0)^T$ is an equilibrium for the model (3.1).

Proposition 3.1. *If the model's parameters $k(t), a(t), l(t), b(t)$ in (3.1) are such that:*

(H_1) $$-a(t) \leq 0 \quad \text{for all } t \in \mathbb{R}_\tau,$$

(H_2) $$a(t)b(t) - (k(t) + l(t))^2 \geq 0 \quad \text{for all } \ t \in \mathbb{R}_\tau,$$

then its zero equilibrium $(x,y)^T = (0,0)^T$ is stable.

Proof. Consider a Lyapunov-like function

$$V(x,y) = \frac{1}{2}\left(x^2 + y^2\right).$$

For the derivative of the above function with respect to the system (3.1), we have

$$\frac{d}{dt}V(x,y) = \frac{dx}{dt}x + \frac{dy}{dt}y = x\left(k(t)y - a(t)x\right) + y\left(l(t)x - b(t)y\right). \tag{3.2}$$

Conditions (H_1) and (H_2) of Proposition 3.1 imply that $\frac{d}{dt}V(x,y)$ is non-negative, i.e., the zero equilibrium $(x,y)^T = (0,0)^T$ is stable by Lyapunov [12,30]. □

Proposition 3.2. *If for the model's parameters $k(t), a(t), l(t), b(t)$ in (3.1) there exist positive constants α, β such that:*

(H_3) $$-a(t) \leq -\alpha \quad \text{for all } t \in \mathbb{R}_\tau,$$

(H_4) $$a(t)b(t) - (k(t) + l(t))^2 \geq \beta \quad \text{for all } t \in \mathbb{R}_\tau,$$

then its zero equilibrium $(x,y)^T = (0,0)^T$ is asymptotically stable.

Proof. If we apply again the Lyapunov-like function

$$V(x,y) = \frac{1}{2}\left(x^2 + y^2\right),$$

its derivative with respect to the system (3.1) will be strictly negative under the conditions (H_3) and (H_4) of Proposition 3.2, which guarantees the asymptotic stability (by Lyapunov) of the zero equilibrium $(x,y)^T = (0,0)^T$ [12,30]. □

Next, we will rewrite system (3.1) in the matrix form $\frac{dw}{dt} = A(t)w$ and consider the initial value problem (IVP)

$$\frac{dw}{dt} = A(t)w, \quad w(t_0) = w_0, \tag{3.3}$$

where the initial time $t_0 \in \mathbb{R}_\tau$, $w = (x,y)^T$ and $A(t)$ is a (2×2) bounded weapon matrix. We will study the stability of the equilibrium state $(x,y)^T = (0,0)^T$ of (3.3) for parametric perturbations of the weapons matrix. The initial value $w(t_0)$ denotes the levels of the two countries' armament at time $t = t_0$, the beginning of the interaction between the two actors.

Let at the moment $t = t^* \in \mathbb{R}_\tau$ the matrix $A(t^*)$ has an eigenvalue λ, that satisfies $Re\lambda \leq -\alpha$, i.e., the equilibrium state $(x,y)^T = (0,0)^T$ of (3.3) with the weapon matrix $A(t^*)$ is asymptotically stable.

System (3.3) can be represented in the form

$$\frac{dw}{dt} = A(t^*)w + \Delta A(t)w, \quad w(t^*) = w^*, \tag{3.4}$$

where we denote $\Delta A(t) = A(t) - A(t^*)$.

For system (3.4) we will consider a Lyapunov-like function $V(w) = w^T w$ and the following conditions:

(H_5) $$w^T\left(A^T(t^*) + A(t^*)\right)w \leq -\lambda_M w^T w \quad \text{for all } t \in \mathbb{R}_\tau;$$

(H_6) $$w^T\left(\Delta A^T(t) + \Delta A(t)\right)w \leq \mu_M(t) w^T w \quad \text{for all } t \in \mathbb{R}_\tau,$$

where λ_M is the greatest eigenvalue of the matrix $A^T(t^*) + A(t^*)$ and $\mu_M(t)$ is the greatest eigenvalue of the matrix $\Delta A^T(t) + \Delta A(t)$ for all $t \in \mathbb{R}_\tau$.

Proposition 3.3. *If conditions (H_5) and (H_6) are satisfied for the model's parameters in (3.4), and for any $\varepsilon > 0$ there exists a $\delta = \delta(t^*, \varepsilon) > 0$ such that:*

$$\exp\left(-\lambda_M(t - t^*) + \int_{t^*}^{t} \mu_M(s)ds\right) < \frac{\varepsilon}{\delta} \text{ for all } t \in \mathbb{R}_\tau,$$

then the zero equilibrium $w = 0$ of (3.4) is stable.

Proof. For the total derivative of the positive definite Lyapunov function $V(w) = w^T w$ with respect to system (3.4) under the hypotheses (H_5) and (H_6), we have

$$\frac{d}{dt}V(w(t)) \leq (-\lambda_M + \mu_M(t))V(w(t)) \text{ for all } t \in \mathbb{R}_\tau. \tag{3.5}$$

From estimate (3.5) it follows that

$$V(w(t)) \leq V(w((t^*)) \exp\left(\int_{t^*}^{t}(-\lambda_M + \mu_M(s))ds\right) \text{ for all } t \in \mathbb{R}_\tau.$$

Hence, if the conditions of Proposition 3.3 are met, then the equilibrium state $w = 0$ of system (3.4) is stable under parametric perturbations of the weapons matrix $A(t^*)$. □

4 Stability conditions for a neutral linear approximation of the model

In this section, we will consider the case where at time $t_r \in \mathbb{R}_\tau$ the linear approximation of system (2.3) is neutral stable. In this case,

$$k(t_r)y - a(t_r)x = 0, \qquad l(t_r)x - b(t_r)y = 0, \tag{4.1}$$

and, the mathematical model of confrontation between two countries (2.3) takes the form

$$\frac{dx}{dt} = g(t,x,y), \quad x(t_r) = x_0^r; \qquad \frac{dy}{dt} = h(t,x,y), \quad y(t_r) = y_0^r \tag{4.2}$$

where $t \geq t_r$.

Let us assume that the hostility functions $g(t,x,y)$, $h(t,x,y)$ are essentially nonlinear (do not contain linear approximations). Denote by

$$(x(t),y(t))^T = (x(t;t_r,x_0^r),y(t;t_r,y_0^r))^T$$

the solution of the model (4.2).

System (4.2) will be unstable if its solutions $(x(t),y(t))^T$ increase for all $t > t_r$ and for arbitrarily small initial conditions $(x_0^r,y_0^r)^T$ at $t = t_r$.

We associate the positive and bounded Lyapunov function $V(x,y) = xy$, $x > 0$, $y > 0$ to the system of equations (4.2) and prove that the following statement is true.

Proposition 4.1. *Let the hostility functions in the mathematical model of confrontation* (4.2) *are such that* $g(t,x,y) \neq 0$ *and* $h(t,x,y) \neq 0$ *for all* $t \geq t_r$. *If there exists a positive function* $\theta_0 = \theta_0(t)$, *with* $\int_0^\infty \theta(s)ds = \infty$ *for which the inequality*

$$g(t,x,y)y + h(t,x,y)x \geq \theta_0(t)xy, \tag{4.3}$$

holds for any $t \geq t_r$, $x,y > 0$, *then the system* (4.2) *is unstable.*

Proof. When inequality (4.3) is satisfied, the total derivative of the Lyapunov function $V(x,y) = xy$ on the solutions of system (4.2) is an increasing function

$$V(x(t),y(t)) \geq V(x(t_r),y(t_r)) \exp\left(\int_{t_r}^t \theta_0(s)ds\right)$$

for any $t \geq t_r$.

It follows that for a given $\varepsilon > 0$, for any $\delta > 0$ such that $x(t_r)y(t_r) < \delta$, there exists a value $t^* > t_r$ such that $x(t^*)y(t^*) > \varepsilon$. Consequently, the system (4.2) is unstable according to the first theorem on Lyapunov instability (see [12]). □

Remark 4.2. If the condition $g(t,x,y)y + h(t,x,y)x > 0$ hold, then the system (4.2) may be stable.

Next, we consider the dynamics in model (4.2) for $(t,x,y) \in \mathbb{R}_\tau \times \mathbb{R}_+ \times \mathbb{R}_+$ under the following assumption:

(H_7) There exist non-negative integrable functions $\psi_1(t)$, $\psi_2(t)$ such that for $(t,x,y) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+$ we have

(a) $$|g(t,x,y)y| \le \psi_1(t)(xy)^p,$$

(b) $$|h(t,x,y)x| \le \psi_2(t)(xy)^q,$$

(c) $$g(t,x_0^r,y_0^r) \neq 0,\ h(t,x_0^r,y_0^r) \neq 0,$$

where $1 < p,\ 1 \le q < \infty$.

Let us introduce the following definition (cf. [30]).

Definition 4.3. The arms race between two countries, described by the system of equations (4.2) is:

(a) β-bounded, if there exists a constant $\beta > 0$ such that

$$x(t;t_r,x_0^r)y(t;t_r,y_0^r) < \beta$$

for all $t \ge t_r$, where β may depend on each solution;

(b) equi-β-bounded, if for any $\alpha > 0$ and $t_r \in \mathbb{R}$ there exists a constant $\bar{\beta}(t_r,\alpha) > 0$ such that $x_0^r y_0^r < \alpha$ implies $x(t;t_r,x_0^r)y(t;t_r,y_0^r) < \bar{\beta}(t_r,\alpha)$ for all $t \ge t_r$;

(c) uniformly β-bounded, if $\bar{\beta}(t_r,\alpha)$ in Definition 4.3 (b) is independent on t_r.

For the Lyapunov function $V(x,y) = xy$ the following result holds.

Lemma 4.4. *Assume that for the model (4.2) the hypothesis (H_7) holds and the inequality*

$$N(t,t_r) = (p+q-2)\left[(V(x_0^r,y_0^r))^{p-1}\int_{t_r}^{t}\psi_1(s)ds + (V(x_0^r,y_0^r))^{q-1}\int_{t_r}^{t}\psi_2(s)ds\right] < 1 \tag{4.4}$$

is satisfied for $t \ge t_r$.

Then, for the Lyapunov function $V(x,y)$ with respect to the solutions of (4.2), the following estimate

$$V(x(t),y(t))) \le V(x_0^r,y_0^r)\,(1-N(t,t_r))^{-\frac{1}{p+q-2}} \tag{4.5}$$

holds for all $t \in \mathbb{R}_\tau$.

Proof. If conditions (a)–(c) of the hypothesis (H_7) are satisfied for the function $V(x,y)$, it is not difficult to obtain the integral inequality

$$V(x(t),y(t)) \le V(x_0^r,y_0^r) + \int_{t_r}^{t}(\psi_1(s)V^p(x(s),\ y(s))ds + \psi_2(s)V^q(x(s),\ y(s)))ds, \quad t \ge t_r. \tag{4.6}$$

Next, we represent the resulting inequality in the pseudo-linear form

$$\begin{aligned} &V(x(t),y(t)) \\ &\quad \le V(x_0^r,y_0^r) + \int_{t_r}^{t}\left(\psi_1(s)V(x(s),y(s))^{p-1} + \psi_2(s)V(x(s),y(s))^{q-1}\right)V(x(s),y(s))\ ds \end{aligned} \tag{4.7}$$

for $t \ge t_r$.

Applying the Gronwall–Bellman lemma [16,17] to inequality (4.7), we obtain the estimate

$$V(x(t),y(t)) \leq V(x_0^r,y_0^r)\exp\left[\int_{t_r}^{t}(\psi_1(s)V(x(s),y(s))^{p-1}+\psi_2(s)V(x(s),y(s))^{q-1})ds\right] \tag{4.8}$$

for all $t \geq t_r$.

From inequality (4.8), we have

$$\begin{aligned}(V(x(t),y(t)))^{p-1} &\leq (V(x_0^r,y_0^r))^{p-1}\exp\left[(p-1)\int_{t_0}^{t}(\psi_1(s)(V(x(s),y(s)))^{p-1}\right.\\ &\qquad\left.+\psi_2(s)(V(x(s),y(s)))^{q-1})ds\right],\\ (V(x(t),y(t)))^{q-1} &\leq (V(x_0^r,y_0^r))^{q-1}\exp\left[(q-1)\int_{t_0}^{t}(\psi_1(s)(V(x(s),y(s)))^{p-1}\right.\\ &\qquad\left.+\psi_2(s)(V(x(s),y(s)))^{q-1})ds\right]\end{aligned} \tag{4.9}$$

for all $t \geq t_r$.

For $p > 1$ and $q \geq 1$, we have from (4.9) that

$$\begin{aligned}(V(x(t),y(t)))^{p-1} &\leq (V(x_0^r,y_0^r))^{p-1}\exp\left[(p+q-2)\int_{t_0}^{t}(\psi_1(s)(V(x(s),y(s)))^{p-1}\right.\\ &\qquad\left.+\psi_2(s)(V(x(s),y(s)))^{q-1})ds\right],\\ (V(x(t),y(t)))^{q-1} &\leq (V(x_0^r,y_0^r))^{q-1}\exp\left[(p+q-2)\int_{t_0}^{t}(\psi_1(s)(V(x(s),y(s)))^{p-1}\right.\\ &\qquad\left.+\psi_2(s)(V(x(s),y(s)))^{q-1})ds\right]\end{aligned} \tag{4.10}$$

for $t \geq t_r$.

Multiplying both sides of the first inequality from system (4.10) by $(p+q-2)\psi_1(t)$ and the second by $(p+q-2)\psi_2(t)$, we obtain

$$\begin{aligned}&-(V(x(t),y(t)))^{p-1}\psi_1(t)(p+q-2)\exp\left[-(p+q-2)\int_{t_r}^{t}(\psi_1(s)(V(x(s),y(s)))^{p-1}\right.\\ &\qquad\left.+\psi_2(s)(V(x(s),y(s)))^{q-1})ds\right] \geq -(V(x_0^r,y_0^r))^{p-1}(p+q-2)\psi_1(t),\\ &-(V(x(t),y(t)))^{q-1}\psi_2(t)(p+q-2)\exp\left[-(p+q-2)\int_{t_r}^{t}(\psi_1(s)(V(x(s),y(s)))^{p-1}\right.\\ &\qquad\left.+\psi_2(s)(V(x(s),y(s)))^{q-1})ds\right] \geq -(V(x_0^r,y_0^r))^{q-1}(p+q-2)\psi_2(t)\end{aligned} \tag{4.11}$$

for all $t \geq t_r$.

Hence,

$$\begin{aligned}&\frac{d}{dt}\exp\left[-(p+q-2)\int_{t_r}^{t}(\psi_1(s)(V(x(s),y(s)))^{p-1}+\psi_2(s)(V(x(s),y(s)))^{q-1})ds\right]\\ &\qquad\geq -(V(x_0^r,y_0^r))^{p-1}(p+q-2)\psi_1(t)-(V(x_0^r,y_0^r))^{q-1}(p+q-2)\psi_2(s)\end{aligned} \tag{4.12}$$

for all $t \geq t_r$.

Integrating both sides of inequality (4.12) from t_r to t, we obtain the estimate

$$\exp\left[(p+q-2)\int_{t_r}^{t}(\psi_1(s)(V(x(s),y(s)))^{p-1}+\psi_2(s)(V(x(s),y(s)))^{q-1})ds\right] \leq (1-N(t,t_r))^{-1} \tag{4.13}$$

for all $t \in \mathbb{R}_\tau$ such that $N(t,t_r) < 1$.

Taking into account the estimate (4.13), the estimate (4.8) implies

$$(V(x(t),y(t)))^{p+q-2}\,(V(x_0^r,y_0^r))^{-(p+q-2)} \leq (1-N(t,t_r))^{-1}$$

or, finally,

$$V(x(t),y(t)) \leq V(x_0^r,y_0^r)(1-N(t,t_r))^{-\frac{1}{p+q-2}} \tag{4.14}$$

for all $t \in \mathbb{R}_\tau$ for which $N(t,t_r) < 1$. □

Remark 4.5. The proof of Lemma 4.4 takes into account the estimation technique for nonlinear integral inequalities established in [16,17]. The technique is applied to the nonlinear integral inequality (4.6).

Lemma 4.4 allows us to establish new boundedness conditions for the arms race of two opposite countries.

Proposition 4.6. *Assume that conditions of Lemma 4.4 for the model (4.2) are met. In addition, assume that for initial weapon levels x_0^r, y_0^r for any $\alpha > 0$, $x_0^r y_0^r < \alpha$ we have:*

(1) $\bar{N}(t,t_r) < 1$ for all $t \geq t_r$;

(2) for a given $\bar{\beta} = \bar{\beta}(\alpha) > 0$ we have

$$\left(1-\bar{N}(t,t_r)\right)^{-\frac{1}{p+q-2}} < \frac{\bar{\beta}(\alpha)}{\alpha}$$

for all $t \geq t_r$, where

$$\bar{N}(t,t_r) = (p+q-2)\left[(\alpha)^{p-1}\int_{t_r}^{t}\psi_1(s)ds + (\alpha)^{q-1}\int_{t_r}^{t}\psi_2(s)ds\right] \tag{4.15}$$

Then the arms race of the two opposing countries is uniformly β-bounded.

Proof. Proposition 4.6 follows directly from estimate (4.5). □

5 Stability analysis of the generalized Richardson model

The unlimited growth of hostility between the countries N_1 and N_2 (lack of restrictions (2.4)) can lead to a real conflict with unpredictable consequences. Here, we will consider some cases of stability (instability) of the equilibrium state in model (2.3) with time-varying hostility.

We will consider the system of equations (2.3) in which the functions $g(t,x,y)$ and $h(t,x,y)$ are defined in the region $t > 0$ and $x \leq H$, $y \leq H$, where H is a positive constant, are continuous and satisfy the condition under which equations (2.3) have a unique solution under given initial conditions. The functions $g(t,x,y)$ and $h(t,x,y)$, in contrast to the linear approximation of system (2.3), always remain unknown. We will assume that they are small and satisfy the next condition:

(H_8) In the region $t > 0$ and $x \leq H$, $y \leq H$, the hostility functions $g(t,x,y)$, $h(t,x,y)$ are such that

$$g(t,x,y) \neq 0, \qquad h(t,x,y) \neq 0, \qquad t \in \mathbb{R}_\tau.$$

The linear approximation of system (2.3)

$$\frac{dx}{dt} = k(t)y - a(t)x, \quad x(t_0) = x_0, \qquad \frac{dy}{dt} = l(t)x - b(t)y, \quad y(t_0) = y_0, \tag{5.1}$$

has a zero solution, the stability (instability) of which we will study further under the influence of hostility functions.

Definition 5.1. The state $x = y = 0$ of the system (5.1) will be stable under the influence of hostility $g(t,x,y)$, $h(t,x,y)$ if for any $\varepsilon > 0$ and $t_0 \geq 0$ one can specify three numbers $\eta_1(t_0,\varepsilon)$, $\eta_2(t_0,\varepsilon)$, and $\eta_3(t_0,\varepsilon)$ such that any solution $(x(t,t_0,x_0),y(t,t_0,y_0))^T$ satisfying the initial condition $|x_0 + y_0| < \eta_1(t_0,\varepsilon)$ with hostility $g(t,x,y)$, $h(t,x,y)$, satisfying in the region $t > t_0$ and $|x+y| < H$ the condition $|g(t,x,y)| < \eta_2(t_0,\varepsilon)$, $|h(t,x,y)| < \eta_3(t_0,\varepsilon)$, satisfies the inequality $(x(t,t_0,x_0) + y(t,t_0,y_0)) < \varepsilon$ for all $t > t_0$.

We will show that the following statement holds.

Proposition 5.2. *Assume that for the model* (2.3) *hypothesis* (H_8) *holds and there exist positive function* $\theta_1(t)$ *such that for* $t \in \mathbb{R}_\tau$, $x, y > 0$ *the next inequality*

$$g(t,x,y)x + h(t,x,y)y + (k(t)+l(t))xy - a(t)x^2 - b(t)y^2 \leq \theta_1(t)(x^2+y^2), \quad \text{holds for } \int_{t_0}^{\infty} \theta_1(s)ds = L_1 < \infty \tag{5.2}$$

then the state equilibrium $(x,y)^T$=$(0,0)^T$ *of the system* (5.1) *is stable under the influence of hostility.*

Proof. The stability of the system (2.3) will be analyzed by the use of the Lyapunov function $2V(x,y) = (x^2+y^2)$.

The total derivative of the function $V(x,y)$ with respect to system (2.3) has the form

$$\frac{d}{dt}V(x,y) = x\frac{dx}{dt} + y\frac{dy}{dt} = g(t,x,y)x + h(t,x,y)y + (k(t)+l(t))xy - a(t)x^2 - b(t)y^2. \tag{5.3}$$

From this equality and condition (5.2) follows the estimate

$$\begin{aligned} V(x(t),y(t)) &\leq V(x(t_0),y(t_0)) \exp\left(\int_{t_0}^{t} 2\theta_1(s)ds\right) \\ &\leq V(x(t_0),y(t_0)) \exp^{2L_1}, \qquad t \geq t_0. \end{aligned}$$

The stability of the equilibrium state $(x,y)^T = (0,0)^T$ of the system (5.1) follows from the fact that the function $V(x(t),y(t))$ does not increase on the solutions of system (2.3) for any $t \geq t_0$. □

Corollary 5.3. *If the equilibrium state* $(x,y)^T = (0,0)^T$ *of the linear approximation* (5.1) *of system* (2.3) *is asymptotically stable, then it is stable under the influence of hostility* $g(t,x,y)$, $h(t,x,y)$.

Example 5.4. Let us consider the system of equations (2.3) with the linear approximation (5.1) which has the following parameters: $a(t) = b(t) = k(t) = l(t) = 1 + t$ for $t \geq t_\tau > 0$ and hostility functions $g(t,x,y)$ and $h(t,x,y)$ satisfying the conditions (2.4).

We introduce the notation $R(t,x,y) = g(t,x,y)x + h(t,x,y)y \leq \eta_2(t_0,\varepsilon)x + \eta_3(t_0,\varepsilon)y$ for $t \geq t_\tau > 0$ in the region $t > t_0$ and $|x+y| < H$.

The total derivative of the function $2V(x,y) = \left(x^2+y^2\right)$ with respect to system (2.3) has the form

$$\begin{aligned}\frac{d}{dt}V(x(t),y(t))|_{(2.3)} &= x\frac{dx}{dt} + y\frac{dy}{dt} = \frac{d}{dt}V(x(t),y(t))|_{(5.1)} + R(t,x,y)\\ &= -(1+t)(x-y)^2 + R(t,x,y).\end{aligned} \tag{5.4}$$

We assume that the hypothesis (H_8) holds for the hostility functions $g(t,x,y)$ and $h(t,x,y)$ and calculate $\bar{\eta} = \max\{\eta_2(t_0,\varepsilon), \eta_3(t_0,\varepsilon)\}$. Then if $\bar{\eta} < \frac{(1+t)(x-y)^2}{x+y}$, then

$$\frac{d}{dt}V(x(t),y(t))|_{(2.3)} \leq -(1+t)(x-y)^2 + \bar{\eta}(x+y) < 0. \tag{5.5}$$

Hence, by Lyapunov's first theorem on stability it follows that the zero equilibrium of (5.1) is stable under the influence of hostility.

Next, we establish the conditions for the instability of the equilibrium state $(x,y)^T = (0,0)^T$ of the system of equations (5.1) under the assumption (H_8)

Proposition 5.5. *Assume that for the model (2.3) hypothesis (H_8) holds and there exist positive function $\theta_2 = \theta_2(t)$ such that for $t \in \mathbb{R}_\tau,\ x,y > 0$ the next inequality*

$$\begin{aligned}g(t,x,y)y + h(t,x,y)x - (a(t)+b(t))xy + l(t)x^2 + k(t)y^2 \geq \theta_2(t)xy,\\ \text{holds for } \int_{t_0}^{\infty}\theta_2(s)ds = \infty,\end{aligned} \tag{5.6}$$

then the state equilibrium $(x,y)^T$=$(0,0)^T$ of the system (2.3) is unstable under the influence of hostility.

Proof. The instability of the system (2.3) will be analyzed by the use of the Lyapunov function $V(x,y) = xy$.

The total derivative of the function $V(x,y)$ with respect to system (2.3) has the form

$$\begin{aligned}\frac{d}{dt}V(x,y) &= \frac{dx}{dt}y + x\frac{dy}{dt}\\ &= g(t,x,y)y + h(t,x,y)x - (a(t)+b(t))xy + l(t)x^2 + k(t)y^2.\end{aligned} \tag{5.7}$$

If condition (5.6) is satisfied, then the equilibrium state $(x,y)^T = (0,0)^T$ of the system (5.1) is unstable under the influence of hostility due to the increase of the function $V(x,y) = xy$ on the solutions of system (2.3):

$$V(x(t),y(t)) \geq V(x(t_0),y(t_0))\exp\left(\int_{t_0}^{t}\theta_2(s)ds\right),$$

for $t \to \infty$ (see [12] and Proposition 4.1). □

Example 5.6. Let us consider the system of equations (2.3) with the linear approximation (5.1) which has the following parameters: $a(t) = b(t) = 1/2$, $k(t) = l(t) = 1$ for $t \geq 0$.

Consider hostility functions $g(t,x,y)$ and $h(t,x,y)$ that satisfy hypothesis (H_8), and introduce the notation $Q(t,x,y) = g(t,x,y)y + h(t,x,y)x \leq \eta_2(t_0,\varepsilon)x + \eta_3(t_0,\varepsilon)y$ for $t \geq t_\tau > 0$ in the region $t > t_0$ and $|x+y| < H$.

For the total derivative of the function $V(x,y) = xy$ with respect to system (2.3) we have

$$\begin{aligned}\frac{d}{dt}V(x(t),y(t))|_{(2.3)} = y\frac{dx}{dt} + x\frac{dy}{dt} &= \frac{d}{dt}V(x(t),y(t))|_{(5.1)} + Q(t,x,y)\\ &= x^2 + y^2 - xy + Q(t,x,y) \geq Q(t,x,y) + xy.\end{aligned} \tag{5.8}$$

If there exists a positive function $\theta_2 = \theta_2(t)$, such that $Q(t,x,y) \geq (\theta_2 - 1)xy$, then

$$\frac{d}{dt}V(x(t),y(t))|_{(2.3)} \geq \theta_2(t)xy \quad \text{for all } t \in \mathbb{R}_+,\ x,y > 0.$$

Hence, by Proposition 5.5 it follows that the zero equilibrium of (5.1) is unstable.

Next, we will consider the case, where at time $t_c \in \mathbb{R}_\tau$ the hostility functions $g(t,x,y)$, $h(t,x,y)$ are equivalent in influence to the level of threats, i.e. the following condition holds.

(H_9) The hostility functions $g(t,x,y)$, $h(t,x,y)$ are such that

$$g(t,x,y) - k(t)y \equiv 0, \quad h(t,x,y) - l(t)x \equiv 0, \quad t \in \mathbb{R}_\tau,\ x,y \in \mathbb{R}_+.$$

Proposition 5.7. *Assume that for the model (2.3) hypotheses (H_8) and (H_9) are satisfied.*
Then, the levels of armament of opposing countries are decreasing exponentially.

Proof. If the assumption (H_9) is satisfied, then the mathematical model (2.3) degenerates into the model

$$\frac{dx}{dt} = -a(t)x, \quad x(t_c) = x_{0c}; \qquad \frac{dy}{dt} = -b(t)y, \quad y(t_c) = y_{0c}. \tag{5.9}$$

Using a Lyapunov function of the form $V(x,y) = xy$ it is not difficult to show that

$$x(t;t_c,x_{0c})y(t;t_c,y_{0c}) = x_{0c}y_{0c}\exp\left(-(a(t)+b(t))(t-t_c)\right) \tag{5.10}$$

or

$$x(t;t_c,x_{0c}) = x_{0c}\exp\left(-a(t-t_c)\right), \qquad y(t;t_c,y_{0c}) = y_{0c}\exp\left(-b(t-t_c)\right) \tag{5.11}$$

for all $t \geq t_c \in \mathbb{R}_\tau$.

The statement of Proposition 5.7 follows from (5.7) and representations (5.8). □

Remark 5.8. The case considered in Proposition 5.7 may indicate some kind of emergency situation, since in the normal mode of coexistence of opposing countries such a phenomenon is not observed.

Finally, in this section we will consider the case where at time $t_c \in \mathbb{R}_\tau$ the hostility functions $g(t,x,y)$, $h(t,x,y)$ are equivalent in terms of the impact of the level of expenditures of each country on maintaining its armament, i.e., the following condition holds.

(H_{10}) The hostility functions $g(t,x,y)$, $h(t,x,y)$ are negative and

$$-g(t,x,y) + a(t)x \equiv 0, \qquad -h(t,x,y) + b(t)y \equiv 0, \qquad t \in \mathbb{R}_\tau,\ x,y \in \mathbb{R}_+.$$

Proposition 5.9. *Assume that for the model (2.3) hypotheses (H_8) and (H_{10}) are satisfied, and for all $t \geq t_c \in \mathbb{R}_\tau$ each country determines its level of armament based on the level of armament of the opposite side without taking into account their own economic expenses.*

Then, the zero equilibrium $(x, y)^T = (0, 0)^T$ of (2.3) is unstable.

Proof. If the assumption (H_{10}) is satisfied, then the mathematical model (2.3) degenerates into the model

$$\frac{dx}{dt} = k(t)y, \quad x(t^c) = x_0^c; \qquad \frac{dy}{dt} = l(t)x, \quad y(t^c) = y_0^c.$$

For the Lyapunov function $V(x, y) = \frac{1}{2}(x^2 + y^2)$ with respect to the system (5.9), we have

$$V(x(t), y(t)) = V(x_0^c, y_0^c) \exp(k(t) + l(t))(t - t_c)$$

for all $t \geq t_c \in \mathbb{R}_\tau$, which implies the statement of Proposition 5.9. □

Remark 5.10. The case considered in Proposition 5.9 may correspond to a global militarization of countries involved in the confrontation.

6 Stability of weapon vector evasion from the equilibrium vector

We rewrite the system of equations (2.3) in the form

$$\frac{dw}{dt} = A(t)w + B(t, w), \qquad w(t_0) = w_0, \tag{6.1}$$

where $w \in \mathbb{R}^2_+$ is the vector of weapons of the two countries, $A(t) = [a_{ij}(t)]$, is a (2×2) matrix with entries $a_{11}(t) = -a(t), a_{12}(t) = k(t), a_{21}(t) = l(t), a_{22}(t) = -b(t)$, which characterizes the armaments of the two countries changing at time t, $B(t, w) = (g(t, w), h(t, w))^T$ denotes a vector function, nonlinear in w, which characterizes the public opinion about the race weapons and hostility between the two countries.

Suppose that for the two opposing countries there is only one weapon balance beam (ray) (which is equilibrium vector of (6.1)) on the form $L = \{w_e \in K : w_e = \lambda e\}$, where $e = (1, 1) \in \mathbb{R}^2_+$, λ is a positive constant, K is a cone in $\mathbb{R}^2_+$.

Also, let us assume that the vector functions $A(t)w$ and $B(t, w)$ in the system of equations (6.1) meet the conditions.

(H_{11}) There exist nonnegative functions $f_1(t)$, $f_2(t)$ such that

- (a) $d[A(t)w, L] \leq f_1(t)d[w, L]$ for all $t \in \mathbb{R}_\tau$, and $w \in \mathbb{R}^2_+$;
- (b) $d[B(t, w), L] \leq f_2(t)d^\alpha[w, L]$ for all $t \in \mathbb{R}_\tau$ $\alpha > 1$ and $w \in \mathbb{R}^2_+$,

where $d[\cdot, \cdot]$ is the distance in $\mathbb{R}^2_+$.

(H_{12}) For $t \in \mathbb{R}_\tau$, we have

$$N(t, t_0) = (\alpha - 1)d^{\alpha-1}[w_0, L] \int_{t_0}^{t} f_2(s) \exp\left((\alpha - 1)\int_{t_0}^{s} f_1(\tau)d\tau\right) ds < 1. \tag{6.2}$$

In this section, using a nonlinear integral inequality, we will establish an estimate of the deviation of the weapon vector $w(t)$ from the equilibrium ray weapons L of system (6.1).

Lemma 6.1. *Assume that for the system (6.1) hypotheses (H_{11}) and (H_{12}) hold.*
Then, the deviation of the arms vector from the equilibrium ray is estimated by the inequality

$$d[w(t;t_0,w_0),L] \leq d[w_0,L]\exp\int_{t_0}^{t} f_1(s)ds\,(1-N(t,t_0))^{-\frac{1}{\alpha-1}} \tag{6.3}$$

for $t \in \mathbb{R}_\tau$.

Proof. From (6.1), we have

$$w(t;t_0,w_0) = w_0 + \int_{t_0}^{t} (A(s)w(s) + B(s,w(s))ds. \tag{6.4}$$

Hence, using the estimates in (H_{11}), we obtain

$$\begin{aligned} d[w(t;t_0,w_0),L] &= d[w_0,L] + \int_{t_0}^{t} d[(A(s)w(s) + B(s,w(s)),L]ds \\ &\leq d[w_0,L] + \int_{t_0}^{t} (f_1(s)d[w(s,t_0,w_0),L] + f_2(s)d^{\alpha}[w(s,t_0,w_0),L])ds. \end{aligned} \tag{6.5}$$

From inequality (6.5) it follows

$$d[w(t;t_0,w_0),L] \leq d[w_0,L] + \int_{t_0}^{t} (f_1(s)d[w(s),L] + f_2(s)d^{\alpha}[w(s),L]))\,ds \tag{6.6}$$

for all $t \in \mathbb{R}_\tau$.

We will represent inequality (6.6) in the pseudo-linear form

$$d[w(t;t_0,w_0),L] \leq d[w_0,L] + \int_{t_0}^{t} \left(f_1(s) + f_2(s)d^{\alpha-1}[w(s),L]\right) d[w(s;t_0,w_0),L]ds \tag{6.7}$$

for all $t \in \mathbb{R}_\tau$.

Applying the Gronwall–Bellman lemma to inequality (6.7), we obtain an estimate in the form

$$d[w(t;t_0,w_0),L] \leq d[w_0,L]\exp\left(\int_{t_0}^{t} \left(f_1(s) + f_2(s)d^{\alpha-1}[w(s),L]\right) ds\right) \tag{6.8}$$

for all $t \in \mathbb{R}_\tau$.

Next, we will apply integral inequalities from [14,15] to evaluate

$$\exp\left(\int_{t_0}^{t} \left(f_1(s) + f_2(s)d^{\alpha-1}[w(s),L]\right) ds\right). \tag{6.9}$$

in (6.8).

Namely, we represent inequality (6.8) in the form

$$\begin{aligned} &d^{\alpha-1}[w(t;t_0,w_0),L] \\ &\quad\leq d^{\alpha-1}[w_0,L]\exp\left((\alpha-1)\int_{t_0}^{t} \left(f_1(s) + f_2(s)d^{\alpha-1}[w(s;t_0,w_0),L]\right) ds\right). \end{aligned} \tag{6.10}$$

Multiplying both parts of inequality (6.10) by the negative factor

$$-(\alpha-1)f_2(t)\exp\left(-(\alpha-1)\int_{t_0}^{t} f_2(s)d^{\alpha-1}[w(s;t_0,w_0),L])ds\right),$$

we obtain

$$-d^{\alpha-1}[w(t;t_0,w_0),L](\alpha-1)f_2(t)\exp\left(-(\alpha-1)\int_{t_0}^{t}f_2(s)d^{\alpha-1}[w(s;t_0,w_0),L]ds\right)$$
$$\geq -(\alpha-1)d^{\alpha-1}[w_0,L]f_2(t)\exp\left((\alpha-1)\int_{t_0}^{t}f_1(s)ds\right). \tag{6.11}$$

The inequality (6.11) implies

$$\frac{d}{dt}\left\{\exp\left[-(\alpha-1)\int_{t_0}^{t}f_2(s)d^{\alpha-1}[w(s;t_0,w_0),L]ds\right]\right\}$$
$$\geq -(\alpha-1)d^{\alpha-1}[w_0,L]f_2(t)\exp\Big((\alpha-1)\int_{t_0}^{t}f_1(s)ds\Big). \tag{6.12}$$

Integrating the above inequality from t_0 to $t\in\mathbb{R}_\tau$, we have

$$\exp\left(-(\alpha-1)\int_{t_0}^{t}f_2(s)d^{\alpha-1}[w(s;t_0,w_0),L]ds\right)$$
$$\geq 1-(\alpha-1)d^{\alpha-1}[w_0,L]\int_{t_0}^{t}f_2(s)\exp\left(-(\alpha-1)\int_{t_0}^{t}f_1(s)d^{\alpha-1}[w(s;t_0,w_0),L]ds\right), \tag{6.13}$$

where $\exp\{t_0\}=1$.

Hence,

$$\exp\left[(\alpha-1)\int_{t_0}^{t}f_2(s)d^{\alpha-1}[w(s;t_0,w_0),L]ds\right]$$
$$\leq\left\{1-(\alpha-1)d^{\alpha-1}[w_0,L]\int_{t_0}^{t}f_2(s)\exp\left[(\alpha-1)\int_{t_0}^{s}f_1(\tau)d\tau\right]ds\right\}^{-1} \tag{6.14}$$

for all $t\in\mathbb{R}_\tau$.

Combining (6.10) and (6.14), we get

$$d^{\alpha-1}[w(t;t_0,w_0),L]\leq\frac{d^{\alpha-1}[w_0,L]\exp\left[(\alpha-1)\int_{t_0}^{t}f_1(s)ds\right]}{1-N(t,t_0)},$$

from which, taking into account (H_{12}), the deviation of the weapon vector $w(t)$ from the ray of equilibrium L is estimated by the inequality (6.3). □

Proposition 6.2. *Assume that for the system (6.1) hypotheses (H_{11}) and (H_{12}) hold. If for any $\varepsilon>0$ there exists a $\delta=\delta(t_0,\varepsilon)>0$ such that:*

(1) $d[w_0,L]<\delta$,

(2) $\overline{N}(t,t_0)=(\alpha-1)\delta^{\alpha-1}\int_{t_0}^{t}f_2(s)\exp\left((\alpha-1)\int_{t_0}^{s}f_1(\tau)d\tau\right)ds<1$ *for all* $t\in\mathbb{R}_\tau$,

(3) $\exp\left(\int_{t_0}^{t}f_1(s)ds\right)\left(1-\overline{N}(t,t_0)\right)^{-\frac{1}{\alpha-1}}<\frac{\varepsilon}{\delta}$, $t\in\mathbb{R}_\tau$,

then

$$d[w(t;t_0,w_0),L]<\varepsilon,\qquad t\in\mathbb{R}_\tau. \tag{6.15}$$

Proof. Under the conditions of Proposition 6.2 the deviation of the weapon vector from the equilibrium beam is estimated by inequality (6.3), from which the statement of Proposition 6.2 follows. □

Remark 6.3. Conditions (1)–(3) of Proposition 6.2 are sufficient for the stability of the deviation of the weapon vector of two countries in model (6.1) from the weapon equilibrium ray L.

7 Concluding remarks

In this paper, the influence of the hostility factor between countries on the stability of the equilibrium state is studied in detail for the non-autonomous model (2.3) of confrontation between two countries. Namely:

- the criteria for stability (asymptotic stability) are established under zero hostility;
- the criteria for stability (instability) of the arms race between two countries are established under the condition of neutral stability of the zero equilibrium state in the linear approximation of the proposed model (2.3);
- the criteria for stability of parametric disturbances of the armament matrix are established in the linear approximation of system (2.3);
- under various assumptions about the hostility between the two countries changing over time, the criteria for stability (instability) of the zero equilibrium state are derived;
- for the non-autonomous quasi-linear model of confrontation between two countries, a new estimate of the deviation of the armament vector from the equilibrium ray is obtained and the conditions for the stability of the equilibrium state are indicated.

The approaches to the analysis of the influence of hostility on the stability of the equilibrium state of the extended model (2.3) proposed in this paper are new and contribute to the development of the theory. It is also expected that this contribution will stimulate further improvement and analysis of arms race models.

Directions for expanding and increasing the flexibility of models include taking into account delay factors in the linear approximation of model (2.3), assessing the influence of control in functions describing hostility, including impulse disturbance. It is of interest to use fractional analysis or sequential modeling. An interesting direction for future research is also to consider an arms race model with more than two countries and with periodic coefficients in the linear approximation. Studying discrete versions of model (2.3) is also important for numerical modeling.

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Positive solutions for a degenerate p-Kirchhoff problem with singular and sublinear nonlinearities

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Abstract. In this paper, we study the following p-Kirchhoff-type elliptic problem:

$$\begin{cases} -M\left(\int_\Omega |\nabla u|^p\right) \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = \dfrac{f(x)}{u^\gamma} + g(x)\,u^q & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $1 < p < N$, $0 < \gamma \leq 1$, $0 < q < p-1$ and $M : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function satisfying additional hypotheses. The data f and g belonging to suitable Lebesgue space. Our approaches are based on an approximation scheme and the pseudomonotone operators theory.

Keywords: nonlocal elliptic equations, singular nonlinearity, pseudomonotone operators theory, weighted eigenvalue problems.

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1 Introduction

Let Ω be a bounded open subset in $\mathbb{R}^N$ with $1 < p < N$. We consider the following non local elliptic problem

$$\begin{cases} -M\left(\|u\|^p\right)\Delta_p u = \frac{f(x)}{u^\gamma} + g(x)\,u^q & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\|u\|^p = \int_\Omega |\nabla u|^p\,dx$, $0 < \gamma \leq 1$ and $M : \mathbb{R}^+ = [0, +\infty[\to \mathbb{R}^+$ is a continuous nondecreasing function satisfying the following structural assumption:

$$\exists\, m_0 > 0,\ \mu > 1 :\ M(t) \geq m_0\, t^{\mu-1},\ \forall t \in [0,1]. \tag{1.2}$$

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Observe that condition (1.2) is trivially satisfied in the non-degenerate case, i.e., when $M(0) > 0$. A prototypical example of a Kirchhoff-type function satisfying (1.2) is:

$$M(t) = a + brt^{r-1}, \qquad a, b \geq 0,\ a + b > 0,\ r \geq 1,\ t \geq 0.$$

Such a condition has been previously utilized by Pucci et al. in [30].

We assume the source terms f and g satisfy the following conditions:

$$f, g \geq 0, \quad \underline{h} := \inf(f, g) \not\equiv 0, \tag{1.3}$$

$$f \in L^m(\Omega), \quad \text{with } m = \frac{pN}{N(p-1) + p + \gamma(N-p)} = \begin{cases} \left(\frac{p^*}{1-\gamma}\right)' & \text{if } 0 < \gamma < 1, \\ 1 & \text{if } \gamma = 1. \end{cases}, \tag{1.4}$$

$$g \in L^{\left(\frac{p*}{1+q}\right)'}(\Omega). \tag{1.5}$$

Due to the nonlocal term $M\left(\|u\|^p\right)$, the first equation in (1.1) is no longer a pointwise identity. This introduces significant mathematical challenges, making the study of such problems particularly interesting.

For $p = 2$, the operator $M\left(\|u\|^p\right)\Delta_p$ arises naturally in the Kirchhoff equation:

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = 0,$$

which generalizes the classical D'Alembert wave equation for vibrating strings:

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0, \qquad x \in (0, L),\ t > 0.$$

For more on this physical background, see [11,23–25,27] and the references therein.

Before presenting the details of our results, we briefly review existing results for singular and nonlocal problems.

In the local case, i.e., $M \equiv 1$, the problem (1.1) is closely related to the foundational work in [10], where the authors showed the existence of a solution for the following nonlinear singular elliptic equation

$$\begin{cases} -\Delta u = \frac{\lambda}{u^\gamma} + u^q & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a bounded domain with smooth boundary $\partial\Omega$, $\lambda > 0$, and the exponent q of the sublinear term satisfies $0 < q < 1$.

Giacomoni, Schindler, and Takáč in [18] extended this analysis to the quasilinear setting:

$$\begin{cases} -\Delta_p u = \frac{\lambda}{u^\gamma} + u^q & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $0 < \gamma < 1$, $1 < p < N$ and $p - 1 < q < p^* - 1$, with $p^* = \frac{Np}{N-p}$ denotes the critical Sobolev exponent for the embedding $W_0^{1,p}(\Omega)$ in to $L^t(\Omega)$ for every $t \in [1, p^*]$.

In [15], the author considered:

$$\begin{cases} -\Delta_p u = \frac{f(x)}{u^\gamma} + g(x)u^q & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.6}$$

where $0 \leq q \leq p-1$, f and g are nonnegative functions belonging to suitable Lebesgue spaces. The author proved that

- If $q < p-1$, $0 < \gamma < 1$, $f \in \left(\frac{p^*}{1-\gamma}\right)'$, and $g \in L^{\left(\frac{p^*}{1+q}\right)'}(\Omega)$, then the solution u belongs to $W_0^{1,p}(\Omega)$.
- if $\gamma = 1$, $f \in L^1(\Omega)$, and $g \in L^{\left(\frac{p^*}{1+q}\right)'}(\Omega)$, then $u \in W_0^{1,p}(\Omega)$.

In the nonlocal setting, Corrêa et al. [12] studied:

$$\begin{cases} -M\left(\|u\|^p\right)\Delta_p u = \frac{f(x)}{u^\gamma} + u^q & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.7}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $2 \leq p \leq N$, $0 < \gamma, q < 1$, $f : \bar{\Omega} \to \mathbb{R}$ is continuous function with $f > 0$ on $\bar{\Omega}$, and $M : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function satisfying

- there are $m_0 > 0$ and $\theta_1 > 0$ such that $M(t) \geq m_0$ $\forall t \geq \theta_1$;
- $\theta_2 = \sup\{t > 0;\ M(t) = 0\} > 0$.

The authors consider an approximation of problem (1.7) by a regularized problem. They then prove that there is at least one solution in $W_0^{1,p}(\Omega)$ by employing the Galerkin method along with a variant of Brouwer's fixed point theorem [22, Lemme 4.3]. Subsequently, they pass to the limit to obtain a solution of (1.7).

Motivated by the above works, in this paper we investigate the existence and regularity of solutions to problem (1.1), extending some results from [15] to the degenerate nonlocal case. To the best of our knowledge, this work provides the first treatment of a degenerate Kirchhoff-type problem involving both singular and sublinear nonlinearities, under optimal integrability assumptions on the source terms f and g.

Before presenting our main results, we first define what we mean by a weak solution to (1.1). Following Giacomoni et al. [18], the solutions of (1.1) are understood in the following sense

Definition 1.1. A weak solution to problem (1.1) is a function $u \in W_0^{1,p}(\Omega)$ such that

$$\forall \omega \subset\subset \Omega,\ \exists c_\omega > 0 :\ u \geq c_\omega > 0 \text{ in } \omega, \tag{1.8}$$

and

$$M\left(\|u\|^p\right)\int_\Omega |\nabla u|^{p-2}\nabla u \nabla \varphi = \int_\Omega \left(\frac{f(x)}{u^\gamma} + g(x)\,u^q\right)\varphi, \qquad \forall \varphi \in C_0^\infty(\Omega). \tag{1.9}$$

Remark 1.2. If u is a weak solution to problem (1.1) in the sense of Definition 1.1, then for every test function $\varphi \in C_0^\infty(\Omega)$, the right-hand side of the weak formulation (1.9) is finite. Indeed, the local positivity condition (1.8) implies

$$\int_\Omega \frac{f\varphi}{u^\gamma} \leq \frac{\|\varphi\|_{L^\infty(\Omega)}}{c_w^\gamma} \int_\Omega f < +\infty.$$

Hence, $\frac{f(x)}{u^\gamma} \in L^1_{loc}(\Omega)$, and consequently,

$$\frac{f(x)}{u^\gamma} + g(x)\, u^q \in L^1_{loc}(\Omega).$$

Therefore, the weak formulation (1.9) is well-defined.

2 Statements of the main result

Here is the main result of this paper.

Theorem 2.1. *Let us assume that* (1.2)–(1.3) *hold true,* $g \in L^{\left(\frac{p^*}{1+q}\right)'}(\Omega)$, *and* $f \in L^m(\Omega)$, *with*

$$m = \frac{pN}{N(p-1) + p + \gamma(N-p)}. \tag{2.1}$$

Then there exists at least one weak solution to problem (1.1).

Remark 2.2. Note that

$$\frac{pN}{N(p-1) + p + \gamma(N-p)} = \begin{cases} \left(\frac{p^*}{1-\gamma}\right)' & \text{if } 0 < \gamma < 1, \\ 1 & \text{if } \gamma = 1. \end{cases}$$

The structure of the rest of this paper is as follows. Section 3 introduces the necessary notations and preliminary results that will be used throughout our analysis. Section 4 establishes the existence of a solution to the approximated problem by employing pseudomonotone operator theory. Section 5 derives essential a priori estimates for the approximate solution, which play a crucial role in proving the main result. Section 6 proves the main result by taking the limit of the sequence of approximated problems. Finally, Section 7 examines additional regularity properties of the obtained solution.

3 Notations and mathematical background

We begin by establishing the necessary framework for our analysis. Let Ω be a bounded smooth domain in $\mathbb{R}^N$ with $1 < p < N$. The truncation function $T_k : \mathbb{R} \to \mathbb{R}$ at level $k > 0$ is defined as

$$T_k(s) = \max\{-k, \min\{s, k\}\}.$$

We now present fundamental results concerning the spectrum of the p-Laplacian operator with weights. For further details on this topic, the reader is referred to [3,17,21,28].

3.1 Weighted eigenvalue problems

Let $V(x) \geq 0$ be a measurable function, not identically zero, $V(x) \in L^\infty(\Omega)$. Let us consider the problem:

$$\begin{cases} u \in W_0^{1,p}(\Omega),\ u \not\equiv 0, \\ -\text{div}\left(|\nabla u|^{p-2}\nabla u\right) = \lambda V(x)|u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{P_λ}$$

We define the functionals $\mathcal{A}$ and $\mathcal{B}$ as follows:

$$\begin{cases} \mathcal{A}(w) = \dfrac{1}{p}\|w\|_{W_0^{1,p}(\Omega)}, \\ \mathcal{B}(w) = \dfrac{1}{p}\displaystyle\int_\Omega V(x)|w(x)|^p\,dx, \end{cases}$$

for all $w \in W_0^{1,p}(\Omega)$.

Definition 3.1 ([3]). A real number $\lambda \in \mathbb{R}$ is called an eigenvalue if there exists a function u such that the pair (u, λ) is a solution to problem (P_λ). Such a function (non-trivial by definition) is called an eigenfunction associated with λ. The first eigenvalue λ_1 is defined by:

$$\lambda_1 = \inf\left\{\mathcal{A}(w) :\ w \in W_0^{1,p}(\Omega),\ \text{and } \mathcal{B}(w) = 1\right\}.$$

Lemma 3.2 ([14, 19, 33]). *Let $u \in W_0^{1,p}(\Omega)\setminus\{0\}$ be an eigenfunction associated to λ. Then $u \in L^\infty(\Omega)$.*

Lemma 3.3 ([3, Lemma 3.1]). *If u is an eigenfunction of (P_λ), then $u \in \mathcal{C}^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$. Moreover, if $u \geq 0$, then $u > 0$ in Ω and $\frac{\partial u}{\partial \nu} < 0$ on $\partial\Omega$, where ν denotes the unit exterior normal vector to $\partial\Omega$.*

Lemma 3.4 ([3, Lemma 3.3]). *Every eigenfunction u associated with λ_1 has constant sign, i.e., either $u > 0$ or $u < 0$ in Ω.*

Lemma 3.5 ([3, Theorem 3.1]). *The first eigenvalue λ_1 is simple, i.e., if u and v are two eigenfunctions associated with λ_1, then $u = \alpha v$ for some $\alpha \in \mathbb{R}$.*

3.2 Weak comparison principle

The following weak comparison principle will be essential for our subsequent analysis. For the proof, we refer to [28, Lemma A.0.7] or the more general version in [26, Lemma 3].

Lemma 3.6. *Let $u_1, u_2 \in W^{1,p}(\Omega)$ satisfying*

$$\begin{cases} -\Delta_p u_1 \leq -\Delta_p u_2, & \text{in } \Omega,\ (\text{in the weak sense}) \\ u_1 \leq u_2, & \text{on } \partial\Omega. \end{cases}$$

Then, $u_1 \leq u_2$ a.e. in Ω.

3.3 Pseudomonotone operator theory

The following operator classes are fundamental to our nonlinear analysis framework. We recall these standard definitions from [37, Definition 26.1].

Definition 3.7. Let $A : X \to X'$ be an operator on the real reflexive Banach space X.

1. A is said to be hemicontinuous iff the real function
$$t \mapsto \langle A(u + tv), w \rangle,$$
is continuous on $[0, 1]$ for all u, v, $w \in X$.
2. A is said to be strongly continuous iff $u_n \rightharpoonup u$ implies $A(u_n) \to A(u)$.
3. A is called to be bounded iff A maps bounded sets into bounded sets.
4. A is said to be monotone iff
$$\langle A(u) - A(v), u - v \rangle \geq 0 \quad \text{for all } u, v \in X.$$
5. A is called coercive if
$$\lim_{\|u\| \to \infty} \frac{\langle A(u), u \rangle}{\|u\|} = +\infty.$$

The following properties of the p-Laplacian operator are well-known, see [32, Theorem 17.11].

Lemma 3.8. *The p-Laplacian operator $-\Delta_p : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is monotone, coercif and continuous.*

The following definition, originally introduced by Brézis [8], plays a fundamental role in our existence proof.

Definition 3.9. [37, Definition 27.5] Let $A : X \to X'$ be an operator on the real reflexive Banach space X. The operator A is called pseudomonotone iff $u_n \rightharpoonup u$ as $n \to \infty$ and
$$\limsup_{n \to \infty} \langle A(u_n), u_n - u \rangle \leq 0,$$
implies that
$$\langle A(u), u - w \rangle \leq \liminf_{n \to \infty} \langle A(u_n), u_n - w \rangle.$$

The following result relates pseudomonotonicity to the sum of a monotone hemicontinuous operator and a strongly continuous operator, see [37, Proposition 27.6].

Lemma 3.10. *Let $A : X \to X'$ be an operator on the real reflexive Banach space X. If A is monotone and hemicontinuous and B is strongly continuous, then $A + B$ is pseudomonotone.*

For our existence proof, we rely on the following surjectivity result due to Brézis [8]. The proof can be found in [31, Theorem 2.6].

Lemma 3.11. *Let X be real, separable, and reflexive Banach space with $\dim X = \infty$. If $A : X \to X'$ is pseudomonotone, bounded, and coercive then A is surjective; this means, for any $f \in X'$, there is at least one solution to the equation*
$$A(u) = f.$$

3.4 Nemytskii's operators

We begin by recalling the definition of Nemytskii's operator.

Definition 3.12 ([1, 2]). Let $F : \Omega \times \mathbb{R} \to \mathbb{R}$. The Nemytskii operator associated to F is the map $\mathcal{N}_F$ defined on $\mathcal{M}(\Omega)$ (the set of all measurable real valued functions defined on Ω) by setting

$$\mathcal{N}_F(u(x)) = F(x, u(x)).$$

We recall the following fundamental result about Nemytskii operators [1, Theorem 2.2], with a more general version can be found in [2, Theorem 1.7].

Lemma 3.13. *Let $\alpha, \beta \geq 1$, and $F : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function. Assume that there exist a positive function $a_1 \in L^{\beta}(\Omega)$ and a constant $a_2 > 0$ such that*

$$|F(x,t)| \leq a_1(x) + a_2|t|^{\frac{\alpha}{\beta}}, \quad a.e\ x \in \Omega,\ \forall t \in \mathbb{R}.$$

Then the Nemystskii's operator $\mathcal{N}_F$ associated to the function F is continuous from $L^{\alpha}(\Omega)$ to $L^{\beta}(\Omega)$.

4 The approximated problem

We approximate the problem (1.1) by the following non-singular and non-degenerate problem

$$\begin{cases} -M_n\left(\|u_n\|^p\right)\Delta_p u_n = \dfrac{f_n}{\left(u_n + \frac{1}{n}\right)^{\gamma}} + g_n\, u_n^q & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where $f_n = T_n(f)$, $g_n = T_n(g)$ and

$$M_n(t) = M(t) + \frac{1}{n}, \qquad \forall t \in \mathbb{R}^+. \tag{4.2}$$

Lemma 4.1. *Assume that (1.2)–(1.5) hold. Then, problem (4.1) has at least one nonnegative weak solution $u_n \in W_0^{1,p}(\Omega) \cap L^{\infty}(\Omega)$.*

Proof. Our proof relies on pseudomonotone operator theory. We define the operators $\mathcal{A}_n$, $\mathcal{F}_n : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ by

$$\langle \mathcal{A}_n(u), v\rangle = M_n\left(\|u\|^p\right) \int_{\Omega} |\nabla u|^{p-2}\nabla u \cdot \nabla v, \tag{4.3}$$

and

$$\langle \mathcal{F}_n(u), v\rangle = \int_{\Omega} \mathcal{N}_F(u) v\, dx, \tag{4.4}$$

where $\mathcal{N}_F$ is the Nemyskii operator associated to the function F defined by

$$F(x,t) := -\frac{f_n(x)}{\left(t^+ + \frac{1}{n}\right)^{\gamma}} - g_n(x)(t^+)^q \quad \text{a.e. } x \in \Omega,\ \forall t \in \mathbb{R},$$

where $t^+ := \max(t, 0)$. Next, we set

$$\mathcal{B}_n(u) = \mathcal{A}_n(u) + \mathcal{F}_n(u), \tag{4.5}$$

We now verify that $\mathcal{B}_n$ satisfies the hypotheses of Lemma 3.11. The proof proceeds in several steps.

Step 1: Hemicontinuity of $\mathcal{A}_n$. This follows directly from the continuity of the p-Laplacian operator (see Lemma 3.8) and the continuity of the function M_n.

Step 2: Monotonicity of $\mathcal{A}_n$. Let $u, v \in W_0^{1,p}(\Omega)$. Then,

$$\begin{aligned}\langle \mathcal{A}_n(u) - \mathcal{A}_n(v), u - v \rangle &= \langle M_n(\|u\|^p)\Delta_p u + M_n(\|v\|^p)\Delta_p v, u - v \rangle \\ &= M_n(\|u\|^p)\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla(u - v) \\ &\quad - M_n(\|v\|^p)\int_\Omega |\nabla v|^{p-2}\nabla v \cdot \nabla(u - v) \\ &= M_n(\|u\|^p)\|u\|^p - M_n(\|u\|^p)\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v \\ &\quad + M_n(\|v\|^p)\|v\|^p - M_n(\|v\|^p)\int_\Omega |\nabla v|^{p-2}\nabla v \cdot \nabla u.\end{aligned} \tag{4.6}$$

Applying Young's inequality:

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v \le \frac{1}{p'}\|u\|^p + \frac{1}{p}\|v\|^p, \tag{4.7}$$

and

$$\int_\Omega |\nabla v|^{p-2}\nabla v \cdot \nabla u \le \frac{1}{p'}\|v\|^p + \frac{1}{p}\|u\|^p. \tag{4.8}$$

Combining (4.6), (4.7) and (4.8) yields:

$$\begin{aligned}\langle \mathcal{A}_n(u) - \mathcal{A}_n(v), u - v \rangle &\ge M_n(\|u\|^p)\|u\|^p\left(1 - \frac{1}{p'}\right) - \frac{1}{p}M_n(\|u\|^p)\|v\|^p \\ &\quad + M_n(\|v\|^p)\|v\|^p\left(1 - \frac{1}{p'}\right) - \frac{1}{p}M_n(\|v\|^p)\|u\|^p \\ &= \frac{1}{p}\left(M_n(\|u\|^p) - M_n(\|v\|^p)\right)\left(\|u\|^p - \|v\|^p\right) \\ &= \frac{1}{p}\left(M(\|u\|^p) - M(\|v\|^p)\right)\left(\|u\|^p - \|v\|^p\right) \\ &\ge 0,\end{aligned} \tag{4.9}$$

where the last inequality holds because M is nondecreasing function.

Step 3: Strong continuity of $\mathcal{F}_n$. Note that F satisfies the growth condition

$$|F(x,t)| \le C(n)\left(1 + (t^+)^{(q+1)-1}\right), \tag{4.10}$$

where $C(n) > 0$ is a constant depending on n.

Let $(u_k)_{k\in\mathbb{N}} \subset W_0^{1,p}(\Omega)$ be a sequence converging weakly to $u \in W_0^{1,p}(\Omega)$. By the Sobolev compact embedding theorem, we have

$$u_k \to u \quad \text{strongly in } L^r(\Omega), \text{ for all } 1 \le r < p^*. \tag{4.11}$$

Since $q + 1 < p < p^*$, it follows that

$$u_k \to u \quad \text{strongly in } L^{q+1}(\Omega).$$

By Lemma 3.13, the Nemytskii operator $\mathcal{N}_F$ associated with the function F is continuous from $L^{q+1}(\Omega)$ into $L^{(q+1)'}(\Omega)$. Hence,

$$\mathcal{N}_F(u_k) \to \mathcal{N}_F(u) \quad \text{strongly in } L^{(q+1)'}(\Omega).$$

Moreover, since $(q+1)' > (p^*)'$ (due to $q+1 < p^*$), we further deduce

$$\mathcal{N}_F(u_k) \to \mathcal{N}_F(u) \quad \text{strongly in } L^{(p^*)'}(\Omega). \tag{4.12}$$

Now, let $\varphi \in W_0^{1,p}(\Omega)$. Applying Hölder's inequality and the Sobolev embedding theorem, we obtain

$$\begin{aligned}\left|\langle \mathcal{F}_n(u_k) - \mathcal{F}_n(u), \varphi \rangle\right| &\leq \int_\Omega |\mathcal{N}_F(u_k) - \mathcal{N}_F(u)||\varphi|\, dx \\ &\leq \|\mathcal{N}_F(u_k) - \mathcal{N}_F(u)\|_{L^{(p^*)'}(\Omega)} \|\varphi\|_{L^{p^*}(\Omega)} \\ &\leq \mathcal{S}_p \|\mathcal{N}_F(u_k) - \mathcal{N}_F(u)\|_{L^{(p^*)'}(\Omega)} \|\varphi\|_{W_0^{1,p}(\Omega)},\end{aligned}$$

where $\mathcal{S}_p$ is the optimal Sobolev constant for the embedding $W_0^{1,p}(\Omega)$ into $L^{p^*}(\Omega)$.

Consequently,

$$\|\mathcal{F}_n(u_k) - \mathcal{F}_n(u)\|_{W^{-1,p'}(\Omega)} \leq \mathcal{S}_p \|\mathcal{N}_F(u_k) - \mathcal{N}_F(u)\|_{L^{(p^*)'}(\Omega)}. \tag{4.13}$$

Combining (4.12) and (4.13), we conclude that

$$\mathcal{F}_n(u_k) \to \mathcal{F}_n(u) \quad \text{strongly in } W^{-1,p'}(\Omega).$$

Step 4: Pseudomonotonicity of $\mathcal{B}_n$. Since the operator $\mathcal{A}_n$ is monotone, hemicontinuous, and the operator $\mathcal{F}_n$ is strongly continuous, by Lemma 3.10, we deduce that $\mathcal{B}_n$ is pseudomonotone operator.

Step 5: Boundedness of $\mathcal{B}_n$. It suffices to prove that both $\mathcal{A}_n$ and $\mathcal{F}_n$ are bounded operators.

1. **Boundedness of $\mathcal{A}_n$.** Let $R > 0$ and consider $u \in W_0^{1,p}(\Omega)$ with $\|u\| \leq R$. For any test function $v \in W_0^{1,p}(\Omega)$, we apply assumptions (1.2), (4.2) and Hölder's inequality to obtain:

$$\begin{aligned}\left|\langle \mathcal{A}_n(u), v \rangle\right| &\leq (M(\|u\|^p) + 1) \int_\Omega |\nabla u|^{p-1} |\nabla v|\, dx \\ &\leq (M(R^p) + 1)\, R^{p-1} \|v\|_{W_0^{1,p}(\Omega)}.\end{aligned}$$

   Hence,

$$\|\mathcal{A}_n(u)\|_{W^{-1,p'}(\Omega)} \leq (M(R^p) + 1)\, R^{p-1}.$$

   This establishes that $\mathcal{A}_n$ maps bounded subsets of $W_0^{1,p}(\Omega)$ to bounded subsets of $W^{-1,p'}(\Omega)$.

2. **Boundedness of $\mathcal{F}_n$.** For any $u, v \in W_0^{1,p}(\Omega)$, we use the growth condition (4.10) and the fact that $p' < \frac{p}{q}$ (since $q < p-1$), we have

$$\begin{aligned}\left|\langle \mathcal{F}_n(u), v \rangle\right| &\leq C \int_\Omega (1 + |u|^q)\, v\, dx \\ &\leq C_1 \|v\|_{L^p(\Omega)} + \left(\int_\Omega u^{p'q}\right)^{\frac{1}{p'}} \|v\|_{L^p(\Omega)} \\ &\leq C_1 \|v\|_{L^p(\Omega)} + C_2 \|u\|^q_{L^p(\Omega)} \|v\|_{L^p(\Omega)} \\ &\leq C_3 \left(1 + \|u\|^q_{W_0^{1,p}(\Omega)}\right) \|v\|_{W_0^{1,p}(\Omega)}\end{aligned}$$

Consequently,

$$\|\mathcal{F}_n(u)\|_{W^{-1,p'}(\Omega)} \leq C_3 \left(1 + \|u\|^q_{W_0^{1,p}(\Omega)}\right).$$

Therefore, $\mathcal{F}_n$ maps bounded subsets of $W_0^{1,p}(\Omega)$ into bounded subsets of $W^{-1,p'}(\Omega)$.

Step 6: Coercivity of $\mathcal{B}_n$. Let $u \in W_0^{1,p}(\Omega)$. Using the growth condition (4.10), we obtain

$$\begin{aligned}
\langle \mathcal{B}_n(u), u \rangle &= \langle \mathcal{A}_n(u), u \rangle + \langle \mathcal{F}_n(u), u \rangle \\
&= M_n\left(\|u\|^p\right)\|u\|^p - \int_\Omega F(x,u)u \\
&\geq \frac{1}{n}\|u\|^p - C_1\|u\| - C_2\|u\|^{q+1} \\
&\geq \|u\|^p \left(\frac{1}{n} - C_1\|u\|^{1-p} - C_2\|u\|^{q+1-p}\right).
\end{aligned}$$

Since $q < p-1$, it follows that

$$\lim_{\|u\|\to\infty} \frac{\langle \mathcal{B}_n(u), u \rangle}{\|u\|} = +\infty.$$

The operator $\mathcal{B}$ satisfies all assumptions of Lemma 3.11. Consequently, there exists $u_n \in W_0^{1,p}(\Omega)$ such that $\mathcal{B}_n(u) = 0$, meaning u_n weakly solves:

$$\begin{cases} -M_n\left(\|u_n\|^p\right)\Delta_p u_n = \dfrac{f_n}{\left(u_n^+ + \frac{1}{n}\right)^\gamma} + g_n\left(u_n^+\right)^q & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.14}$$

Non-negativity and regularity Choosing u_n^- as a test function in the weak formulation of (4.14) yields:

$$-\int_\Omega |\nabla u_n^-|^p = \frac{1}{M_n\left(\|u_n\|^p\right)} \int_\Omega \left(\frac{f_n}{\left(u_n^+ + \frac{1}{n}\right)^\gamma} + g_n\left(u_n^+\right)^q\right) u_n^- \geq 0, \tag{4.15}$$

which implies $u_n^- = 0$ a.e. in Ω, and thus $u_n \geq 0$.

Furthermore, by [19, Proposition 1.2], we have u_n belongs to $L^t(\Omega)$ for any $t \in [1, +\infty)$. Since the right-hand side of (4.1) belongs to $L^\beta(\Omega)$ with $\beta > \frac{N}{p}$, the L^∞-regularity result of [34, Théorème 4.2] implies that $u_n \in L^\infty(\Omega)$. This completes the proof of Lemma 4.1. □

5 A priori estimates

Lemma 5.1. *Let $(u_n)_n$ be a sequence of nonnegative solutions to problem (4.1). Assume that*

$$g \in L^{\left(\frac{p*}{1+q}\right)'}(\Omega) \quad \text{and} \quad f \in L^m(\Omega),$$

where

$$m = \frac{pN}{N(p-1) + p + \gamma(N-p)}.$$

Then there exists a constant $L > 0$, independent of n, such that

$$\|u_n\|_{W_0^{1,p}(\Omega)} \leq L \quad \text{for all } n \in \mathbb{N}^*. \tag{5.1}$$

Proof. We argue by contradiction. Suppose that (5.1) does not hold. Then, there exists a subsequence, still denoted by (u_n), such that

$$\|u_n\|^p_{W_0^{1,p}(\Omega)} \to +\infty \quad \text{as } n \to +\infty. \tag{5.2}$$

Consequently, for n sufficiently large, we have

$$\|u_n\|^p_{W_0^{1,p}(\Omega)} \geq 1. \tag{5.3}$$

Since the function M is nondecreasing and satisfies assumption (1.2), it follows from (5.3) that

$$\|u_n\|^p_{W_0^{1,p}(\Omega)} \geq M(1) \geq m_0. \tag{5.4}$$

Testing (4.1) with u_n, and using (5.4), we obtain

$$m_0\|u_n\|^p_{W_0^{1,p}(\Omega)} \leq \int_\Omega \frac{f_n\, u_n}{\left(u_n + \frac{1}{n}\right)^\gamma} + \int_\Omega g_n\, u_n^{1+q}. \tag{5.5}$$

We now distinguish two cases depending on the value of γ.

Case 1: $0 < \gamma < 1$. In this case, the exponent m satisfies $1 < m < \frac{N}{p}$.

Since $f_n \leq f$ and $g_n \leq g$, we apply Hölder inequality to both terms on the right-hand side of (5.5):

$$m_0\|u_n\|^p_{W_0^{1,p}(\Omega)} \leq \|f\|_{L^{\left(\frac{p^*}{1-\gamma}\right)'}(\Omega)} \|u_n\|^{1-\gamma}_{L^{p^*}(\Omega)} + \|g\|_{L^{\left(\frac{p^*}{1+q}\right)'}(\Omega)} \|u_n\|^{q+1}_{L^{p^*}(\Omega)}. \tag{5.6}$$

Applying Sobolev's inequality on the left-hand side, we derive:

$$Sm_0\|u_n\|^p_{L^{p^*}(\Omega)} \leq \|f\|_{L^{\left(\frac{p^*}{1-\gamma}\right)'}(\Omega)} \|u_n\|^{1-\gamma}_{L^{p^*}(\Omega)} + \|g\|_{L^{\left(\frac{p^*}{1+q}\right)'}(\Omega)} \|u_n\|^{q+1}_{L^{p^*}(\Omega)}. \tag{5.7}$$

Since $0 < 1-\gamma < q+1 < p$, it follows from inequality (5.7) that the sequence (u_n) is uniformly bounded in $L^{p^*}(\Omega)$.

Letting n tends to infinity in (5.6), we arrive at a contradiction: the right-hand side remains bounded, while the left-hand side diverges to infinity as a consequence of assumption (5.2).

Case 2: $\gamma = 1$. In this case $m = 1$. From (5.5) and $f_n \leq f$, we have:

$$m_0\|u_n\|^p_{W_0^{1,p}(\Omega)} \leq \|f\|_{L^1(\Omega)} + \int_\Omega g_n\, u_n^{1+q}. \tag{5.8}$$

Using the fact that $g_n \leq g$, and Hölder inequality on the right hand side of (5.8), we obtain

$$m_0\|u_n\|^p_{W_0^{1,p}(\Omega)} \leq \|f\|_{L^1(\Omega)} + \|g\|_{L^{\left(\frac{p^*}{1+q}\right)'}(\Omega)} \|u_n\|^{q+1}_{L^{p^*}(\Omega)}. \tag{5.9}$$

Again, applying Sobolev's inequality on the right-hand side of (5.9), we get

$$m_0\|u_n\|^p_{W_0^{1,p}(\Omega)} \leq \|f\|_{L^1(\Omega)} + \mathcal{S}^{q+1}\|g\|_{L^{\left(\frac{p^*}{1+q}\right)'}(\Omega)} \|u\|^{q+1}_{W_0^{1,p}(\Omega)}. \tag{5.10}$$

Since $q+1 < p$, inequality (5.10) implies that the sequence (u_n) is uniformly bounded in $W_0^{1,p}(\Omega)$, contradicting (5.2).

In both cases, we reach a contradiction, which completes the proof of Lemma 5.1. □

Lemma 5.2. *Let u_n be a nonnegative solution of problem (4.1). Then, for any $\omega \subset\subset \Omega$, there exists a constant $c_\omega > 0$ such that*

$$u_n \geq c_\omega \quad \text{in } \omega, \text{ for all } n \in \mathbb{N}^*. \tag{5.11}$$

Proof. Using (5.1) and the fact that M is a nondecreasing function, we obtain

$$M_n(\|u_n\|^p) \leq M(L) = M_\infty, \quad \forall n \in \mathbb{N}^*. \tag{5.12}$$

Thus, in the sense of distributions, we have

$$\begin{aligned} -\Delta_p u_n &\geq \frac{T_1(f) + T_1(g)}{M_\infty} \left(\frac{1}{(u_n+1)^\gamma} + u_n^q \right) \\ &\geq \frac{\min\left(T_1(f) + T_1(g)\right)}{M_\infty} \left(\frac{1}{(u_n+1)^\gamma} + u_n^q \right) \\ &\geq \frac{c_0}{M_\infty} T_1(\underline{h}) = V(x), \end{aligned} \tag{5.13}$$

where we have used the fact the function $\frac{1}{t^\gamma+1} + t^q$, attains a positive minimum on $[0, +\infty[$.

Now consider the problem:

$$\begin{cases} -\Delta_p v = V(x) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.14}$$

and the weighted eigenvalue problem:

$$\begin{cases} -\Delta_p \phi_1 = \lambda_1 V(x) |\phi_1|^{p-2} \phi_1 & \text{in } \Omega, \\ \phi_1 = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.15}$$

where λ_1 is the first eigenvalue of $-\Delta_p$ with weight V, and $\phi_1 \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ is its associated eigenfunction. By Lemma 3.3, $\phi_1 \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$ and $\phi_1(x) > 0$ for all $x \in \Omega$. Choose $\eta > 0$ sufficiently small so that

$$\lambda_1 \eta^{p-1} \|\phi_1\|_{L^\infty(\Omega)}^{p-1} \leq 1. \tag{5.16}$$

Then, $\underline{v} = \eta\phi_1$ is a subsolution for problem (5.14), since

$$-\Delta_p \underline{v} = -\Delta_p(\eta\phi_1) = \eta^{p-1}\lambda_1 V(x) \phi_1^{p-1} \leq V(x) \quad \text{in } \Omega.$$

By the weak comparison principle, we obtain

$$u_n(x) \geq v(x) \geq \eta\phi_1(x) \quad \text{a.e. in } \Omega. \tag{5.17}$$

Since $\phi_1 > 0$ in Ω and $\omega \subset\subset \Omega$, there exists $c_\omega = \min_\omega \phi_1 > 0$ such that

$$u_n(x) \geq c_\omega > 0 \quad \text{for a.e. } x \in \omega \text{ for all } n \in \mathbb{N}^*. \tag{5.18}$$

This completes the proof of Lemma 5.2. □

Lemma 5.3. *Let u_n be a nonnegative solution of (4.1). Then there exists a constant $\Lambda > 0$, independent of n, such that*

$$M\left(\|u_n\|^p\right) \geq \Lambda > 0 \quad \text{for all } n \in \mathbb{N}^*. \tag{5.19}$$

Proof. Let ϕ_1 be as in the proof of Lemma 5.2. By normalizing ϕ_1, we may assume that

$$\|\phi_1\|^p_{L^p(\Omega)} = \int_\Omega \phi_1^p \, dx = 1. \tag{5.20}$$

Using the normalization (5.20), the uniform positivity (5.17), and Poincaré's inequality, we obtain

$$\eta^p = \eta^p \int_\Omega \phi_1^p dx \leq \int_\Omega u_n^p dx \leq C_\Omega \|u_n\|^p_{W_0^{1,p}(\Omega)}.$$

This yields

$$\|u_n\|^p_{W_0^{1,p}(\Omega)} \geq \frac{\eta^p}{C_\Omega}. \tag{5.21}$$

Since M is nondecreasing, it follows from (5.21) that

$$M\left(\|u_n\|^p\right) \geq M\left(\frac{\eta^p}{C_\Omega}\right). \tag{5.22}$$

Next, we choose $\eta > 0$ sufficiently small so that it satisfies both (5.16) and

$$0 < \frac{\eta^p}{C_\Omega} \leq 1.$$

Applying the growth condition (1.2) on M, we get

$$M\left(\frac{\eta^p}{C_\Omega}\right) \geq m_0 \left(\frac{\eta^p}{C_\Omega}\right)^{\mu-1}. \tag{5.23}$$

Combining (5.22) and (5.23), we conclude that

$$M\left(\|u_n\|^p\right) \geq m_0 \left(\frac{\eta^p}{C_\Omega}\right)^{\mu-1} = \Lambda.$$

This completes the proof. □

6 Proof of the main results

By Lemma 5.1, the sequence (u_n) is bounded in $W_0^{1,p}(\Omega)$. Consequently, there exists a function $u \in W_0^{1,p}(\Omega)$ such that, up to subsequence,

$$u_n \to u \quad \text{weakly in} \quad W_0^{1,p}(\Omega),$$

and

$$u_n \to u \quad \text{almost everywhere in } \Omega.$$

Moreover, using (5.11), we deduce that u satisfies (1.8).

Furthermore, by (5.19) and (5.11), the sequence

$$\frac{1}{M_n\left(\|u_n\|^p\right)} \left(\frac{f_n}{\left(u_n + \frac{1}{n}\right)^\gamma} + g_n \, u_n^q \right),$$

is bounded in $L^1_{\text{loc}}(\Omega)$. By Theorem 2.1 of [5], it follows that

$$\nabla u_n \to \nabla u \quad \text{almost everywhere in } \Omega.$$

We now prove that u satisfies (1.9). Let $\varphi \in C_0^\infty(\Omega)$. The inequality (5.11) implies

$$0 \leq \left| \frac{f_n \varphi}{\left(u_n + \frac{1}{n}\right)^\gamma} \right| \leq \left| \frac{f_n \varphi}{u_n^\gamma} \right| \leq \frac{\|\varphi\|_{L^\infty(\Omega)}}{c_\omega^\gamma} f,$$

where ω is the compact support of the function φ. By the Lebesgue dominated convergence theorem,

$$\lim_{n\to+\infty} \int_\Omega \frac{f_n \varphi}{\left(u_n + \frac{1}{n}\right)^\gamma} = \int_\Omega \frac{f\varphi}{u^\gamma}. \tag{6.1}$$

Since $g \in L^{\frac{p^*}{p^*-q-1}}(\Omega)$ and (u_n) is bounded in $L^{p^*}(\Omega)$, Vitali's theorem (see [36, p. 333]) ensures

$$\int_\Omega g_n \, u_n^q \varphi \to \int_\Omega g u^q \varphi. \tag{6.2}$$

Indeed, for any measurable subset $E \subset \Omega$, we have

$$\begin{aligned}
\int_E g_n \, u_n^q \varphi \, dx &\leq \|\varphi\|_{L^\infty(\Omega)} \int_E |g(x)| \, |u_n|^q \, dx \\
&\leq \|\varphi\|_{L^\infty(\Omega)} \|g\|_{L^{\frac{p^*}{p^*-q-1}}(\Omega)} \left(\int_E |u_n|^{\frac{qp^*}{q+1}} \, dx \right)^{\frac{q+1}{p^*}} \\
&\leq \|\varphi\|_{L^\infty(\Omega)} \|g\|_{L^{\frac{p^*}{p^*-q-1}}(\Omega)} \|u_n\|_{L^{p^*}(\Omega)} |E|^{\frac{1}{p^*}} \to 0 \quad \text{as } |E| \to 0.
\end{aligned}$$

Thus, $\left(g_n \, u_n^q \varphi\right)_n$ is equi-integrable, and since it converges a.e. to $g \, u^q \varphi$, (6.2) holds by Vitali's theorem.

Combining (6.1) and (6.2), we obtain

$$\lim_{n\to+\infty} \int_\Omega \left(\frac{f_n}{\left(u_n + \frac{1}{n}\right)^\gamma} + g_n \, u_n^q \right) \varphi = \int_\Omega \left(\frac{f}{u^\gamma} + g \, u^q \right) \varphi. \tag{6.3}$$

Recalling that u_n satisfies the identity

$$\int_\Omega |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla \varphi = \frac{1}{M_n\left(\|u_n\|^p\right)} \int_\Omega \left(\frac{f_n}{\left(u_n + \frac{1}{n}\right)^\gamma} + g_n \, u_n^q \right) \varphi, \tag{6.4}$$

for every $\varphi \in C_0^\infty(\Omega)$. By Lemma 5.3, the sequence (u_n) is bounded in $W_0^{1,p}(\Omega)$, so up to a subsequence,

$$\|u_n\| \longrightarrow \theta_0. \tag{6.5}$$

Passing to the limit in (6.4) using (6.3) and (6.5), we conclude

$$M(\theta_0^p) \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi = \int_\Omega \left(\frac{f}{u^\gamma} + g u^q \right) \varphi, \ \forall \varphi \in C_0^\infty(\Omega). \tag{6.6}$$

To complete the proof of (1.9), it remains to show that

$$M(\theta_0^p) = M\left(\|u\|^p\right).$$

Let $\varphi_\varepsilon \in C_0^\infty(\Omega)$ such that $\varphi_\varepsilon \to u$ strongly in $W_0^{1,p}(\Omega)$. Testing (6.6) with φ_ε and taking liminf (using Fatou's Lemma) yields

$$M(\theta_0^p)\|u\|^p \geq \int_\Omega f u^{1-\gamma} + \int_\Omega g u^{q+1}. \tag{6.7}$$

On the other hand, testing the weak formulation (4.1) with u_n gives

$$M_n\left(\|u_n\|^p\right)\|u_n\|^p = \int_\Omega \frac{f_n\, u_n}{\left(u_n+\frac{1}{n}\right)^\gamma} + \int_\Omega g_n\, u_n^{q+1}. \tag{6.8}$$

To pass to the limit in the right side of (6.8), we distinguish two cases based on the value of γ.

Case 1: $0<\gamma<1$. Since $f \in L^{\frac{p^*}{p^*+\gamma-1}}(\Omega)$ and $g \in L^{\frac{p^*}{p^*-q-1}}(\Omega)$, it follows from [6, Theorem 2.5.7] that for every for every $\varepsilon>0$, there exists $\delta>0$ such that for every measurable set $E\subset\Omega$ with $|E|\le\delta$, we have

$$\int_E |f(x)|^{\frac{p^*}{p^*+\gamma-1}}\,dx \le \varepsilon^{\frac{p^*}{p^*+\gamma-1}}, \quad \int_E |g(x)|^{\frac{p^*}{p^*-q-1}}\,dx \le \varepsilon^{\frac{p^*}{p^*-q-1}},$$

Applying Hölder's inequality, we obtain

$$\begin{aligned}\int_E \left(\frac{f_n\, u_n}{\left(u_n+\frac{1}{n}\right)^\gamma} + g_n\, u_n^{q+1}\right)dx &\le \|u_n\|^{1-\gamma}_{L^{p*}(\Omega)}\left(\int_E |f(x)|^{\frac{p^*}{p^*+\gamma-1}}\,dx\right)^{\frac{p^*+\gamma-1}{p^*}} \\ &\quad + \|u_n\|^{q+1}_{L^{p*}(\Omega)}\left(\int_E |g(x)|^{\frac{p^*}{p^*-q-1}}\,dx\right)^{\frac{p^*-q-1}{p^*}} \\ &\le C\varepsilon,\end{aligned} \tag{6.9}$$

where the constant $C>0$ is independent of n, thanks to the uniform boundedness of (u_n) in $W_0^{1,p}(\Omega)$ and the continuous Sobolev embedding.

From (6.9), we deduce that the sequences

$$\left(\frac{f_n\, u_n}{\left(u_n+\frac{1}{n}\right)^\gamma}\right)_n \quad \text{and} \quad \left(g_n\, u_n^{q+1}\right)_n$$

are equi-integrable. Therefore, by Vitali's convergence theorem, we obtain

$$\int_\Omega \frac{f_n\, u_n}{\left(u_n+\frac{1}{n}\right)^\gamma}\,dx \to \int_\Omega f u^{1-\gamma}, \tag{6.10}$$

and

$$\int_\Omega g_n\, u_n^{q+1}\,dx \to \int_\Omega g u^{q+1}\,dx. \tag{6.11}$$

Combining (6.10) and (6.11), it follows that

$$\int_\Omega \frac{f_n\, u_n}{\left(u_n+\frac{1}{n}\right)^\gamma}\,dx + \int_\Omega g_n\, u_n^{q+1}\,dx \to \int_\Omega f u^{1-\gamma} + \int_\Omega g u^{q+1}\,dx. \tag{6.12}$$

Case 2: $\gamma=1$. In this case, observe that

$$\frac{f_n\, u_n}{u_n+\frac{1}{n}} \le f,$$

and

$$\frac{f_n\, u_n}{u_n+\frac{1}{n}} \to f \quad \text{almost everywhere in } \Omega.$$

Hence, by the Lebesgue dominated convergence theorem, we deduce

$$\int_\Omega \frac{f_n\, u_n}{u_n + \frac{1}{n}}\, dx \to \int_\Omega f\, dx. \tag{6.13}$$

Combining (6.13) with the already established convergence (6.11), we obtain

$$\int_\Omega \frac{f_n\, u_n}{u_n + \frac{1}{n}}\, dx + \int_\Omega g_n\, u_n^{q+1}\, dx \to \int_\Omega f\, dx + \int_\Omega g u^{q+1}\, dx. \tag{6.14}$$

Using (6.5) in conjunction with either (6.12) or (6.14) (depending on the value of γ), we can pass to the limit in (6.8). This leads to the following identity:

$$M\left(\theta_0^p\right)\theta_0^p = \int_\Omega f u^{1-\gamma} + \int_\Omega g\, u^{q+1}. \tag{6.15}$$

Comparing (6.7) and (6.15), we obtain $M(\theta_0^p)\|u\|^p \geq M\left(\theta_0^p\right)\theta_0^p$, which implies $\|u\| \geq \theta_0$, since $M(\theta_0) \geq \Lambda > 0$.

On the other hand, since u_n converges weakly to u in $W_0^{1,p}(\Omega)$, and given that $\|u_n\|$ converges to θ_0, we have

$$\theta_0 = \liminf_{n\to+\infty} \|u_n\| \geq \|u\|.$$

Thus, $\|u\| = \theta_0$, and the proof of Theorem 2.1 is complete.

7 Regularity results

In this section, we investigate how the regularity of a solution u to problem (1.1) depends on the summability of f and g.

We begin with a key lemma that extends the class of admissible test functions in the weak formulation (1.9) from $\mathcal{C}_0^\infty(\Omega)$ to $W_0^{1,p}(\Omega)$.

Lemma 7.1. *Let $u \in W_0^{1,p}(\Omega)$ be a weak solution of the problem (1.1). Then, the formulation (1.9) holds for all test functions $W_0^{1,p}(\Omega)$, i.e.,*

$$M\left(\|u\|^p\right)\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla\varphi = \int_\Omega \left(\frac{f(x)}{u^\gamma} + g(x)u^q\right)\varphi, \tag{7.1}$$

for every $\varphi \in W_0^{1,p}(\Omega)$.

Proof. The proof follows the approach of [18, Lemma A.5] and also [16, Lemma 5.1]. Let $u \in W_0^{1,p}(\Omega)$ be a weak solution of the problem (1.1). Given $\varphi \in W_0^{1,p}(\Omega)$, there exists a sequence $(\varphi_\varepsilon)_{\varepsilon>0} \subset \mathcal{C}_0^\infty(\Omega)$ such that

$$\varphi_\varepsilon \to \varphi \quad \text{strongly in } W_0^{1,p}(\Omega). \tag{7.2}$$

Define $\psi_\varepsilon = |\varphi_\varepsilon - \varphi|$. By density, for each fixed $\varepsilon > 0$, there exists a sequence of nonnegative functions $\left(\psi_{\varepsilon\eta}\right)_{\eta>0} \subset \mathcal{C}_0^\infty(\Omega)$ such that

$$\psi_{\varepsilon\eta} \to \psi_\varepsilon \quad \text{in } W_0^{1,p}(\Omega) \text{ and a.e. in } \Omega.$$

Testing (1.9) with $\psi_{\varepsilon\eta}$ yields

$$M\left(\|u\|^p\right)\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla\psi_{\varepsilon\eta}=\int_\Omega\left(\frac{f(x)}{u^\gamma}+g(x)u^q\right)\psi_{\varepsilon\eta}.$$

Taking the lim inf as $\eta\to 0$ and applying Fatou's lemma to the right-hand side, we obtain

$$\int_\Omega\left(\frac{f(x)}{u^\gamma}+g(x)u^q\right)\psi_\varepsilon\le M\left(\|u\|^p\right)\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla\psi_\varepsilon \tag{7.3}$$

Using Hölder's inequality on the right-hand side of (7.3), we deduce

$$\int_\Omega\left(\frac{f(x)}{u^\gamma}+g(x)u^q\right)|\varphi_\varepsilon-\varphi|\le M\left(\|u\|^p\right)\|u\|^{p-1}\|\psi_\varepsilon\|.$$

Passing to the limit as $\varepsilon\to 0$, we conclude that

$$\int_\Omega\left(\frac{f(x)}{u^\gamma}+g(x)u^q\right)\varphi_\varepsilon\to\int_\Omega\left(\frac{f(x)}{u^\gamma}+g(x)u^q\right)\varphi. \tag{7.4}$$

From (7.2), it follows that

$$M\left(\|u\|^p\right)\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla\varphi_\varepsilon\to M\left(\|u\|^p\right)\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla\varphi. \tag{7.5}$$

Combining (7.4) and (7.5), we obtain the desired result. □

Theorem 7.2. *Assume that $f\in L^m(\Omega)$, with $m>\frac{N}{p}$, and $g\in L^\infty(\Omega)$. Let u be a positive solution of (1.1). Then, $u\in L^\infty(\Omega)$.*

Proof. The proof follows the approach in [14,29,35]. For $k\ge 1$, define the truncation function

$$G_k(t)=t-T_k(t)=\begin{cases}k & \text{if } t>k,\\ t & \text{if } -k\le t\le k,\\ -k & \text{if } t<-k.\end{cases}$$

For $s>0$ and $h>0$, define the test function

$$w=G_k(u)\min\left\{G_k(u)^{ps},h^p\right\}.$$

Testing the weak formulation (7.1) with w, and using the uniform lower bound $M\left(\|u\|^p\right)\ge\Lambda$ from Lemma 5.3 (see (5.19)), we obtain

$$\begin{aligned}\Lambda\int_\Omega |\nabla G_k(u)|^p\min\left\{G_k(u)^{ps},h^p\right\}+ps\int_{\{0\le G_k(u)^s\le h\}}G_k(u)^{ps}|\nabla G_k(u)|^p\\ \le\int_\Omega\left(\frac{f}{u^\gamma}+g\,u^q\right)G_k(u)\min\left\{G_k(u)^{ps},h^p\right\}.\end{aligned}$$

Since $u\ge 1$ on the set $\{u\ge k\}$, we deduce that

$$\begin{aligned}\Lambda\int_\Omega\left|\nabla\left(G_k(u)\min\left\{G_k(u)^s,h\right\}\right)\right|^p\le\int_\Omega fG_k(u)\min\left\{G_k(u)^{ps},h^p\right\}\\ +\|g\|_{L^\infty(\Omega)}\int_\Omega G_k(u)^p\min\left\{G_k(u)^{ps},h^p\right\}.\end{aligned}$$

Since $f \in L^{\frac{N}{p}}(\Omega)$, the absolute continuity of the integral (see [6, Theorem 2.5.7]) implies that for every $\varepsilon > 0$, there exists $\delta > 0$ such that for any measurable set $E \subset \Omega$ with $|E| \leq \delta$, we have

$$\int_E |f(x)|^{\frac{N}{p}} dx \leq \varepsilon^{\frac{N}{p}}. \tag{7.6}$$

Now, by the Chebyshev inequality (see [7, Lemma 20.1])), for any $c > 0$,

$$|\{f(x) > c\}| \leq \frac{1}{c^{\frac{N}{p}}} \int_\Omega |f(x)|^{\frac{N}{p}} dx.$$

Then choosing c sufficiently large, we conclude that

$$|\{f > c\}| \leq \delta. \tag{7.7}$$

Combining (7.6) and (7.7), it follows that

$$\left(\int_{\{f \geq c\}} |f|^{\frac{N}{p}} \right)^{\frac{p}{N}} < \varepsilon.$$

Assume $G_k(u) \in L^{p(s+1)}(\Omega)$. Applying the Sobolev inequality on the left-hand side and the Hölder inequality on the right hand-side, we obtain

$$\begin{aligned} &S\Lambda \left(\int_\Omega \left| G_k(u) \min\{G_k(u)^s, h\} \right|^{p^*} \right)^{\frac{p}{p^*}} \\ &\leq h^p |\Omega|^{\frac{p-1}{p^*}} \left(\int_{\{f \geq c\}} |f|^{\frac{N}{p}} \right)^{\frac{p}{N}} \|G_k(u)\|_{L^{p^*}(\Omega)} + \left(c + \|g\|_{L^\infty(\Omega)} \right) \int_\Omega G_k(u)^{p(s+1)} \\ &\leq \varepsilon h^p |\Omega|^{\frac{p-1}{p^*}} \|G_k(u)\|_{L^{p^*}(\Omega)} + \left(c + \|g\|_{L^\infty(\Omega)} \right) \int_\Omega G_k(u)^{p(s+1)}. \end{aligned}$$

Choosing $\varepsilon < \frac{1}{h^p |\Omega|^{\frac{p-1}{p^*}}}$, we derive

$$\left(\int_\Omega \left| G_k(u) \min\{G_k(u)^s, h\} \right|^{p^*} \right)^{\frac{p}{p^*}} \leq \frac{1}{S\Lambda} \|G_k(u)\|_{L^{p^*}(\Omega)} + \frac{c + \|g\|_{L^\infty(\Omega)}}{S\Lambda} \|G_k(u)\|^{p(s+1)}_{L^{p(s+1)}(\Omega)}.$$

Taking $h \to \infty$, we obtain

$$\left(\int_\Omega |G_k(u)|^{p^*(s+1)} \right)^{\frac{p}{p^*}} \leq \frac{1}{S\Lambda} \|G_k(u)\|_{L^{p^*}(\Omega)} + \frac{c + \|g\|_{L^\infty(\Omega)}}{S\Lambda} \|G_k(u)\|^{p(s+1)}_{L^{p(s+1)}(\Omega)}. \tag{7.8}$$

Since $u \in W_0^{1,p}(\Omega)$, it follows from Sobolev's embedding theorem that $G_k(u) \in L^{p^*}(\Omega)$. Therefore, the bootstrap inequality (7.8) holds for $s = s_0 = 0$.

Now, set $s = s_1$ in (7.8) such that

$$p(s_1 + 1) = p^* = \frac{pN}{N - p},$$

which gives

$$s_1 + 1 = (s_0 + 1)N/(N - p).$$

This implies $G_k(u) \in L^{p^*(s_1+1)}(\Omega)$.

We now iterate this argument. Suppose $G_k(u) \in L^{p^*(s_{i-1}+1)}(\Omega)$; then, applying the same reasoning, we obtain

$$G_k(u) \in L^{p^*(s_i+1)}(\Omega), \quad \text{where } s_i + 1 = N(s_{i-1}+1)/(N-p).$$

The sequence $(s_i)_{i\in\mathbb{N}}$ is positive (since $s_1 > 0$) and increasing because $s_i + 1 = N(s_{i-1} + 1)/(N-p) > s_{i-1}+1$. Thus, $(s_i)_{i\in\mathbb{N}}$ approaches a limit l as $i \to \infty$. Suppose, for contradiction, that l is finite. Then, passing to the limit in the recurrence relation gives

$$l + 1 = N(l+1)/(N-p),$$

which is impossible since $\frac{N}{N-p} > 1$. Hence $l = +\infty$, and we conclude that

$$G_k(u) \in L^\rho(\Omega) \quad \text{for all } \rho \in (1, \infty).$$

Next, we take $G_k(u)$ as a test function in the weak formulation (7.1). Using the uniform positivity estimate $M(\|u\|^p) \geq \Lambda$ from Lemma 5.3 (see (5.19)), we obtain

$$\Lambda \int_\Omega |\nabla G_k(u)|^p \leq \int_\Omega \left(\frac{f}{u^\gamma} + g u^q \right) G_k(u).$$

Since $u \geq k \geq 1$ on the set $\{u \geq k\}$, it follows that

$$\int_\Omega |\nabla G_k(u)|^p \leq \frac{1}{\Lambda} \int_\Omega (f + g G_k(u)^q)\, G_k(u).$$

Because $f + g G_k(u)^q \in L^\alpha(\Omega)$ for some $\alpha > \frac{N}{p}$, we can apply Stampacchia's L^∞-regularity result (see [34]) to conclude that $u \in L^\infty(\Omega)$. □

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Configurations of quadratic systems possessing three distinct infinite singularities and one or more invariant parabolas

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Abstract. Denote by **QS** the class of all non-degenerate planar quadratic differential systems and by **QSP** the subclass of **QS** of all systems possessing at least one invariant parabola. In this paper we consider the subfamily of **QSP** defined by the condition $\eta \neq 0$, which means the presence of three distinct infinite singularities real or complex. We denote this subfamily by $\mathbf{QSP}_{(\eta\neq 0)}$. We investigate all possible configurations of invariant parabolas and invariant straight lines which systems in $\mathbf{QSP}_{(\eta\neq 0)}$ could possess and their geometric properties encoded in such configurations. The classification presented here is taken modulo the action of the group of real affine transformations and time rescaling and it is given in terms of affine invariant polynomials. It yields a total of 144 distinct configurations. The obtained classification is an algorithm which makes it possible for any given real quadratic differential system in $\mathbf{QSP}_{(\eta\neq 0)}$ to specify its configuration of invariant parabolas and straight lines. This work will prove helpful in studying the integrability of the systems in $\mathbf{QSP}_{(\eta\neq 0)}$.

Keywords: quadratic differential systems, algebraic solution, configuration of algebraic solutions, invariant parabolas, affine invariant polynomials, group action.

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1 Introduction and statement of main results

For every planar differential system of the form

$$\frac{dx}{dt} = P(x,y), \qquad \frac{dy}{dt} = Q(x,y), \tag{1.1}$$

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where $P, Q \in \mathbb{R}[x, y]$, that is, P and Q are polynomials in x and y with real coefficients, we associate the vector field

$$X = P(x, y)\frac{\partial}{\partial x} + Q(x, y)\frac{\partial}{\partial y}.$$

The *degree* of such a system is defined as the integer $m = \max(\deg P, \deg Q)$. In particular, system (1.1) is called a *quadratic differential system* when $m = 2$; here, **QS** denotes the entire class of real quadratic differential systems. From now on, we assume that P and Q are coprime polynomials. Otherwise, by an appropriate rescaling of time, system (1.1) can be reduced to a linear or constant system. Quadratic differential systems satisfying this coprimality condition are referred to as *non-degenerate quadratic systems*.

Quadratic systems appear in various research fields including models of population dynamics [6], fluid dynamics [9], control systems [11] and even quantum dynamics [2]. They are also of theoretical interest because we have open problems on these systems stated more than a century ago; see for example [1] for a bibliographical survey.

Given $f \in \mathbb{C}[x, y]$, we say that the algebraic curve $f(x, y) = 0$ is an *invariant algebraic curve* of systems (1.1) if there exists $K \in \mathbb{C}[x, y]$ (it is called cofactor of the invariant curve $f = 0$) such that

$$P\frac{\partial f}{\partial x} + Q\frac{\partial f}{\partial y} = Kf.$$

Quadratic systems with invariant algebraic curves have been studied extensively by many authors. For example, Druzhkova [10] (1968) presented necessary and sufficient conditions on the coefficients of a quadratic system, as well as on the coefficients of a conic, for the conic to be an invariant curve of the system. Christopher [7] (1989) provided a normal form for quadratic systems possessing invariant parabolas. Qin Yuan-xum [18] (1996) investigated quadratic systems having an ellipse as a limit cycle. Cairó and Llibre [5] (2002) studied quadratic systems with invariant algebraic conics in the context of Darboux integrability. Schlomiuk and Vulpe [19, 21] (2004, 2008) classified quadratic systems with invariant straight lines of total multiplicity at least four, according to their geometric configurations. Many other works have also contributed to this topic.

The primary objective of this research is to study non-degenerate quadratic systems that possess invariant conics. Irreducible affine conics over the real field $\mathbb{R}$ include hyperbolas, ellipses, and parabolas. These conics can be distinguished by analyzing their behavior at infinity. Specifically, a hyperbola is a real irreducible affine conic with two distinct real points at infinity. A parabola, in contrast, has a single real point at infinity, where the conic meets the line at infinity with multiplicity two. An ellipse, on the other hand, has two complex conjugate points at infinity.

The classifications of **QS** with invariant hyperbolas [14, 15] and with invariant ellipses [13, 16] were obtained in previous works. In this study, we focus on the class **QSP** of non-degenerate quadratic differential systems that possess an invariant parabola. The novel contribution of this work lies in adopting a global approach, employing global tools such as the theory of invariant polynomials for differential systems.

The group of real affine transformations combined with time rescaling acts on the class **QS**. Consequently, modulo this group action, quadratic systems in this class depend on five essential parameters. The same group also acts on the subclass **QSP**, and, modulo the group action, systems in **QSP** depend on at most three parameters. To ensure our study is intrinsic and independent of any particular normal form representation, we employ invariant polynomials and geometric invariants to carry out the desired classification.

In the paper [24], the necessary and sufficient conditions for a non-degenerate quadratic system in **QS** to have invariant parabolas are established. Moreover, [24] presents invariant criteria that determine the number, position, and multiplicity of such parabolas.

The present paper is a continuation of [24]. More precisely, using the conditions from that work, we classify all possible configurations of invariant parabolas and invariant lines that a system in $\mathbf{QSP}_{(\eta\neq 0)}$ may possess. The investigation of configurations within the family $\mathbf{QSP}_{(\eta=0)}$ is currently in progress.

A key concept in this study is the notion of a *configuration of algebraic invariant curves* for a polynomial differential system. This concept was first introduced in [22], with an earlier version focusing solely on invariant lines presented in [19]. Following Darboux's definition, an *algebraic solution* of a polynomial differential system is an algebraic invariant curve defined by an irreducible polynomial over $\mathbb{C}$.

Definition 1.1. A *configuration of invariant algebraic curves* of a real polynomial differential system is defined as a finite set of algebraic invariant curves of the system, each endowed with its multiplicity, together with the real singularities, whether finite or infinite, located on these curves, each also endowed with its multiplicity.

It is worth noting that [8] introduces various notions of multiplicity for an algebraic invariant curve, including infinitesimal, integrable, algebraic, geometric, and holonomic multiplicities. In this work, we adopt the concept of *geometric multiplicity*, defined via perturbations within the family **QS** as follows.

Definition 1.2. An invariant conic

$$\Phi(x,y) = p + qx + ry + sx^2 + 2vxy + uy^2 = 0,$$

with $(s,v,u) \neq (0,0,0)$ and parameters $(p,q,r,s,v,u) \in \mathbb{C}^6$, for a quadratic vector field X, is said to have *multiplicity* m if there exists a sequence of real quadratic vector fields $\{X_k\}$ converging to X (under the topology induced by their coefficients on the sphere S^{11}) such that each X_k admits m distinct (complex) invariant conics

$$\Phi_k^1 = 0, \ \ldots, \ \Phi_k^m = 0,$$

all converging to $\Phi = 0$ as $k \to \infty$ (under the topology induced by their coefficients on the sphere S^5). Moreover, this property does not hold for $m+1$. When an invariant conic $\Phi(x,y) = 0$ has multiplicity one, it is called *simple*.

We note that two non-equivalent systems, modulo the group action, may have the "same configuration" of invariant parabolas and straight lines. Therefore, it is necessary to define when two configurations are considered "the same" or equivalent.

Definition 1.3. Suppose we have two systems (S_1) and (S_2) in **QSP**, each with a finite number of singularities (finite or infinite), a finite set of invariant parabolas

$$\mathcal{P}_i : g_i(x,y) = 0, \quad i = 1,\ldots,k,$$

of (S_1) (respectively

$$\mathcal{P}'_i : g'_i(x,y) = 0, \quad i = 1,\ldots,k,$$

of (S_2)), and a finite set (possibly empty) of invariant straight lines

$$\mathcal{L}_j : f_j(x,y) = 0, \quad j = 1,\ldots,k',$$

of (S_1) (respectively

$$\mathcal{L}'_j : f'_j(x,y) = 0, \quad j = 1, \ldots, k',$$

of (S_2)). We say that the configurations $\mathcal{C}_1$ and $\mathcal{C}_2$ of parabolas and lines of these systems are *equivalent* if there exist one-to-one correspondences

$$\phi_p : \{\mathcal{P}_i\} \to \{\mathcal{P}'_i\} \quad \text{and} \quad \phi_l : \{\mathcal{L}_j\} \to \{\mathcal{L}'_j\}$$

such that:

(i) ϕ_p and ϕ_l preserve multiplicities of parabolas and lines, and map real invariant curves to real invariant curves, and complex invariant curves to complex invariant curves;

(ii) for each parabola $\mathcal{P} : g(x,y) = 0$ in $\mathcal{C}_1$ (respectively each line $\mathcal{L} : f(x,y) = 0$), there is a one-to-one correspondence between the real singularities on $\mathcal{P}$ (respectively on $\mathcal{L}$) and the real singularities on $\phi_p(\mathcal{P})$ (respectively on $\phi_l(\mathcal{L})$), preserving both their multiplicities and locations;

(iii) furthermore, consider the total curves

$$\mathcal{F} : \prod_i G_i(X,Y,Z) \cdot \prod_j F_j(X,Y,Z) \cdot Z = 0,$$

and

$$\mathcal{F}' : \prod_i G'_i(X,Y,Z) \cdot \prod_j F'_j(X,Y,Z) \cdot Z = 0,$$

where $G_i(X,Y,Z) = 0$ and $F_j(X,Y,Z) = 0$ (respectively $G'_i(X,Y,Z) = 0$ and $F'_j(X,Y,Z) = 0$) are the projective completions of $\mathcal{P}_i$ and $\mathcal{L}_j$ (respectively $\mathcal{P}'_i$ and $\mathcal{L}'_j$). Then there is a correspondence ψ between the singularities of $\mathcal{F}$ and $\mathcal{F}'$, preserving their multiplicities as singularities of the total curves.

Our main results are summarized in the following theorem.

Main Theorem. *Consider the class $\mathbf{QSP}_{(\eta\neq 0)}$ of all non-degenerate quadratic differential systems (1.1) possessing an invariant parabola and three distinct infinite singularities (real or complex).*

(A) *This family is classified according to the configurations of invariant parabolas and invariant straight lines, resulting in 144 distinct configurations. Among these, 112 configurations belong to the subclass $\mathbf{QSP}_{(\eta>0)}$ and 32 to the subclass $\mathbf{QSP}_{(\eta<0)}$. This geometric classification is illustrated in Figures 1.1 and 1.2, and is characterized by necessary and sufficient invariant conditions presented in Diagrams 1.1, 1.2, and 1.3.*

(B) *Using 70 invariant polynomials, we derive the bifurcation diagram in the space $\mathbb{R}^{12}$ of the coefficients of systems in $\mathbf{QSP}_{(\eta\neq 0)}$. These diagrams, shown in Diagrams 1.1, 1.2, and 1.3, classify the systems according to their configurations of invariant parabolas and straight lines. Furthermore, the diagrams provide an algorithmic procedure to compute the configuration of any quadratic differential system with an invariant parabola, irrespective of its chosen normal form.*

(C) *Figures 1.1 and 1.2 present all possible configurations for systems in the subclasses $\mathbf{QSP}_{(\eta>0)}$ and $\mathbf{QSP}_{(\eta<0)}$, respectively. We prove that all 144 configurations are realizable within $\mathbf{QSP}_{(\eta\neq 0)}$, and that these configurations are topologically distinct. This proof, based on geometric invariants, is provided in Subsection 3.4 and is illustrated in Diagrams 3.1 and 3.2.*

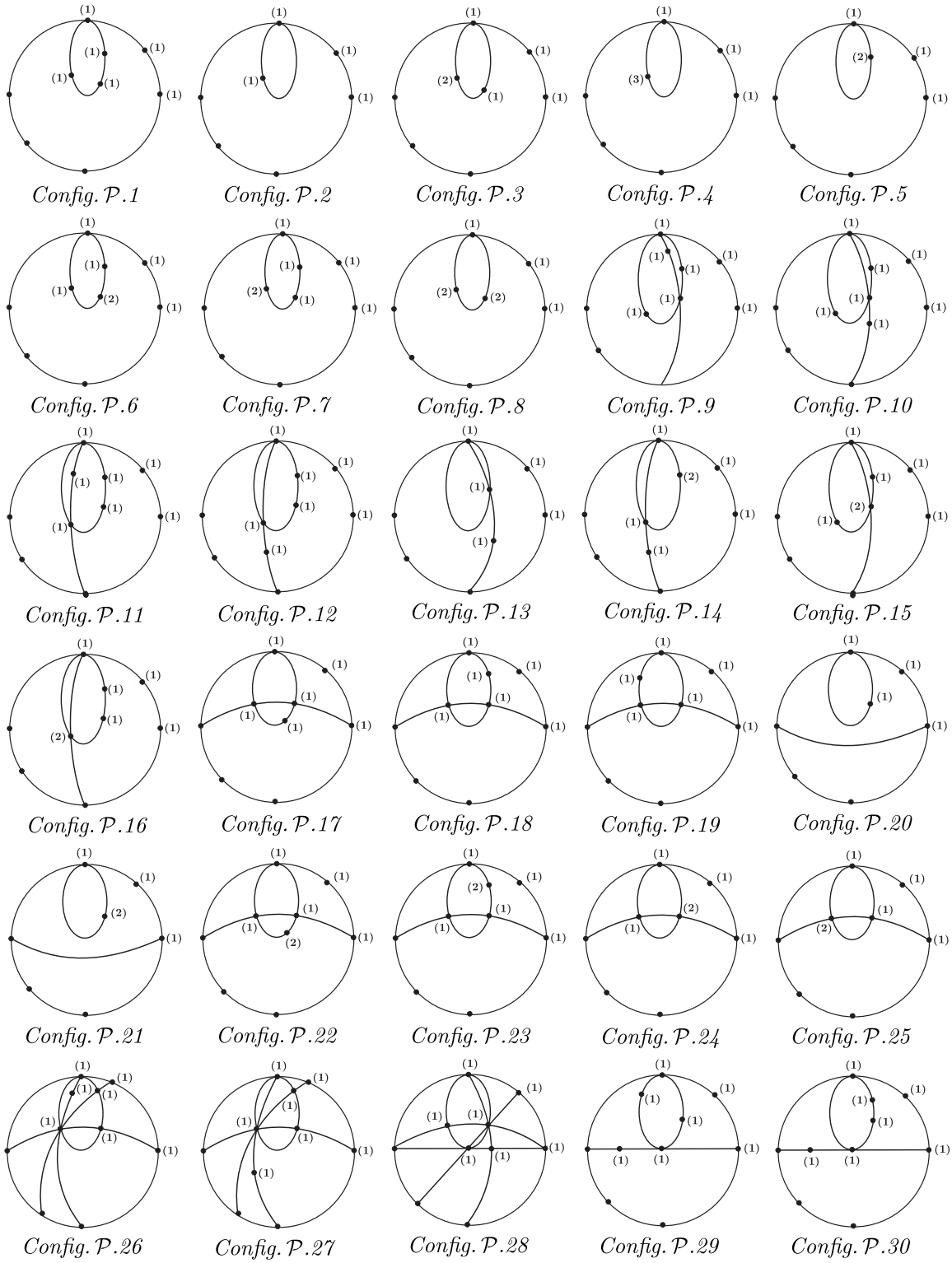


Figure 1.1: Configurations of systems in **QSP** in the case $\eta > 0$.

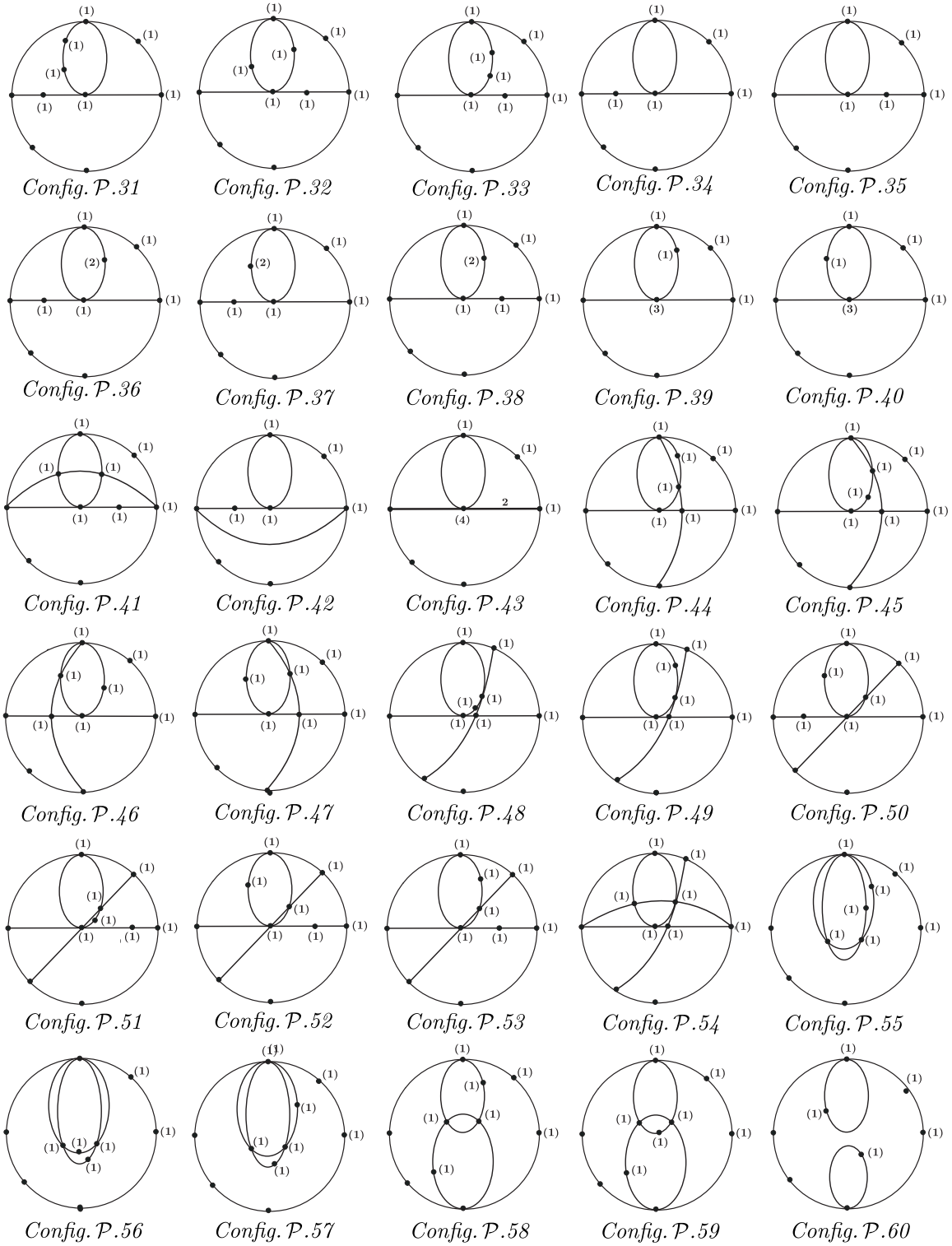


Figure 1.1 *(continued):* Configurations of systems in **QSP** in the case $\eta > 0$.

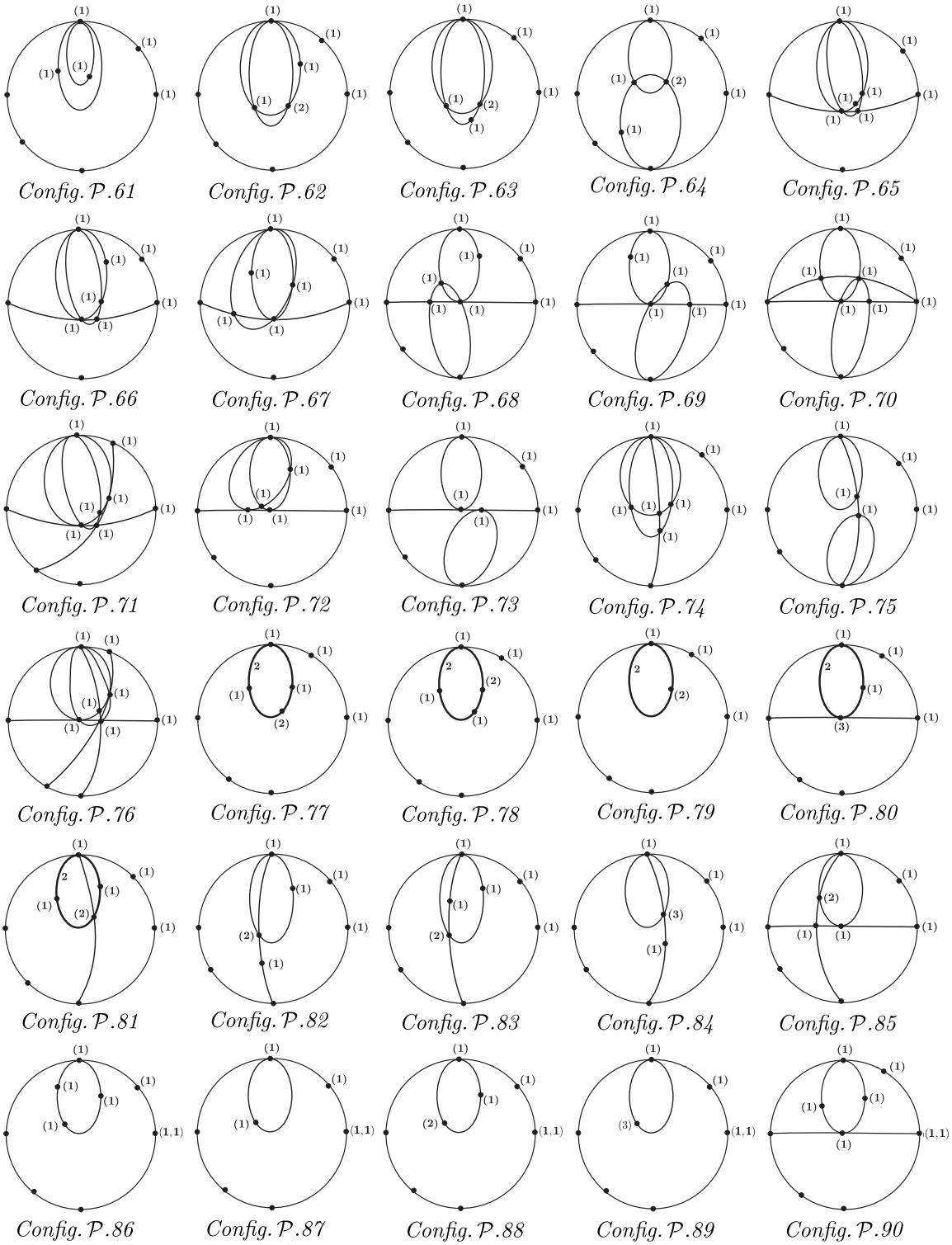


Figure 1.1 *(continued):* Configurations of systems in **QSP** in the case $\eta > 0$.

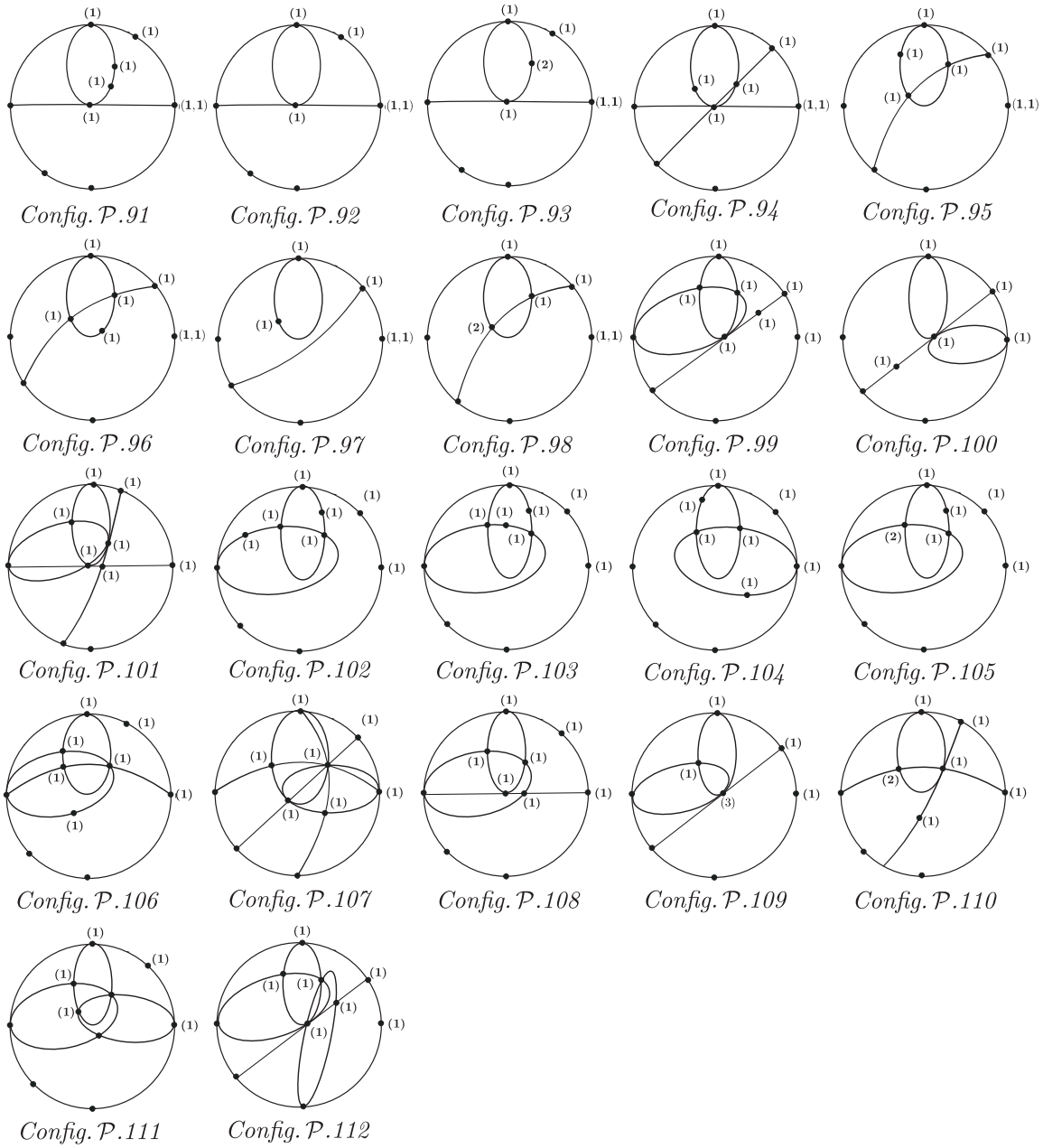


Figure 1.1 *(continued):* Configurations of systems in **QSP** in the case $\eta > 0$.

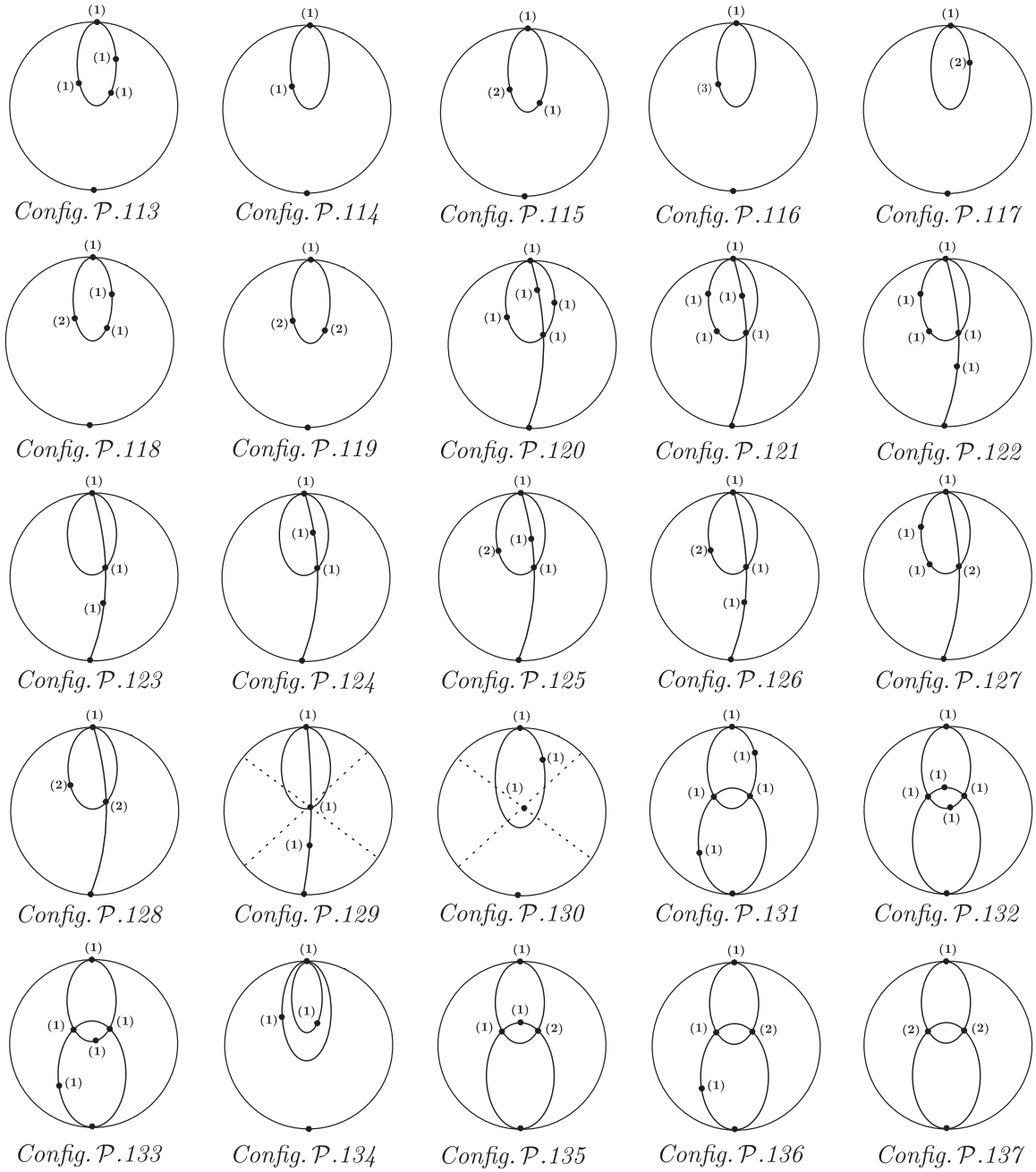


Figure 1.2: Configurations of systems in **QSP** in the case $\eta < 0$.

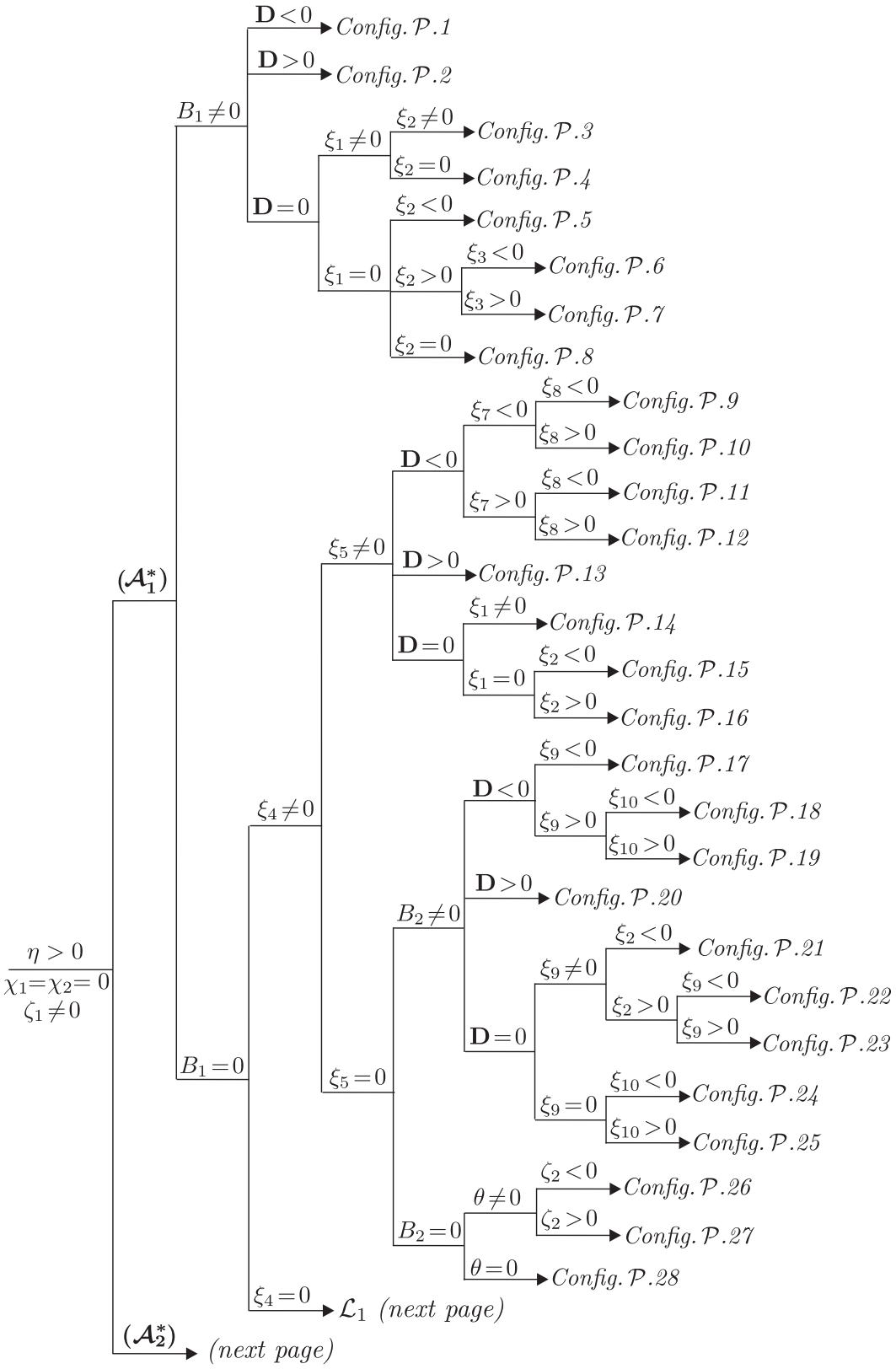


Diagram 1.1: Conditions for the configurations of systems in **QSP** in the case $\eta > 0$, $\zeta_1 \neq 0$.

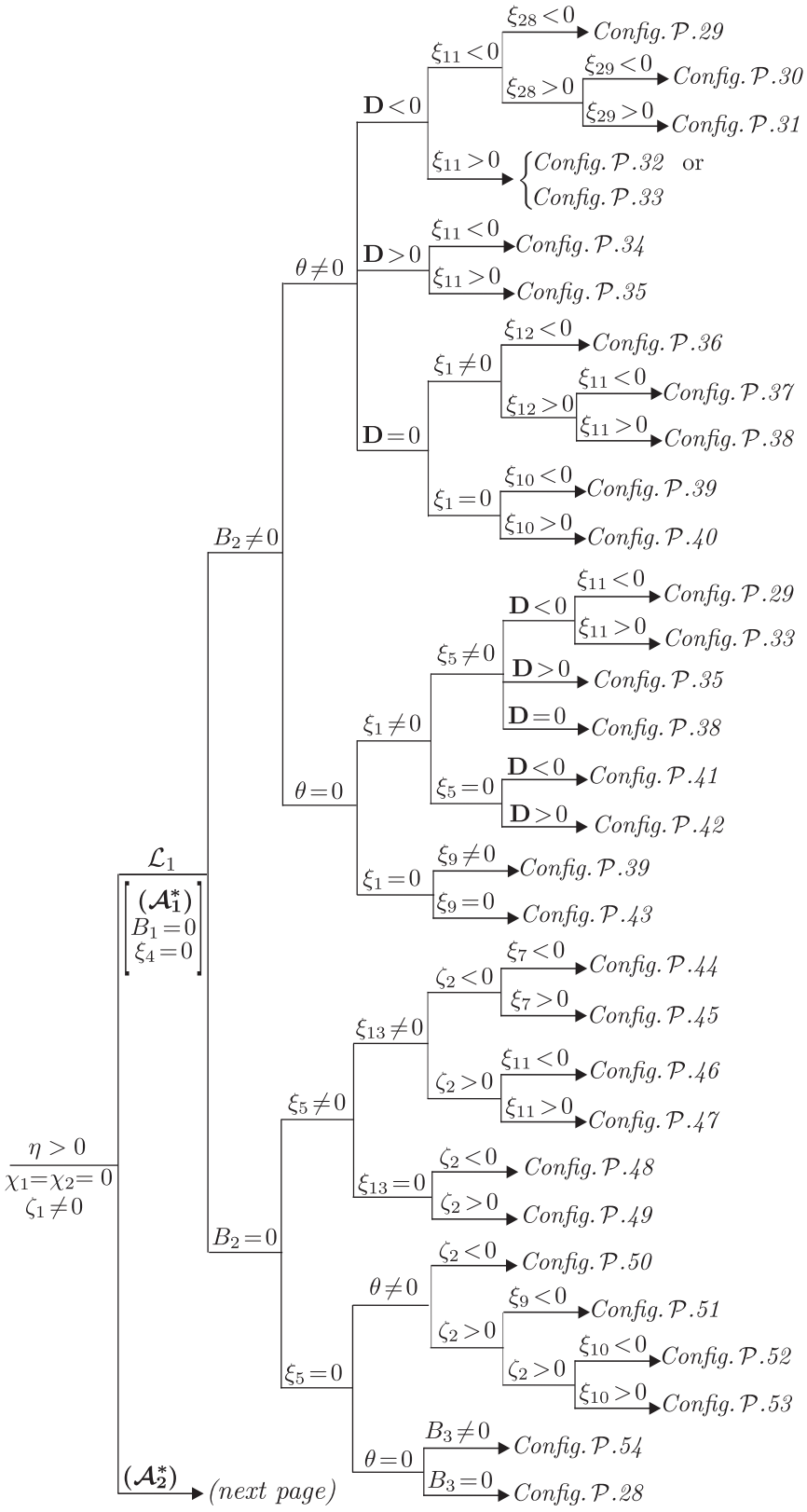


Diagram 1.1 *(continued):* Conditions for the configurations of systems in **QSP** in the case $\eta > 0$, $\zeta_1 \neq 0$.

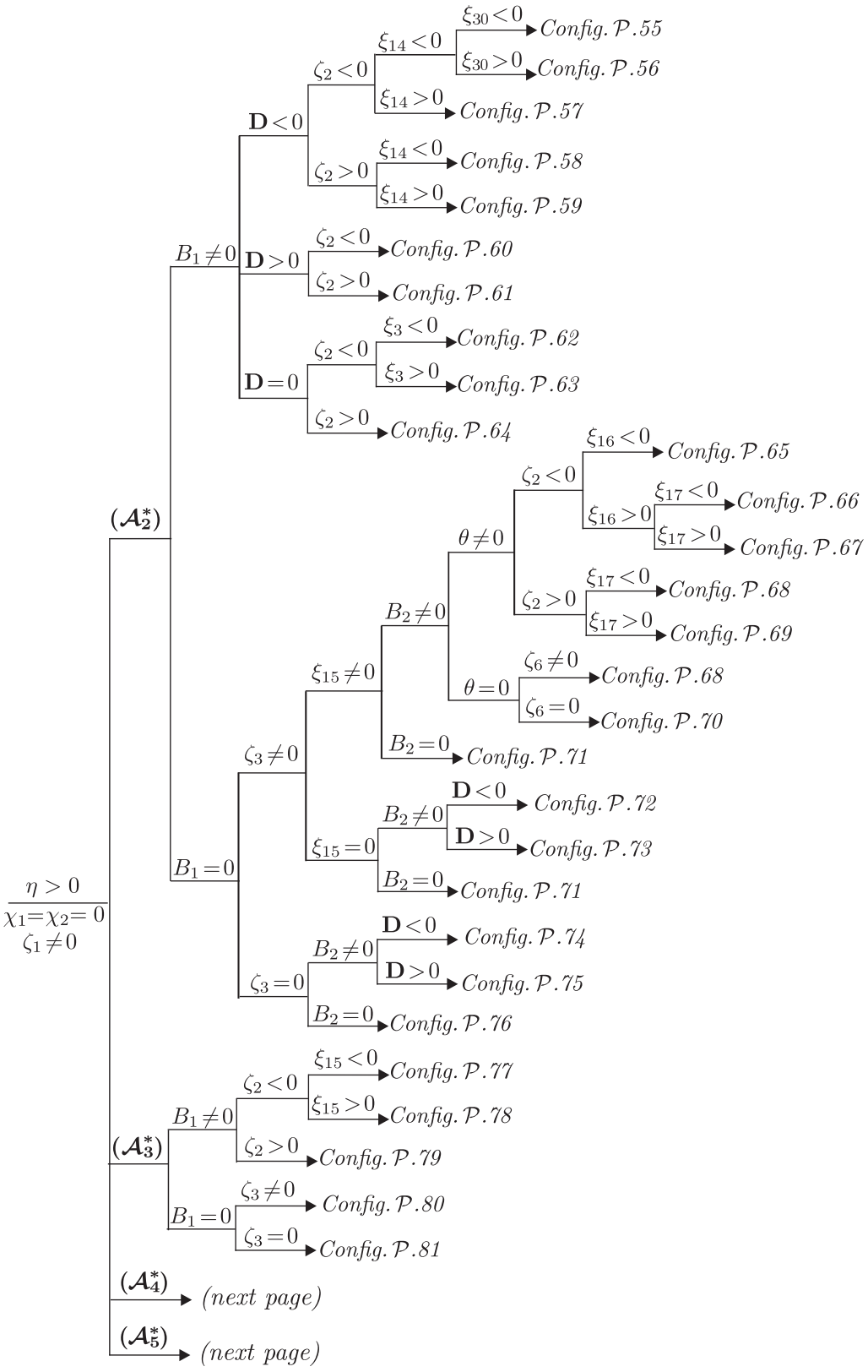


Diagram 1.1 *(continued):* Conditions for the configurations of systems in **QSP** in the case $\eta > 0$, $\zeta_1 \neq 0$.

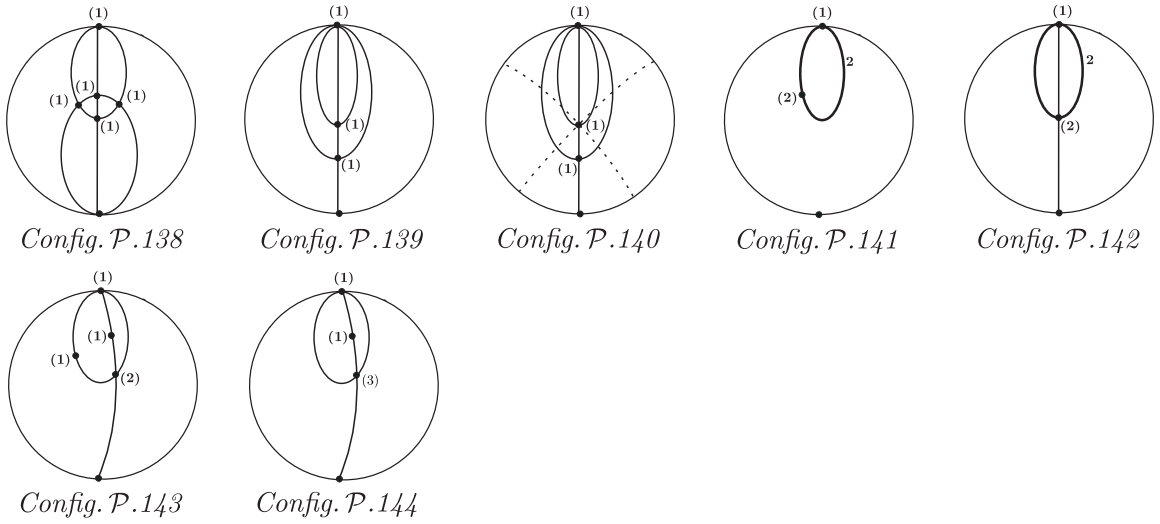


Figure 1.2 *(continued):* Configurations of systems in **QSP** in the case $\eta < 0$.

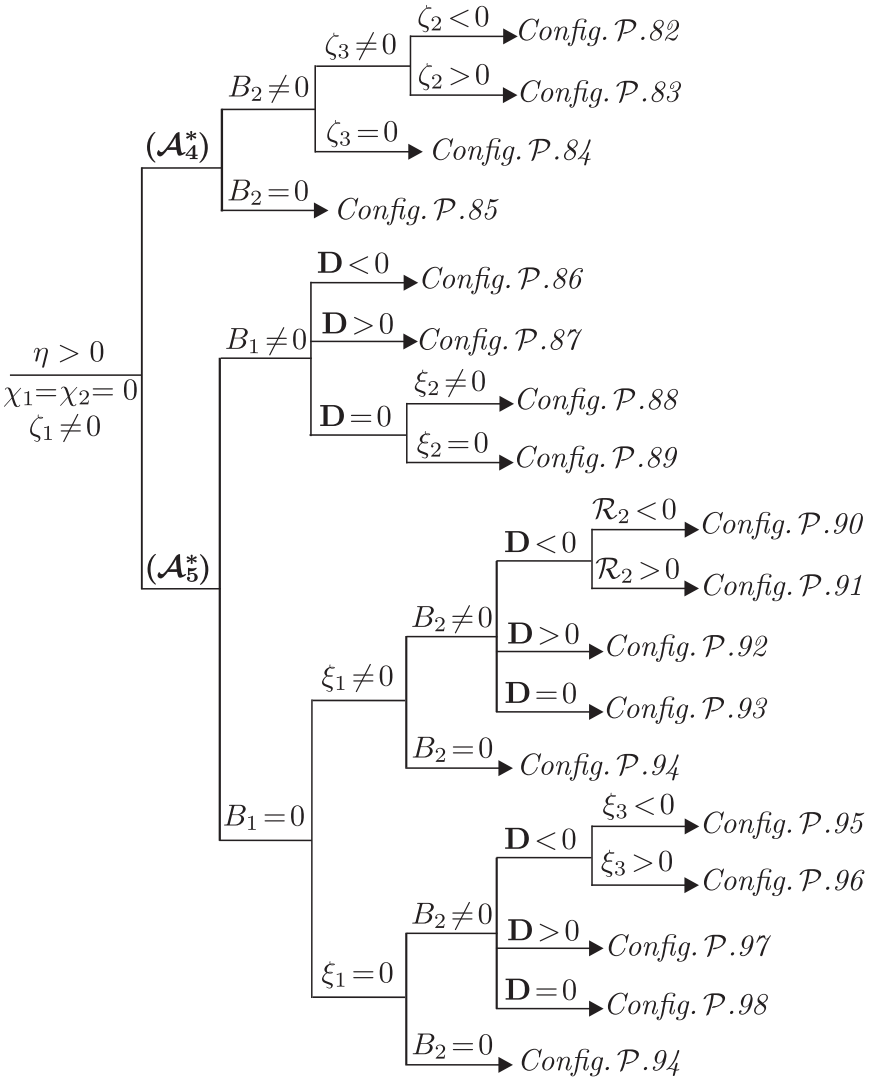


Diagram 1.1 *(continued):* Conditions for the configurations of systems in **QSP** in the case $\eta > 0$, $\zeta_1 \neq 0$.

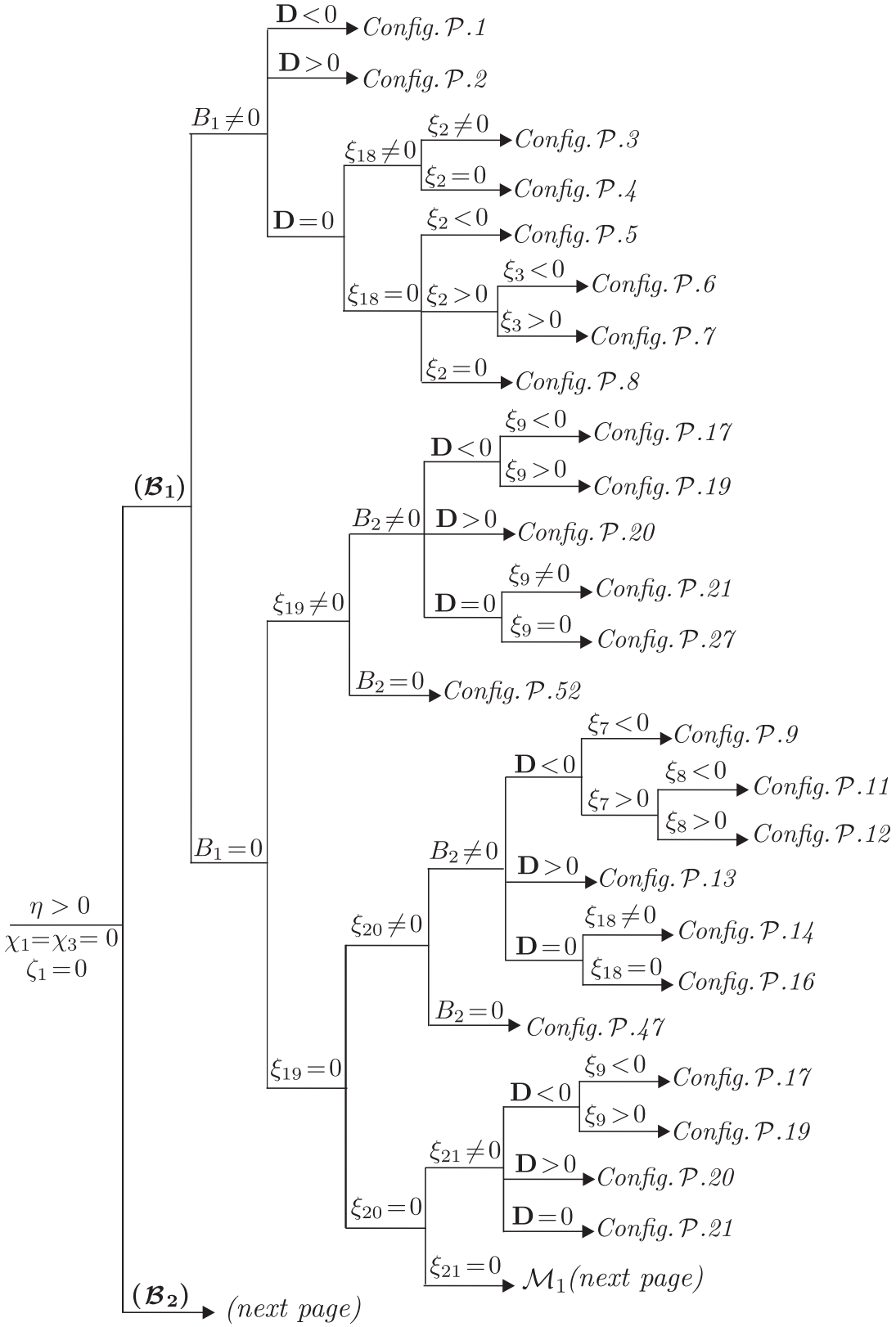


Diagram 1.2: Conditions for the configurations of systems in **QSP** in the case $\eta > 0$, $\zeta_1 = 0$.

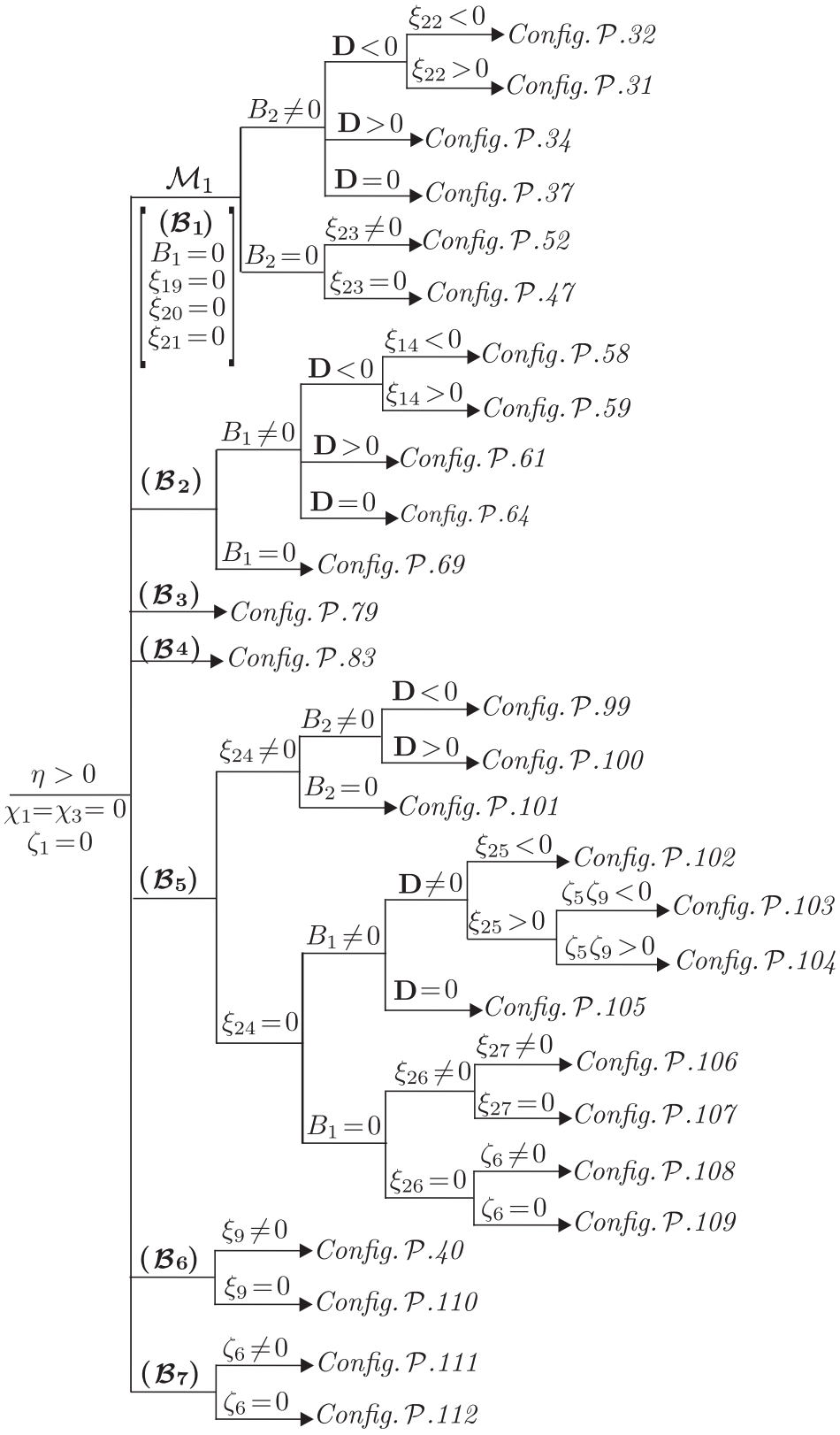


Diagram 1.2 *(continued):* Conditions for the configurations of systems in **QSP** in the case $\eta > 0$, $\zeta_1 = 0$.

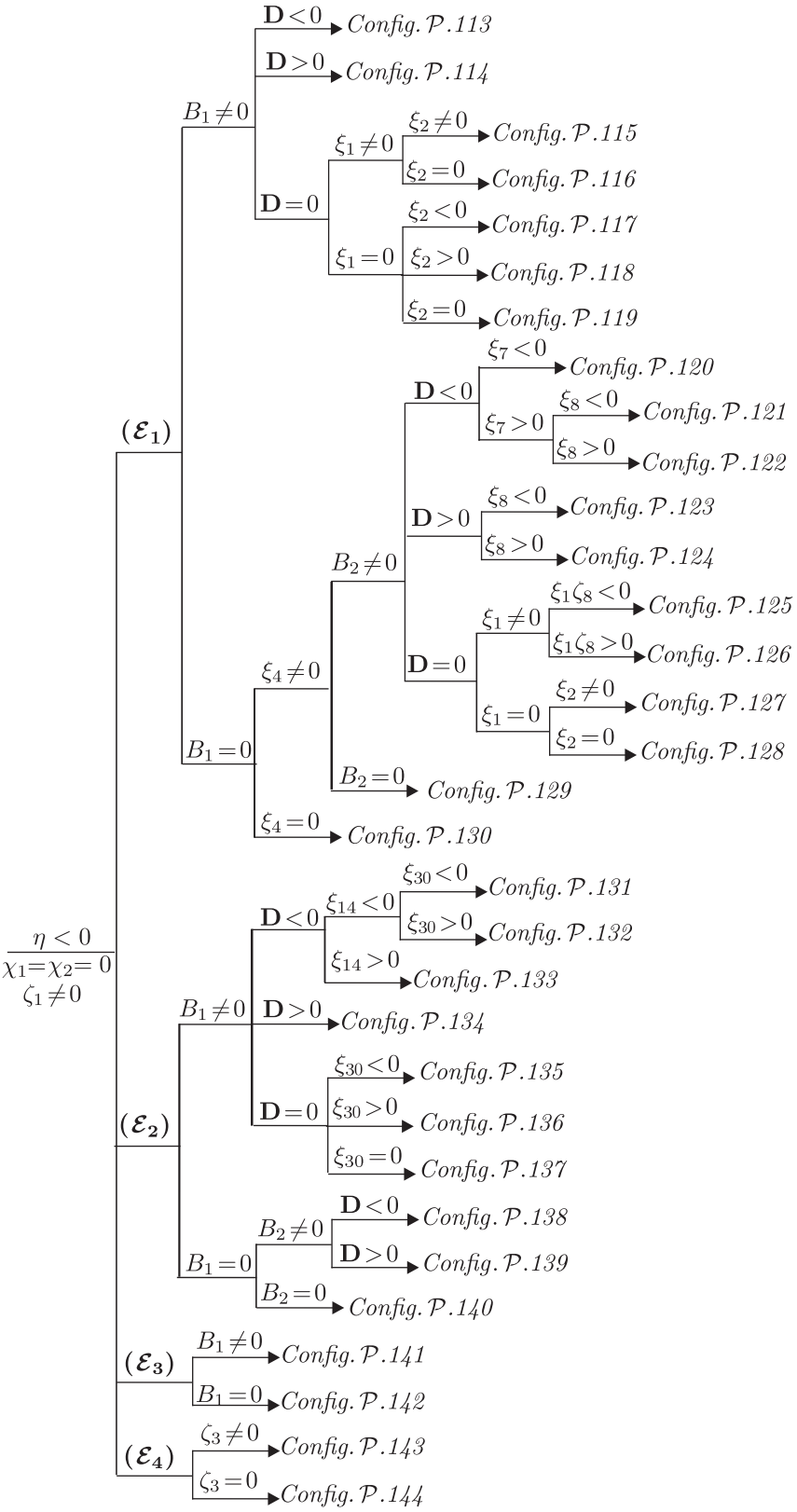


Diagram 1.3: Conditions for the configurations of systems in **QSP** in the case $\eta < 0$.

Remark 1.4. Every branch of the graphs in Diagrams 1.1, 1.2, and 1.3 terminates at a unique configuration *Config.* $\mathcal{P}.1$ through *Config.* $\mathcal{P}.144$, with a single exception. Specifically, in Diagram 1.1 (page 11), there is a branch that leads to an indeterminacy between two configurations: either *Config.* $\mathcal{P}.32$ or *Config.* $\mathcal{P}.33$. We are convinced that there exists an invariant polynomial capable of distinguishing between these two configurations, although we have not yet identified it. The determination of such an invariant remains an open problem.

2 Preliminaries

Consider real quadratic systems of the form:

$$\begin{aligned} \frac{dx}{dt} &= p_0 + p_1(x,y) + p_2(x,y) \equiv P(x,y), \\ \frac{dy}{dt} &= q_0 + q_1(x,y) + q_2(x,y) \equiv Q(x,y), \end{aligned} \tag{2.1}$$

where p_i and q_i $(i = 0,1,2)$ are homogeneous polynomials of degree i in x and y:

$$\begin{aligned} &p_0 = a_{00}, \quad && p_1(x,y) = a_{10}x + a_{01}y, \quad && p_2(x,y) = a_{20}x^2 + 2a_{11}xy + a_{02}y^2, \\ &q_0 = b_{00}, \quad && q_1(x,y) = b_{10}x + b_{01}y, \quad && q_2(x,y) = b_{20}x^2 + 2b_{11}xy + b_{02}y^2. \end{aligned}$$

Such a system (2.1) can be identified with a point in $\mathbb{R}^{12}$. Let

$$\tilde{a} = (a_{00}, a_{10}, a_{01}, a_{20}, a_{11}, a_{02}, b_{00}, b_{10}, b_{01}, b_{20}, b_{11}, b_{02}),$$

and consider the polynomial ring $\mathbb{R}[a_{00}, \ldots, a_{02}, b_{00}, \ldots, b_{02}, x, y]$, which we denote by $\mathbb{R}[\tilde{a}, x, y]$.

It is known that the group $Aff(2,\mathbb{R})$ of affine transformations of the plane acts on the set **QS** of all quadratic differential systems (2.1) (cf. [20]). For every subgroup $G \subseteq Aff(2,\mathbb{R})$, there is an induced action of G on **QS**.

We can identify the set **QS** with a subset of $\mathbb{R}^{12}$ via the map **QS** $\to \mathbb{R}^{12}$, which associates to each system (2.1) the 12-tuple $\tilde{a} = (a_{00}, \ldots, b_{02})$ of its coefficients.

To study this group action, we associate to the systems certain polynomials in x, y and the system parameters that transform in a controlled way under the action. These are known as GL-comitants, T-comitants, and CT-comitants. For detailed definitions and constructions of these comitants, we refer the reader to [20] (see also [1]).

2.1 The main invariant polynomials associated to invariant parabolas

We single out the following five polynomials, which serve as basic ingredients in constructing invariant polynomials for systems (2.1):

$$\begin{aligned} C_i(\tilde{a}, x, y) &= y p_i(x,y) - x q_i(x,y), \quad (i = 0,1,2), \\ D_i(\tilde{a}, x, y) &= \frac{\partial p_i}{\partial x} + \frac{\partial q_i}{\partial y}, \quad (i = 1,2). \end{aligned} \tag{2.2}$$

As shown in [23], these polynomials, which are linear in the coefficients of systems (2.1), are GL-comitants of these systems.

For $f, g \in \mathbb{R}[\tilde{a}, x, y]$, we define the *transvectant of index* k of (f,g) by

$$(f,g)^{(k)} = \sum_{h=0}^{k} (-1)^h \binom{k}{h} \frac{\partial^k f}{\partial x^{k-h} \partial y^h} \cdot \frac{\partial^k g}{\partial x^h \partial y^{k-h}}.$$

The resulting polynomial $(f,g)^{(k)}$ belongs to $\mathbb{R}[\tilde{a}, x, y]$ (cf. [12, 17]).

Proposition 2.1 (see [25]). *Any GL-comitant of systems (2.1) can be constructed from the elements (2.2) by using the operations $+$, $-$, $\times$, and by applying the differential operation $(*,*)^{(k)}$.*

Remark 2.2. We point out that the elements (2.2) generate the entire set of GL-comitants, and consequently also the full set of affine comitants and T-comitants, since any affine comitant and any T-comitant can be constructed from GL-comitants using the same operations: $+$, $-$, $\times$, and $(*,*)^{(k)}$.

We construct the following GL-comitants of second degree with respect to the coefficients of the initial systems:

$$\begin{aligned} &T_1 = (C_0, C_1)^{(1)}, \quad && T_2 = (C_0, C_2)^{(1)}, \quad && T_3 = (C_0, D_2)^{(1)}, \\ &T_4 = (C_1, C_1)^{(2)}, \quad && T_5 = (C_1, C_2)^{(1)}, \quad && T_6 = (C_1, C_2)^{(2)}, \\ &T_7 = (C_1, D_2)^{(1)}, \quad && T_8 = (C_2, C_2)^{(2)}, \quad && T_9 = (C_2, D_2)^{(1)}. \end{aligned} \tag{2.3}$$

Using these GL-comitants, together with the polynomials defined in (2.2), we construct additional invariant polynomials. To enable the direct computation of the required invariant polynomials for each canonical system, we now define a family of T-comitants expressed in terms of the polynomials C_i $(i = 0, 1, 2)$ and D_j $(j = 1, 2)$:

$$\begin{aligned}
\hat{A} &= \left(C_1, T_8 - 2T_9 + D_2^2\right)^{(2)}/144, \\
\widehat{D} &= \left[2C_0(T_8 - 8T_9 - 2D_2^2) + C_1(6T_7 - T_6 - (C_1, T_5)^{(1)} + 6D_1(C_1D_2 - T_5) - 9D_1^2C_2\right]/36, \\
\widehat{E} &= \left[D_1(2T_9 - T_8) - 3\,(C_1, T_9)^{(1)} - D_2(3T_7 + D_1D_2)\right]/72, \\
\widehat{F} &= \left[6D_1^2(D_2^2 - 4T_9) + 4D_1D_2(T_6 + 6T_7) + 48C_0\,(D_2, T_9)^{(1)} - 9D_2^2T_4 + 288D_1\widehat{E}\right. \\
&\quad \left. - 24\left(C_2, \widehat{D}\right)^{(2)} + 120\left(D_2, \widehat{D}\right)^{(1)} - 36C_1\,(D_2, T_7)^{(1)} + 8D_1\,(D_2, T_5)^{(1)}\right]/144, \\
\widehat{K} &= (T_8 + 4T_9 + 4D_2^2)/72, \\
\widehat{H} &= (8T_9 - T_8 + 2D_2^2)/72, \\
\widehat{B} &= \Big\{16D_1\,(D_2, T_8)^{(1)}\,(3C_1D_1 - 2C_0D_2 + 4T_2) + 32C_0\,(D_2, T_9)^{(1)}\,(3D_1D_2 - 5T_6 + 9T_7) \\
&\quad + 2\,(D_2, T_9)^{(1)}\,\left(27C_1T_4 - 18C_1D_1^2 - 32D_1T_2 + 32\,(C_0, T_5)^{(1)}\right) \\
&\quad + 6\,(D_2, T_7)^{(1)}\,[8C_0(T_8 - 12T_9) - 12C_1(D_1D_2 + T_7) \\
&\quad + D_1(26C_2D_1 + 32T_5) + C_2(9T_4 + 96T_3)] \\
&\quad + 6\,(D_2, T_6)^{(1)}\,[32C_0T_9 - C_1(12T_7 + 52D_1D_2) - 32C_2D_1^2] + 48D_2\,(D_2, T_1)^{(1)}\,\left(2D_2^2 - T_8\right) \\
&\quad - 32D_1T_8\,(D_2, T_2)^{(1)} + 9D_2^2T_4\,(T_6 - 2T_7) - 16D_1\,(C_2, T_8)^{(1)}\,\left(D_1^2 + 4T_3\right) \\
&\quad + 12D_1\,(C_1, T_8)^{(2)}\,(C_1D_2 - 2C_2D_1) + 6D_1D_2T_4\,\left(T_8 - 7D_2^2 - 42T_9\right) \\
&\quad + 12D_1\,(C_1, T_8)^{(1)}\,(T_7 + 2D_1D_2) + 96D_2^2\left[D_1\,(C_1, T_6)^{(1)} + D_2\,(C_0, T_6)^{(1)}\right] - \\
&\quad - 16D_1D_2T_3\,\left(2D_2^2 + 3T_8\right) - 4D_1^3D_2\,\left(D_2^2 + 3T_8 + 6T_9\right) + 6D_1^2D_2^2\,(7T_6 + 2T_7) \\
&\quad - 252D_1D_2T_4T_9\Big\}/(2^8 3^3),
\end{aligned}$$

These polynomials, together with those defined in (2.2) and (2.3), will serve as fundamental building blocks for constructing affine invariant polynomials for systems (2.1).

The following 42 affine invariants, labeled $A_1, \ldots, A_{42}$, constitute a minimal polynomial basis of affine invariants up to degree 12. This result was established in [4], where the invariants $A_1, \ldots, A_{42}$ were explicitly constructed using the aforementioned building blocks:

$$
\begin{aligned}
A_1 &= \hat{A},\\
A_2 &= (C_2, \widehat{D})^{(3)}/12,\\
A_3 &= \left[C_2, D_2)^{(1)}, D_2\right)^{(1)}, D_2\right)^{(1)}/48,\\
A_4 &= (\widehat{H}, \widehat{H})^{(2)},\\
A_5 &= (\widehat{H}, \widehat{K})^{(2)}/2,\\
A_6 &= (\widehat{E}, \widehat{H})^{(2)}/2,\\
A_7 &= \left[C_2, \widehat{E})^{(2)}, D_2\right)^{(1)}/8,\\
A_8 &= \left[\widehat{D}, \widehat{H})^{(2)}, D_2\right)^{(1)}/8,\\
A_9 &= \left[\widehat{D}, D_2)^{(1)}, D_2\right)^{(1)}, D_2\right)^{(1)}/48,\\
A_{10} &= \left[\widehat{D}, \widehat{K})^{(2)}, D_2\right)^{(1)}/8,\\
A_{11} &= (\widehat{F}, \widehat{K})^{(2)}/4,\\
A_{12} &= (\widehat{F}, \widehat{H})^{(2)}/4,\\
A_{13} &= \left[C_2, \widehat{H})^{(1)}, \widehat{H}\right)^{(2)}, D_2\right)^{(1)}/24,\\
A_{14} &= (\widehat{B}, C_2)^{(3)}/36,\\
A_{15} &= (\widehat{E}, \widehat{F})^{(2)}/4,\\
A_{16} &= \left[\widehat{E}, D_2)^{(1)}, C_2\right)^{(1)}, \widehat{K}\right)^{(2)}/16,\\
A_{17} &= \left[\widehat{D}, \widehat{D})^{(2)}, D_2\right)^{(1)}, D_2\right)^{(1)}/64,\\
A_{18} &= \left[\widehat{D}, \widehat{F})^{(2)}, D_2\right)^{(1)}/16,\\
A_{19} &= \left[\widehat{D}, \widehat{D})^{(2)}, \widehat{H}\right)^{(2)}/16,\\
A_{20} &= \left[C_2, \widehat{D})^{(2)}, \widehat{F}\right)^{(2)}/16,\\
A_{21} &= \left[\widehat{D}, \widehat{D})^{(2)}, \widehat{K}\right)^{(2)}/16,\\
A_{22} &= \left[C_2, \widehat{D})^{(1)}, D_2\right)^{(1)}, D_2\right)^{(1)}, D_2\right)^{(1)} D_2\right)^{(1)}/1152,\\
A_{23} &= \left[\widehat{F}, \widehat{H})^{(1)}, \widehat{K}\right)^{(2)}/8,\\
A_{24} &= \left[C_2, \widehat{D})^{(2)}, \widehat{K}\right)^{(1)}, \widehat{H}\right)^{(2)}/32,\\
A_{25} &= \left[\widehat{D}, \widehat{D})^{(2)}, \widehat{E}\right)^{(2)}/16,\\
A_{26} &= (\widehat{B}, \widehat{D})^{(3)}/36,\\
A_{27} &= \left[\widehat{B}, D_2)^{(1)}, \widehat{H}\right)^{(2)}/24,\\
A_{28} &= \left[C_2, \widehat{K})^{(2)}, \widehat{D}\right)^{(1)}, \widehat{E}\right)^{(2)}/16,\\
A_{29} &= \left[\widehat{D}, \widehat{F})^{(1)}, \widehat{D}\right)^{(3)}/96,\\
A_{30} &= \left[C_2, \widehat{D})^{(2)}, \widehat{D}\right)^{(1)}, \widehat{D}\right)^{(3)}/288,\\
A_{31} &= \left[\widehat{D}, \widehat{D})^{(2)}, \widehat{K}\right)^{(1)}, \widehat{H}\right)^{(2)}/64,
\end{aligned}
$$

$$
\begin{aligned}
A_{32} &= \big[\widehat{D},\widehat{D})^{(2)},D_2\big)^{(1)},\widehat{H}\big)^{(1)},D_2\big)^{(1)}/64,\\
A_{33} &= \big[\widehat{D},D_2)^{(1)},\widehat{F}\big)^{(1)},D_2\big)^{(1)},D_2\big)^{(1)}/128,\\
A_{34} &= \big[\widehat{D},\widehat{D})^{(2)},D_2\big)^{(1)},\widehat{K}\big)^{(1)},D_2\big)^{(1)}/64,\\
A_{35} &= \big[\widehat{D},\widehat{D})^{(2)},\widehat{E}\big)^{(1)},D_2\big)^{(1)},D_2\big)^{(1)}/128,\\
A_{36} &= \big[\widehat{D},\widehat{E})^{(2)},\widehat{D}\big)^{(1)},\widehat{H}\big)^{(2)}/16,\\
A_{37} &= \big[\widehat{D},\widehat{D})^{(2)},\widehat{D}\big)^{(1)},\widehat{D}\big)^{(3)}/576,\\
A_{38} &= \big[C_2,\widehat{D})^{(2)},\widehat{D}\big)^{(2)},\widehat{D}\big)^{(1)},\widehat{H}\big)^{(2)}/64,\\
A_{39} &= \big[\widehat{D},\widehat{D})^{(2)},\widehat{F}\big)^{(1)},\widehat{H}\big)^{(2)}/64,\\
A_{40} &= \big[\widehat{D},\widehat{D})^{(2)},\widehat{F}\big)^{(1)},\widehat{K}\big)^{(2)}/64,\\
A_{41} &= \big[C_2,\widehat{D})^{(2)},\widehat{D}\big)^{(2)},\widehat{F}\big)^{(1)},D_2\big)^{(1)}/64,\\
A_{42} &= \big[\widehat{D},\widehat{F})^{(2)},\widehat{F}\big)^{(1)},D_2\big)^{(1)}/16.
\end{aligned}
$$

In the above list, the bracket "[" is used as a typographical device to avoid writing up to five consecutive parentheses "(" in some of the expressions.

Using the elements of the minimal polynomial basis listed above, we construct two groups of affine invariant polynomials. The first group contains invariant polynomials associated with the existence of an invariant parabola for a quadratic system, and they are:

$$
\begin{aligned}
\chi_1 &= 32A_3 + 45A_4 - 160A_5;\\
\chi_2 &= 32A_8(14A_8 - 48A_9 + 37A_{10} + 24A_{11}) + 16A_5(76A_{17} + 74A_{18} + 313A_{19} - 80A_{20}\\
&\quad - 167A_{21}) + A_4(160A_2^2 + 368A_{18} - 3363A_{19} + 736A_{20} + 2109A_{21}) + 32(17A_{10}^2\\
&\quad + 27A_{10}A_{11} + 24A_{11}^2 - 48A_9A_{12} + 51A_{10}A_{12} + 24A_{11}A_{12} + 288A_6A_{14} - 96A_7A_{14});\\
\chi_3 &= 6520480A_{20}(407A_{18} - 2253A_{21}) + 24A_{18}(1057715458A_{19} + 5944853225A_{21})\\
&\quad + 28800A_{14}(1872476A_{25} - 122259A_{26}) + 144A_{12}(3620283092A_{29} - 1554910481A_{30})\\
&\quad + 1440A_{15}(107225339A_{25} - 19561440A_{26}) - 72A_{11}(8198511476A_{29} - 2965514443A_{30})\\
&\quad + 652048(4544A_{18}^2 + 125A_{20}^2 - 8955A_2A_{42}) - 9(264364688A_{19}^2 + 39417454842A_{19}A_{21}\\
&\quad - 54474141921A_{21}^2) + 3448898760A_{19}A_{20};\\
\chi_4 &= 62713A_{10}^2 + 45787A_{10}A_{11} - 157928A_{11}^2 + 81202A_{10}A_{12} + A_{19}353474A_{11}A_{12} - 145848A_{12}^2\\
&\quad + 64320A_7A_{15} + 28600A_5A_{17};\\
\zeta_1 &= 13A_4 - 24A_5;\\
\zeta_2 &= -A_4;\\
\zeta_3 &= 16A_5 - 17A_4;\\
\zeta_4 &= 9A_1A_4 - 7A_1A_5 - 2A_{16};\\
\zeta_5 &= 166A_8 + 384A_9 - 1120A_{10} - 512A_{11} - 62A_{12};\\
\zeta_6 &= A_6;\\
\zeta_7 &= 40(71436A_7A_{20} - 640883A_7A_{21} + 1008622A_1A_{32}) + 12A_{12}(3585035A_{14} + 14919259A_{15})\\
&\quad - 5(8092193A_{10} + 15970731A_{11})A_{14} - (129780821A_{10} + 269944167A_{11})A_{15};\\
\zeta_8 &= A_2;
\end{aligned}
$$

$$\begin{aligned}
\zeta_9 &= 1040(2256A_7A_{15}+143A_3A_{21})-264(162941A_{10}+315202A_{11})A_{12}\\
&\quad +3A_{11}(25887132A_{10}+24385177A_{11})+20603609A_{10}^2+24896016A_{12}^2;\\
\zeta_{10} &= 250A_1^2+34A_{11}-41A_{12};\\
\mathcal{R}_1 &= 531A_2A_4-1472A_2A_5-8352A_1A_6+320A_{22}-3216A_{23}+1488A_{24};\\
\mathcal{R}_2 &= 15A_{10}-10A_8-6A_9;\\
\mathcal{R}_3 &= 4800(6650951968A_{14}A_{15}-2382132830A_{14}^2-9860550485A_{15}^2)+1600(4765089473A_{11}\\
&\quad -7838161089A_{12})A_{20}+640(15664652914A_{11}-50944340271A_{12})A_{18}\\
&\quad -6(20392663986679A_{10}+34357804389813A_{11}-739275727012A_{12})A_{21}\\
&\quad +3(46944212550227A_{10}+83455057317969A_{11}-22899810934956A_{12})A_{19};\\
\mathcal{R}_4 &= 251A_1^2+25A_{12};\\
\mathcal{R}_7 &= 62250A_1^2+8956A_9-46223A_{10}-50129A_{11}+14766A_{12}.
\end{aligned}$$

The invariant polynomials from the second group are responsible for the classification of the configurations of invariant parabolas and lines. They are:

$$\begin{aligned}
\xi_1 &= 342A_1^2A_2+A_2(35A_{10}-15A_8-16A_9+97A_{11}-83A_{12})-48A_1(4A_{14}+3A_{15})\\
&\quad +16(2A_{32}+A_{33}-3A_{34})+90A_{31};\\
\xi_2 &= -A_{19};\\
\xi_3 &= 12(49836514A_8^2-40804544A_8A_9-63384469A_8A_{10}-4515985A_{10}^2+93824435A_8A_{11}\\
&\quad -23552547A_{10}A_{11}+51595312A_{11}^2+202411827A_1^2A_{12})-763176315A_4A_{21}\\
&\quad -16(30603408A_9A_{12}+10917387A_7A_{14}+14011860A_7A_{15}-75865539A_5A_{17}\\
&\quad -115398446A_5A_{18}-54568383A_5A_{21})-4(86656770A_6A_{14}+404823654A_6A_{15}\\
&\quad -68396637A_5A_{19}+25391678A_5A_{20})-6A_{12}(154041735A_8+47473233A_{10}\\
&\quad -170661233A_{11}+202411827A_{12});\\
\xi_4 &= 800(175A_2A_5A_7-336A_1A_3A_8-16500A_{13}A_{14}-9300A_{13}A_{15}-47001A_6A_{22}\\
&\quad +39861A_7A_{23}-3150A_6A_{24}-10242A_7A_{24}+168792A_5A_{28})+240(173478A_8A_{16}\\
&\quad +128774A_{10}A_{16}+151602A_{11}A_{16}+134102A_{12}A_{16}+8799A_4A_{27}-134102A_5A_{27})\\
&\quad -1879552(3A_9A_{16}-A_7A_{22})+75(50400A_6A_{23}-646151A_4A_{28});\\
\xi_5 &= 2000(802A_{13}A_{14}+315A_6A_{23}-210A_6A_{24})+320(28A_1A_3A_{11}-13757A_8A_{16}\\
&\quad -11282A_{12}A_{16}+3336A_7A_{24}+11282A_5A_{27})+80(16038A_{13}A_{15}-30398A_{10}A_{16}\\
&\quad -36154A_{11}A_{16}+46738A_6A_{22}-45142A_7A_{23}-162339A_5A_{28})+151552(3A_9A_{16}\\
&\quad -A_7A_{22})-15A_4(28392A_{27}-313721A_{28});\\
\xi_6 &= 1536(16671538A_7A_{14}-5655800A_{11}^2-5655800A_{11}A_{12}-134975925A_6A_{15}\\
&\quad +14236220A_7A_{15})+128(42330182A_8A_9+279065017A_8A_{11}-857954A_8A_{12}\\
&\quad +138313062A_9A_{12}-633595086A_6A_{14}-35417298A_5A_{20})+64(171565045A_8^2\\
&\quad +343921603A_5A_{17})-32(1111806317A_8A_{10}+256225409A_{10}^2+874265715A_{10}A_{11}\\
&\quad +2536914399A_{10}A_{12}-936841383A_5A_{18})-16A_5(2168875001A_{19}+1048355233A_{21})\\
&\quad +A_4(26458433203A_{19}-4734012269A_{21});\\
\xi_7 &= -A_4\big[3200A_{12}(14657A_8-1615148A_{10}+318175A_{11})-640(388968A_9^2-7748782A_{10}^2\\
&\quad -592379A_9A_{12})-160(13079737A_8A_{10}-27509045A_8A_{11}-63353923A_9A_{11}\\
&\quad -16215395A_{10}A_{11}-36662125A_{11}^2)+4121433952A_9A_{10}\big];
\end{aligned}$$

$$
\begin{aligned}
\xi_8 = & -A_4\big[512A_9(1275434A_{10}+2193137A_{11}-170333A_{12})-1280(30087A_9^2+424036A_{10}^2\\
&+1052798A_{10}A_{11}+48550A_{11}^2+61603A_8A_{12})-640(608587A_8A_{10}+248041A_8A_{11}\\
&+430261A_{10}A_{12}+525475A_{11}A_{12})\big];\\
\xi_9 = & -A_4\big[48(675908847A_8A_9+1141726617A_9A_{12}+7216376855A_{10}A_{12}\\
&-4015621128A_6A_{14}+3915909450A_7A_{15})-12(16745223889A_8^2+5997051735A_8A_{11}\\
&-26372062499A_{10}A_{11}+2601951027A_8A_{12}-7916516650A_7A_{14}-30105649725A_6A_{15}\\
&+20512413539A_5A_{17}-1497206278A_4A_{19}-4791714129A_4A_{21})+2(220220676003A_8A_{10}\\
&+58687175103A_{10}^2+14685562719A_{11}^2+9716839839A_{11}A_{12}-219193688911A_5A_{18}\\
&-4467110471A_5A_{20})+3A_5(36033875127A_{19}-37652431103A_{21})\big];\\
\xi_{10} = & A_4\big[48(568199091031A_8A_9-248186616391A_9A_{10}+314207594667A_9A_{11}\\
&+5804879973A_9A_{12}-3905825755777A_{10}A_{12}-2095407390920A_6A_{14}\\
&-546799764750A_7A_{15})+12(6550908482493A_8^2-3402501855145A_8A_{11}\\
&-3448022811579A_{10}A_{11}+2284925158471A_8A_{12}+2482932379806A_7A_{14}\\
&-11017448610465A_6A_{15}+5894909506479A_5A_{17})-2(131290745988327A_8A_{10}\\
&-17334476527245A_{10}^2-11980168965A_{11}^2+21428060568795A_{11}A_{12}\\
&-62352140313275A_5A_{18}+3924064256285A_5A_{20})-3(2258722903315A_5A_{19}\\
&+9533558573843A_4A_{21}-10218122423819A_5A_{21})\big];\\
\xi_{11} = & \zeta_1\zeta_2\xi_6;\\
\xi_{12} = & 1288A_1^2+117A_{10}+351A_{11}-352A_{12};\\
\xi_{13} = & 61A_2^2-20A_{17}-8A_{18}+24A_{19}-28A_{20}+12A_{21};\\
\xi_{14} = & 9854A_{11}-3005A_8-3296A_9+13578A_{10}-991A_{12};\\
\xi_{15} = & 8A_5-9A_4;\\
\xi_{16} = & (525A_8-4448A_9+10554A_{10}-1378A_{11}+8087A_{12});\\
\xi_{17} = & 10005A_8+9856A_9-38348A_{10}-27404A_{11}+8371A_{12};\\
\xi_{18} = & 2240(15452233775A_{14}^2+742923092360A_{14}A_{15}-145263086200A_{15}^2\\
&+10151798384A_{11}A_{18}-68919094926A_{12}A_{18}-14663220305A_{11}A_{20}\\
&+7194838365A_{12}A_{20})+16A_{19}(88266907919051A_8+12824946044853A_{11}\\
&+119819326860153A_{12})-7A_{21}(1380736713246 37A_{10}+258358507987439A_{11}\\
&-32813284182036A_{12});\\
\xi_{19} = & 429A_9(629A_{10}+1275A_{11}-900A_{12})+100(2145A_8A_{11}-1595A_5A_{17}-2970A_5A_{18}\\
&+2886A_2A_{23}-559A_2A_{24});\\
\xi_{20} = & 4A_2(47A_2^2-468A_{18}+3478A_{19}+9A_{20})-9189A_2A_{21}+12(-682A_1A_{25}+2592A_1A_{26}\\
&+395A_{38}+35A_{39});\\
\xi_{21} = & 24(675906A_{40}-672409A_{39}+6578A_{41}+110106A_{42})-73404A_2(74A_{18}+A_{20})\\
&+4(99911A_2^3-2048846A_{38})-15133791A_2A_{21};\\
\xi_{22} = & 84A_{12}-68A_{10}-141A_{11};\\
\xi_{23} = & 5A_8-3A_9;\\
\xi_{24} = & 625A_2(12A_2A_3-775A_1A_6)-62(13500A_1^2+275A_8-276A_9)A_9+10A_3(2561A_{17}\\
&+3240A_{18}+2550A_{19});
\end{aligned}
$$

$$
\begin{aligned}
\xi_{25} =\ & -(46A_{18}+537A_{19}+134A_{20});\\
\xi_{26} =\ & 41A_1A_2+16A_{14}-18A_{15};\\
\xi_{27} =\ & A_1;\\
\xi_{28} =\ & 64(72137434664A_8^2+3322490880A_9^2-58216412276A_{10}^2-217656099219A_{10}A_{11}\\
& -63098236389A_{11}^2-250756327503A_{10}A_{12}-71858710389A_{11}A_{12}+96A_9(449920640A_{11}\\
& +1009660963A_{12})+6A_8(21795888048A_9-66020231422A_{10}-21118997424A_{11}\\
& +2573485725A_{12}))-384(62739943233A_6A_{14}-27065693406A_7A_{14}+7592410800A_6A_{15}\\
& -10442342780A_7A_{15})+A_4(2998959134256A_{17}+4635359414448A_{18}+1132776129074A_{19}\\
& -1187818900002A_{20}-5542617623395A_{21})+32A_3(19078937382A_{20}+81853956367A_{21});\\
\xi_{29} =\ & 497213324620A_8^2-1001736600522A_{10}^2-870653569536A_9A_{11}+337754949134A_{11}^2\\
& +A_8(2170429037822A_{10}-1858453397512A_9+2112595332132A_{11}-304022217484A_{12})\\
& -987799827976A_9A_{12}+949933240214A_{11}A_{12}+A_{10}(-648979472052A_{11}\\
& +956487534504A_{12})-4(125652578829A_6A_{14}+240347919318A_7A_{14}\\
& -775425835368A_6A_{15}+405563103412A_7A_{15})-A_4(197626785161A_{20}\\
& +1540932760870A_{21})+A_5(1910970964424A_{17}+2668708281714A_{18}+182967974851A_{19}\\
& +280452031438A_{20}+2136843181298A_{21});\\
\xi_{30} =\ & 3512A_{10}-1695A_8-544A_9+4576A_{11}-3329A_{12}.
\end{aligned}
$$

2.2 Results involving the use of polynomial invariants

A few more definitions and results, which play an important role in the proof of part (A) of the Main Theorem, are needed. We do not prove these results here but indicate where they can be found.

Consider the differential operator $\mathcal{L} = x \cdot L_2 - y \cdot L_1$, introduced in [3], acting on $\mathbb{R}[\tilde{a}, x, y]$, where the operators L_1 and L_2 are given by

$$
\begin{aligned}
L_1 &= 2a_{00}\frac{\partial}{\partial a_{10}} + a_{10}\frac{\partial}{\partial a_{20}} + \frac{1}{2}a_{01}\frac{\partial}{\partial a_{11}} + 2b_{00}\frac{\partial}{\partial b_{10}} + b_{10}\frac{\partial}{\partial b_{20}} + \frac{1}{2}b_{01}\frac{\partial}{\partial b_{11}},\\
L_2 &= 2a_{00}\frac{\partial}{\partial a_{01}} + a_{01}\frac{\partial}{\partial a_{02}} + \frac{1}{2}a_{10}\frac{\partial}{\partial a_{11}} + 2b_{00}\frac{\partial}{\partial b_{01}} + b_{01}\frac{\partial}{\partial b_{02}} + \frac{1}{2}b_{10}\frac{\partial}{\partial b_{11}}.
\end{aligned}
$$

Using the differential operator $\mathcal{L}$ introduced above and the affine invariant

$$
\mu_0 = \frac{\operatorname{Res}_x\left(p_2(\tilde{a}, x, y), q_2(\tilde{a}, x, y)\right)}{y^4},
$$

we construct the following family of polynomials:

$$
\mu_i(\tilde{a}, x, y) = \frac{1}{i!}\mathcal{L}^{(i)}(\mu_0), \quad i = 1, \ldots, 4,
$$

where the iterated operator $\mathcal{L}^{(i)}$ is defined recursively by

$$
\mathcal{L}^{(i)}(\mu_0) = \mathcal{L}\big(\mathcal{L}^{(i-1)}(\mu_0)\big), \quad \text{with } \mathcal{L}^{(0)}(\mu_0) = \mu_0.
$$

These polynomials μ_i are $GL(2,\mathbb{R})$-comitants of the quadratic systems (2.1), as established in [3]. Their geometric interpretation is detailed in Lemma 5.2 of [1].

From these invariant polynomials, one constructs the affine invariant polynomials **D** and **R**, which characterize the existence of multiple finite singularities in quadratic differential systems:

$$\mathbf{D} = \frac{1}{48}\left[3\big((\mu_3,\mu_3)^{(2)},\mu_2\big)^{(2)} - \big(6\mu_0\mu_4 - 3\mu_1\mu_3 + \mu_2^2,\mu_4\big)^{(4)}\right], \quad \mathbf{R} = 3\mu_1^2 - 8\mu_0\mu_2,$$

where $(*,*)^{(k)}$ denotes the bilinear operation transvectant.

Next, we construct the following T-comitants (for the definition of T-comitants, see [20]), which play a fundamental role in characterizing the existence of invariant straight lines for systems (2.1):

$$B_3(\tilde{a},x,y) = (C_2,\widehat{D})^{(1)} = \text{Jacob}(C_2,\widehat{D}),$$
$$B_2(\tilde{a},x,y) = (B_3,B_3)^{(2)} - 6B_3(C_2,\widehat{D})^{(3)},$$
$$B_1(\tilde{a}) = \frac{\text{Res}_x(C_2,\widehat{D})}{y^9} = -2^{-9}3^{-8}\,(B_2,B_3)^{(4)}.$$

The following result, whose proof can be found in [19], provides a necessary condition for the existence of invariant straight lines in quadratic differential systems.

Lemma 2.3 (see [19]). *For system (2.1) to possess invariant straight lines in one, two, or three distinct directions in the affine plane, it is necessary that the following conditions hold, respectively:*

$$B_1 = 0, \quad B_2 = 0, \quad B_3 = 0.$$

In order to detect the presence of parallel invariant straight lines, we require the following invariant polynomials:

$$N(\tilde{a},x,y) = D_2^2 + T_8 - 2T_9 = 9\widehat{N},$$
$$\theta(\tilde{a}) = 2A_5 - A_4 \quad \big(\equiv \text{Discriminant}\left(N(\tilde{a},x,y)\right)/1296\big).$$

With these definitions, the following necessary condition holds.

Lemma 2.4 (see [19]). *A necessary condition for the existence of one couple (respectively two couples) of parallel invariant straight lines in system (2.1), corresponding to a parameter vector $\tilde{a} \in \mathbb{R}^{12}$, is that $\theta(\tilde{a}) = 0$ (respectively $N(\tilde{a},x,y) = 0$).*

Next, we introduce some important GL-comitants relevant to the study of invariant conics. Let us consider

$$C_2(\tilde{a},x,y) = y\,p_2(\tilde{a},x,y) - x\,q_2(\tilde{a},x,y),$$

which defines a cubic binary form in x and y. Using this form, we define the following polynomials:

$$\eta = \text{Discrim}[C_2], \qquad M = \text{Hessian}[C_2].$$

It is worth noting (see [23]) that the invariant polynomials C_2, η, and M are responsible for controlling the number of infinite singularities (real or complex) of system (2.1).

Remark 2.5. In order to describe the various kinds of multiplicities for infinite singularities, we use the concepts and notations introduced in [19]. Thus, we denote by (a,b) the maximum number a (respectively, b) of infinite (respectively, finite) singularities that can be obtained by perturbation of a multiple infinite singularity. In this case, we say that an infinite singular point *has multiplicity* (a,b).

In this paper, we consider only the case $\eta \neq 0$, that is, $\eta > 0$ or $\eta < 0$. In the first case, according to [23], a quadratic system possesses at infinity three real distinct singularities, whereas in the second case, it possesses one real and two complex singularities.

In [24], necessary and sufficient conditions for a system to belong to the family **QSP** of quadratic systems possessing at least one invariant parabola, in terms of invariant polynomials, are determined.

We extract from [24] only the information related to the case $\eta \neq 0$, and for this, we need some notations.

Definition 2.6. By the *direction of an invariant parabola* of a quadratic system (S), we mean the direction of its axis of symmetry, which intersects the invariant line $Z = 0$ at an infinite singular point of (S).

In order to distinguish the invariant parabolas that a quadratic system could have, we use the following notations:

- $\cup$ for a simple invariant parabola;
- $\Cup$ for two simple invariant parabolas in the same direction (they could intersect);
- $\cup\!\subset$ for two simple invariant parabolas in different directions;
- $\cup^2$ for one double invariant parabola;
- $\Cup\subset$ for three simple invariant parabolas: two in one direction and one in another direction.

The proof of the next proposition can be found in [24].

Proposition 2.7. *Assume that for a non-degenerate arbitrary quadratic system, the conditions $\eta > 0$ and $\zeta_1 \neq 0$ are satisfied. Then, this system could possess invariant parabolas only in one direction. More exactly, it could only possess one of the following sets of invariant parabolas: $\cup$, $\Cup$ and $\cup^2$. Moreover, this system has one of the above sets of parabolas if and only if $\chi_1 = \chi_2 = 0$ and one of the following sets of conditions are satisfied, correspondingly:*

$$
\begin{array}{lll}
(\mathcal{A}_1) & \zeta_2 \neq 0,\ \zeta_3 \neq 0,\ \zeta_4 \neq 0,\ \mathcal{R}_1 \neq 0 & \Rightarrow \cup; \\
(\mathcal{A}_2) & \zeta_2 \neq 0,\ \zeta_3 \neq 0,\ \zeta_4 = 0,\ \mathcal{R}_2 \neq 0,\ \zeta_5 \neq 0 & \Rightarrow \Cup; \\
(\mathcal{A}_3) & \zeta_2 \neq 0,\ \zeta_3 \neq 0,\ \zeta_4 = 0,\ \mathcal{R}_2 \neq 0,\ \zeta_5 = 0 & \Rightarrow \cup^2; \\
(\mathcal{A}_4) & \zeta_2 \neq 0,\ \zeta_3 \neq 0,\ \zeta_4 = 0,\ \mathcal{R}_2 = 0,\ \zeta_5 \neq 0 & \Rightarrow \cup; \\
(\mathcal{A}_5) & \zeta_2 \neq 0,\ \zeta_3 = 0,\ \zeta_4 \neq 0,\ \mathcal{R}_1 \neq 0 & \Rightarrow \cup; \\
(\mathcal{A}_6) & \zeta_2 \neq 0,\ \zeta_3 = 0,\ \zeta_4 = 0,\ \mathcal{R}_2 \neq 0\ \zeta_5 \neq 0 & \Rightarrow \Cup; \\
(\mathcal{A}_7) & \zeta_2 \neq 0,\ \zeta_3 = 0,\ \zeta_4 = 0,\ \mathcal{R}_2 \neq 0,\ \zeta_5 = 0 & \Rightarrow \cup^2; \\
(\mathcal{A}_8) & \zeta_2 \neq 0,\ \zeta_3 = 0,\ \zeta_4 = 0,\ \mathcal{R}_2 = 0,\ \zeta_5 \neq 0 & \Rightarrow \cup; \\
(\mathcal{A}_9) & \zeta_2 = 0,\ \zeta_6 \neq 0,\ \mathcal{R}_1 = 0,\ \mathcal{R}_2 \neq 0 & \Rightarrow \cup.
\end{array}
$$

Furthermore, in the case of the existence of an invariant parabola, a system with $\eta > 0$ and $\zeta_1 \neq 0$ could be brought via an affine transformation and time rescaling to the following canonical form:

$$
\dot{x} = m + nx - \frac{1}{2}(1+g)y + gx^2 + xy, \quad \dot{y} = 2mx + 2ny + (g-1)xy + 2y^2 \tag{2.4}
$$

possessing the invariant parabola $\Phi(x, y) = x^2 - y = 0$.

However, examining the conditions $(\mathcal{A}_1)$–$(\mathcal{A}_9)$, we detect that some of the sets of these conditions could be joined. More exactly, we observe that $(\mathcal{A}_1)$ and $(\mathcal{A}_5)$ contain the same conditions except for the condition involving ζ_3. Therefore, by eliminating the conditions involving only ζ_3, we obtain a new set of conditions which we denote by $(\mathcal{A}_1^*)$: $\zeta_2 \neq 0, \quad \zeta_4 \neq 0, \quad \mathcal{R}_1 \neq 0$.

We perform the analogous operation on $(\mathcal{A}_2)$ and $(\mathcal{A}_6)$ (respectively $(\mathcal{A}_3)$ and $(\mathcal{A}_7)$; $(\mathcal{A}_4)$ and $(\mathcal{A}_8)$), resulting in the new conditions $(\mathcal{A}_2^*)$ (respectively $(\mathcal{A}_3^*)$; $(\mathcal{A}_4^*)$).

Thus, we can replace the first part of Proposition 2.7 obtaining the following one:

Proposition 2.7*. *Assume that for a non-degenerate arbitrary quadratic system, the conditions $\eta > 0$ and $\zeta_1 \neq 0$ are satisfied. Then, this system could possess invariant parabolas only in one direction. More exactly, it could only possess one of the following sets of invariant parabolas: $\bigcup$, $\Cup$ and $\bigcup^2$. Moreover, this system has one of the above sets of parabolas if and only if $\chi_1 = \chi_2 = 0$ and one of the following sets of conditions are satisfied, correspondingly:*

$$
\begin{array}{lll}
(\mathcal{A}_1^*) & \zeta_2 \neq 0, \zeta_4 \neq 0, \mathcal{R}_1 \neq 0 & \Rightarrow \bigcup; \\
(\mathcal{A}_2^*) & \zeta_2 \neq 0, \zeta_4 = 0, \mathcal{R}_2 \neq 0, \zeta_5 \neq 0 & \Rightarrow \Cup; \\
(\mathcal{A}_3^*) & \zeta_2 \neq 0, \zeta_4 = 0, \mathcal{R}_2 \neq 0, \zeta_5 = 0 & \Rightarrow \bigcup^2; \\
(\mathcal{A}_4^*) & \zeta_2 \neq 0, \zeta_4 = 0, \mathcal{R}_2 = 0, \zeta_5 \neq 0 & \Rightarrow \bigcup; \\
(\mathcal{A}_5^*) & \zeta_2 = 0, \ \zeta_6 \neq 0, \mathcal{R}_1 = 0, \mathcal{R}_2 \neq 0 & \Rightarrow \bigcup.
\end{array}
$$

The proof of the next two propositions could also be found in [24].

Proposition 2.8. *Assume that for a non-degenerate arbitrary quadratic system, the conditions $\eta > 0$ and $\zeta_1 = 0$ are satisfied. Then, this system could possess invariant parabolas in one or two directions. More exactly, it could only possess one of the following sets of invariant parabolas: $\bigcup$, $\Cup$, $\bigcup^2$, $\bigcup\subset$ and $\Cup\subset$. Moreover, this system has one of the above sets of invariant parabolas if and only if $\chi_1 = \chi_3 = 0$ and one of the following sets of conditions are satisfied, correspondingly:*

$$
\begin{array}{lll}
(\mathcal{B}_1) & \chi_4 \neq 0, \zeta_7 \neq 0, \mathcal{R}_3 \neq 0 & \Rightarrow \bigcup; \\
(\mathcal{B}_2) & \chi_4 \neq 0, \zeta_7 = 0, \mathcal{R}_4 \neq 0, \zeta_8 \neq 0 & \Rightarrow \Cup; \\
(\mathcal{B}_3) & \chi_4 \neq 0, \zeta_7 = 0, \mathcal{R}_4 \neq 0, \zeta_8 = 0 & \Rightarrow \bigcup^2; \\
(\mathcal{B}_4) & \chi_4 \neq 0, \zeta_7 = 0, \mathcal{R}_4 = 0 & \Rightarrow \bigcup; \\
(\mathcal{B}_5) & \chi_4 = 0, \zeta_5 \neq 0, \zeta_9 \neq 0 & \Rightarrow \bigcup\subset; \\
(\mathcal{B}_6) & \chi_4 = 0, \zeta_5 \neq 0, \zeta_9 = 0, \zeta_{10} \neq 0 & \Rightarrow \bigcup; \\
(\mathcal{B}_7) & \chi_4 = 0, \zeta_5 = 0, \zeta_6 \neq 0 & \Rightarrow \Cup\subset.
\end{array}
$$

Furthermore, in the case of the existence of an invariant parabola, a system with $\eta > 0$ and $\zeta_1 = 0$ could be brought via an affine transformation and time rescaling to the systems (2.4) with $g = 2$.

Proposition 2.9. *Assume that for a non-degenerate arbitrary quadratic system, the conditions $\eta < 0$ is satisfied. Then, this system could only possess one of the following sets of invariant parabolas: $\bigcup$, $\Cup$ and $\bigcup^2$. Moreover, this system has one of the above sets of invariant parabolas if and only if $\chi_1 = \chi_2 = 0$ and $\zeta_1 \neq 0$ and one of the following sets of conditions are satisfied, correspondingly:*

$$
\begin{array}{lll}
(\mathcal{E}_1) & \zeta_4 \neq 0, \mathcal{R}_1 \neq 0 & \Rightarrow \bigcup; \\
(\mathcal{E}_2) & \zeta_4 = 0, \mathcal{R}_7 \neq 0, \zeta_5 \neq 0 & \Rightarrow \Cup; \\
(\mathcal{E}_3) & \zeta_4 = 0, \mathcal{R}_7 \neq 0, \zeta_5 = 0 & \Rightarrow \bigcup^2; \\
(\mathcal{E}_4) & \zeta_4 = 0, \mathcal{R}_7 = 0, \zeta_5 \neq 0 & \Rightarrow \bigcup.
\end{array}
$$

Furthermore, in the case of the existence of an invariant parabola, a system with $\eta < 0$ could be brought via an affine transformation and time rescaling to the following canonical form:

$$\dot{x} = m + (2n-1)x/2 + gx^2 - gy/2 - xy, \quad \dot{y} = 2mx - x^2 + 2ny + gxy - 2y^2, \tag{2.5}$$

with $C_2 = x(x^2+y^2)$, *possessing the invariant parabola* $\Phi(x,y) = x^2 - y = 0$.

3 The proof of the Main Theorem

The statement $(\boldsymbol{B})$ follows directly from the form of the conditions given in Diagrams 1.1, 1.2, and 1.3. These conditions can be evaluated for any point $a \in \mathbb{R}^{12}$ corresponding to a quadratic system satisfying $\eta \neq 0$.

In order to prove the statement $(\boldsymbol{A})$ of the Main Theorem, we must examine the sets of conditions provided by each one of Propositions 2.7*, 2.8, and 2.9.

3.1 Systems in $\mathbf{QSP}_{(\eta>0)}$ with the condition $\zeta_1 \neq 0$

In what follows, we examine the configurations of the systems in $\mathbf{QSP}_{(\eta>0)}$ in each one of the cases provided by Proposition 2.7*. According to Proposition 2.7, we consider the canonical form (2.4), i.e., the systems:

$$\dot{x} = m + nx - \frac{1}{2}(1+g)y + gx^2 + xy, \quad \dot{y} = 2mx + 2ny + (g-1)xy + 2y^2, \tag{3.1}$$

possessing the invariant parabola $\Phi(x,y) = x^2 - y = 0$.

3.1.1 The statement $(\mathcal{A}_1^*)$

For systems (3.1), we calculate:

$$\begin{aligned}
\zeta_1 &= 2(g-2)(3+g), \quad \zeta_2 = 4g(1+g),\\
\zeta_4 &= (g-2)(3+g)(1+7g+15g^2+9g^3-4m+2n+6gn)/16,\\
\mathcal{R}_1 &= -15g(1+g)(g-2)(3+g)(1+7g+15g^2+9g^3-4m+2n+6gn)/2,\\
B_1 &= m(g+8m+4n)(gn-2m-n)(1+2g+g^2-4m+2n+2gn)\\
&\quad \times (g+2g^2+g^3+4m+2n+2gn)/4.
\end{aligned} \tag{3.2}$$

We consider two cases: $B_1 \neq 0$ and $B_1 = 0$.

The case $B_1 \neq 0$**.** Then, according to Lemma 2.3, systems (3.1) could not possess any invariant line.

We examine the finite singularities of these systems. Following [1, Proposition 5.1], we calculate the invariant polynomial $\mathbf{D} = 48F_1^2F_2$, where

$$\begin{aligned}
F_1 &= -4m^2 + 2(g+1)m(g^2-2n) - (g+1)^2n^2,\\
F_2 &= 108m^2 + 2(g-1)m\big(1-2g+g^2-18n\big) + n^2\big(16n-1+2g-g^2\big).
\end{aligned} \tag{3.3}$$

Thus, we discuss two subcases: $\mathbf{D} \neq 0$ and $\mathbf{D} = 0$.

The subcase $\mathbf{D} \neq 0$. We determine that systems (3.1) possess four finite singularities $M_i(x_i, y_i)$ $(i = 1, 2, 3, 4)$ with the coordinates

$$
\begin{aligned}
x_1 &= -\frac{2m+n+gn}{g(1+g)},\ y_1 = \frac{2m}{1+g};\ \ x_2 = \frac{1}{6\mathcal{Z}^{1/3}}\big[\mathcal{Y} + (1-g)\mathcal{Z}^{1/3} + \mathcal{Z}^{2/3}\big],\\
y_2 &= \frac{1}{36\mathcal{Z}}\big[3(\mathcal{Y}+4n)\mathcal{Z} + \mathcal{Y}^2\mathcal{Z}^{1/3} - 2(g-1)\mathcal{Y}\mathcal{Z}^{2/3} - 2(g-1)\mathcal{Z}^{4/3} + \mathcal{Z}^{5/3};\\
x_3 &= \frac{1}{12\mathcal{Z}^{1/3}}\big[-(1+i\sqrt{3})\mathcal{Y} + 2(1-g)\mathcal{Z}^{1/3} - (1-i\sqrt{3})\mathcal{Z}^{2/3}\big],\\
y_3 &= -\frac{1}{72\mathcal{Z}}\big[-6(\mathcal{Y}+4n)\mathcal{Z} + (1-i\sqrt{3})\mathcal{Y}^2\mathcal{Z}^{1/3} - 2(1+i\sqrt{3})(g-1)\mathcal{Y}\mathcal{Z}^{2/3}\\
&\quad - 2(1-i\sqrt{3})(g-1)\mathcal{Z}^{4/3} + (1+i\sqrt{3})\mathcal{Z}^{5/3}\big];\\
x_4 &= \frac{1}{12\mathcal{Z}^{1/3}}\big[(-1+i\sqrt{3})\mathcal{Y} + 2(1-g)\mathcal{Z}^{1/3} - (1+i\sqrt{3})\mathcal{Z}^{2/3}\big],\\
y_4 &= -\frac{1}{72\mathcal{Z}}\big[-6(\mathcal{Y}+4n)\mathcal{Z} + (1+i\sqrt{3})\mathcal{Y}^2\mathcal{Z}^{1/3} - 2(1-i\sqrt{3})(g-1)\mathcal{Y}\mathcal{Z}^{2/3}\\
&\quad - 2(1+i\sqrt{3})(g-1)\mathcal{Z}^{4/3} + (1-i\sqrt{3})\mathcal{Z}^{5/3}\big],
\end{aligned}
\tag{3.4}
$$

where

$$
\mathcal{Z} = 1 - 3g + 3g^2 - g^3 - 108m - 18n + 18gn + 6\sqrt{3}\sqrt{F_2}, \quad \mathcal{Y} = (1-g)^2 - 12n.
$$

Calculations yield:

$$
\Phi(x_2, y_2) = \Phi(x_3, y_3) = \Phi(x_4, y_4) = 0, \quad \Phi(x_1, y_1) = -\frac{F_1}{g^2(1+g)^2},
$$

and therefore we deduce that three singularities M_2, M_3 and M_4 of systems (3.1) are located on the invariant parabola. Moreover, M_1 is located outside the parabola and could belong to it if and only if the condition $F_1 = 0$ holds, where F_1 is given in (3.3). However, we have $\mathbf{D} = 48F_1^2F_2 \neq 0$, and hence on the parabola we always have exactly three distinct singularities.

On the other hand, according to [1, Proposition 5.1], if $\mathbf{D} > 0$, systems (3.1) possess two real and two complex finite singularities. For $\mathbf{D} < 0$, we could either have four real or four complex finite singularities. However, since M_1 is a real singular point for these systems, we conclude that in the case $\mathbf{D} < 0$ we have four real finite distinct singularities.

Thus, since the real singularity M_1 is outside the invariant parabola and all other three finite singularities on the parabola (real or complex) are distinct and furthermore we could not have any invariant line, we arrive at the configuration *Config.* $\mathcal{P}$.1 if $\mathbf{D} < 0$ and *Config.* $\mathcal{P}$.2 if $\mathbf{D} > 0$.

The subcase $\mathbf{D} = 0$. This implies $F_1F_2 = 0$, and for systems (3.1), we calculate:

$$
\xi_1 = -6\zeta_4 F_1 \ \Rightarrow\ F_1 = 0 \ \Leftrightarrow\ \xi_1 = 0.
$$

So we examine two possibilities: $\xi_1 \neq 0$ and $\xi_1 = 0$.

1: The possibility $\xi_1 \neq 0$. Then $F_1 \neq 0$, and therefore the condition $\mathbf{D} = 0$ implies $F_2 = 0$.

We observe that the polynomial F_2 is quadratic with respect to the parameter m, and we calculate

$$
\text{Discrim}\,[F_2, m] = 4(1 - 2g + g^2 - 12n)^3.
$$

Therefore, since the parameters m, n, and g of systems (3.1) must be real, we conclude that the condition $1-2g+g^2-12n\geq 0$ has to be fulfilled. Setting a new parameter v by $1-2g+g^2-12n=v^2\geq 0$, we get $n=\left[(g-1)^2-v^2\right]/12$, and then we calculate

$$F_2=\frac{1}{432}\left[216m-(1-g+v)^2(g-1+2v)\right]\left[216m-(1-g-v)^2(g-1-2v)\right]=0$$

and due to the change $v\mapsto -v$, we may force the first factor to vanish. Then we obtain

$$m=\frac{(1-g+v)^2(g-1+2v)}{216},$$

and considering the expressions for the parameters m and n, we arrive at the two-parameter family of systems

$$\begin{aligned}\dot{x}&=\frac{(1-g+v)^2(g-1+2v)}{216}+\frac{(g-1)^2-v^2}{12}\,x-\frac{1}{2}(1+g)y+gx^2+xy,\\ \dot{y}&=\frac{(1-g+v)^2(g-1+2v)}{108}\,x+\frac{(g-1)^2-v^2}{6}\,y+(g-1)xy+2y^2,\end{aligned}\tag{3.5}$$

possessing the invariant parabola $\Phi(x,y)=x^2-y=0$. We observe that for the above systems, we have the following conditions on the parameters g and v:

$$\begin{aligned}&\zeta_1\zeta_2\zeta_4\mathcal{R}_1\neq 0\ \Leftrightarrow\ g(g-2)(1+g)(3+g)(2+4g-v)(4+8g+v)\neq 0;\\ &\xi_1\neq 0\ \Leftrightarrow\ (g-2)(3+g)(g-1-v)(2+g-v)(2+4g-v)(4+8g+v)^2\\ &\qquad\times(4-2g-2g^2-4v-8gv+v^2)\neq 0;\\ &B_1\neq 0\ \Leftrightarrow\ (g-1-v)(2+g-v)(2+4g-v)(2g-2+v)(1+2g+v)(4+2g+v)\\ &\qquad\times(g-1+2v)(2+g+2v)\neq 0.\end{aligned}\tag{3.6}$$

We determine that systems (3.5) possess three finite singularities $M_i(x_i,y_i)$ $(i=1,2,3)$ with the coordinates

$$\begin{aligned}&x_1=\frac{1-g+v}{6},\ y_1=\frac{(1-g+v)^2}{36};\ \ x_2=\frac{1-g-2v}{6},\ y_2=\frac{(1-g-2v)^2}{36};\\ &x_3=\frac{(1-g+v)(5g^2-4-g+4v+5gv-v^2)}{54g(1+g)},\ y_3=\frac{(1-g+v)^2(g-1+2v)}{108(1+g)}.\end{aligned}\tag{3.7}$$

We calculate

$$\begin{aligned}&\Phi(x_1,y_1)=\Phi(x_2,y_2)=0,\\ &\Phi(x_3,y_3)=\frac{(g-1-v)^2(2+g-v)^2(4-2g-2g^2-4v-8gv+v^2)}{2916g^2(1+g)^2},\end{aligned}$$

and we conclude that the singular points M_1 and M_2 are located on the invariant parabola.

On the other hand, considering the conditions (3.6), we obtain that M_3 will be located on $\Phi(x,y)=0$ if and only if

$$\alpha=4-2g(1+g)-4v-8gv+v^2=0.$$

However, considering (3.6), we conclude that $\alpha\neq 0$ (due to $\xi_1\neq 0$), and hence the singularity M_3 is not located on the invariant parabola in the considered case.

We claim that M_1 is a multiple singularity of systems (3.5). Indeed, applying the corresponding translation, we could place M_1 at the origin of coordinates and arrive at the systems

$$\begin{aligned}\dot{x} &= -\frac{(g-v-1)(4g-v+2)}{18}x - \frac{4g-v+2}{6}y + gx^2 + xy,\\ \dot{y} &= \frac{(g-v-1)^2(2g+v-2)}{54}x + \frac{(g-v-1)(2g+v-2)}{18}y + (g-1)xy + 2y^2,\end{aligned}$$

where $M_0(0,0)$ is a singularity of the above systems corresponding to the singularity M_2.

Considering [1, Lemma 5.2], we calculate the following invariant polynomials: $\mu_4 = \mu_3 = 0$, and

$$\mu_2 = -\frac{1}{324}v(g-v-1)(g-v+2)\big[(2g+v-2)x+6y\big]\big[g(g-v-1)x+(2+4g-v)y\big].$$

Therefore, by [1, Lemma 5.2, statement (ii)], the point M_0 is of multiplicity at least 2. We observe that due to the condition $\xi_1 \neq 0$, we have $\mu_2 = 0$ if and only if $v = 0$. In this case, we calculate

$$\mu_2 = 0, \quad \mu_1 = \frac{1}{27}(g-1)(g+2)\big[g(g-1)x+2(2g+1)y\big] \neq 0,$$

due to $\xi_1 \neq 0$. According to [1, Lemma 5.2, statement (ii)], $M_0(0,0)$ is a double point if $v \neq 0$, and it is a triple one if $v = 0$.

On the other hand, for systems (3.5), we have

$$\xi_2 = \frac{1}{209952}(g-1-v)^2(2+g-v)^2v^2\alpha^2,$$

and due to the conditions (3.6), we conclude that the condition $v = 0$ is equivalent to $\xi_2 = 0$.

Thus, for systems (3.5), we have the configuration *Config.* $\mathcal{P}$.3 if $\xi_2 \neq 0$ and *Config.* $\mathcal{P}$.4 if $\xi_2 = 0$.

2: The possibility $\xi_1 = 0$. This implies $F_1 = 0$, and since the polynomial F_1 is quadratic with respect to the parameter m, we calculate

$$\text{Discrim}\,[F_1, m] = 4g^2(1+g)^2(g^2-4n).$$

Since $g(g+1) \neq 0$ (due to $\zeta_2 \neq 0$), we must have $g^2 - 4n \geq 0$. So we set a new parameter u as follows: $g^2 - 4n = u^2 \geq 0$, and we get $n = (g^2-u^2)/4$. Then, calculation yields

$$F_1 = -\frac{1}{16}\big[8m-(1+g)(g+u)^2\big]\big[8m-(1+g)(g-u)^2\big] = 0,$$

and due to the change $u \mapsto -u$, we may force the second factor to vanish. In this case, we obtain

$$m = \frac{(1+g)(g-u)^2}{8},$$

and considering the expression for the parameters m and n, we arrive at the two-parameter family of systems

$$\begin{aligned}\dot{x} &= \frac{(1+g)(g-u)^2}{8} + \frac{g^2-u^2}{4}x - \frac{1}{2}(1+g)y + gx^2 + xy,\\ \dot{y} &= \frac{(1+g)(g-u)^2}{4}x + \frac{g^2-u^2}{2}y + (g-1)xy + 2y^2,\end{aligned} \tag{3.8}$$

possessing the invariant parabola $\Phi(x,y) = x^2 - y = 0$. We observe that for the above systems, we have the following condition on the parameters g and u:

$$\begin{gathered}\zeta_1\zeta_2\zeta_4\mathcal{R}_1 \neq 0 \;\Leftrightarrow\; (g-2)g(1+g)(3+g)(1+2g+u)(1+5g+5g^2-u-2gu) \neq 0;\\ B_1 \neq 0 \;\Leftrightarrow\; g(1+g)(g-u)(1+g-u)(1+2g-u)(-1+u)(1+u) \neq 0.\end{gathered} \tag{3.9}$$

We determine that systems (3.8) possess three finite singularities $M_i(x_i,y_i)$ $(i=1,2,3)$ with the coordinates

$$\begin{gathered}x_1 = \frac{u-g}{2},\; y_1 = \frac{(u-g)^2}{4};\;\; x_{2,3} = \frac{1}{4}\big(1-u \pm \sqrt{Z_1}\,\big),\\ y_{2,3} = \frac{1}{8}\big[1-2g-2g^2+2gu+u^2 \mp (u-1)\sqrt{Z_1}\,\big],\;\; Z_1 = -4g^2+4g(-1+u)+(1+u)^2.\end{gathered} \tag{3.10}$$

We calculate

$$\Phi(x_1,y_1) = \Phi(x_2,y_2) = \Phi(x_3,y_3) = 0,$$

and therefore all three singularities are located on the invariant parabola.

We point out that M_1 is a multiple singularity of systems (3.8). Indeed, applying the corresponding translation, we could place M_1 at the origin of coordinates and arrive at the systems

$$\begin{aligned}\dot{x} &= -\frac{1}{2}g(g-u)x + \frac{1}{2}(u-2g-1)y + gx^2 + xy,\\ \dot{y} &= \frac{1}{2}g(g-u)^2x + \frac{1}{2}(2g-u+1)(g-u)y + (g-1)xy + 2y^2,\end{aligned}$$

where $M_0(0,0)$ is a singularity of the above systems corresponding to the singularity M_1.

Considering [1], we calculate the invariant polynomials μ_4, μ_3, μ_2, and we obtain $\mu_4 = \mu_3 = 0$, and

$$\mu_2 = \frac{1}{2}g(g+1)(g-u)(g-u+1)\big[g(g-u)x^2 + (2g-1-u)xy + 2y^2\big] \neq 0,$$

due to the conditions (3.9). By [1, Lemma 5.2, statement (ii)], the point M_0 is of multiplicity exactly 2.

On the other hand, it is clear that the singularities M_2 and M_3 could be complex (respectively real; coinciding) if $Z_1 < 0$ (respectively $Z_1 > 0$; $Z_1 = 0$). We observe that for systems (3.8), we have:

$$\xi_2 = g^2(1+g)^2(g-u)^2(1+g-u)^2Z_1,$$

and due to the conditions (3.9), we conclude that the sign of Z_1 is governed by the invariant polynomial ξ_2. So we discuss three cases: $\xi_2 < 0$, $\xi_2 > 0$, and $\xi_2 = 0$.

2.1: *The case* $\xi_2 < 0$. This implies $Z_1 < 0$, and then systems (3.5) possess only one real singular point M_1 (which is double), and evidently we get the configuration *Config.* $\mathcal{P}$.5.

2.2: *The case* $\xi_2 > 0$. Then $Z_1 > 0$, and this implies the existence of three real singularities, and we have to determine the position of the double point with respect to the simple ones. So we calculate

$$(x_3-x_1)(x_2-x_1) = \frac{(g-u)(1+g-u)}{2} \equiv \frac{\alpha_1}{2},\;\; \operatorname{sign}\big((x_3-x_1)(x_2-x_1)\big) = \operatorname{sign}(\alpha_1), \tag{3.11}$$

where $\alpha_1 \neq 0$ due to $B_1 \neq 0$. This means that the singularity M_1 could not coalesce with one of the singularities M_2 or M_3.

On the other hand, for systems (3.5), calculations yield:

$$\xi_3 = \frac{27249129}{2} g^2(1+g)^2\alpha_1^3 Z_1.$$

So, due to the conditions (3.9), we deduce that sign $(\xi_3) =$ sign $(\alpha_1 Z_1)$.

Therefore, in the case $\xi_3 < 0$, the double singular point M_1 is located on the parabola between M_2 and M_3, and we arrive at the configuration *Config.* $\mathcal{P}$.6. If $\xi_3 > 0$, we evidently get the configuration *Config.* $\mathcal{P}$.7.

2.3: *The case* $\xi_2 = 0$. Then $Z_1 = 0$, which implies the coalescence of the singularities M_2 and M_3. Therefore, systems (3.5) possess two double singularities located on the invariant parabola. So we obtain the configuration *Config.* $\mathcal{P}$.8.

It remains to mention that the case $u = 0$ (i.e., when the discriminant of F_1 vanishes) is included in the previous examination because the condition $u \neq 0$ was not necessary. So, in this case, we obtain the same configurations for the corresponding conditions, respectively.

The case $B_1 = 0$**.** Considering (3.2), we observe that the condition $B_1 = 0$ splits into five conditions at the coefficient level. However, due to an affine transformation, we can reduce this number. More precisely, we have the following lemma.

Lemma 3.1. *The condition* $(g+8m+4n)(1+2g+g^2-4m+2n+2gn) = 0$ *for systems* (3.1) *can be transformed into the condition* $m(gn-2m-n) = 0$ *via an affine transformation.*

Proof. Applying to systems (3.1) the transformation

$$x_1 = -x + \frac{1}{2}, \quad y_1 = -x + y + \frac{1}{4},$$

we obtain the systems

$$\begin{aligned}
\dot{x}_1 &= -\frac{1}{8}(g+8m+4n) + \frac{1}{4}(1+2g+4n)x_1 + \frac{g}{2}y_1 - (1+g)x_1^2 + x_1y_1,\\
\dot{y}_1 &= -\frac{1}{4}(g+8m+4n)x_1 + \frac{1}{2}(1+2g+4n)y_1 - (g+2)x_1y_1 + 2y_1^2.
\end{aligned}$$

So, setting the new parameters

$$\begin{aligned}
m_1 &= -\frac{1}{8}(g+8m+4n), \qquad n_1 = \frac{1}{4}(1+2g+4n), \qquad g_1 = -(1+g) \;\Rightarrow\\
m &= -\frac{1}{8}(g_1+8m_1+4n_1), \qquad n = \frac{1}{4}(1+2g_1+4n_1), \qquad g = -(1+g_1),
\end{aligned} \tag{3.12}$$

we obtain the family of systems

$$\dot{x}_1 = m_1 + n_1x_1 - \frac{1+g_1}{2}y_1 + g_1x_1^2 + x_1y_1, \quad \dot{y}_1 = 2m_1x_1 + 2n_1y_1 + (g_1-1)x_1y_1 + 2y_1^2.$$

which has the same form as (3.1).

Then, considering (3.12), calculations yield:

$$g+8m+4n = -8m_1, \quad 1+2g+g^2-4m+2n+2gn = 2(2m_1+n_1-g_1n_1),$$

and this completes the proof of the lemma. □

Thus, by Lemma 3.1, in order to examine the condition $B_1 = 0$, it is sufficient to consider the condition

$$m(gn - 2m - n)(g + 2g^2 + g^3 + 4m + 2n + 2gn) = 0.$$

In order to determine the invariant conditions that distinguish the three possibilities provided by the above equality, for systems (3.1) we calculate:

$$\begin{aligned} \xi_4 &= 21 \cdot 2^6 5^4 m(g + 8m + 4n)\zeta_4, \\ \xi_5 &= -14 \cdot 5^5 (gn - 2m - n)(1 + 2g + g^2 - 4m + 2n + 2gn)\zeta_4. \end{aligned} \tag{3.13}$$

Hence, due to $\zeta_4 \neq 0$, the condition $\xi_4 = 0$ is equivalent to $m(g + 8m + 4n) = 0$ (which implies $B_1 = 0$), whereas the condition $\xi_5 = 0$ is equivalent to $(gn - 2m - n)(1 + 2g + g^2 - 4m + 2n + 2gn) = 0$ (which also implies $B_1 = 0$).

The subcase $\xi_4 \neq 0$. Then $m(g + 8m + 4n) \neq 0$, and we consider two possibilities: $\xi_5 \neq 0$ and $\xi_5 = 0$.

1: The possibility $\xi_5 \neq 0$. In this case, we have $(gn - 2m - n)(1 + 2g + g^2 - 4m + 2n + 2gn) \neq 0$, and therefore the condition $B_1 = 0$ implies $g + 2g^2 + g^3 + 4m + 2n + 2gn = 0$. This yields $m = -(1 + g)(g + g^2 + 2n)/4$, and we get the family of systems

$$\begin{aligned} \dot{x} &= -\frac{1}{4}(1 + g - 2x)(g + g^2 + 2n + 2gx + 2y), \\ \dot{y} &= -\frac{(1 + g)(g + g^2 + 2n)}{2}\, x + 2ny + (g - 1)xy + 2y^2 \end{aligned} \tag{3.14}$$

possessing the invariant line $x = (g + 1)/2$. For these systems, we calculate

$$\begin{aligned} B_2 &= -81g^2(1 + g)^2(g + g^2 + 2n)(1 + 4g + 2g^2 + 4n)(1 + 2g + g^2 + 4n)^2 x^4, \\ \xi_4 &= 13125g(1 + g)(g - 2)(3 + g)(1 + 2g)(g + g^2 + 2n)(1 + 4g + 2g^2 + 4n)(1 + 6g + 5g^2 + 4n), \\ \xi_5 &= -21875(g - 2)g(1 + g)(3 + g)(1 + 2g)(1 + 2g + g^2 + 4n)^2(1 + 6g + 5g^2 + 4n)/16, \end{aligned}$$

and we observe that the condition $\xi_4\xi_5 \neq 0$ implies $B_2 \neq 0$.

Then, by Lemma 2.3, besides the invariant line $x = (g + 1)/2$, systems (3.14) could not possess invariant lines in other directions. However, they could have a parallel invariant line, and by Lemma 2.4, for this to occur it is necessary that $\theta = 0$. This condition implies $(g - 1)(g + 2) = 0$. A straightforward calculation shows that neither of the conditions $g = 1$ nor $g = -2$ could imply the appearance of an additional parallel invariant line.

Next, we determine that systems (3.14) possess four finite singularities $M_i(x_i, y_i)$ $(i = 1, 2, 3, 4)$ with the coordinates

$$\begin{gathered} x_1 = \frac{1 + g}{2},\ y_1 = \frac{(1 + g)^2}{4};\quad x_2 = \frac{1 + g}{2},\ y_2 = -\frac{g + g^2 + 2n}{2};\quad x_{3,4} = \frac{1}{2}\big(-g \pm \sqrt{Z_2}\big), \\ y_{3,4} = \frac{1}{2}\big(-g - 2n \mp g\sqrt{Z_2}\big),\quad Z_2 = -(2g + g^2 + 4n). \end{gathered} \tag{3.15}$$

We determine that the singularities M_1, M_3, and M_4 are located on the invariant parabola. At the same time, M_1 and M_2 are located on the invariant line $x = (g + 1)/2$, and M_1 is the point of intersection of this invariant line with the parabola.

In order to determine the relative position of the singularities M_1 and M_2 on the vertical invariant line, we calculate

$$y_2 - y_1 = -\frac{1+4g+3g^2+4n}{4} \equiv -\frac{\alpha_2}{4} \;\Rightarrow\; \text{sign}\,(y_2-y_1) = -\text{sign}\,(\alpha_2). \tag{3.16}$$

Since the singularities M_3 and M_4 are either complex, real, or coinciding depending on the value of Z_2, we need to distinguish these conditions using affine invariant polynomials. For systems (3.14), we calculate:

$$\begin{aligned} \zeta_4 &= \frac{1}{16}(g-2)(3+g)(1+2g)(1+6g+5g^2+4n) \equiv \frac{1}{16}(g-2)(3+g)(1+2g)\beta_2, \\ \mathbf{D} &= -\frac{3}{4}g^4(1+g)^4\beta_2^2\alpha_2^2 Z_2, \quad \zeta_2 = 4g(1+g), \end{aligned} \tag{3.17}$$

and due to $\zeta_2\zeta_4 \neq 0$, we conclude that $\mathbf{D} = 0$ is equivalent to $\alpha_2 Z_2 = 0$. Moreover, if $\mathbf{D} \neq 0$, then $\text{sign}\,(\mathbf{D}) = -\text{sign}\,(Z_2)$. Thus, we discuss three cases: $\mathbf{D} < 0$, $\mathbf{D} > 0$, and $\mathbf{D} = 0$.

1.1: *The case* $\mathbf{D} < 0$. This implies $Z_2 > 0$, and systems (3.14) possess four real singularities. Clearly, it is necessary to know the position of the real singularities $M_{3,4}$ with respect to M_1, all located on the invariant parabola. We calculate:

$$\begin{gathered} (x_3-x_1)(x_4-x_1) = \frac{\beta_2}{4}, \quad (x_3-x_1)+(x_4-x_1) = -(1+2g), \\ \text{sign}\,\big((x_3-x_1)(x_4-x_1)\big) = \text{sign}\,(\beta_2), \quad \text{sign}\,\big((x_1-x_3)+(x_1-x_4)\big) = -\text{sign}\,(1+2g). \end{gathered}$$

We observe that $\beta_2 \neq 0$ due to the condition $\zeta_4 \neq 0$, and moreover $\alpha_2 \neq 0$ due to $\mathbf{D} \neq 0$.

On the other hand, we need the invariant polynomials that govern the signs of β_2 and α_2. Thus, for systems (3.14), we calculate:

$$\xi_7 = 1174627500\, g^2(1+g)^2(1+2g)^2\alpha_2^2\beta_2 Z_2, \quad \xi_8 = 61822500 g^2(1+g)^2(1+2g)^2\alpha_2\beta_2^2 Z_2.$$

Due to the conditions $\zeta_4 \neq 0$ and $\mathbf{D} < 0$, which imply $g(1+g)(1+2g)\alpha_2\beta_2 \neq 0$ and $Z_2 > 0$ (also ensuring $\xi_7\xi_8 \neq 0$), we have the following relations:

$$\text{sign}\,(\beta_2) = \text{sign}\,(\xi_7), \quad \text{sign}\,(\alpha_2) = \text{sign}\,(\xi_8).$$

Thus, considering the above relations, in the case $\mathbf{D} < 0$ we detect the following configurations:

$$\begin{array}{lll} \xi_7 < 0, \xi_8 < 0 & \Rightarrow & (x_3-x_1)(x_4-x_1) < 0, y_2 > y_1 \;\Rightarrow\; \textit{Config.}\,\mathcal{P}.9; \\ \xi_7 < 0, \xi_8 > 0 & \Rightarrow & (x_3-x_1)(x_4-x_1) < 0, y_2 < y_1 \;\Rightarrow\; \textit{Config.}\,\mathcal{P}.10; \\ \xi_7 > 0, \xi_8 < 0 & \Rightarrow & (x_3-x_1)(x_4-x_1) > 0, y_2 > y_1 \;\Rightarrow\; \textit{Config.}\,\mathcal{P}.11; \\ \xi_7 > 0, \xi_8 > 0 & \Rightarrow & (x_3-x_1)(x_4-x_1) > 0, y_2 < y_1 \;\Rightarrow\; \textit{Config.}\,\mathcal{P}.12. \end{array}$$

1.2: *The case* $\mathbf{D} > 0$. Then $Z_2 < 0$, and we claim that this condition implies $\alpha_2 > 0$. Indeed, supposing the contrary (i.e., $\alpha_2 < 0$), we must have $Z_2 + \alpha_2 < 0$. However, calculations yield:

$$Z_2 + \alpha_2 = -(2g+g^2+4n) + (1+4g+3g^2+4n) = (1+g)^2 + g^2 > 0. \tag{3.18}$$

The contradiction obtained proves our claim.

Therefore, since M_3 and M_4 are complex, we arrive at the configuration *Config.* $\mathcal{P}.13$.

1.3: *The case* $\mathbf{D} = 0$. Considering (3.17), we deduce that, due to $\zeta_2\zeta_4 \neq 0$, the condition $\mathbf{D} = 0$ implies $\alpha_2 Z_2 = 0$.

On the other hand, for systems (3.14), we calculate:

$$\xi_1 = \frac{3}{32} g^2 (1+g)^2 (g-2)(3+g)(1+2g)\alpha_2\beta_2.$$

Thus, due to $\zeta_2\zeta_4 \neq 0$ (i.e., $g(1+g)(g-2)(3+g)(1+2g)\beta_2 \neq 0$), we obtain that the condition $\alpha_2 = 0$ is equivalent to $\xi_1 = 0$. Therefore, we discuss two subcases: $\xi_1 \neq 0$ and $\xi_1 = 0$.

1.3.1: *The subcase* $\xi_1 \neq 0$. In this case, we have $\alpha_2 \neq 0$, and the condition $\mathbf{D} = 0$ implies $Z_2 = 0$. Then M_3 and M_4 coalesce, producing a double point located on the invariant parabola. Considering (3.18), we deduce that the condition $Z_2 = 0$ implies $\alpha_2 > 0$.

Thus, it is not difficult to determine that, in this case, we arrive at the configuration *Config.* $\mathcal{P}.14$.

1.3.2: *The subcase* $\xi_1 = 0$. This implies $\alpha_2 = 0$, and, as we have mentioned earlier (see formulas (3.16)), in this case, we get $y_2 = y_1$, and hence the intersection point M_1 of the invariant line $x = (g+1)/2$ with the parabola becomes a double singularity of systems (3.14). Moreover, the position of the real singularities M_3 and M_4 with respect to M_1 depends on the sign of β_2.

Thus, the condition $\alpha_2 = 0$ implies $n = -(1+g)(1+3g)/4$, and then we obtain

$$\beta_2 = 2g(1+g), \quad \zeta_2 = 4g(1+g) \;\Rightarrow\; \operatorname{sign}(\beta_2) = \operatorname{sign}(\zeta_2).$$

Therefore, in the case $\alpha_2 = 0$ (i.e., $\xi_1 = 0$), we obtain the following two configurations:

$$\begin{array}{lll} \zeta_2 < 0 & \Rightarrow & (x_3 - x_1)(x_4 - x_1) < 0,\ y_2 = y_1 \quad \Rightarrow \textit{ Config. } \mathcal{P}.15; \\ \zeta_2 > 0 & \Rightarrow & (x_3 - x_1)(x_4 - x_1) > 0,\ y_2 = y_1 \quad \Rightarrow \textit{ Config. } \mathcal{P}.16. \end{array}$$

2: The possibility $\xi_5 = 0$. Considering (3.13) and the condition $\zeta_4 \neq 0$, we obtain that the condition $\xi_5 = 0$ implies

$$(gn - 2m - n)(1 + 2g + g^2 - 4m + 2n + 2gn) = 0.$$

On the other hand, according to Lemma 3.1, it is sufficient to examine the condition given by the first factor, because the condition defined by the second factor can be brought to the first one via an affine transformation.

Thus, in what follows, we assume that for systems (3.1), the condition $gn - 2m - n = 0$ holds. Then $m = n(g-1)/2$, and we arrive at the family of systems

$$\dot{x} = \frac{n(g-1)}{2} + nx - \frac{1}{2}(1+g)y + gx^2 + xy, \quad \dot{y} = (n+y)(gx - x + 2y), \tag{3.19}$$

which possess the invariant line $y = -n$ and the invariant parabola $\Phi(x,y) = x^2 - y = 0$. For these systems, we calculate

$$\begin{aligned} B_2 &= -81g^2(1+4n)\left[(1+g)^2 + 4n\right]^2 y^4/2, \\ \xi_4 &= 26250(g-1)g(1+g)(g-2)(3+g)n(1+4n)(1+6g+9g^2+4n), \end{aligned} \tag{3.20}$$

and we consider two cases: $B_2 \neq 0$ and $B_2 = 0$.

2.1: *The case* $B_2 \neq 0$. In this case, by Lemma 2.3, systems (3.19) cannot possess invariant lines in other directions than the invariant line $y = -n$. However, by Lemma 2.4, these systems could possess an invariant line parallel to the existing one if $\theta = -8(g-1)(2+g) = 0$. Thus,

due to $\xi_4 \neq 0$, this condition implies $g = -2$. However, in this case, systems (3.19) do not have any invariant line parallel to $y = -n$.

Next, we determine that systems (3.19) possess the finite singularities $M_i(x_i, y_i)$ $(i = 1, 2, 3, 4)$ with coordinates

$$\begin{gathered} x_1 = \sqrt{-n},\ y_1 = -n;\ \ x_2 = -\sqrt{-n},\ y_2 = -n;\ \ x_3 = \frac{1-g}{2},\ y_3 = \frac{(1-g)^2}{4}; \\ x_4 = -\frac{2n}{1+g},\ y_4 = \frac{n(g-1)}{1+g}. \end{gathered} \tag{3.21}$$

We observe that the singular points M_1, M_2, and M_3 lie on the invariant parabola $\Phi(x, y) = x^2 - y = 0$. Moreover, M_1 and M_2 are the points of intersection between the parabola and the invariant line $y = -n$, and since $n \neq 0$ (due to $\xi_4 \neq 0$), they are either complex (for $n > 0$) or real (for $n < 0$).

On the other hand, for systems (3.19), calculations yield:

$$\mathbf{D} = 48g^4n^3(1 - g^2 + 4n)^2(1 - 2g + g^2 + 4n)^2 \equiv 48g^4n^3\,\alpha_3^2\,\beta_3^2,$$

and it is clear that, in the case $\mathbf{D} \neq 0$, we have $\operatorname{sign}(\mathbf{D}) = \operatorname{sign}(n)$.

To determine the position of the singular point M_4, we calculate

$$\Phi(x_4, y_4) = \frac{n\alpha_3}{(1+g)^2},$$

and since $n \neq 0$ (due to $\xi_4 \neq 0$), we deduce that the singular point M_4 lies on the invariant parabola if and only if $\alpha_3 = 0$.

To examine the configurations of the systems, we consider three subcases: $\mathbf{D} < 0$, $\mathbf{D} > 0$, and $\mathbf{D} = 0$.

2.1.1: *The subcase* $\mathbf{D} < 0$. Then $n < 0$, and the singular points M_1 and M_2 are real. In order to determine the position of the singularity M_3 with respect to the real singularities M_1 and M_2, we calculate

$$\begin{gathered} (x_3 - x_1)(x_3 - x_2) = \frac{1}{4}\left[(1-g)^2 + 4n\right] \equiv \beta_3, \quad (x_3 - x_1) + (x_3 - x_2) = 1 - g; \\ \operatorname{sign}\left((x_3 - x_1)(x_3 - x_2)\right) = \operatorname{sign}(\beta_3), \quad \operatorname{sign}\left((x_3 - x_1) + (x_3 - x_2)\right) = \operatorname{sign}(1 - g). \end{gathered}$$

We observe that $\alpha_3\beta_3 \neq 0$ due to $\mathbf{D} \neq 0$, and we need to determine the invariant polynomials responsible for the signs of β_3 and $g - 1$. Calculations yield:

$$\begin{aligned} \xi_9 &= 22359252960\,g^6(1+g)^2(1+2g+g^2+4n)^2\beta_3, \\ \xi_{10} &= 24814861965\,(g-1)g^2(1+g)^2(1+2g+g^2+4n)^2(1+6g+9g^2+4n)^2/2. \end{aligned} \tag{3.22}$$

Taking into account the condition $\xi_4 B_2 \neq 0$ and from (3.20), we deduce that $\operatorname{sign}(\beta_3) = \operatorname{sign}(\xi_9)$ and $\operatorname{sign}(g - 1) = \operatorname{sign}(\xi_{10})$.

Thus, considering the above relations, in the case $\mathbf{D} < 0$, we arrive at the following configurations:

$$\begin{aligned} \xi_9 < 0 &\ \Rightarrow\ (x_3 - x_1)(x_3 - x_2) < 0 &&\ \Rightarrow\ \textit{Config. } \mathcal{P}.17; \\ \xi_9 > 0,\ \xi_{10} < 0 &\ \Rightarrow\ (x_3 - x_1) > 0,\ (x_3 - x_2) > 0 &&\ \Rightarrow\ \textit{Config. } \mathcal{P}.18; \\ \xi_9 > 0,\ \xi_{10} > 0 &\ \Rightarrow\ (x_3 - x_1) < 0,\ (x_3 - x_2) < 0 &&\ \Rightarrow\ \textit{Config. } \mathcal{P}.19. \end{aligned}$$

2.1.2: *The subcase* $\mathbf{D} > 0$. Then $n > 0$, and the singular points M_1 and M_2 are complex. Therefore, due to the condition $\alpha_3 \neq 0$, we arrive at the configuration *Config.* $\mathcal{P}.20$.

2.1.3: *The subcase* $\mathbf{D} = 0$. Since $n \neq 0$ (due to $\xi_4 \neq 0$), this implies $\alpha_3\,\beta_3 = 0$, and we have to distinguish two cases: $\beta_3 \neq 0$ and $\beta_3 = 0$. From (3.22) we observe that, due to $\xi_4 B_2 \neq 0$, the condition $\xi_9 = 0$ is equivalent to $\beta_3 = 0$.

2.1.3.1: *The possibility* $\xi_9 \neq 0$. Then $\beta_3 \neq 0$, and the condition $\mathbf{D} = 0$ implies $\alpha_3 = 0$, yielding $n = (g^2-1)/4$. Considering (3.21), we observe that in this case the singular point M_4 coalesces with M_3, producing a double singular point on the invariant parabola. Thus, the finite singularities of systems (3.19) have the following coordinates:

$$x_1 = \frac{\sqrt{1-g^2}}{2}, \quad y_1 = \frac{1-g^2}{4}; \quad x_2 = -\frac{\sqrt{1-g^2}}{2}, \quad y_2 = \frac{1-g^2}{4};$$
$$x_3 = x_4 = \frac{1-g}{2}, \quad y_3 = y_4 = \frac{(1-g)^2}{4}.$$

We note that in this case $\beta_3 = 2g(g-1)$, and it is necessary to determine, in an invariant way, the signs of the expressions $1-g^2$ and $g(g-1)$. For systems (3.19) with $n = (g^2-1)/4$, we calculate:

$$\xi_2 = \frac{1}{2}(1-g^2)^3 g^4, \quad \xi_9 = 44718505920(g-1)g^9(1+g)^4.$$

We observe that $\operatorname{sign}(\xi_2) = \operatorname{sign}(1-g^2)$ and $\operatorname{sign}(\xi_9) = \operatorname{sign}(g(g-1))$.

Thus, in the case $\alpha_3 = 0$ which implies $\mathbf{D} = 0$ (and the existence of a double real singularity on the invariant parabola), we obtain the following configurations:

$$\begin{array}{lll} \xi_2 < 0 & \Rightarrow\ M_1 \text{ and } M_2 \text{ are complex} & \Rightarrow\ \textit{Config. } \mathcal{P}.21; \\ \xi_2 > 0,\ \xi_9 < 0 & \Rightarrow\ (x_3-x_1)(x_3-x_2) < 0 & \Rightarrow\ \textit{Config. } \mathcal{P}.22; \\ \xi_2 > 0,\ \xi_9 > 0 & \Rightarrow\ (x_3-x_1) > 0,\ (x_3-x_2) > 0 & \Rightarrow\ \textit{Config. } \mathcal{P}.23. \end{array}$$

2.1.3.2: *The possibility* $\xi_9 = 0$. This implies $\beta_3 = 0$, and hence we get $n = -(g-1)^2/4$. We observe that in this case, considering (3.21), we obtain:

$$x_1 = \frac{1}{2}\sqrt{(g-1)^2}, \quad y_1 = \frac{(g-1)^2}{4}; \quad x_2 = -\frac{1}{2}\sqrt{(g-1)^2}, \quad y_2 = \frac{(g-1)^2}{4};$$
$$x_3 = \frac{1-g}{2}, \quad y_3 = \frac{(g-1)^2}{4}; \quad x_4 = \frac{(g-1)^2}{2(1+g)}, \quad y_4 = \frac{(g-1)^3}{4(1+g)}.$$

We observe that the singular point M_3 either coincides with M_1 or with M_2. Since x_1 is positive and x_2 is negative, we conclude that M_3 coalesces with M_1 if $1-g > 0$, and with M_2 if $1-g < 0$.

On the other hand, for systems (3.19) with $n = -(g-1)^2/4$, we have:

$$\xi_{10} = 12705209326080(g-1)g^6(1+g)^4,$$

and hence we obtain $\operatorname{sign}(\xi_{10}) = \operatorname{sign}(g-1)$. Therefore, it is not difficult to determine that we obtain the configuration *Config.* $\mathcal{P}.24$ if $\xi_{10} < 0$, and *Config.* $\mathcal{P}.25$ if $\xi_{10} > 0$.

2.2: *The case* $B_2 = 0$. Since $\xi_4 \neq 0$ (i.e., $g(1+4n) \neq 0$), considering (3.20), this condition implies $(1+g)^2 + 4n = 0$.

Then we get $n = -(1+g)^2/4$, which leads to the family of systems:

$$\dot{x} = -\frac{1}{8}(1+g-2x)(-1+g^2+4gx+4y), \quad \dot{y} = -\frac{1}{4}(1+2g+g^2-4y)(-x+gx+2y), \tag{3.23}$$

which possess the following three invariant affine lines:

$$1+g-2x = 0, \quad 1+2g+g^2-4y = 0, \quad 1-g^2-4x+4y = 0.$$

For these systems, we have $B_2 = B_3 = 0$, and we see that they possess invariant lines in three distinct directions. However, parallel invariant lines may appear, and by Lemma 2.4, for this to occur, it is necessary that $\theta = 0$. Thus, we discuss two subcases: $\theta \neq 0$ and $\theta = 0$.

2.2.1: *The subcase* $\theta \neq 0$. For the above systems, we calculate the coordinates of the finite singularities $M_i(x_i, y_i)$ $(i = 1, 2, 3, 4)$:

$$x_1 = \frac{1+g}{2}, \quad y_1 = \frac{(1+g)^2}{4}; \quad x_2 = -\frac{1+g}{2}, \quad y_2 = \frac{(1+g)^2}{4};$$
$$x_3 = \frac{1-g}{2}, \quad y_3 = \frac{(1-g)^2}{4}; \quad x_4 = \frac{1+g}{2}, \quad y_4 = \frac{1-g^2}{4}.$$

We observe that the singular point M_1 is the intersection point of all three invariant lines, as well as lying on the invariant parabola. Since this point, together with M_4, lies on the vertical invariant line $1 + g - 2x = 0$, the relative positions of these two points are crucial for determining the configurations of systems (3.23). Thus, we calculate:

$$y_4 - y_1 = -\frac{g(g+1)}{2} \quad \Rightarrow \quad \text{sign}\,(y_4 - y_1) = -\text{sign}\,\big(g(g+1)\big).$$

We also point out that the position of the vertical invariant line $x = (g+1)/2$ is important, and we must consider sign $(g+1)$.

On the other hand, for systems (3.23), we calculate: $\zeta_2 = 4g(1+g)$, and then we determine the following configurations:

$$\begin{array}{lll} \zeta_2 < 0 \text{ (i.e., } -1 < g < 0) & \Rightarrow \ x_1 > 0,\ y_4 > y_1 & \Rightarrow \ \textit{Config. } \mathcal{P}.26; \\ \zeta_2 > 0 \text{ and } g < -1 & \Rightarrow \ x_1 < 0,\ y_4 < y_1 & \Rightarrow \ \textit{Config. } \mathcal{P}.27; \\ \zeta_2 > 0 \text{ and } g > 0 & \Rightarrow \ x_1 > 0,\ y_4 < y_1 & \Rightarrow \ \simeq \textit{Config. } \mathcal{P}.27. \end{array}$$

2.2.2: *The subcase* $\theta = 0$. This condition implies $(g-1)(g+2) = 0$.

If $g = 1$, we arrive at the system:

$$\dot{x} = (x-1)(x+y), \quad \dot{y} = 2(y-1)y, \tag{3.24}$$

which possesses four invariant affine lines: $x = 1$, $y = 0$, $y = 1$, and $y = x$. Therefore, it is easy to determine that this system corresponds to the configuration *Config.* $\mathcal{P}.28$.

Assuming $g = -2$, we arrive at the system:

$$\dot{x} = \frac{1}{8}(1+2x)(3-8x+4y), \quad \dot{y} = -\frac{1}{4}(4y-1)(3x-2y),$$

which, via the transformation $x_1 = -x + 1/2$, $y_1 = -x + y + 1/4$, can be brought to the system (3.24), thus also corresponding to configuration *Config.* $\mathcal{P}.28$.

The subcase $\xi_4 = 0$**.** Considering (3.13) and the condition $\zeta_4 \neq 0$, we obtain that the condition $\xi_4 = 0$ implies:

$$m(g + 8m + 4n) = 0.$$

On the other hand, according to Lemma 3.1, it is sufficient to examine the condition $m = 0$ because the condition $g + 8m + 4n = 0$ can be brought to $m = 0$ via an affine transformation.

Thus, setting $m = 0$, we arrive at the family of systems:

$$\dot{x} = nx - \frac{1}{2}(1+g)y + gx^2 + xy, \quad \dot{y} = y(2n - x + gx + 2y), \tag{3.25}$$

which possess the invariant line $y = 0$ and the invariant parabola $\Phi(x, y) = x^2 - y = 0$. It is clear that the invariant line $y = 0$ is tangent to the invariant parabola at the origin.

We determine that for the above systems, the condition $\zeta_1\zeta_2\zeta_4\mathcal{R}_1 \neq 0$ implies:

$$g(1+g)(g-2)(3+g)(1+3g)(1+4g+3g^2+2n) \neq 0. \tag{3.26}$$

For systems (3.25), we calculate:

$$\begin{gathered} B_2 = -\frac{81}{2}(1+g)^2(1+g+2n)(g+g^2+2n)(g+4n)y^4, \\ \theta = -8(g-1)(2+g). \end{gathered} \tag{3.27}$$

We now consider two possibilities: $B_2 \neq 0$ and $B_2 = 0$.

1: *The possibility* $B_2 \neq 0$. Then, besides the invariant line $y = 0$, systems (3.25) cannot possess invariant lines in other directions. However, there could exist an invariant line parallel to $y = 0$, and by Lemma 2.4, a necessary condition for this is $\theta = 0$. Therefore, we discuss two cases: $\theta \neq 0$ and $\theta = 0$.

1.1: *The case* $\theta \neq 0$. The systems (3.25) possess four finite singularities $M_i(x_i, y_i)$ $(i = 1, 2, 3, 4)$ with coordinates:

$$\begin{gathered} x_1 = 0, \quad y_1 = 0; \\ x_2 = -\frac{n}{g}, \quad y_2 = 0; \\ x_{3,4} = \frac{1}{4}\big(1 - g \pm \sqrt{Z_3}\,\big), \quad y_{3,4} = \frac{1}{8}\big[(1-g)^2 - 8n \pm (1-g)\sqrt{Z_3}\,\big], \\ Z_3 = (1-g)^2 - 16n. \end{gathered} \tag{3.28}$$

We observe that $\Phi(x_3, y_3) = \Phi(x_4, y_4) = 0$, meaning that the singular points M_3 and M_4 lie on the invariant parabola. Moreover, the singularity M_2 lies on the invariant line $y = 0$ and coalesces with M_1 if and only if $n = 0$.

The singularities M_3 and M_4 are complex (respectively, real) if $Z_3 < 0$ (respectively, $Z_3 > 0$), and they coincide (producing a multiple singular point) when $Z_3 = 0$.

On the other hand, for systems (3.25), we have:

$$\mathbf{D} = 48(1+g)^4n^6(-1+2g-g^2+16n) = -48(1+g)^4n^6Z_3,$$

and we proceed to discuss three subcases: $\mathbf{D} < 0$, $\mathbf{D} > 0$, and $\mathbf{D} = 0$.

1.1.1: *The subcase* $\mathbf{D} < 0$. Then $Z_3 > 0$, and therefore the finite singularities M_3 and M_4 are real and distinct [1, Proposition 5.1]. To determine their positions on the parabola with respect to the singularity M_1, we calculate:

$$\begin{gathered} (x_3 - x_1)(x_4 - x_1) = n, \quad (x_3 - x_1) + (x_4 - x_1) = (1-g)/2; \\ \operatorname{sign}\big((x_3 - x_1)(x_4 - x_1)\big) = \operatorname{sign}(n), \quad \operatorname{sign}\big((x_3 - x_1) + (x_4 - x_1)\big) = \operatorname{sign}(1-g). \end{gathered} \tag{3.29}$$

We point out that $g - 1 \neq 0$ (due to $\theta \neq 0$), and the sign of $1 - g$ is only relevant when $n > 0$ (i.e., when $(x_3 - x_1)(x_4 - x_1) > 0$).

Furthermore:

$$x_2 - x_1 = -n/g \quad \Rightarrow \quad \operatorname{sign}(x_2 - x_1) = -\operatorname{sign}(gn).$$

For systems (3.25), calculations yield:

$$\xi_{11} = -95982880gn(g-2)^2(1+g)^2(3+g)^2(1+3g)^2(1+4g+3g^2+2n)^2,$$
$$\xi_{28} = 3244620(1+g)^2(3+g)^2(1+3g)^2(1+4g+3g^2+2n)^2(g+4n),$$
$$\xi_{29} = 1700495253(g-1)g(1+g)^2(3+g)(1+3g)(1+4g+3g^2+2n)^2(g+4n)/16.$$

Remark 3.2. We observe that due to the condition (3.26), we have $\xi_{11} \neq 0$, with $\text{sign}\,(\xi_{11}) = -\text{sign}\,(gn)$. If $\xi_{11} < 0$ (i.e., $gn > 0$), then $\text{sign}\,(\xi_{28}) = \text{sign}\,(g+4n)$. Moreover, in the case $g > 0$ and $n > 0$, we also have $\text{sign}\,(\xi_{29}) = \text{sign}\,(g-1)$.

Thus, considering the above relations in the case $\mathbf{D} < 0$, we obtain the following configurations:

$\xi_{11} < 0$, $n < 0$ (then $g < 0$) $\Rightarrow$ $(x_3 - x_1)(x_4 - x_1) < 0$, $x_2 < x_1$ $\Rightarrow$ *Config.* $\mathcal{P}$.29;
$\xi_{11} < 0$, $n > 0$ (then $g > 0$), $g < 1$ $\Rightarrow$ $x_3 > x_1$, $x_4 > x_1$, $x_2 < x_1$ $\Rightarrow$ *Config.* $\mathcal{P}$.30;
$\xi_{11} < 0$, $n > 0$ (then $g > 0$), $g > 1$ $\Rightarrow$ $x_3 < x_1$, $x_4 < x_1$, $x_2 < x_1$ $\Rightarrow$ *Config.* $\mathcal{P}$.31;
$\xi_{11} > 0$, $n < 0$ (then $g > 0$) $\Rightarrow$ $(x_3 - x_1)(x_4 - x_1) < 0$, $x_2 > x_1$ $\Rightarrow$ *Config.* $\mathcal{P}$.32;
$\xi_{11} > 0$, $n > 0$ (then $g < 0$) $\Rightarrow$ $x_3 > x_1$, $x_4 > x_1$, $x_2 > x_1$ $\Rightarrow$ *Config.* $\mathcal{P}$.33.

Taking into account Remark 3.2, we obtain the following invariant conditions:

$\xi_{11} < 0$, $\xi_{28} < 0$ $\Rightarrow$ *Config.* $\mathcal{P}$.29;
$\xi_{11} < 0$, $\xi_{28} > 0$, $\xi_{29} < 0$ $\Rightarrow$ *Config.* $\mathcal{P}$.30;
$\xi_{11} < 0$, $\xi_{28} > 0$, $\xi_{29} > 0$ $\Rightarrow$ *Config.* $\mathcal{P}$.31;
$\xi_{11} > 0$ $\Rightarrow$ *Config.* $\mathcal{P}$.32, or *Config.* $\mathcal{P}$.33.

1.1.2: *The subcase* $\mathbf{D} > 0$. Then $Z_3 < 0$, and hence the finite singularities M_3 and M_4 are complex. This condition also implies $n > 0$, meaning that the singular point $M_2(-n/g, 0)$ cannot coalesce with $M_1(0,0)$. Moreover, its position relative to M_1 depends on the sign of the parameter g.

Since $n > 0$, we have:

$$\text{sign}\,(\xi_{11}) = -\text{sign}\,(gn) = -\text{sign}\,(g).$$

Therefore, we obtain *Config.* $\mathcal{P}$.34 if $\xi_{11} < 0$, and *Config.* $\mathcal{P}$.35 if $\xi_{11} > 0$.

1.1.3: *The subcase* $\mathbf{D} = 0$. Considering (3.26), we deduce that $\mathbf{D} = 0$ implies $nZ_3 = 0$. For systems (3.25), we calculate:

$$\xi_1 = 3(g-2)(1+g)^2(3+g)(1+3g)n^2(1+4g+3g^2+2n)/8.$$

By (3.26), the condition $n = 0$ is equivalent to $\xi_1 = 0$. Thus, we examine two possibilities: $\xi_1 \neq 0$ and $\xi_1 = 0$.

1.1.3.1: *The possibility* $\xi_1 \neq 0$. Then $\mathbf{D} = 0$ implies $Z_3 = 0$. From (3.28), we get $n = (1-g)^2/16 \neq 0$, and thus:

$$x_1 = 0, \quad y_1 = 0; \quad x_2 = -\frac{(1-g)^2}{16g}, \quad y_2 = 0;$$
$$x_3 = x_4 = \frac{1-g}{4}, \quad y_3 = y_4 = \frac{(1-g)^2}{16};$$
$$\text{sign}\,(x_2 - x_1) = -\text{sign}\,(g), \quad \text{sign}\,(x_3 - x_1) = \text{sign}\,(1-g).$$

Therefore, there is a double singular point on the invariant parabola. For the parameter g, the bifurcation values are $g \in \{0, 1\}$.

For systems (3.25) with $n = (1-g)^2/16$ (i.e., $Z_3 = 0$), we compute:

$$\xi_{11} = -\frac{2999465}{32}(g-2)^2(g-1)^2g(1+g)^2(3+g)^2(1+3g)^2(3+5g)^4, \quad \theta = -8(g-1)(2+g),$$
$$\zeta_2 = 4g(1+g), \quad \zeta_4 = (g-2)(3+g)(1+3g)(3+5g)^2/128, \quad \xi_{12} = g(g-1)^3\psi_1(g)/256,$$

where $\psi_1(g) = 1105 + 1774g + 961g^2$. Observing that Discrim $[\psi_1(g), g] = -1100544 < 0$, we note that $\psi_1(g)$ does not vanish for real g.

Considering the condition $\zeta_4\theta \neq 0$, we conclude:

$$\text{sign}\,(\xi_{11}) = -\text{sign}\,(g), \quad \text{sign}\,(\xi_{12}) = \text{sign}\,(g(g-1)).$$

Thus, we determine the following configurations:

$$\begin{array}{lcll} \xi_{12} < 0 & \Rightarrow & x_2 < x_1,\, x_3 > x_1 & \Rightarrow \textit{ Config. } \mathcal{P}.36; \\ \xi_{12} > 0,\, \xi_{11} < 0 & \Rightarrow & x_2 < x_1,\, x_3 > x_1 & \Rightarrow \textit{ Config. } \mathcal{P}.37; \\ \xi_{12} > 0,\, \xi_{11} > 0 & \Rightarrow & x_2 > x_1,\, x_3 > x_1 & \Rightarrow \textit{ Config. } \mathcal{P}.38. \end{array}$$

1.1.3.2: *The possibility* $\xi_1 = 0$. In this case, $n = 0$, and the singular point $M_2(-n/g, 0)$ coalesces with $M_1(0,0)$. Moreover, one of the singularities M_3 or M_4 also coalesces with M_1, resulting in a triple finite singularity at $M_1(0,0)$. There could still remain a distinct simple singularity (M_3 or M_4), whose position depends on sign $(1-g)$ (see (3.29)).

For $n = 0$, systems (3.25) satisfy:

$$\xi_{10} = 24814861965(g-1)g^2(1+g)^6(1+3g)^4/2,$$

so that: sign (ξ_{10}) = sign $(g-1)$. Therefore:

$\xi_{10} < 0 \;\Rightarrow$ *Config.* $\mathcal{P}.39$;
$\xi_{10} > 0 \;\Rightarrow$ *Config.* $\mathcal{P}.40$.

1.2: *The case* $\theta = 0$. This condition implies $(g-1)(g+2) = 0$. For systems (3.25), we calculate:

$$\begin{array}{c} \xi_1 = 3(g-2)(1+g)^2(3+g)(1+3g)n^2(1+4g+3g^2+2n)/8, \\ \xi_5 = -21875(g-1)(1+g)(g-2)(3+g)(1+3g)n(1+g+2n)(1+4g+3g^2+2n)/8. \end{array} \tag{3.30}$$

We now discuss two subcases: $\xi_1 \neq 0$ and $\xi_1 = 0$.

1.2.1: *The subcase* $\xi_1 \neq 0$. This implies $n \neq 0$, and considering (3.26) and $B_2 \neq 0$ (i.e., $1+g+2n \neq 0$), we conclude that the condition $g = 1$ is equivalent to $\xi_5 = 0$. Thus, we analyze two possibilities: $\xi_5 \neq 0$ and $\xi_5 = 0$.

1.2.1.1: *The possibility* $\xi_5 \neq 0$. Then $g - 1 \neq 0$, and the condition $\theta = 0$ implies $g = -2$. Straightforward calculations show that for $g = -2$, the systems (3.25) do not possess any invariant line parallel to $y = 0$.

For $g = -2$, the systems (3.25) yield:

$$\begin{array}{c} \mathbf{D} = 48n^6(16n-9), \\ \xi_{11} = 76786304000n(5+2n)^2, \\ \zeta_4 = 5(5+2n)/4. \end{array}$$

Thus, sign $(\xi_{11}) = \text{sign}\,(n)$, and sign $(\mathbf{D}) = \text{sign}\,(16n-9)$. Moreover, since $n \neq 0$, we conclude that $\mathbf{D} = 0$ corresponds to $16n - 9 = 0$.

Therefore, considering that $g = -2 < 0$ and applying the previous analysis for systems (3.25), we determine the following configurations:

$$\begin{array}{lll} \mathbf{D} < 0, \xi_{11} < 0 & \Rightarrow & \textit{Config.}\ \mathcal{P}.29; \\ \mathbf{D} < 0, \xi_{11} > 0 & \Rightarrow & \textit{Config.}\ \mathcal{P}.33; \\ \mathbf{D} > 0 & \Rightarrow & \textit{Config.}\ \mathcal{P}.35; \\ \mathbf{D} = 0 & \Rightarrow & \textit{Config.}\ \mathcal{P}.38. \end{array}$$

1.2.1.2: *The possibility* $\xi_5 = 0$. Then $g = 1$, leading to the family of systems:

$$\dot{x} = nx - y + x^2 + xy, \quad \dot{y} = 2y(n + y), \tag{3.31}$$

which possess the additional invariant line $y + n = 0$.

From (3.28), we find:

$$\begin{array}{c} x_1 = 0, \quad y_1 = 0; \\ x_2 = -n, \quad y_2 = 0; \\ x_{3,4} = \pm\sqrt{-n}, \quad y_3 = y_4 = -n. \end{array}$$

We observe that the invariant line $y = -n$ intersects the invariant parabola $\Phi(x, y) = x^2 - y = 0$ at two points $M_{3,4}(\pm\sqrt{-n}, -n)$, which are distinct due to $\xi_1 \neq 0$ (i.e., $n \neq 0$). These points are real if $n < 0$ and complex if $n > 0$.

Calculating the invariant $\mathbf{D} = 12288n^7$, we get sign $(\mathbf{D}) = \text{sign}\,(n)$. Therefore:

$\mathbf{D} < 0 \Rightarrow$ *Config.* $\mathcal{P}.41$;
$\mathbf{D} > 0 \Rightarrow$ *Config.* $\mathcal{P}.42$.

1.2.2: *The subcase* $\xi_1 = 0$. This implies $n = 0$, and for systems (3.25) we obtain:

$$B_2 = -81g^2(1+g)^4y^4/2, \quad \xi_9 = 5589813240(g-1)^2g^6(1+g)^6.$$

Therefore, since $B_2 \neq 0$, we conclude that the condition $\xi_9 = 0$ is equivalent to $g - 1 = 0$. Thus, we discuss two cases: $\xi_9 \neq 0$ and $\xi_9 = 0$.

1.2.2.1: *The case* $\xi_9 \neq 0$. Then, $\theta = 0$ implies $g = -2$, and we obtain the system:

$$\dot{x} = \frac{y}{2} - 2x^2 + xy, \quad \dot{y} = -y(3y - 2x).$$

This system belongs to the family (3.25) with $n = 0$ (i.e., $\xi_1 = 0$) and $g = -2$. As previously shown (see point ***1.1.3.2*** on page 41), in the case $n = 0$, the singular point $M_2(-n/g, 0)$ coalesces with $M_1(0,0)$. Moreover, one of the singular points M_3 or M_4 also coalesces with $M_1(0,0)$, yielding a triple finite singularity at $M_1(0,0)$.

On the other hand, since sign $(\xi_{10}) = \text{sign}\,(g-1)$, for $g = -2$ we get $\xi_{10} < 0$. Therefore, the system has the configuration *Config.* $\mathcal{P}.39$.

1.2.2.2: *The case* $\xi_9 = 0$. Then $g = 1$, and we obtain the systems (3.31). For $n = 0$, the invariant line $y = -n$ coalesces with $y = 0$, producing a double invariant line. Moreover, all finite singular points coalesce, generating a quadruple singularity at $M_1(0,0)$. As a result, the system has the configuration *Config.* $\mathcal{P}.43$.

2: *The possibility* $B_2 = 0$. We first make a remark.

Remark 3.3. The condition $B_2 = 0$ implies $n \neq 0$ for systems (3.25). Indeed, in the case $n = 0$, for systems (3.25), we obtain:

$$B_2 = -81g^2(1+g)^4y^4/2 \neq 0$$

due to the condition $\zeta_2 \neq 0$ (i.e., $g(g+1) \neq 0$).

Thus, $n \neq 0$, and since $g+1 \neq 0$, considering (3.27), we get the condition:

$$(1+g+2n)(g+g^2+2n)(g+4n) = 0.$$

Considering (3.30), we examine two cases: $\xi_5 \neq 0$ and $\xi_5 = 0$.

2.1: *The case* $\xi_5 \neq 0$. Then, by (3.30), we get $1+g+2n \neq 0$, and hence the condition $B_2 = 0$ implies: $(g+g^2+2n)(g+4n) = 0$.

On the other hand, for systems (3.25), we calculate:

$$\xi_{13} = 27(1+g)^2(3+g)(1+3g)n^2(g+4n)/4,$$

and considering Remark 3.3 and the condition (3.26), we conclude that $\xi_{13} = 0$ is equivalent to $g+4n = 0$.

2.1.1: *The subcase* $\xi_{13} \neq 0$. Then $g+4n \neq 0$, and therefore $B_2 = 0$ implies $g+g^2+2n = 0$. In this case, we get $n = -g(g+1)/2$, and we arrive at the following family of systems:

$$\dot{x} = (2x-1-g)(gx+y)/2, \quad \dot{y} = -y(g+g^2+x-gx-2y), \tag{3.32}$$

which possess the invariant lines $y = 0$ and $x = (g+1)/2$.

Considering Lemma 2.3, for these systems we calculate:

$$B_3 = -3(g-1)g(1+g)^2(1+2g)x^2y^2/4, \quad \theta = -8(g-1)(g+2),$$
$$\xi_5 = -21875(g-2)(g-1)^2g(1+g)^4(3+g)(1+2g)(1+3g)/16.$$

We observe that $B_3 \neq 0$ due to $\xi_5 \neq 0$, and hence by Lemma 2.3, the above systems cannot have any invariant line in the third direction.

However, according to Lemma 2.4, we could have parallel invariant lines if $\theta = 0$. Due to $B_3 \neq 0$ (i.e., $g-1 \neq 0$), the condition $\theta = 0$ is equivalent to $g+2 = 0$. It is straightforward to check that for $g = -2$, systems (3.32) do not have any invariant line parallel to $y = 0$ or to $x = (g+1)/2$.

The systems (3.32) possess the following finite singularities $M_i(x_i, y_i)$, $(i = 1,2,3,4)$, with coordinates:

$$x_1 = 0, \quad y_1 = 0;\ x_2 = \frac{1+g}{2}, \quad y_2 = 0;\ x_3 = -g, \quad y_3 = g^2;\ x_4 = \frac{1+g}{2}, \quad y_4 = \frac{(1+g)^2}{4}.$$

We observe that the invariant line $x = (g+1)/2$ intersects the invariant parabola at point M_4 and the invariant line $y = 0$ at singularity M_2. Furthermore, the invariant line $y = 0$ is tangent to the parabola at M_1.

To determine the relative positions of the line $x = (g+1)/2$ and the singularities, we calculate:

$$x_2 - x_1 = \frac{1+g}{2},\ x_3 - x_1 = -g,\ x_3 - x_4 = -\frac{1+3g}{2},$$
$$\operatorname{sign}(x_2 - x_1) = \operatorname{sign}(1+g),\ \operatorname{sign}(x_3 - x_1) = -\operatorname{sign}(g),\ \operatorname{sign}(x_3 - x_4) = -\operatorname{sign}(1+3g).$$

Thus, for the parameter g, we have the following possible bifurcation values: $g \in \{-1, -1/3, 0\}$.

On the other hand, for systems (3.32), we calculate:

$$\zeta_2 = 4g(g+1), \quad \xi_7 = 1174627500g^4(1+g)^7(1+2g)^2(1+3g),$$
$$\xi_{11} = 47991440(g-2)^2g^2(1+g)^5(3+g)^2(1+2g)^2(1+3g)^2$$

and we observe that:

$$\text{sign}\,(\zeta_2) = \text{sign}\,(g(g+1)), \quad \text{sign}\,(\xi_7) = \text{sign}\,((g+1)(1+3g)), \quad \text{sign}\,(\xi_{11}) = \text{sign}\,(g+1).$$

Moreover, in the case $\zeta_2 < 0$, we have $-1 < g < 0$ (i.e., $g+1 > 0$), and then:

$$\text{sign}\,(\xi_7) = \text{sign}\,(1+3g).$$

Thus, considering the above relations, we obtain the following configurations:

$\zeta_2 < 0, \xi_7 < 0$ (i.e., $-1 < g < -1/3$) $\Rightarrow$ $x_2 > x_1, x_3 > x_1, x_3 > x_4 \Rightarrow$ *Config.* $\mathcal{P}.44$;
$\zeta_2 < 0, \xi_7 > 0$ (i.e., $-1/3 < g < 0$) $\Rightarrow$ $x_2 > x_1, x_1 < x_3 < x_4 \Rightarrow$ *Config.* $\mathcal{P}.45$;
$\zeta_2 > 0, \xi_{11} < 0$ (i.e., $g < -1$) $\Rightarrow$ $x_2 < x_1, x_3 > x_1 \Rightarrow$ *Config.* $\mathcal{P}.46$;
$\zeta_2 > 0, \xi_{11} > 0$ (i.e., $g > 0$) $\Rightarrow$ $x_2 > x_1, x_3 < x_1 \Rightarrow$ *Config.* $\mathcal{P}.47$.

2.1.2: *The subcase* $\xi_{13} = 0$. This implies $g + 4n = 0$ (i.e., $n = -g/4$) and we arrive at the family of systems

$$\dot{x} = -g(1+g)x/2 - (1+g)y/2 + gx^2 + xy, \quad \dot{y} = -y(g + g^2 + x - gx - 2y), \tag{3.33}$$

possessing the invariant lines $y = 0$ and $y = x - 1/4$. For these systems we have

$$\xi_5 = 21875(g-2)g(1+g)(g-1)(2+g)(3+g)(1+2g)(1+3g)(2+3g)/128,$$
$$B_3 = 3g(1+g)(1+2g)(x-y)^2y^2/8, \quad \theta = -8(g-1)(2+g),$$

and since $\xi_5 \neq 0$, we obtain $B_3\theta \neq 0$. So, by Lemmas 2.3 and 2.4, we conclude that the above systems could not have a third invariant line.

The systems (3.33) possess the finite singularities $M_i(x_i, y_i)$ $(i = 1, 2, 3, 4)$ with coordinates

$$x_1 = 0,\ y_1 = 0; \quad x_2 = 1/4,\ y_2 = 0; \quad x_3 = 1/2,\ y_3 = 1/4; \quad x_4 = -g/2,\ y_4 = g^2/4;$$
$$\text{sign}\,(x_4 - x_1) = -\text{sign}\,(g), \quad x_4 - x_3 = -(g+1)/2 \ \Rightarrow\ \text{sign}\,(x_4 - x_3) = -\text{sign}\,(g+1).$$

It could be checked directly that the invariant line $y = x - 1/4$ is tangent to the invariant parabola at the singular point $M_3(1/2, 1/4)$. Therefore, considering the above relations, we obtain the following configurations:

$\zeta_2 < 0$ (i.e., $-1 < g < 0$) $\Rightarrow$ $x_4 > x_1, x_4 < x_3$ $\Rightarrow$ *Config.* $\mathcal{P}.48$;
$\zeta_2 > 0$ and $g < -1$ $\Rightarrow$ $x_4 > x_1, x_4 > x_3$ $\Rightarrow$ *Config.* $\mathcal{P}.49$;
$\zeta_2 > 0$ and $g > 0$ $\Rightarrow$ $x_4 < x_1, x_4 < x_3$ $\Rightarrow$ $\simeq$ *Config.* $\mathcal{P}.49$.

2.2: *The case* $\xi_5 = 0$. Considering (3.30), the conditions (3.26) and Remark 3.3 imply $(g-1)(1+g+2n) = 0$, and we examine two subcases: $\theta \neq 0$ and $\theta = 0$.

2.2.1: *The subcase* $\theta \neq 0$. Then $g - 1 \neq 0$, and we get $1 + g + 2n = 0$. Therefore, $n = -(1+g)/2 \neq 0$, and we arrive at the family of systems

$$\dot{x} = -(1+g)(x+y)/2 + gx^2 + xy, \quad \dot{y} = -y(1 + g + x - gx - 2y), \tag{3.34}$$

possessing the invariant lines $y = 0$ and $y = x$, and the finite singularities $M_i(x_i, y_i)$ $(i = 1, 2, 3, 4)$ with the coordinates:

$$x_1 = 0, \quad y_1 = 0; \quad x_2 = \frac{1+g}{2g}, \quad y_2 = 0; \quad x_3 = 1, \quad y_3 = 1; \quad x_4 = -\frac{1+g}{2}, \quad y_4 = \frac{(1+g)^2}{4}.$$

On the other hand, considering Lemma 2.3, we calculate B_3 and we have:

$$B_3 = 3(g-1)(1+g)^2(x-y)^2y^2/4 \neq 0,$$

due to the conditions (3.26) and $\theta \neq 0$. Then, by Lemma 2.3, we could not have any invariant line in the third direction. Moreover, by Lemma 2.4, we could not have parallel invariant lines due to $\theta \neq 0$.

Next, considering the coordinates of the finite singularities of these systems, it follows immediately:

$$\begin{gathered}\operatorname{sign}(x_4 - x_1) = -\operatorname{sign}(1+g), \quad \operatorname{sign}(x_2 - x_1) = \operatorname{sign}\big(g(1+g)\big), \\ x_4 - x_3 = -(g+3)/2 \;\Rightarrow\; \operatorname{sign}(x_4 - x_3) = -\operatorname{sign}(g+3).\end{gathered}$$

We remark that $g(g+1)(g+3) \neq 0$ due to the condition (3.26), and hence for the parameter g we have the following possible bifurcation values: $g \in \{-3, -1, 0\}$.

On the other hand, for systems (3.34), we calculate:

$$\begin{gathered}\zeta_2 = 4g(1+g), \quad \xi_9 = 5589813240(g-1)^2g^2(1+g)^9(3+g), \\ \xi_{10} = -223333757685(g-1)^2g^2(1+g)^6(2+g)(1+3g)^2/2,\end{gathered}$$

and hence we have

$$\operatorname{sign}(\zeta_2) = \operatorname{sign}\big(g(1+g)\big), \quad \operatorname{sign}(\xi_9) = \operatorname{sign}\big((1+g)(3+g)\big), \quad \operatorname{sign}(\xi_{10}) = -\operatorname{sign}(2+g).$$

Remark 3.4. We observe that the conditions $\zeta_2 > 0$ and $\xi_9 > 0$ imply either $g > 0$ or $g < -3$. In order to distinguish these two possibilities, we use the invariant ξ_{10} even though this invariant does not vanish in the bifurcation values of g.

Considering the above remark, we arrive at the following configurations:

$\zeta_2 < 0$ (i.e., $-1 < g < 0$)	$\Rightarrow$ $x_2 < x_1$, $x_4 < x_1$	$\Rightarrow$ *Config.* $\mathcal{P}.50$;
$\zeta_2 > 0$, $\xi_9 < 0$ (i.e., $-3 < g < -1$)	$\Rightarrow$ $x_2 > x_1$, $x_1 < x_4 < x_3$	$\Rightarrow$ *Config.* $\mathcal{P}.51$;
$\zeta_2 > 0$, $\xi_9 > 0$, $\xi_{10} < 0$ (i.e., $g > 0$)	$\Rightarrow$ $x_2 > x_1$, $x_3 < x_1$	$\Rightarrow$ *Config.* $\mathcal{P}.52$;
$\zeta_2 > 0$, $\xi_9 > 0$, $\xi_{10} > 0$ (i.e., $g < -3$)	$\Rightarrow$ $x_2 > x_1$, $x_4 > x_3$	$\Rightarrow$ *Config.* $\mathcal{P}.53$.

2.2.2: *The subcase* $\theta = 0$. This implies $(g-1)(g+2) = 0$, and we discuss two possibilities: $B_3 \neq 0$ and $B_3 = 0$.

2.2.2.1: *The possibility* $B_3 \neq 0$. We claim that in this case we get the same configuration either if $g = 1$ or if $g = -2$.

Indeed, assume first $g = -2$. Then, calculations yield

$$\begin{gathered}\xi_5 = -328125\,n(2n-1)(5+2n)/2, \quad B_2 = -162(1+n)(2n-1)^2y^4, \\ B_3 = 3y^2\left[n(4n-5)x^2 + 2(1+n)xy - (1+n)y^2\right]/2,\end{gathered} \tag{3.35}$$

and the condition $\xi_5 = B_2 = 0$ gives us $n = 1/2$. This leads to the system

$$\dot{x} = (x+y)/2 - 2x^2 + xy, \quad \dot{y} = y(1 - 3x + 2y), \tag{3.36}$$

possessing three invariant affine lines: $y = 0$, $y = x$ and $y = x - 1/4$. Then, this system has the configuration equivalent to *Config.* $\mathcal{P}.54$.

Suppose now $g = 1$. Then, we have

$$\xi_5 = 0, \quad B_2 = -648(1+n)^2(1+4n)y^4, \quad B_3 = -3(1+n)y^2\left(4nx^2+2xy-y^2\right),$$

and due to $B_3 \neq 0$, the condition $B_2 = 0$ implies $n = -1/4$. In this case, we arrive at the system

$$\dot{x} = -x/4 - y + x^2 + xy, \quad \dot{y} = y(4y-1)/2,$$

which via the affine transformation $x_1 = -x + 1/2, \quad y_1 = -x + y + 1/4$ could be brought to system (3.36). Thus, our claim is proved and we get the configuration *Config.* $\mathcal{P}.54$.

2.2.2.2: *The possibility* $B_3 = 0$. Considering (3.35), we conclude that the condition $g = -2$ implies $B_3 \neq 0$, and hence the condition $\theta = 0$ gives us $g = 1$. In this case, we arrive at the system

$$\dot{x} = (x-1)(x+y), \quad \dot{y} = 2(y-1)y,$$

possessing four invariant affine lines: $x = 1$, $y = 0$, $y = 1$ and $y = x$. Therefore, it is easy to determine that this system possesses the configuration equivalent to *Config.* $\mathcal{P}.28$.

3.1.2 The statement $(\mathcal{A}_2^*)$

According to the statement $(\mathcal{A}_2^*)$ of Proposition 2.7*, for systems (3.1) the condition $\zeta_4 = 0$ must hold. Considering (3.2), we obtain

$$(g-2)(3+g)(1+7g+15g^2+9g^3-4m+2n+6gn) = 0,$$

and since $(g-2)(3+g) \neq 0$ (due to $\zeta_1 \neq 0$), we get

$$m = \frac{1}{4}(1+3g)(1+4g+3g^2+2n).$$

Then, we arrive at the two-parameter family of systems

$$\begin{aligned} \dot{x} &= \frac{1}{4}(1+3g)(1+4g+3g^2+2n) + nx - \frac{1}{2}(1+g)y + gx^2 + xy, \\ \dot{y} &= \frac{1}{2}(1+3g)(1+4g+3g^2+2n)x + 2ny + (g-1)xy + 2y^2 \end{aligned} \tag{3.37}$$

possessing the following two invariant parabolas: $\Phi_1(x,y) = x^2 - y = 0$ and

$$\begin{aligned} \Phi_2(x,y) = &-(1+4g+3g^2+2n)(1+4g+3g^2+4n) + 2(1+g)(1+4g+3g^2+4n)x \\ &+ 4g(1+g)x^2 - 2(1+6g+5g^2+4n)y = 0. \end{aligned} \tag{3.38}$$

Following the statement $(\mathcal{A}_2^*)$, for the above systems we calculate

$$\begin{aligned} \zeta_1 &= 2(g-2)(3+g), \quad \zeta_2 = 4g(1+g), \quad \zeta_4 = 0, \\ \zeta_5 &= 19(g-2)(3+g)(1+4g+3g^2+4n)^2/4, \\ \mathcal{R}_2 &= -(g-2)(3+g)(8+27g+27g^2)(1+6g+5g^2+4n)/16, \\ B_1 &= g(1+g)(1+2g)(1+3g)(2+3g)(1+4g+3g^2+2n)(1+6g+5g^2+4n) \\ &\quad \times (1+6g+6g^2+4n)(1+6g+9g^2+4n)(5+14g+9g^2+4n)/32. \end{aligned} \tag{3.39}$$

According to Lemma 2.3, for the existence of an invariant line of systems (3.37), the condition $B_1 = 0$ is necessary.

The case $B_1 \neq 0$**.** Then we could not have any invariant line. The systems (3.37) possess four finite singularities $M_i(x_i, y_i)$ $(i = 1, 2, 3, 4)$ with the coordinates

$$\begin{aligned} x_1 &= -\frac{1+3g}{2},\ y_1 = \frac{(1+3g)^2}{4}; \quad x_2 = -\frac{(1+g)\,(1+3g)^2 + 4(1+2g)\,n}{2g(1+g)}, \\ y_2 &= \frac{(1+3g)\,(1+4g+3g^2+2n)}{2(1+g)}; \quad x_{3,4} = \frac{1}{2}\big(1+g \pm \sqrt{Z_4}\,\big), \\ y_{3,4} &= -\frac{1}{2}\big(2g+2g^2+2n \mp (g+1)\sqrt{Z_4}\,\big), \quad Z_4 = -(1+6g+5g^2+4n). \end{aligned} \tag{3.40}$$

In order to determine the position of the finite singularities with respect to the parabolas $\Phi_1(x, y) = 0$ and $\Phi_2(x, y) = 0$, we calculate

$$\Phi_1(x_1, y_1) = \Phi_1(x_3, y_3) = \Phi_1(x_4, y_4) = 0; \quad \Phi_2(x_2, y_2) = \Phi_2(x_3, y_3) = \Phi_2(x_4, y_4) = 0. \tag{3.41}$$

Therefore, the singularities M_3 and M_4 are the finite intersection points of these two invariant parabolas. We observe that the points of intersection of the invariant parabolas are complex if $Z_4 < 0$ and they are real if $Z_4 > 0$.

On the other hand, for systems (3.37) we calculate:

$$\mathbf{D} = -3Z_4(1+4g+3g^2+4n)^2\,\alpha_4^2\,\beta_4^2/4, \tag{3.42}$$

where

$$\alpha_4 = 5+22g+21g^2+4n, \quad \beta_4 = (1+g)(1+3g)(1+6g+7g^2) + 4(1+2g)^2 n. \tag{3.43}$$

So if $\mathbf{D} \neq 0$, then $\operatorname{sign}(\mathbf{D}) = -\operatorname{sign}(Z_4)$ and we discuss three possibilities: $\mathbf{D} < 0$, $\mathbf{D} > 0$ and $\mathbf{D} = 0$.

1: *The possibility* $\mathbf{D} < 0$. Then $Z_4 > 0$ and systems (3.37) possess four real singularities and it is necessary to know the positions of the singularities $M_{3,4}$ with respect to M_1 and M_2. We calculate

$$\begin{aligned} (x_1-x_3)(x_1-x_4) &= \frac{\alpha_4}{4}, \quad (x_1-x_3)+(x_1-x_4) = -2(1+2g), \\ (x_2-x_3)(x_2-x_4) &= -\frac{Z_4\,\beta_4}{4g^2(1+g)^2}, \quad (x_2-x_3)+(x_2-x_4) = -\frac{(1+2g)\,Z_4}{g(1+g)}. \end{aligned}$$

Therefore, considering the condition $Z_4 > 0$, we obtain

$$\begin{gathered} \operatorname{sign}\big((x_1-x_3)(x_1-x_4)\big) = \operatorname{sign}(\alpha_4), \quad \operatorname{sign}\big((x_1-x_3)+(x_1-x_4)\big) = -\operatorname{sign}(1+2g) \quad \text{if } \alpha_4 > 0; \\ \operatorname{sign}\big((x_2-x_3)(x_2-x_4)\big) = -\operatorname{sign}(\beta_4), \\ \operatorname{sign}\big((x_2-x_3)+(x_2-x_4)\big) = -\operatorname{sign}\big(g(1+g)(1+2g)\big) \quad \text{if } \beta_4 < 0, \end{gathered}$$

where $1+2g \neq 0$ due to $B_1 \neq 0$.

We need invariant polynomials governing the signs of α_4 and β_4. For systems (3.37), we calculate:

$$\xi_{14} = 1235\alpha_4\beta_4/2, \quad \xi_{30} = 1235\big[Z_4\,\beta_4 - g^2(1+g)^2\,\alpha_4\big]/4, \quad \zeta_2 = 4g(1+g),$$

and we have

$$\operatorname{sign}(\xi_{14}) = \operatorname{sign}(\alpha_4\beta_4), \quad \operatorname{sign}(\zeta_2) = \operatorname{sign}\big(g(1+g)\big).$$

Moreover, in the case $\xi_{14} < 0$ (i.e. $\alpha_4\beta_4 < 0$) and $\mathbf{D} < 0$ (i.e. $Z_4 > 0$), we obtain

$$\operatorname{sign}(\xi_{30}) = \operatorname{sign}\left(Z_4\beta_4 - g^2(1+g)^2\alpha_4\right) = \operatorname{sign}(\beta_4).$$

On the other hand, considering the form of the invariant parabola $\Phi_2(x,y) = 0$, we have

$$\begin{aligned} y = & -\frac{1}{Z_4}\left[(1+g)(1+4g+3g^2+4n)x - \frac{1}{2}(1+4g+3g^2+2n)(1+4g+3g^2+4n)\right] \\ & -\frac{2g(1+g)}{Z_4}x^2. \end{aligned} \tag{3.44}$$

Therefore, since $Z_4 > 0$, we deduce that the invariant parabolas $\Phi_1(x,y) = 0$ and $\Phi_2(x,y) = 0$ have the same point at infinity on the Poincaré disk if $\zeta_2 < 0$, and opposite points at infinity if $\zeta_2 > 0$. So we consider these two cases separately.

1.1: *The case* $\zeta_2 < 0$. Then $g(g+1) < 0$ and considering the above relations, in this case we obtain the following configurations:

$\xi_{14} < 0$, $\beta_4 < 0$, $2g+1 < 0$ (i.e. $\beta_4 < 0$, $\alpha_4 > 0$, $-1 < g < -1/2$) $\Rightarrow$
$x_2 - x_3 > 0$, $x_2 - x_4 > 0$, $x_1 - x_3 > 0$, $x_1 - x_4 > 0$ $\Rightarrow$ *Config.* $\mathcal{P}$*.55*;
$\xi_{14} < 0$, $\beta_4 < 0$, $2g+1 > 0$ (i.e. $\beta_4 < 0$, $\alpha_4 > 0$, $-1/2 < g < 0$) $\Rightarrow$
$x_2 - x_3 < 0$, $x_2 - x_4 < 0$, $x_1 - x_3 < 0$, $x_1 - x_4 < 0$ $\Rightarrow \simeq$ *Config.* $\mathcal{P}$*.55*;
$\xi_{14} < 0$, $\beta_4 > 0$ (i.e. $\beta_4 > 0$, $\alpha_4 < 0$, $-1 < g < 0$) $\Rightarrow$
$(x_2 - x_3)(x_2 - x_4) < 0$, $(x_1 - x_3)(x_1 - x_4) < 0$ $\Rightarrow$ *Config.* $\mathcal{P}$*.56*;
$\xi_{14} > 0$, $\beta_4 < 0$, $2g+1 < 0$ (i.e. $\beta_4 < 0$, $\alpha_4 < 0$, $-1 < g < -1/2$) $\Rightarrow$
$x_2 - x_3 > 0$, $x_2 - x_4 > 0$, $(x_1 - x_3)(x_1 - x_4) < 0$ $\Rightarrow$ *Config.* $\mathcal{P}$*.57*;
$\xi_{14} > 0$, $\beta_4 < 0$, $2g+1 > 0$ (i.e. $\beta_4 < 0$, $\alpha_4 < 0$, $-1/2 < g < 0$) $\Rightarrow$
$x_2 - x_3 < 0$, $x_2 - x_4 < 0$, $(x_1 - x_3)(x_1 - x_4) < 0$ $\Rightarrow \simeq$ *Config.* $\mathcal{P}$*.57*;
$\xi_{14} > 0$, $\beta_4 > 0$, $2g+1 < 0$ (i.e. $\beta_4 > 0$, $\alpha_4 > 0$, $-1 < g < -1/2$) $\Rightarrow$
$(x_2 - x_3)(x_2 - x_4) < 0$, $x_1 - x_3 > 0$, $x_1 - x_4 > 0$ $\Rightarrow \simeq$ *Config.* $\mathcal{P}$*.57*;
$\xi_{14} > 0$, $\beta_4 > 0$, $2g+1 > 0$ (i.e. $\beta_4 > 0$, $\alpha_4 > 0$, $-1/2 < g < 0$) $\Rightarrow$
$(x_2 - x_3)(x_2 - x_4) < 0$, $x_1 - x_3 < 0$, $x_1 - x_4 < 0$ $\Rightarrow \simeq$ *Config.* $\mathcal{P}$*.57*.

1.2: *The case* $\zeta_2 > 0$. Then $g(g+1) > 0$ and we obtain the following configurations:

$\xi_{14} < 0$, $\beta_4 < 0$, $2g+1 < 0$ (i.e. $\beta_4 < 0$, $\alpha_4 > 0$, $g < -1$) $\Rightarrow$
$x_2 - x_3 < 0$, $x_2 - x_4 < 0$, $x_1 - x_3 > 0$, $x_1 - x_4 > 0$ $\Rightarrow$ *Config.* $\mathcal{P}$*.58*;
$\xi_{14} < 0$, $\beta_4 < 0$, $2g+1 > 0$ (i.e. $\beta_4 < 0$, $\alpha_4 > 0$, $g > 0$) $\Rightarrow$
$x_2 - x_3 > 0$, $x_2 - x_4 > 0$,$x_1 - x_3 < 0$, $x_1 - x_4 < 0$ $\Rightarrow \simeq$ *Config.* $\mathcal{P}$*.58*;
$\xi_{14} > 0$, $\beta_4 < 0$, $2g+1 < 0$ (i.e. $\beta_4 < 0$, $\alpha_4 < 0$, $g < -1$) $\Rightarrow$
$x_2 - x_3 < 0$, $x_2 - x_4 < 0$, $(x_1 - x_3)(x_1 - x_4) < 0$ $\Rightarrow$ *Config.* $\mathcal{P}$*.59*;
$\xi_{14} > 0$, $\beta_4 < 0$, $2g+1 > 0$ (i.e. $\beta_4 < 0$, $\alpha_4 < 0$, $g > 0$) $\Rightarrow$
$x_2 - x_3 > 0$, $x_2 - x_4 > 0$, $(x_1 - x_3)(x_1 - x_4) < 0$ $\Rightarrow \simeq$ *Config.* $\mathcal{P}$*.59*;
$\xi_{14} > 0$, $\beta_4 > 0$, $2g+1 < 0$ (i.e. $\beta_4 > 0$, $\alpha_4 > 0$, $g < -1$) $\Rightarrow$
$(x_2 - x_3)(x_2 - x_4) < 0$, $x_1 - x_3 > 0$, $x_1 - x_4 > 0$ $\Rightarrow \simeq$ *Config.* $\mathcal{P}$*.59*;
$\xi_{14} > 0$, $\beta_4 > 0$, $2g+1 > 0$ (i.e. $\beta_4 > 0$, $\alpha_4 > 0$, $g < -1$) $\Rightarrow$
$(x_2 - x_3)(x_2 - x_4) < 0$, $x_1 - x_3 < 0$, $x_1 - x_4 < 0$ $\Rightarrow \simeq$ *Config.* $\mathcal{P}$*.59*.

Applying the Mathematica function "FindInstance" (or "Reduce") we detect that the conditions $\mathbf{D} < 0$, $\zeta_2 > 0$, $\xi_{14} < 0$, and $\beta_4 > 0$ (i.e. $Z_4 > 0$, $g(g+1) > 0$, $\alpha_4 < 0$, and $\beta_4 > 0$) are incompatible.

We observe that in both cases (i.e. $\zeta_2 < 0$ and $\zeta_2 > 0$) the configurations do not depend on the sign $(1+2g)$. As a result we obtain the following lemma.

Lemma 3.5. *Assume that for the systems* (3.37) *the condition* $\mathbf{D} < 0$ *holds. Then, these systems exhibit the following configurations if and only if the respective conditions are satisfied:*

$$\begin{array}{ll} \zeta_2 < 0,\ \xi_{14} < 0,\ \xi_{30} < 0 & \Leftrightarrow \textit{ Config. } \mathcal{P}.55; \\ \zeta_2 < 0,\ \xi_{14} < 0,\ \xi_{30} > 0 & \Leftrightarrow \textit{ Config. } \mathcal{P}.56; \\ \zeta_2 < 0,\ \xi_{14} > 0 & \Leftrightarrow \textit{ Config. } \mathcal{P}.57; \\ \zeta_2 > 0,\ \xi_{14} < 0 & \Leftrightarrow \textit{ Config. } \mathcal{P}.58; \\ \zeta_2 > 0,\ \xi_{14} > 0 & \Leftrightarrow \textit{ Config. } \mathcal{P}.59. \end{array}$$

2: *The possibility* $\mathbf{D} > 0$. Then $Z_4 < 0$ and the systems (3.37) possess only two real singularities: M_1 (located on the parabola $\Phi_1(x,y) = 0$) and M_2 (located on the parabola $\Phi_2(x,y) = 0$). As mentioned earlier, the direction of the second invariant parabola depends on the sign of $g(1+g)$ (see (3.44)).

Thus, considering the condition $\mathbf{D} > 0$ (i.e., $Z_4 < 0$), we have the configuration *Config.* $\mathcal{P}.60$ if $\zeta_2 < 0$ (i.e., $g(g+1) < 0$) and *Config.* $\mathcal{P}.61$ if $\zeta_2 > 0$ (i.e., $g(g+1) > 0$).

3: *The possibility* $\mathbf{D} = 0$. From (3.42), (3.39), and the condition $\zeta_5\mathcal{R}_2 \neq 0$ (i.e., $Z_4(1+4g+3g^2+4n) \neq 0$), it follows that $\mathbf{D} = 0$ implies $\alpha_4\beta_4 = 0$. This yields the next lemma.

Lemma 3.6. *For systems* (3.37)*, the condition* $\beta_4 = 0$ *can be transformed by an affine change of variables into the condition* $\alpha_4 = 0$.

Proof. Apply to the systems (3.37) the transformation

$$x_1 = \delta x - \frac{(1+g)(1+4g+3g^2+4n)}{2Z_4}, \quad y_1 = \delta y - \frac{(1+g)(1+3g)(1+4g+3g^2+4n)}{4Z_4}, \\ t_1 = \frac{1}{\delta}, \quad \delta = -\frac{2g(1+g)}{Z_4}. \tag{3.45}$$

Define

$$n_1 = -\frac{(1+g)(1+11g+31g^2+21g^3+4n+20gn)}{4(1+6g+5g^2+4n)} \quad \Rightarrow \\ n = -\frac{(1+g)(1+11g+31g^2+21g^3+4n_1+20gn_1)}{4(1+6g+5g^2+4n_1)}. \tag{3.46}$$

Then we arrive at the family

$$\begin{aligned} \dot{x}_1 &= \frac{1}{4}(1+3g)(1+4g+3g^2+2n_1) + n_1x_1 - \frac{1}{2}(1+g)y_1 + gx_1^2 + x_1y_1, \\ \dot{y}_1 &= \frac{1}{2}(1+3g)(1+4g+3g^2+2n_1)x_1 + 2n_1y_1 + (g-1)x_1y_1 + 2y_1^2. \end{aligned} \tag{3.47}$$

Observe that this family coincides with (3.37) up to notation of variables and parameters. Calculating for the above system, we get:

$$\begin{aligned} \beta_4(g,n_1) &= (1+g)(1+3g)(1+6g+7g^2) + 4(1+2g)^2n_1 \\ &= \frac{g^2(1+g)^2(5+22g+21g^2+4n)}{1+6g+5g^2+4n} = -\frac{g^2(1+g)^2\alpha_4(g,n)}{Z_4}, \\ \alpha_4(g,n_1) &= 5+22g+21g^2+4n_1 \\ &= \frac{4(1+10g+34g^2+46g^3+21g^4+4n+16gn+16g^2n)}{1+6g+5g^2+4n} = -\frac{4\beta_4(g,n)}{Z_4}. \end{aligned}$$

Since $g(g+1)Z_4 \neq 0$, the condition $\beta_4(g,n_1)=0$ (respectively $\alpha_4(g,n_1)=0$) for systems (3.47) implies $\alpha_4(g,n)=0$ (respectively $\beta_4(g,n)=0$) for the original systems (3.37). This completes the proof. □

Thus, in what follows, we assume that the condition

$$\alpha_4 = 5+22g+21g^2+4n=0$$

holds, which implies

$$n=-\frac{(1+3g)(5+7g)}{4}.$$

We then obtain the following family of systems:

$$\begin{aligned}\dot{x} =& -\frac{1}{8}(1+3g)^2(3+5g)-\frac{1}{4}(1+3g)(5+7g)x-\frac{1}{2}(1+g)y+gx^2+xy,\\ \dot{y} =& -\frac{1}{4}(1+3g)^2(3+5g)x-\frac{1}{2}(1+3g)(5+7g)y+(g-1)xy+2y^2,\end{aligned} \tag{3.48}$$

possessing the two invariant parabolas $\Phi_1(x,y)=x^2-y=0$ and

$$\begin{aligned}\Phi_2(x,y) =& -(1+3g)^2(2+3g)(3+5g)-4(1+g)(1+3g)(2+3g)x\\ &+4g(1+g)x^2+8(1+2g)^2y=0.\end{aligned}$$

Considering the coordinates of the finite singularities (3.40), we observe that for $\alpha_4=0$, the singular point M_4 coalesces with M_1, producing a double finite singularity. Thus, systems (3.48) possess three finite singularities $M_i(x_i,y_i)$, $i=1,2,3$ (with M_1 being double), with coordinates

$$\begin{aligned}&x_1=-\frac{1+3g}{2}, && y_1=\frac{(1+3g)^2}{4}; && x_2=\frac{(1+3g)(4+13g+11g^2)}{2g(1+g)},\\ &y_2=-\frac{(1+3g)^2(3+5g)}{4(1+g)}; && x_3=\frac{3+5g}{2}, && y_3=\frac{(3+5g)^2}{4}.\end{aligned}$$

Using (3.41), we see that the singular points $M_1(\equiv M_4)$ and M_3 are the intersection points of the invariant parabolas, while M_2 lies on the parabola $\Phi_2(x,y)=0$. We calculate

$$\begin{aligned}(x_2-x_1)(x_2-x_3) &= \frac{(1+2g)^2(1+3g)(2+3g)(2+7g+7g^2)}{g^2(1+g)^2},\\ (x_2-x_1)+(x_2-x_3) &= \frac{4(1+2g)^3}{g(1+g)}\neq 0 \quad \text{if } (x_2-x_1)(x_2-x_3)>0.\end{aligned}$$

Since the discriminant $\mathrm{Discrim}\,[2+7g+7g^2,g]=-7<0$, it follows that

$$\begin{aligned}\operatorname{sign}\big((x_2-x_1)(x_2-x_3)\big) &= \operatorname{sign}\big((1+3g)(2+3g)\big),\\ \operatorname{sign}\big((x_2-x_1)+(x_2-x_3)\big) &= \operatorname{sign}\big(g(1+g)(1+2g)\big).\end{aligned}$$

Note that $\operatorname{sign}(1+2g)$ is only relevant if $(1+3g)(2+3g)>0$.

On the other hand, for systems (3.48) we have

$$\begin{aligned}&\zeta_2=4g(1+g), \quad \zeta_5=19(g-2)(3+g)(1+3g)^2(2+3g)^2,\\ &\xi_3=217993032\,g(1+g)(1+3g)^3(2+3g)^3(2+7g+7g^2)^2,\end{aligned}$$

and since $\zeta_2\zeta_5 \neq 0$, it follows that

$$\operatorname{sign}(\xi_3) = \operatorname{sign}\big(g(1+g)(1+3g)(2+3g)\big), \quad \operatorname{sign}(\zeta_2) = \operatorname{sign}(g(1+g)).$$

We claim that $\zeta_2 > 0$ implies $\xi_3 > 0$. Indeed, suppose $\zeta_2 > 0$ but $\xi_3 < 0$. This implies $(1+3g)(2+3g) < 0$, i.e., $-2/3 < g < -1/3$. Hence, $-1 < g < 0$, which contradicts $\zeta_2 > 0$. This proves the claim.

Therefore, considering the above relations, for systems (3.48) we obtain the following configurations:

$$\begin{array}{ll}
\zeta_2 < 0, \xi_3 < 0, 2g+1 < 0 \text{ (i.e. } -1 < g < -2/3) & \Rightarrow x_2 - x_1 > 0,\ x_2 - x_3 > 0 \Rightarrow \textit{Config. } \mathcal{P}.62;\\
\zeta_2 < 0, \xi_3 < 0, 2g+1 > 0 \text{ (i.e. } -1/3 < g < 0) & \Rightarrow x_2 - x_1 < 0,\ x_2 - x_3 < 0 \Rightarrow \simeq \textit{Config. } \mathcal{P}.62;\\
\zeta_2 < 0, \xi_3 > 0 \text{ (i.e. } -2/3 < g < -1/3) & \Rightarrow (x_2 - x_1)(x_2 - x_3) < 0 \quad \Rightarrow \textit{Config. } \mathcal{P}.63.\\
\zeta_2 > 0, 2g+1 < 0 \text{ (i.e. } g < -1) & \Rightarrow x_2 - x_1 < 0,\ x_2 - x_3 < 0 \Rightarrow \textit{Config. } \mathcal{P}.64;\\
\zeta_2 > 0, 2g+1 > 0 \text{ (i.e. } g > 0) & \Rightarrow x_2 - x_1 > 0,\ x_2 - x_3 > 0 \Rightarrow \simeq \textit{Config. } \mathcal{P}.64.
\end{array}$$

We observe that these configurations do not depend on the sign of $2g+1$. Hence, we arrive at the following lemma.

Lemma 3.7. *Assume that for systems* (3.37) *the condition* $\mathbf{D} = 0$ *holds. Then these systems possess the following configurations if and only if the corresponding conditions are satisfied:*

$$\begin{array}{ll}
\zeta_2 < 0, \xi_3 < 0 & \Leftrightarrow \textit{Config. } \mathcal{P}.62;\\
\zeta_2 < 0, \xi_3 > 0 & \Leftrightarrow \textit{Config. } \mathcal{P}.63;\\
\zeta_2 > 0 & \Leftrightarrow \textit{Config. } \mathcal{P}.64.
\end{array}$$

The case $B_1 = 0$. Considering (3.39) and the condition $\zeta_2\mathcal{R}_2 \neq 0$ (i.e., $g(g+1)(1+6g+5g^2+4n) \neq 0$), we conclude that the condition $B_1 = 0$ is equivalent to

$$\begin{aligned}
&(1+2g)(1+3g)(2+3g)(1+4g+3g^2+2n)(1+6g+6g^2+4n)\\
&\times(1+6g+9g^2+4n)(5+14g+9g^2+4n) = 0.
\end{aligned} \tag{3.49}$$

However, through suitable transformations, we can reduce the number of cases arising from the condition $B_1 = 0$. This is stated in the following lemma.

Lemma 3.8. *The condition* (3.49) *can be transformed, via affine transformations and time rescaling, into the simpler condition*

$$(1+2g)(1+3g)(1+4g+3g^2+2n) = 0. \tag{3.50}$$

Proof. We prove this lemma in two steps: *(i)* by applying a transformation that replaces the line $y = 0$ with $y = x$ while preserving the invariant parabola $\Phi_1(x,y) = x^2 - y = 0$; and *(ii)* by applying a transformation that maps the invariant parabola $\Phi_2(x,y) = 0$ (see (3.38)) to the parabola $\Phi_1(x,y) = x^2 - y = 0$.

Step (i): Applying to systems (3.37) the change of variables

$$x_1 = -x + 1/2, \quad y_1 = -x + y + 1/4,$$

we obtain the transformed system

$$\begin{aligned}
\dot{x}_1 &= -\frac{1}{8}(2+3g)(1+6g+6g^2+4n) + \frac{1}{4}(1+2g+4n)x_1 + \frac{g}{2}y_1 - (1+g)x_1^2 + x_1y_1,\\
\dot{y}_1 &= -\frac{1}{4}(2+3g)(1+6g+6g^2+4n)x_1 + \frac{1}{2}(1+2g+4n)y_1 - (g+2)x_1y_1 + 2y_1^2.
\end{aligned}$$

Setting new parameters

$$\begin{aligned} n_1 &= \frac{1}{4}(1+2g+4n), \quad g_1 = -(1+g), \quad \Rightarrow \\ n &= \frac{1}{4}(1+2g_1+4n_1), \quad g = -(1+g_1), \end{aligned} \tag{3.51}$$

the system becomes

$$\begin{aligned} \dot{x}_1 &= \frac{1}{4}(1+3g_1)(1+4g_1+3g_1^2+2n_1) + n_1x_1 - \frac{1+g_1}{2}y_1 + g_1x_1^2 + x_1y_1, \\ \dot{y}_1 &= \frac{1}{2}(1+3g_1)(1+4g_1+3g_1^2+2n_1)x_1 + 2n_1y_1 + (g_1-1)x_1y_1 + 2y_1^2, \end{aligned}$$

which has the same form as (3.37).

Moreover, calculations show that

$$\begin{gathered} 2+3g = -(1+3g_1), \quad 1+6g+6g^2+4n = 2(1+4g_1+3g_1^2+2n_1), \\ 5+14g+9g^2+4n = 1+6g_1+9g_1^2+4n_1, \end{gathered}$$

thus reducing the condition (3.49) to

$$(1+2g)(1+3g)(1+4g+3g^2+2n)(1+6g+9g^2+4n) = 0.$$

Step (ii): As shown in the proof of Lemma 3.6, via the transformation (3.45), systems (3.37) can be brought to the canonical form (3.47) but with a new parameter n_1 given by (3.46). Calculations give

$$1+6g+9g^2+4n_1 = \frac{8g^2(1+4g+3g^2+2n)}{4(1+6g+5g^2+4n)}.$$

Since $g \neq 0$, the condition $1+6g+9g^2+4n = 0$ can be transformed into $1+4g+3g^2+2n = 0$. Consequently, we arrive at the condition (3.50), completing the proof of Lemma 3.8. □

For systems (3.37), we compute

$$\zeta_3 = 8(1+2g)^2, \quad \xi_{15} = 2(1+3g)(2+3g),$$

and distinguish two subcases: $\zeta_3 \neq 0$ and $\zeta_3 = 0$.

1: *The subcase* $\zeta_3 \neq 0$. Then $1+2g \neq 0$, and considering (3.50) together with Lemma 3.8, to satisfy the condition $B_1 = 0$ it is sufficient to consider $(1+3g)(1+4g+3g^2+2n) = 0$. Clearly, we must distinguish two possibilities: $\xi_{15} \neq 0$ and $\xi_{15} = 0$.

1.1: *The possibility* $\xi_{15} \neq 0$. Then $1+3g \neq 0$, and in this case we obtain $1+4g+3g^2+2n = 0$, i.e., $n = -(1+g)(1+3g)/2$. This leads to the following one-parameter family of systems:

$$\begin{aligned} \dot{x} &= -\frac{1}{2}(1+g)(1+3g)\,x - \frac{1}{2}(1+g)\,y + gx^2 + xy, \\ \dot{y} &= -\,y\left(1+4g+3g^2+x-gx-2y\right), \end{aligned} \tag{3.52}$$

which, besides the invariant parabolas

$$\Phi_1(x,y) = x^2 - y = 0, \quad \Phi_2(x,y) = (1+g)(1+3g)\,x - 2gx^2 - (1+g)\,y = 0,$$

also possess the invariant line $y = 0$.

Using Lemmas 2.3 and 2.4, we compute:

$$\theta = -8(g-1)(2+g), \quad B_2 = \frac{243}{2} g(1+g)^4(1+2g)^2(2+3g)\, y^4. \tag{3.53}$$

Thus, we need to consider the cases $B_2 \neq 0$ and $B_2 = 0$.

1.1.1: *The case* $B_2 \neq 0$. Then, by Lemma 2.3, the system cannot have invariant lines in directions other than $y = 0$. However, by Lemma 2.4, parallel invariant lines could exist if $\theta = 0$.

1.1.1.1: *The subcase* $\theta \neq 0$. The systems (3.52) possess four finite singularities $M_i(x_i, y_i)$ for $i = 1, 2, 3, 4$, with coordinates:

$$\begin{aligned} &x_1 = 0, \quad y_1 = 0; \quad x_2 = \frac{(1+g)(1+3g)}{2g}, \quad y_2 = 0; \\ &x_3 = 1+g, \quad y_3 = (1+g)^2; \quad x_4 = -\frac{1+3g}{2}, \quad y_4 = \frac{(1+3g)^2}{4}. \end{aligned}$$

From the conditions stated in the statement $(\mathcal{A}_2^*)$, for systems (3.52) we have:

$$\zeta_1\zeta_2\zeta_5\mathcal{R}_2 \neq 0 \quad \Leftrightarrow \quad g(1+g)(g-2)(3+g)(1+3g)(8+27g+27g^2) \neq 0. \tag{3.54}$$

We observe that the invariant parabolas intersect at two points: M_1 and M_3. Moreover, the invariant line $y = 0$ has a contact point at M_1 with the parabola $\Phi_1(x, y) = 0$ and two intersection points, M_1 and M_2, with the parabola $\Phi_2(x, y) = 0$.

Thus, three finite singularities are fixed by the intersections of the invariant curves, with their positions determined by the value of the parameter g.

On the other hand, the singular point M_4 lies on the invariant parabola $\Phi_1(x, y) = 0$ and is a floating singularity. Therefore, we must determine its position relative to the other two singularities on the same invariant curve. To this end, we calculate:

$$(x_4 - x_1)(x_4 - x_3) = \frac{1}{4}(1+3g)(3+5g), \quad (x_4 - x_1) + (x_4 - x_3) = -2(1+2g).$$

Therefore, we obtain:

$$\begin{aligned} \operatorname{sign}\big((x_4 - x_1)(x_4 - x_3)\big) &= \operatorname{sign}\big((1+3g)(3+5g)\big), \\ \operatorname{sign}\big((x_4 - x_1) + (x_4 - x_3)\big) &= -\operatorname{sign}(1+2g). \end{aligned}$$

We observe that the direction of the second invariant parabola depends on $\operatorname{sign}(g(g+1))$.

Furthermore, for systems (3.52), calculations yield:

$$\begin{aligned} \zeta_2 = 4g(g+1), \quad \xi_{16} &= \frac{3705}{2} g^2(1+g)^2(1+3g)(3+5g), \\ \xi_{17} &= \frac{3705}{4}(1+g)^2(1+2g)(1+3g)(3+5g). \end{aligned}$$

Thus, we have:

$$\begin{aligned} \operatorname{sign}(\zeta_2) = \operatorname{sign}\big(g(g+1)\big), \quad \operatorname{sign}(\xi_{16}) &= \operatorname{sign}\big((1+3g)(3+5g)\big), \\ \operatorname{sign}(\xi_{17}) &= \operatorname{sign}\big((1+2g)(1+3g)(3+5g)\big). \end{aligned}$$

Therefore, we establish the following configurations:

$\zeta_2 < 0, \xi_{16} < 0$ (i.e. $-3/5 < g < -1/3$) $\Rightarrow (x_4 - x_1)(x_4 - x_3) < 0 \Rightarrow$ *Config.* $\mathcal{P}.65$;
$\zeta_2 < 0, \xi_{16} > 0, \xi_{17} < 0$ (i.e. $-1 < g < -3/5$) $\Rightarrow x_4 > x_1, x_4 > x_3 \Rightarrow$ *Config.* $\mathcal{P}.66$;
$\zeta_2 < 0, \xi_{16} > 0, \xi_{17} > 0$ (i.e. $-1/3 < g < 0$) $\Rightarrow x_4 < x_1, x_4 < x_3 \Rightarrow$ *Config.* $\mathcal{P}.67$;
$\zeta_2 > 0, \xi_{17} < 0$ (i.e. $g < -1$) $\Rightarrow x_4 > x_1, x_4 > x_3 \Rightarrow$ *Config.* $\mathcal{P}.68$;
$\zeta_2 > 0, \xi_{17} > 0$ (i.e. $g > 0$) $\Rightarrow x_4 < x_1, x_4 < x_3 \Rightarrow$ *Config.* $\mathcal{P}.69$.

1.1.1.2: *The subcase* $\theta = 0$. This condition implies $(g-1)(g+2) = 0$, and since for systems (3.52) we have

$$\zeta_6 = \frac{(g-1)(1+g)(2+5g+5g^2)}{8},$$

we consider two possibilities: $\zeta_6 \neq 0$ and $\zeta_6 = 0$.

1.1.1.2.1: *The possibility* $\zeta_6 \neq 0$. In this case, the condition $\theta = 0$ implies $g = -2$. We then obtain the system:

$$\dot{x} = -\frac{5x}{2} + \frac{y}{2} - 2x^2 + xy, \quad \dot{y} = -y\,(5 + 3x - 2y)\,,$$

which, besides the invariant parabolas:

$$\Phi_1(x,y) = x^2 - y = 0, \quad \Phi_2(x,y) = 5x + 4x^2 + y = 0,$$

possesses only one invariant line: $y = 0$. This means that the condition $g = -2$ does not imply the appearance of any additional parallel invariant line.

Since $g = -2 < -1$, we conclude that this case corresponds to configuration *Config.* $\mathcal{P}.68$ (as detected earlier).

1.1.1.2.2: *The possibility* $\zeta_6 = 0$. Then $\theta = 0$ implies $g = 1$, and we arrive at the system:

$$\dot{x} = -4x - y + x^2 + xy, \quad \dot{y} = 2y(y-4),$$

which possesses the invariant lines $y = 0$ and $y = 4$, as well as the invariant parabolas:

$$\Phi_1(x,y) = x^2 - y = 0, \quad \Phi_2(x,y) = -4x + x^2 + y = 0.$$

In this case, we obtain configuration *Config.* $\mathcal{P}.70$.

1.1.2: *The case* $B_2 = 0$. Considering (3.53) and (3.54), the condition $B_2 = 0$ implies $g = -2/3$, and we arrive at the system:

$$\dot{x} = \frac{x}{6} - \frac{y}{6} - \frac{2x^2}{3} + xy, \quad \dot{y} = \frac{1}{3}y(1 - 5x + 6y), \tag{3.55}$$

which possesses the invariant lines $y = 0$ and $y = x - 1/4$, as well as the invariant parabolas:

$$\Phi_1(x,y) = x^2 - y = 0, \quad \Phi_2(x,y) = x - 4x^2 + y = 0.$$

We observe that the invariant line $y = x - 1/4$ is tangent to the invariant parabola $\Phi_1(x,y) = 0$ at the point $M_4(1/2, 1/4)$ and also tangent to the parabola $\Phi_2(x,y) = 0$ at the point $M_2(1/4, 0)$. Therefore, in this case, we obtain configuration *Config.* $\mathcal{P}.71$.

Thus, we have proved the following lemma.

Lemma 3.9. *Assume that for systems (3.37) the conditions $B_1 = 0$ and $\zeta_3\xi_{15} \neq 0$ hold. Then these systems possess the following configurations, provided the corresponding conditions are satisfied:*

$$\begin{array}{ll}
B_2 \neq 0,\ \theta \neq 0,\ \zeta_2 < 0,\ \xi_{16} < 0 & \Rightarrow \textit{Config. } \mathcal{P}.65; \\
B_2 \neq 0,\ \theta \neq 0,\ \zeta_2 < 0,\ \xi_{16} > 0,\ \xi_{17} < 0 & \Rightarrow \textit{Config. } \mathcal{P}.66; \\
B_2 \neq 0,\ \theta \neq 0,\ \zeta_2 < 0,\ \xi_{16} > 0,\ \xi_{17} > 0 & \Rightarrow \textit{Config. } \mathcal{P}.67; \\
B_2 \neq 0,\ \theta \neq 0,\ \zeta_2 > 0,\ \xi_{17} < 0 & \Rightarrow \textit{Config. } \mathcal{P}.68; \\
B_2 \neq 0,\ \theta \neq 0,\ \zeta_2 > 0,\ \xi_{17} > 0 & \Rightarrow \textit{Config. } \mathcal{P}.69; \\
B_2 \neq 0,\ \theta = 0,\ \zeta_6 \neq 0 & \Rightarrow \textit{Config. } \mathcal{P}.68; \\
B_2 \neq 0,\ \theta = 0,\ \zeta_6 = 0 & \Rightarrow \textit{Config. } \mathcal{P}.70; \\
B_2 = 0 & \Rightarrow \textit{Config. } \mathcal{P}.71.
\end{array}$$

1.2: *The possibility $\xi_{15} = 0$.* Then, considering the proof of Lemma 3.8, we may assume $1 + 3g = 0$. This yields $g = -1/3$, leading to the following one-parameter family of systems:

$$\dot{x} = nx - \frac{y}{3} - \frac{x^2}{3} + xy, \quad \dot{y} = \frac{2}{3}y\,(3n - 2x + 3y)\,, \tag{3.56}$$

which, besides the invariant parabolas:

$$\Phi_1(x,y) = x^2 - y = 0, \quad \Phi_2(x,y) = 9n^2 - 6nx + x^2 + (9n-1)y = 0,$$

also possesses the invariant line $y = 0$.

Considering Lemmas 2.3 and 2.4, we compute:

$$B_2 = -8(1+3n)(9n-1)(12n-1)y^4/9, \quad \mathbf{D} = 4096n^6(9n-1)/243,$$
$$\theta = 160/9 \neq 0, \quad \zeta_5 = -4256n^2/9, \quad \mathcal{R}_2 = 28(9n-1)/81.$$

Thus, we discuss the cases $B_2 \neq 0$ and $B_2 = 0$.

1.2.1: *The case $B_2 \neq 0$.* By Lemmas 2.3 and 2.4, no additional invariant line can exist.

Moreover, considering (3.40), the systems (3.56) possess four finite singularities $M_i(x_i, y_i)$ for $i = 1, 2, 3, 4$, with coordinates:

$$x_1 = 0,\ y_1 = 0; \quad x_2 = 3n,\ y_2 = 0;\ x_{3,4} = \frac{1}{3}\left(1 \pm \sqrt{1-9n}\right), \quad y_{3,4} = \frac{1}{9}\left(2 - 9n \pm 2\sqrt{1-9n}\right).$$

We observe that in this case $n(1-9n) \neq 0$ due to $\zeta_5\mathcal{R}_2 \neq 0$, ensuring all finite singularities are distinct. Moreover, the invariant line $y = 0$ is tangent to the parabola $\Phi_1(x,y) = 0$ at the singular point $M_1(0,0)$ and tangent to $\Phi_2(x,y) = 0$ at $M_2(3n,0)$.

Since $\mathbf{D} \neq 0$ and $\operatorname{sign}(1-9n) = -\operatorname{sign}(\mathbf{D})$, we examine two subcases: $\mathbf{D} < 0$ and $\mathbf{D} > 0$.

1.2.1.1: *The subcase $\mathbf{D} < 0$.* Then $1 - 9n > 0$ (i.e., $n < 1/9$), and we arrive at configuration *Config. $\mathcal{P}.72$*, independent of the relative position of the singularity $M_2(3n,0)$ with respect to $M_1(0,0)$.

1.2.1.2: *The subcase $\mathbf{D} > 0$.* This implies $1 - 9n < 0$, so the singularities M_3 and M_4 become complex. In this case, we arrive at configuration *Config. $\mathcal{P}.73$*.

1.2.2: *The case $B_2 = 0$.* Since $9n - 1 \neq 0$ due to $\mathcal{R}_2 \neq 0$, the condition $B_2 = 0$ implies:

$$(1+3n)(12n-1) = 0.$$

1.2.2.1: *The subcase* $1+3n=0$. Then $n=-1/3$, leading to the system:

$$\dot{x}=-\frac{1}{3}x-\frac{1}{3}y-\frac{1}{3}x^2+xy,\quad \dot{y}=\frac{2}{3}y(-1-2x+3y).$$

Applying the transformation $x_1=(1-x)/4$, $y_1=(y-x)/4$, and $t_1=4t$, we recover system (3.55), corresponding to configuration *Config.* $\mathcal{P}.71$.

1.2.2.2: *The subcase* $12n-1=0$. This implies $n=1/12$, and we arrive at the system:

$$\dot{x}=\frac{1}{12}x-\frac{1}{3}y-\frac{1}{3}x^2+xy,\quad \dot{y}=\frac{1}{6}y(1-8x+12y).$$

Applying the affine transformation $x_1=-x+1/2$, $y_1=y-x+1/4$, the system is brought to the form (3.55), again corresponding to configuration *Config.* $\mathcal{P}.71$.

2: *The subcase* $\zeta_3=0$. Then $g=-1/2$, which implies $B_1=0$. This leads to the following one-parameter family of systems:

$$\dot{x}=(4x-1)\,(8n-1-4x+8y)\,/32,\quad \dot{y}=(x-8nx+32ny-24xy+32y^2)/16, \tag{3.57}$$

which possess the following two invariant parabolas: $\Phi_1(x,y)=x^2-y=0$, and

$$\Phi_2(x,y)=(8n-1)(16n-1)-4(16n-1)x+16x^2+8(16n-3)y=0,$$

as well as the invariant line $x=1/4$.

For the above systems, we calculate:

$$\zeta_5=-475(16n-1)^2/256,\quad \mathcal{R}_2=125(16n-3)/1024,$$

and according to the conditions given by statement $(\mathcal{A}_2^*)$, we require $\zeta_5\mathcal{R}_2\neq 0$. This implies $(16n-1)(16n-3)\neq 0$, and by Proposition 2.7*, the invariant parabolas $\Phi_1(x,y)=0$ and $\Phi_2(x,y)=0$ are distinct.

For these systems, we compute:

$$B_1=0,\quad B_2=-\frac{81}{2048}(8n-1)^2(1+16n)^2x^4,\quad \theta=18\neq 0, \tag{3.58}$$

and since $\theta\neq 0$, by Lemma 2.4, these systems cannot possess an invariant line parallel to $x=1/4$.

On the other hand, according to Lemma 2.3, the existence of an invariant line in another direction would require $B_2=0$. Thus, we consider the possibilities $B_2\neq 0$ and $B_2=0$.

2.1: The possibility $B_2\neq 0$. The systems (3.57) possess four finite singularities $M_i(x_i,y_i)$ $(i=1,2,3,4)$ with coordinates:

$$\begin{gathered} x_1=\frac{1}{4},\ y_1=\frac{1}{16};\quad x_2=\frac{1}{4},\ y_2=\frac{1-8n}{8};\\ x_{3,4}=\frac{1}{4}\left(1\pm\sqrt{3-16n}\right),\quad y_{3,4}=\frac{1}{8}\left(2-8n\pm\sqrt{3-16n}\right). \end{gathered} \tag{3.59}$$

We observe that the invariant parabolas intersect at two points: M_3 and M_4. The invariant line $x=1/4$ intersects the parabola $\Phi_1(x,y)=0$ at M_1 and the parabola $\Phi_2(x,y)=0$ at M_2.

Thus, all four finite singularities are located at the intersections of invariant curves, and their positions are determined by the parameter n. Furthermore, the singularities M_3 and M_4 are real if $3-16n>0$, and complex if $3-16n<0$.

For systems (3.57), we compute:

$$\mathbf{D} = 3(16n-3)^3(16n-1)^2/1048576 \neq 0,$$

due to $\zeta_5\mathcal{R}_2 \neq 0$. Therefore, sign $(\mathbf{D})$ = sign $(16n-3)$.

Thus, for $B_2 \neq 0$, we conclude: if $\mathbf{D} < 0$, the system corresponds to configuration *Config.* $\mathcal{P}.74$, and if $\mathbf{D} > 0$, it corresponds to configuration *Config.* $\mathcal{P}.75$.

2.2: The possibility $B_2 = 0$. From (3.58), this condition implies $(8n-1)(1+16n) = 0$. We analyze two cases: $(1+16n) \neq 0$ and $(1+16n) = 0$.

If $8n-1=0$, i.e., $n = 1/8$, we obtain the system:

$$\dot{x} = -(4x-1)(x-2y)/8, \quad \dot{y} = y(1-6x+8y)/4, \tag{3.60}$$

which possesses the following five invariant curves (two parabolas and three invariant lines):

$$\Phi_1(x,y) = x^2 - y, \quad \Phi_2(x,y) = -x + 4x^2 - 2y, \quad x = 1/4, \quad y = 0, \quad y = x - 1/4.$$

For this system, we obtain configuration *Config.* $\mathcal{P}.76$.

In the case $1+16n=0$, we get $n = -1/16$, leading to the system:

$$\dot{x} = -(4x-1)(3+8x-16y)/64, \quad \dot{y} = -(16y-1)(3x-4y)/32,$$

which can be brought to system (3.60) via the affine transformation and time rescaling:

$$x_1 = x/2 + 1/8, \quad y_1 = y/2 - 1/32, \quad t_1 = 2t.$$

Thus, in this case, we again obtain configuration *Config.* $\mathcal{P}.76$.

3.1.3 The statement $(\mathcal{A}_3^*)$

In this case, the conditions $\zeta_4 = \zeta_5 = 0$, together with (3.39) and the assumption $\zeta_1 \neq 0$ (i.e., $(g-2)(g+3) \neq 0$), yield the condition $1 + 4g + 3g^2 + 4n = 0$, which implies:

$$n = -\frac{1}{4}(1+g)(1+3g).$$

This leads to the following family of systems:

$$\begin{aligned} \dot{x} &= \frac{1}{8}(1+g)(1+3g)^2 - \frac{1}{4}(1+g)(1+3g)x - \frac{1}{2}(1+g)y + gx^2 + xy, \\ \dot{y} &= \frac{1}{4}(1+g)(1+3g)^2x - \frac{1}{2}(1+g)(1+3g)y + (g-1)xy + 2y^2, \end{aligned} \tag{3.61}$$

which possess the invariant parabola $\Phi(x,y) = x^2 - y = 0$ with multiplicity 2.

Following statement $(\mathcal{A}_3^*)$, for the above systems we calculate:

$$\begin{aligned} \zeta_1 &= 2(g-2)(g+3), \quad \zeta_2 = 4g(1+g), \quad \zeta_4 = \zeta_5 = 0, \\ \mathcal{R}_2 &= -\frac{g(1+g)(g-2)(g+3)\left(8+27g+27g^2\right)}{8}, \\ B_1 &= \frac{g^4(1+g)^4(1+2g)(1+3g)^3(2+3g)^3}{8}. \end{aligned} \tag{3.62}$$

Since the quadratic polynomial $8 + 27g + 27g^2$ has a negative discriminant, for systems (3.61) we have:

$$\zeta_1\zeta_2\mathcal{R}_2 \neq 0 \quad \Rightarrow \quad g(1+g)(g-2)(g+3) \neq 0. \tag{3.63}$$

According to Lemma 2.3, the existence of an invariant line for systems (3.61) requires the condition $B_1 = 0$. Thus, we analyze two cases: $B_1 \neq 0$ and $B_1 = 0$.

The case $B_1 \neq 0$**.** Then, no invariant line can exist. The systems (3.61) possess three finite singularities $M_i(x_i, y_i)$, for $i = 1, 2, 3$, with coordinates:

$$\begin{aligned} x_1 = -\frac{1+3g}{2}, \quad y_1 = \frac{(1+3g)^2}{4}; \\ x_{2,3} = \frac{1}{2}\big(1+g \pm \sqrt{-2g(1+g)}\,\big), \quad y_{2,3} = \frac{g+1}{4}\big(1-g \pm 2\sqrt{-2g(1+g)}\,\big). \end{aligned} \tag{3.64}$$

We point out that M_1 is a multiple singularity of systems (3.61). Indeed, by applying the corresponding translation, M_1 can be placed at the origin, yielding the transformed system:

$$\begin{aligned} \dot{x} &= -\frac{1}{2}g(3g+1)x - (2g+1)y + gx^2 + xy, \\ \dot{y} &= \frac{1}{2}g(3g+1)^2x + (2g+1)(3g+1)y + (g-1)xy + 2y^2, \end{aligned}$$

where $M_0(0,0)$ is the singularity corresponding to M_1.

Following [1], we compute the corresponding invariant polynomials at M_0: $\mu_4 = \mu_3 = 0$, and

$$\mu_2 = \frac{1}{2}g(g+1)(3g+1)(3g+2)\left[g(1+3g)x^2 + 4gxy + 2y^2\right] \neq 0,$$

due to $B_1 \neq 0$. By [1, Lemma 5.2, statement (ii)], the point M_0 has multiplicity exactly 2.

We also observe that:

$$\Phi(x_1, y_1) = \Phi(x_2, y_2) = \Phi(x_3, y_3) = 0,$$

meaning that all three singularities lie on the invariant parabola.

The nature (real or complex) of M_2 and M_3 depends on the sign of $g(g+1) \neq 0$ (since $\zeta_2 \neq 0$). Given that $\zeta_2 = 4g(g+1)$, we analyze two subcases: $\zeta_2 < 0$ and $\zeta_2 > 0$.

1: *The subcase* $\zeta_2 < 0$. This implies $g(g+1) < 0$, i.e., $-1 < g < 0$. In this case, all three singularities on the invariant parabola are real. We need to determine the relative position of the double singularity M_1 with respect to the simple singularities M_2 and M_3.

From (3.64), we calculate:

$$(x_2 - x_1)(x_3 - x_1) = (1+3g)(2+3g)/2 \;\Rightarrow\; \operatorname{sign}\big((x_2 - x_1)(x_3 - x_1)\big) = \operatorname{sign}\big((1+3g)(2+3g)\big).$$

On the other hand, for systems (3.61) we obtain:

$$\xi_{15} = 2(1+3g)(2+3g) \neq 0,$$

due to $B_1 \neq 0$.

Therefore, if $\xi_{15} < 0$, the double point M_1 lies between the singularities M_2 and M_3, yielding configuration *Config.* $\mathcal{P}.77$, and if $\xi_{15} > 0$, the double point M_1 lies outside the curvilinear interval (M_2, M_3), yielding configuration *Config.* $\mathcal{P}.78$.

2: *The subcase* $\zeta_2 > 0$. Then $g(g+1) > 0$, so the singularities M_2 and M_3 are complex. In this case, we get only configuration *Config.* $\mathcal{P}.79$.

The case $B_1 = 0$. Considering (3.62) together with the condition (3.63), we conclude that $B_1 = 0$ is equivalent to the condition $(1+2g)(1+3g)(2+3g) = 0$. For systems (3.61), we also have $\zeta_3 = 8(1+2g)^2$, so we consider two subcases: $\zeta_3 \neq 0$ and $\zeta_3 = 0$.

1: *The subcase $\zeta_3 \neq 0$.* Then $1+2g \neq 0$, and the condition $B_1 = 0$ implies $(1+3g)(2+3g) = 0$.

If $g = -1/3$, we arrive at the system:

$$\dot{x} = -(y + x^2 - 3xy)/3, \quad \dot{y} = -2y(2x - 3y)/3, \tag{3.65}$$

which possesses the invariant line $y = 0$, tangent to the double invariant parabola at the singular point $M_1(0,0)$. In this case, the singular point M_3 has coalesced with M_1, producing a triple singularity. Therefore, we obtain configuration *Config.* $\mathcal{P}$*.80*.

Now, consider $g = -2/3$. Then, the system becomes:

$$\dot{x} = (1 + 2x - 4y - 16x^2 + 24xy)/24, \quad \dot{y} = (x + 2y - 20xy + 24y^2)/12,$$

which, via the affine transformation $x_1 = -x + 1/2$, $y_1 = y - x + 1/4$, can be brought to system (3.65), also possessing configuration *Config.* $\mathcal{P}$*.80*.

2: *The subcase $\zeta_3 = 0$.* Then $g = -1/2$, and we arrive at the system:

$$\dot{x} = -(4x-1)(1+8x-16y)/64, \quad \dot{y} = (x + 4y - 48xy + 64y^2)/32,$$

which possesses the invariant line $x = 1/4$ and the double invariant parabola $\Phi_1(x,y) = x^2 - y = 0$. In this case, the singular points M_1 and M_2 have coalesced, and it is straightforward to verify that this yields configuration *Config.* $\mathcal{P}$*.81*.

Thus, we have proved the following lemma.

Lemma 3.10. *Assume that for a quadratic system the conditions $(\mathcal{A}_3^*)$ are satisfied. Then, the system possesses one of the following configurations, if and only if the corresponding conditions below hold:*

$$\begin{array}{lll}
B_1 \neq 0,\ \zeta_2 < 0,\ \xi_{15} < 0 & \Rightarrow & \textit{Config. } \mathcal{P}\textit{.77;} \\
B_1 \neq 0,\ \zeta_2 < 0,\ \xi_{15} > 0 & \Rightarrow & \textit{Config. } \mathcal{P}\textit{.78;} \\
B_1 \neq 0,\ \zeta_2 > 0 & \Rightarrow & \textit{Config. } \mathcal{P}\textit{.79;} \\
B_1 = 0,\ \zeta_3 \neq 0 & \Rightarrow & \textit{Config. } \mathcal{P}\textit{.80;} \\
B_1 = 0,\ \zeta_3 = 0 & \Rightarrow & \textit{Config. } \mathcal{P}\textit{.81.}
\end{array}$$

3.1.4 The statement $(\mathcal{A}_4^*)$

In this case, the conditions $\zeta_4 = \mathcal{R}_2 = 0$ hold. Considering (3.39) and the condition $\zeta_1 \neq 0$ (i.e., $(g-2)(g+3) \neq 0$), we get:

$$(8 + 27g + 27g^2)(1 + 6g + 5g^2 + 4n) = 0.$$

However, the discriminant of the quadratic polynomial $8 + 27g + 27g^2$ equals $-135 < 0$, so this factor cannot vanish. Thus, we obtain:

$$1 + 6g + 5g^2 + 4n = 0 \ \Rightarrow\ n = -\frac{1}{4}(1+g)(1+5g).$$

This leads to the following family of systems:

$$\begin{aligned}
\dot{x} &= \frac{1}{8}(1 + g - 2x)(1 + 4g + 3g^2 - 4gx - 4y), \\
\dot{y} &= \frac{1}{4}(1+g)^2(1+3g)\,x - \frac{1}{2}(1+g)(1+5g)y + (g-1)xy + 2y^2,
\end{aligned} \tag{3.66}$$

which possess the invariant parabola $\Phi(x,y) = x^2 - y = 0$ and the invariant line $x = (g+1)/2$.

For the above systems, following the statement $(\mathcal{A}_4^*)$, we calculate:

$$\begin{aligned} &\zeta_1 = 2(g-2)(3+g), \;\; \zeta_2 = 4g(1+g), \\ &\zeta_4 = 0 = \mathcal{R}_2, \;\; \zeta_5 = 19(g-2)g^2(1+g)^2(3+g), \\ &B_1 = 0, \;\; B_2 = -648g^5(1+g)^5(1+3g)(2+3g)x^4. \end{aligned} \tag{3.67}$$

Therefore, for systems (3.61), we have:

$$\zeta_1\zeta_2\zeta_5 \neq 0 \;\Rightarrow\; g(1+g)(g-2)(3+g) \neq 0. \tag{3.68}$$

We now discuss two possibilities: $B_2 \neq 0$ and $B_2 = 0$.

1: *The case* $B_2 \neq 0$. In this situation, by Lemma 2.3, systems (3.66) cannot possess any invariant line in other direction than $x = (g+1)/2$.

On the other hand, by Lemma 2.4, these systems could possess an invariant line parallel to the existing one if $\theta = (g-1)(g+2) = 0$. However, a straightforward computation shows that neither $g = 1$ nor $g = -2$ leads to the appearance of an additional parallel invariant line.

Systems (3.66) possess three finite singularities $M_i(x_i, y_i)$ $(i = 1,2,3)$ with coordinates:

$$x_1 = \frac{1+g}{2}, \quad y_1 = \frac{(1+g)^2}{4}; \quad x_2 = \frac{1+g}{2}, \quad y_2 = \frac{1}{4}(1+g)(1+3g);$$
$$x_3 = -\frac{1+3g}{2}, \quad y_3 = \frac{(1+3g)^2}{4}.$$

We observe that the invariant line $x = (g+1)/2$ intersects the invariant parabola at the singular point M_1, and we claim that M_1 is a multiple singularity of systems (3.66). Indeed, applying the corresponding translation, we can place M_1 at the origin of coordinates, resulting in the system:

$$\dot{x} = gx^2 + xy, \quad \dot{y} = g(1+g)^2x - g(1+g)y + (g-1)xy + 2y^2,$$

where $M_0(0,0)$ is the singularity corresponding to M_1.

Following [1], we compute the invariant polynomials: $\mu_4 = \mu_3 = 0$, and:

$$\mu_2 = g^2(1+g)^2(1+2g)x(gx+y).$$

Due to condition (3.68), we have that $\mu_2 = 0$ if and only if $1+2g = 0$. By [1, Lemma 5.2, statement (ii)], the point M_0 has multiplicity exactly 2 if $1+2g \neq 0$, and it is triple if $g = -1/2$.

For systems (3.66), we have $\zeta_3 = 8(1+2g)^2$, so we now consider two cases: $\zeta_3 \neq 0$ and $\zeta_3 = 0$.

1.1: *The subcase* $\zeta_3 \neq 0$. Then $1+2g \neq 0$, and therefore M_1 is a double singular point of the system, with all three singularities distinct.

Moreover, the singularity M_2 lies on the invariant line, while M_3 lies on the invariant parabola. To determine the relative positions of these points with respect to the double singularity M_1, we compute:

$$\begin{aligned} y_2 - y_1 = g(1+g)/2 \;&\Rightarrow\; \operatorname{sign}(y_2 - y_1) = \operatorname{sign}\big(g(g+1)\big), \\ x_3 - x_1 = -(1+2g) \;&\Rightarrow\; \operatorname{sign}(x_3 - x_1) = -\operatorname{sign}(1+2g), \end{aligned}$$

and we observe that for systems (3.66), we have $\zeta_2 = 4g(g+1)$, hence sign $(\zeta_2) =$ sign $\big(g(g+1)\big)$.

Thus, we arrive at the following configurations:

$$\begin{array}{lll}
\zeta_2 < 0,\ 2g+1 < 0 \ \text{(i.e., } -1 < g < -1/2) & \Rightarrow y_2 < y_1,\ x_3 > x_1 & \Rightarrow \textit{Config. } \mathcal{P}.82;\\
\zeta_2 < 0,\ 2g+1 > 0 \ \text{(i.e., } -1/2 < g < 0) & \Rightarrow y_2 < y_1,\ x_3 < x_1 & \Rightarrow \simeq \textit{Config. } \mathcal{P}.82;\\
\zeta_2 > 0,\ 2g+1 < 0 \ \text{(i.e., } g < -1) & \Rightarrow y_2 > y_1,\ x_3 > x_1 & \Rightarrow \textit{Config. } \mathcal{P}.83;\\
\zeta_2 > 0,\ 2g+1 > 0 \ \text{(i.e., } g > 0) & \Rightarrow y_2 > y_1,\ x_3 < x_1 & \Rightarrow \simeq \textit{Config. } \mathcal{P}.83.
\end{array}$$

1.2: *The subcase* $\zeta_3 = 0$. Then $g = -1/2$, leading to the system:

$$\dot{x} = -(4x-1)(-1+8x-16y)/64, \quad \dot{y} = (-x+12y-48xy+64y^2)/32,$$

which possesses the invariant line $x = 1/4$ and the invariant parabola $\Phi_1(x,y) = x^2 - y = 0$. In this case, the singular points M_1 and M_3 coalesce, resulting in the configuration *Config.* $\mathcal{P}.84$.

2: *The case* $B_2 = 0$. Considering (3.67) and the condition (3.68), we conclude that $B_2 = 0$ is equivalent to $(1+3g)(2+3g) = 0$.

If $g = -1/3$, we arrive at the system:

$$\dot{x} = -(3x-1)(x-3y)/9, \quad \dot{y} = 2y(1-6x+9y)/9, \tag{3.69}$$

which possesses the additional invariant line $y = 0$, tangent to the invariant parabola at the singular point $M_3(0,0)$ and intersecting the invariant line $x = 1/3$ at the singular point $M_2(1/3,0)$. Therefore, we obtain the configuration *Config.* $\mathcal{P}.85$.

Assume now $g = -2/3$. This leads to the system:

$$\dot{x} = -(6x-1)(8x-1-12y)/72, \quad \dot{y} = (-x+14y-60xy+72y^2)/36,$$

which, via the affine transformation $x_1 = -x + 1/2$, $y_1 = -x + y + 1/4$, can be brought to system (3.69), also corresponding to configuration *Config.* $\mathcal{P}.85$.

Thus, we have proved the following lemma.

Lemma 3.11. *Assume that for a quadratic system the conditions* $(\mathcal{A}_4^*)$ *are satisfied. Then this system possesses one of the following configurations if and only if the corresponding conditions are satisfied, respectively:*

$$\begin{array}{ll}
B_2 \neq 0,\ \zeta_3 \neq 0,\ \zeta_2 < 0 & \Rightarrow \textit{Config. } \mathcal{P}.82;\\
B_2 \neq 0,\ \zeta_3 \neq 0,\ \zeta_2 > 0 & \Rightarrow \textit{Config. } \mathcal{P}.83;\\
B_2 \neq 0,\ \zeta_3 = 0 & \Rightarrow \textit{Config. } \mathcal{P}.84;\\
B_2 = 0 & \Rightarrow \textit{Config. } \mathcal{P}.85.
\end{array}$$

3.1.5 The statement $(\mathcal{A}_5^*)$

According to Proposition 2.7*, the condition $\zeta_2 = 0$ holds. Considering (3.2), we obtain $g(g+1) = 0$.

Following the proof of Lemma 3.1, we conclude that the condition $g + 1 = 0$ can be reduced, via an affine transformation (see formulas (3.12)), to the condition $g = 0$.

Therefore, it suffices to study only the case $g = 0$. In this case, we arrive at the following two-parameter family of systems:

$$\dot{x} = m + nx - y/2 + xy, \quad \dot{y} = 2mx + 2ny - xy + 2y^2, \tag{3.70}$$

which possesses the invariant parabola $\Phi(x,y) = x^2 - y = 0$.

Considering the statement $(\mathcal{A}_5^*)$ for these systems, we calculate:

$$\begin{aligned} &\zeta_1 = -12,\ \ \zeta_2 = 0,\ \ \zeta_6 = (2m+n)/2,\ \ \mathcal{R}_1 = 0,\ \ \mathcal{R}_2 = 6(2m+n),\\ &B_1 = 2m(4m-1-2n)(2m+n)^3. \end{aligned} \tag{3.71}$$

Remark 3.12. Following [1, Lemma 5.2], for systems (3.70), we compute:

$$\mu_0 = 0, \quad \mu_1 = -2(2m+n)y \neq 0,$$

due to the condition $\zeta_6 \neq 0$. Therefore, according to [1, Lemma 5.2, statement (i)], we conclude that one of the singular points of systems (3.70) has gone to infinity and has coalesced with the infinite singularity $N[1:0:0]$, producing an infinite singularity of multiplicity $(1,1)$ (see Remark 2.5).

The case $B_1 \neq 0$. We observe that the family of systems (3.70) is a subfamily of (3.1) defined by the condition $g = 0$. Considering the finite singularities of (3.1) given in (3.4), we remark that in the case $g = 0$, the singular point $M_1(x_1, y_1)$ with coordinates

$$x_1 = -\frac{2m+n+gn}{g(1+g)}, \quad y_1 = \frac{2m}{1+g},$$

has moved to infinity. According to Remark 3.12, this singularity coalesced with the infinite singularity $N[1:0:0]$, producing an infinite singularity of multiplicity $(1,1)$.

Therefore, systems (3.70) possess three finite singularities $M_i(\tilde{x}_i, \tilde{y}_i)$ for $i = 2,3,4$, where, from (3.4), we have

$$\tilde{x}_i = x_i\big|_{g=0}, \quad \tilde{y}_i = y_i\big|_{g=0}, \quad i = 2,3,4.$$

Taking into consideration [1, Proposition 5.1], for systems (3.70), we compute:

$$\mu_0 = 0, \quad \mathbf{D} = 48(2m+n)^4\big(108m^2 - 2m + 36mn - n^2 + 16n^3\big), \quad \mathbf{R} = 12(2m+n)^2y^2.$$

We observe that $\mathbf{R} \neq 0$ due to $\zeta_6 \neq 0$, and by [1, Proposition 5.1], we have three distinct real finite singularities if $\mathbf{D} < 0$, and one real and two complex if $\mathbf{D} > 0$. Considering the point at infinity of multiplicity $(1,1)$, we arrive at *Config. $\mathcal{P}$.86* if $\mathbf{D} < 0$, and *Config. $\mathcal{P}$.87* if $\mathbf{D} > 0$.

Now assume that for systems (3.70), the condition $\mathbf{D} = 48(F_1')^2F_2' = 0$ holds, where, from (3.3), we have:

$$F_1' = -(2m+n)^2, \quad F_2' = -2m + 108m^2 + 36mn - n^2 + 16n^3.$$

Since $F_1' \neq 0$ (due to $\zeta_6 \neq 0$), we conclude that the condition $\mathbf{D} = 0$ is equivalent to $F_2' = 0$.

Introducing a new parameter v, as in the generic case (see page 29), we obtain $n = (1 - v^2)/12$, $m = (1+v)^2(2v-1)/216$, and we arrive at the following one-parameter family of systems:

$$\begin{aligned} \dot{x} &= \frac{(1+v)^2(2v-1)}{216} - \frac{v^2-1}{12}\,x - \frac{1}{2}y + xy,\\ \dot{y} &= \frac{(1+v)^2(2v-1)}{108}\,x - \frac{v^2-1}{6}\,y - xy + 2y^2, \end{aligned} \tag{3.72}$$

which is a subfamily of (3.8) defined by $g = 0$.

The systems (3.5) possess three finite singularities given in (3.7), and M_1 is a multiple singularity (of multiplicity at least two). We observe that for $g = 0$, the singular point M_3

has gone to infinity and, according to Remark 3.12, this singularity coalesced with the infinite singularity $N[1:0:0]$, producing an infinite singularity of multiplicity $(1,1)$.

Thus, the systems (3.72) possess at most two different finite singularities $M_1(\tilde{x}_1,\tilde{y}_1)$ (multiple) and $M_2(\tilde{x}_2,\tilde{y}_2)$, where, from (3.7), we have:

$$\tilde{x}_1=\frac{1+v}{6},\quad \tilde{y}_1=\frac{(1+v)^2}{36};\quad \tilde{x}_2=\frac{1-2v}{6},\quad \tilde{y}_2=\frac{(1-2v)^2}{36}.$$

We observe that M_2 coalesces with the double point M_1 if and only if $v=0$.

Considering that $B_1\neq 0$ (i.e., systems (3.72) do not possess any invariant line), we conclude that the configuration is *Config.* $\mathcal{P}$*.88* if $v\neq 0$, and *Config.* $\mathcal{P}$*.89* if $v=0$.

Moreover, for systems (3.72), we compute:

$$\xi_2=\frac{1}{209952}v^2(v-2)^6(1+v)^2,\quad \zeta_6=\frac{1}{108}(v-2)^2(1+v),$$

and due to $\zeta_6\neq 0$, we conclude that the condition $v\neq 0$ is equivalent to $\xi_2\neq 0$. Therefore, we obtain configuration *Config.* $\mathcal{P}$*.88* if $\xi_2\neq 0$, and *Config.* $\mathcal{P}$*.89* if $\xi_2=0$.

The case $B_1=0$**.** Considering (3.71) and the condition $\zeta_6\neq 0$ (i.e., $2m+n\neq 0$), we deduce that the condition $B_1=0$ is equivalent to $m(4m-1-2n)=0$.

On the other hand, for systems (3.70), we calculate:

$$\xi_1=9(4m-1-2n)(2m+n)^2/4$$

and, due to $\zeta_6\neq 0$, we conclude that the condition $4m-1-2n=0$ is equivalent to $\xi_1=0$. Thus, we discuss two subcases: $\xi_1\neq 0$ and $\xi_1=0$.

1: *The subcase* $\xi_1\neq 0$. In this case, the condition $B_1=0$ yields $m=0$, which leads to the one-parameter family of systems:

$$\dot{x}=\frac{1}{2}(2nx-y+2xy),\quad \dot{y}=y(2n-x+2y),\tag{3.73}$$

possessing the invariant parabola $\Phi(x,y)=x^2-y=0$ and the invariant line $y=0$. Calculations yield:

$$B_2=-324n^2(1+2n)y^4$$

and, according to Lemma 2.3, we discuss two possibilities: $B_2\neq 0$ and $B_2=0$.

1.1: The possibility $B_2\neq 0$. We determine that systems (3.73) possess three finite singularities $M_i(x_i,y_i)$ $(i=1,2,3)$ with coordinates:

$$x_1=0,\quad y_1=0;\quad x_{2,3}=\frac{1}{4}\left(1\pm\sqrt{1-16n}\right),\quad y_{2,3}=\frac{1}{8}\left(1-8n\pm\sqrt{1-16n}\right).\tag{3.74}$$

According to Remark 3.12, the fourth finite singularity has coalesced with an infinite one, resulting in a singular point of multiplicity $(1,1)$.

We observe that M_1 is the tangency point between $y=0$ and the invariant parabola, and that M_2 and M_3 are either real, complex, or coincident, depending on the value of $1-16n$.

To determine the relative position of the real singularities M_2 and M_3 with respect to M_1, we calculate:

$$(x_2-x_1)(x_3-x_1)=n,\quad (x_2-x_1)+(x_3-x_1)=1/2>0.$$

On the other hand, for systems (3.73), we calculate:

$$\mathbf{D} = -48n^6(1-16n), \quad \mathcal{R}_2 = 6n,$$

and therefore, due to $\mathcal{R}_2 \neq 0$, we have $\text{sign}\,(\mathcal{R}_2) = \text{sign}\,(n)$ and $\text{sign}\,(\mathbf{D}) = -\text{sign}\,(1-16n)$.

Thus, in the case $B_2 \neq 0$, we arrive at the following four configurations:

$$\begin{array}{ll} \mathbf{D} < 0, \mathcal{R}_2 < 0 & \Rightarrow \textit{Config. } \mathcal{P}.90; \\ \mathbf{D} < 0, \mathcal{R}_2 > 0 & \Rightarrow \textit{Config. } \mathcal{P}.91; \\ \mathbf{D} > 0 & \Rightarrow \textit{Config. } \mathcal{P}.92; \\ \mathbf{D} = 0 & \Rightarrow \textit{Config. } \mathcal{P}.93. \end{array}$$

1.2: The possibility $B_2 = 0$. This implies that $(1+2n)n = 0$, and since $n \neq 0$ (due to $\zeta_6 = n/2 \neq 0$), we get $1+2n = 0$. Thus, $n = -1/2$, and we obtain the system:

$$\dot{x} = \frac{1}{2}(-x-y+2xy), \quad \dot{y} = y(-1-x+2y), \tag{3.75}$$

possessing two invariant lines: $y = 0$ and $y = x$. Considering (3.74), we get three real finite singularities, leading to configuration *Config.* $\mathcal{P}.94$.

2: *The subcase* $\xi_1 = 0$. This implies $m = (1+2n)/4$, and we arrive at the one-parameter family of systems:

$$\dot{x} = \frac{1}{4}(1+2n+4nx-2y+4xy), \quad \dot{y} = \frac{1}{2}(x+2nx+4ny-2xy+4y^2), \tag{3.76}$$

possessing the invariant parabola $\Phi(x,y) = x^2 - y = 0$ and the invariant line $y = x - (2n+1)/2$. Calculations yield:

$$B_2 = -81(1+2n)(1+4n)^2(x-y)^4$$

and, considering Lemma 2.3, we examine two possibilities: $B_2 \neq 0$ and $B_2 = 0$.

2.1: The possibility $B_2 \neq 0$. The above systems possess three finite singularities $M_i(x_i, y_i)$ $(i = 1,2,3)$ with coordinates:

$$x_1 = -\frac{1}{2}, \quad y_1 = \frac{1}{4}; \quad x_{2,3} = \frac{1}{2}\left(1 \pm \sqrt{-(1+4n)}\right), \quad y_{2,3} = \frac{1}{2}\left(-2n \pm \sqrt{-(1+4n)}\right).$$

The singularities M_2 and M_3 are the intersection points between the invariant line $y = x - (2n+1)/2$ and the invariant parabola, and they are real (respectively, complex or coincident) if $1+4n < 0$ (respectively, $1+4n > 0$, $1+4n = 0$).

Again, in the case of real singularities, we calculate:

$$(x_2 - x_1)(x_3 - x_1) = (5+4n)/4, \quad (x_2 - x_1) + (x_3 - x_1) = 2 > 0.$$

Moreover, for systems (3.76), calculations give:

$$\mathbf{D} = 3(1+4n)^5(5+4n)^2/4, \quad \xi_3 = 8164197(1+4n)^3(5+4n)/2,$$

and therefore, for $\mathbf{D} \neq 0$, we have $\text{sign}\,(\mathbf{D}) = \text{sign}\,(1+4n)$ and $\text{sign}\,(\xi_3) = \text{sign}\,\big((1+4n)(5+4n)\big)$.

Thus, in the case $B_2 \neq 0$ and $\mathbf{D} \neq 0$, we arrive at the following configurations:

$$\begin{array}{ll} \mathbf{D} < 0, \xi_3 < 0 & \Rightarrow \textit{Config. } \mathcal{P}.95; \\ \mathbf{D} < 0, \xi_3 > 0 & \Rightarrow \textit{Config. } \mathcal{P}.96; \\ \mathbf{D} > 0 & \Rightarrow \textit{Config. } \mathcal{P}.97. \end{array}$$

Now, assume that $\mathbf{D} = 0$. This implies $(1+4n)(5+4n) = 0$. Since $\zeta_6 = (1+4n)/4 \neq 0$, we get $5+4n = 0$, meaning that one of the singularities M_2 or M_3 coalesces with M_1. Thus, we arrive at configuration *Config. $\mathcal{P}$.98*.

2.2: The possibility $B_2 = 0$. This implies $(1+2n)(1+4n) = 0$, and since $1+4n \neq 0$ (due to $\zeta_6 \neq 0$), we get $1+2n = 0$. Thus, $n = -1/2$, and we arrive again at system (3.75), possessing configuration *Config. $\mathcal{P}$.94*.

Thus, we have proved the following lemma.

Lemma 3.13. *Assume that for a quadratic system the conditions $(\mathcal{A}_5^*)$ are satisfied. Then this system possesses one of the following configurations if and only if the corresponding conditions are satisfied, respectively:*

$$
\begin{array}{ll}
B_1 \neq 0,\ \mathbf{D} < 0 & \Rightarrow \textit{ Config. } \mathcal{P}.86; \\
B_1 \neq 0,\ \mathbf{D} > 0 & \Rightarrow \textit{ Config. } \mathcal{P}.87; \\
B_1 \neq 0,\ \mathbf{D} = 0,\ \xi_2 \neq 0 & \Rightarrow \textit{ Config. } \mathcal{P}.88; \\
B_1 \neq 0,\ \mathbf{D} = 0,\ \xi_2 = 0 & \Rightarrow \textit{ Config. } \mathcal{P}.89; \\
B_1 = 0,\ \xi_1 \neq 0,\ B_2 \neq 0,\ \mathbf{D} < 0,\ \mathcal{R}_2 < 0 & \Rightarrow \textit{ Config. } \mathcal{P}.90; \\
B_1 = 0,\ \xi_1 \neq 0,\ B_2 \neq 0,\ \mathbf{D} < 0,\ \mathcal{R}_2 > 0 & \Rightarrow \textit{ Config. } \mathcal{P}.91; \\
B_1 = 0,\ \xi_1 \neq 0,\ B_2 \neq 0,\ \mathbf{D} > 0 & \Rightarrow \textit{ Config. } \mathcal{P}.92; \\
B_1 = 0,\ \xi_1 \neq 0,\ B_2 \neq 0,\ \mathbf{D} = 0 & \Rightarrow \textit{ Config. } \mathcal{P}.93; \\
B_1 = 0,\ \xi_1 \neq 0,\ B_2 = 0 & \Rightarrow \textit{ Config. } \mathcal{P}.94; \\
B_1 = 0,\ \xi_1 = 0,\ B_2 \neq 0,\ \mathbf{D} < 0,\ \xi_3 < 0 & \Rightarrow \textit{ Config. } \mathcal{P}.95; \\
B_1 = 0,\ \xi_1 = 0,\ B_2 \neq 0,\ \mathbf{D} < 0,\ \xi_3 > 0 & \Rightarrow \textit{ Config. } \mathcal{P}.96; \\
B_1 = 0,\ \xi_1 = 0,\ B_2 \neq 0,\ \mathbf{D} > 0 & \Rightarrow \textit{ Config. } \mathcal{P}.97; \\
B_1 = 0,\ \xi_1 = 0,\ B_2 \neq 0,\ \mathbf{D} = 0 & \Rightarrow \textit{ Config. } \mathcal{P}.98; \\
B_1 = 0,\ \xi_1 = 0,\ B_2 = 0 & \Rightarrow \textit{ Config. } \mathcal{P}.94.
\end{array}
$$

Since all the statements provided by Proposition 2.7* have been considered, this proposition is proved. □

3.2 Systems in QSP$_{(\eta>0)}$ with the condition $\zeta_1 = 0$

In what follows, we examine each of the statements $(\mathcal{B}_1)$ to $(\mathcal{B}_7)$ given by Proposition 2.8.

According to this proposition, a system satisfying the conditions provided by one of the statements $(\mathcal{B}_1)$ to $(\mathcal{B}_7)$ can be brought to the form:

$$\dot{x} = m + nx - \frac{3y}{2} + 2x^2 + xy, \quad \dot{y} = 2mx + 2ny + xy + 2y^2, \tag{3.77}$$

and this system possesses the invariant parabola $\Phi_1(x,y) = x^2 - y = 0$.

3.2.1 The statement $(\mathcal{B}_1)$

According to this statement, for systems (3.77) we calculate $\chi_3 = 0$ and

$$\chi_4 = 61875\,\mathcal{U}_1\mathcal{U}_3, \quad \zeta_7 = -\frac{52875}{2}\,\mathcal{U}_1\mathcal{U}_2\mathcal{U}_3, \quad \mathcal{R}_3 = \quad 3850561006875\,\mathcal{U}_1\mathcal{U}_2\mathcal{U}_3, \tag{3.78}$$

where

$$\mathcal{U}_1 = 1 + 4m + 2n, \quad \mathcal{U}_2 = 4m - 147 - 14n,$$
$$\mathcal{U}_3 = 18m + 1372m^2 - 84mn + 27n^2 + 144n^3.$$

On the other hand, following Lemma 2.3, we calculate:

$$B_1 = m(2m-n)(2m+3n+9)(4m-6n-9)(1+4m+2n) \tag{3.79}$$

and, considering Lemma 2.3, we discuss two cases: $B_1 \neq 0$ and $B_1 = 0$.

The case $B_1 \neq 0$. We observe that the family of systems (3.77) is a subfamily of (3.1) defined by the condition $g = 2$. Therefore, it is clear that systems (3.77) possess four finite singularities $M_i(\tilde{x}_i, \tilde{y}_i)$ $(i = 1,2,3,4)$, where, considering (3.4), we have:

$$\tilde{x}_i = x_i\big|_{\{g=2\}}, \quad \tilde{y}_i = y_i\big|_{\{g=2\}}, \quad i = 1,2,3,4.$$

On the other hand, for systems (3.77), we have:

$$\mathbf{D} = 48\widetilde{F}_1^2\widetilde{F}_2, \quad \widetilde{F}_1 = F_1\big|_{\{g=2\}}, \quad \widetilde{F}_2 = F_2\big|_{\{g=2\}},$$

where F_1 and F_2 are given in (3.3).

As it was proved for the family (3.1), in the case $\mathbf{D} \neq 0$, these systems possess only two distinct configurations: *Config.* $\mathcal{P}.1$ if $\mathbf{D} < 0$, and *Config.* $\mathcal{P}.2$ if $\mathbf{D} > 0$. We obtain the same two configurations in the particular case $g = 2$, because this value of the parameter g is not a bifurcation value for these two configurations.

Assume now that $\mathbf{D} = 0$. This implies $\widetilde{F}_1\widetilde{F}_2 = 0$, and we have to distinguish which factor vanishes. We point out that the invariant polynomial ξ_1, which governed the condition $F_1 = 0$ for systems (3.1) in the generic case (i.e., $g \neq 2$), vanishes for $g = 2$. Therefore, we must use another invariant polynomial, and for systems (3.77) we calculate:

$$\xi_{18} = 17969284698750\, \mathcal{U}_2\, \mathcal{U}_3\, \widetilde{F}_1.$$

Therefore, due to the condition $\zeta_7 \neq 0$, we obtain that the condition $\widetilde{F}_1 = 0$ is equivalent to $\xi_{18} = 0$. Thus, we examine two possibilities: $\xi_{18} \neq 0$ and $\xi_{18} = 0$.

1: The possibility $\xi_{18} \neq 0$. Then $\widetilde{F}_1 \neq 0$ and hence the condition $\mathbf{D} = 0$ implies $\widetilde{F}_2 = 0$. Following the investigation of the family of systems (3.1) in the particular case $g = 2$, we arrive at the systems (3.5), which for $g = 2$ become:

$$\begin{aligned} \dot{x} &= \frac{1}{216}(2v+1)(v-1)^2 - \frac{1}{12}(v^2-1)x - \frac{3y}{2} + 2x^2 + xy, \\ \dot{y} &= \frac{1}{108}(2v+1)(v-1)^2 x - \frac{1}{6}(v^2-1)y + xy + 2y^2. \end{aligned} \tag{3.80}$$

For the above systems, we calculate:

$$\xi_2 = 2^{-5}3^{-8}(v-4)^2(v-1)^2v^2(v^2-20v-8)^2,$$
$$\xi_{18} = 998293594375(v-10)(v-4)^4(v-1)^4(20+v)^2(4+5v)^2(v^2-20v-8)/3188646,$$

and we observe that, due to $\xi_{18} \neq 0$, the condition $\xi_2 = 0$ is equivalent to $v = 0$.

Therefore, following the examination of the two-parameter family of systems (3.1), we conclude that the one-parameter family of systems (3.80), in the case $B_1\xi_{18} \neq 0$, possesses the configuration *Config.* $\mathcal{P}.3$ if $\xi_2 \neq 0$ and *Config.* $\mathcal{P}.4$ if $\xi_2 = 0$.

2: The possibility $\xi_{18} = 0$. Then we have $\widetilde{F}_1 = 0$, which implies $\mathbf{D} = 0$. Following the investigation of the family of systems (3.1) in the particular case $g = 2$, we arrive at the systems (3.8), which for $g = 2$ become:

$$\begin{aligned} \dot{x} =& \frac{3}{8}(u-2)^2 - \frac{1}{4}(u^2-4)x - \frac{3y}{2} + 2x^2 + xy, \\ \dot{y} =& \frac{3}{4}(u-2)^2 x - \frac{1}{2}(u^2-4)y + xy + 2y^2. \end{aligned} \tag{3.81}$$

For the above systems, we calculate:

$$\chi_4 = -556875(u-3)^5(u-2)^2(13+u)/4, \quad \xi_2 = 9(u-3)^2(u-2)^2\widetilde{Z}_1/2,$$
$$\xi_3 = 490484322\,\tilde{\alpha}_1^3\,\widetilde{Z}_1, \quad \widetilde{Z}_1 = Z_1\big|_{\{g=2\}}, \quad \tilde{\alpha}_1 = \alpha_1\big|_{\{g=2\}},$$

where Z_1 and α_1 are the polynomials defined for systems (3.8) (see (3.10) and (3.11)).

We observe that, due to the condition $\chi_4 \neq 0$, we have:

$$\operatorname{sign}(\xi_2) = \operatorname{sign}(\widetilde{Z}_1), \quad \operatorname{sign}(\xi_3) = \operatorname{sign}(\widetilde{Z}_1\tilde{\alpha}_1),$$

and, following the examination of the two-parameter family of systems (3.8), we conclude that the one-parameter family of systems (3.81), in the case $B_1 \neq 0$, possesses the following configurations if and only if the corresponding conditions are satisfied:

$$\begin{array}{ll} \xi_2 < 0 & \Leftrightarrow \textit{Config. } \mathcal{P}.5; \\ \xi_2 > 0,\ \xi_3 < 0 & \Leftrightarrow \textit{Config. } \mathcal{P}.6; \\ \xi_2 > 0,\ \xi_3 > 0 & \Leftrightarrow \textit{Config. } \mathcal{P}.7; \\ \xi_2 = 0 & \Leftrightarrow \textit{Config. } \mathcal{P}.8. \end{array}$$

The case $B_1 = 0$. Considering (3.79) and the condition $\chi_4 \neq 0$ (i.e., $4m + 2n + 1 \neq 0$), we observe that the condition $B_1 = 0$ is equivalent to:

$$m(2m-n)(2m+3n+9)(4m-6n-9) = 0.$$

For systems (3.77), calculations yield:

$$\begin{aligned} \xi_{19} &= -12870000\,m\,(2m-n)\,(2m+3n+9), \\ \xi_{20} &= -540\,m\,(2m-n)\,\mathcal{U}_1\mathcal{U}_3, \quad \xi_{21} = -110106\,m\,\mathcal{U}_1\mathcal{U}_2\mathcal{U}_3, \end{aligned} \tag{3.82}$$

and we consider two subcases: $\xi_{19} \neq 0$ and $\xi_{19} = 0$.

The subcase $\xi_{19} \neq 0$. Then $m(2m-n)(2m+3n+9) \neq 0$, and therefore the condition $B_1 = 0$ yields $4m - 6n - 9 = 0$. This implies $m = \frac{3(3+2n)}{4}$, and we arrive at the one-parameter family of systems:

$$\begin{aligned} \dot{x} &= (9 + 6n + 4nx + 8x^2 - 6y + 4xy)/4, \\ \dot{y} &= (9x + 6nx + 4ny + 2xy + 4y^2)/2, \end{aligned} \tag{3.83}$$

which possesses the invariant line $y = x - \frac{3+2n}{2}$ and four finite singularities $M_i(x_i, y_i)$ $(i = 1, 2, 3, 4)$ with coordinates:

$$x_1 = -\frac{3}{2}, \quad y_1 = \frac{9}{4}; \quad x_2 = -\frac{3+4n}{4}, \quad y_2 = \frac{3+2n}{4};$$
$$x_{3,4} = \frac{1}{2}\left(1 \pm \sqrt{-(5+4n)}\right), \quad y_{3,4} = \frac{1}{2}\left(-2-2n \pm \sqrt{-(5+4n)}\right).$$

We determine that the singularities M_1, M_3, and M_4 lie on the invariant parabola. Moreover, M_3 and M_4 are the points of intersection of the invariant line $y = x - (3+2n)/2$ with the invariant parabola $\Phi_1(x,y) = x^2 - y = 0$. We calculate:

$$\Phi_1(x_2,y_2) = \frac{1}{16}(4n-3)(5+4n),$$

and we conclude that M_2 lies on the parabola if and only if $4n-3=0$, because for systems (3.83), we have:

$$\chi_4\zeta_7\mathcal{R}_3 \neq 0 \;\Leftrightarrow\; (5+4n)(9+4n)(69+4n) \neq 0. \tag{3.84}$$

In order to determine the position of the singularity M_1 with respect to M_3 and M_4 (when they are real), we calculate:

$$(x_1-x_3)(x_1-x_4) = (21+4n)/4 \;\Rightarrow\; \operatorname{sign}\big((x_1-x_3)(x_1-x_4)\big) = \operatorname{sign}(21+4n),$$
$$(x_1-x_3)+(x_1-x_4) = -4 < 0.$$

Thus, we observe that for the parameter n, the following possible bifurcation values arise: $n \in \{-21/4, -5/4, 3/4\}$. Moreover, we point out that due to the condition (3.84), the inequality $5+4n \neq 0$ must hold (i.e., $n \neq -5/4$), and hence the singularities M_3 and M_4 cannot coincide.

On the other hand, according to Lemma 2.3, for the existence of an invariant line in a direction different from $y = x$, the condition $B_2 = 0$ is necessary. For systems (3.83), we calculate:

$$B_2 = -729(3+2n)(9+4n)^2(x-y)^4,\;\; \mathbf{D} = 243(4n-3)^2(5+4n)^3(21+4n)^2/4,$$
$$\xi_9 = 16299895407840(9+4n)^2(21+4n),$$

and in the case $\mathbf{D} \neq 0$, we have:

$$\operatorname{sign}(\mathbf{D}) = \operatorname{sign}(5+4n), \quad \operatorname{sign}(\xi_9) = \operatorname{sign}(21+4n).$$

Considering Lemma 2.3, we examine two possibilities: $B_2 \neq 0$ and $B_2 = 0$.

1: *The possibility* $B_2 \neq 0$. We discuss two cases: $\mathbf{D} \neq 0$ and $\mathbf{D} = 0$.

1.1: *The case* $\mathbf{D} \neq 0$. In this case, all four finite singular points of systems (3.83) are distinct. Considering the bifurcation values of the parameter n mentioned above, for systems (3.83) we obtain the following configurations (depending on the parameter n):

$\mathbf{D} < 0$, $\xi_9 < 0$ (i.e., $n < -21/4$) $\Rightarrow \simeq$ *Config.* $\mathcal{P}.17$;
$\mathbf{D} < 0$, $\xi_9 > 0$ (i.e., $-21/4 < n < -5/4$) $\Rightarrow \simeq$ *Config.* $\mathcal{P}.19$;
$\mathbf{D} > 0$ (i.e., $n > -5/4$) $\Rightarrow \simeq$ *Config.* $\mathcal{P}.20$.

1.2: *The case* $\mathbf{D} = 0$. Then, due to the condition (3.84), we get $(4n-3)(21+4n) = 0$, and we observe that the condition $21+4n = 0$ is governed by the invariant polynomial ξ_9. Therefore, we arrive at the configuration *Config.* $\mathcal{P}.21$ if $\xi_9 \neq 0$ and *Config.* $\mathcal{P}.25$ if $\xi_9 = 0$.

2: *The possibility* $B_2 = 0$. Considering the condition (3.84), we get $n = -3/2$, which leads to the system:

$$\dot{x} = \frac{1}{2}(-3x+4x^2-3y+2xy), \quad \dot{y} = y(-3+x+2y),$$

possessing two invariant lines $y = x$ and $y = 0$, in addition to the invariant parabola. Therefore, in this case, we obtain an equivalent configuration to *Config.* $\mathcal{P}.52$.

The subcase $\xi_{19} = 0$. Then $m(2m-n)(2m+3n+9) = 0$, and we examine two possibilities: $\xi_{20} \neq 0$ and $\xi_{20} = 0$.

1: *The possibility* $\xi_{20} \neq 0$. In this case, considering (3.82), we obtain $m(2m-n) \neq 0$, and therefore $2m+3n+9 = 0$. This implies $m = -3(3+n)/2$, and we arrive at the one-parameter family of systems:

$$\dot{x} = \frac{1}{2}(2x-3)(3+n+2x+y), \quad \dot{y} = -3(3+n)x + 2ny + xy + 2y^2, \tag{3.85}$$

which is a subfamily of (3.14) defined by the condition $g = 2$. The family (3.14) was investigated earlier, and considering (3.15), (3.16) and (3.17) for $g = 2$ (i.e., for systems (3.85)), we have:

$$\begin{gathered} Z_2 = -4(2+n), \quad \alpha_2 = 21+4n, \quad \beta_2 = 33+4n, \\ B_2 = -5832(3+n)(9+4n)^2(17+4n)x^4, \quad \mathbf{D} = -972Z_2\alpha_2^2\beta_2^2, \\ B_3 = -9(9+4n)x^2(12x^2+4nx^2-24xy-8nxy-5y^2)/2, \\ \xi_7 = 1057164750000Z_2\alpha_2^2\beta_2, \quad \xi_8 = 55640250000Z_2\alpha_2\beta_2^2, \\ \xi_{18} = -7277560302993750(6+n)(9+4n)(57+4n)\alpha_2\beta_2. \end{gathered}$$

We observe that for the parameter n, the following possible bifurcation values arise: $n \in \{-33/4, -21/4, -2\}$. Considering (3.78) for systems (3.85), we obtain:

$$\chi_4\zeta_7\mathcal{R}_3 \neq 0 \;\Leftrightarrow\; (6+n)(9+4n)(17+4n)(57+4n)\beta_2 \neq 0, \tag{3.86}$$

and therefore $\beta_2 B_3 \neq 0$. Moreover, from the above condition we deduce that $\xi_{18} = 0$ if and only if $\alpha_2 = 0$.

Thus, in the case $B_2 \neq 0$, and following the investigation of the family (3.14) for $g = 2$, we get the following configurations (depending on the parameter n):

$\mathbf{D} < 0$, $\xi_7 < 0$ (i.e., $n < -33/4$)	$\Rightarrow$ *Config.* $\mathcal{P}.9$;
$\mathbf{D} < 0$, $\xi_7 > 0$, $\xi_8 < 0$ (i.e., $-33/4 < n < -21/4$)	$\Rightarrow$ *Config.* $\mathcal{P}.11$;
$\mathbf{D} < 0$, $\xi_7 > 0$, $\xi_8 > 0$ (i.e., $-21/4 < n < -2$)	$\Rightarrow$ *Config.* $\mathcal{P}.12$;
$\mathbf{D} > 0$ (i.e., $n > -2$)	$\Rightarrow$ *Config.* $\mathcal{P}.13$;
$\mathbf{D} = 0$, $\xi_{18} \neq 0$ (i.e., $n = -2$)	$\Rightarrow$ *Config.* $\mathcal{P}.14$;
$\mathbf{D} = 0$, $\xi_{18} = 0$ (i.e., $n = -21/4$)	$\Rightarrow$ *Config.* $\mathcal{P}.16$.

Assuming $B_2 = 0$, and considering the condition (3.86), we get $n = -3$ and arrive at the system:

$$\dot{x} = \frac{1}{2}(2x-3)(2x+y), \quad \dot{y} = y(-6+x+2y),$$

which possesses two invariant lines $x = 3/2$ and $y = 0$, in addition to the invariant parabola. Therefore, for this system, we obtain the configuration *Config.* $\mathcal{P}.47$.

2: *The possibility* $\xi_{20} = 0$. Then, from (3.82), we obtain $m(2m-n) = 0$, and we discuss two cases: $\xi_{21} \neq 0$ and $\xi_{21} = 0$.

2.1: *The case* $\xi_{21} \neq 0$. Then $m \neq 0$, and we obtain $m = n/2$. This leads to the following one-parameter family of systems:

$$\dot{x} = \frac{n}{2} + nx - \frac{3y}{2} + 2x^2 + xy, \quad \dot{y} = (n+y)(x+2y), \tag{3.87}$$

which is a subfamily of (3.19) defined by the condition $g = 2$. The family (3.19) was investigated earlier, and considering (3.21) for $g = 2$ (i.e., for systems (3.87)), we conclude that these systems possess four singularities $M_i(x_i, y_i)$ $(i = 1, 2, 3, 4)$ with coordinates:

$$x_1 = \sqrt{-n}, \quad y_1 = -n; \quad x_2 = -\sqrt{-n}, \quad y_2 = -n;$$
$$x_3 = -\frac{1}{2}, \quad y_3 = \frac{1}{4}; \quad x_4 = -\frac{2n}{3}, \quad y_4 = \frac{n}{3}.$$

For these systems, we calculate:

$$\theta = -32 \neq 0, \quad B_2 = -162(1+4n)(9+4n)^2y^4, \quad \chi_4 = 61875n(1+4n)(9+4n)(1+36n),$$

and therefore $B_2 \neq 0$ due to $\chi_4 \neq 0$. Following the examination of the configurations of systems (3.19) for $g = 2$, we obtain:

$$\alpha_3 = 4n - 3, \quad \beta_3 = 1 + 4n, \quad \mathbf{D} = 768n^3\alpha_3^2\beta_3^2, \quad \xi_9 = 3219732426240(9+4n)^2\beta_3,$$

and due to $\chi_4 \neq 0$, in the case $\mathbf{D} \neq 0$, we have:

$$\operatorname{sign}(\mathbf{D}) = \operatorname{sign}(n), \quad \operatorname{sign}(\xi_9) = \operatorname{sign}(\beta_3),$$

and the condition $\alpha_3 = 0$ is equivalent to $\mathbf{D} = 0$.

Thus, we arrive at the following configurations (depending on the parameter n):

$$\begin{aligned}
&\mathbf{D} < 0, \xi_9 < 0 \text{ (i.e., } n < -1/4) && \Rightarrow \textit{ Config. } \mathcal{P}.17;\\
&\mathbf{D} < 0, \xi_9 > 0 \text{ (i.e., } -1/4 < n < 0) && \Rightarrow \textit{ Config. } \mathcal{P}.19;\\
&\mathbf{D} > 0 \text{ (i.e., } n > 0, n \neq 3/4) && \Rightarrow \textit{ Config. } \mathcal{P}.20;\\
&\mathbf{D} = 0 \text{ (i.e., } n = 3/4) && \Rightarrow \textit{ Config. } \mathcal{P}.21.
\end{aligned}$$

2.2: *The case* $\xi_{21} = 0$. Then $m = 0$, and we obtain the one-parameter family of systems:

$$\dot{x} = nx - \frac{3y}{2} + 2x^2 + xy, \quad \dot{y} = y(2n + x + 2y), \tag{3.88}$$

which is a subfamily of (3.25) defined by the condition $g = 2$. The family (3.25) was investigated earlier, and considering (3.28) for $g = 2$ (i.e., for systems (3.88)), we have:

$$Z_3 = 1 - 16n, \quad B_2 = -1458(3+n)(1+2n)(3+2n)y^4, \quad \theta = -32 \neq 0,$$
$$B_3 = -9n(7+4n)x^2y^2/2 - 9(3+n)xy^3 + 9(3+n)y^4/2 \neq 0,$$
$$\mathbf{D} = -3888n^6Z_3, \quad \xi_{22} = 1050n, \quad \chi_4 = 556875n^2(1+2n)(3+16n).$$

So, due to the condition $\chi_4 \neq 0$, in the case $\mathbf{D} \neq 0$, we have:

$$\operatorname{sign}(\mathbf{D}) = -\operatorname{sign}(Z_3), \quad \operatorname{sign}(\xi_{22}) = \operatorname{sign}(n).$$

Thus, in the case $B_2 \neq 0$, and following the investigation of the family (3.25) for $g = 2$, we get the following configurations (depending on the parameter n):

$$\begin{aligned}
&\mathbf{D} < 0, \xi_{22} < 0 \text{ (i.e., } n < 0) && \Rightarrow \textit{ Config. } \mathcal{P}.32;\\
&\mathbf{D} < 0, \xi_{22} > 0 \text{ (i.e., } 0 < n < 1/16) && \Rightarrow \textit{ Config. } \mathcal{P}.31;\\
&\mathbf{D} > 0 \text{ (i.e., } n > 1/16) && \Rightarrow \textit{ Config. } \mathcal{P}.34;\\
&\mathbf{D} = 0 \text{ (i.e., } n = 1/16) && \Rightarrow \textit{ Config. } \mathcal{P}.37.
\end{aligned}$$

Assume now $B_2 = 0$. The condition $\chi_4 \neq 0$ implies $1 + 2n \neq 0$, and we get $(3+2n)(3+n) = 0$. Since for systems (3.88) we have $\xi_{23} = -225(3+n)/4$, we arrive at configuration *Config.* $\mathcal{P}.52$ in the case $\xi_{23} \neq 0$ (i.e., $n = -3/2$), and *Config.* $\mathcal{P}.47$ in the case $\xi_{23} = 0$ (i.e., $n = -3$).

3.2.2 The statements $(\mathcal{B}_2)$, $(\mathcal{B}_3)$, and $(\mathcal{B}_4)$

According to Proposition 2.8, all these three statements share the common condition $\zeta_7 = 0$. Considering (3.78) and the requirement $\chi_4 \neq 0$, we must impose the condition $\mathcal{U}_2 = 4m - 147 - 14n = 0$, leading to $m = 7(21 + 2n)/4$. This gives rise to the following one-parameter family:

$$\begin{aligned} \dot{x} &= (147 + 14n + 4nx + 8x^2 - 6y + 4xy)/4, \\ \dot{y} &= (147x + 14nx + 4ny + 2xy + 4y^2)/2, \end{aligned} \tag{3.89}$$

for which we calculate:

$$\begin{aligned} \chi_4 &= 61875(37 + 4n)(69 + 4n)(357 + 4n)(301 + 36n), \\ \zeta_8 &= 5(21 + 4n)^2/4, \quad \mathcal{R}_4 = 19500(33 + 4n). \end{aligned} \tag{3.90}$$

Thus, following Proposition 2.8, we distinguish three possibilities: $\zeta_8\mathcal{R}_4 \neq 0$ (statement $(\mathcal{B}_2)$), $\zeta_8 = 0$ (statement $(\mathcal{B}_3)$), and $\mathcal{R}_4 = 0$ (statement $(\mathcal{B}_4)$). We examine each possibility in turn.

The possibility $\zeta_8\mathcal{R}_4 \neq 0$. Then $33 + 4n \neq 0$, and systems (3.89) possess the following two invariant parabolas:

$$\Phi_1(x, y) = x^2 - y = 0, \quad \Phi_2(x, y) = -(21+2n)(21+4n) + 6(21+4n)x + 24x^2 - 2(33+4n)y = 0.$$

We observe that systems (3.89) form a subfamily of (3.37) defined by the condition $g = 2$. The family (3.37) was investigated earlier, and considering (3.40) and (3.43) for $g = 2$ (i.e., for systems (3.89)), we have:

$$\begin{gathered} Z_4 = -(33 + 4n), \quad \alpha_4 = 133 + 4n, \quad \beta_4 = 861 + 100n, \quad \theta = -32 \neq 0, \\ \mathbf{D} = -3(21 + 4n)^2 Z_4 \alpha_4^2 \beta_4^2/4, \quad \zeta_2 = 24 > 0, \quad \xi_{14} = 1235\,\alpha_4\beta_4/2, \\ B_1 = 105(21 + 2n)(33 + 4n)(37 + 4n)(49 + 4n)(69 + 4n)/2. \end{gathered}$$

We observe that due to $\zeta_8\mathcal{R}_4 \neq 0$, we have $Z_4(21 + 4n) \neq 0$, and, in the case $\mathbf{D} \neq 0$, we obtain:

$$\operatorname{sign}(\mathbf{D}) = -\operatorname{sign}(Z_4), \quad \operatorname{sign}(\xi_{14}) = \operatorname{sign}(\alpha_4\beta_4).$$

Moreover, the direction of the invariant parabola $\Phi_2(x, y) = 0$ depends on $\operatorname{sign}(33 + 4n)$.

According to Lemma 2.3, the existence of an invariant line in systems (3.89) requires $B_1 = 0$. Thus, we consider two cases: $B_1 \neq 0$ and $B_1 = 0$.

The case $B_1 \neq 0$. Then no invariant line exists. For the parameter n, we have three possible bifurcation values: $n \in \{-133/4, -861/100, -33/4\}$. However, due to $\mathcal{R}_4 \neq 0$, we must have $Z_4 \neq 0$, i.e., $n \neq -33/4$.

Considering these possible bifurcation values, in the case $B_1 \neq 0$, systems (3.89) lead to the following configurations (depending on the parameter n):

$$\begin{array}{ll} \mathbf{D} < 0,\ \xi_{14} < 0 \text{ (i.e., } -133/4 < n < -861/100) & \Rightarrow \textit{ Config. } \mathcal{P}.58; \\ \mathbf{D} < 0,\ \xi_{14} > 0,\ \beta_4 < 0 \text{ (i.e., } n < -133/4) & \Rightarrow \textit{ Config. } \mathcal{P}.59; \\ \mathbf{D} < 0,\ \xi_{14} > 0,\ \beta_4 > 0 \text{ (i.e., } -861/100 < n < -33/4) & \Rightarrow \textit{ Config. } \mathcal{P}.59; \\ \mathbf{D} > 0 \text{ (i.e., } n > -33/4) & \Rightarrow \textit{ Config. } \mathcal{P}.61; \\ \mathbf{D} = 0,\ \beta_4 \neq 0 \text{ (i.e., } n = -133/4) & \Rightarrow \textit{ Config. } \mathcal{P}.64; \\ \mathbf{D} = 0,\ \beta_4 = 0 \text{ (i.e., } n = -861/100) & \Rightarrow \textit{ Config. } \mathcal{P}.64. \end{array}$$

We may summarize the above as:

$$\begin{array}{ll} \mathbf{D} < 0, \xi_{14} < 0 & \Rightarrow \textit{Config. } \mathcal{P}.58; \\ \mathbf{D} < 0, \xi_{14} > 0 & \Rightarrow \textit{Config. } \mathcal{P}.59; \\ \mathbf{D} > 0 & \Rightarrow \textit{Config. } \mathcal{P}.61; \\ \mathbf{D} = 0 & \Rightarrow \textit{Config. } \mathcal{P}.64. \end{array}$$

The case $B_1 = 0$. Considering (3.90) and the condition $\chi_4\zeta_8\mathcal{R}_4 \neq 0$, the condition $B_1 = 0$ is equivalent to $49 + 4n = 0$, i.e., $n = -49/4$. The corresponding system (3.89) possesses the invariant line $y = 49/4$, leading to a configuration equivalent to *Config.* $\mathcal{P}.69$.

The possibility $\zeta_8 = 0$. From (3.90), this condition implies $n = -21/4$, and we arrive at the system:

$$\begin{aligned} \dot{x} &= (147 - 42x - 12y + 16x^2 + 8xy)/8, \\ \dot{y} &= (147x - 42y + 4xy + 8y^2)/4. \end{aligned} \tag{3.91}$$

For $n = -21/4$, we find:

$$\Phi_1(x, y) = x^2 - y = 0, \quad \Phi_2(x, y) = 24(x^2 - y) = 0,$$

i.e., the system has a double invariant parabola. Moreover, in this case, there is one real singular point $M_1(x_1, y_1)$ and two complex singularities $M_{2,3}(x_{2,3}, y_{2,3})$, with:

$$x_1 = -\frac{7}{2}, \quad y_1 = \frac{49}{4}; \quad x_{2,3} = \frac{3}{2} \pm i\sqrt{3}, \quad y_{2,3} = -\frac{3}{4} \pm 3i\sqrt{3}.$$

We point out that M_1 is a double singularity of system (3.91), located on the double invariant parabola $\Phi_1(x, y) = x^2 - y = 0$. Therefore, we arrive at configuration *Config.* $\mathcal{P}.79$.

The possibility $\mathcal{R}_4 = 0$. From (3.90), this condition implies $n = -33/4$, and we obtain the system:

$$\dot{x} = (2x - 3)(-21 + 8x + 4y)/8, \quad \dot{y} = (63x - 66y + 4xy + 8y^2)/4.$$

For $n = -33/4$, the second invariant parabola becomes the reducible conic $\Phi_2(x, y) = 6(2x - 3)^2 = 0$. Thus, the system possesses the invariant line $x = 3/2$ and the invariant parabola $\Phi_1(x, y) = x^2 - y = 0$, yielding a configuration equivalent to *Config.* $\mathcal{P}.83$.

3.2.3 The statement $(\mathcal{B}_5)$

According to Proposition 2.8, we must have the condition $\chi_4 = 0$ and $\zeta_5\zeta_9 \neq 0$. Thus, for systems (3.77) we compute:

$$\begin{aligned} \chi_4 = 61875(1 + 4m + 2n)\mathcal{V}, \quad \xi_{24} = 140625(4m - 14n - 147)\mathcal{V}/16, \\ \zeta_5 = 25(196m - 46n - 3)(4m - 14n - 147)/16, \\ \zeta_9 = -2970000(4m - 14n - 147)\mathcal{W}, \end{aligned} \tag{3.92}$$

where

$$\mathcal{V} = 18m + 1372m^2 - 84mn + 27n^2 + 144n^3, \quad \mathcal{W} = 10m + 196m^2 - 88mn + 15n^2.$$

We consider two cases: $\xi_{24} \neq 0$ and $\xi_{24} = 0$.

The case $\xi_{24} \neq 0$. Then $\mathcal{V} \neq 0$ and the condition $1+4m+2n=0$ implies $m=-(1+2n)/4$. This leads to the one-parameter family of systems

$$\dot{x} = -\frac{1}{4}(2n+1) + nx - \frac{3y}{2} + 2x^2 + xy, \quad \dot{y} = -\frac{1}{2}(2n+1)x + 2ny + xy + 2y^2, \tag{3.93}$$

which possess the invariant line $y = x - \frac{1}{4}$ and two invariant parabolas:

$$\Phi_1(x,y) = x^2 - y = 0, \quad \Phi_2(x,y) = -1 - 2n + 2(1+4n)x - 2(-1+4n)y - 8y^2 = 0.$$

For these systems, we have

$$\begin{gathered}\zeta_5 = 25(37+4n)(13+36n), \quad \xi_{24} = -140625(5+4n)^2(37+4n)(13+36n)/16,\\ \zeta_9 = 11\cdot 30^4(1+4n)(37+4n)(13+36n), \quad B_1 = 0, \quad \theta = -32 \neq 0,\\ B_2 = -81(1+2n)(1+4n)(17+4n)(x-y)^4, \quad \mathbf{D} = 3(1+4n)(5+4n)^6/4.\end{gathered} \tag{3.94}$$

Following Lemma 2.3, we consider two subcases: $B_2 \neq 0$ and $B_2 = 0$.

1: *The subcase $B_2 \neq 0$.* Then, by Lemmas 2.3 and 2.4 (since $\theta \neq 0$), we conclude that systems (3.93) can have only one invariant line (namely, $y = x - 1/4$).

Systems (3.93) have four finite singularities $M_i(x_i, y_i)$, $i = 1, \ldots, 4$, with coordinates

$$\begin{gathered}x_1 = \frac{1}{2}, \quad y_1 = \frac{1}{4}; \quad x_2 = \frac{1-4n}{12}, \quad y_2 = -\frac{1+2n}{6};\\ x_{3,4} = -\frac{1}{2}\left(1 \pm \sqrt{-1-4n}\right), \quad y_{3,4} = \frac{1}{2}\left(-2n \pm \sqrt{-1-4n}\right).\end{gathered}$$

Note that the invariant parabolas intersect at three points: M_1, M_3, and M_4. The singularities M_3 and M_4 may be real or complex depending on $1+4n \neq 0$ (due to $\zeta_9 \neq 0$). Moreover, the direction of the parabola $\Phi_2(x,y) = 0$ also depends on $1+4n$.

It is easy to verify that the invariant line $y = x - 1/4$ is tangent to both invariant parabolas at M_1. The singularity M_2 lies on this invariant line, and

$$\Phi_1(x_2, y_2) = \frac{1}{144}(5+4n)^2 \neq 0, \quad \Phi_2(x_2, y_2) = -\frac{1}{18}(5+4n)^2 \neq 0,$$

due to $\xi_{24} \neq 0$. Considering (3.94), $\mathbf{D} \neq 0$ since $\zeta_9\xi_{24} \neq 0$ and $\operatorname{sign}(\mathbf{D}) = \operatorname{sign}(5+4n)$.

To understand the position of M_2 relative to M_1, we compute

$$x_2 - x_1 = \frac{1-4n}{12} - \frac{1}{2} = -\frac{5+4n}{12} \quad \Rightarrow \quad \operatorname{sign}(x_2 - x_1) = -\operatorname{sign}(5+4n).$$

Hence, all finite singularities except M_2 are fixed as intersection points of the invariant curves, and their positions depend on n. In the case $B_2 \neq 0$, the possible bifurcation values for n are $-5/4$ and $-1/4$.

Thus, for $B_2 \neq 0$ in systems (3.93) we have the following configurations:

$$\begin{aligned}&\mathbf{D} < 0 \text{ and } n < -5/4 && \Rightarrow \textit{Config. } \mathcal{P}.99;\\ &\mathbf{D} < 0 \text{ and } n > -5/4 && \Rightarrow \simeq \textit{Config. } \mathcal{P}.99;\\ &\mathbf{D} > 0 \text{ (i.e., } n > -1/4) && \Rightarrow \textit{Config. } \mathcal{P}.100.\end{aligned}$$

So, *Config.* $\mathcal{P}.99$ occurs if $\mathbf{D} < 0$, and *Config.* $\mathcal{P}.100$ if $\mathbf{D} > 0$.

2: *The subcase $B_2 = 0$.* From $\zeta_9 \neq 0$, we get $(1+2n)(17+4n) = 0$.

If $1+2n=0$, then $n=-1/2$, yielding the system

$$\dot{x}=(-x+4x^2-3y+2xy)/2,\quad \dot{y}=y(-1+x+2y),$$

which has four invariant curves (two parabolas and two lines):

$$\Phi_1(x,y)=x^2-y=0,\quad \Phi_2(x,y)=x-3y+4y^2=0,\quad y=x-1/4,\quad y=0.$$

This corresponds to configuration *Config. $\mathcal{P}$.101*.

If $n=-17/4$, we get

$$\dot{x}=(-3+2x)(-5+8x+4y)/8,\quad \dot{y}=(15x-34y+4xy+8y^2)/4,$$

which has invariant lines $y=x-1/4$ and $x=3/2$, and invariant parabolas

$$\Phi_1(x,y)=x^2-y=0,\quad \Phi_2(x,y)=15-64x+72y-16y^2=0.$$

The line $x=3/2$ is tangent to $\Phi_2(x,y)=0$ at $M_4(3/2,9/4)$. Thus, in this case, we get a configuration equivalent to *Config. $\mathcal{P}$.101*.

The case $\zeta_{24}=0$. This implies $\mathcal{V}=0$ and we calculate

$$\text{Discrim}\,[\mathcal{V},m]=36(1-28n)(3+28n)^2\equiv\gamma(n).$$

Since $m,n\in\mathbb{R}$, the condition $\gamma(n)\geq 0$ is necessary.

We claim that $\zeta_5\neq 0$ implies $3+28n\neq 0$. Indeed, setting $n=-3/28$ we get

$$\mathcal{V}=\frac{(2744m+27)^2}{5488}=0\quad\Rightarrow\quad m=-\frac{27}{2744},$$

which implies $\zeta_5=0$. Thus, the claim is proved.

Therefore, the condition $1-28n\geq 0$ is necessary for $\mathcal{V}$ to have real roots. Setting a new parameter u as $1-28n=u^2\geq 0$, we have $n=(1-u^2)/28$ and

$$\mathcal{V}=\frac{1}{5488}(15+2744m+24u+3u^2-6u^3)(15+2744m-24u+3u^2+6u^3)=0.$$

By symmetry ($u\mapsto -u$), we may assume the second factor vanishes, yielding

$$m=-3(u-1)^2(5+2u)/2744.$$

This leads to the one-parameter family of systems

$$\begin{aligned}\dot{x}&=-\frac{3(u-1)^2(5+2u)}{2744}-\frac{u^2-1}{28}x-\frac{3}{2}y+2x^2+xy,\\ \dot{y}&=-\frac{3(u-1)^2(5+2u)}{1372}x-\frac{u^2-1}{14}y+xy+2y^2,\end{aligned}\tag{3.95}$$

which possess two invariant parabolas:

$$\Phi_1(x,y)=x^2-y=0,\quad \Phi_2(x,y)=3(u-1)^4+112(u-1)^3x+1176(u-1)^2y-38416y^2=0.$$

For these systems,

$$\begin{gathered}\zeta_5 = 25(u-50)(u-22)(u^2-4)(3u-10)(3u+46)/38416,\\ \zeta_9 = 2227500(u-50)(u-22)(u^2-4)(u-1)^3(3u-10)(3u+46)/823543,\\ \zeta_{10} = -15(354160+48u-1336u^2+64u^3+27u^4-19u^5+3u^6)/38416,\\ \mathbf{D} = -3^5 7^{-18}(u-1)^6(6+u)^8(34+u)^2(2+5u)^2.\end{gathered} \tag{3.96}$$

Also, for (3.95) we have

$$B_1 = -3^3 2^{-5} 7^{-15}(u-22)(u-8)^3(u-1)^3(6+u)^3(13+u)(20+u)(5+2u)(3u-10)(4+3u). \tag{3.97}$$

Following Lemma 2.3, we consider two subcases: $B_1 \neq 0$ and $B_1 = 0$.

1: *The subcase* $B_1 \neq 0$. Then, by Lemma 2.3, systems (3.95) cannot possess any invariant line.

The systems (3.95) have four finite singularities $M_i(x_i, y_i)$, $i = 1,2,3,4$, with coordinates

$$\begin{gathered}x_1 = -\frac{5+2u}{14}, \quad y_1 = \frac{(5+2u)^2}{196}; \quad x_2 = \frac{(u-1)(22+26u+u^2)}{1372}, \quad y_2 = -\frac{(u-1)^2(5+2u)}{1372};\\ x_3 = \frac{1-u}{14}, \quad y_3 = \frac{(1-u)^2}{196}; \quad x_4 = \frac{3(u-1)}{14}, \quad y_4 = \frac{9(u-1)^2}{196}.\end{gathered} \tag{3.98}$$

The invariant parabolas intersect at points M_3 and M_4. The singularities M_1 and M_2 lie on $\Phi_1 = 0$ and $\Phi_2 = 0$, respectively. The direction of $\Phi_2 = 0$ depends on $u-1$, and since $\zeta_9 \neq 0$, we have $u \neq 1$.

Because $\mathbf{D} = 0$ indicates a multiple singularity, we consider the possibilities $\mathbf{D} \neq 0$ and $\mathbf{D} = 0$.

1.1: *The possibility* $\mathbf{D} \neq 0$. Then all singularities are distinct. To determine the relative positions of M_1 and M_2 with respect to M_3 and M_4 (intersection points of the parabolas), we compute:

$$\begin{aligned}x_3 - x_1 &= \frac{6+u}{14}, & x_3 - x_2 &= -\frac{(u-1)(6+u)(20+u)}{1372},\\ x_4 - x_1 &= \frac{2+5u}{14}, & x_4 - x_2 &= -\frac{(u-8)(u-1)(34+u)}{1372}.\end{aligned}$$

Moreover, the singular point M_2, which lies on the parabola $\Phi_2(x,y) = 0$, can be located either above or below its axis $y = y_v$, where y_v is the ordinate of the vertex of this parabola. For $\Phi_2(x,y) = 0$, we have $y_v = 3(u-1)^2/196$ and then we calculate

$$y_2 - y_v = -(u-1)^2(13+u)/686 \quad \Rightarrow \quad \operatorname{sign}(y_2 - y_v) = -\operatorname{sign}(13+u).$$

Additionally,

$$\Phi_1(x_2, y_2) = \frac{(u-1)^2(6+u)^3(34+u)}{1882384}, \quad \Phi_2(x_1, y_1) = -(6+u)^3(2+5u).$$

Possible bifurcation values for u are $\{-34, -20, -13, -6, -\frac{2}{5}, 1, 8\}$. Since $\zeta_5\zeta_9 B_1 \mathbf{D} \neq 0$, we have:

$$(u-8)(u-1)(u+20)(u+13)(u+6)(u+34)(2+5u) \neq 0.$$

Thus, in the case $B_1 \neq 0$ and $\mathbf{D} \neq 0$, systems (3.95) yield the following configurations:

$$\begin{array}{lll}
u<-34 & \Rightarrow x_3>x_4,\ x_1>x_3,\ x_2<x_4,\ y_2>y_v & \Rightarrow \textit{Config. } \mathcal{P}.102;\\
-34<u<-20 & \Rightarrow x_3>x_4,\ x_1>x_3,\ x_4<x_2<x_3,\ y_2>y_v & \Rightarrow \textit{Config. } \mathcal{P}.103;\\
-20<u<-13 & \Rightarrow x_3>x_4,\ x_1>x_3,\ x_4<x_3<x_2,\ y_2>y_v & \Rightarrow \simeq \textit{Config. } \mathcal{P}.103;\\
-13<u<-6 & \Rightarrow x_3>x_4,\ x_1>x_3,\ x_4<x_3<x_2,\ y_2<y_v & \Rightarrow \simeq \textit{Config. } \mathcal{P}.103;\\
-6<u<-2/5 & \Rightarrow x_3>x_4,\ x_4<x_1<x_3,\ x_4<x_2<x_3,\ y_2<y_v & \Rightarrow \simeq \textit{Config. } \mathcal{P}.103;\\
-2/5<u<1 & \Rightarrow x_3>x_4,\ x_1<x_3,\ x_4<x_2<x_3,\ y_2<y_v & \Rightarrow \simeq \textit{Config. } \mathcal{P}.102;\\
1<u<8 & \Rightarrow x_3>x_4,\ x_1<x_3,\ x_4<x_2<x_3,\ y_2<y_v & \Rightarrow \textit{Config. } \mathcal{P}.104;\\
u>8 & \Rightarrow x_3>x_4,\ x_1<x_3,\ x_4<x_2<x_3,\ y_2<y_v & \Rightarrow \simeq \textit{Config. } \mathcal{P}.104.
\end{array}$$

We can summarize these as

$$\begin{array}{lll}
\textit{Config. } \mathcal{P}.103 & \Leftrightarrow & (u+34)(5u+2)<0;\\
\textit{Config. } \mathcal{P}.102 & \Leftrightarrow & (u+34)(5u+2)>0 \text{ and } u-1<0;\\
\textit{Config. } \mathcal{P}.104 & \Leftrightarrow & (u+34)(5u+2)>0 \text{ and } u-1>0.
\end{array}$$

On the other hand, for systems (3.95) we have

$$\xi_{25}=2^{-3}7^{-9}5913(u-1)^3(6+u)^4(34+u)(2+5u),$$
$$\zeta_5\zeta_9=2^{-2}7^{-11}3^45^611(u-1)^3(-50+u)^2(-22+u)^2(-2+u)^2(2+u)^2(-10+3u)^2(46+3u)^2,$$

and due to $\mathbf{D}\zeta_5\zeta_9\neq 0$ we have

$$\xi_{25}\neq 0,\quad \operatorname{sign}(\xi_{25})=\operatorname{sign}\big((u-1)(34+u)(2+5u)\big),\quad \operatorname{sign}(\zeta_5\zeta_9)=\operatorname{sign}(u-1).$$

This leads to the following invariant conditions:

$$\begin{array}{lll}
\xi_{25}<0 & \Leftrightarrow & \textit{Config. } \mathcal{P}.102;\\
\xi_{25}>0,\ \zeta_5\zeta_9<0 & \Leftrightarrow & \textit{Config. } \mathcal{P}.103;\\
\xi_{25}>0,\ \zeta_5\zeta_9>0 & \Leftrightarrow & \textit{Config. } \mathcal{P}.104.
\end{array}$$

1.2: *The possibility* $\mathbf{D}=0$. Considering the condition $B_1\neq 0$, this implies $(34+u)(2+5u)=0$. Taking into consideration the position of the invariant parabolas and the coordinates (3.98) of the singularities of systems (3.95), we obtain:

$$\begin{array}{lll}
u=-34 & \Rightarrow x_3>x_4,\ x_1>x_3,\ x_2=x_4,\ y_2>y_v & \Rightarrow \textit{Config. } \mathcal{P}.105;\\
u=-2/5 & \Rightarrow x_3>x_4,\ x_1=x_4,\ x_4<x_2<x_3,\ y_2<y_v & \Rightarrow \simeq \textit{Config. } \mathcal{P}.105.
\end{array}$$

So, we deduce that in the case $B_1\neq 0$ and $\mathbf{D}=0$ we get the unique configuration *Config. $\mathcal{P}$.105*.

2: *The subcase* $B_1=0$. Considering (3.97) and (3.96), we conclude that due to $\zeta_9\neq 0$ the condition $B_1=0$ is equivalent to

$$(u-8)(6+u)(13+u)(20+u)(5+2u)(4+3u)=0.$$

However, we could decrease the number of factors.

Remark 3.14. We remark that in the case $u-1\neq 0$ (i.e., when the second parabola exists), applying the transformation

$$x_1=\frac{343}{(u-1)^3}y-\frac{21}{4(u-1)},\quad y_1=\frac{343}{(u-1)^3}x+\frac{147}{4(u-1)^2},\quad t_1=\frac{(u-1)^3}{343}t,$$

we arrive at a family of systems of the same form (3.95):

$$\dot{x}_1=-\frac{3(u_1-1)^2(5+2u_1)}{2744}-\frac{u_1^2-1}{28}x_1-\frac{3y_1}{2}+2x_1^2+x_1y_1,$$
$$\dot{y}_1=-\frac{3(u_1-1)^2(5+2u_1)}{1372}x_1-\frac{u_1^2-1}{4}y_1+x_1y_1+2y_1^2,$$

with the new parameter $u_1=(48+u)/(u-1)$ (then $u=(48+u_1)/(u_1-1)$).

Considering Remark 3.14 and the relation $u = (48 + u_1)/(u_1 - 1)$, we calculate

$$u + 13 = \frac{7(5 + 2u_1)}{u_1 - 1}, \qquad u + 20 = \frac{7(3u_1 + 4)}{u_1 - 1}.$$

So, to determine the configurations given by the condition $B_1 = 0$, it is sufficient to consider the conditions provided by the equality

$$(u - 8)(6 + u)(5 + 2u)(4 + 3u) = 0.$$

2.1: *The possibility* $u = -4/3$. This leads to the system

$$\dot{x} = (-1 - 2x + 144x^2 - 108y + 72xy)/72, \quad \dot{y} = (x + 2y)(-1 + 36y)/36,$$

possessing the invariant line $y = 1/36$ and two invariant parabolas:

$$\Phi_1(x, y) = x^2 - y = 0, \quad \Phi_2(x, y) = 1 - 16x + 72y - 432y^2.$$

We determine that the configuration of the above system corresponds to *Config.* $\mathcal{P}$*.106*.

2.2: *The possibility* $u = 8$. This leads to the system

$$\dot{x} = (2x - 3)(3 + 8x + 4y)/8, \quad \dot{y} = (x + 2y)(4y - 9)/4,$$

possessing three invariant lines $y = 9/4$, $y = x + 3/4$ and $x = 3/2$ and two invariant parabolas:

$$\Phi_1(x, y) = x^2 - y = 0, \quad \Phi_2(x, y) = 3 + 16x + 24y - 16y^2.$$

We observe that all five invariant curves intersect at the singular point $M_4(3/2, 9/4)$. So we get the configuration *Config.* $\mathcal{P}$*.107*.

2.3: *The possibility* $u = -5/2$. In this case we arrive at the system

$$\dot{x} = (-3x + 32x^2 - 24y + 16xy)/16, \quad \dot{y} = y(-3 + 8x + 16y)/4,$$

possessing the invariant line $y = 0$ and two invariant parabolas:

$$\Phi_1(x, y) = x^2 - y = 0, \quad \Phi_2(x, y) = 3 - 32x + 96y - 256y^2.$$

We observe that the invariant line $y = 0$ is tangent to the parabola $\Phi_1(x, y) = 0$ at the point $M_1(0, 0)$ and intersects the second parabola at $M_2(3/32, 0)$. In this case, we have the configuration *Config.* $\mathcal{P}$*.108*.

2.4: *The possibility* $u = -6$. In this case we get the system

$$\dot{x} = (3 - 10x + 16x^2 - 12y + 8xy)/8, \quad \dot{y} = (3x - 10y + 4xy + 8y^2)/4,$$

possessing the invariant line $y = x - 1/4$ and the invariant parabolas

$$\Phi_1(x, y) = x^2 - y = 0, \quad \Phi_2(x, y) = 3 - 16x + 24y - 16y^2 = 0.$$

Considering the coordinates (3.98) of the singularities of systems (3.95), we observe that for $u = -6$ the singular points M_2, M_3 and M_1 coalesce producing a triple singular point. Moreover, this triple singularity is a point of tangency of the invariant line $y = x - 1/4$ with both parabolas. As a result, we get the configuration *Config.* $\mathcal{P}$*.109*.

On the other hand, for systems (3.95) we have

$$\zeta_9 = \frac{2227500}{823543}(u-50)(u-22)(u^2-4)(u-1)^3(3u-10)(3u+46),$$
$$\zeta_6 = -\frac{3}{1372}(6+u)(292-52u+5u^2), \quad \text{Discrim}\,[292-52u+5u^2] = -3136 < 0,$$
$$\xi_{26} = \frac{3}{941192}(u-50)(u^2-4)(u-1)(3u+46)(6+u)(13+u)(5+2u),$$
$$\xi_{27} = \frac{1}{2744}(u-8)(380+52u+9u^2), \quad \text{Discrim}\,[380+52u+9u^2] = -10976 < 0.$$

We observe that, due to $\zeta_9 \neq 0$, the condition $\xi_{26} \neq 0$ is equivalent to $(6+u)(13+u)(5+2u) \neq 0$. Moreover, considering Remark 3.14, we conclude that for $\xi_{26} = 0$ we may assume $(6+u)(5+2u) = 0$ because the condition $13+u = 0$ could be brought to $5+2u = 0$ via an affine transformation and time rescaling.

Thus, in the case $B_1 = 0$, systems (3.95) possess the following configurations if and only if the corresponding conditions are satisfied:

$$\begin{array}{lcl}
\xi_{26} \neq 0,\ \xi_{27} \neq 0\ (\text{then } u = -4/3) & \Leftrightarrow & \textit{Config. } \mathcal{P}.106;\\
\xi_{26} \neq 0,\ \xi_{27} = 0\ (\text{then } u = 8) & \Leftrightarrow & \textit{Config. } \mathcal{P}.107;\\
\xi_{26} = 0,\ \zeta_6 \neq 0\ (\text{then } u = -5/2) & \Leftrightarrow & \textit{Config. } \mathcal{P}.108;\\
\xi_{26} = 0,\ \zeta_6 = 0\ (\text{then } u = -6) & \Leftrightarrow & \textit{Config. } \mathcal{P}.109.
\end{array}$$

3.2.4 The statement $(\mathcal{B}_6)$

In this case the condition $\chi_4 = \zeta_9 = 0$ must be fulfilled. Considering (3.92), due to the condition $\zeta_5 \neq 0$, we obtain that $\zeta_9 = 0$ is equivalent to $\mathcal{W} = 0$. Straightforward calculations give us that the systems of equations $\chi_4 = 0$ and $\mathcal{W} = 0$ could have only the following solutions $\mathcal{S}_i = (m_i, n_i)$ $(i = 1,2,3,4)$:

$$\mathcal{S}_1 = (0,0), \quad \mathcal{S}_2 = \left(-\frac{1}{8}, -\frac{1}{4}\right), \quad \mathcal{S}_3 = \left(-\frac{5}{72}, -\frac{13}{36}\right), \quad \mathcal{S}_4 = \left(-\frac{27}{2944}, -\frac{3}{8}\right).$$

However, we have

$$\chi_4(\mathcal{S}_i) = \zeta_9(\mathcal{S}_i) = 0, \quad i = 1,2,3,4,$$
$$\zeta_5(\mathcal{S}_1) \neq 0, \quad \zeta_5(\mathcal{S}_2) \neq 0, \quad \zeta_5(\mathcal{S}_3) = \zeta_5(\mathcal{S}_4) = 0,$$

and hence only the solutions $\mathcal{S}_1$ and $\mathcal{S}_2$ satisfy the conditions of statement $(\mathcal{B}_6)$. Therefore, we examine only these two solutions.

We observe that each one of them gives us a concrete system (without parameters), and it remains to construct the corresponding system having a single fixed configuration of the invariant parabolas and lines.

For systems (3.77), we calculate:

$$\begin{aligned}
\xi_9 = 3^6 17252510\big[&65536m^4 - 32m^3(6131+3252n) - 16m^2(-32110-7953n+484n^2)\\
&+ 6m(10221-53292n+5540n^2+4336n^3)\\
&- 9(-2304-7857n-12140n^2+836n^3+240n^4)\big].
\end{aligned}$$

We obtain that $\xi_9(\mathcal{S}_2) = 0$ and $\xi_9(\mathcal{S}_1) \neq 0$, and then we examine two cases: $\xi_9 \neq 0$ and $\xi_9 = 0$.

The case $\xi_9 \neq 0$. Then we consider the solution $\mathcal{S}_1$, i.e., $m = n = 0$. In this case, we arrive at the system

$$\dot{x} = -\frac{3y}{2} + 2x^2 + xy, \quad \dot{y} = y(x + 2y), \tag{3.99}$$

possessing the invariant line $y = 0$ and the invariant parabola $\Phi(x, y) = x^2 - y = 0$.

This system possesses the following two singular points: $M_1(0, 0)$ and $M_2(-1/2, 1/4)$. We observe that the point M_1 is the point of tangency of the invariant line with the parabola. Moreover, this point is a triple singularity of system (3.99), because we have

$$\mu_4 = \mu_3 = \mu_2 = 0, \quad \mu_1 = -3(x + 2y) \neq 0,$$

and, by [1, Lemma 5.2, statement (ii)], the point M_1 is of multiplicity exactly 3. As a result, we get the configuration *Config.* $\mathcal{P}.40$.

The case $\xi_9 = 0$. In this case we get the solution $\mathcal{S}_2$, i.e., $m = -1/8$ and $n = -1/4$. Then, we arrive at the system

$$\dot{x} = -\frac{1}{8} - \frac{x}{4} - \frac{3y}{2} + 2x^2 + xy, \quad \dot{y} = \frac{1}{4}(4y - 1)(x + 2y),$$

possessing the invariant lines $y = 1/4$ and $y = x - 1/4$ and the invariant parabola $\Phi(x, y) = x^2 - y = 0$.

As a result, we get the configuration *Config.* $\mathcal{P}.110$.

3.2.5 The statement $(\mathcal{B}_7)$

In this case, the condition $\chi_4 = \zeta_5 = 0$ and $\zeta_6 \neq 0$ must be fulfilled. Straightforward calculations give us that the systems of equations $\chi_4 = 0$ and $\zeta_5 = 0$ could have only the following solutions $\widetilde{\mathcal{S}}_i = (m_i, n_i)$ $(i = 1, \ldots, 6)$:

$$\widetilde{\mathcal{S}}_1 = \left(-\frac{27}{2744}, -\frac{3}{28}\right), \quad \widetilde{\mathcal{S}}_2 = \left(-\frac{2205}{8}, -\frac{357}{4}\right), \quad \widetilde{\mathcal{S}}_3 = \left(\frac{539}{72}, -\frac{301}{36}\right),$$
$$\widetilde{\mathcal{S}}_4 = \left(\frac{35}{8}, -\frac{37}{4}\right), \quad \widetilde{\mathcal{S}}_5 = \left(-\frac{5}{72}, -\frac{13}{36}\right), \quad \widetilde{\mathcal{S}}_6 = \left(-\frac{189}{8}, -\frac{69}{4}\right).$$

We split these solutions into two sets:

$$\mathcal{G}_1 = \{\widetilde{\mathcal{S}}_1, \widetilde{\mathcal{S}}_2, \widetilde{\mathcal{S}}_3\}, \quad \mathcal{G}_2 = \{\widetilde{\mathcal{S}}_4, \widetilde{\mathcal{S}}_5, \widetilde{\mathcal{S}}_6\}.$$

Lemma 3.15. *Assume that the conditions of statement* $(\mathcal{B}_7)$ *are satisfied and then the system of equations* $\chi_4 = \zeta_5 = 0$ *generates six solutions* $\widetilde{\mathcal{S}}_i = (m_i, n_i)$ $(i = 1, \ldots, 6)$ *given above. In this case, the invariant polynomial* ξ_6 *distinguishes the set* $\mathcal{G}_1$ *from the set* $\mathcal{G}_2$.

Proof. To prove this lemma it is sufficient to evaluate ξ_6 for the elements of each one of the sets. For systems (3.77), we calculate:

$$\xi_6 = 2^9 3^4 2877985 m(1 + 4m + 2n)(-147 + 50m + 61n + 8n^2),$$

and we obtain

$$\xi_6(\widetilde{\mathcal{S}}_i) \neq 0, \quad i = 1, 2, 3, \qquad \xi_6(\widetilde{\mathcal{S}}_j) = 0, \quad j = 4, 5, 6,$$

and we complete the proof of the lemma. □

According to the above lemma, we discuss two cases: $\xi_6 \neq 0$ and $\xi_6 = 0$.

The case $\xi_6 \neq 0$. Then we have to examine the elements of the first set $\mathcal{G}_1$.

1: *The subcase* $\widetilde{\mathcal{S}}_1$. Then we have $m = -27/2744$ and $n = -3/28$, and we get the system

$$\dot{x} = 2x^2 + xy - \frac{3x}{28} - \frac{3y}{2} - \frac{27}{2744}, \quad \dot{y} = xy - \frac{27x}{1372} + 2y^2 - \frac{3y}{14}, \tag{3.100}$$

possessing the following three invariant parabolas: $\Phi_1(x,y) = x^2 - y = 0$ and

$$\Phi_2(x,y) = 3 + 112x + 1176y - 38416y^2 = 0, \quad \Phi_3(x,y) = -243 + 3024x - 10584y + 38416y^2 = 0.$$

We observe that the singular point $M_1(3/14, 9/196)$ is the point of intersection of all three invariant parabolas. So we get the configuration *Config.* $\mathcal{P}$*.111*.

Next, we prove that the systems generated by $\widetilde{\mathcal{S}}_2$ and $\widetilde{\mathcal{S}}_3$ could be brought to system (3.100) via an affine transformation and a time rescaling.

Consider first the solution $\widetilde{\mathcal{S}}_2$, i.e., $m = -2205/8$ and $n = -357/4$. This leads to the system

$$\dot{x} = 2x^2 + xy - \frac{357x}{4} - \frac{3y}{2} - \frac{2205}{8}, \quad \dot{y} = xy - \frac{2205x}{4} + 2y^2 - \frac{357y}{2},$$

which via the transformation

$$x_1 = -\frac{3}{28} + \frac{y}{343}, \quad y_1 = \frac{3}{196} + \frac{x}{343}, \quad t_1 = 343t,$$

is brought to the system (3.100).

Analogously, taking the solution $\widetilde{\mathcal{S}}_3$, i.e., $m = 539/72$ and $n = -301/36$, we arrive at the system

$$\dot{x} = 2x^2 + xy - \frac{301x}{36} - \frac{3y}{2} + \frac{539}{72}, \quad \dot{y} = xy + \frac{539x}{36} + 2y^2 - \frac{301y}{18},$$

which via the transformation

$$x_1 = \frac{9}{28} - \frac{27y}{343}, \quad y_1 = \frac{27}{196} - \frac{27x}{343}, \quad t_1 = -\frac{343t}{27},$$

is brought to the system (3.100).

The case $\xi_6 = 0$. Then we have to examine the elements of the second set $\mathcal{G}_2$.

1: *The subcase* $\widetilde{\mathcal{S}}_4$. Then we have $m = 35/8$ and $n = -37/4$, and we get the system

$$\dot{x} = 2x^2 + xy - \frac{37x}{4} - \frac{3y}{2} + \frac{35}{8}, \quad \dot{y} = xy + \frac{35x}{4} + 2y^2 - \frac{37y}{2}, \tag{3.101}$$

possessing the invariant line $y = x - 1/4$ and three invariant parabolas: $\Phi_1(x,y) = x^2 - y = 0$ and

$$\Phi_2(x,y) = 5 - 12x + 3x^2 + y = 0, \quad \Phi_3(x,y) = -35 + 144x - 152y + 16y^2 = 0.$$

We observe that the singular point $M_1(1/2, 1/4)$ is the point of intersection of all four invariant curves. So we get the configuration *Config.* $\mathcal{P}$*.112*.

Next, we prove that the systems generated by $\widetilde{\mathcal{S}}_5$ and $\widetilde{\mathcal{S}}_6$ could be brought to system (3.101) via an affine transformation and a time rescaling.

Consider first the solution $\widetilde{\mathcal{S}}_5$, i.e., $m = -5/72$ and $n = -13/36$. This leads to the system

$$\dot{x} = 2x^2 + xy - \frac{13x}{36} - \frac{3y}{2} - \frac{5}{72}, \quad \dot{y} = -\frac{5x}{36} - \frac{13y}{18} + xy + 2y^2,$$

which via the transformation

$$x_1 = \frac{11}{4} - 9y, \quad y_1 = \frac{19}{4} - 9x, \quad t_1 = -\frac{t}{9},$$

is brought to the system (3.101).

Analogously, taking the solution $\widetilde{\mathcal{S}}_6$, i.e., $m = -189/8$ and $n = -69/4$, we arrive at the system

$$\dot{x} = 2x^2 + xy - \frac{69x}{4} - \frac{3y}{2} - \frac{189}{8}, \quad \dot{y} = xy - \frac{189x}{4} + 2y^2 - \frac{69y}{2},$$

which via the transformation

$$x_1 = 2 - \frac{x}{3}, \quad y_1 = 7 - \frac{y}{3}, \quad t_1 = -3t,$$

is brought to the system (3.101).

3.3 Configurations of systems in $\mathbf{QSP}_{(\eta<0)}$

In what follows, we examine the configurations of the systems in $\mathbf{QSP}_{(\eta<0)}$ for each of the cases provided by Proposition 2.9. According to this proposition, we consider the canonical form (2.5), i.e., the systems

$$\dot{x} = m + (2n-1)x/2 + gx^2 - gy/2 - xy, \quad \dot{y} = 2mx - x^2 + 2ny + gxy - 2y^2, \tag{3.102}$$

with $C_2 = x(x^2+y^2)$, possessing the invariant parabola $\Phi(x,y) = x^2 - y = 0$.

3.3.1 The statement $(\mathcal{E}_1)$

For systems (3.102), we calculate

$$\begin{aligned} \zeta_4 &= (25+g^2)(3g+9g^3-4m-6gn)/16, \\ \mathcal{R}_1 &= 15(1+g^2)(25+g^2)(3g+9g^3-4m-6gn)/2. \end{aligned} \tag{3.103}$$

The case $B_1 \neq 0$**.** Then, according to Lemma 2.3, systems (3.102) cannot possess any invariant line.

Let us examine the finite singularities of these systems. Following [1, Proposition 5.1], we calculate the invariant polynomial $\mathbf{D} = 12F_1'^2F_2'$, where

$$\begin{aligned} F_1' &= -2gm - 2g^3m + 4m^2 - n - g^2n - 4gmn + g^2n^2, \\ F_2' &= 8 - g^2 - 72gm + 8g^3m + 432m^2 - 48n + 4g^2n + 144gmn + 96n^2 - 4g^2n^2 - 64n^3, \end{aligned} \tag{3.104}$$

and we discuss two subcases: $\mathbf{D} \neq 0$ and $\mathbf{D} = 0$.

The subcase $\mathbf{D} \neq 0$. The systems (3.102) possess four finite singularities $M_i(x_i, y_i)$, $(i = 1, 2, 3, 4)$, with coordinates

$$x_1 = -\frac{gn-2m}{g^2+1},\ y_1 = \frac{2gm+n}{g^2+1};\quad x_2 = \frac{1}{6\mathcal{Y}^{1/3}}\left[\mathcal{Y}^{2/3} + \mathcal{Y}^{1/3}g - 3g^2 + 2\mathcal{Z}\right],$$

$$y_2 = \frac{1}{6\left(\sqrt[3]{\mathcal{W}}\,(2\mathcal{Z}-3g^2) + 4g\mathcal{W}^{2/3} + \mathcal{W}\right)}\Big[3\sqrt{3F_2'}\mathcal{Z} + 3\mathcal{W}^{2/3}\left(g^3 + 10gn - 5g + 12m\right)$$
$$+ \mathcal{W}^{1/3}\mathcal{Z}\left(2\mathcal{Z} - 3g^2\right) + 3\Big(g\left(g^4 + 22g^2n - 11g^2 + 84n^2 - 84n + 21\right) + 36m\mathcal{Z}\Big) + g\mathcal{W}^{4/3}\Big];$$

$$x_3 = \frac{1}{12\mathcal{Y}^{1/3}}\left[\left(-1 + i\sqrt{3}\right)\mathcal{Y}^{2/3} + 2g\mathcal{Y}^{1/3} - \left(1 + i\sqrt{3}\right)\left(2\mathcal{Z} - 3g^2\right)\right],$$

$$y_3 = \frac{1}{-48g\mathcal{Y}^{2/3} + 6\left(1+i\sqrt{3}\right)\mathcal{Y}^{1/3}\left(2\mathcal{Z}-3g^2\right) + 6\left(1-i\sqrt{3}\right)\mathcal{Y}}\Big[-6\mathcal{Y}^{2/3}\left(g^3 + 10gn - 5g + 12m\right)$$
$$+ \left(1 + i\sqrt{3}\right)\mathcal{Y}^{4/3}g + \left(1 + i\sqrt{3}\right)\mathcal{Y}^{1/3}\mathcal{Z}\left(2\mathcal{Z} - 3g^2\right)$$
$$+ \left(1 - i\sqrt{3}\right)\mathcal{Y}\mathcal{Z} + \left(1 - i\sqrt{3}\right)g\left(2\mathcal{Z} - 3g^2\right)^2\Big];$$

$$x_4 = \frac{1}{12\mathcal{Y}^{1/3}}\left[\left(-1 - i\sqrt{3}\right)\mathcal{Y}^{2/3} + 2g\mathcal{Y}^{1/3} - \left(1 - i\sqrt{3}\right)\left(2\mathcal{Z} - 3g^2\right)\right],$$

$$y_4 = \frac{1}{-48g\mathcal{Y}^{2/3} + 6\left(1-i\sqrt{3}\right)\mathcal{Y}^{1/3}\left(2\mathcal{Z}-3g^2\right) + 6\left(1+i\sqrt{3}\right)\mathcal{Y}}\Big[-6\mathcal{Y}^{2/3}\left(g^3 + 10gn - 5g + 12m\right)$$
$$+ \left(1 - i\sqrt{3}\right)\mathcal{Y}^{4/3}g + \left(1 - i\sqrt{3}\right)\mathcal{Y}^{1/3}\mathcal{Z}\left(2\mathcal{Z} - 3g^2\right)$$
$$+ \left(1 + i\sqrt{3}\right)\mathcal{Y}\mathcal{Z} + \left(1 + i\sqrt{3}\right)g\left(2\mathcal{Z} - 3g^2\right)^2\Big].$$

In the above expressions for the singularities, we use the following notations:

$$\mathcal{Y} = g^3 + 18gn - 9g + 108m + 3\sqrt{\mathcal{X}},\quad \mathcal{Z} = -3 + 2g^2 + 6n,$$
$$\mathcal{W} = -9g + g^3 + 108m + 18gn + 3\sqrt{3F_2'},$$

where

$$\mathcal{X} = 24 - 3g^2 - 216gm + 24g^3m + 1296m^2 - 144n + 12g^2n + 432gmn + 288n^2 - 12g^2n^2 - 192n^3.$$

Calculations yield:

$$\Phi(x_2, y_2) = \Phi(x_3, y_3) = \Phi(x_4, y_4) = 0,\quad \Phi(x_1, y_1) = \frac{F_1'}{(1+g^2)^2},$$

and therefore the three singularities M_2, M_3, and M_4 of systems (3.102) lie on the invariant parabola. Moreover, M_1 is located outside the parabola and would belong to the parabola if and only if the condition $F_1' = 0$ holds, where F_1' is given in (3.104). However, we have $\mathbf{D} = 12F_1'^2F_2' \neq 0$, and hence there are always exactly three simple singularities on the parabola.

On the other hand, according to [1, Proposition 5.1], if $\mathbf{D} > 0$, systems (3.102) possess two real and two complex finite singularities. For $\mathbf{D} < 0$, we could have either four real or four complex finite singularities. However, since M_1 is a real singular point for these systems, we conclude that in the case $\mathbf{D} < 0$, we must have four real finite distinct singularities.

Thus, since the real singularity M_1 is outside the invariant parabola and all three finite singularities on the parabola (whether real or complex) are distinct, and furthermore we cannot have any invariant line, we arrive at the configuration *Config.* $\mathcal{P}$.113 if $\mathbf{D} < 0$, and at *Config.* $\mathcal{P}$.114 if $\mathbf{D} > 0$.

The subcase D = 0. This implies that $F_1'F_2' = 0$, and we calculate:

$$\xi_1 = -6\zeta_4 F_1'.$$

Therefore, we deduce that, due to $\zeta_4 \neq 0$, the condition $F_1' = 0$ is equivalent to $\xi_1 = 0$. Thus, we examine two possibilities: $\xi_1 \neq 0$ and $\xi_1 = 0$.

1: *The possibility* $\xi_1 \neq 0$. In this case, the condition $\mathbf{D} = 0$ implies $F_2' = 0$. Since this polynomial is quadratic in the parameter m, we calculate

$$\text{Discrim}\,[F_2', m] = 64(g^2 + 12n - 6)^3.$$

Therefore, as the parameters m, n, and g of systems (3.102) must be real, the condition $g^2 + 12n - 6 \geq 0$ must hold. Introducing a new parameter v such that $g^2 + 12n - 6 = v^2 \geq 0$, we get $n = (6 - g^2 + v^2)/12$. Then, we compute

$$F_2' = \frac{1}{108}\left[216m - (g+v)^2(g-2v)\right]\left[216m - (g-v)^2(g+2v)\right] = 0,$$

and, due to the symmetry under the change $v \mapsto -v$, we may assume the first factor vanishes. This yields:

$$m = (g-2v)(g+v)^2/216.$$

Considering the expressions for the parameters m and n, we arrive at the following two-parameter family of systems:

$$\begin{aligned} \dot{x} &= \frac{(g-2v)(g+v)^2}{216} - \frac{g^2-v^2}{12}\,x - \frac{g}{2}y + gx^2 - xy, \\ \dot{y} &= \frac{(g-2v)(g+v)^2}{108}\,x + \frac{6-g^2+v^2}{6}\,y + gxy - 2y^2, \end{aligned} \tag{3.105}$$

possessing the invariant parabola $\Phi(x,y) = x^2 - y = 0$.

We observe that for the above systems, the following conditions on the parameters g and v hold:

$$\begin{aligned} &\xi_1 \neq 0 \;\Leftrightarrow\; (8g-v)^2(4g+v)\left(2g^2 - 8gv - v^2 + 18\right)\left(g^2 + 2gv + v^2 + 9\right) \neq 0; \\ &B_1 \neq 0 \;\Leftrightarrow\; (2g-v)(4g+v)(36 + 4g^2 - 4gv + v^2)(g^2 + 2gv + v^2 + 9) \\ &\qquad \times (g^2 - 4gv + 4v^2 + 9) \neq 0. \end{aligned} \tag{3.106}$$

We determine that systems (3.105) possess three finite singularities $M_i(x_i, y_i)$ $(i = 1,2,3)$ with coordinates:

$$\begin{gathered} x_1 = \frac{-27g + 5g^3 - 6gv^2 - v^3}{54(1+g^2)}, \quad y_1 = \frac{54 - 9g^2 + g^4 + 9v^2 - 3g^2v^2 - 2gv^3}{108(1+g^2)}, \\ x_2 = \frac{g-2v}{6}, \quad y_2 = \frac{(g-2v)^2}{36}, \quad x_3 = \frac{g+v}{6}, \quad y_3 = \frac{(g+v)^2}{36}. \end{gathered}$$

We calculate:

$$\Phi(x_2, y_2) = \Phi(x_3, y_3) = 0, \quad \Phi(x_1, y_1) = -\frac{\left(2g^2 - 8gv - v^2 + 18\right)\left(g^2 + 2gv + v^2 + 9\right)^2}{2916(1+g^2)^2},$$

and conclude that the singular points M_2 and M_3 lie on the invariant parabola, while M_1 lies outside the parabola due to the conditions (3.106).

We claim that M_3 is a multiple singularity of systems (3.105). Indeed, applying the corresponding translation, we can place M_3 at the origin, arriving at the systems:

$$\begin{aligned}\dot{x} &= \frac{1}{18}(g+v)(4g+v)x - \frac{1}{6}(4g+v)y + gx^2 - xy,\\ \dot{y} &= \frac{1}{54}(g+v)\left(2g^2+gv-v^2-18\right)x + \frac{1}{18}\left(v^2-2g^2-gv+18\right)y + gxy - x^2 - 2y^2,\end{aligned}$$

where $M_0(0,0)$ corresponds to the singularity M_3.

Following [1], we calculate the following invariant polynomials: $\mu_4 = \mu_3 = 0$, and

$$\mu_2 = \frac{1}{324}v\left[(g+v)^2+9\right]\left[(v-2g)x+6y\right]\left[(3-g^2-gv)x+(4g+v)y\right].$$

By [1, Lemma 5.2, statement (ii)], the point M_0 has multiplicity at least 2. We observe that, due to the condition $B_1 \neq 0$, we have $\mu_2 = 0$ if and only if $v = 0$. In this case, we calculate:

$$\mu_2 = 0, \quad \mu_1 = -\frac{1}{27}\left[(5g^4+27)x - 32g^3y\right] \neq 0.$$

Thus, according to [1, Lemma 5.2, statement (ii)], we have a double point if $v \neq 0$ and a triple point if $v = 0$.

On the other hand, for systems (3.105), we calculate:

$$\xi_2 = \frac{1}{209952}v^2\left(18+2g^2-8gv-v^2\right)^2\left(9+g^2+2gv+v^2\right)^2,$$

and due to (3.106), we conclude that the condition $v = 0$ is equivalent to $\xi_2 = 0$.

Thus, for systems (3.105), we obtain configuration *Config. $\mathcal{P}$.115* if $\xi_2 \neq 0$ and *Config. $\mathcal{P}$.116* if $\xi_2 = 0$.

2: *The possibility* $\xi_1 = 0$. We obtain $F_1' = 0$, and since this polynomial is quadratic in the parameter m, we calculate:

$$\text{Discrim}\,[F_1', m] = 4(1+g^2)^2(g^2+4n).$$

It is clear that for the existence of real solutions of the equation $F_1' = 0$, the condition $g^2+4n \geq 0$ must hold.

Thus, we introduce a new parameter u such that $g^2+4n = u^2 \geq 0$, leading to $n = (u^2-g^2)/4$. Then, calculations yield:

$$F_1' = -\frac{1}{16}\left[8m-(g-u)(2+g^2-gu)\right]\left[8m-(g+u)(2+g^2+gu)\right] = 0,$$

and, due to symmetry under $u \mapsto -u$, we may assume the first factor vanishes. This gives:

$$m = (g-u)(2+g^2-gu)/8.$$

Considering the expressions for m and n, we arrive at the following two-parameter family of systems:

$$\begin{aligned}\dot{x} &= \frac{(g-u)(2+g^2-gu)}{8} + \frac{u^2-g^2-2}{4}x - \frac{g}{2}y + gx^2 - xy,\\ \dot{y} &= \frac{(g-u)(2+g^2-gu)}{4}x - \frac{g^2-u^2}{2}y + gxy - 2y^2,\end{aligned} \tag{3.107}$$

possessing the invariant parabola $\Phi(x,y) = x^2 - y = 0$.

The systems (3.107) possess three finite singularities $M_i(x_i, y_i)$ $(i = 1,2,3)$ with coordinates:

$$x_1 = \frac{g-u}{2}, \quad y_1 = \frac{(g-u)^2}{4};$$
$$x_{2,3} = \frac{1}{4}\left(u \pm \sqrt{Y_1}\right), \quad y_{2,3} = \frac{1}{8}\left[u^2 + 2gu - 2g^2 - 4 \pm u\sqrt{Y_1}\right],$$
$$Y_1 = u^2 + 4gu - 4g^2 - 8.$$

We calculate:

$$\Phi(x_1, y_1) = \Phi(x_2, y_2) = \Phi(x_3, y_3) = 0,$$

and therefore all three singularities are located on the invariant parabola.

Moreover, we point out that M_1 is a singularity of systems (3.107) with multiplicity at least 2. Indeed, applying a translation that places M_1 at the origin, we arrive at the system:

$$\dot{x} = \frac{1}{2}(g^2 - gu - 1)x + \frac{1}{2}(u - 2g)y + gx^2 - xy,$$
$$\dot{y} = \frac{1}{2}(g-u)(g^2 - gu - 1)x + \frac{1}{2}(g-u)(2g-u)y - x^2 + gxy - 2y^2,$$

where $M_0(0,0)$ is the singularity corresponding to M_1.

Following [1], we calculate the following invariant polynomials: $\mu_4 = \mu_3 = 0$, and:

$$\mu_2 = \frac{1}{2}(g^2+1)\left[(g-u)^2 + 1\right]\left[(g^2 - gu + 1)x^2 + (u - 2g)xy + 2y^2\right].$$

We observe that $\mu_2 \neq 0$, and by [1, Lemma 5.2, statement (ii)], the point M_0 has multiplicity exactly 2.

On the other hand, the singularities M_2 and M_3 may be either real or complex, depending on the value of Y_1. To determine the position of the double singularity M_1 relative to M_2 and M_3 when they are real (i.e., $Y_1 > 0$), we calculate:

$$(x_2 - x_1)(x_3 - x_1) = \frac{(g-u)^2 + 1}{2} > 0.$$

Therefore, in the case $Y_1 > 0$, both singularities M_2 and M_3 lie on the same side of the double point M_1.

It is clear that for $Y_1 = 0$, the points M_2 and M_3 coalesce, and we obtain two double points located on the invariant parabola.

For systems (3.107), calculations yield:

$$\xi_2 = \frac{1}{8}(1+g^2)^2\left[1 + (g-u)^2\right]^2 Y_1,$$

and hence $\operatorname{sign}(\xi_2) = \operatorname{sign}(Y_1)$ whenever $\xi_2 \neq 0$.

Thus, we conclude that systems (3.107) possess configuration *Config. $\mathcal{P}$.117* if $\xi_2 < 0$, *Config. $\mathcal{P}$.118* if $\xi_2 > 0$, and *Config. $\mathcal{P}$.119* if $\xi_2 = 0$.

The case $B_1 = 0$. For systems (3.102), we calculate:

$$B_1 = -\frac{1}{64}\left[g + g^3 + 4m - 2gn\right]\left[(g - 8m)^2 + (1 - 4n)^2\right]\Psi(g,m,n),$$

where

$$\Psi(g,m,n) = 16m^2 + 8gm(3+2n) + (4+g^2)(1+g^2+4n+4n^2).$$

On the other hand, we calculate:

$$\xi_4 = \frac{13125}{2}(25+g^2)\left(3g+9g^3-4m-6gn\right)\left[(g-8m)^2+(1-4n)^2\right],$$
$$\zeta_4 = \frac{(25+g^2)\left(3g+9g^3-4m-6gn\right)}{16},$$

and, due to $\zeta_4 \neq 0$, we deduce that $\xi_4 = 0$ is equivalent to $(g-8m)^2+(1-4n)^2 = 0$.

Thus, we examine two subcases: $\xi_4 \neq 0$ and $\xi_4 = 0$.

The subcase $\xi_4 \neq 0$. Then, the condition $B_1 = 0$ implies either:

$$g + g^3 + 4m - 2gn = 0, \quad \text{or} \quad \Psi(g,m,n) = 0.$$

We state the following lemma.

Lemma 3.16. *For systems (3.102), if $\xi_4 \neq 0$, then the condition $B_1 = B_2 = 0$ is equivalent to $\Psi(g,m,n) = 0$.*

Proof. Assume first that the condition $\Psi(g,m,n) = 0$ holds. We calculate the discriminant of Ψ with respect to m:

$$\text{Discrim}\,[\Psi, m] = -64(g^2-4n-2)^2 \leq 0.$$

Thus, for real solutions in m, the necessary condition is $g^2-4n-2 = 0$, which gives $n = (g^2-2)/4$. Substituting back, we obtain:

$$\Psi(g,m) = \frac{1}{4}\left(4g+g^3+8m\right)^2, \quad B_2 = \left(4g+g^3+8m\right)^2\phi(g,m,x,y),$$

where $\phi(g,m,x,y)$ is a quartic polynomial in x and y. Thus, clearly, $\Psi(g,m,n) = 0$ implies $B_2 = 0$.

Conversely, assume now that for systems (3.102), the conditions $B_1 = B_2 = 0$ and $\xi_4 \neq 0$ hold, but suppose (for contradiction) that $\Psi(g,m,n) \neq 0$. Then, the condition $B_1 = 0$ yields $g + g^3 + 4m - 2gn = 0$, and solving for m, we get $m = -g(g^2-2n+1)/4$.

Next, we calculate:

$$\Psi(g,n) = (1+g^2)\left(g^2-4n-2\right)^2,$$
$$B_2 = -\frac{81}{2}(1+g^2)^2\left(g^2-4n-2\right)^2\left[4g^4+g^2(8-16n)+(1-4n)^2\right]x^4,$$
$$\text{Discrim}\left[4g^4+g^2(8-16n)+(1-4n)^2, n\right] = -256g^2 < 0,$$

where the last inequality holds because $\zeta_4 = g(g^2+25)(5g^2-4n+2)/8 \neq 0$. Therefore, for $B_2 = 0$, it must follow that $\Psi(g,n) = 0$, leading to a contradiction with our assumption that $\Psi(g,m,n) \neq 0$. This completes the proof of Lemma 3.16. □

Therefore, in what follows we discuss two possibilities: $B_2 \neq 0$ and $B_2 = 0$.

1: *The possibility* $B_2 \neq 0$. Then, by Lemma 3.16, we have $\Psi(g,m,n) \neq 0$, and the condition $B_1 = 0$ implies $g + g^3 + 4m - 2gn = 0$. Solving for m, we obtain $m = -g(g^2 - 2n + 1)/4$, leading to the two-parameter family of systems:

$$\begin{aligned} \dot{x} &= -\frac{1}{4}g(g^2 - 2n + 1) + \frac{1}{2}(2n-1)x - \frac{g}{2}y + gx^2 - xy, \\ \dot{y} &= -\frac{1}{2}g(g^2 - 2n + 1)x + 2ny - x^2 + gxy - 2y^2, \end{aligned} \tag{3.108}$$

possessing the invariant line $L_1(x,y) = 2x + g = 0$, in addition to the invariant parabola $\Phi(x,y) = x^2 - y = 0$.

For these systems, we calculate:

$$\zeta_4 = \frac{g(g^2+25)(5g^2 - 4n + 2)}{8}, \quad \theta = -8(g^2+9).$$

Since $\theta \neq 0$, by Lemma 2.4, systems (3.108) cannot have an invariant line parallel to $2x + g = 0$. Moreover, by Lemma 2.3, there cannot be invariant lines in other directions because $B_2 \neq 0$.

The systems (3.108) possess four finite singularities $M_i(x_i, y_i)$, $i = 1,2,3,4$, with coordinates:

$$x_1 = -\frac{g}{2}, \quad y_1 = \frac{g^2}{4}; \quad x_2 = -\frac{g}{2}, \quad y_2 = \frac{2n - g^2}{2};$$
$$x_{3,4} = \frac{1}{2}\left(g \pm \sqrt{Y_2}\right), \quad y_{3,4} = \frac{1}{2}\left(2n - 1 \pm g\sqrt{Y_2}\right), \quad Y_2 = 4n - g^2 - 2.$$

We compute:

$$\Phi(x_1,y_1) = \Phi(x_3,y_3) = \Phi(x_4,y_4) = L_1(x_1,y_1) = L_1(x_2,y_2) = 0,$$
$$\Phi(x_2,y_2) = (3g^2 - 4n)/4,$$

therefore, M_1 is the intersection point of the invariant line with the invariant parabola. Moreover, M_2 lies on the invariant line and belongs to the invariant parabola if and only if $3g^2 - 4n = 0$. Finally, M_3 and M_4 may be real, complex, or coinciding, depending on the value of Y_2, and they always lie on the invariant parabola.

To determine the relative positions of the finite singularities, we calculate:

$$(x_3 - x_1)(x_4 - x_1) = (5g^2 - 4n + 2)/4 = \gamma_1/4, \quad y_2 - y_1 = (4n - 3g^2)/4 = \delta_1/4.$$

Thus, when $Y_2 > 0$, M_2 and M_3 lie on the same side (respectively, on opposite sides) of M_1 if $\gamma_1 > 0$ (respectively, $\gamma_1 < 0$). Notice that $\gamma_1 \neq 0$ due to $\zeta_4 \neq 0$.

Also, $y_2 > y_1$ if $\delta_1 > 0$, $y_2 < y_1$ if $\delta_1 < 0$, and $y_2 = y_1$ if $\delta_1 = 0$, in which case the intersection point of the invariant line and the parabola is a double singular point of systems (3.108).

We compute the invariant polynomial $\mathbf{D}$ responsible for the existence of multiple finite singularities:

$$\mathbf{D} = -3(g^2+1)^4 Y_2 \gamma_1^2 \delta_1^2/4, \quad \zeta_4 = g(25 + g^2)\gamma_1/8,$$
$$\xi_7 = 4698510000\, g^2(1+g^2)^2 Y_2 \gamma_1 \delta_1^2, \quad \xi_8 = -247290000\, g^2(1+g^2)^2 Y_2 \delta_1 \gamma_1^2.$$

Thus, since $\zeta_4 \neq 0$, in the case $\mathbf{D} \neq 0$ we have:

$$\operatorname{sign}(\mathbf{D}) = -\operatorname{sign}(Y_2), \quad \operatorname{sign}(\xi_7) = \operatorname{sign}(Y_2\gamma_1), \quad \operatorname{sign}(\xi_8) = -\operatorname{sign}(Y_2\delta_1),$$

and we examine two cases: $\mathbf{D} \neq 0$ and $\mathbf{D} = 0$.

1.1: *The case* $\mathbf{D} \neq 0$. Then $Y_2 \neq 0$, and systems (3.108) have four distinct finite singularities.

Remark 3.17. We observe that $\gamma_1 + \delta_1 = 2(g^2+1) > 0$. Therefore, the conditions $\gamma_1 < 0$ and $\delta_1 < 0$ are incompatible.

Considering Remark 3.17, in the case $\mathbf{D} \neq 0$, systems (3.108) exhibit the following configurations:

$$\begin{array}{lll}
\mathbf{D} < 0, \xi_7 < 0 & \Rightarrow\ (x_3 - x_1)(x_4 - x_1) < 0,\ y_2 > y_1 & \Rightarrow\ \textit{Config. } \mathcal{P}.120; \\
\mathbf{D} < 0, \xi_7 > 0, \xi_8 < 0 & \Rightarrow\ (x_3 - x_1)(x_4 - x_1) > 0,\ y_2 > y_1 & \Rightarrow\ \textit{Config. } \mathcal{P}.121; \\
\mathbf{D} < 0, \xi_7 > 0, \xi_8 > 0 & \Rightarrow\ (x_3 - x_1)(x_4 - x_1) > 0,\ y_2 < y_1 & \Rightarrow\ \textit{Config. } \mathcal{P}.122; \\
\mathbf{D} > 0, \xi_8 < 0 & \Rightarrow\ y_2 < y_1 & \Rightarrow\ \textit{Config. } \mathcal{P}.123; \\
\mathbf{D} > 0, \xi_8 > 0 & \Rightarrow\ y_2 > y_1 & \Rightarrow\ \textit{Config. } \mathcal{P}.124.
\end{array}$$

1.2: *The case* $\mathbf{D} = 0$. Since $\gamma_1 \neq 0$ (due to $\zeta_4 \neq 0$), this condition implies $Y_2\delta_1 = 0$, and we calculate:

$$\xi_1 = -3g(1+g^2)^2(25+g^2)\gamma_1\delta_1/16.$$

Thus, since $\gamma_1 \neq 0$, the condition $\delta_1 = 0$ is equivalent to $\xi_1 = 0$. We consider two subcases: $\xi_1 \neq 0$ and $\xi_1 = 0$.

1.2.1: *The subcase* $\xi_1 \neq 0$. Then $Y_2 = 0$ (i.e., $n = (g^2+2)/4$), so M_3 and M_4 coalesce into a double singular point on the parabola. Moreover, the position of M_2 is determined by δ_1. For these systems, with $n = (g^2+2)/4$, we calculate:

$$\xi_1\zeta_8 = -3g^4(1+g^2)^3(25+g^2)\delta_1/2, \quad \zeta_4 = g^3(25+g^2)/2.$$

Since $\zeta_4 \neq 0$, we have $\operatorname{sign}(\xi_1\zeta_8) = -\operatorname{sign}(\delta_1)$. Therefore, in the case $\mathbf{D} = 0$ and $\xi_1 \neq 0$, we have configuration *Config.* $\mathcal{P}.125$ if $\xi_1\zeta_8 < 0$, and *Config.* $\mathcal{P}.126$ if $\xi_1\zeta_8 > 0$.

1.2.2: *The subcase* $\xi_1 = 0$. This condition implies $n = (3g^2)/4$, where M_2 and M_1 coalesce. Note that in this case, $\gamma_1 = 2(g^2+1) > 0$.

For $n = 3g^2/4$, we have:

$$Y_2 = 2(g^2-1), \quad \xi_2 = (g^2-1)(1+g^2)^4,$$

and clearly, $\xi_2 = 0$ if and only if $Y_2 = 0$, which corresponds to M_3 and M_4 also coalescing, yielding two double singularities on the invariant parabola.

Thus, for $\mathbf{D} = \xi_1 = 0$, we get *Config.* $\mathcal{P}.127$ if $\xi_2 \neq 0$, and *Config.* $\mathcal{P}.128$ if $\xi_2 = 0$.

2: *The possibility* $B_2 = 0$. Thus, $B_1 = B_2 = 0$, and by Lemma 3.16, the condition $\Psi(g,m,n) = 0$ holds. Referring to the proof of Lemma 3.16, we get:

$$g^2 - 4n - 2 = 0 \quad \Rightarrow \quad n = (g^2-2)/4, \qquad 4g + g^3 + 8m = 0 \quad \Rightarrow \quad m = -g(g^2+4)/8.$$

This leads to the one-parameter family of systems:

$$\begin{aligned}
\dot{x} &= -\frac{1}{8}g(g^2+4) + \frac{1}{4}(g^2-4)x - \frac{g}{2}y + gx^2 - xy, \\
\dot{y} &= -\frac{1}{4}g(g^2+4)x + \frac{1}{2}(g^2-2)y - x^2 + gxy - 2y^2,
\end{aligned} \tag{3.109}$$

possessing three invariant lines:

$$L_1(x,y) = 2x + g = 0, \quad L_{2,3}(x,y) = 4(y \pm ix) - g(g \mp 2i) = 0,$$

besides the invariant parabola.

We find that systems (3.109) possess four finite singularities $M_i(x_i, y_i)$, $i = 1,2,3,4$, with coordinates:

$$x_1 = -\frac{g}{2}, \quad y_1 = \frac{g^2}{4}; \quad x_2 = -\frac{g}{2}, \quad y_2 = -\frac{g^2+2}{4};$$
$$x_{3,4} = \frac{1}{2}(g \pm 2i), \quad y_{3,4} = \frac{1}{4}(g \pm 2i)^2.$$

We calculate:

$$\Phi(x_1, y_1) = \Phi(x_3, y_3) = \Phi(x_4, y_4) = 0, \quad \Phi(x_2, y_2) = (g^2+1)/2,$$
$$L_1(x_1, y_1) = L_1(x_2, y_2) = L_2(x_1, y_1) = L_2(x_3, y_3) = L_3(x_4, y_4) = 0.$$

Thus, M_1 is the intersection point of all three invariant lines with the invariant parabola. Additionally, M_2 lies on the invariant line but not on the parabola, since $g^2+1 \neq 0$. Moreover, since $y_2 - y_1 = -(g^2+1)/2 < 0$, M_2 lies below M_1 along the vertical line $L_1 = 0$.

Hence, systems (3.109) possess the unique configuration *Config.* $\mathcal{P}$*.129*.

The subcase $\zeta_4 = 0$. This condition implies $(g - 8m)^2 + (1 - 4n)^2 = 0$, yielding $m = g/8$ and $n = 1/4$. Thus, we obtain the following one-parameter family of systems:

$$\begin{aligned} \dot{x} &= \frac{g}{8} - \frac{x}{4} - \frac{gy}{2} + gx^2 - xy, \\ \dot{y} &= \frac{gx}{4} + \frac{y}{2} - x^2 + gxy - 2y^2, \end{aligned} \tag{3.110}$$

which, in addition to the invariant parabola $\Phi(x, y) = x^2 - y = 0$, possess two complex invariant lines given by $L_{1,2}(x, y) = 4(y \pm ix) - 1 = 0$.

For these systems, we calculate:

$$\zeta_4 = g(25+g^2)(1+9g^2)/16, \qquad B_3 = -3g(1+g^2)(x^2+y^2)^2/4,$$

and since $\zeta_4 \neq 0$ (i.e., $g \neq 0$), we have $B_3 \neq 0$. Thus, according to Lemma 2.3, these systems cannot have an invariant line in the third (real) direction.

The systems (3.110) possess four finite singularities $M_i(x_i, y_i)$, $i = 1,2,3,4$, with coordinates:

$$x_1 = 0, \quad y_1 = \frac{1}{4}; \qquad x_2 = \frac{g}{2}, \quad y_2 = \frac{g^2}{4}; \qquad x_{3,4} = \pm\frac{i}{2}, \quad y_{3,4} = -\frac{1}{4}.$$

We compute:

$$\Phi(x_2, y_2) = \Phi(x_3, y_3) = \Phi(x_4, y_4) = 0, \quad \Phi(x_1, y_1) = -\frac{1}{4},$$

and observe that the real singularity M_1 is the intersection point of the two complex invariant lines and lies outside the invariant parabola. The second real singular point M_2 lies on the parabola, and its position depends on the real parameter $g \neq 0$.

As a result, we arrive at a single configuration: *Config.* $\mathcal{P}$*.130*.

3.3.2 The statement $(\mathcal{E}_2)$

According to Proposition 2.9, in this case the conditions $\zeta_4 = 0$ and $\mathcal{R}_7\zeta_5 \neq 0$ must hold. From (3.103), the condition $\zeta_4 = 0$ implies:

$$3g + 9g^3 - 4m - 6gn = 0 \quad \Rightarrow \quad m = 3g(3g^2 - 2n + 1)/4,$$

which leads to the following family of systems:

$$\begin{aligned}\dot{x} &= \frac{3}{4}g\left(3g^2-2n+1\right)+\frac{1}{2}(2n-1)x-\frac{gy}{2}+gx^2-xy,\\ \dot{y} &= \frac{3}{2}g\left(3g^2-2n+1\right)x+2ny-x^2+gxy-2y^2,\end{aligned} \tag{3.111}$$

possessing two invariant parabolas: the canonical one, $\Phi_1(x,y)=x^2-y=0$, and a second parabola,

$$\Phi_2(x,y)=(3g^2-4n)(1+3g^2-2n)+2g(3g^2-4n)x-4(1+g^2)x^2+2(2+5g^2-4n)y=0.$$

For the systems (3.111), we calculate:

$$\begin{aligned}\zeta_5 &= \frac{19}{4}(g^2+25)(3g^2-4n)^2, \quad \mathcal{R}_7=16120(3g^2+1)(5g^2-4n+2),\\ \theta &= -8(g^2+9), \quad B_1=-g(g^2+1)(9g^2+1)(5g^2-4n+2)\Psi_2\Psi_3/32,\end{aligned} \tag{3.112}$$

where

$$\Psi_2(g,n)=81g^4+g^2(28-72n)+4(1+2n)^2, \quad \Psi_3(g,n)=36g^4+g^2(16-48n)+(1-4n)^2.$$

According to Lemma 2.3, systems (3.111) may possess at least one invariant line only if $B_1=0$. Thus, we proceed to examine two possibilities: $B_1\neq 0$ and $B_1=0$.

The possibility $B_1\neq 0$. The systems (3.111) possess four finite singularities $M_i(x_i,y_i)$ $(i=1,2,3,4)$ with coordinates:

$$\begin{gathered}x_1=\frac{3g}{2}, \quad y_1=\frac{9g^2}{4};\\ x_2=\frac{g(3+9g^2-8n)}{2(1+g^2)}, \quad y_2=\frac{9g^4+g^2(3-6n)+2n}{2(1+g^2)};\\ x_{3,4}=\frac{1}{2}\left(-g\pm\sqrt{Y_3}\right), \quad y_{3,4}=\frac{1}{2}\left(2n-1-2g^2\mp g\sqrt{Y_3}\right),\\ Y_3=4n-5g^2-2.\end{gathered} \tag{3.113}$$

We calculate:

$$\Phi_1(x_1,y_1)=\Phi_1(x_3,y_3)=\Phi_1(x_4,y_4)=0, \quad \Phi_2(x_2,y_2)=\Phi_2(x_3,y_3)=\Phi_2(x_4,y_4)=0,$$

therefore, the singularities M_3 and M_4 are the points of intersection of both invariant parabolas. Moreover, the point M_1 lies on the parabola $\Phi_1=0$, while M_2 lies on $\Phi_2=0$.

To determine the relative positions of M_1 and M_2 with respect to M_3 and M_4 (when $Y_3>0$), we calculate:

$$\begin{gathered}(x_3-x_1)(x_4-x_1)=(2+21g^2-4n)/4=\gamma_2/4,\\ (x_3-x_2)(x_4-x_2)=-\frac{Y_3}{4(1+g^2)^2}\left[21g^4+2g^2(5-8n)+1\right]=-\frac{Y_3}{4(1+g^2)^2}\delta_2,\\ (x_3-x_1)+(x_4-x_1)=-4g, \quad (x_3-x_2)+(x_4-x_2)=\frac{2gY_3}{1+g^2}.\end{gathered}$$

We observe the following sign relations:

$$\operatorname{sign}\big((x_3-x_1)+(x_4-x_1)\big)=-\operatorname{sign}(g),\quad \operatorname{sign}\big((x_3-x_2)+(x_4-x_2)\big)=\operatorname{sign}(gY_3). \tag{3.114}$$

Thus, we deduce that the point M_1 (respectively M_2) lies on the invariant parabola $\Phi_1=0$ (respectively $\Phi_2=0$) between M_3 and M_4 if and only if $\gamma_2<0$ (respectively $Y_3\delta_2>0$).

From these relations, we state the following remark.

Remark 3.18. Assume that, for the singularities given in (3.113), the following conditions hold:

$$(x_3-x_1)(x_4-x_1)>0,\quad (x_3-x_2)(x_4-x_2)>0.$$

Then, when $Y_3>0$, the condition

$$\big[(x_3-x_1)+(x_4-x_1)\big]\big[(x_3-x_2)+(x_4-x_2)\big]>0$$

is impossible.

This follows directly from the sign relations in (3.114).

For systems (3.111), we also calculate:

$$\mathbf{D}=-3(3g^2-4n)^2Y_3\gamma_2^2\delta_2^2/4,\quad \xi_{14}=1235\gamma_2\delta_2/2,\quad \xi_{30}=1235\left[Y_3\delta_2-(1+g^2)^2\gamma_2\right]/2. \tag{3.115}$$

We observe that, when $\mathbf{D}\neq 0$,

$$\operatorname{sign}(\mathbf{D})=-\operatorname{sign}(Y_3),\quad \operatorname{sign}(\xi_{14})=\operatorname{sign}(\gamma_2\delta_2).$$

Moreover, in the case where $\xi_{14}<0$ (i.e., $\gamma_2\delta_2<0$) and $\mathbf{D}<0$ (i.e., $Y_3>0$), we find:

$$\operatorname{sign}(\xi_{30})=\operatorname{sign}\left[Y_3\delta_2-(1+g^2)^2\gamma_2\right]=\operatorname{sign}(\delta_2).$$

Thus, we proceed to analyze two cases: $\mathbf{D}\neq 0$ and $\mathbf{D}=0$.

The case $\mathbf{D}\neq 0$. We observe that, in the case $\mathbf{D}>0$, the singular points M_3 and M_4 are complex, and therefore in this situation it is not necessary to distinguish the signs of the polynomials γ_2 and δ_2.

Thus, taking into account Remark 3.18 and the information discussed above, we detect that in the case $\mathbf{D}\neq 0$ the systems admit the following configurations:

$\mathbf{D}<0,\ \gamma_2<0,\ \delta_2>0 \Rightarrow (x_3-x_1)(x_4-x_1)<0,\ (x_3-x_2)(x_4-x_2)<0 \Rightarrow$ *Config.* $\mathcal{P}.132$;
$\mathbf{D}<0,\ \gamma_2>0,\ \delta_2<0 \Rightarrow (x_3-x_1)(x_4-x_1)>0,\ (x_3-x_2)(x_4-x_2)>0 \Rightarrow$ *Config.* $\mathcal{P}.131$;
$\mathbf{D}<0,\ \gamma_2<0,\ \delta_2<0 \Rightarrow (x_3-x_1)(x_4-x_1)<0,\ (x_3-x_2)(x_4-x_2)>0 \Rightarrow$ *Config.* $\mathcal{P}.133$;
$\mathbf{D}<0,\ \gamma_2>0,\ \delta_2>0 \Rightarrow (x_3-x_1)(x_4-x_1)>0,\ (x_3-x_2)(x_4-x_2)<0 \Rightarrow \simeq$ *Config.* $\mathcal{P}.133$;
$\mathbf{D}>0 \Rightarrow$ *Config.* $\mathcal{P}.134$.

Note that in the case $\gamma_2\delta_2>0$, both conditions lead to equivalent configurations to *Config.* $\mathcal{P}.133$. Thus, we deduce that for $\mathbf{D}\neq 0$, the systems (3.111) realize the following configurations if and only if the corresponding invariant conditions hold:

$\mathbf{D}<0,\ \xi_{14}<0,\ \xi_{30}<0 \Rightarrow$ *Config.* $\mathcal{P}.131$;
$\mathbf{D}<0,\ \xi_{14}<0,\ \xi_{30}>0 \Rightarrow$ *Config.* $\mathcal{P}.132$;
$\mathbf{D}<0,\ \xi_{14}>0 \Rightarrow$ *Config.* $\mathcal{P}.133$;
$\mathbf{D}>0 \Rightarrow$ *Config.* $\mathcal{P}.134$.

The case D = 0. Considering the values of the invariant polynomials obtained above and the conditions $\zeta_5 \neq 0$ (i.e., $3g^2 - 4n \neq 0$) and $\mathcal{R}_7 \neq 0$ (i.e., $Y_3 \neq 0$), we deduce that the condition $\mathbf{D} = 0$ implies $\gamma_2\delta_2 = 0$. From (3.115), this is equivalent to $\xi_{14} = 0$.

We claim that, in the case $\xi_{14} = 0$, the systems (3.111) exhibit configuration *Config. $\mathcal{P}$.135* if $\xi_{30} < 0$, configuration *Config. $\mathcal{P}$.136* if $\xi_{30} > 0$, and configuration *Config. $\mathcal{P}$.137* if $\xi_{30} = 0$.

Indeed, assume $\xi_{14} = 0$, that is, $\gamma_2\delta_2 = 0$. To prove our claim, we examine both possible subcases.

1: *The subcase $\delta_2 = 0$.* This condition implies:

$$n = \frac{1 + 10g^2 + 21g^4}{16g^2}, \quad \gamma_2 = \frac{(7g^2 - 1)(9g^2 + 1)}{4g^2}.$$

We determine that, under this condition, the singular point M_2 coalesces with M_4. If in addition $\gamma_2 = 0$, then M_1 coalesces with M_3, producing two double singularities on the parabola.

Thus, for systems (3.111), if $\gamma_2 > 0$, they realize *Config. $\mathcal{P}$.135*, if $\gamma_2 < 0$, they realize *Config. $\mathcal{P}$.136*, and if $\gamma_2 = 0$, they realize *Config. $\mathcal{P}$.137*.

Considering (3.115), for $\delta_2 = 0$ we observe that $\operatorname{sign}(\xi_{30}) = -\operatorname{sign}(\gamma_2)$, which confirms the claim for this subcase.

2: *The subcase $\gamma_2 = 0$.* This implies $n = (2 + 21g^2)/4$, and under this condition, the singular point M_1 coalesces with M_3. We calculate:

$$\delta_2 = (1 - 7g^2)(1 + 9g^2).$$

It is straightforward to verify that, in this case, if $\delta_2 < 0$, the configuration is *Config. $\mathcal{P}$.135*, if $\delta_2 > 0$, the configuration is *Config. $\mathcal{P}$.136*, and if $\delta_2 = 0$, the configuration is *Config. $\mathcal{P}$.137*.

Finally, from (3.115), for $\gamma_2 = 0$ we find $Y_3 = 16g^2 > 0$, so $\operatorname{sign}(\xi_{30}) = \operatorname{sign}(\delta_2)$, which completes the proof of our claim.

The possibility $B_1 = 0$. From (3.112), this condition implies $g(5g^2 - 4n + 2)\Psi_2\Psi_3 = 0$. We claim that, due to the condition $\mathcal{R}_7 \neq 0$, the equality $B_1 = 0$ is equivalent to $g = 0$.

Indeed, assuming $g \neq 0$, we compute:

$$\operatorname{Discrim}[\Psi_2, n] = -4096g^2 < 0, \quad \operatorname{Discrim}[\Psi_3, n] = -256g^2 < 0,$$

and hence the equations $\Psi_2 = 0$ and $\Psi_3 = 0$ cannot have real solutions with respect to the parameter n. This completes the proof of our claim.

Thus, we conclude $g = 0$ and arrive at the following one-parameter family of systems:

$$\dot{x} = \frac{1}{2}x\,(2n - 2y - 1), \quad \dot{y} = -x^2 + 2n\,y - 2y^2, \tag{3.116}$$

which possess the invariant line $x = 0$ and the invariant parabolas:

$$\Phi_1(x, y) = x^2 - y = 0, \quad \Phi_2(x, y) = x^2 + (2n - 1)y - n(2n - 1) = 0.$$

For these systems, we calculate:

$$\zeta_5 = 1900n^2, \quad \mathcal{R}_7 = -32240(2n - 1), \quad B_2 = -162(2n + 1)^2(4n - 1)^2x^4.$$

We now discuss two cases: $B_2 \neq 0$ and $B_2 = 0$.

The case $B_2 \neq 0$**.** By Lemma 2.3, in this case the system cannot possess invariant lines in any other direction. Considering the condition $g = 0$ in (3.113), we find that the systems (3.116) possess four finite singularities $M_i(x_i, y_i)$ $(i = 1, 2, 3, 4)$ with coordinates:

$$x_1 = 0, \quad y_1 = 0; \quad x_2 = 0, \quad y_2 = n;$$
$$x_{3,4} = \frac{1}{2}\big(\pm\sqrt{Y_3}\big), \quad y_{3,4} = \frac{1}{2}(2n-1), \quad Y_3 = 2(2n-1).$$

We observe that the invariant line $x = 0$ intersects the invariant parabola $\Phi_1 = 0$ at point M_1, and intersects the parabola $\Phi_2 = 0$ at point M_2.

For these systems we have:

$$\mathbf{D} = -12n^2Y_3^3 \quad \Rightarrow \quad \text{sign}\,(\mathbf{D}) = -\text{sign}\,(Y_3).$$

Since all singular points are fixed as intersection points of invariant curves, and $\mathbf{D} \neq 0$ (due to $\zeta_5\mathcal{R}_7 \neq 0$), we conclude that these systems realize configuration *Config.* $\mathcal{P}$*.138* if $\mathbf{D} < 0$, and configuration *Config.* $\mathcal{P}$*.139* if $\mathbf{D} > 0$.

The case $B_2 = 0$**.** This condition implies $(2n+1)(4n-1) = 0$.

Assume first $4n - 1 = 0$, that is, $n = 1/4$. Then, we arrive at the system:

$$\dot{x} = -\frac{1}{4}x(4y+1), \quad \dot{y} = \frac{1}{2}\left(-2x^2 - 4y^2 + y\right), \tag{3.117}$$

which possesses three invariant lines: $L_1(x, y) = x = 0$ and $L_{2,3}(x, y) = y \pm ix - 1/4 = 0$, along with the invariant parabolas $\Phi_1(x, y) = x^2 - y = 0$ and $\Phi_2(x, y) = -4x^2 + 2y - 1/2 = 0$.

We observe that the point $M_2(0, 1/4)$ corresponds to the intersection of the above complex lines. As a result, this system exhibits the configuration *Config.* $\mathcal{P}$*.140*.

Now, if $2n + 1 = 0$ (i.e., $n = -1/2$), we arrive at the system:

$$\dot{x} = -x(1+y), \quad \dot{y} = -x^2 - y - 2y^2,$$

which can be transformed into system (3.117) via the affine transformation and time rescaling:

$$x_1 = x/2, \quad y_1 = y/2 + 1/4, \quad t_1 = 2t,$$

and hence also realizes *Config.* $\mathcal{P}$*.140*.

3.3.3 The statement $(\mathcal{E}_3)$

According to Proposition 2.9, in this case the conditions $\zeta_4 = \zeta_5 = 0$ and $\mathcal{R}_7 \neq 0$ hold. Considering (3.112), the condition $\zeta_5 = 0$ implies:

$$3g^2 - 4n = 0 \quad \Rightarrow \quad n = \frac{3g^2}{4},$$

and we obtain the following one-parameter family of systems:

$$\begin{aligned} \dot{x} &= \frac{3}{8}g\,(3g^2+2) + \frac{1}{4}(3g^2-2)x - \frac{g\,y}{2} + g\,x^2 - x\,y, \\ \dot{y} &= \frac{3}{4}g\,(3g^2+2)x + \frac{3g^2}{2}y - x^2 + g\,x\,y - 2y^2, \end{aligned} \tag{3.118}$$

which, according to Proposition 2.9, possess $\Phi(x,y) = x^2 - y = 0$ as a double invariant parabola.

For these systems we compute:

$$\zeta_4 = \zeta_5 = 0, \quad \mathcal{R}_7 = 32240(1+g^2)(1+3g^2) \neq 0,$$
$$\theta = -8(g^2+9) \neq 0, \quad B_1 = -\frac{g\,(1+g^2)^4\,(1+9g^2)^3}{4}.$$

According to Lemma 2.3, systems (3.118) can possess at least one invariant line only if $B_1 = 0$. Thus, we discuss two cases: $B_1 \neq 0$ and $B_1 = 0$.

The case $B_1 \neq 0$. Systems (3.118) possess the following three finite singularities $M_i(x_i, y_i)$ $(i = 1,2,3)$ with coordinates:

$$\begin{gathered} x_1 = \frac{3g}{2}, \quad y_1 = \frac{9g^2}{4}; \\ x_{2,3} = -\frac{1}{2}\left[g \pm i\sqrt{2(1+g^2)}\right], \quad y_{2,3} = \frac{1}{4}\left[-2 - g^2 \pm 2gi\sqrt{2(1+g^2)}\right]. \end{gathered} \tag{3.119}$$

It is clear that the real singular point M_1 is a double point, as it lies on the double invariant parabola. This fact can be checked directly.

Moreover, the complex singular points M_2 and M_3 also lie on the invariant parabola, though this is not relevant for the classification of the configuration. We deduce that in the case $B_1 \neq 0$, systems (3.118) realize a single configuration: *Config.* $\mathcal{P}$*.141*.

The case $B_1 = 0$**.** This condition implies $g = 0$. Thus, system (3.118) with $g = 0$ possesses an additional invariant line: $x = 0$. Considering the singularities (3.119) evaluated at $g = 0$, we arrive at the unique configuration *Config.* $\mathcal{P}$*.142*.

3.3.4 The statement $(\mathcal{E}_4)$

According to Proposition 2.9, in this case the conditions $\zeta_4 = \mathcal{R}_7 = 0$ and $\zeta_5 \neq 0$ hold. Considering (3.112), the condition $\mathcal{R}_7 = 0$ implies:

$$5g^2 - 4n + 2 = 0 \quad \Rightarrow \quad n = \frac{2+5g^2}{4}.$$

Thus, we arrive at the following one-parameter family of systems:

$$\begin{aligned} \dot{x} &= \frac{1}{8}(g+2x)\left(3g^2 + 4g\,x - 4y\right), \\ \dot{y} &= \frac{3g^3 x}{4} + \frac{1}{2}(5g^2+2)y - x^2 + g\,x\,y - 2y^2, \end{aligned} \tag{3.120}$$

which possess the invariant parabola $\Phi(x,y) = x^2 - y = 0$ and the invariant line $2x + g = 0$. For these systems we calculate:

$$B_2 = -648(1+g^2)^5(1+9g^2)x^4 \neq 0,$$

and by Lemma 2.3, no additional invariant lines can exist.

Systems (3.120) have three finite singularities $M_i(x_i, y_i)$ $(i = 1, 2, 3)$ with coordinates:

$$x_1 = -\frac{g}{2}, \quad y_1 = \frac{g^2}{4}; \quad x_2 = -\frac{g}{2}, \quad y_2 = \frac{1}{4}(3g^2 + 2); \quad x_3 = \frac{3g}{2}, \quad y_3 = \frac{9g^2}{2}.$$

We claim that the singular point M_1 is a multiple singularity of systems (3.120). Indeed, applying a suitable translation to place M_1 at the origin, we arrive at the system:

$$\begin{aligned} \dot{x} &= x(gx - y), \\ \dot{y} &= g(1 + g^2)x - x^2 + (1 + g^2)y + g\,x\,y - 2y^2, \end{aligned}$$

where $M_0(0, 0)$ corresponds to the singularity M_1.

Following [1], we compute the following invariant polynomials:

$$\mu_4 = \mu_3 = 0, \quad \mu_2 = 2g(1 + g^2)^2 x(gx - y),$$
$$\mu_1 = -(1 + g^2)(x + 5g^2 x - 4gy),$$

and by [1, Lemma 5.2, statement (ii)], the point M_0 has multiplicity at least 2. Notice that $\mu_2 = 0$ if and only if $g = 0$, but in this case $\mu_1 \neq 0$. Therefore, according to [1, Lemma 5.2, statement (ii)], M_1 is a double point if $g \neq 0$ and a triple point if $g = 0$. This classification is governed by the invariant polynomial $\zeta_3 = 32g^2$.

We also note that the multiple singularity M_1 is the intersection point between the invariant line $2x + g = 0$ and the invariant parabola. Moreover, the singularity M_3 lies on the same invariant parabola, and M_3 coalesces with M_1 when $g = 0$, producing a triple finite singularity in systems (3.120).

On the other hand, the singularity M_2 lies on the invariant line above the point M_1, since $y_2 - y_1 = (g^2 + 1)/2 > 0$. Therefore, we conclude that systems (3.120) realize configuration *Config.* $\mathcal{P}$*.143* if $\zeta_3 \neq 0$, and *Config.* $\mathcal{P}$*.144* if $\zeta_3 = 0$.

As all the cases have now been examined, we conclude that statement $(\boldsymbol{B})$ of the Main Theorem is fully proved.

3.4 Geometric invariants and the proof of the statement (C)

In this subsection, we complete the proof of the Main Theorem by showing that all 144 configurations of invariant parabolas and invariant lines that we have constructed are non-equivalent according to Definition 1.3. To achieve this, we define the invariants that distinguish the configurations within the family $\mathbf{QSP}_{(\eta \neq 0)}$ into 144 *distinct* cases. We believe that these invariants are among the most suitable for describing the geometric phenomena specific to this class.

The basic algebraic-geometric notions that we will use here include the concept of an integer-valued r-cycle and its type:

Definition 3.19. Let V be an irreducible algebraic variety of dimension n over a field K. A *cycle of dimension* r, or an *r-cycle*, on V is a formal sum

$$\sum_W m(W)W,$$

where each W is a subvariety of V of dimension r that is not contained in the singular locus of V, $m(W) \in \mathbb{Z}$, and only finitely many of the coefficients $m(W)$ are nonzero. The *degree of an r-cycle* is the sum $\sum_W m(W)$. An $(n-1)$-cycle is called a *divisor*.

Definition 3.20. The *type of an r-cycle* is the set of all ordered pairs (n_1, n_2), where n_1 is a coefficient appearing in the r-cycle (i.e., $n_1 = m(W)$ for some W), and n_2 is the number of subvarieties W in the cycle for which $m(W) = n_1$.

Definition 3.21. We define a set of numerical and geometric invariants that allow us to distinguish the obtained configurations:

(a) $(M_{\mathcal{P}}, m_{\mathcal{L}})$ = (total multiplicity of invariant parabolas, total multiplicity of invariant affine lines);

(b) $\tau^f_{\mathcal{P}}$ = the type of the divisor of the multiplicities of finite singularities of the system located on the parabolas;

(c) $t^f_{\mathcal{P}\cap\mathcal{L}}$ = the type of the divisor of the intersection multiplicities (in the finite plane) between $\mathcal{P}$ and $\mathcal{L}$;

(d) τ^{∞}_s = the type of the divisor of the multiplicities of singularities on the line at infinity;

(e) $M^n_{\mathcal{L}\cap\mathcal{P}}$ = the maximum number of real finite intersection points between an invariant line and a parabola;

(f) $M^m_{\mathcal{P}}$ = the maximum multiplicity among the invariant parabolas;

(g) Consider the finite segment of a parabola delimited by two singularities s_1 and s_2 of the system, having inside another singularity s_0. Let $m(s_0)$ denote the multiplicity of s_0. We define the invariant $I(s_0)$ for this configuration, assigning it the value 1 if s_0 is an intersection point between the parabola and a line, and 0 otherwise;

(h) $m^f(\mathcal{P}\cap\mathcal{L})$ = the multiplicity of the intersection point (considered as a singularity of the system) between the parabola and an affine line;

(i) Consider an invariant line passing through the infinite point of the parabola. This line has another finite intersection point with the parabola, splitting it into two branches. We introduce the invariant $\{n^1_{\mathcal{P}}, n^2_{\mathcal{P}}\}$, where $n^i_{\mathcal{P}}$ denotes the number of singularities of the system located on each open branch of the parabola;

(j) $m^{\mathbb{R}}_s(\mathcal{P})$ = the total multiplicity of real singularities of the system located on the parabola;

(k) $n^s_{in}(\mathcal{P})$ = the total number of real singularities located inside the union of all domains delimited by the parabolas;

(l) If an invariant line intersects a parabola at two distinct real finite points, we obtain a domain $\mathcal{D}^f_{\mathcal{P}\cap\mathcal{L}}$ bounded by the line segment and the arc of the parabola not passing through its infinite point. We denote by $\tau(\mathcal{D}^f_{\mathcal{P}\cap\mathcal{L}})$ the type of the divisor of the of the multiplicities of singularities of the system located on the boundary of $\mathcal{D}^f_{\mathcal{P}\cap\mathcal{L}}$;

(m) $n^f_s(\mathcal{F})$: the number of finite singular points of the curve

$$\mathcal{F} : \mathcal{P}(X, Y, Z) \cdot L_1(X, Y, Z) \cdot L_2(X, Y, Z) = 0$$

that have multiplicity at least 2.

(n) Suppose an affine line intersects the parabola $x^2 - y = 0$ at two finite points or has a point of tangency with it. Let Δ^b denote the curvilinear triangle on the Poincaré disk, with vertex at the infinite point of the parabola, bounded by the closed branch of the parabola, the adjacent segment of the invariant line, and the open arc a_∞ containing a singular point at infinity in its interior. We denote by $n(\Delta^b)$ (respectively, $m(\Delta^b)$) the number (respectively, the total multiplicity) of singularities located on Δ^b;

(o) Assume we have a real system possessing two real invariant parabolas. In this case, two distinct possibilities occur in the real projective plane:

(1) The points at infinity of the two parabolas are distinct,

(2) The points at infinity coincide.

Clearly, the positions of the points at infinity (P_1^∞, P_2^∞) separate these two cases. If $P_1^\infty \neq P_2^\infty$, we are in case (1). If $P_1^\infty = P_2^\infty$, we are in case (2), which further splits into two subcases on the Poincaré disk (PD):

(i) The two points at infinity coincide on PD,

(ii) The two points are opposite on PD.

We define the invariant $J(P_1^\infty, P_2^\infty)$ of the group acting on the systems viewed on the Poincaré disk, assigning the value 0 to case (1), the value 1 to case $(2.i)$, and the value 2 to case $(2.ii)$;

(p) If two invariant parabolas intersect at two distinct finite points, we obtain a finite domain $\mathcal{D}^f_{\mathcal{P}_1\cap\mathcal{P}_2}$ delimited by the closed branches of the parabolas. We denote by $\tau(\mathcal{D}^f_{\mathcal{P}_1\cap\mathcal{P}_2})$ the type of the divisor of the multiplicities of singularities of the system located on the boundary of $\mathcal{D}^f_{\mathcal{P}_1\cap\mathcal{P}_2}$;

(q) Assume that two real invariant parabolas have two distinct points at infinity, $\mathcal{P}_1^\infty$ and $\mathcal{P}_2^\infty$, which are not opposite on PD. Let a_1^∞ and a_2^∞ denote the open arcs on the circumference of PD determined by $\mathcal{P}_1^\infty$ and $\mathcal{P}_2^\infty$. Assume that within the interior of arc a_1^∞ (respectively, a_2^∞), there are n_1 (respectively, n_2) singularities at infinity. We define a new invariant: $m(\mathcal{P}_1^\infty, \mathcal{P}_2^\infty) = \min\{n_1, n_2\}$.

The set of numerical and geometric invariants introduced in Definition 3.21 provides the necessary criteria to distinguish all 144 non-equivalent configurations of systems in the family **QSP**. Each invariant encodes specific geometric information regarding the arrangement of invariant parabolas, affine lines, and singularities, both finite and at infinity.

Figures 3.1 and 3.2 illustrate the complete bifurcation diagram associated with the Main Theorem. The diagrams are organized according to the sign of the parameter η, with Figure 3.1 representing the case $\eta > 0$ and Figure 3.2 corresponding to $\eta < 0$.

Each branch in these diagrams reflects a bifurcation step governed by one or more of the invariants from Definition 3.21, capturing the sequence of geometric changes that distinguish one configuration from another. Together, the invariants and the diagrams provide a complete and systematic classification of all phase portraits in the family **QSP**.

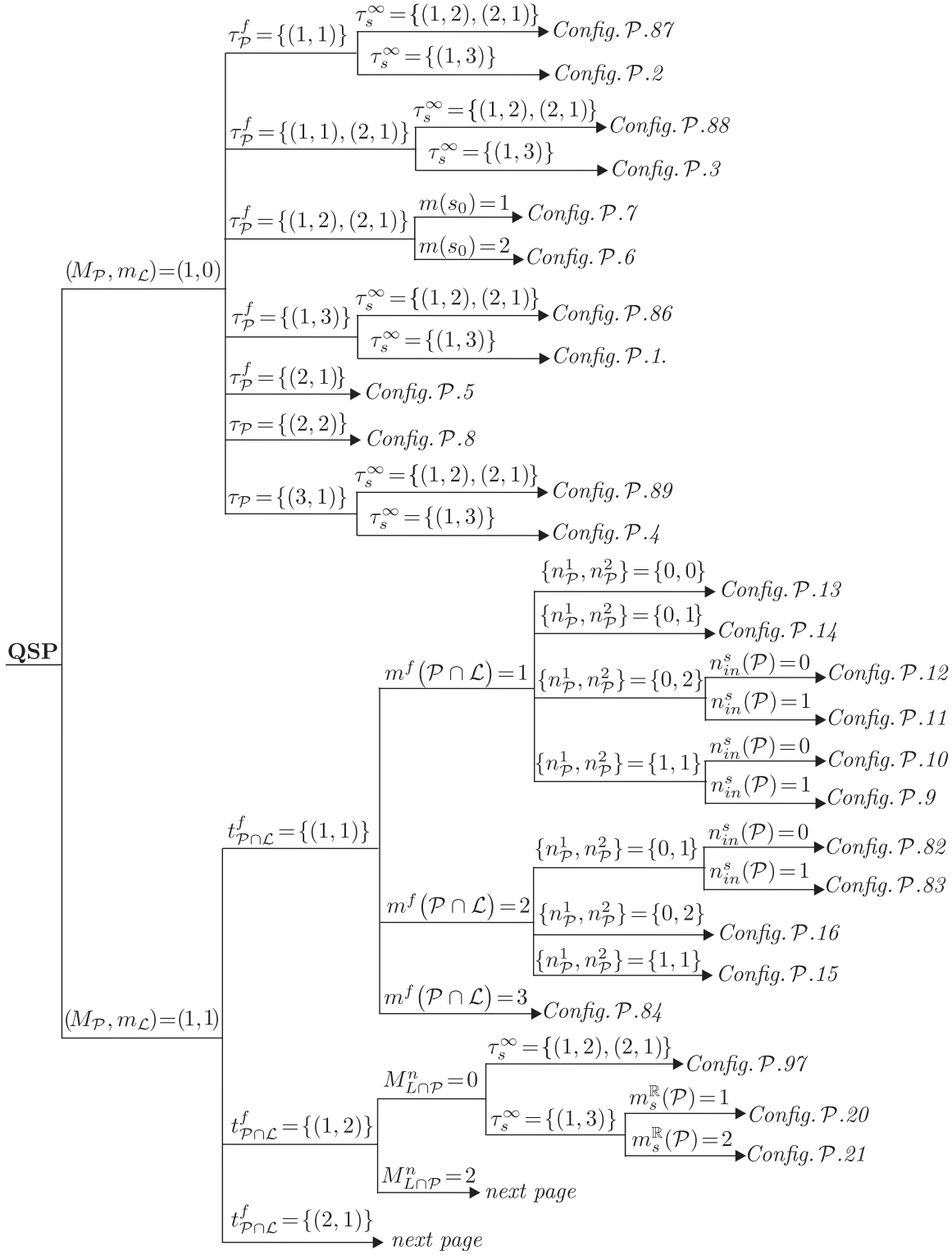


Diagram 3.1: Non-equivalent configurations of systems in **QSP** (the case $\eta > 0$).

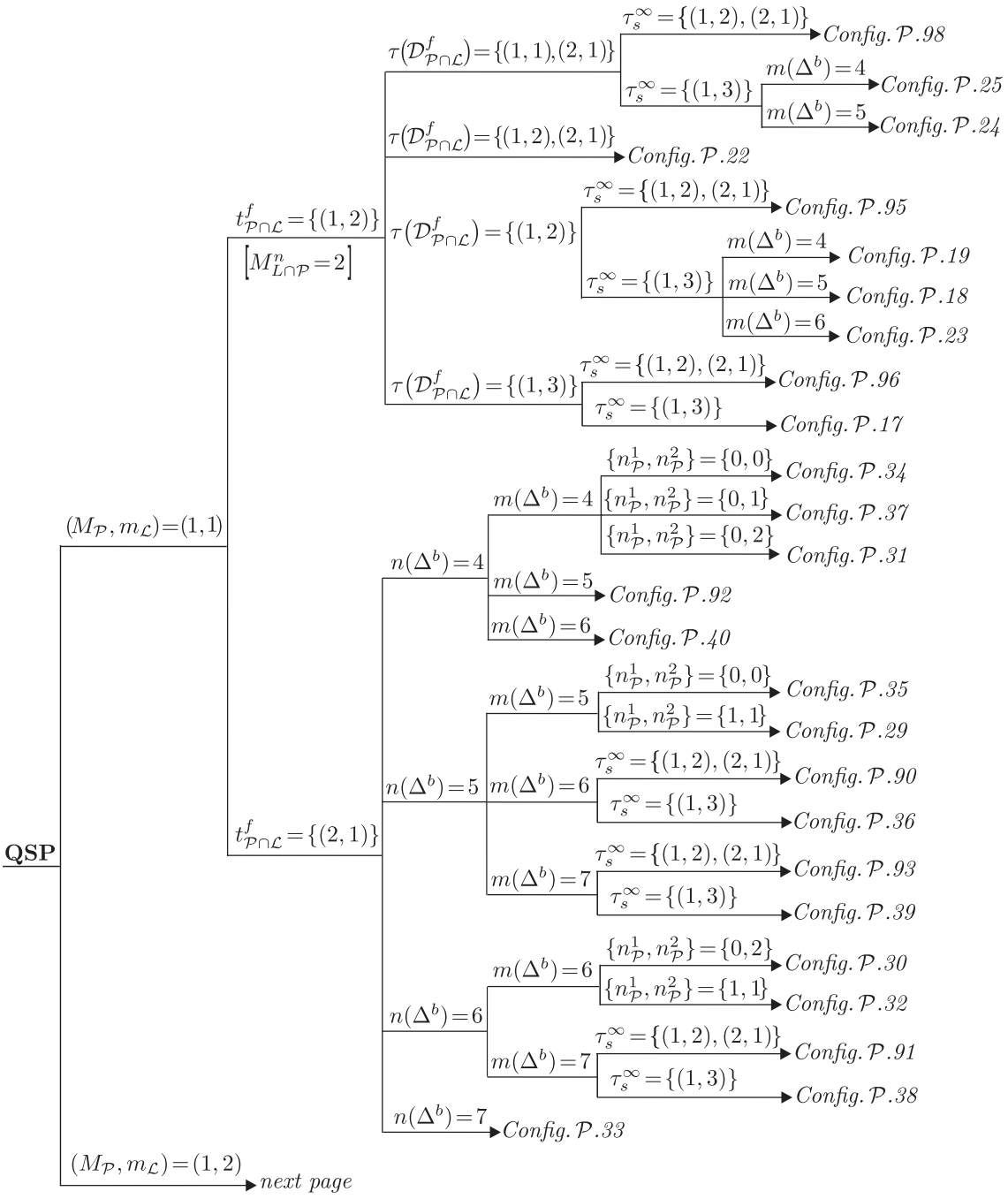


Diagram 3.1 *(continued)*: Non-equivalent configurations of systems in **QSP** (the case $\eta > 0$).

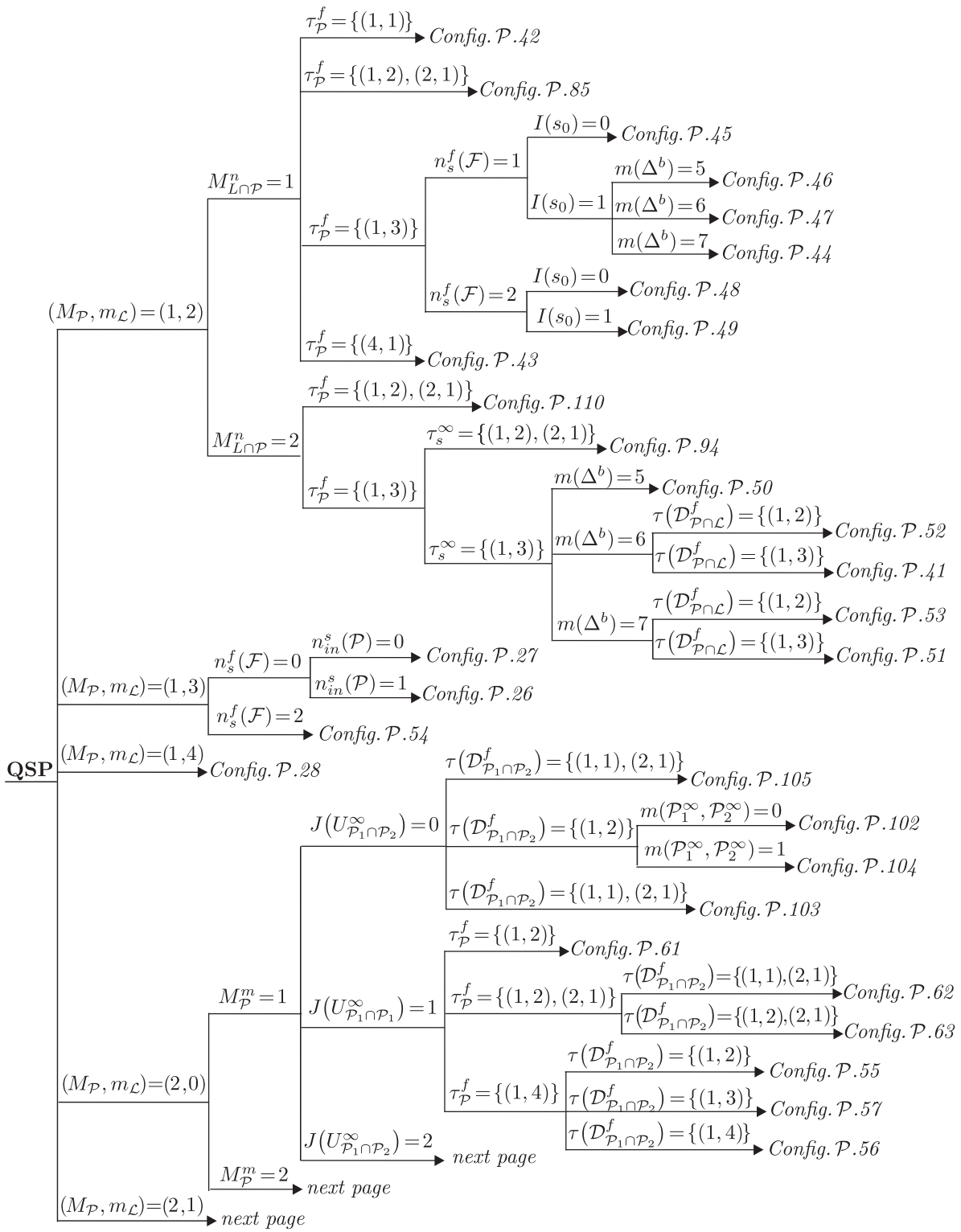


Diagram 3.1 *(continued)*: Non-equivalent configurations of systems in **QSP** (the case $\eta > 0$).

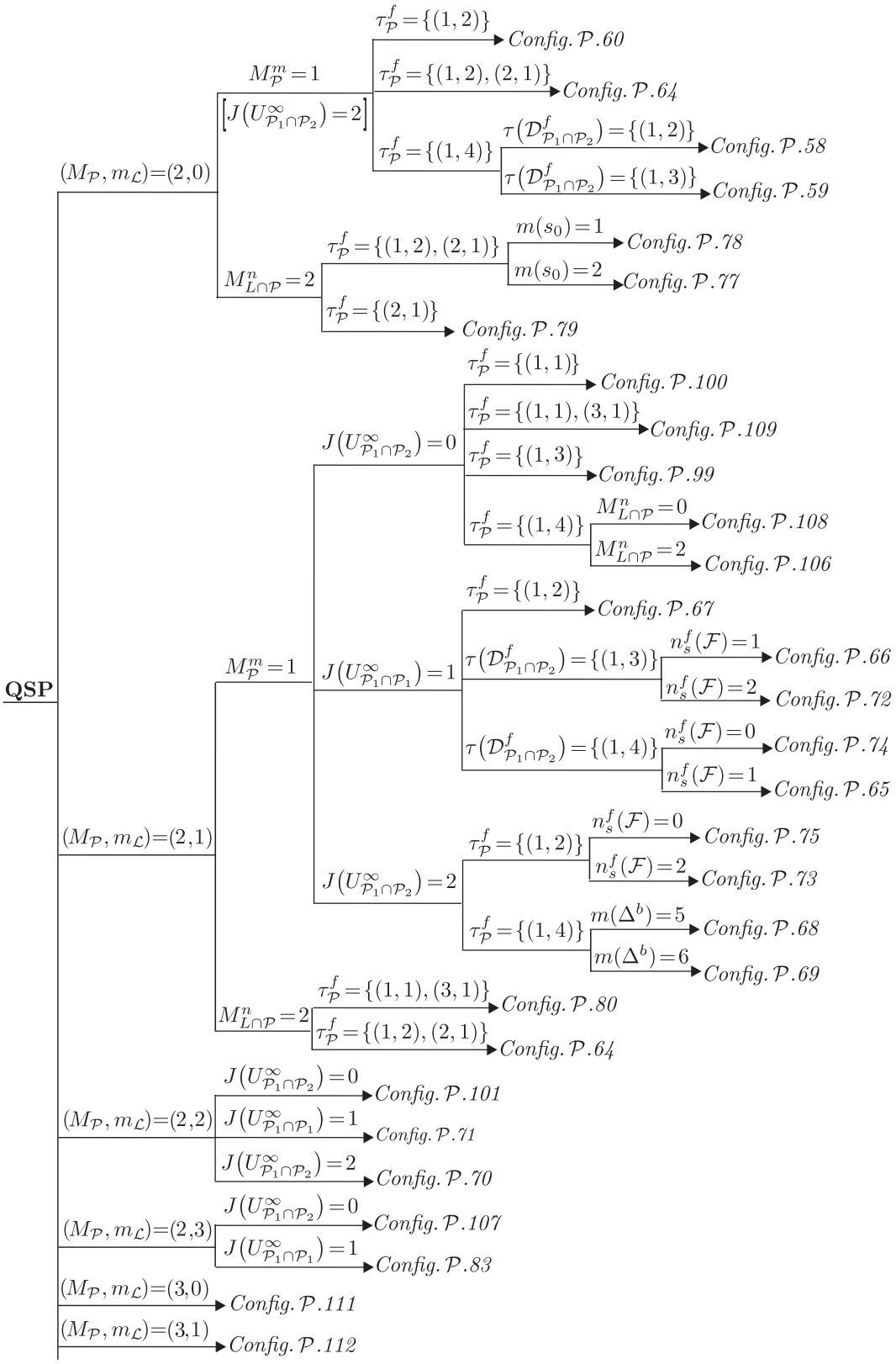


Diagram 3.1 *(continued)*: Non-equivalent configurations of systems in **QSP** (the case $\eta > 0$).

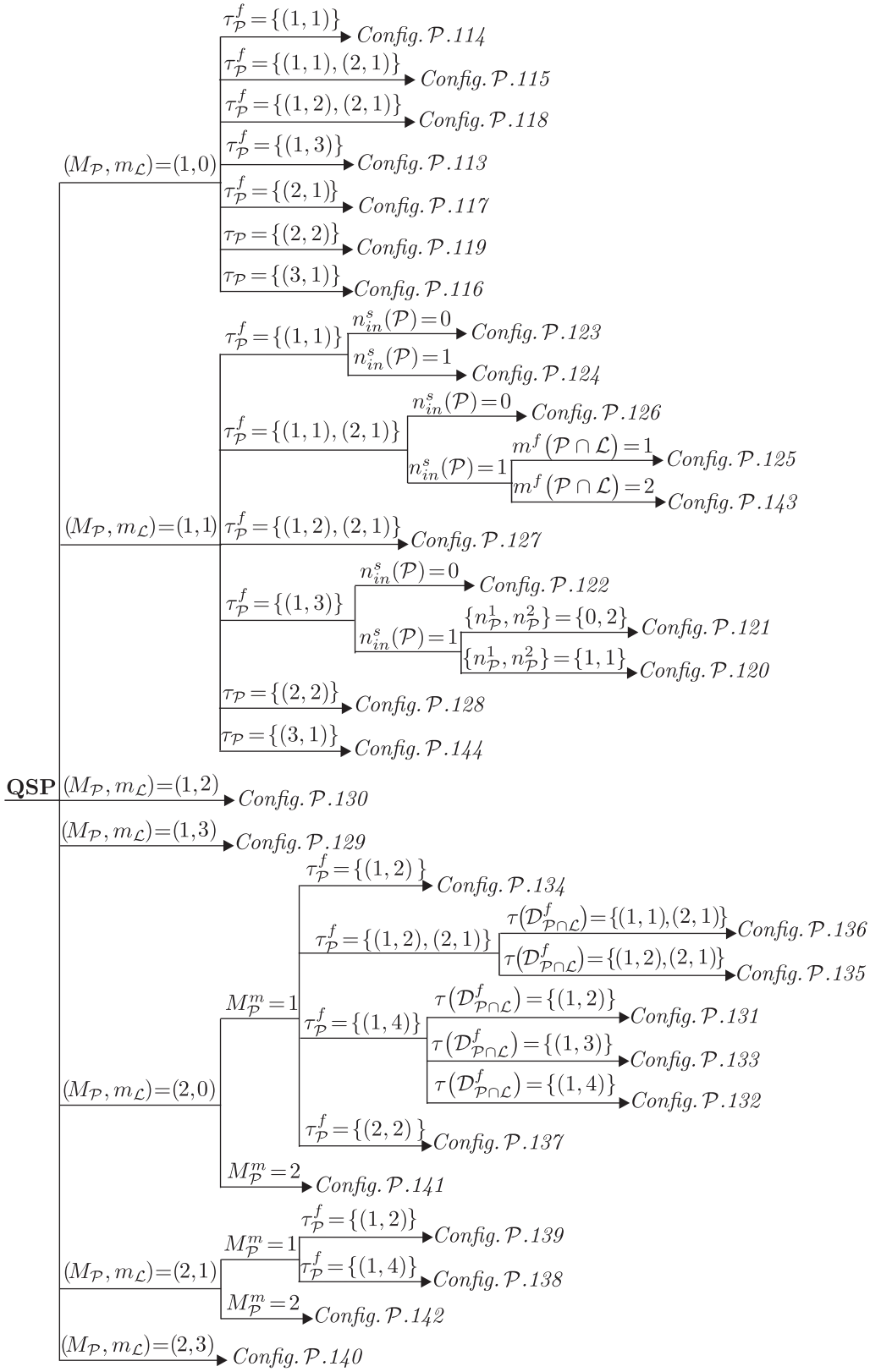


Diagram 3.2: Non-equivalent configurations of systems in **QSP** (the case $\eta < 0$).

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Resonance phenomena in the second differential order equations with nonlinear impulsive effects

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Abstract. In this paper, we discuss the resonance phenomena in linear and weakly nonlinear impulsive differential equations. Using a topological approach, we obtain the Landesman–Lazer type necessary and sufficient conditions for the existence of periodic solutions and extend the Fredholm alternative for linear equations with impulses. Moreover, we show the coexistence of periodic and unbounded solutions of weakly nonlinear impulsive differential equations through the use of the successor map and a Lyapunov function.

Keywords: impulsive differential equations, resonance phenomena, periodic solutions, Fredholm alternative, Lyapunov function.

2020 Mathematics Subject Classification: 34C25, 34C11.

1 Introduction

In this paper, we focus on the second-order impulsive ordinary differential equations

$$\begin{cases} u'' + f(t,u) = 0, \\ \Delta u(t_j) = J_j(u(t_j^-), u'(t_j^-)), \\ \Delta u'(t_j) = Q_j(u(t_j^-), u'(t_j^-)), \ j = \pm 1, \pm 2, \dots, \end{cases} \tag{1.1}$$

where $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous, 2π-periodic with respect to first variable, $0 \le t_1 < \cdots < t_k < 2\pi$, $\Delta u(t_j) = u(t_j^+) - u(t_j^-)$, $\Delta u'(t_j) = u'(t_j^+) - u'(t_j^-)$, J_j, $Q_j : \mathbb{R}^2 \to \mathbb{R}$ are continuous for $j = \pm 1, \pm 2, \dots$ Moreover, we assume that the impulsive time is 2π-periodic, that is, $t_{j+k} = t_j + 2\pi$, for $j = \pm 1, \pm 2, \dots$

There are many interesting results on the existence and multiplicity of periodic solutions of impulsive differential equations. Various methodologies include fixed point theory [12–14], the Leray–Schauder continuation method of topological degree [5,19], the variational method

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[4, 15, 21, 22], the Poincaré–Birkhoff twist theorem [8, 18], and Moser's twist theorem for the Lagrange stability of impulsive differential equations [16, 20].

Regarding the existence of periodic solutions for impulsive differential equations, several significant results have been achieved. However, compared to non-impulsive differential equations, many unresolved issues remain that are worthy of further investigation. For the linear equation

$$u'' + n^2 u = p(t), \tag{1.2}$$

where $p(t) \in C([0, 2\pi])$ is 2π-period. The so-called phenomenon of linear resonance arises whenever n is a positive integer. In other words, periodic solutions and unbounded solutions cannot coexist for the linear equation (1.2). This implies that all solutions of (1.2) are either periodic or unbounded. This is also known as the Fredholm alternative. However, slightly different equations do not have such strong conclusions. For instance, Alonso and Ortega [2] proved that similar results fail to hold for piecewise linear equations

$$u'' + au^+ - bu^- = p(t),$$

where $u^+ = \max\{u, 0\}$, $u^- = \max\{-u, 0\}$ and a, b are different and satisfy

$$\frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}} \in \mathbb{Q}. \tag{1.3}$$

They deduced that periodic and unbounded solutions can coexist when (1.3) holds and $a \neq b$ in [2].

Meanwhile, under the influence of impulses, the resonance phenomenon in linear equations has attracted significant attention. Similarly, the coexistence of periodic solutions with unbounded solutions in weakly linear equations has been a focus of recent studies. Recently, Drábek and Langerová [6] obtained the Fredholm alternative type results for linear equations with bounded impulses. They applied critical point theory to provide the necessary and sufficient conditions for the existence of a solution when solutions satisfying the impulse conditions in the derivative:

$$\Delta u(t_j) = 0, \qquad \Delta u'(t_j) = \tilde{I}_j(u(t_j^-)),$$

$$\lim_{s \to \pm\infty} \tilde{I}_j(s) = \tilde{I}_j(\pm\infty), \qquad \tilde{I}_j(+\infty) < \tilde{I}_j(s) < \tilde{I}_j(-\infty), \qquad j = 1, \ldots, k.$$

In their work [15], they also mentioned employing similar approaches to analyze the necessary and sufficient conditions for the periodic solutions of linear equations. However, we note that when applying variational methods or critical point theory to impulsive equations, due to the limitations of these methods, it is necessary to transform the periodic solution or boundary value problem into a functional critical point. In such cases, the impulses must satisfy

$$\Delta u(t_j) = 0, \qquad \Delta u'(t_j) = \tilde{I}_j(u(t_j^-)).$$

An interesting question is whether we can overcome this limitation and obtain the necessary and sufficient conditions for the existence of periodic solutions of linear equations under more general impulsive conditions. One of the main results of this paper is to relax the impulse restrictions established in [6], via topological framework, allowing for impulses in both the function and its derivative.

We consider the following impulsive equation

$$\begin{cases} u''(t) + n^2 u(t) = p(t), \quad t \in [0, 2\pi],\ t \neq t_j; \\ \Delta u(t_j) = J_j(u'(t_j^-)), \quad \Delta u'(t_j) = Q_j(u(t_j^-)), \quad j = 1, \dots, k; \\ u(0) = u(2\pi), \quad u'(0) = u'(2\pi). \end{cases} \tag{1.4}$$

Suppose that

(H_1) For $j = 1, \dots, k$, there exist limits

$$\lim_{s \to \pm\infty} J_j(s) = J_j(\pm\infty), \qquad \lim_{s \to \pm\infty} Q_j(s) = Q_j(\pm\infty),$$

and the following inequalities hold:

$$J_j(+\infty) < J_j(s) < J_j(-\infty), \qquad Q_j(-\infty) < Q_j(s) < Q_j(+\infty), \qquad s \in \mathbb{R}.$$

That gives our Landesman–Lazer type condition.

(H_2) For any $\theta \in \mathbb{R}$, $p(t)$ satisfies

$$\begin{aligned} & n \sum_{j=1}^{k} J_j(-\infty) \cos^+(nt_j + \theta) - n \sum_{j=1}^{k} J_j(+\infty) \cos^-(nt_j + \theta) + \sum_{j=1}^{k} Q_j(+\infty) \sin^-(nt_j + \theta) \\ & - \sum_{j=1}^{k} Q_j(-\infty) \sin^+(nt_j + \theta) > \int_0^{2\pi} \sin(nt + \theta) p(t) dt > n \sum_{j=1}^{k} J_j(+\infty) \cos^+(nt_j + \theta) \\ & - n \sum_{j=1}^{k} J_j(-\infty) \cos^-(nt_j + \theta) - \sum_{j=1}^{k} Q_j(+\infty) \sin^+(nt_j + \theta) + \sum_{j=1}^{k} Q_j(-\infty) \sin^-(nt_j + \theta). \end{aligned}$$

Theorem 1.1. *Assume that* (H_1) *holds. Then* (1.4) *has a* 2π*-periodic solution if and only if* $p(t)$ *satisfies* (H_2)*. Moreover, if* (1.4) *has no periodic solutions then every solution must be unbounded.*

Remark 1.2. Theorem 1.1 provides the necessary and sufficient conditions for impulses in both the function and its derivative, and also discusses unbounded solutions, extending the results in [6].

And we illustrate our results with an example.

Example 1.3. Let $n = 2$ in (1.4), the impulse time $t_1 = \frac{\pi}{4}$, $t_2 = \frac{\pi}{2}$ and the impulse condition is of the form $J_j(u') = -\arctan u'$, $Q_j(u) = \arctan u$, $j = 1, 2$. Then (1.4) has a 2π-periodic solution if $p(t) = \cos 2t + \sin 2t$.

Furthermore, the Fredholm alternative does not hold for nonlinear equations. There is no similar theorem even for piecewise linear equations, as shown by Alonso and Ortega [2]. Fonda and Mawhin [7] proved all solutions of $J\dot{u} = \nabla \mathcal{H}(u) + f(u) + p(t)$ are bounded in the past, and those with sufficiently large amplitude are unbounded in the future. Qian [17] showed that the coexistence of periodic solution and unbounded solution, the infinity of large amplitude subharmonics for asymmetric weakly nonlinear oscillator There is no similar theorem even for piecewise linear equations by Alonso and Ortega [2]. Fonda and Mawhin [7] proved all solutions of $J\dot{u} = \nabla \mathcal{H}(u) + f(u) + p(t)$ are bounded in the past, and those with sufficiently large amplitude are unbounded in the future. Qian [17] showed that the coexistence

of periodic solution and unbounded solution, the infinity of large amplitude subharmonics for asymmetric weakly nonlinear oscillator

$$x'' + a^2x^+ - b^2x^- + h(x) = p(t),$$

with $h(\pm\infty) = 0$ and $xh(x) \to +\infty (x \to \infty)$. For the other discussion of the coexistence of unbounded solutions and periodic solutions, we refer the reader to [3, 9, 10]. Thus the coexistence of periodic solution and unbounded solution is a typical nonlinear phenomenon. We consider the following weakly nonlinear impulsive equation

$$\begin{cases} u''(t) + n^2u(t) + h(u(t)) = p(t), \quad t \in [0, 2\pi],\ t \neq t_j; \\ \Delta u(t_j) = J_j(u(t_j^-), u'(t_j^-)), \quad \Delta u'(t_j) = Q_j(u(t_j^-), u'(t_j^-)), \quad j = 1, \dots, k, \end{cases} \tag{1.5}$$

where $0 < t_1 < \cdots < t_k < 2\pi$, $h : C[0, 2\pi] \to C[0, 2\pi]$, $p(t) \in C([0, 2\pi])$ is 2π-periodic and $J_j, Q_j : \mathbb{R}^2 \to \mathbb{R}$ are continuous, $j = 1, \dots, k$. Suppose that

(D_1) $h(u)$ satisfies

(i) $h(\pm\infty) = \lim_{u\to\pm\infty} h(u) = 0$,

(ii) $\lim_{|u|\to\infty} uh(u) = +\infty$.

(D_2) Impulsive terms satisfy

$$J_j(u, u') = O\left(\frac{1}{r}\right), \qquad Q_j(u, u') = O\left(\frac{1}{r}\right), \qquad j = 1, \dots, k,$$

where $r = \sqrt{|u|^2 + |u'|^2}$. $f = O\left(\frac{1}{r}\right)$ for $r \to \infty$ means that $f/\left(\frac{1}{r}\right)$ is bounded for all sufficiently large r.

Theorem 1.4. *Assume that (D_1) and (D_2) hold. Then there exists $q_0 > 0$, such that (1.5) has both 2π-periodic solution and unbounded solution provided that $p(t)$ satisfies*

$$0 < M_p < q_0,$$

where

$$M_p = \max_{\theta\in[0,2\pi]} \left| \int_0^{2\pi} p(t) \sin(nt + \theta) dt \right|.$$

Example 1.5. Let $h(u) = u^{-\frac{1}{3}}$ and the impulse condition is given by $Q_j(u, u') = J_j(u, u') = \frac{1}{\sqrt{u^2+u'^2}}$. Then (1.5) has both 2π-periodic solution and unbounded solution.

The rest of the paper is organized as follows. In Section 2, we discuss the necessary and sufficient conditions for the existence of periodic solutions of linear impulsive equations via constant variation method and degree theory. Moreover, we prove that all solutions are unbounded when the periodic solution does not exist by the method of variation of constants. In Section 3, we consider weakly nonlinear impulsive equations and introduce the successor map to prove the existence of a periodic solution. The proof of the existence of the unbounded solution is mainly using Poincaré map to construct a Liapunov function that makes the solution grow infinitely along the orbit. So that the solution of the equation is unbounded.

2 Resonance and the existence of periodic solutions

In this section, we discuss a linear equation with impulses and provide necessary and sufficient conditions for the existence of a 2π-periodic solution under assumptions (H_1) and (H_2). Furthermore, we prove the existence of unbounded solutions and then complete the proof of Theorem 1.1.

Consider the linear periodic boundary value problem

$$\begin{cases} u''(t) + n^2 u(t) = p(t), \quad t \in [0, 2\pi],\ t \neq t_j; \\ \Delta u(t_j) = J_j(u(t_j^-), u'(t_j^-)), \quad \Delta u'(t_j) = Q_j(u(t_j^-), u'(t_j^-)),\ j = 1, \dots, k; \\ u(0) = u(2\pi), \quad u'(0) = u'(2\pi). \end{cases} \tag{2.1}$$

where $0 < t_1 < \cdots < t_k < 2\pi$, $p(t) \in C(\mathbb{R}, \mathbb{R})$, $p(t)$ is 2π-periodic and the impulses J_j, $Q_j : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $j = 1, \dots, k$, are continuous.

2.1 The necessary condition

For a general case, we consider the following impulsive equation

$$\begin{cases} x'(t) = Ax(t) + P(t), \quad t \neq t_j; \\ \Delta x(t_j) = I_j(x(t_j^-)), \quad j = 1, \dots, k; \\ x(0) = x(2\pi), \end{cases} \tag{2.2}$$

where 2×2 matrix A is constant, two-dimensional vector $P(t)$ is continuous, $\Delta x(t_j) = x(t_j^+) - x(t_j^-)$, I_j are continuous, $j = 1, \dots, k$.

For simplicity, we set $t_0 = 0$ and $t_{k+1} = 2\pi$. It is well known that the solutions of (2.2) which include instantaneous impulses result in jump discontinuities in both velocity and position at times $t = t_j$, $j = 1, \dots, k$. In other words, any motion of the solution is the same as the motion of the corresponding equation without impulses until it meets the next impulse time. Hence, we first consider the differential equation without impulses

$$x'(t) = Ax(t) + P(t). \tag{2.3}$$

Fixed any $j \in \{0, 1, \dots, k\}$. Suppose that equation (2.3) satisfies the initial condition $x(t_j) = x(t_j^+)$. By the method of variation of parameters, we have

$$x(t) = \Phi(t) \left(\Phi^{-1}(t_j^+) x(t_j^+) + \int_{t_j}^{t} \Phi^{-1}(s) P(s) ds \right), \qquad t \in (t_j^+, t_{j+1}^-),$$

where $\Phi(t)$ is the fundamental matrix of (2.3) that satisfies $\Phi(0) = \Phi(2\pi) = E$. In particular,

$$x(t_{j+1}^-) = \Phi(t_{j+1}^-) \left(\Phi^{-1}(t_j^+) x(t_j^+) + \int_{t_j}^{t_{j+1}^-} \Phi^{-1}(s) P(s) ds \right).$$

Next, we present the expression for the solutions of the impulsive differential equation (2.2). Note that $\Phi(t_j^+) = \Phi(t_j) = \Phi(t_j^-)$, since $\Phi(t)$ is continuous. Substituting $x(t_1^-)$ into the

expression for $x(t_2^-)$ yields

$$\begin{aligned} x(t_2^-) &= \Phi(t_2^-)\left(\Phi^{-1}(t_1^+)x(t_1^+) + \int_{t_1^+}^{t_2^-} \Phi^{-1}(s)P(s)ds\right) \\ &= \Phi(t_2^-)\left(\Phi^{-1}(t_1^+)\left(x(t_1^-) + x(t_1^+) - x(t_1^-)\right) + \int_{t_1^+}^{t_2^-} \Phi^{-1}(s)P(s)ds\right) \\ &= \Phi(t_2^-)\left(x(0) + \int_0^{t_2^-} \Phi^{-1}(s)P(s)ds + \Phi^{-1}(t_1^+)I_1(x(t_1^-))\right). \end{aligned}$$

Repeating the above steps by substituting $x(t_2^-)$ into the expression for $x(t_3^-)$, and so on, we obtain

$$x(t) = \Phi(t)\left(x(0) + \int_0^t \Phi^{-1}(s)P(s)ds + \sum_{t>t_j} \Phi^{-1}(t_j)I_j(x(t_j^-))\right). \tag{2.4}$$

In particular,

$$\begin{aligned} x(2\pi) &= \Phi^{-1}(2\pi)\left(x(0) + \int_0^{2\pi} \Phi^{-1}(s)P(s)ds + \sum_{j=1}^{k} \Phi^{-1}(t_j)I_j(x(t_j^-))\right) \\ &= x(0) + \int_0^{2\pi} \Phi^{-1}(s)P(s)ds + \sum_{j=1}^{k} \Phi^{-1}(t_j)I_j(x(t_j^-)). \end{aligned} \tag{2.5}$$

Since $x(0) = x(2\pi)$, it follows that

$$\int_0^{2\pi} \Phi^{-1}(s)P(s)ds + \sum_{j=1}^{k} \Phi^{-1}(t_j)I_j(x(t_j^-)) = 0. \tag{2.6}$$

Recall that the equation (2.1). We set $x_1(t) = u(t)$, $x_2(t) = u'(t)$, $x(t) = (x_1(t), x_2(t))^T$, $P(t) = (0, p(t))^T$, $I_j = (J_j, Q_j)^T$, $j = 1, \ldots, k$, and $A = \begin{pmatrix} 0 & 1 \\ -n^2 & 0 \end{pmatrix}$. Obviously,

$$\Phi(t) = \begin{pmatrix} \cos nt & \frac{1}{n}\sin nt \\ -n\sin nt & \cos nt \end{pmatrix}.$$

Then, plugging these expression back into (2.6), we obtain

$$\begin{aligned} \int_0^{2\pi} p(t)\sin nt dt &= n\sum_{j=1}^{k} J_j(u(t_j^-), u'(t_j^-))\cos nt_j - \sum_{j=1}^{k} Q_j(u(t_j^-), u'(t_j^-))\sin nt_j, \\ \int_0^{2\pi} p(t)\cos nt dt &= -n\sum_{j=1}^{k} J_j(u(t_j^-), u'(t_j^-))\sin nt_j - \sum_{j=1}^{k} Q_j(u(t_j^-), u'(t_j^-))\cos nt_j. \end{aligned}$$

We now assume that the impulses affect only one component of the system. Consider the following linear periodic boundary value problem

$$\begin{cases} u''(t) + n^2u(t) = p(t), \quad t \in [0, 2\pi],\ t \neq t_j; \\ \Delta u(t_j) = J_j(u'(t_j^-)), \quad \Delta u'(t_j) = Q_j(u(t_j^-)), \quad j = 1, \ldots, k; \\ u(0) = u(2\pi), \quad u'(0) = u'(2\pi). \end{cases} \tag{2.7}$$

where the impulses satisfies (H_1).

Similarly, we have

$$\int_0^{2\pi} p(t)\sin ntdt = n\sum_{j=1}^{k} J_j(u'(t_j^-))\cos nt_j - \sum_{j=1}^{k} Q_j(u(t_j^-))\sin nt_j,$$
$$\int_0^{2\pi} p(t)\cos ntdt = -n\sum_{j=1}^{k} J_j(u'(t_j^-))\sin nt_j - \sum_{j=1}^{k} Q_j(u(t_j^-))\cos nt_j.$$

Choose any $\theta \in \mathbb{R}$. Multiplying the above two expressions by $\cos\theta$ and $\sin\theta$, respectively, and adding them together, we obtain

$$\begin{aligned}\int_0^{2\pi} p(t)\sin(nt+\theta)dt &= n\sum_{j=1}^{k} J_j(u'(t_j^-))\cos(nt_j+\theta) - \sum_{j=1}^{k} Q_j(u(t_j^-))\sin(nt_j+\theta)\\ &= n\sum_{j=1}^{k} J_j(u'(t_j^-))(\cos^+(nt_j+\theta) - \cos^-(nt_j+\theta))\\ &\quad - \sum_{j=1}^{k} Q_j(u(t_j^-))(\sin^+(nt_j+\theta) - \sin^-(nt_j+\theta)),\end{aligned}$$

where $f^+ = \max\{f,\ 0\}$ and $f^- = \max\{0,\ -f\}$.

By combining (H_1), we finally obtain

$$\begin{aligned}&n\sum_{j=1}^{k} J_j(-\infty)\cos^+(nt_j+\theta) - n\sum_{j=1}^{k} J_j(+\infty)\cos^-(nt_j+\theta)\\ &\qquad + \sum_{j=1}^{k} Q_j(+\infty)\sin^-(nt_j+\theta) - \sum_{j=1}^{k} Q_j(-\infty)\sin^+(nt_j+\theta)\\ &> \int_0^{2\pi} \sin(nt+\theta)p(t)dt\\ &> n\sum_{j=1}^{k} J_j(+\infty)\cos^+(nt_j+\theta) - n\sum_{j=1}^{k} J_j(-\infty)\cos^-(nt_j+\theta)\\ &\qquad - \sum_{j=1}^{k} Q_j(+\infty)\sin^+(nt_j+\theta) + \sum_{j=1}^{k} Q_j(-\infty)\sin^-(nt_j+\theta),\end{aligned}$$

which is exactly assumption (H_2).

2.2 The sufficient condition

For the Banach space

$$X = \{x : [0,2\pi] \to \mathbb{R}^2 :\ x(t) \text{ is continuous for } t \neq t_j,\ x(t_j^-),\ x(t_j^+) \text{ exist,}\\ \text{and } x(t_j^-) = x(t_j),\ j = 1,\dots,k,\ x(0) = x(2\pi)\},$$

we define the norm as

$$\|x\|_X = \sup_{t\in B}\left(\sum_{i=1}^{2} x_i(t)^2\right)^{\frac{1}{2}},$$

where $B = [0,2\pi]\backslash\{t_1, t_2, \ldots, t_k\}$.

The Banach space X is reflexive. Indeed, for any $x \in X$, since the interval $[0,2\pi]$ is measurable, $x(t)$ is a measurable function in it. It is obvious that $\int_0^{2\pi} x^2(t)dt < \infty$ which implies $X \subset L^2[0,2\pi]$. $L^2[0,2\pi]$ is a reflexive Banach space and so is X.

To establish the existence of a periodic solution to (2.7), we make use the homotopy invariance property of Leray–Schauder degree. Consider the following equation

$$\begin{cases} u''(t) + (n^2 - (1-\lambda)\delta)u(t) = \lambda p(t), \quad t \in [0,2\pi],\ t \neq t_j; \\ \Delta u(t_j) = \lambda J_j(u'(t_j^-)), \quad \Delta u'(t_j) = \lambda Q_j(u(t_j^-)), \quad j = 1, \ldots, k; \\ u(0) = u(2\pi), \quad u'(0) = u'(2\pi). \end{cases} \tag{2.8}$$

and the corresponding homotopy $F_\lambda : X \to X$, defined as

$$F_\lambda x = x(0) + \frac{2\pi - t}{2\pi}\left(\int_0^{2\pi} (\lambda P(s) + A_\lambda x(s))ds + \sum_{j=1}^{k} \lambda I_j(x(t_j^-))\right)$$
$$+ \int_0^t (\lambda P(s) + A_\lambda x(s))ds + \sum_{t>t_j}^{k} \lambda I_j(x(t_j^-)),$$

where $A_\lambda = \begin{pmatrix} 0 & 1 \\ -(n^2-(1-\lambda)\delta) & 0 \end{pmatrix}$, $\lambda \in [0,1]$ and $\delta \in (0, \frac{1}{2})$ is sufficiently small so that

$$(n-1)^2 < n^2 - (1-\lambda)\delta < n^2.$$

It is well known that the 2π-periodic solutions of (2.8) correspond to the fixed points of the operator F_λ.

Indeed, $F_\lambda(M)$ is an equicontinuous family of functions. Let $M \subset X$ be a bounded set. For any $x \in M$, $F_\lambda(x(t))$ is continuous, for $t \neq t_j$, $j = 1, \ldots, k$. Thus, $F_\lambda(x(t))$ is uniformly continuous on each closed subinterval with $0, t_1, \ldots, t_k, 2\pi$ as the end point. For any $x \in M$, $x(t)$ is bounded since $x(t)$ is continuous in $[0,2\pi]$, $t \neq t_j$. Clearly, $P(t)$ is bounded. Note that the boundedness of $J_j(x(t))$ and $Q_j(x(t))$ implies the boundedness of $I_j(x(t))$, $j = 1, \ldots, k$. We conclude from (2.8) that $F_\lambda x$ is bounded and hence uniformly bounded. By the Arzelà–Ascoli's theorem, we conclude that $F_\lambda(M)$ is relatively compact and, in consequence, the operator F_λ is completely continuous.

It remains to show that there exists $R_0 > 0$, such that for all $R \geq R_0$ and $\lambda \in [0,1]$, if $x \in X$, $\|x\| = R$, then $x - F_\lambda x \neq 0$.

We perform the proof via contradiction. Assume that there exist sequences $x_m \in X$, $\|x_m\| \to \infty$, $m \to \infty$ and $\lambda_m \in [0,1]$ such that $x_m = F_{\lambda_m} x_m$. Define $y_m = \frac{x_m}{\|x_m\|}$. Then the equation is equivalent to

$$F_{\lambda_m} y_m = \frac{x(0)}{\|x_m\|} + \frac{2\pi - t}{2\pi}\left(\int_0^{2\pi} \left(\lambda_m \frac{P(s)}{\|x_m\|} + A_{\lambda_m} y_m(s)\right) ds + \sum_{j=1}^{k} \lambda_m \frac{I_j(x_m(t_j^-))}{\|x_m\|}\right)$$
$$+ \int_0^t \left(\lambda_m \frac{P(s)}{\|x_m\|} + A_{\lambda_m} y_m(s)\right) ds + \sum_{t>t_j}^{k} \lambda_m \frac{I_j(x_m(t_j^-))}{\|x_m\|} = y_m. \tag{2.9}$$

Clearly, $\frac{x(0)}{\|x_m\|} \to 0$ and $\frac{P(t)}{\|x_m\|} \to 0$. By (H_1), we see that $\frac{J_j(u_m'(t_j^-))}{\|x_m\|} \to 0$, $\frac{Q_j(u_m(t_j^-))}{\|x_m\|} \to 0$ and so $\frac{I_j(x_m(t_j^-))}{\|x_m\|} \to 0$.

Since the Banach space X is reflexive and the boundedness of $\{y_m\}$, there exists a weakly convergence subsequence which is still denoted by $\{y_m\}$. Assume that $y_m \rightharpoonup y \in X$ and $\lambda_m \to \lambda \in [0,1]$. Since F_λ is compact, we have $F_\lambda y_m \to F_\lambda y$. It then follows from (2.9) that $y_m \to y$ strongly in X.

Passing to the limit as $m \to \infty$, we obtain from (2.9) that

$$F_\lambda y = \frac{2\pi - t}{2\pi} \int_0^{2\pi} A_\lambda y(t)dt + \int_0^t A_\lambda y(t)dt = y,$$

and equivalent to

$$\begin{cases} u'' + (n^2 - (1-\lambda)\delta)u = 0, \\ u(0) = u(2\pi), \quad u'(0) = u'(2\pi). \end{cases}$$

where $y = (u, u')$, $\|y\| = 1$, $\lambda \in [0,1]$. By the fact that $(n-1)^2 < n^2 - (1-\lambda)\delta < n^2$, the above equation has a solution if and only if $\lambda = 1$ and $u(t) = c_0 \sin(nt + \theta_0)$ where $c_0 > 0$, $\theta_0 \in \mathbb{R}$. The condition $c_0 > 0$ can be ensured by choosing an appropriate θ_0. Therefore, we obtain

$$\frac{u_m}{\|x_m\|} \to c_0 \sin(nt + \theta_0), \qquad m \to \infty.$$

We have

$$u_m''(t) + (n^2 - (1-\lambda_m)\delta)u_m(t) = \lambda_m p(t), \tag{2.10}$$

$$u_m(t_j^+) - u_m(t_j^-) = \lambda_m J_j(u_m'(t_j^-)),\ u_m'(t_j^+) - u_m'(t_j^-) = \lambda_m Q_j(u_m(t_j^-)),\ j = 1, \dots, k, \tag{2.11}$$

which follows from $x_m - F_{\lambda_m} x_m = 0$.

Choose the test function $v : [0, 2\pi] \to R$ that is continuously differentiable and satisfies $v(0) = v(2\pi)$, $v'(0) = v'(2\pi)$. Multiplying equation (2.10) by $v(t)$ and integrating over B, we get

$$\int_0^{2\pi} u_m''(t)v(t)dt + (n^2 - (1-\lambda_m)\delta) \int_0^{2\pi} u_m(t)v(t)dt = \int_0^{2\pi} \lambda_m p(t)v(t)dt, \tag{2.12}$$

where

$$\int_0^{2\pi} u_m''(t)v(t)dt = \int_0^{t_1^-} u_m''(t)v(t)dt + \sum_{j=1}^{k-1} \int_{t_j^+}^{t_{j+1}^-} u_m''(t)v(t)dt + \int_{t_k^+}^{2\pi} u_m''(t)v(t)dt.$$

Then, using integration by parts, we obtain

$$\begin{aligned} \int_{t_j^+}^{t_{j+1}^-} u_m''(t)v(t)dt &= u_m'(t)v(t) \,|_{t_j^+}^{t_{j+1}^-} - \int_{t_j^+}^{t_{j+1}^-} u_m'(t)v'(t)dt \\ &= u_m'(t_{j+1}^-)v(t_{j+1}) - u_m'(t_j^+)v(t_j) - u_m(t_{j+1}^-)v'(t_{j+1}) \\ &\quad + u_m(t_j^+)v'(t_j) + \int_{t_j^+}^{t_{j+1}^-} u_m(t)v''(t)dt. \end{aligned} \tag{2.13}$$

Combining (2.11), (2.12) and (2.13), we obtain

$$\begin{aligned} &\int_0^{2\pi} u_m(t)v''(t)dt + (n^2 - (1-\lambda_m)\delta) \int_0^{2\pi} u_m(t)v(t)dt + \lambda_m \sum_{j=1}^{k} J_j(u'(t_j^-))v'(t_j) \\ &\quad - \lambda_m \sum_{j=1}^{k} Q_j(u_m(t_j^-))v(t_j) = \int_0^{2\pi} \lambda_m p(t)v(t)dt. \end{aligned}$$

Now, choose $v(t) = c_0 \sin(nt + \theta_0)$, then,

$$\begin{aligned} &-(1-\lambda_m)\delta \int_0^{2\pi} u_m(t)\sin(nt+\theta_0)dt + n\lambda_m \sum_{j=1}^{k} J_j(u'_m(t_j^-))\cos(nt_j+\theta_0) \\ &\quad -\lambda_m \sum_{j=1}^{k} Q_j(u_m(t_j^-))\sin(nt_j+\theta_0) = \int_0^{2\pi} \lambda_m p(t)\sin(nt+\theta_0)dt. \end{aligned} \tag{2.14}$$

Notice that $\dfrac{u_m}{\|x_m\|} \to c_0 \sin(nt+\theta_0),\ c_0 > 0$, which implies that

$$\int_0^{2\pi} u_m(t)\sin(nt+\theta_0)dt > 0, \qquad m \gg 1. \tag{2.15}$$

From (2.14) and (2.15), it follows that

$$\int_0^{2\pi} p(t)\sin(nt+\theta_0)dt \le n\sum_{j=1}^{k} J_j(u'_m(t_j^-))\cos(nt_j+\theta_0) - \sum_{j=1}^{k} Q_j(u_m(t_j^-))\sin(nt_j+\theta_0).$$

Passing to the limit as $m \to \infty$, we finally obtain

$$\begin{aligned} \int_0^{2\pi} p(t)\sin(nt+\theta_0)dt \le{}& n\sum_{j=1}^{k} J_j(+\infty)\cos^+(nt_j+\theta_0) - n\sum_{j=1}^{k} J_j(-\infty)\cos^-(nt_j+\theta_0) \\ &-\sum_{j=1}^{k} Q_j(+\infty)\sin^+(nt_j+\theta_0) + \sum_{j=1}^{k} Q_j(-\infty)\sin^-(nt_j+\theta_0), \end{aligned}$$

a contradiction with the assumption (H_2).

We can now choose an appropriate $R > 0$, such that $x - F_\lambda x \neq 0$, for $x \in X$ with $\|x\| = R$. Define $B_R = \{x \in X \mid \|x\| < R\}$. We say that $x(t) \in \partial B_R$ if there exists a $t_0 \in [0, 2\pi]$, such that $x(t_0) = R$.

Let $x_1(t) = r\cos\varphi,\ x_2(t) = r\sin\varphi$, we have

$$|r'(t)| = \left|\frac{x_1x'_1 + x_2x'_2}{\sqrt{x_1^2+x_2^2}}\right| = \left|\frac{(1-n^2+(1-\lambda)\delta)x_1(t)x_2(t) + \lambda p(t)x_2(t)}{\sqrt{x_1^2+x_2^2}}\right| \le n^2 r(t) + P,$$

where $P = \max_{t\in[0,2\pi]}|p(t)|$, which implies that for sufficiently large $x_1^2 + x_2^2$, there exists a $B_P > 0$, such that

$$(r(t_0)+B_P)\mathrm{e}^{-n^2(t-t_0)} \le r(t) + B_P \le (r(t_0)+B_P)\mathrm{e}^{n^2(t-t_0)}, \qquad t \in [t_0, t_0+2\pi].$$

This shows that $x(t_0) \to +\infty \Leftrightarrow \|x\| \to +\infty$.

To summarize: there exists a $B_R > 0$ such that for each $x(t) \in \partial B_R$, $x - F_\lambda x \neq 0$, $\lambda \in [0,1]$. Now, by the Theorem 3 of [11], we have

$$\deg(I - F_1; B_R, 0) = \deg(I - F_0; B_R, 0) = (-1)^2 \deg_{\mathrm{B}}(A_0|_{\mathbb{R}^2}; B_R \cap \mathbb{R}^2, 0),$$

where $\deg(I - F_0; B_R, 0)$ denotes the Leray–Schauder degree, and $\deg_{\mathrm{B}}(A_0|_{\mathbb{R}^2}; B_R \cap \mathbb{R}^2, 0)$ denotes the Brouwer degree.

Let

$$A_0x : \mathbb{R}^2 \to \mathbb{R}^2, \qquad (x_1, x_2) \longmapsto (x_2, -(n^2-\delta)x_1),$$

where $x = (x_1, x_2)^T$.

It is easy to verify that $A_0 x$ has exactly one zero in $B_R \cap \mathbb{R}^2$. Therefore,

$$\deg_B(A_0|_{\mathbb{R}^2}; B_R \cap \mathbb{R}^2, 0) = \operatorname{sgn}(\det(A_0)) = \operatorname{sgn}(n^2 - \delta) = 1,$$

and consequently,

$$\deg(I - F_1; B_R, 0) = \deg(I - F_0; B_R, 0) = \deg_B(I - F_0; B_R, 0) = 1.$$

Hence, the original system has at least one 2π-periodic solution.

Finally, we have the following result.

Proposition 2.1. *Assume that* (H_1) *holds. Then* (2.7) *has a* 2π*-periodic solution if and only if* $p(t)$ *satisfies* (H_2).

Remark 2.2. Suppose that

(H_1') For $j = 1, \ldots, k$, there exist limits

$$\lim_{s \to \pm\infty} J_j(s) = J_j(\pm\infty), \qquad \lim_{s \to \pm\infty} Q_j(s) = Q_j(\pm\infty),$$

and the following inequalities hold:

$$J_j(-\infty) < J_j(s) < J_j(+\infty), \quad Q_j(+\infty) < Q_j(s) < Q_j(-\infty).$$

(H_2') For any $\theta \in \mathbb{R}$, $p(t)$ satisfies

$$\begin{aligned}
&n \sum_{j=1}^{k} J_j(-\infty) \cos^+(nt_j + \theta) - n \sum_{j=1}^{k} J_j(+\infty) \cos^-(nt_j + \theta) \\
&\quad + \sum_{j=1}^{k} Q_j(+\infty) \sin^-(nt_j + \theta) - \sum_{j=1}^{k} Q_j(-\infty) \sin^+(nt_j + \theta) \\
&< \int_0^{2\pi} \sin(nt + \theta) p(t) dt \\
&< n \sum_{j=1}^{k} J_j(+\infty) \cos^+(nt_j + \theta) - n \sum_{j=1}^{k} J_j(-\infty) \cos^-(nt_j + \theta) \\
&\quad - \sum_{j=1}^{k} Q_j(+\infty) \sin^+(nt_j + \theta) + \sum_{j=1}^{k} Q_j(-\infty) \sin^-(nt_j + \theta).
\end{aligned}$$

Proposition 2.3. *Assume that* (H_1') *holds. Then* (2.7) *has a* 2π*-periodic solution if and only if* $p(t)$ *satisfies* (H_2').

The proof of this proposition can be completed using a method similar to the one presented above.

2.3 The existence of unbounded solutions

This section is to prove that all solutions of (2.7) are unbounded when the periodic solution does not exist, namely

Proposition 2.4. *Assume that (H_1) holds. There exists $\theta_0 \in \mathbb{R}$ such that $p(t)$ satisfies either*

$$\begin{aligned}\int_0^{2\pi} p(t)\sin(nt+\theta_0)dt \geq\ & n\sum_{j=1}^{k} J_j(-\infty)\cos^+(nt_j+\theta_0) - n\sum_{j=1}^{k} J_j(+\infty)\cos^-(nt_j+\theta_0)\\ & + \sum_{j=1}^{k} Q_j(+\infty)\sin^-(nt_j+\theta_0) - \sum_{j=1}^{k} Q_j(-\infty)\sin^+(nt_j+\theta_0),\end{aligned}$$

or

$$\begin{aligned}\int_0^{2\pi} p(t)\sin(nt+\theta_0)dt \leq\ & n\sum_{j=1}^{k} J_j(+\infty)\cos^+(nt_j+\theta_0) - n\sum_{j=1}^{k} J_j(-\infty)\cos^-(nt_j+\theta_0)\\ & - \sum_{j=1}^{k} Q_j(+\infty)\sin^+(nt_j+\theta_0) + \sum_{j=1}^{k} Q_j(-\infty)\sin^-(nt_j+\theta_0).\end{aligned}$$

Then all solutions of (2.7) are unbounded (in the phase plane).

Proof. We denote by $u(t)$ a solution of (2.7). Setting $x(t) = (u(t), u'(t))^T$. Then for any $m \in \mathbb{N}$, by (2.4) we get

$$\begin{aligned}nu(2m\pi) &= nu(0) - \int_0^{2m\pi} p(t)\sin ntdt + n\sum_{j=1}^{mk} J_j(u'(t_j^-))\cos nt_j - \sum_{j=1}^{mk} Q_j(u(t_j^-))\sin nt_j,\\ u'(2m\pi) &= u'(0) + \int_0^{2m\pi} p(t)\cos ntdt + n\sum_{j=1}^{mk} J_j(u'(t_j^-))\sin nt_j + \sum_{j=1}^{mk} Q_j(u(t_j^-))\cos nt_j.\end{aligned}$$

Multiply the two equations above by $-\cos\theta_0, \sin\theta_0$, respectively, and add them together to get

$$\begin{aligned}&u'(2m\pi)\sin\theta_0 - nu(2m\pi)\cos\theta_0\\ &\quad= u'(0)\sin\theta_0 - nu(0)\cos\theta_0 + \int_0^{2m\pi} p(t)\sin(nt+\theta_0)dt\\ &\qquad - n\sum_{j=1}^{mk} J_j(u'(t_j^-))\cos(nt_j+\theta_0) + \sum_{j=1}^{mk} Q_j(u(t_j^-))\sin(nt_j+\theta_0).\end{aligned}$$

Hence, we immediately deduce that

$$\begin{aligned}&\left|u'(2m\pi)\sin\theta_0 - nu(2m\pi)\cos\theta_0\right|\\ &\quad\geq \left|\int_0^{2m\pi} p(t)\sin(nt+\theta_0)dt - n\sum_{j=1}^{mk} J_j(u'(t_j^-))\cos(nt_j+\theta_0) + \sum_{j=1}^{mk} Q_j(u(t_j^-))\sin(nt_j+\theta_0)\right|\\ &\qquad - \left|u'(0)\sin\theta_0 - nu(0)\cos\theta_0\right|.\end{aligned}$$

Note that

$$\int_0^{2m\pi} p(t)\sin(nt+\theta_0)dt = m\int_0^{2\pi} p(t)\sin(nt+\theta_0)dt,$$

since $p(t)\sin(nt+\theta_0)$ is 2π-periodic.

Assume that the $u(t)$ is bounded. By (H_1), for all t_j, $j = 1, 2, \ldots$, we have

$$J_j(+\infty) < J_j(u'(t_j^-)) < J_j(-\infty), \qquad Q_j(-\infty) < Q_j(u(t_j^-)) < Q_j(+\infty).$$

and

$$\xi_0 = \min\left\{ n\sum_{j=1}^{k} |J_j(\pm\infty) - J_j(u'(t_j^-))|,\ \sum_{j=1}^{k} |Q_j(\pm\infty) - Q_j(u(t_j^-))| \right\} > 0.$$

For $t \in [2l\pi, 2(l+1)\pi]$, $l = 1, 2, \ldots, m-1$,

$$\begin{aligned}
\int_0^{2\pi} \sin(nt+\theta_0)p(t)dt &= \int_{2l\pi}^{2(l+1)\pi} \sin(nt+\theta_0)p(t)dt \\
&\geq n\sum_{j=(l-1)k}^{lk} J_j(-\infty)\cos^+(nt_j+\theta_0) - n\sum_{j=(l-1)k}^{lk} J_j(+\infty)\cos^-(nt_j+\theta_0) \\
&\quad + \sum_{j=(l-1)k}^{lk} Q_j(+\infty)\sin^-(nt_j+\theta_0) - \sum_{j=(l-1)k}^{lk} Q_j(-\infty)\sin^+(nt_j+\theta_0) \\
&> n\sum_{j=(l-1)k}^{lk} J_j(u'(t_j^-))\cos(nt_j+\theta_0) - \sum_{j=(l-1)k}^{lk} Q_j(u(t_j^-))\sin(nt_j+\theta_0),
\end{aligned}$$

it follows that

$$\int_0^{2\pi} p(t)\sin(nt+\theta_0)dt - n\sum_{j=(l-1)k}^{lk} J_j(u'(t_j^-))\cos(nt_j+\theta_0) + \sum_{j=(l-1)k}^{lk} Q_j(u(t_j^-))\sin(nt_j+\theta_0) > \xi_0.$$

Adding the m inequalities above, we get

$$m\int_0^{2\pi} p(t)\sin(nt+\theta_0)dt - n\sum_{j=1}^{mk} J_j(u'(t_j^-))\cos(nt_j+\theta_0) + \sum_{j=1}^{mk} Q_j(u(t_j^-))\sin(nt_j+\theta_0) > m\xi_0,$$

therefore,

$$|u'(2m\pi)\sin\theta_0 - nu(2m\pi)\cos\theta_0| \geq m\xi_0 - |u'(0)\sin\theta_0 - nu(0)\cos\theta_0|.$$

Since

$$n\|x(2m\pi)\|_X \geq n(u^2(2m\pi) + u'^2(2m\pi))^{\frac{1}{2}} \geq |u'(2m\pi)\sin\theta_0 - nu(2m\pi)\cos\theta_0|,$$

which yields

$$n||x(2m\pi)||_X \geq m\xi_0 - |u'(0)\sin\theta_0 - nu(0)\cos\theta_0|,$$

hence

$$n\|x(2m\pi)\|_X \to +\infty, \qquad m \to +\infty,$$

which implies that $x(t)$ is unbounded in $[0, +\infty)$. □

By combining Proposition 2.1 and Proposition 2.4, we have thus proved the Theorem 1.1.

3 Coexistence of periodic and unbounded solutions

In this section, we consider the coexistence of periodic and unbounded solutions of the following weakly nonlinear impulsive differential equations

$$\begin{cases} u''(t) + n^2u(t) + h(u(t)) = p(t), \quad t \in [0, 2\pi],\ t \neq t_j; \\ \Delta u(t_j) = J_j(u(t_j^-), u'(t_j^-)), \quad \Delta u'(t_j) = Q_j(u(t_j^-), u'(t_j^-)), \quad j = 1, \ldots, k; \end{cases} \tag{3.1}$$

where $0 < t_1 < \cdots < t_k < 2\pi$, $p(t)$ is 2π-periodic and continuous, h is continuous and J_j, $Q_j : \mathbb{R}^2 \to \mathbb{R}$, $j = 1, \ldots, k$, are continuous. Assume that (D_1), (D_2) hold.

Let $P(t) = (0, p(t) - h(u(t)))^T$ and initial value $(u(\tau;\tau,w), u'(\tau;\tau,w)) = (0, w)$. We conclude from (2.4) that

$$\begin{aligned} u(t) =& \frac{w}{n}\sin(nt - n\tau) + \frac{1}{n}\int_\tau^t \sin(nt - ns)(p(s) - h(u(s)))ds \\ &+ \sum_{t>t_j} J_j(u(t_j^-), u'(t_j^-))\cos(nt - nt_j) + \frac{1}{n}\sum_{t>t_j} Q_j(u(t_j^-), u'(t_j^-))\sin(nt - nt_j), \\ u'(t) =& w\cos(nt - n\tau) + \int_\tau^t \cos(nt - ns)(p(s) - h(u(s)))ds \\ &- n\sum_{t>t_j} J_j(u(t_j^-), u'(t_j^-))\sin(nt - nt_j) + \sum_{t>t_j} Q_j(u(t_j^-), u'(t_j^-))\cos(nt - nt_j). \end{aligned} \tag{3.2}$$

The proof of Theorem 1.4 is presented as following. We divide the proof in two steps. The first one is to prove the existence of the periodic solution using the successor map. The second step is to construct an auxiliary Lyapunov function that makes the solution grow infinitely along the orbit, which is motivated by [1]. So that the solution is unbounded.

3.1 The existence of a periodic solution

We assume that $h(u)$ is locally Lipschitzian which guarantees the uniqueness of the solutions for the initial value problem associated with (3.1), if necessary. Write $g(u) = n^2u + h(u)$, $G(u) = \int_0^u g(s)ds$. Consider the following equation

$$\begin{cases} u' = v, \\ v' = -g(u) + \lambda p(t), \quad t \in [0, 2\pi],\ t \neq t_j; \\ \Delta u(t_j) = \lambda J_j(u(t_j^-), v(t_j^-)), \\ \Delta v(t_j) = \lambda Q_j(u(t_j^-), v(t_j^-)),\ j = 1, \ldots, k, \end{cases} \tag{3.3}$$

where $\lambda \in [0, 1]$. Notice that (3.3) is equivalent to (3.1) when $\lambda = 1$.

Let $(u(t;\tau,w), v(t;\tau,w))$ denote the solution of (3.3) with the initial point $(0, w)$, $w \gg 1$. We define τ_1 as the time when $(u(t;\tau,w), v(t;\tau,w))$ intersects the positive v-axis again, and write $w(\tau_1;\tau,w) = w_1$.

The successor map is defined as $S : (\tau, v) \mapsto (\tau_1, w_1)$. S is well defined for $|w| \gg 1$, and the uniqueness of the solution for the initial value problem guarantees that S is continuous and one to one. The periodicity of $p(t)$ implies that $S(\tau + 2\pi, w) = S(\tau, w) + (2\pi, 0)$.

By the boundedness of J_j, Q_j and h, we have

Lemma 3.1. $\tau_1 - \tau = \frac{2\pi}{n} + O(\frac{1}{r})$.

Now, let $u(t) = u(t;\tau,v)$, $v(t) = v(t;\tau,v)$, $H(t) = \frac{1}{2}v^2(t) + G(u(t))$, and $l(t) = \sqrt{2H(t)}$. Then we have

$$|l'(t)| = \left|\frac{v(t)}{l(t)}p(t)\right| \leq |p(t)| < P, \qquad t \neq t_j,\ j = 1, \ldots, k,$$

where $P = \max_{t\in[0,2\pi]}\{|p(t)|\}$.

From this, it follows that

$$|l(t) - l(t')| \leq P|t - t'|,$$

when $t, t' \in (t_j, t_{j+1})$, $j = 1, \ldots, k-1$, or $t, t' \in (0, t_1)$, or $t, t' \in (t_k, 2\pi)$.

When $t = t_j$, by (D_2), $u(t_j^+) - u(t_j^-) = O(\frac{1}{r})$ and $v(t_j^+) - v(t_j^-) = O(\frac{1}{r})$ for large w. Then for sufficiently small $\varepsilon > 0$, there exists $w_0' > 0$, such that for $w > w_0'$, $u(t_j^+) = u(t_j^-) \pm \varepsilon, v(t_j^+) = v(t_j^-) \pm \varepsilon$. Therefore,

$$\begin{aligned} H(t_j^+) &= \frac{1}{2}v^2(t_j^+) + G(u(t_j^+)) = \frac{1}{2}(v(t_j^-) \pm \varepsilon)^2 + G(u(t_j^-) \pm \varepsilon) \\ &= H(t_j^-) + \varepsilon_1, \end{aligned}$$

where $\varepsilon_1 = \pm v(t_j^-)\varepsilon + \int_{u(t_j^-)}^{u(t_j^-)\pm\varepsilon} g(s)ds + \varepsilon^2$.

In conclusion, for sufficiently large $|l(\tau)|$, we have

$$|l(t) - l(\tau)| \le 2(P+1)|t - \tau|.$$

Thus, we obtain the following

Lemma 3.2.

$$|l(t) - w| \le 2(P+1)T, \qquad |t - \tau| \le T,\ w \ge w_0.$$

Further, we can prove the following lemma.

Lemma 3.3.

$$w_1 - w = \int_\tau^{\tau + \frac{2\pi}{n}} \cos(ns - n\tau)p(s) + O\left(\frac{1}{r}\right), \qquad w \gg 1;$$

$$\begin{aligned} \tau_1 - \tau = {} & \frac{\pi}{n} - \frac{1}{nw}\int_\tau^{\tau+\frac{2\pi}{n}} \sin(n\tau - ns)(h - p(s))ds \\ & - \frac{1}{w}\lambda \sum_{t>t_j} J_j(u(t_j^-), u'(t_j^-)) \cos(n\tau_1 - nt_j) \\ & - \frac{1}{nw}\lambda \sum_{t>t_j} Q_j(u(t_j^-), u'(t_j^-)) \sin(nt - nt_j) + O\left(\frac{1}{r^2}\right), \qquad w \gg 1. \end{aligned}$$

Proof. Without loss of generality, we can assume that there are $k_1 \le k$ impulses in the interval $[\tau, \tau_1]$. The corresponding impulse times are denoted by $t_1, t_2, \ldots, t_{k_1}$.

Step 1. We estimate $w_1 - w$.

By the nature of definite integral, we have

$$w_1 - w = l(\tau_1) - l(\tau) = \int_\tau^{\tau_1} \frac{v(s)p(s)}{l(s)}ds + \sum_{j=1}^{k_1} l(t_j^+) - l(t_j^-).$$

We now begin by estimating $\sum_{j=1}^{k_1} l(t_j^+) - l(t_j^-)$. By definition, we have

$$\begin{aligned} l(t_j^+) - l(t_j^-) &= \sqrt{v^2(t_j^+) + 2G(u(t_j^+))} - \sqrt{v^2(t_j^-) + 2G(u(t_j^-))} \\ &= \frac{(v(t_j^+) + v(t_j^-))(v(t_j^+) - v(t_j^-)) + 2\int_{u(t_j^-)}^{u(t_j^+)} g(s)ds}{\sqrt{v^2(t_j^+) + 2G(u(t_j^+))} + \sqrt{v^2(t_j^-) + 2G(u(t_j^-))}}. \end{aligned}$$

According to the mean value theorem for integrals and (i) of (D_1), there exists M_j such that $\int_{u(t_j^-)}^{u(t_j^+)} h(s)ds = M_j(u(t_j^+) - u(t_j^-))$. Hence, we have

$$\int_{u(t_j^-)}^{u(t_j^+)} g(s)ds = \int_{u(t_j^-)}^{u(t_j^+)} n^2 u + h(s)ds = \left(\frac{1}{2}n^2(u(t_j^+) + u(t_j^-)) + M_j\right)(u(t_j^+) - u(t_j^-)\Big),$$

it follows that

$$\begin{aligned}\sum_{j=1}^{k_1} l(t_j^+) - l(t_j^-) = &\sum_{j=1}^{k_1} \frac{(v(t_j^+) + v(t_j^-))\lambda Q_j(u(t_j^-), u'(t_j^-))}{\sqrt{v^2(t_j^+) + 2G(u(t_j^+))} + \sqrt{v^2(t_j^-) + 2G(u(t_j^-))}} \\ &+ \frac{2(\frac{1}{2}n^2(u(t_j^+) + u(t_j^-)) + M_j)\lambda J_j(u(t_j^-), u'(t_j^-))}{\sqrt{v^2(t_j^+) + 2G(u(t_j^+))} + \sqrt{v^2(t_j^-) + 2G(u(t_j^-))}}.\end{aligned}$$

This implies that

$$\sum_{j=1}^{k_1} l(t_j^+) - l(t_j^-) = O\left(\frac{1}{r}\right), \qquad w \gg 1, \tag{3.4}$$

by (D_2).

Combining (3.4), Lemma 3.1 and Lemma 3.2, the further estimate for $w_1 - w$ is given by

$$\begin{aligned}w_1 - w &= \int_\tau^{\tau_1} \frac{v(s)p(s)}{l(s)}ds + O\left(\frac{1}{r}\right) \\ &= \int_\tau^{\tau_1} \left(w\cos(ns - n\tau) + \int_\tau^s (\cos(ns - n\xi)(p(\xi) - h))\, d\xi\right)\frac{p(s)}{l(s)}ds + O\left(\frac{1}{r}\right) \\ &= \int_\tau^{\tau_1} \frac{w\cos(ns - n\tau)p(s)}{l(s)}ds + O\left(\frac{1}{r}\right) \\ &= \int_\tau^{\tau + \frac{2\pi}{n}} \cos n(s - \tau)p(s)ds + O\left(\frac{1}{r}\right), \qquad w \gg 1.\end{aligned}$$

Step 2. The estimation of $\tau_1 - \tau$ is given below.

By the definition of τ_1 and (3.2), we have

$$\begin{aligned}\frac{w}{n}\sin(n\tau_1 - n\tau) = &\frac{1}{n}\int_\tau^{\tau_1} \sin(n\tau_1 - ns)(h - p(s))ds \\ &- \lambda\sum_{t>t_j} J_j(u(t_j^-), u'(t_j^-))\cos(n\tau_1 - nt_j) \\ &- \frac{\lambda}{n}\sum_{t>t_j} Q_j(u(t_j^-), u'(t_j^-))\sin(nt - nt_j).\end{aligned}$$

Using the Taylor expansion, we obtain

$$\begin{aligned}\frac{w}{n}\sin(n\tau_1 - n\tau) &= \frac{w}{n}\sin n\left(\frac{\pi}{n} - \tau_1 + \tau\right) \\ &= w\left(\frac{\pi}{n} - \tau_1 + \tau + O\left(\frac{1}{r^3}\right)\right).\end{aligned}$$

It follows from Lemma 3.1 that

$$\begin{aligned}\tau_1 - \tau = &\frac{\pi}{n} - \frac{1}{nw}\int_\tau^{\tau + \frac{2\pi}{n}} \sin(n\tau - ns)(h - p(s))ds - \frac{1}{w}\lambda\sum_{t>t_j} J_j(u(t_j^-), u'(t_j^-))\cos(n\tau_1 - nt_j) \\ &- \frac{1}{nw}\lambda\sum_{t>t_j} Q_j(u(t_j^-), u'(t_j^-))\sin(nt - nt_j) + O\left(\frac{1}{r^2}\right), \qquad w \gg 1.\end{aligned}$$

The proof of Lemma 3.2 is then completed. □

Let

$$\phi(t-\tau) = \frac{1}{n}\sin(nt-n\tau),$$

$$\psi(t) = \frac{1}{n}\int_\tau^t \sin(nt-ns)(p(s)-h)ds + \sum_{t>t_j} J_j(u(t_j^-),u'(t_j^-))\cos(nt-nt_j)$$

$$+\frac{1}{n}\sum_{t>t_j} Q_j(u(t_j^-),u'(t_j^-))\sin(nt-nt_j).$$

It is easy to observe that $\psi(t)$ is bounded by (D_1) and (D_2).

Lemma 3.4. *Assume that (ii) of (D_1) holds. For any $K>0$, there exists $w_0 = w(K) > 0$, such that*

$$\int_\tau^{\tau+\frac{2\pi}{n}} h(w\phi(s-\tau)+\psi(s))\sin n(s-\tau)ds \geq \frac{K}{w}, \qquad w \geq w_0.$$

Proof. Without loss of generality, we take $\tau = 0$. It is sufficient to prove the inequality:

$$\int_0^{\frac{2\pi}{n}} h\left(\frac{\omega}{n}\sin ns+\psi(s)\right)\sin nsds \geq \frac{K}{w}, \qquad w \geq w_0.$$

By definition of ψ, $\psi(t)$ is bounded and $\psi(t)\to 0$ as $t\to 0$.

By condition (D_2), there exists $\delta = O(\frac{1}{r}) > 0$, such that, for $\frac{2\pi}{n} > t > \frac{2\pi}{n}-\delta$, $h(\frac{\omega}{n}\sin nt+\psi(t))\sin nt > 0$, which together with (D_1) yields

$$\int_0^{\frac{2\pi}{n}} h\left(\frac{\omega}{n}\sin ns+\psi(s)\right)\sin nsds \geq \int_\delta^{\frac{2\pi}{n}-\delta} h\left(\frac{\omega}{n}\sin ns+\psi(s)\right)\sin nsds - O\left(\frac{1}{r}\right).$$

Introducing the transformation $\frac{\omega}{n}\sin nt+\psi(t) = \frac{\omega}{n}\sin n\xi$. Then, there exists $\omega \gg 1$, such that

$$h\left(\frac{\omega}{n}\sin n\xi\right)\frac{\omega}{n}\sin n\xi \geq 4M_h.$$

Therefore,

$$\begin{aligned}\int_\delta^{\frac{2\pi}{n}-\delta} h\left(\frac{\omega}{n}\sin ns+\psi(s)\right)\sin nsds &= \frac{1}{\omega}\int_{\xi^-}^{\xi^+} h\left(\frac{\omega}{n}\sin n\xi\right)\frac{\omega\sin n\xi - n\psi(t)}{\omega\cos nt+\psi'(t)}\omega\cos n\xi d\xi\\ &\geq \frac{n}{4\omega}\int_{\xi^-}^{\xi^+} h\left(\frac{\omega}{n}\sin n\xi\right)\frac{\omega}{n}\sin n\xi\cos n\xi d\xi\\ &\geq \frac{\bar{M}M_h}{\omega} := \frac{K}{\omega}\end{aligned}$$

where $\bar{M} = \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}}\cos n\xi d(n\xi)$. □

Introduce the Banach space

$$X = \{x:\ [0,2\pi]\to\mathbb{R}^2:\ x(t) \text{ is continuous for } t\neq t_j,\ x(t_j^-),\ x(t_j^+)\text{exist,}\\ \text{and } x(t_j^-) = x(t_j) j = 1,\dots,k,\ x(0) = x(2\pi)\},$$

and the operator $F_\lambda : X \to X$

$$F_\lambda x = x(0) + \frac{2\pi - t}{2\pi}\left(\int_0^{2\pi}(\lambda P(t) + Ax(t) - H(x(t)))dt + \lambda\sum_{j=1}^{k} I_j(x(t_j^-))\right)$$
$$+ \int_0^t (\lambda P(s) + Ax(s) - H(x(s)))ds + \lambda\sum_{t>t_j} I_j(x(t_j^-)),$$

where $H(x(t)) = (0, h(u(t)))^T$. According to the above we can know F_λ is completely continuous.

It is clear that (3.3) is equivalent to a fixed point problem, that is $x = F_\lambda x$. We now apply the topological degree to prove the existence of periodic solutions. Let

$$\Sigma = \{(x, \lambda) \in X \times [0,1] \mid F_\lambda x = x\}.$$

Lemma 3.5. *If there is a bounded open set $\Omega \subset X$, such that*

$$\Sigma \cap \partial\Omega = \emptyset,$$

then $\deg(F_\lambda, \Omega, 0) = \deg(F_0, \Omega, 0) = 1$, *and thus, the equation* (3.3) *has at least one 2π-periodic solution.*

Similar to the proof of proposition 2.1, we have $\deg(F_0, \Omega, 0) = 1$. We now seek an appropriate bounded open set. Let

$$\Omega(k) = \left\{(u, v) \in X \mid \omega(u, v) = \frac{1}{2}v^2(t) + G(u(t)) < k\right\}.$$

On the one hand, $(u(t), v(t)) \in \Sigma \cap \partial(\Omega(k))$, for sufficiently large k, then there exists $t_0 \in [0, 2\pi]$, such that $\frac{1}{2}v^2(t_0) + G(u(t_0)) = k$. By Lemma 3.2, $(u(t), v(t)) \in \mathbb{R}^2$ is in a bounded set by

$$\left\{(u, v) \in X \mid k_- < \frac{1}{2}v^2(t) + G(u(t)) < k_+\right\},$$

where $\sqrt{k_\pm} = \sqrt{k} \pm 4(P+1)\pi$. Thus $(u(t), v(t))$ starts from $u(\tau) = 0$, $v(\tau) = w$, $\tau \in [0, 2\pi]$, $w \in [k_-, k_+]$. The 2π-periodicity implies that $u(\tau + 2\pi) = 0$, $v(\tau + 2\pi) = w$. Thus,

$$\tau_m = \tau + 2\pi,$$

for some $m \in \mathbb{N}$.

On the other hand, by using the variation of constant formula, we obtain $u(t) = w\phi(t) + \psi(t)$. Combining this with Lemma 3.4, we have

$$\tau_n - \tau = 2\pi - \frac{1}{nw}\int_\tau^{\tau+2\pi} \sin(n\tau_1 - ns)(h - p(s))ds + O\left(\frac{1}{w^2}\right)$$
$$\leq 2\pi - \frac{K}{w^2} + \frac{M_p'}{w} + O\left(\frac{1}{w^2}\right) < 2\pi,$$

where K is sufficiently large, M_p' is sufficiently small, such that for some k,

$$\frac{K}{w^2} - \frac{M_p'}{w} - O\left(\frac{1}{w^2}\right) > 0,$$

for $w \in [k_-, k_+]$.

Thus, we find that the bounded open set $\Omega(k)$ satisfies the condition of Lemma 3.5 and we have

Proposition 3.6. *Assume that* (D_1) *and* (D_2) *hold. Then there exists* $q_0 > 0$*, such that the equation* (1.5) *has* 2π*-periodic solution provided that* $p(t)$ *satisfies*

$$M_p < q_0,$$

where

$$M_p = \max_{\theta\in[0,2\pi]} \left| \int_0^{2\pi} p(t)\sin(nt+\theta)dt \right|.$$

3.2 The existence of unbounded solutions

Let us continue our analysis about the above equation. Given a function $p(t)$, we shown in the previous section that there exists 2π-periodic solution with

$$\max\left\{ \int_0^{2\pi} p(t)\sin(nt+\phi)dt \ : \ \phi\in\mathbb{R} \right\} > 0,$$

within a sufficiently large disk in the phase plane. Accordingly, we fix some $\phi\in\mathbb{R}$ such that

$$d_* := \frac{1}{2}\int_0^{2\pi} p(t)\sin(nt+\phi)dt > 0.$$

Let $(u(t;\zeta), u'(t;\zeta))$ be the solution of (3.1) with the initial condition $\zeta = (\omega, v)\in\mathbb{R}^2$. The associated Poincaré map is defined by $\mathcal{P} : (\omega, v) \to (u(2\pi;\zeta), u'(2\pi;\zeta))$. To investigate the existence of unbounded solutions, we introduce the auxiliary function

$$V : \ \mathbb{R}^2 \to \mathbb{R}, \qquad V(\zeta) := v\sin\phi - n\omega\cos\phi.$$

First, we verify the following property of V: there exists a constant $r_* > 0$, such that, for $|\zeta| \geq r_*$,

$$V(\mathcal{P}(\zeta)) \geq V(\zeta) + d_*. \tag{3.5}$$

Multiplying both sides of (3.1) by $\sin(nt+\phi)$ and integrating both sides at $[0, 2\pi]$, we get

$$\begin{aligned}\int_0^{2\pi} u''(t;\zeta)\sin(nt+\phi)dt = &\int_0^{2\pi} p(t)\sin(nt+\phi)dt - n^2\int_0^{2\pi} u(t;\zeta)\sin(nt+\phi)dt\\ &- \int_0^{2\pi} h(u(t;\zeta))\sin(nt+\phi)dt.\end{aligned}$$

Using integration by parts we obtain

$$\begin{aligned}\int_0^{2\pi} u''(t;\zeta)\sin(nt+\phi)dt = V(F(\zeta)) - V(\zeta) + n\sum_{j=1}^{k} J_j(u(t_j^-), u'(t_j^-))\cos(nt_j+\phi)\\ - \sum_{j=1}^{k} Q_j(u(t_j^-), u'(t_j^-))\sin(nt_j+\phi) - n^2\int_0^{2\pi} u(t;\zeta)\sin(nt+\phi)dt.\end{aligned}$$

It follows that

$$\begin{aligned}V(F(\zeta)) - V(\zeta) = &\int_0^{2\pi} p(t)\sin(nt+\phi)dt - \int_0^{2\pi} h(u(t;\zeta))\sin(nt+\phi)dt\\ &- n\sum_{j=1}^{k} J_j(u(t_j^-), u'(t_j^-))\cos(nt_j+\phi) + \sum_{j=1}^{k} Q_j(u(t_j^-), u'(t_j^-))\sin(nt_j+\phi).\end{aligned}$$

We now estimate the second term on the right-hand side of the above equation.

Lemma 3.7. *There exists a sufficiently large $r_* > 0$, such that $\zeta = (\omega, v) > r_*$ satisfying*

$$\left| \int_0^{2\pi} h(u(t;\zeta)) \sin(nt + \phi) dt \right| < \frac{d_*}{2}. \tag{3.6}$$

Proof. By (D_1), there exists $u_* > 0$, such that for $|u(t;\zeta)| > u_*$, $|h(u(t;\zeta))| < \frac{d_*}{8\pi}$. Denote the upper bound of $|h|$ by M_h.

We take the transformation

$$u(t;\zeta) = r(t;\zeta)\cos\theta(t;\zeta), \qquad u'(t;\zeta) = r(t;\zeta)\sin\theta(t;\zeta).$$

For $r(t;\zeta) \geq 1$, we have

$$|r'(t;\zeta)| \leq (n^2 + M_h) r(t;\zeta), \qquad \frac{1}{2} \leq \theta'(t;\zeta) \leq (n^2 + M_h), \qquad t \neq t_1, \ldots, t_k.$$

Analysis similar to that in the proof of Lemma 3.1 shows that, for $t \in [0, 2\pi], t \neq t_1, \ldots, t_k$,

$$\mathrm{e}^{-2(n^2+M_h)\pi} r(0;\zeta) \leq r(t;\zeta) \leq \mathrm{e}^{2(n^2+M_h)\pi} r(0;\zeta), \tag{3.7}$$

$$-2(n^2 + M_h)\pi < \theta(2\pi;\zeta) - \theta(0;\zeta) < 0. \tag{3.8}$$

Let $\delta_* > 0$ be sufficiently small such that $|\cos\theta(t;\zeta)| \leq \cos(\frac{\pi}{2} - \delta_*)$ for $\theta(t;\zeta) \in [m\pi + \frac{\pi}{2} - \delta_*, m\pi + \frac{\pi}{2} + \delta_*]$, $m \in \mathbb{Z}$. Define

$$E_* = \left\{ t \in [0, 2\pi] : \theta(t;\zeta) \in \left[m\pi + \frac{\pi}{2} - \delta_*, m\pi + \frac{\pi}{2} + \delta_* \right], m \in \mathbb{Z} \right\}.$$

By (3.8), we have $\mathrm{mes}(E_*) \leq 8(n^2 + M_h)\delta_*$ and

$$|u(t;\zeta)| \geq r(t;\zeta)\cos\left(\frac{\pi}{2} - \delta_*\right), \qquad t \in [0, 2\pi] \setminus E_*.$$

By (3.7), we take

$$\delta_* = \frac{d_*}{32(n^2 + M_h)M_h}, \qquad r = |\zeta| \geq r_* = \frac{u_* \mathrm{e}^{2(n^2+M_h)\pi}}{\cos(\frac{\pi}{2} - \delta_*)},$$

then

$$\begin{aligned}
&\left| \int_0^{2\pi} h(u(t;\zeta)) \sin(nt + \phi) dt \right| \\
&\quad \leq \left| \int_{t \in E_*} h(u(t;\zeta)) \sin(nt + \phi) dt \right| + \left| \int_{t \in [0,2\pi] \setminus E_*} h(u(t;\zeta)) \sin(nt + \phi) dt \right| \\
&\quad \leq \mathrm{mes}(E_*) M_h + 2\pi \cdot \frac{d_*}{8\pi} \leq \frac{d_*}{4} + \frac{d_*}{4} = \frac{d_*}{2}.
\end{aligned}$$

□

On the other hand, from (D_2), it follows that

$$\left| -n \sum_{j=1}^{k} J_j(u(t_j^-), u'(t_j^-)) \cos(nt_j + \phi) + \sum_{j=1}^{k} Q_j(u(t_j^-), u'(t_j^-)) \sin(nt_j + \phi) \right| \leq \frac{d_*}{2},$$

for $r(t;\zeta)$ large enough.

Combining (3.7) and the boundedness of impulses, we can take $|\zeta| \geq r_*$ large enough such that the above inequality holds. Then

$$V(\mathcal{P}(\zeta)) - V(\zeta) \geq 2d_* - \frac{d_*}{2} - \frac{d_*}{2} = d_*.$$

Furthermore, we notice that V is bounded above on the bounded set $|\zeta| \leq r_*$, i.e.

$$\overline{V_*} := \sup\{V(\zeta) : |\zeta| \leq r_*\} < +\infty.$$

Write $r^* = n\overline{V_*}^{1/2}\mathrm{e}^{4(n^2+M_h)\pi}$. For $|\zeta_0| \geq r^*$ and $\zeta_0 = (-\cos\phi, \sin\phi)|\zeta_0|$, we can see (3.5) holds. It follows that

$$V(\mathcal{P}(\zeta_0)) \geq V(\zeta_0) + d_* > V(\zeta_0) = \cos^2\phi + n\sin^2\phi > \overline{V_*},$$

hence that $|\mathcal{P}(\zeta_0)| \geq r_*$ by the definition of $\overline{V_*}$.

Then let us denote by $\zeta_1 = \mathcal{P}(\zeta_0)$ the new initial point. Likewise,

$$V(\mathcal{P}(\zeta_1)) \geq V(\zeta_1) + d_* \geq V(\zeta_0) + 2d_*.$$

Similar arguments apply to the case $m \in \mathbb{N}$. Then we have

$$V(\mathcal{P}(\zeta_{m+1})) \geq V(\zeta_m) + d_* \geq V(\zeta_0) + (m+1)d_*,$$

which implies that $\zeta_m \to \infty$, $m \to \infty$. If otherwise, then this contradicts the fact that V is bounded above on the bounded set. From this, we conclude that if $|\zeta_0| \geq r^*$ and $\zeta_0 = (-\cos\phi, \sin\phi)|\zeta_0|$, then $(u(t;\zeta_0), u'(t;\zeta_0))$ is an unbounded solution of (3.1).

If $|\zeta_0| \geq r^*$ but $\zeta_0 \neq (-\cos\phi, \sin\phi)|\zeta_0|$, by $\theta'(t;\zeta_0) \geq \frac{1}{2}$ and $r^* = n\overline{V_*}^{1/2}\mathrm{e}^{4(n^2+M_h)\pi}$, then we find a $\tau_0 \in [0, 4\pi)$ such that $(u(\tau_0;\zeta_0), u'(\tau_0;\zeta_0)) = (-\cos\phi, \sin\phi)r(\tau_0;\zeta_0)$ and $r(\tau_0;\zeta_0) \geq r_*$. Similarly, let $\bar{\zeta}_m = \mathcal{P}^m(\bar{\zeta}_0)$. By the same method as shown before, we have $\bar{\zeta}_m \to \infty$, $m \to \infty$, which implies that $(u(t;\zeta_0), u'(t;\zeta_0))$ is also an unbounded solution. Therefore, we have the following assertion.

Proposition 3.8. *Assume that (D_1) and (D_2) hold. Then there exists $q_0 > 0$, such that the equation (3.1) has an unbounded solution provided that $p(t)$ satisfies*

$$0 < M_p < q_0.$$

Combining Proposition 3.6 and Proposition 3.8, we have thus proved the Theorem 1.4.

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Existence and uniqueness of positive periodic solutions of a certain functional differential equation

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Abstract. In our work, we prove the existence and uniqueness of positive 1-periodic solutions of a first order differential equation that contains a piecewise constant term. Such equations admit periodic solutions and hence they can be used for numerical modelling of temperature changes in a space surrounded by environment with different temperature. For some types of functional equations, it is possible to use the comparison principle method of differential equations to obtain existence, uniqueness, and asymptotic stability results.

Keywords: functional differential equations, periodic solutions, existence, uniqueness.

2020 Mathematics Subject Classification: 34K13, 39B22.

1 Introduction

In our paper, we analyze positive 1-periodic solutions of functional differential equation

$$x'(t) = f(x(t)) + x(\lfloor t \rfloor)g(t - \lfloor t \rfloor), \qquad t \geq 0 \tag{1.1}$$

where $\lfloor . \rfloor : [0, \infty) \to \mathbb{N} \cup \{0\}$ is the standard floor function, i.e. $\lfloor t \rfloor = n$ if $t \in [n, n+1)$ for some $n \in \mathbb{N} \cup \{0\}$. We will assume that the function f is negative for positive x, and equals zero at zero, and g is positive over the interval $[0, 1]$ (see conditions 1) and 2) below for more precise assumptions).

The problem (1.1) represents a numeric model that describes temperature changes in a space bounded by outside environment. A solution of problem represents temperature of the space cooled by colder environment; the negative function f represents cooling up to zero. The space is simultaneously warmed by a heating agent inside; the positive function g represents heating. Moreover, the function g is multiplied by the temperature values at discrete points so that the actual temperature is affected by previous temperature.

In this work, we use the comparison principle for differential equations to prove the existence, uniqueness and asymptotic stability (see Theorem 3.6) of positive 1-periodic solution of

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the equation (1.1) under certain assumptions on the functions f and g. In some other works (e.g. [6]), authors analyzed an explicit form of solutions of a similar problem. In [5], a special type of Gronwall lemma was used. For more results and methods used regarding analyzing functional equations of this type, we refer to [4] and references therein. Next, linear functional equations similar to (1.1) have been studied by many authors. The floor function $\lfloor . \rfloor$ that appears in (1.1) was used in the automatic control theory in the frame of so-called sampled-data approach; see e.g. [10], [11]. In [3], [8], and [9], formulas of representation of solutions were found using the idea to the direction of the Floquet theory; the idea of such approach is to reduce equations in linear case to equations with finite-dimensional fundamental systems. In some other works (e.g. [6]), authors analyzed an explicit form of solutions. General theory of functional differential equations proposed in the books [1] and [2] was built namely on the idea of finite-dimensional fundamental systems. We also note that in [13], such delays are used in semi-discretization methods for the study of stability analysis of linear time-periodic and time-delay systems.

2 Preliminaries

Throughout the whole paper, for the functions f, g in the equation (1.1), we will assume the following conditions:

1) $f : [0, \infty) \to (-\infty, 0]$ is continuously differentiable, $f(0) = 0$ and $f(x) < 0$ for $x > 0$,

2) $g : [0, 1] \to (0, \infty)$ is continuous.

It is important to note that the derivative of solution x of the equation (1.1) need not exist at $t = n$ for $n \in \mathbb{N}$. In such case, x is not continuously differentiable at these points. However, one-sided derivatives of x at $t = n$ do exist, since in (1.1), one can pass to the limit for $t \to n^+$ or $t \to n^-$. Due to these facts, we state the following definition of solutions of (1.1).

We say that a continuous function $x : [0, \infty) \to \mathbb{R}$ is a solution of the equation (1.1) if for every $n \in \mathbb{N} \cup \{0\}$, the function x is continuously differentiable over $(n, n + 1)$, and the equation (1.1) is satisfied for $t \in (n, n + 1)$. Necessarily, there exist finite one-sided derivatives of x at points n and $n + 1$.

In order to study $1-$periodic solutions of (1.1), we will consider the problem

$$x'(t) = f(x(t)) + x(0)g(t), \qquad t \in [0, 1], \tag{2.1}$$

$$x(0) = x(1), \tag{2.2}$$

where $x'(0)$ and $x'(1)$ are the corresponding one-sided derivatives. Obviously, for every initial condition $x(0) > 0$, there exists a unique positive solution x of the equation (2.1) defined over the whole interval $[0, 1]$. The equation (1.1) has a 1-periodic solution if and only if there exists a solution of problem (2.1)–(2.2).

Let g satisfy the condition 2). We introduce some notation we will use throughout the whole paper. Denote $g_1 = \min_{t\in[0,1]} g(t)$, $g_2 = \max_{t\in[0,1]} g(t)$, and $g_3 = \int_0^1 g(t)\,\mathrm{d}t$. Thus, it holds $g_2 \geq g_3 \geq g_1 > 0$ and if g is nonconstant over $[0, 1]$, then strict inequalities are valid.

3 Results

In this section, we present our results. In the proofs, we use standard methods of analysis of solutions of ODE such as the comparison method.

Lemma 3.1. *Let x be a solution of the equation (2.1), $x(0) > 0$, and g is differentiable and nonincreasing over the interval $[0,1]$. Then the function x attains its minimum either at $t = 0$ or $t = 1$.*

Proof. Observe that since $x(0) > 0$ and $g > 0$ over $[0,1]$, the corresponding solution x of (2.1) remains positive over $[0,1]$ (this is not obviously true for every continuous function g; see Remark 3.3). Also, if there is a continuously differentiable function y such that

$$y'(t) > f(y(t)) + y(0)g(t), \qquad t \in (0,1]$$

with $y(0) = x(0)$, then necessarily $y(t) > x(t)$ for $t \in (0,1]$. These assertions can be proven by using standard methods; see e.g. [12].

First, we assume that $g' < 0$ over $[0,1]$ and proceed by a contradiction. Let $t_0 \in (0,1)$ be the point of minimum of x. Hence $x'(t_0) = 0$ and differentiating (2.1) with respect to t, we obtain

$$x''(t) = f'(x(t))x'(t) + x(0)g'(t). \tag{3.1}$$

This means that $x''(t_0) < 0$ and so t_0 cannot be a point of minimum.

Now, we assume that g is differentiable and nonincreasing over $[0,1]$. Denote $g_n(t) = g(t) - \frac{t}{n}$ for $n \in \mathbb{N}$ large enough so that $g_n > 0$ over $[0,1]$ (say $n \geq n_0$). Let x_n be the corresponding solution of

$$x_n'(t) = f(x_n(t)) + x(0)g_n(t)$$

that satisfies $x_n(0) = x(0)$. Then x_n is bounded by zero from below and by x from above, and the sequence $\{x_n\}_{n \geq n_0}$, x_n converges uniformly to x over $[0,1]$. Since $g_n' < 0$, we have $x_n(t) \geq \min\{x_n(0), x_n(1)\}$ by the previous part of the proof. Hence $x(t) \geq \min\{x(0), x(1)\}$. □

Remark 3.2. Using arguments from the proof of Lemma 3.1, one can prove that if there holds (2.2) and $g' < 0$ over $[0,1]$, then x attains its maximum at a unique point $t_0 \in (0,1)$.

Remark 3.3. It is well known that if g is not positive, then a solution x that starts at some $x(0) > 0$ may cross the zero solution. Necessary and sufficient conditions of this phenomenon for linear delay differential equations were obtained in [7]. For convenience of readers, we can consider the equation

$$x'(t) = -x(t) + x(0)(-2t - 1).$$

Observe that the function $x(t) = \frac{1}{2} - t$ is a solution that crosses zero.

Lemma 3.4. *Let $f(x) = -\eta x$ where $\eta > 0$. Then there exists a unique $\eta^* = \eta^*(g) > 0$ such that for every positive solution x of (2.1), the following assertions are true:*

i) If $\eta > \eta^*$, then $x(1) < x(0)$.

ii) If $\eta < \eta^*$, then $x(1) > x(0)$.

iii) If $\eta = \eta^*$, then $x(1) = x(0)$.

Proof. Let x be a positive solution of the equation

$$x'(t) = -\eta x(t) + x(0)g(t), \qquad t \in [0,1] \tag{3.2}$$

that starts at some $x(0) > 0$. Multiplying the equation with $e^{\eta t}$ and integrating over $[0,1]$, we obtain

$$x(1) = x(0)e^{-\eta}\left(1 + \int_0^1 e^{\eta t} g(t)\,\mathrm{d}t\right).$$

Now, we discuss when there holds

$$e^{-\eta}\left(1+\int_0^1 e^{\eta t}g(t)\,\mathrm{d}t\right)=1 \tag{3.3}$$

or equivalently, $F(\eta):=\int_0^1 e^{\eta t}(g(t)-\eta)\,\mathrm{d}t=0$.

Let us prove that there is a unique solution of the equation $F(\eta)=0$. Note that F is a continuous function and $F(g_1)>0>F(g_2)$. Hence there exists at least one solution $\eta^*\in(g_1,g_2)$ of (3.3) and obviously, there is no solution beyond (g_1,g_2). Thus, every positive solution x of (3.2) for $\eta=\eta^*$ satisfies $x(0)=x(1)$. If there exists a root $\eta_1>\eta^*$ of (3.3) and y is a positive solution of problem

$$y'(t)=-\eta_1 y(t)+y(0)g(t),$$

then $y'(t)<-\eta^* y(t)+y(0)g(t)$. We can choose x, a solution of (3.2) with $\eta=\eta^*$ such that $x(0)=y(0)$, then $y(1)<x(1)=x(0)=y(0)$ what is a contradiction. If $\eta_1<\eta^*$, then a similar argument can be used. We conclude that $\eta_1\neq\eta^*$ cannot be a root of (3.3).

We turn to prove *i*). If $\eta>\eta^*$, then $F(\eta)<0$, hence $e^{-\eta}\left(1+\int_0^1 e^{\eta t}g(t)\,\mathrm{d}t\right)<1$ and $x(1)<x(0)$. The assertions *ii*) and *iii*) can be proven similarly. □

Remark 3.5. From the proof of Lemma 3.4, it is clear that $\eta^*\in(g_1,g_2)$. If moreover, g is nonconstant, nonincreasing and differentiable over $[0,1]$, then $\eta^*\in(g_1,g_3)$. Indeed, integrating by parts, we get

$$F(g_3)=\int_0^1 e^{g_3 t}(g(t)-g_3)\,\mathrm{d}t=\left[e^{g_3 t}\left(G(t)-g_3 t\right)\right]_0^1-g_3\int_0^1 e^{g_3 t}(G(t)-g_3 t)\,\mathrm{d}t,$$

where $G(t)=\int_0^t g(s)\,\mathrm{d}s$. Note that $G(0)=0$ and $G(1)=g_3$. Since G is a function concave over $[0,1]$ and not equal to the line that connects the points $[0,0]$ and $[1,g_3]$, it holds $G(t)>g_3 t$ for $t\in(0,1)$. We conclude that $F(g_3)<0$ and since $F(g_2)<0$, then necessarily, $\eta^*\in(g_1,g_3)$.

Theorem 3.6. *Let f,g satisfy the conditions 1), 2), and let η^* be as in Lemma 3.4. Assume that*

$$\lim_{x\to 0^+}\frac{f(x)}{x}>-\eta^*>\lim_{x\to\infty}\frac{f(x)}{x}. \tag{3.4}$$

Then there exists a positive solution x of problem (2.1)–(2.2), hence a 1-periodic solution of (1.1).

Moreover, if the function $\frac{f(x)}{x}$ is decreasing on interval $(0,\infty)$, then the positive 1-periodic solution x is unique and for every solution y of (1.1) such that $y(0)>0$, it holds

$$\lim_{t\to\infty}(x(t)-y(t))=0. \tag{3.5}$$

Proof. Note that if x,y are two solutions of (2.1) and $x(0)>y(0)>0$, then $x(t)>y(t)$ for $t\in[0,1]$. Indeed, it holds $x'(t)-y'(t)=a(t,x,y)(x(t)-y(t))+(x(0)-y(0))g(t)$ for $a(t,x,y)=\int_0^1 f'(sx(t)+(1-s)y(t))\,\mathrm{d}s$. Thus,

$$x(t)-y(t)=(x(0)-y(0))\left(e^{\int_0^t a(s)\,\mathrm{d}s}+\int_0^t e^{\int_s^t a(v)\,\mathrm{d}v}g(s)\,\mathrm{d}s\right)$$

and hence, $x>y$ over $[0,1]$ due to the condition 2).

Observe that if there exist two positive solutions $x_{-,+}$ of (2.1) such that $x_-<x_+$ over $[0,1]$, $x_-(0)<x_-(1)$, and $x_+(0)>x_+(1)$, then there is a solution of (2.1)–(2.2) that lies between x_-

and x_+. In fact, if we define a sequence $\{x_n\}_{n\in\mathbb{N}}$ such that $x_1(0) = x_-(1)$ and $x_{n+1}(0) = x_n(1)$ for $n \in \mathbb{N}$, then $x_-(t) < x_1(t) < x_2(t) < \ldots < x_+(t)$ for $t \in [0,1]$, since two distinct positive solutions of (2.1) do not intersect each other. The sequence of functions $\{x_n\}_{n\in\mathbb{N}}$ is increasing with respect to n and bounded by x_+ from above hence convergent to some solution of (2.1)-(2.2). In the following, our goal will be to find such functions x_- and x_+.

Due to the assumption $\lim_{x\to 0^+} \frac{f(x)}{x} > -\eta^*$, there holds $f(x) > -\eta x$ for some $\eta < \eta^*$ close to η^* and for all $x \in (0,\varepsilon)$ where $\varepsilon = \varepsilon(\eta) > 0$ is sufficiently small. If $x(0) \in \left(0, \frac{\varepsilon}{2}\min\{1, \frac{1}{g_2}\}\right)$ and x is the corresponding solution of (2.1), then $x' < x(0)g_2 \le \frac{\varepsilon}{2}$. Thus, the function x is bounded by ε. If y is the solution of (2.1) with $f(x) = -\eta x$ and $y(0) = x(0)$, then we can similarly prove that $y(t) \in (0,\varepsilon)$ for $t \in [0,1]$ and so $y < x$ over $(0,1]$. Due to Lemma 3.4, it holds $y(1) > y(0)$ therefore $x(1) > x(0)$ and we can denote $x_- := x$.

Now, we turn to find a solution x_+. Observe that if the initial condition $x(0) > 0$ is arbitrarily large the the value $\min_{t\in[0,1]} x(t)$ is also arbitrarily large where x is the corresponding solution of (2.1). Assume by contradiction that there is a sequence $\{x_n(0)\}_{n\in\mathbb{N}}$ of initial values such that $x_n(0) \to \infty$ as $n \to \infty$ and $\min_{t\in[0,1]} x_n(t)$ is bounded with respect to n. Let $t_n \in [0,1]$ be such that $x_n(t_n) = \min_{t\in[0,1]} x_n(t)$. Clearly, $t_n > 0$ for all n sufficiently large, since $x_n(0)$ is unbounded, and so $x_n'(t_n) \le 0$. On the other hand, the sequence $\{f(x_n(t_n))\}_{n\in\mathbb{N}}$ is bounded from below by some negative constant K and we conclude that

$$0 \ge x_n'(t_n) = f(x_n(t_n)) + x_n(0)g(t_n) \ge K + x_n(0)g_1.$$

This is not true if n is large enough.

The assumption (3.4) implies that $f(x) < -\eta^* x$ for all $x \in (x_0,\infty)$ for some x_0 large enough. We choose $x(0) > x_0$ large enough so that $x(t) > x_0$ for $t \in [0,1]$ where x is the corresponding solution of (2.1). Let y be the solution of equation

$$y'(t) = -\eta^* y(t) + y(0)g(t)$$

with $y(0) = x(0)$. Lemma 3.4 implies that $y(1) = y(0)$ and y lies entirely in (x_0,∞) (increasing $x(0)$ if necessary). Consequently, $y > x$ over $(0,1]$, $x(1) < y(1) = x(0)$, and we can choose $x_+ = x$.

Let x be a fixed positive solution of problem (2.1)–(2.2). We prove that x is unique. Let $\alpha \in (0,1)$ and denote $z(t) = \alpha x(t)$ for $t \in [0,1]$. Since the function $\frac{f(x)}{x}$ is decreasing on $(0,\infty)$, it holds

$$\frac{f(\alpha\xi)}{\alpha\xi} > \frac{f(\xi)}{\xi} \quad \text{for } \xi > 0$$

and so $\alpha f(x(t)) < f(\alpha x(t))$. Thus, we obtain

$$z'(t) = \alpha(f(x(t)) + x(0)g(t)) < f(z(t)) + z(0)g(t).$$

If y is the solution of (2.1) equipped with $y(0) = z(0) = \alpha x(0)$, then $y(t) > z(t)$ over $(0,1]$. Hence $y(1) > \alpha x(1) = \alpha x(0) = y(0)$. Thus, any solution of (2.1) that lies between x and the zero solution cannot be extended to a 1-periodic solution of (1.1).

If $\alpha > 1$, then similar arguments imply that $y(1) < y(0)$ where y is the solution of (2.1) satisfying $y(0) = \alpha x(0)$. Hence the solution x is the unique positive 1-periodic solution.

The assertion (3.5) is a consequence of above arguments. Let y be as in the previous part of the proof. If $y(0) < x(0)$, then the sequence $\{y(n)\}_{n\in\mathbb{N}}$ is increasing and necessarily converges to $x(0)$ so there holds (3.5). Otherwise y would converge to a 1-periodic solution different from x which is not possible. The case $y(0) > x(0)$ is similar. □

Remark 3.7. If $f(x) > -\eta^* x$ or $f(x) < -\eta^* x$ for all $x > 0$, then one can use arguments as in the proof of Theorem 3.6 and Lemma 3.4 to prove that there is no positive solution of problem (2.1)–(2.2).

Theorem 3.8. *Let $x_0 > 0$ and assume that one of the cases is true:*

i) It holds either $f(x_0) > -g_1 x_0$ or $f(x_0) < -g_2 x_0$.

ii) It holds $f(x_0) < -g_3 x_0$, f, g are decreasing.

Then there is no solution of the problem (2.1)–(2.2) with $x(0) = x_0$.

Proof. Let $i)$ be valid. Assume first that $f(x_0) > -g_1 x_0$ and there is a solution x of (2.1)–(2.2) satisfying $x(0) = x_0$. Let y be the solution of equation

$$y' = f(y) + x_0 g_1, \qquad y(0) = x_0.$$

Since the equation is autonomous, y is either strict monotone or constant. From $g \geq g_1$, we have $x \geq y$ over $[0,1]$. On the other hand, $f(y(0)) = f(x_0) > -x_0 g_1$ and so y is increasing on some neighbourhood of 0, hence increasing on $[0,1]$. Hence $x(1) > x_0 = x(0)$. The case $f(x_0) < -g_2 x_0$ is analogous.

Assume that $ii)$ is valid. Lemma 3.1 implies that if x is a solution of (2.1)–(2.2), then $x(t) \geq x_0$ for $t \in (0,1)$, hence $f(x(t)) \leq f(x_0)$. Integrating the equation (2.1) over $[0,1]$, it holds

$$0 = \int_0^1 f(x(t))\,\mathrm{d}t + x_0 \int_0^1 g(t)\,\mathrm{d}t \leq f(x_0) + x_0 g_3,$$

a contradiction. □

4 Example

Consider the equation

$$x'(t) = -\frac{x(t)}{3} - x^2(t) + x(\lfloor t \rfloor) e^{-(t - \lfloor t \rfloor)}, \qquad t \geq 0. \tag{4.1}$$

Note that the functions $f(x) = -\frac{x}{3} - x^2$, $x \geq 0$, and $g(t) = e^{-t}$, $t \in [0,1]$ satisfy the conditions 1), 2). One can easily check that all of the assumptions of Theorems 3.6 and 3.8 are true so that there is a unique positive 1-periodic solution x. Due to Remark 3.5, η^* lies in the interval $(e^{-1}, 1 - e^{-1})$. Theorem 3.8 provides us a bound for the starting value $x(0) := x_0$. From the part $i)$, we come to inequality $-\frac{x_0}{3} - x_0^2 > -e^{-1} x_0$, or equivalently, $x_0 > e^{-1} - \frac{1}{3}$. We can use the part $ii)$ and we obtain $-\frac{x_0}{3} - x_0^2 < -(1 - e^{-1}) x_0$, or, $x_0 > \frac{2}{3} - e^{-1}$. Hence for $x(0) \notin [e^{-1} - \frac{1}{3}, \frac{2}{3} - e^{-1}]$, then there is no positive 1-periodic solution of (4.1).

The equation (3.3) can be solved numerically and we find that $\eta^* \approx 0.601$. In the Figure 4.1, we see two solutions $x_{-,+}$ of (4.1) over the interval $[0, 100]$. For x_-, we choose $x_-(0) = 0.03 < e^{-1} - \frac{1}{3}$ and for x_+, $x_-(0) = 0.3 > \frac{2}{3} - e^{-1}$. From the proof of Theorem 3.6, we know that the values $x_-(n)$ and $x_+(n)$ for integer n converge to a value that is the initial condition for the periodic solution. Numerical computations show that $x_-(100) \approx 0.256$ and $x_+(100) - x_-(100) \approx 1.64 \times 10^{-9}$. This means that $x_-(100)$ approximates the initial value of the 1-periodic solution with error 1.64×10^{-9}. As observed in Remark 3.2, the graphs of both solutions consist of smooth parts of solutions of (1.1) defined on intervals $[n-1, n]$

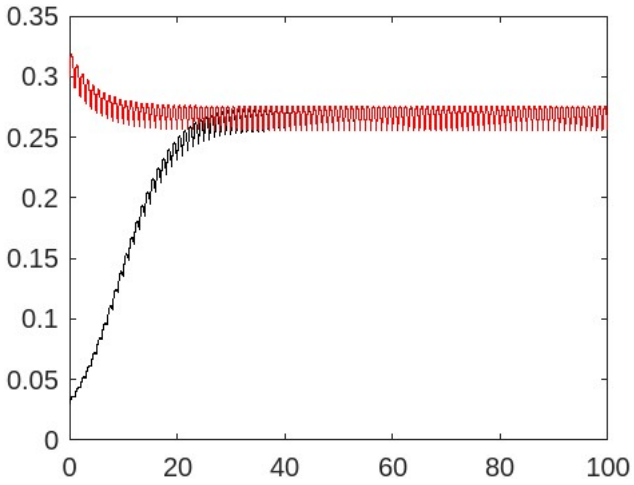


Figure 4.1: Comparison of solutions x_- (black line) and x_+ (red line).

for $n = 1, \dots, 100$ that attain local maximum at exactly one interior point on each interval $[n-1, n]$ and attain local minima at points $n-1$ and n.

The problem (4.1) describes temperature changes in a space bounded by outside cold environment for particular functions f and g. We see that for such f, g, the optimal initial temperature is approximately equal to 0.256 and repeats regularly at every positive integer point. If the initial temperature is higher or lower, then, after some time, it becomes close to optimal due to the asymptotic stability of the periodic solution in sense of (3.5).

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Sign-changing solution for critical Schrödinger–Poisson system with concave perturbation

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Abstract. In this paper, we consider the following Schrödinger–Poisson system with critical-concave nonlinearity:

$$\begin{cases} -\Delta u + l(x)\phi u = |u|^4 u + \mu h(x)|u|^{q-2}u & \text{in } \mathbb{R}^3, \\ -\Delta \phi = l(x)u^2 & \text{in } \mathbb{R}^3, \end{cases}$$

where $\mu > 0$, $1 < q < 2$ and the functions l, h satisfy some mild conditions. By constraining the energy functional of the above problem on a closed subset of the sign-changing Nehari manifold, we obtain a sign-changing solution with positive energy for $\mu > 0$ small enough.

Keywords: Schrödinger–Poisson system, critical-concave nonlinearity, Nehari manifold, sign-changing solution.

2020 Mathematics Subject Classification: 35J20, 35B33, 35B38, 35D30.

1 Introduction and main results

This paper focuses on standing waves of the following nonlinear Schrödinger–Poisson system:

$$\begin{cases} i\psi_t = -\Delta\psi + \phi(x)\psi - g(x,\psi), \\ -\Delta\phi = |\psi|^2, \end{cases} \tag{1.1}$$

where $g \in C(\mathbb{R}^3 \times \mathbb{C}, \mathbb{R})$ and $\psi : \mathbb{R}^3 \times [0,T] \to \mathbb{C}$. Such systems arise as electromagnetic field models in quantum mechanics and has a strong physical background. The first equation of system (1.1) describes the evolution of the quantum state of microscopic particles (such as electrons) over time, while the second equation (i.e. Poisson equation) is often used to describe the relationship between the electrostatic potential and the charge distribution. The interaction of charge particles with electromagnetic field can be described by coupling the

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nonlinear Schrödinger equation and the Poisson equation. For more physical backgrounds of Schrödinger–Poisson system, we recommend readers to refer to [2,6].

In the past decades, many mathematicians studied the standing wave solution of type $\psi(x,t) = e^{i\lambda t}u(x)$ for system (1.1), which corresponds to a solution u of the elliptic system

$$\begin{cases} -\Delta u + u + \phi u = g(x,u) & \text{in } \mathbb{R}^3, \\ -\Delta \phi = u^2 & \text{in } \mathbb{R}^3. \end{cases} \tag{1.2}$$

For the case of $g(x,u) = |u|^{p-2}u$, D'Aprile and Mugnai [7] showed that system (1.2) has no nontrivial solution once $p \geq 6$ or $p \leq 2$ by establishing a related Pohožaev identity, Ruiz [18] showed that system (1.2) does not admit any nontrivial solution if $p \leq 3$ and has positive solution if $p \in (3,6)$, in this case ground state solution was further proved by Azzollini and Pomponio in [1]. Afterwards, Schrödinger–Poisson systems with Sobolev critical exponent and subcritical perturbation were investigated by many scholars. For example, see the cases of superlinear perturbations in [1, 25], the cases of sublinear perturbations in [13] and the references therein. As we will recall below, various existence results of sign-changing solutions for Schrödinger–Poisson systems were widely studied in [4,5,8–12,15,17,19–21,23,24,26,27]. More precisely, for the Schrödinger–Poisson system

$$\begin{cases} -\Delta u + V(x)u + \phi u = K(x)f(u) & \text{in } \mathbb{R}^3, \\ -\Delta \phi = u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.3}$$

Ianni [10] studied the case of $V = K = 1$, $f(u) = |u|^{p-2}u$ and $p \in [4,6)$, by the heat flow method he got a radially nodal solution changing sign exactly $k-1$ times for every integer $k \geq 2$. Meanwhile, Kim and Seok [12] proved the similar results to [10] for $p \in (4,6)$ by using the variational methods. Wang and Zhou assumed $K = 1$, $f(u) = |u|^{p-2}u$, $p \in (4,6)$ and V satisfies compactness condition in [21], where they proved a sign-changing solution of system (1.3) by combining the constraint minimization argument and the Brouwer degree theory. In [19], by using a quantitative deformation lemma, Shuai and Wang extended the results in [21] to the case of general nonlinearity f, which satisfies an increasing condition. If V, K satisfy some vanishing conditions, Liang et al. [15] proved system (1.3) has a sign-changing solution. Chen and Tang [4] assumed that $f \in C(\mathbb{R},\mathbb{R})$ satisfies a weaker condition than the increasing condition, by using a direct approach they obtained a ground state sign-changing solution with precisely two nodal domains. After this, Chen and Tang assumed that f satisfies asymptotically cubic or super-cubic conditions in [5], where they proved a sign-changing solution with precisely two nodal domains under mild assumptions on asymptotic behaviors of V, K. When $K = 1$ and f satisfies super-cubic conditions, via the method of invariant sets of descending flow, Liu et al. [17] proved that system (1.3) has infinitely many sign-changing solutions. In [26], Zhong and Tang studied the system

$$\begin{cases} -\Delta u + V(x)u + \phi u = |u|^2 u + \mu h(x)u & \text{in } \mathbb{R}^3, \\ -\Delta \phi = u^2 & \text{in } \mathbb{R}^3, \end{cases}$$

where $V \in C(\mathbb{R}^3,\mathbb{R}^+)$ satisfies a compactness condition and $h \in L^{\frac{3}{2}}(\mathbb{R}^3) \setminus \{0\}$ is nonnegative, by using the sign-changing Nehari manifold they got a ground state sign-changing solution for any $\mu \in (0,\mu_1)$, with μ_1 denoting the first eigenvalue of $-\Delta u + u = \mu h(x)u$ in $H^1(\mathbb{R}^3)$. In

[8], Gu et al. considered the following system with subquadratic or quadratic nonlinearity:

$$\begin{cases} -\Delta u + V(x)u + \lambda\phi u = f(u) & \text{in } \mathbb{R}^3, \\ -\Delta\phi = u^2 & \text{in } \mathbb{R}^3, \end{cases}$$

where $V \in C(\mathbb{R}^3, \mathbb{R}^+)$ is radial if f is subquadratic and coercive if f is quadratic, by following the ideas of [17] they verified the existence and multiplicity of sign-changing solution for $\lambda > 0$ small enough.

As for the existence of sign-changing solutions for the critical Schrödinger–Poisson system

$$\begin{cases} -\Delta u + V(x)u + K(x)\phi u = |u|^4 u + g(x,u) & \text{in } \mathbb{R}^3, \\ -\Delta\phi = K(x)u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.4}$$

there are seldom results proved in [9, 11, 20, 27]. In [27], Zhong and Tang studied the case of $V = 1$ and $g(x,u) = \mu h(x)u$, under some mild conditions on K, g, by combining the variational methods and sign-changing Nehari manifold they obtained a ground state sign-changing solution of system (1.4) for any $\mu \in (0, \mu_1)$, where μ_1 is the first eigenvalue of $-\Delta u + u = \mu h(x)u$ in $H^1(\mathbb{R}^3)$. When $K = 1$, $f(x,u) = \mu f(u)$ satisfies an increasing condition and V satisfies compactness condition, Wang et al. [20] proved that system (1.4) has a least-energy sign-changing solution for $\mu > 0$ large enough. Later, Kang et al. [11] concerned with the following Schrödinger–Poisson system with steep well potential:

$$\begin{cases} -\Delta u + (\lambda V(x) + 1)u + \phi u = |u|^4 u + |u|^{p-2}u & \text{in } \mathbb{R}^3, \\ -\Delta\phi = u^2 & \text{in } \mathbb{R}^3, \end{cases}$$

where $p \in (5, 6)$, by combining constrained minimization arguments and Hofer's deformation lemma, they proved the existence and concentration of ground sign-changing solution for $\lambda > 0$ large enough. Subsequently, Huang et al. [9] extended the results in [11] to the case of general perturbed nonlinearity.

Recently, sign-changing solutions on the Schrödinger–Poisson systems with concave-convex nonlinearity have attracted wide attention. Yang and Ou [24] studied the Schrödinger–Poisson system

$$\begin{cases} -\Delta u + \dfrac{1}{4\pi}\displaystyle\int_\Omega \frac{u^2(y)}{|x-y|}dy u = |u|^{q-2}u + \mu|u|^{p-2}u & \text{in } \Omega, \\ u(x) = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

where $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary $\partial\Omega$, $1 < p < 2$ and $4 < q < 6$. By using constrained variational method and quantitative deformation lemma, they proved that there exists some constant $\mu^* > 0$ such that system (1.5) has a sign-changing solution with positive energy for any $\mu < \mu^*$. After this work, Yang and Tang [23] studied the Schrödinger–Poisson system

$$\begin{cases} -\Delta u + V(x)u + \phi u = |u|^{p-2}u + \mu K(x)|u|^{q-2}u & \text{in } \mathbb{R}^3, \\ -\Delta\phi = u^2 & \text{in } \mathbb{R}^3, \end{cases}$$

where $1 < q < 2$, $4 < p < 6$, $K \in L^{\frac{6}{6-q}}(\mathbb{R}^3)$ and $V \in C(\mathbb{R}^3, \mathbb{R})$ satisfies a weakly coercive assumption. By constructing a nonempty closed subset of the sign-changing Nehari manifold,

they obtained a sign-changing solution with positive energy provided $\mu < \mu^*$ for some $\mu^* > 0$. Inspired by the above works, we are interested in whether the critical-concave Schrödinger–Poisson system has sign-changing solution. In this paper, we consider sign-changing solution of the following Schrödinger–Poisson system:

$$\begin{cases} -\Delta u + l(x)\phi u = |u|^4 u + \mu h(x)|u|^{q-2}u & \text{in } \mathbb{R}^3, \\ -\Delta \phi = l(x)u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.6}$$

where $1 < q < 2$, $\mu > 0$, $0 < l(\cdot) \in L^2(\mathbb{R}^3) \cap L^3(\mathbb{R}^3)$ and the potential function h satisfies the condition

(H) $0 < h(\cdot) \in L^{\frac{6}{6-q}}(\mathbb{R}^3)$ and there exists some $\beta \in \left(3 - \frac{2q}{3}, 3 - \frac{q}{2}\right)$ such that $h(x) \geq \frac{1}{|x|^\beta}$ for all $|x| \leq 1$.

As is well known, the Poisson equation is uniquely solvable. Indeed, for any $u \in D^{1,2}(\mathbb{R}^3)$, define the linear functional

$$L_u(v) = \int_{\mathbb{R}^3} l(x)u^2 v dx, \qquad \forall\, v \in D^{1,2}(\mathbb{R}^3).$$

It follows from $l \in L^2(\mathbb{R}^3)$, the Hölder and Sobolev inequalities that

$$|L_u(v)| \leq |l|_2 |u|_6^2 |v|_6 \leq S^{-\frac{3}{2}} |l|_2 \|u\|^2 \|v\|.$$

Then, by the Lax–Milgram theorem, there exists a unique $\phi_u \in D^{1,2}\left(\mathbb{R}^3\right)$ such that $-\Delta\phi_u = l(x)u^2$ in $\left[D^{1,2}(\mathbb{R}^3)\right]^*$, where

$$\phi_u(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{l(y)u^2(y)}{|x-y|} dy.$$

Therefore, system (1.6) can be reduced to the semilinear Schrödinger equation with a non-local term

$$-\Delta u + l(x)\phi_u u = |u|^4 u + \mu h(x)|u|^{q-2}u \quad \text{in } \mathbb{R}^3. \tag{1.7}$$

Then, we say that $u \in D^{1,2}\left(\mathbb{R}^3\right)$ satisfying (1.7) is a weak solution of system (1.6) for short. Moreover, the solution u of system (1.6) is called as a sign-changing solution of system (1.6) once $u^\pm \neq 0$, where

$$u^+(x) := \max\{u, 0\} \quad \text{and} \quad u^-(x) := \min\{u, 0\}.$$

By the variational methods, weak solution of system (1.6) correspond to critical point of the functional

$$\mathcal{I}_\mu(u) = \frac{1}{2}\|u\|^2 + \frac{1}{4}\int_{\mathbb{R}^3} l(x)\phi_u u^2 dx - \frac{1}{6}\int_{\mathbb{R}^3} |u|^6 dx - \frac{\mu}{q}\int_{\mathbb{R}^3} h(x)|u|^q dx, \qquad \forall\, u \in D^{1,2}(\mathbb{R}^3).$$

Based on [14, Lemma 2.3], it is standard to verify that the functional $\mathcal{I}_\mu$ is well-defined in $D^{1,2}(\mathbb{R}^3)$ and is of class C^1. Naturally, we introduce the Nehari manifold of system (1.6) as follows:

$$\mathcal{N}_\mu = \left\{ u \in D^{1,2}\left(\mathbb{R}^3\right) \setminus \{0\} : \left\langle \mathcal{I}'_\mu(u), u \right\rangle = 0 \right\}.$$

As we will see, $\mathcal{N}_\mu$ is closely related to the behavior of the fiber map $\psi_u : t \to \mathcal{I}_\mu(tu)$, which is defined for any $u \in D^{1,2}(\mathbb{R}^3)$ and $t > 0$ by

$$\psi_u(t) = \frac{t^2}{2}\|u\|^2 + \frac{t^4}{4}\int_{\mathbb{R}^3} l(x)\phi_u u^2 dx - \frac{t^6}{6}|u|_6^6 - \frac{\mu t^q}{q}\int_{\mathbb{R}^3} h(x)|u|^q dx.$$

By a simple calculation, we obtain

$$\psi_u'(t) = t\|u\|^2 + t^3\int_{\mathbb{R}^3} l(x)\phi_u u^2 dx - t^5|u|_6^6 - \mu t^{q-1}\int_{\mathbb{R}^3} h(x)|u|^q dx$$

and

$$\psi_u''(t) = \|u\|^2 + 3t^2\int_{\mathbb{R}^3} l(x)\phi_u u^2 dx - 5t^4|u|_6^6 - (q-1)\mu t^{q-2}\int_{\mathbb{R}^3} h(x)|u|^q dx.$$

Clearly, for any $u \in D^{1,2}(\mathbb{R}^3) \setminus \{0\}$, there holds $\psi_u'(t) = 0$ iff $tu \in \mathcal{N}_\mu$. Specially, $\psi_u'(1) = 0$ iff $u \in \mathcal{N}_\mu$. To search for sign-changing solutions of system (1.6), we introduce the sign-changing Nehari manifold

$$\mathcal{M}_\mu = \left\{u \in D^{1,2}(\mathbb{R}^3) : u^\pm \neq 0,\ \langle \mathcal{I}_\mu'(u), u^\pm\rangle = 0\right\}.$$

Now we present our main result of this paper.

Theorem 1.1. *Assume (H) holds, then there exists some $\mu^* > 0$ such that system (1.6) possesses a sign-changing solution u_μ with positive energy for any $\mu \in (0, \mu^*)$.*

Remark 1.2. Our result extends the works [23, 24] to critical-concave Schrödinger–Poisson systems. The methods of studying subcritical concave-convex case in [23, 24] are invalid for critical-concave case, we need to search for a sign-changing Palais–Smale sequence and estimate the corresponding level less than a prescribed threshold to recover the loss of compactness. As we know, the Nehari manifold is a good choice to look for two positive solutions of system (1.6) by splitting the manifold into two parts and using minimization arguments on each parts. However, this idea cannot be directly used to prove sign-changing solutions, since we may not verify that the corresponding parts of the sign-changing Nehari manifold are complete. Thus we work in a closed subset of the one part of $\mathcal{M}_\mu$. Moreover, we propose a question whether system (1.6) has (ground state) sign-changing solution with negative energy.

The rest of this paper focus on proving Theorem 1.1 in Sect. 2. Throughout the present paper, we acknowledge the following notations:

- $L^p(\mathbb{R}^3)$ is the usual Lebesgue space with the norm $|u|_p = \left(\int_{\mathbb{R}^3}|u|^p dx\right)^{\frac{1}{p}}$ for $p \in [1, +\infty)$.
- $C_0^\infty(\mathbb{R}^3)$ consists of infinitely times differentiable functions with compact support in $\mathbb{R}^3$.
- $D^{1,2}(\mathbb{R}^3)$ is the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm $\|u\| = \left(\int_{\mathbb{R}^3}|\nabla u|^2 dx\right)^{\frac{1}{2}}$.
- The best Sobolev constant $S = \inf\left\{\|u\|^2 : u \in D^{1,2}(\mathbb{R}^3) \setminus \{0\}, |u|_6 = 1\right\}$.
- For $r > 0$ and $y \in \mathbb{R}^3$, $\mathbb{B}_r(y) := \left\{x \in \mathbb{R}^3 : |x - y| < r\right\}$ and $\mathbb{B}_r^c(y) = \mathbb{R}^3 \setminus \mathbb{B}_r(y)$.
- C_i $(i \in \mathbb{N}_+)$ denotes positive constant and $\mathbb{R}^+ = (0, +\infty)$.
- $o(1)$ denotes a quantity tending to 0 as $n \to \infty$.

2 Proof of Theorem 1.1

In this section, we will present some preliminary lemmas before completing the proof of our main result. For any $u \in D^{1,2}(\mathbb{R}^3)$ with $u^{\pm} \neq 0$, let the function f_u: $[0,+\infty) \times [0,+\infty) \to \mathbb{R}$ be defined as $f_u(s,t) = \mathcal{I}_\mu(su^+ + tu^-)$. From the Fubini theorem, we get

$$\begin{aligned} f_u(s,t) = {} & \frac{s^2}{2}\|u^+\|^2 + \frac{s^4}{4}\int_{\mathbb{R}^3} l(x)\phi_{u^+}|u^+|^2dx - \frac{s^6}{6}|u^+|_6^6 - \frac{\mu s^q}{q}\int_{\mathbb{R}^3} h(x)|u^+|^q dx \\ & + \frac{t^2}{2}\|u^-\|^2 + \frac{t^4}{4}\int_{\mathbb{R}^3} l(x)\phi_{u^-}|u^-|^2dx - \frac{t^6}{6}|u^-|_6^6 - \frac{\mu t^q}{q}\int_{\mathbb{R}^3} h(x)|u^-|^q dx \\ & + \frac{1}{2}s^2t^2\int_{\mathbb{R}^3} l(x)\phi_{u^+}|u^-|^2dx. \end{aligned}$$

Through direct computations, we have

$$\begin{aligned} \frac{\partial f_u}{\partial s}(s,t) = {} & s\|u^+\|^2 + s^3\int_{\mathbb{R}^3} l(x)\phi_{u^+}|u^+|^2dx + st^2\int_{\mathbb{R}^3} l(x)\phi_{u^+}|u^-|^2dx \\ & - s^5|u^+|_6^6 - \mu s^{q-1}\int_{\mathbb{R}^3} h(x)|u^+|^q dx, \end{aligned} \tag{2.1}$$

$$\begin{aligned} \frac{\partial f_u}{\partial t}(s,t) = {} & t\|u^-\|^2 + t^3\int_{\mathbb{R}^3} l(x)\phi_{u^-}|u^-|^2dx + ts^2\int_{\mathbb{R}^3} l(x)\phi_{u^+}|u^-|^2dx \\ & - t^5|u^-|_6^6 - \mu t^{q-1}\int_{\mathbb{R}^3} h(x)|u^-|^q dx, \end{aligned} \tag{2.2}$$

$$\begin{aligned} \frac{\partial^2 f_u}{\partial s^2}(s,t) = {} & \|u^+\|^2 + 3s^2\int_{\mathbb{R}^3} l(x)\phi_{u^+}|u^+|^2dx + t^2\int_{\mathbb{R}^3} l(x)\phi_{u^+}|u^-|^2dx \\ & - 5s^4|u^+|_6^6 - \mu(q-1)s^{q-2}\int_{\mathbb{R}^3} h(x)|u^+|^q dx, \end{aligned} \tag{2.3}$$

$$\begin{aligned} \frac{\partial^2 f_u}{\partial t^2}(s,t) = {} & \|u^-\|^2 + 3t^2\int_{\mathbb{R}^3} l(x)\phi_{u^-}|u^-|^2dx + s^2\int_{\mathbb{R}^3} l(x)\phi_{u^+}|u^-|^2dx \\ & - 5t^4|u^-|_6^6 - \mu(q-1)t^{q-2}\int_{\mathbb{R}^3} h(x)|u^-|^q dx \end{aligned} \tag{2.4}$$

and

$$\frac{\partial^2 f_u}{\partial s\partial t}(s,t) = \frac{\partial^2 f_u}{\partial t\partial s}(s,t) = 2st\int_{\mathbb{R}^3} l(x)\phi_{u^+}|u^-|^2dx. \tag{2.5}$$

It is easy to see that

$$\frac{\partial f_u}{\partial s}(s,t) = \left\langle \mathcal{I}'_\mu(su^+ + tu^-), u^+ \right\rangle \quad \text{and} \quad \frac{\partial f_u}{\partial t}(s,t) = \left\langle \mathcal{I}'_\mu(su^+ + tu^-), u^- \right\rangle,$$

which indicate that, for any $u \in D^{1,2}(\mathbb{R}^3)$ with $u^{\pm} \neq 0$, $su^+ + tu^- \in \mathcal{M}_\mu$ if and only if $\frac{\partial f_u}{\partial s}(s,t) = 0$ and $\frac{\partial f_u}{\partial t}(s,t) = 0$, that is

$$\mathcal{M}_\mu = \left\{ u \in D^{1,2}(\mathbb{R}^3) : u^{\pm} \neq 0,\ \frac{\partial f_u}{\partial s}(1,1) = 0,\ \frac{\partial f_u}{\partial t}(1,1) = 0 \right\}.$$

Moreover, for any $u \in \mathcal{M}_\mu$, by (2.1)–(2.4) we have

$$\begin{aligned}\frac{\partial^2 f_u}{\partial s^2}(1,1) &= \|u^+\|^2 + 3\int_{\mathbb{R}^3} l(x)\phi_{u^+}|u^+|^2dx + \int_{\mathbb{R}^3} l(x)\phi_{u^+}|u^-|^2dx \\ &\quad - 5|u^+|_6^6 - \mu(q-1)\int_{\mathbb{R}^3} h(x)|u^+|^q dx \\ &= (2-q)\|u^+\|^2 + (4-q)\int_{\mathbb{R}^3} l(x)\phi_{u^+}|u^+|^2dx \\ &\quad + (2-q)\int_{\mathbb{R}^3} l(x)\phi_{u^+}|u^-|^2dx - (6-q)|u^+|_6^6 \\ &= -4\|u^+\|^2 - 2\int_{\mathbb{R}^3} l(x)\phi_{u^+}|u^+|^2dx - 4\int_{\mathbb{R}^3} l(x)\phi_{u^+}|u^-|^2dx \\ &\quad - \mu(q-6)\int_{\mathbb{R}^3} h(x)|u^+|^q dx \end{aligned} \tag{2.6}$$

and

$$\begin{aligned}\frac{\partial^2 f_u}{\partial t^2}(1,1) &= \|u^-\|^2 + 3\int_{\mathbb{R}^3} l(x)\phi_{u^-}|u^-|^2dx + \int_{\mathbb{R}^3} l(x)\phi_{u^+}|u^-|^2dx \\ &\quad - 5|u^-|_6^6 - \mu(q-1)\int_{\mathbb{R}^3} h(x)|u^-|^q dx \\ &= (2-q)\|u^-\|^2 + (4-q)\int_{\mathbb{R}^3} l(x)\phi_{u^-}|u^-|^2dx \\ &\quad + (2-q)\int_{\mathbb{R}^3} l(x)\phi_{u^+}|u^-|^2dx - (6-q)|u^-|_6^6 \\ &= -4\|u^-\|^2 - 2\int_{\mathbb{R}^3} l(x)\phi_{u^-}|u^-|^2dx - 4\int_{\mathbb{R}^3} l(x)\phi_{u^+}|u^-|^2dx \\ &\quad - \mu(q-6)\int_{\mathbb{R}^3} h(x)|u^-|^q dx. \end{aligned} \tag{2.7}$$

To begin with, we give some useful properties for the non-local term in the following

Lemma 2.1. *For the unique solution ϕ_u of the Poisson equation, there hold*

(1) $\phi_u(\cdot) \geq 0$ *for any* $u \in D^{1,2}(\mathbb{R}^3)$;

(2) $\phi_{tu} = t^2\phi_u$ *for any* $t > 0$ *and* $u \in D^{1,2}(\mathbb{R}^3)$;

(3) $\int_{\mathbb{R}^3} l(x)\phi_u u^2 dx \leq |\phi_u|_6 \, |lu^2|_{\frac{6}{5}} \leq S^{-1}|l|_2^2|u|_6^4$;

(4) $\int_{\mathbb{R}^3} l(x)\phi_u u^2 dx \leq |\phi_u|_6 \, |lu^2|_{\frac{6}{5}} \leq S^{-1}|l|_3^2|u|_4^4$ *for any* $u \in L^4(\mathbb{R}^3)$;

(5) *if* $u_n \rightharpoonup u$ *in* $D^{1,2}\left(\mathbb{R}^3\right)$, *then* $\phi_{u_n} \rightharpoonup \phi_u$ *in* $D^{1,2}\left(\mathbb{R}^3\right)$;

(6) *if* $u_n \rightharpoonup u$ *in* $D^{1,2}\left(\mathbb{R}^3\right)$, *then* $\int_{\mathbb{R}^3} l(x)\phi_{u_n}u_n^2 dx \to \int_{\mathbb{R}^3} l(x)\phi_u u^2 dx$.

Proof. (1)–(4) are easily verified. (5) and (6) were proved in [14], we omit the details here. □

Lemma 2.2. *There exists some $\mu_1 > 0$ such that, for any $u \in D^{1,2}(\mathbb{R}^3)$ with $u^\pm \neq 0$, there holds that*

(1) *if $\mu \in (0, \mu_1)$, then $f_u(s,t)$ has exactly two critical points $0 < s_1(t) < s_2(t)$ for any fixed $t \geq 0$, where $s_1(t)$ is the minimum point and $s_2(t)$ is the maximum point;*

(2) *if $\mu \in (0, \mu_1)$, then $f_u(s,t)$ has exactly two critical points $0 < t_1(s) < t_2(s)$ for any fixed $s \geq 0$, where $t_1(s)$ is the minimum point and $t_2(s)$ is the maximum point.*

Proof. (1) For any $u \in D^{1,2}(\mathbb{R}^3)$ with $u^{\pm} \neq 0$, by (2.1) we get

$$\frac{\partial f_u}{\partial s}(s,t) = s^{q-1}\bigg(s^{2-q}\|u^+\|^2 + s^{4-q}\int_{\mathbb{R}^3} l(x)\phi_{u^+}|u^+|^2dx + s^{2-q}t^2\int_{\mathbb{R}^3} l(x)\phi_{u^+}|u^-|^2dx - s^{6-q}|u^+|_6^6 - \mu\int_{\mathbb{R}^3} h(x)|u^+|^q dx\bigg).$$

Define the function $\beta_t(s) : [0,+\infty) \to \mathbb{R}$ by

$$\beta_t(s) = s^{2-q}\|u^+\|^2 + s^{4-q}\int_{\mathbb{R}^3} l(x)\phi_{u^+}|u^+|^2dx + s^{2-q}t^2\int_{\mathbb{R}^3} l(x)\phi_{u^+}|u^-|^2dx - s^{6-q}|u^+|_6^6 - \mu\int_{\mathbb{R}^3} h(x)|u^+|^q dx.$$

For any fixed $t \geq 0$, we deduce $\frac{\partial f_u(s,t)}{\partial s} = 0$ iff $\beta_t(s) = 0$ on $(0,+\infty)$. By direct calculation, we derive

$$\begin{aligned}\beta_t'(s) &= (2-q)s^{1-q}\|u^+\|^2 + (4-q)s^{3-q}\int_{\mathbb{R}^3} l(x)\phi_{u^+}|u^+|^2dx \\ &\quad + (2-q)s^{1-q}t^2\int_{\mathbb{R}^3} l(x)\phi_{u^+}|u^-|^2dx - (6-q)s^{5-q}|u^+|_6^6 \\ &= s^{1-q}\bigg[(2-q)\|u^+\|^2 + (4-q)s^2\int_{\mathbb{R}^3} l(x)\phi_{u^+}|u^+|^2dx \\ &\quad + (2-q)t^2\int_{\mathbb{R}^3} l(x)\phi_{u^+}|u^-|^2dx - (6-q)s^4|u^+|_6^6\bigg].\end{aligned}$$

Then, for any fixed $t \geq 0$, it is easy to obtain that $\beta_t(s)$ has exactly one critical point $s_{\max} > 0$ and β_t is increasing in $(0, s_{\max})$ while decreasing in $(s_{\max}, +\infty)$. Moreover, we define $\gamma_t(s) : [0,+\infty) \to \mathbb{R}$ by

$$\gamma_t(s) = s^{2-q}\|u^+\|^2 - s^{6-q}|u^+|_6^6 - \mu\int_{\mathbb{R}^3} h(x)|u^+|^q dx.$$

By direct computation, we have

$$\gamma_t'(s) = (2-q)s^{1-q}\|u^+\|^2 - (6-q)s^{5-q}|u^+|_6^6.$$

Obviously, γ_t takes the maximum at $\bar{s} = \big[\frac{(2-q)\|u^+\|^2}{(6-q)|u^+|_6^6}\big]^{\frac{1}{4}}$ and

$$\max_{s>0}\gamma_t(s) = \gamma_t(\bar{s}) = \frac{4}{6-q}\left(\frac{2-q}{6-q}\right)^{\frac{2-q}{4}}\frac{\|u^+\|^{\frac{6-q}{2}}}{|u^+|_6^{\frac{6-3q}{2}}} - \mu\int_{\mathbb{R}^3} h(x)|u^+|^q dx.$$

Denote

$$\mu_1^+ = \frac{4}{6-q}\left(\frac{2-q}{6-q}\right)^{\frac{2-q}{4}}\inf_{u\in D^{1,2}(\mathbb{R}^3),u^+\neq 0}\frac{\|u^+\|^{\frac{6-q}{2}}}{|u^+|_6^{\frac{6-3q}{2}}\int_{\mathbb{R}^3} h(x)|u^+|^q dx}.$$

It follows from (H), the Hölder and Sobolev inequalities that

$$\mu_1^+ \geq \frac{4}{6-q}\left(\frac{2-q}{6-q}\right)^{\frac{2-q}{4}}\frac{S^{\frac{6-q}{4}}}{|h|_{\frac{6}{6-q}}} =: \mu_1. \tag{2.8}$$

Then, once $\mu \in (0, \mu_1)$, we can deduce $\beta_t(s_{\max}) \geq \beta_t(\bar{s}) > \gamma_t(\bar{s}) > 0$ for any $u \in D^{1,2}(\mathbb{R}^3)$ with $u^+ \neq 0$. As a consequence, for any fixed $t \geq 0$, if $\mu \in (0, \mu_1)$, there exist exactly $s_2(t) > s_1(t) > 0$ such that $\beta_t(s_1(t)) = 0$ and $\beta_t(s_2(t)) = 0$, which imply $\frac{\partial f_u}{\partial s}(s_1(t), t) = \frac{\partial f_u}{\partial s}(s_2(t), t) = 0$. Furthermore, we can easily deduce that f_u is decreasing in $s \in (0, s_1(t))$, increasing in $s \in (s_1(t), s_2(t))$ and decreasing in $s \in (s_2(t), +\infty)$. Thus, $s_1(t)$ is the minimum point and $s_2(t)$ is the maximum point, (1) is verified.

(2) Similar to the proof of (1). We define $\beta_s(t) : [0, +\infty) \to \mathbb{R}$ by

$$\begin{aligned}\beta_s(t) = {} & t^{2-q}\|u^-\|^2 + t^{4-q}\int_{\mathbb{R}^3} l(x)\phi_{u^-}|u^-|^2dx \\ & + t^{2-q}s^2\int_{\mathbb{R}^3} l(x)\phi_{u^+}|u^-|^2dx - t^{6-q}|u^-|_6^6 - \mu\int_{\mathbb{R}^3} h(x)|u^-|^q dx.\end{aligned}$$

For any fixed $s \geq 0$, we deduce $\frac{\partial f_u(s,t)}{\partial t} = 0$ iff $\beta_s(t) = 0$. It can be verified that $\beta_s(t)$ has exactly one critical point $t_{\max} > 0$ and β_s is increasing in $(0, t_{\max})$ while decreasing in $(t_{\max}, +\infty)$. Denote

$$\mu_1^- = \frac{4}{6-q}\left(\frac{2-q}{6-q}\right)^{\frac{2-q}{4}} \inf_{u \in D^{1,2}(\mathbb{R}^3), u^- \neq 0} \frac{\|u^-\|^{\frac{6-q}{2}}}{|u^-|_6^{\frac{6-3q}{2}} \int_{\mathbb{R}^3} h(x)|u^-|^q dx}.$$

It is clear that $\mu_1^+ = \mu_1^- \geq \mu_1$. Repeating the discussion in the proof of (1), we may obtain $\beta_s(t_{\max}) > 0$ for any $\mu \in (0, \mu_1)$. Then, for any fixed $s \geq 0$, if $\mu \in (0, \mu_1)$, there exist exactly $t_2(s) > t_1(s) > 0$ such that $\beta_s(t_1(s)) = \beta_s(t_2(s)) = 0$, which imply $\frac{\partial f_u}{\partial t}(s, t_1(s)) = \frac{\partial f_u}{\partial t}(s, t_2(s)) = 0$. Besides, it is easy to see that f_u is decreasing in $t \in (0, t_1(s))$, increasing in $t \in (t_1(s), t_2(s))$ and decreasing in $t \in (t_2(s), +\infty)$. Naturally, $t_1(s)$ is the minimum point and $t_2(s)$ is the maximum point. Thus, (2) is proved. □

Lemma 2.3. *If $0 < \mu < \mu_1$, then $\frac{\partial^2 f_u}{\partial s^2}(1,1) \neq 0$ and $\frac{\partial^2 f_u}{\partial t^2}(1,1) \neq 0$ for any $u \in \mathcal{M}_\mu$.*

Proof. For any $u \in \mathcal{M}_\mu$, according to Lemma 2.2-(1), $f_u(s,1)$ has exactly two critical points $s_1(1)$ and $s_2(1)$, with $\frac{\partial^2 f_u}{\partial s^2}(s_1(1),1) > 0$ and $\frac{\partial^2 f_u}{\partial s^2}(s_2(1),1) < 0$. Since $u \in \mathcal{M}_\mu$, it follows that $\frac{\partial f_u}{\partial s}(1,1) = 0$. This implies either $s_1(1) = 1$ or $s_2(1) = 1$. Consequently, $\frac{\partial^2 f_u}{\partial s^2}(1,1) \neq 0$. Analogously, by Lemma 2.2-(2) we conclude $\frac{\partial^2 f_u}{\partial t^2}(1,1) \neq 0$. Thus the proof is completed. □

In order to find sign-changing solution of system (1.6), we introduce the following two sets

$$\mathcal{M}_\mu^- = \left\{u \in \mathcal{M}_\mu : \frac{\partial^2 f_u}{\partial s^2}(1,1) < 0,\ \frac{\partial^2 f_u}{\partial t^2}(1,1) < 0\right\}$$

and

$$\mathcal{M}_\mu^* = \left\{u \in \mathcal{M}_\mu : \frac{\partial^2 f_u}{\partial s^2}(1,1) < 0,\ \frac{\partial^2 f_u}{\partial t^2}(1,1) < 0,\ \psi_u''(1) < 0\right\}. \tag{2.9}$$

In the following, we will use the properties of f_u to show that the set $\mathcal{M}_\mu^* \neq \emptyset$ and $\mathcal{M}_\mu^* = \mathcal{M}_\mu^-$.

Lemma 2.4. *There exists some constant $c > 0$, independent of μ, such that $\|u^\pm\| \geq c$ for all $u \in \mathcal{M}_\mu^-$. Moreover, the functional $\mathcal{I}_\mu$ is coercive on $\mathcal{M}_\mu^*$.*

Proof. For any $u \in \mathcal{M}_\mu^-$, from $\frac{\partial^2 f_u}{\partial s^2}(1,1) < 0$, $\frac{\partial^2 f_u}{\partial t^2}(1,1) < 0$, (2.6), (2.7) and the Sobolev inequality, it follows that

$$\begin{aligned}(2-q)\|u^\pm\|^2 &< (2-q)\|u^\pm\|^2 + (4-q)\int_{\mathbb{R}^3} l(x)\phi_{u^\pm}|u^\pm|^2dx + (2-q)\int_{\mathbb{R}^3} l(x)\phi_{u^+}|u^-|^2dx\\ &< (6-q)|u^\pm|_6^6 \le (6-q)S^{-3}\|u^\pm\|^6,\end{aligned}$$

which shows

$$\|u^\pm\| > \left[\frac{(2-q)S^3}{6-q}\right]^{\frac{1}{4}} =: c > 0.$$

Moreover, for any $u \in \mathcal{M}_\mu^*$, by the Hölder inequality one has

$$\begin{aligned}\mathcal{I}_\mu(u) &= \mathcal{I}_\mu(u) - \frac{1}{4}\langle \mathcal{I}'_\mu(u), u\rangle\\ &= \frac{1}{4}\|u\|^2 + \frac{1}{12}|u|_6^6 - \mu\left(\frac{1}{q}-\frac{1}{4}\right)\int_{\mathbb{R}^3} h(x)|u|^q dx\\ &\ge \frac{1}{4}\|u\|^2 + \frac{1}{12}|u|_6^6 - \mu\left(\frac{1}{q}-\frac{1}{4}\right)|h|_{\frac{6}{6-q}}|u|_6^q\\ &\ge \frac{1}{4}\|u\|^2 + \min_{t\ge 0}\left[\frac{1}{12}t^6 - \mu\left(\frac{1}{q}-\frac{1}{4}\right)|h|_{\frac{6}{6-q}}t^q\right]\\ &\ge \frac{1}{4}\|u\|^2 - \mu^{\frac{6}{6-q}}\frac{6-q}{6}\left[\left(\frac{1}{q}-\frac{1}{4}\right)|h|_{\frac{6}{6-q}}(2q)^{\frac{q}{6}}\right]^{\frac{6}{6-q}}.\end{aligned} \tag{2.10}$$

As a consequence, we deduce that $\mathcal{I}_\mu$ is coercive on $\mathcal{M}_\mu^*$. Thus this lemma is proved. □

Lemma 2.5. *There exists $\mu_2 \in (0,\mu_1)$ such that, once $\mu \in (0,\mu_2)$, for any $u \in D^{1,2}(\mathbb{R}^3)$ with $u^\pm \neq 0$, there exists a unique pair $(s_u,t_u) \in \mathbb{R}^+\times\mathbb{R}^+$ satisfying $s_u u^+ + t_u u^- \in \mathcal{M}_\mu^-$. Moreover,*

$$\mathcal{I}_\mu(s_u u^+ + t_u u^-) = \max_{s,t>0}\mathcal{I}_\mu(su^+ + tu^-).$$

Proof. Let $0 < \mu < \mu_1$. For any $u \in D^{1,2}(\mathbb{R}^3)$ with $u^\pm \neq 0$, by Lemma 2.2, we recall that $\frac{\partial f_u}{\partial s}$ satisfies

(a) $\frac{\partial f_u}{\partial t}(s,t_2(s)) = 0$ for all $s \ge 0$,

(b) $\frac{\partial^2 f_u}{\partial t^2}(s,t_2(s)) < 0$ for all $s \ge 0$,

(c) $\frac{\partial f_u}{\partial t}(s,t)$ is continuous and has continuous partial derivatives in $[0,+\infty)\times[0,+\infty)$.

Then, applying the implicit function theorem, we have that $\frac{\partial f_u}{\partial t}(s,t) = 0$ determines an implicit function $t_2(s)$ with continuous derivative on $[0,+\infty)$. Similarly, we may deduce that $\frac{\partial f_u}{\partial s}(s,t) = 0$ determines an implicit function $s_2(t)$ with continuous derivative on $[0,+\infty)$. For any $s \ge 0$, from $\frac{\partial f_u}{\partial t}(s,t_2(s)) = 0$ and $\frac{\partial f_u}{\partial t}(s,t) \le 0$ for sufficiently large $t > 0$, we claim that

$$t_2(s) < s \quad \text{for } s > 0 \text{ large enough.} \tag{2.11}$$

If not, for some sufficiently large s, it follows from the definition (2.2) that $\frac{\partial f_u}{\partial t}(s,t_2(s)) < 0$, which contradicts $\frac{\partial f_u}{\partial t}(s,t_2(s)) = 0$. Analogously, there holds that

$$s_2(t) < t \quad \text{for } t > 0 \text{ large enough.} \tag{2.12}$$

Hence, by (2.11), (2.12), $t_2(0) > 0$, $s_2(0) > 0$, the continuity of $t_2(s)$ and $s_2(t)$, we conclude the curves of $t_2(s)$ and $s_2(t)$ intersect at some point $(s_u, t_u) \in \mathbb{R}^+ \times \mathbb{R}^+$. Naturally, $\frac{\partial f_u}{\partial s}(s_u, t_u) = \frac{\partial f_u}{\partial t}(s_u, t_u) = 0$. In addition, from (2.5) and (b) it follows that

$$t_2'(s) = -\frac{\frac{\partial^2 f_u}{\partial t \partial s}(s, t_2(s))}{\frac{\partial^2 f_u}{\partial t^2}(s, t_2(s))} > 0 \quad \text{for any } s > 0.$$

Therefore, for any $s > 0$, we obtain that the function t_2 is increasing in $s \in (0, +\infty)$. Similarly, the function s_2 is increasing in $t \in (0, +\infty)$. Hence, there exists unique pair $(s_u, t_u) \in \mathbb{R}^+ \times \mathbb{R}^+$ such that

$$\frac{\partial f_u}{\partial s}(s_u, t_u) = \frac{\partial f_u}{\partial t}(s_u, t_u) = 0$$

and

$$\frac{\partial^2 f_u}{\partial s^2}(s_u, t_u) < 0, \qquad \frac{\partial^2 f_u}{\partial t^2}(s_u, t_u) < 0,$$

that is, $s_u u^+ + t_u u^- \in \mathcal{M}_\mu^-$. Moreover, we prove (s_u, t_u) is the unique maximum point of f_u on $(\mathbb{R}^+)^2$. For this, it suffices to show $(1, 1)$ is the unique maximum point of f_u on $(\mathbb{R}^+)^2$ for any $u \in \mathcal{M}_\mu^-$. Denote

$$\bar{\mu}_2 = \frac{4S^{\frac{q}{2}}}{(6-q)q|h|_{\frac{6}{6-q}}} \left(\frac{qc^2}{2-q}\right)^{\frac{2-q}{2}}.$$

Set $\mu_2 = \min\{\mu_1, \bar{\mu}_2\}$ and let $\mu \in (0, \mu_2)$. For any $u \in \mathcal{M}_\mu^-$, we consider the Hessian matrix

$$H(u) = \begin{pmatrix} \frac{\partial^2 f_u}{\partial s^2}(1,1) & \frac{\partial^2 f_u}{\partial s \partial t}(1,1) \\ \frac{\partial^2 f_u}{\partial t \partial s}(1,1) & \frac{\partial^2 f_u}{\partial t^2}(1,1) \end{pmatrix}.$$

With (2.5)−(2.7) in hand, we deduce from Lemma 2.4 and the Hölder inequality that the determinant

$$\begin{aligned}
\det H(u) = & \left[4\|u^+\|^2 + \int_{\mathbb{R}^3} \left(2l(x)\phi_{u^+}|u^+|^2 + 4l(x)\phi_{u^+}|u^-|^2\right) dx - \mu(6-q)\int_{\mathbb{R}^3} h(x)|u^+|^q dx\right] \\
& \times \left[4\|u^-\|^2 + \int_{\mathbb{R}^3} \left(2l(x)\phi_{u^-}|u^-|^2 + 4l(x)\phi_{u^+}|u^-|^2\right) dx - \mu(6-q)\int_{\mathbb{R}^3} h(x)|u^-|^q dx\right] \\
& - 4\left(\int_{\mathbb{R}^3} l(x)\phi_{u^+}|u^-|^2 dx\right)^2 \\
\geq & \left[4\|u^+\|^2 + 4\int_{\mathbb{R}^3} l(x)\phi_{u^+}|u^-|^2 dx - \mu(6-q)S^{-\frac{q}{2}}|h|_{\frac{6}{6-q}}\|u^+\|^q\right] \\
& \times \left[4\|u^-\|^2 + 4\int_{\mathbb{R}^3} l(x)\phi_{u^+}|u^-|^2 dx - \mu(6-q)S^{-\frac{q}{2}}|h|_{\frac{6}{6-q}}\|u^-\|^q\right] \\
& - 4\left(\int_{\mathbb{R}^3} l(x)\phi_{u^+}|u^-|^2 dx\right)^2 \\
\geq & \left\{2\|u^+\|^2 + 4\int_{\mathbb{R}^3} l(x)\phi_{u^+}|u^-|^2 dx - \frac{4-2q}{q}\left[\frac{\mu(6-q)q|h|_{\frac{6}{6-q}}}{4S^{\frac{q}{2}}}\right]^{\frac{2}{2-q}}\right\}
\end{aligned}$$

$$
\begin{aligned}
&\times\left\{2\|u^-\|^2+4\int_{\mathbb{R}^3}l(x)\phi_{u^+}|u^-|^2dx-\frac{4-2q}{q}\left[\frac{\mu(6-q)q|h|_{\frac{6}{6-q}}}{4S^{\frac{q}{2}}}\right]^{\frac{2}{2-q}}\right\}\\
&-4\left(\int_{\mathbb{R}^3}l(x)\phi_{u^+}|u^-|^2dx\right)^2\\
\geq\ &4\int_{\mathbb{R}^3}l(x)\phi_{u^+}|u^-|^2dx\left\{2c^2-\frac{4-2q}{q}\left[\frac{\mu(6-q)q|h|_{\frac{6}{6-q}}}{4S^{\frac{q}{2}}}\right]^{\frac{2}{2-q}}\right\}\\
&+4\int_{\mathbb{R}^3}l(x)\phi_{u^+}|u^-|^2dx\left\{2c^2-\frac{4-2q}{q}\left[\frac{\mu(6-q)q|h|_{\frac{6}{6-q}}}{4S^{\frac{q}{2}}}\right]^{\frac{2}{2-q}}\right\}\\
&+\left\{2c^2-\frac{4-2q}{q}\left[\frac{\mu(6-q)q|h|_{\frac{6}{6-q}}}{4S^{\frac{q}{2}}}\right]^{\frac{2}{2-q}}\right\}\times\left\{2c^2-\frac{4-2q}{q}\left[\frac{\mu(6-q)q|h|_{\frac{6}{6-q}}}{4S^{\frac{q}{2}}}\right]^{\frac{2}{2-q}}\right\}\\
>\ &0,
\end{aligned}
$$

which together with $\frac{\partial^2 f_u}{\partial s^2}(1,1)<0$ implies that $H(u)$ is negative definite. Consequently, there holds that $\mathcal{I}_\mu(u^++u^-)=\max_{s,t>0}\mathcal{I}_\mu(su^++tu^-)$ for any $u\in\mathcal{M}_\mu^-$. Therefore, the proof is completed. □

Lemma 2.6. *Once $\mu\in(0,\mu_2)$, then $\mathcal{M}_\mu^*\neq\emptyset$ and $\mathcal{M}_\mu^*=\mathcal{M}_\mu^-$.*

Proof. It is obvious that $\mathcal{M}_\mu^*\subset\mathcal{M}_\mu^-$. Moreover, for any $u\in\mathcal{M}_\mu^-$, it follows from the proof of Lemma 2.5 that $(1,1)$ is the unique maximum point of $f_u(s,t)$. Observe $f_u(r,r)=\psi_u(r)$, we know that 1 is a maximum point of ψ_u. Naturally, $\psi_u''(1)<0$. By the arbitrariness of $u\in\mathcal{M}_\mu^-$, we derive $\mathcal{M}_\mu^-\subset\mathcal{M}_\mu^*$. That is, $\mathcal{M}_\mu^-=\mathcal{M}_\mu^*$. Then, due to Lemma 2.5, we know $\mathcal{M}_\mu^*\neq\emptyset$. Thus the proof is completed. □

In the forthcoming lemmas, we construct a minimizing sequence of Palais–Smale type for

$$m_\mu^*=\inf_{u\in\mathcal{M}_\mu^*}\mathcal{I}_\mu(u).$$

First, we estimate the level m_μ^*. As we know, the extremal functions of the best Sobolev constant S are

$$U_\varepsilon(\cdot)=\frac{3^{\frac{1}{4}}\varepsilon^{\frac{1}{2}}}{(\varepsilon^2+|\cdot|^2)^{\frac{1}{2}}},\qquad \varepsilon>0.$$

Choose two cut-off functions $\varphi,\psi\in C_0^\infty\left(\mathbb{R}^3,[0,1]\right)$ such that,

$$\varphi(x)=\begin{cases}1, & x\in\mathbb{B}_1(0),\\ 0, & x\in\mathbb{B}_2^c(0),\end{cases}$$

and

$$\psi(x)=\begin{cases}0, & x\in\mathbb{B}_2(0),\\ 1, & x\in\mathbb{B}_4(0)\setminus\mathbb{B}_3(0),\\ 0, & x\in\mathbb{B}_5^c(0).\end{cases}$$

Define $u_\varepsilon(x) = \varphi(x)U_\varepsilon(x)$, $v_\varepsilon(x) = -\psi(x)U_\varepsilon(x)$ and $w_\varepsilon(x) = u_\varepsilon(x) + v_\varepsilon(x)$. Clearly, $w_\varepsilon^+(x) = u_\varepsilon(x)$ and $w_\varepsilon^-(x) = v_\varepsilon(x)$. Following the similar calculations to [3], we conclude that, for $\varepsilon > 0$ small enough,

$$\begin{cases} \int_{\mathbb{R}^3} |\nabla u_\varepsilon|^2 dx = S^{\frac{3}{2}} + O(\varepsilon), \\ \int_{\mathbb{R}^3} |u_\varepsilon|^6 dx = S^{\frac{3}{2}} + O(\varepsilon^3), \\ \int_{\mathbb{R}^3} |u_\varepsilon|^4 dx = O(\varepsilon), \\ \int_{\mathbb{R}^3} |\nabla v_\varepsilon|^2 dx = O(\varepsilon), \\ \int_{\mathbb{R}^3} |v_\varepsilon|^6 dx \in \left[C_1\varepsilon^3, C_2\varepsilon^3\right]. \end{cases} \tag{2.13}$$

Lemma 2.7. *There exists $\mu_3 \in (0, \mu_2)$ such that $0 < m_\mu^* < \frac{1}{3}S^{\frac{3}{2}} - D\mu^{\frac{6}{6-q}}$ for any $\mu \in (0, \mu_3)$, where $D = \frac{6-q}{6}\left[\left(\frac{1}{q} - \frac{1}{4}\right)|h|_{\frac{6}{6-q}}(2q)^{\frac{q}{6}}\right]^{\frac{6}{6-q}}$.*

Proof. Firstly, let

$$\mu < \min\left\{\mu_2, \frac{4q}{(4-q)|h|_{\frac{6}{6-q}}(2q)^{\frac{q}{6}}}\left[\frac{3c^2}{4(6-q)}\right]^{\frac{6-q}{6}}\right\}.$$

From (2.10) and Lemma 2.4, it follows that

$$\mathcal{I}_\mu(u) \geq \frac{1}{4}c^2 - \mu^{\frac{6}{6-q}}\frac{6-q}{6}\left[\left(\frac{1}{q} - \frac{1}{4}\right)|h|_{\frac{6}{6-q}}(2q)^{\frac{q}{6}}\right]^{\frac{6}{6-q}} \geq \frac{1}{8}c^2, \quad \forall\, u \in \mathcal{M}_\mu^*,$$

which implies $m_\mu^* > 0$. Due to Lemma 2.5, there exist some $s_\varepsilon, t_\varepsilon > 0$ such that $s_\varepsilon u_\varepsilon + t_\varepsilon v_\varepsilon \in \mathcal{M}_\mu^*$. Observe $\operatorname{spt} u_\varepsilon \cap \operatorname{spt} v_\varepsilon = \emptyset$, we have

$$\begin{aligned} \mathcal{I}_\mu(s_\varepsilon u_\varepsilon + t_\varepsilon v_\varepsilon) = {} & \frac{s_\varepsilon^2}{2}\|u_\varepsilon\|^2 + \frac{s_\varepsilon^4}{4}\int_{\mathbb{R}^3} l(x)\phi_{u_\varepsilon}u_\varepsilon^2 dx - \frac{s_\varepsilon^6}{6}|u_\varepsilon|_6^6 - \frac{\mu s_\varepsilon^q}{q}\int_{\mathbb{R}^3} h(x)|u_\varepsilon|^q dx \\ & + \frac{t_\varepsilon^2}{2}\|v_\varepsilon\|^2 + \frac{t_\varepsilon^4}{4}\int_{\mathbb{R}^3} l(x)\phi_{v_\varepsilon}v_\varepsilon^2 dx - \frac{t_\varepsilon^6}{6}|v_\varepsilon|_6^6 - \frac{\mu t_\varepsilon^q}{q}\int_{\mathbb{R}^3} h(x)|v_\varepsilon|^q dx \\ & + \frac{1}{2}\int_{\mathbb{R}^3} l(x)\phi_{s_\varepsilon u_\varepsilon}|t_\varepsilon v_\varepsilon|^2 dx. \end{aligned} \tag{2.14}$$

By the semigroup property of convolution (see [16, Theorem 5.9]) and the Young inequality, we have

$$\begin{aligned} & \int_{\mathbb{R}^3} l(x)\phi_{s_\varepsilon u_\varepsilon}|t_\varepsilon v_\varepsilon|^2 dx \\ & = \frac{1}{4\pi}\int_{\mathbb{R}^3}\left(\frac{1}{|x|} * l(x)|s_\varepsilon u_\varepsilon|^2\right) l(x)|t_\varepsilon v_\varepsilon|^2 dx \\ & = \frac{1}{4\pi}\int_{\mathbb{R}^3}\left(\frac{1}{|x|^2} * l(x)|s_\varepsilon u_\varepsilon|^2\right)\left(\frac{1}{|x|^2} * l(x)|t_\varepsilon v_\varepsilon|^2\right) dx \\ & \leq \frac{1}{8\pi}\int_{\mathbb{R}^3}\left(\frac{1}{|x|^2} * l(x)|s_\varepsilon u_\varepsilon|^2\right)^2 + \left(\frac{1}{|x|^2} * l(x)|t_\varepsilon v_\varepsilon|^2\right)^2 dx \\ & = \frac{1}{8\pi}\int_{\mathbb{R}^3}\left(\frac{1}{|x|} * l(x)|s_\varepsilon u_\varepsilon|^2\right) l(x)|s_\varepsilon v_\varepsilon|^2 + \left(\frac{1}{|x|} * l(x)|t_\varepsilon u_\varepsilon|^2\right) l(x)|t_\varepsilon v_\varepsilon|^2 dx \\ & = \frac{s_\varepsilon^4}{2}\int_{\mathbb{R}^3} l(x)\phi_{u_\varepsilon}u_\varepsilon^2 dx + \frac{t_\varepsilon^4}{2}\int_{\mathbb{R}^3} l(x)\phi_{v_\varepsilon}v_\varepsilon^2 dx. \end{aligned}$$

Then, (2.14) can be reduced to

$$\mathcal{I}_\mu(s_\varepsilon u_\varepsilon + t_\varepsilon v_\varepsilon) \le \frac{s_\varepsilon^2}{2}\int_{\mathbb{R}^3}|\nabla u_\varepsilon|^2dx + \frac{s_\varepsilon^4}{2}\int_{\mathbb{R}^3} l(x)\phi_{u_\varepsilon}|u_\varepsilon|^2dx - \frac{s_\varepsilon^6}{6}|u_\varepsilon|_6^6 - \frac{\mu s_\varepsilon^q}{q}\int_{\mathbb{R}^3} h(x)|u_\varepsilon|^q dx + \frac{t_\varepsilon^2}{2}\int_{\mathbb{R}^3}|\nabla v_\varepsilon|^2dx + \frac{t_\varepsilon^4}{2}\int_{\mathbb{R}^3} l(x)\phi_{v_\varepsilon}|v_\varepsilon|^2dx - \frac{t_\varepsilon^6}{6}|v_\varepsilon|_6^6. \tag{2.15}$$

For convenience, we introduce

$$\mathcal{J}_1(\varepsilon) = \frac{s_\varepsilon^2}{2}\int_{\mathbb{R}^3}|\nabla u_\varepsilon|^2dx + \frac{s_\varepsilon^4}{2}\int_{\mathbb{R}^3} l(x)\phi_{u_\varepsilon}u_\varepsilon^2dx - \frac{s_\varepsilon^6}{6}|u_\varepsilon|_6^6 - \frac{\mu s_\varepsilon^q}{q}\int_{\mathbb{R}^3} h(x)|u_\varepsilon|^q dx,$$
$$\mathcal{J}_2(\varepsilon) = \frac{t_\varepsilon^2}{2}\int_{\mathbb{R}^3}|\nabla v_\varepsilon|^2dx + \frac{t_\varepsilon^4}{2}\int_{\mathbb{R}^3} l(x)\phi_{v_\varepsilon}v_\varepsilon^2dx - \frac{t_\varepsilon^6}{6}|v_\varepsilon|_6^6.$$

By Lemma 2.1-(3) and (2.13), for $\varepsilon > 0$ small enough, we get

$$\begin{aligned}\mathcal{J}_1(\varepsilon) &\le s_\varepsilon^2 S^{\frac{3}{2}} + s_\varepsilon^4|l|_2^2 - \frac{s_\varepsilon^6}{12}S^{\frac{3}{2}},\\ \mathcal{J}_2(\varepsilon) &\le C_3 t_\varepsilon^2\varepsilon + C_4 t_\varepsilon^4\varepsilon^2 - C_1 t_\varepsilon^6\varepsilon^3\\ &= C_3\left(t_\varepsilon\sqrt{\varepsilon}\right)^2 + C_4\left(t_\varepsilon\sqrt{\varepsilon}\right)^4 - C_1\left(t_\varepsilon\sqrt{\varepsilon}\right)^6.\end{aligned} \tag{2.16}$$

Define $\mathcal{G}(t) = C_3t^2 + C_4t^4 - C_1t^6$ for $t \ge 0$. We infer that $\mathcal{G}$ has exactly one critical point $t_{\max} > 0$ and $\mathcal{G}(t_{\max}) > \mathcal{G}(0) = 0$. Hence,

$$\mathcal{J}_2(\varepsilon) \le \max_{t\ge 0}\mathcal{G}(t) = \mathcal{G}(t_{\max}). \tag{2.17}$$

Noting $m_\mu^* \le \mathcal{I}_\mu(s_\varepsilon u_\varepsilon + t_\varepsilon v_\varepsilon)$, we deduce from (2.15)−(2.17) that s_ε is bounded from above and below by positive constants. Namely, there exist $S_1, S_2 > 0$ such that $S_1 \le s_\varepsilon \le S_2$ for $\varepsilon > 0$ small enough. From Lemma 2.5, we know $t_\varepsilon = t_2(s_\varepsilon)$ and t_ε is continuous on $[0, +\infty)$. Therefore, t_ε is also bounded from above. Then, by (2.15), (2.13) and Lemma 2.1-(4) we can obtain that, for $\varepsilon > 0$ small enough,

$$\begin{aligned}&\mathcal{I}_\mu(s_\varepsilon u_\varepsilon + t_\varepsilon v_\varepsilon)\\ &\le \max_{s>0}\left(\frac{s^2}{2}\|u_\varepsilon\|^2 - \frac{s^6}{6}|u_\varepsilon|_6^6\right) + \frac{S_2^4}{2}\int_{\mathbb{R}^3} l(x)\phi_{u_\varepsilon}u_\varepsilon^2dx - \frac{\mu S_1^q}{q}\int_{\mathbb{R}^3} h(x)|u_\varepsilon|^q dx + O(\varepsilon)\\ &\le \frac{\|u_\varepsilon\|^3}{3|u_\varepsilon|_6^3} + \frac{1}{2}S_2^4S^{-1}|l|_3^2|u_\varepsilon|_4^4 - \frac{\mu S_1^q}{q}\int_{\mathbb{R}^3} h(x)|u_\varepsilon|^q dx + O(\varepsilon)\\ &\le \frac{1}{3}\frac{\left[S^{\frac{3}{2}} + O(\varepsilon)\right]^{\frac{3}{2}}}{\left[S^{\frac{3}{2}} + O(\varepsilon^3)\right]^{\frac{1}{2}}} - \frac{\mu S_1^q}{q}\int_{\mathbb{R}^3} h(x)|u_\varepsilon|^q dx + O(\varepsilon)\\ &\le \frac{1}{3}S^{\frac{3}{2}} + C_5\varepsilon - \frac{\mu S_1^q}{q}\int_{\mathbb{R}^3} h(x)|u_\varepsilon|^q dx.\end{aligned}$$

Recalling that $h(x) \geq \frac{1}{|x|^\beta}$ for all $|x| \leq 1$ and some $\beta > 3 - \frac{2q}{3}$, if letting $\varepsilon = \mu^{\frac{6}{6-q}}$, we conclude

$$\begin{aligned}
&C_5\varepsilon - \frac{\mu S_1^q}{q}\int_{\mathbb{R}^3} h(x)|u_\varepsilon|^q dx \\
&\leq C_5\varepsilon - \frac{\mu S_1^q}{q}\int_{\mathbb{B}_1(0)} \frac{3^{\frac{q}{4}}\varepsilon^{\frac{q}{2}}}{|x|^\beta(\varepsilon^2+|x|^2)^{\frac{q}{2}}}dx \\
&= C_5\varepsilon - \mu\varepsilon^{3-\frac{q}{2}-\beta}\frac{S_1^q}{q}\int_{\mathbb{B}_{\frac{1}{\varepsilon}}(0)} \frac{3^{\frac{q}{4}}}{|x|^\beta(1+|x|^2)^{\frac{q}{2}}}dx \\
&\leq C_5\varepsilon - C_6\mu\varepsilon^{3-\frac{q}{2}-\beta} \\
&= \mu^{\frac{6}{6-q}}\left[C_5 - C_6\mu^{1+\frac{6}{6-q}(2-\frac{q}{2}-\beta)}\right] \\
&< -D\mu^{\frac{6}{6-q}}
\end{aligned}$$

for $\mu > 0$ small enough. Thus, there exists some small

$$\mu_3 < \min\left\{\mu_2, \frac{4q}{(4-q)|h|_{\frac{6}{6-q}}(2q)^{\frac{q}{6}}}\left[\frac{3c^2}{4(6-q)}\right]^{\frac{6-q}{6}}\right\}$$

such that $m_\mu^* \leq \mathcal{I}_\mu(s_\varepsilon u_\varepsilon + t_\varepsilon v_\varepsilon) < \frac{1}{3}S^{\frac{3}{2}} - D\mu^{\frac{2}{2-q}}$ for any $\mu \in (0, \mu_3)$, the proof is completed. □

Lemma 2.8. *There exists some $\mu^* \in (0, \mu_3)$ such that m_μ^* has a minimizing sequence $\{u_n\} \subset \mathcal{M}_\mu^*$ satisfying $\mathcal{I}'_\mu(u_n) \xrightarrow{n} 0$ in $[D^{1,2}(\mathbb{R}^3)]^*$ for any $\mu \in (0, \mu^*)$.*

Proof. Observe that $\mathcal{I}_\mu$ is coercive on $\mathcal{M}_\mu^*$ due to Lemma 2.4, so $\mathcal{I}_\mu$ is bounded from blow on $\mathcal{M}_\mu^*$. It follows from Lemmas 2.3 and 2.4 that $\mathcal{M}_\mu^*$ is complete. Then, by the Ekeland variational principle, there exists a minimizing sequence $\{u_n\} \subset \mathcal{M}_\mu^*$ such that

$$\begin{aligned}
&m_\mu^* \leq \mathcal{I}_\mu(u_n) \leq m_\mu^* + \frac{1}{n}, \\
&\mathcal{I}_\mu(v) \geq \mathcal{I}_\mu(u_n) - \frac{1}{n}\|v - u_n\|, \qquad \forall\, v \in M_\mu^*.
\end{aligned} \tag{2.18}$$

For each n and $\varphi \in D^{1,2}(\mathbb{R}^3)$, we define the functions $h_n^\pm : \mathbb{R}^3 \to \mathbb{R}$ by

$$\begin{aligned}
h_n^\pm(t,s,\tau) = &\int_{\mathbb{R}^3} \left|\nabla(u_n + t\varphi + su_n^+ + \tau u_n^-)^\pm\right|^2 dx \\
&+ \int_{\mathbb{R}^3} l(x)\phi_{u_n+t\varphi+su_n^+\tau u_n^-}|(u_n + t\varphi + su_n^+ + \tau u_n^-)^\pm|^2 dx \\
&- \int_{\mathbb{R}^3} \left|(u_n + t\varphi + su_n^+ + \tau u_n^-)^\pm\right|^6 dx - \mu\int_{\mathbb{R}^3} h(x)\left|(u_n + t\varphi + su_n^+ + \tau u_n^-)^\pm\right|^q dx.
\end{aligned}$$

It is clear that $h_n^\pm(0,0,0) = 0$ and $h_n^\pm(t,s,\tau)$ are of class C^1. Let $v_n = u_n + t\varphi + su_n^+ + \tau u_n^-$. We have

$$\begin{aligned}
\frac{\partial h_n^+(t,s,\tau)}{\partial s} = &2\int_{\mathbb{R}^3} \nabla v_n^+ \cdot \nabla u_n^+ dx + 2\int_{\mathbb{R}^3} l(x)\phi_{v_n}v_n^+u_n^+ dx \\
&+ 2\int_{\mathbb{R}^3} l(x)\phi_{v_n^+}v_n u_n^+ dx - 6\int_{\mathbb{R}^3} |v_n^+|^5 u_n^+ dx - \mu q\int_{\mathbb{R}^3} h(x)|v_n^+|^{q-1}u_n^+ dx.
\end{aligned}$$

Because of $\{u_n\} \subset \mathcal{M}_\mu^*$, there holds

$$\frac{\partial h_n^+(0,0,0)}{\partial s} = 2\int_{\mathbb{R}^3} l(x)\phi_{u_n^+}|u_n^+|^2dx - 4|u_n^+|_6^6 + \mu(2-q)\int_{\mathbb{R}^3} h(x)|u_n^+|^q dx.$$

Through direct calculation, we have

$$\begin{aligned}\frac{\partial h_n^+(t,s,\tau)}{\partial \tau} &= 2\int_{\mathbb{R}^3} \nabla v_n^+ \cdot \nabla u_n^- dx + 2\int_{\mathbb{R}^3} l(x)\phi_{v_n} v_n^+ u_n^- dx \\ &\quad + 2\int_{\mathbb{R}^3} l(x)\phi_{v_n^+} v_n u_n^- dx - 6\int_{\mathbb{R}^3} |v_n^+|^5 u_n^- dx - \mu q\int_{\mathbb{R}^3} h(x)|v_n^+|^{q-1}u_n^- dx,\end{aligned}$$

which implies

$$\frac{\partial h_n^+(0,0,0)}{\partial \tau} = 2\int_{\mathbb{R}^3} l(x)\phi_{u_n^+}|u_n^-|^2dx > 0. \tag{2.19}$$

Analogously, we can obtain

$$\frac{\partial h_n^-(0,0,0)}{\partial \tau} = 2\int_{\mathbb{R}^3} l(x)\phi_{u_n^-}|u_n^-|^2dx - 4|u_n^-|_6^6 + \mu(2-q)\int_{\mathbb{R}^3} h(x)|u_n^-|^q dx \tag{2.20}$$

and

$$\frac{\partial h_n^-(0,0,0)}{\partial s} = 2\int_{\mathbb{R}^3} l(x)\phi_{u_n^-}|u_n^+|^2dx > 0.$$

Consider the Jacobi determinant

$$\det M = \begin{vmatrix} \frac{\partial h^+(0,0,0)}{\partial s} & \frac{\partial h^+(0,0,0)}{\partial \tau} \\ \frac{\partial h^-(0,0,0)}{\partial s} & \frac{\partial h^-(0,0,0)}{\partial \tau} \end{vmatrix}.$$

It is easy to see that

$$\begin{aligned}\det M &= \left[2\int_{\mathbb{R}^3} l(x)\phi_{u_n^+}|u_n^+|^2dx - 4|u_n^+|_6^6 + \mu(2-q)\int_{\mathbb{R}^3} h(x)|u_n^+|^q dx\right] \\ &\quad \times \left[2\int_{\mathbb{R}^3} l(x)\phi_{u_n^-}|u_n^-|^2dx - 4|u_n^-|_6^6 + \mu\,(2-q)\int_{\mathbb{R}^3} h(x)|u_n^-|^q dx\right] \\ &\quad - 4\left(\int_{\mathbb{R}^3} l(x)\phi_{u_n^-}|u_n^+|^2dx\right)^2.\end{aligned}$$

Since $\{u_n\} \subset \mathcal{M}_\mu^*$, there results that

$$\begin{aligned}\|u_n^+\|^2 + \int_{\mathbb{R}^3} l(x)\phi_{u_n^+}|u_n^+|^2dx + \int_{\mathbb{R}^3} l(x)\phi_{u_n^-}|u_n^+|^2dx - |u_n^+|_6^6 - \mu\int_{\mathbb{R}^3} h(x)|u_n^+|^q dx = 0, \\ \|u_n^-\|^2 + \int_{\mathbb{R}^3} l(x)\phi_{u_n^-}|u_n^-|^2dx + \int_{\mathbb{R}^3} l(x)\phi_{u_n^+}|u_n^-|^2dx - |u_n^-|_6^6 - \mu\int_{\mathbb{R}^3} h(x)|u_n^-|^q dx = 0.\end{aligned}$$

Therefore,

$$\begin{aligned}\det M = &\left[-2\|u_n^+\|^2 - 2\int_{\mathbb{R}^3} l(x)\phi_{u_n^+}|u_n^-|^2dx - 2|u_n^+|_6^6 + \mu(4-q)\int_{\mathbb{R}^3} h(x)|u_n^+|^q dx\right] \\ &\times \left[-2\|u_n^-\|^2 - 2\int_{\mathbb{R}^3} l(x)\phi_{u_n^+}|u_n^-|^2dx - 2|u_n^-|_6^6 + \mu(4-q)\int_{\mathbb{R}^3} h(x)|u_n^-|^q dx\right] \\ &- 4\left(\int_{\mathbb{R}^3} l(x)\phi_{u_n^-}|u_n^+|^2dx\right)^2 \\ \geq\ &4\|u_n^+\|^2\|u_n^-\|^2 - 2\mu(4-q)\left(\|u_n^-\|^2 + |u_n^-|_6^6 + \int_{\mathbb{R}^3} l(x)\phi_{u_n^+}|u_n^-|^2dx\right)\int_{\mathbb{R}^3} h(x)|u_n^+|^q dx \\ &- 2\mu(4-q)\left(\|u_n^+\|^2 + |u_n^+|_6^6 + \int_{\mathbb{R}^3} l(x)\phi_{u_n^+}|u_n^-|^2dx\right)\int_{\mathbb{R}^3} h(x)|u_n^-|^q dx.\end{aligned} \tag{2.21}$$

Since Lemma 2.7 and (2.10) imply

$$\frac{1}{3}S^{\frac{3}{2}} \geq \frac{1}{4}\|u_n\|^2 - \mu_3^{\frac{6}{6-q}}\frac{6-q}{6}\left[\left(\frac{1}{q}-\frac{1}{4}\right)|h|_{\frac{6}{6-q}}(2q)^{\frac{q}{6}}\right]^{\frac{6}{6-q}},$$

there exists some $C_7 > 0$, independent of μ, such that $\sup_n \|u_n\| \leq C_7$. Then, by the Hölder inequality and Lemma 2.1-(3), we have

$$|u_n^{\pm}|_6^6 \leq S^{-3}\|u_n^{\pm}\|^6 \leq S^{-3}C_7^6$$

and

$$\int_{\mathbb{R}^3} h(x)|u_n^{\pm}|^q dx \leq |h|_{\frac{6}{6-q}}|u_n^{\pm}|_6^q \leq S^{-\frac{q}{2}}|h|_{\frac{6}{6-q}}\|u_n^{\pm}\|^q \leq S^{-\frac{q}{2}}|h|_{\frac{6}{6-q}}C_7^q,$$
$$\int_{\mathbb{R}^3} l(x)\phi_{u_n^+}|u_n^-|^2 dx \leq \left|\phi_{u_n^+}\right|_6\left|l(x)|u_n^-|^2\right|_{\frac{6}{5}} \leq S^{-\frac{1}{2}}|l|_2\left\|\phi_{u_n^+}\right\||u_n^-|_6^2 \leq S^{-3}|l|_2^2C_7^4.$$

Then, if letting $\mu < \mu^*$, where

$$\mu^* = \frac{c^4 S^{\frac{q}{2}}}{2(4-q)|h|_{\frac{6}{6-q}}C_7^{2+q}(1+S^{-3}C_7^4+S^{-3}|l|_2^2C_7^2)},$$

by (2.21) and Lemma 2.4, we obtain

$$\det M \geq 2c^4 > 0. \tag{2.22}$$

Therefore, by the implicit function theorem, there exist δ_n and two functions $s_n(t), \tau_n(t) \in C^1((-\delta_n, \delta_n), \mathbb{R})$ such that $s_n(0) = \tau_n(0)$ and for any $t \in (-\delta_n, \delta_n)$,

$$h_n^{\pm}(t, s_n(t), \tau_n(t)) = 0, \tag{2.23}$$

which means that

$$v_n = u_n + t\varphi + s_n(t)u_n^+ + \tau_n(t)u_n^- \in \mathcal{M}_\mu \quad \text{for } t \in (-\delta_n, \delta_n).$$

Further, according to (2.3) and (2.4), by the continuity, there exists some $\delta_n' \in (0, \delta_n)$ such that $\frac{\partial^2 f_{v_n}}{\partial s^2}(1,1) < 0$ and $\frac{\partial^2 f_{v_n}}{\partial t^2}(1,1) < 0$, i.e.

$$v_n = u_n + t\varphi + s_n(t)u_n^+ + \tau_n(t)u_n^- \in \mathcal{M}_\mu^* \quad \text{for } t \in (-\delta_n', \delta_n'). \tag{2.24}$$

We assert that both $\left\{\frac{s_n'(0)}{\|\varphi\|}\right\}$ and $\left\{\frac{\tau_n'(0)}{\|\varphi\|}\right\}$ are bounded for any $\varphi \in D^{1,2}(\mathbb{R}^3) \setminus \{0\}$. It follows from (2.23) that

$$\begin{cases} \dfrac{\partial h_n^+(0,0,0)}{\partial t} + \dfrac{\partial h_n^+(0,0,0)}{\partial s}s_n'(0) + \dfrac{\partial h_n^+(0,0,0)}{\partial \tau}\tau_n'(0) = 0, \\ \dfrac{\partial h_n^-(0,0,0)}{\partial t} + \dfrac{\partial h_n^-(0,0,0)}{\partial s}s_n'(0) + \dfrac{\partial h_n^-(0,0,0)}{\partial \tau}\tau_n'(0) = 0, \end{cases}$$

where

$$\begin{aligned}\frac{\partial h_n^{\pm}(0,0,0)}{\partial t} &= 2\int_{\mathbb{R}^3}\nabla u_n^{\pm}\cdot\nabla\varphi dx + 2\int_{\mathbb{R}^3} l(x)\phi_{u_n^{\pm}}u_n\varphi dx \\ &\quad + 2\int_{\mathbb{R}^3} l(x)\phi_{u_n}u_n^{\pm}\varphi dx - 6\int_{\mathbb{R}^3}|u_n^{\pm}|^5\varphi dx - \mu q\int_{\mathbb{R}^3}h(x)|u_n^{\pm}|^{q-1}\varphi dx.\end{aligned} \tag{2.25}$$

By Cramer's rule, there results

$$s_n'(0) = \frac{1}{\det M}\left(\frac{\partial h_n^-}{\partial t}\frac{\partial h_n^+}{\partial \tau} - \frac{\partial h_n^+}{\partial t}\frac{\partial h_n^-}{\partial \tau}\right)\Big|_{(0,0,0)}. \tag{2.26}$$

Recalling $\sup_n \|u_n\| \le C_7$, from (2.19), (2.20), (2.25), Lemma 2.1 and the Hölder inequality we deduce

$$\left|\left(\frac{\partial h_n^+}{\partial t}\frac{\partial h_n^-}{\partial \tau} - \frac{\partial h_n^-}{\partial t}\frac{\partial h_n^+}{\partial \tau}\right)\Big|_{(0,0,0)}\right| \le C_8\|\varphi\|.$$

Then, (2.22) and (2.26) imply that $|s_n'(0)| \le C_9\|\varphi\|$ for any $\varphi \in D^{1,2}(\mathbb{R}^3)\setminus\{0\}$. Similarly, we deduce that $|\tau_n'(0)| \le C_{10}\|\varphi\|$ for any $\varphi \in D^{1,2}(\mathbb{R}^3)\setminus\{0\}$. Due to (2.18), there holds that

$$\mathcal{I}_\mu(u_n) - \mathcal{I}_\mu\left(u_n + t\varphi + s_n(t)u_n^+ + \tau_n(t)u_n^-\right) \le \frac{1}{n}\|t\varphi + s_n(t)u_n^+ + \tau_n(t)u_n^-\|.$$

Dividing t in both sides of the above inequality and letting $t \to 0^+$, by $\sup_n \|u_n\| \le C_7$, $|s_n'(0)| \le C_9\|\varphi\|$ and $|\tau_n'(0)| \le C_{10}\|\varphi\|$ we have

$$\left\langle \mathcal{I}_\mu'(u_n), \varphi\right\rangle \le \frac{1}{n}\left\|\varphi + s_n'(0)u_n^+ + \tau_n'(0)u_n^-\right\| \le \frac{C_{11}}{n}\|\varphi\|, \quad \forall\ \varphi \in D^{1,2}(\mathbb{R}^3),$$

which implies

$$\left|\left\langle \mathcal{I}_\mu'(u_n), \varphi\right\rangle\right| \le \frac{C_{11}}{n}\|\varphi\|, \quad \forall\ \varphi \in D^{1,2}(\mathbb{R}^3).$$

Consequently, we obtain that $\mathcal{I}_\mu'(u_n) \to 0$ as $n \to \infty$. Thus, the proof is completed. □

Lemma 2.9. *For any $\mu \in (0, \mu^*)$, every sequence $\{u_n\} \subset \mathcal{M}_\mu^*$ such that $\mathcal{I}_\mu(u_n) \xrightarrow{n} m_\mu^*$ and $\mathcal{I}_\mu'(u_n) \xrightarrow{n} 0$ has a convergent subsequence. Moreover, up to a subsequence, $u_n^\pm \xrightarrow{n} u_\mu^\pm$ in $D^{1,2}(\mathbb{R}^3)$.*

Proof. Since Lemma 2.4 implies that $\mathcal{I}_\mu$ is coercive on $\mathcal{M}_\mu^*$ and Lemma 2.7 implies $m_\mu^* < +\infty$, we deduce that $\{u_n\}$ is bounded in $D^{1,2}(\mathbb{R}^3)$. Up to a subsequence, there exists some $u_\mu \in D^{1,2}(\mathbb{R}^3)$ such that, as $n \to \infty$,

$$u_n \rightharpoonup u_\mu \qquad \text{in } D^{1,2}(\mathbb{R}^3)\cap L^6(\mathbb{R}^3), \tag{2.27}$$

$$u_n(x) \to u_\mu(x) \quad \text{a.e. in } \mathbb{R}^3. \tag{2.28}$$

By using standard arguments, we derive $\mathcal{I}_\mu'(u_\mu) = 0$. From (2.28), it easily follows that $|u_n|^q \rightharpoonup |u_\mu|^q$ in $L^{\frac{6}{q}}(\mathbb{R}^3)$ in the sense of subsequence, which and $h \in L^{\frac{6}{6-q}}(\mathbb{R}^3)$ imply

$$\int_{\mathbb{R}^3} h(x)|u_n|^q dx \xrightarrow{n} \int_{\mathbb{R}^3} h(x)|u_\mu|^q dx. \tag{2.29}$$

Let $v_n = u_n - u_\mu$. By the Brézis–Lieb lemma [22, Lemma 1.32], (2.29) and Lemma 2.1-(6), we have

$$\begin{aligned} m_\mu^* + o(1) = \mathcal{I}_\mu(u_n) &= \frac{1}{2}\|u_n\|^2 + \frac{1}{4}\int_{\mathbb{R}^3} l(x)\phi_{u_n}u_n^2 dx - \frac{1}{6}|u_n|_6^6 - \frac{\mu}{q}\int_{\mathbb{R}^3} h(x)|u_n|^q dx \\ &= \frac{1}{2}\|v_n\|^2 - \frac{1}{6}|v_n|_6^6 + \mathcal{I}_\mu(u_\mu) \end{aligned} \tag{2.30}$$

and

$$o(1) = \left\langle \mathcal{I}'_\mu(u_n), u_n \right\rangle = \|u_n\|^2 + \int_{\mathbb{R}^3} l(x)\phi_{u_n}u_n^2 dx - |u_n|_6^6 - \mu\int_{\mathbb{R}^3} h(x)|u_n|^q dx$$
$$= \|v_n\|^2 - |v_n|_6^6. \tag{2.31}$$

Denote $\limsup_{n\to\infty}\|v_n\|^2 = a$. By (2.31) and the definition of the best Sobolev constant, we get $a \le S^{-3}a^3$, which implies either $a = 0$ or $a \ge S^{\frac{3}{2}}$. We claim that $a \ge S^{\frac{3}{2}}$ can't occur. From $\mathcal{I}'_\mu(u_\mu) = 0$, (2.30), (2.31) and (2.10), it follows that

$$m^*_\mu = \lim_{n\to\infty}\left[\frac{1}{3}\|v_n\|^2 + \mathcal{I}_\mu(u_\mu) - \frac{1}{4}\langle \mathcal{I}'_\mu(u_\mu), u_\mu\rangle\right] \ge \frac{1}{3}S^{\frac{3}{2}} - D\mu^{\frac{6}{6-q}},$$

which together with Lemma 2.7 gives a contradiction. Therefore, $a = 0$. That is, $u_n \to u_\mu$ in $D^{1,2}(\mathbb{R}^3)$.

Further, we prove $u_n^\pm \to u_\mu^\pm$ in $D^{1,2}(\mathbb{R}^3)$. It follows from $\{u_n\} \subset \mathcal{M}^*_\mu$ and $\mathcal{I}'_\mu(u_\mu) = 0$ that

$$\begin{cases} \|u_n^\pm\|^2 + \int_{\mathbb{R}^3} l(x)\phi_{u_n}|u_n^\pm|^2 dx = |u_n^\pm|_6^6 + \mu\int_{\mathbb{R}^3} h(x)|u_n^\pm|^q dx, \\ \|u_\mu^\pm\|^2 + \int_{\mathbb{R}^3} l(x)\phi_{u_\mu}|u_\mu^\pm|^2 dx = |u_\mu^\pm|_6^6 + \mu\int_{\mathbb{R}^3} h(x)|u_\mu^\pm|^q dx. \end{cases} \tag{2.32}$$

It is easy to verify that

$$|u_n^\pm(x) - u_\mu^\pm(x)| \le |u_n(x) - u_\mu(x)|, \qquad \forall\, x \in \mathbb{R}^3, \tag{2.33}$$

which and $u_n \to u_\mu$ in $D^{1,2}(\mathbb{R}^3)$ lead to

$$\left||u_n^\pm|_6 - |u_\mu^\pm|_6\right|^6 \le \left|u_n^\pm - u_\mu^\pm\right|_6^6 \le |u_n - u_\mu|_6^6 \le S^{-3}\|u_n - u_\mu\|^6 \xrightarrow{n} 0. \tag{2.34}$$

Noting $u_n \to u_\mu$ in $L^6(\mathbb{R}^3)$, we deduce $|u_n^\pm|^q \rightharpoonup |u_\mu^\pm|^q$ in $L^{\frac{6}{q}}(\mathbb{R}^3)$ in the sense of subsequence. Therefore,

$$\int_{\mathbb{R}^3} h(x)|u_n^\pm|^q dx \xrightarrow{n} \int_{\mathbb{R}^3} h(x)|u_\mu^\pm|^q dx. \tag{2.35}$$

Moreover, since $l \in L^2(\mathbb{R}^3)$ and $|u_n - u_\mu|_6 \to 0$ implies that $\left||u_n^\pm|^2 - |u_\mu^\pm|^2\right|^{\frac{6}{5}} \rightharpoonup 0$ in $L^{\frac{5}{2}}(\mathbb{R}^3)$ up to a subsequence, by the Hölder inequality and Lemma 2.1-(3) we have

$$\begin{aligned}\left|\int_{\mathbb{R}^3} l(x)\phi_{u_n}(|u_n^\pm|^2 - |u_\mu^\pm|^2)dx\right| &\le |\phi_{u_n}|_6\left(\int_{\mathbb{R}^3}|l(x)|^{\frac{6}{5}}\left||u_n^\pm|^2 - |u_\mu^\pm|^2\right|^{\frac{6}{5}}dx\right)^{\frac{5}{6}} \\ &\le S^{-1}|l|_2|u_n|_6^2\left(\int_{\mathbb{R}^3}|l(x)|^{\frac{6}{5}}\left||u_n^\pm|^2 - |u_\mu^\pm|^2\right|^{\frac{6}{5}}dx\right)^{\frac{5}{6}} \\ &\xrightarrow{n} 0. \end{aligned} \tag{2.36}$$

Observe $|lu_\mu^\pm|^2 \in L^{\frac{6}{5}}(\mathbb{R}^3)$, we deduce from (2.27) and Lemma 2.1-(5) that

$$\left|\int_{\mathbb{R}^3} l(x)(\phi_{u_n} - \phi_{u_\mu})|u_\mu^\pm|^2 dx\right| \xrightarrow{n} 0,$$

by which and (2.36) we get

$$\int_{\mathbb{R}^3} l(x)\phi_{u_n}|u_n^{\pm}|^2dx \xrightarrow{n} \int_{\mathbb{R}^3} l(x)\phi_{u_\mu}|u_\mu^{\pm}|^2dx. \tag{2.37}$$

Now, by combining (2.32), (2.34), (2.35) and (2.37) we conclude $\|u_n^{\pm}\|^2 \to \|u_\mu^{\pm}\|^2$, which together with (2.27) implies $u_n^{\pm} \xrightarrow{n} u_\mu^{\pm}$ in $D^{1,2}(\mathbb{R}^3)$. Thus the lemma is proved. □

Next, with the above preliminary lemmas in hand, we end the proof of our main result.

Proof of Theorem 1.1. For any $\mu \in (0, \mu^*)$, due to Lemmas 2.8 and 2.7, there exists a sequence $\{u_n\} \subset \mathcal{M}_\mu^*$ such that, as $n \to \infty$,

$$\mathcal{I}_\mu(u_n) \to m_\mu^* \in \left(0, \frac{1}{3}S^{\frac{3}{2}} - D\mu^{\frac{6}{6-q}}\right) \quad \text{and} \quad \mathcal{I}'_\mu(u_n) \to 0 \text{ in } \left[D^{1,2}(\mathbb{R}^3)\right]^*.$$

According to Lemma 2.9, up to a subsequence, there exists some u_μ satisfying $u_n \to u_\mu$ and $u_n^{\pm} \to u_\mu^{\pm}$ in $D^{1,2}(\mathbb{R}^3)$ as $n \to \infty$. Therefore, we get $\mathcal{I}'_\mu(u_\mu) = 0$ and $\mathcal{I}_\mu(u_\mu) = m_\mu^*$. Moreover, Lemma 2.4 says $\|u_n^{\pm}\| \geq c > 0$, which implies $u_\mu^{\pm} \neq 0$. By combining the above arguments, we know that u_μ is a sign-changing solution of system (1.6) with positive energy. Thus Theorem 1.1 is proved. □

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Existence and uniqueness of crossing periodic solutions of a discontinuous mixed-type Duffing equation

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Abstract. In this paper, we investigate the existence and uniqueness of crossing periodic solutions for a class of mixed-type second order Duffing equations with discontinuity and asymmetry. By the Poincaré–Bohl theorem, we obtain several existence criteria of regular periodic solutions which are crossing. We also present a uniqueness result.

Keywords: discontinuity, Duffing equation, Poincaré–Bohl theorem, crossing periodic solution, existence and uniqueness.

2020 Mathematics Subject Classification: 34A36, 34A34, 34A12, 34C25.

1 Introduction

Second order Duffing equation

$$x'' + g(x) = p(t), \tag{1.1}$$

is one of the typical models both in ODE and forced vibrations, where g is an elastic force and p is an external excitation. There have been many interesting results reported concerning the existence of periodic solutions. The methods involve the Poincaré–Birkhoff twist theorem, the Poincaré–Bohl theorem, averaging theory, variational method and the Leray–Schauder continuation method of topological degree etc., see for example [3–6, 9, 22–26] and the references therein.

In the paper, we are concerned with Eq. (1.1) with discontinuous vector field, in which g presents a discontinuity at $\alpha \in \mathbb{R}$ and it satisfies asymmetric conditions, and then we called it as discontinuous mixed-type Duffing equations. During the last decades, the amount of publications on the discontinuous differential equations is vast. We refer readers to consult classical monographs [8, 19] for the general theory, and literatures [7, 11, 12, 17, 18] for some related research. In [8], the author extend discontinuous differential equations to differential inclusions, he revealed many aspects of discontinuous differential equations and addressed existence and uniqueness of solutions. More results on the differential inclusion can be found

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in [1]. Inspired by these, we consider the existence and uniqueness of periodic solutions for the following second order Duffing equations allowing for discontinuities at 0

$$x'' + g_{\pm}(x) = p(t), \qquad x \neq 0, \tag{1.2}$$

where $p(t) = p(t + 2\pi)$, $g_{\pm}(x)$ are of class C^1 and satisfy the asymmetric growth condition

$$0 \leq \lim_{x\to-\infty} \frac{g_-(x)}{x} < +\infty, \qquad \lim_{x\to+\infty} \frac{g_+(x)}{x} = +\infty. \tag{1.3}$$

Many existing literatures, whether continuous or not, focused on periodic solution problem of Duffing equations with symmetric growth condition. But there is no study about the discontinuous Duffing equation with the asymmetric condition. According to the relationship of linear growth satisfied by the elastic force g, the Duffing equations can be divided into three categories:

(S_p) Superlinear growth: $\lim_{|x|\to\infty} \frac{g(x)}{x} = +\infty$;

(S_m) Semilinear growth: $0 < \liminf_{|x|\to\infty} \frac{g(x)}{x} \leq \limsup_{|x|\to\infty} \frac{g(x)}{x} < \infty$;

(S_b) Sublinear growth: $\lim_{|x|\to\infty} \frac{g(x)}{x} = 0$.

On the other hand, when g satisfies different linear growth conditions for $x \in \mathbb{R}$, it means that the processing method is also different from the one of symmetric Duffing equations. Recently, there are a few papers on the periodic solution problem of discontinuous symmetric Duffing equations (with impulsive perturbations or discontinuous vector field), in which the elastic force g satisfies one of the above linear growth conditions, see [2,10,12–16,18,20,21] for example and the references therein. However, in this paper we consider the existence criteria of crossing periodic solutions of (1.2) when $g_{\pm}(x)$ satisfy different linear growth conditions (1.3) for $x \in \mathbb{R}$. And a uniqueness theorem is also presented.

To obtain the desirable results, we first assume that $g_{\pm}$ satisfy the following conditions.

$(H1)$ Let $g_+ \in C^1([0, +\infty), \mathbb{R})$, $g_- \in C^1((-\infty, 0], \mathbb{R})$, and the behavior of $g_{\pm}$ at the discontinuity point $x = 0$ satisfies that the left and right limits

$$g_- = \lim_{x\to0^-} g_-(x), \qquad g_+ = \lim_{x\to0^+} g_+(x)$$

exist and are finite, and satisfy $g_- < 0 < g_+$.

$(H2)$ There are two real numbers d_1 and d_2 with $d_1 < 0 < d_2$ such that

$$x \leq d_1 \Rightarrow g_-(x) < 0, \qquad x \geq d_2 \Rightarrow g_+(x) > 0.$$

$(H3)$ There are positive constants l_-, l_+, L_- and A_1 such that the asymptotic growth of $g_{\pm}$ is controlled by

$$l_- \leq \frac{g_-(x)}{x} \leq L_-, \quad x \leq -A_1; \qquad \frac{g_+(x)}{x} \geq l_+, \quad x \geq A_1. \tag{1.4}$$

Moreover, there exists an integer $m > 0$ that satisfies

$$\frac{2}{m+1} < \frac{1}{\sqrt{L_-}} \leq \frac{1}{\sqrt{l_-}} + \frac{1}{\sqrt{l_+}} < \frac{2}{m}. \tag{1.5}$$

Then by applying the Poincaré–Bohl theorem to the discontinuous Duffing equation (1.2) with the asymmetric growth condition, we obtain several existence criteria of crossing periodic solutions as follows.

Theorem 1.1. *Assume that* $(H1)$–$(H3)$ *hold. Then* (1.2) *has at least one* 2π *crossing periodic solution.*

Note that (1.4) means that g_- satisfies semilinear growth for $x \le -A_1$, while g_+ can satisfy both semilinear growth and superlinear growth when $x \ge A_1$. Further we have that

$$\lim_{x\to-\infty} g_-(x) = -\infty, \qquad \lim_{x\to+\infty} g_+(x) = +\infty. \tag{1.6}$$

Theorem 1.2. *In Theorem 1.1, if* $(H3)$ *is replaced as* (1.6) *and*

$(H3)'$ *There exist* $L_- \in (0, \frac{1}{4})$ *and* $A_1 > 0$ *such that*

$$\frac{g_-(x)}{x} \le L_-, \qquad x \le -A_1.$$

Then (1.2) *has at least one* 2π *crossing periodic solution.*

Theorem 1.3. *In Theorem 1.1, if* $(H3)$ *is replaced as*

$(H3)^*$ *There are positive constants* l_-, L_-, A *and an integer* $m > 0$ *such that* $\lim_{x\to+\infty} \frac{g_+(x)}{x} = +\infty$, *and*

$$\frac{1}{4}m^2 < l_- \le \frac{g_-(x)}{x} \le L_- < \frac{1}{4}(m+1)^2, \qquad x \le -A. \tag{1.7}$$

Then (1.2) *has at least one* 2π *crossing periodic solution.*

Moreover, for the completeness of the paper we further provide a uniqueness criterion.

Theorem 1.4. *Assume that* $g_\pm$ *are Lipschitz continuous and satisfy* $0 < \frac{g_\pm(x)-g_\pm(y)}{x-y} < 1$ *for any* $x, y \in \mathbb{R} \setminus \{0\}$. *Then* (1.2) *has at most one* 2π *crossing periodic solution.*

This paper is organized as follows. In Section 2, we present some useful preliminaries. In Sections 3–4, we give the behavior of solutions for the autonomous and non-autonomous systems, respectively. Then by applying the Poincaré–Bohl theorem, we obtain several existence criteria of 2π crossing periodic solutions. And we also prove the uniqueness. Moreover, to demonstrate the effectiveness of the obtained results, we further provide an example in Section 5. Concluding remarks are outlined in Section 6.

2 Preliminaries

Consider the second order Duffing equation

$$x'' + g(x) = p(t), \tag{2.1}$$

where $p : \mathbb{R} \to \mathbb{R}$ is continuous and 2π-periodic, and denote by

$$E = \max_{t\in[0,2\pi]}\{|p(t)|\}, \qquad \overline{E} = \max_{t\in[0,2\pi]}\{p(t)\}, \qquad \underline{E} = \min_{t\in[0,2\pi]}\{p(t)\}. \tag{2.2}$$

While $g : \mathbb{R} \to \mathbb{R}$ is continuous except for a discontinuity point at $x = 0$. Assume that $\Sigma_0 = \{(x,y) \in \mathbb{R}^2 : x = 0\}$ divide $g(x)$ of the form

$$g(x) = \begin{cases} g_+(x), & x > 0, \\ g_-(x), & x < 0. \end{cases} \tag{2.3}$$

Here Σ_0 is called as a discontinuity line on the plane, and we denote by

$$\begin{aligned} \Sigma_0^+ &= \{(x,y) : x = 0, y > 0\}, \qquad & \Sigma_0^- &= \{(x,y) : x = 0, y < 0\}, \\ \Sigma_+ &= \{(x,y) \in \mathbb{R}^2 : x > 0\}, \qquad & \Sigma_- &= \{(x,y) \in \mathbb{R}^2 : x < 0\}. \end{aligned}$$

Then $\Sigma_0 = \Sigma_0^+ \cup \{O\} \cup \Sigma_0^-$, $\mathbb{R}^2 = \Sigma_+ \cup \Sigma_0 \cup \Sigma_-$, and the unit normal vector perpendicular to Σ_0 is taken to be $\mathbf{n}^T = (1,0)$.

We recall an existence result of periodic solutions from the Poincaré–Bohl theorem.

Theorem 2.1 ([26, Theorem 1.2 of Chapter VI]). *Suppose that $\mathcal{F} : \mathbb{D} \to \mathbb{R}^2$ is a continuous mapping, where $\mathbb{D}(\subset \mathbb{R}^2)$ is a closed bounded region including the origin as an interior point, the boundary $\partial\mathbb{D}$ is a piecewise smooth simple closed curve. For any $p \in \partial\mathbb{D}$ then the image $q = \mathcal{F}(p)$ satisfies $\overrightarrow{Oq} \neq \lambda\overrightarrow{Op}$, where $\lambda \geq 1$ is a constant. Then $\mathcal{F}$ has at least one fixed point in $\mathbb{D}$.*

It is sufficient to prove that any orbit which starts from $p = (x(0), x'(0)) \in \partial\mathbb{D}$ and moves to $q = (x(2\pi), x'(2\pi)) \in \mathbb{R}^2 \setminus \mathbb{D}$, then the points p and q are not on the same ray starting from the origin.

In what follows, we consider the equivalent system of (2.1) and (2.3)

$$\begin{cases} x' = y, \\ y' = -g_+(x) + p(t), \quad x > 0; \end{cases} \qquad \begin{cases} x' = y, \\ y' = -g_-(x) + p(t), \quad x < 0. \end{cases} \tag{2.4}$$

Let $\mathbf{V}_\pm(t;x,y) = (y, -g_\pm(x) + p(t))^T$ denote the vector field of (2.4). The second component of $\mathbf{V}_\pm$ is discontinuous on Σ_0, so by using Filippov theory to define orbits of (2.4) when they intersect with Σ_0 such that the orbits can be concatenated in a natural way. Concretely, the set-valued extension as follows

$$z'(t) \in \mathbf{V}(t;x,y) = \begin{cases} \mathbf{V}_+(t;x,y), & x > 0, \\ \overline{co}\{\mathbf{V}_+(t;x,y), \mathbf{V}_-(t;x,y)\}, & x = 0, \\ \mathbf{V}_-(t;x,y), & x < 0, \end{cases} \tag{2.5}$$

where $z(t) \doteq (x(t), y(t))^T$, $\mathbf{V}(t;x,y) \doteq \mathbf{V}(t;x(t),y(t))$, and $\overline{co}(A)$ denotes the smallest closed convex set containing A. The convex set with $\mathbf{V}_-$ and $\mathbf{V}_+$ is of the form

$$\overline{co}\{\mathbf{V}_+, \mathbf{V}_-\} = \{(1-q)\mathbf{V}_- + q\mathbf{V}_+, \ \forall q \in [0,1]\},$$

here q is a parameter which defines the convex combination. The extension of (2.4) into the convex differential inclusion (2.5) is known as the Filippov's convex method. The existence of solutions of (2.5) is guaranteed with an upper semi-continuity of set-valued functions, and the existence of solutions of an initial value problem (IVP for short)

$$z'(t) \in \mathbf{V}(t;x,y), \qquad z(0) = (x_0, y_0)^T$$

combines with Filippov's convex method define the solutions of (2.4). For more details, we refer readers to consult [1,8] and the references therein.

In this paper, we deal with the periodic solutions that appear when the discontinuity line Σ_0 is considered, and the solution $t \to (x(t), x'(t))$ vanishes twice per period. These solutions are built piecewise, from restrictions to some subintervals of solutions of $x'' + g_+(x) = p(t)$ (*resp.* $x'' + g_-(x) = p(t)$) vanishing at the bounds of the corresponding subinterval. Now we give the following definitions concerning what kind of solutions we are looking for.

Definition 2.2 ([12])**.** Let I be an interval of $\mathbb{R}$ and $x : I \to \mathbb{R}$ be any solution of (2.4). We say that x is singular if there exists $t^* \in I$ such that $x(t^*) = x'(t^*) = 0$. A non-singular solution is said to be regular.

Definition 2.3. We say that $x : \mathbb{R} \to \mathbb{R}$ is a regular solution of (2.4) if it is continuous to t, and there exists a strictly increasing sequence $(t_j)_{j\in\mathbb{Z}}$, with no accumulation points, such that $x(t_j) = 0, x'(t_j) \neq 0$ and the restriction of x on each interval $[t_{j-1}, t_j]$ is twice continuously differentiable. Moreover, the limits $\lim_{t\to t_j^-} x''(t)$ and $\lim_{t\to t_j^+} x''(t)$ exist and are finite.

By $(H1)$, we assume the existence of solutions to the IVP of (2.4) with $z(0) = (x_0, y_0)^T$. And from the above statements, any solution of IVP with the initial condition $(x_0, y_0) \notin \Sigma_0$ is locally unique, because $\mathbf{V}_+(t; x, y)$ and $\mathbf{V}_-(t; x, y)$ are of class C^1. While for the uniqueness problem of IVP with the initial condition on Σ_0, there are three basic ways in which the vector field around Σ_0 can behave as follows.

(1) Transversal intersection.

Any solution of IVP with the initial condition $(x_0, y_0) \in \Sigma_+$/or $(x_0, y_0) \in \Sigma_-$, exists and is unique. A necessary condition for the transversal intersection at Σ_0 is

$$[\mathbf{n}^T\mathbf{V}_-(t; x, y)] \cdot [\mathbf{n}^T\mathbf{V}_+(t; x, y)] > 0, \qquad (x, y) \in \Sigma_0,$$

where $\mathbf{n}^T\mathbf{V}_-$ and $\mathbf{n}^T\mathbf{V}_+$ are projections of $\mathbf{V}_-$ and $\mathbf{V}_+$ on the normal vector to Σ_0.

(2) Attraction sliding mode (i.e. Σ_0 attracts the solution). Here the solution will hit Σ_0 but cannot leave it and then move along Σ_0.

Any solution of IVP exists and is unique in the forward time. During the sliding mode, the solution will continue along Σ_0 with the vector field $\widetilde{\mathbf{V}}$ being given by

$$\widetilde{\mathbf{V}} = \alpha\mathbf{V}_+ + (1 - \alpha)\mathbf{V}_-, \qquad \alpha = \frac{n^T\mathbf{V}_-}{n^T(\mathbf{V}_- - \mathbf{V}_+)},$$

where the parameter α is chosen such that the vector field $\widetilde{\mathbf{V}}$ lies along Σ_0. An attraction sliding mode at Σ_0 occurs if

$$\mathbf{n}^T\mathbf{V}_-(t; x, y) > 0, \quad \mathbf{n}^T\mathbf{V}_+(t; x, y) < 0, \qquad (x, y) \in \Sigma_0.$$

(3) Repulsion sliding mode (i.e. the vector field is repulsing from Σ_0). Here the solution is diverging from Σ_0.

The IVP with the initial condition on Σ_0 has three possible solutions. In fact, a solution which starts close to Σ_0 will move away from it, but a solution emanating on Σ_0 can stay on Σ_0, obeying Filippov's solution, or leave Σ_0 by entering either Σ_+ or Σ_-. Hence the solution exists but is not unique in forward time.

Remark 2.4. Note that the solution of (2.4) passing through the origin is not unique. For example, we consider

$$g_+(0) - p(t) < 0, \qquad g_-(0) - p(t) > 0.$$

A qualitative sketch of the dynamics is as shown in Figure 2.1. There are two orbits passing through the origin: one is contained in Σ_+ involving the system with g_+, the other is contained in Σ_- involving the system with g_-, but it is not clear that which one should be the unique solution.

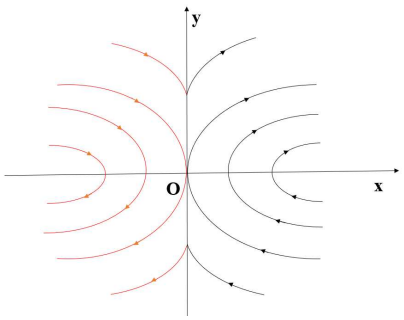


Figure 2.1: Schematic diagram

Definition 2.5. For any given $(0, y) \in \Sigma_0$, if

$$[\mathbf{n}^T \mathbf{V}_-(t; 0^-, y)] \cdot [\mathbf{n}^T \mathbf{V}_+(t; 0^+, y)] \leq 0,$$

$(0, y)$ is called to be *a sliding point*. A set of sliding points is called as a sliding set. We say $(0, y)$ a *crossing point*, if

$$[\mathbf{n}^T \mathbf{V}_-(t; 0^-, y)] \cdot [\mathbf{n}^T \mathbf{V}_+(t; 0^+, y)] > 0.$$

Then the origin is the unique possible sliding point on Σ_0, and it is an isolated and singular sliding point.

Definition 2.6. *A crossing periodic orbit* is defined as a closed orbit with no points in common with the sliding set.

In conclusion, any regular orbit of (2.4) not passing through the origin on the phase plane exists and is unique. In the next text, we focus on the crossing solutions (or orbits) of (2.4) exception for the origin. To be precise, we discuss large-amplitude *crossing periodic orbits* of (2.4). They intersect with Σ_0 transversally, *i.e.* the transversal intersection case. For more information about the existence and uniqueness of solutions of discontinuous differential equations, please see [8, 11, 17] for example and the references therein.

3 Autonomous system

The autonomous equation

$$x'' + g_\pm(x) = 0, \qquad x \neq 0, \tag{3.1}$$

which is equivalent to the system

$$\begin{cases} x' = y, \\ y' = -g_+(x), \quad x > 0; \end{cases} \qquad \begin{cases} x' = y, \\ y' = -g_-(x), \quad x < 0. \end{cases} \tag{3.2}$$

Let $G(x) = \int_0^x g_\pm(u)du$ with $G(0) = 0$, it follows from $(H3)$ that

$$\lim_{x\to-\infty} G(x) = \lim_{x\to+\infty} G(x) = +\infty. \tag{3.3}$$

Then the function G has a minimum on $\mathbb{R}$, by adding a suitable constant we can assume without loss of generality that $G(x) \geq 0$ for every $x \in \mathbb{R}$.

For the system (3.2) we have a Hamiltonian function $H : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined as

$$H(x, y) = \frac{1}{2}y^2 + G(x).$$

Note that H is not of class C^1 in $\mathbb{R}^2$, since G is continuous but not differentiable at $x = 0$. The orbits of the crossing solutions are symmetric with respect to the horizontal axis, and they lie on the level sets of the Hamiltonian function, i.e.,

$$L^c = \{(x, y) \in \mathbb{R}^2 : H(x, y) = c\}.$$

The level sets L^c are compact, and consequently the crossing solutions of (3.1) are globally defined.

We are now concentrate on the dynamics both near the origin $O(0, 0)$.

Lemma 3.1. *The origin is a local center for (3.2); if $c > 0$ is small sufficiently, the level set L^c is a closed curve, star-shaped with respect to the origin, corresponding to the crossing periodic solution of (3.2). The same is true for $c > 0$ being large enough.*

Proof. By $(H1)$, there exists a $\delta > 0$ such that for every $x \in [-\delta, 0) \cup (0, \delta]$ we have that

$$xg_\pm(x) > x^2.$$

Let $z(t) = (x(t), y(t))$ be a crossing solution of (3.2) with $x(t) \in [-\delta, 0) \cup (0, \delta]$ for some $t \in \mathbb{R}$. Making the following polar coordinates transformation

$$x(t) = r(t)\cos\theta(t), \qquad y(t) = r(t)\sin\theta(t), \tag{3.4}$$

where $r : [0, 2\pi] \to (0, +\infty)$ and $\theta : [0, 2\pi] \to \mathbb{R}$ are continuous functions, we have that

$$-\theta'(t) = \frac{x'(t)y(t) - x(t)y'(t)}{x^2(t) + y^2(t)} = \frac{x(t)g_\pm(x(t)) + y^2(t)}{x^2(t) + y^2(t)} \geq 1. \tag{3.5}$$

We observe that for $x \in [-\delta, 0) \cup (0, \delta]$,

$$\nabla H(x, y) \cdot (x, y)^T = xg_\pm(x) + y^2 > 0.$$

Hence near the origin, $H(x, y)$ is strictly increasing along the rays emanating from the origin, at least in a small neighborhood of the origin. Then a solution starting with initial point $(x(0), y(0)) \neq (0, 0)$ sufficiently near the origin, on a given ray, will remain in the strip $\{(x, y) : |x| < \delta\}$ for every $t \in [0, 2\pi]$, and during this time it will perform an entire rotation, returning to the initial ray, by (3.5). Since $H(x, y)$ is constant along the orbit, it must indeed return to the same initial point. This proves that the small-amplitude crossing solutions of (3.2) are periodic and from (3.5) they have star-shaped orbits with respect to the origin.

We next consider a large-amplitude crossing solutions of (3.2). From $(H2)$ and set $M = \max\{|xg_{\pm}(x)| : x \in [d_1, d_2]\}$. Let $(x(t), y(t))$ be a crossing solution of (3.2) with $(x(t), y(t)) \neq [d_1, d_2] \times [-\sqrt{M}, \sqrt{M}]$ and $x(t) \neq 0$ for some $t \in \mathbb{R}$. Then it follows from (3.4) that

$$-\theta'(t) \begin{cases} \geq \frac{-M+y^2(t)}{x^2(t)+y^2(t)} > 0, & x(t) \in [d_1, d_2], \\ > \frac{y^2(t)}{x^2(t)+y^2(t)} \geq 0, & x(t) \notin [d_1, d_2]. \end{cases}$$

The same argument shows that far from the origin then $H(x, y)$ is strictly increasing along the rays emanating from the origin. Let

$$\overline{S} = \max\{H(x, y) : (x, y) \in [d_1, d_2] \times [-\sqrt{M}, \sqrt{M}]\},$$

and set $K = \sqrt{2\overline{S}}$. By $(H3)$ there exist $\widehat{d}_1, \widehat{d}_2$ with $\widehat{d}_1 \leq d_1 < 0 < d_2 \leq \widehat{d}_2$ such that

$$x \notin [\widehat{d}_1, \widehat{d}_2] \Rightarrow G(x) > \overline{S}.$$

We claim that if (x, y) is a crossing solution of (3.2) with $(x(0), y(0)) \notin [\widehat{d}_1, \widehat{d}_2] \times [-K, K]$, then $(x(t), y(t)) \notin [d_1, d_2] \times [-\sqrt{M}, \sqrt{M}]$ for every $t \in \mathbb{R}$. Once the claim is proved, the above argument used for the small-amplitude crossing solutions can be applied, and consequently the large-amplitude crossing solutions of (3.2) are periodic and have star-shaped orbits with respect to the origin.

We let (x, y) be a crossing solution of (3.2) such that $(x(0), y(0)) \notin [\widehat{d}_1, \widehat{d}_2] \times [-K, K]$. There are two cases. Either $x(0) \notin [\widehat{d}_1, \widehat{d}_2]$, in which case $G(x(0)) > \overline{S}$, hence

$$H(x(t), y(t)) = H(x(0), y(0)) \geq G(x(0)) > \overline{S}, \qquad t \in \mathbb{R};$$

or $x(0) \in [\widehat{d}_1, \widehat{d}_2]$ and $y(0) \notin [-K, K]$, in which case $\frac{1}{2}y^2(0) > \overline{S}$, hence

$$H(x(t), y(t)) = H(x(0), y(0)) \geq \frac{1}{2}y^2(0) > \overline{S}, \qquad t \in \mathbb{R}.$$

By the definition $\overline{S}$, in both cases then $(x(t), y(t)) \notin [d_1, d_2] \times [-\sqrt{M}, \sqrt{M}]$ for every $t \in \mathbb{R}$. □

For every positive numbers $c_1 < c_2$, define the set

$$A(c_1, c_2) = \left\{(x, y) \in \mathbb{R}^2 : c_1 \leq \frac{1}{2}y^2 + G(x) \leq c_2\right\}.$$

Then it follows from Lemma 3.1 that when $c_1 > 0$ is small enough and $c_2 > c_1$ is large enough, the set $A(c_1, c_2)$ is an annulus with strictly star-shaped boundary curves. Moreover, all crossing solutions of (3.2) with the associated initial value

$$(x(0), y(0)) = (x_0, y_0) \in \mathbb{R} \times \mathbb{R}$$

are regular, with the only exception for $(x_0, y_0) = (0, 0)$, in which case the solution is not defined. Further we agree that the origin will be considered as a singular point defined as in Definition 2.2.

Remark 3.2. If we consider c_1 and c_2 satisfying the above properties as fixed in Lemma 3.1. It is easy to see that we can choose a constant $\rho > 1$ such that for every $(x, y) \in \mathbb{R}^2$

$$\frac{1}{2}y^2 + G(x) \geq c_1 \Rightarrow x^2 + y^2 > \rho^{-1},$$
$$\frac{1}{2}y^2 + G(x) \leq c_2 \Rightarrow x^2 + y^2 < \rho.$$

Lemma 3.1 implies that all regular solutions (or regular orbits) of (3.1), exception for the origin, cross Σ_0 transversally in a clockwise fashion. For example see Figure 3.1: crossing periodic orbits of $x'' + x + \operatorname{sgn}(x) = 0$. It shows that the crossing periodic orbits are nonsmooth but continuous at the intersection points of the orbits and Σ_0.

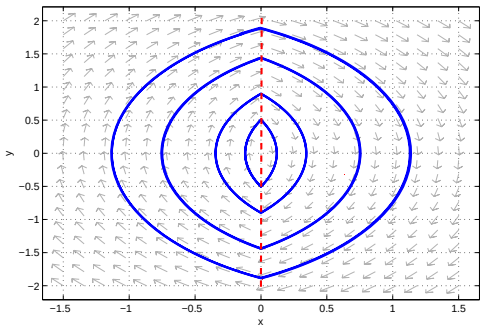


Figure 3.1: Crossing periodic orbits of $x'' + x + \operatorname{sgn}(x) = 0$. Red dashed line denotes Σ_0.

4 Perturbed system

We rewrite the equivalent system associated to (2.1) and (2.3) as follows

$$\begin{cases} x' = y, \\ y' = -g_+(x) + p(t), \quad x > 0; \end{cases} \qquad \begin{cases} x' = y, \\ y' = -g_-(x) + p(t), \quad x < 0. \end{cases} \tag{4.1}$$

Any function $z(t) = (x(t), y(t)) : [0, 2\pi] \to \mathbb{R} \times \mathbb{R}$ such that $(x(t), y(t)) \neq (0,0)$ for every $t \in [0, 2\pi]$. By the polar coordinates transformation (3.4), the resulting equation of (4.1) for $r(t) > 0$ and $\theta(t) \neq \frac{\pi}{2} + k\pi, k \in \mathbb{Z}$ is of the form

$$\begin{cases} \dfrac{dr(t)}{dt} = r\cos\theta\sin\theta + [p(t) - g_\pm(r\cos\theta)]\sin\theta, \\ \dfrac{d\theta(t)}{dt} = -\sin^2\theta + \dfrac{1}{r}[p(t) - g_\pm(r\cos\theta)]\cos\theta. \end{cases} \tag{4.2}$$

Correspondingly, we let $r(t) = r(t; r_0, \theta_0), \theta(t) = \theta(t; r_0, \theta_0)$ be the solution of (4.2), which satisfies $(r(0), \theta(0)) \doteq (r_0, \theta_0) \neq (0,0)$, and $x_0 = r_0\cos\theta_0,\ y_0 = r_0\sin\theta_0$.

Denote by $\mathbb{D}(R_0) \doteq \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < R_0^2\}$ and $\mathbb{D}[R_0] \doteq \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq R_0^2\}$ are, respectively, the open and closed discs in the plane (*i.e.* two-dimensional balls), and $\mathbb{S}_{R_0} = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = R_0^2\}$, with center the origin $O(0,0)$ and radius $R_0 > 0$.

Note that any solution function $z(t) = (x(t), y(t))$ of (4.1) is supposed to be defined on a maximal interval of existence. By (1.6) and the boundedness of $p(t)$, it is easy to show that global existence occurs that $z(t)$ is defined for all $t \in \mathbb{R}$.

Lemma 4.1. *The solution $z(t) = (x(t), y(t))$ of (4.1) is defined on the whole t-axis.*

Proof. Define a function $W(x,y):\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ as

$$W(x,y)=\frac{1}{2}y^2+G(x)-G_{\min}+1,$$

where $G_{\min}=\min\{G(x):x\in\mathbb{R}\}$. It follows that $W(x,y)>\frac{1}{2}y^2+\frac{1}{2}>0$ for every $(x,y)\in\mathbb{R}^2$, and

$$\lim_{x^2+y^2\to+\infty}W(x,y)=+\infty. \tag{4.3}$$

We set $W(t)\doteq W(z(t))=W(x(t),y(t))$ with $W(t_0)=W_0=W(z_0)=W(x_0,y_0)$. When $x(t)\neq 0$ for every $t\in\mathbb{R}$, consider the derivative of W along the solution $z(t)=(x(t),y(t))$ of (4.1) one has that

$$\left|\frac{dW(t)}{dt}\right|=|y(t)p(t)|\le E|y(t)|\le EW(t).$$

Using the elementary differential inequalities we have that

$$W(t)\le W_0e^{E|t-t_0|},$$

for t in a maximal interval containing t_0. Further we have proved that $W(z(t))$ is bounded for t in any bounded interval. And then there is no blow-up at finite time and the global existence is proved. □

Lemma 4.2. *There exists $R_0>0$ such that when $r>R_0$, we have that $\frac{d\theta(t)}{dt}<0$ for all $t\in[0,2\pi]$.*

Proof. By (1.6), there exists $N>0$ sufficiently large such that

$$g_+(x)>\overline{E},\quad x\ge N,\qquad g_-(x)<\underline{E},\quad x\le -N,$$

where $\overline{E}$ and $\underline{E}$ come from (2.2). Let $R_0>N$. When $r>R_0$ and $|r\cos\theta|\ge N$ one has that

$$\frac{d\theta(t)}{dt}=-\sin^2\theta-\frac{[g_\pm(r\cos\theta)-p(t)]\cos^2\theta}{r\cos\theta}<0.$$

While for $r>R_0$ and $|r\cos\theta|<N$ with $r\cos\theta\neq 0$, one has that

$$\frac{d\theta(t)}{dt}<-\sin^2\theta+\frac{|p(t)-g_\pm(r\cos\theta)|}{r}<-\frac{R_0^2-N^2-R_0\delta}{R_0^2},$$

where $\delta=E+\max_{|x|\le N}|g_\pm(x)|$ with E being from (2.2). Further, $\frac{d\theta(t)}{dt}<0$ for $t\in[0,2\pi]$ as long as $R_0>\frac{1}{2}(\delta+\sqrt{4N^2+\delta^2})$. □

Lemma 4.2 implies that there exists $R_0>0$ sufficiently large such that the regular orbits of (4.1) rotate clockwise outside of $\mathbb{S}_{R_0}$ as t ($\in[0,2\pi]$) increases.

Let $|w|=|(a,b)|\doteq\sqrt{a^2+b^2}$, we denote the usual Euclidean norm of the point $w=(a,b)\in\mathbb{R}^2$. Next we prove that any solution $z(t)=(x(t),y(t))$ of (4.1) possesses an elastic property as follows.

Lemma 4.3. *For each $\alpha>0$, there is $\beta=\beta(\alpha)>0$ such that $|z(0)|\ge\beta$ implies that*

$$|z(t)|>\alpha,\qquad t\in[0,2\pi].$$

Proof. Let $W(x,y)=\frac{1}{2}y^2+G(x)$, where $G(x)$ is shown as in (3.3). By (1.6), there exists $X>0$ such that $G(x)>0$ for $|x|>X$. Further, $W(x,y)>0$ for $|x|>X$ and (4.3) holds. We set $G=\min_{x\in\mathbb{R}}\{G(x)\}\le 0$. Consider any solution $z(t;z_0)\doteq(x(t;x_0,y_0),y(t;x_0,y_0))$ of (4.1) passing through $z_0=(x_0,y_0)$, and we denote by $W(t)\doteq W(x(t;x_0,y_0),y(t;x_0,y_0))$ satisfying $W(0)\doteq W_0=W(x_0,y_0)$. When $x(t)\neq 0$, along the solution of (4.1) one has that

$$\frac{dW(t)}{dt}=y(t)p(t)\ge -E\sqrt{2(W(t)-G)},$$

where E is from (2.2). Further

$$\sqrt{2(W(t)-G)}\ge\sqrt{2(W_0-G)}-Et,\qquad t\in[0,2\pi].$$

Let $\gamma(\alpha)=\max\{W(x,y):x^2+y^2\le\alpha^2\}$. By (4.3), there exists $\beta=\beta(\alpha)(>\alpha)$ sufficiently large such that for $x^2+y^2\ge\beta^2$ then

$$2(W(x,y)-G)>(\sqrt{2(\gamma(\alpha)-G)}+2E\pi)^2.$$

Hence $x_0^2+y_0^2\ge\beta^2$, we have that

$$\begin{aligned}\sqrt{2(W(x(t;x_0,y_0),y(t;x_0,y_0))-G)}&\ge\sqrt{2(W_0-G)}-Et\\&>\sqrt{2(\gamma(\alpha)-G)}+2E\pi-Et\ge\sqrt{2(\gamma(\alpha)-G)},\ t\in[0,2\pi].\end{aligned}$$

This implies that $x^2(t;x_0,y_0)+y^2(t;x_0,y_0)>\alpha^2$ for $t\in[0,2\pi]$. □

Lemmas 4.2–4.3 imply that there exists $\alpha>0$ sufficiently large such that the regular orbits of (4.1) rotate clockwise in an annulus as t ($\in[0,2\pi]$) increases.

Consider any solution $(x(t),y(t))$ of (4.1) with the initial point $(x(0),y(0))=(x_0,y_0)\in\Sigma_-$. In the polar coordinates, the solution is denoted by $(r(t),\theta(t))$ satisfying $(r_0,\theta_0)\doteq(r(0),\theta(0))$. By Lemma 4.2, the corresponding regular orbit rotates in a clockwise fashion in Σ_-. Assume that it reaches Σ_0^+ at $t=t_0(>0)$, that is $(x(t_0),y(t_0))=(0,y_{t_0})\in\Sigma_0^+$. Next we present a result, and the discussion for $(x_0,y_0)\in\Sigma_+$ is the same and so omitted here.

Lemma 4.4. *There is $\epsilon_0>0$ such that the solution of (4.1) with $(x(t_0+\epsilon_0),y(t_0+\epsilon_0))\in\Sigma_+$ satisfies*

$$\lim_{\epsilon_0\to 0^+}(x(t_0+\epsilon_0),y(t_0+\epsilon_0))=(0,y_{t_0}).$$

Proof. By (4.2), the solution $(x(t_0+\epsilon_0),y(t_0+\epsilon_0))\in\Sigma_+$ satisfies

$$\int_{\frac{\pi}{2}}^{\theta(t_0+\epsilon_0)}\frac{d\theta}{-\sin^2\theta+\frac{1}{r}[p(t)-g_+(r\cos\theta)]\cos\theta}=\int_{t_0}^{t_0+\epsilon_0}dt=\epsilon_0,\tag{4.4}$$

$$\int_{y_{t_0}}^{r(t_0+\epsilon_0)}\frac{dr}{r\cos\theta\sin\theta+[p(t)-g_+(r\cos\theta)]\sin\theta}=\int_{t_0}^{t_0+\epsilon_0}dt=\epsilon_0.\tag{4.5}$$

We rewrite (4.4) as the form

$$\int_{\frac{\pi}{2}}^{\theta(t_0+\epsilon_0)}\frac{d\theta}{-\sin^2\theta+\frac{1}{r(\theta)}[p(t)-g_+(r(\theta)\cos\theta)]\cos\theta}=\epsilon_0.$$

Further, when ϵ_0 is small enough, (4.4) becomes

$$\int_{\frac{\pi}{2}}^{\theta(t_0+\epsilon_0)} \frac{d\theta}{-\sin^2\theta} = \epsilon_0. \tag{4.6}$$

It is easy to find $\theta^*(t_0+\epsilon_0)$ such that (4.6) holds. And on (r,θ) coordinate plane, $\theta^*(r,t_0+\epsilon_0)$ is a continuous curve close to $\theta=\frac{\pi}{2}$ for a given ϵ_0.

Similarly, we can find the corresponding $r^*(\theta,t_0+\epsilon_0)$ such that (4.5) holds. And on (r,θ) coordinate plane, $r^*(\theta)$ is a continuous curve near $r=y_{t_0}$ for a given ϵ_0.

Put $\theta^*(r,t_0+\epsilon_0)$ and $r^*(\theta,t_0+\epsilon_0)$ into the (r,θ) coordinate plane for the same given ϵ_0. Then there is at least one intersection point $(r^*(t_0+\epsilon_0),\theta^*(t_0+\epsilon_0))$. And the solution of (4.4)–(4.5) with this intersection point as an initial value is the result. □

The above lemmas prompt us to provide the following definition, which can be used to define the Poincaré mapping of (4.1) on the plane.

Definition 4.5. For each $(0,y)\in\Sigma_0^+$, let $(0,-p_R(y))\in\Sigma_0^-$ be the first intersection point of the regular orbit starting from $(0,y)$ and Σ_0. Then the planar mapping $P_R:(0,y)\to(0,-p_R(y))$ is called as a right Poincaré mapping. Similarly, for each $(0,-y)\in\Sigma_0^-$ let $(0,p_L(y))\in\Sigma_0^+$ be the first intersection point of the regular orbit starting from $(0,-y)$ and Σ_0, and we call $P_L:(0,-y)\to(0,p_L(y))$ as a left Poincaré mapping.

By the continuous dependence of solutions on the initial values, P_L and P_R are continuous. Consequently, the component functions p_L and p_R are also continuous. For a regular orbit of (4.1) starting from a given point $(0,y)\in\Sigma_0^+$, by Definition 4.5 it intersects with Σ_0^- at $(0,-p_R(y))$ and Σ_0^+ then at $(0,p_L(p_R(y)))$. Clearly the regular orbit is periodic if and only if $y=p_L(p_R(y))$.

Further, the Poincaré mapping $\mathcal{P}$ of (4.1) is well defined as follows

$$\begin{aligned}\mathcal{P}(\doteq P_L\circ P_R):\mathbb{R}^2\setminus\{O\}&\to\mathbb{R}^2\setminus\{O\},\\ (x_0,y_0)&\to(x(2\pi;x_0,y_0),y(2\pi;x_0,y_0)),\\ (r_0,\theta_0)&\to(r(2\pi;r_0,\theta_0),\theta(2\pi;r_0,\theta_0)),\end{aligned}$$

where $(x_0,y_0)\neq(0,0)$ and $(r_0,\theta_0)\neq(0,0)$. Moreover, note that $\mathcal{P}$ is continuous, and fixed points of $\mathcal{P}$ correspond to the initial points of the regular periodic solutions.

Based on the above several lemmas, we now give the proofs of Theorems 1.1–1.4.

Proof of Theorem 1.1. By $(H3)$, there exists $A\geq A_1$ such that

$$g_+(x)>\overline{E},\quad x\geq A,\qquad g_-(x)<\underline{E},\quad x\leq -A.$$

By Lemmas 4.2–4.3, there exists $R_0>0$ large enough such that regular orbits of (4.1) rotate clockwise outside $\mathbb{S}_{R_0}(\doteq\{(x,y):x^2+y^2=R_0^2\})$. We consider any regular orbit $L_{M_0}(\subset\{(x,y):x^2+y^2>R_0^2\})$ which starts from $M_0(r_0,\theta_0)$ at $t=0$, where $r_0=r(0)$ and $\theta_0=\theta(0)$. Assume that $R_0>A$, $M_0\in\{(x,y):x>A\}$, and L_{M_0} successively intersects with $\{x=A\},\Sigma_0^-,\{x=-A\},\{x=-A\},\Sigma_0^+,\{x=A\}$ and $\{\theta=\theta_0\}$ at points M_1,M_2,M_3,M_4,M_5,M_6 and M_7 (see Figure 4.1), and the corresponding moments and arguments are denoted by $\theta_i=\theta(t_i)$ for $i=1,2,3,4,5,6,7$. We next estimate the time of L_{M_0} rotating clockwise a circle around $\mathbb{S}_{R_0}$ when $R_0>0$ is sufficiently large.

According to Lemmas 4.2–4.3, we choose $\varepsilon > 0$ small sufficiently and $R_0 = \frac{E}{\varepsilon} > A$. Let

$$\begin{aligned}\psi_\pm &= \sin^2\theta + l_\pm \cos^2\theta - \varepsilon,\\ \varphi_- &= \sin^2\theta + L_- \cos^2\theta + \varepsilon.\end{aligned}$$

Then $\psi_\pm > 0$ and $\varphi_- > 0$ when ε is sufficiently small. By the second equality of (4.2),

$$\begin{aligned}-\varphi_- < \theta' < -\psi_-, \qquad & x \le -A,\\ \theta' < -\psi_+, \qquad & x \ge A,\end{aligned}$$

where $\theta' = \frac{d\theta(t)}{dt}$. When $x \ge A$, one has that

$$\begin{aligned}t_1 &= \int_{\theta_0}^{\theta_1} \frac{d\theta}{\theta'} < \int_{\theta_1}^{\theta_0} \frac{d\theta}{\psi_+},\\ t_7 - t_6 &= \int_{\theta_6}^{\theta_7} \frac{d\theta}{\theta'} < \int_{\theta_7}^{\theta_6} \frac{d\theta}{\psi_+} = \int_{\theta_7+2\pi}^{\theta_6+2\pi} \frac{d\theta}{\psi_+}.\end{aligned}$$

Since $\theta_7 + 2\pi = \theta_0$, it follows that

$$t_7 - t_6 + t_1 < \int_{\theta_1}^{\theta_6+2\pi} \frac{d\theta}{\psi_+} < \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\theta}{\psi_+} = \frac{\pi}{\sqrt{(l_+ - \varepsilon)(1-\varepsilon)}}.$$

Similarly, for $x \le -A$ then

$$t_4 - t_3 < \int_{-\frac{3\pi}{2}}^{-\frac{\pi}{2}} \frac{d\theta}{\psi_-} = \frac{\pi}{\sqrt{(l_- - \varepsilon)(1-\varepsilon)}}.$$

While for $|x| \le A$, it follows that

$$\begin{aligned}|\theta'| &\ge \sin^2\theta - \frac{|g_\pm(x)| + |p(t)|}{r}\\ &\ge \frac{R_0^2 - A^2}{R_0^2} - \frac{\delta}{R_0} = 1 - \frac{\delta\varepsilon}{E} - \frac{A^2\varepsilon}{E^2},\end{aligned}$$

where $\delta = E + \max_{|x|\le A} |g_\pm(x)|$. Note that the time T that L_{M_0} rotates a circle around $\mathbb{S}_{R_0}$ satisfies

$$T < \frac{\pi}{\sqrt{(l_- - \varepsilon)(1-\varepsilon)}} + \frac{\pi}{\sqrt{(l_+ - \varepsilon)(1-\varepsilon)}} + \left(\int_{\theta_1}^{\theta_2} + \int_{\theta_2}^{\theta_3} + \int_{\theta_4}^{\theta_5} + \int_{\theta_5}^{\theta_6}\right) \frac{d\theta}{\theta'}.$$

It is obvious that $1 - \frac{\delta\varepsilon}{E} - \frac{A^2\varepsilon}{E^2} \to 1$ as $\varepsilon \to 0$. So when $0 < \varepsilon$ is sufficiently small,

$$\begin{aligned}\left|\left(\int_{\theta_1}^{\theta_2} + \int_{\theta_2}^{\theta_3} + \int_{\theta_4}^{\theta_5} + \int_{\theta_5}^{\theta_6}\right) \frac{d\theta}{\theta'}\right| &\le \left(\int_{\theta_1}^{\theta_2} + \int_{\theta_2}^{\theta_3} + \int_{\theta_4}^{\theta_5} + \int_{\theta_5}^{\theta_6}\right) \frac{d\theta}{|\theta'|}\\ &\le \left(\int_{\theta_1}^{\theta_2} + \int_{\theta_2}^{\theta_3} + \int_{\theta_4}^{\theta_5} + \int_{\theta_5}^{\theta_6}\right) \frac{d\theta}{1 - \frac{\delta\varepsilon}{E} - \frac{A^2\varepsilon}{E^2}} \le \frac{4\sigma}{1 - \frac{\delta\varepsilon}{E} - \frac{A^2\varepsilon}{E^2}},\end{aligned}$$

where σ is an angle shown as in Figure 4.1. Since $R_0 = \frac{E}{\varepsilon} \to +\infty$ as $\varepsilon \to 0$, and then $\sigma \to 0$. Hence when R_0 is large sufficiently (*i.e.* $\varepsilon \to 0$), it follows from (1.5) that

$$T < \frac{2\pi}{m}.$$

On the other hand, when $x \le -A$ it follows from $\theta' \ge -\varphi_-$ that

$$T > t_4 - t_3 > \int_{\theta_3}^{\theta_4} \frac{d\theta}{\theta'} > \int_{\theta_4}^{\theta_3} \frac{d\theta}{\varphi_-} = \int_{-\frac{3\pi}{2}}^{-\frac{\pi}{2}} \frac{d\theta}{\varphi_-} - \left(\int_{-\frac{3\pi}{2}}^{\theta_4} + \int_{\theta_3}^{-\frac{\pi}{2}} \right) \frac{d\theta}{\varphi_-}$$
$$= \frac{\pi}{\sqrt{(L_- + \varepsilon)(1+\varepsilon)}} - \left(\int_{-\frac{3\pi}{2}}^{\theta_4} + \int_{\theta_3}^{-\frac{\pi}{2}} \right) \frac{d\theta}{\varphi_-},$$

and

$$\left| \left(\int_{-\frac{3\pi}{2}}^{\theta_4} + \int_{\theta_3}^{-\frac{\pi}{2}} \right) \frac{d\theta}{\varphi_-} \right| \le \frac{2\sigma}{\min\{1, L_-\}} \to 0 \quad \text{as } \varepsilon \to 0.$$

In conclusion, when $R_0 > 0$ is large sufficiently we have that

$$\frac{2\pi}{m+1} < T < \frac{2\pi}{m}.$$

This implies that the number of rotation of any regular orbit of (4.1) in $[0, 2\pi]$ is greater than m but less than $m+1$. Hence by Theorem 2.1, (4.1) has at least one 2π crossing periodic solution. The proof is complete. □

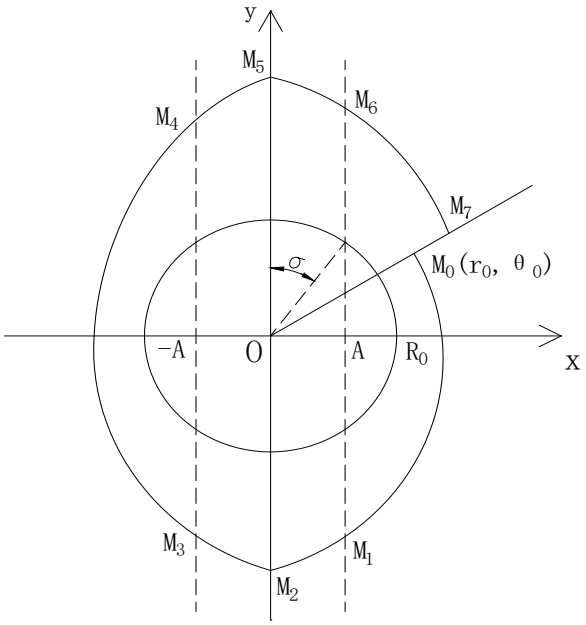


Figure 4.1: Schematic diagram

Proof of Theorem 1.2. By $(H3)'$, there is $A(> A_1)$ large enough such that

$$\theta' \doteq \frac{d\theta}{dt} > -(\sin^2\theta + L_- \cos^2\theta + \varepsilon), \qquad x \le -A.$$

Further, one has that

$$T > \int_{\theta_3}^{\theta_4} \frac{d\theta}{\theta'} > \int_{\theta_4}^{\theta_3} \frac{d\theta}{\sin^2\theta + L_- \cos^2\theta + \varepsilon}$$
$$= \int_{-\frac{3\pi}{2}}^{-\frac{\pi}{2}} \frac{d\theta}{\sin^2\theta + L_- \cos^2\theta + \varepsilon} - \left(\int_{-\frac{3\pi}{2}}^{\theta_4} + \int_{\theta_3}^{-\frac{\pi}{2}} \right) \frac{d\theta}{\sin^2\theta + L_- \cos^2\theta + \varepsilon},$$

and

$$\left|\left(\int_{-\frac{3\pi}{2}}^{\theta_4}+\int_{\theta_3}^{-\frac{\pi}{2}}\right)\frac{d\theta}{\sin^2\theta+L_-\cos^2\theta+\varepsilon}\right|\le\frac{2\sigma}{L_-},$$

where σ is the same as in the proof of Theorem 1.1. Further, by $0<L_-<\frac{1}{4}$, when $R_0>0$ is sufficiently large one has that

$$T>\frac{\pi}{\sqrt{(L_-+\varepsilon)(1+\varepsilon)}}>2\pi.$$

This implies that the number of rotation for any regular orbit rotating clockwise outside $\mathbb{S}_{R_0}$ in $[0,2\pi]$ is less than one. Hence by Theorem 2.1, the conclusion holds. □

Proof of Theorem 1.3. By $\lim\limits_{x\to+\infty}\dfrac{g_+(x)}{x}=+\infty$, there are $A_1(>A)$ and $l_+>0$ large enough such that

$$\frac{g_+(x)}{x}\ge l_+,\qquad x\ge A_1.$$

When $m\ge1$, by (1.7) we can find certain arbitrary large l_+ such that

$$\frac{1}{\sqrt{l_-}}+\frac{1}{\sqrt{l_+}}<\frac{2}{m}.$$

Hence all conditions in Theorem 1.1 hold, and when $m=0$ it is Theorem 1.2. □

Proof of Theorem 1.4. Let $(x_1(t),y_1(t))$ and $(x_2(t),y_2(t))$ be any two 2π crossing periodic solutions, and denote by

$$u(t)=x_1(t)-x_2(t),\qquad v(t)=y_1(t)-y_2(t).$$

When $x_i(t)\neq0$ for $i=1,2$, one has that

$$\begin{aligned}\frac{du(t)}{dt}&=v(t),\\ \frac{dv(t)}{dt}&=-[g_\pm(x_1(t))-g_\pm(x_2(t))],\end{aligned}\tag{4.7}$$

where $g_\pm(x_1(t))-g_\pm(x_2(t))$ are 2π-periodic. Making a polar coordinates transformation

$$u(t)=r(t)\cos\theta(t),\qquad v(t)=r(t)\sin\theta(t),$$

then (4.7) is transformed into the form

$$\begin{aligned}\frac{dr(t)}{dt}&=r\sin\theta\cos\theta-[g_\pm(x_1(t))-g_\pm(x_2(t))]\sin\theta,\\ \frac{d\theta(t)}{dt}&=-\sin^2\theta-\frac{1}{r}[g_\pm(x_1(t))-g_\pm(x_2(t))]\cos\theta.\end{aligned}\tag{4.8}$$

Let $(r(t;\theta_0,r_0),\theta(t;\theta_0,r_0))$ be any regular solution of (4.8) with the initial value condition $(\theta_0,r_0)\neq(0,0)$. Then $r(t)>0$ for $t\in[0,2\pi]$. Note that

$$0<\frac{1}{r}[g_\pm(x_1(t))-g_\pm(x_2(t))]\cos\theta=\frac{g_\pm(x_1(t))-g_\pm(x_2(t))}{x_1(t)-x_2(t)}\cos^2\theta<\cos^2\theta.$$

So we have that

$$-2\pi<\theta(2\pi)-\theta(0)=\int_{\theta(0)}^{\theta(2\pi)}\theta'dt<0.$$

Further (4.7) has no nontrivial 2π crossing periodic solution, and then $u(t)\equiv0$. □

5 An example

Let $g_+(x) = e^x + c_1$ and $g_-(x) = ax - c_2$ in (4.1), where $c_1 \geq 0, c_2 > 0, a \in (\frac{1}{3}, \frac{1}{2})$. Choosing $l_1 = \frac{1}{3}, L_- = \frac{1}{2}$ and $A_1 > 0$ sufficiently large. It is easy to verify $(H1)$–$(H3)$ as follows.

$(H1)$ $g_+ \in C^1([0, +\infty), \mathbb{R})$, $g_- \in C^1((-\infty, 0], \mathbb{R})$, and

$$\lim_{x\to 0^-} (ax - c_2) = -c_2 < 0, \qquad \lim_{x\to 0^+} (e^x + c_1) = 1 + c_1 > 0.$$

$(H2)$ $g_-(x) < 0$ for $x < 0$ and $g_+(x) > 0$ for $x > 0$, and $\lim_{x\to-\infty}(ax - c_2) = -\infty$, $\lim_{x\to+\infty}(e^x + c_1) = +\infty$.

$(H3)$ When $A_1 > 0$ is large sufficiently, there holds

$$\frac{1}{3} < \lim_{x\to-\infty} \frac{g_-(x)}{x} < \frac{1}{2}, \qquad \lim_{x\to+\infty} \frac{g_+(x)}{x} = +\infty.$$

Moreover, we can choose l_+ large arbitrary such that when $m = 1$ we have that

$$1 < \frac{1}{\sqrt{L_-}} \leq \frac{1}{\sqrt{l_-}} + \frac{1}{\sqrt{l_+}} < 2.$$

Hence by using Theorem 1.1, it follows that (4.1) has at least one 2π crossing periodic solution.

6 Concluding remarks

In this paper, we have studied the existence and uniqueness of crossing periodic solutions for a mixed-type second order discontinuous Duffing equation (1.2), where $g_\pm$ satisfy some asymmetric growth conditions for $x \in \mathbb{R}$. However in the existing literatures [2, 10, 12, 13], the authors considered the periodic solution problem of discontinuous symmetric Duffing equation. In [2, 12], the authors considered $g_\pm$ being some specific functions and obtained the existence of periodic solutions. In [2], $g_\pm = x(1 - \frac{1}{\sqrt{a^2+x^2}})$ with $a = 0$ being a discontinuous case; in [12] $g_\pm = \eta\,\mathrm{sgn}(y)$ with $\eta \in \mathbb{R}, \eta > 0$. While in [10, 13], by the Poincaré–Birkhoff theorem, the authors studied the multiplicity of periodic solutions with large-amplitude. So to some sense, our results are improvement and generalization of the ones in the literatures [2, 10, 12, 13].

We first presented the definitions of regular solutions and crossing periodic solutions (or orbits) of (1.2), and state the existence and uniqueness of any regular solution to initial value problem of (1.2) exception for the origin. Next, we analyzed the behavior of regular solutions (or orbits) for the autonomous and non-autonomous systems of (1.2), respectively. After some technical lemmas, by giving the definition of left and right Poincaré mappings, we constructed the Poincaré mapping of (1.2) on the whole plane. Then by using the Poincaré–Bohl theorem (*i.e*, Theorem 2.1), we found a 2π crossing periodic solution. And several corollaries are presented by giving different asymmetric linear growth conditions to $g_\pm$ as $|x| \to +\infty$. Next, to demonstrate the effectiveness of the obtained results, we further provided an example. Finally, a uniqueness criterion of the crossing periodic solution was also given when $g_\pm$ satisfy a kind of Lipschitz condition on the pieces of the nonlinearity.

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Infinitely many solutions for nonlinear elliptic problems in the whole space

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Abstract. This paper investigates the existence of infinitely many weak solutions for a nonlinear elliptic p-Laplacian equation defined in the whole space $\mathbb{R}^N$. In particular, the equation is parameter-dependent and a range of positive parameters in which the equation admits such solutions is provided. Moreover, some particular cases in which the solutions turn out to be nonnegative are presented. The study is conducted via variational methods and critical points theory.

Keywords: nonlinear p-Laplacian equation, unbounded domain, critical points theory, infinitely many solutions.

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1 Introduction

The p-Laplacian differential operator, denoted by $\Delta_p(u) = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, has been studied extensively by many authors in the latest years with various techniques. Precisely, there is a wide literature on elliptic p-Laplacian problems in bounded domains, while the results are fewer when the problems are defined in unbounded domains or in the whole space. Indeed, considering elliptic equations in the whole space $\mathbb{R}^N$ increases the technical difficulties of the study and, among the results, we mention Ambrosetti–García Azorero–Peral [1], Barletta–Candito–Marano–Perera [3], Drábek–Huang [6], Guarnotta–Livrea–Marano [7] and the references therein. Inspired by this issues, the authors of the present paper considered in [2] the following nonlinear elliptic problem

$$\begin{cases} -\Delta_p u + a(x)|u|^{p-2}u = \lambda f(x,u) & \text{in } \mathbb{R}^N, \\ \lim_{|x|\to\infty} u(x) = 0, \end{cases} \tag{1.1}$$

where $p > N$, $\lambda > 0$ is a parameter, $a\colon \mathbb{R}^N \to \mathbb{R}$ is a $L^\infty(\mathbb{R}^N)$-function with $\operatorname{ess\,inf}_{\mathbb{R}^N} a > 0$ and $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a L^1-Carathéodory function. Despite the lack of compactness of the embedding of Sobolev spaces in appropriate function spaces, the authors are able to obtain

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results in line with the bounded domain case; precisely, some results on the existence of one and two non-zero solutions are presented. As far as we know, [2] is the first note for the case $p > N$ in the whole space.

The present paper is intended to continue the research carried out in [2], studying the existence of infinitely many solutions for problem (1.1) through variational methods and critical points theory. In particular, the main tool of our investigation is a critical points theorem due to Bonanno [4, Theorem 7.4]. By requiring a suitable oscillation of the primitive of the nonlinear term, we get the existence a sequence of pairwise distinct weak solutions that turn out to be nonnegative with an additional assumption on the nonlinearity. To give an idea of the results that we obtain, we present here the following particular case.

Theorem 1.1. *Let $g\colon \mathbb{R} \to \mathbb{R}$ be a continuous nonnegative function and set $G(t) = \int_0^t g(\xi)\mathrm{d}\xi$ for any $t \in \mathbb{R}$. Given $p > N$, assume that*

$$\liminf_{\xi\to+\infty} \frac{G(\xi)}{\xi^p} = 0 \quad \textit{and} \quad \limsup_{\xi\to+\infty} \frac{G(\xi)}{\xi^p} = +\infty.$$

Then, for any $a \in L^\infty(\mathbb{R}^N)$ with $\operatorname{ess\,inf}_{\mathbb{R}^N} a > 0$ and for any non-null and nonnegative function $h \in L^1(\mathbb{R}^N)$, the following problem

$$\begin{cases} -\Delta_p u + a(x)|u|^{p-2}u = h(x)g(u), & x \in \mathbb{R}^N, \\ \lim_{|x|\to\infty} u(x) = 0, \end{cases}$$

admits a sequence of pairwise distinct nonnegative weak solutions.

The paper is organized as follows. In Section 2, we describe the variational setting of the problem and in Theorem 2.2 we recall the abstract theorem that we use in the proof of our main results, which are presented in Section 3. In particular, Theorem 3.1 and Theorem 3.3 ensure the existence of infinitely many weak solutions for problem (1.1) requiring two different behavior on the nonlinearity. Finally, Section 4 deals with the particular case where the nonlinear term is of separable variable type and also provides an example.

2 Variational framework

Given $p > N$, we consider the Sobolev space $W^{1,p}(\mathbb{R}^N)$ endowed with the usual norm

$$\|u\|_{1,p} = \|u\|_p + \|\nabla u\|_p,$$

where $\|\cdot\|_p$ denotes the classical norm in the Lebesgue space $L^p(\mathbb{R}^N)$. It is well known that $W^{1,p}(\mathbb{R}^N)$ is continuously embedded in $L^\infty(\mathbb{R}^N)$ (see Morrey theorem in Brezis [5, Theorem 9.12]). Hence, there exists a constant $c > 0$ such that

$$\|u\|_\infty \le c\|u\|_{1,p} \quad \text{for all } u \in W^{1,p}(\mathbb{R}^N),$$

and an optimal numerical estimate of the embedding constant c can be found in Proposition 2.2 in Amoroso–Bonanno–Perera [2], see also Remark 2.1. However, for our investigation we equip the space with the following equivalent norm

$$\|u\| = \left(\int_{\mathbb{R}^N} |\nabla u(x)|^p dx + \int_{\mathbb{R}^N} a(x)|u(x)|^p dx \right)^{\frac{1}{p}},$$

where $a\colon \mathbb{R}^N \to \mathbb{R}$ is such that

$$a \in L^\infty(\mathbb{R}^N) \quad \text{and} \quad a_- = \operatorname*{ess\,inf}_{\mathbb{R}^N} a > 0.$$

By Amoroso–Bonanno–Perera [2, Lemma 2.1] one has

$$\|u\|_\infty \leq C\|u\| \quad \text{for all } u \in W^{1,p}(\mathbb{R}^N), \tag{2.1}$$

where

$$C = \left(\frac{1}{a_-}\right)^{\frac{p-N}{p^2}} 2^{\frac{p-1}{p}} \left(\frac{\Gamma\left(1+\frac{N}{2}\right)}{\pi^{\frac{N}{2}}}\right)^{\frac{1}{p}} \frac{1}{N^{\frac{N}{p^2}}} \left(\frac{p-1}{p-N}\right)^{\frac{N(p-1)}{p^2}}, \tag{2.2}$$

with $\Gamma(s) = \int_0^\infty e^{-t}t^{s-1}\mathrm{d}t$ being the well known gamma function. Note that $W_0^{1,p}(\mathbb{R}^N)$ coincides with $W^{1,p}(\mathbb{R}^N)$, hence if $u \in W^{1,p}(\mathbb{R}^N)$ then it holds that

$$\lim_{|x|\to\infty} u(x) = 0.$$

Therefore, studying problem (1.1) is equivalent to studying the following one

$$-\Delta_p u + a(x)|u|^{p-2}u = \lambda f(x,u) \quad \text{in } \mathbb{R}^N, \qquad u \in W^{1,p}(\mathbb{R}^N), \tag{P_λ}$$

that means finding $u \in W^{1,p}(\mathbb{R}^N)$ such that

$$\int_{\mathbb{R}^N} \left(|\nabla u|^{p-2}\nabla u\nabla v + a(x)|u|^{p-2}uv\right) dx = \lambda \int_{\mathbb{R}^N} f(x,u)v\,\mathrm{d}x,$$

for all $v \in W^{1,p}(\mathbb{R}^N)$.

To this end, we suppose that $f\colon \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a L^1-Carathéodory function, i.e.

(i) $f(\cdot,t)$ is measurable for all $t \in \mathbb{R}$,

(ii) $f(x,\cdot)$ is continuous for almost all $x \in \mathbb{R}^N$,

(iii) for all $\sigma > 0$ the function $\sup_{|t|\leq\sigma} |f(\cdot,t)|$ belongs to $L^1(\mathbb{R}^N)$.

Hence, summarizing our hypotheses, we assume that

(H) $p > N, a \in L^\infty(\mathbb{R}^N)$ such that $a_- = \operatorname{ess\,inf}_{\mathbb{R}^N} a > 0$ and $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is an L^1-Carathéodory function.

Our aim is to find weak solutions for problem (P_λ), more precisely we say that $u \in W^{1,p}(\mathbb{R}^N)$ is a weak solution of (P_λ) if

$$\int_{\mathbb{R}^N} \left(|\nabla u|^{p-2}\nabla u\nabla v + a(x)|u|^{p-2}uv\right)\,\mathrm{d}x = \lambda \int_{\mathbb{R}^N} f(x,u)v\,\mathrm{d}x$$

for any $v \in W^{1,p}(\mathbb{R}^N)$. Consider the functionals $\Phi, \Psi\colon W^{1,p}(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$\Phi(u) = \frac{1}{p}\|u\|^p, \qquad \Psi(u) = \int_{\mathbb{R}^N} F(x,u(x))\,\mathrm{d}x,$$

where $F(x,t) = \int_0^t f(x,\xi)\,\mathrm{d}\xi$ for any $(x,t) \in \mathbb{R}^N \times \mathbb{R}$. Then, the functional $I_\lambda = \Phi - \lambda\Psi$ is the so-called energy functional associated to problem (P_λ). It is well known that Φ, Ψ, I_λ are Gâteaux differentiable functionals and it holds that

$$\Phi'(u)(v) = \int_{\mathbb{R}^N} \left(|\nabla u|^{p-2}\nabla u\nabla v + a(x)|u|^{p-2}uv\right)\,\mathrm{d}x,$$
$$\Psi'(u)(v) = \int_{\mathbb{R}^N} f(x,u)v\,\mathrm{d}x,$$

for every $u, v \in W^{1,p}(\mathbb{R}^N)$. Hence, $u \in W^{1,p}(\mathbb{R}^N)$ is a weak solution for (P_λ) if and only if u is a critical point of I_λ, i.e. $I'_\lambda(u)(v) = 0$ for all $v \in W^{1,p}(\mathbb{R}^N)$. Therefore, our aim is to find critical points of the energy functional I_λ, for some $\lambda > 0$.

Our approach is based on variational methods and critical point theory. In particular, our main tool is Theorem 7.4 of Bonanno [4], that we recall here together with the definition of $(PS)^{[r]}$-condition.

Definition 2.1. Let $(X, \|\cdot\|)$ be a Banach space, X^* its dual and $I_\lambda : X \to \mathbb{R}$ a Gâteaux differentiable functional, with $\lambda > 0$. Fix $r \in]-\infty, \infty]$. We say that I_λ satisfies the Palais–Smale condition cut-off upper at r (in short, $(PS)^{[r]}$-condition), if any sequence $\{u_n\} \subseteq X$ such that

(P_1) $I_\lambda(u_n)$ is bounded,

(P_2) $\lim_{n\to\infty} \|I_\lambda(u_n)\|_{X^*} = 0$,

(P_3) $\Phi(u_n) < r$,

has a convergent subsequence in X.

Clearly, if I_λ satisfies the classical (PS)-condition, which involves only conditions (P_1)–(P_2), then it satisfies also the $(PS)^{[r]}$-condition.

Now, let $\Phi, \Psi : X \to \mathbb{R}$ be two functions and for all $r > \inf_X \Phi$, set

$$\varphi(r) = \inf_{u \in \Phi^{-1}((-\infty, r))} \frac{\left(\sup_{v \in (\Phi^{-1}(-\infty, r))} \Psi(v)\right) - \Psi(u)}{r - \Phi(u)}, \tag{2.3}$$

and

$$\alpha := \liminf_{r \to +\infty} \varphi(r), \qquad \beta := \liminf_{r \to (\inf_X \Phi)^+} \varphi(r). \tag{2.4}$$

Theorem 2.2. *Let X be a real Banach space and let $\Phi, \Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functions with Φ bounded from below.*

(a) *If $\alpha < +\infty$ and for each $\lambda \in \left(0, \frac{1}{\alpha}\right)$ the function $I_\lambda = \Phi - \lambda\Psi$ satisfies $(PS)^{[r]}$-condition for all $r \in \mathbb{R}$, then, for each $\lambda \in \left(0, \frac{1}{\alpha}\right)$, the following alternative holds:*

either

(a_1) *I_λ possesses a global minimum, or*

(a_2) *there is a sequence $\{u_n\}$ of critical points (local minima) of I_λ such that $\lim_{n\to+\infty} \Phi(u_n) = +\infty$.*

(b) *If $\beta < +\infty$ and for each $\lambda \in \left(0, \frac{1}{\beta}\right)$ the function $I_\lambda = \Phi - \lambda\Psi$ satisfies $(PS)^{[r]}$-condition for some $r > \inf_X \Phi$, then, for each $\lambda \in \left(0, \frac{1}{\beta}\right)$, the following alternative holds:*

either

(b_1) *there is a global minimum of Φ which is a local minimum of I_λ, or*

(b_2) *there is a sequence of pairwise distinct critical points (local minima) of I_λ sucht that $\lim_{n\to+\infty} \Phi(u_n) = \inf_X \Phi$.*

3 Main results

In this section, we present our main result on the existence of infinitely many weak solutions for problem (P_λ), by requiring an appropriate oscillation of the primitive of the nonlinearity and providing also a range for the parameter λ for which the problem admits such solutions.

Fix an arbitrary open ball $B(x_0, R)$ of center x_0 and radius $R > 0$ and set

$$K_R = \frac{1}{C^p} \frac{\Gamma\left(1+\frac{N}{2}\right)}{\pi^{\frac{N}{2}}} \left(\frac{R^{p-N}}{2^p - 2^{p-N} + R^p \|a\|_\infty} \right), \tag{3.1}$$

where C is given in (2.2); we underline that K_R does not depend on the choice of x_0. Furthermore, put

$$A = \liminf_{\xi \to +\infty} \frac{\int_{\mathbb{R}^N} \max_{|t| \le \xi} F(x,t)\, \mathrm{d}x}{\xi^p},$$

$$B = \limsup_{\xi \to +\infty} \frac{\int_{B\left(x_0, \frac{R}{2}\right)} F(x,\xi)\, \mathrm{d}x}{\xi^p},$$

and

$$\lambda_1 = \frac{1}{B} \frac{1}{p\, C^p K_R}, \qquad \lambda_2 = \frac{1}{A} \frac{1}{p\, C^p}. \tag{3.2}$$

Theorem 3.1. *Let* (H) *be satisfied and assume that*

(h_1) $F(x,t) \ge 0$ *for all* $x \in \mathbb{R}^N, t \ge 0$;

(h_2) $\liminf_{\xi \to +\infty} \frac{\int_{\mathbb{R}^N} \max_{|t| \le \xi} F(x,t)\, \mathrm{d}x}{\xi^p} < K_R \limsup_{\xi \to +\infty} \frac{\int_{B(x_0, \frac{R}{2})} F(x,\xi)\, \mathrm{d}x}{\xi^p}$.

Then, for each $\lambda \in (\lambda_1, \lambda_2)$ *the problem* (P_λ) *admits a sequence of pairwise distinct weak solutions.*

Proof. Our aim is to find a sequence of weak solutions for problem (P_λ), that are critical points of the energy functional $I_\lambda = \Phi - \lambda \Psi$ defined in Section 2, by using part (a) of Theorem 2.2. The functionals Φ and Ψ satisfy the regularity assumptions requested in Theorem 2.2 and in [2, Lemma 2.3] we already proved that I_λ satisfies the $(PS)^{[r]}$-condition for every $r > 0$. So, first we need to verify that $\alpha < +\infty$, where α is given in (2.4). Fix $\lambda \in (\lambda_1, \lambda_2)$ and let $\{s_n\}_{n \in \mathbb{N}} \subseteq \mathbb{R}$ be a sequence such that $\lim_{n \to +\infty} s_n = +\infty$ and

$$A = \lim_{n \to +\infty} \frac{\int_{\mathbb{R}^N} \max_{|t| \le s_n} F(x,t)\, \mathrm{d}t}{s_n^p}.$$

Set $r_n = \frac{s_n^p}{pC^p}$ for all $n \in \mathbb{N}$. From (2.1) it follows that for any $u \in W^{1,p}(\mathbb{R}^N)$ such that $\Phi(u) = \frac{1}{p}\|u\|^p < r_n$, one has

$$\|u\|_\infty \le C\|u\| \le C(pr_n)^{\frac{1}{p}} = s_n \quad \text{for all } n \in \mathbb{N}.$$

Hence, we have that

$$\Phi^{-1}((-\infty, r_n)) \subset \left\{ u \in W^{1,p}(\mathbb{R}^N) : \|u\|_\infty \le s_n \right\} \quad \text{for all } n \in \mathbb{N}.$$

Therefore, by (2.3) it holds that

$$\begin{aligned}\varphi(r_n) &= \inf_{u\in\Phi^{-1}((-\infty,r_n))} \frac{\left(\sup_{v\in(\Phi^{-1}(-\infty,r_n))}\Psi(v)\right)-\Psi(u)}{r_n-\Phi(u)}\\ &\leq \frac{\left(\sup_{v\in(\Phi^{-1}(-\infty,r_n))}\Psi(v)\right)-\Psi(0)}{r_n-\Phi(0)} \leq \frac{\sup_{\|v\|_\infty\leq s_n}\int_{\mathbb{R}^N}F(x,v)\,\mathrm{d}x}{r_n}\\ &\leq p\,C^p\,\frac{\int_{\mathbb{R}^N}\max_{|t|\leq s_n}F(x,t)\,\mathrm{d}x}{s_n^p}.\end{aligned}$$

So, we get

$$\alpha=\liminf_{n\to+\infty}\varphi(r_n)\leq p\,C^p\,A.$$

From assumption (h_2), that can be read as $A<K_RB$, it follows that $A<+\infty$ and so $\alpha<+\infty$. Consequently, either (a_1) or (a_2) holds.

In order to get the existence of a sequence of critical points of I_λ, we prove that it does not posses a global minimum, i.e. that it is unbounded from below. Consider the fixed open ball $B(x_0,R)$ of center x_0 and radius $R>0$ (see (3.1)) and let $\{b_n\}_{n\in\mathbb{N}}\subseteq$ be a sequence such that $\lim_{n\to+\infty}b_n=+\infty$ and

$$B=\lim_{n\to+\infty}\frac{\int_{B\left(x_0,\frac{R}{2}\right)}F(x,b_n)\,\mathrm{d}x}{b_n^p}. \tag{3.3}$$

Set $S=B\left(x_0,R\right)\setminus B\left(x_0,\frac{R}{2}\right)$ and for any $n\in\mathbb{N}$ consider the following function

$$w_n(t)=\begin{cases}0 & \text{if}\quad x\in\mathbb{R}^N\setminus B\left(x_0,R\right),\\ \dfrac{2b_n}{R}(R-|x-x_0|) & \text{if}\quad x\in S,\\ b_n & \text{if}\quad x\in B\left(x_0,\frac{R}{2}\right).\end{cases}$$

Clearly, for every $n\in\mathbb{N}$ it holds that $w_n\in W^{1,p}(\mathbb{R}^N)$ (see Brezis [5, Remark 4(ii), p. 265] or Motreanu–Motreanu–Papageorgiou[8, Proposition 1.10, p. 3]) and

$$\begin{aligned}\|w_n\|^p &= \int_{\mathbb{R}^N}\left(|\nabla w_n|^p+a(x)|w_n|^p\right)\mathrm{d}x\\ &= \int_S\left(\left|\frac{2b_n}{R}\right|^p+a(x)\left|\frac{2b_n}{R}(R-|x-x_0|^p)\right|\right)\mathrm{d}x+\int_{B\left(x_0,\frac{R}{2}\right)}a(x)b_n^p\mathrm{d}x\\ &\leq \int_S\left(\left|\frac{2b_n}{R}\right|^p+\|a\|_\infty\left|\frac{2b_n}{R}\frac{R}{2}\right|^p\right)\mathrm{d}x+\|a\|_\infty\int_{B\left(x_0,\frac{R}{2}\right)}b_n^p\mathrm{d}x\\ &\leq b_n^p\left[\left(\frac{2^p}{R^p}+\|a\|_\infty\right)\mathrm{meas}(S)+\|a\|_\infty\,\mathrm{meas}\left(B\left(x_0,\frac{R}{2}\right)\right)\right],\end{aligned}$$

where

$$\begin{aligned}\mathrm{meas}(S) &= \mathrm{meas}\left(B\left(x_0,R\right)\right)-\mathrm{meas}\left(B\left(x_0,\frac{R}{2}\right)\right)\\ &= \frac{\pi^{\frac{N}{2}}}{\Gamma\left(1+\frac{N}{2}\right)}R^N-\frac{\pi^{\frac{N}{2}}}{\Gamma\left(1+\frac{N}{2}\right)}\left(\frac{R}{2}\right)^N\\ &= \frac{2^N-1}{2^N}\frac{\pi^{\frac{N}{2}}}{\Gamma\left(1+\frac{N}{2}\right)}R^N.\end{aligned}$$

Hence, we obtain

$$\begin{aligned}\|w_n\|^p &\leq b_n^p \frac{\pi^{\frac{N}{2}}}{\Gamma\left(1+\frac{N}{2}\right)}\left[\left(\frac{2^p}{R^p}+\|a\|_\infty\right)\frac{2^N-1}{2^N}R^N+\|a\|_\infty\frac{R^N}{2^N}\right]\\ &= b_n^p \frac{\pi^{\frac{N}{2}}}{\Gamma\left(1+\frac{N}{2}\right)}\left[\frac{2^p-2^{p-N}+R^p\|a\|_\infty}{R^{p-N}}\right],\end{aligned}$$

which leads to

$$\Phi(w_n)=\frac{1}{p}\|w_n\|^p \leq \frac{b_n^p}{p}\frac{\pi^{\frac{N}{2}}}{\Gamma\left(1+\frac{N}{2}\right)}\left[\frac{2^p-2^{p-N}+R^p\|a\|_\infty}{R^{p-N}}\right]=\frac{b_n^p}{p}\frac{1}{K_R C^p}, \tag{3.4}$$

see (3.1). On the other hand, by (h_1) we know that

$$\begin{aligned}\Psi(w_n) &= \int_{\mathbb{R}^N} F(x,w_n)\mathrm{d}x\\ &= \int_S F\left(x,\frac{2b_n}{R}(R-|x-x_0|)\right)\mathrm{d}x+\int_{B\left(x_0,\frac{R}{2}\right)} F(x,b_n)\mathrm{d}x\\ &\geq \int_{B\left(x_0,\frac{R}{2}\right)} F(x,b_n)\mathrm{d}x.\end{aligned} \tag{3.5}$$

Combining together (3.4) and (3.5), we get

$$I_\lambda(w_n)=\Phi(w_n)-\lambda\Psi(w_n)\leq \frac{b_n^p}{p}\frac{1}{K_R C^p}-\lambda\int_{B\left(x_0,\frac{R}{2}\right)} F(x,b_n)\mathrm{d}x, \tag{3.6}$$

for all $\lambda>0, n\in\mathbb{N}$. Now, taking (3.3) into account we have two possibilities:

1. $B=+\infty$
   In this case, fix $M>\frac{1}{\lambda}\frac{1}{p}\frac{1}{K_R C^p}$; from (3.3) there exists $\nu_M>0$ such that
   $$\int_{B\left(x_0,\frac{R}{2}\right)} F(x,b_n)\mathrm{d}x > Mb_n^p \quad \text{for all } n>\nu_M.$$
   Then, from (3.6) it holds that
   $$I_\lambda(w_n)\leq b_n^p\left(\frac{1}{p}\frac{1}{K_R C^p}-\lambda M\right) \quad \text{for all } n>\nu_M.$$
   By the choice of M, one has
   $$\lim_{n\to+\infty} I_\lambda(w_n)=-\infty.$$

2. $B<+\infty$
   In this case, fix $\varepsilon\in\left(0,B-\frac{1}{\lambda p}\frac{1}{K_R C^p}\right)$; from (3.3) there exists $\nu_\varepsilon>0$ such that
   $$\int_{B\left(x_0,\frac{R}{2}\right)} F(x,b_n)\mathrm{d}x > (B-\varepsilon)b_n^p \quad \text{for all } n>\nu_\varepsilon.$$
   Therefore, from (3.6) one has
   $$I_\lambda(w_n)\leq b_n^p\left(\frac{1}{p}\frac{1}{K_R C^p}-\lambda(B-\varepsilon)\right) \quad \text{for all } n>\nu_\varepsilon.$$
   By the choice of ε, it follows that
   $$\lim_{n\to+\infty} I_\lambda(w_n)=-\infty.$$

We underline that the choices of M and ε are allowed thanks to assumption (h_2), which also implies that (λ_1, λ_2) is nonempty and

$$(\lambda_1, \lambda_2) \subseteq \left(0, \frac{1}{\alpha}\right).$$

So, we get that the energy functional I_λ is unbounded from below and part (a_2) of Theorem 2.2 ensures that for each $\lambda \in (\lambda_1, \lambda_2)$ the functional $I_\lambda = \Phi - \lambda\Psi$ admits a sequence $\{u_n\}_{n\in\mathbb{N}} \subset W^{1,p}(\mathbb{R}^N)$ of critical points such that $\lim_{n\to+\infty} \|u_n\| = +\infty$, which are weak solutions of the problem (P_λ). □

Requiring an additional assumption, we can get the existence of nonnegative weak solutions, as the following result ensures.

Theorem 3.2. *Let (H) be satisfied. Suppose that $f(x,0) \geq 0$ for any $x \in \mathbb{R}^N$ and assume that (h_1)–(h_2) hold. Then, for each $\lambda \in (\lambda_1, \lambda_2)$, where λ_1, λ_2 are defined in (3.2), the problem (P_λ) admits a sequence of pairwise distinct nonnegative weak solutions.*

Proof. By applying Theorem 3.1 we get the existence of a sequence of pairwise distinct weak solutions and [2, Lemma 2.2] ensures that the solutions are nonnegative. □

Clearly, being $\Phi(\cdot) = \frac{1}{p}\|\cdot\|^p$, one has that

$$\inf_{W^{1,p}(\mathbb{R}^N)} \Phi = \min_{W^{1,p}(\mathbb{R}^N)} \Phi = \Phi(0) = 0.$$

Hence, exploiting part (b) of Theorem 2.2 and arguing as in the proof of Theorem 3.1, we can state the following alternative result.

Theorem 3.3. *Let (H) be satisfied and assume that*

(h_1') $F(x,t) \geq 0$ *for all* $x \in \mathbb{R}^N, t \geq 0$;

(h_2') $\liminf_{\xi\to 0^+} \frac{\int_{\mathbb{R}^N} \max_{|t|\leq\xi} F(x,t)\,dx}{\xi^p} < K_R \limsup_{\xi\to 0^+} \frac{\int_{B\left(x_0,\frac{R}{2}\right)} F(x,\xi)\,dx}{\xi^p}$.

Then, for each $\lambda \in (\lambda_1', \lambda_2')$, with

$$\lambda_1' = \frac{1}{p\,C^p K_R} \limsup_{\xi\to 0^+} \frac{\xi^p}{\int_{B\left(x_0,\frac{R}{2}\right)} F(x,\xi)\,dx}, \quad \lambda_2' = \frac{1}{p\,C^p} \liminf_{\xi\to 0^+} \frac{\xi^p}{\int_{\mathbb{R}^N} \max_{|t|\leq\xi} F(x,t)\,dx},$$

the problem (P_λ) admits a sequence of pairwise distinct weak solutions which uniformly converges to zero.
In addition, if $f(x,0) \geq 0$ for any $x \in \mathbb{R}^N$, the solutions are nonnegative.

4 Particular cases

This section deals with some consequences of the main results, depending on particular nonlinear terms. Precisely, when the nonlinear term is with separated variables, i.e. of the type $f(x,u) = h(x)g(u)$, the problem (P_λ) becomes the following one

$$-\Delta_p u + a(x)|u|^{p-2}u = \lambda h(x)g(u) \quad \text{in}\mathbb{R}^N, \qquad u \in W^{1,p}(\mathbb{R}^N), \tag{$\tilde{P}_\lambda$}$$

where we suppose that

($\tilde{\mathrm{H}}$) $p > N, a \in L^\infty(\mathbb{R}^N)$ such that $a_- = \operatorname{ess\,inf}_{\mathbb{R}^N} a > 0$, $h \in L^1(\mathbb{R}^N)$ such that $h \not\equiv 0, h \geq 0$ and $g \in C(\mathbb{R}), g \geq 0$.

In this case, the energy functional related to problem ($\tilde{\mathrm{P}}_\lambda$) is

$$\tilde{I}_\lambda(u) = \Phi(u) - \lambda\tilde{\Psi}(u) \quad \text{for all } u \in W^{1,p}(\mathbb{R}^N),$$

where Φ is defined in Section 2 and

$$\tilde{\Psi}(u) = \int_{\mathbb{R}^N} h(x)G(u(x))\mathrm{d}x, \quad \text{with} \quad G(t) = \int_0^t g(\xi)\mathrm{d}\xi \quad \text{for all } t \in \mathbb{R}.$$

Clearly, $u \in W^{1,p}(\mathbb{R}^N)$ is a critical point of $\tilde{I}_\lambda$ if and only if is a weak solution for problem ($\tilde{\mathrm{P}}_\lambda$). Set

$$\tilde{K}_R = K_R \int_{B\left(x_0, \frac{R}{2}\right)} h(x)\mathrm{d}x, \tag{4.1}$$

where K_R is given by (3.1), and put

$$\tilde{\lambda}_1 = \frac{1}{p\,C^p\tilde{K}_R} \limsup_{\xi\to+\infty} \frac{\xi^p}{G(\xi)}, \qquad \tilde{\lambda}_2 = \frac{1}{p\,C^p\|h\|_1} \liminf_{\xi\to+\infty} \frac{\xi^p}{G(\xi)}.$$

Theorem 4.1. *Let* ($\tilde{H}$) *be satisfied and assume that*

($\tilde{h}_2$) $\liminf_{\xi\to+\infty} \frac{G(\xi)}{\xi^p} < \frac{\tilde{K}_R}{\|h\|_1} \limsup_{\xi\to+\infty} \frac{G(\xi)}{\xi^p}$.

Then, for each $\lambda \in \left(\tilde{\lambda}_1, \tilde{\lambda}_2\right)$ *the problem* ($\tilde{\mathrm{P}}_\lambda$) *admits a sequence of pairwise distinct nonnegative weak solutions.*

Proof. Arguing as in the proof of Theorem 3.1, taking into account that g is nonnegative, we get that

$$\alpha = \liminf_{n\to+\infty} \varphi(r_n) \leq p\,C^p\,\|h\|_1 \lim_{n\to+\infty} \frac{G(s_n)}{s_n},$$

and $\alpha < +\infty$ from hypothesis ($\tilde{h}_2$). Furthermore, one has that

$$\tilde{I}_\lambda(w_n) \leq \frac{b_n^p}{p} \frac{1}{K_R\,C^p} - \lambda G(b_n) \int_{B\left(x_0, \frac{R}{2}\right)} h(x)\mathrm{d}x,$$

for all $\lambda > 0, n \in \mathbb{N}$. So, it yields to $\tilde{I}_\lambda(w_n) \to -\infty$ as $n \to +\infty$ thanks to the following:

1. choose $M > \frac{1}{\lambda}\frac{1}{p}\frac{1}{\tilde{K}_R C^p}$ if $\limsup_{\xi\to+\infty} \frac{G(\xi)}{\xi^p} = +\infty$;
2. fix $\varepsilon \in \left(0, \limsup_{\xi\to+\infty} \frac{G(\xi)}{\xi^p} - \frac{1}{\lambda p}\frac{1}{\tilde{K}_R C^p}\right)$ if $\limsup_{\xi\to+\infty} \frac{G(\xi)}{\xi^p} < +\infty$.

Then, we obtain that for each $\lambda \in \left(\tilde{\lambda}_1, \tilde{\lambda}_2\right)$ the problem ($\tilde{\mathrm{P}}_\lambda$) admits a sequence $\{u_n\}_{n\in\mathbb{N}} \subset W^{1,p}(\mathbb{R}^N)$ of weak solutions such that $\lim_{n\to+\infty} \|u_n\| = +\infty$. Finally, since h and g are nonnegative functions, by [2, Lemma 2.2] we know that the weak solutions are nonnegative. □

Clearly, Theorem 1.1 is a special case of Theorem 4.1. In the following, we give a version of Theorem 3.3.

Theorem 4.2. *Let* ($\tilde{H}$) *be satisfied and assume that*

$(\tilde{h}_2')$ $\liminf_{\xi\to 0^+} \frac{G(\xi)}{\xi^p} < \frac{\tilde{K}_R}{\|h\|_1} \limsup_{\xi\to 0^+} \frac{G(\xi)}{\xi^p}$.

Then, for each $\lambda \in \left(\tilde{\lambda}_1', \tilde{\lambda}_2'\right)$, *with*

$$\tilde{\lambda}_1' = \frac{1}{p\, C^p \tilde{K}_R} \limsup_{\xi\to 0^+} \frac{\xi^p}{G(\xi)}, \qquad \tilde{\lambda}_2' = \frac{1}{p\, C^p \|h\|_1} \liminf_{\xi\to 0^+} \frac{\xi^p}{G(\xi)}.$$

the problem ($\tilde{P}_\lambda$) *admits a sequence of pairwise distinct nonnegative weak solutions which uniformly converges to zero.*

Finally, we provide an example of applicability of our results.

Example 4.3. Let $k > 1$ be an arbitrary constant and consider

$$h(x) = e^{-|x|^2} \quad \text{for all } x \in \mathbb{R}^N,$$

and

$$g(t) = \begin{cases} p(1+t)^{p-1}\left(1+k-\cos(k\ln(1+t)) + \frac{k}{p}\sin(k\ln(1+t))\right) & \text{if } t \geq 0, \\ pk & \text{if } t < 0. \end{cases}$$

Clearly, $h\colon \mathbb{R}^N \to \mathbb{R}$ is non-null, positive and $h \in L^1(\mathbb{R}^N)$ with $\|h\|_1 = \pi^{\frac{N}{2}}$, while $g\colon \mathbb{R} \to \mathbb{R}$ is continuous and positive. Moreover, one has

$$G(t) = \begin{cases} (1+t)^p\,(1+k-\cos(k\ln(1+t))) - k & \text{if } t \geq 0, \\ pkt & \text{if } t < 0. \end{cases}$$

By choosing the arbitrary ball $B(0,1)$, see (3.1) and (4.1), we have that

$$\tilde{K}_1 = \frac{N\,\gamma\left(\frac{N}{2}, \frac{1}{4}\right)}{C^p\,(2^{p+1} - 2^{p+1-N} + 2\|a\|_\infty)},$$

where $\gamma(s,x) = \int_0^x e^{-t} t^{s-1} \mathrm{d}t$ is the lower incomplete gamma function. Furthermore, it holds that

$$\liminf_{\xi\to+\infty} \frac{G(\xi)}{\xi^p} = 1-k, \qquad \limsup_{\xi\to+\infty} \frac{G(\xi)}{\xi^p} = 1+k.$$

Then, it is easy to verify that hypothesis ($\tilde{h}_2$) holds and Theorem 4.1 ensures that for any $\lambda \in \left(\frac{2^{p+1}-2^{p+1-N}+2\|a\|_\infty}{Np(1+k)\gamma(\frac{N}{2},\frac{1}{4})}, \frac{1}{pC^p\pi^{\frac{N}{2}}(1-k)}\right)$ and for any $a \in L^\infty(\mathbb{R}^N)$ with $\operatorname{ess\,inf}_{\mathbb{R}^N} a > 0$, the following equation

$$-\Delta_p u + a(x)|u|^{p-2}u = \lambda h(x) g(u), \quad x \in \mathbb{R}^N,$$

admits a sequence of pairwise distinct nonnegative weak solutions in $W^{1,p}(\mathbb{R}^N)$.

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Quadratic systems with two invariant real straight lines and an invariant parabola

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Abstract. After the linear differential systems in the plane the easiest ones are the quadratic polynomial differential systems. Due to their nonlinearity and also to their many applications these systems have been studied by many authors. Let **QS** denote the set of all planar quadratic polynomial differential systems, or simply *quadratic systems*, and let **QSR2P** denote the subset of **QS** having two finite invariant real straight lines and an invariant parabola. We classify the phase portraits in the Poincaré disc of all the quadratic systems in the class **QSR2P**.

Keywords: quardratic systems, planar polynomial differential systems, global phase portrait, Poincaré disc.

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1 Introduction

The study of the quadratic systems started at the early twentieth century. Coppel in [10] mentions that in 1904 Büchel [3] published the first article on quadratic systems. Two classical surveys on quadratic systems were published in 1966 by Coppel [10], and in 1982 by Chicone and Tian [8].

Quadratic systems have been studied intensively during these last decades obtaining many good results on them, see for instance the books [1,28] and the references cited therein. More than one thousand papers have been published on the quadratic systems, but we are still far from completely understanding them.

We consider the following class of planar polynomial differential systems

$$\dot{x} = P(x,y), \quad \dot{y} = Q(x,y), \tag{1.1}$$

where $P, Q \in \mathbb{R}[x,y]$, being $\mathbb{R}[x,y]$ the ring of real polynomials in the two variables x and y. System (1.1) is said to be quadratic if the highest degree of the polynomials P and Q is two, that is, if $\max\{\deg P, \deg Q\} = 2$. Such quadratic systems include a wide range of planar

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dynamical models, including classical examples such as certain Lotka–Volterra systems in population dynamics and Liénard-type systems in control theory.

In this work we restrict our attention to non-degenerate quadratic systems, meaning that the polynomials P and Q are coprime. This assumption guarantees that the quadratic system cannot be reduced to linear or constant differential systems.

The study of quadratic systems has been a central topic in the qualitative theory of dynamical systems, because they exhibit structurally rich dynamics despite their relatively simple algebraic form. These systems may admit multiple invariant algebraic curves, limit cycles, and different types of equilibria. The classification and characterisation of such systems remains an active area of research.

Quadratic systems having invariant conics, i.e., invariant algebraic curves of degree 2, have been studied by several authors, see [2, 5, 14, 16–18, 24–27]. Quadratic systems having invariant algebraic curves of degree 3 have been analysed in [4, 7, 12]. Quadratic systems having invariant straight lines with multiplicity 4 have been classified in [22, 23]. Quadratic systems having invariant straight lines with multiplicity 5 have been classified in [20, 21].

The objective of this paper is to classify the quadratic systems possessing two invariant real straight lines and an invariant parabola. Building on previous classifications of quadratic systems with invariant algebraic curves of degrees two and three [6, 12], this study forms part of a broader effort to investigate systems possessing degree-four invariant curves. Within this framework, we focus on a specific subclass and contribute to its classification. By deriving normal forms, analysing equilibria, and classifying their global phase portraits via Poincaré compactification, we integrate algebraic, geometric, and topological methods to provide all their different topological phase portraits in the Poincaré disc.

2 Preliminaries

We begin by recalling some fundamental notions used throughout the paper.

Let $X = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y}$ be the polynomial vector field associated with the polynomial differential system (1.1).

2.1 First integrals and integrating factors.

Let U be an open subset of $\mathbb{R}^2$ and let $H : U \to \mathbb{R}$ be a non-constant C^1 function. We say that H is a *first integral* of the vector field X or of the system (1.1) if $H(x(t), y(t))$ is constant on all the solutions $(x(t), y(t))$ of system (1.1) contained in U. Note that H is a first integral if and only if

$$\frac{dH}{dt} = \frac{\partial H}{\partial x}P(x,y) + \frac{\partial H}{\partial y}Q(x,y) = 0.$$

Let $R : U \to \mathbb{R}$ be a C^1 non-zero function. The function R is an *integrating factor* of the vector field X or of the system (1.1) on U if one of the following three equivalent conditions holds on U:

$$\frac{\partial(RP)}{\partial x} = -\frac{\partial(RQ)}{\partial y}, \qquad \operatorname{div}(RP, RQ) = 0, \qquad XR = -R\operatorname{div}(P,Q).$$

Here $\operatorname{div}(P,Q) = \dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y}$. The *first integral H associated to the integrating factor R* is given by

$$H(x,y) = \int R(x,y)P(x,y)\,dy + h(x),$$

where the function h is chosen such that $\frac{\partial H}{\partial x} = -RQ$. Then

$$\dot{x} = RP = \frac{\partial H}{\partial y}, \qquad \dot{y} = RQ = -\frac{\partial H}{\partial x}.$$

2.2 Invariant algebraic curves.

We recall that system (1.1) admits an *invariant algebraic curve* $f = f(x,y) = 0$ if

$$Xf = \frac{\partial f}{\partial x}P(x,y) + \frac{\partial f}{\partial y}Q(x,y) = K(x,y)f(x,y) = Kf, \tag{2.1}$$

where K is a polynomial called the *cofactor* of the invariant algebraic curve $f(x,y) = 0$. This terminology reflects the fact that if an orbit of the system has a point on the algebraic curve $f(x,y) = 0$, then the equality (2.1) forces that the entire orbit is contained in the algebraic curve $f(x,y) = 0$.

2.3 Extactic polynomial.

Let X be the vector field associated with the polynomial differential system (1.1) of degree m. The *n–th extactic polynomial of* X, $E_n(X)$, is defined by the determinant

$$\det \begin{pmatrix} v_1 & v_2 & \cdots & v_l \\ X(v_1) & X(v_2) & \cdots & X(v_l) \\ \vdots & \vdots & \ddots & \vdots \\ X^{l-1}(v_1) & X^{l-1}(v_2) & \cdots & X^{l-1}(v_l) \end{pmatrix},$$

where $v_1, v_2, \ldots, v_l$ form a basis of $\mathbb{R}_n[x,y]$, the $\mathbb{R}$–vector space of polynomials in $\mathbb{R}[x,y]$ of degree at most n, $l = (n+1)(n+2)/2$, $X^0(v_i) = v_i$ and $X^j(v_i) = X^{j-1}(X(v_i))$.

Note that the definition of the extactic polynomial is independent of the choice of basis in $\mathbb{R}_n[x,y]$.

Proposition 2.1 (Proposition 5.2 in [9]). *Let $f = 0$ be an invariant algebraic curve of degree n of the vector field X. Then the polynomial f divides the polynomial $E_n(X)$.*

If the invariant algebraic curve $f = f(x,y) = 0$ in Proposition 2.1 satisfies that f^k divides the polynomial $E_n(X)$ but f^{k-1} does not divide it, then by definition the *multiplicity* of the invariant algebraic curve f is k. Geometrically, this means that within the class of all polynomial vector fields of degree m, one can perturb the vector field X into a family X_ε, such that each X_ε admits k invariant algebraic curves, all of them tending to the curve $f = 0$ when $\varepsilon \to 0$. For further details see [9].

2.4 Poincaré compactification.

The Poincaré compactification is a classical tool for analysing the dynamics of planar polynomial differential systems in a neighbourhood of infinity.

Let X be a polynomial vector field of degree d and

$$\mathbb{S}^2 = \{(y_1, y_2, y_3) \in \mathbb{R}^3 : y_1^2 + y_2^2 + y_3^2 = 1\}$$

be the unit sphere in $\mathbb{R}^3$. Let π_0 be the plane tangent to $\mathbb{S}^2$ at $(0,0,1)$, then we identify the plane π_0 with $\mathbb{R}^2$. Consider the central projection $f : \mathbb{R}^2 \to \mathbb{S}^2$, which maps the point $(x,y,1)$ in plane π_0 to the two intersection points of the straight line through $(x,y,1)$ and $(0,0,0)$ with the sphere $\mathbb{S}^2$. Then $\mathbb{S}^1 = \{y \in \mathbb{S}^2 : y_3 = 0\}$ corresponds to the infinity of $\mathbb{R}^2$. Thus the differential Df maps the vector field X in $\mathbb{R}^2$ to X' in $\mathbb{S}^2 \setminus \mathbb{S}^1$. The new vector field X' consists of two symmetric copies of X with respect to the origin of coordinates.

Next we multiply X' by y_3^d to get a vector field $p(X)$ defined in the whole sphere $\mathbb{S}^2$. This process is called the *Poincaré compactification*. Furthermore, the dynamical behavior of $p(X)$ near $\mathbb{S}^1$ corresponds to behavior of X in a neighborhood of infinity.

Since $\mathbb{S}^2$ is a curved manifold we use local charts on the surface of $\mathbb{S}^2$ to facilitate the computations. More precisely, we map the surface of $\mathbb{S}^2$ into a group of local charts (U_i, ϕ_i) and (V_i, ψ_i) that are identical to $\mathbb{R}^2$ for $i = 1,2,3$, where

$$U_i = \{(y_1,y_2,y_3) \in \mathbb{S}^2 : y_i > 0\}, \quad V_i = \{(y_1,y_2,y_3) \in \mathbb{S}^2 : y_i < 0\},$$
$$\phi_i(y_1,y_2,y_3) = \psi_i(y_1,y_2,y_3) = \left(\frac{y_j}{y_i}, \frac{y_k}{y_i}\right), \text{ with } j < k, j,k \neq i.$$

The expressions of $p(X)$ in the local charts (U_1,ϕ_1), (U_2,ϕ_2), and (U_3,ϕ_3) are

$$\dot{u} = v^d\left[-uP\left(\frac{1}{v},\frac{u}{v}\right) + Q\left(\frac{1}{v},\frac{u}{v}\right)\right], \quad \dot{v} = -v^{d+1}P\left(\frac{1}{v},\frac{u}{v}\right).$$
$$\dot{u} = v^d\left[P\left(\frac{u}{v},\frac{1}{v}\right) - uQ\left(\frac{u}{v},\frac{1}{v}\right)\right], \quad \dot{v} = -v^{d+1}Q\left(\frac{u}{v},\frac{1}{v}\right),$$
$$\dot{u} = P(u,v), \quad \dot{v} = Q(u,v),$$

respectively. In the local chart (V_i,ψ_i) the expression of $p(X)$ is the same as in (U_i,ϕ_i) multiplied by $(-1)^{d-1}$ for $i = 1,2,3$. Furthermore, the straight line $v = 0$ in local charts U_1, U_2, V_1, V_2 corresponds to the equator $\mathbb{S}^1$ of $\mathbb{S}^2$, i.e., to the infinity of $\mathbb{R}^2$.

We note that an *infinite equilibrium* of X under the Poincaré compactification is an equilibrium of $p(X)$ in $\mathbb{S}^1$. Moreover, if y is an equilibrium in $\mathbb{S}^1$, then $-y$ is also an equilibrium. When the degree d of X is odd, these two equilibria have the same stability, and if d is even they have opposite stability. Due to the symmetry of $p(X)$ in $\mathbb{S}^2$, it is sufficient to investigate equilibria at infinity in the chart (U_1,ϕ_1) and checking if the origin of chart (U_2,ϕ_2) is also an equilibrium or not.

Since the integral curves in $\mathbb{S}^2$ are symmetric with respect to the origin, we only need to study the flow of $p(X)$ in the closed northern hemisphere, i.e., when $y_3 \geqslant 0$, projecting this part of $\mathbb{S}^2$ onto plane $y_3 = 0$ via a projection $\pi : (y_1,y_2,y_3) \mapsto (y_1,y_2,0)$ gives the so-called *Poincaré disc* $\mathbb{D}$.

For more details on the Poincaré compactification see Chapter 5 of [11].

2.5 Topological equivalence of two polynomial vector fields.

Two Poincaré compactifications $p(X_1)$ and $p(X_2)$ on $\mathbb{D}$ are *topologically equivalent* if there exists a homomorphism $h : \mathbb{D} \to \mathbb{D}$ preserving $\mathbb{S}^1$ and mapping the orbits of $\pi(p(X_1))$ onto the orbits of $\pi(p(X_2))$ preserving or reversing orientation of all the orbits.

The set of separatrices Σ_X of a Poincaré compactification $\pi(p(X))$ consists of all the orbits on $\mathbb{S}^1$, finite equilibria, limit cycles, and the two orbits of the boundary of all hyperbolic sectors associated with the finite and infinite equilibria. For more details, see [13,15].

The open connected regions of $\mathbb{D} \setminus \Sigma_X$ are called *canonical regions* of $\pi(p(X))$. The set Σ_X and one orbit inside each canonical region form the *separatrix configuration* Σ'_X. We denote by S the number of separatrices in the phase portrait of $\pi(p(X))$ on $\mathbb{D}$, and by R the number of its canonical regions. When the circle at infinity in the Poincaré disc is filled with equilibria we do not compute them in the number S.

Two separatrix configurations Σ'_1 and Σ'_2 are *topologically equivalent* if and only if there exists a homomorphism $h : \Sigma'_1 \to \Sigma'_2$ such that $h(\Sigma'_1) = \Sigma'_2$. Hence by the following theorem which was proved by Markus [13], Neumann [15] and Peixoto [19] independently, it suffices to determine when two phase portraits in the Poincaré disc are topologically equivalent by analysing their separatrix configurations.

Theorem 2.2. *Two Poincaré compactified polynomial vector fields $\pi(p(X_1))$ and $\pi(p(X_2))$ with finitely many separatrices are topologically equivalent if and only if their separatrix configurations Σ'_{X_1} and Σ'_{X_2} are topologically equivalent.*

2.6 Quadratic systems with an invariant parabola

The following proposition provides the classification of the quadratic systems having an invariant parabola.

Lemma 2.3 (Proposition 4 in [6])**.** *A quadratic system having an invariant parabola after an affine change of coordinates and a rescaling of the time can be written in the following form*

$$\dot{x} = \frac{b}{2}xy - \frac{a}{2}\left(y - x^2\right) - (p + qx + ry), \quad \dot{y} = by^2 + c\left(y - x^2\right) - 2x(p + qx + ry), \qquad \text{(P)}$$

where a, b, c, p, q, r are real parameters.

3 Main results

In the following two theorems, we present the normal forms of all planar quadratic systems possessing two invariant real straight lines and an invariant parabola, together with their corresponding phase portraits in the Poincaré disc.

Theorem 3.1. *By doing an affine change of variables and a rescaling of the time all the quadratic systems having either two invariant real straight lines and an invariant parabola $f_j = f_j(x, y) = 0$ with cofactors $K_j = K_j(x, y)$ for $j = 1, 2, 3$, or having one real straight line of multiplicity two and an invariant parabola, can be written as one of the following systems, where $H = H(x, y)$ is a first integral and $R = R(x, y)$ is an integrating factor of the corresponding quadratic system.*

$S(1)$ $\dot{x} = x + 1, \quad \dot{y} = x^2 + 2x + y,$
$f_1 = y - x^2, \quad K_1 = 1,$
$f_2 = x + 1, \quad$ *with multiplicity two,* $\quad K_2 = 1,$
$H = \frac{f_1}{f_2}.$

$S(2)$ $\dot{x} = -4x^2 - 2ax + 2y - a + \frac{1}{2}, \quad \dot{y} = -4xy + (1 - 2a)\,x - 4ay,$
$f_1 = y - x^2, \quad K_1 = -4(2x + a),$
$f_2 = 4x + 4y + 1, \quad K_2 = -4x - 4a + 2,$
$f_3 = 4ax + 4y + 2a - 1, \quad K_3 = -2(2x + a), \quad H = \frac{f_1}{f_3^2}.$

$S(3)$ $\dot{x} = -x^2 + (a+b)\,x - ab, \quad \dot{y} = (2b-1)\,x^2 - 2xy + (2a+1)\,y - 2abx,$
$f_1 = y - x^2, \quad K_1 = -2x + 2a + 1,$
$f_2 = x - a, \quad K_2 = b - x,$
$f_3 = x - b, \quad K_3 = a - x, \quad H = f_1^{a-b}\, f_2\, f_3^{2b-2a-1},$
where $(2a - 2b + 1) \neq 0$.

$S(4)$ $\dot{x} = 8(a+1)\,x^2 + 8xy + 2(4ab + 4a + 4b + 3)\,x + 4(2b+1)\,y + (4ab + 2a + 2b + 1),$
$\dot{y} = 8(a + 2b + 1)\,x^2 + 8(2a + 2b + 3)\,xy + 2(4ab + 2a + 2b + 1)\,x + 4(4ab + 2a + 1)\,y + 16y^2,$
$f_1 = y - x^2, \quad K_1 = 16(a+1)\,x + 16y + 4(4ab + 2a + 1),$
$f_2 = 4x + 4y + 1, \quad K_2 = 16(a + b + 1)\,x + 16y + 4(4ab + 2a + 2b + 1),$
$f_3 = 2x + 2b + 1, \quad K_3 = 8(a+1)\,x + 8y + 2(2a+1),$
$R = \frac{f_2^{(a-2b)/(2b)}}{f_1^{(a+b)/(2b)} f_3^2},$
where $ab \neq 0$.

$S(5)$ $\dot{x} = 4(a - b + 2)\,x^2 + 8xy + 2(b+3)\,x - 4(a - b - 1)\,y + (b+1),$
$\dot{y} = 4(2 - a + b)\,x^2 + 24xy + 16y^2 + 2(b+1)\,x + 4(a+1)\,y,$
$f_1 = y - x^2, \quad K_1 = 8(a - b + 2)\,x + 16y + 4(a+1),$
$f_2 = 4x + 4y + 1, \quad K_2 = 4(4x + 4y + b + 1),$
$f_3 = 4x + 4y + b + 1, \quad K_3 = 4(4x + 4y + 1),$
$R = \frac{1}{f_1 \sqrt{f_2 f_3}},$
where $a - b \neq 0$.

$S(6)$ $\dot{x} = 8(ab + 2a - b^2 + 4)\,x^2 + 8(b+2)^2\,xy - 2(b^2 - 4a - 12)\,x - 4(ab - 2b^2 - 6b + 2a - 4)\,y - (ab - 2a + 2b - 4),$
$\dot{y} = 16(2 + b)\,x^2 + 8(ab + 2a + 6b + 12)\,xy + 16(b+2)^2\,y^2 - 2(ab - 2a + 2b - 4)\,x - 4(b^2 - 4a + 4b - 4)\,y,$
$f_1 = y - x^2, \quad K_1 = 16(ab + 2a - b^2 + 4)\,x + 16(b+2)^2\,y - 4(b^2 - 4a + 4b - 4),$
$f_2 = 4x + 4y + 1, \quad K_2 = 8(ab + 2a + 2b - b^2 + 8)\,x + 16(b+2)^2\,y - 4(ab - 2a + 2b - 4),$
$f_3 = 8x + 4(b+2)\,y + (2 - b), \quad K_3 = 8(a+4)(b+2)\,x + 16(b+2)^2\,y + 8(a+2),$
$R = \frac{f_1^{(a-3b)/(2b-2a)}\, f_2^{b/(b-a)}}{f_3^2},$
where $(a - b)(b + 2) \neq 0$.

$S(7)$ $\dot{x} = 8(a+1)\,x^2 + 8xy + 2(ab + 2a + 2b + 3)\,x - 4(a - b - 1)\,y + (ab + a + b + 1),$
$\dot{y} = 8(b+1)\,x^2 + 8(a + b + 3)\,xy + 16y^2 + (2ab + 2a + 2b + 2)\,x + 4(ab + 2a + 1)\,y,$
$f_1 = y - x^2, \quad K_1 = 16(a+1)\,x + 16y + 4(ab + 2a + 1),$
$f_2 = 4x + 4y + 1, \quad K_2 = 8(a + b + 2)\,x + 16y + 4(ab + a + b + 1),$
$f_3 = 4(b+1)\,x + 4y + (b+1)^2, \quad K_3 = 8(a+2)\,x + 16y + 4(a+1),$
$R = \frac{1}{\sqrt{f_1 f_2 f_3}},$
where $a(a - b) \neq 0$.

$S(8)$ $\dot{x} = 8(a - b + 1)\,x^2 + 8xy - 2(4b^2 - 4a - 3)\,x + 4(2b+1)\,y - (2a+1)(4b^2 - 1),$
$\dot{y} = 8(a + b + 1)\,x^2 + 8(2a + 3)\,xy + 16y^2 - 2(8ab^2 + 4b^2 - 2a - 1)\,x - 4(4b^2 + 2b - 2a - 1)\,y,$
$f_1 = y - x^2, \quad K_1 = 16(a - b + 1)\,x + 16y - 4(4b^2 + 2b - 2a - 1),$
$f_2 = 4b^2 - 1 - 4x - 4y, \quad K_2 = 16(a+1)\,x + 16y + 4(2a+1),$

$f_3 = 2x + 2b + 1, \quad K_3 = 8(a - b + 1)\,x + 8y - 2(4ab - 2a + 2b - 1),$
$R = f_1^{-(a+3b)/(2b)} f_2^{(a-3b)/(2b)} f_3,$
where $b(a - b) \neq 0$.

Here $a, b \in \mathbb{R}$.

Theorem 3.2. *The global phase portrait of each system from Theorem 3.1 in the Poincaré disc is topologically equivalent to one of the phase portraits shown in Figure 3.1. Below each phase portrait in the Poincaré disc appear its name, the number of its separatrices S, the number of its canonical regions R, and the system associated with this phase portrait.*

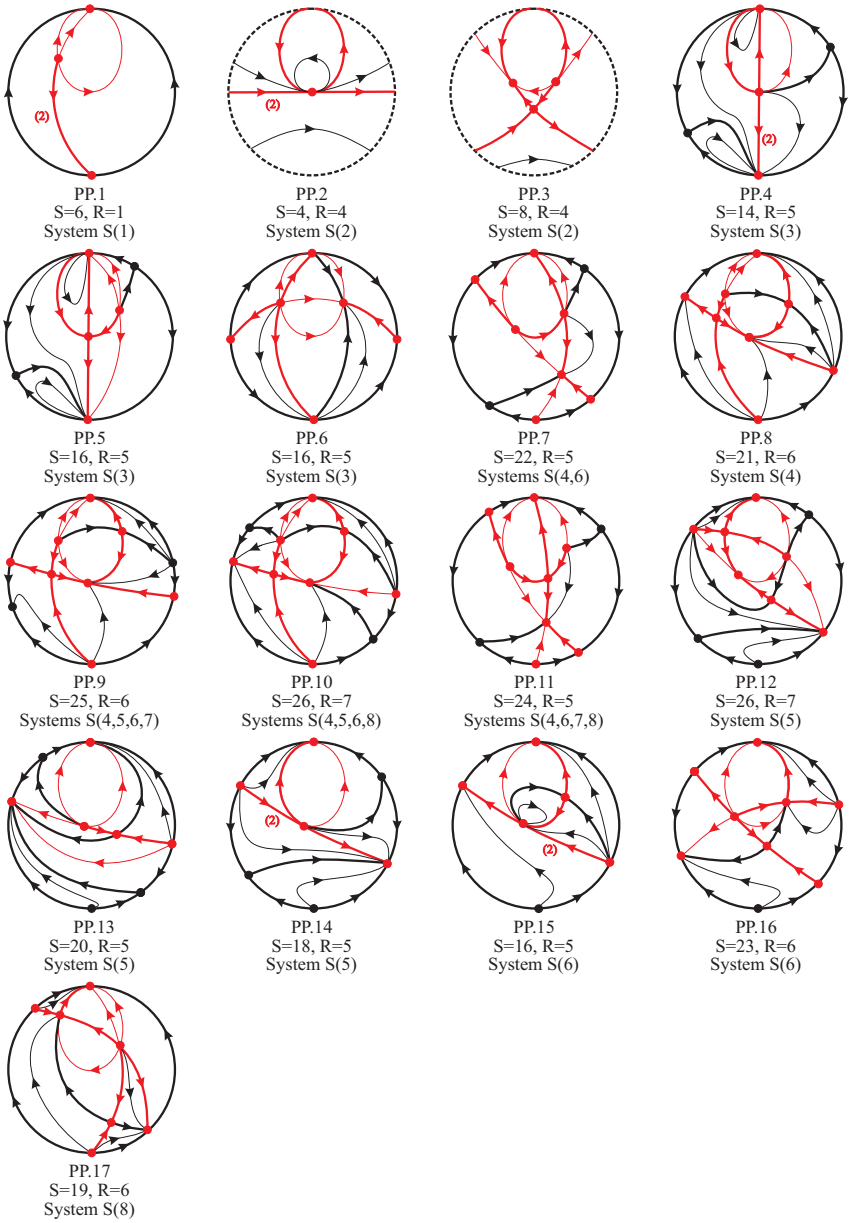


Figure 3.1: Phase portraits in the Poincaré disc of the systems in Theorem 3.1. Separatrices are represented by thick black or red lines, invariant curves are drawn in red, while trajectories within canonical regions are drawn with thin black lines.

4 Structure of the proof

To simplify the proofs of Theorems 3.1 and 3.2 we divide the original system (P) into two separate cases: $b = 0$ and $b = 1$. The overall structure of the analysis for each case is summarised in Figures 4.1 and 4.2.

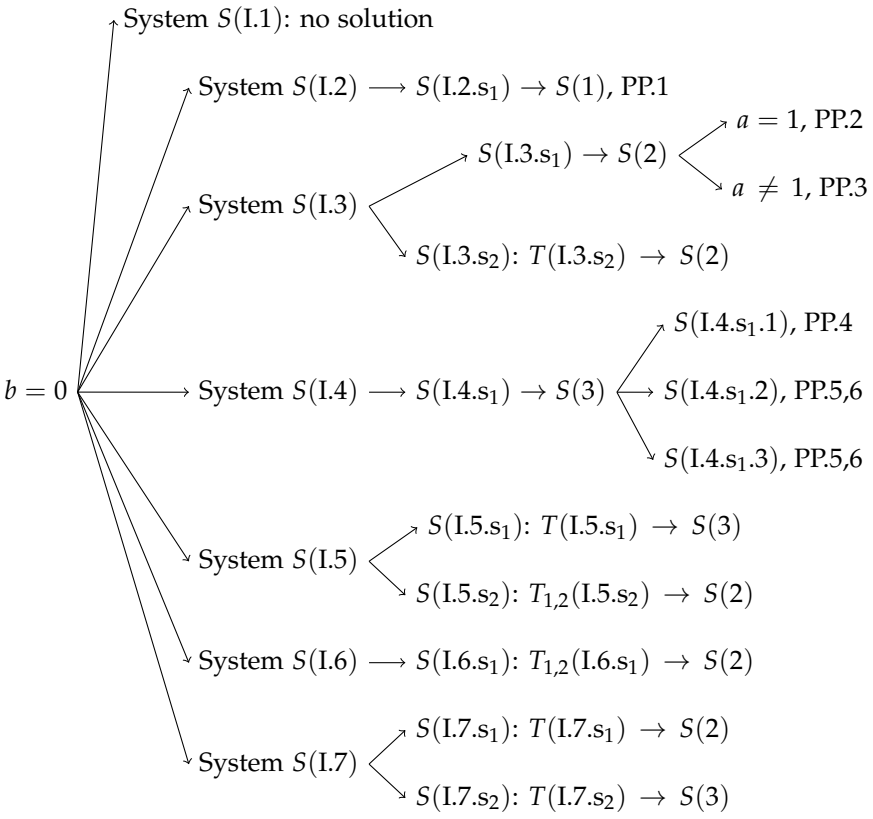


Figure 4.1: Structure of the proof of Theorem 3.1 when $b = 0$.

For each system in Theorem 3.1 we first impose the existence of one invariant straight line to the system, obtaining in total 12 systems, denoted by $S(\text{I}.i)$ and $S(\text{II}.i)$ for $b = 0$ and $b = 1$, respectively. Then, for each of these systems, we impose the existence of a second invariant straight line, obtaining new systems denoted by $S(\text{I}.i.\text{s}_j)$ and $S(\text{II}.i.\text{s}_j)$ which already have an invariant parabola and two invariant straight lines.

Sometimes one of those systems having an invariant parabola and two invariant straight lines is particular cases of another system having an invariant parabola and two invariant straight lines. For example, the notation

$$S(\text{I}.3.\text{s}_2) : T(\text{I}.3.\text{s}_2) \to S(2)$$

indicates that the system $S(\text{I}.3.\text{s}_2)$ can be obtained from system $S(2)$ via the transformation $T(\text{I}.3.\text{s}_2)$, and can therefore be regarded as a particular or an equivalent case. Consequently, when computing the phase portraits, it suffices to consider only the systems that cannot be obtained from other systems.

We shall identify only eight distinct systems whose phase portraits need to be studied. For each of these systems we list the relevant parameter conditions corresponding to different combinations of equilibria, along with the associated phase portrait labels. For instance, under system $S(2)$, the condition "$a = 1$, PP.2" means that when $a = 1$, the corresponding phase portrait is labelled as PP.2.

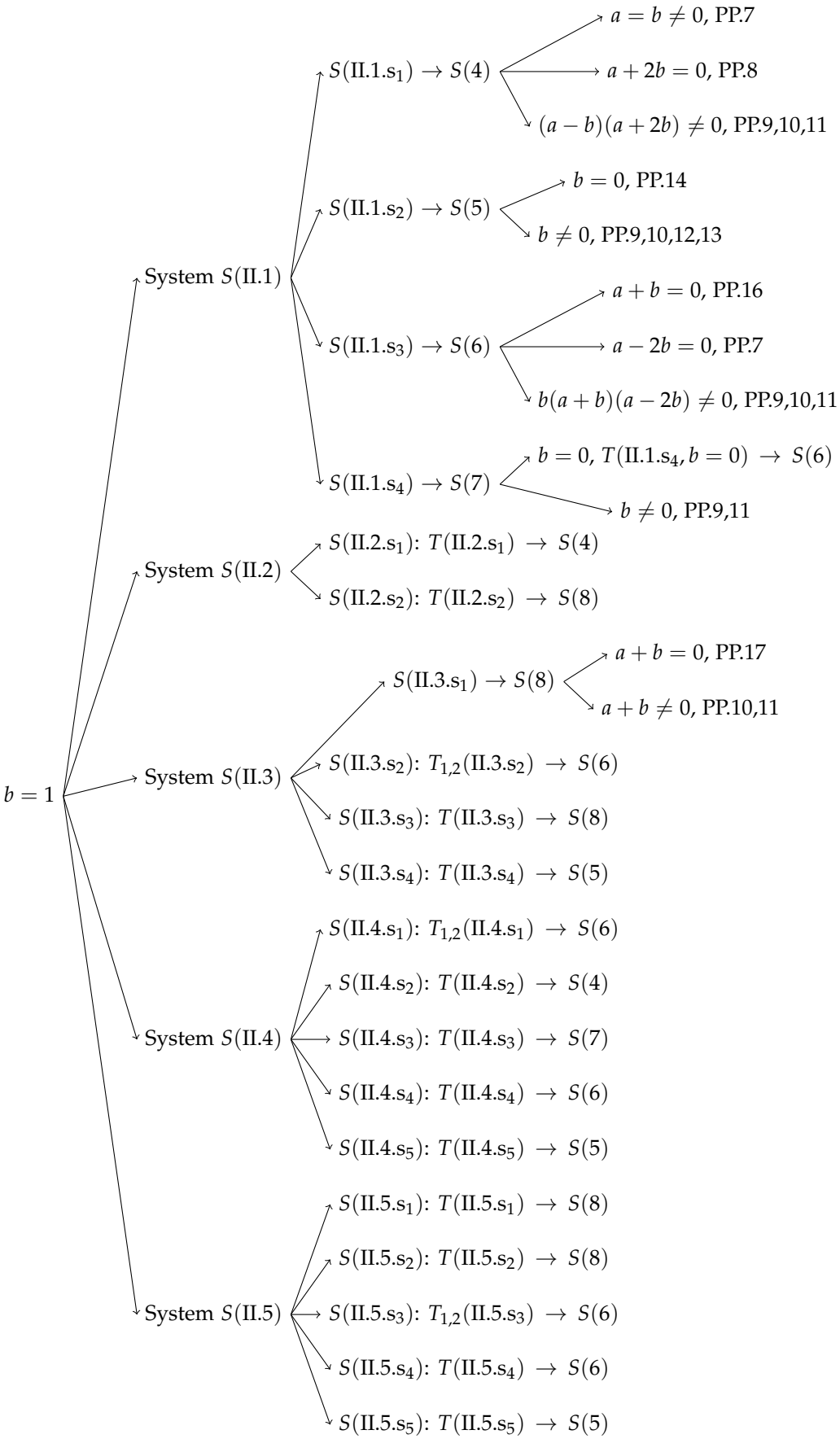


Figure 4.2: Structure of the proof of Theorem 3.1 when $b = 1$.

5 Proof of Theorem 3.1

Proof. Our objective is to determine the general forms of all systems in **QSR2P**. The proof begins with system (P) in Lemma 2.3, which admits an invariant parabola defined by $f_1 = y - x^2 = 0$.

The approach is first to force the existence of one additional invariant straight line $f_2 = 0$ in system (P). By applying the definition of invariant algebraic curve, we obtain all possible systems with one invariant straight line. After that, for each system, we force the existence of a second invariant straight line $f_3 = 0$. In this way, we obtain all systems that admit two invariant straight lines $f_2 = 0$ and $f_3 = 0$, and the invariant parabola $f_1 = 0$.

To facilitate the computations in the study of system (P) we consider the two distinct cases $b = 0$ and $b = 1$. Note that when $b \neq 0$, after rescaling the time, we can assume $b = 1$; the other parameters of the new system are divided by b, but we keep using the same letters to denote them.

Case I: $b = 0$. If $a = 0$, the existence of the invariant parabola requires $c \neq 0$. In this case, after performing the change of parameters $(p, q, r) \to (cp, cq, cr)$ together with the time rescaling $t \to \dfrac{t}{c}$, the system (P) simplifies to

$$\dot{x} = -qx - ry - p, \quad \dot{y} = (y - x^2) - 2x(qx + ry + p). \tag{P.1}$$

The existence of one invariant straight line $f_2 = 0$ in system (P.1) leads to two distinct subsystems, denoted by $S(\text{I}.k)$ for $k = 1, 2$.

If $a \neq 0$, after doing the change of parameters $(p, q, r, c) \to \left(\frac{ap}{2}, \frac{aq}{2}, \frac{ar}{2}, -\frac{ac}{2}\right)$ and the time rescaling $t \to \frac{-2t}{a}$ the system (P) reduces to

$$\dot{x} = y - x^2 + qx + ry + p, \quad \dot{y} = c(y - x^2) + 2x(qx + ry + p). \tag{P.2}$$

The existence of an invariant straight line $f_2 = 0$ in system (P.2) leads to five distinct subsystems, denoted by $S(\text{I}.k)$ for $k = 3, 4, 5, 6, 7$.

Case II: $b = 1$. The system (P) reduces to

$$\dot{x} = \frac{1}{2}xy - \frac{a}{2}(y - x^2) - (qx + ry + p), \quad \dot{y} = y^2 + c(y - x^2) - 2x(qx + ry + p). \tag{P.3}$$

Forcing the existence of the straight line $f_2 = 0$ produces five subsystems, denoted by $S(\text{II}.k)$ for $k = 1, 2, 3, 4, 5$.

Thus, we have obtained a total of twelve systems. To each of these systems we force the existence of a second invariant straight line $f_3 = 0$ in order to find all possible systems in the class **QSR2P**.

The proof is organised into the following fourteen steps:

- **Step 1**: Derivation of the systems $S(\text{I}.1)$ through $S(\text{I}.7)$,
- **Step 2**: Derivation of the systems $S(\text{II}.1)$ through $S(\text{II}.5)$,
- **Steps 3–14**: To each of the above twelve systems we force the existence of a second invariant straight line $f_3 = 0$ for generating all systems in **QSR2P**.

Step 1. Suppose that system (P.1) possesses an invariant straight line $f_2 = a_1x + a_2y + a_0 = 0$ with cofactor $K_2 = k_1x + k_2y + k_0$. Substituting f_2, K_2 and the system (P.1) into (2.1), we obtain

two solutions

$$\text{I.s}_1 = \left\{ \begin{array}{l} a_0 = \dfrac{a_1^2}{4a_2},\ k_0 = \dfrac{a_1k_1}{2a_2} + 1,\ k_2 = 0,\ p = -\dfrac{a_1(2a_2 + a_1k_1)}{8a_2^2}, \\ q = -\dfrac{a_2 + a_1k_1}{2a_2},\ r = -\dfrac{k_1}{2} \end{array} \right\},$$

$$\text{I.s}_2 = \left\{ a_2 = 0,\ k_1 = 0,\ k_2 = 0,\ p = -\frac{a_0k_0}{a_1},\ q = -k_0,\ r = 0 \right\}.$$

Since $a_2 \neq 0$ in solution (I.s$_1$), substituting it into the system (P.1), after doing the transformation $a_1 \to 2a_1a_2$ and the time rescaling $t \to 2t$, we obtain the system

$$\begin{array}{ll} S(\text{I.1}) & \dot{x} = (2a_1k_1 + 1)x + k_1y + a_1 + a_1^2k_1, \\ & \dot{y} = 4a_1k_1x^2 + 2k_1xy + (2a_1 + 2a_1^2k_1)x + 2y, \\ & f_2 = 2a_1x + y + a_1^2, \quad K_2 = 2k_1x + 2k_1a_1 + 2. \end{array}$$

For the solution (I.s$_2$), we have $a_1k_0 \neq 0$. Indeed, if $k_0 = 0$, then we must have $p = q = r = 0$, which implies $\dot{x} = 0$. Substituting (I.s$_2$) into the system (P.1), after doing the transformation $a_0 \to \frac{a_1a_0}{k_0}$, we obtain the system

$$\begin{array}{ll} S(\text{I.2}) & \dot{x} = k_0x + a_0, \quad \dot{y} = (2k_0 - 1)x^2 + 2a_0x + y, \\ & f_2 = k_0x + a_0, \quad K_2 = k_0. \end{array}$$

Suppose that the system (P.2) possesses an invariant straight line $f_2 = 0$ with cofactor K_2 as defined previously. To simplify the computations, we consider two cases for the parameter a_1 in f_2: $a_1 = 1$ and $a_1 = 0$. Of course, if $a_1 \neq 0$ by rescaling of the time we can set $a_1 = 1$.
Case 1. $a_1 = 1, f_2 = x + a_2y + a_0$. Substituting f_2, K_2, and the system (P.2) into (2.1), we obtain the following three solutions

$$\text{I.s}_3 = \left\{ a_0 = \frac{1}{4a_2},\ c = \frac{2a_2k_0 - k_1 - 2}{2a_2},\ k_2 = 0,\ p = \frac{k_0}{4a_2},\ q = \frac{2a_2k_0 + k_1}{4a_2},\ r = \frac{k_1}{2} \right\},$$

$$\text{I.s}_4 = \{ a_2 = 0,\ k_1 = -1,\ k_2 = 0,\ p = a_0k_0,\ q = k_0 - a_0,\ r = -1 \},$$

$$\text{I.s}_5 = \left\{ c = -\frac{1}{a_2},\ k_1 = 2a_2k_0,\ k_2 = 0,\ p = a_0k_0,\ q = k_0,\ r = a_2k_0 \right\}.$$

For the solution (I.s$_3$), we have $a_2 \neq 0$. Substituting (I.s$_3$) into the system (P.2), after doing the transformation $(x,\ y,\ t,\ k_0) \to \left(\frac{x}{a_2},\ \frac{y}{a_2^2},\ 4a_2t,\ \frac{k_0}{a_2}\right)$, we obtain the system

$$\begin{array}{ll} S(\text{I.3}) & \dot{x} = -4x^2 + (2k_0 + k_1)x + (2k_1 + 4)y + k_0, \\ & \dot{y} = (4k_1 + 4)x^2 + 2k_0x + 4k_1xy + (4k_0 - 2k_1 - 4)y, \\ & f_2 = 4x + 4y + 1, \quad K_2 = 4(k_1x + k_0). \end{array}$$

Substituting the solution (I.s$_4$) into system the (P.2), we obtain the system $*S(\text{I.4})$

$$\dot{x} = -x^2 + (k_0 - a_0)\,x + a_0k_0, \quad \dot{y} = (2k_0 - 2a_0 - c)\,x^2 + 2a_0k_0\,x - 2xy + cy.$$

Here we have $c + 2a_0 \neq 0$. Otherwise, if $c + 2a_0 = 0$, the system becomes

$$\dot{x} = -(x + a_0)(x - k_0), \quad \dot{y} = 2(x + a_0)(k_0x - y),$$

and can be reduced to a linear differential system.

Doing the transformation $(x,\ y,\ t,\ a_0,\ c,\ k_0) \to \left(c_1x,\ c_1^2y,\ \frac{t}{c_1},\ a_0c_1,\ c_1 - 2a_0,\ k_0c_1\right)$ with $c_1 \neq 0$ to system $*S(\text{I.4})$ we obtain

$$S(\text{I.4})\ \dot{x} = -x^2 + (k_0 - a_0)\,x + a_0k_0, \quad \dot{y} = (2k_0 - 1)\,x^2 + 2a_0k_0\,x - 2xy - (2a_0 - 1)\,y,$$
$$f_2 = x + a_0, \quad K_2 = k_0 - x.$$

For the solution (I.s_5), we have $a_2 \neq 0$. Substituting (I.s_5) into the system (P.2), after doing the transformation $(x,\ y,\ t,\ k_0,\ a_0) \to \left(\frac{x}{a_2},\ \frac{y}{a_2^2},\ a_2t,\ \frac{k_0}{a_2},\ \frac{a_0}{a_2}\right)$, we obtain the system

$$S(\text{I.5})\ \dot{x} = -x^2 + k_0x + (k_0 + 1)\,y + a_0k_0, \quad \dot{y} = (2k_0 + 1)\,x^2 + 2a_0k_0\,x + 2k_0xy - y,$$
$$f_2 = x + y + a_0, \quad K_2 = k_0(2x + 1).$$

Case 2. $a_1 = 0$, $f_2 = a_2y + a_0$. Substituting f_2, K_2, and the system (P.2) into (2.1), we obtain two solutions

$$\text{I.s}_6 = \left\{a_0 = 0,\ k_0 = c,\ k_2 = 0,\ p = 0,\ q = \frac{c}{2},\ r = \frac{k_1}{2}\right\},$$
$$\text{I.s}_7 = \left\{c = 0,\ k_0 = 0,\ k_2 = 0,\ p = \frac{a_0k_1}{2a_2},\ q = 0,\ r = \frac{k_1}{2}\right\}.$$

For the solution (I.s_6), we have $k_1 \neq 2$. Indeed, substituting (I.s_6) and $k_1 = 2$ into the system (P.2) we obtain a linear differential system. Taking (I.s_6) into (P.2), doing the parameter change $k_1 \to k_1 - 2$ and the time rescaling $t \to 2t$ we obtain

$$S(\text{I.6})\ \dot{x} = -2x^2 + cx + k_1y, \quad \dot{y} = (2k_1 - 4)\,xy + 2cy,$$
$$f_2 = y, \quad K_2 = (2k_1 - 4)\,x + 2c.$$

For the solution (I.s_7), we have $a_2 \neq 0$. Substituting (I.s_7) into the system (P.2), after doing the transformation $(x,\ y,\ t,\ a_0) \to \left(\frac{x}{a_2},\ \frac{y}{a_2^2},\ 2a_2t,\ \frac{a_0}{a_2k_1}\right)$, we obtain the system

$$S(\text{I.7})\ \dot{x} = -2x^2 + (k_1 + 2)\,y + a_0, \quad \dot{y} = 2k_1xy + 2a_0x,$$
$$f_2 = k_1y + a_0, \quad K_2 = 2k_1x.$$

Step 2. Suppose that system (P.3) possesses an invariant straight line $f_2 = 0$ with cofactor K_2 as defined previously. In order to simplify the computations, we consider two cases for the parameter a_1 in f_2: $a_1 = 1$ and $a_1 = 0$.

Case 1. $a_1 = 1$, $f_2 = x + a_2y + a_0$. Substituting f_2, K_2, and the system (P.3) into (2.1), we obtain three solutions

$$\text{II.s}_1 = \left\{\begin{array}{l} a = 2a_2c - 2a_2k_0 + k_1,\ a_0 = \dfrac{1}{4a_2},\ k_2 = 1,\ p = -\dfrac{k_0}{4a_2},\ q = -\dfrac{2a_2k_0 + k_1}{4a_2}, \\ r = -\dfrac{2a_2k_1 + 1}{4a_2} \end{array}\right\},$$
$$\text{II.s}_2 = \left\{a = 2k_1,\ a_2 = 0,\ k_2 = \frac{1}{2},\ p = -a_0k_0,\ q = -k_0 - a_0k_1,\ r = -\frac{a_0 + 2k_1}{2}\right\},$$
$$\text{II.s}_3 = \left\{\begin{array}{l} a = \dfrac{2a_2^2c - 2a_0a_2 + 1}{a_2},\ k_1 = \dfrac{4a_2^2k_0 + 1}{2a_2},\ k_2 = 1,\ p = -a_0k_0, \\ q = -\dfrac{a_0 + 2a_2k_0}{2a_2},\ r = -\dfrac{2a_2^2k_0 + 1}{2a_2} \end{array}\right\}.$$

For the solution (II.s_1), we have $a_2 \neq 0$. Substituting (II.s_1) into the system (P.3), after doing the transformation $(x, y, t, k_0, k_1, c) \to \left(\frac{x}{a_2},\ \frac{y}{a_2^2},\ 4a_2^2t,\ \frac{2k_0}{a_2^2},\ \frac{k_1}{a_2},\ \frac{c}{a_2^2}\right)$, we obtain the system

$$S(\text{II.1})\ \dot{x} = (4c - 8k_0 + 2k_1)\,x^2 + 2xy + (4k_0 + k_1)\,x + (8k_0 - 4c + 1)\,y + 2k_0,$$
$$\dot{y} = (8k_0 - 4c + 2k_1)\,x^2 + (2 + 4k_1)\,xy + 4y^2 + 4k_0x + 4cy,$$
$$f_2 = 4(x + y) + 1, \quad K_2 = 4(k_1x + y + 2k_0).$$

Substituting (II.s_2) into the system (P.3) and doing the time rescaling $t \to 2t$, it simplifies to system $*S(\text{II.2})$

$$\begin{aligned}\dot{x} &= 2k_1x^2 + xy + (2k_0 + 2a_0k_1)\,x + a_0y + 2a_0k_0,\\ \dot{y} &= (4a_0k_1 + 4k_0 - 2c)\,x^2 + (2a_0 + 4k_1)\,xy + 2y^2 + 4a_0k_0x + 2c\,y.\end{aligned}$$

Here we have $c - 2k_0 \neq 0$. If $c - 2k_0 = 0$, the system becomes

$$\dot{x} = (x + a_0)(2k_1x + y + 2k_0), \quad \dot{y} = 2(a_0x + y)(2k_1x + y + 2k_0),$$

and can be reduced to a linear differential system.

Doing the following transformation to system $*S(\text{II.2})$

$$(x, y, t, a_0, c, k_0, k_1) \to \left(\sqrt{c_1}\,x,\ c_1y,\ \frac{t}{c_1},\ \sqrt{c_1}\,a_0,\ c_1(2k_0 + 1),\ c_1k_0,\ \sqrt{c_1}\,k_1\right)$$

yields

$$\begin{aligned}S(\text{II.2})\ \ &\dot{x} = 2k_1x^2 + xy + a_0y + (2k_0 + 2a_0k_1)\,x + 2a_0k_0,\\ &\dot{y} = (4a_0k_1 - 2)\,x^2 + (2a_0 + 4k_1)\,xy + 2y^2 + 4a_0k_0x + (2 + 4k_0)\,y,\\ &f_2 = x + a_0, \quad K_2 = 2k_1x + y + 2k_0.\end{aligned}$$

For the solution (II.s_3), we have $a_2 \neq 0$. Substituting (II.s_3) into the system (P.3), after doing the transformation $(x, y, t, k_0, c, a_0) \to \left(\frac{x}{a_2},\ \frac{y}{a_2^2},\ 2a_2^2t,\ \frac{k_0}{a_2^2},\ \frac{c}{a_2^2},\ \frac{a_0}{a_2}\right)$, it reduces to

$$\begin{aligned}S(\text{II.3})\ \ &\dot{x} = (2c - 2a_0 + 1)\,x^2 + xy + (a_0 + 2k_0)\,x + (2a_0 - 2c + 2k_0)\,y + 2a_0k_0,\\ &\dot{y} = (2a_0 - 2c + 4k_0)\,x^2 + (4k_0 + 2)\,xy + 2y^2 + 4a_0k_0x + 2cy,\\ &f_2 = x + y + a_0, \quad K_2 = (4k_0 + 1)\,x + 2y + 2k_0.\end{aligned}$$

Case 2. $a_1 = 0$, $f_2 = a_2y + a_0$. Since $a_2 \neq 0$, we set $a_2 = 1$ and then $f_2 = y + a_0$. Substituting f_2, K_2, and the system (P.3) into (2.1), we obtain two solutions

$$\begin{aligned}\text{II.s}_4 &= \left\{a_0 = 0,\ k_0 = c,\ k_2 = 1,\ p = 0,\ q = -\frac{c}{2},\ r = -\frac{k_1}{2}\right\},\\ \text{II.s}_5 &= \left\{c = a_0,\ k_0 = 0,\ k_2 = 1,\ p = -\frac{a_0k_1}{2},\ q = -\frac{a_0}{2},\ r = -\frac{k_1}{2}\right\}.\end{aligned}$$

Substituting the solution (II.s_4) into the system (P.3) and doing the time rescaling $t \to 2t$ yields system $*S(\text{II.4})$

$$\dot{x} = ax^2 + xy + cx + (k_1 - a)\,y, \qquad \dot{y} = 2k_1xy + 2y^2 + 2cy.$$

Here we have $k_1 - a \neq 0$. If $k_1 - a = 0$, the system reduces to

$$\dot{x} = x(ax + y + c), \quad \dot{y} = 2y(ax + y + c),$$

and can be reduced to a linear differential system.

Applying the transformation $(x, y, t, a, c, k_1) \to \left(kx,\ k^2y,\ \frac{t}{k^2},\ ak,\ ck^2,\ k(a+1)\right)$ to the system $*S(\text{II.4})$ yields

$$\begin{aligned}S(\text{II.4})\ \ &\dot{x} = ax^2 + xy + cx + y, \quad \dot{y} = (2a + 2)\,xy + 2y^2 + 2cy,\\ &f_2 = y, \quad K_2 = 2(a + 1)\,x + 2y + 2c.\end{aligned}$$

Substituting the solution $(\mathrm{II.s}_5)$ into the system (P.3) and doing the time rescaling $t \to 2t$ yields system $*S(\mathrm{II.5})$

$$\dot{x} = ax^2 + xy + a_0x + (k_1 - a)\,y + a_0k_1, \qquad \dot{y} = 2k_1xy + 2y^2 + 2a_0k_1x + 2a_0y.$$

Here we have $a \neq 0$. If $a = 0$, the system becomes

$$\dot{x} = (x + k_1)(y + a_0), \quad \dot{y} = 2(y + a_0)(k_1x + y),$$

and it can be reduced to a linear differential system.

Applying the transformation $(x, y, t, a_0, k_1) \to \left(ax,\ a^2y,\ \frac{t}{a^2},\ a_0a^2,\ k_1a\right)$ to the system $*S(\mathrm{II.5})$ yields

$$\begin{aligned} S(\mathrm{II.5})\ \ & \dot{x} = x^2 + xy + a_0x + (k_1 - 1)\,y + a_0k_1, \quad \dot{y} = 2k_1xy + 2y^2 + 2a_0k_1x + 2a_0y, \\ & f_2 = y + a_0, \quad K_2 = 2(k_1x + y). \end{aligned}$$

Step 3. Suppose that system $S(\mathrm{I.1})$ possesses an invariant straight line $f_3 = b_1x + b_2y + b_0 = 0$ with cofactor $K_3 = l_1x + l_2y + l_0$. Substituting f_3, K_3, and the system $S(\mathrm{I.1})$ into (2.1), we obtain the following four solutions

$$\begin{aligned} \mathrm{I.1.s}_1 &= \left\{b_0 = a_1^2b_2,\ b_1 = 2a_1b_2,\ l_0 = 2a_1k_1 + 2,\ l_1 = 2k_1,\ l_2 = 0\right\}, \\ \mathrm{I.1.s}_2 &= \left\{b_0 = a_1b_1,\ b_2 = 0,\ k_1 = 0,\ l_0 = 1,\ l_1 = 0,\ l_2 = 0\right\}, \\ \mathrm{I.1.s}_3 &= \left\{b_0 = \frac{a_1b_1}{2},\ b_2 = \frac{b_1}{2a_1},\ k_1 = 0,\ l_0 = 2,\ l_1 = 0,\ l_2 = 0\right\}, \\ \mathrm{I.1.s}_4 &= \left\{a_1 = 0,\ b_0 = 0,\ b_2 = 0,\ k_1 = 0,\ l_0 = 1,\ l_1 = 0,\ l_2 = 0\right\}. \end{aligned}$$

Solutions $(\mathrm{I.1.s}_k), k = 2, 3, 4$ correspond to linear systems, so we discard them. Solution $(\mathrm{I.1.s}_1)$ corresponds to a system with a double invariant straight line. By requiring that f_2^2 be a factor of the extactic polynomial E_1 we obtain the following polynomial system

$$2a_1^3k_1^2 + 4a_1^2k_1 + a_1 = 0, \quad 6a_1^2k_1^2 + 8a_1k_1 + 1 = 0, \quad 6a_1k_1^2 + 4k_1 = 0,\ 2k_1^2 = 0.$$

However, there are no solutions for the above polynomial system.

Step 4. Imposing an invariant straight line $f_3 = 0$ with cofactor K_3 on $S(\mathrm{I.2})$, and substituting f_3, K_3, and $S(\mathrm{I.2})$ into (2.1), we obtain the following three solutions

$$\begin{aligned} \mathrm{I.2.s}_1 &= \left\{a_0 = \frac{b_0l_0}{b_1},\ b_2 = 0,\ k_0 = l_0,\ l_1 = 0,\ l_2 = 0\right\}, \\ \mathrm{I.2.s}_2 &= \left\{a_0 = \frac{b_1}{4b_2},\ b_0 = \frac{b_1^2}{4b_2},\ k_0 = \frac{1}{2},\ l_0 = 1,\ l_1 = 0,\ l_2 = 0\right\}, \\ \mathrm{I.2.s}_3 &= \left\{a_0 = 0,\ b_0 = 0,\ b_1 = 0,\ k_0 = \frac{1}{2},\ l_0 = 1,\ l_1 = 0,\ l_2 = 0\right\}. \end{aligned}$$

Solutions $(\mathrm{I.2.s}_2), (\mathrm{I.2.s}_3)$ correspond to linear systems. Solution $(\mathrm{I.2.s}_1)$ corresponds to a system with a double invariant straight line. By requiring that f_2^2 be a factor of the extactic polynomial E_1 we obtain the following polynomial equations

$$\frac{b_0^2l_0(l_0 - 1)}{a_1b_1^2} = 0, \quad \frac{l_0(1 - l_0)}{a_1} = 0,$$

and two solutions $l_0 = 0$ and $l_0 = 1$. If $l_0 = 0$, the system degenerates to

$$\dot{x} = 0, \quad \dot{y} = y - x^2.$$

If $l_0 = 1$, taking it into the solution (I.2.s_1) we obtain

$$\text{I.2.s}_{1.1} = \left\{ a_0 = \frac{b_0}{b_1},\ b_2 = 0,\ k_0 = 1,\ l_0 = 1,\ l_1 = 0,\ l_2 = 0 \right\}.$$

Substituting $(\text{I.2.s}_{1.1})$ into system $S(\text{I.2})$, we obtain the system $S(\text{I.2.s}_{1.1})$:

$$\dot{x} = x + \frac{b_0}{b_1}, \qquad \dot{y} = x^2 + \frac{2b_0 x}{b_1} + y.$$

If $b_0 \neq 0$, set $a \neq 0$. Doing the transformation $(x, y, b_0) \to (ax,\ a^2 y,\ a b_1)$, the system $S(\text{I.2.s}_{1.1})$ reduces to

$$\begin{aligned} S(1)\ \ & \dot{x} = x + 1, \quad \dot{y} = x^2 + 2x + y, \\ & f_1 = y - x^2, \quad K_1 = 1, \\ & f_2 = x + 1, \quad K_2 = 1, \quad H = \frac{f_1}{f_2}. \end{aligned}$$

Here and in what follows, H denotes a first integral.

If $b_0 = 0$, applying the translation $(x, y) \to (x + 1,\ y - 1)$ to the system $S(\text{I.2.s}_{1.1})$, it reduces to $S(1)$.

Step 5. We impose that $S(\text{I.3})$ possesses an invariant straight line $f_3 = 0$ with cofactor K_3. To simplify the computations, we consider the parameter b_1 in f_3 in two cases: $b_1 = 1$ and $b_1 = 0$.

Case 1. $b_1 = 1$, so that $f_3 = x + b_2 y + b_0$. Substituting f_3, K_3, and $S(\text{I.3})$ into (2.1), eliminating the solutions corresponding to linear systems, and treating the solutions corresponding to systems with a double invariant straight line as in $S(\text{I.1})$ and $S(\text{I.2})$, we eventually obtain two solutions

$$\begin{aligned} \text{I.3.s}_1 &= \left\{ b_0 = \frac{1}{2} - \frac{b_2}{4},\ k_0 = \frac{1}{2} - \frac{1}{b_2},\ k_1 = -1,\ l_0 = -\frac{2}{b_2},\ l_1 = -4,\ l_2 = 0 \right\}, \\ \text{I.3.s}_2 &= \left\{ b_0 = \frac{1}{4b_2},\ k_0 = -\frac{1}{2b_2},\ k_1 = -1,\ l_0 = -2,\ l_1 = -4,\ l_2 = 0 \right\}. \end{aligned}$$

Substituting (I.3.s_1) into $S(\text{I.3})$, we obtain the system $S(\text{I.3.s}_1)$:

$$\dot{x} = -4x^2 - \frac{2}{b_2} x + 2y - \frac{1}{b_2} + \frac{1}{2}, \qquad \dot{y} = -4xy - \left(\frac{2}{b_2} - 1 \right) x - \frac{4}{b_2} y.$$

Doing the transformation $b_2 \to \frac{1}{a}$, the system reduces to

$$\begin{aligned} S(2)\ \ & \dot{x} = -4x^2 - 2ax + 2y - a + \frac{1}{2}, \quad \dot{y} = -4xy + (1 - 2a)\, x - 4ay, \\ & f_1 = y - x^2, \quad K_1 = -4(2x + a), \\ & f_2 = 4x + 4y + 1, \quad K_2 = -4x - 4a + 2, \\ & f_3 = 4ax + 4y + 2a - 1, \quad K_3 = -2(2x + a), \quad H = \frac{f_1}{f_3^2}. \end{aligned}$$

Substituting (I.3.s_2) into $S(\text{I.3})$, we obtain the system $S(\text{I.3.s}_2)$:

$$\dot{x} = -4x^2 - \left(\frac{1}{b_2} + 1\right) x + 2y - \frac{1}{2b_2}, \qquad \dot{y} = -4xy - \frac{1}{b_2} x - \left(\frac{2}{b_2} + 2\right) y.$$

However, applying the parameter change $T(\text{I.3.s}_2) : a \to \frac{b_2+1}{2b_2}$ to system $S(2)$, the system reduces to $S(\text{I.3.s}_2)$.

Case 2. $b_1 = 0$, so that $f_3 = b_2 y + b_0$. Following the same process as in **Case 1**, we obtain two solutions

$$\text{I.3.s}_3 = \{b_0 = 0,\ k_0 = 0,\ k_1 = -1,\ l_0 = -2,\ l_1 = -4,\ l_2 = 0\},$$
$$\text{I.3.s}_4 = \left\{b_0 = -\frac{b_2}{4},\ k_0 = \frac{1}{2},\ k_1 = -1,\ l_0 = 0,\ l_1 = -4,\ l_2 = 0\right\}.$$

However, the system associated with solution (I.3.s_3) is the case $b_2 = 2$ of system $S(\text{I.3.s}_1)$, while the solution (I.3.s_4) arises when $b_2 = -1$ in system $S(\text{I.3.s}_2)$.

Step 6. Substituting f_3, K_3, and $S(\text{I.4})$ into (2.1), and following the same process as in $S(\text{I.3})$, we obtain a solution

$$\text{I.4.s}_1 = \{b_0 = -k_0,\ b_2 = 0,\ l_0 = -a_0,\ l_1 = -1,\ l_2 = 0\}.$$

Substituting (I.4.s_1) into $S(\text{I.4})$ yields system $S(\text{I.4.s}_1)$:

$$\dot{x} = -x^2 + (k_0 - a_0)\, x + a_0 k_0, \qquad \dot{y} = (2k_0 - 1)\, x^2 - 2xy + (1 - 2a_0)\, y + 2a_0 k_0 x.$$

Applying the parameter change $(a_0, k_0) \to (-a,\ b)$ to the above system yields

$$\begin{aligned} S(3)\ \ & \dot{x} = -x^2 + (a+b)\, x - ab, \quad \dot{y} = (2b-1)\, x^2 - 2xy + (2a+1)\, y - 2abx, \\ & f_1 = y - x^2, \quad K_1 = -2x + 2a + 1, \\ & f_2 = x - a, \quad K_2 = b - x, \\ & f_3 = x - b, \quad K_3 = a - x, \quad H = f_1^{a-b}\, f_2\, f_3^{2b-2a-1}. \end{aligned}$$

Here we have $2a - 2b + 1 \neq 0$. If $2a - 2b + 1 = 0$, the system becomes

$$\dot{x} = -\frac{1}{2}(2x - 2a - 1)(x - a), \quad \dot{y} = (2x - 2a - 1)(ax - y),$$

and can be reduced to a linear differential system.

Step 7. Substituting f_3, K_3, and $S(\text{I.5})$ into (2.1), we obtain two solutions

$$\text{I.5.s}_1 = \{a_0 = b_0 - b_0^2,\ b_2 = 0,\ k_0 = -1,\ l_0 = b_0 - 1,\ l_1 = -1,\ l_2 = 0\},$$
$$\text{I.5.s}_2 = \left\{a_0 = \frac{2b_2 - 1}{4b_2^2},\ b_0 = \frac{1}{4b_2},\ k_0 = -\frac{1}{2},\ l_0 = \frac{1 - 2b_2}{2b_2},\ l_1 = -1,\ l_2 = 0\right\}.$$

Substituting (I.5.s_1) and (I.5.s_2) into $S(\text{I.5})$, we obtain systems $S(\text{I.5.s}_1)$ and $S(\text{I.5.s}_2)$:

$$\dot{x} = -x^2 - x + b_0^2 - b_0, \qquad \dot{y} = -x^2 - 2xy + (2b_0^2 - 2b_0)\, x - y.$$

$$\dot{x} = -x^2 - \frac{1}{2}x + \frac{1}{2}y + \frac{1}{8b_2^2} - \frac{1}{4b_2}, \qquad \dot{y} = -xy + \left(\frac{1}{4b_2^2} - \frac{1}{2b_2}\right) x - y,$$

respectively. Applying the transformations $T(\text{I.5.s}_1)$ to system $S(3)$, it reduces to $S(\text{I.5.s}_1)$, where

$$T(\text{I.5.s}_1) : (x, y, t, a, b) \to \left(a + \frac{x + b_0}{2b_0 - 1},\ a^2 + \frac{b_0^2 + 2a(b_0 + x) - y}{2b_0 - 1},\ (2b_0 - 1)^2 t,\ a,\ a + 1\right).$$

For the system $S(\text{I.5.s}_2)$, if $2b_2-1\neq 0$, then applying the transformation $T_1(\text{I.5.s}_2)$ to system $S(2)$ yields $S(\text{I.5.s}_2)$ with $b_2\neq\frac{1}{2}$; if $2b_2-1=0$, then applying the transformation $T_2(\text{I.5.s}_2)$ to system $S(2)$ yields $S(\text{I.5.s}_2)$ with $b_2=\frac{1}{2}$, where

$$T_1(\text{I.5.s}_2):\quad (x,y,t,a)\to\left(\frac{b_2x}{2b_2-1},\ \frac{b_2^2y}{(2b_2-1)^2},\ \frac{(2b_2-1)t}{4b_2},\ \frac{b_2}{2b_2-1}\right).$$

$$T_2(\text{I.5.s}_2):\quad (x,y,t,a)\to\left(\frac{x}{2},\ \frac{y}{4},\ \frac{t}{2},\ \frac{1}{2}\right).$$

Step 8. Substituting f_3, K_3, and $S(\text{I.6})$ into (2.1), we obtain a solution

$$\text{I.6.s}_1=\{b_0=0,\ c=-b_2,\ k_1=1,\ l_0=-b_2,\ l_1=-2,\ l_2=0\}.$$

Substituting (I.6.s_1) into $S(\text{I.6})$, the system $S(\text{I.6})$ simplifies to $S(\text{I.6.s}_1)$:

$$\dot{x}=-2x^2-b_2x+y,\qquad \dot{y}=-2xy-2b_2y.$$

If $b_2\neq 0$, then applying the transformation $T_1(\text{I.6.s}_1)$ to $S(2)$ yields $S(\text{I.6.s}_1)$; if $b_2=0$, then applying the transformation $T_2(\text{I.6.s}_1)$ to $S(2)$ yields $S(\text{I.6.s}_1)$ with $b_2=0$, where

$$T_1(\text{I.6.s}_1):\quad (x,y,t,a)\to\left(\frac{x}{2b_2},\ \frac{y}{4b_2^2},\ b_2t,\ \frac{1}{2}\right).$$

$$T_2(\text{I.6.s}_1):\quad (x,y,t,a)\to\left(-x-\frac{1}{2},\ x+y+\frac{1}{4},\ -\frac{t}{2},\ 1\right).$$

Step 9. Substituting f_3, K_3, and $S(\text{I.7})$ into (2.1), we obtain two solutions

$$\text{I.7.s}_1=\left\{a_0=\frac{b_2^2}{4},\ b_0=\frac{b_2}{4},\ k_1=-1,\ l_0=b_2,\ l_1=-2,\ l_2=0\right\},$$
$$\text{I.7.s}_2=\{a_0=2b_0^2,\ b_2=0,\ k_1=-2,\ l_0=2b_0,\ l_1=-2,\ l_2=0\}.$$

Substituting (I.7.s_1) and (I.7.s_2) into $S(\text{I.7})$, we obtain the systems $S(\text{I.7.s}_1)$ and $S(\text{I.7.s}_2)$

$$\dot{x}=-2x^2+y+\frac{b_2^2}{4},\qquad \dot{y}=-2xy+\frac{b_2^2}{2}x,$$

$$\dot{x}=-2x^2+2b_0^2,\qquad \dot{y}=-4xy+4b_0^2x,$$

respectively. Applying the transformation $T(\text{I.7.s}_1)$ to system $S(2)$ yields $S(\text{I.7.s}_1)$, where

$$T(\text{I.7.s}_1):\quad (x,y,t,a)\to\left(-x+\frac{b_2}{2}-\frac{1}{2},\ (1-b_2)\,x+y+\frac{1}{4}(b_2-1)^2,\ -\frac{t}{2},\ 1-b_2\right).$$

Similarly, applying the transformation $T(\text{I.7.s}_2)$ to $S(3)$ yields $S(\text{I.7.s}_2)$, where

$$T(\text{I.7.s}_2):\quad (x,y,t,a,b)\to\left(a+\frac{x+b_0}{2b_0},\ a^2-b_0^2+\frac{(2a+1)(x+b_0)}{2b_0}+y,\ -4b_0t,\ a,\ a+1\right).$$

Step 10. Substituting f_3, K_3, and $S(\text{II.1})$ into (2.1), we obtain four solutions

$$\text{II.1.s}_1 = \left\{ b_2 = 0,\ c = b_0(l_0 - \frac{1}{2}) + \frac{1}{4},\ k_1 = b_0 + l_0,\ k_0 = \frac{l_0 b_0}{2},\ l_1 = 2l_0 + 1,\ l_2 = 2 \right\},$$

$$\text{II.1.s}_2 = \{ b_0 = 2k_0,\ b_2 = 1,\ k_1 = 1,\ l_0 = 1,\ l_1 = 4,\ l_2 = 4 \},$$

$$\text{II.1.s}_3 = \left\{ \begin{array}{l} b_0 = \dfrac{2 - b_2}{4},\ k_0 = \dfrac{(2 - b_2)(4c + 1)}{16},\ k_1 = \dfrac{4b_2^2 c + b_2^2 - 2b_2 + 4}{4b_2}, \\ l_0 = 2c + \dfrac{1}{2},\ l_1 = \dfrac{b_2^2 + 4b_2^2 c + 2}{b_2},\ l_2 = 4 \end{array} \right\},$$

$$\text{II.1.s}_4 = \left\{ \begin{array}{l} b_0 = \dfrac{1}{4b_2},\ c = \dfrac{8b_2k_0 + 8b_2^2k_0 - 1}{4b_2},\ k_1 = \dfrac{8b_2^2k_0 + 1}{2b_2},\ l_0 = 8b_2k_0, \\ l_1 = 16b_2k_0 + 2,\ l_2 = 4 \end{array} \right\}.$$

Substituting (II.1.k) into $S(\text{II.1})$, we obtain the systems $S(\text{II.1.k})$ for $k = \text{s}_1,\ \text{s}_2,\ \text{s}_3,\ \text{s}_4$.

$$\begin{aligned} \dot{x} &= (2l_0 + 1)\,x^2 + 2xy + (b_0 + l_0 + 2b_0l_0)\,x + 2b_0y + b_0l_0, \\ \dot{y} &= (4b_0 + 2l_0 - 1)\,x^2 + (4b_0 + 4l_0 + 2)\,xy + 4y^2 + 2b_0l_0x + (4b_0l_0 - 2b_0 + 1)\,y. \end{aligned} \tag{$S(\text{II.1.s}_1)$}$$

$$\begin{aligned} \dot{x} &= 2(2c - 4k_0 + 1)\,x^2 + 2xy + (4k_0 + 1)\,x + (8k_0 - 4c + 1)\,y + 2k_0, \\ \dot{y} &= 2(4k_0 - 2c + 1)\,x^2 + 6xy + 4y^2 + 4k_0x + 4cy. \end{aligned} \tag{$S(\text{II.1.s}_2)$}$$

$$\begin{aligned} \dot{x} &= \left(\frac{2}{b_2} + b_2 + 4b_2c - 2\right)x^2 + 2xy + \left(\frac{1}{b_2} + 2c\right)x - \left(\frac{b_2}{2} + 2b_2c - 2\right)y \\ &\quad + \left(c - \frac{b_2c}{2} - \frac{b_2}{8} + \frac{1}{4}\right), \\ \dot{y} &= \frac{2}{b_2}x^2 + \left(\frac{4}{b_2} + b_2 + 4b_2c\right)xy + \left(2c - \frac{b_2}{4} - b_2c + \frac{1}{2}\right)x + 4y^2 + 4cy. \end{aligned} \tag{$S(\text{II.1.s}_3)$}$$

$$\begin{aligned} \dot{x} &= 16b_2k_0\,x^2 + 2xy + \left(4k_0 + 4b_2k_0 + \frac{1}{2b_2}\right)x + \left(\frac{1}{b_2} - 8b_2k_0 + 1\right)y + 2k_0, \\ \dot{y} &= \frac{2}{b_2}x^2 + \left(\frac{2}{b_2} + 16b_2k_0 + 2\right)xy + 4y^2 + 4k_0x + \left(8k_0 + 8b_2k_0 - \frac{1}{b_2}\right)y. \end{aligned} \tag{$S(\text{II.1.s}_4)$}$$

Applying the parameter change $(b_0, l_0) \to \left(b + \frac{1}{2},\ a + \frac{1}{2}\right)$ and the time rescaling $t \to 4t$ to $S(\text{II.1.s}_1)$, we obtain

S(4) $\dot{x} = 8(a+1)\,x^2 + 8xy + 2(4ab + 4a + 4b + 3)\,x + 4(2b+1)\,y + (4ab + 2a + 2b + 1)$,
$\dot{y} = 8(a + 2b + 1)\,x^2 + 8(2a + 2b + 3)\,xy + 2(4ab + 2a + 2b + 1)\,x + 4(4ab + 2a + 1)\,y + 16y^2$,
$f_1 = y - x^2, \quad K_1 = 16(a+1)\,x + 16y + 4(4ab + 2a + 1)$,
$f_2 = 4x + 4y + 1, \quad K_2 = 16(a + b + 1)\,x + 16y + 4(4ab + 2a + 2b + 1)$,
$f_3 = 2x + 2b + 1, \quad K_3 = 8(a+1)\,x + 8y + 2(2a + 1)$,
$R = \dfrac{f_2^{(a-2b)/(2b)}}{f_1^{(a+b)/(2b)} f_3^2}$.

Here R is an integrating factor of the system and $ab \neq 0$. If $a = 0$, the system becomes

$$\dot{x} = (4x + 4y + 1)(2x + 2b + 1), \quad \dot{y} = 2(4x + 4y + 1)((2b + 1)x + 2y).$$

If $b = 0$, the system becomes

$$\dot{x} = (4(a+1)x + 4y + 2a + 1)(2x + 1), \quad \dot{y} = 2(4(a+1)x + 4y + 2a + 1)(x + 2y).$$

In both cases these systems can be reduced to linear differential systems.

Applying the parameter change $(c, k_0) \to \left(\frac{1+a}{4}, \frac{1+b}{8}\right)$ and the time rescaling $t \to 4t$ to $S(\text{II.1.s}_2)$, we obtain

S(5) $\dot{x} = 4(a-b+2)\,x^2 + 8xy + 2(b+3)\,x - 4(a-b-1)\,y + (b+1)$,
$\dot{y} = 4(2-a+b)\,x^2 + 24xy + 16y^2 + 2(b+1)\,x + 4(a+1)\,y$,
$f_1 = y - x^2, \quad K_1 = 8(a-b+2)\,x + 16y + 4(a+1)$,
$f_2 = 4x + 4y + 1, \quad K_2 = 4(4x + 4y + b + 1)$,
$f_3 = 4x + 4y + b + 1, \quad K_3 = 4(4x + 4y + 1)$,
$R = \dfrac{1}{f_1\sqrt{f_2}\,f_3}$.

Here we have $a - b \neq 0$. If $a - b = 0$, the system becomes

$$\dot{x} = (4x + 4y + b + 1)(2x + 1), \quad \dot{y} = 2(4x + 4y + b + 1)(x + 2y),$$

and the system can be reduced to a linear differential system.

Applying the parameter change $(c, b_2) \to \left(\frac{a+2}{(b+2)^2} - \frac{1}{4}, \frac{b+2}{2}\right)$ and the time rescaling $t \to 4(b+2)^2 t$ to $S(\text{II.1.s}_3)$, we obtain

S(6) $\dot{x} = 8(ab + 2a - b^2 + 4)\,x^2 + 8(b+2)^2\,xy - 2(b^2 - 4a - 12)\,x - 4(ab - 2b^2 - 6b + 2a - 4)\,y - (ab - 2a + 2b - 4)$,
$\dot{y} = 16(2+b)\,x^2 + 8(ab + 2a + 6b + 12)\,xy + 16(b+2)^2\,y^2 - 2(ab - 2a + 2b - 4)\,x - 4(b^2 - 4a + 4b - 4)\,y$,
$f_1 = y - x^2, \quad K_1 = 16(ab + 2a - b^2 + 4)\,x + 16(b+2)^2\,y - 4(b^2 - 4a + 4b - 4)$,
$f_2 = 4x + 4y + 1, \quad K_2 = 8(ab + 2a + 2b - b^2 + 8)\,x + 16(b+2)^2\,y - 4(ab - 2a + 2b - 4)$,
$f_3 = 8x + 4(b+2)\,y + (2-b), \quad K_3 = 8(a+4)(b+2)\,x + 16(b+2)^2\,y + 8(a+2)$,
$R = \dfrac{f_1^{(a-3b)/(2b-2a)} f_2^{b/(b-a)}}{f_3^2}$.

Here we have $(a-b)(b+2) \neq 0$. Indeed, if $a - b = 0$, the system becomes

$$\dot{x} = (a+2)(8x + 4(a+2)y - a + 2)(2x + 1), \quad \dot{y} = 2(a+2)(8x + 4(a+2)y - a + 2)(x + 2y).$$

If $b + 2 = 0$, the system becomes

$$\dot{x} = 4(a+2)(2x + 1), \quad \dot{y} = 8(a+2)(x + 2y).$$

Applying the parameter change $(k_0, b_2) \to \left(\frac{(b+1)(a+1)}{8}, \frac{1}{b+1}\right)$ and the time rescaling $t \to 4t$ to $S(\text{II.1.s}_4)$, we obtain

S(7) $\dot{x} = 8(a+1)\,x^2 + 8xy + 2(ab + 2a + 2b + 3)\,x - 4(a-b-1)\,y + (ab + a + b + 1)$,
$\dot{y} = 8(b+1)\,x^2 + 8(a + b + 3)\,xy + 16y^2 + (2ab + 2a + 2b + 2)\,x + 4(ab + 2a + 1)\,y$,
$f_1 = y - x^2, \quad K_1 = 16(a+1)\,x + 16y + 4(ab + 2a + 1)$,
$f_2 = 4x + 4y + 1, \quad K_2 = 8(a + b + 2)\,x + 16y + 4(ab + a + b + 1)$,
$f_3 = 4(b+1)\,x + 4y + (b+1)^2, \quad K_3 = 8(a+2)\,x + 16y + 4(a+1)$,
$R = \dfrac{1}{\sqrt{f_1}\,f_2\,f_3}$.

Here we have $a(a-b)\neq 0$. If $a=0$ the system becomes

$$\dot{x}=(4x+4y+1)(2x+b+1),\quad \dot{y}=2(4x+4y+1)((b+1)x+2y).$$

If $a-b=0$, the system becomes

$$\dot{x}=(4(a+1)x+4y+(a+1)^2)(2x+1),\quad \dot{y}=2(4(a+1)x+4y+(a+1)^2)(x+2y).$$

Step 11. Substituting f_3, K_3, and $S(\mathrm{II}.2)$ into (2.1), we obtain two solutions

$$\mathrm{II}.2.\mathrm{s}_1=\left\{\begin{array}{l} b_0=\dfrac{1}{4b_2},\ k_0=\dfrac{2a_0b_2+4b_2^2-1}{8b_2^2(2a_0b_2-1)},\ k_1=\dfrac{2b_2(a_0+b_2)-1}{2b_2(2a_0b_2-1)},\\ l_0=\dfrac{a_0(2a_0b_2+4b_2^2-1)}{b_2(2a_0b_2-1)},\ l_1=\dfrac{4(1+a_0^2)b_2^2-1}{b_2(2a_0b_2-1)},\ l_2=2 \end{array}\right\},$$

$$\mathrm{II}.2.\mathrm{s}_2=\left\{\begin{array}{l} b_0=a_0-a_0^2b_2,\ k_0=\dfrac{(a_0b_2-1)(a_0-2b_2-2a_0^2b_2)}{2b_2(2a_0b_2-1)},\ k_1=\dfrac{2b_2(a_0+b_2)-1}{2b_2(2a_0b_2-1)},\\ l_0=\dfrac{2b_2+2a_0^2b_2-a_0}{b_2(2a_0b_2-1)},\ l_1=\dfrac{4(a_0^2+1)b_2^2-1}{b_2(2a_0b_2-1)},\ l_2=2 \end{array}\right\}.$$

Substituting $(\mathrm{II}.2.k)$ into $S(\mathrm{II}.2)$, we obtain the systems $S(\mathrm{II}.2.k)$ for $k=\mathrm{s}_1,\ \mathrm{s}_2$.

$$\begin{aligned}\dot{x}=&\frac{1}{4b_2^2(2a_0b_2-1)}\left(4b_2(2a_0b_2+2b_2^2-1)\,x^2+4b_2^2(2a_0b_2-1)\,xy+4a_0b_2^2(2a_0b_2-1)\,y\right.\\&\left.+(8a_0^2b_2^2+8a_0b_2^3+4b_2^2-2a_0b_2-1)\,x+a_0(2a_0b_2+4b_2^2-1)\right),\\ \dot{y}=&\frac{1}{4b_2^2(2a_0b_2-1)}\left(8(2a_0^2b_2^2-a_0b_2+b_2^2)\,x^2+8b_2(2a_0^2b_2^2+a_0b_2+2b_2^2-1)\,xy\right.\\&\left.+8b_2^2(2a_0b_2-1)\,y^2+2a_0(2a_0b_2+4b_2^2-1)\,x+2(8a_0b_2^3+2a_0b_2-1)\,y\right).\end{aligned}\tag{$S(\mathrm{II}.2.\mathrm{s}_1)$}$$

$$\begin{aligned}\dot{x}=&\frac{1}{b_2(2a_0b_2-1)}\left((2a_0b_2+2b_2^2-1)\,x^2+b_2(2a_0b_2-1)\,xy+a_0b_2(2a_0b_2-1)\,y\right.\\&\left.-(2a_0^3b_2^2-5a_0^2b_2+2a_0-2b_2)\,x-(2a_0^4b_2^2-3a_0^3b_2+2a_0^2b_2^2+a_0^2-2a_0b_2)\right),\\ \dot{y}=&\frac{1}{b_2(2a_0b_2-1)}\left(2(2a_0^2b_2-a_0+b_2)\,x^2+(4a_0^2b_2^2+2a_0b_2+4b_2^2-2)\,xy\right.\\&+2(2a_0b_2^2-b_2)\,y^2-(4a_0^4b_2^2+4a_0^2b_2^2-6a_0^3b_2-4a_0b_2+2a_0^2)\,x\\&\left.-(4a_0^3b_2^2-6a_0^2b_2+2a_0-2b_2)\,y\right).\end{aligned}\tag{$S(\mathrm{II}.2.\mathrm{s}_2)$}$$

Doing the transformation $T(\mathrm{II}.2.\mathrm{s}_1)$ to system $S(4)$ yields $S(\mathrm{II}.2.\mathrm{s}_1)$, where

$$T(\mathrm{II}.2.\mathrm{s}_1):\quad (x,y,t,a,b)\to\left(b_2x,\ b_2^2y,\ \frac{t}{8b_2^2},\ \frac{2b_2^2}{2a_0b_2-1},\ a_0b_2-\frac{1}{2}\right).$$

Applying the transformation

$$T_1(\mathrm{II}.2.\mathrm{s}_2):\quad (x,y,a_0)\to\left(x-\frac{2ab_2+1}{2b_2},\ \frac{(b_2-1)\,x}{b_2}+y+\frac{(1+2ab_2)^2}{4b_2^2},\ \frac{2ab_2+1}{2b_2}\right),$$

to $S(\mathrm{II}.2.\mathrm{s}_2)$ we obtain the system $S(\mathrm{II}.2.\mathrm{s}_2.1)$.

$$\dot{x}=\frac{a+1}{a}\,x^2+xy-2x,\qquad \dot{y}=\frac{2a^2+a+1}{a}\,x^2+\frac{2a^2+3a+2}{a}\,xy+2y^2-2y.$$

In **Step 12** we will prove that this system is a special case of system $S(8)$.
Step 12. Substituting f_3, K_3, and $S(\text{II}.3)$ into (2.1), we obtain four solutions

$$\text{II}.3.\text{s}_1 = \left\{ \begin{array}{l} a_0 = (1-b_0)b_0,\ b_2 = 0,\ c = \dfrac{b_0}{2} - b_0^2 + k_0,\ l_0 = 2(1-b_0)k_0, \\ l_1 = 1 - b_0 + 2k_0,\ l_2 = 1 \end{array} \right\},$$

$$\text{II}.3.\text{s}_2 = \left\{ \begin{array}{l} a_0 = \dfrac{2b_2-1}{4b_2^2},\ b_0 = \dfrac{1}{4b_2},\ c = 2k_0 - \dfrac{1}{4b_2^2},\ l_0 = \left(4 - \dfrac{2}{b_2}\right) k_0, \\ l_1 = 4k_0 - \dfrac{1}{b_2} + 2,\ l_2 = 2 \end{array} \right\},$$

$$\text{II}.3.\text{s}_3 = \left\{ \begin{array}{l} a_0 = -\dfrac{(2b_2k_0-1)(2b_2k_0+b_2-1)}{b_2^2},\ b_0 = 2k_0(1-2b_2k_0), \\ c = \dfrac{b_2 + 8b_2k_0 - 8b_2^2k_0^2 - 2}{2b_2^2},\ l_0 = \dfrac{b_2 + 2b_2k_0 - 1}{b_2^2}, \\ l_1 = 4k_0 - \dfrac{1}{b_2} + 2,\ l_2 = 2 \end{array} \right\},$$

$$\text{II}.3.\text{s}_4 = \left\{ b_0 = \frac{1}{4},\ b_2 = 1,\ k_0 = \frac{1}{4},\ l_0 = 2a_0,\ l_1 = 2,\ l_2 = 2 \right\}.$$

Substituting $(\text{II}.3.k)$ into $S(\text{II}.3)$, we obtain the systems $S(\text{II}.3.k)$ for $k = \text{s}_1,\ \text{s}_2,\ \text{s}_3,\ \text{s}_4$.

$$\begin{aligned} \dot{x} &= (2k_0 - b_0 + 1)\,x^2 + xy + b_0 y - (b_0^2 - b_0 - 2k_0)\,x - 2b_0k_0(b_0 - 1), \\ \dot{y} &= (2k_0 + b_0)\,x^2 + (4k_0 + 2)\,xy + 2y^2 - 4b_0k_0(b_0 - 1)\,x - (2b_0^2 - b_0 - 2k_0)\,y. \end{aligned} \tag{$S(\text{II}.3.\text{s}_1)$}$$

$$\begin{aligned} \dot{x} &= \frac{1}{4b_2^2}\left(4b_2(4b_2k_0 + b_2 - 1)\,x^2 + 4b_2^2\,xy - 4b_2(2b_2k_0 - 1)\,y + (8b_2^2k_0 + 2b_2 - 1)\,x \right. \\ &\quad \left. +2k_0(2b_2 - 1)\right), \end{aligned} \tag{$S(\text{II}.3.\text{s}_2)$}$$

$$\dot{y} = \frac{1}{4b_2^2}\left(4b_2\,x^2 + 8b_2^2(2k_0 + 1)\,xy + 8b_2^2\,y^2 + 4k_0(2b_2 - 1)\,x + 2(8b_2^2k_0 - 1)\,y\right).$$

$$\begin{aligned} \dot{x} &= \frac{1}{b_2^2}\left(b_2(4b_2k_0 + b_2 - 1)\,x^2 + b_2^2\,xy - b_2(2b_2k_0 - 1)\,y - (4b_2^2k_0^2 - 4b_2k_0 - b_2 + 1)\,x \right. \\ &\quad \left. -8b_2^2k_0^3 - 4b_2^2k_0^2 + 8b_2k_0^2 + 2b_2k_0 - 2k_0\right), \\ \dot{y} &= \frac{1}{b_2^2}\left(b_2\,x^2 + 2b_2^2(2k_0 + 1)\,xy + 2b_2^2\,y^2 - 4k_0(4b_2^2k_0^2 + 2b_2^2k_0 - 4b_2k_0 - b_2 + 1)\,x \right. \\ &\quad \left. -(8b_2^2k_0^2 - 8b_2k_0 - b_2 + 2)\,y\right). \end{aligned} \tag{$S(\text{II}.3.\text{s}_3)$}$$

$$\begin{aligned} \dot{x} &= -(2a_0 - 2c - 1)\,x^2 + xy + \left(a_0 + \frac{1}{2}\right)x + \left(2a_0 - 2c + \frac{1}{2}\right)y + \frac{a_0}{2}, \\ \dot{y} &= (2a_0 - 2c + 1)\,x^2 + 3xy + 2y^2 + a_0 x + 2cy. \end{aligned} \tag{$S(\text{II}.3.\text{s}_4)$}$$

Applying the parameter change $(b_0, k_0) \to \left(b + \frac{1}{2},\ \frac{2a+1}{4}\right)$ and the time rescaling $t \to 8t$ to $S(\text{II}.3.\text{s}_1)$, we obtain

$$\begin{aligned} \text{S(8)}\quad & \dot{x} = 8(a - b + 1)\,x^2 + 8xy - 2(4b^2 - 4a - 3)\,x + 4(2b + 1)\,y - (2a + 1)(4b^2 - 1), \\ & \dot{y} = 8(a + b + 1)\,x^2 + 8(2a + 3)\,xy + 16y^2 - 2(8ab^2 + 4b^2 - 2a - 1)\,x - 4(4b^2 + 2b - 2a - 1)\,y, \\ & f_1 = y - x^2, \quad K_1 = 16(a - b + 1)\,x + 16y - 4(4b^2 + 2b - 2a - 1), \end{aligned}$$

$$f_2 = 4b^2 - 1 - 4x - 4y, \quad K_2 = 16(a+1)\,x + 16y + 4(2a+1),$$
$$f_3 = 2x + 2b + 1, \quad K_3 = 8(a-b+1)\,x + 8y - 2(4ab - 2a + 2b - 1),$$
$$R = f_1^{-(a+3b)/(2b)} f_2^{(a-3b)/(2b)} f_3.$$

Here we have $b(a-b) \neq 0$. If $b = 0$ the system becomes

$$\dot{x} = (4(a+1)x + 4y + 2a + 1)(2x+1), \quad \dot{y} = 2(4(a+1)x + 4y + 2a + 1)(x + 2y).$$

If $a - b = 0$, the system becomes

$$\dot{x} = (4x + 4y - 4a^2 + 1)(2x + 2a + 1), \quad \dot{y} = 2(4x + 4y - 4a^2 + 1)((2a+1)x + 2y).$$

Applying transformation $T_2(\text{II.2.s}_2)$ to $S(8)$ yields $S(\text{II.2.s}_2.1)$.

$$T_2(\text{II.2.s}_2): \quad (x, y, t, a, b) \to \left(x - a - \frac{1}{2},\ y + \frac{1}{4}(2a+1)^2,\ \frac{t}{8},\ \frac{1}{a} + a,\ a\right).$$

For solution (II.3.s_2), if $2b_2 - 1 \neq 0$, then applying the transformation $T_1(\text{II.3.s}_2)$ to $S(6)$ yields $S(\text{II.3.s}_2)$; if $2b_2 - 1 = 0$, then applying the transformation $T_2(\text{II.3.s}_2)$ to $S(6)$ yields $S(\text{II.3.s}_2)$ with $b_2 = \dfrac{1}{2}$, where

$$T_1(\text{II.3.s}_2): \quad (x, y, t, a, b) \to$$
$$\left(x + \frac{1-b_2}{2b_2},\ y + \frac{(b_2-1)(b_2 - 4b_2x - 1)}{4b_2^2},\ \frac{(2b_2-1)^2 t}{32b_2^2},\ \frac{2b_2(4k_0-1)}{2b_2-1},\ \frac{2(1-b_2)}{2b_2-1}\right).$$

$$T_2(\text{II.3.s}_2): \quad (x, y, t, a, b) \to \left(\frac{x}{2},\ \frac{y}{4},\ \frac{t}{32},\ 8k_0 - 2,\ 2\right).$$

Applying the transformation $T(\text{II.3.s}_3)$ to $S(8)$ yields $S(\text{II.3.s}_3)$, where

$$T(\text{II.3.s}_3): \quad (t, a, b) \to \left(\frac{t}{8},\ 2k_0 - \frac{1}{2},\ \frac{1 - 2b_2k_0}{b_2} - \frac{1}{2}\right).$$

For the solution (II.3.s_4) applying the transformation $T(\text{II.3.s}_4)$ to $S(5)$ yields $S(\text{II.3.s}_4)$, where

$$T(\text{II.3.s}_4): \quad (t, a, b) \to \left(\frac{t}{8},\ 4c - 1,\ 4a_0 - 1\right).$$

Step 13. Substituting f_3, K_3, and $S(\text{II.4})$ into (2.1), we obtain five solutions

$$\text{II.4.s}_1 = \left\{a = \frac{1}{b_2} - 2,\ b_0 = 0,\ c = -\frac{1}{b_2},\ l_0 = -\frac{1}{b_2},\ l_1 = \frac{1}{b_2} - 2,\ l_2 = 2\right\},$$
$$\text{II.4.s}_2 = \{a = l_1,\ b_0 = 1,\ b_2 = 0,\ c = l_1,\ l_0 = 0,\ l_2 = 1\},$$
$$\text{II.4.s}_3 = \left\{a = \frac{1}{b_2} - 2,\ b_0 = \frac{1}{4b_2},\ c = \frac{1-2b_2}{4b_2^2},\ l_0 = 0,\ l_1 = \frac{1}{b_2} - 2,\ l_2 = 2\right\},$$
$$\text{II.4.s}_4 = \{a = -2,\ b_0 = 0,\ c = 0,\ l_0 = 0,\ l_1 = -2,\ l_2 = 2\},$$
$$\text{II.4.s}_5 = \left\{a = -1,\ c = \frac{b_0}{b_2},\ l_0 = 0,\ l_1 = 0,\ l_2 = 2\right\}.$$

Substituting $(\text{II.4.}k)$ into $S(\text{II.4})$, we obtain the systems $S(\text{II.4.}k)$ for $k = \text{s}_1,\ \text{s}_2,\ \text{s}_3,\ \text{s}_4,\ \text{s}_5$. In system $S(\text{II.4.s}_5)$ we replace $\frac{b_0}{b_2}$ by c.

$$\dot{x} = -\frac{2b_2 - 1}{b_2}\,x^2 + xy - \frac{1}{b_2}x + y, \qquad \dot{y} = \frac{-2(b_2-1)}{b_2}\,xy + 2y^2 - \frac{2}{b_2}y. \qquad (S(\text{II.4.s}_1))$$

$$\dot{x} = (x+1)(l_1 x + y), \qquad \dot{y} = 2y\left((l_1+1)\,x + y + l_1\right). \tag{S(II.4.s$_2$)}$$

$$\dot{x} = -\frac{2b_2-1}{b_2}\,x^2 + xy - \frac{2b_2-1}{4b_2^2}\,x + y, \qquad \dot{y} = \frac{-2(b_2-1)}{b_2}\,xy + 2y^2 - \frac{2b_2-1}{2b_2^2}\,y. \tag{S(II.4.s$_3$)}$$

$$\dot{x} = -2x^2 + xy + y, \qquad \dot{y} = -2xy + 2y^2. \tag{S(II.4.s$_4$)}$$

$$\dot{x} = -x^2 + xy + cx + y, \qquad \dot{y} = 2y^2 + 2cy. \tag{S(II.4.s$_5$)}$$

Applying the following transformations T(II.4.s$_2$) to $S(4)$, T(II.4.s$_3$) to $S(7)$, T(II.4.s$_4$) to $S(6)$, and T(II.4.s$_5$) to $S(5)$ yields S(II.4.s$_2$), S(II.4.s$_3$), S(II.4.s$_4$), and S(II.4.s$_5$), respectively.

$$T(\text{II.4.s}_2): \quad (x,y,t,b,a) \to \left(-x-\frac{1}{2},\ x+y+\frac{1}{4},\ \frac{t}{8},\ -1,\ -l_1\right).$$

$$T(\text{II.4.s}_3): \quad (x,y,t,a,b) \to \left(\frac{1}{2}(x-1),\ -\frac{1}{4}(2x-y-1),\ \frac{t}{2},\ \frac{1}{2b_2}-1,\ \frac{1}{2b_2}\right).$$

$$T(\text{II.4.s}_4): \quad (x,y,t,a,b) \to \left(\frac{1}{2}(x-1),\ -\frac{1}{4}(2x-y-1),\ \frac{t}{8},\ -2,\ 0\right).$$

$$T(\text{II.4.s}_5): \quad (x,y,t,a,b) \to \left(\frac{1}{2}(x-1),\ -\frac{1}{4}(2x-y-1),\ \frac{t}{2},\ c-1,\ c\right).$$

For solution (II.4.s$_1$), if $2b_2+1 \neq 0$, then applying the transformation T_1(II.4.s$_1$) to system $S(6)$ yields S(II.4.s$_1$); if $2b_2+1=0$, then applying the transformation T_2(II.4.s$_1$) to $S(6)$ yields S(II.4.s$_1$) with $b_2=-\frac{1}{2}$, where

$$T_1(\text{II.4.s}_1): \quad (x,y,t,a,b) \to \left(\frac{1}{2}(x-1),\ -\frac{1}{4}(2x-y-1),\ \frac{(2b_2+1)^2 t}{32b_2^2},\ -2,\ -\frac{2}{2b_2+1}\right).$$

$$T_2(\text{II.4.s}_1): \quad (x,y,t,a,b) \to \left(-\frac{1}{2}(x+1),\ \frac{1}{4}(2x+y+1),\ \frac{t}{2},\ 0,\ -1\right).$$

Step 14. Substituting f_3, K_3, and S(II.5) into (2.1), we obtain five solutions

$$\text{II.5.s}_1 = \left\{a_0 = -b_0^2,\ b_2 = 0,\ k_1 = b_0+1,\ l_0 = -b_0(b_0+1),\ l_1 = 1,\ l_2 = 1\right\},$$

$$\text{II.5.s}_2 = \left\{a_0 = -\frac{(b_2-1)^2}{4b_2^2},\ b_0 = \frac{1-b_2^2}{4b_2},\ k_1 = \frac{b_2+1}{2b_2},\ l_0 = \frac{b_2-1}{2b_2^2},\ l_1 = 1,\ l_2 = 2\right\},$$

$$\text{II.5.s}_3 = \left\{a_0 = -\frac{1}{4b_2^2},\ b_0 = \frac{1}{4b_2},\ k_1 = \frac{b_2+1}{2b_2},\ l_0 = -\frac{b_2+1}{2b_2^2},\ l_1 = 1,\ l_2 = 2\right\},$$

$$\text{II.5.s}_4 = \left\{a_0 = 0,\ b_0 = 0,\ k_1 = \frac{1}{2},\ l_0 = 0,\ l_1 = 1,\ l_2 = 2\right\},$$

$$\text{II.5.s}_5 = \left\{b_0 = 0,\ k_1 = 0,\ l_0 = 2a_0,\ l_1 = 0,\ l_2 = 2\right\}.$$

Substituting (II.5.k) into S(II.5), we obtain systems S(II.5.k) for $k = \text{s}_1,\ \text{s}_2,\ \text{s}_3,\ \text{s}_4,\ \text{s}_5$.

$$\begin{aligned} \dot{x} &= x^2 + xy - b_0^2 x + b_0 y - b_0^2(b_0+1), \\ \dot{y} &= 2(b_0+1)\,xy + 2y^2 - 2b_0^2(b_0+1)\,x - 2b_0^2 y. \end{aligned} \tag{S(II.5.s$_1$)}$$

$$\begin{aligned} \dot{x} &= \frac{1}{8b_2^3}\left(8b_2^3x^2 + 8b_2^3xy - 2b_2(b_2-1)^2x - 4b_2^2(b_2-1)y - b_2^3 + b_2^2 + b_2 - 1\right), \\ \dot{y} &= \frac{1}{8b_2^3}\left(8b_2^2(b_2+1)\,xy + 16b_2^3y^2 - 2(b_2-1)^2(b_2+1)\,x - 4b_2(b_2-1)^2 y\right). \end{aligned} \tag{S(II.5.s$_2$)}$$

$$\begin{aligned}\dot{x} &= \frac{1}{8b_2^3}\left(8b_2^3x^2+8b_2^3xy-2b_2x-4b_2^2(b_2-1)\,y-(b_2+1)\right),\\ \dot{y} &= \frac{1}{8b_2^3}\left(8b_2^2(b_2+1)\,xy+16b_2^3y^2-2(b_2+1)\,x-4b_2y\right).\end{aligned} \tag{S(II.5.s$_3$)}$$

$$\dot{x} = x^2-\frac{1}{2}y+xy, \qquad \dot{y} = xy+2y^2. \tag{S(II.5.s$_4$)}$$

$$\dot{x} = x^2+xy+a_0x-y, \qquad \dot{y} = 2y^2+2a_0y. \tag{S(II.5.s$_5$)}$$

Applying the following transformations T(II.5.s$_1$) to $S(8)$, T(II.5.s$_2$) to $S(8)$, T(II.5.s$_4$) to $S(6)$, and T(II.5.s$_5$) to $S(5)$ yields S(II.5.s$_1$), S(II.5.s$_2$), S(II.5.s$_4$), and S(II.5.s$_5$), respectively.

$$T(\text{II.5.s}_1): \quad (x,y,t,a,b)\to\left(-x-\frac{1}{2},\ x+y+\frac{1}{4},\ \frac{t}{8},\ -b_0-1,\ -b_0\right).$$

$$T(\text{II.5.s}_2): \quad (x,y,t,a,b)\to \left(\frac{x}{2}+\frac{1-2b_2}{4b_2},\ \frac{1-2b_2}{4b_2}\,x+\frac{y}{4}+\frac{(2b_2-1)^2}{16b_2^2},\ \frac{t}{2},\ \frac{b_2-2}{4b_2},\ -\frac{1}{4}\right).$$

$$T(\text{II.5.s}_4): \quad (x,y,a,b)\to\left(-x-\frac{1}{2},\ x+y+\frac{1}{4},\ \frac{t}{32},\ -2,\ 0\right).$$

$$T(\text{II.5.s}_5): \quad (x,y,t,a,b)\to\left(-\frac{1}{2}(x+1),\ \frac{1}{4}(2x+y+1),\ \frac{t}{2},\ a_0-1,\ a_0\right).$$

For the solution (II.5.s$_3$), if $2b_2-1\neq 0$, then applying the transformation T_1(II.5.s$_3$) to system $S(6)$ yields S(II.5.s$_3$); if $2b_2-1=0$, then applying the transformation T_2(II.5.s$_3$) to $S(6)$ yields S(II.5.s$_3$) with $b_2=\dfrac{1}{2}$, where

$$T_1(\text{II.5.s}_3): \quad (x,y,t,a,b)\to \left(\frac{x}{2}-\frac{2b_2-1}{4b_2},\ \frac{1-2b_2}{4b_2}\,x+\frac{y}{4}+\frac{(2b_2-1)^2}{16b_2^2},\ \frac{(2b_2-1)^2t}{32b_2^2},\ \frac{2(b_2+1)}{2b_2-1},\ \frac{2}{2b_2-1}\right).$$

$$T_2(\text{II.5.s}_3): \quad (x,y,t,a,b)\to\left(-\frac{x}{2}-1,\ x+\frac{y}{4}+1,\ \frac{t}{2},\ -\frac{3}{2},\ -1\right).$$

This finishes the proof. □

6 Proof of Theorem 3.2

Proof. **System** $S(1)$**.** The system together with its invariant curves is

$$\dot{x}=x+1, \quad \dot{y}=x^2+2x+y.$$

$$f_1=y-x^2, \quad f_2=x+1.$$

It has only one finite equilibrium at $P_1=(-1,1)$ with eigenvalues $1,1$. In what follows, we use the notation $P_1=(-1,1):1,1$ to denote the equilibrium point and its corresponding eigenvalues.

The Poincaré compactification of this system in the chart U_1 (with coordinates (u,v)) is

$$\dot{u}=-uv^2+2v+1, \quad \dot{v}=-v^3-v^2.$$

Hence, there are no infinite equilibria in the chart U_1.

The Poincaré compactification of this system in the chart U_2 is

$$\dot{u} = -u^3 - 2u^2v + v^2, \quad \dot{v} = -u^2v - 2uv^2 - v^2.$$

The infinite equilibrium point O_2 at the origin of the local chart U_2 is linearly zero. Doing the vertical blow-up $(u, v) = (u_1, u_1v_1)$ and a rescaling of the time we eliminate the common factor u_1 between $\dot{u}_1$ and $\dot{v}_1$ obtaining the system

$$\dot{u}_1 = -2u_1^2v_1 + u_1v_1^2 - u_1^2, \qquad \dot{v}_1 = -v_1^3 - v_1^2.$$

It has two infinite equilibria on the straight line $u_1 = 0$:

$$(0, 0) : 0, 0; \quad (0, -1) : -1, 1.$$

However, the origin in the coordinates (u_1, v_1) is again a linearly zero equilibrium. Since $u_1 = 0$ is a characteristic direction of the equilibrium $(0, 0)$, we do the twist $(u_1, v_1) = (u_2 - v_2, v_2)$ in order to avoid the loss of information when we do a vertical blow-up, in this way we obtain the system

$$\dot{u}_2 = -4v_2^3 + 5u_2v_2^2 - 2u_2^2v_2 - u_2^2 + 2u_2v_2 - 2v_2^2, \qquad \dot{v}_2 = -v_2^3 - v_2^2.$$

Doing the vertical blow-up $(u_2, v_2) = (u_3, u_3v_3)$ and a rescaling of time we eliminate the common factor u_3 between $\dot{u}_3$ and $\dot{v}_3$, obtaining the system

$$\begin{aligned}\dot{u}_3 &= -4u_3^2v_3^3 + 5u_3^2v_3^2 - 2u_3^2v_3 - 2u_3v_3^2 + 2u_3v_3 - u_3,\\ \dot{v}_3 &= 4u_3v_3^4 - 6u_3v_3^3 + 2u_3v_3^2 + 2v_3^3 - 3v_3^2 + v_3.\end{aligned}$$

This system has three equilibria on the straight line $u_3 = 0$:

$$(0, 0) : -1, 1; \quad \left(0, \frac{1}{2}\right) : -\frac{1}{2}, -\frac{1}{2}; \quad (0, 1) : -1, 1.$$

Doing the blowing-down $B(\text{I.2.s}_1)$ (Figure 6.1) we obtain the local phase portrait of the infinite equilibrium O_2.

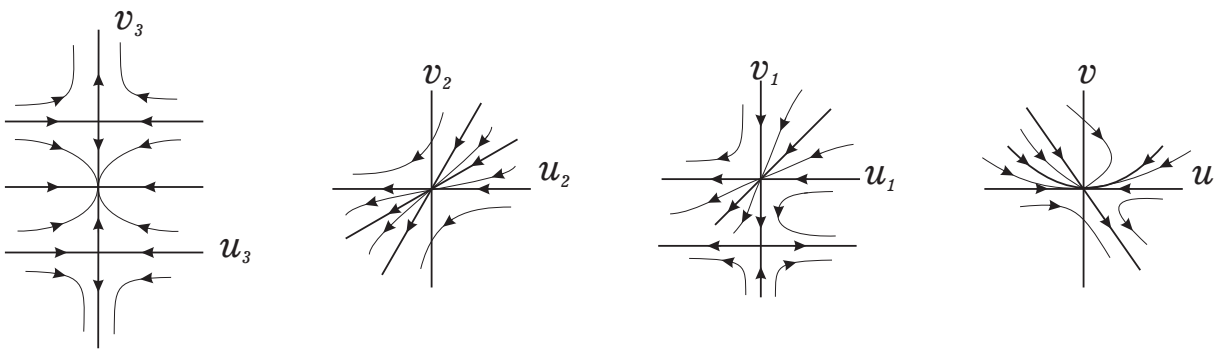


Figure 6.1: $B(\text{I.2.s}_1)$: Blowing-down at the origin O_2 of the chart U_2 for system $S(1)$.

Since $y - x^2 = 0$ and $x + 1 = 0$ are invariant algebraic curves of system $S(1)$, we obtain the configuration $C(\text{I.2.s}_1)$ (see Figure 6.2).

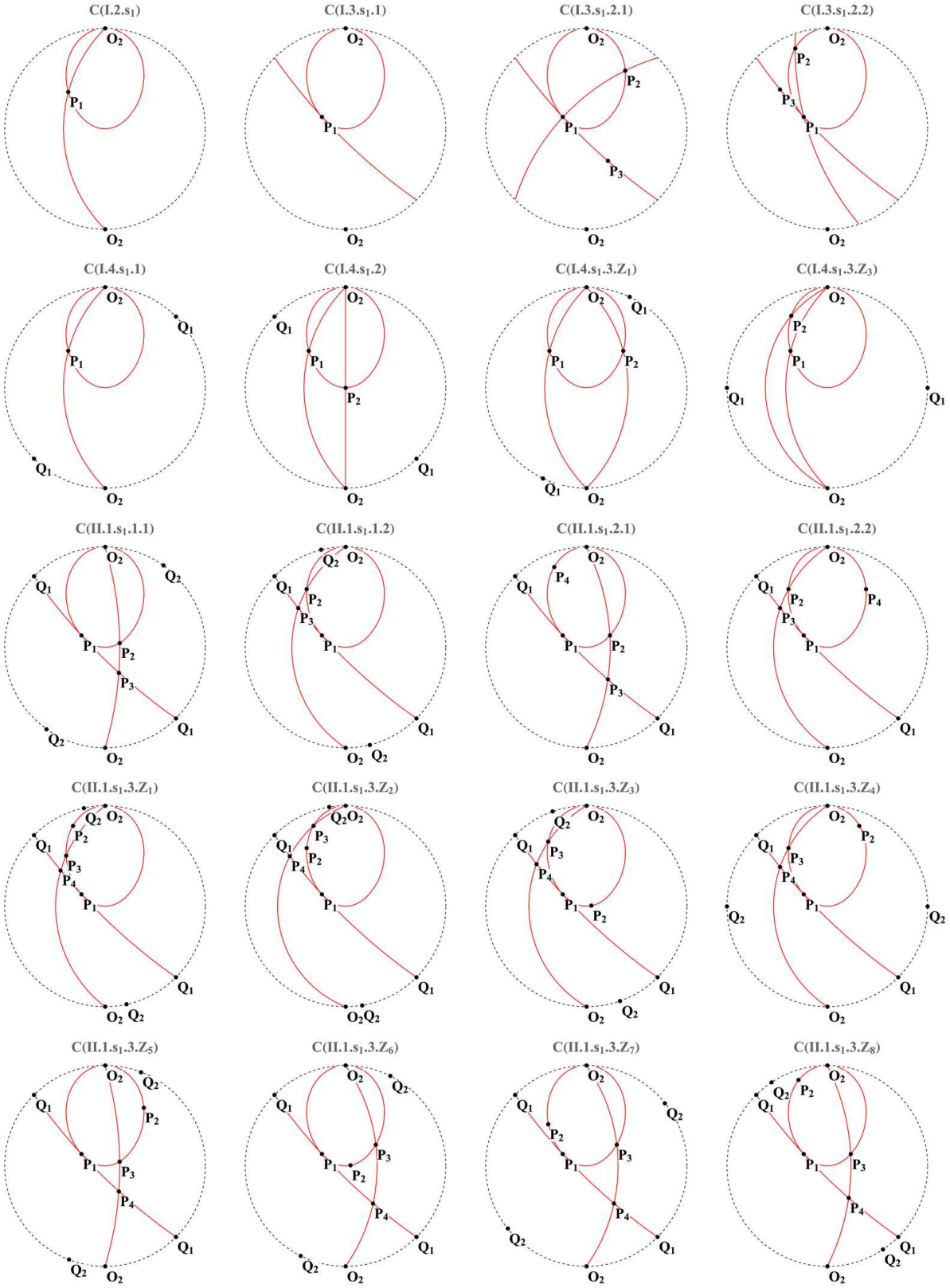


Figure 6.2: Configurations of the invariant algebraic curves of the systems of Theorem 3.1. Here P_i for $i = 1,2,3,4$ are equilibria in the plane $\mathbb{R}^2$, Q_i for $i = 1,2$ are equilibria in the chart U_1 and O_2 is the origin of the chart U_2.

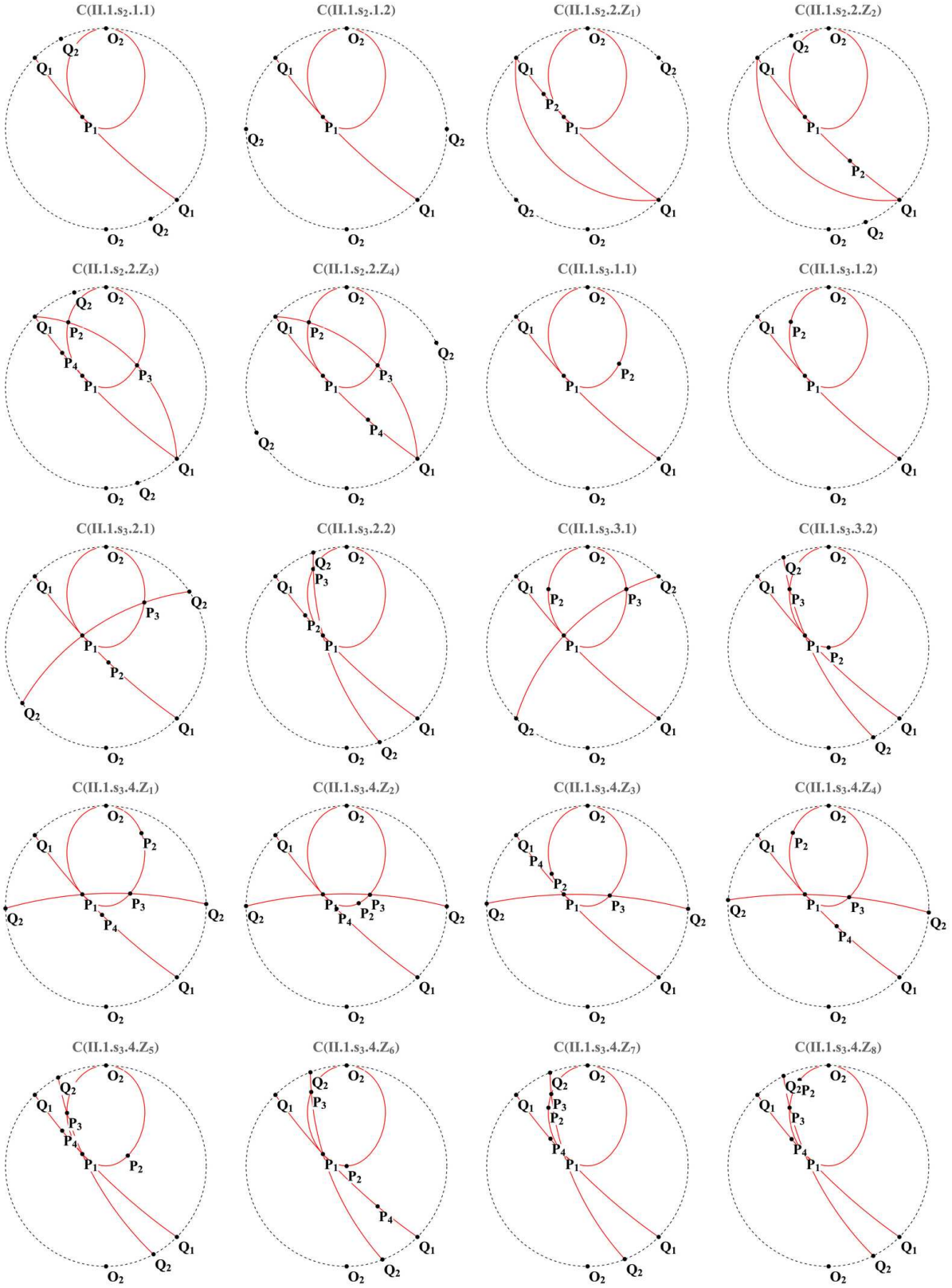


Figure 6.3: Configurations of the invariant algebraic curves of the systems of Theorem 3.1. Here P_i for $i = 1,2,3,4$ are equilibria in the plane $\mathbb{R}^2$, Q_i for $i = 1,2$ are equilibria in the chart U_1 and O_2 is the origin of the chart U_2.

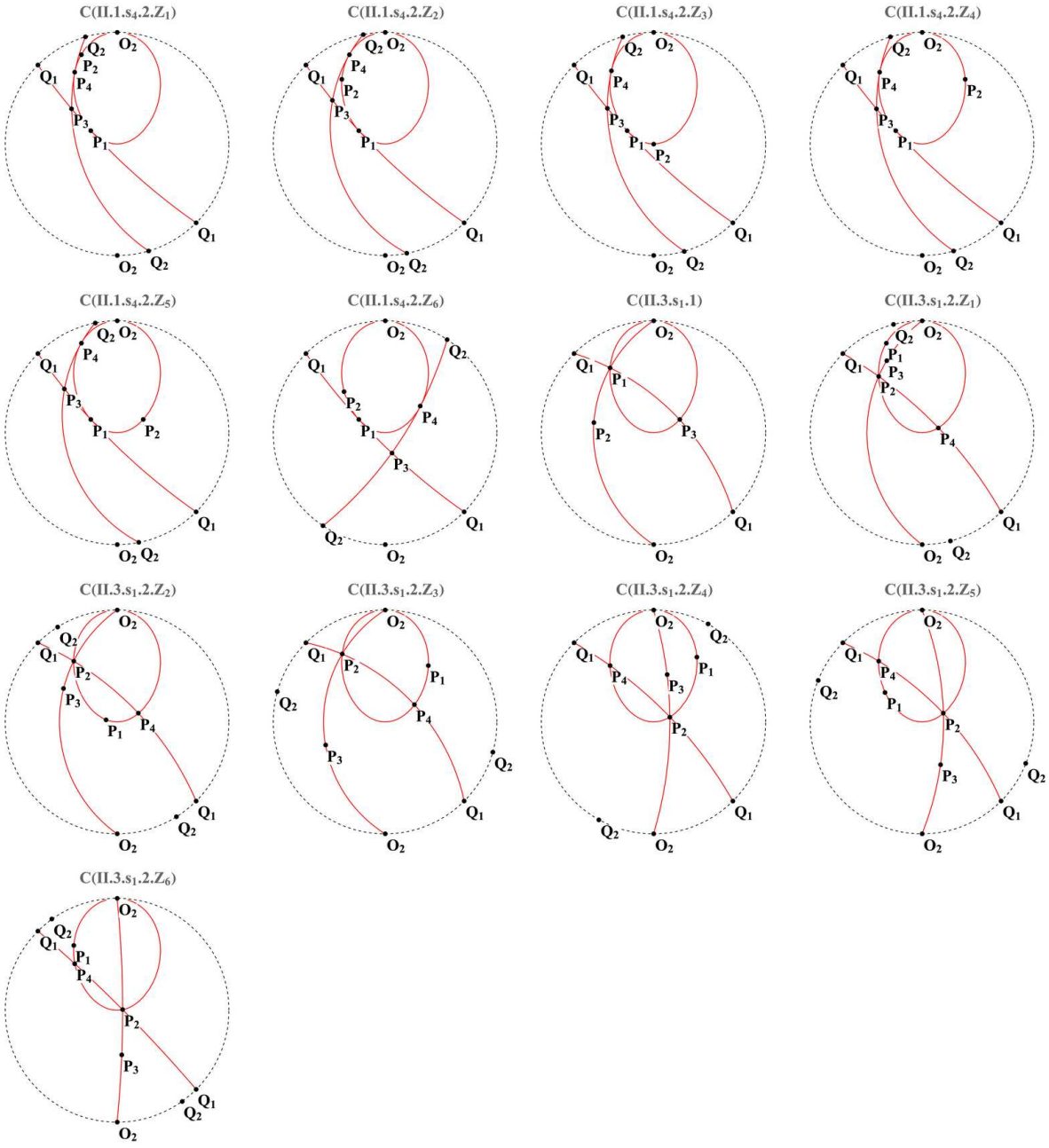


Figure 6.4: Configurations of the invariant algebraic curves of the systems of Theorem 3.1. Here P_i for $i = 1,2,3,4$ are equilibria in the plane $\mathbb{R}^2$, Q_i for $i = 1,2$ are equilibria in the chart U_1 and O_2 is the origin of the chart U_2.

Taking into account the blowing-down $B(\mathrm{I.2.s}_1)$ and the configuration $C(\mathrm{I.2.s}_1)$, we obtain the phase portrait PP.1 of Figure 3.1.

System $S(2)$**.** The system together with its invariant curves is

$$\dot{x} = -4x^2 - 2ax + 2y - a + \frac{1}{2}, \quad \dot{y} = -4xy + (1-2a)\,x - 4ay.$$

$$f_1 = y - x^2, \quad f_2 = 4x + 4y + 1, \quad f_3 = 4ax + 4y + 2a - 1.$$

We divide the study of the phase portraits of system $S(2)$ into two cases: $a = 1$ and $a \neq 1$.

Case 1. $a = 1$**.** Then system $S(2)$ reduces to the system

$$\dot{x} = -4x^2 - 2x + 2y - \frac{1}{2}, \qquad \dot{y} = -4xy - x - 4y.$$

This system has a finite equilibrium at $P_1 = \left(-\frac{1}{2}, \frac{1}{4}\right) : 0, 0$. It is a nilpotent equilibrium and satisfies condition (4.iii.iii2) of Theorem 3.5 in [11], then its phase portrait is formed by one hyperbolic and one elliptic sector.

The Poincaré compactification of this system in the chart U_1 is

$$\dot{u} = \frac{1}{2}uv^2 - 2u^2v - 2uv - v, \qquad \dot{v} = \frac{1}{2}v^3 - 2uv^2 + 2v^2 + 4v.$$

Every point at infinity is an equilibrium. Removing the common factor v between $\dot{u}$ and $\dot{v}$, we obtain the system

$$\dot{u} = -2u^2 + \frac{1}{2}uv - 2u - 1, \qquad \dot{v} = \frac{1}{2}v^2 - 2uv + 2v + 4.$$

It has no equilibria on $v = 0$.

The Poincaré compactification in the chart U_2 is

$$\dot{u} = u^2v + 2uv - \frac{1}{2}v^2 + 2v, \qquad \dot{v} = uv^2 + 4uv + 4v^2.$$

Removing the common factor v between $\dot{u}$ and $\dot{v}$, we obtain the system

$$\dot{u} = u^2 + 2u - \frac{1}{2}v + 2, \qquad \dot{v} = uv + 4u + 4v.$$

There is an orbit tangent to the line at infinity at the origin O_2.

Since $y - x^2 = 0$ and $4x + 4y + 1 = 0$ are invariant curves of this system, we obtain the configuration $C(\mathrm{I.3.s}_{1.1})$ shown in Figure 6.2. Based on this configuration and the local phase portrait of the equilibrium P_1, we obtain the phase portrait PP.2 of Figure 3.1.

Case 2. $a \neq 1$**.** System $S(2)$ has three finite equilibria and their corresponding eigenvalues are

$$P_1 = \left(-\frac{1}{2}, \frac{1}{4}\right) : -4(a-1), -2(a-1);$$
$$P_2 = \left(-\frac{1}{2}(2a-1), \frac{1}{4}(2a-1)^2\right) : 2(a-1), 4(a-1);$$
$$P_3 = \left(-\frac{a}{2}, \frac{1}{4}(2a-1)\right) : -2(a-1), 2(a-1).$$

The Poincaré compactification of this system in the chart U_1 is

$$\dot{u} = \frac{2a-1}{2}uv^2 - 2u^2v - 2auv + (1-2a)v, \qquad \dot{v} = -2uv^2 + 2av^2 + \frac{2a-1}{2}v^3 + 4v.$$

Every point at infinity is an equilibrium.

The Poincaré compactification of this system in the chart U_2 is

$$\dot{u} = (2a-1)u^2v + 2auv + \left(\frac{1}{2} - a\right)v^2 + 2v, \qquad \dot{v} = (2a-1)uv^2 + 4av^2 + 4uv.$$

Removing the common factor v, we find an orbit tangent to the line at infinity at the origin O_2. Since $y - x^2 = 0$, $4x + 4y + 1 = 0$ and $4ax + 4y + 2a - 1 = 0$ are invariant algebraic curves of the system, we obtain the configurations $C(\text{I.3.s}_{1.1})$ and $C(\text{I.3.s}_{1.2})$, as shown in Figure 6.2.

Table 6.1 shows the local phase portraits of the finite equilibria and the corresponding configurations and phase portraits in the Poincaré disc according to the parameter a.

Zone	P_1	P_2	P_3	Configuration	PP
$a > 1$	Ns	Nu	S	$C(\text{I.3.s}_1\text{.2.2})$	PP.3
$a < 1$	Nu	Ns	S	$C(\text{I.3.s}_1\text{.2.1})$	PP.3

Table 6.1: Types of the finite and infinite equilibria, together with their corresponding local phase portraits, configurations, and global phase portraits in the Poincaré disc for Case 2 of the system $S(2)$. The abbreviations "Ns", "Nu", and "S" denote "stable node", "unstable node" and "saddle", respectively.

System $S(3)$**.** The system together with its invariant curves is

$$\dot{x} = -x^2 + (a+b)\,x - ab, \quad \dot{y} = (2b-1)\,x^2 - 2xy + (2a+1)\,y - 2abx,$$

$$f_1 = y - x^2, \quad f_2 = x - a, \quad f_3 = x - b,$$

where $(2a - 2b + 1) \neq 0$.

We consider the following four cases: 1. $a = b \neq 0$; 2. $a = b = 0$; 3. $ab = 0$ and $a^2 + b^2 \neq 0$; 4. $ab(a-b) \neq 0$.

Case 1. $a = b \neq 0$ and $(2a - 2b + 1) \neq 0$. Let $c \neq 0$. The transformation $(x,\ y,\ t,\ a,\ b) \to \left(-\frac{x}{c},\ \frac{y}{c^2},\ \frac{t}{c},\ \frac{1}{c},\ \frac{1}{c}\right)$ reduces the system to $S(\text{I.4.s}_{1.1})$:

$$\dot{x} = (x+1)^2, \quad \dot{y} = (2-c)x^2 + 2xy + 2x + (c+2)y, \quad f_1 = y - x^2, \quad f_2 = x + 1.$$

We now analyse the phase portraits of the system $S(\text{I.4.s}_{1.1})$. There is a finite equilibrium $P_1 = (-1,\ 1) : 0,\ c$. According to Theorem 2.19 of [11], the point is a saddle-node.

The Poincaré compactification of this system in the chart U_1 is

$$\dot{u} = -uv^2 + cuv + u + 2v - c + 2, \qquad \dot{v} = -v^3 - 2v^2 - v.$$

There is an equilibrium $Q_1 = (c+2,\ 0) : -1,\ 1$.

The Poincaré compactification of this system in the chart U_2 is

$$\dot{u} = (c-2)u^3 - 2u^2v - u^2 - cuv + v^2, \qquad \dot{v} = (c-2)u^2v - 2uv^2 - 2uv - (c+2)v^2.$$

The origin O_2 is linearly zero. Doing a vertical blow-up $(u, v) \to (u_1, u_1v_1)$ on the system, followed by a time rescaling $(\dot{u}_1, \dot{v}_1) \to \left(\frac{\dot{u}_1}{u_1}, \frac{\dot{v}_1}{u_1}\right)$, we obtain the system:

$$\dot{u}_1 = cu_1^2 - 2u_1^2v_1 + u_1v_1^2 - (c+2)u_1v_1 - u_1, \qquad \dot{v}_1 = -v_1^3 - 2v_1^2 - v_1.$$

There are two equilibria on $u_1 = 0$:

$$(0,-1) : 0,\ c; \quad (0,0) : -1,\ -1.$$

According to Theorem 2.19 of [11], $(0,-1)$ is a saddle-node.

By carrying out the blow-downs $B(\text{I.4.s}_{1.1}.k)$ for $k = 1, 2$, we obtain the local phase portrait of O_2.

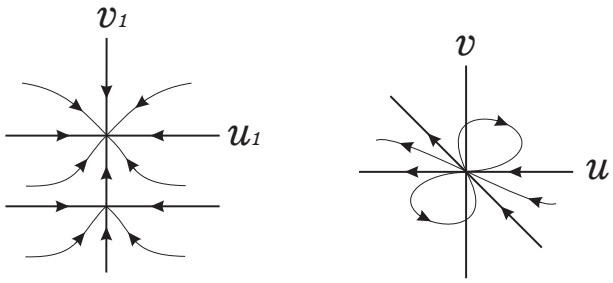


Figure 6.5: $B(\text{I.4.s}_1.1.1)$: Blowing-down at O_2 of the system $S(\text{I.4.s}_{1.1})$ with $c < 0$.

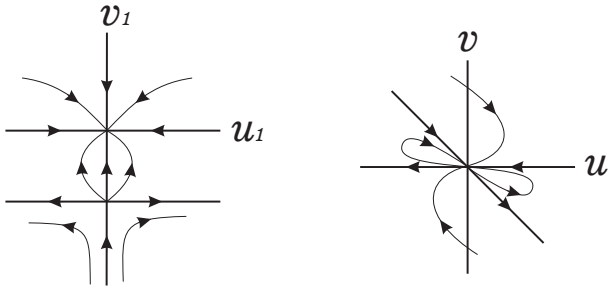


Figure 6.6: $B(\text{I.4.s}_1.1.2)$: Blowing-down at O_2 of the system $S(\text{I.4.s}_{1.1})$ with $c > 0$.

Table 6.2 shows the local phase portraits of the finite and infinite equilibria. By analysing the local behaviour at O_2, we obtain the phase portraits in the Poincaré disc.

Zone	P_1	P_2	Q_1	O_2	Blow-up	Configuration	PP
$c < 0$	SN	S	S	LN	$B(\text{I.4.s}_1.1.1)$	$C(\text{I.4.s}_1.1)$	PP.4
$c > 0$	SN	S	S	LN	$B(\text{I.4.s}_1.1.2)$	$C(\text{I.4.s}_1.1)$	PP.4

Table 6.2: Types of the finite and infinite equilibria, together with their corresponding local phase portraits, configurations, and global phase portraits in the Poincaré disc for Case 1 of the system $S(3)$. The abbreviations "SN" and "LN" denote "saddle-node" and "linearly zero equilibrium", respectively.

Case 2. $a = b = 0$ and $(4a - 2b + 1)(2a + 1) \neq 0$. Then system $S(3)$ simplifies to

$$\dot{x} = -x^2, \quad \dot{y} = -x^2 - 2xy + y.$$

Taking $c = -1$ in the system $S(\text{I.4.s}_1.1)$ and applying the transformation $(x, y) \to (x - 1, y - 2x + 1)$, we obtain the above system. Thus, it is a particular case of Case 1 with $a = b = -1$.

Case 3. $ab = 0$ and $(a^2 + b^2)(2a - 2b + 1) \neq 0$. If $b \neq 0$, then system $S(3)$ for $a = 0$ simplifies to $S(\text{I.4.s}_1.a = 0)$

$$\dot{x} = -x(x - b), \quad \dot{y} = (2b - 1)x^2 - 2xy + y.$$

Let $c \neq 0$, doing the transformation $(x, y, t, b) \to \left(-\frac{x}{c}, \frac{y}{c^2}, ct, \frac{1}{c}\right)$ to $S(\text{I.4.s}_1.a = 0)$ yields $S(\text{I.4.s}_1.2)$:

$$\dot{x} = x(x+1), \quad \dot{y} = (2-c)x^2 + 2xy + cy,$$
$$f_1 = y - x^2, \quad f_2 = x, \quad f_3 = x + 1.$$

Since $2a - 2b + 1 \neq 0$, $a = 0$ and $b = \dfrac{1}{c}$, here we have $c(c-2) \neq 0$.

If $b = 0$ then system $S(3)$ simplifies to $S(\text{I.4.s}_1.b = 0)$

$$\dot{x} = -x(x-a), \quad \dot{y} = -x^2 - 2xy + (2a+1)y.$$

Let $c - 2 \neq 0$, doing the transformation $(x, y, t, a) \to \left(\frac{-x}{c-2}, \frac{y}{(c-2)^2}, (c-2)t, \frac{1}{c-2}\right)$ to $S(\text{I.4.s}_1.b = 0)$ yields $S(\text{I.4.s}_1.2)$. Thus, we only have to consider system $S(\text{I.4.s}_1.2)$ in this case.

The system $S(\text{I.4.s}_1.2)$ has two finite equilibria

$$P_1 = (-1,\ 1) : -1,\ c-2; \quad P_2 = (0,\ 0) : 1,\ c.$$

The Poincaré compactification of this system in the chart U_1 is

$$\dot{u} = (c-1)uv + u - c + 2, \qquad \dot{v} = -v^2 - v.$$

Hence, there is an infinite equilibrium $Q_1 = (c-2, 0) : -1, 1$ in the chart U_1.

The Poincaré compactification of this system in the chart U_2 is

$$\dot{u} = (c-2)u^3 - u^2 + (1-c)uv, \qquad \dot{v} = (c-2)u^2v - 2uv - cv^2.$$

The origin O_2 is linearly zero. Since $u = 0$ is a characteristic direction of the equilibrium, we perform the twist $(u, v) \to (u_1 - v_1, v_1)$, and obtain the following system:

$$\dot{u}_1 = (c-2)u_1^3 + 2(2-c)u_1^2v_1 + (c-2)u_1v_1^2 - u_1^2 + (1-c)u_1v_1,$$
$$\dot{v}_1 = (c-2)u_1^2v_1 - 2(c-2)u_1v_1^2 + (c-2)v_1^3 - 2u_1v_1 + (2-c)v_1^2.$$

By doing the vertical blow-up $(u_1, v_1) \to (u_2, u_2v_2)$ and eliminating the common factor u_2 between $\dot{u}_2$ and $\dot{v}_2$ doing a time rescaling, we obtain the system

$$\dot{u}_2 = (c-2)u_2^2v_2^2 - 2(c-2)u_2^2v_2 + (1-c)u_2v_2 + (c-2)u_2^2 - u_2, \qquad \dot{v}_2 = v_2^2 - v_2.$$

This system has two equilibria on the straight line $u_2 = 0$

$$(0,\ 0) : -1,\ -1; \quad (0,\ 1) : 1,\ -c.$$

By doing the blow-downs $B(\text{I.4.s}_1.2.k)$ for $k = 1, 2, 3$, we obtain the local phase portrait of O_2.

Since $y - x^2 = 0$, $x = 0$, and $x + 1 = 0$ are invariant algebraic curves of the system, we obtain the configuration $C(\text{I.4.s}_1.2)$.

Table 6.3 shows the local phase portraits of the finite and infinite equilibria, the process of blowing-down at O_2, and the corresponding phase portraits in the Poincaré disc.

Case 4. $ab(a-b)(2a-2b+1) \neq 0$. Assume that $c(c-1)(d-2c) \neq 0$, doing the following transformation to system $S(3)$

$$(x, y, t, a, b) \to \left(\frac{-x}{d-2c}, \frac{y}{(d-2c)^2}, (d-2c)t, \frac{c}{d-2c}, \frac{1}{d-2c}\right),$$

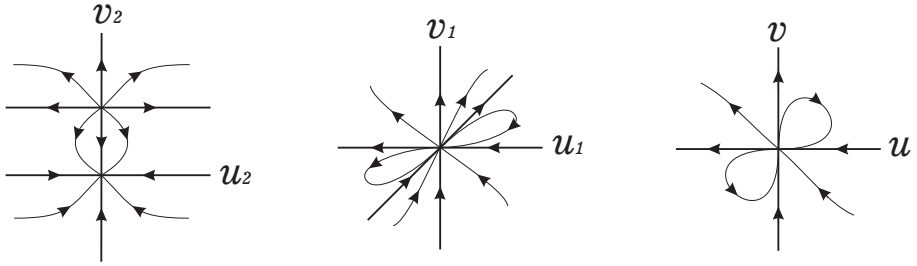


Figure 6.7: $B(\text{I.4.s}_1\text{.2.1})$: Blowing-down at O_2 of the system $S(\text{I.4.s}_1\text{.2})$ with $c < 0$.

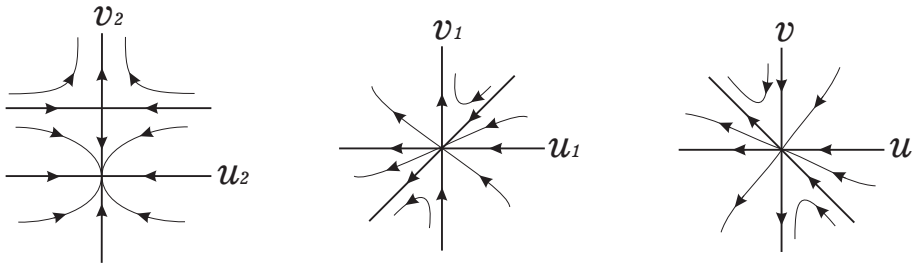


Figure 6.8: $B(\text{I.4.s}_1\text{.2.2})$: Blowing-down at O_2 of the system $S(\text{I.4.s}_1\text{.2})$ with $0 < c < 2$.

we obtain the system $S(\text{I.4.s}_1\text{.3})$

$$\dot{x} = (x+1)(x+c), \quad \dot{y} = (2c-d+2)\,x^2 + 2xy + 2cx + dy,$$
$$f_1 = y - x^2, \quad f_2 = x + c, \quad f_3 = x + 1.$$

Since $(2a - 2b + 1) \neq 0$ in system $S(3)$, it follows that $(d-2) \neq 0$ in $S(\text{I.4.s}_1\text{.3})$. This further implies that the full parameter condition is $c(c-1)(d-2c)(d-2) \neq 0$.

The system has two finite equilibria

$$P_1 = (-1,\ 1) : c-1,\ d-2; \quad P_2 = (-c,\ c^2) : 1-c,\ d-2c.$$

The Poincaré compactification of this system in the chart U_1 is

$$\dot{u} = -cuv^2 - (c-d+1)uv + 2cv + u + 2c - d + 2, \qquad \dot{v} = -cv^3 - (c+1)v^2 - v.$$

Hence, there is an infinite equilibrium $Q_1 = (d-2c-2, 0) : (-1, 1)$ in the local chart U_1.

The Poincaré compactification of this system in the chart U_2 is

$$\dot{u} = (d-2c-2)u^3 - 2cu^2v - u^2 + (c-d+1)uv + cv^2,$$
$$\dot{v} = (d-2c-2)u^2v - 2cuv^2 - 2uv - dv^2.$$

The origin is linearly zero. By doing the vertical blow-up $(u, v) \to (u_1, u_1v_1)$ and eliminating the common factor u_1 between $\dot{u}_1$ and $\dot{v}_1$ doing a time rescaling, we obtain the system

$$\dot{u}_1 = (d-2c-2)u_1^2 - 2cu_1^2v_1 + cu_1v_1^2 + (c-d+1)u_1v_1 - u_1,$$
$$\dot{v}_1 = -cv_1^3 - (c+1)v_1^2 - v_1.$$

This system has three equilibria on the straight line $u_1 = 0$

$$(0,\ -1) : 1-c,\ d-2; \quad (0,\ 0) : -1,\ -1; \quad \left(0,\ -\frac{1}{c}\right) : \frac{c-1}{c},\ \frac{d-2c}{c}.$$

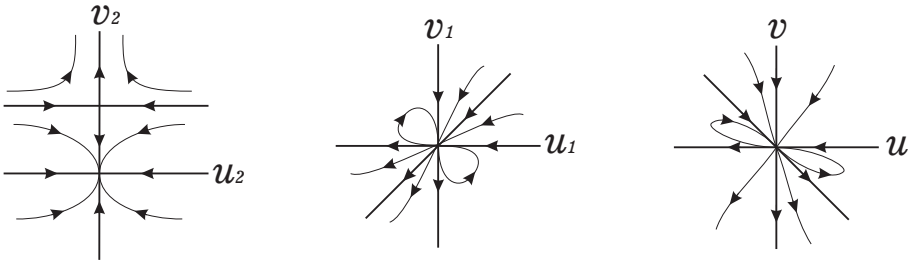


Figure 6.9: $B(\text{I.4.s}_1\text{.2.3})$: Blowing-down at O_2 of the system $S(\text{I.4.s}_1\text{.2})$ with $c > 2$.

Zone	P_1	P_2	Q_1	O_2	Blow-up	Configuration	PP
$c < 0$	Ns	S	S	LN	$B(\text{I.4.s}_1\text{.2.1})$	$C(\text{I.4.s}_1\text{.2})$	PP.5
$0 < c < 2$	Ns	Nu	S	LN	$B(\text{I.4.s}_1\text{.2.2})$	$C(\text{I.4.s}_1\text{.2})$	PP.6
$c > 2$	S	Nu	S	LN	$B(\text{I.4.s}_1\text{.2.3})$	$C(\text{I.4.s}_1\text{.2})$	PP.5

Table 6.3: Types of the finite and infinite equilibria, together with their corresponding local phase portraits, configurations, and global phase portraits in the Poincaré disc for Case 3 of the system $S(3)$.

According to the expressions of the eigenvalues, and under the condition $c(c-1)(2c-d)(d-2) \neq 0$, we divide the parameter space (c,d) into nine regions Z_k for $k = 1,2,\ldots,9$, as shown in Figure 6.10.

By doing the blow-downs $B(\text{I.4.s}_1\text{.3.}Z_k)$ for $k = 1,2,4,5,8,9$, we obtain the local phase portrait at O_2.

Since $y - x^2 = 0, x + c = 0$, and $x + 1 = 0$ are invariant curves of the system, we obtain the configurations $C(\text{I.4.s}_1\text{.3.}Z_1)$ and $C(\text{I.4.s}_1\text{.3.}Z_3)$.

Table 6.4 shows the types of finite and infinite equilibria, the process of blowing-down at O_2, and the corresponding global phase portraits in different regions.

Zone	P_1	P_2	Q_1	O_2	Blow-up	Configuration	PP
Z_1	S	Nu	S	LN	$B(\text{I.4.s}_1\text{.3.}Z_1)$	$C(\text{I.4.s}_1\text{.3.}Z_1)$	PP.5
Z_2	S	Nu	S	LN	$B(\text{I.4.s}_1\text{.3.}Z_2)$	$C(\text{I.4.s}_1\text{.3.}Z_1)$	PP.5
Z_3	Nu	S	S	LN	$B(\text{I.4.s}_1\text{.3.}Z_2)$	$C(\text{I.4.s}_1\text{.3.}Z_3)$	PP.5
Z_4	Nu	Ns	S	LN	$B(\text{I.4.s}_1\text{.3.}Z_4)$	$C(\text{I.4.s}_1\text{.3.}Z_3)$	PP.6
Z_5	S	Ns	S	LN	$B(\text{I.4.s}_1\text{.3.}Z_5)$	$C(\text{I.4.s}_1\text{.3.}Z_3)$	PP.5
Z_6	Ns	S	S	LN	$B(\text{I.4.s}_1\text{.3.}Z_5)$	$C(\text{I.4.s}_1\text{.3.}Z_1)$	PP.5
Z_7	Ns	Nu	S	LN	$B(\text{I.4.s}_1\text{.3.}Z_4)$	$C(\text{I.4.s}_1\text{.3.}Z_1)$	PP.6
Z_8	Ns	S	S	LN	$B(\text{I.4.s}_1\text{.3.}Z_8)$	$C(\text{I.4.s}_1\text{.3.}Z_1)$	PP.5
Z_9	Ns	Nu	S	LN	$B(\text{I.4.s}_1\text{.3.}Z_9)$	$C(\text{I.4.s}_1\text{.3.}Z_1)$	PP.6

Table 6.4: Types of the finite and infinite equilibria, together with their corresponding local phase portraits, configurations, and global phase portraits in the Poincaré disc for Case 4 of system $S(3)$.

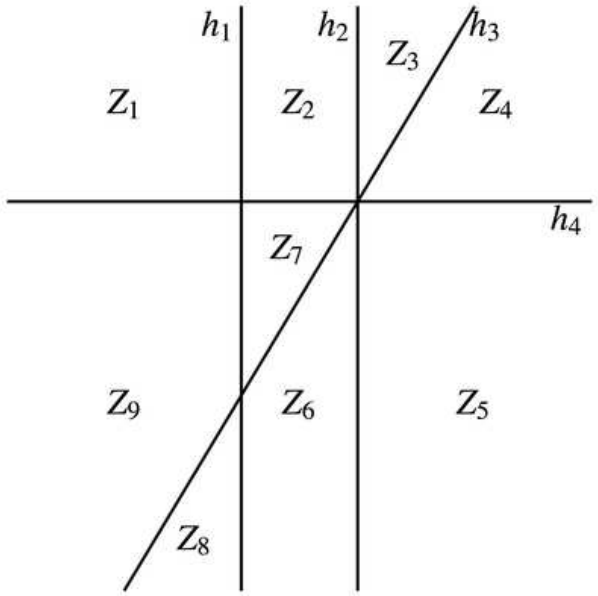


Figure 6.10: Zones of the parameter space (c, d), bounded by the straight lines $h_1 : c = 0$, $h_2 : c = 1$, $h_3 : d = 2c$, and $h_4 : d = 2$.

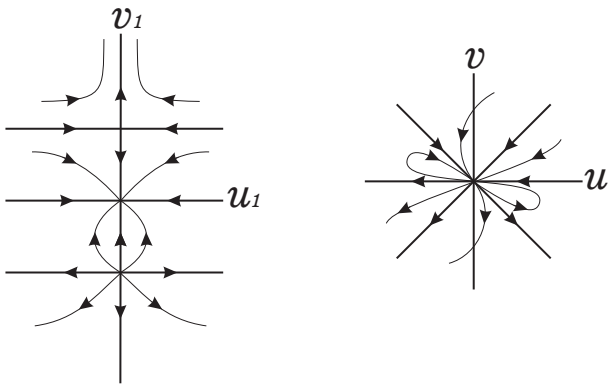


Figure 6.11: $B(\text{I.4.s}_1\text{.3.}Z_1)$: Blowing-down at O_2 of the system $S(\text{I.4.s}_1\text{.3})$ with (c, d) in Z_1.

System $S(4)$**.** The system together with its invariant curves is

$$\dot{x} = 8(a+1)\,x^2 + 8xy + 2(4ab+4a+4b+3)\,x + 4(2b+1)\,y + (4ab+2a+2b+1),$$
$$\dot{y} = 8(a+2b+1)\,x^2 + 8(2a+2b+3)\,xy + 2(4ab+2a+2b+1)\,x + 4(4ab+2a+1)\,y + 16y^2,$$

$$f_1 = y - x^2, \quad f_2 = 4x + 4y + 1, \quad f_3 = 2x + 2b + 1,$$

where $ab \neq 0$.

We consider the following three cases: 1. $a = b \neq 0$; 2. $a + 2b = 0$; 3. $(a - b)(a + 2b) \neq 0$.

Case 1. $a = b \neq 0$. The system simplifies to

$$\dot{x} = 8(b+1)x^2 + 8xy + 2(2b+1)(2b+3)x + 4(2b+1)y + (2b+1)^2,$$
$$\dot{y} = 8(3b+1)x^2 + 8(4b+3)xy + 2(2b+1)^2x + 4(4b^2+2b+1)y + 16y^2.$$

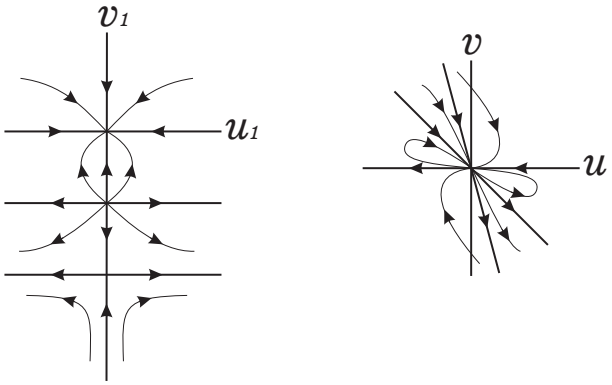


Figure 6.12: $B(\text{I.4.s}_1\text{.3}.Z_2)$: Blowing-down of the origin on chart U_2 for system $S(\text{I.4.s}_1\text{.3})$ with (c,d) in Z_2, Z_3.

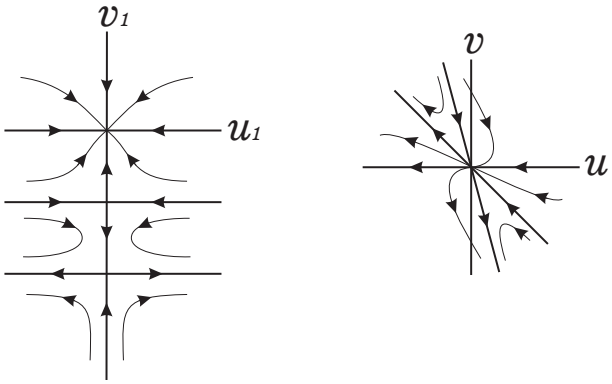


Figure 6.13: $B(\text{I.4.s}_1\text{.3}.Z_4)$: Blowing-down at O_2 of the system $S(\text{I.4.s}_1\text{.3})$ with (c,d) in Z_4, Z_7.

There are three equilibria

$$P_1 = \left(-\frac{1}{2},\ \frac{1}{4}\right) : 8b^2,\ 16b^2; \quad P_2 = \left(-\frac{1}{2}(2b+1),\ \frac{1}{4}(2b+1)^2\right) : 0,\ 16b^2,$$
$$P_3 = \left(-b-\frac{1}{2},\ \frac{b}{4}\right) : -16b^2,\ -8b^2.$$

The Poincaré compactification in the chart U_1 is

$$\begin{aligned}\dot{u} &= 8u^2 - 4(2b+1)u^2v + 2(4b^2-4b-1)uv - (2b+1)^2uv^2 + 8(3b+2)u \\ &\quad + 2(2b+1)^2v + 8 + 24b, \\ \dot{v} &= -(2b+1)^2v^3 - 4(2b+1)uv^2 - 2(2b+1)(2b+3)v^2 - 8uv - 8(b+1)v.\end{aligned}$$

There are two equilibria:

$$Q_1 = (-1,\ 0) : -8b,\ 24b; \quad Q_2 = (-3b-1,\ 0) : -24b,\ 16b.$$

The Poincaré compactification in the chart U_2 is

$$\begin{aligned}\dot{u} &= -8(3b+1)u^3 - 2(2b+1)^2u^2v - 8(3b+2)u^2 - 2(4b^2-4b-1)uv - 8u \\ &\quad + (2b+1)^2v^2 + 4(2b+1)v, \\ \dot{v} &= -8(3b+1)u^2v - 2(2b+1)^2uv^2 - 8(4b+3)uv - 4(4b^2+2b+1)v^2 - 16v.\end{aligned}$$

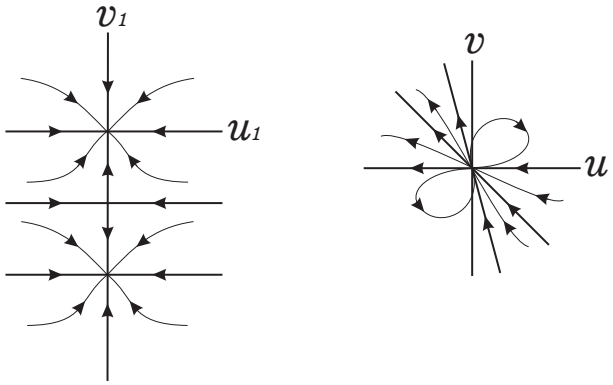


Figure 6.14: $B(\text{I.4.s}_1\text{.3.}Z_5)$: Blowing-down at O_2 of the system $S(\text{I.4.s}_1\text{.3})$ with (c, d) in Z_5, Z_6.

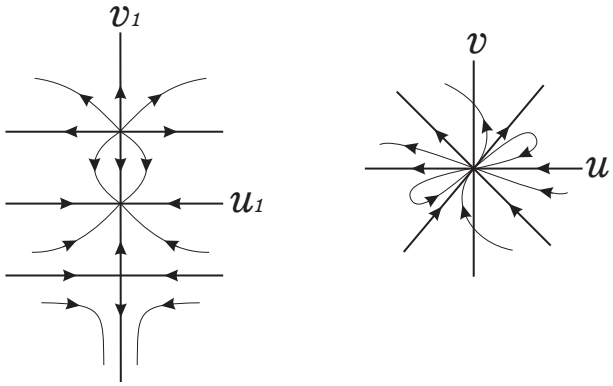


Figure 6.15: $B(\text{I.4.s}_1\text{.3.}Z_8)$: Blowing-down at O_2 of the system $S(\text{I.4.s}_1\text{.3})$ with (c, d) in Z_8.

The origin is an equilibrium with eigenvalues: $-16, -8$.

Since $y - x^2 = 0, 4x + 4y + 1 = 0$, and $2x + 2b + 1 = 0$ are invariant curves of the system, we obtain the configurations $C(\text{II.1.s}_1\text{.1.1})$ and $C(\text{II.1.s}_1\text{.1.2})$.

Table 6.5 shows the different types of finite and infinite equilibria, the corresponding configurations, and the associated global phase portraits.

Zone	P_1	P_2	P_3	Q_1	O_2	Configuration	PP
$a = b < 0$	Nu	SN	Ns	S	Ns	$C(\text{II.1.s}_1\text{.1.1})$	PP.7
$a = b > 0$	Nu	SN	Ns	S	Ns	$C(\text{II.1.s}_1\text{.1.2})$	PP.7

Table 6.5: Types of the finite and infinite equilibria, together with their corresponding local phase portraits, configurations, and global phase portraits in the Poincaré disc for Case 1 of the system $S(4)$.

Case 2. $a + 2b = 0$ and $ab \neq 0$. The system simplifies to

$$\dot{x} = -8(2b-1)x^2 + 8xy - 2(8b^2 + 4b - 3)x + 4(2b+1)y - (2b+1)(4b-1),$$
$$\dot{y} = 8x^2 - 8(2b-3)xy - 2(2b+1)(4b-1)x - 4(8b^2 + 4b - 1)y + 16y^2.$$

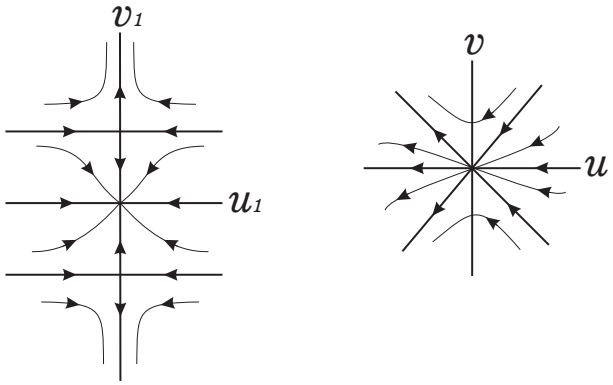


Figure 6.16: $B(\text{I.4.s}_1\text{.3.}Z_9)$: Blowing-down at O_2 of the system $S(\text{I.4.s}_1\text{.3})$ with (c,d) in Z_9.

This system has four equilibria

$$
\begin{aligned}
P_1 &= \left(-\frac{1}{2},\ \frac{1}{4}\right) : -32b^2,\ -16b^2;\\
P_2 &= \left(-\frac{1}{2}(2b+1),\ \frac{1}{4}(2b+1)^2\right) : 16b^2,\ 24b^2;\\
P_3 &= \left(-b-\frac{1}{2},\ b+\frac{1}{4}\right) : -16b^2,\ 16b^2;\\
P_4 &= \left(-\frac{1}{2}(1-4b),\ \frac{1}{4}(1-4b)^2\right) : -32b^2,\ 48b^2.
\end{aligned}
$$

The Poincaré compactification in the chart U_1 is

$$
\begin{aligned}
\dot{u} &= -4(2b+1)u^2v + (2b+1)(4b-1)uv^2 - 2(8b^2+4b+1)uv + 8u^2 + 16u\\
&\quad -2(2b+1)(4b-1)v + 8,\\
\dot{v} &= (2b+1)(4b-1)v^3 - 4(2b+1)uv^2 + 2(8b^2+4b-3)v^2 - 8uv + 8(2b-1)v.
\end{aligned}
$$

There is an equilibrium $Q_1 = (-1,0) : 0, 16b$.

The Poincaré compactification in the chart U_2 is

$$
\begin{aligned}
\dot{u} &= -8u^3 + 2(2b+1)(4b-1)u^2v - 16u^2 + 2(8b^2+4b+1)uv - 8u\\
&\quad -(2b+1)(4b-1)v^2 + 4(2b+1)v,\\
\dot{v} &= -8u^2v + 8(2b-3)uv + 2(2b+1)(4b-1)uv^2 + 4(8b^2+4b-1)v^2 - 16v.
\end{aligned}
$$

The origin is an equilibrium with eigenvalues: $-16, -8$.

Since $y - x^2 = 0, 4x + 4y + 1 = 0$, and $2x + 2b + 1 = 0$ are invariant curves of the system, we obtain the configurations $C(\text{II.1.s}_1\text{.2.1})$ and $C(\text{II.1.s}_1\text{.2.2})$.

Table 6.6 shows the types of finite and infinite equilibria, the corresponding configurations, and the associated global phase portraits.

Zone	P_1	P_2	P_3	P_4	Q_1	O_2	Configuration	PP
$a=-2b>0$	Ns	Nu	S	S	SN	Ns	$C(\text{II.1.s}_1\text{.2.1})$	PP.8
$a=-2b<0$	Ns	Nu	S	S	SN	Ns	$C(\text{II.1.s}_1\text{.2.2})$	PP.8

Table 6.6: Types of the finite and infinite equilibria, together with their corresponding local phase portraits, configurations, and global phase portraits in the Poincaré disc for Case 2 of the system $S(4)$.

Case 3. $ab(a-b)(a+2b)\neq 0$. Then system $S(4)$ has four finite equilibria

$$\begin{aligned}
P_1&=\left(-\frac{1}{2},\ \frac{1}{4}\right):8ab,\ 16ab;\\
P_2&=\left(-\frac{1}{2}(2a+1),\ \frac{1}{4}(2a+1)^2\right):16ab,\ 8a(a-b);\\
P_3&=\left(-\frac{1}{2}(2b+1),\ \frac{1}{4}(2b+1)^2\right):8b(b-a),\ 16b^2;\\
P_4&=\left(-\frac{1}{2}(2b+1),\ \frac{1}{4}(4b+1)\right):-8ab,\ -16b^2.
\end{aligned}$$

The Poincaré compactification in the chart U_1 is

$$\begin{aligned}
\dot u&=-4(2b+1)u^2v-(2a+1)(2b+1)uv^2+2(4ab-4b-1)uv+2(2a+1)(2b+1)v,\\
&\quad +8u^2+8(a+2b+2)u+8(a+2b+1)\\
\dot v&=-4(2b+1)uv^2-(2a+1)(2b+1)v^3-2(4ab+4a+4b+3)v^2-8uv-8(a+1)v.
\end{aligned}$$

The system in the chart U_1 has two infinite equilibria

$$Q_1=(-1,\ 0):-8a,\ 8(a+2b);\quad Q_2=(-a-2b-1,\ 0):-8(a+2b),\ 16b.$$

The Poincaré compactification in the chart U_2 is

$$\begin{aligned}
\dot u&=-8(a+2b+1)u^3-2(2a+1)(2b+1)u^2v-8(a+2b+2)u^2+2(4b+1-4ab)uv\\
&\quad -8u+(2a+1)(2b+1)v^2+4(2b+1)v,\\
\dot v&=-2(2a+1)(2b+1)uv^2-4(4ab+2a+1)v^2-8(a+2b+1)u^2v\\
&\quad -8(2a+2b+3)uv-16v.
\end{aligned}$$

The origin O_2 is an infinite equilibrium with eigenvalues $-16,-8$.

According to the expressions of the eigenvalues, and under the condition $ab(a-b)(a+2b)\neq 0$, we divide the parameter space (a,b) into eight regions Z_k for $k=1,\dots,8$, as shown in Figure 6.17.

Since $y-x^2=0$, $4x+4y+1=0$, and $2x+2b+1=0$ are invariant algebraic curves of the system, we obtain the configurations $C(\text{II.1.s}_1\text{.3.}Z_k)$ for $k=1,2,\dots,8$.

Table 6.7 shows the types of finite and infinite equilibria, the corresponding configurations, and the corresponding phase portraits in the Poincaré disc of Figure 3.1.

System $S(5)$**.** The system together with its invariant curves is

$$\begin{aligned}
\dot x&=4(a-b+2)\,x^2+8xy+2(b+3)\,x-4(a-b-1)\,y+(b+1),\\
\dot y&=4(2-a+b)\,x^2+24xy+16y^2+2(b+1)\,x+4(a+1)\,y,
\end{aligned}$$

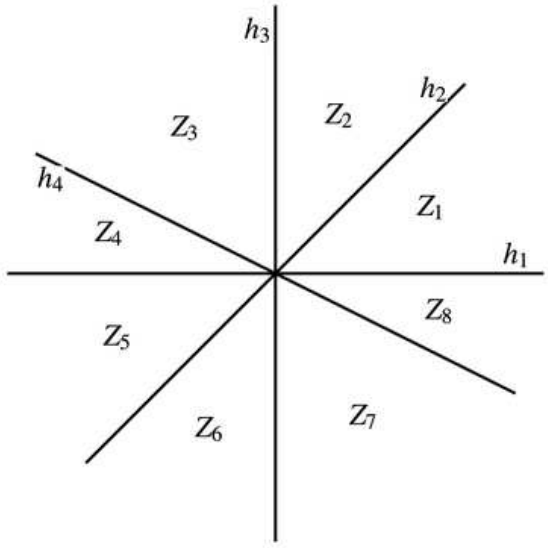


Figure 6.17: Zones of the parameter space (a, b), bounded by the straight lines $h_1 : b = 0$, $h_2 : a - b = 0$, $h_3 : a = 0$, and $h_4 : a + 2b = 0$.

$$f_1 = y - x^2, \quad f_2 = 4x + 4y + 1, \quad f_3 = 4x + 4y + b + 1,$$

where $a - b \neq 0$.

We consider two cases: 1. $b = 0$; 2. $b \neq 0$.

Case 1. $b = 0$ and $a - b \neq 0$. System $S(5)$ simplifies to

$$\begin{aligned} \dot{x} &= 4(a+2)x^2 + 8xy + 6x - 4(a-1)y + 1, \\ \dot{y} &= -4(a-2)x^2 + 24xy + 2x + 4(1+a)y + 16y^2. \end{aligned}$$

There is a finite equilibrium $P_1 = \left(-\frac{1}{2},\ \frac{1}{4}\right) : (0,\ 0)$. It satisfies condition (4.i.i2) in Theorem 3.5 of [11] and is therefore a saddle-node.

The Poincaré compactification in the chart U_1 is

$$\begin{aligned} \dot{u} &= 4(a-1)u^2v + 2(2a-1)uv - 4(a-4)u + 8u^2 - uv^2 + 2v - 4(a-2), \\ \dot{v} &= 4(a-1)uv^2 - 8uv - 4(a+2)v - 6v^2 - v^3. \end{aligned}$$

Hence, the system has two infinite equilibria in the chart U_1

$$P_1 = (-1,\ 0) : -4a,\ -4a; \quad P_2 = \left(\frac{1}{2}(a-2),\ 0\right) : -8a,\ 4a.$$

The Poincaré compactification in the chart U_2 is

$$\begin{aligned} \dot{u} &= 4(a-2)u^3 + 4(a-4)u^2 - 2u^2v - 2(2a-1)uv - 8u + v^2 - 4(a-1)v, \\ \dot{v} &= 4(a-2)u^2v - 2uv^2 - 24uv - 4(a+1)v^2 - 16v. \end{aligned}$$

The origin O_2 is an infinite equilibrium with eigenvalues: $-16, -8$.

Since $y - x^2 = 0$, $4x + 4y + 1 = 0$ are invariant curves of the system, we obtain the configurations $C(\text{II.1.s}_2\text{.1.1})$ and $C(\text{II.1.s}_2\text{.1.2})$.

Zone	P_1	P_2	P_3	P_4	Q_1	Q_2	O_2	Configuration	PP
Z_1	Nu	Nu	S	Ns	S	S	Ns	$C(\text{II.1.s1.3.Z}_1)$	PP.11
Z_2	Nu	S	Nu	Ns	S	S	Ns	$C(\text{II.1.s1.3.Z}_2)$	PP.11
Z_3	Ns	S	Nu	S	Nu	S	Ns	$C(\text{II.1.s1.3.Z}_3)$	PP.10
Z_4	Ns	S	Nu	S	S	Nu	Ns	$C(\text{II.1.s1.3.Z}_4)$	PP.9
Z_5	Nu	Nu	S	Ns	S	S	Ns	$C(\text{II.1.s1.3.Z}_5)$	PP.11
Z_6	Nu	S	Nu	Ns	S	S	Ns	$C(\text{II.1.s1.3.Z}_6)$	PP.11
Z_7	Ns	S	Nu	S	Ns	S	Ns	$C(\text{II.1.s1.3.Z}_7)$	PP.10
Z_8	Ns	S	Nu	S	S	Ns	Ns	$C(\text{II.1.s1.3.Z}_8)$	PP.9

Table 6.7: Types of the finite and infinite equilibria, together with their corresponding local phase portraits, configurations, and global phase portraits in the Poincaré disc for Case 3 of the system $S(4)$.

Zone	P_1	Q_1	Q_2	O_2	Configuration	PP
$b=0, a<0$	Ni	Nu	S	Ns	$C(\text{II.1.s}_2\text{.1.1})$	PP.14
$b=0, a>0$	Ni	Ns	S	Ns	$C(\text{II.1.s}_2\text{.1.2})$	PP.14

Table 6.8: Types of the finite and infinite equilibria, together with their corresponding local phase portraits, configurations, and global phase portraits in the Poincaré disc for Case 1 of the system $S(5)$. Here "Ni" stands for "nilpotent".

Table 6.8 shows the phase portraits of the finite and infinite equilibria, the corresponding configurations, and the associated global phase portraits in Figure 3.1.

Case 2. $b(a-b)\neq 0$. If $b<0$, then system $S(5)$ has four finite equilibria

$$
\begin{aligned}
P_1 &= \left(-\frac{1}{2},\ \frac{1}{4}\right) : 2b,\ 4b;\\
P_2 &= \left(-\frac{1}{2}\left(\sqrt{-b}+1\right),\ \frac{1}{4}\left(1+2\sqrt{-b}-b\right)\right) : -4(a-b)\sqrt{-b},\ -4b;\\
P_3 &= \left(\frac{1}{2}\left(\sqrt{-b}-1\right),\ \frac{1}{4}\left(1-2\sqrt{-b}-b\right)\right) : 4(a-b)\sqrt{-b},\ -4b;\\
P_4 &= \left(\frac{-a}{2(a-b)},\ \frac{a+b}{4(a-b)}\right) : -2b,\ 4b.
\end{aligned}
$$

If $b>0$, then the system $S(5)$ has only two finite equilibria P_1 and P_4.

The Poincaré compactification in the chart U_1 is

$$
\begin{aligned}
\dot{u} &= 4(a-b-1)u^2v-(b+1)uv^2+2(2a-b-1)uv+8u^2-4(a-b-4)u\\
&\quad +2(b+1)v-4(a-b-2),\\
\dot{v} &= 4(a-b-1)uv^2-8uv-(b+1)v^3-(2b+6)v^2-4(a-b+2)v.
\end{aligned}
$$

There are two equilibria

$$
P_1=(-1,\ 0) : 4(b-a),\ 4(b-a);\quad P_2=\left(\frac{a-b-2}{2},\ 0\right) : 8(b-a),\ 4(a-b).
$$

The Poincaré compactification in the chart U_2 is

$$\begin{aligned}\dot{u} &= 4(a-b-2)u^3 - 2(b+1)u^2v + 4(a-b-4)u^2 - 2(2a-b-1)uv - 8u \\ &\quad +(b+1)v^2 - 4(a-b-1)v, \\ \dot{v} &= 4(a-b-2)u^2v - 2(b+1)uv^2 - 24uv - 4(a+1)v^2 - 16v.\end{aligned}$$

The origin is an equilibrium with eigenvalues: $-16, -8$.

According to the expressions of the eigenvalues, and under the condition $b \neq 0$, we divide the parameter space (a, b) into four regions Z_k for $k = 1, 2, 3, 4$, as shown in Figure 6.18.

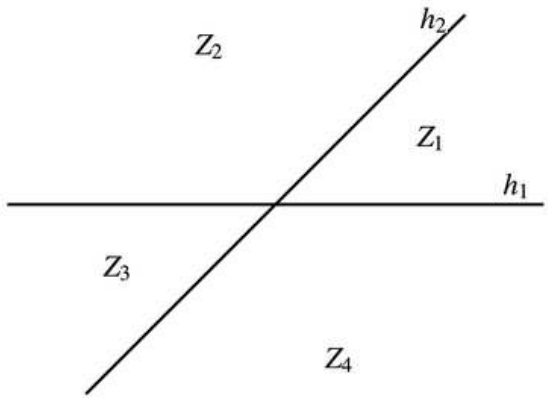


Figure 6.18: Zones of the parameter space (a, b), bounded by the straight lines $h_1 : b = 0$, $h_2 : a - b = 0$.

Since $y - x^2 = 0, 4x + 4y + 1 = 0$, and $4x + 4y + b + 1 = 0$ are invariant curves of the system, we obtain the configurations $C(\text{II.1.s}_2\text{.1.}Z_k)$ for $k = 1, 2, 3, 4$.

Table 6.9 shows the types of finite and infinite equilibria, the corresponding configurations, and the associated global phase portraits.

Zone	P_1	P_2	P_3	P_4	Q_1	Q_2	O_2	Configuration	PP
Z_1	Nu	–	–	S	Ns	S	Ns	$C(\text{II.1.s}_2\text{.2.}Z_1)$	PP.13
Z_2	Nu	–	–	S	Nu	S	Ns	$C(\text{II.1.s}_2\text{.2.}Z_2)$	PP.13
Z_3	Ns	Nu	S	S	Nu	S	Ns	$C(\text{II.1.s}_2\text{.2.}Z_3)$	PP.9,10,12
Z_4	Ns	S	Nu	S	Ns	S	Ns	$C(\text{II.1.s}_2\text{.2.}Z_4)$	PP.9,10,12

Table 6.9: Types of the finite and infinite equilibria, together with their corresponding local phase portraits, configurations, and global phase portraits in the Poincaré disc for Case 2 of the system $S(5)$.

System $S(6)$**.** The system together with its invariant curves is

$$\begin{aligned}\dot{x} &= 8(ab+2a-b^2+4)\,x^2+8(b+2)^2\,xy-2(b^2-4a-12)\,x\\ &\quad -4(ab-2b^2-6b+2a-4)\,y-(ab-2a+2b-4),\\ \dot{y} &= 16(2+b)\,x^2+8(ab+2a+6b+12)\,xy+16(b+2)^2\,y^2-2(ab-2a+2b-4)\,x\\ &\quad -4(b^2-4a+4b-4)\,y,\end{aligned}$$

$$f_1 = y-x^2, \quad f_2 = 4x+4y+1, \quad f_3 = 8x+4(b+2)\,y+(2-b),$$

where $(a-b)(b+2)\neq 0$.

We distinguish four cases: $1.b=0$; $2.a+b=0$; $3.a-2b=0$; $4.b(a+b)(a-2b)\neq 0$.

Case 1. $b=0$ and $(a-b)(b+2)\neq 0$. System $S(6)$ simplifies to

$$\begin{aligned}\dot{x} &= 16(a+2)x^2+32xy+8(a+3)x+8(2-a)y+2a+4,\\ \dot{y} &= 32x^2+16(a+6)xy+4(a+2)x+16(a+1)y+64y^2.\end{aligned}$$

This system has two finite equilibria

$$P_1=\left(-\frac{1}{2},\,\frac{1}{4}\right):0,\;0; \quad P_2=\left(-\frac{1}{4}(a+2),\,\frac{1}{16}(a+2)^2\right):-4a^2,\;2a^2.$$

The point P_1 satisfies the condition (4.iii.iii2) in Theorem 3.5 of [11]. Therefore, its phase portrait is formed by one hyperbolic and one elliptic sector.

The Poincaré compactification of this system in the chart U_1 is

$$\begin{aligned}\dot{u} &= 8(a-2)u^2v-2(a+2)uv^2+8(a-1)uv+32u^2+64u+4(a+2)v+32,\\ \dot{v} &= -2(a+2)v^3+8(a-2)uv^2-8(a+3)v^2-32uv-16(a+2)v.\end{aligned}$$

Hence, there is an infinite equilibrium $Q_1=(-1,0):-16a,0$ in the chart U_1.

The Poincaré compactification of the system in the chart U_2 is

$$\begin{aligned}\dot{u} &= -32u^3-4(a+2)u^2v-64u^2-8(a-1)uv-32u+2(a+2)v^2-8(a-2)v,\\ \dot{v} &= -32u^2v-4(a+2)uv^2-16(a+6)uv-16(a+1)v^2-64v.\end{aligned}$$

The origin O_2 is an infinite equilibrium with eigenvalues $-64, -32$.

Since $y-x^2=0$, $4x+4y+1=0$ are invariant algebraic curves for this system, we obtain the configurations $C(\text{II.1.s}_3\text{.1.1})$ and $C(\text{II.1.s}_3\text{.1.2})$ and the phase portraits in the Poincaré disc indicated in Table 6.10.

Zone	P_1	P_2	Q_1	O_2	Configuration	PP
$b=0,\ a<0$	Ni	S	SN	Ns	$C(\text{II.1.s}_3\text{.1.1})$	PP.15
$b=0,\ a>0$	Ni	S	SN	Ns	$C(\text{II.1.s}_3\text{.1.2})$	PP.15

Table 6.10: Types of the finite and infinite equilibria, together with their corresponding local phase portraits, configurations, and global phase portraits in the Poincaré disc for Case 1 of the system $S(6)$.

Case 2. $a+b=0$ and $b(a-b)(b+2)\neq 0$. System $S(6)$ simplifies to

$$\begin{aligned}\dot{x} &= -16(b-1)(b+2)x^2+8(b+2)^2xy-2(b-2)(b+6)x+4(b+2)(3b+2)y+(b-2)^2,\\ \dot{y} &= 16(b+2)x^2-8(b-6)(b+2)xy+2(b-2)^2x-4(b^2+8b-4)y+16(b+2)^2y^2.\end{aligned}$$

Since $b+2 \neq 0$, this system has three finite equilibria

$$P_1 = \left(-\frac{1}{2},\ \frac{1}{4}\right) : 8b^2,\ 16b^2; \quad P_2 = \left(-\frac{b+6}{6(b+2)},\ -\frac{b-6}{12(b+2)}\right) : -8b^2,\ \frac{16b^2}{3};$$
$$P_3 = \left(\frac{b-2}{2(b+2)},\ \frac{(b-2)^2}{4(b+2)^2}\right) : -16b^2,\ 0.$$

The Poincaré compactification of this system in the chart U_1 is

$$\begin{aligned}\dot{u} &= -4(3b+2)(b+2)u^2v - (b-2)^2uv^2 - 2(b^2+12b+4)uv + 8(b+2)^2u^2,\\ &\quad +8(b+2)(b+4)u + 2(b-2)^2v + 16(b+2)\\ \dot{v} &= -4(b+2)(3b+2)uv^2 - 8(b+2)^2uv - (b-2)^2v^3 + 2(b-2)(b+6)v^2\\ &\quad +16(b+2)(b-1)v.\end{aligned}$$

Since $b+2 \neq 0$, the system has two infinite equilibria in the chart U_1

$$Q_1 = (-1,\ 0) : 24b(b+2),\ -8b(b+2); \quad Q_2 = \left(\frac{-2}{b+2},\ 0\right) : 16b(b+2),\ 8b(b+2).$$

The Poincaré compactification of the system in the chart U_2 is

$$\begin{aligned}\dot{u} &= -16(b+2)u^3 - 2(b-2)^2u^2v - 8(b+2)(b+4)u^2 + 2(b^2+12b+4)uv\\ &\quad -8(b+2)^2u + (b-2)^2v^2 + 4(b+2)(3b+2)v,\\ \dot{v} &= -16(b+2)u^2v - 2(b-2)^2uv^2 + 8(b-6)(b+2)uv + 4(b^2+8b-4)v^2 - 16(b+2)^2v.\end{aligned}$$

The origin O_2 is an infinite equilibrium with eigenvalues: $-16(b+2)^2, -8(b+2)^2$.

Since $y - x^2 = 0$, $4x + 4y + 1 = 0$, and $8x + 8y + 4by - b + 2 = 0$ are invariant algebraic curves of the system, we obtain the configurations $C(\text{II.1.s}_3\text{.2.1})$ and $C(\text{II.1.s}_3\text{.2.2})$.

Table 6.11 shows the local phase portraits of the finite and infinite equilibria, the corresponding configurations and phase portraits in the Poincaré disc.

Zone	P_1	P_2	P_3	Q_1	Q_2	O_2	Configuration	PP
$b < -2$ or $b > 0$	Nu	S	SN	S	Nu	Ns	$C(\text{II.1.s}_3\text{.2.1})$	PP.16
$-2 < b < 0$	Nu	S	SN	S	Ns	Ns	$C(\text{II.1.s}_3\text{.2.2})$	PP.16

Table 6.11: Types of the finite and infinite equilibria, together with their corresponding local phase portraits, configurations, and global phase portraits in the Poincaré disc for Case 2 of the system $S(6)$.

Case 3. $a - 2b = 0$ and $b(a-b)(b+2) \neq 0$. System $S(6)$ simplifies to

$$\begin{aligned}\dot{x} &= 8(b+2)^2x^2 + 8(b+2)^2xy - 2(b^2-8b-12)x + 8(b+2)y - 2(b-2)(b+1),\\ \dot{y} &= 16(b+2)x^2 + 16(b+2)(b+3)xy - 4(b-2)(b+1)x - 4(b^2-4b-4)y + 16(b+2)^2y^2.\end{aligned}$$

Since $b+2 \neq 0$, this system has three finite equilibria

$$P_1 = \left(-\frac{1}{2},\ \frac{1}{4}\right) : -8b^2,\ -4b^2; \quad P_2 = \left(-\frac{b+1}{b+2},\ \frac{(b+1)^2}{(b+2)^2}\right) : -4b^2,\ 6b^2;$$
$$P_3 = \left(\frac{b-2}{2(b+2)},\ \frac{(b-2)^2}{4(b+2)^2}\right) : 8b^2,\ 12b^2,$$

The Poincaré compactification of this system in the chart U_1 is

$$\begin{aligned}\dot{u} &= -8(b+2)u^2v + 2(b-2)(b+1)uv^2 - 2(b^2+4)uv + 8(b+2)^2u^2 + 8(b+2)(b+4)u\\ &\quad -4(b-2)(b+1)v + 16(b+2),\\ \dot{v} &= -8(b+2)^2uv - 8(b+2)uv^2 + 2(b-2)(b+1)v^3 + 2(b^2-8b-12)v^2 - 8(b+2)^2v.\end{aligned}$$

Since $b+2 \neq 0$, there are two infinite equilibria in the chart U_1

$$Q_1 = (-1,\ 0) : -8b(b+2),\ 0; \quad Q_2 = \left(-\frac{2}{b+2},\ 0\right) : -8b(b+2),\ 8b(b+2).$$

The Poincaré compactification of this system in the chart U_2 is

$$\begin{aligned}\dot{u} &= -16(b+2)u^3 + 4(b-2)(b+1)u^2v - 8(b+2)(b+4)u^2 + 2(b^2+4)uv\\ &\quad -8(b+2)^2u - 2(b-2)(b+1)v^2 + 8(b+2)v,\\ \dot{v} &= -16(b+2)u^2v + 4(b-2)(b+1)uv^2 - 16(b+2)(b+3)uv + 4(b^2-4b-4)v^2\\ &\quad -16(b+2)^2v.\end{aligned}$$

The origin O_2 is an infinite equilibrium with eigenvalues: $-16(b+2)^2, -8(b+2)^2$.

Since $y - x^2 = 0$, $4x + 4y + 1 = 0$, and $8x + 8y + 4by - b + 2 = 0$ are invariant algebraic curves of the system, we obtain the configurations $C(\text{II.1.s}_3\text{.3.1})$ and $C(\text{II.1.s}_3\text{.3.2})$.

Table 6.12 shows the local phase portraits of the finite and infinite equilibria, the corresponding configurations and phase portraits in the Poincaré disc.

Zone	P_1	P_2	P_3	Q_1	Q_2	O_2	Configuration	PP
$b > 0$ or $b < -2$	Ns	S	Nu	SN	S	Ns	$C(\text{II.1.s}_3\text{.3.1})$	PP.7
$-2 < b < 0$	Ns	S	Nu	SN	S	Ns	$C(\text{II.1.s}_3\text{.3.2})$	PP.7

Table 6.12: Types of the finite and infinite equilibria, together with their corresponding local phase portraits, configurations, and global phase portraits in the Poincaré disc for Case 3 of the system $S(6)$.

Case 4. $b(a-2b)(a+b)(a-b)(b+2) \neq 0$. Then system $S(6)$ has four finite equilibria

$$\begin{aligned}P_1 &= \left(-\frac{1}{2},\ \frac{1}{4}\right) : 8b(b-a),\ 4b(b-a);\\ P_2 &= \left(-\frac{a+2}{2(b+2)},\ \frac{(a+2)^2}{4(b+2)^2}\right) : -4(a-b)^2,\ 2(a-b)(a+b);\\ P_3 &= \left(\frac{b-2}{2(b+2)},\ \frac{(b-2)^2}{4(b+2)^2}\right) : 4b(a+b),\ 8(a-b)b;\\ P_4 &= \left(\frac{b(b+4)-2a}{2(a-2b)(b+2)},\ -\frac{a(b-2)+4b}{4(a-2b)(b+2)}\right) : -\frac{4(a-b)^2b}{a-2b},\ 4(a-b)b,\end{aligned}$$

The Poincaré compactification in the chart U_1 is

$$\begin{aligned}\dot{u} &= 4(a-2b-2)(b+2)u^2v + (a+2)(b-2)uv^2 - 2(b^2-4a+8b+4)uv\\ &\quad +8(b+2)^2u^2 + 8(b+2)(b+4)u - 2(a+2)(b-2)v + 16(b+2),\\ \dot{v} &= (a+2)(b-2)v^3 + 2(b^2-4a-12)v^2 + 4(a-2b-2)(b+2)uv^2\\ &\quad -8(a-b+2)(b+2)v - 8(b+2)^2uv.\end{aligned}$$

Since $b+2 \neq 0$, there are two equilibria

$$Q_1 = (-1,\ 0) : 8(2b-a)(b+2),\ -8b(b+2);$$
$$Q_2 = \left(-\frac{2}{b+2},\ 0\right) : 8(b-a)(b+2),\ 8b(b+2),$$

The Poincaré compactification in the chart U_2 is

$$\begin{aligned}
\dot{u} &= -8(b+2)^2u - 8(b+2)(b+4)u^2 - 16(b+2)u^3 - 4(a-2b-2)(b+2)v \\
&\quad +2(b^2-4a+8b+4)uv - (a+2)(b-2)v^2 + 2(a+2)(b-2)u^2v, \\
\dot{v} &= -16(b+2)u^2v - 8(a+6)(b+2)uv + 2(a+2)(b-2)uv^2 \\
&\quad +4(b^2-4a+4b-4)v^2 - 16(b+2)^2v.
\end{aligned}$$

The origin O_2 is an infinite equilibrium with eigenvalues: $-16(b+2)^2, -8(b+2)^2$.

According to the expressions of the eigenvalues, and under the condition $b(a+b)(a-b)(a+2b)(b+2) \neq 0$, we divide the parameter space (a,b) into twelve regions Z_k for $k = 1,2,\ldots,12$, as shown in Figure 6.19.

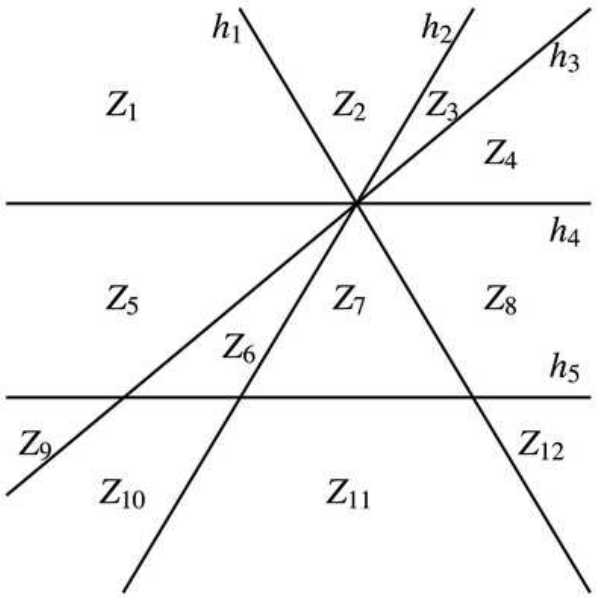


Figure 6.19: Zones of the parameter space (a,b), bounded by the straight lines $h_1 : a+b=0$, $h_2 : a-b=0$, $h_3 : a-2b=0$, $h_4 : b=0$, and $h_5 : b+2=0$.

Since $y-x^2=0$, $4x+4y+1=0$, and $8x+8y+4by-b+2=0$ are invariant algebraic curves of the system, we obtain the configurations $C(\text{II.1.s}_3\text{.4.}Z_k)$ for $k = 1,2,\ldots,8$.

Table 6.13 shows the local phase portraits of the finite and infinite equilibria, the corresponding configurations, and phase portraits in the Poincaré disc.

System $S(7)$**.** The system together with its invariant curves is

$$\begin{aligned}
\dot{x} &= 8(a+1)\,x^2 + 8xy + 2(ab+2a+2b+3)\,x - 4(a-b-1)\,y + (ab+a+b+1), \\
\dot{y} &= 8(b+1)\,x^2 + 8(a+b+3)\,xy + 16y^2 + (2ab+2a+2b+2)\,x + 4(ab+2a+1)\,y,
\end{aligned}$$

$$f_1 = y - x^2, \quad f_2 = 4x+4y+1, \quad f_3 = 4(b+1)\,x + 4y + (b+1)^2,$$

Zone	P_1	P_2	P_3	P_4	Q_1	Q_2	O_2	Configuration	PP
Z_1	Nu	S	Ns	S	S	Nu	Ns	$C(\text{II.1.s}_3\text{.4.}Z_1)$	PP.9
Z_2	Nu	Ns	S	S	S	Nu	Ns	$C(\text{II.1.s}_3\text{.4.}Z_2)$	PP.9
Z_3	Ns	S	Nu	Nu	S	S	Ns	$C(\text{II.1.s}_3\text{.4.}Z_3)$	PP.11
Z_4	Ns	S	Nu	S	Ns	S	Ns	$C(\text{II.1.s}_3\text{.4.}Z_4)$	PP.10
Z_5	Ns	S	Nu	S	Nu	S	Ns	$C(\text{II.1.s}_3\text{.4.}Z_5)$	PP.10
Z_6	Ns	S	Nu	Nu	S	S	Ns	$C(\text{II.1.s}_3\text{.4.}Z_6)$	PP.11
Z_7	Nu	Ns	S	S	S	Ns	Ns	$C(\text{II.1.s}_3\text{.4.}Z_7)$	PP.9
Z_8	Nu	S	Ns	S	S	Ns	Ns	$C(\text{II.1.s}_3\text{.4.}Z_8)$	PP.9
Z_9	Ns	S	Nu	S	Ns	S	Ns	$C(\text{II.1.s}_3\text{.4.}Z_4)$	PP.10
Z_{10}	Ns	S	Nu	Nu	S	S	Ns	$C(\text{II.1.s}_3\text{.4.}Z_3)$	PP.11
Z_{11}	Nu	Ns	S	S	S	Nu	Ns	$C(\text{II.1.s}_3\text{.4.}Z_2)$	PP.9
Z_{12}	Nu	S	Ns	S	S	Nu	Ns	$C(\text{II.1.s}_3\text{.4.}Z_1)$	PP.9

Table 6.13: Types of the finite and infinite equilibria, together with their corresponding local phase portraits, configurations, and global phase portraits in the Poincaré disc for Case 4 of the system $S(6)$.

where $a(a-b) \neq 0$.

We divide the study of this system into two cases: $1.b = 0$; $\quad 2.b \neq 0$.

Case 1. $b = 0$ and $a(a-b) \neq 0$. System $S(7)$ simplifies to

$$\dot{x} = 8(a+1)x^2 + 8xy + 2(2a+3)x - 4(a-1)y + a + 1,$$
$$\dot{y} = 8x^2 + 8(a+3)xy + 2(a+1)x + 16y^2 + 4(2a+1)y.$$

Doing the transformation $(a, b, t) \to \left(2a,\ 0,\ \frac{t}{4}\right)$ on $S(6)$ we obtain the above system.

Case 2. $ab(a-b) \neq 0$. Then system $S(7)$ has four finite equilibria

$$P_1 = \left(-\frac{1}{2}, \frac{1}{4}\right) : 2ab,\ 4ab; \quad P_2 = \left(-\frac{1}{2}(a+1),\ \frac{1}{4}(a+1)^2\right) : -4a(a-b),\ 2a(a-b);$$
$$P_3 = \left(-\frac{1}{4}(b+2),\ \frac{b+1}{4}\right) : -2ab,\ 2(a-b)b;$$
$$P_4 = \left(-\frac{1}{2}(b+1),\ \frac{1}{4}(b+1)^2\right) : 4b(b-a),\ 2b(b-a),$$

The Poincaré compactification of system $S(7)$ in the chart U_1 is

$$\dot{u} = 4(a-b-1)u^2v - (a+1)(b+1)uv^2 + 2(ab+2a-2b-1)uv + 8u^2 + 8(b+2)u$$
$$+2(a+1)(b+1)v + 8(b+1),$$
$$\dot{v} = -(a+1)(b+1)v^3 + 4(a-b-1)uv^2 - 2(ab+2a+2b+3)v^2 - 8uv - 8(a+1)v,$$

Hence, there are two infinite equilibria in the chart U_1

$$Q_1 = (-1,\ 0) : -8a,\ 8b; \quad Q_2 = (-b-1,\ 0) : -8(a-b),\ -8b.$$

The Poincaré compactification of system $S(7)$ in the chart U_2 is

$$\dot{u} = -8(b+1)u^3 - 8(b+2)u^2 - 2(a+1)(b+1)u^2v - 2(ab+2a-2b-1)uv - 8u$$
$$+(a+1)(b+1)v^2 + 4(b-a+1)v,$$
$$\dot{v} = -8(b+1)u^2v - 2(a+1)(b+1)uv^2 - 8(a+b+3)uv - 4(ab+2a+1)v^2 - 16v.$$

The origin O_2 is an infinite equilibrium with eigenvalues: $-16, -8$.

According to the expressions of the eigenvalues, and under the condition $ab(a-b) \neq 0$, we divide the parameter space (a,b) into six regions Z_k for $k = 1, 2, \ldots, 6$, as shown in Figure 6.20.

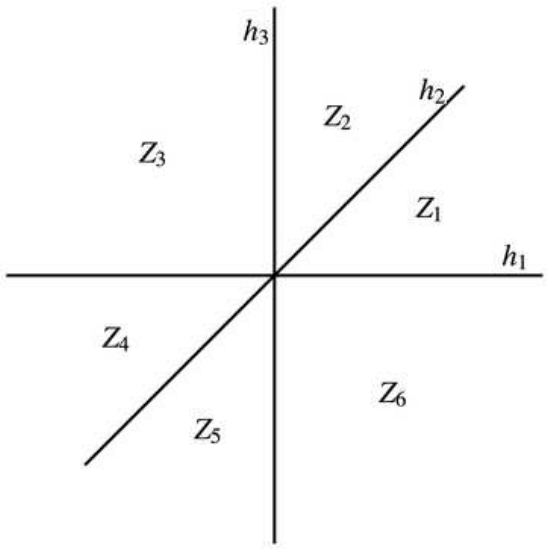


Figure 6.20: Zones of parameter space (a,b), bounded by the straight lines h_1 : $b = 0$, $h_2 : a - b = 0$, and $h_3 : a = 0$.

Since $y - x^2 = 0$, $4x + 4y + 1 = 0$, and $b^2 + 4x + 4bx + 4y + 2b + 1 = 0$ are invariant algebraic curves of the system, we obtain the configurations $C(\text{II.1.s}_4\text{.2.}Z_k)$ for $k = 1, \ldots, 6$.

Table 6.14 shows the local phase portraits of the finite and infinite equilibria, the corresponding configurations, and phase portraits in the Poincaré disc.

Zone	P_1	P_2	P_3	P_4	Q_1	Q_2	O_2	Configuration	PP
Z_1	Nu	S	S	Ns	S	Ns	Ns	$C(\text{II.1.s}_4\text{.2.}Z_1)$	PP.9
Z_2	Nu	S	Ns	Nu	S	S	Ns	$C(\text{II.1.s}_4\text{.2.}Z_2)$	PP.11
Z_3	Ns	S	S	Nu	Nu	S	Ns	$C(\text{II.1.s}_4\text{.2.}Z_3)$	PP.9
Z_4	Nu	S	S	Ns	S	Nu	Ns	$C(\text{II.1.s}_4\text{.2.}Z_4)$	PP.9
Z_5	Nu	S	Ns	Nu	S	S	Ns	$C(\text{II.1.s}_4\text{.2.}Z_5)$	PP.11
Z_6	Ns	S	S	Nu	Ns	S	Ns	$C(\text{II.1.s}_4\text{.2.}Z_6)$	PP.9

Table 6.14: Types of the finite and infinite equilibria, together with their corresponding local phase portraits, configurations, and global phase portraits in the Poincaré disc for Case 2 of the system $S(7)$.

System $S(8)$**.** The system together with its invariant curves is

$$\begin{aligned}
\dot{x} &= 8(a-b+1)\,x^2 + 8xy - 2(4b^2 - 4a - 3)\,x + 4(2b+1)\,y - (2a+1)(4b^2-1),\\
\dot{y} &= 8(a+b+1)\,x^2 + 8(2a+3)\,xy + 16y^2 - 2(8ab^2 + 4b^2 - 2a - 1)\,x\\
&\quad -4(4b^2 + 2b - 2a - 1)\,y,
\end{aligned}$$

$$f_1 = y - x^2, \quad f_2 = 4x + 4y - 4b^2 + 1, \quad f_3 = 2x + 2b + 1,$$

where $b(a-b) \neq 0$.

For studying this system we consider the two cases: $1.a+b=0$; $\quad 2.a+b\neq 0$.

Case 1. $a+b=0$ and $b(a-b)\neq 0$. The system simplifies to:

$$\begin{aligned}\dot{x} &= -8(2b-1)\,x^2+8xy-2(2b-1)(2b+3)\,x+4(2b+1)\,y+(2b-1)^2(2b+1),\\ \dot{y} &= 8x^2-8(2b-3)\,xy+2(2b-1)^2(2b+1)\,x-4(4b^2+4b-1)\,y+16y^2.\end{aligned}$$

The system has three finite equilibria

$$\begin{aligned}P_1 &= \left(-\frac{1}{2}(2b+1),\ \frac{1}{4}(2b+1)^2\right) : 32b^2,\ 32b^2;\\ P_2 &= \left(-\frac{1}{2}(2b+1),\ \frac{1}{4}(1+4b-4b^2)\right) : -32b^2,\ 16b^2;\\ P_3 &= \left(\frac{1}{2}(2b-1),\ \frac{1}{4}(2b-1)^2\right) : -32b^2,\ 0,\end{aligned}$$

The Poincaré compactification of this system in the chart U_1 is

$$\begin{aligned}\dot{u} &= -4(2b+1)\,u^2v-(2b-1)^2(2b+1)\,uv^2-2(2b+1)^2\,uv+2(2b-1)^2(2b+1)\,v\\ &\quad +8u^2+16u+8,\\ \dot{v} &= -(2b-1)^2(2b+1)\,v^3-4(2b+1)\,uv^2+2(2b-1)(2b+3)\,v^2-8uv+8(2b-1)\,v.\end{aligned}$$

Hence, there is an infinite equilibrium $Q_1=(-1,0):0,16b$ in the local chart U_1. According to Theorem 2.19 of [11], this equilibrium is a saddle–node.

The Poincaré compactification of this system in the chart U_2 is

$$\begin{aligned}\dot{u} &= -8u^3-2(2b-1)^2(2b+1)\,u^2v+2(2b+1)^2\,uv+(2b-1)^2(2b+1)\,v^2-16u^2\\ &\quad -8u+4(2b+1)\,v,\\ \dot{v} &= -8u^2v-2(2b-1)^2(2b+1)\,uv^2+8(2b-3)\,uv+4(4b^2+4b-1)\,v^2-16v.\end{aligned}$$

The origin O_2 is an infinite equilibrium with eigenvalues: $-16,-8$.

Since $y-x^2=0$, $4x+4y-4b^2+1=0$, and $2x+2b+1=0$ are invariant algebraic curves of the system, we obtain the configurations $C(\text{II.3.s}_1.1)$.

Table 6.15 shows the local phase portraits of the finite and infinite equilibria, the corresponding configurations, and phase portraits in the Poincaré disc.

Zone	P_1	P_2	P_3	Q_1	O_2	Configuration	PP
$a+b=0$	Nu	S	SN	SN	Ns	$C(\text{II.3.s}_1.1)$	PP.17

Table 6.15: Types of the finite and infinite equilibria, together with their corresponding local phase portraits, configurations, and global phase portraits in the Poincaré disc for Case 1 of the system $S(8)$.

Case 2. $b(a-b)(a+b)\neq 0$. Then system $S(8)$ has four finite equilibria

$$P_1=\left(-\frac{1}{2}(2a+1),\ \frac{1}{4}(2a+1)^2\right):16(a-b)b,\ 8(a-b)(a+b);$$
$$P_2=\left(-\frac{1}{2}(2b+1),\ \frac{1}{4}(2b+1)^2\right):16b(b-a),\ 16b(b-a);$$
$$P_3=\left(-\frac{1}{2}(2b+1),\ \frac{1}{4}(4ab+4b+1)\right):-8b(a-b),\ 16b(a-b);$$
$$P_4=\left(\frac{1}{2}(2b-1),\ \frac{1}{4}(2b-1)^2\right):16(a-b)b,\ 16b(a+b).$$

The Poincaré compactification of this system in the chart U_1 is

$$\begin{aligned}\dot{u}&=-4(2b+1)\,u^2v+(4b^2-1)(2a+1)\,uv^2-2(2b+1)^2\,uv+8u^2+8(a+b+2)\,u\\&\quad-2(4b^2-1)(2a+1)\,v+8(a+b+1),\\\dot{v}&=(4b^2-1)(2a+1)\,v^3-4(2b+1)\,uv^2+2(4b^2-4a-3)\,v^2-8uv-8(a-b+1)\,v.\end{aligned}$$

Hence, there are two infinite equilibria in the local chart U_1

$$Q_1=(-1,\ 0):8(b-a),\ 8(a+b);\quad Q_2=(-a-b-1,\ 0):-8(a+b),\ 16b,$$

The Poincaré compactification of this system in the chart U_2 is

$$\begin{aligned}\dot{u}&=-8(a+b+1)\,u^3+2(2a+1)(4b^2-1)\,u^2v-8(a+b+2)\,u^2\\&\quad+2(2b+1)^2\,uv-8u-(4b^2-1)(2a+1)\,v^2+4(2b+1)\,v,\\\dot{v}&=-8(a+b+1)\,u^2v+2(2a+1)(4b^2-1)\,uv^2-8(2a+3)\,uv\\&\quad+4(4b^2-2a+2b-1)\,v^2-16v.\end{aligned}$$

The origin O_2 is an infinite equilibrium with eigenvalues: $-16,-8$.

According to the expressions of the eigenvalues, and under the condition $b(a+b)(a-b)\neq 0$, we divide the parameter space (a,b) into six regions Z_k for $k=1,2,\ldots,6$, as shown in Figure 6.21.

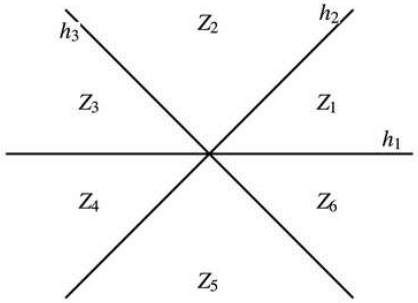


Figure 6.21: Zones of the parameter space (a,b), bounded by the straight lines $h_1:b=0$, $h_2:a-b=0$, and $h_3:a+b=0$.

Since $y - x^2 = 0$, $4x + 4y - 4b^2 + 1 = 0$, and $2x + 2b + 1 = 0$ are invariant algebraic curves of the system, we obtain the configurations $C(\text{II.3.s}_1\text{.2.}Z_k)$ for $k = 1, \ldots, 6$.

Table 6.16 shows the local phase portraits of the finite and infinite equilibria, the corresponding configurations, and phase portraits in the Poincaré disc.

Zone	P_1	P_2	P_3	P_4	Q_1	Q_2	O_2	Configuration	PP
Z_1	Nu	Ns	S	Nu	S	S	Ns	$C(\text{II.3.s}_1\text{.2.}Z_1)$	PP.11
Z_2	Ns	Nu	S	S	Nu	S	Ns	$C(\text{II.3.s}_1\text{.2.}Z_2)$	PP.10
Z_3	S	Nu	S	Ns	S	Nu	Ns	$C(\text{II.3.s}_1\text{.2.}Z_3)$	PP.10
Z_4	Nu	Ns	S	Nu	S	S	Ns	$C(\text{II.3.s}_1\text{.2.}Z_4)$	PP.11
Z_5	Ns	Nu	S	S	Ns	S	Ns	$C(\text{II.3.s}_1\text{.2.}Z_5)$	PP.10
Z_6	S	Nu	S	Ns	S	Ns	Ns	$C(\text{II.3.s}_1\text{.2.}Z_6)$	PP.10

Table 6.16: Types of the finite and infinite equilibria, together with their corresponding local phase portraits, configurations, and global phase portraits in the Poincaré disc for Case 2 of the system $S(8)$.

This completes the proof. □

7 Conclusion

This study systematically classifies the phase portraits in the Poincaré disc of the quadratic systems with one or two invariant real straight lines taking into account that their total multiplicity be two and an invariant parabola. The distinct phase portraits in the Poincaré disc of these class of quadratic systems are given in Theorem 3.2. The results contribute to a broader understanding of quadratic systems with algebraic invariant curves, offering new insights into their global dynamics and bifurcations.

As a natural extension of this work, future research will focus on the classification of phase portraits for quadratic systems with one or two invariant real straight lines taking into account that their total multiplicity be two and an invariant ellipse or hyperbola. This will ultimately lead to a complete classification of all planar quadratic differential systems possessing two invariant straight lines and an irreducible conic. This classification will provide a unified framework for analysing the global dynamics of this important class of quadratic systems.

8 Data and code availability

All symbolic computations supporting Theorems 3.1 and 3.2 are openly available at the following public repository:

- Llibre, J. & Ou, H. (2025). *Mathematica computations for the parabola paper (Theorems 3.1 and 3.2).* Repository: `https://github.com/Ovason/R2P` (accessed on December 4, 2025).

The repository contains the Mathematica notebooks ordered exactly as in the manuscript.

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Multiple solutions for a fractional (p,t)-Laplacian system with logarithmic nonlinearities

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Abstract. This paper is concerned with the existence and multiplicity of a ground state solution for a class of fractional (p,t)-type systems involving logarithmic nonlinearities with sign-changing coefficients, which are obtained by variational methods. More precisely, by combining Nehari manifold and fibering map method. The paper proves two results: The first one provides the existence of a ground-state solution for the proposed problem. Under a different set of hypotheses with respect to the first result, a second result is obtained, which provides the existence of at least two nontrivial ground state solutions.

Keywords: logarithmic nonlinearity, fibering map, Nehari manifold, (p,t) fractional operator, multiplicity, mountain pass theorem.

2020 Mathematics Subject Classification: 35K55, 35R11, 47G20.

1 Introduction and main results

The study of elliptic equations and systems involving fractional operators (see [14]), as well as, fractional (p,t) Laplacians has energized many applications in nonlinear mathematical and physical analysis. In recent years, much attention has been paid to research on very delicate readings about the existence and multiplicity of ground state solutions for such elliptic problems with different types of nonlinearities. This paper deals with nonhomogeneous systems of equations involving fractional operators of type (p,t) with logarithmic coupling terms with coefficients changing sign, specifically we study the following equation

$$\begin{cases} (-\Delta)_p^{s_1}u + (-\Delta)_t^{s_2}u = \lambda H_1(x)|u|^{\theta-2}u\ln|u| + \dfrac{q}{r+q}R_1(x)|v|^r|u|^{q-2}u & \text{in } \Omega, \\ (-\Delta)_p^{s_1}v + (-\Delta)_t^{s_2}v = \mu H_2(x)|v|^{\theta-2}v\ln|v| + \dfrac{r}{r+q}R_2(x)|u|^r|v|^{q-2}v & \text{in } \Omega, \\ u = v = 0, \qquad \mathbb{R}^N \setminus \Omega, \end{cases} \tag{P}$$

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where $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz boundary, $s \in (0,1)$, $N > ps$, $\lambda, \mu > 0$, $0 < s_2 < s_1 < 1 < t < p < \theta < p^*_{s_1} = \frac{pN}{N-ps_1}$, $\lambda, \mu > 0$ are two parameters, $r > 1, q > 1$ satisfies $q + r < p^*_{s_1}$ and the additional weights $H_1, H_2, R_1, R_2 \in C(\bar{\Omega})$ are such that: $R_1(x), R_2(x)$ are positive functions and $H_1(x), H_2(x)$ are sign-changing functions. The operators $(-\Delta)^{s_1}_p$ and $(-\Delta)^{s_2}_t$ represents, both, fractional p-Laplacian operator, a generalization for the fractional Laplacian $(-\Delta)^s, 0 < s_i < 1$ for $p = 2$, defined in a integral way as

$$(-\Delta)^{s_i} u(x) := \frac{c(n,s_i)}{2} \int_{\mathbb{R}^N} \frac{2u(x) - (x+y) - u(x-y)}{|y|^{n+2s_i}} dy, \qquad x \in \mathbb{R}^N,$$

where $c(n,s)$ is a positive normalizing constant, and another fractional operator.

Specifically, the operators $(-\Delta)^i_{p_i}$ are defined, up to a normalization constant, by the formula

$$(-\Delta)^{s_i}_{p_i} u(x) := \lim_{\varepsilon \to 0^+} 2 \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|u(x) - u(y)|^{p_i - 2}(u(x) - u(y))}{|x-y|^{N+s_i p_i}} dy,$$

where $u \in u \in C^\infty_0(\mathbb{R}^N)$ and $B_\varepsilon(x)$ denotes ball in $\mathbb{R}^N$ with centre x and radius ε. When $s_1 = s_2$ the equation reduces to a (p,t)-type Laplacian problem which appears in a more general reaction-diffusion system

$$u_t = \nabla \cdot (f(u)\nabla u) + h(x,u) \tag{1.1}$$

where $f(u) = |\nabla u|^{p-2}\nabla u + |\nabla u|^{t-2}\nabla u$. Such problems have a wide range of applications in physics and related sciences such as biophysics, plasma physics, and chemical reaction design, etc. where u describes a concentration, and the first term on the right-hand side of (1.1) corresponds to a diffusion with a diffusion coefficient $f(u)$; the term $g(x,u)$ stands for the reaction, related to sources and energy-loss processes. On the other hand, since logarithmic nonlinearities play a significant role in depicting the mathematical and physical phenomena, they have received much attention in PDEs (see [19,20,22] and references therein).

In [19], the authors studied the existence and multiplicity of a class of fractional Laplacian systems with logarithmic nonlinearity in which three types of weights with certain regularity are involved.

$$\begin{cases} (-\Delta)^s u = \lambda g_1(x) u \ln(|u|) + \dfrac{p}{p+q} b(x)|v|^q |u|^{p-2}u & \text{in } \Omega, \\ (-\Delta)^t v = \mu g_2(x) v \ln(|v|) + \dfrac{r}{q+r} b(x)|u|^r |v|^{q-2}v & \text{in } \Omega, \\ u = v = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz boundary, $s, t \in (0,1)$, $N > \max\{2s, 2t\}$, $\lambda, \mu > 0$, $2 < q + p < \min\left\{\frac{2N}{N-2s}, \frac{2N}{N-2t}\right\}$, and the additional weights $g_1, g_2, b \in C(\overline{\Omega})$ are such that: $b(x)$ are positive functions and $g_1(x), g_2(x)$ are sign-changing functions.

In [20], the authors studied a class of systems of equations where they showed the existence and multiplicity of solutions for a mixed local-nonlocal system with logarithmic nonlinearities

$$\begin{cases} (-\Delta)^s u_j + u_j = \lambda_j a_j(x) u_j \ln|u_j| + \sum_{i \neq j} \beta_{ij} |u_j|^{q-2} u_j & \text{in } \Omega, \\ u = v = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega \end{cases} \tag{1.3}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz boundary, $s \in (0,1)$, $N > 4$, λ_j are parameters, $\beta_{ij} > 0$ for all $1 \leq i < j \leq k$, $\beta_{ij} = \beta_{ji}$ for all $i, j = 1, \ldots, k$, $a_j \in C(\overline{\Omega})$. When

$1 < 2q < 2 < 2^* = \frac{2N}{N-2}$ and a_j they are functions that change sign, they obtained two different solutions using Nehari Method. When $2 < q < 2 < 2^* = \frac{2N}{N-2}$ and a_j are positive constant functions, the existence of the ground state solution is obtained using the minimization method.

In [13], the authors studied a class of fractional Laplacian systems where they showed the existence of solutions using the Nehari method and multiplicity of solutions using the Lusternik–Schnirelmann category, with polynomial nonlinearity.

$$\begin{cases} (-\Delta)^s u = \lambda |u|^{p-2}u + \dfrac{2\alpha}{\alpha+\beta}|v|^{\beta}|u|^{\alpha-2}u & \text{in } \Omega, \\ (-\Delta)^s v = \mu |u|^{p-2}u + \dfrac{2\beta}{\alpha+\beta}|u|^{\alpha}|v|^{\beta-2}v & \text{in } \Omega, \\ u = v = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega \end{cases} \tag{1.4}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, $s \in (0,1)$, $N > 4s$, $\lambda, \mu > 0$ are parameters, $\alpha + \beta = \frac{2N}{N-2s}$ is the critical Sobolev exponent.

In [5], the authors showed the existence of solutions for the fractional critical system (p,q) Laplacian using variational method

$$\begin{cases} (-\Delta)_p^{s_1} u + (-\Delta)_q^{s_2} u = \lambda Q(x)|u|^{r-2}u + \dfrac{2\alpha}{\alpha+\beta}|v|^{\beta}|u|^{\alpha-2}u & \text{in } \Omega, \\ (-\Delta)_p^{s_1} v + (-\Delta)_q^{s_2} v = \mu Q(x)|u|^{r-2}u + \dfrac{2\beta}{\alpha+\beta}|u|^{\alpha}|v|^{\beta-2}v & \text{in } \Omega, \\ u = v = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega \end{cases} \tag{1.5}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, $0 < s_2 < s_1 < 1$, $1 < q < p < r < p_{s_1}^*$, $N > p_{s_1}$, $\lambda, \mu > 0$ are parameters and $\alpha > 1$ $\beta > 1$ satisfy $\alpha + \beta = p_{s_1}^*$ with $\frac{Np}{N-ps_1}$ is the critical Sobolev exponent, and $(-\Delta)_t^s$ is the fractional t-Laplacian operator.

In [22] was concerned with the existence and asymptotic behavior of normalized ground states solutions for the following coupled Schrödinger system with logarithmic terms:

$$\begin{cases} -\Delta u_1 + \omega_1 u_1 = \mu_1 u_1 \log u_1^2 + \dfrac{p}{p+q}|u_2|^q |u_1|^{p-2}u_1, & \text{in } \Omega, \\ -\Delta u_2 + \omega_2 u_2 = \mu_2 u_2 \log u_2^2 + \dfrac{q}{p+q}|u_1|^p |u_2|^{q-2}u_2, & \text{in } \Omega, \\ \int_\Omega |u_i|^2 dx = \rho_i, \qquad i = 1,2, \end{cases} \tag{1.6}$$

where $\Omega = \mathbb{R}^N$ or $\Omega \subset \mathbb{R}^N (N \geq 3)$ is a bounded smooth domain, $\omega_i \in \mathbb{R}, \mu_i, \rho_i > 0, i = 1,2$. Moreover, $p, q \geq 1, 2 \leq p + q \leq 2^*$ where $2^* := \frac{2N}{N-2}$.

In [11] the authors demonstrate existence of solutions for fractional (p,q) Laplacian systems

$$\begin{cases} (-\Delta)_p^{s_1} u + (-\Delta)_q^{s_2} u = \lambda |u|^{r-2} + \dfrac{\alpha\eta}{\eta+\nu}|u|^{\eta-2}|v|^{\nu}u & \text{in } \Omega, \\ (-\Delta)_p^{s_1} v + (-\Delta)_q^{s_2} v = \mu |v|^{r-2} + \dfrac{\alpha\nu}{\nu+\eta}|v|^{\nu-2}|u|^{\eta}v & \text{in } \Omega, \\ u = v = 0, \quad x \in \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.7}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $0 < s_2 < s_1 < 1 < q < r < p < p_{s_1}^* = \frac{pN}{N-ps_1}$, $\lambda, \mu > 0$ are two parameters, $r > 1, q > 1$ satisfies $q + r = p_{s_1}^*$.

In [21] the authors studied the existence of least energy solutions to the following fractional Kirchhoff problem with logarithmic nonlinearity

$$\begin{cases} M([u]_{s,t}^{p})(-\Delta)_{p}^{s}u = h(x)|u|^{\theta p-2}u\ln|u| + \lambda|u|^{q-2}u, & x \in \Omega \\ u = 0, \quad x \in \mathbb{R}^{N} \setminus \Omega, \end{cases} \tag{1.8}$$

where $s \in (0,1)$, $1 < p < Np$, $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz boundary, $M([u]_{s,p}^{p}) = [u]_{s,p}^{(\theta-1)p}$, with $\theta \geq 1$ and $u, h \in C(\overline{\Omega})$ may change sign, $\lambda > 0$ and $q \in (1, p_{s_1}^{*})$.

On the other hand, parabolic and hyperbolic type equations with logarithmic nonlinearity have been studied extensively in recent years. Here we only refer some results involving fractional Laplacian. [2] considered the Cauchy problem of the following Schrödinger equation

$$iu_t - (-\Delta)^{s}u + u\log|u|^2 = 0, \qquad x \in \Omega,\ t > 0. \tag{1.9}$$

The existence of global solutions was obtained by using a compactness method. Moreover, the author obtained the existence of ground states by the Nehari manifold approach.

The main tool used in this paper is the so-called fibering method introduced by Pohozaev [15], [16] and [17], and applied to a single equation of p-Laplacian type by Drábek and Pohozaev in [12]. Bozhkov and Mitidieri [3] used this method to study the existence for multiple solution for a quasilinear system. Fibering method is very used to proof existence of multiple solution for a large class of equations. For example, Brown and Wu [4] used this method to show the existence of at least two positive solutions for the semilinear elliptic boundary-value problem. For more recent applying of fibering method we indicate [1], used to show existence of multiple solution for a class of Schrödinger equations involving indefinite weight functions, and other interesting work involving the fractional operator is [7].

Inspired by references [9, 10, 13, 19, 20, 22], in this article, we investigate the existence and multiplicity of ground state solutions for problem (P) in both subcritical and critical cases. One goal of this present work is to establish the existence of a nontrivial ground state solution using variational methods. We remark that there are several types of difficulty in deriving the existence results:

(1) Furthermore, we need to deal with essential difficulties due to the fact that the logarithmic term $f(u) = |u|^{\theta-2}u\ln|u|$ neither satisfies the monotonicity condition nor the Ambrosetti–Rabinowitz (A-R) condition, we manage to overcome these difficulties by combining the Brezis–Lieb lemma with some delicate estimates of this logarithmic term.

(2) Since the weights $H_1, H_2 \in C(\bar{\Omega})$ change sign, the study of this type of problem becomes difficult, to overcome this difficulty we use the fibers in the Nehari manifold.

(3) The operator $(-\Delta)_{p}^{s_1} + (-\Delta)_{t}^{s_2}$ does not have homogeneity properties, which makes it difficult to solve.

(4) This system is a broader class of the system studied in [9].

The main results of this paper is writing as follows:

Theorem 1.1. *Problem (P) has a nontrivial ground state solution in W for $0 < s_2 < s_1 < 1 < t < p < \theta < p_{s_1}^{*}$ and H_1, H_2 being positive functions in $C(\overline{\Omega})$.*

Theorem 1.2. *Problem (P) has a nontrivial ground state solution in W for $0 < s_2 < s_1 < 1 < t < p < \theta < p^*_{s_1}$ and H_1, H_2 two sign-changing functions in $C(\overline{\Omega})$.*

The paper is organized as follows. In Section 2 we study the variational framework. In Section 3 we study Nehari Manifold and fibering map analysis. In Section 4 we prove Theorem 1.1. In Section 5 we prove Theorem 1.2.

2 The variational framework

This paper starts with an introduction to the fractional Sobolev space, where the solutions for problem (P) lies. After this introduction, some technical results that will be use to proof the next results that will be shown in the others chapters.

For this, let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary, for $p \in (1, +\infty)$, $s_1 \in (0,1)$ and $N > s_1 p$, we define the fractional Sobolev space.

$$W^{s_1,p}(\Omega) := \left\{ u \in L^p(\Omega) : \ \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps_1}} \in L^p(\Omega) \times L^p(\Omega) \right\}. \tag{2.1}$$

The space $W^{s_1,p}(\Omega)$ is an intermediate Banach space between $L^p(\Omega)$ and $W^{1,p}(\Omega)$, endowed with the natural norm

$$\|u\|_{W^{s_1,p}} = \left(\int_\Omega |u|^p dx \right)^{1/p} + [u]^p_{s_1},$$

where

$$[u]^p_{s_1,p} = \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps_1}} dxdy$$

is the so-called Gagliardo seminorm of u.

We set $\Omega^c = \mathbb{R}^N \setminus \Omega$ and $\mathcal{Q}^c = \mathbb{R}^N \setminus \Omega^c \times \Omega^c$. We define the following set

$$X^{s_1,p}(\Omega) := \left\{ u : \mathbb{R}^N \to \mathbb{R} \text{ is measurable, } u|_\Omega \in L^p(\Omega), \text{ and } \int_{\mathcal{Q}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+s_1 p}} dxdy < +\infty \right\}.$$

The space $X^{s_1,p}(\Omega)$ is endowed with the norm defined by

$$\|u\|_{X^{s_1,p}(\Omega)} = \left(\int_\Omega |u|^p dx \right)^{1/p} + \left(\int_{\mathcal{Q}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps_1}} dxdy \right)^{1/p}.$$

Also, we define the space

$$X_0^{s_1,p}(\Omega) := \{ u \in X^{s_1,p}(\Omega) : \ u = 0 \ a.e \text{ in } \Omega^c \},$$

or equivalently as $\overline{C_0^\infty(\Omega)}^{X^{s_1,p}(\Omega)}$. It is well-known that for $p > 1$, $X_0^{s_1,p}(\Omega)$ is a uniformly convex Banach space endowed with the norm

$$\|u\|_{s_1,p} = \left(\int_{\mathcal{Q}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps_1}} dxdy \right)^{1/p}.$$

Since $u = 0$ in $\mathbb{R}^N \setminus \{0\}$, the above integral can be extended to all of $\mathbb{R}^N$.

The embedding $X_0^{s_1,p}(\Omega) \hookrightarrow L^\kappa(\Omega)$ is continuous for any $\kappa \in [0, p^*_{s_1}]$ and compact for $\kappa \in [1, p^*_{s_1})$.

Lemma 2.1 ([8]). *Let $1 \leq t \leq p$, $0 < s_2 < s_1 < 1$. Then, $W^{s_1,p}(\Omega) \subset W^{s_2,t}(\Omega)$.*

Lemma 2.2 ([8]). *Let $1 \leq t \leq p$, $0 < s_2 < s_1 < 1$ and and Ω be a smooth bounded domain in $\mathbb{R}^N$ where $N > s_1 p$. Then, $X_0^{s_1,p}(\Omega) \subset X_0^{s_2,t}(\Omega)$ and there exists $C = C(\Omega, t, p, s_1, s_2)$ such that*

$$\|u\|_{s_2,t} \leq C\|u\|_{s_1,p}, \qquad \forall u \in X_0^{s_1,p}(\Omega). \tag{2.2}$$

Let $W_p := X_0^{s_1,p}(\Omega) \times X_0^{s_1,p}(\Omega)$ and $W_t := X_0^{s_2,t}(\Omega) \times X_0^{s_2,t}(\Omega)$ be reflexive Banach spaces equipped, respectively, with the norms

$$\|(u,v)\|_{s_1,p} = \left(\|u\|_{s_1,p}^p + \|v\|_{s_1,p}^p\right)^{1/p}, \qquad \|(u,v)\|_{s_2,t} = \left(\|u\|_{s_2,t}^t + \|v\|_{s_2,t}^t\right)^{1/t}.$$

In this paper, we work over the space $W := W_p \bigcap W_t$ endowed the norm $\|(u,v)\| := \|(u,v)\|_{s_1,p} + \|(u,v)\|_{s_2,t}$.

The following lemma is very important.

Lemma 2.3 ([8, Theorem 6.7]). *W is continuously embedding on $L^{s_1,p}(\Omega) \times L^{s_2,t}(\Omega)$.*

Lemma 2.4. *Let $S_{s_1,p}$ the best fractional critical Sobolev constant of embedding $X_0^{s_1,p}(\Omega)$ to $L^{p_{s_1}^*}(\Omega)$*

$$S_{s_1,p} := \inf_{u \in X_0^{s,p}(\Omega)\setminus\{0\}} \left\{ \frac{\|u\|_{s_1,p}^p}{\left(\int_\Omega |u|^{p_s^*} dx\right)^{p/p_s^*}} \right\}. \tag{2.3}$$

and the definition of $S_{q,r}$ is as follows:

$$S_{q,r} := \inf_{(u,v)\in W\setminus\{0\}} \left\{ \frac{\|(u,v)\|_{s_1,p}^p}{\left(\int_\Omega |u|^q |v|^r dx\right)^{p/p_{s_1}^*}} \right\}, \tag{2.4}$$

Then,

$$S_{q,r} := \left[\left(\frac{q}{r}\right)^{r/(q+r)} + \left(\frac{r}{q}\right)^{q/(q+r)} \right] S_{s_1,p}.$$

Lemma 2.5. *Let $\xi, \eta \in \mathbb{R}^N$. Then,*

$$|\xi - \eta|^p \leq \begin{cases} 2^p(|\xi|^{p-2}\xi + |\eta|^{p-2}\eta)(\xi - \eta), & \text{for } p \geq 2, \\ \dfrac{1}{(p-1)} \dfrac{\left[(|\xi|^{p-2}\xi + |\eta|^{p-2}\eta)(\xi - \eta)\right]^{p/2}}{(|\xi|^p + |\eta|^p)^{(p-2)/p}} & \text{for } 1 < p < 2. \end{cases} \tag{2.5}$$

The proof of this lemma can be found in [18, Lemma 6].

Now we will use some technical results that involve the logarithmic term. This result represents an important part of the continuation of this paper

Lemma 2.6 ([10, Lemma 2.6]). *Let ρ be a positive real number. Then we have the following inequalities*

$$t^p \ln t \leq \frac{t^{p+\rho}}{e\rho} \quad \text{for all } 1 \leq t \tag{2.6}$$

and

$$|t^p \ln t| \leq \frac{1}{e\rho} \quad \text{for all } 0 < t < 1, \tag{2.7}$$

where e is Euler logarithm basis.

Proceeding with the introduction, we will now make some definitions about the variational framework.

Definition 2.7. We say that $(u,v) \in W$ is a weak solution of problem (P) if

$$\begin{aligned}
&\int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+ps_1}}dxdy \\
&\quad + \int_{\Omega\times\Omega} \frac{|v(x)-v(y)|^{p-2}(v(x)-v(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+ps_1}}dxdy \\
&\quad + \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{N+ts_2}}dxdy \\
&\quad + \int_{\Omega\times\Omega} \frac{|v(x)-v(y)|^{p-2}(v(x)-v(y))(\phi(x)-\phi(y))}{|x-y|^{N+ts_2}}dxdy \\
&= \lambda\int_\Omega H_1(x)|u|^{\theta-2}u\ln|u|\varphi dx + \mu\int_\Omega H_2(x)|v|^{\theta-2}v\ln|v|\phi dx \\
&\quad + \frac{q}{q+r}\int_\Omega R_1(x)|v|^r|u|^{q-2}u\varphi dx + \frac{r}{q+r}\int_\Omega R_2(x)|u|^q|v|^{r-2}v\phi dx,
\end{aligned}$$

for any $(\varphi,\phi) \in W$.

Because of the above definition, we consider the energy functional for problem (P), $E : W \to \mathbb{R}$ defined by

$$\begin{aligned}
E(u,v) = \frac{1}{p}\|(u,v)\|^p_{s_1,p} + \frac{1}{t}\|(u,v)\|^t_{s_2,t} - \frac{\lambda}{\theta}\int_\Omega H_1(x)|u|^\theta\ln|u|dx + \frac{\lambda}{\theta^2}\int_\Omega H_1(x)|u|^\theta dx \\
- \frac{\mu}{\theta}\int_\Omega H_2(x)|v|^\theta\ln|v|dx + \frac{\mu}{\theta^2}\int_\Omega H_2(x)|v|^\theta dx - \frac{2}{q+r}\int_\Omega (R_1+R_2)(x)|u|^q|v|^r dx.
\end{aligned}$$

We also consider the auxiliary functional

$$\begin{aligned}
I(u,v) = \|(u,v)\|^p_{s_1,p} + \|(u,v)\|^t_{s_2,t} - \lambda\int_\Omega H_1(x)|u|^\theta\ln|u|dx \\
- \mu\int_\Omega H_2(x)|v|^\theta\ln|v|dx - \int_\Omega (R_1+R_2)(x)|u|^q|v|^r dx.
\end{aligned}$$

As a consequence of embedding on Remark 2.3 and Lemma 2.4, we obtain that the functional I is well-defined and $E, I \in C^1(W,\mathbb{R})$. Furthermore, we have that the derivatives I' and E' in the Fréchet sense are given as

$$\begin{aligned}
E'(u,v)(\varphi,\phi) = &\int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+ps_1}}dxdy \\
&+ \int_{\Omega\times\Omega} \frac{|v(x)-v(y)|^{p-2}(v(x)-v(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+ps_1}}dxdy \\
&+ \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{N+ts_2}}dxdy \\
&+ \int_{\Omega\times\Omega} \frac{|v(x)-v(y)|^{p-2}(v(x)-v(y))(\phi(x)-\phi(y))}{|x-y|^{N+ts_2}}dxdy \\
&- \lambda\int_\Omega H_1(x)|u|^{\theta-2}u\ln|u|\varphi dx - \mu\int_\Omega H_2(x)|v|^{\theta-2}v\ln|v|\phi dx \\
&- \frac{q}{q+r}\int_\Omega R_1(x)|v|^r|u|^{q-2}u\varphi dx - \frac{r}{q+r}\int_\Omega R_2(x)|u|^q|v|^{r-2}v\phi dx
\end{aligned}$$

and

$$\begin{aligned}
I'(u,v)(\varphi,\phi) = {} & p\int_{\Omega\times\Omega}\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+ps_1}}dxdy \\
& + p\int_{\Omega\times\Omega}\frac{|v(x)-v(y)|^{p-2}(v(x)-v(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+ps_1}}dxdy \\
& + t\int_{\Omega\times\Omega}\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{N+t_2}}dxdy \\
& + t\int_{\Omega\times\Omega}\frac{|v(x)-v(y)|^{p-2}(v(x)-v(y))(\phi(x)-\phi(y))}{|x-y|^{N+ts_2}}dxdy \\
& - \lambda\theta\int_\Omega H_1(x)|u|^{\theta-2}u\varphi\ln|u|dx - \lambda\int_\Omega H_1(x)|u|^{\theta-2}u\varphi dx \\
& - \mu\theta\int_\Omega H_2(x)|v|^{\theta-2}v\phi\ln|v|dx - \mu\int_\Omega H_2(x)|v|^{\theta-2}v\phi dx \\
& - (q+r)\Big[\int_\Omega R_1(x)|v|^r|u|^{q-2}u\varphi dx + \int_\Omega R_2(x)|u|^q|v|^{r-2}v\phi\Big]dx
\end{aligned}$$

for all $(\varphi,\phi)\in W$.

We now prove the Lebesgue dominated convergence theorem for the logarithm function.

Lemma 2.8 ([9, Lemma 2.6]). *Assume that (u_n,v_n) is bounded in W such that u_n converges to u a.e. and v_n converges to v in Ω . Then*

$$\lim_{n\to\infty}\int_\Omega H_1(x)|u_n|^\theta\ln|u_n|dx = \int_\Omega H_1(x)|u|^\theta\ln|u|\,dx, \quad \text{for } \theta\in(2,p_{s_1}^*),$$

and

$$\lim_{n\to\infty}\int_\Omega H_2(x)|v_n|^\theta\ln|v_n|dx = \int_\Omega H_2(x)|v|^\theta\ln|v|\,dx, \quad \text{for } \theta\in(2,p_{s_1}^*).$$

Lemma 2.9 ([6, Lemma 3.2]). *Let the sequences (u_n) and (v_n) be in $W_0^{s_1,p}(\Omega)$ such that $u_n\rightharpoonup u$, $v_n\rightharpoonup v$ in $W_0^{s_1,p}(\Omega)$ and $u_n(x)\to u(x)$, $v_n(x)\to v(x)$ a.e. in Ω. Then,*

$$\lim_{n\to\infty}\int_\Omega\left(|u_n|^q|v_n|^r - |u_n-u|^q|v_n-v|^r\right)dx = \int_\Omega|u|^q|v|^r\,dx.$$

Lemma 2.10. *Let $(u,v)\in W\setminus\{(0,0)\}$. Then*

$$\begin{aligned}
&\int_\Omega(\lambda H_1(x)|u|^\theta\ln|u| + \mu H_2(x)|v|^\theta\ln|v|)\,dx \\
&\quad\le(\lambda C_{H_1}+\mu C_{H_2})L\left(\|(u,v)\|_{s_1,p}^p + \|(u,v)\|_{s_2,t}^t\right) \\
&\qquad + \ln\|(u,v)\|\int_\Omega(\lambda H_1(x)|u|^\theta + \mu H_2(x)|v|^\theta)\,dx
\end{aligned}$$

where

$$L := \frac{|\Omega|}{e\theta} + \frac{1}{e(p_{s_1}^*-\theta)}S_{p_{s_1}^*}^{\frac{p_{p_{s_1}^*}}{}}, \qquad C_{H_1} := \max_{x\in\overline{\Omega}}|H_1(x)|, \qquad C_{H_2} := \max_{x\in\overline{\Omega}}|H_2(x)|$$

and $S>0$ denote the best constants of embeddings from $W_0^{s_1,p}(\Omega)$ to $L^{p_{s_1}^*}(\Omega)$.

Proof. Let us consider $\Omega = \Omega_1 \cup \Omega_2$, using integration properties over Ω, where $\Omega_1 = \{x \in \Omega : |u(x)| \leq \|(u,v)\|\}$ and $\Omega_2 = \{x \in \Omega : |u(x)| > \|(u,v)\|\}$. Then

$$\int_\Omega H_1(x)|u|^\theta \ln \frac{|u(x)|}{\|(u,v)\|}dx = \int_{\Omega_1} H_1(x)|u|^\theta \ln \frac{|u(x)|}{\|(u,v)\|}dx + \int_{\Omega_2} H_1(x)|u|^\theta \ln \frac{|u(x)|}{\|(u,v)\|}dx.$$

Using (2.7) by direct calculation gives that

$$\int_{\Omega_1} H_1(x)|u|^\theta \ln \frac{|u(x)|}{\|(u,v)\|}dx \quad \leq \quad C_{H_1}\|(u,v)\|^\theta \int_\Omega \frac{1}{e\theta}dx = C_{H_1}\|(u,v)\|^\theta |\Omega|\frac{1}{e\theta}.$$

By Lemma 2.6 for $\rho = p^*_{s_1} - \theta > 0$ and direct calculation

$$\begin{aligned}
\int_{\Omega_2} H_1(x)|u|^\theta \ln \frac{|u(x)|}{\|(u,v)\|}dx &\leq \frac{C_{H_1}}{e(p^*_{s_1}-\theta)} \int_\Omega |u|^\theta \left[\frac{|u(x)|}{\|(u,v)\|}\right]^{p^*_{s_1}-\theta} dx \\
&\leq \frac{C_{H_1}}{e(p^*_{s_1}-\theta)} \frac{1}{\|(u,v)\|^{p^*_{s_1}-\theta}} \int_\Omega |u|^{p^*_{s_1}} dx \\
&\leq \frac{C_{H_1}}{e(p^*_{s_1}-\theta)} \frac{1}{\|(u,v)\|^{p^*_{s_1}-\theta}} S^{p^*_{s_1}}_{p^*_{s_1}} \|u\|^{p^*_{s_1}}_{p^*_{s_1}} \\
&\leq \frac{C_{H_1}}{e(p^*_{s_1}-\theta)} S^{p^*_{s_1}}_{p^*_{s_1}} \|(u,v)\|^\theta.
\end{aligned}$$

Consequently, we get

$$\int_\Omega |u|^\theta \ln \left[\frac{|u(x)|}{\|(u,v)\|}\right]dx \leq C_{H_1}\left[\frac{|\Omega|}{e\theta} + \frac{S^{p^*_{s_1}}_{p^*_{s_1}}}{e(p^*_{s_1}-\theta)}\right]\|(u,v)\|^\theta.$$

On the other hand,

$$\begin{aligned}
&\lambda \int_\Omega H_1(x)|u|^\theta \ln|u|dx \\
&\quad = \lambda \int_\Omega H_1(x)|u|^\theta \ln\left[\frac{|u(x)|}{\|(u,v)\|}\right]dx + \ln(\|(u,v)\|)\int_\Omega \lambda H_1(x)|u|^\theta dx \\
&\quad \leq \lambda C_{H_1}\left[\frac{|\Omega|}{e\theta} + \frac{S^{p^*_{s_1}}}{e(p^*_{s_1}-\theta)}\right]\|(u,v)\|^\theta + C_{H_1}\ln(\|(u,v)\|)\int_\Omega \lambda|u|^\theta dx. \qquad (2.8)
\end{aligned}$$

Similarly,

$$\mu\int_\Omega H_2(x)|v|^\theta \ln|v|dx \leq \mu C_{H_2}\left[\frac{|\Omega|}{e\theta} + \frac{S^{p^*_{s_1}}}{e(p^*_t-\theta)}\right]\|(u,v)\|^\theta + C_{H_2}\ln(\|(u,v)\|)\int_\Omega \mu|v|^\theta dx. \quad (2.9)$$

Combining (2.8) and (2.9) follows the result. □

3 Nehari manifold and fibering map analysis

Continuing with the paper, for The main tool used in this paper is the so-called fibering method, introduced by Pohozaev [15], [16] and [17]. In this section, we assume $\lambda, \mu > 0$, and the functions $H_1, H_2, R_1, R_2 \in C(\overline{\Omega})$.

We define the Nehari Manifolds as:

$$\mathcal{N} := \{(u,v) \in W \setminus \{(0,0)\} \ \mid \ I(u,v) = 0\}.$$

For all $(u,v) \in \mathcal{N}$, we have $(u,v) \neq 0$ and

$$\|(u,v)\|^p_{s_1,p} + \|(u,v)\|^t_{s_2,t} = \lambda \int_\Omega H_1(x)|u|^\theta \ln|u|dx + \mu \int_\Omega H_2(x)|v|^\theta \ln|v|dx + \int_\Omega (R_1(x)+R_2(x))|u|^q|v|^r dx.$$

Now we define the fibering map $\Phi_{(u,v)} : (0,\infty) \to \mathbb{R}$, given by $\Phi_{(u,v)}(k) := E(k(u,v))$, $k > 0$, it is

$$\begin{aligned}\Phi_{(u,v)}(k) = {} & \frac{k^p}{p}\|(u,v)\|^p_{s_1,p} + \frac{k^t}{t}\|(u,v)\|^t_{s_2,t} - \frac{\lambda k^\theta}{\theta}\ln|k|\int_\Omega H_1(x)|u|^\theta dx - \frac{\lambda k^\theta}{\theta}\int_\Omega H_1(x)|u|^\theta \ln|u|dx \\ & + \frac{\lambda k^\theta}{\theta^2}\int_\Omega H_1(x)|u|^\theta dx - \frac{\mu k^\theta}{\theta}\ln|k|\int_\Omega H_2(x)|v|^\theta dx - \frac{\mu k^\theta}{\theta}\int_\Omega H_2(x)|v|^\theta \ln|v|dx \\ & + \frac{\mu k^\theta}{\theta^2}\int_\Omega H_2(x)|v|^\theta dx - \frac{k^{p^*_{s_1}}}{p^*_{s_1}}\int_\Omega (R_1(x)+R_2(x))|u|^q|v|^r dx. \end{aligned} \tag{3.1}$$

Differentiating (3.1) with respect to t, we have

$$\begin{aligned}\Phi'_{(u,v)}(k) = {} & k^{p-1}\|(u,v)\|^p_{s_1,p} + k^{t-1}\|(u,v)\|^t_{s_2,t} - \lambda k^{\theta-1}\int_\Omega H_1(x)|u|^\theta \ln|ku|dx \\ & - \mu k^{\theta-1}\int_\Omega H_2(x)|v|^\theta \ln|kv|dx - k^{p^*_{s_1}-1}\int_\Omega (R_1(x)+R_2(x))|u|^q|v|^r dx. \end{aligned} \tag{3.2}$$

Again, differentiating (3.2) with respect to t, we get

$$\begin{aligned}\Phi''_{(u,v)}(k) = {} & (p-1)k^{p-2}\|(u,v)\|^p_{s_1,p} + (t-1)k^{t-2}\|(u,v)\|^t_{s_2,t} - (\theta-1)k^{\theta-2}\lambda\int_\Omega H_1(x)|u|^\theta \ln|ku|dx \\ & - (\theta-1)k^{\theta-2}\mu\int_\Omega H_2(x)|v|^\theta \ln|kv|dx - \lambda k^{\theta-2}\int_\Omega H_1(x)|u|^\theta dx \\ & - \mu k^{\theta-2}\int_\Omega H_2(x)|v|^\theta dx - (p^*_{s_1}-1)k^{p^*_{s_1}-2}\int_\Omega (R_1(x)+R_2(x))|u|^q|v|^r dx. \end{aligned}$$

Thus, one can easily verify that for all $(u,v) \in \mathcal{N}$, $\Phi'_{(u,v)}(1) = 0$ and

$$\begin{aligned}\Phi''_{(u,v)}(1) = {} & (p-1)\|(u,v)\|^p_{s_1,p} + (t-1)\|(u,v)\|^t_{s_2,t} - \lambda(\theta-1)\int_\Omega H_1(x)|u|^\theta \ln|u|dx \\ & - \mu(\theta-1)\int_\Omega H_2(x)|v|^\theta \ln|v|dx - \lambda\int_\Omega H_1(x)|u|^\theta dx - \mu\int_\Omega H_2(x)|v|^\theta dx \\ & - (p^*_{s_1}-1)\int_\Omega (R_1(x)+R_2(x))|u|^q|v|^r dx, \end{aligned}$$

which implies that

$$\begin{aligned}\Phi''_{(u,v)}(1) = {} & (p-p^*_{s_1})\|(u,v)\|^p_{s_1,p} + (t-p^*_{s_1})\|(u,v)\|^t_{s_2,t} - (\theta-p^*_{s_1})\lambda\int_\Omega H_1(x)|u|^\theta \ln|u|dx \\ & - (\theta-p^*_{s_1})\mu\int_\Omega H_2(x)|v|^\theta \ln|v|dx - \lambda\int_\Omega H_1(x)|u|^\theta dx - \mu\int_\Omega H_2(x)|v|^\theta dx. \end{aligned}$$

As a consequence of the previously calculus, it's make possible to rewrite the Nehari manifold $\mathcal{N}$ as

$$\mathcal{N} = \mathcal{N}^+ \cup \mathcal{N}^0 \cup \mathcal{N}^-, \tag{3.3}$$

where

$$\mathcal{N}^+ = \Big\{(u,v)\in\mathcal{N} \;\mid\; \Phi''_{(u,v)}(1)>0\Big\} = \Big\{(tu,tv)\in W\setminus(0,0) \;\mid\; \Phi'_{(u,v)}(t)=0,\ \Phi''_{(u,v)}(t)>0\Big\};$$
$$\mathcal{N}^0 = \Big\{(u,v)\in\mathcal{N} \;\mid\; \Phi''_{(u,v)}(1)=0\Big\} = \Big\{(tu,tv)\in W\setminus(0,0) \;\mid\; \Phi'_{(u,v)}(t)=0,\ \Phi''_{(u,v)}(t)=0\Big\};$$
$$\mathcal{N}^- = \Big\{(u,v)\in\mathcal{N} \;\mid\; \Phi''_{(u,v)}(1)<0\Big\} = \Big\{(tu,tv)\in W\setminus(0,0) \;\mid\; \Phi'_{(u,v)}(t)=0,\ \Phi''_{(u,v)}(t)<0\Big\}.$$

Lemma 3.1. *Assume $(u,v)\in W\setminus\{(0,0)\}$ and $k>0$. Then $(ku,kv)\in\mathcal{N}$ if, and only if, $\Phi'_{(u,v)}(k)=0$.*

Proof. If $k(u,v)\in\mathcal{N}$, for $k>0$ we have $I(ku,kv)=0$. Then, we get

$$0 = k^p\|(u,v)\|^p_{s_1,p} + k^t\|(u,v)\|^t_{s_2,t} - \lambda k^\theta\int_\Omega H_1(x)|u|^\theta\ln|ku|dx - \mu k^{p^*_{s_1}}\int_\Omega (R_1+R_2)(x)|u|^q|v|^r dx.$$

Dividing the above equation for $k>0$, we have

$$\begin{aligned}0 = {} & k^{p-1}\|(u,v)\|^p_{s_1,p} + k^{t-1}\|(u,v)\|^t_{s_2,t} - \lambda k^{\theta-1}\int_\Omega H_1(x)|u|^\theta\ln|ku|dx\\ & - \mu k^{\theta-1}\int_\Omega H_2(x)|u|^\theta\ln|kv|dx - k^{p^*_{s_1}-1}\int_\Omega (R_1+R_2)(x)|u|^q|v|^r dx = \Phi'(k).\end{aligned}$$

This completes the proof. □

Lemma 3.2. *If (u,v) is a local minimizer for the functional E on $\mathcal{N}$, with $(u,v)\notin\mathcal{N}^0$. Then, $E'(u,v)=0$.*

Proof. By the assumption for $u\in\mathcal{N}$, applying Lagrange's multipliers, there exists $\gamma\in\mathbb{R}$ such that

$$E'(u,v)(u,v) = \gamma I'(u,v)(u,v). \tag{3.4}$$

But because of $(u,v)\in\mathcal{N}$, we get

$$\begin{aligned}E'(u,v)(u,v) = {} & \|(u,v)\|^p_{s_1,p} + \|(u,v)\|^t_{s_2,t} - \lambda\int_\Omega H_1(x)|u|^\theta\ln|u|dx - \mu\int_\Omega H_2(x)|v|^\theta\ln|v|dx\\ & - \int_\Omega (R_1+R_2)(x)|u|^q|v|^r dx = 0.\end{aligned}$$

Thus, $\gamma I'(u,v)(u,v)=0$. Now since $(u,v)\in\mathcal{N}^0$, we get

$$\begin{aligned}I'(u,v)(u,v) = {} & (p-p^*_{s_1})\|(u,v)\|^p_{s_1,p} + (t-p^*_{s_1})\|(u,v)\|^t_{s_2,t}\\ & - (\theta-p^*_{s_1})\lambda\int_\Omega H_1(x)|u|^\theta\ln|u|dx - (\theta-p^*_{s_1})\mu\int_\Omega H_2(x)|v|^\theta\ln|v|dx\\ & - \lambda\int_\Omega H_1(x)|u|^\theta dx - \mu\int_\Omega H_2(x)|v|^\theta dx = \Phi''_{(u,v)}(1).\end{aligned}$$

Thus, $\gamma\neq 0$, and $E'(u,v)=0$. □

Lemma 3.3. *Let $0<\lambda C_{H_1}+\mu C_{H_2}<\frac{1}{L}$ and $2<q+r<p^*_{s_1}$. If $R_1,R_2\in L^\infty(\Omega)$ are non-negative functions satisfying $C_B<K$, where $C_B:=\max_{x\in\overline{\Omega}}|R_1(x)+R_2(x)|$ then, for any $(u,v)\in W\setminus\{(0,0)\}$, we get*

1) *If $\lambda\int_\Omega H_1(x)|u|^\theta+\mu\int_\Omega H_2(x)|v|^\theta dx\geq 0$, then there exists a unique $k_{(u,v)}>0$ such that $\Phi'_{(u,v)}(k_{(u,v)})=0$ and $k_{(u,v)}(u,v)\in\mathcal{N}^-$. Moreover,*

$$E(k_{(u,v)}u,k_{(u,v)}v) = \sup_{k>0}E(ku,kv).$$

2) *If* $\lambda \int_\Omega H_1(x)|u|^\theta + \mu \int_\Omega H_2(x)|v|^\theta dx < 0$, *then there exists* $k_1, k_2 > 0$ *such that* $0 < k_1 < k_{\max} < k_2 < \infty$ *where* $\Phi'_{(u,v)}(k_1) = 0 = \Phi'_{(u,v)}(k_2)$ *witch* $(k_1u, k_1v) \in \mathcal{N}^+$ *and* $(k_2u, k_2v) \in \mathcal{N}^-$. *Moreover,*

$$E(k_1u, k_1v) = \inf_{0<k<k_{\max}} E(ku, kv) \quad \text{and} \quad E(k_2u, k_2v) = \sup_{k>0} E(ku, kv).$$

Proof. For $k > 0$, we defined a function $f : \mathbb{R} \to W$ given by

$$f(k) = k^{p-p^*_{s_1}} \Big[\|(u,v)\|^p_{s_1,p} + \|(u,v)\|^t_{s_2,t} - \lambda \ln|k| \int_\Omega H_1(x)|u|^\theta dx - \lambda \int_\Omega H_1(x)|u|^\theta \ln|u| dx$$
$$- \mu \ln|k| \int_\Omega H_2(x)|v|^\theta dx - \mu \int_\Omega H_2(x)|v|^\theta \ln|v| dx \Big].$$

We can rewrite $f(k)$, for $k > 0$, as

$$f(k) = k^{p-p^*_{s_1}} \left(\|(u,v)\|^p_{s_1,p} + \|(u,v)\|^t_{s_2,t} \right) - k^{p-p^*_{s_1}} \ln|k| \lambda \int_\Omega H_1(x)|u|^\theta dx$$
$$- k^{p-p^*_{s_1}} \lambda \int_\Omega H_1(x)|u|^\theta \ln|u| dx - k^{p-p^*_{s_1}} \ln|k| \lambda \int_\Omega H_2(x)|v|^\theta dx$$
$$- k^{p-p^*_{s_1}} \lambda \int_\Omega H_2(x)|v|^\theta \ln|v| dx.$$

A direct computation shows that

$$f'(k) = k^{p-p^*_{s_1}-1} \Big[(p - p^*_{s_1}) \left(\|(u,v)\|^p_{s_1,p} + \|(u,v)\|^t_{s_2,t} \right) - \lambda (p - p^*_{s_1}) \int_\Omega H_1(x)|u|^\theta dx$$
$$- \lambda (p - p^*_{s_1}) \int_\Omega H_1(x)|u|^\theta \ln|u| dx$$
$$- \mu \int_\Omega H_2(x)|v|^\theta dx - \mu (p - p^*_{s_1}) \int_\Omega H_2(x)|v|^\theta \ln|v| dx$$
$$- (p - p^*_{s_1}) \ln|k| \Big(\lambda \int_\Omega H_1(x)|u|^\theta dx + \mu \int_\Omega H_2(x)|v|^\theta dx \Big) \Big]$$

Now we analyze all the possibilities:

i) If $\lambda \int_\Omega H_1(x)|u|^\theta dx + \mu \int_\Omega H_2(x)|v|^\theta dx > 0$, then $f \in C(0,\infty)$ and because $p^*_{s_1} > p > t$, we have

$$\lim_{k\to 0^+} f(k) = +\infty \quad \text{and} \quad \lim_{k\to\infty} f(k) = 0.$$

Then there exists a unique minimum point $k_{\min} > 0$ such that $f'(k_{\min}) = 0$. Because $k_{\min} > 0$,

$$(p - p^*_{s_1}) \ln|k_{\min}| \Big[\lambda \int_\Omega H_1(x)|u|^\theta dx + \mu \int_\Omega H_2(x)|v|^\theta dx \Big]$$
$$= (p - p^*_{s_1}) \left[\|(u,v)\|^p_{s_1,p} + \|(u,v)\|^t_{s_2,t} \right]$$
$$- \lambda \int_\Omega H_1(x)|u|^\theta dx - \lambda (p - p^*_{s_1}) \int_\Omega H_1(x)|u|^\theta \ln|u| dx$$
$$- \mu \int_\Omega H_2(x)|u|^\theta dx - \mu (p - p^*_{s_1}) \int_\Omega H_2(x)|v|^\theta \ln|v| dx.$$

This implies that

$$k_{\min} = \exp\left[\frac{\|(u,v)\|^p_{s_1,p} + \|(u,v)\|^t_{s_2,t} - \int_\Omega [\lambda H_1(x)|u|^\theta \ln|u| + \mu H_2(x)|v|^\theta \ln|v|] dx}{\lambda \int_\Omega H_1(x)|u|^\theta dx + \mu \int_\Omega H_2(x)|v|^\theta dx} - \frac{1}{p - p^*_{s_1}} \right]. \tag{3.5}$$

Obviously, f is decreasing on $(0,k_{\min})$ and increasing on $(k_{\min},\infty)$. Then, because of equation (3.5), the fact that $k_{\min}>0$ and that

$$\lambda\int_\Omega H_1(x)|u|^\theta dx+\mu\int_\Omega H_2(x)|v|^\theta dx>0,$$

we get

$$\begin{aligned}f(k_{\min})=k_{\min}^{p-p^*_{s_1}}\Bigg[&\|(u,v)\|^p_{s_1,p}+\|(u,v)\|^t_{s_2,t}\\&-\ln|k_{\min}|\left(\lambda\int_\Omega H_1(x)|u|^\theta dx+\mu\int_\Omega H_2(x)|v|^\theta dx\right)\\&-\left(\lambda\int_\Omega H_1(x)|u|^\theta\ln|u|dx+\mu\int_\Omega H_2(x)|v|^\theta\ln|v|dx\right)\Bigg]\\=k_{\min}^{p-p^*_{s_1}}&\left[\frac{1}{p-p^*_{s_1}}\left(\lambda\int_\Omega H_1(x)|u|^\theta dx+\mu\int_\Omega H_2(x)|v|^\theta dx\right)\right]<0.\end{aligned}$$

Since we have

$$\int_\Omega(R_1+R_2)(x)|u|^q|v|^r dx>0,\quad\text{and}\quad\lambda\int_\Omega H_1(x)|u|^\theta\ln|u|dx+\mu\int_\Omega H_2(x)|v|^\theta\ln|v|dx>0,$$

and because

$$\begin{aligned}\Phi'_{(u,v)}(k)=k^{p^*_{s_1}+\theta-p-1}\Big[&f(k)-k^{2p-p^*_{s_1}-\theta}\|(u,v)\|^p_{s_1,p}+k^{2t-p^*_{s_1}-\theta}\|(u,v)\|^t_{s_1,t}\\&-k^{p-p^*_{s_1}}\left(\|(u,v)\|^p_{s_1,p}+\|(u,v)\|^t_{s_2,t}\right)-k^{p-\theta}\int_\Omega(R_1+R_2)(x)|u|^q|v|^r dx\Big],\end{aligned}$$

there exists a unique $k_{(u,v)}$ such that $0<k_{(u,v)}<k_{\min}$ such that

$$\begin{aligned}f(k_{(u,v)})=k_{(u,v)}^{p-p^*_{s_1}}\Big[&\|(u,v)\|^p_{s_1,p}+\|(u,v)\|^t_{s_2,t}-\lambda\ln|k_{(u,v)}|\int_\Omega H_1(x)|u|^\theta dx\\&-\lambda\int_\Omega H_1(x)|u|^\theta\ln|u|dx-\mu\ln|k_{(u,v)}|\int_\Omega H_2(x)|v|^\theta dx\\&-\mu\int_\Omega H_2(x)|v|^\theta\ln|v|dx\Big]\end{aligned}$$

and $f'(k_{(u,v)})<0$, we get $k_{(u,v)}(u,v)\in\mathcal{N}^-$. Moreover, it follows from $f(k)<f(k_{(u,v)})$, for all $k>k_{(u,v)}$ and $f(k)>f(k_{(u,v)})$, for all $k<k_{(u,v)}$, that

$$E(k_{(u,v)}(u,v))=\sup_{k>0}E(ku,kv).$$

ii) If $\lambda\int_\Omega H_1(x)|u|^\theta dx+\mu\int_\Omega H_2(x)|v|^\theta dx=0$, it follow from Lemma 2.10 there exists a unique $k_{(u,v)}>0$ such that $k_{(u,v)}(u,v)\in\mathcal{N}^-$, and

$$E(k_{(u,v)})=\sup_{k>0}E(ku,kv).$$

iii) If $\lambda \int_\Omega H_1(x)|u|^\theta dx + \mu \int_\Omega H_2(x)|v|^\theta dx < 0$, then $f \in C(0,\infty)$, $\lim_{k\to 0^+} f(k) = -\infty$ and $\lim_{k\to\infty} f(k) = 0$. Then f has a unique maximum point $k_{\max} > 0$ which is given by

$$k_{\max} = \exp\left[\frac{\|(u,v)\|^p_{s_1,p} + \|(u,v)\|^t_{s_2,t} - \int_\Omega [\lambda H_1(x)|u|^\theta \ln|u| + \mu H_2(x)|v|^\theta \ln|v|]dx}{\lambda \int_\Omega H_1(x)|u|^\theta dx + \mu \int_\Omega H_2(x)|v|^\theta dx} - \frac{1}{p - p^*_{s_1}}\right].$$

Moreover, f is increasing on $(0, k_{\max})$ and decreasing on $(k_{\max}, \infty)$. By Lemma 2.10, we get

$$k_{\max} \geq \exp\left[\frac{(1 - (\lambda C_{H_1} + \mu C_{H_2})L)\left(\|(u,v)\|^p_{s_1,p} + \|(u,v)\|^t_{s_2,t}\right)}{\lambda \int_\Omega H_1(x)|u|^\theta dx + \mu \int_\Omega H_2(x)|v|^\theta dx} - \ln\|(u,v)\| - \frac{1}{p - p^*_{s_1}}\right].$$

So,

$$k^{p-p^*_{s_1}}_{\max} \geq \exp\left[(p - p^*_{s_1})\frac{(1 - (\lambda C_{H_1} + \mu C_{H_2})L)\left(\|(u,v)\|^p_{s_1,p} + \|(u,v)\|^t_{s_2,t}\right)}{\lambda \int_\Omega H_1(x)|u|^\theta dx + \mu \int_\Omega H_2(x)|v|^\theta dx} - 1\right] \|(u,v)\|^{p^*_{s_1}}.$$

It's known that the following inequality holds:

$$\exp(k-1) \geq k, \qquad \forall\, k \geq 0.$$

Then

$$k^{p-p^*_{s_1}}_{\max} \geq (p - p^*_{s_1})\frac{1 - (\lambda C_{H_1} + \mu C_{H_2})L}{\lambda \int_\Omega H_1(x)|u|^\theta dx + \mu \int_\Omega H_2(x)|v|^\theta dx}\|(u,v)\|^{p^*_{s_1}}.$$

Therefore

$$\begin{aligned} f(k_{\max}) &= k^{p-p^*_{s_1}}_{\max}\frac{\lambda \int_\Omega H_1(x)|u|^\theta dx + \mu \int_\Omega H_2(x)|v|^\theta dx}{p - p^*_{s_1}} \\ &\geq (1 - (\lambda C_{H_1} + \mu C_{H_2})L)\|(u,v)\|^{p^*_{s_1}}. \end{aligned}$$

Because $\left(k^{2p-p^*_{s_1}-\theta}_{(u,v)} + k^{p-p^*_{s_1}}_{(u,v)}\right)\left(\|(u,v)\|^p_{s_1,p} + \|(u,v)\|^t_{s_2,t}\right) + k^{p-\theta}_{(u,v)} \int_\Omega (R_1 + R_2)(x)|u|^q|v|^r dx > 0$, by Hölder's inequality,

$$\int_\Omega (R_1 + R_2)(x)|u|^p|v|^r dx \leq C_B S^{\frac{p}{p^*_{s_1}}}_{q,r}\|(u,v)\|^{p^*_{s_1}},$$

for all $(u,v) \in W$, where $C_B = \max_{x\in\overline{\Omega}} |R_1(x) + R_2(x)|$.

Then

$$(1 - (\lambda C_{H_1} + \mu C_{H_2})L)S^{\frac{p}{p^*_{s_1}}}_{q,r} > C_B,$$

it implies that

$$\begin{aligned} f(k_{\max}) &> k^{2p-p^*_{s_1}-\theta}_{(u,v)}\|(u,v)\|^p_{s_1,p} + k^{2t-p^*_{s_1}-\theta}_{(u,v)}\|(u,v)\|^t_{s_2,t} + k^{p-\theta}_{(u,v)}\int_\Omega (R_1 + R_2)(x)|u|^q|v|^r dx \\ &> 0 = \lim_{k\to\infty} f(k), \end{aligned}$$

and it shows us that there exists k_1, k_2 for which

$$0 < k_1 < k_{\max} < k_2 < \infty,$$

such that

$$f(k_1) = k_{(u,v)}^{2p-p_{s_1}^*-\theta}\|(u,v)\|_{s_1,p}^p + k_{(u,v)}^{2t-p_{s_1}^*-\theta}\|(u,v)\|_{s_2,t}^t + k_{(u,v)}^{p-\theta}\int_\Omega (R_1+R_2)(x)|u|^q|v|^r dx$$
$$> 0 = \lim_{k\to\infty} f(k) = f(k_2).$$

It shows us that $\Phi'_{(u,v)}(k_1) = 0 = \Phi'_{(u,v)}(k_2)$. Moreover, f is increasing on $(0, k_{\max})$ and decreasing on $(k_{\max}, \infty)$. So

$$k_1(u,v) \in \mathcal{N}^+ \quad \text{and} \quad k_2(u,v) \in \mathcal{N}^-.$$

Furthermore, we have for all $k \in [k_1, k_2]$, that

$$f(k) \geq k_{(u,v)}^{2p-p_{s_1}^*-\theta}\|(u,v)\|_{s_1,p}^p + k_{(u,v)}^{2t-p_{s_1}^*-\theta}\|(u,v)\|_{s_2,t}^t + k_{(u,v)}^{p-\theta}\int_\Omega (R_1+R_2)(x)|u|^q|v|^r dx$$
$$> 0 = \lim_{k\to\infty} f(k),$$

and

$$f(k) < k_{(u,v)}^{2p-p_{s_1}^*-\theta}\|(u,v)\|_{s_1,p}^p + k_{(u,v)}^{2t-p_{s_1}^*-\theta}\|(u,v)\|_{s_2,t}^t + k_{(u,v)}^{p-\theta}\int_\Omega (R_1+R_2)(x)|u|^q|v|^r dx,$$

for all $k \in \mathbb{R}_+^* \setminus [k_1, k_2]$.

Thus

$$E(k_1(u,v)) = \inf_{0<k\leq k_{\max}} E(k(u,v)) \qquad \text{and,} \qquad E(k_2(u,v)) = \sup_{k>0} E(k(u,v)).$$

The proof is complete. □

The next remark will help in understanding the next results.

Remark 3.4. In what follows, we define

$$M := \frac{|\Omega| S_{q,r}^{\frac{p_{s_1}^*}{p}}}{p_{s_1}^* - p}, \quad \text{and} \quad K := \frac{1-(\lambda C_{H_1} + \mu C_{H_2})(L+M)}{S_{q,r}^{\frac{p_{s_1}^*}{p}}}.$$

Lemma 3.5. *If $0 < \lambda C_{H_1} + \mu C_{H_2} < \frac{1}{L+M}$ and $C_B < K$, then $\mathcal{N}^0 = \emptyset$.*

Proof. Arguing by contradiction, let $(u,v) \in \mathcal{N}^0$. Then $I(u,v) = 0$, and

$$\left(\|(u,v)\|_{s_1,p}^p + \|(u,v)\|_{s_2,t}^t\right) - \lambda\int_\Omega H_1(x)|u|^\theta dx - \mu\int_\Omega H_2(x)|v|^\theta dx$$
$$= \int_\Omega (R_1+R_2)(x)|u|^q|v|^r dx \quad (3.6)$$

and

$$0 = (p-p_{s_1}^*)\|(u,v)\|_{s_1,p}^p + (t-p_{s_1}^*)\|(u,v)\|_{s_2,t}^t - (\theta-p_{s_1}^*)\lambda\int_\Omega H_1(x)|u|^\theta \ln|u| dx$$
$$- (\theta-p_{s_1}^*)\mu\int_\Omega H_2(x)|v|^\theta \ln|v| dx - \lambda\int_\Omega H_1(x)|u|^\theta dx - \mu\int_\Omega H_2(x)|v|^\theta dx. \quad (3.7)$$

From equations (3.6) and (3.7), and Lemma 2.10, we get

$$\begin{aligned}
&\left(\|(u,v)\|_{s_1,p}^p + \|(u,v)\|_{s_2,t}^t\right) - \int_\Omega (R_1+R_2)(x)|u|^\theta dx \\
&\quad \leq (\lambda C_{H_1} + \mu C_{H_2}) L \left(\|(u,v)\|_{s_1,p}^p + \|(u,v)\|_{s_2,t}^t\right) \\
&\qquad + \ln\|(u,v)\| \left(\lambda \int_\Omega H_1(x)|u|^\theta dx + \mu \int_\Omega H_2(x)|v|^\theta dx\right). \qquad (3.8)
\end{aligned}$$

Then, again by (3.7)

$$\begin{aligned}
&(1-(\lambda C_{H_1}+\mu C_{H_2})L)\left(\|(u,v)\|_{s_1,p}^p + \|(u,v)\|_{s_2,t}^t\right) \\
&\quad \leq \ln\|(u,v)\| \left[\lambda \int_\Omega H_1(x)|u|^\theta dx + \mu \int_\Omega H_2(x)|v|^\theta dx\right] + \int_\Omega (R_1+R_2)(x)|v|^q|v|^r dx \\
&\quad \leq \ln\|(u,v)\| \left[\lambda \int_\Omega H_1(x)|u|^\theta dx + \mu \int_\Omega H_2(x)|v|^\theta dx\right] \\
&\qquad + \frac{1}{p-p_{s_1}^*}\left[-\lambda(p-\theta)\int_\Omega H_1(x)|u|^\theta \ln|u| dx\right. \\
&\qquad \left. -\mu(p-\theta)\int_\Omega H_2(x)|v|^\theta \ln|v| dx + \lambda\int_\Omega H_1(x)|u|^\theta dx + \mu\int_\Omega H_2(x)|v|^\theta dx\right].
\end{aligned}$$

That is,

$$\begin{aligned}
&(1-(\lambda C_{H_1}+\mu C_{H_2})L)\left(\|(u,v)\|_{s_1,p}^p + \|(u,v)\|_{s_2,t}^t\right) \\
&\quad \leq \ln\|(u,v)\| \left(\lambda \int_\Omega H_1(x)|u|^\theta dx + \mu \int_\Omega H_2(x)|v|^\theta dx\right) \\
&\qquad + \frac{1}{p-p_{s_1}^*}\left[-\lambda(p-\theta)\int_\Omega H_1(x)|u|^\theta \ln|u| dx - \mu(p-\theta)\int_\Omega H_2(x)|v|^\theta \ln|v| dx\right] \\
&\qquad + \frac{\lambda C_{H_1}+\mu C_{H_2}}{p_{s_1}^*-p}\left[|\Omega|^{p_s^*} S_{q,r}^{\frac{p_{s_1}^*}{p}}\right]\left(\|(u,v)\|_{s_1,p}^p + \|(u,v)\|_{s_2,t}^t\right), \qquad (3.9)
\end{aligned}$$

which means that

$$\begin{aligned}
&\left[1-(\lambda C_{H_1}+\mu C_{H_2})L - \frac{\lambda C_{H_1}+\mu C_{H_2}}{p_{s_1}^*-p}\left[|\Omega|^{p_s^*} S_{q,r}^{\frac{p_{s_1}^*}{p}}\right]\right]\left(\|(u,v)\|_{s_1,p}^p + \|(u,v)\|_{s_2,t}^t\right) \\
&\quad \leq \ln\|(u,v)\| \left[\lambda \int_\Omega H_1(x)|u|^\theta dx + \mu \int_\Omega H_2(x)|v|^\theta dx\right] \\
&\qquad + \frac{1}{p-p_{s_1}^*}\left(-\lambda(p-\theta)\int_\Omega H_1(x)|u|^\theta \ln|u| dx - \mu(p-\theta)\int_\Omega H_2(x)|v|^\theta \ln|v| dx\right).
\end{aligned}$$

Using again equation (3.7), in view of $p < p_{s_1}^*$, we have

$$\begin{aligned}
0 > &\ln\|(u,v)\| \left(\lambda \int_\Omega H_1(x)|u|^\theta dx + \mu \int_\Omega H_2(x)|v|^\theta dx\right) \\
&+ \frac{1}{p-p_{s_1}^*}\left(-\lambda(p-\theta)\int_\Omega H_1(x)|u|^\theta \ln|u| dx - \mu(p-\theta)\int_\Omega H_2(x)|v|^\theta \ln|v| dx\right).
\end{aligned}$$

Thus, it follows from $\lambda C_{H_1} + \mu C_{H_2} < \frac{1}{L+M}$ that $\|(u,v)\| \leq 1$, where, here we consider M as in Remark 3.4.

Otherwise, using (3.9), and one more time (3.6) and (3.7),

$$\begin{aligned}
&\left[1-(\lambda C_{H_1}+\mu C_{H_2})L\right]\left(\|(u,v)\|_{s_1,p}^{p}+\|(u,v)\|_{s_2,t}^{t}\right)\\
&\quad\leq \ln\|(u,v)\|\Big(\lambda\int_\Omega H_1(x)|u|^{\theta}dx+\mu\int_\Omega H_2(x)|v|^{\theta}dx\Big)+\int_\Omega (R_1+R_2)(x)|u|^{q}|v|^{r}dx\\
&\quad\leq (p-p_{s_1}^{*})\ln\|(u,v)\|\int_\Omega (R_1+R_2)(x)|u|^{q}|v|^{r}dx\\
&\qquad+(\lambda C_{H_1}+\mu C_{H_2})M\left(\|(u,v)\|_{s_1,p}^{p}+\|(u,v)\|_{s_2,t}^{t}\right).
\end{aligned}\tag{3.10}$$

Now, we can note that

$$\int_\Omega (R_1+R_2)(x)|u|^{q}|v|^{r}dx\leq C_B S_{q,r}^{\frac{p_{s_1}^{*}}{p}}\|(u,v)\|^{p_{s_1}^{*}},\tag{3.11}$$

thus

$$\left[1-(\lambda C_{H_1}+\mu C_{H_2})(L+M)-C_B S_{q,r}^{\frac{p_{s_1}^{*}}{p}}\right]\left(\|(u,v)\|_{s_1,p}^{p}+\|(u,v)\|_{s_2,t}^{t}\right)\leq 0.$$

Since $(\lambda C_{H_1}+\mu C_{H_2})<\frac{1}{L+M}$, and

$$C_B<K:=\frac{1-(\lambda C_{H_1}+\mu C_{H_2})(L+M)}{S_{q,r}^{\frac{p_{s_1}^{*}}{p}}},$$

it follows that $\|(u,v)\|=0$. Then $(u,v)=(0,0)$, which is a contradiction. Therefore $\mathcal{N}^0=\emptyset$. □

For the next result, we consider

$$\Lambda_{\lambda,\mu}:=\min\left\{1,\left(\frac{1-(\lambda C_{H_1}+\mu C_{H_2})L-C_B S_{q,r}^{\frac{p_{s_1}^{*}}{p}}}{C_B S_{q,r}^{\frac{p_{s_1}^{*}}{p}}}\right)^{\frac{1}{p_{s_1}^{*}-p}}\right\}.$$

Lemma 3.6. *If $0<\lambda C_{H_1}+\mu C_{H_2}<\frac{1}{L+M}$ and $C_B<K$, then $\|(U,V)\|\geq\Lambda_{\lambda,\mu}$, for all $(U,V)\in\mathcal{N}^{-}$ and $\|(u,v)\|\leq 1$, for all $(u,v)\in\mathcal{N}^{+}$.*

Proof. Let $(U,V)\in\mathcal{N}^{-}$. Then

$$\begin{aligned}
&\int_\Omega (R_1+R_2)(x)|U|^{q}|V|^{r}dx\\
&\quad=\left(\|(u,v)\|_{s_1,p}^{p}+\|(u,v)\|_{s_2,t}^{t}\right)-\lambda\int_\Omega H_1(x)|U|^{\theta}\ln|U|dx-\mu\int_\Omega H_2(x)|V|^{\theta}\ln|V|dx,
\end{aligned}\tag{3.12}$$

and

$$\begin{aligned}
&(p-p_{s_1}^{*})\|(U,V)\|_{s_1,p}^{p}+(t-p_{s_1}^{*})\|(U,V)\|_{s_2,t}^{t}\\
&\quad<\lambda\int_\Omega H_1(x)|U|^{\theta}dx+\mu\int_\Omega H_2(x)|V|^{\theta}dx\\
&\qquad-(p_{s_1}^{*}-\theta)\Big(\lambda\int_\Omega H_1(x)|U|^{\theta}\ln|U|dx+\mu\int_\Omega H_2(x)|V|^{\theta}\ln|V|dx\Big).
\end{aligned}\tag{3.13}$$

Similarly to equation (3.10) in Lemma 3.5, we have

$$\begin{aligned}
&\left[1-(\lambda C_{H_1}+\mu C_{H_2})L\right]\|(u,v)\left(\|_{s_1,p}^{p}+\|(u,v)\|_{s_2,t}^{t}\right)\\
&\quad\leq\ln\|(U,V)\|\left(\lambda\int_\Omega H_1(x)|U|^{\theta}dx+\mu\int_\Omega H_2(x)|V|^{\theta}dx\right)+C_B S_{p_{s_1}^{*}}^{q}S_{p_{s_1}^{*}}^{r}\|(U,V)\|^{p_{s_1}^{*}}.
\end{aligned}\tag{3.14}$$

We consider two cases:

i) If $\lambda \int_\Omega H_1(x)|U|^\theta dx + \mu \int_\Omega H_2(x)|V|^\theta dx > 0$, then $\|(U,V)\| \geq 1$, because otherwise, being $\|(U,V)\| < 1$, we have

$$[1-(\lambda C_{H_1}+\mu C_{H_2})L]\left(\|(u,v)\|^p_{s_1,p}+\|(u,v)\|^t_{s_2,t}\right) < C_B S^q_{p^*_{s_1}} S^r_{p^*_{s_1}} \|(U,V)\|^{p^*_{s_1}},$$

and together with $C_B < K$, we get

$$\|(U,V)\| > \left(\frac{1-(\lambda C_{H_1}+\mu C_{H_2})L}{C_B S^q_{p^*_{s_1}} S^r_{p^*_{s_1}}}\right)^{\frac{1}{p^*_{s_1}-p}} > 1, \tag{3.15}$$

which is a contradiction. Thus $\|(U,V)\| \geq 1$.

ii) If $\lambda \int_\Omega H_1(x)|U|^\theta dx + \mu \int_\Omega H_2(x)|V|^\theta dx < 0$, we have two more cases to analyse:

2.1) If $\|(U,V)\| > 1$, similar to case i) above, we get the same equation (3.15).

2.2) If $\|(U,V)\| < 1$, we have by equations (3.12) and (3.13),

$$\begin{aligned}
&[1-(\lambda C_{H_1}+\mu C_{H_2})L]\left(\|(u,v)\|^p_{s_1,p}+\|(u,v)\|^t_{s_2,t}\right)\\
&\quad < C_B S^q_{p^*_{s_1}} S^r_{p^*_{s_1}} \|(U,V)\|^p\\
&\qquad + (p-\theta)\left(\lambda \int_\Omega H_1(x)|U|^\theta \ln|U| dx + \mu \int_\Omega H_2(x)|V|^\theta \ln|V| dx\right)\\
&\qquad + C_B S^q_{p^*_{s_1}} S^r_{p^*_{s_1}} \|(U,V)\|^{p^*_{s_1}}.
\end{aligned}$$

So, because $p \leq \theta$,

$$\|(U,V)\| > \left(\frac{1-(\lambda C_{H_1}+\mu C_{H_2})L - C_B S^q_{p^*_{s_1}} S^r_{p^*_{s_1}}}{C_B S^q_{p^*_{s_1}} S^r_{p^*_{s_1}}}\right)^{\frac{1}{p^*_{s_1}-p}} > 1.$$

Thus, $\|(U,V)\| \geq \Lambda_{\lambda,\mu}$, for all $(U,V) \in \mathcal{N}^-$.

Now, if $(u,v) \in \mathcal{N}^+$, a similar discussion shows us that $\|(u,v)\| \leq 1$. □

Lemma 3.7. *If $0 < \lambda C_{H_1} + \mu C_{H_2} < \frac{1}{L+M}$ and $C_B < K$, then $\mathcal{N}^-$ is a closed subset o W.*

Proof. The proof follows directly from Lemmas 3.5 and 3.6. □

Lemma 3.8. *If $0 < \lambda C_{H_1} + \mu C_{H_2} < \frac{1}{L+M}$. Then the functional E is bounded from below on $\mathcal{N}$.*

Proof. Because of the relationship between the functionals E and I, for $(u,v) \in \mathcal{N}$, we have

$$\begin{aligned}
E(u,v) = &\left(\frac{1}{p}-\frac{1}{\theta}\right)\left(\|(u,v)\|^p_{s_1,p}+\|(u,v)\|^t_{s_2,t}\right) + \frac{1}{\theta^2}\left(\lambda\int_\Omega H_1(x)|u|^\theta dx + \mu\int_\Omega H_2(x)|v|^\theta dx\right)\\
&- \left(\frac{1}{p^*_{s_1}}-\frac{1}{\theta}\right)\int_\Omega (R_1+R_2)(x)|u|^q|v|^r dx.
\end{aligned}$$

Because $(u,v) \in \mathcal{N}$, we have also

$$\begin{aligned}
&\int_\Omega (R_1+R_2)(x)|u|^q|v|^r dx\\
&\quad = \left(\|(u,v)\|^p_{s_1,p}+\|(u,v)\|^t_{s_2,t}\right) - \lambda\int_\Omega H_1(x)|u|^\theta \ln|u| dx - \mu\int_\Omega H_2(x)|v|^\theta \ln|v| dx.
\end{aligned}$$

So

$$\begin{aligned}E(u,v) = &\left(\frac{1}{p}-\frac{1}{p^*_{s_1}}\right)\|(u,v)\|^p_{s_1,p}+\left(\frac{1}{t}-\frac{1}{p^*_{s_1}}\right)\|(u,v)\|^t_{s_2,t}\\&+\frac{1}{\theta^2}\Big(\lambda\int_\Omega H_1(x)|u|^\theta dx+\mu\int_\Omega H_2(x)|v|^\theta dx\Big)\\&+\left(\frac{1}{\theta}-\frac{1}{p^*_{s_1}}\right)\left(\lambda\int_\Omega H_1(x)|u|^\theta\ln|u|dx+\mu\int_\Omega H_2(x)|v|^\theta\ln|v|dx\right).\end{aligned}$$

Now, concerning to exposed above, we consider two cases:

i) If $\lambda\int_\Omega H_1(x)|u|^\theta dx+\mu\int_\Omega H_2(x)|v|^\theta dx>0$, because $0<t<p<\theta<p^*_{s_1}$, it follows that

$$E(u,v)>\left(\frac{1}{p}-\frac{1}{p^*_{s_1}}\right)\|(u,v)\|^p_{s_1,p}+\left(\frac{1}{t}-\frac{1}{p^*_{s_1}}\right)\|(u,v)\|^t_{s_2,t}\geq 0.$$

ii) If $\lambda\int_\Omega H_1(x)|u|^\theta dx+\mu\int_\Omega H_2(x)|v|^\theta dx\leq 0$, it follows from Lemma 2.10 that

$$\begin{aligned}E(u,v) = &\left(\frac{1}{p}-\frac{1}{p^*_{s_1}}\right)[1-(\lambda C_{H_1}+\mu C_{H_2})L]\|(u,v)\left(\|^p_{s_1,p}+\|(u,v)\|^t_{s_2,t}\right)\\&+\left[\frac{1}{\theta^2}-\left(\frac{1}{\theta}-\frac{1}{p^*_{s_1}}\right)\ln\|(u,v)\|\right]\left(\lambda\int_\Omega H_1(x)|u|^\theta dx+\mu\int_\Omega H_2(x)|v|^\theta dx\right).\end{aligned}$$

Now, if $\frac{1}{\theta^2}-\left(\frac{1}{\theta}-\frac{1}{p^*_{s_1}}\right)\ln\|(u,v)\|\geq 0$, then $E(u,v)\geq 0$.

Otherwise, if $\frac{1}{\theta^2}-\left(\frac{1}{\theta}-\frac{1}{p^*_{s_1}}\right)\ln\|(u,v)\|<0$, there exists a constant $C>0$ such that

$$\begin{aligned}E(u,v)&\geq\left(\frac{1}{p}-\frac{1}{p^*_{s_1}}\right)[1-(\lambda C_{H_1}+\mu C_{H_2})L]\left(\|(u,v)\|^p_{s_1,p}+\|(u,v)\|^t_{s_2,t}\right)\\&\quad+\left[\frac{1}{\theta^2}-\left(\frac{1}{\theta}-\frac{1}{p^*_{s_1}}\right)\ln\|(u,v)\|\right]\left(\lambda\int_\Omega H_1(x)|u|^\theta dx+\mu\int_\Omega H_2(x)|v|^\theta dx\right)\\&\geq\left(\frac{1}{p}-\frac{1}{p^*_{s_1}}\right)[1-(\lambda C_{H_1}+\mu C_{H_2})L]\left(\|(u,v)\|^p_{s_1,p}+\|(u,v)\|^t_{s_2,t}\right)\\&\quad-C\left[\frac{1}{\theta^2}-\left(\frac{1}{\theta}-\frac{1}{p^*_{s_1}}\right)\ln\|(u,v)\|\right]\geq -C.\end{aligned}$$

Thus, E is bounded from below on $\mathcal{N}$. □

4 Ground state solution for positive weight function

After the introductory part, in the two following sections we establish the existence of a ground state solution for problem (P). Specifically in this section, we carry on with the weights functions $H_1, H_2 > 0$, it is H_1, H_2 being non sign-changing functions. For this, we use the Mountain Pass theorem to show the existence of a level $c_* \in W$, where the functional E satisfy the $(PS)_{c_*}$ condition.

The version of Mountain Pass Theorem used in this paper is given in the below lemma.

Lemma 4.1. *Let $H_1, H_2 \in C(\overline{\Omega})$ and $H_1(x), H_2(x) > 0$ for all $x \in \Omega$. Then there exist $\eta, \zeta > 0$, such that*

(*i*) $E(u,v) \geq \eta > 0$ *for all* $\|(u,v)\| = \zeta$,

(*ii*) *there exists* $(u,v) \in W$ *such that* $E(u,v) < 0$ *if* $\|(u,v)\| > \zeta$.

Proof. Because of the definition of E and inequality (3.11) and $1 < q < p < p + r = p^*_{s_1}$, we have

$$\begin{aligned} E(u,v) \geq & \left[\frac{1}{p} - \frac{1}{\theta}(\lambda C_{H_1} + \mu C_{H_2})L\right] \|(u,v)\|^p_{s_1,p} + \left[\frac{1}{t} - \frac{1}{\theta}(\lambda C_{H_1} + \mu C_{H_2})L\right] \|(u,v)\|^t_{s_2,t} \\ & - \frac{1}{\theta}\Big(\ln\|(u,v)\| - \frac{1}{\theta}\Big)\left[\lambda \int_\Omega H_1(x)|u|^\theta dx + \mu \int_\Omega H_2(x)|v|^\theta dx\right] - C_B S_{q,r}^{\frac{p^*_s}{p}} \|(u,v)\|^{p^*_{s_1}} \\ \geq & \left[\frac{1}{p} - \frac{1}{\theta}(\lambda C_{H_1} + \mu C_{H_2})L\right] \|(u,v)\|^p_{s_1,p} \\ & - \frac{1}{\theta}\Big(\ln\|(u,v)\| - \frac{1}{\theta}\Big)\left[\lambda \int_\Omega H_1(x)|u|^\theta dx + \mu \int_\Omega H_2(x)|v|^\theta dx\right] - C_B S_{q,r}^{\frac{p^*_s}{p}} \|(u,v)\|^{p^*_{s_1}}, \end{aligned}$$

which implies that, for all $(u,v) \in W$ with $0 < \|(u,v)\| \leq 1$, we have

$$E(u,v) \geq \left[\frac{1}{p} - \frac{1}{\theta}(\lambda C_{H_1} + \mu C_{H_2})L\right] \|(u,v)\|^p_{s_1,p} - C_B S_{q,r}^{\frac{p^*_{s_1}}{p}} \|(u,v)\|^{p^*_{s_1}}.$$

Choosing $\zeta \in (0,1]$ small enough, such that

$$\left(\frac{1}{p} - \frac{1}{\theta}(\lambda C_{H_1} + \mu C_{H_2})L\right) - C_B S_{q,r}^{\frac{p^*_{s_1}}{p}} \|(u,v)\|^{p^*_{s_1}} \zeta^{p^*_{s_1}} > 0,$$

we have

$$E(u,v) \geq \left[\left(\frac{1}{p} - \frac{1}{\theta}(\lambda C_{H_1} + \mu C_{H_2})L\right) - C_B S_{q,r}^{\frac{p^*_{s_1}}{p}} \|(u,v)\|^{p^*_{s_1}} \zeta^{p^*_{s_1}}\right] \zeta^p > 0,$$

for all $(u,v) \in W$, with $\|(u,v)\| = \zeta$. Thus (i) holds.

Otherwise, for all $(u,v) \in W \setminus \{0,0\}$ and $k > 0$, we have

$$\begin{aligned} E(ku,kv) = k^{p^*_{s_1}} \Bigg[& \frac{k^{p-p^*_{s_1}}}{p} \|(u,v)\|^p_{s_1,p} + \frac{k^{t-p^*_{s_1}}}{t} \|(u,v)\|^t_{s_2,t} - \frac{k^{\theta-p^*_{s_1}} \ln|k|}{\theta} \int_\Omega H_1(x)|u|^\theta dx \\ & - \frac{k^{\theta-p^*_{s_1}}}{\theta} \lambda \int_\Omega H_1(x)|u|^\theta \ln|u| dx + \frac{k^{\theta-p^*_{s_1}}}{\theta^2} \lambda \int_\Omega H_1(x)|u|^\theta dx \\ & - \frac{k^{\theta-p^*_{s_1}} \ln|k|}{\theta} \int_\Omega H_2(x)|v|^\theta dx - \frac{k^{\theta-p^*_{s_1}}}{\theta} \mu \int_\Omega H_2(x)|v|^\theta \ln|v| dx \\ & + \frac{k^{\theta-p^*_{s_1}}}{\theta^2} \mu \int_\Omega H_2(x)|v|^\theta dx - \frac{1}{p^*_{s_1}} \int_\Omega (R_1 + R_2)(x)|u|^q|v|^r dx \Bigg], \end{aligned}$$

and because $1 < \theta < q < p < p^*_{s_1}$ it implies that there exists $k_0 > 0$ large enough such that $\|(k_0u, k_0v)\| > \zeta$ and $E(k_0u, k_0v) < 0$. So, taking $(u,v) = (k_0u, k_0v)$, item (ii) holds □

Lemma 4.2. *Let* (u_n, v_n) *be a* $(PS)_{c_*}$ *sequence of the functional* E. *Then, a sequence* (u_n, v_n) *is bounded in* W.

Proof. For $c_* \in \mathbb{R}$, we assume that $\{(u_n, v_n)\}_n \subset W$ with $\|(u_n, v_n)\| > 1$ is a $(PS)_{c_*}$ sequence with, it is $E(u_n, v_n) \to c_*$ and $E'(u_n, v_n)(u_n, v_n) \to 0$, as $n \to \infty$. On the other hand, since that (u_n, v_n) is a $(PS)_{c_*}$ sequence for functional E, we have

$$\begin{aligned} c_* + o_n(1) &\geq E(u_n, v_n) - \frac{1}{p} E'(u_n, v_n)(u_n, v_n) \\ &= \left(\frac{1}{t} - \frac{1}{p}\right) \|(u, v)\|^t_{s_2,p} \\ &\quad + \left(\frac{1}{p} - \frac{1}{\theta}\right) \left[\lambda \int_\Omega H_1(x)|u|^\theta \ln|u| dx + \mu \int_\Omega H_2(x)|v|^\theta \ln|v| dx\right] \\ &\quad + \frac{1}{\theta^2} \left[\lambda \int_\Omega H_1(x)|u|^\theta dx + \mu \int_\Omega H_2(x)|v|^\theta dx\right] \\ &\geq \frac{1}{\theta^2} \left[\lambda \int_\Omega H_1(x)|u_n|^\theta dx + \mu \int_\Omega H_2(x)|v_n|^\theta dx\right]. \end{aligned} \tag{4.1}$$

We have also

$$\begin{aligned} c_* + o_n(1) &\geq E(u_n, v_n) - \frac{1}{p^*_{s_1}} E'(u_n, v_n)(u_n, v_n) \\ &= \left(\frac{1}{p} - \frac{1}{p^*_{s_1}}\right) \|(u_n, v_n)\|^p_{s_1,p} + \left(\frac{1}{t} - \frac{1}{p^*_{s_1}}\right) \|(u_n, v_n)\|^t_{s_2,t} \\ &\quad + \frac{1}{\theta^2} \left[\lambda \int_\Omega H_1(x)|u_n|^\theta dx + \mu \int_\Omega H_2(x)|v_n|^\theta dx\right] \\ &\quad - \left(\frac{1}{p} - \frac{1}{p^*_{s_1}}\right) \left[\lambda \int_\Omega H_1(x)|u_n|^\theta \ln u_n dx + \mu \int_\Omega H_2(x)|v_n|^\theta \ln v_n dx\right] \\ &\geq \left(\frac{1}{p} - \frac{1}{p^*_{s_1}}\right) \|(u_n, v_n)\|^p_{s_1,p} \\ &\quad - \left(\frac{1}{p} - \frac{1}{p^*_{s_1}}\right) \left[\lambda \int_\Omega H_1(x)|u_n|^\theta \ln u_n + \mu \int_\Omega H_2(x)|v_n|^\theta \ln v_n\right]. \end{aligned} \tag{4.2}$$

Combining equations (4.1) and (4.2), Lemma 2.6 and Lemma 2.10, we have

$$\begin{aligned} \left[\left(\frac{1}{p} - \frac{1}{p^*_{s_1}}\right) [1 - (\lambda C_{H_1} + \mu C_{H_2}) L]\right] \|(u, v)\| &\leq \left(\frac{1}{p} - \frac{1}{p^*_{s_1}}\right) \frac{\theta^2 [(u_n, v_n)]^\sigma}{e\sigma} [c_* + o(1)] \\ &\leq C \left(1 + \|(u_n, v_n)\|^{1+\sigma}\right) + o(1), \end{aligned}$$

where $\sigma \in (0, p-1)$. Thus $\{(u_n, v_n)\}_n$ is bounded in W. □

Lemma 4.3. *Let (u_n, v_n) be a $(PS)_{c_*}$ sequence of the functional E. Then, functional E satisfies the $(PS)_{c_*}$ condition at any level c_*.*

Proof. By Lemmas 2.8 and 2.9, we get

$$\lim_{n\to\infty} \int_\Omega H_1(x) u |u_n|^{\theta-1} \ln|u| dx = \int_\Omega H_1(x)|u|^\theta \ln|u| dx, \tag{4.3}$$

$$\lim_{n\to\infty} \int_\Omega H_2(x) v |v_n|^{\theta-1} \ln|v| dx = \int_\Omega H_2(x)|v|^\theta \ln|v| dx, \tag{4.4}$$

$$\lim_{n\to\infty}\int_\Omega (R_1+R_2)(x)|u_n|^{q-2}u_n|v_n|^r(u_n-u)dx=0, \tag{4.5}$$

$$\lim_{n\to\infty}\int_\Omega (R_1+R_2)(x)|v_n|^{r-2}v_n|u_n|^\theta(v_n-v)dx=0. \tag{4.6}$$

Note that

$$\begin{aligned}\|(u_n-u,v_n-v)\|^p &= (E'(u_n,v_n)-E'(u,v))(u_n-u,v_n-v)\\ &+\lambda\int_\Omega H_1(x)\left(|u_n|^{\theta-1}\ln|u_n|-|u|^{\theta-1}\ln|u|\right)(u_n-u)dx\\ &+\mu\int_\Omega H_2(x)\left(|v_n|^{\theta-1}\ln|v_n|-|v|^{\theta-1}\ln|v|\right)(v_n-v)dx\\ &+\frac{q}{p^*_{s_1}}\int_\Omega (R_1+R_2)(x)\left(|v_n|^r u_n^{q-2}u_n-|v|^r|u|^{q-2}u\right)(u_n-u)dx\\ &+\frac{r}{p^*_{s_1}}\int_\Omega (R_1+R_2)(x)\left(|u_n|^q v_n^{r-2}v_n-|u|^q|v|^{r-2}v^{r-2}v\right)(v_n-v)dx.\end{aligned}$$

Since $(E'(u_n,v_n)-E'(u,v))(u_n-u,v_n-v)\to 0$ as $n\to\infty$, it follow from equations (4.3), (4.4), (4.5), (4.6) and Lemma 2.5, that $(u_n,v_n)\to(u,v)$ in W. This yields the proof. □

Now we can prove the first of the main results in the paper. More precisely, in what follows, we prove Theorem 1.1.

4.1 Proof of Theorem 1.1

Define $\mathfrak{N}:=\{(u,v)\in W\setminus\{0,0\}\mid E'(u,v)(u,v)=0\}$. By the previous lemmas, $\mathfrak{N}$ is nonempty. Let $(u,v)\in\mathfrak{N}$, from Lemma 2.10, we have

$$\begin{aligned}0&=\|(u,v)\|^p_{s_1,p}+\|(u,v)\|^t_{s_2,t}-\left(\lambda\int_\Omega H_1(x)|u|^\theta\ln|u|dx+\mu\int_\Omega H_2(x)|v|^\theta\ln|v|dx\right)\\ &\quad-\int_\Omega (R_1+R_2)(x)|u|^q|v|^r dx\\ &\geq (1-(\lambda C_{H_1}+\mu C_{H_2})L)\left(\|(u,v)\|^p_{s_1,p}+\|(u,v)\|^t_{s_2,t}\right)\\ &\quad-\ln\|(u,v)\|\left(\lambda\int_\Omega H_1(x)|u|^\theta\ln|u|dx+\mu\int_\Omega H_2(x)|v|^\theta\ln|v|dx\right)\\ &\quad-\int_\Omega (R_1+R_2)(x)|u|^q|v|^r dx.\end{aligned} \tag{4.7}$$

If $\|(u,v)\|\leq 1$, it follows from the above equation (4.7) that

$$[1-(\lambda C_{H_1}+\mu C_{H_2})]\left(\|(u,v)\|^p_{s_1,p}+\|(u,v)\|^t_{s_2,t}\right)\leq\int_\Omega (R_1+R_2)(x)|u|^q|v|^r dx\leq C_B S_{q,r}^{\frac{p^*_{s_1}}{p}}.$$

Then,

$$\|(u,v)\|\geq\left[\frac{1-(\lambda C_{H_1}+\mu C_{H_2})L}{C_B S_{q,r}^{\frac{p^*_{s_1}}{p}}}\right]^{\frac{1}{p^*_{s_1}-p}},$$

which implies that

$$\int_\Omega (R_1+R_2)(x)|u|^q|v|^r dx \geq [1-(\lambda C_{H_1}+\mu C_{H_2})L]\left(\|(u,v)\|^p_{s_1,p}+\|(u,v)\|^t_{s_2,t}\right)$$
$$\geq \left[\frac{[1-(\lambda C_{H_1}+\mu C_{H_2})L]^{p^*_{s_1}}}{\left(C_B S_{q,r}^{\frac{p^*_{s_1}}{p}}\right)^p}\right]^{\frac{1}{p^*_{s_1}-p}}. \tag{4.8}$$

Define

$$\tilde{C} := \left[\frac{[1-(\lambda C_{H_1}+\mu C_{H_2})L]^{p^*_{s_1}}}{\left(C_B S_{q,r}^{\frac{p^*_{s_1}}{p}}\right)^p}\right]^{\frac{1}{p^*_{s_1}-p}},$$

and

$$\mathfrak{C} := \inf\{E(u,v) \mid (\tilde{U}_n,\tilde{V}_n)\in\mathfrak{N}\},$$

then $\mathfrak{C}>0$, because otherwise, there exists $\{(\tilde{U}_n,\tilde{V}_n)\}_n\subset\mathfrak{N}$ such that $E'(\tilde{U}_n,\tilde{V}_n)\to 0$.

It follows from

$$E(u,v) = \left(\frac{1}{p}-\frac{1}{\theta}\right)\|(u,v)\|^p_{s_1,p}+\left(\frac{1}{t}-\frac{1}{\theta}\right)\|(u,v)\|^t_{s_2,t}+\frac{1}{\theta}I(u,v)$$
$$+\frac{1}{\theta^2}\left(\lambda\int_\Omega H_1(x)|u|^\theta dx+\mu\int_\Omega H_2(x)|v|^\theta dx\right)$$
$$-\left(\frac{1}{p^*_{s_1}}-\frac{1}{\theta}\right)\int_\Omega (R_1+R_2)(x)|u|^q|v|^r dx,$$

that

$$\frac{1}{\theta^2}\left(\lambda\int_\Omega H_1(x)|\tilde{U}_n|^\theta dx+\mu\int_\Omega H_2(x)|\tilde{V}_n|^\theta dx\right)-\left(\frac{1}{p^*_{s_1}}-\frac{1}{p}\right)\int_\Omega (R_1+R_2)(x)|\tilde{U}_n|^q|\tilde{V}_n|^r dx\to 0,$$

when $n\to\infty$, and by Lemma 4.2. But, from equation (4.8) it follows that

$$\int_\Omega (R_1+R_2)(x)|\tilde{U}_n|^q|\tilde{V}_n|^r dx\geq\tilde{C}>0,$$

and it implies that $0\geq\tilde{C}>0$, which is a contradiction. Thus, $\mathfrak{C}>0$.

Finally, let $\{(u_n,v_n)\}_n\subset\mathfrak{N}$ be a minimizing sequence. Then $E'(u_n,v_n)(u_n,v_n)=0$ and $\lim_{n\to\infty}E(u_n,v_n)=\mathfrak{C}>0$. Again by Lemma 4.2, there exists $(u_0,v_0)\in W\setminus\{0,0\}$ such that

$$(u_n,v_n)\to(u_0,v_0)$$

in W. Hence $E(u_0,v_0)=\mathfrak{C}$ and $E'(u_0,v_0)=0$, and it means that (u_0,v_0) is a nontrivial ground state solution of problem (P).

5 Nontrivial solutions for sign-changing weight functions

To conclude this paper, we now proceed with the proof of the existence of solutions to the problem (P), but now for the cases in which the weight functions H_1, H_2 are sign-changing functions.

Lemma 5.1. *E has a nontrivial and nonnegative minimizer on $\mathcal{N}^+$.*

Proof. The proof of this lemma will be shown in two steps.

Step 1. The strong convergence of minimizing sequence. By Lemma 3.8, we get

$$c^+ = \inf_{(u,v)\in\mathcal{N}^+} E(u,v).$$

Claim: $c^+ < 0$. Indeed, for each $(u,v) \in \mathcal{N}^+$, we get

$$\begin{aligned}\|(u,v)\|^p_{s_1,p} + \|(u,v)\|^t_{s_2,t} &= \lambda\int_\Omega H_1(x)|u|^\theta \ln|u|dx + \mu\int_\Omega H_2(x)|v|^\theta \ln|v|dx \\ &\quad + \int_\Omega (R_1(x)+R_2(x))|u|^q|v|^r dx \end{aligned} \tag{5.1}$$

and there holds

$$\begin{aligned}E(u,v) &= \frac{1}{p}\|(u,v)\|^p_{s_1,p} + \frac{1}{t}\|(u,v)\|^t_{s_2,t} - \frac{\lambda}{\theta}\int_\Omega H_1(x)|u|^\theta\ln|u|dx + \frac{\lambda}{\theta^2}\int_\Omega H_1(x)|u|^\theta dx \\ &\quad - \frac{\mu}{\theta}\int_\Omega H_2(x)|v|^\theta\ln|v|dx + \frac{\mu}{\theta^2}\int_\Omega H_2(x)|v|^\theta dx - \frac{1}{p^*_{s_1}}\int_\Omega (R_1(x)+R_2(x))|u|^q|v|^r dx \\ &= \frac{1}{p}\left[\lambda\int_\Omega H_1(x)|u|^\theta\ln|u|dx + \mu\int_\Omega H_2(x)|v|^\theta\ln|v|dx + \int_\Omega (R_1(x)+R_2(x))|u|^q|v|^r dx\right] \\ &\quad - \frac{\lambda}{\theta}\int_\Omega H_1(x)|u|^\theta\ln|u|dx + \frac{\lambda}{\theta^2}\int_\Omega H_1(x)|u|^\theta dx - \frac{\mu}{\theta}\int_\Omega H_2(x)|v|^\theta\ln|v|dx \\ &\quad + \frac{\mu}{\theta^2}\int_\Omega H_2(x)|v|^\theta dx - \frac{1}{p^*_{s_1}}\int_\Omega (R_1(x)+R_2(x))|u|^q|v|^r dx \\ &= \left(\frac{1}{p}-\frac{1}{\theta}\right)\left[\lambda\int_\Omega H_1(x)|u|^\theta\ln|u|dx + \mu\int_\Omega H_2(x)|v|^\theta\ln|v|dx\right] \\ &\quad + \frac{1}{\theta^2}\left[\lambda\int_\Omega H_1(x)|u|^\theta dx + \mu\int_\Omega H_2(x)|v|^\theta dx\right] \\ &\quad + \left(\frac{1}{\theta}-\frac{1}{p^*_{s_1}}\right)\int_\Omega (R_1(x)+R_2(x))|u|^q|v|^r dx.\end{aligned}$$

Using the following inequality

$$\begin{aligned}&(p-\theta)\left[\lambda\int_\Omega H_1(x)|u_n|^\theta\ln|u|dx + \mu\int_\Omega H_2(x)|v_n|^\theta\ln|v|dx\right] \\ &\quad > (p^*_{s_1}-p)\int_\Omega (R_1(x)+R_2(x))|u|^q|v|^r dx + \lambda\int_\Omega H_1(x)|u|^\theta dx + \mu\int_\Omega H_2(x)|v|^\theta dx\end{aligned}$$

and $p < \theta$ we have

$$\begin{aligned}&\lambda\int_\Omega H_1(x)|u_n|^\theta\ln|u|dx + \mu\int_\Omega H_2(x)|v_n|^\theta\ln|v|dx \\ &\quad < -\frac{p^*_{s_1}-p}{\theta-p}\int_\Omega (R_1(x)+R_2(x))|u|^q|v|^r dx \\ &\qquad - \frac{1}{\theta-p}\left[\lambda\int_\Omega H_1(x)|u|^\theta dx + \mu\int_\Omega H_2(x)|v|^\theta dx\right].\end{aligned} \tag{5.2}$$

Using (5.2) implies that

$$\begin{aligned}
E(u,v) &= \left(\frac{1}{p}-\frac{1}{\theta}\right)\left[\lambda\int_\Omega H_1(x)|u|^\theta \ln|u|dx+\mu\int_\Omega H_2(x)|v|^\theta\ln|v|dx\right]\\
&\quad+\frac{1}{\theta^2}\left[\lambda\int_\Omega H_1(x)|u|^\theta dx+\mu\int_\Omega H_2(x)|v|^\theta dx\right]\\
&\quad+\left(\frac{1}{p}-\frac{1}{p^*_{s_1}}\right)\int_\Omega (R_1(x)+R_2(x))|u|^q|v|^r dx\\
&< (p-p^*_{s_1})\left(\frac{1}{\theta-p}-\frac{1}{p(p^*_{s_1})}\right)\int_\Omega (R_1(x)+R_2(x))|u|^q|v|^r dx\\
&\quad+\left(\frac{1}{\theta^2}-\frac{1}{\theta-p}\right)\left[\lambda\int_\Omega H_1(x)|u|^\theta dx+\mu\int_\Omega H_2(x)|v|^\theta dx\right]\\
&< 0.
\end{aligned}$$

Let $\{(u_n,v_n)\}\subset\mathcal{N}^+$ be a minimizing sequence. Then,

$$\begin{aligned}
\|(u,v)\|^p_{s_1,p}+\|(u,v)\|^t_{s_2,t} &= \lambda\int_\Omega H_1(x)|u_n|^\theta\ln|u|dx+\mu\int_\Omega H_2(x)|v_n|^\theta\ln|v|dx\\
&\quad+\int_\Omega (R_1(x)+R_2(x))|u|^q|v|^r dx, \qquad (5.3)
\end{aligned}$$

and using $\Phi''_{(u_n,v_n)}(1)>0$,

$$\begin{aligned}
\Phi''_{(u_n,v_n)}(1) &= \|(u,v)\|^p_{s_1,p}+\|(u,v)\|^t_{s_2,t}-\lambda\int_\Omega H_1(x)|u_n|^\theta\ln|u_n|dx-\mu\int_\Omega H_2(x)|v|^\theta\ln|v|dx\\
&\quad-\int_\Omega (R_1+R_2)(x)|u_n|^q|v_n|^r dx-\lambda\int_\Omega H_1(x)|u_n|^\theta dx\\
&\quad-\mu\int_\Omega H_2(x)|v_n|^\theta dx+(p-p^*_{s_1})\int_\Omega (R_1(x)+R_2(x))|u|^q|v|^r dx\\
&> 0. \qquad (5.4)
\end{aligned}$$

This implies

$$-\lambda\int_\Omega H_1(x)|u_n|^\theta dx-\mu\int_\Omega H_2(x)|v_n|^\theta dx+(p-p^*_{s_1})\int_\Omega (R_1(x)+R_2(x))|u|^q|v|^r dx>0. \qquad (5.5)$$

Since $\mathcal{N}^+$ is bounded by Lemma 3.7, up to subsequence we assume that

$$\begin{cases}
(u_n,v_n)\rightharpoonup(u^+,v^+) & \text{in } W;\\
u_n\to u^+,\ v_n\to v^+ & \text{strongly in } L^t(\Omega),\ \text{for } 1\le t<p^*_s\\
u_n(x)\to u^+(x),\ v_n(x)\to v^+(x) & \text{a.e in } \Omega.
\end{cases} \qquad (5.6)$$

Similarly to Lemma 4.3, we get

$$\lim_{n\to\infty}\int_\Omega H_1(x)u_n|u_n|^\theta\ln|u_n|dx=\int_\Omega H_1(x)|u^+|^\theta\ln|u^+|dx, \qquad (5.7)$$

$$\lim_{n\to\infty}\int_\Omega H_2(x)v_n|v_n|^\theta\ln|v_n|dx=\int_\Omega H_2(x)|v^+|^\theta\ln|v^+|dx, \qquad (5.8)$$

$$\lim_{n\to\infty}\int_\Omega R_1(x)|u_n|^q|v_n|^r dx=\int_\Omega R_2(x)|u^+|^q|v^+|^r dx, \qquad (5.9)$$

$$\lim_{n\to\infty}\int_\Omega R_2(x)|u_n|^q|v_n|^r dx=\int_\Omega R_2(x)|u^+|^q|v^+|^r dx. \qquad (5.10)$$

If $(u_n, v_n) \rightharpoonup (u^+, v^+)$ in W, then

$$\|(u^+, v^+)\|^2 \leq \liminf_{n\to+\infty} \|(u_n, v_n)\|^2. \tag{5.11}$$

This implies that

$$\begin{aligned}
&\|(u^+, v^+)\|^p_{s_1,p} + \|(u^+, v^+)\|^t_{s_2,t} - \lambda \int_\Omega H_1(x)|u^+|^\theta \ln|u^+|dx - \mu \int_\Omega H_1(x)|v^+|^\theta \ln|v^+|dx \\
&\quad - \int_\Omega R_1(x)|u^+|^q|v^+|^r dx - \int_\Omega R_2(x)|u^+|^q|v^+|^r dx \\
&< \liminf_{n\to+\infty} \left[\|(u_n, v_n)\|^2 - \lambda \int_\Omega H_1(x)|u_n|^\theta \ln|u_n|dx - \mu \int_\Omega H_1(x)|v_n|^\theta \ln|v_n|dx \right. \\
&\quad \left. - \int_\Omega R_1(x)|u_n|^q|v_n|^r dx - \int_\Omega R_2(x)|u_n|^q|v_n|^r dx \right] = 0.
\end{aligned} \tag{5.12}$$

Now we prove that for (u^+, v^+) there exists $0 < t_{(u^+,v^+)} \neq 1$ such that

$$t_{(u^+,v^+)}(u^+, v^+) \in \mathcal{N}^+.$$

Since $c^+ < 0$, we can show that $(u^+, v^+) \neq (0, 0)$. By (5.5), we deduce that

$$\lambda \int_\Omega R_1(x)|u^+|^q dx + \mu \int_\Omega R_2(x)|v^+|^r dx < 0,$$

then by Lemma 3.3 there exists $t_{(u^+,v^+)} > 0$ such that

$$t_{(u^+,v^+)}(u^+, v^+) \in \mathcal{N}^+ \quad \text{and} \quad \Psi'_{(u^+,v^+)}(t_{(u^+,v^+)}) = 0$$

By (5.12), $\Psi'_{(u^+,v^+)}(1)) < 0$. Thus, $t_{(u^+,v^+)} \neq 1$. Note that $t_{(u^+,v^+)}(u^+, v^+)$ is minimizer of $g(t) = E(\tau(u^+, \tau v^+)$. Thus,

$$E(t_{(u^+,v^+)}u^+, t_{(u^+,v^+)}v^+) < E(u^+, v^+)) \leq \lim_{n\to+\infty} E(u_n, v_n) = \inf_{(u,v)\in\mathcal{N}^+} E(u, v),$$

which is absurd. Therefore, we obtain $(u_n, v_n) \to (u^+, v^+)$ in W.

Step 2. Existence of nonnegative minimizers. From $(u_n, v_n) \to (u^+, v^+)$ in W and Lemma 3.8, we get

$$-\lambda \int_\Omega H_1(x)|u^+|^\theta dx - \mu \int_\Omega H_2(x)|v^+|^\theta dx + (p - p^*_{s_1}) \int_\Omega (R_1 + R_2)(x)|u^+|^q|v^+|^r dx > 0. \tag{5.13}$$

Thus, we obtain that $(u^+, v^+) \in \mathcal{N}^+$. This gives that (u^+, v^+) is a minimizer of E on $\mathcal{N}^+$. We can prove that $(|u^+|, |v^+|)$ is also a minimizer in $\mathcal{N}^+$. Since

$$E(|u^+|, |v^+|) \leq E(u^+, v^+)$$

and because (5.13) holds for $(|u^+|, |v^+|)$, one can show that

$$\begin{aligned}
&\|(u^+, v^+)\|^p_{s_1,p} + \|(u^+, v^+)\|^t_{s_2,t} - \lambda \int_\Omega H_1(x)|u^+|^\theta \ln|u^+|dx - \mu \int_\Omega H_1(x)|v^+|^\theta \ln|v^+|dx \\
&= \int_\Omega R_1(x)|u^+|^q|v^+|^r dx + \int_\Omega R_2(x)|u^+|^q|v^+|^r dx.
\end{aligned} \tag{5.14}$$

Using $[|u^+|]_s^p \leq [u^+]_s^p$, we obtain

$$\|(u^+,v^+)\|_{s_1,p}^p + \|(u^+,v^+)\|_{s_2,t}^t - \lambda\int_\Omega H_1(x)|u^+|^\theta \ln|u^+|dx - \mu\int_\Omega H_1(x)|v^+|^\theta \ln|v^+|dx$$
$$\leq \int_\Omega R_1(x)|u^+|^q|v^+|^r dx + \int_\Omega R_2(x)|u^+|^q|v^+|^r dx. \tag{5.15}$$

If

$$\|(u^+,v^+)\|_{s_1,p}^p + \|(u^+,v^+)\|_{s_2,t}^t - \lambda\int_\Omega H_1(x)|u^+|^\theta \ln|u^+|dx - \mu\int_\Omega H_1(x)|v^+|^\theta \ln|v^+|dx$$
$$< \int_\Omega R_1(x)|u^+|^q|v^+|^r dx + \int_\Omega R_2(x)|u^+|^q|v^+|^r dx, \tag{5.16}$$

then $\Psi'_{(|u^+|,|v^+|)}(1)) < 0$. For $(|u^+|,|v^+|)$, by Lemma 3.3 there exist $t_{(|u^+|,|v^+|)} \in \mathcal{N}^+$ and $\Psi'_{(|u^+|,|v^+|)}t_{(|u^+|,|v^+|)} = 0$. Thus, $t_{(|u^+|,|v^+|)} \neq 1$. For other hand $t_{(|u^+|,|v^+|)}$ is a minimizer of

$$\xi(t) := E(u^+,v^+).$$

Thus,

$$E(t_{(|u^+|,|v^+|)}|u^+|, t_{(|u^+|,|v^+|)}|v^+| \leq E(|u^+|,|v^+|) \leq E(u^+,v^+) = \inf_{(u,v)\in\mathcal{N}^+} E(u,v),$$

which is absurd. Thus, $(|u^+|,|v^+|) \in \mathcal{N}^+$ and

$$E(|u^+|,|v^+|) = \inf_{(u,v)\in\mathcal{N}^+} E(u,v).$$

In conclusion, we get nonnegative minimizer E on $\mathcal{N}^+$. □

Lemma 5.2. *E has a nontrivial and nonnegative minimizer on $\mathcal{N}^-$.*

Proof. By Lemma 3.8, we know $c^- := \inf_{(u,v)\in\mathcal{N}^-} E(u,v)$ is attained. Let $\{(u_n,v_n)\}_n \subset \mathcal{N}^-$ be a minimizing sequence such that $E(u_n,v_n) \to c^-$. Then

$$\|(u_n,v_n)\|_{s_1,p}^p + \|(u_n,v_n)\|_{s_2,t}^t - \lambda\int_\Omega H_1(x)|u_n|^\theta dx - \mu\int_\Omega H_2(x)|v_n|^\theta dx$$
$$= \int_\Omega (R_1+R_2)(x)|u_n|^q|v_n|^r dx, \tag{5.17}$$

and,

$$0 > \lambda(p-\theta)\int_\Omega H_1(x)|u_n|^\theta \ln|u_n|dx + \mu(p-\theta)\int_\Omega H_2(x)|v_n|^\theta \ln|v_n|dx$$
$$- \lambda(p-\theta)\int_\Omega H_1(x)|u_n|^\theta dx - \mu(p-\theta)\int_\Omega H_2(x)|v_n|^\theta dx$$
$$- (p_{s_1}^* - p)\int_\Omega (R_1+R_2)(x)|u_n|^q|v_n|^r dx. \tag{5.18}$$

We claim that $\{(u_n,v_n)\}_n$ is bounded in $\mathcal{N}^-$.

Without loss of generality, we assume that $\|(u_n,v_n)\| \geq 1$. Then

$$c^- + o_n(1) = E(u_n,v_n) - \frac{1}{p}E'(u_n,v_n)(u_n,v_n)$$
$$= \frac{1}{\theta^2}\left(\lambda\int_\Omega H_1(x)|u_n|^\theta dx + \mu\int_\Omega H_2(x)|v_n|^\theta dx\right)$$
$$+ \left(\frac{1}{p} - \frac{1}{p_{s_1}^*}\right)\int_\Omega (R_1+R_2)(x)|u_n|^q|v_n|^r dx. \tag{5.19}$$

So

$$c^- + o_n(1) \geq \frac{1}{\theta^2}\left(\lambda \int_\Omega H_1(x)|u_n|^\theta dx + \mu \int_\Omega H_2(x)|v_n|^\theta dx\right). \tag{5.20}$$

If $\lambda \int_\Omega H_1(x)|u_n|^\theta dx + \mu \int_\Omega H_2(x)|v_n|^\theta dx \geq 0$, then from equations (5.19), (5.20), we have

$$\|(u,v)\| \leq \left(\frac{1}{\frac{1}{p} - \frac{1}{\theta}(\lambda C_{H_1} + \mu C_{H_2})L}\right)^{\frac{1}{p-2}}.$$

If, $\lambda \int_\Omega H_1(x)|u_n|^\theta dx + \mu \int_\Omega H_2(x)|v_n|^\theta dx < 0$, from equation (5.20), we have

$$\begin{aligned} c^- + o_n(1) &= E(u_n, v_n) \\ &\geq \left[\frac{1}{p} - \frac{1}{\theta}(\lambda C_{H_1} + \mu C_{H_2})L\right]\left(\|(u_n,v_n)\|^p_{s_1,p} + \|(u_n,v_n)\|^t_{s_2,t}\right) \\ &\quad + \left[\frac{1}{\theta^2} - \left(\frac{1}{\theta} - \frac{1}{p^*_{s_1}}\right)\ln\|(u_n,v_n)\|\right]\left(\lambda \int_\Omega H_1(x)|u_n|^\theta dx + \mu \int_\Omega H_2(x)|v_n|^\theta dx\right). \end{aligned}$$

We have two cases to analyze:

i) If $\frac{1}{\theta^2} - \left(\frac{1}{\theta} - \frac{1}{p^*_{s_1}}\right)\ln\|(u_n,v_n)\| \leq 0$, we have $c^- + o_n(1) = \left(\frac{1}{p} - \frac{1}{p^*_{s_1}}\right)[1 - (\lambda C_{H_1} + \mu C_{H_2})L]$ $\left(\|(u_n,v_n)\|^p_{s_1,p} + \|(u_n,v_n)\|^t_{s_2,t}\right)$, which implies that

$$\|(u_n,v_n)\| \leq \left[\frac{p(p^*_{s_1})(c^- + o_n(1))}{(p^*_{s_1} - p)[1 - (\lambda C_{H_1} + \mu C_{H_2})L]}\right]^{\frac{1}{p}}.$$

ii) If $\frac{1}{\theta^2} - \left(\frac{1}{\theta} - \frac{1}{p^*_{s_1}}\right)\ln\|(u_n,v_n)\| > 0$, then $\|(u_n,v_n)\| < \exp\left(\frac{p^*_{s_1} - \theta}{\theta^3(p^*_{s_1})}\right)$. So there exists $C > 0$ such that $\|(u_n,v_n)\| < C$.

Thus, every minimizer sequence of E on $\mathcal{N}^-$ is bounded.
Now, since $\{(u_n,v_n)\}$ is bounded in $\mathcal{N}^-$, up to a subsequence, we may assume that

$$\begin{aligned} &(u_n,v_n) \rightharpoonup (u_-,v_-) && \text{weakly in } W; \\ &(u_n,v_n) \to (u_-,v_-) && \text{strongly in } L^\nu(\Omega) \times L^\tau(\Omega); \\ &(u_n,v_n) \to (u_+,v_+) && \text{a.e in } \Omega, \end{aligned}$$

where $1 < \nu, \tau < \min\{p^*_{s_1}, p^*_{s_1}\}$. It implies that

$$\begin{aligned} \lim_{n\to\infty}\int_\Omega H_1(x)|u_n|^\theta \ln|u_n|dx &= \int_\Omega H_1(x)|u_-|^\theta \ln|u_-|dx; \\ \lim_{n\to\infty}\int_\Omega H_2(x)|v_n|^\theta \ln|v_n|dx &= \int_\Omega H_2(x)|v_-|^\theta \ln|v_-|dx; \\ \lim_{n\to\infty}\int_\Omega H_1(x)|u_n|^\theta dx &= \int_\Omega H_1(x)|u_-|^\theta dx; \\ \lim_{n\to\infty}\int_\Omega H_1(x)|v_n|^\theta dx &= \int_\Omega H_2(x)|v_-|^\theta dx. \end{aligned}$$

An easy computation, combined to equation (5.18), shows that

$$I(u_-,v_-) \leq \liminf_{n\to\infty} I(u_n,v_n) = 0.$$

It means that $(u_-, v_-) \in \mathcal{N}^-$. Now, by Lemma 3.3, item $[i]$, there exists $k_{(u,v)} > 0$ such that $I'_{(u_-,v_-)}k_{(u_-,v_-)} = 0$ and $k_{(u_-,v_-)} \neq 1$. Since $(u_n, v_n) \not\to (u_-, v_-)$, we get

$$k_{(u_-,v_-)}(u_n, v_n) \not\to k_{(u_-,v_-)}(u_-, v_-)$$

in W. Another easy computation shows that

$$E(k_{(u_-,v_-)}u_-, k_{(u_-,v_-)}v_-) \leq E(k_{(u_-,v_-)}u_n, k_{(u_-,v_-)}v_n).$$

Now, observe that the function $z(k) := E(ku_n, kv_n)$ attains its maximum at $k = k_{(u_-,v_-)}$. So

$$\begin{aligned} E(k_{(u_-,v_-)}u_-, k_{(u_-,v_-)}v_-) &< \liminf_{n\to\infty} E(k_{(u_-,v_-)}u_n, k_{(u_-,v_-)}v_n) \\ &\leq \lim_{n\to\infty} E(u_n, v_n) = \inf_{(u,v)\in\mathcal{N}^-} E(u,v), \end{aligned}$$

which is a contradiction. Thus $(u_n, v_n) \to (u_-, v_-)$ in W.

Because $(u_-, v_-) \in \mathcal{N}^-$, then (u_-, v_-) is a minimizer of E on $\mathcal{N}^-$. Moreover, a similar discussion as Theorem 5.1 – *Step* 2 – one can show that $(|u_-|, |v_-|)$ is a minimizer of E on $\mathcal{N}^-$. This yields the proof. □

Now is possible to proof Theorem 1.2.

5.1 Proof of Theorem 1.2

From Lemmas 5.1 and 5.2, E has two non-negative minimizers $(u_+, v_+) \in \mathcal{N}^+$ and $(u_-, v_-) \in \mathcal{N}^-$. Then, from Lemma 3.2, E has two non-negative critical points on W, which is non-trivial and non-negative local least energy solution of problem (P). This two solution are distinct, because is obviously that $\mathcal{N}^- \cap \mathcal{N}^+ = \emptyset$.

We claim that (u_+, v_+) and (u_-, v_-) are not semi-trivial solution.

Supposing, otherwise, $v_+ = 0$ in (u_+, v_+), we get that u_+ is a non-trivial solution of the problem

$$\begin{cases} (-\Delta)_p^{s_1}u + (-\Delta)_t^{s_2}u = \lambda H_1(x)|u|^{\theta-2}u\ln|u| \quad \text{in } \Omega, \\ (u,v) \in W_0^{s_1,p}(\Omega) \times W_0^{s_2,t}(\Omega). \end{cases} \tag{P'}$$

Then $[u_+]_{s_1,p}^p + [u_+]_{s_2,t}^t = \lambda \int_\Omega H_1(x)|u_+|^\theta \ln|u_+|dx$, and because $(u_+, 0) \in \mathcal{N}^+$ and $\Phi''_{(u,v)}(1) > 0$, we get

$$\lambda(p-\theta)\int_\Omega H_1(x)|u_+|^\theta dx < \lambda \int_\Omega H_1(x)|u_+|^\theta dx.$$

Because $p < \theta$, we have $\int_\Omega H_1(x)|u_+|^\theta dx < 0$.

Now we choose $w \in W_0^{t,p}(\Omega) \setminus \{0\}$ such that $\int_\Omega H_1(x)|w|^\theta dx < 0$.

For (u_+, w), by Lemma 3.3, there exists a unique $k_1 > 0$ such that $k_1(u_+, w) \in \mathcal{N}^+$. Moreover, we have

$$k_{\max} = \exp\left[\frac{\|u_+\|_{s_1,p}^p + \|u_+\|_{s_2,t}^t - \left[\int_\Omega \lambda H_1(x)|u_+|^\theta \ln|u_+|^\theta + \mu H_2(x)|w|^\theta \ln|w|^\theta\right]dx}{\lambda\int_\Omega H_1(x)^\theta dx + \mu\int_\Omega H_2(x)|w|^\theta dx} - \frac{1}{p - p^*_{s_1}}\right],$$

and

$$E(k_1u_+, k_1w) = \inf_{0<k<k_{\max}} E(ku_+, kw).$$

It follows that

$$c^+ \leq E(k_1 u_+, k_1 w) < E(u_+, w) < E(u_+, 0) = c^+,$$

which is a contradiction. Thus (u_+, v_+) is not a semi-trivial solution for problem (P).

Otherwise, (u_-, v_-) is not a semi-trivial solution for problem (P), by using the same above argument, but this time assuming $v_- = 0$. In this case $(u_-, 0)$ is a nontrivial solution for problem (P') and $\int_\Omega H_1(x)|u_-|^\theta dx > 0$ and $w \in W_0^{t,p}(\Omega) \setminus \{0\}$ is taking such that $\lambda \int_\Omega H_1(x)|u_-|^\theta dx + \mu \int_\Omega H_2(x)|w|^\theta dx < 0$.

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Existence and nonexistence of positive solutions of nonlinear elliptic equations involving singular term and convection term

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Abstract. This paper establishes the existence and nonexistence of positive solutions for a class of Dirichlet problems involving singular nonlinearities and gradient-dependent terms. The nonlinearities considered are subcritical in the Sobolev sense. Our approach is based on an approximation scheme.

Keywords: quasilinear elliptic equation, positive solution, singular nonlinearity, dependence on the gradient, convection term.

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1 Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a C^2-boundary, $N \geq 2$, and $0 < \alpha < 1$. In this paper, we investigate the existence of at least one positive solution to the following singular elliptic equation with Dirichlet boundary conditions, driven by the p-Laplacian operator $-\Delta_p u = -\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$, with $1 < p < \infty$:

$$\begin{cases} -\Delta_p u = \lambda\left(\dfrac{1}{u^\alpha} + |\nabla u|^{r_2}\right) + f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $0 < \lambda < \lambda^*$ is a parameter, r_2 is a constant such that $0 < r_2 < p-1$, and f is a continuous nonlinearity satisfying

$$0 \leq f(x,t) \leq a_2|t|^{r_1}, \tag{1.2}$$

with constants $a_2 > 0$ and $r_1 \in (0, p-1) \cup (p-1, p^*-1)$.

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We denote by p^* the critical Sobolev exponent, that is, $p^* = \frac{Np}{N-p}$ if $1 < p < N$, and $p^* \in (p, \infty)$ if $p = N$.

In recent years, singular elliptic equations have posed considerable analytical challenges. A vast body of literature has been devoted to such problems, particularly from a theoretical standpoint. For example, the following problem has been extensively investigated:

$$\begin{cases} -\Delta_p u = \eta(x) u^{-\alpha} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where $\eta(x) \geq 0$ in Ω and $0 < \alpha < 1$. Canino, Sciunzi, and Trombetta [6] established the existence and uniqueness of solutions to (1.3) for $p \neq 2$ under various configurations. For the particular case $p = 2$, see, for instance, [14,20,27].

Giacomoni, Schindler, and Takáč [13] studied the existence and multiplicity of weak solutions for the following p-Laplacian equation:

$$\begin{cases} -\Delta_p u = \lambda u^{-\delta} + u^q & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $1 < p < \infty$, $p - 1 < q \leq p^* - 1$, $\lambda > 0$, and $0 < \delta < 1$.

Further results concerning singular problems without convection terms can be found in [3, 5,12,25] and the references therein.

Elliptic problems involving convection terms have also attracted substantial attention, as they introduce additional analytical difficulties. For instance, Francesca and Puglisi [11], employing the subsolution–supersolution method, truncation techniques, nonlinear regularity theory, the Leray–Schauder alternative principle, and tools from set-valued analysis, proved the existence of a positive solution for the problem,

$$\begin{cases} -\Delta u = f(x, u(x), \nabla u(x)) + g(x, u(x)) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satisfies an appropriate growth condition, and the semilinear function $g : \Omega \times (0, \infty) \to \mathbb{R}$ is singular at $s = 0$, that is,

$$\lim_{s \to 0^+} g(x, s) = +\infty.$$

It is worth noting that condition (1.2) in the present paper is considerably less restrictive than the assumptions imposed on f in [11] (see also [23] and the references therein).

Faria, Miyagaki, and Motreanu [10] established the existence of a positive solution for the following quasilinear elliptic problem involving the (p, q)-Laplacian and a convection term, by means of the Galerkin method and a Schauder basis:

$$\begin{cases} -\Delta_p u - \mu \Delta_q u = f(x, u, \nabla u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Additional works concerning problems with convection terms can be found in [4,26,31] and the references therein.

Beyond the study of variational and nonvariational elliptic problems, the analytical framework of this paper is also connected to broader developments in partial differential equations with irregular coefficients and singular terms. Ragusa [30] proved existence and regularity results for elliptic boundary value problems under vanishing mean oscillation (VMO) assumptions on the coefficients, highlighting how solvability can be preserved even in the absence of smoothness. Polidoro and Ragusa [28] extended these ideas to degenerate ultraparabolic equations with singular lower-order terms, establishing a Harnack inequality and proving the existence of a Green's function for hypoelliptic operators. Kozhanov and Shipina [18] examined loaded elliptic equations and linear inverse problems, where source terms must be simultaneously recovered with the solution itself. Guariglia [15] developed a fractional-calculus framework for the Lerch zeta function, contributing to the analytic study of operators with nonlocal and fractional structures. The work of Guariglia and Silvestrov [16] successfully bridges harmonic analysis with fractional differential operators in complex domains. They achieved this by introducing a new fractional-wavelet analysis tailored for the spaces of positive definite distributions ($\mathcal{D}'(\mathbb{C})$). These contributions collectively advance the analytic study of nonlocal, fractional, and singular operators relevant to modern elliptic theory.

Unlike the cited works, which only address singular nonlinearities without gradient dependence or equations with regular lower-order terms, our study is unique because it simultaneously incorporates a singular term of the form $u^{-\alpha}$ and a convection term involving $|\nabla u|^{r_2}$. This interaction destroys the problem's variational structure and introduces substantial analytical challenges, in obtaining uniform estimates and ensuring the positivity of weak solutions. To address these difficulties, we develop an approximation scheme based on Galerkin approximations using a Schauder basis and non-variational *a priori* bounds, and pass to the limit to obtain existence results. This strategy not only circumvents the lack of a variational framework but also extends the existing theory to a broader class of nonlinear singular elliptic problems. A key contribution of this study is the proof of existence for an equation with a singular nonlinearity where the exponent r_1 of $|u|$ satisfies $r_1 < p^* - 1$.

We now state the main results of the paper.

Theorem 1.1. *Assume that f is a continuous function satisfying (1.2). Then there exists a constant $\lambda^* > 0$ such that, for every $\lambda \in (0, \lambda^*)$, problem (1.1) admits at least one positive solution $u_\lambda \in W_0^{1,p}(\Omega)$.*

Proposition 1.2. *Assume that $f(t) = |t|^{r_1}$, where $r_1 \in (p-1, p^*-1)$, and define*

$$\lambda^* = \sup\{\lambda > 0 : (1.1) \text{ admits a solution } u_\lambda \text{ in } W_0^{1,p}(\Omega)\}.$$

Then $\lambda^* < \infty$.

The paper is organized as follows. In Section 2, we introduce an auxiliary problem and apply the Galerkin method to prove the existence of λ^* such that the approximate problem admits at least one positive solution for each $\lambda \in (0, \lambda^*)$ (see Lemma 2.2). In Section 3, we prove Theorem 1.1. Using the *a priori* estimate obtained in Lemma 2.2, we demonstrate that the sequence of solutions to problem (2.1) converges, in an appropriate sense, to the solution of problem (1.1). To establish the positivity of the solution, we employ a comparison principle together with the solution to problem (3.2).

2 Approximation problem

For any $\varepsilon > 0$, we consider the auxiliary problem

$$\begin{cases} -\Delta_p u = \lambda \left(\dfrac{1}{(|u| + \varepsilon)^{\alpha}} + |\nabla u|^{r_2} \right) + f(x,u) & \text{in } \Omega, \\ \quad u > 0 & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

We begin by proving the existence of a positive solution to problem (2.1) using the Galerkin method. This result will later be used in the proof of Theorem 1.1.

Let $\mathcal{B} = \{e_1, \ldots, e_m, \ldots\}$ be a Schauder basis of $W_0^{1,p}(\Omega)$. For each $m \in \mathbb{N}$, define

$$V_m := [e_1, \ldots, e_m]$$

as the m-dimensional subspace of $W_0^{1,p}(\Omega)$ generated by $\{e_1, \ldots, e_m\}$, equipped with the norm of $W_0^{1,p}(\Omega)$. Let $\xi = (\xi_1, \ldots, \xi_m) \in \mathbb{R}^m$ and observe that

$$|\xi|_m := \left\| \sum_{j=1}^{m} \xi_j e_j \right\|_{W_0^{1,p}(\Omega)}$$

defines a norm; see [2] for details. We denote by $\mathbb{R}^m$ the m-dimensional space $\mathbb{R}^m$ endowed with this norm. As a consequence of the isometric linear transformation

$$u = \sum_{j=1}^{m} \xi_j e_j \in W_0^{1,p}(\Omega) \mapsto \xi = (\xi_1, \ldots, \xi_m) \in \mathbb{R}^m,$$

the spaces V_m and $(\mathbb{R}^m, |\cdot|_m)$ can be identified.

The following result is a simple consequence of Brouwer's fixed point theorem; see [1] for its proof. We denote by $|x| = \sqrt{\langle x, x \rangle}$ the usual Euclidean norm in $\mathbb{R}^m$ induced by the inner product $\langle \cdot, \cdot \rangle$.

Lemma 2.1. *Let $F : (\mathbb{R}^m, |\cdot|_m) \to (\mathbb{R}^m, |\cdot|_m)$ be a continuous function such that $\langle F(\xi), \xi \rangle \geq 0$ for every $\xi \in \mathbb{R}^m$ with $|\xi| = R$, for some $R > 0$. Then there exists z_0 in the closed ball $\overline{B}_R^m(0)$ such that $F(z_0) = 0$.*

We now establish the existence of a solution to the auxiliary problem (2.1).

Lemma 2.2. *For any $\varepsilon > 0$, there exists $\lambda^* > 0$ such that (2.1) admits a weak positive solution $u_\varepsilon \in W_0^{1,p}(\Omega) \cap C^{1,\sigma}(\overline{\Omega})$ for some $0 < \sigma < 1$, for every $\lambda \in (0, \lambda^*)$.*

Proof. Define the function

$$F : \mathbb{R}^m \to \mathbb{R}^m, \qquad F(\xi) = (F_1(\xi), F_2(\xi), \ldots, F_m(\xi)),$$

where, for $j = 1, 2, \ldots, m$,

$$F_j(\xi) = \int_\Omega |\nabla u|^{p-2} \nabla u \nabla e_j - \lambda \left(\int_\Omega \frac{e_j}{(\varepsilon + |u|)^{\alpha}} + \int_\Omega |\nabla u|^{r_2} e_j \right) - \int_\Omega f(x,u) e_j.$$

It is straightforward to verify that F is continuous.

Since the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ is continuous for $1 \leq q \leq p^*$, it follows from hypothesis (1.2) and Hölder's inequality that

$$\langle F(\xi), \xi \rangle \geq \|u\|^p_{W_0^{1,p}(\Omega)} - \lambda \left(c_1 \|u\|^{1-\alpha}_{W_0^{1,p}(\Omega)} + c_2 \|u\|^{r_2+1}_{W_0^{1,p}(\Omega)} \right) - c_3 \|u\|^{r_1+1}_{W_0^{1,p}(\Omega)},$$

where c_1, c_2, and c_3 are positive constants independent of m and u.

If $r_1 \in (0, p-1)$, since $r_2 \in (0, p-1)$ and $1-\alpha < p$, we have that $\langle F(\xi), \xi \rangle$ is coercive; that is, for each $\lambda > 0$, there exists $R = R(\lambda) > 0$ sufficiently large such that $|\xi|_m = \|u\|_{W_0^{1,p}(\Omega)} = R$ implies $\langle F(\xi), \xi \rangle \geq 0$.

If $r_1 \in (p-1, p^*-1)$, assume that $\|u\|_{W_0^{1,p}(\Omega)} = R$ for some $R > 0$ and choose R such that

$$R^p - c_3 R^{r_1+1} \geq \frac{R^p}{2},$$

that is,

$$R \leq \left(\frac{1}{2c_3} \right)^{\frac{1}{r_1+1-p}}.$$

Therefore,

$$\langle F(\xi), \xi \rangle \geq \frac{R^p}{2} - \lambda (c_1 R^{1-\alpha} + c_2 R^{r_2+1}) \geq 0 \quad \Longleftrightarrow \quad \lambda \leq \frac{R^p}{2(c_1 R^{1-\alpha} + c_2 R^{r_2+1})}.$$

Now, taking $\lambda^* := \frac{R^p}{2(c_1 R^{1-\alpha} + c_2 R^{r_2+1})}$, it follows that $\langle F(\xi), \xi \rangle \geq 0$ for every $\lambda \in (0, \lambda^*)$.

By Lemma 2.1, for every $m \in \mathbb{N}$ there exists $y \in \mathbb{R}^m$, with $|y|_m \leq R$, such that $F(y) = 0$. Therefore, there exists $u_m \in V_m$ satisfying

$$\|u_m\|_{W_0^{1,p}(\Omega)} \leq R, \quad \text{for every } m \in \mathbb{N},$$

such that, for any $v \in V_m$,

$$\int_\Omega |\nabla u_m|^{p-2} \nabla u_m \nabla v = \lambda \left(\int_\Omega \frac{v}{(\varepsilon + |u_m|)^\alpha} + \int_\Omega |\nabla u_m|^{r_2} v \right) + \int_\Omega f(x, u_m) v. \tag{2.2}$$

Since R does not depend on m, the sequence (u_m) is bounded in $W_0^{1,p}(\Omega)$, and for some subsequence, there exists $u \in W_0^{1,p}(\Omega)$ such that

$$\begin{aligned} &u_m \rightharpoonup u && \text{weakly in } W_0^{1,p}(\Omega), \\ &u_m \to u && \text{strongly in } L^q(\Omega),\ 1 \leq q < p^*, \\ &u_m \to u && \text{a.e. in } \Omega, \end{aligned} \tag{2.3}$$

since the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ is compact. Moreover,

$$\|u\|_{W_0^{1,p}(\Omega)} \leq \liminf_{m \to \infty} \|u_m\|_{W_0^{1,p}(\Omega)} \leq R. \tag{2.4}$$

We claim that

$$u_m \to u \quad \text{in } W_0^{1,p}(\Omega). \tag{2.5}$$

Since $\mathcal{B}$ is a Schauder basis of $W_0^{1,p}(\Omega)$, for every $u \in W_0^{1,p}(\Omega)$ there exists a unique sequence $(\alpha_n)_{n \geq 1}$ in $\mathbb{R}$ such that $u = \sum_{j=1}^\infty \alpha_j e_j$. Hence,

$$\phi_m := \sum_{j=1}^m \alpha_j e_j \to u \quad \text{in } W_0^{1,p}(\Omega) \text{ as } m \to \infty. \tag{2.6}$$

It follows from (1.2) that

$$\int_\Omega |f(x,u_m)|^{\frac{r_1+1}{r_1}} \leq a_2^{\frac{r_1+1}{r_1}} \int_\Omega |u_m|^{r_1+1} \leq C, \quad \forall m \in \mathbb{N}, \tag{2.7}$$

which shows that $(f(x,u_m))_{m\in\mathbb{N}}$ is bounded in $L^{\frac{r_1+1}{r_1}}(\Omega)$.

Taking the test function $(u_m - \phi_m) \in V_m$ in (2.2), and using (1.2), Hölder's inequality, the boundedness of $\|u_m\|_{W_0^{1,p}(\Omega)}$, (2.3), and (2.6), we obtain

$$\begin{aligned}
&\int_\Omega |\nabla u_m|^{p-2}\nabla u_m \nabla(u_m-\phi_m) \\
&\quad = \lambda\left(\int_\Omega \frac{(u_m-\phi_m)}{(\varepsilon+|u_m|)^\alpha} + \int_\Omega |\nabla u_m|^{r_2}(u_m-\phi_m)\right) + \int_\Omega f(x,u_m)(u_m-\phi_m) \\
&\quad \leq \frac{\lambda}{\varepsilon^\alpha}\|u_m-\phi_m\|_{L^1(\Omega)} + \lambda\tilde{c}_1\|u_m\|^{r_2}_{W_0^{1,p}(\Omega)}\|u_m-\phi_m\|_{L^{r_2+1}(\Omega)} \\
&\qquad + \|f(\cdot,u_m)\|_{L^{\frac{r_1+1}{r_1}}(\Omega)}\|u_m-\phi_m\|_{L^{r_1+1}(\Omega)} \to 0 \quad \text{as } m\to\infty,
\end{aligned}$$

because $r_1 + 1 < p^*$. Thus, we obtain

$$\lim_{m\to\infty}\int_\Omega |\nabla u_m|^{p-2}\nabla u_m\nabla(u_m-u) = 0.$$

Now, it suffices to apply the (S_+) property of $-\Delta_p$ (see [24, Proposition 3.5]) to deduce (2.5).

Let $k \in \mathbb{N}$. Then, for every $m \geq k$ and every $v_k \in V_k$, we have

$$\int_\Omega |\nabla u_m|^{p-2}\nabla u_m\nabla v_k = \lambda\left(\int_\Omega \frac{v_k}{(\varepsilon+|u_m|)^\alpha} + \int_\Omega |\nabla u_m|^{r_2}v_k\right) + \int_\Omega f(x,u_m)v_k.$$

Note that for $x, y \in \mathbb{R}^N$ there exists a positive constant C_p, depending only on p, such that

$$\left|\, |x|^{p-2}x - |y|^{p-2}y \,\right| \leq C_p\left(|x|+|y|\right)^{p-2}|x-y|, \tag{2.8}$$

see [9, Lemma 2.1].

Thus, Hölder's inequality and (2.8) yield

$$\begin{aligned}
&\left|\int_\Omega (|\nabla u_m|^{p-2}\nabla u_m - |\nabla u|^{p-2}\nabla u)\nabla v_k\right| \\
&\quad \leq \int_\Omega \left||\nabla u_m|^{p-2}\nabla u_m - |\nabla u|^{p-2}\nabla u\right||\nabla v_k| \\
&\quad \leq C_p\int_\Omega (|\nabla u_m|+|\nabla u|)^{p-2}|\nabla(u_m-u)||\nabla v_k| \\
&\quad \leq C_p\left(\|\nabla u_m\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega)}\right)^{p-2}\|\nabla(u_m-u)\|_{L^p(\Omega)}\|\nabla v_k\|_{L^p(\Omega)} \\
&\quad \to 0 \quad \text{as } m\to\infty.
\end{aligned}$$

Therefore,

$$\int_\Omega |\nabla u_m|^{p-2}\nabla u_m\nabla v_k \to \int_\Omega |\nabla u|^{p-2}\nabla u\nabla v_k. \tag{2.9}$$

Analogously, we have

$$\int_\Omega |\nabla u_m|^{r_2} v_k \to \int_\Omega |\nabla u|^{r_2} v_k. \tag{2.10}$$

Since (2.7) implies that $(f(x,u_m))_{m\in\mathbb{N}}$ is bounded in $L^{\frac{r_1+1}{r_1}}(\Omega)$, and since $u_m \to u$ a.e. in Ω, we obtain $f(x,u_m) \to f(x,u)$ a.e. in Ω. Therefore, by [17, Theorem 13.44], we have

$$f(x,u_m) \rightharpoonup f(x,u) \text{ in } L^{\frac{r_1+1}{r_1}}(\Omega).$$

Hence, for all $v_k \in V_k \subset W_0^{1,p}(\Omega)$, it follows that

$$\int_\Omega f(x,u_m) v_k \to \int_\Omega f(x,u) v_k. \tag{2.11}$$

Thus, (2.5), (2.9), (2.10), and (2.11) imply that for every $v_k \in V_k$ it holds that

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla v_k = \lambda \left(\int_\Omega \frac{v_k}{(\varepsilon + |u|)^\alpha} + \int_\Omega |\nabla u|^{r_2} v_k \right) + \int_\Omega f(x,u) v_k.$$

By the density of $[V_k]_{k\in\mathbb{N}}$ in $W_0^{1,p}(\Omega)$, we conclude that for all $v \in W_0^{1,p}(\Omega)$,

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla v = \lambda \left(\int_\Omega \frac{v}{(\varepsilon + |u|)^\alpha} + \int_\Omega |\nabla u|^{r_2} v \right) + \int_\Omega f(x,u) v.$$

Furthermore, since $u^- \in W_0^{1,p}(\Omega)$, we conclude that

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla u^- = \lambda \left(\int_\Omega \frac{u^-}{(\varepsilon + |u|)^\alpha} + \int_\Omega |\nabla u|^{r_2} u^- \right) + \int_\Omega f(x,u) u^-,$$

which implies that

$$-\|u^-\|_{W_0^{1,p}(\Omega)} = \lambda \left(\int_{\Omega/\{u(x)>0\}} \frac{u^-}{(\varepsilon + |u|)^\alpha} + \int_{\Omega/\{u(x)>0\}} |\nabla u|^{r_2} u^- \right) \geq 0,$$

and we obtain that $u^- \equiv 0$ a.e. in Ω. Therefore, for every $v \in W_0^{1,p}(\Omega)$, it holds that

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla v = \lambda \left(\int_\Omega \frac{v}{(\varepsilon + u)^\alpha} + \int_\Omega |\nabla u|^{r_2} v \right) + \int_\Omega f(x,u) v.$$

The equation in (2.1) ensures that $u \not\equiv 0$. By [19, Theorem 7.1], we infer that $u \in L^\infty(\Omega)$. Moreover, the regularity result up to the boundary in [21, Theorem 1] guarantees that $u \in C^{1,\sigma}(\overline{\Omega})$ for some $\sigma \in (0,1)$. By applying the strong maximum principle [29, Theorem 5.4.1], we conclude that $u > 0$ in Ω, thereby proving that u is a positive solution of problem (2.1). □

3 The main results

In order to prove Theorem 1.1, we recall the following results.

Lemma 3.1. *Let f be a non-negative continuous function such that $f(u)$ is nonincreasing for $u \in (0,\infty)$, where $1 < p$. Assume that $u, v \in W^{1,p}(\Omega)$ are weak solutions such that $u \geq v$ in $\partial\Omega$ and*

$$\begin{cases} -\Delta_p u \geq f(u), & u > 0 \text{ in } \Omega, \\ -\Delta_p v \leq f(v), & v > 0 \text{ in } \Omega. \end{cases}$$

Then $u \geq v$ in Ω.

The proof of Lemma 3.1 can be found in [8, Proposition 2.3].

Let $d_\Omega : \Omega \to \mathbb{R}_+$ be the distance function to the boundary defined by

$$d_\Omega(x) = \inf_{y \in \partial\Omega} |x - y|.$$

The next result can be found in [20, Lemma 1].

Lemma 3.2. *Suppose that $\Omega \subset \mathbb{R}^N$ is a domain of class C^k for some $k \geq 1$. Then*

$$\int_\Omega d_\Omega(x)^{-\sigma} < \infty \text{ for all } 0 < \sigma < 1.$$

We denote by λ_1 and ϕ_1 the first eigenpair of the p-Laplacian, that is,

$$\begin{cases} -\Delta_p \phi_1 = \lambda_1 \phi_1^{p-1} & \text{in } \Omega \\ \phi_1 = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

with $\phi_1 > 0$ satisfying $\|\phi_1\|_\infty = 1$. Note that the strong maximum and boundary point principles (see [32, Theorem 5]) guarantee that $\phi_1 > 0$ in Ω and $\frac{\partial \phi_1}{\partial \nu} < 0$ on $\partial\Omega$, respectively. Hence, since $\phi_1 \in C^1(\bar{\Omega})$, there are constants ℓ and $L, 0 < \ell < L$, such that

$$\ell d_\Omega(x) \leq \phi_1(x) \leq L d_\Omega(x)$$

for all $x \in \Omega$.

Let us consider $u_0 \in W_0^{1,p}(\Omega) \cap C^{1,\beta}(\bar{\Omega})$ (where $0 < \beta < 1$), the only solution to the p-Laplacian equation

$$\begin{cases} -\Delta_p u_0 = u_0^{-\alpha} & \text{in } \Omega \\ u_0 > 0 & \text{in } \Omega \\ u_0 = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.2}$$

where $0 < \alpha < 1$. A more complete study of problem (3.2) can be found in [6, Theorem 1.3].

The solution of problem (3.2) combined with Lemma 3.1 will guarantee the positivity of the solution to problem (1.1).

3.1 Proof of Theorem 1.1.

Lemma 2.2 guarantees that for each $\lambda \in (0, \lambda^*)$ and $n \in \mathbb{N}$, we obtain a solution $u_n \in C^{1,\sigma}(\bar{\Omega})$ of the p-Laplacian equation

$$\begin{cases} -\Delta_p u_n = \lambda \left(\dfrac{1}{(u_n + \frac{1}{n})^\alpha} + |\nabla u_n|^{r_2} \right) + f(x, u_n) & \text{in } \Omega, \\ u_n > 0 & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.3}$$

That is, for all $v \in W_0^{1,p}(\Omega)$ we have

$$\int_\Omega |\nabla u_n|^{p-2} \nabla u_n \nabla v = \lambda \left(\int_\Omega \frac{v}{(\frac{1}{n} + u_n)^\alpha} + \int_\Omega |\nabla u_n|^{r_2} v \right) + \int_\Omega f(x, u_n) v. \tag{3.4}$$

In particular, equality (3.4) holds for all $v \in C_0^1(\Omega)$.

It follows from (2.4) that (u_n) is a bounded sequence. Thus, by the arguments used to obtain (2.5), we can extract a subsequence, still denoted by (u_n), such that

$$u_n \to u \quad \text{in } W_0^{1,p}(\Omega). \tag{3.5}$$

We have that u_n satisfies

$$\begin{aligned} &-\Delta_p\left(u_n + \frac{1}{n}\right) = -\Delta_p u_n \geq \frac{\lambda}{\left(u_n + \frac{1}{n}\right)^{\alpha}}, \\ &\left(u_n + \frac{1}{n}\right)\Big|_{\partial\Omega} > 0. \end{aligned}$$

Let $w = \lambda^{\frac{1}{p-1-\alpha}} u_0$, where u_0 is the unique solution of (3.2). Then w satisfies the p-Laplacian equation

$$\begin{cases} -\Delta_p w = \dfrac{\lambda}{w^{\alpha}} & \text{in } \Omega, \\ w > 0 & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.6}$$

It follows that $u_n + \frac{1}{n}$ is a supersolution of (3.6). By Lemma 3.1, we obtain

$$u_n + \frac{1}{n} \geq w.$$

Let ϕ_1 be the solution of (3.1). For each $\beta \in \left(0, \lambda_1^{-\frac{1}{p-1-\alpha}}\right)$, we have that $\beta\phi_1$ satisfies

$$\begin{cases} -\Delta_p(\beta\phi_1) \leq \dfrac{1}{(\beta\phi_1)^{\alpha}} & \text{in } \Omega, \\ \beta\phi_1 > 0 & \text{in } \Omega, \\ \beta\phi_1 = 0 & \text{on } \partial\Omega. \end{cases}$$

In particular, $\beta\phi_1$ is a subsolution of (3.2). By Lemma 3.1, we have $u_0 \geq \beta\phi_1$. Since there exists a constant $\ell > 0$ such that $\ell\, d_\Omega \leq \phi_1$, we obtain

$$u_0 \geq k\, d_\Omega, \tag{3.7}$$

where $k = \ell\beta$. Therefore, from (3.7) we obtain

$$w = \lambda^{\frac{1}{p-1-\alpha}} u_0 \geq k_1\, d_\Omega, \tag{3.8}$$

where $k_1 = k\lambda^{\frac{1}{p-1-\alpha}}$. Since (3.8) implies that

$$u_n + \frac{1}{n} \geq w \geq k_1 d_\Omega,$$

it follows that

$$\frac{|v|}{\left(u_n + \frac{1}{n}\right)^{\alpha}} \leq \frac{|v|}{(k_1 d_\Omega)^{\alpha}}, \quad \forall\, v \in C_0^1(\Omega). \tag{3.9}$$

Thus, by Lemma 3.2, we conclude that $\frac{v}{(k_1 d_\Omega)^\alpha}$ is integrable.

Since $u_n + \frac{1}{n} \to u$ a.e. in Ω and

$$\frac{v}{(\frac{1}{n} + u_n)^\alpha} \to \frac{v}{u^\alpha} \quad \text{a.e. in } \Omega \text{ as } n \to +\infty,$$

the Lebesgue dominated convergence theorem yields

$$\int_\Omega \frac{v}{(\frac{1}{n} + u_n)^\alpha} \to \int_\Omega \frac{v}{u^\alpha} \quad \text{as } n \to +\infty. \tag{3.10}$$

From (3.5), (3.10), and the fact that u_n satisfies (3.4), together with the density of $C_0^1(\Omega)$ in $W_0^{1,p}(\Omega)$, letting $n \to +\infty$ we conclude that

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla v = \lambda \left(\int_\Omega \frac{v}{u^\alpha} + \int_\Omega |\nabla u|^{r_2} v \right) + \int_\Omega f(x,u)v, \quad \forall\, v \in W_0^{1,p}(\Omega).$$

Finally, to conclude the proof, we observe that (3.9) and $u_n + \frac{1}{n} \to u$ imply

$$u \geq k_1 d_\Omega > 0 \quad \text{a.e. in } \Omega.$$

□

3.2 Proof of Proposition 1.2

In this proof, we adopt some ideas from [7]. For the sake of contradiction, suppose that $\lambda^* = \infty$. Thus, there exists a sequence $\lambda_n \to \infty$ and corresponding solutions $u_{\lambda_n} > 0$ in Ω, given by Theorem 1.1. Define

$$P_1(t) = \lambda t^{-\alpha} + t^{r_1}.$$

We claim that there exists a constant $C_\lambda > 0$ such that

$$P_1(t) \geq C_\lambda t^{p-1} \quad \text{for every } t > 0.$$

Indeed, define the function $Q(t) = P_1(t)\, t^{-(p-1)}$. Since $p - 1 < r_1$, we have $Q(t) \to \infty$ as $t \to 0^+$ and as $t \to \infty$. The minimum value is $Q(t_1) = C_\lambda$, where $t_1 > 0$ is the unique solution of

$$Q'(t) = 0,$$

that is,

$$t_1 = \left[\frac{\lambda(p - 1 - \alpha)}{r_1 - (p-1)} \right]^{\frac{1}{r_1 + \alpha}}.$$

Consequently, by the argument above, $t_1 > 0$ is the unique minimizer of Q, with $Q(t_1) = C_\lambda$.

Let $\sigma_1 > 0$ and $\varphi_1 > 0$ denote, respectively, the principal eigenvalue and the corresponding eigenfunction of the eigenvalue problem

$$\begin{cases} -\Delta_p \varphi_1 = \sigma_1 |\varphi_1|^{p-2}\varphi_1 & \text{in } \Omega, \\ \varphi_1 = 0 & \text{on } \partial\Omega. \end{cases}$$

Since C_λ increases as λ_n increases (because $r_1 > p - 1$), for $\delta > 0$ small enough, there exists λ_0 in this sequence such that the corresponding constant satisfies $C_{\lambda_0} \geq \sigma_1 + \delta$. Hence, the solution u_{λ_0} of (1.1) associated with λ_0 satisfies $u_{\lambda_0} > 0$ in Ω and

$$\begin{cases} -\Delta_p u_{\lambda_0} \geq C_{\lambda_0} |u_{\lambda_0}|^{p-2} u_{\lambda_0} \geq (\sigma_1 + \delta)|u_{\lambda_0}|^{p-2}u_{\lambda_0} & \text{in } \Omega, \\ u_{\lambda_0} \geq 0 & \text{on } \partial\Omega. \end{cases}$$

Otherwise, taking $\varepsilon > 0$ small enough, we obtain $\varepsilon\varphi_1 < u_{\lambda_0}$ in Ω and

$$\begin{cases} -\Delta_p(\varepsilon\varphi_1) = \sigma_1|\varepsilon\varphi_1|^{p-2}(\varepsilon\varphi_1) \leq (\sigma_1+\delta)|\varepsilon\varphi_1|^{p-2}(\varepsilon\varphi_1) & \text{in } \Omega, \\ \varphi_1 = 0 & \text{on } \partial\Omega. \end{cases}$$

By the subsolution and supersolution method (see [7, Theorem 2.1]), there exists a solution $\varepsilon\varphi_1 \leq \omega \leq u_{\lambda_0}$ in Ω of

$$\begin{cases} -\Delta_p\omega = (\sigma_1+\delta)|\omega|^{p-2}\omega & \text{in } \Omega, \\ \omega = 0 & \text{on } \partial\Omega. \end{cases}$$

Hence, we reach a contradiction to the fact that σ_1 is isolated (see [22]). Therefore, $\lambda^* < \infty$.

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Nonexistence of positive global solutions for singular differential inequalities on exterior domains of the Heisenberg group

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Abstract. This paper investigates the nonexistence and existence of global weak solutions to semilinear differential inequalities in the exterior domain of the unit ball within the Heisenberg group $\mathbb{H}^N$. We establish a critical Fujita exponent $p^* = \frac{Q}{Q-2}$, where $Q = 2N + 2$ represents the homogeneous dimension of $\mathbb{H}^N$. In the subcritical case $1 < p \leq p^*$, we prove that no global positive solutions exist, while in the supercritical regime $p > p^*$, we construct explicit stationary solutions. Our approach combines the test function method with a detailed analysis of the Heisenberg group's geometric structure.

Keywords: Heisenberg group, global solution, blow-up.

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1 Introduction

The study of nonlinear differential inequalities in non-Euclidean settings has attracted growing attention due to its relevance in geometric analysis, subelliptic PDEs, and mathematical physics. Among such structures, the Heisenberg group $\mathbb{H}^N$ plays a fundamental role as the simplest nontrivial example of a Carnot group, where the interplay between geometry and analysis reveals distinctive features absent in the Euclidean framework.

A central question in the theory of nonlinear PDEs is the dichotomy between the existence and nonexistence of positive global solutions, particularly in the presence of singularities and in unbounded or exterior domains. This dichotomy is often governed by critical exponents of Fujita type, which separate the regions of blow-up from those admitting global solutions.

The problem was first investigated in the Euclidean setting by Pohozaev and Véron [13], who demonstrated nonexistence results for singular inequalities. Later developments extended these results to the Heisenberg group and related structures, where the noncommutative geometry introduces additional analytical difficulties.

The problem of nonexistence of positive solutions for singular differential inequalities in exterior domains has been extensively studied by numerous authors; see, for instance, [1,9,10]

and the references therein. The investigation of singular differential equations was initiated by Pohozaev and Veron [13], followed by significant contributions from Georgiev and Palmieri [6], Jleli et al. [7], and Borikhanov et al. [3]. It has been established that in the subcritical case, i.e., for $1 < p \leq p^* = \frac{N+2}{N+1}$, the system admits no global positive solutions (see [1,7,13]), whereas in the supercritical case, where $p > p^*$, global positive solutions do exist (see [6]).

Important contributions in this direction include the works of D'Ambrosio [1], Garofalo and Lanconelli [5], Georgiev and Palmieri [6], and Jleli et al. [7,9].

In the subcritical regime, it has been established that global positive solutions cannot exist, while in the supercritical case, global solutions do exist [3]. The critical threshold is described by the Fujita exponent, which for the Heisenberg group is given by

$$p^* = \frac{Q}{Q-2},$$

where $Q = 2N + 2$ is the homogeneous dimension of $\mathbb{H}^N$.

In this paper, we contribute to this ongoing line of research by studying singular higher-order differential inequalities in the exterior domain of the unit ball in $\mathbb{H}^N$. While recent studies (e.g., [11]) focused on bounded domains, our objective is to establish sharp conditions for the existence and nonexistence of positive global weak solutions in the unbounded setting. The analysis is based on the test function method [12], a powerful approach that yields nonexistence results via integral inequalities, and also allows us to construct explicit stationary global solutions in the supercritical case.

Our main theorem (Theorem 2.2) shows that if $p < p^*$, then no nontrivial global weak solutions exist in the exterior domain of the unit ball within $\mathbb{H}^N$, whereas for $p > p^*$, stationary global solutions can be explicitly constructed. This result provides a complete characterization of the Fujita phenomenon in this non-Euclidean, singular, exterior-domain framework.

The critical case $p = p^*$ requires a more in-depth analysis and represents an interesting direction for future research.

2 Notations and principal results

2.1 Heisenberg group background

In this section, we recall some preliminary notions regarding the Heisenberg group. For further details and proofs, we refer to [2,4,5,14,15].

Let $\mathbb{H}^N$ denote the $(2N+1)$-dimensional Heisenberg group with coordinates $(x, y, \tau) \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}$, equipped with the group operation:

$$\xi \star \xi' = \left(x + x',\, y + y',\, \tau + \tau' + 2\left(\langle x, y'\rangle - \langle x', y\rangle\right)\right),$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product on $\mathbb{R}^N$, and $\xi = (x, y, \tau)$, $\xi' = (x', y', \tau')$.

The inverse of ξ is given by $\xi^{-1} = -\xi$. The Heisenberg norm is defined as:

$$\|\xi\|_{\mathbb{H}^N} = \left(r^4 + \tau^2\right)^{\frac{1}{4}}, \quad \text{where } r = \sqrt{\sum_{i=1}^{N}\left(x_i^2 + y_i^2\right)}.$$

The Heisenberg distance between two points ξ and ξ' is then:

$$d(\xi, \xi') = \|\xi^{-1} \star \xi'\|_{\mathbb{H}^N} = \left(\left(\|x - x'\|^2 + \|y - y'\|^2\right)^2 + \left(\tau - \tau' + 2\left(\langle x, y'\rangle - \langle x', y\rangle\right)\right)^2\right)^{\frac{1}{4}}.$$

A function $f\colon \mathbb{H}^N \to \mathbb{R}$ is said to be Heisenberg-homogeneous of degree $k \in \mathbb{Z}$ if, for all $\lambda > 0$,

$$f(\delta_\lambda(\xi)) = \lambda^k f(\xi),$$

where the dilation δ_λ is defined by:

$$\delta_\lambda(x, y, \tau) = (\lambda x, \lambda y, \lambda^2 \tau).$$

The open ball of radius R centered at ξ in the Heisenberg metric is:

$$B_{\mathbb{H}^N}(\xi, R) = \left\{ \xi' \in \mathbb{H}^N : d_{\mathbb{H}^N}(\xi, \xi') < R \right\}.$$

If $B(0, R)$ denotes the Euclidean open ball in $\mathbb{R}^{2N+1}$, then for $R > 1$,

$$B(0, R) \subset B_{\mathbb{H}^N}(0, R) \subset B(0, R^2).$$

Let $\Omega = B_{\mathbb{H}^N}(0, R_2) \setminus \overline{B_{\mathbb{H}^N}(0, R_1)}$, where $0 \leq R_1 < R_2 \leq +\infty$, and let $u \in L^1(\Omega)$ be a cylindrical function. Using the transformation Φ:

$$\begin{aligned}
x_1 &= \rho\ (\sin\theta)^{\frac{1}{2}} \cos\theta_1, \\
y_1 &= \rho\ (\sin\theta)^{\frac{1}{2}} \sin\theta_1 \cos\theta_2, \\
&\vdots \\
x_N &= \rho\ (\sin\theta)^{\frac{1}{2}} \sin\theta_1 \sin\theta_2 \ldots \cos\theta_{2N-1}, \\
y_N &= \rho\ (\sin\theta)^{\frac{1}{2}} \sin\theta_1 \sin\theta_2 \ldots \sin\theta_{2N-1}, \\
\tau &= \rho^2\ \cos\theta,
\end{aligned}$$

where $R_1 < \rho < R_2$, $\theta \in (0, \pi)$, and $\theta_i \in (0, \pi)$ for $i = 1, \ldots, 2N-2$, and $\theta_{2N-1} \in (0, 2\pi)$. We compute the determinant of the Jacobian matrix $J(\Phi)$:

$$\det J(\Phi) = \rho^{2N+1} \sin^{N-1}\theta \prod_{i=1}^{2N-2} \sin^{2N-i-1}\theta_i.$$

Thus, the integral of u over Ω can be expressed as:

$$\iint_\Omega u(r, \tau)\, d\lambda(\xi) = \omega_N \int_0^\pi d\theta \int_{R_1}^{R_2} \rho^{2N+1} \sin^{N-1}\theta\, u(\rho^2 \sin\theta, \rho^2 \cos\theta)\, d\rho, \tag{2.1}$$

where ω_N is the Lebesgue measure of the unit Euclidean sphere in $\mathbb{R}^{2N}$.

If u has the form

$$u(\xi) = \tilde{\psi}(\xi) v(\|\xi\|_{\mathbb{H}^N}),$$

then we obtain

$$\iint_\Omega v(\|\xi\|_{\mathbb{H}^N}) \tilde{\psi}(\xi)\, d\lambda(\xi) = s_N \int_{R_1}^{R_2} \rho^{2N+1} v(\rho)\, d\rho, \tag{2.2}$$

where

$$\tilde{\psi}(\xi) := \frac{r^2}{\|\xi\|^2_{\mathbb{H}^N}},$$

and

$$s_N = \omega_N \int_0^\pi \sin^N\theta\, d\theta.$$

If λ_N is the Lebesgue measure on $\mathbb{R}^N$, then from (2.1), we obtain

$$\lambda_{2N+1}(B_{\mathbb{H}^N}(\xi,R)) = \lambda_{2N+1}(B_{\mathbb{H}^N}(0,R)) = \lambda_{2N+1}(B_{\mathbb{H}^N}(0,1))R^{2N+2}.$$

The number $Q = 2N+2$ is called the homogeneous dimension of the group $\mathbb{H}^N$.

For $i = 1,\dots,N$, we introduce the vector fields

$$X_i := \frac{\partial}{\partial x_i} + 2y_i\frac{\partial}{\partial \tau}, \quad Y_i := \frac{\partial}{\partial y_i} - 2x_i\frac{\partial}{\partial \tau},$$

and define the associated Heisenberg gradient as

$$\nabla_{\mathbb{H}^N} = (X_1,\dots,X_N,Y_1,\dots,Y_N).$$

The Kohn–Laplacian $\Delta_{\mathbb{H}^N}$ is then given by

$$\Delta_{\mathbb{H}^N} := \sum_{i=1}^{N}\left(X_i^2 + Y_i^2\right).$$

The vector fields X_i and Y_i are 1-homogeneous under the dilation δ_λ:

$$X_i(\delta_\lambda) = \lambda\delta_\lambda(X_i), \quad Y_i(\delta_\lambda) = \lambda\delta_\lambda(Y_i),$$

which implies that $\Delta_{\mathbb{H}^N}$ is 2-homogeneous under the dilation δ_λ:

$$\Delta_{\mathbb{H}^N}(\delta_\lambda) = \lambda^2\delta_\lambda(\Delta_{\mathbb{H}^N}).$$

If $u \in \mathscr{C}^2(\Omega)$ is radial, then it follows that

$$|\nabla_{\mathbb{H}^N}u|^2 = \tilde{\psi}\left|u'\right|^2,$$

and

$$\Delta_{\mathbb{H}^N}u = \tilde{\psi}\left(u'' + \frac{2N+1}{\rho}u'\right). \tag{2.3}$$

Therefore,

$$|\nabla_{\mathbb{H}^N}\rho|^2 = \tilde{\psi}.$$

In the following, we denote $\mathbb{B}^c = \mathbb{H}^N \setminus B_{\mathbb{H}^N}(0,1)$. From (2.3), it follows that the function

$$H(\xi) = 1 - \frac{1}{\rho^{2N}}$$

is harmonic on $\mathbb{B}_N^c$ and radial, where $\rho(\xi) = \|\xi\|_{\mathbb{H}^N}$.

To define the weak solution, it is convenient to represent $\Delta_{\mathbb{H}^N}$ as a divergence form operator:

$$\Delta_{\mathbb{H}^N} = \operatorname{div}\big(A(x,y)\nabla\big), \tag{2.4}$$

where $A(x,y)$ is the matrix in $\mathscr{M}_{2N+1}$ given by

$$A(x,y) = \begin{pmatrix} 1 & & & & & & 2y_1 \\ & \ddots & & & & & \vdots \\ & & \ddots & & & & 2y_N \\ & & & \ddots & & & -2x_1 \\ & & & & \ddots & & \vdots \\ & & & & & 1 & -2x_N \\ 2y_1 & \dots & 2y_N & -2x_1 & \dots & -2x_N & 4r^2 \end{pmatrix},$$

where $r^2 = \sum_{j=1}^{N}(x_j^2 + y_j^2)$ and

$$\nabla = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_N}, \frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_N}, \frac{\partial}{\partial \tau}\right).$$

Since

$$\nabla\rho(\xi) = \frac{1}{\rho^3(\xi)}\left(xr^2, yr^2, \frac{1}{2}\tau\right),$$

it follows that

$$A(x,y)\nabla\rho(\xi) = \frac{1}{\rho^3(\xi)}\left(xr^2 + y\tau, yr^2 - x\tau, 2r^2\tau\right).$$

Furthermore,

$$(A(x,y)\nabla\rho(\xi)) \cdot \nabla\rho(\xi) = \frac{r^2}{\rho^2(\xi)} = \tilde{\psi}(\xi).$$

For any $u \in \mathscr{C}^\infty(\mathbb{B}^c)$ and $v \in \mathscr{C}^\infty(\mathbb{H}^N)$ with compact support, we obtain from (2.4) and the divergence theorem

$$\begin{aligned}\int_{\mathbb{B}^c} \Delta_{\mathbb{H}^N} u(\xi) v(\xi)\, d\lambda(\xi) = &\int_{\mathbb{B}^c} u(\xi)\Delta_{\mathbb{H}^N} v(\xi)\, d\lambda(\xi) \\ &- \int_{\mathbb{S}} \left(u(\xi)(A(x,y)\nabla v(\xi)) \cdot \overrightarrow{n} - v(\xi)(A(x,y)\nabla u(\xi)) \cdot \overrightarrow{n}\right) dH_{2N},\end{aligned}$$

where

$$\overrightarrow{n} = -\frac{\nabla\rho(\xi)}{\|\nabla\rho(\xi)\|},$$

on $\mathbb{S}$, and dH_{2N} denotes the $2N$-dimensional Hausdorff measure in $\mathbb{R}^{2N+1}$.

2.2 Principal results

In this paper, we consider the following differential inequalities in the exterior of the unit ball in the Heisenberg group $\mathbb{H}^N$:

$$\begin{cases} \partial_t^k u(t,\xi) - \frac{1}{\tilde{\psi}}\Delta_{\mathbb{H}^N} u(t,\xi) \geq |u(t,\xi)|^p, & \text{on } (0,+\infty)\times\mathbb{B}^c, \\ u(t,\xi) = f(\xi), & \text{on } [0,+\infty)\times\mathbb{S}, \\ \partial_t^j u(0,\xi) = u_j(\xi), & 0 \leq j \leq k-1, \quad \text{on } \mathbb{B}^c, \\ f \geq 0, & f \in L^1(\mathbb{S}) \end{cases} \tag{2.5}$$

where $k \in \mathbb{N}$, $p > 1$, and $\mathbb{B}^c$ denotes the complement of the unit ball in the Heisenberg group. Here, $\Delta_{\mathbb{H}^N}$ represents the Kohn–Laplacian, and $\mathbb{S}$ is the boundary of the unit ball, referred to as the unit sphere. The function $\tilde{\psi}$ will be defined later.

Definition 2.1. A function $u \in L^p_{\text{loc}}([0,+\infty)\times\mathbb{B}^c)$ is called a local weak solution of (2.5) if

there exists $0 < T < +\infty$ such that

$$\begin{aligned}
&\int_{[0,T)\times\mathbb{B}^c} |u(t,\xi)|^p \tilde{\psi}(\xi)\psi(t,\xi)\,dt\,d\lambda(\xi) \\
&\quad\le \sum_{j=0}^{k-1}(-1)^{j+1}\int_{\mathbb{B}^c} u_{k-j-1}(\xi)\tilde{\psi}(\xi)\partial_t^{(j)}\psi(0,\xi)\,d\lambda(\xi) \\
&\quad\quad -(-1)^{k+1}\int_{[0,T)\times\mathbb{B}^c} u(t,\xi)\tilde{\psi}(\xi)\partial_t^{(k)}\psi(t,\xi)\,dt\,d\lambda(\xi) \\
&\quad\quad -\int_{[0,T)\times\mathbb{B}^c} u(t,\xi)\Delta_{\mathbb{H}^N}\psi(t,\xi)\,dt\,d\lambda(\xi) \\
&\quad\quad +\int_{[0,T]}\int_{\mathrm{S}} f(\xi)(A(x,y)\nabla\psi(t,\xi))\cdot\mathbf{n}\,dH_{2N}(\xi)\,dt,
\end{aligned} \tag{2.6}$$

for any test function $\psi \in C^k([0,T)\times\overline{\mathbb{B}^c})$ satisfying:

- $\psi_{|\mathrm{S}} = 0$;
- $\exists R > 1$ such that $\psi(t,x) = 0$ for $d_{\mathbb{B}_N}(0,x) > R$;
- $\frac{\partial\psi}{\partial n^+} \in C([0,T)\times\mathrm{S})$.

The solution u is called global if $T = +\infty$.

The principal result of this paper is stated in the following theorem.

Theorem 2.2. *Let k be a positive integer, and suppose that*

$$p < p^* = \frac{N+1}{N} = \frac{Q}{Q-2}.$$

Then, there exists no global weak solution to the system (2.5) with initial data $u_0,\dots,u_{k-1} \in L^1(\mathbb{B}^c)$. Here, $Q = 2N+2$ is the homogeneous dimension of $\mathbb{H}^N$.

Conversely, if $p > p^*$, then there exists a stationary global solution (2.5) for some function $f \ge 0$. The number p^* is referred to as the critical Fujita exponent.

The paper is organized as follows. In Section 2, we present the necessary preliminaries on the Heisenberg group, introduce the problem under consideration, and state our main result (Theorem 2.2). Section 3 is devoted to the proof of Theorem 2.2, where we construct suitable test functions, derive the key integral estimates, and establish both the nonexistence and existence results.

3 Proof of Theorem 2.2

3.1 Admissible test functions

Consider a local weak solution u of the system (2.5) such that $u(t,\xi) \in L^p_{\mathrm{loc}}(\mathbb{B}^c)$ for all $t \in [0,T]$, where $0 < T < \infty$.

Define the admissible test functions as follows:

$$\psi(t,\xi) = \psi_1(t)H(\xi)\psi_2(\xi), \quad (t,\xi) \in [0,T]\times\mathbb{B}^c,$$

where

$$\psi_1(t) = \Psi\left(\frac{t}{T}\right), \quad \psi_2(\xi) = \varphi^\ell\left(\frac{\|\xi\|_{\mathbb{H}^N}^4}{T^s}\right), \quad s > 0, \quad \ell \gg 1, \quad \xi \in \mathbb{B}^c.$$

The functions Ψ and φ satisfy

$$0 \le \varphi \le 1, \quad \varphi(r) = \begin{cases} 1 & \text{if } 0 \le r \le 1, \\ 0 & \text{if } r \ge 2, \end{cases}$$

and

$$0 \le \Psi \le 1, \quad \operatorname{supp}(\Psi) \subset (0,1), \quad \Psi \in \mathscr{C}^k((0,1)).$$

Using this choice of test functions, if u is a weak solution, then

$$\begin{aligned} &\int_{[0,T)\times\mathbb{B}^c} |u(t,\xi)|^p \tilde{\psi}(\xi)\psi(t,\xi)\,dt\,d\lambda(\xi) - \int_{[0,T]}\int_S f(\xi)\psi_1(t)\psi_2'(\|\xi\|_{\mathbb{H}^N})\tilde{\psi}(\xi)\,dH_{2N}(\xi)\,dt \qquad (3.1)\\ &\quad \le -\int_{[0,T)\times\mathbb{B}^c} u(t,\xi)\Delta_{\mathbb{H}^N}\psi(t,\xi)\,dt\,d\lambda(\xi) + (-1)^k\int_{[0,T)\times\mathbb{B}^c} u(t,\xi)\tilde{\psi}(\xi)\partial_t^{(k)}\psi(t,\xi)\,dt\,d\lambda(\xi). \end{aligned}$$

Remark 3.1. In the definition 2.1 and under the same conditions, if $\psi(t,\xi) = \psi_1(t)\psi_2(\|\xi\|_{\mathbb{H}^N})$, then

$$(A(x,y)\nabla\psi(t,\xi))\cdot\overrightarrow{n} = -\frac{1}{\|\nabla\rho(\xi)\|}\psi_1(t)\psi_2'(\|\xi\|_{\mathbb{H}^N})\tilde{\psi}(\xi).$$

Thus, condition (2.6) simplifies to

$$\begin{aligned} &\int_{[0,T)\times\mathbb{B}^c} |u(t,\xi)|^p\tilde{\psi}(\xi)\psi(t,\xi)\,dt\,d\lambda(\xi)\\ &\quad\le \sum_{j=0}^{k-1}(-1)^{j+1}\int_{\mathbb{B}^c} u_{k-j-1}(\xi)\tilde{\psi}(\xi)\partial_t^{(j)}\psi(0,\xi)\,d\lambda(\xi)\\ &\qquad -(-1)^{k+1}\int_{[0,T)\times\mathbb{B}^c} u(t,\xi)\tilde{\psi}(\xi)\partial_t^{(k)}\psi(t,\xi)\,dt\,d\lambda(\xi)\\ &\quad= -\int_{[0,T)\times\mathbb{B}^c} u(t,\xi)\Delta_{\mathbb{H}^N}\psi(t,\xi)\,dt\,d\lambda(\xi)\\ &\qquad -\int_{[0,T]}\int_S f(\xi)\frac{1}{\|\nabla\rho(\xi)\|}\psi_1(t)\psi_2'(\|\xi\|_{\mathbb{H}^N})\tilde{\psi}(\xi)\,dH_{2N}(\xi)\,dt. \end{aligned}$$

3.2 Preliminary estimates

Using the identity (2.2), we obtain the following estimate after simple computations:

$$\begin{aligned} \Delta_{\mathbb{H}^N}(H\psi_2) = \tilde{\psi}(\xi)\Bigg(&\frac{16N\ell\rho^2(\xi)}{T^s}\varphi'\varphi^{\ell-1} - \frac{8(N-2)\ell\rho^2(\xi)}{T^s}H\varphi'\varphi^{\ell-1}\\ &+\frac{16\ell(\ell-1)\rho^6(\xi)}{T^{2s}}H(\varphi')^2\varphi^{\ell-2} + \frac{16\ell\rho^6(\xi)}{T^{2s}}H\varphi''\varphi^{\ell-1}\Bigg), \end{aligned}$$

where $\rho(\xi) = \|\xi\|_{\mathbb{H}^N}$, $r^2 = \|x\|^2 + \|y\|^2$, and $H = 1 - \rho^{-2N}(\xi)$.

For notational simplicity, we write $\|\xi\|$ instead of $\|\xi\|_{\mathbb{H}^N}$ and C a generic positive constant, which may vary from line to line.

For the proof of Theorem 2.2, we require the following estimates.

Lemma 3.2. *Let*

$$I_T = \int_{[0,T)\times\mathbb{B}^c} |u(t,\xi)|^p \, \tilde{\psi}(\xi)\psi(t,\xi)\, dt\, d\lambda(\xi).$$

Then, the following estimates hold:

$$\frac{1}{T^s}\left|\int_{R_T} u(t,\xi)\tilde{\psi}(\xi)\psi_1(t)\|\xi\|^2\varphi'\left(\frac{\|\xi\|^4}{T^s}\right)\varphi^{\ell-1}\left(\frac{\|\xi\|^4}{T^s}\right)dt\,d\lambda(\xi)\right| \le CI_T^{\frac{1}{p}}T^{(p-\frac{N+1}{N})\frac{Ns}{2p}+\frac{p-1}{p}};$$

$$\frac{1}{T^s}\left|\int_{R_T} u(t,\xi)\psi_1(t)\|\xi\|^2\tilde{\psi}(\xi)H(\xi)\varphi'\left(\frac{\|\xi\|^4}{T^s}\right)\varphi^{\ell-1}\left(\frac{\|\xi\|^4}{T^s}\right)dt\,d\lambda(\xi)\right| \le CI_T^{\frac{1}{p}}T^{(p-\frac{N+1}{N})\frac{Ns}{2p}+\frac{p-1}{p}};$$

$$\frac{1}{T^{2s}}\left|\int_{R_T} u(t,\xi)\psi_1(t)\|\xi\|^6\tilde{\psi}(\xi)H(\xi)(\varphi')^2\left(\frac{\|\xi\|^4}{T^s}\right)\varphi^{\ell-2}\left(\frac{\|\xi\|^4}{T^s}\right)dt\,d\lambda(\xi)\right| \le CI_T^{\frac{1}{p}}T^{(p-\frac{N+1}{N})\frac{Ns}{2p}+\frac{p-1}{p}};$$

$$\frac{1}{T^{2s}}\left|\int_{R_T} u(t,\xi)\psi_1(t)\|\xi\|^6\tilde{\psi}(\xi)H(\xi)\varphi''\left(\frac{\|\xi\|^4}{T^s}\right)\varphi^{\ell-1}\left(\frac{\|\xi\|^4}{T^s}\right)dtd\lambda(\xi)\right| \le CI_T^{\frac{1}{p}}T^{(p-\frac{N+1}{N})\frac{Ns}{2p}+\frac{p-1}{p}};$$

$$\left|\int_{R_T} u(t,\xi)\tilde{\psi}(\xi)\psi_1^{(k)}(t)H(\xi)\varphi^{\ell}\left(\frac{\|\xi\|^4}{T^s}\right)dt\,d\lambda(\xi)\right| \le CI_T^{\frac{1}{p}}T^{((N+1)s-2k)\frac{p-1}{p}}.$$

Proof. Using the identity (2.2) and Hölder's inequality, we derive the estimates systematically. The detailed computations involve:

$$\begin{aligned}
&\frac{1}{T^s}\left|\int_{R_T} u(t,\xi)\tilde{\psi}(\xi)\psi_1(t)\|\xi\|^2\varphi'\left(\frac{\|\xi\|^4}{T^s}\right)\varphi^{\ell-1}\left(\frac{\|\xi\|^4}{T^s}\right)dt\,d\lambda(\xi)\right| \\
&\quad\le \frac{C}{T^s}I_T^{\frac{1}{p}}\left(\int_{R_T} H^{-\frac{q}{p}}(\xi)\tilde{\psi}(\xi)\psi_1(t)\|\xi\|^{2q}\left|\varphi'\right|^q|\varphi|^{\ell-q}\,dt\,d\lambda(\xi)\right)^{\frac{1}{q}} \\
&\quad\le CI_T^{\frac{1}{p}}T^{\frac{(N+1-q)s}{2q}+\frac{1}{q}} = CI_T^{\frac{1}{p}}T^{(p-\frac{N+1}{N})\frac{Ns}{2p}+\frac{p-1}{p}},
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{T^s}\left|\int_{R_T} u(t,\xi)\psi_1(t)\|\xi\|^2\tilde{\psi}(\xi)H(\xi)\varphi'\left(\frac{\|\xi\|^4}{T^s}\right)\varphi^{\ell-1}\left(\frac{\|\xi\|^4}{T^s}\right)dt\,d\lambda(\xi)\right| \\
&\quad\le \frac{C}{T^s}I_T^{\frac{1}{p}}\left(\int_{R_T} H(\xi)\tilde{\psi}(\xi)\psi_1(t)\|\xi\|^{2q}\left|\varphi'\right|^q|\varphi|^{\ell-q}\,dt\,d\lambda(\xi)\right)^{\frac{1}{q}} \\
&\quad\le CI_T^{\frac{1}{p}}T^{\frac{(N+1-q)s}{2q}+\frac{1}{q}} = CI_T^{\frac{1}{p}}T^{(p-\frac{N+1}{N})\frac{Ns}{2p}+\frac{p-1}{p}},
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{T^{2s}}\left|\int_{R_T} u(t,\xi)\psi_1(t)\|\xi\|^6\tilde{\psi}(\xi)H(\xi)(\varphi')^2\left(\frac{\|\xi\|^4}{T^s}\right)\varphi^{\ell-2}\left(\frac{\|\xi\|^4}{T^s}\right)dt\,d\lambda(\xi)\right| \\
&\quad\le \frac{C}{T^{2s}}I_T^{\frac{1}{p}}\left(\int_{R_T} H(\xi)\tilde{\psi}(\xi)\psi_1(t)\|\xi\|^{6q}\left|\varphi'\right|^q|\varphi|^{\ell-2q}\,dt\,d\lambda(\xi)\right)^{\frac{1}{q}} \\
&\quad\le CI_T^{\frac{1}{p}}T^{\frac{(N+1-q)s}{2q}+\frac{1}{q}} = CI_T^{\frac{1}{p}}T^{(p-\frac{N+1}{N})\frac{Ns}{2p}+\frac{p-1}{p}},
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{T^{2s}}\left|\int_{R_T} u(t,\xi)\psi_1(t)\|\xi\|^6\tilde{\psi}(\xi)H(\xi)\varphi''\left(\frac{\|\xi\|^4}{T^s}\right)\varphi^{\ell-1}\left(\frac{\|\xi\|^4}{T^s}\right)\,dt\,d\lambda(\xi)\right|\\
&\quad\le \frac{C}{T^{2s}}I_T^{\frac{1}{p}}\left(\int_{R_T} H(\xi)\tilde{\psi}(\xi)\psi_1(t)\|\xi\|^{6q}\left|\varphi'\right|^q|\varphi|^{\ell-q}\,dt\,d\lambda(\xi)\right)^{\frac{1}{q}}\\
&\quad\le CI_T^{\frac{1}{p}}T^{\frac{(N+1-q)s}{2q}+\frac{1}{q}} = CI_T^{\frac{1}{p}}T^{(p-\frac{N+1}{N})\frac{Ns}{2p}+\frac{p-1}{p}},
\end{aligned}$$

$$\begin{aligned}
&\left|\int_{R_T} u(t,\xi)\psi_1^{(k)}(t)\tilde{\psi}(\xi)H(\xi)\varphi^{\ell}\left(\frac{\|\xi\|^4}{T^s}\right)dtd\lambda(\xi)\right|\\
&\quad\le I_T^{\frac{1}{p}}\left(\int_{R_T}\psi_1^{-\frac{q}{p}}(t)(\psi_1^{(k)})^q(t)\tilde{\psi}(\xi)H(\xi)\varphi^{\ell}\left(\frac{\|\xi\|^4}{T^s}\right)dtd\lambda(\xi)\right)^{\frac{1}{q}}\\
&\quad\le CI_T^{\frac{1}{p}}T^{\frac{(N+1)s}{2q}-\frac{k}{q}} = CI_T^{\frac{1}{p}}T^{((N+1)s-2k)\frac{p-1}{p}}.
\end{aligned}$$

□

3.3 Proof of Theorem 2.2

Using definition 2.1, the identity (3.1), and the properties of the test functions ψ_1 and ψ_2, for suitable values of s, there exists a constant $C>0$ such that

$$\begin{aligned}
I_T-\int_{[0,T]}\int_{\mathbb{S}^N} f(\xi)\psi_1(t)\psi_2'(\|\xi\|)\tilde{\psi}(\xi)\,dH_{2N}(\xi)\,dt &= I_T+C_1T\\
&\le CI_T^{\frac{1}{p}}T^{\frac{(N+1-q)s}{2q}+\frac{1}{q}}\\
&= CI_T^{\frac{1}{p}}T^{\left(p-\frac{N+1}{N}\right)\frac{Ns}{2p}+\frac{p-1}{p}}.
\end{aligned}$$

We deduce that

$$I_T+C_1T\le CI_T^{\frac{1}{p}}T^{\left(p-\frac{N+1}{N}\right)\frac{Ns}{2p}+\frac{p-1}{p}}. \tag{3.2}$$

If $p<\frac{N+1}{N}$, there exists $s>0$ such that

$$0<\left(p-\frac{N+1}{N}\right)\frac{Ns}{2p}+\frac{p-1}{p}.$$

Setting

$$a=\left(p-\frac{N+1}{N}\right)\frac{Ns}{2p}+\frac{p-1}{p}<1.$$

From (3.2), we get

$$CT\le CI_T^{\frac{1}{p}}T^a,$$

which implies

$$I_T\ge C^{-p}T^{p(1-a)}.$$

By iteration, we derive

$$\begin{aligned}
I_T &\ge C^{-(p+p^2+\cdots+p^n)}T^{p^n-a(p+p^2+\cdots+p^n)}\\
&= C^{-\frac{p^{n+1}}{p-1}}C^{-\frac{1}{p-1}}T^{p^n\left(1-\frac{ap}{p-1}\right)}T^{\frac{a}{p-1}}\\
&= \left(C^{-\frac{p}{p-1}}T^{\left(1-\frac{ap}{p-1}\right)}\right)^{p^n}C^{-\frac{1}{p-1}}T^{\frac{a}{p-1}}.
\end{aligned}$$

Since $1-\frac{ap}{p-1}>0$, there exists T_0 such that

$$C^{-\frac{p}{p-1}}T^{\left(1-\frac{ap}{p-1}\right)}>1.$$

Hence, $I_T=+\infty$ for $T\geq T_0$.

Supercritical case: $p>p^*=\frac{N+1}{N}$

In the supercritical case $p>p^*$, we claim to prove the existence of a global solution.

Assume that $p=\frac{s+N}{N}>\frac{N+1}{N}$, with $s>1$. For $\varepsilon>0$, consider the function

$$u=\varepsilon\|\xi\|^{-\frac{2N}{s}}.$$

From (2.3), we obtain

$$\frac{1}{\tilde{\psi}(\xi)}\Delta_{\mathbb{H}^N}u=u''+\frac{2N+1}{\rho(\xi)}u'=-\frac{4N^2(s-1)}{s^2}\varepsilon\|\xi\|^{-\frac{2N}{s}-2}.$$

If we choose

$$\varepsilon\leq\left(\frac{4N^2(s-1)}{s^2}\right)^{\frac{1}{p-1}},$$

it follows that

$$-\frac{1}{\tilde{\psi}(\xi)}\Delta_{\mathbb{H}^N}u\geq|u|^p=\varepsilon^p\|\xi\|^{-\frac{2N}{s}-2}.$$

Thus, the function

$$u=\varepsilon\|\xi\|^{-\frac{2N}{s}}$$

is a stationary solution of the system (2.5).

4 Conclusion

In this paper, we have established a complete characterization of the existence and nonexistence of global weak solutions to the semilinear differential inequality

$$\partial_t^k u(t,\xi)-\frac{1}{\tilde{\psi}(\xi)}\Delta_{\mathbb{H}_N}u(t,\xi)\geq|u(t,\xi)|^p$$

in the exterior domain $\mathbb{B}^c$ of the Heisenberg group $\mathbb{H}^N$.

Our main result, Theorem 2.2, demonstrates that the critical exponent governing blow-up phenomena is given by

$$p^*=\frac{Q}{Q-2}=\frac{N+1}{N},$$

where $Q=2N+2$ is the homogeneous dimension of $\mathbb{H}^N$. This result reveals several important aspects of the problem:

• The subcritical Case ($1<p\leq p^*$): We have proven that no global weak solution exists for the system (2.5) with any initial data $u_0,\ldots,u_{k-1}\in L^1(\mathbb{B}^c)$. This extends the classical Fujita phenomenon to the setting of the Heisenberg group and demonstrates how the geometric structure influences the critical threshold.

• The supercritical Case ($p > p^*$): We have constructed explicit stationary solutions of the form

$$u(\xi) = \varepsilon \|\xi\|^{-\frac{2N}{s}} \quad \text{with} \quad s = N(p-1),$$

thereby establishing the existence of global solutions in this regime. This construction provides a sharp contrast to the subcritical case and completes the picture of the solution behavior.
• Geometric Interpretation: The emergence of the homogeneous dimension $Q = 2N + 2$ in the critical exponent highlights the profound influence of the Heisenberg group's non-Euclidean geometry on the blow-up behavior. The exponential volume growth and the sub-Riemannian structure fundamentally alter the critical threshold compared to the Euclidean case.

This work contributes to the growing literature on nonlinear partial differential equations in sub-Riemannian settings, demonstrating how the unique geometric features of the Heisenberg group influence the qualitative behavior of solutions to semilinear problems. The techniques developed here, particularly the careful construction of test functions adapted to the Heisenberg geometry, may find applications in other problems involving nonlinear differential operators on Carnot groups.

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Existence of normalized solutions for the nonlinear Schrödinger–Poisson–Boltzmann system

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Abstract. In this paper, we consider the following nonlinear Schrödinger–Poisson–Boltzmann (SPB) system under an L^2-norm constraint

$$\begin{cases} -\Delta u + \lambda u + \phi u = |u|^{p-2}u & \text{in } \mathbb{R}^3, \\ -\Delta \phi + \kappa^2 \phi = 4\pi u^2 & \text{in } \mathbb{R}^3, \\ \|u\|_2 = a, \end{cases}$$

where $p \in \left(2, \frac{10}{3}\right) \setminus \{3\}$, $\lambda \in \mathbb{R}$, $a > 0$ is a prescribed constant and $\kappa > 0$ is a parameter. We prove that the above system admits a positive ground state normalized solution with the Lagrange multiplier $\lambda > 0$ when either $2 < p < 3$ and $a > 0$ is sufficiently small, or $3 < p < \frac{10}{3}$ and $a > 0$ is large enough. Moreover, we prove that when $2 < p < 3$, $\kappa > 0$ and $a > 0$ is small enough, the ground state normalized solutions are radially symmetric up to translation, and as $\kappa \to 0$, they converge to a radially symmetric ground state normalized solution of the Schrödinger–Poisson–Slater system under the same L^2-norm constraint.

Keywords: Schrödinger–Poisson–Boltzmann system, ground state, constrained minimization, radial symmetry, limit behavior.

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1 Introduction

1.1 Physical background

This paper concerns the study of ground state normalized solutions for the following nonlinear Schrödinger–Poisson–Boltzmann (SPB) system under the L^2-norm constraint $\|u\|_2 = a$

$$\begin{cases} -\Delta u + \lambda u + \phi u = |u|^{p-2}u & \text{in } \mathbb{R}^3, \\ -\Delta \phi + \kappa^2 \phi = 4\pi u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.1}$$

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where $a > 0$ is a prescribed constant, $\kappa > 0$ is a parameter, $\lambda \in \mathbb{R}$ and $p \in \left(2, \frac{10}{3}\right) \setminus \{3\}$. System (1.1) arises from seeking the standing wave solutions $\Psi(x,t) = e^{i\lambda t}u(x)$ to the following system

$$\begin{cases} i\partial_t \Psi + \Delta_x \Psi - \phi\Psi + |\Psi|^{p-2}\Psi = 0 & \text{in } \mathbb{R}^3, \\ -\Delta\phi + \kappa^2\phi = 4\pi|\Psi|^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.2}$$

where $\Psi : \mathbb{R}^3 \times [0,\infty) \to \mathbb{C}$ represents the wave function and $\phi : \mathbb{R}^3 \to \mathbb{R}$ denotes the electrostatic potential in the Poisson–Boltzmann equation. Therefore, system (1.2) describes a self-consistent quantum-electrostatic system, in which the wave function Ψ and the electrostatic potential ϕ mutually influence each other through nonlinear term and source term, collectively determining the dynamical behavior of this system.

The Poisson–Boltzmann (PB) equation describes an electric double layer model developed independently by Gouy [16] and Chapman [10], which explains the interaction between the ions in the solution and the charged layer. Subsequently, Andelman [1] proposed the Poisson–Boltzmann theory, in which the Poisson–Boltzmann equation is employed to depict the distribution of electrostatic potential within electrolyte solutions. Blossey [4] made further improvements to this theory. Since the electrolyte solutions formed after the dissolution of 1:1 salts (such as NaCl) play a crucial role in many biochemical reactions and physical processes, according to the Boltzmann distribution and Gauss's law, we obtain the following Poisson–Boltzmann equation for 1:1 salts

$$\Delta\phi = \kappa^2 \sinh\phi,$$

where ϕ denotes the electrostatic potential, Δ is the Laplace operator, κ is the inverse of Debye-Hückel screening length λ_D which satisfies

$$\kappa^2 = \frac{2n_0e^2}{\varepsilon k_B T},$$

where ε stands for the dielectric constant, k_B denotes the Boltzmann constant, T represents the absolute temperature, e is the elementary charge, and n_0 corresponds to the ionic concentration in a neutral electrolyte solution of (1:1) salt (see [4, Section 1.2]). In many physical situations, when the value of the electrostatic potential ϕ is relatively small, the above equation can be linearized into the following equation

$$\Delta\phi = \kappa^2\phi,$$

which not only demonstrates its convenience in the process of mathematical solving, but also has extensive and significant applications in fields such as biophysics, electrochemistry, and colloid chemistry. Moreover, the above linearized Poisson–Boltzmann equation is a classic example of the homogeneous modified Helmholtz equation, where the Laplacian of a function is directly proportional to the function itself (see [4, Section 1.3]). For more physical details, one can refer to the papers [1,4,9,21] and the references therein.

1.2 Problems and main results

In order to explore the existence of solutions to the SPB system (1.1), we would like to recall two distinct approaches in terms of the frequency λ. The first one is to fix $\lambda \in \mathbb{R}$ and search for the coupled solution (u,ϕ) of system (1.1), which is referred to the fixed frequency problem.

In this situation, any solution to system (1.1) can be identified as a critical point of the action functional $F_\lambda(u,\phi)$ on $H^1(\mathbb{R}^3)$, where

$$F_\lambda(u,\phi) := \frac{1}{2}\|\nabla u\|_2^2 + \frac{\lambda}{2}\|u\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^3}\phi u^2 \mathrm{d}x - \frac{1}{16\pi}\|\nabla\phi\|_2^2 - \frac{\kappa^2}{16\pi}\|\phi\|_2^2 - \frac{1}{p}\|u\|_p^p,$$

and the potential ϕ is given by

$$\phi(x) := \left(\frac{e^{-\kappa|\cdot|}}{|\cdot|} * u^2\right)(x) = \int_{\mathbb{R}^3}\frac{e^{-\kappa|x-y|}}{|x-y|}u^2(y)\mathrm{d}y.$$

Due to the presence of ϕu, system (1.1) is classified as a nonlocal problem. A typical example of such nonlocal problems is the Schrödinger–Poisson–Slater system, and a substantial body of literature has investigated various properties of solutions to this system over the past decades. For further related results, we refer to [31,32,38] and the references therein.

Moreover, Han, Huh and Seok [18] studied the following Schrödinger equation coupled with a neutral scalar field N

$$\begin{cases} -\frac{1}{2m}\Delta u + \frac{q}{4m^2}|u|^2 u + (1+\frac{\kappa q}{2m})Nu + \omega u = 0 & \text{in } \mathbb{R}^2, \\ \left(-\Delta + \kappa^2 q^2\right)N + q(1+\frac{\kappa q}{2m})u^2 = 0 & \text{in } \mathbb{R}^2, \end{cases} \tag{1.3}$$

where $m, \kappa, q > 0$ respectively represent the mass of the particle, the Chern–Simons coupling constant and the Maxwell coupling constant. Since $N := -q\left(1+\frac{\kappa q}{2m}\right)\left(G^{\kappa q} * u^2\right)$, where $G^{\kappa q}$ is the Yukawa potential in two dimension (see [26, Theorem 6.23]), (1.3) is also a nonlocal problem. They proved the existence of standing wave solutions for system (1.3) in the radially symmetric space $H_r^1(\mathbb{R}^2)$ by applying Mountain-Pass theorem. Inspired by [18], Kang, Liu and Tang [23] considered the problem (1.3) in non-radial symmetric space $H^1(\mathbb{R}^2)$. By combining the Nehari manifold, Moser iteration and some analytical skills, they got a positive ground state solution to (1.3), which is classical and spherically symmetric.

D'Avenia and Siciliano [11] explored the following nonlinear Schrödinger–Bopp–Podolsky system

$$\begin{cases} -\Delta u + \omega u + q^2\tilde{\phi}u = |u|^{p-2}u & \text{in } \mathbb{R}^3, \\ -\Delta\tilde{\phi} + a^2\Delta^2\tilde{\phi} = 4\pi u^2 & \text{in } \mathbb{R}^3, \end{cases}$$

where $u, \tilde{\phi} : \mathbb{R}^3 \to \mathbb{R}$, a, $\omega > 0$, $p > 0$ and $q \neq 0$. Depending on the two parameters q and p, the authors established both existence and nonexistence results using variational methods. Moreover, they demonstrated that these solutions converge to those of the classical Schrödinger–Poisson–Slater system as $a \to 0$ in the radial case. However, as far as we know, there is scarcely any literature dedicated to studying the existence of solutions for the SPB system (1.1).

Alternatively, it is of great interest to study solutions to system (1.1) having prescribed L^2-norm. Namely, for any given mass $a > 0$, we consider solutions to system (1.1) under the L^2-norm constraint

$$S_a := \left\{u \in H^1(\mathbb{R}^3) : \|u\|_2 = a\right\}, \tag{1.4}$$

which is called the fixed mass problem. Such solutions are often referred to as the normalized solutions for the following system which is given by

$$\begin{cases} -\Delta u + \lambda u + \phi u = |u|^{p-2}u & \text{in } \mathbb{R}^3, \\ -\Delta\phi + \kappa^2\phi = 4\pi u^2 & \text{in } \mathbb{R}^3, \\ \|u\|_2 = a, \end{cases} \tag{1.5}$$

and the frequency $\lambda \in \mathbb{R}$ cannot be fixed any longer and appears as a Lagrange multiplier.

For the sake of simplicity, by applying the reduction argument introduced in [11,23], we simplify the system (1.5) to the following nonlocal equation with the constraint S_a

$$-\Delta u + \lambda u + \phi_u u = |u|^{p-2} u \quad \text{in } \mathbb{R}^3, \tag{1.6}$$

where $\phi_u := \phi \in H^1(\mathbb{R}^3)$ uniquely solves

$$-\Delta \phi + \kappa^2 \phi = 4\pi u^2 \quad \text{in } \mathbb{R}^3$$

(see Section 2.3). Therefore, for given $a > 0$, to search for the coupled solution (u, ϕ, λ) of system (1.5) is equivalent to find the solution pair (u, λ) of (1.6) under the constraint S_a, where $\|u\|_2 = a$ and $\phi_u = \phi$ in (1.6). In this case, the normalized solution u of (1.6) can be obtained as a critical point of the constrained functional $I|_{S_a}$, which is given by

$$\begin{aligned} I(u) &:= \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u u^2 \mathrm{d}x - \frac{1}{p}\|u\|_p^p \\ &= \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{e^{-\kappa|x-y|}}{|x-y|} u^2(x)u^2(y)\mathrm{d}x\mathrm{d}y - \frac{1}{p}\|u\|_p^p, \end{aligned} \tag{1.7}$$

and λ is the corresponding Lagrange multiplier.

We deduce that $\bar{p} := \frac{10}{3}$ is the mass critical exponent and $2^* = 6$ is the Sobolev critical exponent for $I|_{S_a}$ by scaling argument and the Gagliardo–Nirenberg inequality (see Lemma 2.1). For further clarification, we agree that the mass subcritical case, mass critical case and mass supercritical case mean that $2 < p < \frac{10}{3}$, $p = \bar{p} = \frac{10}{3}$ and $\frac{10}{3} < p < 6$, respectively.

When $2 < p < \frac{10}{3}$, namely, the mass subcritical case, it holds that the constrained functional $I|_{S_a}$ is coercive and bounded from below (see Lemma 2.10). This leads to the following global minimization problem

$$I_{a^2} := \inf_{u \in S_a} I(u). \tag{1.8}$$

In particular, if $(u, \phi, \lambda) \in S_a \times H^1(\mathbb{R}^3) \times \mathbb{R}$ is a coupled solution to system (1.5) such that $I(u) = I_{a^2}$, then $u \in S_a$ is the ground state normalized solution of (1.5).

The first study on normalized solutions to the Schrödinger–Poisson–Slater system was conducted by Sánchez and Soler [33], who established the foundational framework for this problem. Specifically, they studied the following system in $\mathbb{R}^3$

$$\begin{cases} -\frac{1}{2}\Delta u + Vu - C_S |u|^{\frac{2}{3}} u = \beta u \qquad \lim_{|x|\to\infty} u(x) = 0, \\ -\Delta V = u^2 \qquad \lim_{|x|\to\infty} V(x) = 0, \\ \|u\|_2 = M > 0, \end{cases}$$

where $V := \frac{1}{4\pi|x|} * u^2 \in \mathcal{D}^{1,2}(\mathbb{R}^3)$ and C_S denotes the Slater constant. The corresponding energy functional is

$$E(u) := \frac{1}{2}\|\nabla u\|_2^2 + \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{1}{8\pi|x-y|} u^2(x)u^2(y)\mathrm{d}x\mathrm{d}y - \frac{3C_S}{4}\|u\|_{\frac{8}{3}}^{\frac{8}{3}}.$$

By applying the concentration-compactness argument in [28,29] and nonzero weak convergence after translations in [26], they proved that when $M > 0$ is sufficiently small, all the minimizing sequences for I_M are compact, where

$$I_M := \inf\left\{E(u) : u \in H^1(\mathbb{R}^3), \|u\|_2 = M\right\}.$$

Kikuchi [24] proved that if C_S is sufficiently large, then the infimum of the minimization problem $I(\mu) := \inf\left\{\mathcal{E}(v) : v \in H^1(\mathbb{R}^3), \|u\|_2^2 = \mu\right\}$ is achieved for any $\mu > 0$, where

$$\mathcal{E}(v) := \frac{1}{2}\|\nabla v\|_2^2 + \frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{1}{|x-y|}v^2(x)v^2(y)\mathrm{d}x\mathrm{d}y - \frac{C_S}{3}\|v\|_3^3.$$

More generally, Bellazzini and Siciliano [6, 7] explored the following Schrödinger–Poisson equation

$$-\Delta u + \varphi u - |u|^{p-2}u = \omega u \quad \text{in } \mathbb{R}^3, \tag{1.9}$$

where $\omega \in \mathbb{R}$, $p \in \left(2, \frac{10}{3}\right)\setminus\{3\}$ and $\varphi := \frac{1}{|x|} * u^2$ satisfying $-\Delta\varphi = 4\pi u^2$. They considered the following minimization problem

$$\mathcal{I}_\rho := \inf_{u\in B_\rho}\mathcal{I}(u), \tag{1.10}$$

where

$$\mathcal{I}(u) := \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{1}{|x-y|}u^2(x)u^2(y)\mathrm{d}x\mathrm{d}y - \frac{1}{p}\|u\|_p^p$$

and

$$B_\rho := \left\{u \in H^1(\mathbb{R}^3) : \|u\|_2 = \rho\right\},$$

and they proved that $\mathcal{I}_\rho$ admits a minimizer when either $3 < p < \frac{10}{3}$ and $\rho > 0$ is large enough, or $2 < p < 3$ and $\rho > 0$ is sufficiently small. Subsequently, Georgiev, Prinari and Visciglia [13] studied the radial symmetry of the minimizers of problem (1.10) up to translation provided that $2 < p < 3$ and $\rho > 0$ is sufficiently small. When $p \in [3, \frac{10}{3}]$, Jeanjean and Luo [22] specified a threshold value of $\rho > 0$ that separates the existence and nonexistence of minimizers for the minimization problem (1.10).

Based on the above research results, by applying minimizing method, He, Li and Chen [17] considered the following nonlinear Schrödinger–Bopp–Podolsky system with prescribed mass

$$\begin{cases} -\Delta u + \omega u + \tilde{\phi}u = |u|^{p-2}u & \text{in } \mathbb{R}^3, \\ -\Delta\tilde{\phi} + a^2\Delta^2\tilde{\phi} = 4\pi u^2 & \text{in } \mathbb{R}^3, \\ \|u\|_2 = \rho > 0, \end{cases} \tag{1.11}$$

where $\omega > 0$, $a > 0$, $p \in (2, \frac{10}{3})$ and $\tilde{\phi} := \frac{1-e^{-|x|/a}}{|x|} * u^2$. When $a = 1$, they respectively proved the existence of positive normalized solutions of system (1.11) when $2 < p < 3$ and ρ is sufficiently small, as well as when $3 < p < \frac{10}{3}$ and $\rho > 0$ is sufficiently large. Applying critical point theory, minimization method and some ideas borrowed from [6, 7, 13], de Paula Ramos and Siciliano [12] proved that when $p \in (2, \frac{10}{3})\setminus\{3\}$ and $a > 0$ is a parameter, system (1.11) has a ground state normalized solution. Meanwhile, under the assumptions that $p \in (2, \frac{14}{5})$ and $\rho > 0$ is sufficiently small, they demonstrated that the ground state normalized solution is radially symmetric up to translation, and these solutions converge to a ground state normalized solution of the Schrödinger–Poisson–Slater system as $a \to 0$.

In the light of [11] and [12], Hernandez and Siciliano [20] studied the existence and multiplicity of solutions for the following Schrödinger–Bopp–Podolsky system under an L^2-norm constraint

$$\begin{cases} -\Delta u + \tilde{\phi}u = \omega u & \text{in } \Omega, \\ -\Delta\tilde{\phi} + a^2\Delta^2\tilde{\phi} = u^2 & \text{in } \Omega, \\ u = \tilde{\phi} = \Delta\tilde{\phi} = 0 & \text{on } \partial\Omega, \\ \int_\Omega u^2\mathrm{d}x = 1, \end{cases}$$

where Ω is an open bounded and smooth domain in $\mathbb{R}^3$, and $a > 0$ is the Bopp–Podolsky parameter. The unknowns are $u, \tilde{\phi} : \Omega \to \mathbb{R}$ and $\omega \in \mathbb{R}$. By applying variational methods and topological invariants from Ljusternik–Schnirelmann theory in the critical point theory, they established that for any $a > 0$, there exist infinitely many nontrivial solutions with diverging energy and norm-divergent behavior. Moreover, they also showed that the ground state solutions of the above system converge to a ground state solution of the corresponding Schrödinger–Poisson system under the same L^2-norm constraint as $a \to 0$.

Motivated by [17] and [12], Li and Zhang [25] proved the existence, asymptotic behavior and the multiplicity of normalized solutions for the following Schrödinger–Bopp–Podolsky system with a critical term

$$\begin{cases} -\Delta u + \tilde{\phi} u = \lambda u + \mu |u|^{p-2} u + u^5 & \text{in } \mathbb{R}^3, \\ -\Delta \tilde{\phi} + \Delta^2 \tilde{\phi} = 4\pi u^2 & \text{in } \mathbb{R}^3, \\ \|u\|_2^2 = m^2 > 0, \end{cases} \tag{1.12}$$

where $\lambda \in \mathbb{R}$, $2 < p < 6$ and $\mu > 0$ is a parameter. For $p \in (\frac{10}{3}, 6)$, by applying Lagrange multipliers argument and Mountain-Pass theorem, they obtained the existence and asymptotic behavior of positive normalized ground state solutions for (1.12). For $p \in (2, \frac{10}{3}]$, they proved the existence of a normalized ground state solution to (1.12) by combining Mountain-Pass theorem with Lebesgue dominated convergence theorem. Moreover, they also verified the multiplicity of normalized solutions to (1.12).

When $\frac{10}{3} < p < 6$, that is, the mass supercritical case, and the minimizing method becomes inapplicable since the constrained functional $I|_{S_a}$ is unbounded from below, that is, $I_{a^2} = -\infty$. Therefore, by applying a Mountain-Pass argument, Bellazzini, Jeanjean and Luo [8] proved that for any $\rho > 0$ sufficiently small, there exist critical points of the constrained functional $\mathcal{I}|_{B_\rho}$ of the Schrödinger–Poisson equation (1.9).

However, when $p = \frac{10}{3}$, that is, the mass critical case, it becomes delicate to say that $I_{a^2} > -\infty$ or $I_{a^2} = -\infty$ since it depends on the range of value of $a > 0$.

Therefore, the purpose of this paper is to consider the existence and limit behavior of the ground state normalized solutions for system (1.5). As far as we know, no relevant results can be found in the existing literature. Next, we state our main results as follows.

Theorem 1.1. *Suppose that $p \in (2, 3)$ and $a, \kappa > 0$. Then there exists $a_0 > 0$, such that for any $a \in (0, a_0)$, system (1.5) admits a ground state normalized solution $(u, \phi_u, \lambda) \in H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3) \times (0, \infty)$ with $u > 0$ in $\mathbb{R}^3$.*

Theorem 1.2. *Suppose that $p \in (3, \frac{10}{3})$ and $a, \kappa > 0$. Then there exists $a_1 > 0$, such that for any $a \in (a_1, \infty)$, system (1.5) admits a ground state normalized solution $(u, \phi_u, \lambda) \in H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3) \times (0, \infty)$ with $u > 0$ in $\mathbb{R}^3$.*

Let $S_a^r := S_a \cap H_r^1(\mathbb{R}^3)$. Then, the following theorem reveals the radial symmetry of the ground state normalized solutions for system (1.5) when $p \in (2, 3)$ and $a > 0$ is sufficiently small.

Theorem 1.3. *Suppose that $p \in (2, 3)$, $\kappa > 0$ and $a_0 > 0$ is obtained in Theorem 1.1. Let $(u, \phi_u, \lambda) \in S_a \times H^1(\mathbb{R}^3) \times (0, \infty)$ be a ground state normalized solution to system (1.5) with $u > 0$. Then there exists $a_2 \in (0, a_0]$ such that up to translation, $(u, \phi_u, \lambda) \in S_a^r \times H_r^1(\mathbb{R}^3) \times (0, \infty)$ for any $a \in (0, a_2)$.*

Our final result concerns the limit behavior of the ground state normalized solutions $(u_\kappa, \phi_\kappa, \lambda_\kappa)$ to system (1.5) as $\kappa \to 0$. For this purpose, we introduce the following Schrödinger–Poisson–Slater system

$$\begin{cases} -\Delta u + \lambda u + \varphi u = |u|^{p-2}u & \text{in } \mathbb{R}^3, \\ -\Delta \varphi = 4\pi u^2 & \text{in } \mathbb{R}^3, \\ \|u\|_2 = a, \end{cases} \tag{1.13}$$

where $\varphi := \frac{1}{|x|} * u^2 \in \mathcal{D}^{1,2}(\mathbb{R}^3)$ and the corresponding energy functional is

$$\mathcal{J}_0(u) := \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{1}{|x-y|}u^2(x)u^2(y)\mathrm{d}x\mathrm{d}y - \frac{1}{p}\|u\|_p^p. \tag{1.14}$$

Theorem 1.4. *Suppose that $p \in (2,3)$, $\kappa > 0$ and $a_2 > 0$ is obtained in Theorem 1.3. Let $\{(u_\kappa, \phi_\kappa, \lambda_\kappa)\} \subset S_a \times H^1(\mathbb{R}^3) \times (0,\infty)$ be a set of ground state normalized solutions to system (1.5) with $u_\kappa > 0$. Then when $a \in (0, a_2)$ is small enough, the Schrödinger–Poisson–Slater system (1.13) admits a ground state normalized solution $(u_0, \varphi_0, \lambda_0) \in S_a^r \times \mathcal{D}_r^{1,2}(\mathbb{R}^3) \times (0,\infty)$ such that, up to a subsequence and up to translation, the following holds*

$$(u_\kappa, \phi_\kappa, \lambda_\kappa) \to (u_0, \varphi_0, \lambda_0) \quad \text{in } S_a^r \times \mathcal{D}_r^{1,2}(\mathbb{R}^3) \times (0,\infty) \quad \text{as } \kappa \to 0. \tag{1.15}$$

1.3 Comments on the Theorems 1.1–1.4

- The proofs of Theorems 1.1 and 1.2 adopt the following approach: if we could prove that the minimizing sequences for the constrained functional $I|_{S_a}$ are precompact, then we conclude that the infimum $I_{a^2} := \inf_{u\in S_a} I(u)$ is achieved. For this end, we apply the abstract framework for constrained minimization problems developed by Bellazzini and Siciliano [6,7]. Furthermore, since it is known that $\varphi := \frac{1}{|x|} * u^2 \in \mathcal{D}^{1,2}(\mathbb{R}^3)$ uniquely solves $-\Delta\varphi = 4\pi u^2$ and $\tilde{\phi} := \frac{1-e^{-|x|/a}}{|x|} * u^2 \in \{\phi \in \mathcal{D}^{1,2}(\mathbb{R}^3) : \Delta\phi \in L^2(\mathbb{R}^3)\}$ uniquely solves $-\Delta\tilde{\phi} + a^2\Delta^2\tilde{\phi} = 4\pi u^2$ for any $u \in H^1(\mathbb{R}^3)$, through a detailed study of operator $-\Delta + \kappa^2$ in the second equation of the SPB system (1.5), we conclude that for any $u \in H^1(\mathbb{R}^3)$, there exists a unique $\phi_u := \frac{e^{-\kappa|x|}}{|x|} * u^2 \in H^1(\mathbb{R}^3)$ satisfying $-\Delta\phi + \kappa^2\phi = 4\pi u^2$. Notably, according to Sections 2.4 and 2.5, by applying the regularity of solutions and the Pohožaev identity of system (1.5) which is obviously different from that of the Schrödinger–Poisson–Slater system (1.13), we further deduce that the ground state normalized solution u and the corresponding Lagrange multiplier λ of system (1.5) are positive, which is a novelty compared with [6,7,12].

- The proof of Theorem 1.3 follows from making small adjustments to the implicit function argument proposed by Georgiev, Prinari and Visciglia [13]. We remark that the argument rests on the study of the behavior of ground state normalized solutions under an appropriate scaling (see Section 3.3). To some extent, this scaling permits one to explore the asymptotic behavior of ground state normalized solutions to Schrödinger–Poisson–Slater system (1.13) as $a \to 0$ by means of solutions to the following semilinear partial differential equation

$$-\Delta u + \omega_0 u = |u|^{p-2}u \quad \text{in } \mathbb{R}^3,$$

where $\omega_0 > 0$ is a certain constant. Through further calculations in this paper, we demonstrate that the range of p can be extended from $(2, \frac{14}{5})$ to $(2,3)$. Therefore, when

$2 < p < 3$ and $a > 0$ is small enough, by applying the above argument, we obtain the radial symmetry of the ground state normalized solutions to system (1.5), which extends the result in [12, Theorem C].

- The proof of Theorem 1.4 is inspired by the results presented in [11, Theorem 1.3], [12, Theorem D] and [20, Theorem 1.3]. Based on Theorem 1.3 and the Sobolev embedding $H_r^1(\mathbb{R}^3) \hookrightarrow \mathcal{D}_r^{1,2}(\mathbb{R}^3)$, we conclude that the set of ground state normalized solutions $\{(u_\kappa, \phi_\kappa, \lambda_\kappa)\}$ is a subset of $S_a^r \times \mathcal{D}_r^{1,2}(\mathbb{R}^3) \times (0,\infty)$. Moreover, by applying Lemmas 3.6, 3.7, Proposition 3.8 and the Sobolev embedding $H_r^1(\mathbb{R}^3) \hookrightarrow\hookrightarrow L^p(\mathbb{R}^3)$ with $p \in (2,6)$, we get the limit behavior of radially symmetric ground state normalized solutions to system (1.5) as $\kappa \to 0$. In conclusion, our findings supplement and extend the results presented in [6,7,12,20].

This paper is organized as follows. In Section 1, we state the main results and make some comments on the proofs of Theorems 1.1–1.4. In section 2, we introduce some preliminary results, which are crucial to our subsequent arguments. Specifically, in Section 2.2, we establish some properties of operator $-\Delta + \kappa^2$, which play a crucial role in our subsequent analysis. Sections 2.3, 2.4 and 2.5 aim at exploring the energy functional, the regularity of solutions, and the Pohožaev identity respectively. Moreover, in Section 2.6, we present some lemmas and propositions for the constrained minimization problem (1.8) developed by Bellazzini and Siciliano [6,7]. Finally, Sections 3.1–3.4 are dedicated to proving Theorems 1.1–1.4 respectively.

We now introduce some notations which will be used in the sequel. Let $H^1(\mathbb{R}^3)$ be the usual Sobolev space endowed with the standard inner product and norm $\|\cdot\|_{H^1}$. $H_r^1(\mathbb{R}^3)$ denotes the usual radially symmetric space. $L^p(\mathbb{R}^3)$ denotes the usual Lebesgue space with the norm $\|\cdot\|_p$ for $1 \le p \le \infty$. $\mathcal{D}^{1,2}(\mathbb{R}^3)$ denotes the Sobolev space defined as the completion of $C_c^\infty(\mathbb{R}^3)$ with respect to the norm $\|\cdot\|_{\mathcal{D}^{1,2}} := \|\nabla \cdot\|_2$. For any $\rho > 0$ and $x \in \mathbb{R}^3$, $B_\rho(x)$ denotes the ball of radius ρ centered at x. As usual, C and C_i $(i = 1,2,3\dots)$ denote various positive constants that may change from line to line but are not essential for the analysis of the problem.

2 Preliminaries

2.1 Some important inequalities

Lemma 2.1 ([35]). *Let $2^* := \frac{2N}{N-2}$, for any $N \ge 3$ and $p \in [2, 2^*)$, there exists a constant $C_{N,p}$ depending on N, p such that the following well-known Gagliardo–Nirenberg inequality holds*

$$\|u\|_{L^p(\mathbb{R}^N)}^p \le C_{N,p}^p \|\nabla u\|_{L^2(\mathbb{R}^N)}^{p\gamma_p} \|u\|_{L^2(\mathbb{R}^N)}^{p-p\gamma_p} \quad \forall u \in H^1(\mathbb{R}^N),$$

where $\gamma_p := \frac{N(p-2)}{2p}$. In particular, when $p = 2^*$, the following Sobolev inequality holds

$$\mathcal{S} \|u\|_{L^{2^*}(\mathbb{R}^N)}^2 \le \|\nabla u\|_{L^2(\mathbb{R}^N)}^2 \quad \forall u \in \mathcal{D}^{1,2}(\mathbb{R}^N),\ N \ge 3,$$

where the constant $\mathcal{S}$ is called the best Sobolev constant.

Lemma 2.2 (Hardy–Littlewood–Sobolev inequality [26]). *Let $p, r > 1$ and $0 < \lambda < N$ with $\frac{1}{p} + \frac{\lambda}{N} + \frac{1}{r} = 2$, and for any $f \in L^p(\mathbb{R}^N)$ and $g \in L^r(\mathbb{R}^N)$, there exists a sharp constant $C_{N,\lambda,p,r}$ such that*

$$\left| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{f(x)g(y)}{|x-y|^\lambda} dxdy \right| \le C_{N,\lambda,p,r} \|f\|_p \|g\|_r .$$

Lemma 2.3 ([19,37]). *Let $\alpha \in (0, N)$, the Riesz potential operator J_α of order α (also referred to as the fractional integral operator) is defined by*

$$J_\alpha(f)(x) := \int_{\mathbb{R}^N} \frac{f(y)}{|x-y|^{N-\alpha}} \mathrm{d}y, \qquad x \in \mathbb{R}^N.$$

When $1 < p < \frac{N}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{N}$, there exists a constant $C_{N,\alpha}$ such that $\|J_\alpha(f)\|_q \le C_{N,\alpha} \|f\|_p$ for any $f \in L^p(\mathbb{R}^N)$.

2.2 The operator $-\Delta + \kappa^2$

In the field of electrostatics, by the Gauss's law (or Poisson equation), for a given charge-density distribution ρ in $\mathbb{R}^3$, the electrostatic potential ϕ satisfies $-\Delta\phi = \rho$. If $\rho = 4\pi\delta_y$ with $y \in \mathbb{R}^3$, where δ_y denotes Dirac's delta measure at y, then it is known that the fundamental solution of $-\Delta\phi = 4\pi\delta_y$ (or Green's function for $-\Delta$) is $G(x-y)$, where $G(x) := \frac{1}{|x|}$ (see [26, Theorem 6.20]). Particularly, for the following Poisson–Boltzmann equation

$$-\Delta\phi + \kappa^2 \sinh(\phi) = \rho \quad \text{in } \mathbb{R}^3, \tag{2.1}$$

where ϕ, κ and ρ are consistent with those in the previous text. According to Section 1.1, when the surface potential ϕ is small enough, (2.1) can be linearized by the following equation

$$-\Delta\phi + \kappa^2\phi = \rho \quad \text{in } \mathbb{R}^3.$$

In particular, when $\kappa > 0$ and $\rho = 4\pi\delta_y$, we find that $G^\kappa(x-y)$ is the fundamental solution of the following equation

$$-\Delta\phi + \kappa^2\phi = 4\pi\delta_y \quad \text{in } \mathbb{R}^3,$$

where

$$G^\kappa(x) := \frac{e^{-\kappa|x|}}{|x|}. \tag{2.2}$$

Namely, G^κ is the Green's function for $-\Delta + \kappa^2$ (see [2, Table 10.1]), which is also referred to as the Yukawa potential in three dimension (see [26, Theorem 6.23]).

Lemma 2.4. *For any $\kappa > 0$, we have $G^\kappa \in L^p(\mathbb{R}^3)$ with $1 \le p < 3$, and $\nabla G^\kappa \in L^1(\mathbb{R}^3)$. Moreover, in the sense of distributions, there holds*

$$-\Delta G^\kappa(x-y) + \kappa^2 G^\kappa(x-y) = 4\pi\delta_y, \tag{2.3}$$

where δ_y denotes Dirac's delta measure at y (often written as $\delta(x-y)$).

Proof. By applying spherical coordinate transformation and variable substitution, we deduce from (2.2) that

$$\begin{aligned}\|G^\kappa\|_p^p &= \int_{\mathbb{R}^3} \frac{e^{-\kappa p|x|}}{|x|^p} \mathrm{d}x = 4\pi \int_0^\infty r^{2-p} e^{-\kappa p r} \mathrm{d}r = \frac{4\pi}{(\kappa p)^{3-p}} \int_0^\infty y^{(3-p)-1} e^{-y} \mathrm{d}y \\ &= \frac{4\pi}{(\kappa p)^{3-p}} \Gamma(3-p),\end{aligned}$$

where $\Gamma(s)$ represents the Gamma function and thus we obtain that $1 \le p < 3$. Since

$$\frac{\partial G^\kappa}{\partial x_i} = -\left(\frac{\kappa x_i}{|x|^2} + \frac{x_i}{|x|^3}\right) e^{-\kappa|x|},\ i = 1,2,3 \quad \text{and} \quad \nabla G^\kappa = -\left(\frac{\kappa x}{|x|^2} + \frac{x}{|x|^3}\right) e^{-\kappa|x|}, \tag{2.4}$$

it follows from spherical coordinate transformation and integration by parts that

$$\int_{\mathbb{R}^3} |\nabla G^\kappa| \, \mathrm{d}x = \int_{\mathbb{R}^3} \left| -\left(\frac{\kappa x}{|x|^2} + \frac{x}{|x|^3} \right) \right| e^{-\kappa|x|} \mathrm{d}x \leq \int_{\mathbb{R}^3} \frac{\kappa e^{-\kappa|x|}}{|x|} \mathrm{d}x + \int_{\mathbb{R}^3} \frac{e^{-\kappa|x|}}{|x|^2} \mathrm{d}x = \frac{8\pi}{\kappa} < \infty.$$

To sum up, for any $\kappa > 0$, $G^\kappa \in L^p(\mathbb{R}^3)$ with $1 \leq p < 3$ and $\nabla G^\kappa \in L^1(\mathbb{R}^3)$.

Next, from [26, Theorem 6.20] or [11, Lemma 3.3], in order to prove (2.3), let $y = 0$ and then we need to show that for any $\psi \in C_c^\infty(\mathbb{R}^3)$, it holds that

$$-\int_{\mathbb{R}^3} G^\kappa(x-0)\Delta\psi \mathrm{d}x + \kappa^2 \int_{\mathbb{R}^3} G^\kappa(x-0)\psi \mathrm{d}x = 4\pi\psi(0). \tag{2.5}$$

Since $G^\kappa \in L^1_{\mathrm{loc}}(\mathbb{R}^3)$, it suffices to show that

$$\lim_{r\to 0^+} I(r) = 4\pi\psi(0), \tag{2.6}$$

where

$$I(r) := -\int_{|x|>r} G^\kappa(x-0)\Delta\psi \, \mathrm{d}x + \kappa^2 \int_{|x|>r} G^\kappa(x-0)\psi \, \mathrm{d}x.$$

Since ψ has compact support, we consider the annulus $A := \{x \in \mathbb{R}^3 : r < |x| < R\}$ for R large enough. Since $-\Delta G^\kappa + \kappa^2 G^\kappa = 0$ in A, by applying a standard integration by parts, we have

$$\begin{aligned}
I(r) &= -\int_A G^\kappa \Delta\psi \, \mathrm{d}x + \kappa^2 \int_A G^\kappa \psi \, \mathrm{d}x \\
&= \int_A \nabla G^\kappa \nabla\psi \, \mathrm{d}x - \int_{|x|=r} G^\kappa \nabla\psi \cdot \nu \, \mathrm{d}\mathbf{S} + \kappa^2 \int_A G^\kappa \psi \, \mathrm{d}x \\
&= -\int_A \psi \Delta G^\kappa \, \mathrm{d}x + \int_{|x|=r} \psi \nabla G^\kappa \cdot \nu \, \mathrm{d}\mathbf{S} - \int_{|x|=r} G^\kappa \nabla\psi \cdot \nu \, \mathrm{d}\mathbf{S} + \kappa^2 \int_A G^\kappa \psi \, \mathrm{d}x \\
&= \int_A \psi \left(-\Delta G^\kappa + \kappa^2 G^\kappa \right) \mathrm{d}x + \int_{|x|=r} \psi \nabla G^\kappa \cdot \nu \, \mathrm{d}\mathbf{S} - \int_{|x|=r} G^\kappa \nabla\psi \cdot \nu \, \mathrm{d}\mathbf{S} \\
&= \int_{|x|=r} \psi \nabla G^\kappa \cdot \nu \, \mathrm{d}\mathbf{S} - \int_{|x|=r} G^\kappa \nabla\psi \cdot \nu \, \mathrm{d}\mathbf{S} := I_1(r) - I_2(r),
\end{aligned}$$

where ν is the unit outward normal to A and $\mathbf{S}$ is the unit sphere in $\mathbb{R}^3$. When considering $I_1(r)$ on the sphere $|x| = r$, it follows from (2.4) that $\nabla G^\kappa \cdot \nu = \frac{\kappa r+1}{r^2} e^{-\kappa r}$. Since ψ is continuous, we get

$$I_1(r) = \int_{|x|=r} \psi \nabla G^\kappa \cdot \nu \, \mathrm{d}\mathbf{S} = \int_{\mathbf{S}_1} \psi(r\sigma)(\kappa r + 1) e^{-\kappa r} \, \mathrm{d}\sigma \to 4\pi\psi(0) \quad \text{as } r \to 0^+,$$

where σ represents a unit vector on the unit sphere surface $\mathbf{S}_1$.

As for $I_2(r)$, since ψ has compact support, there exists some constant $C_1 > 0$ such that $|\nabla\psi \cdot \nu| \leq C_1$. Moreover, $\left| |x|^2 G^\kappa \right| < |x|^{\frac{1}{2}}$ when $|x|$ is small enough. Thus,

$$|I_2(r)| = \left| \int_{|x|=r} G^\kappa \nabla\psi \cdot \nu \, \mathrm{d}\mathbf{S} \right| \leq \int_{|x|=r} |G^\kappa| \, |\nabla\psi \cdot \nu| \, \mathrm{d}\mathbf{S} \leq 4\pi C_1 r^{\frac{1}{2}} \to 0 \quad \text{as } r \to 0^+.$$

Namely, (2.6) has been verified and then (2.3) holds. □

Lemma 2.5. *For any $\kappa > 0$, if $G^\kappa, \nabla G^\kappa \in L^1(\mathbb{R}^3)$ and $f \in L^p(\mathbb{R}^3)$ with $1 \leq p \leq \infty$, the following conclusions hold:*

(i) The function $y \mapsto G^\kappa(x-y)f(y)$ is integrable on $\mathbb{R}^3$. Moreover, let

$$\phi(x) := (G^\kappa * f)(x) = \int_{\mathbb{R}^3} \frac{e^{-\kappa|x-y|}}{|x-y|} f(y)\,\mathrm{d}y. \tag{2.7}$$

Then for any $1 \le p \le \infty$, there holds

$$\phi \in L^p(\mathbb{R}^3) \quad \text{and} \quad \|\phi\|_p = \|G^\kappa * f\|_p \le \|G^\kappa\|_1 \|f\|_p.$$

(ii) In the sense of distributions, ϕ solves

$$-\Delta\phi + \kappa^2\phi = 4\pi f \qquad \text{in } \mathcal{D}'(\mathbb{R}^3), \tag{2.8}$$

where $\mathcal{D}'(\mathbb{R}^3)$ denotes the dual space of $C_c^\infty(\mathbb{R}^3)$. Moreover, ϕ has a distributional derivative, which is given by

$$\partial_i\phi(x) = \int_{\mathbb{R}^3} \frac{\partial G^\kappa(x-y)}{\partial x_i} f(y)\,\mathrm{d}y, \quad i = 1,2,3, \tag{2.9}$$

and $\nabla\phi = \int_{\mathbb{R}^3} \nabla G^\kappa(x-y)f(y)\,\mathrm{d}y \in L^p(\mathbb{R}^3)$ with $1 \le p \le \infty$.

Proof. (i) For any $\kappa > 0$, it follows from Lemma 2.4 that $G^\kappa \in L^1(\mathbb{R}^3)$. Therefore, from [3, Theorem 4.15], it follows that (2.7) is well defined for almost everywhere $x \in \mathbb{R}^3$. Moreover, $\phi \in L^p(\mathbb{R}^3)$ and $\|\phi\|_p = \|G^\kappa * f\|_p \le \|G^\kappa\|_1 \|f\|_p$ with $1 \le p \le \infty$.

(ii) To verify (2.8), we need to show that for any $\psi \in C_c^\infty(\mathbb{R}^3)$, it holds that

$$-\int_{\mathbb{R}^3} \phi\Delta\psi\,\mathrm{d}x + \kappa^2 \int_{\mathbb{R}^3} \phi\psi\,\mathrm{d}x = 4\pi \int_{\mathbb{R}^3} f\psi\,\mathrm{d}x. \tag{2.10}$$

First, let $F(x,y) := G^\kappa(x-y)f(y)\Delta\psi(x)$. Since $G^\kappa \in L^1(\mathbb{R}^3)$, $f \in L^p(\mathbb{R}^3)$ with $1 \le p \le \infty$, and $\psi \in C_c^\infty(\mathbb{R}^3)$, we deduce that $F \in L^1\left(\mathbb{R}^3 \times \mathbb{R}^3\right)$. Next, by substituting (2.7) into the first integral of (2.10) and applying Fubini's theorem, we deduce that

$$-\int_{\mathbb{R}^3} \phi\Delta\psi\,\mathrm{d}x = \int_{\mathbb{R}^3} f(y) \left(-\int_{\mathbb{R}^3} G^\kappa(x-y)\Delta\psi(x)\,\mathrm{d}x \right) \mathrm{d}y. \tag{2.11}$$

It follows from (2.5) that

$$-\int_{\mathbb{R}^3} G^\kappa(x-y)\Delta\psi(x)\,\mathrm{d}x = -\kappa^2 \int_{\mathbb{R}^3} G^\kappa(x-y)\psi(x)\,\mathrm{d}x + 4\pi\psi(y). \tag{2.12}$$

By substituting (2.12) into (2.11), it follows that

$$-\int_{\mathbb{R}^3} \phi\Delta\psi\,\mathrm{d}x = -\kappa^2 \int_{\mathbb{R}^3} f(y) \left(\int_{\mathbb{R}^3} G^\kappa(x-y)\psi(x)\,\mathrm{d}x \right) \mathrm{d}y + 4\pi \int_{\mathbb{R}^3} f(y)\psi(y)\,\mathrm{d}y. \tag{2.13}$$

Next, substituting (2.7) into the second integral of (2.10) and applying Fubini's theorem once again, we have

$$\kappa^2 \int_{\mathbb{R}^3} \phi\psi\,\mathrm{d}x = \kappa^2 \int_{\mathbb{R}^3} f(y) \left(\int_{\mathbb{R}^3} G^\kappa(x-y)\psi(x)\,\mathrm{d}x \right) \mathrm{d}y. \tag{2.14}$$

Combining (2.13) and (2.14), we have

$$-\int_{\mathbb{R}^3} \phi\Delta\psi\,\mathrm{d}x + \kappa^2 \int_{\mathbb{R}^3} \phi\psi\,\mathrm{d}x = 4\pi \int_{\mathbb{R}^3} f(y)\psi(y)\,\mathrm{d}y = 4\pi \int_{\mathbb{R}^3} f\psi\,\mathrm{d}x,$$

thus, (2.10) holds, namely, (2.8) holds true.

To prove (2.9), we start from the definition of distributional derivative. For any test function $\psi \in C_c^\infty(\mathbb{R}^3)$, we have that the distributional derivative $\partial_i\phi(x)$ satisfies

$$\int_{\mathbb{R}^3} \partial_i\phi(x)\psi(x)\,\mathrm{d}x = -\int_{\mathbb{R}^3} \phi(x)\partial_i\psi(x)\,\mathrm{d}x, \qquad i = 1,2,3. \tag{2.15}$$

Since $G^\kappa \in L^1(\mathbb{R}^3)$ and $f \in L^p(\mathbb{R}^3)$ with $1 \le p \le \infty$, by substituting (2.7) into the right side of (2.15) and applying Fubini's theorem, it follows that

$$-\int_{\mathbb{R}^3} \phi(x)\partial_i\psi(x)\,\mathrm{d}x = -\int_{\mathbb{R}^3} f(y)\left(\int_{\mathbb{R}^3} G^\kappa(x-y)\partial_i\psi(x)\,\mathrm{d}x\right)\mathrm{d}y. \tag{2.16}$$

Thus, according to (2.16), by applying integration by parts to $\int_{\mathbb{R}^3} G^\kappa(x-y)\partial_i\psi(x)\,\mathrm{d}x$ and the fact that $G^\kappa(x-y)\psi(x)$ vanishes as x goes to infinity due to the compact support of ψ, we infer that

$$\int_{\mathbb{R}^3} G^\kappa(x-y)\partial_i\psi(x)\,\mathrm{d}x = -\int_{\mathbb{R}^3} \psi(x)\frac{\partial G^\kappa(x-y)}{\partial x_i}\,\mathrm{d}x,$$

and

$$-\int_{\mathbb{R}^3} \phi(x)\partial_i\psi(x)\,\mathrm{d}x = \int_{\mathbb{R}^3} f(y)\left(\int_{\mathbb{R}^3} \psi(x)\frac{\partial G^\kappa(x-y)}{\partial x_i}\,\mathrm{d}x\right)\mathrm{d}y.$$

Since $\nabla G^\kappa \in L^1(\mathbb{R}^3)$, we have $\frac{\partial G^\kappa(x-y)}{\partial x_i} \in L^1(\mathbb{R}^3)$. Then, it follows from Fubini's theorem and (2.15) that

$$\int_{\mathbb{R}^3} \partial_i\phi(x)\psi(x)\,\mathrm{d}x = -\int_{\mathbb{R}^3} \phi(x)\partial_i\psi(x)\,\mathrm{d}x = \int_{\mathbb{R}^3}\left(\int_{\mathbb{R}^3} \frac{\partial G^\kappa(x-y)}{\partial x_i} f(y)\,\mathrm{d}y\right)\psi(x)\mathrm{d}x,$$

which shows that

$$\partial_i\phi(x) = \int_{\mathbb{R}^3} \frac{\partial G^\kappa(x-y)}{\partial x_i} f(y)\,\mathrm{d}y,$$

namely, (2.9) holds true and this completes the proof. □

Lemma 2.6. *For any $u \in H^1(\mathbb{R}^3)$ and $\kappa > 0$, the following conclusions hold:*

(i) There exists a unique $\phi_u \in H^1(\mathbb{R}^3)$ solving the following equation

$$-\Delta\phi + \kappa^2\phi = 4\pi u^2 \tag{2.17}$$

in the weak sense. Moreover, there exists some constant $C > 0$ such that $\|\phi_u\|_{H^1} \le C\|u\|_{H^1}^2$.

(ii) Let $\Phi : H^1(\mathbb{R}^3) \to H^1(\mathbb{R}^3)$ be the mapping defined by $\Phi(u) := \phi_u$. Then Φ maps bounded sets into bounded sets. Moreover, Φ is continuous.

(iii) When $1 \le p \le 3$, it holds that

$$\phi_u := \left(G^\kappa * u^2\right)(x) = \int_{\mathbb{R}^3} \frac{e^{-\kappa|x-y|}}{|x-y|} u^2(y)\mathrm{d}y \in L^p(\mathbb{R}^3)$$

and

$$\nabla\phi_u := \left(\nabla G^\kappa * u^2\right)(x) = \int_{\mathbb{R}^3} \nabla G^\kappa(x-y)u^2(y)\mathrm{d}y \in L^p(\mathbb{R}^3).$$

Moreover, $\|\phi_u\|_2 \le C\|u\|_4^2$ and $\int_{\mathbb{R}^3} \phi_u u^2 \mathrm{d}x \le C\|u\|_4^4$.

(iv) *If $u \in H_r^1(\mathbb{R}^3)$, then $\phi_u \in H_r^1(\mathbb{R}^3)$. For any $\theta > 0$, $\phi_{\theta u} = \theta^2 \phi_u$ and $\phi_{u(\cdot+z)}(x) = \phi_u(x+z)$.*

(v) *If $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$ and $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^3)$ as $n \to \infty$, then, there exists a unique $\phi_u \in H^1(\mathbb{R}^3)$ solving (2.17) in the weak sense such that, up to a subsequence, $\phi_{u_n} \rightharpoonup \phi_u$ in $H^1(\mathbb{R}^3)$,*

$$\|\phi_{u_n}\|_2^2 = \|\phi_{u_n} - \phi_u\|_2^2 + \|\phi_u\|_2^2 + o_n(1)$$

and

$$\|\nabla\phi_{u_n}\|_2^2 = \|\nabla\phi_{u_n} - \nabla\phi_u\|_2^2 + \|\nabla\phi_u\|_2^2 + o_n(1).$$

Furthermore, for any $\psi \in H^1(\mathbb{R}^3)$, the following holds

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} \phi_{u_n} u_n \psi \, \mathrm{d}x = \int_{\mathbb{R}^3} \phi_u u \psi \, \mathrm{d}x. \tag{2.18}$$

Proof. (i) For any given $u \in H^1(\mathbb{R}^3)$, define a linear functional $L_u(v) : H^1(\mathbb{R}^3) \to \mathbb{R}$ by

$$L_u(v) := 4\pi \int_{\mathbb{R}^3} u^2 v \, \mathrm{d}x.$$

Then, from the continuous Sobolev embedding $H^1(\mathbb{R}^3) \hookrightarrow L^4(\mathbb{R}^3)$, we have

$$|L_u(v)| = 4\pi \left| \int_{\mathbb{R}^3} u^2 v \, \mathrm{d}x \right| \le C \|u\|_4^2 \|v\|_2 \le C \|u\|_{H^1}^2 \|v\|_{H^1} < \infty.$$

Hence, $L_u(v)$ is continuous in $H^1(\mathbb{R}^3)$. Now we define the following inner product $\langle u, v\rangle$ on $H^1(\mathbb{R}^3)$

$$\langle u, v\rangle := \int_{\mathbb{R}^3} \nabla u \nabla v \, \mathrm{d}x + \kappa^2 \int_{\mathbb{R}^3} uv \, \mathrm{d}x.$$

Then, from Lemma 2.5 and the Riesz representation theorem, for every $v \in H^1(\mathbb{R}^3)$, there exists a unique $\phi_u \in H^1(\mathbb{R}^3)$ satisfying

$$\int_{\mathbb{R}^3} \nabla\phi_u \nabla v \, \mathrm{d}x + \kappa^2 \int_{\mathbb{R}^3} \phi_u v \, \mathrm{d}x = 4\pi \int_{\mathbb{R}^3} u^2 v \, \mathrm{d}x,$$

which exactly means that ϕ_u solves (2.17) in the weak sense. In particular, let $v = \phi_u$ in the above equation, from Hölder's inequality, Lemma 2.5-(i) and Sobolev embedding theorem, we have that

$$\|\nabla\phi_u\|_2^2 + \kappa^2 \|\phi_u\|_2^2 = 4\pi \int_{\mathbb{R}^3} \phi_u u^2 \, \mathrm{d}x \le 4\pi \|\phi_u\|_2 \left\|u^2\right\|_2 \le 4\pi \|G^\kappa\|_1 \|u\|_4^4 \le C \|u\|_{H^1}^4,$$

which implies that there exists some constant $C > 0$ such that $\|\phi_u\|_{H^1} \le C \|u\|_{H^1}^2$.

(ii) Let $\{u_n\}$ be any bounded sequence in $H^1(\mathbb{R}^3)$. Then there exists $M > 0$ such that for all $n \in \mathbb{N}^+$, it holds that $\|u_n\|_{H^1} \le M$. Thus, according to (i), we have $\|\Phi(u_n)\|_{H^1} = \|\phi_{u_n}\|_{H^1} \le CM^2$, namely, $\Phi : H^1(\mathbb{R}^3) \to H^1(\mathbb{R}^3)$ maps bounded sets into bounded sets.

Next, we claim that if $u_n \to u$ in $H^1(\mathbb{R}^3)$ as $n \to \infty$, then $\Phi(u_n) \to \Phi(u)$ in $H^1(\mathbb{R}^3)$ as $n \to \infty$. Indeed, according to (i), we conclude that both $\Phi(u_n) = \phi_{u_n} \in H^1(\mathbb{R}^3)$ and $\Phi(u) = \phi_u \in H^1(\mathbb{R}^3)$ solve (2.17) in the weak sense. Therefore, for any $v \in H^1(\mathbb{R}^3)$, it holds that

$$\int_{\mathbb{R}^3} \nabla\Phi(u_n) \nabla v \, \mathrm{d}x + \kappa^2 \int_{\mathbb{R}^3} \Phi(u_n) v \, \mathrm{d}x = 4\pi \int_{\mathbb{R}^3} u_n^2 v \, \mathrm{d}x$$

and

$$\int_{\mathbb{R}^3} \nabla\Phi(u) \nabla v \, \mathrm{d}x + \kappa^2 \int_{\mathbb{R}^3} \Phi(u) v \, \mathrm{d}x = 4\pi \int_{\mathbb{R}^3} u^2 v \, \mathrm{d}x.$$

Subtracting the above two equations, we deduce that

$$\int_{\mathbb{R}^3} (\nabla\Phi(u_n) - \nabla\Phi(u)) \nabla v \,\mathrm{d}x + \kappa^2 \int_{\mathbb{R}^3} (\Phi(u_n) - \Phi(u)) v \,\mathrm{d}x = 4\pi \int_{\mathbb{R}^3} \left(u_n^2 - u^2\right) v \,\mathrm{d}x.$$

Let $v = \Phi(u_n) - \Phi(u)$. Then we have

$$\begin{aligned} C \left\|\Phi(u_n) - \Phi(u)\right\|_{H^1}^2 &\leq \left\|\nabla \left(\Phi(u_n) - \Phi(u)\right)\right\|_2^2 + \kappa^2 \left\|\Phi(u_n) - \Phi(u)\right\|_2^2 \\ &= 4\pi \int_{\mathbb{R}^3} \left(u_n^2 - u^2\right) \left(\Phi(u_n) - \Phi(u)\right) \mathrm{d}x. \end{aligned}$$

Since $u_n \to u$ in $H^1(\mathbb{R}^3)$ as $n \to \infty$ and $\Phi(u_n), \Phi(u) \in H^1(\mathbb{R}^3)$, from Hölder's inequality and Sobolev's embedding theorem, we infer that

$$\begin{aligned} &\left| \int_{\mathbb{R}^3} \left(u_n^2 - u^2\right) \left(\Phi(u_n) - \Phi(u)\right) \mathrm{d}x \right| \\ &\leq \int_{\mathbb{R}^3} |u_n + u| \, |u_n - u| \, |\Phi(u_n)| \,\mathrm{d}x + \int_{\mathbb{R}^3} |u_n + u| \, |u_n - u| \, |\Phi(u)| \,\mathrm{d}x \\ &\leq C \left\|u_n + u\right\|_4 \left\|u_n - u\right\|_4 \left\|\Phi(u_n)\right\|_2 + C \left\|u_n + u\right\|_4 \left\|u_n - u\right\|_4 \left\|\Phi(u)\right\|_2 \\ &\leq C \left\|u_n + u\right\|_4 \left\|u_n - u\right\|_{H^1} \left\|\Phi(u_n)\right\|_2 + C \left\|u_n + u\right\|_4 \left\|u_n - u\right\|_{H^1} \left\|\Phi(u)\right\|_2 \to 0 \end{aligned}$$

as $n \to \infty$. Then $\left\|\Phi(u_n) - \Phi(u)\right\|_{H^1}^2 \to 0$, namely, $\Phi(u_n) \to \Phi(u)$ in $H^1(\mathbb{R}^3)$ as $n \to \infty$.

(iii) Fix $u \in H^1(\mathbb{R}^3)$, from Lemma 2.5 and conclusion (i), we have that $\phi_u, \nabla\phi_u \in L^p(\mathbb{R}^3)$ with $1 \leq p \leq 3$. Moreover, from Lemma 2.5-(i) and the fact that $\left\|G^\kappa\right\|_1 = \frac{4\pi}{\kappa^2}$, we deduce that

$$\left\|\phi_u\right\|_2 \leq \left\|G^\kappa\right\|_1 \left\|u^2\right\|_2 = \frac{4\pi}{\kappa^2} \left\|u\right\|_4^2 \leq C \left\|u\right\|_4^2.$$

Then, from Hölder's inequality, we have

$$\int_{\mathbb{R}^3} \phi_u u^2 \mathrm{d}x \leq \left\|\phi_u\right\|_2 \left\|u^2\right\|_2 \leq \frac{4\pi}{\kappa^2} \left\|u\right\|_4^4 \leq C \left\|u\right\|_4^4.$$

(iv) It is only necessary to check that if $u \in H_r^1(\mathbb{R}^3)$, then $\phi_u \in H_r^1(\mathbb{R}^3)$. Let $u \in H_r^1(\mathbb{R}^3)$, and let $O(3)$ denote the three-dimensional orthogonal group. Then for any $A \in O(3)$ and x, $y \in \mathbb{R}^3$, it holds that $|Ax| = |x|$ and $|Ax - Ay| = |x - y|$. Thus, we get that $\phi_u(Ax) = \phi_u(x)$ by applying a change of variable in the integrals.

(v) Since $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$ and $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^3)$, then, up to a subsequence, $u_n \to u$ in $L_{\mathrm{loc}}^r(\mathbb{R}^3)$ with $r \in [1, 6)$ and $u_n(x) \to u(x)$ a.e. in $\mathbb{R}^3$ as $n \to \infty$. Since $\{\phi_{u_n}\}$ is bounded in $H^1(\mathbb{R}^3)$ from (ii), there exists $\bar{\phi}_u \in H^1(\mathbb{R}^3)$ such that, up to a subsequence, $\phi_{u_n} \rightharpoonup \bar{\phi}_u$ in $H^1(\mathbb{R}^3)$, $\phi_{u_n} \to \bar{\phi}_u$ in $L_{\mathrm{loc}}^r(\mathbb{R}^3)$ with $r \in [1, 6)$ and $\phi_{u_n} \to \bar{\phi}_u$ a.e. in $\mathbb{R}^3$. Next, we claim that $\bar{\phi}_u = \phi_u$. Indeed, for any $\psi \in C_c^\infty(\mathbb{R}^3)$, it holds that

$$\int_{\mathbb{R}^3} \nabla\phi_{u_n} \nabla\psi \,\mathrm{d}x + \kappa^2 \int_{\mathbb{R}^3} \phi_{u_n} \psi \,\mathrm{d}x = 4\pi \int_{\mathbb{R}^3} u_n^2 \psi \,\mathrm{d}x.$$

Since $\nabla\phi_{u_n} \rightharpoonup \nabla\bar{\phi}_u$ in $L^2(\mathbb{R}^3)$ and $\nabla\psi \in L^2(\mathbb{R}^3)$, we infer that

$$\int_{\mathbb{R}^3} \nabla\phi_{u_n} \nabla\psi \,\mathrm{d}x \to \int_{\mathbb{R}^3} \nabla\bar{\phi}_u \nabla\psi \,\mathrm{d}x.$$

From $\phi_{u_n} \to \bar{\phi}_u$, $u_n \to u$ in $L_{\mathrm{loc}}^r(\mathbb{R}^3)$ with $r \in [1, 6)$, $\psi \in C_c^\infty(\mathbb{R}^3)$ and Hölder's inequality, we have

$$\left| \int_{\mathbb{R}^3} \left(\phi_{u_n} - \bar{\phi}_u\right) \psi \,\mathrm{d}x \right| \leq \int_{\mathrm{supp}\,\psi} \left|\left(\phi_{u_n} - \bar{\phi}_u\right) \psi\right| \,\mathrm{d}x \leq \left\|\phi_{u_n} - \bar{\phi}_u\right\|_{L^2(\mathrm{supp}\,\psi)} \left\|\psi\right\|_2 \to 0$$

and

$$\begin{aligned}\left|\int_{\mathbb{R}^3}\left(u_n^2-u^2\right)\psi\,\mathrm{d}x\right| &\le \int_{\operatorname{supp}\psi}\left|\left(u_n^2-u^2\right)\psi\right|\,\mathrm{d}x\\ &\le \|u_n+u\|_{L^4(\operatorname{supp}\psi)}\,\|u_n-u\|_{L^4(\operatorname{supp}\psi)}\,\|\psi\|_2\to 0\end{aligned}$$

as $n\to\infty$. Therefore, it holds that

$$\int_{\mathbb{R}^3}\nabla\bar{\phi}_u\nabla\psi\,\mathrm{d}x+\kappa^2\int_{\mathbb{R}^3}\bar{\phi}_u\psi\,\mathrm{d}x=4\pi\int_{\mathbb{R}^3}u^2\psi\,\mathrm{d}x,$$

which implies that $\bar{\phi}_u$ solves (2.17) in the weak sense. By the uniqueness of the solution to (2.17) according to (i), we have $\bar{\phi}_u=\phi_u$. Thus, it follows from $\phi_u\in H^1(\mathbb{R}^3)$ that, up to a subsequence, $\phi_{u_n}\rightharpoonup\phi_u$ in $H^1(\mathbb{R}^3)$, $\phi_{u_n}\to\phi_u$ in $L^r_{\mathrm{loc}}(\mathbb{R}^3)$ with $r\in[1,6)$ and $\phi_{u_n}\to\phi_u$ a.e. in $\mathbb{R}^3$.

We claim that up to a subsequence, $\nabla\phi_{u_n}\to\nabla\phi_u$ a.e. in $\mathbb{R}^3$ as $n\to\infty$. Indeed, it follows from (iii) that

$$\nabla\phi_u=\int_{\mathbb{R}^3}\nabla G^\kappa(x-y)u^2(y)\mathrm{d}y.$$

Thus, from (2.4), we infer that

$$\begin{aligned}|\nabla\phi_{u_n}-\nabla\phi_u| &= \left|\int_{\mathbb{R}^3}\left(u_n^2(y)-u^2(y)\right)\nabla G^\kappa(x-y)\mathrm{d}y\right|\\ &\le \int_{\mathbb{R}^3}\left|u_n^2(y)-u^2(y)\right|\left|\kappa\frac{x-y}{|x-y|^2}+\frac{x-y}{|x-y|^3}\right|e^{-\kappa|x-y|}\mathrm{d}y\\ &\le \int_{\mathbb{R}^3}\left|u_n^2(y)-u^2(y)\right|\frac{\kappa e^{-\kappa|x-y|}}{|x-y|}\mathrm{d}y+\int_{\mathbb{R}^3}\left|u_n^2(y)-u^2(y)\right|\frac{e^{-\kappa|x-y|}}{|x-y|^2}\mathrm{d}y\\ &=: J_n^1+J_n^2.\end{aligned}$$

Let $R\ge 1$. By Hölder's inequality and the facts that $e^{-t}\le 1$ and $e^t\ge\frac{t^2}{2}$ for $t\ge 0$, we deduce that

$$\begin{aligned}J_n^1 &\le \|u_n+u\|_{L^4(B_R(x))}\,\|u_n-u\|_{L^4(B_R(x))}\left(\int_{|y-x|\le R}\frac{\kappa^2e^{-2\kappa|x-y|}}{|x-y|^2}\mathrm{d}y\right)^{\frac{1}{2}}\\ &\quad+\|u_n+u\|_{L^4(B_R^c(x))}\,\|u_n-u\|_{L^4(B_R^c(x))}\left(\int_{|y-x|\ge R}\frac{\kappa^2}{|x-y|^2e^{2\kappa|x-y|}}\mathrm{d}y\right)^{\frac{1}{2}}\\ &\le \|u_n+u\|_{L^4(\mathbb{R}^3)}\,\|u_n-u\|_{L^4(B_R(x))}\left(\int_{|y-x|\le R}\frac{\kappa^2}{|x-y|^2}\mathrm{d}y\right)^{\frac{1}{2}}\\ &\quad+\|u_n+u\|_{L^4(\mathbb{R}^3)}\,\|u_n-u\|_{L^4(B_R^c(x))}\left(\int_{|y-x|\ge R}\frac{1}{2\,|x-y|^4}\mathrm{d}y\right)^{\frac{1}{2}}\end{aligned}$$

and

$$
\begin{aligned}
J_n^2 &= \int_{|y-x|\le R} \left|u_n^2(y)-u^2(y)\right| \frac{e^{-\kappa|x-y|}}{|x-y|^2}\mathrm{d}y + \int_{|y-x|\ge R} \left|u_n^2(y)-u^2(y)\right| \frac{e^{-\kappa|x-y|}}{|x-y|^2}\mathrm{d}y \\
&\le \int_{|y-x|\le R} \frac{|u_n(y)+u(y)|}{|x-y|}\frac{|u_n(y)-u(y)|}{|x-y|}\mathrm{d}y \\
&\quad + \int_{|y-x|\ge R} |u_n(y)+u(y)|\,|u_n(y)-u(y)|\,\frac{e^{-\kappa|x-y|}}{|x-y|}\mathrm{d}y \\
&\le \left(\int_{|y-x|\le R} (|u_n(y)|+|u(y)|)^2\,\frac{1}{|x-y|^2}\mathrm{d}y\right)^{\frac{1}{2}} \left(\int_{|y-x|\le R} |u_n(y)-u(y)|^2\,\frac{1}{|x-y|^2}\mathrm{d}y\right)^{\frac{1}{2}} \\
&\quad + \|u_n+u\|_{L^4(B_R^c(x))}\,\|u_n-u\|_{L^4(B_R^c(x))} \left(\int_{|y-x|\ge R} \frac{1}{|x-y|^2\,e^{2\kappa|x-y|}}\mathrm{d}y\right)^{\frac{1}{2}} \\
&\le C(R)\left(\|u_n\|_{L^2(\mathbb{R}^3)}+\|u\|_{L^2(\mathbb{R}^3)}\right)\|u_n-u\|_{L^2(B_R(x))} \\
&\quad + \|u_n+u\|_{L^4(\mathbb{R}^3)}\,\|u_n-u\|_{L^4(B_R^c(x))} \left(\int_{|y-x|\ge R} \frac{1}{2\kappa^2\,|x-y|^4}\mathrm{d}y\right)^{\frac{1}{2}}.
\end{aligned}
$$

Fix R, then let $n\to\infty$, and subsequently let $R\to\infty$, since $u_n\to u$ in $L^r_{\mathrm{loc}}(\mathbb{R}^3)$ with $r\in[1,6)$, we conclude that both $J_n^1\to 0$ and $J_n^2\to 0$. Thus, $\nabla\phi_{u_n}\to\nabla\phi_u$ a.e. in $\mathbb{R}^3$ as $n\to\infty$. Since $\{\phi_{u_n}\}$ is bounded in $H^1(\mathbb{R}^3)$, $\phi_u\in H^1(\mathbb{R}^3)$ and $\phi_{u_n}\to\phi_u$ a.e. in $\mathbb{R}^3$ as $n\to\infty$, from Brezis–Lieb lemma ([36, Lemma 1.32]), we deduce that

$$
\|\phi_{u_n}\|_2^2 = \|\phi_{u_n}-\phi_u\|_2^2+\|\phi_u\|_2^2+o_n(1) \text{ and } \|\nabla\phi_{u_n}\|_2^2 = \|\nabla\phi_{u_n}-\nabla\phi_u\|_2^2+\|\nabla\phi_u\|_2^2+o_n(1).
$$

Finally, for any $\psi\in C_c^\infty(\mathbb{R}^3)$, we infer that

$$
\begin{aligned}
\left|\int_{\mathbb{R}^3}\left(\phi_{u_n}u_n\psi-\phi_u u\psi\right)\mathrm{d}x\right| &\le \int_{\mathbb{R}^3}\left|\left(\phi_{u_n}-\phi_u\right)u_n\psi\right|\mathrm{d}x+\int_{\mathbb{R}^3}\left|\phi_u\left(u_n-u\right)\psi\right|\mathrm{d}x \\
&\le \|\phi_{u_n}-\phi_u\|_{L^4(\operatorname{supp}\psi)}\,\|u_n\|_4\,\|\psi\|_2+\|\phi_u\|_2\,\|u_n-u\|_{L^4(\operatorname{supp}\psi)}\,\|\psi\|_4 \\
&\to 0
\end{aligned}
$$

as $n\to\infty$, namely, (2.18) holds since $C_c^\infty(\mathbb{R}^3)$ is dense in $H^1(\mathbb{R}^3)$. □

2.3 The functional setting

It is easy to see that the critical points (u,ϕ) of the C^1 functional

$$
F(u,\phi) := \frac{1}{2}\|\nabla u\|_2^2+\frac{1}{2}\int_{\mathbb{R}^3}\phi u^2\mathrm{d}x-\frac{1}{16\pi}\|\nabla\phi\|_2^2-\frac{\kappa^2}{16\pi}\|\phi\|_2^2-\frac{1}{p}\|u\|_p^p
$$

on $H^1(\mathbb{R}^3)\times H^1(\mathbb{R}^3)$ restricted to the sphere S_a are the weak solutions of system (1.5). Since the constrained functional $F|_{S_a}$ is strongly unbounded from below and above, the usual techniques of the critical point theory cannot be used directly. To deal with this issue, we shall reduce the functional F to another functional I depending on the single variable u, following a procedure introduced by Benci and Fortunato in [5] for this kind of problem.

From Lemma 2.6, we find that system (1.5) can be written as the following nonlocal equation on the constraint S_a

$$
-\Delta u+\lambda u+\phi_u u=|u|^{p-2}u \quad \text{in } \mathbb{R}^3,
$$

where $\phi_u = \frac{e^{-\kappa|x|}}{|x|} * u^2 \in H^1(\mathbb{R}^3)$. By multiplying the second equation of (1.5) by $\frac{1}{16\pi}\phi_u$ and integrating in $\mathbb{R}^3$, we obtain that

$$\frac{1}{16\pi}\|\nabla\phi_u\|_2^2 + \frac{\kappa^2}{16\pi}\|\phi_u\|_2^2 = \frac{1}{4}\int_{\mathbb{R}^3}\phi_u u^2 \mathrm{d}x.$$

Then, by substituting the above equation into $F(u,\phi)$, we obtain the energy functional

$$I(u) := \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{4}\int_{\mathbb{R}^3}\phi_u u^2 \mathrm{d}x - \frac{1}{p}\|u\|_p^p$$

with the constraint S_a. Obviously, I is well defined and is of class C^1 in $H^1(\mathbb{R}^3)$. Then, for any $\varphi \in H^1(\mathbb{R}^3)$, we have

$$\langle I'(u), \varphi\rangle = \int_{\mathbb{R}^3}\nabla u\nabla\varphi\,\mathrm{d}x + \int_{\mathbb{R}^3}\phi_u u\varphi\,\mathrm{d}x - \int_{\mathbb{R}^3}|u|^{p-2}u\varphi\,\mathrm{d}x.$$

Hence, the critical point (u,ϕ) of $F|_{S_a}$ is equivalent to the critical point u of $I|_{S_a}$.

2.4 The regularity of solutions

We remark here that the weak solutions are classical solutions. Indeed, let $(u,\phi_u) \in H^1(\mathbb{R}^3)\times H^1(\mathbb{R}^3)$ be a weak solution of system (1.5), namely, u and ϕ_u satisfy (1.6). From (1.6) and the fact that $\phi_u \in L^p(\mathbb{R}^3)$ with $1 \le p \le 3$, it follows that $u \in W^{2,2}_{\mathrm{loc}}(\mathbb{R}^3)$ by applying Sobolev embedding theorem and L^p estimates (see [14, Theorem 9.9]). By reapplying Sobolev embedding theorem to u, we conclude that $u \in C^{0,\alpha}_{\mathrm{loc}}(\mathbb{R}^3)$ with $0 < \alpha \le \frac{1}{2}$. Similarly, we have $\phi_u \in W^{2,3}_{\mathrm{loc}}(\mathbb{R}^3)$, and thus $\phi_u \in C^{0,\alpha}_{\mathrm{loc}}(\mathbb{R}^3)$ with $0 < \alpha \le 1 - \frac{3}{p}$, $p > 3$. Therefore, from (1.6) and Schauder estimates (see [34, Theorem B.1]), we get that $u, \phi_u \in C^{2,\alpha}_{\mathrm{loc}}(\mathbb{R}^3)$.

2.5 The Pohožaev identity

The following lemma presents two forms of the Pohožaev identity of system (1.5), which are widely used in the study of the fixed mass problem.

Lemma 2.7. *Let $2 < p < 6$ and $a, \kappa > 0$. Then, for any nontrivial solution $(u,\phi_u) \in H^1(\mathbb{R}^3)\times H^1(\mathbb{R}^3)$ of system (1.5), the following two Pohožaev identities*

$$\frac{1}{2}\|\nabla u\|_2^2 + \frac{3}{2}\lambda\|u\|_2^2 + \frac{5}{4}\int_{\mathbb{R}^3}\phi_u u^2\mathrm{d}x - \frac{\kappa}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{-\kappa|x-y|}u^2(x)u^2(y)\mathrm{d}x\mathrm{d}y - \frac{3}{p}\|u\|_p^p = 0 \quad (2.19)$$

and

$$P(u) := \|\nabla u\|_2^2 + \frac{1}{4}\int_{\mathbb{R}^3}\phi_u u^2\mathrm{d}x + \frac{\kappa}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{-\kappa|x-y|}u^2(x)u^2(y)\mathrm{d}x\mathrm{d}y - \frac{3(p-2)}{2p}\|u\|_p^p = 0 \quad (2.20)$$

hold, where ϕ_u is defined in Lemma 2.6-(iii).

Proof. For any nontrivial solution $(u,\phi_u) \in H^1(\mathbb{R}^3)\times H^1(\mathbb{R}^3)$ of system (1.5), we have

$$\|\nabla u\|_2^2 + \lambda\|u\|_2^2 + \int_{\mathbb{R}^3}\phi_u u^2\mathrm{d}x - \|u\|_p^p = 0 \quad (2.21)$$

and

$$\|\nabla\phi_u\|_2^2 + \kappa^2\|\phi_u\|_2^2 = 4\pi\int_{\mathbb{R}^3}\phi_u u^2\mathrm{d}x. \quad (2.22)$$

We claim that u satisfies (2.19). Indeed, if (u, ϕ_u) solves system (1.5), from Section 2.4, we have $u, \phi_u \in C^2(B_R)$, where B_R is an arbitrary ball in $\mathbb{R}^3$ centered at the origin with radius R. Thus, from [11, Appendix A.3], we deduce that

$$\int_{B_R} -\Delta u\,(x\cdot\nabla u)\,\mathrm{d}x = -\frac{1}{2}\int_{B_R}|\nabla u|^2\,\mathrm{d}x - \frac{1}{R}\int_{\partial B_R}|x\cdot\nabla u|^2\,\mathrm{d}\sigma + \frac{R}{2}\int_{\partial B_R}|\nabla u|^2\,\mathrm{d}\sigma,$$

$$\int_{B_R}\phi_u u\,(x\cdot\nabla u)\,\mathrm{d}x = -\frac{1}{2}\int_{B_R}u^2\,(x\cdot\nabla\phi_u)\,\mathrm{d}x - \frac{3}{2}\int_{B_R}\phi_u u^2\mathrm{d}x + \frac{R}{2}\int_{\partial B_R}\phi_u u^2\mathrm{d}\sigma,$$

$$\int_{B_R} u\,(x\cdot\nabla u)\,\mathrm{d}x = -\frac{3}{2}\int_{B_R}u^2\mathrm{d}x + \frac{R}{2}\int_{\partial B_R}u^2\mathrm{d}\sigma,$$

and

$$\int_{B_R}|u|^{p-2}u\,(x\cdot\nabla u)\,\mathrm{d}x = -\frac{3}{p}\int_{B_R}|u|^p\,\mathrm{d}x + \frac{R}{p}\int_{\partial B_R}|u|^p\,\mathrm{d}\sigma,$$

where $\mathrm{d}\sigma$ represents the surface area element on the sphere ∂B_R.

By multiplying the first equation of system (1.5) by $x\cdot\nabla u$, multiplying the second equation by $x\cdot\nabla\phi_u$, and integrating over B_R, and according to the above four equations, we get

$$\begin{aligned}
&-\frac{1}{2}\int_{B_R}|\nabla u|^2\,\mathrm{d}x - \frac{1}{R}\int_{\partial B_R}|x\cdot\nabla u|^2\,\mathrm{d}\sigma + \frac{R}{2}\int_{\partial B_R}|\nabla u|^2\,\mathrm{d}\sigma - \frac{3}{2}\lambda\int_{B_R}u^2\mathrm{d}x + \frac{R}{2}\lambda\int_{\partial B_R}u^2\mathrm{d}\sigma\\
&\quad -\frac{1}{2}\int_{B_R}u^2\,(x\cdot\nabla\phi_u)\,\mathrm{d}x - \frac{3}{2}\int_{B_R}\phi_u u^2\mathrm{d}x + \frac{R}{2}\int_{\partial B_R}\phi_u u^2\mathrm{d}\sigma\\
&= -\frac{3}{p}\int_{B_R}|u|^p\,\mathrm{d}x + \frac{R}{p}\int_{\partial B_R}|u|^p\,\mathrm{d}\sigma
\end{aligned}\tag{2.23}$$

and

$$\begin{aligned}
4\pi\int_{B_R}u^2\,(x\cdot\nabla\phi_u)\,\mathrm{d}x &= \int_{B_R}-\Delta\phi_u\,(x\cdot\nabla\phi_u)\,\mathrm{d}x + \kappa^2\int_{B_R}\phi_u\,(x\cdot\nabla\phi_u)\,\mathrm{d}x\\
&= -\frac{1}{2}\int_{B_R}|\nabla\phi_u|^2\,\mathrm{d}x - \frac{1}{R}\int_{\partial B_R}|x\cdot\nabla\phi_u|^2\,\mathrm{d}\sigma + \frac{R}{2}\int_{\partial B_R}|\nabla\phi_u|^2\,\mathrm{d}\sigma\\
&\quad -\frac{3}{2}\kappa^2\int_{B_R}\phi_u^2\,\mathrm{d}x + \frac{R}{2}\kappa^2\int_{\partial B_R}\phi_u^2\,\mathrm{d}\sigma.
\end{aligned}\tag{2.24}$$

Substituting (2.24) into (2.23), we obtain that

$$\begin{aligned}
&-\frac{1}{2}\int_{B_R}|\nabla u|^2\,\mathrm{d}x + \frac{1}{16\pi}\int_{B_R}|\nabla\phi_u|^2\,\mathrm{d}x + \frac{3\kappa^2}{16\pi}\int_{B_R}\phi_u^2\,\mathrm{d}x\\
&\quad -\frac{3}{2}\lambda\int_{B_R}u^2\mathrm{d}x - \frac{3}{2}\int_{B_R}\phi_u u^2\mathrm{d}x + \frac{3}{p}\int_{B_R}|u|^p\,\mathrm{d}x\\
&= \frac{1}{R}\int_{\partial B_R}\left(|x\cdot\nabla u|^2 - \frac{1}{8\pi}|x\cdot\nabla\phi_u|^2\right)\mathrm{d}\sigma - \frac{R}{2}\int_{\partial B_R}\left(|\nabla u|^2 + \lambda u^2 + \phi_u u^2 - \frac{1}{8\pi}|\nabla\phi_u|^2\right)\mathrm{d}\sigma\\
&\quad + \frac{R}{16\pi}\kappa^2\int_{\partial B_R}\phi_u^2\mathrm{d}\sigma + \frac{R}{p}\int_{\partial B_R}|u|^p\,\mathrm{d}\sigma.
\end{aligned}$$

Now, let $R = R_n$ with $R_n \to \infty$ as $n \to \infty$. Then, we have that $B_{R_n} \to \mathbb{R}^3$ and the right-hand side of the above equation tends to zero as $n \to \infty$. Thus, we get that

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^3}|\nabla u|^2\,\mathrm{d}x - \frac{1}{16\pi}\int_{\mathbb{R}^3}|\nabla\phi_u|^2\,\mathrm{d}x - \frac{3\kappa^2}{16\pi}\int_{\mathbb{R}^3}\phi_u^2\mathrm{d}x\\
&\qquad + \frac{3}{2}\lambda\int_{\mathbb{R}^3}u^2\mathrm{d}x + \frac{3}{2}\int_{\mathbb{R}^3}\phi_u u^2\mathrm{d}x - \frac{3}{p}\int_{\mathbb{R}^3}|u|^p\,\mathrm{d}x = 0.
\end{aligned}$$

Then, by inserting (2.22) into the above equation, we infer that

$$\frac{1}{2}\|\nabla u\|_2^2+\frac{3}{2}\lambda\|u\|_2^2+\frac{5}{4}\int_{\mathbb{R}^3}\phi_u u^2\mathrm{d}x-\frac{\kappa^2}{8\pi}\|\phi_u\|_2^2-\frac{3}{p}\|u\|_p^p=0. \tag{2.25}$$

For the convenience of subsequent calculations, we will adopt the method in [11, Appendix A.2] to transform $\|\phi_u\|_2^2$ in (2.25) into another form. For any $f\in L^1(\mathbb{R}^3)$, we define the following Fourier transform

$$\mathcal{F}[f](x):=\frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathbb{R}^3}e^{-ix\cdot y}f(y)\,\mathrm{d}y.$$

For any $f,g\in L^2(\mathbb{R}^3)$, it holds that

$$\mathcal{F}[f*g]=(2\pi)^{\frac{3}{2}}\mathcal{F}[f]\mathcal{F}[g]\quad\text{and}\quad\int_{\mathbb{R}^3}\mathcal{F}[f]\mathcal{F}[g]\,\mathrm{d}x=\int_{\mathbb{R}^3}fg\,\mathrm{d}x.$$

Moreover, when $f\in L^1(\mathbb{R}^3)\cap L^2(\mathbb{R}^3)$, it follows from [26, Theorem 5.3] that $\mathcal{F}[f]\in L^2(\mathbb{R}^3)$ and $\|\mathcal{F}[f]\|_2^2=\|f\|_2^2$. Since

$$\mathcal{F}\left[e^{-\kappa|\cdot|}\right](x)=\sqrt{\frac{2}{\pi}}\frac{2\kappa}{\left(\kappa^2+|x|^2\right)^2}\quad\text{and}\quad\mathcal{F}[G^\kappa](x)=\mathcal{F}\left[\frac{e^{-\kappa|\cdot|}}{|\cdot|}\right](x)=\sqrt{\frac{2}{\pi}}\frac{1}{\kappa^2+|x|^2},$$

we have

$$\mathcal{F}[G^\kappa*G^\kappa]=(2\pi)^{\frac{3}{2}}\mathcal{F}[G^\kappa]\mathcal{F}[G^\kappa]=\frac{4\sqrt{2\pi}}{\left(\kappa^2+|x|^2\right)^2}\quad\text{and}\quad(G^\kappa*G^\kappa)(x)=\frac{2\pi}{\kappa}e^{-\kappa|x|}.$$

Since $\phi_u\in L^p(\mathbb{R}^3)$ with $1\le p\le 3$, we conclude that

$$\begin{aligned}\frac{\kappa^2}{8\pi}\|\phi_u\|_2^2&=\frac{\kappa^2}{8\pi}\int_{\mathbb{R}^3}|\mathcal{F}[\phi_u]|^2\,\mathrm{d}x=\frac{\kappa^2}{8\pi}(2\pi)^{\frac{3}{2}}\int_{\mathbb{R}^3}\mathcal{F}\left[u^2\right]\mathcal{F}[G^\kappa]\mathcal{F}[\phi_u]\,\mathrm{d}x\\&=\frac{\kappa^2}{8\pi}\int_{\mathbb{R}^3}\mathcal{F}\left[u^2\right]\mathcal{F}[G^\kappa*\phi_u]\,\mathrm{d}x=\frac{\kappa^2}{8\pi}\int_{\mathbb{R}^3}\left((G^\kappa*G^\kappa)*u^2\right)u^2\,\mathrm{d}x\\&=\frac{\kappa}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}e^{-\kappa|x-y|}u^2(x)u^2(y)\mathrm{d}x\mathrm{d}y.\end{aligned}$$

Then by substituting the above result into (2.25), we obtain (2.19), and thus (2.20) follows from (2.19) and (2.21). This completes the proof of Lemma 2.7. □

2.6 The abstract minimization problem

In [6] and [7], Bellazzini and Siciliano developed an abstract framework to prove the existence of solutions to constrained minimization problems. We shall follow this framework in our proofs of Theorems 1.1 and 1.2. Next, we define the following functionals which will be frequently used in subsequent results:

$$A(u):=\|\nabla u\|_2^2,\quad B(u):=\int_{\mathbb{R}^3}\phi_u u^2\mathrm{d}x,\quad C(u):=\|u\|_p^p,\quad T(u):=\frac{1}{4}B(u)-\frac{1}{p}C(u).$$

For any $u\in H^1(\mathbb{R}^3)$, $\beta\in\mathbb{R}$ and $\theta>0$, we define $u^\theta\in H^1(\mathbb{R}^3)$ as the scaling

$$u^\theta(x):=\theta^{1-\frac{3}{2}\beta}u\left(\frac{x}{\theta^\beta}\right)\quad\text{for a.e. } x\in\mathbb{R}^3 \tag{2.26}$$

such that $\left\|u^\theta\right\|_2=\theta\|u\|_2$. Then, the following proposition is a corollary of [6, Theorem 2.1].

Proposition 2.8. *If $2 < p < 3$, let T be a C^1 functional on $H^1(\mathbb{R}^3)$ and let $\{u_n\} \subset S_a$ be a minimizing sequence for I_{a^2} such that, up to a subsequence, $u_n \rightharpoonup \bar{u}$ in $H^1(\mathbb{R}^3)$ as $n \to \infty$ with $\bar{u} \in H^1(\mathbb{R}^3)\setminus\{0\}$. Assume $a, \kappa > 0$ and the following conditions are satisfied.*

(i) If $0 < \mu < a$, then $I_{a^2} \le I_{\mu^2} + I_{a^2-\mu^2}$.

(ii) For all $a > 0$, $-\infty < I_{a^2} < 0$ with $I(0) = 0$.

(iii) The function $a \mapsto I_{a^2}$ is continuous and $I_{a^2}/a^2 \to 0$ as $a \to 0^+$.

(iv) Let $\alpha_n := \left(a^2 - \|\bar{u}\|_2^2\right)\|u_n - \bar{u}\|_2^{-2}$, then the functional T satisfies

$$T(u_n - \bar{u}) + T(\bar{u}) = T(u_n) + o_n(1) \tag{2.27}$$

and

$$T\left(\alpha_n(u_n - \bar{u})\right) - T(u_n - \bar{u}) = o_n(1). \tag{2.28}$$

Then, for every $a > 0$, the set $M(a)$ is nonempty, where

$$M(a) := \bigcup_{\mu\in(0,a]} \left\{ u \in S_\mu : I(u) = I_{\mu^2} = \inf_{u\in S_\mu} I(u) \right\}.$$

If we suppose further that for any $u \in M(a)$, there exists $\beta \in \mathbb{R}$ such that for any $\theta > 0$,

$$h_\beta^u(\theta) := I(u^\theta) - \theta^2 I(u)$$

is differentiable and $\left(h_\beta^u\right)'(1) \neq 0$, then, for any $\mu \in (0, a)$, the function $\mu \mapsto I_{\mu^2}/\mu^2$ is monotone decreasing. Thus, for any $\mu \in (0, a)$, the following strong subadditivity inequality holds

$$I_{a^2} = \frac{\mu^2}{a^2} I_{a^2} + \frac{a^2-\mu^2}{a^2} I_{a^2} < I_{\mu^2} + I_{a^2-\mu^2}. \tag{2.29}$$

Therefore, it follows that $\bar{u} \in S_a$ and $I(\bar{u}) = I_{a^2} = \inf_{u\in S_a} I(u) < 0$.

If we suppose further that

$$\left\langle T'(u_n), u_n \right\rangle = O_n(1) \tag{2.30}$$

and

$$\left\langle T'(u_n) - T'(u_m), u_n - u_m \right\rangle = o_n(1) \quad \text{as } n, m \to \infty, \tag{2.31}$$

then, $\|u_n - \bar{u}\|_{H^1} \to 0$ as $n \to \infty$.

Proposition 2.9 ([6, Lemma 2.1])**.** *Let $3 < p < \frac{10}{3}$, $a, \kappa > 0$ and $\mu := \|\bar{u}\|_2 \in (0, a]$. Suppose that $\{u_n\} \subset S_a$ is a minimizing sequence for I_{a^2} such that, up to a subsequence, $u_n \rightharpoonup \bar{u}$ in $H^1(\mathbb{R}^3)$ as $n \to \infty$ with $\bar{u} \in H^1(\mathbb{R}^3)\setminus\{0\}$, and let $T : H^1(\mathbb{R}^3) \to \mathbb{R}$ be a C^1 functional satisfying (2.27)–(2.29), then $\bar{u} \in S_a$ and $I(\bar{u}) = I_{a^2}$. If we further assume that (2.30) and (2.31) also hold, then, $\|u_n - \bar{u}\|_{H^1} \to 0$ as $n \to \infty$.*

After introducing the above abstract framework, we now state some results showing how this framework applies to the minimization problem (1.8).

Lemma 2.10. *Assume that $2 < p < \frac{10}{3}$ and $a, \kappa > 0$, then the functional I is coercive and bounded from below on S_a. Moreover, it holds that $I_{a^2} = \inf_{u\in S_a} I(u) \in (-\infty, 0]$.*

Proof. For any $u \in S_a$, it follows from $\frac{3(p-2)}{2} < 2$ and Lemma 2.1 that

$$I(u) = \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{4}\int_{\mathbb{R}^3}\phi_u u^2 \mathrm{d}x - \frac{1}{p}\|u\|_p^p \geq \frac{1}{2}\|\nabla u\|_2^2 - \frac{C_{3,p}^p a^{\frac{6-p}{2}}}{p}\|\nabla u\|_2^{\frac{3(p-2)}{2}},$$

which implies that I is coercive on S_a and then I is bounded from below on S_a, namely, $I_{a^2} := \inf_{S_a} I(u) > -\infty$. Moreover, for any $u \in S_a$ and $t > 0$, let $u_t(x) := t^{\frac{3}{2}}u(tx)$ for a.e. $x \in \mathbb{R}^3$. Then it holds that $u_t \in S_a$ and

$$\begin{aligned} I_{a^2} \leq I(u_t) &= \frac{t^2}{2}\|\nabla u\|_2^2 + \frac{t}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{e^{-\frac{\kappa}{t}|x-y|}}{|x-y|}u^2(x)u^2(y)\mathrm{d}x\mathrm{d}y - \frac{t^{\frac{3(p-2)}{2}}}{p}\|u\|_p^p \\ &\leq \frac{t^2}{2}\|\nabla u\|_2^2 + \frac{t}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{1}{|x-y|}u^2(x)u^2(y)\mathrm{d}x\mathrm{d}y - \frac{t^{\frac{3(p-2)}{2}}}{p}\|u\|_p^p \to 0 \quad \text{as } t \to 0. \end{aligned}$$

Hence, when $2 < p < \frac{10}{3}$, it holds that $I_{a^2} \in (-\infty, 0]$ for any $a > 0$. □

Lemma 2.11. *If $2 < p < \frac{10}{3}$ and $\kappa > 0$, then I satisfies the following weak subadditivity inequality*

$$I_{a^2} \leq I_{\mu^2} + I_{a^2-\mu^2} \quad \text{for any } 0 < \mu < a.$$

Proof. Now argue as in the [15, Lemma 3.1], and we complete the proof. □

Lemma 2.12. *For any $\kappa > 0$, we have*

(i) If $2 < p < 3$, then there exists $a_0 > 0$ such that $I_{a^2} < 0$ for any $a \in (0, a_0)$. Moreover, it holds that $I_{a^2} < 0$ for any $a > 0$.

(ii) If $3 < p < \frac{10}{3}$, then there exists $a_1 > 0$ such that $I_{a^2} < 0$ for any $a \in (a_1, \infty)$.

Proof. From (2.26), we have $u^\theta(x) := \theta^{1-\frac{3}{2}\beta}u\left(\frac{x}{\theta^\beta}\right)$ for a.e. $x \in \mathbb{R}^3$, where $u \in H^1(\mathbb{R}^3)$, $\beta \in \mathbb{R}$ and $\theta > 0$. Then, from direct calculations, we get that $\left\|u^\theta\right\|_2 = \theta\|u\|_2$ and

$$\begin{aligned} &A(u^\theta) = \left\|\nabla u^\theta\right\|_2^2 = \theta^{2-2\beta}A(u), \quad C(u^\theta) = \left\|u^\theta\right\|_p^p = \theta^{\left(1-\frac{3}{2}\beta\right)p+3\beta}C(u), \\ &B(u^\theta) = \theta^{4-\beta}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{e^{-\kappa\theta^\beta|x-y|}}{|x-y|}u^2(x)u^2(y)\mathrm{d}x\mathrm{d}y < \theta^{4-\beta}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{1}{|x-y|}u^2(x)u^2(y)\mathrm{d}x\mathrm{d}y. \end{aligned}$$

(i) When $2 < p < 3$, fix $u \in S_1$ and $\beta = -2$ such that $\left(1-\frac{3}{2}\beta\right)p + 3\beta = 4p - 6 < 2 - 2\beta = 4 - \beta = 6$, then $u^a \in S_a$. Thus, as $a \to 0^+$, we have

$$I(u^a) < \frac{a^6}{2}A(u) + \frac{a^6}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{1}{|x-y|}u^2(x)u^2(y)\mathrm{d}x\mathrm{d}y - \frac{a^{4p-6}}{p}C(u) = \mathcal{J}_0(u_a) \to 0^-,$$

where $\mathcal{J}_0$ is the energy functional of the Schrödinger–Poisson–Slater system (1.14) in Section 1.2. Then, there exists $a_0 > 0$ small enough such that

$$I_{a^2} < 0 \quad \text{for any } a \in (0, a_0].$$

Let $\bar{a} \in \left(a_0, \sqrt{2}a_0\right]$. For every $a \in (a_0, \bar{a}]$, we have $a^2 - a_0^2 \leq \bar{a}^2 - a_0^2 \leq 2a_0^2 - a_0^2 = a_0^2$ and then we deduce from Lemma 2.11 that

$$I_{a^2} \leq I_{a_0^2} + I_{a^2-a_0^2} < 0,$$

which implies that $I_{a^2} < 0$ for any $a \in (0, \bar{a}]$. Thus, for given $\kappa > 0$, by iterating the above procedure, it follows that $I_{a^2} < 0$ for every $a > 0$ and (i) holds.

(ii) When $3 < p < \frac{10}{3}$, fix $u \in S_1$ and $\beta = -2$ such that $\left(1 - \frac{3}{2}\beta\right) p + 3\beta = 4p - 6 > 2 - 2\beta = 4 - \beta = 6$, then $u^a \in S_a$ and as $a \to \infty$, we have

$$I(u^a) < \frac{a^6}{2} A(u) + \frac{a^6}{4} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{1}{|x-y|} u^2(x) u^2(y) \mathrm{d}x\mathrm{d}y - \frac{a^{4p-6}}{p} C(u) = \mathcal{J}_0(u_a) \to -\infty.$$

Thus, for given $\kappa > 0$, it follows that there exists $a_1 > 0$ such that if $a > a_1$, then

$$I_{a^2} \leq I(u^a) < \mathcal{J}_0(u^a) < 0,$$

and this completes the proof of Lemma 2.12. □

Lemma 2.13. *If a, $\kappa > 0$ and $\{u_n\} \subset S_a$ is a minimizing sequence for I_{a^2} such that, up to a subsequence, $u_n \rightharpoonup \bar{u}$ in $H^1(\mathbb{R}^3)$ as $n \to \infty$, then, up to translation, the following statements hold:*

(i) For any $a > 0$ and $2 < p < 3$, $\bar{u} \neq 0$.

(ii) For any $a \in (a_1, \infty)$ and $3 < p < \frac{10}{3}$, $\bar{u} \neq 0$, where a_1 is obtained in Lemma 2.12.

Proof. It follows from Lemma 2.10 that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$. Let

$$\delta := \lim_{n\to\infty} \left(\sup_{y \in \mathbb{R}^3} \int_{B(y,1)} |u_n|^2 \, \mathrm{d}x \right).$$

If $\delta = 0$, from [29, Lemma I.1] or [36, Lemma 1.21], we conclude that $u_n \to 0$ in $L^p(\mathbb{R}^3)$ as $n \to \infty$ with $p \in (2, 6)$. Hence, from Lemma 2.12-(i), for any $a > 0$ and $2 < p < 3$, it holds that

$$0 \leq \lim_{n\to\infty} I(u_n) = \lim_{n\to\infty} \left(\frac{1}{2} A(u_n) + \frac{1}{4} B(u_n) \right) = I_{a^2} < 0,$$

which leads to a contradiction. Similarly, for any $a \in (a_1, \infty)$ and $3 < p < \frac{10}{3}$, we also obtain a contradiction. Thus, we deduce that $\delta > 0$ in both scenarios (i) and (ii) and then there exists a sequence $\{y_n\} \subset \mathbb{R}^3$ such that

$$\int_{B(y_n,1)} |u_n|^2 \, \mathrm{d}x \geq \frac{\delta}{2} > 0.$$

Let $\tilde{u}_n(\cdot) := u_n(\cdot + y_n)$. Then $\{\tilde{u}_n\} \subset S_a$ is also a bounded minimizing sequence for I_{a^2} and

$$\int_{B(0,1)} |\tilde{u}_n|^2 \, \mathrm{d}x \geq \frac{\delta}{2} > 0 \quad \text{for } n \in \mathbb{N}^+ \text{ large enough.}$$

Hence, it follows that conclusions (i) and (ii) hold. □

Lemma 2.14. *Let $p \in (2, \frac{10}{3})$ and $\{u_n\} \subset S_a$ be a minimizing sequence for I_{a^2} such that, up to a subsequence, $u_n \rightharpoonup \bar{u}$ in $H^1(\mathbb{R}^3)$ as $n \to \infty$ with $\bar{u} \in H^1(\mathbb{R}^3) \setminus \{0\}$. Then the C^1 functional T satisfies (2.27), (2.28) and (2.30). If the strong subadditivity inequality (2.29) holds, then T also satisfies (2.31).*

Proof. From Lemma 2.10 and Lemma 2.6-(i), we know that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$ and $\|\phi_{u_n}\|_{H^1} \leq C\|u_n\|^2_{H^1}$. Therefore, there exists $\bar{u} \in H^1(\mathbb{R}^3)$ such that, up to a subsequence, $u_n \rightharpoonup \bar{u}$ in $H^1(\mathbb{R}^3)$, $u_n \to \bar{u}$ in $L^r_{\text{loc}}(\mathbb{R}^3)$ with $r \in [1,6)$ and $u_n \to \bar{u}$ a.e. in $\mathbb{R}^3$ as $n \to \infty$. From the conclusions (ii) and (v) of Lemma 2.6, we deduce that $\{\phi_{u_n}\}$ is bounded in $H^1(\mathbb{R}^3)$ and there exists a unique $\phi_{\bar{u}} \in H^1(\mathbb{R}^3)$ solving (2.17) in the weak sense such that, up to a subsequence, $\phi_{u_n} \rightharpoonup \phi_{\bar{u}}$ in $H^1(\mathbb{R}^3)$.

We now introduce the following notations:

$$
\begin{aligned}
&G^\kappa(x,y) := G^\kappa(x-y) = \frac{e^{-\kappa|x-y|}}{|x-y|}, \qquad A := \int_{\mathbb{R}^3} \phi_{\bar{u}}\bar{u}^2 \mathrm{d}x,\\
&I_n^{(1)} := \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} G^\kappa(x,y)u_n^2(y)\bar{u}^2(x)\mathrm{d}x\mathrm{d}y = \int_{\mathbb{R}^3} \phi_{u_n}\bar{u}^2\mathrm{d}x,\\
&I_n^{(2)} := \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} G^\kappa(x,y)u_n(y)\bar{u}(y)u_n(x)\bar{u}(x)\mathrm{d}x\mathrm{d}y,\\
&I_n^{(3)} := \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} G^\kappa(x,y)u_n^2(y)u_n(x)\bar{u}(x)\mathrm{d}x\mathrm{d}y = \int_{\mathbb{R}^3} \phi_{u_n}u_n\bar{u}\mathrm{d}x,\\
&I_n^{(4)} := \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} G^\kappa(x,y)u_n(y)\bar{u}(y)\bar{u}^2(x)\mathrm{d}x\mathrm{d}y.
\end{aligned}
$$

From direct calculation we deduce that

$$
B(u_n-\bar{u}) - B(u_n) + B(\bar{u}) = 2I_n^{(1)} + 4I_n^{(2)} - 4I_n^{(3)} - 4I_n^{(4)} + 2A.
$$

Next, we will show that $\lim_{n\to\infty} I_n^{(i)} = A$, $i = 1,2,3,4$. Since $\phi_{u_n} \rightharpoonup \phi_{\bar{u}}$ in $H^1(\mathbb{R}^3)$, we infer that $\phi_{u_n} \rightharpoonup \phi_{\bar{u}}$ in $L^2(\mathbb{R}^3)$. It follows from $\bar{u}^2 \in L^2(\mathbb{R}^3)$ that $\lim_{n\to\infty} I_n^{(1)} = A$. To prove $\lim_{n\to\infty} I_n^{(2)} = A$, let

$$
\hat{v}_n(x) := \int_{\mathbb{R}^3} \frac{e^{-\kappa|x-y|}}{|x-y|} u_n(y)\bar{u}(y)\mathrm{d}y.
$$

Then we claim that $\hat{v}_n \to \phi_{\bar{u}}$ a.e. in $\mathbb{R}^3$ as $n \to \infty$. Indeed, for any $R \geq 1$, by Hölder's inequality, $e^{-t} \leq 1$ and $e^t \geq \frac{t^2}{2}$ for $t \geq 0$, we deduce that

$$
\begin{aligned}
|\hat{v}_n(x) - \phi_{\bar{u}}(x)| &\leq \|u_n-\bar{u}\|_{L^3(B_R(x))}\|\bar{u}\|_{L^6(B_R(x))}\left(\int_{|y-x|\leq R}\frac{e^{-2\kappa|x-y|}}{|x-y|^2}\mathrm{d}y\right)^{\frac{1}{2}}\\
&\quad + \|u_n-\bar{u}\|_{L^3(B_R^c(x))}\|\bar{u}\|_{L^6(B_R^c(x))}\left(\int_{|y-x|\geq R}\frac{1}{|x-y|^2 e^{2\kappa|x-y|}}\mathrm{d}y\right)^{\frac{1}{2}}\\
&\leq \|u_n-\bar{u}\|_{L^3(B_R(x))}\|\bar{u}\|_{L^6(\mathbb{R}^3)}\left(\int_{|y-x|\leq R}\frac{1}{|x-y|^2}\mathrm{d}y\right)^{\frac{1}{2}}\\
&\quad + \|u_n-\bar{u}\|_{L^3(B_R^c(x))}\|\bar{u}\|_{L^6(\mathbb{R}^3)}\left(\int_{|y-x|\geq R}\frac{1}{2\kappa^2|x-y|^4}\mathrm{d}y\right)^{\frac{1}{2}}.
\end{aligned}
$$

which implies that $\hat{v}_n \to \phi_{\bar{u}}$ a.e. in $\mathbb{R}^3$ by first letting $n \to \infty$ (for fixed R) and then letting $R \to \infty$. Since $\{u_n\} \subset S_a$ and $\bar{u} \in H^1(\mathbb{R}^3)$, from Hölder's inequality and Sobolev embedding theorem, we have

$$
\|u_n\bar{u}\|_{\frac{6}{5}} \leq C\|u_n\|_{\frac{12}{5}}\|\bar{u}\|_{\frac{12}{5}} \leq C\|u_n\|_{H^1}\|\bar{u}\|_{H^1} < \infty,
$$

namely, $u_n\bar{u} \in L^{\frac{6}{5}}(\mathbb{R}^3)$. Let $\alpha = 2$, $N = 3$, $f = |u_n\bar{u}|$, $q = 6$ and $p = \frac{6}{5}$ in Lemma 2.3. Then, we have

$$|\hat{v}_n| \leq \int_{\mathbb{R}^3} \frac{e^{-\kappa|x-y|}}{|x-y|} |u_n(y)\bar{u}(y)| \, \mathrm{d}y \leq \int_{\mathbb{R}^3} \frac{|u_n(y)\bar{u}(y)|}{|x-y|} \mathrm{d}y := J_2(|u_n\bar{u}|)(x),$$

and thus it holds that

$$\|\hat{v}_n\|_6 \leq \|J_2(|u_n\bar{u}|)\|_6 \leq C \|u_n\bar{u}\|_{\frac{6}{5}} \leq C \|u_n\|_{H^1} \|\bar{u}\|_{H^1} < \infty,$$

namely, $\hat{v}_n \in L^6(\mathbb{R}^3)$. It follows from Hölder's inequality and Sobolev embedding theorem that

$$\|\hat{v}_n u_n\|_2 \leq \|\hat{v}_n\|_6 \|u_n\|_3 \leq C \|u_n\|_{H^1}^2 \|\bar{u}\|_{H^1} < \infty.$$

Since $\hat{v}_n \to \phi_{\bar{u}}$, $u_n \to \bar{u}$ a.e. in $\mathbb{R}^3$, we have $\hat{v}_n u_n \to \phi_{\bar{u}}\bar{u}$ a.e. in $\mathbb{R}^3$ and $\hat{v}_n u_n \rightharpoonup \phi_{\bar{u}}\bar{u}$ in $L^2(\mathbb{R}^3)$. Due to $\bar{u} \in L^2(\mathbb{R}^3)$, we get $\lim_{n\to\infty} I_n^{(2)} = A$. Let $\psi = \bar{u}$ in Lemma 2.6-(v), and thus it holds that $\lim_{n\to\infty} I_n^{(3)} = A$. Since $\hat{v}_n \to \phi_{\bar{u}}$ a.e. in $\mathbb{R}^3$ and $\hat{v}_n \in L^6(\mathbb{R}^3)$, we get $\hat{v}_n \rightharpoonup \phi_{\bar{u}}$ in $L^6(\mathbb{R}^3)$ as $n \to \infty$. By the fact that $\bar{u}^2 \in L^{\frac{6}{5}}(\mathbb{R}^3)$, we deduce that $\lim_{n\to\infty} I_n^{(4)} = A$. In summary, we have

$$B(u_n - \bar{u}) + B(\bar{u}) = B(u_n) + o_n(1).$$

Moreover, it follows from the Brezis–Lieb lemma that

$$C(u_n - \bar{u}) + C(\bar{u}) = C(u_n) + o_n(1),$$

and thus (2.27) holds.

From Lemma 2.6-(iii) and Hölder's inequality, we infer that

$$B(u_n) = \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 \mathrm{d}x \leq C \|u_n\|_4^4 \leq C \|u_n\|_{H^1}^4 < \infty.$$

It follows from Brezis-Lieb lemma that $\|u_n - \bar{u}\|_2^2 + \|\bar{u}\|_2^2 = \|u_n\|_2^2 + o_n(1) = a^2 + o_n(1)$. Therefore, from Proposition 2.8-(iv), we have $\alpha_n = \left(a^2 - \|\bar{u}\|_2^2\right) \|u_n - \bar{u}\|_2^{-2} \to 1$ as $n \to \infty$, and thus we obtain that

$$\begin{aligned} B(\alpha_n(u_n - \bar{u})) - B(u_n - \bar{u}) &= \left(\alpha_n^4 - 1\right) B(u_n - \bar{u}) = o_n(1), \\ C(\alpha_n(u_n - \bar{u})) - C(u_n - \bar{u}) &= \left(\alpha_n^p - 1\right) C(u_n - \bar{u}) = o_n(1), \end{aligned}$$

thus, (2.28) holds. Since $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$, from Lemma 2.6-(iii), we conclude that

$$\langle T'(u_n), u_n \rangle = \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 \mathrm{d}x - \|u_n\|_p^p \leq \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 \mathrm{d}x \leq C \|u_n\|_4^4 \leq C \|u_n\|_{H^1}^4 < \infty,$$

namely, (2.30) holds.

Since (2.27) and (2.28) hold, if the strong subadditivity inequality (2.29) also holds, from Proposition 2.8, we conclude that $\bar{u} \in S_a$, namely, $\|\bar{u}\|_2^2 = a^2$. Thus, $\|u_n - \bar{u}\|_2 = o_n(1)$ and $\|u_n - u_m\|_2 = o_n(1)$ as $n, m \to \infty$. Notice that $2 < p < \frac{10}{3} < 6$, from Lemma 2.1 and interpolation inequality, we have that

$$\|u_n - u_m\|_p \leq \|u_n - u_m\|_2^{\alpha} \|u_n - u_m\|_6^{1-\alpha} \leq \mathcal{S}^{-\frac{1-\alpha}{2}} \|u_n - u_m\|_2^{\alpha} \|\nabla u_n - \nabla u_m\|_2^{1-\alpha} = o_n(1),$$

where $\mathcal{S}$ denotes the sharp constant and α satisfies $\frac{\alpha}{2}+\frac{1-\alpha}{6}=\frac{1}{p}$, therefore, $\|u_n-u_m\|_p = o_n(1)$. From Hölder's inequality and Sobolev's embedding theorem, we have

$$\begin{aligned}\int_{\mathbb{R}^3}|u_n|^{p-1}|u_n-u_m|\,\mathrm{d}x &\le \left(\int_{\mathbb{R}^3}\left||u_n|^{p-1}\right|^{\frac{p}{p-1}}\mathrm{d}x\right)^{\frac{p-1}{p}}\|u_n-u_m\|_p\\ &= \|u_n\|_p^{p-1}\|u_n-u_m\|_p = o_n(1),\end{aligned}$$

and thus

$$\begin{aligned}&\left|\int_{\mathbb{R}^3}\left(|u_n|^{p-2}u_n-|u_m|^{p-2}u_m\right)(u_n-u_m)\,\mathrm{d}x\right|\\ &\quad\le \int_{\mathbb{R}^3}|u_n|^{p-1}|u_n-u_m|\,\mathrm{d}x+\int_{\mathbb{R}^3}|u_m|^{p-1}|u_n-u_m|\,\mathrm{d}x\\ &\quad\le \|u_n\|_p^{p-1}\|u_n-u_m\|_p+\|u_m\|_p^{p-1}\|u_n-u_m\|_p = o_n(1).\end{aligned}$$

Hence, we have $\langle C'(u_n)-C'(u_m),u_n-u_m\rangle = o_n(1)$ as $n,m\to\infty$. From $\phi_{u_n}\in H^1(\mathbb{R}^3)$, Lemma 2.6-$(i)$ and Sobolev's embedding theorem, we infer that $\phi_{u_n}\in L^6(\mathbb{R}^3)$ and $\|\phi_{u_n}\|_6\le C\|\phi_{u_n}\|_{H^1}\le C\|u_n\|_{H^1}^2$. By Hölder's inequality and $\|u_n-u_m\|_p=o_n(1)$ with $p\in\left(2,\frac{10}{3}\right)$, we have

$$\int_{\mathbb{R}^3}\phi_{u_n}u_n|u_n-u_m|\,\mathrm{d}x\le C\|\phi_{u_n}\|_6\|u_n\|_2\|u_n-u_m\|_3\le C\|u_n\|_{H^1}^2\|u_n\|_2\|u_n-u_m\|_3=o_n(1).$$

Therefore, as $n,m\to\infty$, we have $\langle B'(u_n)-B'(u_m),u_n-u_m\rangle=o_n(1)$, and thus

$$\langle T'(u_n)-T'(u_m),u_n-u_m\rangle=o_n(1),$$

namely, (2.31) holds. □

3 Proofs of main results

Based on the preparatory lemmas presented in Section 2, we now proceed to prove the main results.

3.1 Proof of Theorem 1.1

To prove Theorem 1.1, we need to demonstrate the following two crucial lemmas.

Lemma 3.1. *Assume $2<p<\frac{10}{3}$ and $a,\kappa>0$, let $\{u_n\}\subset S_a$ be a minimizing sequence for I_{a^2}. Then the function $a\mapsto I_{a^2}$ is continuous and $I_{a^2}/a^2\to 0$ as $a\to 0^+$.*

Proof. When $2<p<\frac{10}{3}$, we have $\frac{3(p-2)}{2}<2$. Now arguing as in Step 4 in the proof of Theorem 4.1 in [7], we can complete the proof. □

Lemma 3.2. *If $a,\kappa>0$, $2<p<3$ and $M(a)$ is defined in Proposition 2.8, then for any $u\in M(a)$, there exists $\beta\in\mathbb{R}$ such that for any $\theta>0$,*

$$h_\beta^u(\theta):=I(u^\theta)-\theta^2 I(u)$$

is differentiable and $(h_u)'(1)\neq 0$, where $u^\theta(x):=\theta^{1-\frac{3}{2}\beta}u\left(\frac{x}{\theta^\beta}\right)$.

Proof. For any $u \in M(a)$ and $\beta \in \mathbb{R}$, it follows from (1.7) and the definition of u^θ that

$$\begin{aligned} h_\beta^u(\theta) &= I(u^\theta) - \theta^2 I(u) \\ &= \frac{1}{2}\left(\theta^{(2-2\beta)} - \theta^2\right)\|\nabla u\|_2^2 + \frac{1}{4}\left(\theta^{4-\beta}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{e^{-\kappa\theta^\beta|x-y|}}{|x-y|}u^2(x)u^2(y)\mathrm{d}x\mathrm{d}y - \theta^2\int_{\mathbb{R}^3}\phi_u u^2\mathrm{d}x\right) \\ &\quad - \frac{1}{p}\left(\theta^{(1-\frac{3}{2}\beta)p+3\beta} - \theta^2\right)\|u\|_p^p. \end{aligned}$$

Thus, $h_\beta^u(\theta)$ is differentiable and

$$\begin{aligned} \left(h_\beta^u\right)'(1) &= -\beta\|\nabla u\|_2^2 + \frac{1}{4}\left((2-\beta)\int_{\mathbb{R}^3}\phi_u u^2\mathrm{d}x - \kappa\beta\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}e^{-\kappa|x-y|}u^2(x)u^2(y)\mathrm{d}x\mathrm{d}y\right) \\ &\quad - \frac{1}{p}\left((1-\frac{3}{2}\beta)p + 3\beta - 2\right)\|u\|_p^p. \end{aligned}$$

To prove this lemma, we argue by contradiction. Assume that there exists a sequence $\{u_n\} \subset M(a)$ with $\|u_n\|_2 := \mu_n \le a$ and $\mu_n \to 0$ as $n \to \infty$ such that $\left(h_\beta^{u_n}\right)'(1) = 0$ for all $\beta \in \mathbb{R}$, then we conclude that $I(u_n) = I_{\mu_n^2} \to 0$ as $n \to \infty$ according to Lemma 3.1. Moreover, since (u_n, ϕ_{u_n}) solves system (1.1) under the constraint S_{μ_n}, from Lemma 2.7, we infer that (u_n, ϕ_{u_n}) satisfies the following Pohožaev identity

$$\|\nabla u_n\|_2^2 + \frac{1}{4}\int_{\mathbb{R}^3}\phi_{u_n}u_n^2\mathrm{d}x + \frac{\kappa}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}e^{-\kappa|x-y|}u_n^2(x)u_n^2(y)\mathrm{d}x\mathrm{d}y - \frac{3(p-2)}{2p}\|u_n\|_p^p = 0. \tag{3.1}$$

Let $\beta = \frac{2}{3}$, from $\left(h_{\frac{2}{3}}^{u_n}\right)'(1) = 0$, we deduce that

$$\frac{1}{2}\|\nabla u_n\|_2^2 = \frac{1}{4}\int_{\mathbb{R}^3}\phi_{u_n}u_n^2\mathrm{d}x - \frac{\kappa}{8}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}e^{-\kappa|x-y|}u_n^2(x)u_n^2(y)\mathrm{d}x\mathrm{d}y. \tag{3.2}$$

It follows from (3.2), Lemma 2.6 and Lemma 2.1 that

$$\|\nabla u_n\|_2^2 \le \frac{1}{2}\int_{\mathbb{R}^3}\phi_{u_n}u_n^2\mathrm{d}x \le C\|u_n\|_4^4 \le C\|\nabla u_n\|_2^3\|u_n\|_2. \tag{3.3}$$

Let $\beta = 0$, from $(h_0^{u_n})'(1) = 0$, we have

$$\|u_n\|_p^p = \frac{p}{2(p-2)}\int_{\mathbb{R}^3}\phi_{u_n}u_n^2\mathrm{d}x \le C(p)\|\nabla u_n\|_2^3\|u_n\|_2. \tag{3.4}$$

By (3.3), (3.4), $\|u_n\|_2 = \mu_n \to 0$ and $I(u_n) = I_{\mu_n^2} \to 0$ as $n \to \infty$, we infer that

$$\|\nabla u_n\|_2^2,\ \int_{\mathbb{R}^3}\phi_{u_n}u_n^2\mathrm{d}x,\ \|u_n\|_p^p \to 0 \quad \text{as } n \to \infty. \tag{3.5}$$

It follows from Lemma 2.2 that

$$\begin{aligned} \int_{\mathbb{R}^3}\phi_{u_n}u_n^2\mathrm{d}x &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{e^{-\kappa|x-y|}}{|x-y|}u_n^2(x)u_n^2(y)\mathrm{d}x\mathrm{d}y \\ &\le \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{1}{|x-y|}u_n^2(x)u_n^2(y)\mathrm{d}x\mathrm{d}y \le C\|u_n\|_{\frac{12}{5}}^4. \end{aligned} \tag{3.6}$$

Similar to the Cases (a)–(d) of Step 5 in the proof of Theorem 4.1 in [6], when $2 < p \leq \frac{8}{3}$, by applying (3.3)–(3.6), interpolation inequality and Lemma 2.1, we obtain a contradiction in all four cases. Then, we discuss the case when $\frac{8}{3} < p < 3$. Substituting (3.2) into Pohožaev identity (3.1), we obtain that

$$\frac{1}{4}\int_{\mathbb{R}^3}\phi_{u_n}u_n^2\mathrm{d}x = \frac{p-2}{2p}\|u_n\|_p^p, \tag{3.7}$$

and thus we have

$$I(u_n) = \frac{1}{2}\|\nabla u_n\|_2^2 + \frac{1}{4}\int_{\mathbb{R}^3}\phi_{u_n}u_n^2\mathrm{d}x - \frac{1}{p}\|u_n\|_p^p = \frac{1}{2}\|\nabla u_n\|_2^2 - \frac{4-p}{2p}\|u_n\|_p^p. \tag{3.8}$$

Let $\{w_n\}, \{v_n\} \subset M(a)$ such that $w_n := v_n^{\mu_n} = \mu_n^{1-\frac{3}{2}\beta}v_n\left(\frac{x}{\mu_n^\beta}\right)$, where $\beta \in \mathbb{R}$, $\|v_n\|_2 = 1$ and $\|w_n\|_2 = \mu_n \to 0$ as $n \to \infty$. Moreover, let $C(p) > 0$ such that when $\|\nabla v_n\|_2^2$ is small enough, it holds that $\frac{1}{2}\|\nabla v_n\|_2^2 - \frac{4-p}{2p}\|v_n\|_p^p < -C(p)$. Now, let $\beta = \frac{2(2-p)}{10-3p}$ such that $-2\beta = (1-\frac{3}{2}\beta)p + 3\beta - 2 = \frac{4(p-2)}{10-3p}$. From (3.8), we have

$$\begin{aligned}\frac{I_{\mu_n^2}}{\mu_n^2} \leq \frac{I(w_n)}{\mu_n^2} &= \frac{\mu_n^{-2\beta}}{2}\|\nabla v_n\|_2^2 - \frac{4-p}{2p}\mu_n^{(1-\frac{3}{2}\beta)p+3\beta-2}\|v_n\|_p^p \\ &= \mu_n^{\frac{4(p-2)}{10-3p}}\left(\frac{1}{2}\|\nabla v_n\|_2^2 - \frac{4-p}{2p}\|v_n\|_p^p\right) \leq -C(p)\mu_n^{\frac{4(p-2)}{10-3p}}.\end{aligned} \tag{3.9}$$

From (3.6), (3.8) and interpolation inequality, we infer that

$$\|u_n\|_p^p = \frac{p}{2(p-2)}\int_{\mathbb{R}^3}\phi_{u_n}u_n^2\mathrm{d}x \leq C(p)\|u_n\|_{\frac{12}{5}}^4 \leq C(p)\|u_n\|_2^{4\alpha}\|u_n\|_p^{4(1-\alpha)} = C(p)\mu_n^{4\alpha}\|u_n\|_p^{4(1-\alpha)},$$

where $\alpha := \frac{5p-12}{6(p-2)}$. Since $\frac{8}{3} < p < 3$, namely, $p > 4(1-\alpha)$, we cannot get a contradiction with $\|u_n\|_p \to 0$ as $n \to \infty$. But we deduce from the above inequality that

$$\|u_n\|_p^p \leq C(p)\mu_n^{\frac{2(5p-12)}{3p-8}} \quad \text{and} \quad \frac{I(u_n)}{\mu_n^2} \geq -\frac{1}{p}\frac{\|u_n\|_p^p}{\mu_n^2} \geq -C(p)\mu_n^{\frac{4(p-2)}{3p-8}}. \tag{3.10}$$

Combining (3.9) and (3.10), we infer that

$$-C(p)\mu_n^{\frac{4(p-2)}{3p-8}} \leq \frac{I_{\mu_n^2}}{\mu_n^2} \leq -C(p)\mu_n^{\frac{4(p-2)}{10-3p}},$$

which contradicts $\mu_n \to 0$ as $n \to \infty$ since $\frac{4(p-2)}{3p-8} > \frac{4(p-2)}{10-3p}$ for any $\frac{8}{3} < p < 3$. In summary, for all the cases where $2 < p < 3$, we have derived contradictions. □

Proof of Theorem 1.1. Let $a_0 > 0$ be given by Lemma 2.12 and fix $a \in (0, a_0)$. Let $\{u_n\} \subset S_a$ be a minimizing sequence for $I_{a^2} < 0$ and then from Ekeland's variational principle ([36, Theorem 2.4]), we can know that $\{u_n\} \subset S_a$ is also a Palais–Smale sequence for I. From Lemma 2.10, we know that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$. It follows from Lemma 2.13 that there exists $\bar{u} \in H^1(\mathbb{R}^3)\backslash\{0\}$ such that $\|\bar{u}\|_2 \in (0, a]$, then, up to a subsequence, $u_n \rightharpoonup \bar{u}$ as $n \to \infty$. From Lemma 2.14, we know that (2.27), (2.28), (2.30) and (2.31) are satisfied. Further, Lemmas 2.11, 3.1, 3.2 ensure that the assumptions of Proposition 2.8 hold. Then, it follows from Proposition 2.8 that, up to a subsequence, $u_n \to \bar{u}$ in $H^1(\mathbb{R}^3)$ as $n \to \infty$. Moreover,

$\bar{u} \in S_a$ and $I(\bar{u}) = I_{a^2} < 0$. Therefore, $\bar{u}$ is the critical point of the constrained functional $I|_{S_a}$, that is, $\bar{u}$ is the normalized solution to system (1.5). From Section 2.3 and Lemma 2.7, we know that $\bar{u}$ satisfies the Pohožaev identity (2.20). Since $P(\bar{u}) = P(|\bar{u}|) = 0$, it follows that $\bar{u}$ is nonnegative. From Section 2.4 and the strong maximum principle, we conclude that $\bar{u}$ is a classical solution and $\bar{u} > 0$ in $\mathbb{R}^3$. Moreover, there exists a Lagrange multiplier $\lambda \in \mathbb{R}$ such that (2.21) holds.

When $2 < p < 3$, since $I_{a^2} < 0$ for any $a \in (0, a_0)$ from Lemma 2.12, we claim that there exists some constant $\sigma > 0$ such that $\|\nabla \bar{u}\|_2 > \sigma$. Indeed, if $\|\nabla \bar{u}\|_2 = 0$, from the fact that $\int_{\mathbb{R}^3} \phi_{\bar{u}} \bar{u}^2 \mathrm{d}x \le C \|\bar{u}\|_4^4$ and Lemma 2.1, we obtain that $\int_{\mathbb{R}^3} \phi_{\bar{u}} \bar{u}^2 \mathrm{d}x = \|\bar{u}\|_p^p = 0$ and thus $I_{a^2} = 0$, which contradicts $I(\bar{u}) = I_{a^2} < 0$. When $2 < p \le \frac{12}{5}$, from (2.21), Pohožaev identity (2.20) and $\bar{u} > 0$ in $\mathbb{R}^3$, we infer that

$$\begin{aligned}\lambda \|\bar{u}\|_2^2 &= -\|\nabla \bar{u}\|_2^2 - \int_{\mathbb{R}^3} \phi_{\bar{u}} \bar{u}^2 \mathrm{d}x + \|\bar{u}\|_p^p \\ &= 3\|\nabla \bar{u}\|_2^2 + \kappa \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{-\kappa|x-y|} \bar{u}^2(x) \bar{u}^2(y) \mathrm{d}x\mathrm{d}y + \frac{12-5p}{p} \|\bar{u}\|_p^p > 3\sigma^2 > 0.\end{aligned}$$

When $\frac{12}{5} < p < 3$, let $\sigma := \left(\frac{5p-12}{3p} C_{3,p}^p a_0^{\frac{6-p}{2}} \right)^{\frac{2}{10-3p}} > 0$ and we have

$$\begin{aligned}\lambda \|\bar{u}\|_2^2 &= 3\|\nabla \bar{u}\|_2^2 + \kappa \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{-\kappa|x-y|} \bar{u}^2(x) \bar{u}^2(y) \mathrm{d}x\mathrm{d}y - \frac{5p-12}{p} \|\bar{u}\|_p^p \\ &\ge \|\nabla \bar{u}\|_2^{\frac{3(p-2)}{2}} \left(3\|\nabla \bar{u}\|_2^{\frac{10-3p}{2}} - \frac{5p-12}{p} C_{3,p}^p a_0^{\frac{6-p}{2}} \right) > 0.\end{aligned}$$

Thus, it holds that $\lambda > 0$ when $2 < p < 3$ and we have completed the proof. □

3.2 Proof of Theorem 1.2

Lemma 3.3. *Let $a, \kappa > 0$, $3 < p < \frac{10}{3}$ and a_1 be obtained in Lemma 2.12. Moreover, let $\{u_n\} \subset S_a$ be a minimizing sequence for I_{a^2}. Then, for any $0 < a_1 < \mu < a$, the following strong subadditivity inequality holds*

$$I_{a^2} < I_{\mu^2} + I_{a^2-\mu^2}.$$

Proof. Since $\{u_n\} \subset S_a$ is bounded, we claim that there exist positive constants k_1, k_2 such that

$$0 < k_1 \le A(u_n) = \|\nabla u_n\|_2^2 \le k_2. \tag{3.11}$$

Indeed, if $A(u_n) = o_n(1)$, from Lemma 2.6-(i) and Lemma 2.1, we infer that $B(u_n) = C(u_n) = o_n(1)$ and $I_{a^2} = 0$, which contradicts $I_{a^2} < 0$ from Lemma 2.12-(ii).

Let $u_n^\theta(x) := \theta^{1-\frac{3}{2}\beta} u_n\left(\frac{x}{\theta^\beta}\right)$ for a.e. $x \in \mathbb{R}^3$, then $\|u_n^\theta\|_2 = \theta \|u_n\|_2 = \theta a$. Fix $\beta = -2$ and it holds that

$$\begin{aligned}I\left(u_n^\theta\right) &= \frac{\theta^6}{2} A(u_n) + \frac{\theta^6}{4} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{e^{-\kappa\theta^{-2}|x-y|}}{|x-y|} u_n^2(x) u_n^2(y) \mathrm{d}x\mathrm{d}y - \frac{\theta^{4p-6}}{p} C(u_n) \\ &\le \frac{\theta^6}{2} A(u_n) + \frac{\theta^6}{4} N(u_n) - \frac{\theta^{4p-6}}{p} C(u_n) = \mathcal{J}_0\left(u_n^\theta\right) \\ &= \theta^2 \left(I(u_n) + \frac{1}{2}\left(\theta^4 - 1\right) A(u_n) + \frac{1}{4}\left(\theta^4 N(u_n) - B(u_n)\right) - \frac{1}{p}\left(\theta^{4p-8} - 1\right) C(u_n) \right) \\ &=: \theta^2 \left(I(u_n) + f(\theta, u_n) \right), \end{aligned} \tag{3.12}$$

where $N(u_n) := \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{1}{|x-y|}u_n^2(x)u_n^2(y)\mathrm{d}x\mathrm{d}y$ and $\mathcal{J}_0$ is defined in (1.14). When $a > a_1$, it follows from [7, Lemma 3.2] that

$$\mathcal{J}_0(u_n) = \frac{1}{2}A(u_n) + \frac{1}{4}N(u_n) - \frac{1}{p}C(u_n) < 0. \tag{3.13}$$

Therefore, when $3 < p < \frac{10}{3}$, it follows from (3.11) and (3.13) that

$$f'(\theta, u_n)|_{\theta=1} = 2A(u_n) + N(u_n) - \frac{4p-8}{p}C(u_n) < (6-2p)\,A(u_n) + (3-p)\,N(u_n) < 0. \tag{3.14}$$

Since $(4p-8)(4p-9) > 12$ when $3 < p < \frac{10}{3}$, from (3.11) and (3.13), we derive that

$$\begin{aligned} f''(\theta, u_n)|_{\theta=1} &= 6A(u_n) + 3N(u_n) - \frac{(4p-8)(4p-9)}{p}C(u_n) \\ &< \left(6 - \frac{(4p-8)(4p-9)}{2}\right)A(u_n) + \left(3 - \frac{(4p-8)(4p-9)}{4}\right)N(u_n) < 0. \end{aligned} \tag{3.15}$$

Since $f(1, u_n) = \frac{1}{4}\left(N(u_n) - B(u_n)\right) > 0$ and $f(\theta, u_n) \to -\infty$ as $\theta \to \infty$, we conclude from (3.14) and (3.15) that there exists $\theta_1 > 1$ such that $f(\theta_1, u_n) = 0$. Hence, for any $\theta > \theta_1 > 1$, we get $f(\theta, u_n) < 0$ and from (3.12), we have

$$I_{\theta^2 a^2} \le I\left(u_n^\theta\right) < \theta^2 I(u_n) \to \theta^2 I_{a^2} \quad \text{as } n \to \infty.$$

Therefore, for any $0 < a_1 < \mu < a$, it holds that

$$I_{a^2} = \frac{\mu^2}{a^2} I_{\frac{a^2}{\mu^2}\mu^2} + \frac{a^2-\mu^2}{a^2} I_{\frac{a^2}{a^2-\mu^2}(a^2-\mu^2)} < I_{\mu^2} + I_{a^2-\mu^2}.$$

This completes the proof of Lemma 3.3. □

Proof of Theorem 1.2. Let $a_1 > 0$ be given by Lemma 2.12 and fix $a > a_1$. Let $\{u_n\} \subset S_a$ be a minimizing sequence for $I_{a^2} < 0$ and then from Ekeland's variational principle ([36, Theorem 2.4]), we can know that $\{u_n\} \subset S_a$ is also a Palais–Smale sequence for I. From Lemma 2.10, we know that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$. It follows from Lemma 2.13 that there exists $\bar{u} \in H^1(\mathbb{R}^3)\setminus\{0\}$ such that $\|\bar{u}\|_2 \in (0, a]$, then, up to a subsequence, $u_n \rightharpoonup \bar{u}$ as $n \to \infty$. From Lemma 2.14, we know that (2.27), (2.28), (2.30) and (2.31) are satisfied. Lemma 3.3 guarantees that the strong subadditivity inequality (2.29) holds. Then, it follows from Proposition 2.9 that, up to a subsequence, $u_n \to \bar{u}$ in $H^1(\mathbb{R}^3)$ as $n \to \infty$. Moreover, $\bar{u} \in S_a$ and $I(\bar{u}) = I_{a^2} < 0$ for any $a \in (a_1, \infty)$ from Lemma 2.12. Therefore, $\bar{u}$ is the critical point of the constrained functional $I|_{S_a}$, namely, $\bar{u}$ is the normalized solution to system (1.5). Similar to the proof of Theorem 1.1, we conclude that $\bar{u}$ is a positive classical solution and the corresponding Lagrange multiplier $\lambda > 0$ when $3 < p < \frac{10}{3}$. This completes the proof of Theorem 1.2. □

3.3 Proof of Theorem 1.3

In Section 2.6, we have introduced the scaling $u^\theta(x) := \theta^{1-\frac{3}{2}\beta}u\left(\frac{x}{\theta^\beta}\right)$ for a.e. $x \in \mathbb{R}^3$ and $\left\|u^\theta\right\|_2 = \theta\|u\|_2$. Specifically, when $2 < p < 3$ and $u \in H^1(\mathbb{R}^3)$, let $\beta = \frac{4-2p}{10-3p}$ and $u^a(x) :=$

$a^{\frac{4}{10-3p}}u\left(a^{\frac{2p-4}{10-3p}}x\right)$. Then for any $u \in S_1$, we have $\|u^a\|_2 = a$ and

$$I(u^a) = a^{\frac{12-2p}{10-3p}}\left(\frac{1}{2}\|\nabla u\|_2^2 + \frac{a^{\alpha(p)}}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{e^{-\kappa a^{\frac{4-2p}{10-3p}}|x-y|}}{|x-y|}u^2(x)u^2(y)\mathrm{d}x\mathrm{d}y - \frac{1}{p}\|u\|_p^p\right)$$
$$=: a^{\frac{12-2p}{10-3p}}\mathcal{I}(u),$$

where $\alpha(p) := \frac{8(3-p)}{10-3p} > 0$. Therefore, the minimization problem (1.8) is equivalent to the following one

$$\mathcal{I}_{a^2} := \inf_{u\in S_1}\mathcal{I}(u).$$

According to the arguments in [13], we introduce the following minimization problem

$$K_{a,p} := \inf_{u\in S_1}\mathcal{E}_{a,p}(u),$$

where

$$\mathcal{E}_{a,p}(u) := \frac{1}{2}\|\nabla u\|_2^2 + \frac{a^{\alpha(p)}}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{e^{-\kappa a^{\frac{4-2p}{10-3p}}|x-y|}}{|x-y|}u^2(x)u^2(y)\mathrm{d}x\mathrm{d}y - \frac{1}{p}\|u\|_p^p.$$

We denote by $\mathcal{N}_{a,p}$ the corresponding set of minimizers

$$\mathcal{N}_{a,p} := \left\{u \in S_1 : \mathcal{E}_{a,p}(u) = K_{a,p}\right\}.$$

Thus, Theorem 1.3 is equivalent to the following proposition.

Proposition 3.4. *Let $2 < p < 3$ and a_0 be obtained in Theorem 1.1. Then, there exists $a_2 = a_2(p) \in (0, a_0]$ such that for any $0 < a < a_2$, every function $u \in \mathcal{N}_{a,p}$ is radially symmetric. Consequently, $\phi_u \in H_r^1(\mathbb{R}^3)$ up to translation.*

Remark 3.5. It follows from Lemma 2.6-(iv) and the implicit function argument in [13] that Proposition 3.4 holds. Therefore, the detailed proof of Theorem 1.3 will not be presented here, and readers can refer to [13].

3.4 Proof of Theorem 1.4

In this section, we shall study the limit behavior of the ground state normalized solutions $(u_\kappa, \phi_\kappa, \lambda_\kappa)$ to system (1.5) as $\kappa \to 0$.

Lemma 3.6. *([12, Lemma 3.4]) Let $p \in (2,3)$, $\kappa > 0$ and $a \in (0, a_2)$ with $a_2 > 0$ obtained in Theorem 1.3. Moreover, let $\mathcal{J}_0$ be defined in (1.14) and let*

$$I_\kappa(u_\kappa) := \frac{1}{2}\|\nabla u_\kappa\|_2^2 + \frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{e^{-\kappa|x-y|}}{|x-y|}u_\kappa^2(x)u_\kappa^2(y)\mathrm{d}x\mathrm{d}y - \frac{1}{p}\|u_\kappa\|_p^p.$$

If

$$\left\{u_\kappa \in S_a\setminus\{0\} : I_\kappa(u_\kappa) = I_{\kappa,a^2} := \inf_{u_\kappa\in S_a} I_\kappa(u_\kappa) < 0\right\}$$

is a family of minimizers for the functional I_κ, then it holds that $I_{\kappa,a^2} \to \mathcal{J}_{0,a^2} := \inf_{u\in S_a}\mathcal{J}_0(u) < 0$ as $\kappa \to 0$.

Proof. According to $\frac{e^{-\kappa|x|}}{|x|} \leq \frac{1}{|x|}$, we know that $I_{\kappa,a^2} \leq \mathcal{J}_{0,a^2}$ for any $\kappa \geq 0$. Therefore, it suffices to prove that $\mathcal{J}_{0,a^2} \leq \liminf_{\kappa\to 0} I_{\kappa,a^2}$. Given $\kappa > 0$, we have

$$\mathcal{J}_{0,a^2} \leq \mathcal{J}_0(u_\kappa) = I_\kappa(u_\kappa) + \frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{1-e^{-\kappa|x-y|}}{|x-y|} u_\kappa^2(x)u_\kappa^2(y)\mathrm{d}x\mathrm{d}y =: I_\kappa(u_\kappa) + \frac{1}{4}g(\kappa).$$

In this situation, we only need to prove that $g(\kappa) \to 0$ as $\kappa \to 0$. Fix $\kappa \in (0,1)$. It follows from a change of variable that

$$\begin{aligned} |g(\kappa)| \leq & \int_{\left\{z\in\mathbb{R}^3:|z|\leq\frac{1}{\kappa}\right\}} \frac{1-e^{-\kappa|z|}}{|z|} \int_{\mathbb{R}^3} |u_\kappa(x)|^2 |u_\kappa(x-z)|^2 \,\mathrm{d}x\mathrm{d}z \\ & + \int_{\left\{z\in\mathbb{R}^3:|z|\geq\frac{1}{\kappa}\right\}} \frac{1-e^{-\kappa|z|}}{|z|} \int_{\mathbb{R}^3} |u_\kappa(x)|^2 |u_\kappa(x-z)|^2 \,\mathrm{d}x\mathrm{d}z =: g_1(\kappa) + g_2(\kappa). \end{aligned}$$

For any $u_\kappa \in S_a$, since $1 - e^{-t} \leq t$ for all $t \geq 0$, let $t = \kappa|z| > 0$, we have

$$\begin{aligned} g_1(\kappa) &\leq \int_{\left\{z\in\mathbb{R}^3:|z|\leq\frac{1}{\kappa}\right\}} \kappa \int_{\mathbb{R}^3} |u_\kappa(x)|^2 |u_\kappa(x-z)|^2 \,\mathrm{d}x\mathrm{d}z \\ &\leq \kappa \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |u_\kappa(x)|^2 |u_\kappa(x-z)|^2 \,\mathrm{d}x\mathrm{d}z = \kappa a^4. \end{aligned}$$

For any $u_\kappa \in S_a$, it follows from $1 - e^{-t} \leq 1$ for all $t \geq 0$ that

$$\begin{aligned} g_2(\kappa) &\leq \int_{\left\{z\in\mathbb{R}^3:|z|\geq\frac{1}{\kappa}\right\}} \frac{1}{|z|} \int_{\mathbb{R}^3} |u_\kappa(x)|^2 |u_\kappa(x-z)|^2 \,\mathrm{d}x\mathrm{d}z \\ &\leq \kappa \int_{\left\{z\in\mathbb{R}^3:|z|\geq\frac{1}{\kappa}\right\}} \int_{\mathbb{R}^3} |u_\kappa(x)|^2 |u_\kappa(x-z)|^2 \,\mathrm{d}x\mathrm{d}z \\ &\leq \kappa \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |u_\kappa(x)|^2 |u_\kappa(x-z)|^2 \,\mathrm{d}x\mathrm{d}z = \kappa a^4. \end{aligned}$$

Therefore, it holds that $g_1(\kappa), g_2(\kappa) \to 0$ as $\kappa \to 0$, that is, $g(\kappa) \to 0$ as $\kappa \to 0$, which implies that $I_{\kappa,a^2} \to \mathcal{J}_{0,a^2}$ as $\kappa \to 0$. Moreover, from $p \in (2,3)$, $a \in (0, a_2)$, (1.14), [7] and Lemma 2.12-(i), we have $\mathcal{J}_{0,a^2} < 0$. □

Lemma 3.7 ([12, Lemma 3.5]). *Let $\{f_0\} \cup \{f_\kappa : 0 < \kappa < 1\} \subset L^{\frac{6}{5}}(\mathbb{R}^3)$. Let $\varphi_0 \in \mathcal{D}^{1,2}(\mathbb{R}^3)$ be the unique solution to the following equation*

$$-\Delta\varphi = f_0. \tag{3.16}$$

Given $\kappa \in (0,1)$, let $\phi_\kappa \in H^1(\mathbb{R}^3)$ be the unique solution to the following equation

$$-\Delta\phi + \kappa^2\phi = f_\kappa. \tag{3.17}$$

As $\kappa \to 0$, it holds that

(i) If $f_\kappa \rightharpoonup f_0$ in $L^{\frac{6}{5}}(\mathbb{R}^3)$, then $\phi_\kappa \rightharpoonup \varphi_0$ in $\mathcal{D}^{1,2}(\mathbb{R}^3)$.

(ii) If $f_\kappa \to f_0$ in $L^{\frac{6}{5}}(\mathbb{R}^3)$, then $\phi_\kappa \to \varphi_0$ in $\mathcal{D}^{1,2}(\mathbb{R}^3)$ and $\kappa^2\phi_\kappa \to 0$ in $L^2(\mathbb{R}^3)$.

Proof. (i) It follows from $\phi_\kappa \in H^1(\mathbb{R}^3)$ and Sobolev's embedding theorem that $\|\phi_\kappa\|_6 \leq C\|\phi_\kappa\|_{H^1}$, and thus $\phi_\kappa \in \mathcal{D}^{1,2}(\mathbb{R}^3)$. Fix $\kappa \in (0,1)$, since $\phi_\kappa \in \mathcal{D}^{1,2}(\mathbb{R}^3)$ is the unique solution to (3.17), from Hölder's inequality and Lemma 2.1, we deduce that

$$\|\nabla\phi_\kappa\|_2^2 \leq \|\nabla\phi_\kappa\|_2^2 + \kappa^2\|\phi_\kappa\|_2^2 = \int_{\mathbb{R}^3} f_\kappa\phi_\kappa \mathrm{d}x \leq C\|f_\kappa\|_{\frac{6}{5}}\|\phi_\kappa\|_6 \leq C\|f_\kappa\|_{\frac{6}{5}}\|\nabla\phi_\kappa\|_2, \tag{3.18}$$

thus, $\|\nabla\phi_\kappa\|_2 \leq C\|f_\kappa\|_{\frac{6}{5}}$. Since $f_\kappa \rightharpoonup f_0$ in $L^{\frac{6}{5}}(\mathbb{R}^3)$ as $\kappa \to 0$, we deduce that $\{f_\kappa : 0 < \kappa < 1\}$ is a bounded subset of $L^{\frac{6}{5}}(\mathbb{R}^3)$. It follows from $\phi_\kappa \in \mathcal{D}^{1,2}(\mathbb{R}^3)$ that $\{\phi_\kappa : 0 < \kappa < 1\}$ is a bounded subset of $\mathcal{D}^{1,2}(\mathbb{R}^3)$. In particular, we conclude that for any $\{\kappa_n\}_{n\in\mathbb{N}} \subset (0,1)$ satisfying $\lim_{n\to\infty}\kappa_n = 0$, there exists $\phi_* \in \mathcal{D}^{1,2}(\mathbb{R}^3)$ such that, up to a subsequence, $\phi_{\kappa_n} \rightharpoonup \phi_*$ in $\mathcal{D}^{1,2}(\mathbb{R}^3)$ as $n \to \infty$. Next, it suffices to prove that if $\{\kappa_n\}_{n\in\mathbb{N}} \subset (0,1)$ satisfies $\kappa_n \to 0$ and $\phi_{\kappa_n} \rightharpoonup \phi_*$ in $\mathcal{D}^{1,2}(\mathbb{R}^3)$ as $n \to \infty$, then $\phi_* = \varphi_0$.

Fix $n \in \mathbb{N}^+$, since the function ϕ_{κ_n} is a solution to (3.17), for any $\psi \in C_c^\infty(\mathbb{R}^3)$, it holds that

$$\int_{\mathbb{R}^3} \nabla\phi_{\kappa_n}\nabla\psi \mathrm{d}x + \kappa_n^2 \int_{\mathbb{R}^3} \phi_{\kappa_n}\psi \mathrm{d}x = \int_{\mathbb{R}^3} f_{\kappa_n}\psi \mathrm{d}x. \tag{3.19}$$

Since $\phi_{\kappa_n} \in L^6(\mathbb{R}^3)$, $\phi_{\kappa_n} \rightharpoonup \phi_*$ in $\mathcal{D}^{1,2}(\mathbb{R}^3)$ and $\kappa_n^2\int_{\mathbb{R}^3}\phi_{\kappa_n}\psi \mathrm{d}x \leq \kappa_n^2 C\|\phi_{\kappa_n}\|_6\|\psi\|_{\frac{6}{5}} \to 0$ as $n \to \infty$, from (3.19), we take the limit as $n \to \infty$ to conclude that

$$\int_{\mathbb{R}^3} \nabla\phi_*\nabla\psi \mathrm{d}x = \int_{\mathbb{R}^3} f_0\psi \mathrm{d}x \quad \text{for any } \psi \in C_c^\infty(\mathbb{R}^3). \tag{3.20}$$

Therefore, we infer that $\phi_* = \varphi_0$ according to (3.20) and the uniqueness of the solution $\varphi_0 \in \mathcal{D}^{1,2}(\mathbb{R}^3)$ to (3.16).

(ii) Since $f_\kappa \to f_0$ in $L^{\frac{6}{5}}(\mathbb{R}^3)$, we deduce from (i) that $\phi_\kappa \rightharpoonup \varphi_0$ in $\mathcal{D}^{1,2}(\mathbb{R}^3)$ as $\kappa \to 0$. According to the weak lower semicontinuity of $\|\nabla\cdot\|_2^2$, we conclude that

$$\|\nabla\varphi_0\|_2^2 \leq \liminf_{\kappa\to 0}\|\nabla\phi_\kappa\|_2^2. \tag{3.21}$$

Let $\{\psi_n\}_{n\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}^3)$ be a sequence such that $\psi_n \to \varphi_0$ in $\mathcal{D}^{1,2}(\mathbb{R}^3)$ as $n \to \infty$. Fix $\kappa \in (0,1)$, let $E_\kappa : \mathcal{D}^{1,2}(\mathbb{R}^3) \to \mathbb{R}$ be the energy functional of (3.17), which is given by

$$E_\kappa(\phi) = \frac{1}{2}\|\nabla\phi\|_2^2 + \frac{\kappa^2}{2}\|\phi\|_2^2 - \int_{\mathbb{R}^3} f_\kappa\phi \,\mathrm{d}x.$$

By the facts that $\phi_\kappa \in H^1(\mathbb{R}^3)$ is the unique solution to (3.17) and $H^1(\mathbb{R}^3) \hookrightarrow \mathcal{D}^{1,2}(\mathbb{R}^3)$, we deduce from (i) that $E_\kappa(\phi_\kappa) = \inf_{\phi\in\mathcal{D}^{1,2}(\mathbb{R}^3)} E_\kappa(\phi)$. It follows from $\{\psi_n\}_{n\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}^3)$ and $C_c^\infty(\mathbb{R}^3)$ is dense in $\mathcal{D}^{1,2}(\mathbb{R}^3)$ that

$$\begin{aligned}\frac{1}{2}\|\nabla\phi_\kappa\|_2^2 &= E_\kappa(\phi_\kappa) - \frac{\kappa^2}{2}\|\phi_\kappa\|_2^2 + \int_{\mathbb{R}^3} f_\kappa\phi_\kappa \,\mathrm{d}x \leq E_\kappa(\psi_n) + \int_{\mathbb{R}^3} f_\kappa\phi_\kappa \,\mathrm{d}x \\ &= \frac{1}{2}\|\nabla\psi_n\|_2^2 + \frac{\kappa^2}{2}\|\psi_n\|_2^2 - \int_{\mathbb{R}^3} f_\kappa\left(\psi_n - \phi_\kappa\right)\mathrm{d}x.\end{aligned} \tag{3.22}$$

Since $f_\kappa \to f_0$ in $L^{\frac{6}{5}}(\mathbb{R}^3)$ and $\phi_\kappa \rightharpoonup \varphi_0$ in $\mathcal{D}^{1,2}(\mathbb{R}^3)$ as $\kappa \to 0$, we deduce that

$$\int_{\mathbb{R}^3} f_\kappa\psi_n \mathrm{d}x \to \int_{\mathbb{R}^3} f_0\psi_n \mathrm{d}x \quad \text{and} \quad \int_{\mathbb{R}^3} f_\kappa\phi_\kappa \mathrm{d}x \to \int_{\mathbb{R}^3} f_0\varphi_0 \mathrm{d}x. \tag{3.23}$$

Thus, from (3.22) and (3.23), we infer that

$$\limsup_{\kappa\to 0}\frac{1}{2}\|\nabla\phi_\kappa\|_2^2 \leq \frac{1}{2}\|\nabla\psi_n\|_2^2 - \int_{\mathbb{R}^3} f_0\left(\psi_n - \phi_0\right)\mathrm{d}x.$$

From $\psi_n \to \varphi_0$ in $\mathcal{D}^{1,2}(\mathbb{R}^3)$ as $n \to \infty$, we obtain that

$$\limsup_{\kappa\to 0}\|\nabla\phi_\kappa\|_2^2 \leq \|\nabla\varphi_0\|_2^2 \quad \text{as } n \to \infty. \tag{3.24}$$

It follows from (3.21) and (3.24) that $\|\nabla\phi_\kappa\|_2^2 \to \|\nabla\varphi_0\|_2^2$ as $\kappa \to 0$, and thus $\phi_\kappa \to \varphi_0$ in $\mathcal{D}^{1,2}(\mathbb{R}^3)$ as $\kappa \to 0$.

Finally, since $\varphi_0 \in \mathcal{D}^{1,2}(\mathbb{R}^3)$ is the unique solution to equation (3.16), we conclude that

$$\kappa^2 \|\phi_\kappa\|_2^2 = \int_{\mathbb{R}^3} f_\kappa \phi_\kappa \mathrm{d}x - \|\nabla\phi_\kappa\|_2^2 \to \int_{\mathbb{R}^3} f_0\varphi_0 \mathrm{d}x - \|\nabla\varphi_0\|_2^2 = 0 \quad \text{as } \kappa \to 0,$$

namely, (ii) holds. □

The following proposition shows that if $p \in (2,3)$, then the Lagrange multipliers of solutions to the Schrödinger–Poisson–Slater system (1.13) tend to a certain $\tilde{\lambda} > 0$ as $a \to 0$.

Proposition 3.8 ([13, Proposition 1.3]). *Assume $p \in (2,3)$, $\kappa > 0$ and $\tilde{\lambda} > 0$, when $a \in (0, a_2)$, where a_2 is obtained in Theorem 1.3, for every $\varepsilon > 0$, there exists $a(\varepsilon) \in (0, a_2)$ such that*

$$\sup_{\lambda \in \mathcal{A}_a} |\lambda - \tilde{\lambda}| < \varepsilon \quad \forall\, 0 < a < a(\varepsilon),$$

where

$$\mathcal{A}_a := \{\lambda \in \mathbb{R} : \text{there exists } u \in H^1(\mathbb{R}^3) \text{ such that } (u,\varphi,\lambda) \in H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3) \times \mathbb{R} \text{ is a ground state solution to the Schrödinger–Poisson–Slater system (1.13)}\}.$$

Proof of Theorem 1.4. Since $\kappa > 0$, $2 < p < 3$ and $0 < a < a_2$, according to Theorems 1.1 and 1.3, for any ground state normalized solution $(u_\kappa, \phi_\kappa, \lambda_\kappa)$ to system (1.5) with $u_\kappa \in S_a \setminus \{0\}$ and $I_\kappa(u_\kappa) = I_{\kappa,a^2} < 0$, we deduce that up to translation, $(u_\kappa, \phi_\kappa, \lambda_\kappa) \in S_a^r \times H_r^1(\mathbb{R}^3) \times (0,\infty)$, which implies that $(u_\kappa, \phi_\kappa, \lambda_\kappa) \in S_a^r \times \mathcal{D}_r^{1,2}(\mathbb{R}^3) \times (0,\infty)$.

It follows from Lemma 2.10 that the set $\{u_\kappa \in S_a : I_\kappa(u_\kappa) = I_{\kappa,a^2} < 0\}$ is bounded in $H_r^1(\mathbb{R}^3)$ for any $\kappa > 0$. Let $\{\kappa_n\}_{n\in\mathbb{N}^+} \subset (0,1)$ be a sequence such that $\kappa_n \to 0$ as $n \to \infty$. Then, for any $n \in \mathbb{N}^+$, both $\{u_{\kappa_n}\} \subset S_a^r$ and $\{\phi_{\kappa_n}\} \subset \mathcal{D}_r^{1,2}(\mathbb{R}^3)$ are bounded and thus there exists $u_0 \in H_r^1(\mathbb{R}^3)$ such that, up to a subsequence, $u_{\kappa_n} \rightharpoonup u_0$ in $H_r^1(\mathbb{R}^3)$ as $n \to \infty$, $u_{\kappa_n} \to u_0$ in $L^p(\mathbb{R}^3)$ with $p \in (2,6)$ and $u_{\kappa_n} \to u_0$ a.e. in $\mathbb{R}^3$. Since $u_{\kappa_n}^2 \to u_0^2$ in $L^{\frac{6}{5}}(\mathbb{R}^3)$, from Lemma 3.7, we deduce that as $n \to \infty$, up to a subsequence, $\phi_{\kappa_n} \to \varphi_0$ in $\mathcal{D}_r^{1,2}(\mathbb{R}^3)$, where φ_0 is the unique solution of $-\Delta\varphi = 4\pi u_0^2$ in $\mathbb{R}^3$. Since for any $n \in \mathbb{N}^+$, $(u_{\kappa_n}, \phi_{\kappa_n}, \lambda_{\kappa_n}) \in S_a^r \times \mathcal{D}_r^{1,2}(\mathbb{R}^3) \times (0,\infty)$ is a ground state normalized solution to (1.5), from (2.21), Lemmas 2.1 and 2.6-(iii), we infer that

$$\begin{aligned} |\lambda_{\kappa_n}| &= \frac{1}{a^2}\left|\|u_{\kappa_n}\|_p^p - \|\nabla u_{\kappa_n}\|_2^2 - \int_{\mathbb{R}^3} \phi_{\kappa_n} u_{\kappa_n}^2 \mathrm{d}x\right| \\ &\le \frac{1}{a^2}\left(C\|\nabla u_{\kappa_n}\|_2^{\frac{3(p-2)}{2}} a^{\frac{6-p}{2}} + \|\nabla u_{\kappa_n}\|_2^2 + C\|\nabla u_{\kappa_n}\|_2^3 a\right) < \infty, \end{aligned}$$

namely, $\{\lambda_{\kappa_n}\} \subset (0,\infty)$ is a bounded sequence, and then there exists $\lambda_0 \in [0,\infty)$ such that, up to a subsequence, $\lambda_{\kappa_n} \to \lambda_0$ as $n \to \infty$.

Moreover, since $\mathcal{J}_0$, defined in (1.14), is the functional of the Schrödinger–Poisson–Slater system (1.13), it follows from Lemma 3.6 that $\{u_{\kappa_n}\}$ is a minimizing sequence for $\mathcal{J}_0$ at level $\mathcal{J}_{0,a^2} < 0$. Hence, from $u_{\kappa_n} \rightharpoonup u_0$ in $H_r^1(\mathbb{R}^3)$, $\phi_{\kappa_n} \to \varphi_0$ in $\mathcal{D}_r^{1,2}(\mathbb{R}^3)$, $\lambda_{\kappa_n} \to \lambda_0$, Proposition 2.8, Lemma 2.13 and [7], we conclude that $u_0 \in S_a^r \setminus \{0\}$, $u_{\kappa_n} \to u_0$ in $H_r^1(\mathbb{R}^3)$ as $n \to \infty$ and $\mathcal{J}_0(u_0) = \mathcal{J}_{0,a^2} < 0$. Thus, $(u_0, \varphi_0, \lambda_0) \in S_a^r \times \mathcal{D}_r^{1,2}(\mathbb{R}^3) \times [0,\infty)$ is a ground state normalized solution to the Schrödinger–Poisson–Slater system (1.13) with

$$\lambda_0 := \frac{1}{a^2}\left(\|u_0\|_p^p - \|\nabla u_0\|_2^2 - \int_{\mathbb{R}^3} \varphi_0 u_0^2 \,\mathrm{d}x\right).$$

According to Proposition 3.8, for every $\varepsilon > 0$, there exists $a(\varepsilon) \in (0, a_2)$ such that $\lambda_0 > 0$ when $a \in (0, a_2)$ is sufficiently small. Therefore, we conclude that, up to a subsequence and up to translation, it holds that

$$(u_\kappa, \phi_\kappa, \lambda_\kappa) \to (u_0, \varphi_0, \lambda_0) \quad \text{in } S_a^r \times \mathcal{D}_r^{1,2}(\mathbb{R}^3) \times (0, \infty) \quad \text{as } \kappa \to 0.$$

That is, (1.15) holds. Thus, we have completed the proof of Theorem 1.4. □

Declarations

- **Data availability** Data sharing not applicable to this article as no data sets were generated or analyzed during the current study.
- **Conflict of interests** On behalf of all authors, the corresponding author states that there is no conflict of interest.

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A priori estimates of stable solutions of the general Hardy–Hénon equation in the ball

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Abstract. This paper is devoted to the study of semi-stable radial solutions $u \in H^1(B_1)$ of $-\Delta u = |x|^\alpha f(u)$ in $B_1 \setminus \{0\}$, where $f \in C^1(\mathbb{R})$ is a general nonlinearity, $\alpha > -2$ and B_1 is the unit ball of $\mathbb{R}^N$, $N > 1$. We establish the boundedness of such solutions for dimensions $2 \le N < 10 + 4\alpha$ and sharp pointwise estimates in the case $N \ge 10 + 4\alpha$. In addition, we provide, for this range of dimensions, a large family of semi-stable radially decreasing unbounded $H^1(B_1)$ solutions.

Keywords: Hardy–Hénon equation, regularity estimates, semi-stable solutions.

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1 Introduction

This paper deals with the semi-stability of radial solutions $u \in H^1(B_1)$ of

$$-\Delta u = |x|^\alpha f(u) \quad \text{in } B_1 \setminus \{0\}, \tag{P}$$

where $x = (x_1, \dots, x_N)$ and B_1 the unit ball of $\mathbb{R}^N$ with measure ω_N/N for $N > 1$. In the whole paper we assume $\alpha > -2$, and $f \in C^1(\mathbb{R})$. We deal with the regularity of semi-stable radially symmetric energy solutions $u \in H^1(B_1)$ of (P). Denoting by $r := |x|$ standard arguments show that these radial solutions $u = u(|x|) \equiv u(r)$ are continuous in $r \in (0,1]$ and can be viewed as classical solutions. We denote as usual $u_r := \frac{\langle x, \nabla u \rangle}{|x|}$ and constants by capital letters $C, K, C', K' \dots$ that may vary throughout the discussion and computations, with subscripts that determine the dependence of the constants on other parameters.

The equation (P) corresponds to the Euler–Lagrange equation of the energy functional

$$E(u) := \int_{B_1} \left(\frac{|\nabla u|^2}{2} - |x|^\alpha F(u) \right) dx, \tag{1.1}$$

where $F(t) = \int_0^t f(s)ds$. Since $f \in C^1(\mathbb{R})$ and u is continuous, then $f'(u)$ is continuous. It is meaningful to call a solution $u \in H^1(B_1)$ semi-stable if the second variation of the energy is nonnegative definite, i.e.,

$$d^2E(\varphi, \varphi) = \int_{B_1} \left(|\nabla \varphi|^2 - |x|^\alpha f'(u)\varphi^2 \right) dx \ge 0, \tag{1.2}$$

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for all $\varphi \in C^1(B_1)$ with compact support in $B_1 \setminus \{0\}$.

Equation (P) can be regarded as a generalized version of the so called Hardy–Hénon equation which corresponds to power-type nonlinearity $f(u) = |u|^{p-1}u$. That equation is in particular called the Lane–Emden equation in the autonomous case $\alpha = 0$. Another nomenclature has been used for $f(u) = e^u$, the Hénon–Gelfand problem.

The study of this problem is preceded by the study of the so called Hénon equation whose origins goes back to the celebrated work of S. Chandrasekhar [3, Chapter IV, 87–88] on the stellar structure in polytropic equilibrium in astrophysics.

The pioneering contribution in the autonomous case $\alpha = 0$ is due to [12] where the authors prove that for power-type nonlinearity $f(u) = |u|^{p-1}u$ and posed in the whole space $\mathbb{R}^N$, there is no positive solution for $p \in (1, p_S(N))$, where $p_S(N) := 2^* - 1 = \frac{N+2}{N-2}$ if $N \leq 3$ and $p_S = +\infty$ if $N \leq 2$ is the classical critical Sobolev exponent.

For $p = p_S(N)$, the same equation is known to have (up to translation and rescaling) a unique positive solution, which is radial and explicit (see [2]).

Let now $p_{JL}(N) > p_S(N)$ denote the so called Joseph–Lundgren exponent:

$$p_{JL}(N) = \begin{cases} +\infty & \text{if } N \leq 10, \\ \frac{(N-2)^2 - 4N + 8\sqrt{N-1}}{(N-2)(N-10)} & \text{if } N \geq 11. \end{cases}$$

This exponent can be characterized as follows: for $p \geq p_S(n)$, the explicit function $u_s(x) = C_{p,N}|x|^{-\frac{2}{p-1}}$ for an appropriate constant $C_{p,N}$ depending only on p and N, is a singular solution of the Lane–Emden equation, which is stable if and only if $p \geq p_{JL}(N)$. It was proved in [9] that Lane–Emden equation has no nontrivial finite Morse index solution whenever $1 < p < p_{JL}(N)$, $p \neq p_S(N)$. Through the application of some blow-up analysis technique, such Liouville-type theorems imply interior regularity for solutions of a large class of semilinear elliptic equations: they are known to be equivalent to universal estimates for solutions of

$$-Lu = f(x, u, \nabla u) \quad \text{in } \Omega, \tag{1.3}$$

where L is a uniformly elliptic operator with smooth coefficients, the nonlinearity f scales like $|u|^{p-1}u$ for large values of u, and Ω is an open set of $\mathbb{R}^N$. For precise statements, see the work [17] in the subcritical setting, as well as its adaptation to the supercritical case by [6].

In [10], the authors consider Liouville-type theorems for Hénon–Lane–Emden systems, as well as solutions of the fourth-order Hénon equation with finite Morse index.

For the Hénon–Hardy equation (i.e. $\alpha \neq 0$, $f(u) = |u|^{p-1}u$), there are also an extensive literature. In [5], the authors show the following:

Theorem 1.1 (Dancer, Du, Guo [5])). *Let u be a stable solution of the Hénon–Hardy equation with $\alpha > -2$, $\Omega = \mathbb{R}^N$, and one of the following hypothesis is true: $2 \leq N \leq 10 + 4\alpha$ and $p > 1$; or $N > 10 + 4\alpha$ and $1 < p < p_{JL}(N, \alpha)$; where we have defined*

$$p_{JL}(N, \alpha) = \begin{cases} +\infty & \text{if } 2 \leq N \leq 10 + 4\alpha, \\ \frac{(N-2)^2 - 2(\alpha+2)(\alpha+N) + 2\sqrt{(\alpha+2)^3(\alpha+2N-2)}}{(N-2)(N-4\alpha-10)} & \text{if } N > 10 + 4\alpha. \end{cases}$$

Then $u \equiv 0$. On the contrary, for $p \geq p_{JL}(N, \alpha)$ the equation admits a family of radially symmetric positive stable solutions.

These results were later generalized in [19] for sign-changing solutions where the authors also proved that assuming $0 \in \Omega$, no weak solution exists for $\alpha \leq -2$. As they remark in

their proof, the result remains valid for any positive, convex and nondecreasing nonlinearity f. This indicates that the case $\alpha \leq -2$ carries additional pathological behavior, highlighting the necessity of avoiding that values for α. For $\alpha > -2$, $\Omega = \mathbb{R}^N$ they also prove the following two results:

(i) If $2 \leq N < 10 + 4\alpha$, there is no weak stable solution of the Gelfand–Hénon equation.

(ii) If $2 < N < 10 + 4\alpha^-$ where $\alpha^- = \min\{\alpha, 0\}$, then any weak solution of the Gelfand–Hénon equation has infinite Morse index.

Remark 1.2. The dimension range $2 \leq N < 10 + 4\alpha$ is optimal, since for $N \geq 10 + 4\alpha$

$$u(x) = -(2+\alpha)\log|x| + \log[(2+\alpha)(N-2)],$$

is a radial stable weak solution to the Gelfand–Hénon equation.

For analogous recent results in the fractional/non local setting we refer to [8,11,13–15] and even with a Hardy potential term $|x|^{-2s}u$ on the right-hand side [16] or concerning systems [4,7] and references therein.

In our case, we are mainly concerned with the regularity of semi-stable radial solutions of the general Hénon–Hardy equation, although we think that our results can be further developed in several directions involving extensions of what is known in the autonomous case. For instance, it would be interesting to investigate the existence and regularity of a critical solution of the equation $-\Delta u = \lambda|x|^{\alpha} f(u)$ with the appropriate set of hypothesis over α and f in the spirit of [1]. The content is developed through a series of lemmas and propositions, arriving at the end to the proof of the theorem stated at the beginning of the next section.

2 Statement of the main result and steps of the proof

We directly proceed to state the main result of the paper: a classification of the behavior of semi-stable radial solutions of (P) depending on the dimension.

Theorem 2.1. *Let $N \geq 2$, $f \in C^1(\mathbb{R})$, $\alpha > -2$ and $u \in H^1(B_1)$ be a semi-stable radial solution of (P). Then there exists a constant $C_{\alpha,N}$ depending only on α and N such that:*

(i) if $N < 10 + 4\alpha$, then $\|u\|_{L^\infty(B_1)} \leq C_{\alpha,N}\|u\|_{H^1(B_1\setminus\overline{B_{1/2}})}$;

(ii) if $N = 10 + 4\alpha$, then $|u(r)| \leq C_{\alpha,10+4\alpha}\|u\|_{H^1(B_1\setminus\overline{B_{1/2}})}\left(|\log r| + 1\right)$, $\forall r \in (0,1]$;

(iii) if $N > 10 + 4\alpha$, then $|u(r)| \leq C_{\alpha,N}\|u\|_{H^1(B_1\setminus\overline{B_{1/2}})}\, r^{\gamma(N,\alpha)}$, $\forall r \in (0,1]$,

where we have defined the exponent $\gamma(N,\alpha) := 2 - N/2 + \alpha/2 + \sqrt{(\alpha+2)(\alpha+2N-2)}/2$.

Remark 2.2. Observe that $\gamma(N,\alpha) = 0$ if and only if $N = 10 + 4\alpha$ and $\gamma(N,\alpha) < 0$ if and only if $N > 10 + 4\alpha$ so that the pointwise estimates *ii)* and *iii)* do not give rise to boundedness of u.

We shall divide the proof in several lemmas and propositions.

Lemma 2.3. *Let $N \geq 2$, $f \in C^1(\mathbb{R})$, $\alpha > -2$ and $u \in H^1(B_1)$ be a semi-stable radial solution of (P). Let $v \in C^{0,1}(0,1]$ such that $v(1) = 0$. Then*

$$\int_{r_0}^1 t^{N-1} u_r^2(t) \left(v'(t)^2 + \alpha \frac{v'(t)v(t)}{t} + \left(1 - N - \alpha \frac{N}{2}\right) \frac{v(t)^2}{t^2} \right) dt \geq 0, \tag{2.1}$$

for every $r_0 \in (0,1)$.

Proof. Assume, as a first step, that $v \in C^\infty(0,1)$ has compact support and $v \equiv 0$ in $(0, r_0]$ (later we will prove it for any $v \in C^{0,1}(0,1]$ with $v(1) = 0$). Differentiating (P) with respect to r we obtain

$$(-\Delta u)_r = -\Delta u_r + \frac{N-1}{r^2} u_r = \alpha r^{\alpha-1} f(u) + r^\alpha f'(u) u_r.$$

Multiplying by the radial function $u_r v^2$ and integrating by parts yields

$$\int_{B_1} \left(\nabla u_r \nabla (u_r v^2) + \frac{N-1}{r^2} u_r^2 v^2 \right) dx = \int_{B_1} \left(\alpha r^{\alpha-1} f(u) + r^\alpha f'(u) u_r \right) u_r v^2 dx.$$

Therefore

$$\int_{B_1} \left(v^2 |\nabla u_r|^2 + 2 u_r v \nabla u_r \nabla v + \frac{N-1}{r^2} u_r^2 v^2 \right) dx = \int_{B_1} \left(\alpha r^{\alpha-1} f(u) u_r v^2 + r^\alpha f'(u) u_r^2 v^2 \right) dx. \tag{2.2}$$

On the other hand, since u is stable, we can consider the radial function $u_r v$ (which is a $C^1(B_1)$ function with compact support in $B_1 \setminus \{0\}$) obtaining

$$\int_{B_1} |\nabla (u_r v)|^2 dx \geq \int_{B_1} r^\alpha f'(u) (u_r v)^2 dx. \tag{2.3}$$

Subtracting (2.2) from (2.3) we can assert that

$$\int_{B_1} \left(u_r^2 v_r^2 - \frac{N-1}{r^2} u_r^2 v^2 \right) \geq -\alpha \int_{B_1} r^{\alpha-1} f(u) u_r v^2 dx. \tag{2.4}$$

Let us expand this last term:

$$\begin{aligned} \int_{B_1} r^{\alpha-1} f(u) u_r v^2 dx &= \int_{B_1} (-\Delta u) \frac{u_r v^2}{r} dx = \omega_N \int_{r_0}^1 t^{N-1} \left(-u_{rr} - \frac{N-1}{t} u_r \right) \frac{u_r v^2}{t} dt \\ &= -\frac{\omega_N}{2} \int_{r_0}^1 \left(t^{2N-2} u_r^2 \right)' t^{-N} v^2 dt = \frac{\omega_N}{2} \int_{r_0}^1 t^{2N-2} u_r^2 (t^{-N} v^2)' dt \\ &= \omega_N \int_{r_0}^1 t^{N-1} \left(\frac{-N}{2} \frac{u_r^2 v^2}{t^2} + \frac{u_r^2 v v'}{t} \right) dt. \end{aligned}$$

Combining this and (2.4) we obtain

$$\omega_N \int_{r_0}^1 t^{N-1} \left(u_r^2 v'^2 - \frac{N-1}{t^2} u_r^2 v^2 \right) dt \geq -\alpha\, \omega_N \int_{r_0}^1 t^{N-1} \left(\frac{-N}{2} \frac{u_r^2 v^2}{t^2} + \frac{u_r^2 v v'}{t} \right) dt,$$

which is the desired conclusion.

In fact, by standard density arguments, the above formula is also true if we consider a $C^{0,1}(0,1]$ function vanishing in $(0, r_0] \cup \{1\}$.

Now, for the sake of clarity of the exposition let us denote

$$I(a, b; v) := \int_a^b t^{N-1} u_r^2(t) \left(v'(t)^2 + \alpha \frac{v'(t)v(t)}{t} + \left(1 - N - \alpha \frac{N}{2}\right) \frac{v(t)^2}{t^2} \right) dt. \tag{2.5}$$

For an arbitrary $v \in C^{0,1}(0,1]$ vanishing at $t = 1$, we define a radial truncated function

$$\bar{v}_\varepsilon(t) = \begin{cases} 0 & \text{if } 0 < t < \varepsilon, \\ \frac{v(r_0)}{r_0 - \varepsilon}(t - \varepsilon) & \text{if } \varepsilon \le t \le r_0, \\ v(t) & \text{if } r_0 < t \le 1, \end{cases} \tag{2.6}$$

for any $0 < \varepsilon < r_0$. Now, applying the first step of the lemma to the function $\bar{v}_\varepsilon$ we have that $I(\varepsilon, 1; \bar{v}_\varepsilon) \ge 0$ and thus

$$\begin{aligned} I(r_0, 1; v) &\ge -I(\varepsilon, r_0; \bar{v}_\varepsilon) \\ &= -\left(\frac{v(r_0)}{r_0 - \varepsilon}\right)^2 \int_\varepsilon^{r_0} t^{N-1} u_r(t)^2 \left[1 + \alpha\frac{t-\varepsilon}{t} + \left(1 - N - \alpha\frac{N}{2}\right)\frac{(t-\varepsilon)^2}{t^2}\right] dt. \end{aligned}$$

Observe that since $u \in H^1(B_1)$ the whole right hand side is uniformly bounded for $\epsilon \in (0, r_0)$. Therefore

$$I(r_0, 1; v) \ge -\lim_{\varepsilon \to 0} I(\varepsilon, r_0; \bar{v}_\varepsilon) = -\left(\frac{v(r_0)}{r_0}\right)^2 (2 + \alpha)\left(1 - \frac{N}{2}\right) \int_0^{r_0} t^{N-1} u_r(t)^2 dt \ge 0,$$

as we claimed. □

Remark 2.4. In the last step, it is shown that the assumption $u \in H^1(B_1)$ is essential. Indeed, it is well known (see for instance [1]), that there exists semi-stable weak solutions to $-\Delta u = C_{N,q}(1+u)^{\frac{q-2}{q}}$ in B_1 of the form $u(x) = |x|^q - 1$ such that $u \notin H^1(B_1)$ in the range $q \in \left(-\frac{N}{2} + 2 - \sqrt{N-1},\ -\frac{N}{2} + 1\right]$, $N \ge 3$.

Proposition 2.5. *Let $N \ge 2$, $f \in C^1(\mathbb{R})$, $\alpha > -2$ and $u \in H^1(B_1)$ be a non constant semi-stable radial solution of (P). Then $u_r \ne 0$ in $(0,1]$.*

Proof. Assume by contradiction that there exists $r_1 \in (0,1]$ such that $u_r(r_1) = 0$. From $u \in H^1(B_1)$, $f \in C^1(\mathbb{R})$ and the radial symmetry we know that $u(r) \in C^3((0,1])$. Thus, in particular there is a constant $C_{r_1} > 0$ such that $|u_r(r)| \le C_{r_1}|r - r_1|$ for any $r \in [r_1/2, r_1]$. On the other hand, we take the following test function

$$v(t) = \begin{cases} \frac{t}{r_1 - \varepsilon} & \text{if } 0 < t < r_1 - \varepsilon, \\ \frac{r_1 - t}{\varepsilon} & \text{if } r_1 - \varepsilon \le t \le r_1, \\ 0 & \text{if } r_1 < t \le 1, \end{cases}$$

with $0 < \varepsilon < r_1/2$. Applying Lemma 2.3 we have that $-I(r_0, r_1 - \varepsilon; v) \le I(r_1 - \varepsilon, r_1; v)$ for any $0 < r_0 < r_1 - \varepsilon$. Therefore

$$\begin{aligned} &-\frac{1}{(r_1 - \varepsilon)^2} \int_{r_0}^{r_1 - \varepsilon} t^{N-1} u_r^2(t)(2 + \alpha)\left(1 - \frac{N}{2}\right) dt \\ &\quad \le \int_{r_1 - \epsilon}^{r_1} \frac{t^{N-1} u_r^2(t)}{\varepsilon^2}\left[1 - \alpha\frac{(r_1 - t)}{t} + \left(1 - N - \alpha\frac{N}{2}\right)\frac{(r_1 - t)^2}{t^2}\right] dt \\ &\quad \le K \int_{r_1 - \epsilon}^{r_1} \frac{u_r^2(t)}{\varepsilon^2} dt \le K C_{r_1}^2 \int_{r_1 - \epsilon}^{r_1} \frac{(t - r_1)^2}{\varepsilon^2} dt = \frac{K C_{r_1}^2}{3}\,\varepsilon. \end{aligned}$$

Taking limit as $\varepsilon \to 0$ we have that

$$-\frac{1}{r_1^2}(2 + \alpha)\left(1 - \frac{N}{2}\right) \int_{r_0}^{r_1} t^{N-1} u_r^2(t) dt \le 0,$$

that implies that $u_r \equiv 0$ in $[r_0, r_1]$ if $N > 2$. By the uniqueness of the corresponding Cauchy problem, $u_r \equiv 0$ in $(0, 1]$ which gives the contradiction for $N > 2$. Finally, if $N = 2$ we change the test function by

$$v(t) = \begin{cases} \left(\frac{t}{r_1-\varepsilon}\right)^\beta & \text{if } 0 < t < r_1 - \varepsilon, \\ \frac{r_1-t}{\varepsilon} & \text{if } r_1 - \varepsilon \le t \le r_1, \\ 0 & \text{if } r_1 < t \le 1, \end{cases}$$

with $\beta \in \mathbb{R}$ to be chosen later. With similar arguments we can conclude

$$-\frac{1}{r_1^{2\beta}}(\beta - 1)\,(\beta + \alpha + 1) \int_{r_0}^{r_1} t^{N-1+2\beta-2} u_r^2(t) dt \le 0.$$

Choosing any $\beta \in (-1-\alpha, 1)$, we obtain $u_r \equiv 0$ in $[r_0, r_1]$ which is again a contradiction. □

Lemma 2.6. *Let $N \ge 2$, $f \in C^1(\mathbb{R})$, $\alpha > -2$ and $u \in H^1(B_1)$ be a non constant semi-stable radial solution of (P). Then*

$$\int_{r/2}^{r} u_r^2\, dt \le K_{\alpha,N} \|\nabla u\|^2_{L^2(B_1 \setminus \overline{B_{1/2}})}\, r^{3-N+\alpha+\sqrt{(2+\alpha)(2N-2+\alpha)}},$$

for any $r \in (0, 1]$.

Proof. We first consider the case $0 < r \le 1/2$ and define

$$v(t) = \begin{cases} r^{s-1} t & \text{if } 0 < t < r, \\ t^s & \text{if } r \le t \le 1/2, \\ 2^{1-s}(1-t) & \text{if } 1/2 < t \le 1 \end{cases}$$

with $s \in \mathbb{R}$ to be determined. By Lemma 2.3 applied for $r_0 = r/2$, we have

$$\begin{aligned} 0 \le\ & r^{2s-2} \int_{r/2}^{r} t^{N-1} u_r^2(t)\,(2+\alpha) \left(1 - \frac{N}{2}\right) dt \\ & + \int_{r}^{1/2} t^{N-3+2s} u_r^2(t) \left(s^2 + \alpha s + 1 - N - \alpha \frac{N}{2}\right) dt \\ & + \int_{1/2}^{1} t^{N-1} u_r^2(t)\, 2^{2-2s} \left[1 - \alpha \frac{1-t}{t} + \left(1 - N - \alpha \frac{N}{2}\right) \left(\frac{1-t}{t}\right)^2\right] dt. \end{aligned}$$

Choosing $s_\alpha = -\alpha/2 - \sqrt{(2+\alpha)(2N-2+\alpha)}/2$, the second term is indeed zero. Therefore, we deduce

$$\begin{aligned} & - r^{2s_\alpha - 2} \int_{r/2}^{r} t^{N-1} u_r^2(t) (2+\alpha) \left(1 - \frac{N}{2}\right) dt \\ & \le \int_{1/2}^{1} t^{N-1} u_r^2(t)\, 2^{2-2s_\alpha} \left[1 - \alpha \frac{1-t}{t} + \left(1 - N - \alpha \frac{N}{2}\right) \left(\frac{1-t}{t}\right)^2\right] dt. \end{aligned}$$

As a result

$$\begin{aligned}
&- r^{2s_\alpha-2}\int_{r/2}^{r}\left(\frac{r}{2}\right)^{N-1}u_r^2(t)(2+\alpha)\left(1-\frac{N}{2}\right)dt\\
&\le r^{2s_\alpha-2}\int_{r/2}^{r}t^{N-1}u_r^2(t)(2+\alpha)\left(1-\frac{N}{2}\right)dt\\
&\le \int_{1/2}^{1}t^{N-1}u_r^2(t)\,2^{2-2s_\alpha}\left[1-\alpha\frac{1-t}{t}+\left(1-N-\alpha\frac{N}{2}\right)\left(\frac{1-t}{t}\right)^2\right]dt\\
&\le C\int_{1/2}^{1}t^{N-1}u_r^2(t)dt,
\end{aligned}$$

Finally, we obtain

$$-\frac{(2+\alpha)\left(1-\frac{N}{2}\right)}{2^{N-1}}r^{2s_\alpha-3+N}\int_{r/2}^{r}u_r^2(t)dt\le\frac{C}{\omega_N}\|\nabla u\|^2_{L^2(B_1\setminus\overline{B_{1/2}})}$$

which is the desired conclusion if $N>2$. If $N=2$, we change the test function by

$$v(t)=\begin{cases}r^{-1-\alpha-\beta}t^\beta & \text{if } 0<t<r,\\ t^{-1-\alpha} & \text{if } r\le t\le 1/2,\\ 2^{2+\alpha}(1-t) & \text{if } 1/2<t\le 1.\end{cases}$$

with $\beta\in(-1-\alpha,1)$ as in the previous proposition. We have

$$\begin{aligned}
&-r^{-2-2\alpha-2\beta}\int_{r/2}^{r}t^{-1+2\beta}\,u_r^2(t)(\beta-1)(\beta+\alpha+1)\,dt\\
&\qquad\le\int_{1/2}^{1}t\,u_r^2(t)\,2^{4+2\alpha}\left[1-\alpha\frac{1-t}{t}-(1+\alpha)\left(\frac{1-t}{t}\right)^2\right]dt,
\end{aligned}$$

that gives the desired conclusion once the same argument as above is employed for the last integral.

On the other hand, let us call $R_{\alpha,N}$ a constant such that $0<R_{\alpha,N}<r^{3-N-2s_\alpha}$ for all $r\in[1/2,1]$. Hence, applying the above conclusion for $r=1/2$

$$\begin{aligned}
\int_{r/2}^{r}u_r^2(t)dt&\le\int_{1/4}^{1/2}u_r^2(t)dt+\int_{1/2}^{1}u_r^2(t)dt\\
&\le K_{\alpha,N}\left(\frac12\right)^{3-N-2s_\alpha}\|\nabla u\|^2_{L^2(B_1\setminus\overline{B_{1/2}})}+\frac{1}{\omega_N}\|\nabla u\|^2_{L^2(B_1\setminus\overline{B_{1/2}})}\\
&=\left(K_{\alpha,N}\left(\frac12\right)^{3-N-2s_\alpha}+\frac{1}{\omega_N}\right)\|\nabla u\|^2_{L^2(B_1\setminus\overline{B_{1/2}})}\\
&\le\frac{1}{R_{\alpha,N}}\left[K_{\alpha,N}\left(\frac12\right)^{3-N-2s_\alpha}+\frac{1}{\omega_N}\right]\|\nabla u\|^2_{L^2(B_1\setminus\overline{B_{1/2}})}r^{3-N-2s_\alpha},
\end{aligned}$$

giving rise to the same conclusion again. □

Proposition 2.7. *Let $N\ge2$, $f\in C^1(\mathbb{R})$, $\alpha>-2$ and $u\in H^1(B_1)$ be a non constant semi-stable radial solution of (P). Then*

$$|u(r)-u(r/2)|\le K'_{\alpha,N}\|\nabla u\|_{L^2(B_1\setminus\overline{B_{1/2}})}\,r^{2-N/2+\alpha/2+\sqrt{(2+\alpha)(2N-2+\alpha)}/2},$$

for all $r\in(0,1]$.

Proof. Fix $r \in (0,1/2)$, using the Cauchy–Schwarz inequality and Lemma 2.6, we have the following chain of inequalities:

$$\begin{aligned}|u(r)-u(r/2)| &= \left|\int_{r/2}^{r} u_r(t)\,dt\right| \le \int_{r/2}^{r} |u_r(t)|\,dt \le \left(\int_{r/2}^{r} u_r^2(t)dt\right)^{1/2}\left(\int_{r/2}^{r} dt\right)^{1/2}\\ &\le K_{\alpha,N}\|\nabla u\|_{L^2(B_1\setminus\overline{B_{1/2}})} r^{\frac{3-N+\alpha+\sqrt{(2+\alpha)(2N-2+\alpha)}}{2}}\left(\frac{r}{2}\right)^{\frac{1}{2}}\\ &\le K'_{\alpha,N}\|\nabla u\|^2_{L^2(B_1\setminus\overline{B_{1/2}})} r^{\frac{4-N+\alpha+\sqrt{(2+\alpha)(2N-2+\alpha)}}{2}}.\end{aligned}$$

□

Proof of Theorem 2.1. Following the reasoning of [18], let $r \in (0,1]$, there exists $r_1 \in (1/2,1]$ such that $r = r_1/2^{m-1}$ for some $m \in \mathbb{N}$. By Sobolev embedding in dimension one we observe that $u(r_1) \le \|u\|_{L^\infty(B_1\setminus B_{1/2})} \le \mathfrak{s}_N\|u\|_{H^1(B_1\setminus\overline{B_{1/2}})}$. Therefore

$$\begin{aligned}|u(r)| &\le |u(r_1)-u(r)| + |u(r_1)| \le \sum_{k=1}^{m-1}\left|u\left(\frac{r_1}{2^{k-1}}\right)-u\left(\frac{r_1}{2^k}\right)\right| + |u(r_1)|\\ &\le K'_{\alpha,N}\|\nabla u\|_{L^2(B_1\setminus\overline{B_{1/2}})}\sum_{k=1}^{m-1}\left(\frac{r_1}{2^{k-1}}\right)^{2-\frac{N}{2}+\frac{\alpha+\sqrt{(2+\alpha)(2N-2+\alpha)}}{2}} + \mathfrak{s}_N\|u\|_{H^1(B_1\setminus\overline{B_{1/2}})}\\ &\le \left(K''_{\alpha,N}\sum_{k=1}^{m-1}\left(\frac{r_1}{2^{k-1}}\right)^{2-\frac{N}{2}+\frac{\alpha+\sqrt{(2+\alpha)(2N-2+\alpha)}}{2}} + \mathfrak{s}_N\right)\|u\|_{H^1(B_1\setminus\overline{B_{1/2}})}. \qquad (2.7)\end{aligned}$$

- If $2 \le N < 10+4\alpha$, we have $2-N/2+\alpha/2+\sqrt{(2+\alpha)(2N-2+\alpha)}/2 > 0$. Then

$$\sum_{i=1}^{m-1}\left(\frac{r_1}{2^{i-1}}\right)^{2-\frac{N}{2}+\alpha/2+\sqrt{(2+\alpha)(2N-2+\alpha)}/2} \le \sum_{i=1}^{\infty}\left(\frac{1}{2^{i-1}}\right)^{2-\frac{N}{2}+\alpha/2+\sqrt{(2+\alpha)(2N-2+\alpha)}/2},$$

  which is a convergent series. Hence, equation (2.7) implies statement i) of the theorem.

- If $N = 10+4\alpha$, we have $2-N/2+\alpha/2+\sqrt{(2+\alpha)(2N-2+\alpha)}/2 = 0$. From (2.7) we obtain

$$\begin{aligned}|u(r)| &\le \left(K'_{\alpha,N}(m-1)+\mathfrak{s}_N\right)\|u\|_{H^1(B_1\setminus\overline{B_{1/2}})}\\ &= \left(K'_{\alpha,N}\left(\frac{\log r_1-\log r}{\log 2}\right)+\mathfrak{s}_N\right)\|u\|_{H^1(B_1\setminus\overline{B_{1/2}})}\\ &\le \left(\frac{K'_{\alpha,N}}{\log 2}+\mathfrak{s}_N\right)(|\log r|+1)\|u\|_{H^1(B_1\setminus\overline{B_{1/2}})},\end{aligned}$$

  which proves statement ii).

- Finally, if $N > 10+4\alpha$, we have $2-N/2+\alpha/2+\sqrt{(2+\alpha)(2N-2+\alpha)}/2 < 0$. Then

$$\sum_{i=1}^{m-1}\left(\frac{r_1}{2^{i-1}}\right)^{2-\frac{N}{2}+\frac{\alpha}{2}+\frac{\sqrt{(2+\alpha)(2N-2+\alpha)}}{2}} = \frac{r^{2-\frac{N}{2}+\frac{\alpha}{2}+\frac{\sqrt{(2+\alpha)(2N-2+\alpha)}}{2}} - r_1^{2-\frac{N}{2}+\frac{\alpha}{2}+\frac{\sqrt{(2+\alpha)(2N-2+\alpha)}}{2}}}{(1/2)^{2-\frac{N}{2}+\frac{\alpha}{2}+\frac{\sqrt{(2+\alpha)(2N-2+\alpha)}}{2}}-1}.$$

  From this and again (2.7), we deduce

$$|u(r)| \le \left(\frac{K'_{\alpha,N}}{(1/2)^{2-\frac{N}{2}+\frac{\alpha}{2}+\frac{\sqrt{(2+\alpha)(2N-2+\alpha)}}{2}}-1}+\mathfrak{s}_N\right) r^{2-\frac{N}{2}+\frac{\alpha}{2}+\frac{\sqrt{(2+\alpha)(2N-2+\alpha)}}{2}}\|u\|_{H^1(B_1\setminus\overline{B_{1/2}})},$$

  which completes the proof. □

3 Optimality of Theorem 2.1

These a priori estimates are indeed optimal as one can see with the following examples:

- If $N = 10 + 4\alpha$, then the function $u(x) = |\log|x||$ is a solution to the Hénon–Gelfand equation
$$-\Delta u = (N-2)|x|^{\alpha} e^{u(2+\alpha)} \quad \text{in } B_1 \setminus \{0\},$$
such that $|x|^{\alpha} f'(u) = \frac{(N-2)^2}{4|x|^2}$. This implies that the stability inequality (1.2) attaining exactly the optimal Hardy constant and hence it is a stable solution.

- If $N > 10 + 4\alpha$, recall that we have that for any $\alpha > -2$, $\gamma(\alpha, N) < 0$. We consider the function $u(|x|) = |x|^{\gamma} - 1$ for any
$$\gamma(\alpha, N) \leq \gamma < 0, \tag{3.1}$$
which is the solution to the power-type equation
$$-\Delta u = |x|^{\alpha}(-\gamma)(\gamma + N - 2)(1+u)^{1+\frac{2+\alpha}{-\gamma}} \quad \text{in } B_1 \setminus \{0\}.$$
In this case
$$|x|^{\alpha} f'(u) = \frac{(-\gamma + \alpha + 2)(\gamma + N - 2)}{|x|^2}. \tag{3.2}$$
Taking into account (3.1), (3.2) and comparing again the stability inequality (1.2) with the hardy inequality, the constants satisfies:
$$(-\gamma + \alpha + 2)(\gamma + N - 2) \leq \frac{(N-2)^2}{4},$$
implying the stability of u.

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Local and global solutions for abstract integro-differential equations with state-dependent delay and applications to a system with coupled memory and diffusion

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Abstract. In this work we study the local and global existence, uniqueness and well-posedness of solutions for abstract integro-differential equations with state-dependent delay. We establish these results without requiring the conventional assumption of local Lipschitz continuity in the forcing terms. Furthermore, our goal is to apply the abstract theory to the study of a problem involving a memory and diffusion-coupled integro-differential system with state-dependent delay.

Keywords: integro-differential equations, state-dependent delay, local and global existence of solutions, uniqueness of solutions, local well-posedness, diffusion-coupled system.

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1 Introduction

In this work, we are first interested in the investigation of the existence and uniqueness, regularity, well-posedness, and global existence of solutions, problems that are crucial and challenging when dealing with differential equations involving state-dependent delays. In this article, we study the integro-differential abstract equations modeled by

$$\frac{d}{dt}\left(u(t)+\int_0^t P(t-s)u(s)ds\right) = Au(t)+\int_0^t Q(t-s)u(s)ds+G(t,u(t-\sigma(t,u_t))), t\in[0,a], \tag{1.1}$$

$$u_0=\varphi\in\mathcal{B}=C([-p,0];Z), \tag{1.2}$$

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where A and $Q(t)$ for $t \geq 0$ are closed linear operators defined on a common domain $D(A)$ that is dense in Z, while $(P(t))_{t\geq 0}$ is a family of bounded linear operators on Z. The variable u_t represents the history of $u(\cdot)$ at time t ($u_t \in \mathcal{B}$ and $u_t(\theta) = u(t+\theta)$ for $\theta \in [-p, 0]$, $t > 0$) and $G(\cdot)$ and $\sigma(\cdot)$ are continuous functions.

Secondly, we apply abstract theory to study the existence and uniqueness of mild and classical local and global solutions for a diffusion-coupled integro-differential system with state-dependent delay described by

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left[u(t,x) + \int_0^t (t-s)^{\alpha} e^{-\omega(t-s)} v(s,x)ds\right] \\
&\qquad = \Delta u(t,x) + \int_0^t B(t-s)\Delta v(s,x)ds + \zeta(t)\int_\Omega b(u(\varrho(t,u_t),y))f(x-y)dy, \\
&\frac{\partial}{\partial t}\left[v(t,x) + \int_0^t (t-s)^{\alpha} e^{-\gamma(t-s)} u(s,x)ds\right] \\
&\qquad = \Delta v(t,x) + \int_0^t C(t-s)\Delta u(s,x)ds + g(v(\varrho(t,u_t),x)), \\
&u(t,\cdot)|_{\partial\Omega} = 0, \quad v(t,\cdot)|_{\partial\Omega} = 0 \\
&u(\theta,y) = \varphi(\theta,y), \quad v(\theta,y) = \phi(\theta,y), \qquad \theta \in [-p,0],\ y \in \Omega,
\end{aligned}
$$

for $t \in [0,a]$ and $x \in \Omega$, where $f \in C(\widetilde{\Omega};\mathbb{R})$, $\widetilde{\Omega} = \{x-y : x,y \in \Omega\}$, $g \in C_{\mathrm{Lip}}(\mathbb{R}^n;\mathbb{R}^n)$, $b \in C_{\mathrm{Lip}}(\mathbb{R}^n;\mathbb{R}^n)$, $\varrho \in C_{\mathrm{Lip}}([0,a]\times\mathcal{B};\mathbb{R})$, $d-p \leq \varrho(t,\psi) \leq t$ for all $(t,\psi) \in [0,a]\times\mathcal{B}$ and $\zeta \in C_{\mathrm{Lip}}(\mathbb{R};\mathbb{R})$.

It is well-known that integro-differential equations in abstract spaces play a significant role in the literature due to their ability to model a wide range of important equations arising in physical and biological systems. For example, in the classical theory developed by Gurtin and Pipkin [14], Miller [27], and Nunziato [29], integro-differential equations are employed to describe heat conduction in materials with fading memory. They are also used to model the non-equilibrium thermal response of glass-forming substances with dynamic (time-dependent) heat capacity under fast thermal perturbations (see [28] for more details).

In [2], the author considers a coupled linear integro-differential wave system with memory. In this work, the existence and uniqueness of global solutions are studied, and the uniform stabilization of the total energy is proved as time goes to infinity. Another example of the importance of integro-differential equations is the McKean–Vlasov equation, which has attracted considerable attention due to its broad applicability. It models not only continuous phenomena in areas such as statistical physics, stochastic control, mean field games, financial mathematics, Bayesian inference, and neural networks, but also discontinuous phenomena in mean-field dynamics—such as the motion of particles in polymer dynamics, the onset of financial crises, and discontinuities in control systems [23].

An important framework for analyzing partial integro-differential equations in abstract spaces is the theory of resolvent operators. This theory is widely used to establish the existence, uniqueness, and qualitative properties of solutions to such equations. The resolvent operator approach has been developed in several works. Firstly, by Grimmer et al. [11–13]. Secondly, Da Prato et al. [3,4] and Lunardi [24,25] studied systems with $P \equiv 0$. Recently, Dos Santos et al. [7,8,16] addressed the general case where both $P \neq 0$ and $Q \neq 0$.

An important topic in the theory of differential equations, abstract differential equations with state-dependent delay are a field of great significance due to their applications (see [1,15]) and because their qualitative theory is rather different from the usual theory on functional differential equations with constant time-dependent delay. The literature on state-dependent delay problems is extensive. For ordinary differential equations (ODE) in finite-dimensional

spaces, notable works include [1, 9, 10, 15] and their associated references. For ODEs and partial differential equations (PDEs) with state-dependent delay, one can refer to the papers [17–19, 19–22, 26]. Many works in the literature prove the existence of solutions of integro-differential equations with state-dependent delay [5,7,30,32]. But these papers establish only local solutions by fixed point results using compactness; they do not address uniqueness, the regularity of solutions, or the existence and uniqueness of maximal solutions for integro-differential equations. The central difference emerges from the fact that functions of the form $u \mapsto u(\cdot - \sigma(\cdot, u_{(\cdot)}))$ and $u \mapsto u_{\mu(\cdot,u_{(\cdot)})}$ are not Lipschitz in spaces of continuous functions, and this has implications concerning the uniqueness of solutions and the well-posedness of certain differential equations with state-dependent delay.

Inspired by the ideas from [17–19], we establish the local and global existence, uniqueness and local well-posedness of the problem (1.1)–(1.2) in spaces of Lipschitz functions. In this context, we emphasize the importance of inequalities like the following:

$$\begin{aligned}&\|u(\sigma(\cdot,u(\cdot))) \;-\; v(\sigma(\cdot,v(\cdot)))\|_{C([0,b];Z)}\\&\quad\leq (1+[v]_{C_{\mathrm{Lip}}([-p,b];Z)}[\sigma]_{C_{\mathrm{Lip}}([0,b]\times X;\mathbb{R})})\|u-v\|_{C([0,b];Z)}.\end{aligned}$$

where

$$[v]_{C_{\mathrm{Lip}}([-p,b];Z)} := \sup_{t,s\in[-p,b],t\neq s}\frac{\|v(t)-v(s)\|_Z}{|t-s|}.$$

Moreover, such analysis becomes even more challenging in the context of integro-differential equations, since the associated resolvent operator does not satisfy the semigroup property. This difficulty motivates the present work, whose main contributions can be summarized as follows:

First, we establish existence and uniqueness results—both local and global—as well as well-posedness for the integro-differential problem described above in spaces of Lipschitz functions. This contribution is significant because the absence of the semigroup property in the associated resolvent operator makes the construction of maximal solutions considerably more intricate. Our results therefore extend the theory beyond the scope of classical semigroup-based approaches.

Second, we introduce and rigorously analyze a new class of coupled integro-differential systems that incorporate fading memory terms and diffusion operators with state-dependent delays. To the best of our knowledge, this paper provides the first systematic treatment of such systems, thereby opening new avenues of research in the field.

Third, we address additional aspects that, to date, have not appeared in the literature, reinforcing the novelty and breadth of our contributions.

Finally, the paper is organized into five sections. Section 2 introduces the notation and preliminary results used throughout the study. Section 3 develops the existence and uniqueness theory for problem (1.1)–(1.2), both locally and globally. Section 4 establishes the local well-posedness of (1.1)–(1.2) in the spaces $\mathcal{D}_a$ and $C([0,a];Z)$. Section 5 concludes with the analysis of a non-trivial coupled diffusive integro-differential system with state-dependent delay, further illustrating the scope of our framework.

2 Preliminaries

Let $(Z,\|\cdot\|_Z)$ and $(W,\|\cdot\|_W)$ be Banach spaces. In this work $\mathcal{B} = C([-p,0];Z)$ endowed with the usual norm $\|\cdot\|_{\mathcal{B}}$, $C_{\mathrm{Lip}}([b,c];Z)$ is the space of the Lipschitz functions from $[b,c]$ into Z endowed with the norm

$$\|\cdot\|_{C_{\mathrm{Lip}}([b,c];Z)} = \|\cdot\|_{C([b,c];Z)} + [\cdot]_{C_{\mathrm{Lip}}([b,c];Z)},$$

where $[\xi]_{C_{\text{Lip}}([b,c];Z)} = \sup_{t,s\in[b,c],t\neq s} \frac{\|\xi(t)-\xi(s)\|_Z}{|t-s|}$ and $C^\gamma([b,c];Z)$, with $\gamma \in (0,1)$, is the space of the γ-Hölder functions from $[b,c]$ into Z, endowed with the norm

$$\|\cdot\|_{C^\gamma([b,c];Z)} = \|\cdot\|_{C([b,c];Z)} + [\cdot]_{C^\gamma([b,c];Z)},$$

where $[\xi]_{C^\gamma([b,c];Z)} = \sup_{t,s\in[b,c],t\neq s} \frac{\|\xi(t)-\xi(s)\|_Z}{|t-s|^\gamma}$.

Furthermore, $\mathcal{L}(Z,W)$ is the space of bounded linear operators from Z into W endowed with the operator norm $\|\cdot\|_{\mathcal{L}(Z,W)}$ and we write $\mathcal{L}(Z)$ and $\|\cdot\|$ if $Z = W$. Finally, for $r > 0$ and $z \in Z$, $B_r(z,Z) = \{x \in Z; \|x-z\|_Z \leq r\}$.

As cited in the introduction, our results on existence and uniqueness of solutions are proved without assuming that $G(\cdot)$ and $\sigma(\cdot)$ are locally Lipschitz, they belong to a more general class of functions. The next definition comes from [18].

Definition 2.1. Let Y_i, $i = 1,2$, be Banach spaces and $p \geq 1$. A function $G \in C([c,d]\times Y_1;Y_2)$ is a L^p-Lipschitz function, if there exists an integrable function $[G]_{(\cdot,\cdot)} : [c,d]\times[c,d] \to \mathbb{R}^+$ and a non-decreasing function $\mathcal{W}_G : \mathbb{R}^+ \to \mathbb{R}^+$ such that $[G]_{(t,\cdot)} \in L^p([c,t];\mathbb{R}^+)$ and $[G]_{(\cdot,0)} \in L^p([c,t];\mathbb{R}^+)$ for all $t \in (c,d]$, and

$$\|G(t,x)-G(s,y)\|_{Y_2} \leq \mathcal{W}_G(\max\{\|x\|_{Y_1},\|y\|_{Y_1}\})[G]_{(t,s)}(|t-s|+\|x-y\|_{Y_1}),$$

for all $x,y \in Y_1$ and $c \leq s \leq t \leq d$. We use the notation $L^p_{\text{Lip}}([c,d]\times Y_1;Y_2)$ for the set formed by this type of functions.

The next result can be found in [18, Lemma 1] and there for $p,b > 0$, $u \in C([-p,b];Z)$ and $\sigma \in C([0,a]\times\mathcal{B};[0,p])$, $u^\sigma : [0,b] \to Z$ is the function given by $u^\sigma(t) = u(t-\sigma(t,u_t))$.

Lemma 2.2 ([18, Lemma 1]). *Assume $0 < b \leq a$, $u,v \in C([-p,b];Z)$, $\sigma \in L^r_{\text{Lip}}([0,a]\times\mathcal{B};[0,p])$ and let $\rho = \max\{\|u\|_{C([-p,b];Z)},\|v\|_{C([-p,b];Z)}\}$. If $u_{|_{[-p,0]}} \in C_{\text{Lip}}([-p,0];Z)$ and $u_{|_{[0,b]}} \in C_{\text{Lip}}([0,b];Z)$, then $u \in C_{\text{Lip}}([-p,b];Z)$, $u_{(\cdot)} \in C_{\text{Lip}}([0,b];\mathcal{B})$ and*

$$\begin{aligned}
\max\{[u]_{C_{\text{Lip}}([-p,b];Z)},[u_{(\cdot)}]_{C_{\text{Lip}}([0,b];\mathcal{B})}\} &\leq [u_0]_{C_{\text{Lip}}([-p,0];Z)} + [u_{|_{[0,b]}}]_{C_{\text{Lip}}([0,b];\mathcal{B})},\\
\|u^\sigma(s+h)-u^\sigma(s)\| &\leq [u]_{C_{\text{Lip}}([-p,b];Z)}(1+[\sigma]_{(s+h,s)}\mathcal{W}_\sigma(\rho)(1+[u]_{C_{\text{Lip}}([-p,b];Z)}))h,\\
\|u^\sigma(s)-v^\sigma(s)\| &\leq (1+[u]_{C_{\text{Lip}}([-p,b];Z)}[\sigma]_{(s,s)}\mathcal{W}_\sigma(\rho))\|u-v\|_{C([-p,b];Z)},
\end{aligned}$$

for all $h,s \in [0,b]$ with $s+h \in [0,b]$.

In addition, if $w \in C([-p,b];Z)$, $\alpha \in (0,1)$, $w_{|_{[0,b]}} \in C^\alpha([0,b];Z)$ and $w_0 \in C^\alpha([-p,0];Z)$, then $w_{(\cdot)} \in C^\alpha([0,b];\mathcal{B})$, $w^\sigma \in C^{\alpha^2}([0,b];Z)$ and

$$\begin{aligned}
[w_{(\cdot)}]_{C^\alpha([0,b];\mathcal{B})} &\leq \varpi(w,\alpha) := [w]_{C^\alpha([0,b];X)} + [w_0]_{C^\alpha([-p,0];Z)},\\
[w^\sigma]_{C^{\alpha^2}([0,b];Z)} &\leq [w]_{C^\alpha([-p,b];X)}(b^{1-\alpha} + [\sigma]_{C_{\text{Lip}}([0,a]\times\mathcal{B};[0,p])}(b^{1-\alpha}+[w]_{C^\alpha([-p,b];Z)}))^\alpha,\\
[w^\sigma]_{C^{\alpha^2}([0,b];Z)} &\leq [w]_{C^\alpha([-p,b];Z)}(b^{\alpha-\alpha^2} + [\sigma]^\alpha_{C_{\text{Lip}}([0,a]\times\mathcal{B};[0,p])}(b^{\alpha-\alpha^2}+[w]^\alpha_{C^\alpha([-p,b];Z)})).
\end{aligned}$$

To achieve our results, we mention the main points about the theory of the resolvent operator $(\mathcal{V}(t))_{t\geq 0}$ on Z to the integro-differential abstract Cauchy problem.

$$\frac{d}{dt}\left[u(t)+\int_0^t P(t-s)u(s)\,ds\right] = Au(t)+\int_0^t Q(t-s)u(s)\,ds, \tag{2.1}$$

$$u(0) = z \in Z. \tag{2.2}$$

where $A : D(A) \subseteq Z \to Z$, is a closed linear operator and the notation $[D(A)]$ represents the domain of A endowed with the graph norm, $\|z\|_1 = \|z\|_Z + \|Az\|_W$, $z \in D(A)$, and $Q(t)$ for $t \geq 0$ are closed linear operators defined on a common domain $D(A)$ that is dense in Z, while $P(t)$ for $t \geq 0$ is a family of bounded linear operators on Z.

Next we present the definition of resolvent operator family for the problem (2.1)–(2.2).

Definition 2.3. A one parameter family of bounded linear operators $(\mathcal{V}(t))_{t\geq 0}$ on Z is called a resolvent operator of (2.1)–(2.2) if the following conditions are verified.

(a) The function $\mathcal{V}(\cdot) : [0,\infty) \to \mathcal{L}(Z)$ is strongly continuous, exponentially bounded and $\mathcal{V}(0)z = z$ for all $z \in Z$.

(b) For $z \in D(A)$, $\mathcal{V}(\cdot)z \in C([0,\infty); [D(A)]) \bigcap C^1((0,\infty); Z)$, and

$$\frac{d}{dt}\left[\mathcal{V}(t)z + \int_0^t P(t-s)\mathcal{V}(s)zds\right] = A\mathcal{V}(t)z + \int_0^t Q(t-s)\mathcal{V}(s)zds, \tag{2.3}$$

$$\frac{d}{dt}\left[\mathcal{V}(t)z + \int_0^t \mathcal{V}(t-s)P(s)zds\right] = \mathcal{V}(t)Az + \int_0^t \mathcal{V}(t-s)Q(s)zds, \tag{2.4}$$

for every $t \geq 0$.

The existence of a resolvent operator for problem (2.1)–(2.2) was studied in [7,8]. In these works we considered the following conditions:

(C1) The operator $A : D(A) \subseteq Z \to Z$ is the infinitesimal generator of an analytic semigroup $(T(t))_{t\geq 0}$ on Z, and there are constants $M_0 > 0$ and $\vartheta \in (\pi/2, \pi)$ such that $\rho(A) \supseteq \Lambda_\vartheta = \{\lambda \in \mathbb{C} \setminus \{0\} : |\arg(\lambda)| < \vartheta\}$ and $\|R(\lambda, A)\| \leq M_0|\lambda|^{-1}$ for all $\lambda \in \Lambda_\vartheta$.

(C2) The function $P : [0,\infty) \to \mathcal{L}(Z)$ is strongly continuous and $\widehat{P}(\lambda)z$ is absolutely convergent for $z \in Z$ and $Re(\lambda) > 0$. There exists $\alpha > 0$ and an analytical extension of $\widehat{P}(\lambda)$ (still denoted by $\widehat{P}(\lambda)$) to Λ_ϑ such that $\|\widehat{P}(\lambda)\| \leq N_0|\lambda|^{-\alpha}$ for every $\lambda \in \Lambda_\vartheta$, and $\|\widehat{P}(\lambda)z\| \leq N_1|\lambda|^{-1}\|z\|_1$ for every $\lambda \in \Lambda_\vartheta$ and $z \in D(A)$.

(C3) For all $t \geq 0$, $Q(t) : D(Q(t)) \subseteq Z \to Z$ is a closed linear operator, $D(A) \subseteq D(Q(t))$ and $Q(\cdot)z$ is strongly measurable on $(0,\infty)$ for each $z \in D(A)$. There exists $q(\cdot) \in L^1_{\text{loc}}(\mathbb{R}^+)$ such that $\widehat{q}(\lambda)$ exists for $Re(\lambda) > 0$ and $\|Q(t)z\| \leq q(t)\|z\|_1$ for all $t > 0$ and $z \in D(A)$. Moreover, the operator valued function $\widehat{Q} : \Lambda_{\pi/2} \to \mathcal{L}([D(A)]; Z)$ has an analytical extension (still denoted by $\widehat{Q}$) to Λ_ϑ such that $\|\widehat{Q}(\lambda)z\| \leq \|\widehat{Q}(\lambda)\|\,\|z\|_1$ for all $z \in D(A)$, and $\|\widehat{Q}(\lambda)\| \to 0$ as $|\lambda| \to \infty$.

(C4) There exists a subspace $D \subseteq D(A)$ dense in $[D(A)]$ and positive constants C_i, $i = 1, 2$, such that $A(D) \subseteq D(A)$, $\widehat{Q}(\lambda)(D) \subseteq D(A)$, $\widehat{P}(\lambda)(D) \subseteq D(A)$, $\|A\widehat{Q}(\lambda)z\| \leq C_1\|z\|$ and $\|\widehat{P}(\lambda)z\|_1 \leq C_2|\lambda|^{-\alpha}\|z\|_1$ for every $z \in D$ and all $\lambda \in \Lambda_\vartheta$.

The following result has been established in [8, Theorem 2.1].

Theorem 2.4 ([8, Theorem 2.1])**.** *Assume that conditions* **(C1)–(C4)** *are fulfilled. Then there exists a unique resolvent operator for problem* (2.1)–(2.2).

For the development of this work we will assume that the conditions **(C1)–(C4)** are satisfied.

Remark 2.5. From Theorem 2.4 it follows that there exists a constant $M > 0$ such that the resolvent satisfies $\|\mathcal{V}(t)\| \leq M$, for all $t \in [0, a]$ and there exists a constant $C > 0$ such that $\|A\mathcal{V}(t)\| \leq \frac{C}{t}$ for all $t \in (0, a]$ and $z \in Z$.

Let E be the space consisting of $z \in Z$ such that

$$\mathcal{V}(\cdot)z \in C([0,\infty),[D(A)]) \cap C^1([0,\infty);Z).$$

It is clear that $E \subseteq D(A)$ and $\frac{d}{dt}\mathcal{V}(t)z|_{t=0} = Az - P(0)z$ for $z \in E$. For more details see [8, Remark 2.2]

The following definitions and results are related to the non-homogeneous problem

$$\frac{d}{dt}\left[u(t) + \int_0^t P(t-s)u(s)ds\right] = Au(t) + \int_0^t Q(t-s)u(s)\,ds + G(t), \qquad t \in I = [0,b], \quad (2.5)$$

with initial condition (2.2), where $G : [0,b] \to Z$ is a continuous function.

Definition 2.6. A function $u \in C([0,b];Z)$, $0 < b \leq a$ is called a mild solution of (2.5)–(2.2) on $[0,b]$ if $u(0) = z$ and

$$u(t) = \mathcal{V}(t)z + \int_0^t \mathcal{V}(t-s)G(s)ds, \qquad \forall\, t \in [0,b]. \quad (2.6)$$

Definition 2.7. A function $u : [0,b] \to Z$, $0 < b \leq a$, is called a classical solution of (2.5)–(2.2) on $[0,b]$ if $u \in C([0,b],[D(A)]) \cap C^1((0,b];Z)$, the condition (2.2) holds and the equation (2.5) is verified on $[0,b]$. Moreover, if $u \in C([0,b],[D(A)]) \cap C^1([0,b];Z)$, the function u is said to be a strict solution of (2.5)–(2.2) on $[0,b]$.

Remark 2.8. From the above definition, it is clear that $\mathcal{V}(\cdot)z$ is a classical solution of problem (2.5)–(2.2) on $(0,\infty)$ for $z \in D(A)$ and $G(t) \equiv 0$.

In the work [8] the following results are established for the problem (2.5)–(2.2)

Theorem 2.9 ([8, Theorem 2.4]). *Let $z \in D(A)$. Assume that $G \in C([0,b];Z)$ and $u(\cdot)$ is a classical solution of problem (2.5)–(2.2) on $(0,b]$. Then*

$$u(t) = \mathcal{V}(t)z + \int_0^t \mathcal{V}(t-s)G(s)\,ds, \qquad t \in [0,b]. \quad (2.7)$$

Theorem 2.10 ([8, Lemma 3.1.1]). *If $R(\lambda_0, A)$ is compact for some $\lambda_0 \in \rho(A)$, then $\mathcal{V}(t)$ is compact for all $t > 0$.*

Theorem 2.11 ([8, Theorem 2.6]). *Let $z \in D(A)$ and let $G \in W^{1,1}([0,a];Z)$. Then the mild solution $u(\cdot)$ of problem (2.5)–(2.2) is a classical solution on $[0,a]$. Further, if $z \in E$, then $u(\cdot)$ is a strict solution on $[0,a]$.*

Corollary 2.12 ([8, Corollary 2.2]). *Let $z \in D(A)$ and $G \in C([0,a];Z)$. Let $u(\cdot)$ be the mild solution of problem (2.5)–(2.2). If $u \in C([0,a],[D(A)])$, then $u(\cdot)$ is a classical solution on $(0,a]$.*

3 Existence and uniqueness of solution

In this section we begin by studying the existence and uniqueness of mild solution for the integro-differential problem (1.1)–(1.2). First, we establish some concepts and assumptions that will be used in this work. For this remember that for $p, b > 0$, $u \in C([-p,b];Z)$ and $\sigma \in C([0,a] \times \mathcal{B};[0,p])$, $u^\sigma : [0,b] \to Z$ denotes the function given by $u^\sigma(t) = u(t - \sigma(t,u_t))$.

Definition 3.1. A function $u \in C([-p,b];Z)$, $0 < b \leq a$ is called a mild solution of (1.1)–(1.2) on $[-p,b]$ if $u_0 = \varphi$ on $[-p,0]$ and

$$u(t) = \mathcal{V}(t)\varphi(0) + \int_0^t \mathcal{V}(t-s)G\left(s,u^\sigma(s)\right)ds, \qquad \forall\, t \in [0,b]. \quad (3.1)$$

Definition 3.2. A function $u \in C([-p,b];Z)$, $0 < b \leq a$, is said to be a strict solution (resp. a classical solution) of (1.1)–(1.2) on $[-p,b]$ if $u_{|_{[0,b]}} \in C^1([0,b];Z) \cap C([0,b];[D(A)])$ (resp. $u_{|_{[0,b]}} \in C^1((0,b];Z) \cap C((0,b];[D(A)])$), $u_0 = \varphi$ and $u(\cdot)$ satisfies (1.1) on $[0,b]$ (resp. satisfies (1.1) on $(0,b]$).

To establish the main results of this section, we introduce the following condition.

Condition $H^{q,r}_{G,\sigma,a}$: $q, r \geq 1$, $\Theta(q,r) = \frac{1}{q} + \frac{1}{r} \leq 1$, $G \in L^q_{\text{Lip}}([0,a] \times Z;Z)$ and $\sigma \in L^r_{\text{Lip}}([0,a] \times \mathcal{B};[0,p])$.

Notation 3.3. If condition $H^{q,r}_{G,\sigma,a}$ holds, we use the notations

$$\Sigma_1(b) := \sup_{h,c \in [0,b], h+c \in [0,b]} \|[G]_{(\cdot+h,\cdot)}(1 + [\sigma]_{(\cdot+h,\cdot)})\|_{L^1([0,c])},$$
$$\Sigma_2(b) := \sup_{c \in [0,b]} \|[G]_{(\cdot,0)}(1 + [\sigma]_{(\cdot,0)})\|_{L^1([0,c])}.$$

In addition, if $\Theta(q,r) < 1$, $\Lambda(q,r)$ is the number defined by the relation $\Theta(q,r) + \frac{1}{\Lambda(q,r)} = 1$.

Now we can present the first theorem of this work.

Theorem 3.4. *Suppose $\varphi \in C_{\text{Lip}}([-p,0];Z)$, $\mathcal{V}(\cdot)\varphi(0) \in C_{\text{Lip}}([0,a];Z)$. If condition $H^{q,r}_{G,\sigma,a}$ is satisfied and $\Sigma_1(b) + \Sigma_2(b) \to 0$ as $b \to 0$, then there exists a unique mild solution $u \in C_{\text{Lip}}([-p,b];Z)$ of the problem (1.1)–(1.2) on $[-p,b]$ for some $0 < b \leq a$.*

Proof. Consider $R_\varphi > [\mathcal{V}(\cdot)\varphi(0)]_{C_{\text{Lip}}([0,a];Z)} + \partial|\varphi\|_{C_{\text{Lip}}([-p,0];Z)} + M\|G(0,\varphi(-\sigma(0,\varphi)))\|$. From the hypothesis on $\Sigma_i(\cdot)$, $i = 1,2$, we can choose $0 < b \leq a$ such that

$$[\mathcal{V}(\cdot)\varphi(0)]_{C_{\text{Lip}}([0,a];Z)} + \|\varphi\|_{C_{\text{Lip}}([-p,0];Z)} + M\|G(0,\varphi(-\sigma(0,\varphi)))\| + (1+R_\varphi)^2\mathcal{W}_G(\rho)(1+\mathcal{W}_\sigma(\rho))M\left(\Sigma_1(b)+\Sigma_2(b)\right) < R_\varphi, \tag{3.2}$$

$$(1+R_\varphi)\mathcal{W}_G(\rho)M(1+\mathcal{W}_\sigma(\rho))\Sigma_1(b) < 1 \tag{3.3}$$

with $\rho := R_\varphi(b+p) + \|\varphi\|_{\mathcal{B}}$.

Define the space

$$\mathcal{Z}(b,R_\varphi) = \{u \in C([-p,b];Z) : u_0 = \varphi,\ [u]_{C_{\text{Lip}}([-p,b];Z)} \leq R_\varphi\},$$

endowed with the metric $d(u,v) = \|u-v\|_{C([0,b];Z)}$ and let $\Gamma : \mathcal{Z}(b,R_\varphi) \to C([-p,b];Z)$ be the map defined by $\Gamma u(t) = \varphi(t)$ for $t \in [-p,0]$ and

$$\Gamma u(t) = \mathcal{V}(t)\varphi(0) + \int_0^t \mathcal{V}(t-s)G\left(s,u^\sigma(s)\right)ds, \quad \text{for } t \in [0,b].$$

We state that $\Gamma(\cdot)$ is a contraction. Indeed, for $u \in \mathcal{Z}(b,R_\varphi)$ and $s \in [0,b]$, we note that

$$\begin{aligned}\max\{\|u(s)\|, \|u^\sigma(s)\|, \|u_s\|_{\mathcal{B}}\} &\leq \sup_{\tau \in [-p,b]} \|u(\tau) - \varphi(0)\| + \|\varphi(0)\| \\ &\leq [u]_{C_{\text{Lip}}([-p,b];X)}(b+p) + \|\varphi\|_{\mathcal{B}} \\ &\leq R_\varphi(b+p) + \|\varphi\|_{\mathcal{B}} \leq \rho. \end{aligned} \tag{3.4}$$

Using the former estimate and Lemma 2.2, for $s, h \in [0, b]$ with $s + h \in [0, b]$, we have that

$$\begin{aligned}
&\|G(s+h, u^\sigma(s+h)) - G(s, u^\sigma(s))\| \\
&\quad \le [G]_{(s+h,s)} \mathcal{W}_G(\rho)(h + [u]_{C_{\mathrm{Lip}}([-p,b];Z)}(h + \mathcal{W}_\sigma(\rho)[\sigma]_{(s+h,s)}(h + [u]_{C_{\mathrm{Lip}}([-p,b];Z)}h)) \\
&\quad \le [G]_{(s+h,s)} \mathcal{W}_G(\rho)(1 + R_\varphi(1 + \mathcal{W}_\sigma(\rho)[\sigma]_{(s+h,s)}(1 + R_\varphi)))h \\
&\quad \le (1 + R_\varphi)^2 \mathcal{W}_G(\rho)(1 + \mathcal{W}_\sigma(\rho))[G]_{(s+h,s)}(1 + [\sigma]_{(s+h,s)})h. \qquad (3.5)
\end{aligned}$$

For $0 \le s \le h \le b$ and arguing as above, we also have that

$$\|G(s, u^\sigma(s)) - G(0, \varphi(-\sigma(0,\varphi)))\| \le (1 + R_\varphi)^2 \mathcal{W}_G(\rho)(1 + \mathcal{W}_\sigma(\rho))[G]_{(s,0)}(1 + [\sigma]_{(s,0)})h. \quad (3.6)$$

Using the previous inequalities, for $h, t \in [0, b]$ with $t + h \in [0, b]$, we obtain

$$\begin{aligned}
&\|\Gamma u(t+h) - \Gamma u(t)\| \\
&\quad \le [\mathcal{V}(\cdot)\varphi(0)]_{C_{\mathrm{Lip}}([0,b];Z)}h + \int_0^h \|\mathcal{V}(t+h-s))G(0, \varphi(-\sigma(0,\varphi)))\|ds \\
&\qquad + \int_0^h \|\mathcal{V}(t+h-s))\|\|G(s, u^\sigma(s)) - G(0, \varphi(-\sigma(0,\varphi)))\|ds \\
&\qquad + \int_0^t \|\mathcal{V}(t-s)\|\|G(s+h, u^\sigma(s+h)) - G(s, u^\sigma(s))\|ds \\
&\quad \le [\mathcal{V}(\cdot)\varphi(0)]_{C_{\mathrm{Lip}}([0,b];Z)}h + M\|G(0, \varphi(-\sigma(0,\varphi))\|h \\
&\qquad + (1 + R_\varphi)^2 \mathcal{W}_G(\rho)(1 + \mathcal{W}_\sigma(\rho))M \int_0^h [G]_{(s,0)}(1 + [\sigma]_{(s,0)})sds \\
&\qquad + (1 + R_\varphi)^2 \mathcal{W}_G(\rho)(1 + \mathcal{W}_\sigma(\rho))M \int_0^t [G]_{(s+h,s)}(1 + [\sigma]_{(s+h,s)})hds \\
&\quad \le [\mathcal{V}(\cdot)\varphi(0)]_{C_{\mathrm{Lip}}([0,b];Z)}h + M\|G(0, \varphi(-\sigma(0,\varphi))\|h \\
&\qquad + (1 + R_\varphi)^2 \mathcal{W}_G(\rho)(1 + \mathcal{W}_\sigma(\rho))M(\Sigma_1(b) + \Sigma_2(b))h. \qquad (3.7)
\end{aligned}$$

Then $[\Gamma u]_{C_{\mathrm{Lip}}([0,b];Z)} \le R_\varphi$ and $[\Gamma u]_{C_{\mathrm{Lip}}([-p,b];Z)} \le R_\varphi$ since $\|\varphi\|_{C_{\mathrm{Lip}}([-p,0];Z)} < R_\varphi$. This proves that $\Gamma(\cdot)$ is a $\mathcal{Z}(b, R_\varphi)$-valued function.

Also, from Lemma 2.2, for $u, v \in \mathcal{Z}(b, R_\varphi)$ and $t \in [0, b]$ it follows that

$$\begin{aligned}
&\|\Gamma u(t) - \Gamma v(t)\| \\
&\quad \le \int_0^t \|\mathcal{V}(t-s)\|\|G(s, u^\sigma(s)) - G(s, v^\sigma(s))\|ds \\
&\quad \le \mathcal{W}_G(\rho) \int_0^t M[G]_{(s,s)}\|u^\sigma(s) - v^\sigma(s)\|ds \\
&\quad \le \mathcal{W}_G(\rho) \int_0^t M[G]_{(s,s)}(1 + [u]_{C_{\mathrm{Lip}}([-p,b];Z)}\mathcal{W}_\sigma(\rho)[\sigma]_{(s,s)})\|u - v\|_{C([0,s];Z)}ds \\
&\quad \le \mathcal{W}_G(\rho) \int_0^t M[G]_{(s,s)}(1 + R_\varphi \mathcal{W}_\sigma(\rho)[\sigma]_{(s,s)})\|u - v\|_{C([0,s];Z)}ds \\
&\quad \le (1 + R_\varphi)\mathcal{W}_G(\rho)(1 + \mathcal{W}_\sigma(\rho))M \int_0^t [G]_{(s,s)}(1 + [\sigma]_{(s,s)})ds\|u - v\|_{C([0,b];Z)} \\
&\quad \le (1 + R_\varphi)\mathcal{W}_G(\rho)(1 + \mathcal{W}_\sigma(\rho))\Sigma_1(b)M\|u - v\|_{C([0,b];Z)}, \qquad (3.8)
\end{aligned}$$

and, from (3.3), we conclude that $\Gamma(\cdot)$ is a contraction on $\mathcal{Z}(b, R_\varphi)$ and then there exists a unique mild solution $u \in C_{\mathrm{Lip}}([-p, b]; Z)$ of (1.1)–(1.2) on $[-p, b]$. □

For the next result we will use the Radon–Nikodym property. It is well-known that every reflexive space has the Radon–Nikodym property (abbreviated, RNP) and that there are non reflexive spaces that have the RNP. We refer to [6] several characterizations of the RNP. For this reason, in the sequel, we consider spaces that have the RNP.

Proposition 3.5. *If conditions in Theorem 3.4 are valid, Z has the RNP, the functions $[G]_{(\cdot,\cdot)}$ and $[\sigma]_{(\cdot,\cdot)}$ are continuous, and $u \in C_{\mathrm{Lip}}([-p,b];Z)$ is the unique associated mild solution of the problem (1.1)–(1.2), then $u(\cdot)$ is a classical solution if $\varphi(0) \in D(A)$ and a strict solution if $\varphi(0) \in E$.*

Proof. Note that

$$\begin{aligned}
&\|G(t,u^\sigma(t)) - G(s,u^\sigma(s))\| \\
&\quad \le [G]_{(t,s)}\mathcal{W}_G(\rho)(1+[u]_{C_{\mathrm{Lip}}([-p,b];Z)}(1+\mathcal{W}_\sigma(\rho)[\sigma]_{(t,s)}(1+[u]_{C_{\mathrm{Lip}}([-p,b];Z)}))(t-s) \\
&\quad \le \mathcal{K}|t-s|,
\end{aligned}$$

where $\rho = \|u\|_{C([-p,b];Z)}$ and

$$\mathcal{K} = \sup_{(t,s)\in[0,a]\times[0,a]} ([G]_{(t,s)}\mathcal{W}_G(\rho)(1+[u]_{C_{\mathrm{Lip}}([-p,b];Z)}(1+\mathcal{W}_\sigma(\rho)[\sigma]_{(t,s)}(1+[u]_{C_{\mathrm{Lip}}([-p,b];Z)}))).$$

Consequently, the function $t \mapsto G(t) = G(t,u^\sigma(t))$ is Lipschitz continuous and since the space Z has satisfies the RNP, the function $G(\cdot) \in W^{1,1}([0,b];Z)$. The statement is now consequence of Theorem 2.11. □

In the next corollary, $\Theta(q,r)$ and $\Lambda(q,r)$ are the numbers defined in Notation 3.3.

Corollary 3.6. *Suppose that $\varphi \in C_{\mathrm{Lip}}([-p,0];Z)$, $\mathcal{V}(\cdot)\varphi(0) \in C_{\mathrm{Lip}}([0,a];Z)$ and assume that $\Theta(q,r) < 1$. If there exists $0 < c \le a$ such that*

$$\Sigma_3(c) = \sup_{b\in[0,c],d,h\in[0,b],d+h\le b} \|[G]_{(\cdot+h,\cdot)}\|_{L^q([0,d])}\|(1+[\sigma]_{(\cdot+h,\cdot)})\|_{L^r([0,d])} < \infty.$$

Then there exists a unique mild solution $u \in C_{\mathrm{Lip}}([-p,b];Z)$ of the problem (1.1)–(1.2) on $[-p,b]$ for some $0 < b \le a$. In particular, the assertion is valid if $G(\cdot)$ and $\sigma(\cdot)$ are Lipschitz.

Proof. From Theorem 3.4, it is sufficient to show that $\Sigma_1(b) + \Sigma_2(b) \to 0$ as $b \to 0$. For $b \in [0,c]$, $t,h \in [0,b)$ with $t+h \le b$, it is easy to see that

$$\begin{aligned}
&\int_0^t [G]_{(s+h,s)}(1+[\sigma]_{(s+h,s)})ds \le \Sigma_3(c), \\
&\int_0^t [G]_{(s,0)}(1+[\sigma]_{(s,0)})ds \le \|[G]_{(\cdot,0)}\|_{L^q([0,b])}\|(1+[\sigma]_{(\cdot,0)})\|_{L^r([0,b])},
\end{aligned}$$

which implies that $\Sigma_1(b) + \Sigma_2(b) \to 0$ as $b \to 0$. The last claim is direct. □

Now we will discuss solutions defined on a pre-defined interval $[-p,a]$ and on the whole semi-axis $[-p,\infty)$.

Remark 3.7. If condition $\boldsymbol{H}^{q,r}_{G,\sigma,a}$ is satisfied, for $x > 0$ and $b \in [0,a]$, we assume that

$$\begin{aligned}
\Sigma_{1,b}(x) &:= \sup_{h,c\in[0,b],h+c\in[0,b]} \|\mathcal{W}_{F,\sigma}(\rho(x,\cdot))[G]_{(\cdot+h,\cdot)}(1+[\sigma]_{(\cdot+h,\cdot)})\|_{L^1([0,c])}, \\
\Sigma_{2,b}(x) &:= \sup_{c\in[0,b]} \|\mathcal{W}_{F,\sigma}(\rho(x,\cdot))[G]_{(\cdot,0)}(1+[\sigma]_{(\cdot,0)})\|_{L^1([0,c])},
\end{aligned}$$

where $\mathcal{W}_{F,\sigma}(\rho(x,s)) = \mathcal{W}_G(\rho(x,s))(1+\mathcal{W}_\sigma(\rho(x,s)))$ and $\rho(x,s) = x(s+p) + \|\varphi\|_{\mathcal{B}}$, are finite. Additionally, if the condition $\boldsymbol{H}^{q,r}_{G,\sigma,a}$ is satisfied for all $a > 0$, we use the notations $\Sigma_{1,\infty}(x) := \sup_{b\ge0}\Sigma_{1,b}(x)$ and $\Sigma_{2,\infty}(x) := \sup_{b\ge0}\Sigma_{2,b}(x)$.

From the proof of Theorem 3.4, we deduce the next result.

Proposition 3.8. *Suppose that condition* $\boldsymbol{H}^{q,r}_{G,\sigma,a}$ *is satisfied,* $\varphi \in C_{\mathrm{Lip}}([-p,0];Z)$, $\mathcal{V}(\cdot)\varphi(0) \in C_{\mathrm{Lip}}([0,a];Z)$ *and let* $\mathcal{P}_a : \mathbb{R} \mapsto \mathbb{R}$ *be the function given by*

$$\begin{aligned}\mathcal{P}_a(x) := [\mathcal{V}(\cdot)\varphi(0)]_{C_{\mathrm{Lip}}([0,a];Z)} + \|\varphi\|_{C_{\mathrm{Lip}}([-p,0];Z)} + M\|G(0,\varphi(-\sigma(0,\varphi))\| \\ + (1+x)^2 M(\Sigma_{1,a}(x) + \Sigma_{2,a}(x)) + (1+x)M\Sigma_{1,a}(x)x - x.\end{aligned}$$

If $\mathcal{P}_a(R_\varphi) < 0$ *for some* $R_\varphi > 0$, *then there exists a unique mild solution* $u \in C_{\mathrm{Lip}}([-p,a];Z)$ *of problem* (1.1)–(1.2) *on* $[-p,a]$.

Proof. The demonstration of this proposition follows from the proof of Theorem 3.4 and we will include some details to complete the proof. We remark, from the definition of $\mathcal{P}_a(\cdot)$ and our hypothesis about $\mathcal{P}_a(R_\varphi)$, that

$$\begin{aligned}[\mathcal{V}(\cdot)\varphi(0)]_{C_{\mathrm{Lip}}([0,a];X)} + \|\varphi\|_{C_{\mathrm{Lip}}([-p,0];X)} + M\|G(0,\varphi(-\sigma(0,\varphi))\| \\ + M(1+R_\varphi)^2\left(\Sigma_{1,a}(R_\varphi) + \Sigma_{2,a}(R_\varphi)\right) + M(1+R_\varphi)\Sigma_{1,a}(R_\varphi)R_\varphi < R_\varphi. \quad (3.9)\end{aligned}$$

Consider $\mathcal{Z}(a,R_\varphi)$ and $\Gamma(\cdot)$ as in the proof Theorem 3.4, but using a instead of b and let $u \in \mathcal{Z}(a,R_\varphi)$.

Proceeding as in the proof of Theorem 3.4 and from the definition of $\rho(R_\varphi,s)$, for $s \in [0,a]$ we obtain

$$\max\{\|u(s)\|, \|u^\sigma(s)\|, \|u_s\|_{\mathcal{B}}\} \leq R_\varphi(a+p) + \|\varphi\|_{\mathcal{B}} \leq \rho(R_\varphi,s).$$

From the inequality above and the proof of Theorem 3.4, for $s \in [0,a]$ and $\tau, h \in [0,a]$ with $\tau + h \in [0,a]$, it follows that

$$\begin{aligned}&\|G(\tau+h,u^\sigma(\tau+h)) - G(\tau,u^\sigma(\tau))\| \\ &\quad \leq (1+R_\varphi)^2\mathcal{W}_G(\rho(R_\varphi,\tau)))(1+\mathcal{W}_\sigma(\rho(R_\varphi,\tau)))[G]_{(\tau+h,\tau)}(1+[\sigma]_{(\tau+h,\tau)})h,\end{aligned}$$

$$\begin{aligned}&\|G(s,u^\sigma(s)) - G(0,\varphi(-\sigma(0,\varphi)))\| \\ &\quad \leq (1+R_\varphi)^2\mathcal{W}_G(\rho(R_\varphi,s))(1+\mathcal{W}_\sigma(\rho(R_\varphi,s)))[G]_{(s,0)}(1+[\sigma]_{(s,0)})h.\end{aligned}$$

From the previous inequalities and arguing as in the estimates (3.7) and (3.8) we have that

$$\begin{aligned}\|\Gamma u(t+h) - \Gamma u(t)\| &\leq [\mathcal{V}(\cdot)\varphi(0)]_{C_{\mathrm{Lip}}([0,a];X)}h + M\|G(0,\varphi(-\sigma(0,\varphi))\|h \\ &\quad + (1+R_\varphi)^2 M(\Sigma_{1,a}(R_\varphi) + \Sigma_{2,a,}(R_\varphi))h, \\ \|\Gamma u(t) - \Gamma v(t)\| &\leq (1+R_\varphi)M\Sigma_{1,a}(R_\varphi)\|u-v\|_{C([0,a];X)},\end{aligned}$$

for all $t \in [0,a]$. Then, $\Gamma(\cdot)$ is a contraction on $\mathcal{Z}(a,R_\varphi)$ and we conclude that there exists a unique mild solution $u \in C_{\mathrm{Lip}}([-p,a];X)$ of (1.1)–(1.2) on $[-p,a]$. □

Proceeding as in the proof of the Proposition 3.8, we can demonstrate the next result.

Corollary 3.9. *Let* $\varphi \in C_{\mathrm{Lip}}([-p,0];Z)$ *and* $\mathcal{V}(\cdot)\varphi(0) \in C_{\mathrm{Lip}}([0,a];Z)$.

(a) *Assume that* $G \in C_{\mathrm{Lip}}([0,a] \times B_r(0,X);Z)$ *and that* $\sigma \in C_{\mathrm{Lip}}([0,a] \times B_r(0,\mathcal{B});[0,p])$ *for all* $r > 0$ *and let* $\mathcal{P}_a : \mathbb{R} \mapsto \mathbb{R}$ *be the function given by*

$$\begin{aligned}\mathcal{P}_a(x) = [\mathcal{V}(\cdot)\varphi(0)]_{C_{\mathrm{Lip}}([0,a];Z)} + \|\varphi\|_{C_{\mathrm{Lip}}([-p,0];Z)} + M\|G(0,\varphi(-\sigma(0,\varphi))\| \\ + [(1+x)^2 + (1+x)][G]_{C_{\mathrm{Lip},\rho(x)}}(1+[\sigma]_{C_{\mathrm{Lip},\rho(x)}})Ma - x,\end{aligned}$$

where $\rho(x) = x(a+2p) + \|\varphi\|_{\mathcal{B}}$ *and* $[G]_{C_{\mathrm{Lip},\rho(x)}}, [\sigma]_{C_{\mathrm{Lip},\rho(x)}}$ *are the Lipschitz constants of* $G(\cdot)$ *and* $\sigma(\cdot)$ *on* $[0,a] \times B_{\rho(x)}(0,X)$ *and* $[0,a] \times B_{\rho(x)}(0,\mathcal{B})$. *If* $\mathcal{P}_a(y) < 0$ *for some* $y > 0$, *then there exists a unique mild solution* $u \in C_{\mathrm{Lip}}([-p,a];Z)$ *of* (1.1)–(1.2) *on* $[-p,a]$.

(b) *If $G(\cdot)$ and $\sigma(\cdot)$ are Lipschitz and $\mathcal{P}_a(y) < 0$ for some $y > 0$, where*

$$\begin{aligned}\mathcal{P}_a(x) &= [\mathcal{V}(\cdot)\varphi(0)]_{C_{\mathrm{Lip}}([0,a];Z)} + \|\varphi\|_{C_{\mathrm{Lip}}([-p,0];Z)} + M\|G(0,\varphi(-\sigma(0,\varphi))\| \\ &\quad + (1+x)^2[G]_{C_{\mathrm{Lip}}}(1+[\sigma]_{C_{\mathrm{Lip}}})Ma + (1+x)[G]_{C_{\mathrm{Lip},\rho(x)}}(1+[\sigma]_{C_{\mathrm{Lip},\rho(x)}})Ma - x \\ &= \theta + (1+x)^2\mu_a + (1+x)\mu_a - x, \end{aligned} \tag{3.10}$$

then there exists a unique mild solution $u \in C_{\mathrm{Lip}}([-p,a];Z)$ of (1.1)–(1.2) on $[-p,a]$. In particular, the affirmation holds if $1 > \frac{4\mu_a(\theta+2\mu_a)}{(3\mu_a-1)^2}$.

With respect to the existence and uniqueness of the solutions defined in $[-p,\infty)$, we have the next corollary.

Corollary 3.10. *Assume that $\mathcal{V}(\cdot)\varphi(0) \in C_{\mathrm{Lip}}([0,\infty);Z)$, $\varphi \in C_{\mathrm{Lip}}([-p,0];Z)$, that the condition $\boldsymbol{H}^{q,r}_{G,\sigma,a}$ is satisfied for all $a > 0$ and that $\Sigma_{i,\infty}(x) < \infty$ for $i = 1,2$ and every $x > 0$. Let $\mathcal{P}_\infty : [0,\infty) \mapsto [0,\infty)$ be the function given by*

$$\begin{aligned}\mathcal{P}_\infty(x) &= [\mathcal{V}(\cdot)\varphi(0)]_{C_{\mathrm{Lip}}([0,\infty;Z)} + \|\varphi\|_{C_{\mathrm{Lip}}([-p,0];Z)} + M\|G(0,\varphi(-\sigma(0,\varphi))\| \\ &\quad + (1+x)^2M(\Sigma_{1,\infty}(x) + \Sigma_{2,\infty}(x)) + (1+x)M\Sigma_{1,\infty}(x)x - x. \end{aligned} \tag{3.11}$$

If $\mathcal{P}_\infty(y) < 0$ for some $y > 0$, then there exists a unique locally Lipschitz mild solution $u \in C([-p,\infty);Z)$ of the problem (1.1)–(1.2) on $[-p,\infty)$.

Proof. Given $a > 0$, let $\mathcal{P}_a(\cdot)$ be the function in Proposition 3.8. As $0 < \mathcal{P}_a(x) \leq \mathcal{P}_\infty(x)$ for all $x \geq 0$, it follows that $0 < \mathcal{P}_a(y) < 0$ for all $a > 0$.

From Proposition 3.8, there exists a unique mild solution $u^a \in C_{\mathrm{Lip}}([-p,a];Z)$ of (1.1)–(1.2) on $[-p,a]$. Since the solution is unique and setting $u : [0,\infty) \mapsto X$ as $u(t) = u^a(t)$ for $t \in [0,a]$, we get a unique locally Lipschitz mild solution of (1.1)–(1.2) defined on the semi-axis $[-p,\infty)$. □

Remark 3.11. The first example of Section 5 contains comments with respect to the existence of the numbers $\Sigma_{i,\infty}(x)$ and an application of Corollary 3.10.

In the next result we investigate the existence and uniqueness of a maximal locally Lipschitz mild solution for (1.1)–(1.2). For this purpose, we say that $\eta : [c,d] \to \mathbb{R}$ is a Grönwall's type function, if the Grönwall's inequality is valid for inequalities of the form $\gamma(t) \leq l_1 + l_2\int_c^t \eta(s)\gamma(s)ds$ with $\gamma(\cdot)$ continuous, that is, this inequality implies that $\gamma(t) \leq l_1 e^{l_2\int_0^t \eta(s)ds}$ for all $t \in [c,d]$.

Proposition 3.12. *Suppose that the hypotheses of Theorem 3.4 hold, $q > 1$, $[G]_{(s,s)}$ is a Grönwall-type function, $\mathcal{W}_\infty := \sup_{s\geq 0}\mathcal{W}_G(s)(1+\mathcal{W}_\sigma(s)) < \infty$, and there is $\alpha \in (0,\frac{1}{q'})$, such that*

$$\sup_{t\in[0,a]}\left\|\frac{[G]_{(t,\cdot)}}{(t-\cdot)^{1-\alpha}}[\sigma]^\alpha_{(t,\cdot)}\right\|_{L^1([0,t])} < \infty$$

and that

$$\Sigma_3(c,\nu) := \sup_{h\in[0,\nu],d\in[c,c+\nu]}\|[G]_{(\cdot+h,\cdot)}(1+[\sigma]_{(\cdot+h,\cdot)})\|_{L^1([c-\nu,d])} \to 0, \tag{3.12}$$

as $\nu \downarrow 0$, for all $c \in [0,a]$. Then there exists a unique maximal mild solution $u \in C_{\mathrm{Lip}}([-p,a];Z)$ of the problem (1.1)–(1.2) on $[-p,a]$.

Proof. Let $u \in C_{\text{Lip}}([-p,b];Z)$ be the mild solution obtained in Theorem 3.4. From the demonstration of Theorem 3.4, we can show that there exists $\delta > 0$ and a unique mild solution $v \in C_{\text{Lip}}([-p,b+\delta];Z)$ of (1.1)–(1.2) such that $v = u$ on $[-p,b]$.

Indeed, let R_φ, b and ρ presented in the proof of Theorem 3.4 and assume $a > b$. Consider $R_1 > [\mathcal{V}(\cdot)\varphi(0)]_{C_{\text{Lip}}([b,a];Z)} + 2R_\varphi$. Using (3.12), we can choose $0 < \delta < b$ small such that

$$2R_\varphi + [\mathcal{V}(\cdot)\varphi(0)]_{C_{\text{Lip}}([b,a];X)} + (1+R_1)^2\mathcal{W}_G(\rho_1)(1+\mathcal{W}_\sigma(\rho_1))M\Sigma_3(b,\delta) < R_1,$$

where $\rho_1 := R_1(2b+p) + \|\varphi\|_{\mathcal{B}}$. Now consider the space

$$\mathfrak{Y} = \{w \in C([-p,b+\delta];Z)\,;\, w_{|_{[-p,b]}} = u_{|_{[-p,b]}},\, [w]_{C_{\text{Lip}}([-p,b+\delta];Z)} \le R_1\}, \tag{3.13}$$

endowed with the metric $d(u,v) = \|u-v\|_{C([0,b+\delta];Z)}$ and define the map $\Gamma : \mathfrak{Y} \to C([-p,b+\delta];Z)$ as $(\Gamma w)_{|_{[-p,b]}} = u_{|_{[-p,b]}}$ and

$$\Gamma w(t) = \mathcal{V}(t)\varphi(0) + \int_0^b \mathcal{V}(t-s)G\left(s,u^\sigma(s)\right)ds + \int_b^t \mathcal{V}(t-s)G\left(s,w^\sigma(s)\right)ds, \tag{3.14}$$

for $t \in [b,b+\delta]$.

Let $w \in \mathfrak{Y}$. Then, $[\Gamma w]_{C_{\text{Lip}}([-p,b];Z)} \le R_\varphi$ since $\Gamma w(\cdot) = u(\cdot)$ on $[-p,b]$. We remark that the inequalities (3.4), (3.5) and (3.6) are verified. Moreover, using (3.4), for $s \in [b,b+\delta]$ we obtain

$$\begin{aligned}
&\max\{\|w(s)\|, \|w^\sigma(s)\|, \|w_s\|_{\mathcal{B}}\}\\
&\quad\le \max\{\|w(\cdot)\|_{C([b,b+\delta];Z)}, \|w\|_{C([b-p,b+\delta];Z)}, \|w\|_{C([b-p,b+\delta];Z)}\}\\
&\quad\le \max\{\|w(\cdot)\|_{C([b,b+\delta];Z)}, \|u\|_{C([b-p,b];Z)}\}\\
&\quad\le \max\{\|w-u(b)\|_{C([b,b+\delta];X)} + \|u(b)\|, \|u\|_{C([b-p,b];Z)}\}\\
&\quad\le [w]_{C_{\text{Lip}}([b,b+\delta];Z)}\delta + R_\varphi(b+p) + \|\varphi\|_{\mathcal{B}}\\
&\quad\le R_1\delta + R_\varphi(b+p) + \|\varphi\|_{\mathcal{B}}\\
&\quad\le R_1 b + R_1(b+p) + \|\varphi\|_{\mathcal{B}} \le \rho_1.
\end{aligned}$$

In addition, since $R_\varphi \le R_1$ and $\rho \le \rho_1$ and , for $s,h \in [0,b+\delta]$ with $s+h \in [b,b+\delta]$ and arguing as in the estimate (3.5), it follows that

$$\begin{aligned}
&\|G(s+h,w^\sigma(s+h)) - G(s,w^\sigma(s))\|\\
&\quad\le \mathcal{W}_G(\rho_1)(1+\mathcal{W}_\sigma(\rho_1))(1+R_1)^2[G]_{(s+h,s)}(1+[\sigma]_{(s+h,s)})h.
\end{aligned}$$

Let $t \in [b,b+\delta)$ and $h > 0$ with $t+h \in [b,b+\delta]$. From the above inequalities, the inequalities (3.4)–(3.6), the fact that $0 < h < \delta < b$, and proceeding as in the estimate (3.7), we get

$$\begin{aligned}
&\|\Gamma w(t+h) - \Gamma w(t)\|\\
&\quad\le [\mathcal{V}(\cdot)\varphi(0)]_{C_{\text{Lip}}([b,b+\delta];Z)}h + \int_0^h \|\mathcal{V}(t+h-s))G(0,\varphi(-\sigma(0,\varphi)))\|ds\\
&\qquad + \int_0^h \|\mathcal{V}(t+h-s))\|\|G(s,u^\sigma(s)) - G(0,\varphi(-\sigma(0,\varphi)))\|ds\\
&\qquad + \int_0^t \|\mathcal{V}(t-s)\|\|G(s+h,w^\sigma(s+h)) - G(s,w^\sigma(s))\|ds
\end{aligned}$$

$$
\begin{aligned}
&\leq [\mathcal{V}(\cdot)\varphi(0)]_{C_{\mathrm{Lip}}([b,b+\delta];Z)}h+M\|G(0,\varphi(-\sigma(0,\varphi))\|h\\
&\quad+\mathcal{W}_G(\rho)(1+\mathcal{W}_\sigma(\rho))(1+R_\varphi)^2\int_0^h M[G]_{(s,0)}(1+[\sigma]_{(s,0)})hds\\
&\quad+\int_0^{b-h}\|\mathcal{V}(t-s)(G(s+h,u^\sigma(s+h))-G(s,u^\sigma(s)))\|ds\\
&\quad+\int_{b-h}^t\|\mathcal{V}(t-s)(G(s+h,w^\sigma(s+h))-G(s,w^\sigma(s)))\|ds\\
&\leq [\mathcal{V}(\cdot)\varphi(0)]_{C_{\mathrm{Lip}}([b,b+\delta];Z)}h+M\|G(0,\varphi(-\sigma(0,\varphi))\|h\\
&\quad+\mathcal{W}_G(\rho)(1+\mathcal{W}_\sigma(\rho))(1+R_\varphi)^2\int_0^h M[G]_{(s,0)}(1+[\sigma]_{(s,0)})hds\\
&\quad+\mathcal{W}_G(\rho)(1+\mathcal{W}_\sigma(\rho))(1+R_\varphi)^2\int_0^{b-h} M[G]_{(s+h,s)}(1+[\sigma]_{(s+h,s)})hds\\
&\quad+\mathcal{W}_G(\rho_1)(1+\mathcal{W}_\sigma(\rho_1))(1+R_1)^2\int_{b-h}^t M[G]_{(s+h,s)}(1+[\sigma]_{(s+h,s)})hds\\
&\leq [\mathcal{V}(\cdot)\varphi(0)]_{C_{\mathrm{Lip}}([b,b+\delta];Z)}h+I_1+I_2+I_3\\
&\quad+\mathcal{W}_G(\rho_1)(1+\mathcal{W}_\sigma(\rho_1))(1+R_1)^2\int_{b-h}^t M[G]_{(s+h,s)}(1+[\sigma]_{(s+h,s)})hds.
\end{aligned}
$$

Observing that in the estimate (3.7) the term $I_1+I_2+I_3$ also appears, we deduce that

$$
\begin{aligned}
&\|\Gamma w(t+h)-\Gamma w(t)\|\\
&\quad\leq [\mathcal{V}(\cdot)\varphi(0)]_{C_{\mathrm{Lip}}([b,b+\delta];Z)}h+R_\varphi h\\
&\qquad+\mathcal{W}_G(\rho_1)(1+\mathcal{W}_\sigma(\rho_1))(1+R_1)^2M\int_{b-h}^t[G]_{(s+h,s)}(1+[\sigma]_{(s+h,s)})hds\\
&\quad\leq R_\varphi h+[\mathcal{V}(\cdot)\varphi(0)]_{C_{\mathrm{Lip}}([b,a];Z)}h+\mathcal{W}_G(\rho_1)(1+\mathcal{W}_\sigma(\rho_1))(1+R_1)^2M\Sigma_3(b,\delta)h\leq R_1h,
\end{aligned}
$$

which allows us to conclude that $[w_{|_{[b,b+\delta]}}]_{C_{\mathrm{Lip}}([b,b+\delta];Z)}\leq R_1$ and $[w]_{C_{\mathrm{Lip}}([-p,b+\delta];Z)}\leq R_1$ because $[w_{|_{[-p,b]}}]_{C_{\mathrm{Lip}}([-p,b];Z)}=[u_{|_{[-p,b]}}]_{C_{\mathrm{Lip}}([-p,b];Z)}\leq R_\varphi\leq R_1$. Then, $\Gamma(\cdot)$ is a $\mathfrak{Y}$-valued function.

Moreover, arguing as in (3.8), for $t\in[b,b+\delta]$ and $w,v\in\mathfrak{Y}$ we have that

$$
\begin{aligned}
&\|\Gamma w(t)-\Gamma v(t)\|\\
&\quad\leq\int_b^t M\mathcal{W}_G(\rho_1)[G]_{(s,s)}\|w^\sigma(s)-v^\sigma(s)\|ds\\
&\quad\leq\int_b^t M\mathcal{W}_G(\rho_1)[G]_{(s,s)}(1+[w]_{C_{\mathrm{Lip}}([-p,b+\delta];Z)}\mathcal{W}_\sigma(\rho_1)[\sigma]_{(s,s)})\|w-v\|_{C([0,s];Z)}ds\\
&\quad\leq M(1+R_1)\mathcal{W}_G(\rho_1)(1+\mathcal{W}_\sigma(\rho_1))\int_b^t[G]_{(s,s)}(1+[\sigma]_{(s,s)})ds\|w-v\|_{C([b,b+\delta];Z)}\\
&\quad\leq M(1+R_1)\mathcal{W}_G(\rho_1)(1+\mathcal{W}_\sigma(\rho_1))\Sigma_3(b,\delta)\|w-v\|_{C([b,b+\delta];Z)},
\end{aligned}
$$

and then $\Gamma(\cdot)$ is a contraction and there exists unique mild solution $w\in C_{\mathrm{Lip}}([-p,b+\delta];Z)$ of (1.1)–(1.2) on $[-p,b+\delta]$ such that $w=u$ on $[-p,b]$.

Proceeding this way, we obtain a maximal "locally Lipschitz" mild solution $u\in C(I_{\max};Z)$ of (1.1)–(1.2).

Now let $b_\varphi=\sup I_{\max}$. We state that $I_{\max}=[-p,b_\varphi]$ and that $u\in C_{\mathrm{Lip}}([0,b_\varphi];Z)$. Indeed,

note that $\|u^\sigma(t)\| \leq \|\varphi\|_{\mathcal{B}} + \|u\|_{C([0,t];Z)}$, for all $t \in [0, b_\varphi)$ and then

$$\begin{aligned}\|u(t)\| &\leq \|\mathcal{V}(\cdot)\|\|\varphi(0)\| + \int_0^t \|\mathcal{V}(t-s)G(s,\varphi(-\sigma(0,\varphi)))\|ds \\ &\quad + \int_0^t \|\mathcal{V}(t-s)\|\|G(s,u^\sigma(s)) - G(s,\varphi(-\sigma(0,\varphi)))\|ds \\ &\leq M(\|\varphi\|_{\mathcal{B}} + \int_0^t \|G(s,\varphi(-\sigma(0,\varphi)))\|ds) \\ &\quad + M\int_0^t \mathcal{W}_\infty[G]_{(s,s)}(\|u\|_{C([0,s];Z)} + 2\|\varphi\|_{\mathcal{B}})ds,\end{aligned}$$

that is, $u(\cdot)$ is bounded on $[0, b_\varphi)$ and $G(\cdot, u^\sigma(\cdot)) \in L^q([0,b_\varphi);Z) \cap C([0,b_\varphi);Z)$ because $[G]_{(\cdot,\cdot)}$ is a Grönwall-type function.

Moreover, we can adapt the argument in [3, Proposition 6.3] to the resolvent family and show that $u \in C^\alpha([b,b_\varphi);Z)$ and $\lim_{t\to b_\varphi} u(t)$ exists. Let $v : [-p, b_\varphi] \to Z$ be defined by $v(t) = u(t)$ for $t < b_\varphi$ and $v(b_\varphi) = \lim_{t\to b_\varphi} u(t)$. Obviously, $v \in C([0,b_\varphi];Z), v(\cdot)$ is a mild solution of (1.1)–(1.2) on $[-p, b_\varphi]$ and $v \in C^\alpha([0,b_\varphi];Z)$. From the above, for $t \in (0, b_\varphi]$ and $\rho = [v]_{C^\alpha([-p,b_\varphi);Z)} + \|v\|_{C([-p,b_\varphi];Z)}$, we have that

$$\begin{aligned}&\int_0^t \|A\mathcal{V}(t-s)(G(s,v^\sigma(s)) - G(t,v^\sigma(t)))\|ds \\ &\quad \leq \int_0^t C\mathcal{W}_\infty \frac{[G]_{(t,s)}}{(t-s)}((t-s) + \rho(a^{\alpha-\alpha^2} + \mathcal{W}_\sigma(\rho)^\alpha[\sigma]^\alpha_{(t,s)}(a^{\alpha-\alpha^2}) + \rho^\alpha)(t-s)^{\alpha^2})ds \\ &\quad \leq C\mathcal{W}_\infty \sup_{t\in[0,a]} \left\| [G]_{(t,\cdot)} + \frac{[G]_{(t,\cdot)}}{(t-\cdot)^{1-\alpha^2}}\rho(a^{\alpha-\alpha^2} + \mathcal{W}_\sigma(\rho)^\alpha[\sigma]^\alpha_{(t,\cdot)}(a^{\alpha-\alpha^2}) + \rho^\alpha) \right\|_{L^1([0,t])}. \quad (3.15)\end{aligned}$$

Denote by $\mathcal{L}_\infty$ the right hand side of (3.15) and note that

$$\begin{aligned}v(t) &= \mathcal{V}(t)\varphi(0) + \int_0^t \mathcal{V}(t-s)G(t,v^\sigma(t))ds + \int_0^t \mathcal{V}(t-s)(G(s,v^\sigma(s)) - G(t,v^\sigma(t)))ds \\ &= \mathcal{V}(t)\varphi(0) + \int_0^t \mathcal{V}(s)G(t,v^\sigma(t))ds + \int_0^t \mathcal{V}(t-s)(G(s,v^\sigma(s)) - G(t,v^\sigma(t)))ds. \quad (3.16)\end{aligned}$$

Defining $V(t) = \int_0^t \mathcal{V}(s)G(t,v^\sigma(t))ds$ and using the same argument of the [8, Lemma 2.10] we can show the function $AV(t) : [0,b_\varphi] \to Z$ is continuous. Now we infer

$$\begin{aligned}\sup_{t\in[b,b_\varphi]} \|Av(t)\| &\leq \sup_{t\in[b,b_\varphi]} \left(\|A\mathcal{V}(t)\varphi(0)\| + \left\| A\int_0^t \mathcal{R}(s)G(t,v^\sigma(t))ds \right\| + \mathcal{L}_\infty \right) \\ &\leq \frac{C_1}{b}\|\varphi(0)\| + \|AV(\cdot)\|_{C([0,b_\varphi];Z)} + \mathcal{L}_\infty. \quad (3.17)\end{aligned}$$

We get $v(t) \in D(A)$ for all $t \in (0, b_\varphi]$, and from Corollary 2.12 we conclude that $v(\cdot)$ is a classical solution of (1.1)–(1.2) on $[-p, b_\varphi]$, which implies that $\sup_{t\in[b,b_\varphi]} \|v'(t)\| < \infty$, $v \in C_{\text{Lip}}([-p,b_\varphi];Z)$, $u(\cdot) = v(\cdot)$ on $[0,b_\varphi]$ and $I_{\max} = [-p,b_\varphi]$.

Assume $b_\varphi < a$. This fact allows us to conclude that there exists $\delta' > 0$ and a unique Lipschitz mild solution $w \in C_{\text{Lip}}([-p, b_\varphi + \delta'];Z)$ of (1.1)–(1.2) on $[-p, b_\varphi + \delta']$ such that $w = u$ on $[-p,b_\varphi]$, which is contrary to the maximality of $u(\cdot)$. Thus, $b_\varphi = a$, $u \in C_{\text{Lip}}([-p,a];Z)$ and $u(\cdot)$ is the unique mild solution of (1.1)–(1.2) on $[-p,a]$. □

For the next result we use Schauder's fixed point Theorem to investigate the existence of a mild solution.

Proposition 3.13. *Assume that condition* $\boldsymbol{H}^{q,r}_{G,\sigma,a}$ *holds,* $\varphi \in \mathcal{B}$ *and* $\mathcal{V}(t)$ *is compact for all* $t > 0$. *Then there exists a mild solution* $u \in C([-p,b];Z)$ *of the problem* (1.1)–(1.2) *on* $[-p,b]$ *for some* $0 < b \leq a$.

Proof. Let $r > 0$, $\rho := r + \|\varphi\|_{\mathcal{B}}$ and $\Lambda(r) := \mathcal{W}_G(\rho)\rho$. Since $s \to [G]_{(s,s)}$ is integrable on $[0,t]$ for all $t \in [0,a]$ then

$$\Sigma_4(0,c) := \sup_{t\in[0,c]} \int_0^t [G]_{(s,s)} ds \to 0,$$

as $c \to 0$. Moreover, as $\lim_{t\downarrow 0} \mathcal{V}(t)\varphi(0) = \varphi(0)$, we can choose $b > 0$ such that

$$\|\mathcal{V}(\cdot)\varphi(0) - \varphi(0)\|_{C([0,b];Z)} + M\left(\Sigma_4(0,b)\mathcal{W}_G(\rho)\rho + \|G(\cdot,0)\|_{C([0,a];Z)}b\right) < r. \tag{3.18}$$

Consider the space

$$\mathfrak{Y}(b) = \{u \in C([-p,b];Z) : u_0 = \varphi, \|u(\cdot) - \varphi(0)\|_{C([0,b];Z)} \leq r\},$$

endowed with the metric $d(u,v) = \|u - v\|_{C([0,b];Z)}$ and define $\Gamma : \mathfrak{Y}(b) \to C([-p,b];Z)$ as in the proof of Theorem 3.4.

If $u \in \mathfrak{Y}(b)$, from the definition of $\mathfrak{Y}(b)$, for $s \in [0,b]$ we note that

$$\begin{aligned}\max\{\|u(s)\|, \|u^\sigma(s)\|, \|u_s\|_{\mathcal{B}}\} &\leq \sup_{\tau\in[0,b]} \|u(\tau) - \varphi(0)\| + \|\varphi\|_{\mathcal{B}} \\ &\leq r + \|\varphi\|_{\mathcal{B}} \ = \ \rho,\end{aligned}$$

which shows that

$$\begin{aligned}\|G(s,u^\sigma(s))\| &\leq \|G(s,u^\sigma(s)) - G(s,0)\| + \|G(\cdot,0)\|_{C([0,b];Z)} \\ &\leq [G]_{(s,s)}\mathcal{W}_G(\rho)\|u^\sigma(s)\| + \|G(\cdot,0)\|_{C([0,b];Z)} \\ &\leq [G]_{(s,s)}\mathcal{W}_G(\rho)\rho + \|G(\cdot,0)\|_{C([0,b];Z)}.\end{aligned}$$

Using the previous inequalities, it is easy to see that

$$\begin{aligned}\|\Gamma u(t) - \varphi(0)\| &\leq \|\mathcal{V}(t)\varphi(0) - \varphi(0)\| + M\int_0^t [G]_{(s,s)}\mathcal{W}_G(\rho)\rho ds \\ &\quad + M\|G(\cdot,0)\|_{C([0,b];Z)}b \\ &\leq \|\mathcal{V}(\cdot)\varphi(0) - \varphi(0)\|_{C([0,b];Z)} \\ &\quad + M(\Sigma_4(0,b)\mathcal{W}_G(\rho)\rho + \|G(\cdot,0)\|_{C([0,a];Z)}b) \leq r,\end{aligned}$$

which implies that $\Gamma(\cdot)$ is a $\mathfrak{Y}(b)$-valued function.

Furthermore, for $t \in (0,b]$ and $h > 0$ with $t + h < b$, we obtain

$$\begin{aligned}&\|\Gamma u(t+h) - \Gamma u(t)\| \\ &\quad\leq \|\mathcal{V}(t+h)\varphi(0) - \mathcal{V}(t)\varphi(0)\| \\ &\qquad + \int_0^t \|\mathcal{V}(t+h-s) - \mathcal{V}(t-s)\|([G]_{(s,s)}\mathcal{W}_G(\rho)\rho + \|G(\cdot,0)\|_{C([0,b];Z)})ds \\ &\qquad + \int_t^{t+h} M([G]_{(s,s)}\mathcal{W}_G(\rho)\rho + \|G(\cdot,0)\|_{C([0,b];Z)})ds \\ &\quad\leq \|\mathcal{V}(t+h)\varphi(0) - \mathcal{V}(t)\varphi(0)\| \\ &\qquad + \int_0^t \|\mathcal{V}(t+h-s) - \mathcal{V}(t-s)\|([G]_{(s,s)}\mathcal{W}_G(\rho)\rho + \|G(\cdot,0)\|_{C([0,b];Z)})ds \\ &\qquad + M(\|[G]_{(\cdot,\cdot)}\|_{L^1([0,a])}\mathcal{W}_G(\rho)\rho + \|G(\cdot,0)\|_{C([0,b];Z)})h.\end{aligned}$$

We can write the last inequality in the form

$$\|\Gamma u(t+h) - \Gamma u(t)\| \leq \epsilon_{\varphi}^{1}(t, t+h) + \int_{0}^{t} \epsilon_{\varphi}^{2}(t, t+h, s) G(s) ds,$$

where $G(s) = [G]_{(s,s)} \mathcal{W}_G(\rho)\rho + \|G(\cdot, 0)\|_{C([0,b];Z)}$.

Note that $\epsilon_{\varphi}^{1}(t, t+h) \to 0$ as $h \to 0$ uniformly for $(t, u) \in [0, b] \times \mathfrak{Y}(b)$. Moreover, since $\epsilon_{\varphi}^{2}(t, t+h, s)G(s) \to 0$ for all $s \in [0, t]$ as $h \downarrow 0$, $\epsilon_{\varphi}^{2}(t, t+h, s)G(s) \leq MG(s)$ and $MG(\cdot)$ is integrable on $[0, t]$, we conclude that the second term at the right hand side converge to zero as $h \to 0$, uniformly for $u \in \mathfrak{Y}(b)$.

Then, the set $\{\Gamma u : u \in \mathfrak{Y}(b)\}$ is right equicontinuous at t. Similarly, we show the left equicontinuity at $t \in (0, b]$ and the right equicontinuity at $t = 0$.

Now, we need to prove that $\{\Gamma u(t) : u \in \mathfrak{Y}(b)\}$ is relatively compact in Z for all $t \in [0, b]$. Let $t \in (0, b]$ and $\varepsilon > 0$. By noting that

$$\int_{t-\delta}^{t} \|\mathcal{V}(t-s)\| \|G(s, u^{\sigma}(s))\| ds \leq M\delta^{\frac{1}{q}} \|G(\cdot)\|_{L^q([0,a])},$$

for $u \in \mathfrak{Y}(b)$, we can select $0 < \delta < t$ such that

$$\int_{t-\delta}^{t} \|\mathcal{V}(t-s)\| \|G(s, u^{\sigma}(s))\| ds \leq \varepsilon, \qquad \forall u \in \mathfrak{Y}(b).$$

From the continuity of $\mathcal{V}(\cdot)$ on $[\delta, t]$, we can choose $N \in \mathbb{N}$ and numbers $0 < t_1 < t_2 < \ldots < t_N = t - \delta < t_{N+1} = t$ such that $\|\mathcal{V}(t-s) - \mathcal{V}(t-t_i)\| \leq \frac{\varepsilon}{1+R}$ for all $s \in [t_i, t_{i+1}]$ and $i = 1, \ldots, N-1$.

Under the previous conditions and remarking that

$$\int_{\mu}^{t_{i+1}} \|G(s, u^{\sigma}(s))\| ds \leq \int_{0}^{t} \|G(s)\| ds \leq \Theta := b^{\frac{1}{q}} \|G(\cdot)\|_{L^q([0,a])},$$

for all $i = 1, \ldots, N-1$ and every $\mu < t$, for $u \in \mathfrak{Y}(b)$ we obtain

$$\begin{aligned}
\Gamma u(t) &= \mathcal{V}(t)\varphi(0) + \int_{t_N}^{t} \mathcal{V}(t-s) G(s, u^{\sigma}(s)) ds + \sum_{i=1}^{N-1} \mathcal{V}(t-t_i) \int_{t_i}^{t_{i+1}} G(s, u^{\sigma}(s)) ds \\
&\quad + \sum_{i=1}^{N-1} \int_{t_i}^{t_{i+1}} (\mathcal{V}(t-s) - \mathcal{V}(t-t_i)) G(s, u^{\sigma}(s)) ds \\
&\in \mathcal{V}(t)\varphi(0) + B_{\varepsilon}(0; Z) + \sum_{i=1}^{N-1} \mathcal{V}(t-t_i) B_{\Theta}(0; Z) + \varepsilon B_{\Theta}(0; Z),
\end{aligned}$$

and hence, $\{\Gamma u(t) : u \in \mathfrak{Y}(b)\} \subset D_{\varepsilon} + K_{\varepsilon}$, where the diameter of D_{ε} go to 0 as $\varepsilon \to 0$ and K_{ε} is relatively compact in Z. Then, $\{\Gamma u(t) : u \in \mathfrak{Y}(b)\}$ is relatively compact in Z.

By hypothesis $\boldsymbol{H}_{G,\sigma,a}^{q,r}$ it follows that $\Gamma(\cdot)$ is continuous and from the previous commentaries we can conclude that $\Gamma(\cdot)$ is completely continuous. Therefore, from Schauder's Fixed Point Theorem, there exists a mild solution $u \in C([-p, b]; Z)$ of (1.1)–(1.2) on $[-p, b]$. □

4 Well-posedness

To begin our study on the local well-posedness of the problem (1.1)–(1.2), we first introduce the next concept.

Definition 4.1. Let $(\mathcal{D}, \|\cdot\|_{\mathcal{D}}) \hookrightarrow (\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ be a Banach space and $a > 0$. We say that the problem (1.1)–(1.2) is locally well-posed related to the spaces $(\mathcal{D}, \|\cdot\|_{\mathcal{D}})$ and $C([0,a];Z)$, if for all $\varphi \in \mathcal{D}$ and $r > 0$, there exists $c_\varphi > 0$ such that, for all $\psi \in B_r(\varphi, \mathcal{D})$ there exists a unique mild solution $u(\cdot,\psi) \in C([-p, c_\varphi];Z)$ of the problem (1.1)–(1.2) on $[-p, c_\varphi]$ and

$$\|u(\cdot,\varphi) - u(\cdot,\psi)\|_{C([0,c_\varphi];Z)} \to 0 \quad \text{as } \|\varphi - \psi\|_{\mathcal{D}} \to 0. \tag{4.1}$$

Combining the proofs of Theorem 3.4 and Corollary 3.6, we have the next result, with

$$\mathcal{D} = \mathcal{D}_a = \{\psi \in C_{\text{Lip}}([-p,0];Z) : \mathcal{V}(\cdot)\psi(0) \in C_{\text{Lip}}([0,a];Z),\ \psi(0) \in E\},$$

endowed with the metric $d(\psi_1,\psi_2) = \|\psi_1 - \psi_2\|_{C_{\text{Lip}}([-p,0];Z)} + \|\mathcal{V}(\cdot)(\psi_1(0) - \psi_2(0))\|_{C_{\text{Lip}}([0,a];Z)}$.

Proposition 4.2. *Suppose that condition* $\boldsymbol{H}^{q,r}_{G,\sigma,a}$ *is satisfied and* $\Theta(q,r) < 1$. *Then the problem* (1.1)–1.2 *is locally well-posed related to the spaces* $(\mathcal{D}_a, \|\cdot\|_{\mathcal{D}_a})$ *and* $C([0,a];Z)$.

Proof. Let $\varphi \in \mathcal{D}_a$, $r > 0$ and $R^*_\varphi > 0$ be such that

$$R^*_\varphi > \Lambda := \sup_{\psi \in B_r(\varphi;\mathcal{D}_a)} \left([\mathcal{V}(\cdot)\psi(0)]_{C_{\text{Lip}}([0,a];Z)} + \|\psi\|_{C_{\text{Lip}}([-p,0];Z)} + M\|G(0,\psi(-\sigma(0,\psi)))\| \right).$$

Since $\Sigma_i(c) \to 0$ as $c \to 0$ for $i = 1,2$, we can choose $0 < c_\varphi \le a$ such that

$$\Lambda + (1+R^*_\varphi)^2 \mathcal{W}_G(\rho^*)(1+\mathcal{W}_\sigma(\rho^*))\left(\Sigma_1(c_\varphi) + \Sigma_2(c_\varphi)\right) < R^*_\varphi,$$

where $\rho^* := R^*_\varphi(a+2p) + \|\varphi\|_{\mathcal{B}} + r$.

Consider $\psi \in B_r(\varphi, \mathcal{D}_a)$. Proceeding as in the proof of Theorem 3.4, with $\psi(\cdot)$ instead of $\varphi(\cdot)$, we can show that there exists a unique mild solution $u(\cdot,\psi) \in C_{\text{Lip}}([-p,c_\varphi];Z)$ of the problem (1.1) with initial condition $\psi(\cdot)$. In addition, it follows that $[u(\cdot,\psi)]_{C_{\text{Lip}}([-p,c_\varphi];Z)} \le R^*_\varphi$ and $\|u(\cdot,\psi)\|_{C([-p,c_\varphi];Z)} \le \rho^*$.

We define $\mathcal{K}(\cdot,R^*_\varphi) = [G]_{(\cdot,\cdot)}(1 + [\sigma]_{(\cdot,\cdot)})$. Let $\psi \in B_r(\varphi;\mathcal{D}_a)$, $u(\cdot) = u(\cdot,\varphi)$ and $v(\cdot) = u(\cdot,\psi)$. From the above remarks and the last inequality in Lemma 2.2, for $t \in [0,c_\varphi]$ we get

$$\begin{aligned}
&\|u(t) - v(t)\| \\
&\quad \le \|\mathcal{V}(t)\varphi(0) - \mathcal{V}(t)\psi(0)\| + \int_0^t \|\mathcal{V}(t-s)\|\|G(s,u^\sigma(s)) - G(s,v^\sigma(s))\|ds \\
&\quad \le d(\varphi,\psi) + \mathcal{W}_G(\rho^*)\int_0^t M[G]_{(s,s)}\|u^\sigma(s) - v^\sigma(s)\|ds \\
&\quad \le d(\varphi,\psi) + \mathcal{W}_G(\rho^*)\int_0^t M[G]_{(s,s)}(1 + R^*_\varphi \mathcal{W}_\sigma(\rho^*)[\sigma]_{(s,s)})\|u - v\|_{C([0,s];Z)}ds \\
&\quad \le d(\varphi,\psi) + (1+R^*_\varphi)\mathcal{W}_G(\rho^*)(1+\mathcal{W}_\sigma(\rho^*))M\int_0^t \mathcal{K}(s,R^*_\varphi)\|u - v\|_{C([0,s];Z)}ds,
\end{aligned}$$

and hence,

$$\|u - v\|_{C([0,t];Z)} \le d(\varphi,\psi)e^{(1+R^*_\varphi)\mathcal{W}_G(\rho^*)(1+\mathcal{W}_\sigma(\rho^*))M\|\mathcal{K}(\cdot,R^*_\varphi)\|_{L^1([0,c_\varphi])}}.$$

Therefore, there exists $\Lambda > 0$ such that $\|u(\cdot,\varphi) - v(\cdot,\psi)\|_{C([0,c_\varphi];Z)} \le \Lambda d(\varphi,\psi)$, for all $\psi \in B_r(\varphi;\mathcal{D})$, which completes the proof. □

Remark 4.3. Every result proved in this work can be easily adapted for the integro-differential equations

$$\frac{du(t)}{dt} = Au(t) + \int_0^t Q(t-s)u(s)ds + G(t,u(t-\sigma(t,u_t))), t \in [0,a], \tag{4.2}$$

$$u_0 = \varphi \in \mathcal{B} = C([-p,0];Z), \tag{4.3}$$

where A and $Q(t)$ for $t \geq 0$ are closed linear operators defined on a common domain $D(A)$ that is dense in Z where the functions $G(\cdot)$ and $\sigma(\cdot)$ are continuous. Simply use the resolvent theory for integro-differential equations developed by Grimmer et al. [11–13].

5 Memory and diffusion-coupled system

To finish this paper, we study now the existence of solutions for a problem concerning the coupled memory and diffusive integro-differential system with state-dependent delay. Specifically, we consider the integro-differential system

$$\begin{aligned}
&\frac{\partial}{\partial t}\left[u(t,x)+\int_0^t (t-s)^{\alpha}e^{-\omega(t-s)}v(s,x)ds\right]\\
&\quad=\Delta u(t,x)+\int_0^t B(t-s)\Delta v(s,x)ds+\zeta(t)\int_\Omega b(u(\varrho(t,u_t),y))f(x-y)dy, \qquad (5.1)\\
&\frac{\partial}{\partial t}\left[v(t,x)+\int_0^t (t-s)^{\alpha}e^{-\gamma(t-s)}u(s,x)ds\right]\\
&\quad=\Delta v(t,x)+\int_0^t C(t-s)\Delta u(s,x)ds+g(v(\varrho(t,u_t),x)), \qquad (5.2)\\
&u(t,\cdot)|_{\partial\Omega}=0,\quad v(t,\cdot)|_{\partial\Omega}=0, \qquad (5.3)\\
&u(\theta,y)=\varphi(\theta,y),\quad v(\theta,y)=\phi(\theta,y),\qquad \theta\in[-p,0],\ y\in\Omega, \qquad (5.4)
\end{aligned}$$

for $t \in [0,a]$ and $x \in \Omega$, where $f \in C(\widetilde{\Omega};\mathbb{R})$, $\widetilde{\Omega}=\{x-y : x,y\in\Omega\}$, $g \in C_{\mathrm{Lip}}(\mathbb{R}^n;\mathbb{R}^n)$, $b \in C_{\mathrm{Lip}}(\mathbb{R}^n;\mathbb{R}^n)$, $\varrho \in C_{\mathrm{Lip}}([0,a]\times\mathcal{B};\mathbb{R}), d-p \leq \varrho(t,\psi) \leq t$ for all $(t,\psi)\in[0,a]\times\mathcal{B}$ and $\zeta \in C_{\mathrm{Lip}}(\mathbb{R};\mathbb{R})$.

To show the existence of solutions for the problem (5.1)–(5.4) we consider the operators $A, Q(t) : D(A) \subseteq X \to X$, $t \geq 0$, where $X = L^2(\Omega;\mathbb{R})$ given by $Ax = \Delta x$, for $x \in D(A) = H^2(\Omega)\cap H_0^1(\Omega)$ and $P_\omega(t)x = t^\alpha e^{-\omega t}x$ and $P_\gamma(t)x = t^\alpha e^{-\gamma t}x$ for $x \in X$. The operator A is the infinitesimal generator of an analytic semigroup and for all $\vartheta \in (\pi/2,\pi)$ there exists constant $M_\vartheta > 0$ such that $\|R(\lambda,A)\| \leq M_\vartheta|\lambda|^{-1}$ for all $\lambda \in \Lambda_\vartheta$. We assume that $B(\cdot)$ and $C(\cdot)$ in $L^1_{\mathrm{loc}}(\mathbb{R}^+)$, furthermore $B(\cdot)$ and $C(\cdot)$ has Laplace transformed absolutely convergent for $Re(\lambda) > 0$ which analytical extension to Λ_ϑ and $|\widehat{B}(\lambda)| + |\widehat{C}(\lambda)| = O(\frac{1}{|\lambda|})$, as $|\lambda| \to \infty$. To study this problem, we define the functions $\sigma : [0,\infty)\times\mathcal{B} \to \mathbb{R}$ by $\sigma(t,\psi) = t - \varrho(t,\psi), F_1(t,u)(x) = \zeta(t)\int_\Omega b(u(y))f(x-y)dy$ and $F_2(u)(x) = g(u(x))$.

Let $\mathcal{U} = \binom{u}{v}$ in this problem the space Z is defined by $Z = L^2(\Omega;\mathbb{R})\times L^2(\Omega;\mathbb{R})$ under the norm $\|\binom{u}{v}\| = \left(\int_\Omega |u|^2+|v|^2dx\right)^{\frac{1}{2}}$, it is easy to see that Z is a Hilbert space. We define the $D(\mathcal{A})$ by $D(\mathcal{A}) = \left\{\binom{u}{v}\in X : u,v\in H^2(\Omega)\cap H_0^1(\Omega)\right\}$, and $\mathcal{G} : [0,a]\times Z \to Z$

$$\begin{aligned}
&\frac{\partial}{\partial t}\left[\begin{pmatrix}u(t,x)\\ v(t,x)\end{pmatrix}+\int_0^t\begin{pmatrix}0 & (t-s)^\alpha e^{-\omega(t-s)}\\ (t-s)^\alpha e^{-\gamma(t-s)} & 0\end{pmatrix}\begin{pmatrix}u(s,x)\\ v(s,x)\end{pmatrix}ds\right]\\
&\quad=\begin{pmatrix}\Delta & 0\\ 0 & \Delta\end{pmatrix}\begin{pmatrix}u(t,x)\\ v(t,x)\end{pmatrix}+\int_0^t\begin{pmatrix}0 & B(t-s)\Delta\\ C(t-s)\Delta & 0\end{pmatrix}\begin{pmatrix}u(s,x)\\ v(s,x)\end{pmatrix}ds\\
&\qquad+\begin{pmatrix}F_1(t,u^\sigma(t,x))\\ F_2(t,v^\sigma(t,x))\end{pmatrix}.
\end{aligned}$$

Therefore, we can represent the system (5.1)–(5.4) in the abstract form

$$\frac{\partial}{\partial t}\left(\mathcal{U}(t)+\int_0^t \mathcal{P}(t-s)\mathcal{U}(s)ds\right)=\mathcal{A}\mathcal{U}(t)+\int_0^t \mathcal{Q}(t-s)\mathcal{U}(s)ds+\mathcal{G}(t,\mathcal{U}^\sigma(t)), \tag{5.5}$$

$$\mathcal{U}_0=\begin{pmatrix}\varphi\\ \phi\end{pmatrix}\in\mathcal{B}\times\mathcal{B}, \tag{5.6}$$

where

$$\mathcal{A}=\begin{pmatrix}\Delta & 0\\ 0 & \Delta\end{pmatrix},\ \mathcal{P}(t)=\begin{pmatrix}0 & P_\omega(t)\\ P_\gamma(t) & 0\end{pmatrix},\ \mathcal{Q}(t)=\begin{pmatrix}0 & A(t)\Delta\\ B(t)\Delta & 0\end{pmatrix},\ \text{and}$$

$$\mathcal{G}(t,\mathcal{U}^\sigma(t))=\begin{pmatrix}F_1(t,u^\sigma(t,x))\\ F_2(t,v^\sigma(t,x))\end{pmatrix}.$$

Now we show that $\mathcal{A}$ is a sectorial operator, this fact follows that

$$(\lambda-\mathcal{A})\begin{pmatrix}u_1\\ u_2\end{pmatrix}=\begin{pmatrix}f_1,\\ f_2.\end{pmatrix},$$

this is equivalent to

$$\begin{cases}(\lambda-A)u_1=f_1,\\ (\lambda-A)u_2=f_2.\end{cases}$$

Assume that $|\arg(\lambda)|<\vartheta$. Then $\lambda\in\rho(\mathcal{A})$, and the resolvent is written as

$$\begin{cases}u_1=(\lambda-A)^{-1}f_1,\\ u_2=(\lambda-A)^{-1}f_2.\end{cases}$$

we get

$$\begin{pmatrix}u_1\\ u_2\end{pmatrix}=\begin{pmatrix}(\lambda-A)^{-1} & 0\\ 0 & (\lambda-A)^{-1}\end{pmatrix}\begin{pmatrix}f_1\\ f_2\end{pmatrix}.$$

Now we can show

$$\begin{aligned}\|(\lambda I-\mathcal{A})^{-1}\|&\le\|(\lambda-A)^{-1}\|+\|(\lambda-A)^{-1}\|\\ &\le 2M_\vartheta|\lambda|^{-1}.\end{aligned}$$

The previous argument shows that $\mathcal{A}$ is a sectorial operator.

From the definitions of $\mathcal{Q}(t)$ we have for $\mathcal{U}\in D(\mathcal{A})$

$$\begin{aligned}\|\mathcal{Q}(t)\mathcal{U}\|^2&=\left\|\begin{pmatrix}0 & A(t)\Delta x\\ B(t)\Delta x & 0\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix}\right\|^2\\ &=\int_\Omega \|B(t)\Delta v\|^2+\|A(t)\Delta u\|^2dx\\ &\le (A(t)+B(t))^2\left(\int_\Omega\|\Delta v\|^2dx+\int_\Omega\|\Delta u\|^2dx\right)\\ &\le (A(t)+B(t))^2(\|\mathcal{A}\mathcal{U}\|+\|\mathcal{U}\|)^2.\end{aligned}$$

Therefore

$$\|\mathcal{Q}(t)U\|\le k(t)(\|\mathcal{A}\mathcal{U}\|+\|\mathcal{U}\|), \tag{5.7}$$

where $k(t) = A(t) + B(t)$. From (5.7) we obtain the Laplace transform of $(\mathcal{Q}(t))_{t\geq 0}$ is absolutely convergent for $Re(\lambda) > 0$, admits an analytical extension to Λ_ϑ and

$$\|\widehat{\mathcal{Q}}(\lambda)\|_{\mathcal{L}([D(\mathcal{A}),X])} = O\left(\frac{1}{|\lambda|}\right), \quad \text{as } |\lambda| \to \infty.$$

By the definitions of $\mathcal{P}(t)$ we get for $\mathcal{U} \in Z$

$$\begin{aligned}
\|\mathcal{P}(t)\mathcal{U}\|^2 &= \left\| \begin{pmatrix} 0 & P_\omega(t) \\ P_\gamma(t) & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \right\|^2 \\
&= \int_\Omega \|P_\gamma(t)u\|^2 + \|P_\omega(t)v\|^2 dx \\
&\leq (P_\omega(t) + P_\gamma(t))^2 \left(\int_\Omega \|u\|^2 dx + \int_\Omega \|v\|^2 dx \right) \\
&\leq (P_\omega(t) + P_\gamma(t))^2 \|\mathcal{U}\|^2.
\end{aligned}$$

It is clear that the function $\mathcal{P} : [0,\infty) \to \mathcal{L}(Z)$ is strongly continuous and $\widehat{\mathcal{P}}(\lambda)z$ is absolutely convergent for $z \in Z$ and $Re(\lambda) > 0$. There exists an analytical extension of $\widehat{\mathcal{P}}(\lambda)$ (still denoted by $\widehat{\mathcal{P}}(\lambda)$) to Λ_ϑ such that $\|\widehat{\mathcal{P}}(\lambda)\| \leq N_0|\lambda|^{-\alpha-1}$ for every $\lambda \in \Lambda_\vartheta$, and $\|\widehat{\mathcal{P}}(\lambda)z\| \leq N_1|\lambda|^{-1}\|z\|_1$ for every $\lambda \in \Lambda_\vartheta$ and $z \in D(\mathcal{A})$. As well as $D = C_0^\infty(\Omega) \times C_0^\infty(\Omega)$, where $C_0^\infty(\Omega)$ is the space of infinitely differentiable functions that vanish at $\partial\Omega$. It is easy to see that conditions **(C1)–(C4)** are satisfied with the previous conditions.

By the [8, Theorem 2.] we obtain the next result.

Proposition 5.1. *Assume that the above conditions are fulfilled. Then, there exists an analytical resolvent operator family $(\mathcal{V}(t))_{t\geq 0}$ associated to (5.5)–(5.6).*

Next, we prove that $\mathcal{G}(\cdot)$ is a L^q- Lipschitz function. Assume $(\int_\Omega \int_\Omega f(x-y)^2 dydx) < \infty$. It is easy to see that

$$\begin{aligned}
&\|F_1(t,u) - F_1(s,v)\| \\
&\quad \leq (\|b\|_{C_{\text{Lip}}} \int_\Omega \int_\Omega f(x-y)^2 dydx) m(\Omega))(\|u\|)([\zeta]_{(t,s)} + \zeta(s))(|t-s| + \|u-v\|), \\
&\qquad \|F_2(t,u) - F_2(s,v)\| \leq \|g\|_{C_{\text{Lip}}}(|t-s| + \|u-v\|),
\end{aligned}$$

where $m(\Omega)$ represent the measure of Ω. Thus, $\mathcal{G}(\cdot)$ is a L^q-Lipschitz function, the condition $\boldsymbol{H}^{q,r}_{G,\sigma,a}$ is satisfied with $W_\sigma(r) = 1$, $[\sigma]_{C_{\text{Lip}}} = [\varrho]_{C_{\text{Lip}}} + 1$ and

$$\begin{aligned}
&\mathcal{W}_\mathcal{G}(r) = r + 1, \\
&[\mathcal{G}]_{(t,s)} = (\|b\|_{C_{\text{Lip}}} \int_\Omega \int_\Omega f(x-y)^2 dydx) m(\Omega) + \|g\|_{C_{\text{Lip}}})([\zeta]_{(t,s)} + \zeta(s)).
\end{aligned}$$

We have the next result.

Proposition 5.2. *Assume $q > 1$ and $\mathcal{V}(\cdot)\varphi(0) \in C_{\text{Lip}}([0,a];Z)$.*

(a) If $\sup_{b\in[0,c],d,h\in[0,b],d+h\leq b} \|[\zeta]_{(\cdot+h,\cdot)}\|_{L^q([0,d])} < \infty$, for some $0 < c \leq a$ or $\zeta(\cdot)$ is Lipschitz, then there exists a unique mild solution $u \in C_{\text{Lip}}([-p,b];Z)$ of the problem (5.1)–(5.4) on $[-p,b]$ for some $0 < b \leq a$.

(b) If $\sup_{b\in[0,c],d,h\in[0,b],d+h\leq b} \|[\zeta]_{(\cdot+h,\cdot)}\|_{L^q([0,d])} < \infty$, for some $0 < c \leq a$ or $\zeta(\cdot)$ is Lipschitz, then if $\varphi(0,\cdot)$ and $\phi(0,\cdot) \in H^2(\Omega) \cap H_0^1(\Omega)$, then there exists a unique classical solution $u(\cdot)$ of the problem (5.1)–(5.4) on $[-p,b]$ for some $0 < b \leq a$.

(c) If $\sup_{b\in[0,c],d,h\in[0,b],d+h\leq b}\|[\zeta]_{(\cdot+h,\cdot)}\|_{L^q([0,d])}<\infty$, *for some* $0<c\leq a$ *or* $\zeta(\cdot)$ *is Lipschitz, moreover, if* $\zeta(\cdot)$ *is Lipschitz and* $1>\frac{4\beta(\theta+2\beta)}{(3\beta-1)^2}$. *where* $\theta:=[\mathcal{V}(\cdot)\varphi(0)]_{C_{\mathrm{Lip}}([0,a];Z)}+\|\varphi\|_{C_{\mathrm{Lip}}([-p,0];Z)}+M\|G(0,\varphi(-\sigma(0,\varphi))\|$ *and* $\beta:=(\|b\|_{C_{\mathrm{Lip}}}\int_\Omega\int_\Omega f(x-y)^2dydx)m(\Omega)+\|g\|_{C_{\mathrm{Lip}}})Ma$, *then there exists a unique mild solution* $u\in C_{\mathrm{Lip}}([-p,a];X)$ *of* (5.1)–(5.4) *on* $[-p,a]$.

Proof. The first assumption (a) follows from Corollary 3.6.

It is well-known that every reflexive Banach space has the Radon–Nikodym property (RNP). Since Hilbert spaces are reflexive, they automatically enjoy the RNP. By $\varphi(0,\cdot),\phi(0,\cdot)\in H^2(\Omega)\cap H_0^1(\Omega)$, $[\mathcal{G}]_{(t,s)}$ and $\zeta(\cdot)$ are continuous, the second assumption (b) follows from Proposition 3.5.

The assumption (c), the existence of a Lipschitz mild solution defined on $[-p,a]$ follows from Proposition 3.8 noting that the function $P_a:\mathbb{R}\mapsto\mathbb{R}$ given by

$$\begin{aligned}\mathcal{P}_a(x)&=[\mathcal{V}(\cdot)\varphi(0)]_{C_{\mathrm{Lip}}([0,a];Z)}+\|\varphi\|_{C_{\mathrm{Lip}}([-p,0];Z)}+M\|G(0,\varphi(-\sigma(0,\varphi))\|\\&\quad+(1+x)^2(\|b\|_{C_{\mathrm{Lip}}}\int_\Omega\int_\Omega f(x-y)^2dydx)m(\Omega)+\|g\|_{C_{\mathrm{Lip}}})Ma\\&\quad+(1+x)(\|b\|_{C_{\mathrm{Lip}}}\int_\Omega\int_\Omega f(x-y)^2dydx)m(\Omega)+\|g\|_{C_{\mathrm{Lip}}})Ma\\&=\theta+(1+x)^2\beta+(1+x)\beta-x,\end{aligned}$$

has two positive root if $1>\frac{4\beta(\theta+2\beta)}{(3\beta-1)^2}$, the results follow by Corollary 2.12 □

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7 Conflict of interest

The authors declare that they have no conflicts of interest.

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Existence and nonexistence of solutions for unsteady shrinking disk flow

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Abstract. We analyze a mathematical model recently proposed by Mehmood *et al.* [*J. Appl. Mech. Tech. Phys.* **63**(2022), 782–789] of unsteady fluid flow over a shrinking porous disk. Mehmood *et al.* conjectured that solutions to the model exist only for sufficiently strong suction of fluid through the disk surface. Here we prove that solutions in fact exist for both suction and injection. However, with injection, the solutions are very difficult to determine numerically. In this case, we derive precise bounds on the skin friction coefficient as a function of injection velocity. These bounds are then used to pinpoint the solutions computationally.

Keywords: existence, nonexistence, boundary-layer flow, nonlinear boundary value problem.

2020 Mathematics Subject Classification: 34B15, 76D10.

1 Introduction

Fluid flow induced by a stretching surface is a classic problem in mechanics, first considered by Crane [3] in 1970 for a two dimensional case in which a horizontal boundary stretches away from a fixed point with a velocity proportional to the distance from the fixed point. Since that time, the analysis has been expanded to include mathematical models of various other geometries and configurations (See e.g. [2,5–7,13–16]). For the case of *stretching* surfaces, solutions generally turn out to be unique. (For Crane's [3] original model, see for example [9] or [12].)

In the case of *shrinking* surfaces, the solution sets turn out to be much more varied, and can include parameter regimes with two [1], three (or more) [10], or even a whole continuum of solutions [11]. There can also be ranges of parameters where no solutions exist [10,11].

Recently, Mehmood *et al.* [8] proposed a model for axisymmetric two-dimensional unsteady flow induced by a shrinking porous disk, with accelerated and decelerated flow situations. Using a dimensionless similarity variable transformation, they reduce the governing partial differential equations to an ordinary differential equation boundary value problem.

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Using numerical integration, Mehmood *et al.* [8] analyze the model and make several conjectures regarding the nature of the solutions. The purpose of this note is to study the problem analytically and compare the results to the computational results of [8].

The model involves two physical parameters governing injection/suction of fluid through the disk surface and acceleration/deceleration of the radial coordinate of the disk as it shrinks. Based on numerical integration of the differential equation, Mehmood *et al.* [8] find solutions to the problem only in the case of sufficiently strong suction. These solutions are found for both accelerating and decelerating disks. However, we will prove that, at least in the case of an *accelerating* disk, solutions exist for both suction *and* injection. But it should be noted that, in the case of injection, these solutions are very difficult to pinpoint numerically. Thus we can readily understand how Mehmood *et al.* [8] came to the conclusion that they might not exist.

The numerical search for solutions to many fluid flow problems can be aided by a-priori knowledge about the skin friction coefficient, and for an *accelerated* disk we are able to derive precise bounds on this coefficient. These bounds become increasingly more accurate as the fluid injection velocity is increased, and can be used to guide numerical calculations.

Existence or nonexistence of a solution for a *decelerating* disk is much less clear cut. In this case we prove that for a large portion of parameter space, any solution must contain a region of flow reversal in the radial direction. In other regions of parameter space it does appear that solutions without flow reversal exist. Mehmood *et al.* [8], in fact, find dual solutions for a decelerating disk. However, we conjecture that there are actually infinitely many solutions for a decelerating disk with suction.

The plan of the paper is as follows. In Section 2, we state the problem and consider the case of an accelerated disk. Section 3 explores a decelerated disk. In Section 4, we use the results of the previous sections to help pinpoint the solutions computationally. Even with the precise bounds on the skin friction coefficient derived in Section 2, we still find it difficult to track the solutions using standard numerical integration routines. We conclude by discussing these numerical issues and state some open problems regarding the model.

2 Accelerated disk

The model proposed by Mehmood *et al.* [8] is the following: Find $f(\eta)$ satisfying

$$f''' - \left(2f + \frac{\beta\eta}{2}\right) f'' + f'(f' - \beta) = 0, \quad 0 < \eta < \infty, \tag{2.1}$$

subject to

$$f(0) = S, \quad f'(0) = 1, \quad f'(\infty) = 0, \tag{2.2}$$

where $\beta > 0$ denotes an accelerating disk, $\beta < 0$ denotes a decelerating disk, $S > 0$ corresponds to fluid injection through the disk surface, and $S < 0$ corresponds to fluid suction. The radial and axial fluid velocities are proportional to $f'(\eta)$ and $f(\eta)$, respectively, where η is the dimensionless similarity variable.

To explore the existence or nonexistence of solutions to the boundary value problem (BVP) (2.1–2.2) in β-S parameter space, we will find it useful to consider a family of related initial value problems (IVP), namely, equations (2.1), $(2.2)_1$, and $(2.2)_2$ along with a third initial condition:

$$f''(0) = \alpha. \tag{2.3}$$

This IVP will have a unique local solution for any value of α. The question then becomes: Can α be chosen so that the solution of the IVP exists for all $\eta > 0$ *and* satisfies $(2.2)_3$, giving a solution to the BVP? Our first result along these lines is a topological shooting argument to prove existence for $\beta > 1$.

Theorem 2.1. *If $\beta > 1$ and $S \in \mathbb{R}$, then a solution to the BVP exists. Further, this solution satisfies $f' > 0$, $f'' < 0$ $f''' > 0$, $f^{(4)} < 0$, and $f^{(5)} > 0$ for all $\eta > 0$.*

Proof. For the family of initial value problems (2.1), $(2.2)_1$, $(2.2)_2$ and (2.3) defined above, consider the two sets:

$$\mathcal{A} = \{\alpha < 0 : f'' = 0 \text{ strictly before } f' = 0\}, \tag{2.4}$$

$$\mathcal{B} = \{\alpha < 0 : f' = 0 \text{ strictly before } f'' = 0\}. \tag{2.5}$$

By construction, the two sets are disjoint. We next show that they are nonempty and open as well. For $\mathcal{A}$ this follows directly from continuity of solutions of the IVP in its initial conditions. First note that when $f''(0) = \alpha = 0$, we have $f'''(0) = \beta - 1 > 0$. Thus $f'(\eta; \alpha = 0) > 1$ for $\eta \in (0, \varepsilon]$ for some $\varepsilon > 0$. By continuity of the solutions of the IVP in its initial conditions, for small $\alpha < 0$, we can arrange that $f'(\varepsilon; \alpha) > 1$. But since $f''(0) = \alpha < 0$, $f'(\eta)$ initially decreases below 1. Thus, f' must attain a minimum where $f'' = 0$. Also, for small $\alpha < 0$ we can ensure that $f' > 0$ on $[0, \varepsilon]$. Thus $f'' = 0$ strictly before $f' = 0$ for small $\alpha < 0$. Thus $\mathcal{A}$ is nonempty. Further note that we cannot have $f'' = f''' = 0$ simultaneously, since the ODE (2.1) then implies that $f' \equiv 0$ or $f' \equiv \beta$, which cannot be the case since $f''(0) = \alpha < 0$. Thus, when $f'' = 0$, $f''' \neq 0$ and so $\mathcal{A}$ is open.

To prove that $\mathcal{B}$ is nonempty and open we consider large $\alpha < 0$. We claim that for $\alpha < 0$ and $|\alpha|$ sufficiently large, we must have $f' = 0$ before $f'' = 0$ in the interval $[0, 1]$, say. Suppose for contradiction that the assertion is false, then one of following three possibilities must occur: (1) there exists a first $\eta_0 \in (0, 1]$ such that $f''(\eta_0) = 0$ with $0 < f' < 1$ on $(0, \eta_0]$ and $f'' < 0$ on $(0, \eta_0)$, (2) $0 < f' < 1$ and $f'' < 0$ on all of $(0, 1]$, or (3) $f' = f'' = 0$ simultaneously.

First consider possibility (1). Since $0 < f' < 1$ on $(0, \eta_0] \subset (0, 1]$, on integration and using $(2.2)_1$ we have $S < f < S + 1$ on $(0, \eta_0]$. Next, an integration of (2.1) from 0 to $\eta > 0$ results in

$$f'' = \alpha + \left(2f + \frac{\beta\eta}{2}\right) f' - 2S + \frac{\beta}{2}(f - S) - 3\int_0^\eta f'(t)^2\, dt. \tag{2.6}$$

Using the bounds on f and f' on $(0, \eta_0]$ in (2.6) we have

$$f'' < \alpha + \beta + 2, \quad \forall\, \eta \in (0, \eta_0]. \tag{2.7}$$

Choosing $\alpha < -(\beta + 2)$ in (2.7) implies that $f''(\eta_0) < 0$, contradicting the assumption that $f''(\eta_0) = 0$.

Next, consider possibility (2). In this case the bounds on f, f', and f'' from the previous paragraph hold on the entire interval $(0, 1]$, and choosing $\alpha < -(\beta + 3)$ implies that $f'' < -1$ on $(0, 1]$. On integration we then have that $f'(1) < 0$, contradicting the assumption that $f' > 0$ on $(0, 1]$. Thus possibility (2) cannot occur.

Finally, as we have seen previously, f' and f'' cannot vanish simultaneously, which precludes possibility (3). Thus if $\alpha < -(\beta + 3)$, then $\alpha \in \mathcal{B}$ and $\mathcal{B}$ is nonempty. And since from (2.1) we must have $f'' \neq 0$ when $f' = 0$, we see that $\mathcal{B}$ is open.

Since the interval $(-\infty, 0)$ is connected, it cannot be expressed as the union of $\mathcal{A}$ and $\mathcal{B}$. Thus there exists an α^* with $\alpha^* \notin \mathcal{A}$ and $\alpha^* \notin \mathcal{B}$. Thus for $f(\eta; \alpha^*)$ we cannot have $f'' = 0$

before $f' = 0$ nor can we have $f' = 0$ before $f'' = 0$. Also, as we have seen, $f' = f'' = 0$ simultaneously leads to a contradiction.

Thus we must have $0 < f' < 1$ and $f'' < 0$ for as long as $f(\eta;\alpha^*)$ exists. Let $[0,\overline{\eta})$ denote the maximal interval of existence of $f(\eta;\alpha^*)$. We wish to show that $\overline{\eta} = \infty$. Since $0 < f' < 1$ we have that f' is bounded on $[0,\overline{\eta})$, and on integration we conclude that f is also bounded on $[0,\overline{\eta})$. From (2.6) we then have f'' is bounded on any finite interval as well. Thus the solution can be continued past any finite $\overline{\eta}$ so that $\overline{\eta} = \infty$.

Since $f' > 0$ and $f'' < 0$ for all $\eta > 0$ we have that $f'(\infty) = L$ exists with $0 \leq L < 1$. We wish to show that $L = 0$, giving a solution to the BVP. If $L > 0$, then $0 < L < f' < 1$ for all $\eta > 0$ and using (2.1) we then have

$$f''' - \left(2f + \frac{\beta\eta}{2}\right) f'' = f'(\beta - f') > L(\beta - 1) = K > 0, \quad \forall\, \eta > 0. \tag{2.8}$$

Next note that when $f'' < 0$, f''' cannot increase through zero, since at such a point we would need $f^{(4)} \geq 0$, but a differentiation of (2.1) implies that $f^{(4)} = 3\beta f''/2 < 0$ at a root of f''', giving a contradiction. Thus if f''' were ever to equal zero, it would thereafter remain negative, which would then eventually force f' negative, contradicting the fact that $f' > 0$ for all η. Therefore we need $f''' > 0$ for all $\eta \geq 0$.

Using a similar argument and further differentiations of (2.1) it can be shown that $f^{(4)} < 0$ and $f^{(5)} > 0$ for all $\eta \geq 0$. Therefore, f'' and f''' are both monotonic and bounded, so that their limits as $\eta \to \infty$ exist. However, since $f'(\infty) = L$ exists, we must have $f''(\infty) = f'''(\infty) = 0$.

Next, note that $2L + \beta/2 < 2f' + \beta/2 < 2 + \beta/2$ for all $\eta > 0$ and so on integration from 0 to $\eta > 0$ we have

$$2S + \left(\frac{\beta + 4L}{2}\right)\eta < 2f + \frac{\beta\eta}{2} < 2S + \left(\frac{\beta + 4}{2}\right)\eta, \quad \forall\, \eta > 0. \tag{2.9}$$

Further, note that $2S + (\beta + 4L)\eta/2 > 0$ for η sufficiently large, and so by (2.9), $2f + \beta\eta/2 > 0$ for η sufficiently large.

Using these facts in (2.8), we can find $\hat{\eta}$ sufficiently large so that

$$f'' < \frac{-K}{2\left(2S + \frac{\beta+4}{2}\eta\right)}, \quad \forall\, \eta \geq \hat{\eta}. \tag{2.10}$$

Integrating (2.10) from $\hat{\eta}$ to $\eta > \hat{\eta}$ we have

$$f' < f'(\hat{\eta}) - \frac{K}{\beta + 4}\left[\ln\left(2S + \frac{\beta + 4}{2}\eta\right) - \ln\left(2S + \frac{\beta + 4}{2}\hat{\eta}\right)\right], \quad \forall\, \eta \geq \hat{\eta}, \tag{2.11}$$

which tends to $-\infty$ as $\eta \to \infty$, contradicting $f' \to L > 0$. Thus we must have $f'(\infty;\alpha^*) = L = 0$, giving a solution to the BVP and proving the theorem. □

Physically, this theorem states that for a sufficiently accelerated disk ($\beta > 1$) solutions to the problem exist for both injection ($S > 0$) and suction ($S < 0$) of fluid through the disk surface, as well as for an impermeable disk ($S = 0$).

Based on numerical integration of the problem, Mehmood *et al.* conjectured that for any value of $\beta \in \mathbb{R}$, no solution exists when $S > 0$. Theorem 2.1 above shows that, at least for $\beta > 1$, this conjecture is incorrect. However, when $\beta > 1$ and $S > 0$, the solutions are very difficult to pinpoint numerically, so that a conjecture of their nonexistence is understandable.

Theorem 2.1 guarantees at least one value of α which gives a solution to the BVP, but gives little indication of its value other than $-(\beta+3) < \alpha < 0$. In the next theorem we obtain much more precise bounds on α. This, in turn, gives bounds on the skin friction coefficient, which is proportional to α.

Theorem 2.2. *If $\beta > 1$ and $S > 0$, then the solution to the BVP given in Theorem 2.1 must satisfy $f''(0) = \alpha$ where*

$$\frac{2(1-\beta)(2S^2+\beta+1)}{8S^3+7S\beta+4S} < \alpha < \frac{2S(1-\beta)}{4S^2+3\beta/2}. \tag{2.12}$$

Proof. The bounds on α will involve the initial values of several derivatives of f. Using (2.1) and its derivatives we have

$$f'''(0) = 2S\alpha + \beta - 1, \tag{2.13}$$

$$f^{(4)}(0) = \left(4S^2 + \frac{3\beta}{2}\right)\alpha + 2S(\beta-1), \tag{2.14}$$

and

$$f^{(5)}(0) = (8S^3+7S\beta+4S)\alpha + 2(\beta-1)(2S^2+\beta+1). \tag{2.15}$$

Recall that the solution of Theorem 2.1 satisfies $0 < f' < 1$ and $f'' < 0$ for all $\eta > 0$. In the proof of Theorem 2.1 it was further shown that $f''' > 0$, $f^{(4)} < 0$, and $f^{(5)} > 0$ for all $\eta \geq 0$. In particular, $f^{(4)}(0) < 0$, and $f^{(5)}(0) > 0$. Using these inequalities in (2.14) and (2.15) results in (2.12). □

The coefficient of wall skin friction, C_f, of the fluid is given as [8]:

$$C_f = -Re_x^{-1/2} f''(0) = -Re_x^{-1/2}\alpha, \tag{2.16}$$

where Re_x is the Reynolds number. Thus the results of Theorem 2.2 give ever more precise bounds on the skin friction coefficient as the injection rate ($S > 0$) increases.

Remark 2.3. The results of this section do not touch on the question of uniqueness of a solution, but for the $\beta > 1$ and $S > 0$ case, we can conclude that for *any* solution, α must satisfy the bounds given in Theorem 2.2. To see this, first note that when $\beta > 0$, f' cannot attain a minimum at or below zero. Thus any solution must satisfy $f' > 0$ for all $\eta \geq 0$. Next, if $\alpha \geq 0$ and $\beta > 1$, then f' is initially increasing. In order to satisfy the boundary condition at infinity, f' would have to first attain a maximum where $f''' < 0$. However, as in the proof of Theorem 2.2, f''' could not thereafter increase through zero, and so $f'(\infty) = 0$ could not be achieved.

So far we have been concerned with existence of a solution and its properties, however there are parameter values for which no solution exists. Specifically:

Theorem 2.4. *For $0 < \beta \leq 1$ and $S \geq 0$, the BVP has no solution.*

Proof. Recall that for $\beta > 0$, f' cannot have a minimum at or below 0, thus any solution must satisfy $f' > 0$ for all $\eta \geq 0$. Also, when $\beta > 0$, f''' cannot increase through zero when $f'' < 0$, so that $f''' > 0$ for all $\eta \geq 0$. However $f'''(0) = 2S\alpha + \beta - 1 > 0$ implies that $\alpha > (1-\beta)/2S \geq 0$. This then implies that $\alpha > 0$ and arguing as in the Remark 2.3, f' could not then satisfy the boundary condition at infinity. □

The results of Theorems 2.1 and 2.4 imply that in the case of fluid injection ($S > 0$), solutions to the problem only exist for a sufficiently accelerating disk ($\beta > 1$) whereas no solution exists for insufficient acceleration ($0 < \beta \leq 1$).

3 Decelerated disk

The analysis regarding a decelerating disk is much less comprehensive than the accelerating case. For $\beta < 0$ and $S > 0$, we conjecture that no solution exists, but have only been able to prove this for a relatively small portion of parameter space (see Theorem 3.2 below). However, in the next theorem we show that for a decelerating disk ($\beta < 0$) and sufficient fluid injection ($S > 0$ sufficiently large) there is no solution to the BVP with the property $f' > 0$ for all $\eta \geq 0$. Thus, if a solution does exist in this case, f' must at some point become negative. Physically, this implies a region of flow reversal in the radial direction. This implication is discussed further in Section 4.

Theorem 3.1. *If $-4 \leq \beta < 0$ and $S > 0$, then the BVP has no solution that satisfies $f' > 0$ for all $\eta \geq 0$. Further, if $\beta < -4$ and*

$$S \geq \frac{1}{8}\left[25\beta^4 + \left(\frac{28\beta^3}{3} - 128\beta + \frac{256}{3}\right)(\beta+4)\right]^{\frac{1}{4}} \equiv p(\beta), \tag{3.1}$$

then the BVP has no solution that satisfies $f' > 0$ for all $\eta \geq 0$.

Proof. We will assume the existence of a solution satisfying $f' > 0$ for all $\eta \geq 0$ and derive a contradiction. We first list some properties that such a solution must possess. When $\beta < 0$, from (2.1) we see that f' cannot have a positive minimum. Also note that when $\beta < 0$, f''' cannot decrease through zero when $f'' < 0$. Thus as f' approaches zero from above, we must have $f'' < 0$ and $f''' > 0$ for η large. Thus $f''(\infty) = 0$. Finally, note that the quantity $u = 2f + \beta\eta/2$ is negative for large η since $\beta < 0$ and $f > 0$ grows sublinearly.

If some value of $f''(0) = \alpha \leq 0$ gives a solution to the BVP, then using (2.6) we have

$$f'' = \alpha + \left(2f + \frac{\beta\eta}{2}\right)f' - 2S + \frac{\beta}{2}(f - S) - 3\int_0^\eta f'(t)^2\,dt. \tag{3.2}$$

As η tends to infinity, the left side of (3.2) tends to zero while the right side is strictly negative, giving a contradiction.

If some value of $f''(0) = \alpha > 0$ gives a solution to the BVP, then f' must attain a positive maximum, say at η_2, where $f''(\eta_2) = 0$. Integrating the ODE (2.1) from η_2 to $\eta > \eta_2$ we have

$$f'' + \left(2f(\eta_2) + \frac{\beta\eta_2}{2}\right)f'(\eta_2) = \left(2f + \frac{\beta\eta}{2}\right)f' + \frac{\beta}{2}(f - f(\eta_2)) - 3\int_{\eta_2}^\eta f'(t)^2\,dt. \tag{3.3}$$

As $\eta \to \infty$ and $f'' \to 0$, the right side of (3.3) is strictly negative, implying that

$$2f(\eta_2) + \frac{\beta\eta_2}{2} < 0. \tag{3.4}$$

But the quantity

$$u(\eta) = 2f(\eta) + \frac{\beta\eta}{2}, \tag{3.5}$$

is initially positive since $u(0) = 2S > 0$. Thus a root of $u = 2f + \beta\eta/2$ must occur as f' increases towards its maximum. However, $f' > f'(0) = 1$ as it increases to its max. But if $-4 \leq \beta < 0$, then

$$u'(\eta) = 2f'(\eta) + \frac{\beta}{2} \geq 0, \quad \forall\, \eta \in [0, \eta_2]. \tag{3.6}$$

Thus $u = 2f + \beta\eta/2$ cannot decrease through zero, giving a contradiction. Thus if $-4 \le \beta < 0$ and $S > 0$, the BVP cannot have a solution that satisfies $f' > 0$ for all $\eta > 0$.

If $\beta < -4$, we need additional conditions on S in order to obtain our final contradiction regarding the root of $u = 2f + \beta\eta/2$. Recall that $f''(\eta_2) = 0$ at the maximum of f' and let $\eta_1 < \eta_2$ be the point where $u(\eta_1) = 2f(\eta_1) + \beta\eta_1/2 = 0$ first decreases through zero.

Integrating the ODE (2.1) from zero to η_1 we have

$$\alpha - 2S + \frac{\beta}{2}(f(\eta_1) - S) - 3\int_0^{\eta_1} f'(t)^2\,dt = f''(\eta_1) > 0 \tag{3.7}$$

from which we conclude that

$$\alpha > 2S > 0, \tag{3.8}$$

since $\beta(f(\eta_1) - S)/2$ and $-3\int_0^{\eta_1} f'^2\,dt$ are both strictly negative.

Next, note from (3.8) and the assumption $S \ge p(\beta) > \sqrt{1-\beta}/2$ we have that

$$f'''(0) = 2S\alpha + \beta - 1 > 0, \tag{3.9}$$

so that f'' initially increases above its starting value of α. As f' increases towards its max, f''' must at some first point decrease through zero (and cannot subsequently increase through zero for $\eta \le \eta_2$). Thus, there must exist a point η_0 such that

$$f''(\eta_0) = \alpha \;\text{ with }\; f''(\eta) > \alpha \quad \forall\,\eta \in (0,\eta_0) \;\text{ and }\; f''(\eta) < \alpha \quad \forall\,\eta \in (\eta_0,\eta_2]. \tag{3.10}$$

Note that $\eta_0 < \eta_1$. To see this, note that (2.6) evaluated at η_0 gives

$$\left(2f(\eta_0) - \frac{\beta\eta_0}{2}\right) f'(\eta_0) = 2S - \frac{\beta}{2}(f(\eta_0) - S) + 3\int_0^{\eta_0} f'(t)^2\,dt > 0, \tag{3.11}$$

which implies that $u(\eta_0) = 2f(\eta_0) + \beta\eta_0 > 0$. But $u = 2f + \beta\eta/2$ first decreases through zero at η_1. If $\eta_0 > \eta_1$, then there would have to exist a intermediate point, $\eta^* \in (\eta_1, \eta_0)$, where $u = 2f + \beta\eta/2$ increases back through zero with $f''(\eta^*) > \alpha$ by (3.10). Evaluating (2.6) at such a point gives

$$f''(\eta^*) - \alpha = -2S + \frac{\beta}{2}(f(\eta^*) - S) - 3\int_0^{\eta^*} f'(t)^2\,dt < 0, \tag{3.12}$$

contradicting $f''(\eta^*) > \alpha$. Thus $\eta_0 < \eta_1$.

Next note that when $u(\eta_1) = 2f(\eta_1) + \beta\eta_1/2 = 0$, we need $u'(\eta_1) = 2f'(\eta_1) + \beta/2 \le 0$, thus $f'(\eta_1) \le -\beta/4$. Since f' is increasing on $(0,\eta_1)$ we then have that

$$1 < f'(\eta) \le -\frac{\beta}{4}, \quad \forall\,\eta \in (0,\eta_1). \tag{3.13}$$

Subsequently,

$$\frac{\beta+4}{2} \le 2f' + \frac{\beta}{2}, \quad \forall\,\eta \in (0,\eta_1), \tag{3.14}$$

and on integration from zero to $\eta > 0$ we have

$$2S + \frac{\beta+4}{2}\eta < u(\eta) = 2f(\eta) + \frac{\beta\eta}{2}, \quad \forall\,\eta \in (0,\eta_1). \tag{3.15}$$

Further, note that when $f'' = \alpha$ at $\eta_0 < \eta_1$ we have from (2.1) that

$$\left(2f(\eta_0) + \frac{\beta\eta_0}{2}\right) f''(\eta_0) = f'(\eta_0)(f'(\eta_0) - \beta) + f'''(\eta_0) < f'(\eta_0)(f'(\eta_0) - \beta), \tag{3.16}$$

since $f'''(\eta_0) < 0$. Next, using (3.8) and (3.13) in (3.16) we obtain

$$2f(\eta_0) + \frac{\beta\eta_0}{2} < \frac{5\beta^2}{32S}. \tag{3.17}$$

Combining (3.15) and (3.17) we conclude that

$$2S + \frac{\beta+4}{2}\eta_0 < 2f(\eta_0) + \frac{\beta\eta_0}{2} < \frac{5\beta^2}{32S}, \tag{3.18}$$

or (using $S \geq p(\beta) > \sqrt{5}|\beta|/8$)

$$\eta_0 > \frac{5\beta^2 - 64S^2}{16S(\beta+4)} > 0. \tag{3.19}$$

Next, multiply the ODE (2.1) by f'' and integrate from zero to η_1 to obtain

$$\int_0^{\eta_0} \left(2f(t) + \frac{\beta t}{2}\right) f''(t)^2\, dt = \frac{1}{3}(f'(\eta_0)^3 - 1) - \frac{\beta}{2}(f'(\eta_0)^2 - 1). \tag{3.20}$$

Using (3.13) to bound the right hand side of (3.20) we have

$$\int_0^{\eta_0} \left(2f(t) + \frac{\beta t}{2}\right) f''(t)^2\, dt < \frac{-7\beta^3}{192} - \frac{1}{3} + \frac{\beta}{2}. \tag{3.21}$$

However, using (3.8), (3.15) and (3.19) we also have

$$\frac{25\beta^4 - 4096S^4}{256(\beta+4)} < \int_0^{\frac{5\beta^2-64S^2}{16S(\beta+4)}} \left(2S + \frac{\beta+4}{2}t\right) 4S^2\, dt < \int_0^{\eta_0} \left(2f(t) + \frac{\beta t}{2}\right) f''(t)^2\, dt. \tag{3.22}$$

Combing (3.21) and (3.22) gives

$$\frac{25\beta^4 - 4096S^4}{256(\beta+4)} < \frac{-7\beta^3}{192} - \frac{1}{3} + \frac{\beta}{2} \tag{3.23}$$

which is contradicted if

$$S \geq \frac{1}{8}\left[25\beta^4 + \left(\frac{28\beta^3}{3} - 128\beta + \frac{256}{3}\right)(\beta+4)\right]^{\frac{1}{4}}. \tag{3.24}$$

Thus if $\beta < -4$ and (3.24) holds, then the BVP has no solution with the property $f' > 0$ for all $\eta > 0$. □

Physically, the results of this Theorem 3.1 imply that for a decelerating disk ($\beta < 0$) and sufficiently strong fluid injection ($S > 0$ if $-4 \leq \beta < 0$ and $S > p(\beta)$ if $\beta < -4$), any solution to the BVP must contain a region where $f'(\eta)$ becomes negative. As $f'(\eta)$ is proportional to the fluid velocity in the radial direction, this implies a region of flow reversal in the boundary layer, and such solutions are unlikely to be stable in the full governing PDE and thus are not physically realizable. Thus this theorem gives strong evidence that no solution exists for the parameter values given in the theorem statement. In fact, we can eliminate the possibility of these non-physical solutions mathematically, but only for a small range of parameter space (see Theorem 3.2 below).

Theorem 3.2. *If $-4 \leq \beta < 0$ and $S \geq 3\beta^2/2 - 5\beta/8$, then the BVP has no solution.*

Proof. We will assume a solution exists and derive a contradiction. By Theorem 3.1, for $-4 \leq \beta < 0$, if a solution exists, then f' at some point must become negative. Let η_A be the first point at which f' decreases through zero, $f'(\eta_A) = 0$. Let η_B be the first point, if ever, that the quantity $u = 2f + \beta\eta/2$ decreases through zero, $u(\eta_B) = 2f(\eta_B) + \beta\eta_B/2 = 0$. Let

$$\eta_1 = \min\{\eta_A, \eta_B\}. \tag{3.25}$$

Note that for $\beta < 0$, f' cannot have a minimum at or below $f' = \beta$. Also note that $u' = 2f' + \beta/2 \geq 0$ for $f' \geq -\beta/4$. Thus if a root of $u = 2f + \beta\eta/2$ occurs, it must occur with f' in the range $\beta < f' < -\beta/4$.

With these preliminaries out of the way, we will show that for $-4 \leq \beta < 0$ and $S > 0$ sufficiently large, f' must fall below $f' = \beta$ at some point in the interval $[\eta_1, \eta_1 + 1]$, and thus cannot be a solution to the BVP.

To see this, first consider a value $f''(0) = \alpha \leq 0$ that gives a solution to the BVP. An integration of the ODE (2.1) evaluated at η_1 gives

$$f''(\eta_1) = \alpha - 2S + \frac{\beta}{2}(f(\eta_1) - S) - 3\int_0^{\eta_1} f'(t)^2\,dt < -2S, \tag{3.26}$$

since each term of the right hand side of (3.26) is nonpositive.

In order to satisfy the boundary condition $f'(\infty) = 0$, f' must attain a first negative minimum at some point $\eta_2 > \eta_1$ with

$$\beta < f', \quad \forall \eta \in [\eta_1, \eta_2]. \tag{3.27}$$

Integrating (3.27) from η_1 to $\eta > \eta_1$ and multiplying by $\beta/2 < 0$ we have

$$\frac{\beta}{2}(f - f(\eta_1)) < \frac{\beta^2}{2}(\eta - \eta_1), \quad \forall \eta \in [\eta_1, \eta_2]. \tag{3.28}$$

Again using (3.27), we also have

$$\frac{5\beta}{2} < u' = 2f' + \frac{\beta}{2}, \quad \forall \eta \in [\eta_1, \eta_2]. \tag{3.29}$$

Integrating (3.29) from η_1 to $\eta > \eta_1$ we have

$$\frac{5\beta}{2}(\eta - \eta_1) < 2f(\eta_1) + \frac{\beta\eta_1}{2} + \frac{5\beta}{2}(\eta - \eta_1) < 2f + \frac{\beta\eta}{2}, \quad \forall \eta \in [\eta_1, \eta_2]. \tag{3.30}$$

Taking the worst case scenario where $2f + \beta\eta/2 < 0$ and $f' < 0$ on $(\eta_1, \eta_2]$ in (3.30) we derive the bound

$$\left(2f + \frac{\beta\eta}{2}\right) f' < \frac{5\beta^2}{2}(\eta - \eta_1), \quad \forall \eta \in (\eta_1, \eta_2]. \tag{3.31}$$

Next, integrating the ODE (2.1) from η_1 to $\eta > \eta_1$ we have

$$f'' = f''(\eta_1) + \left(2f + \frac{\beta\eta}{2}\right) f' + \frac{\beta}{2}(f - f'(\eta_1)) - 3\int_{\eta_1}^{\eta} f'(t)^2\,dt. \tag{3.32}$$

Using (3.26), (3.28), and (3.31) in (3.32) we then have

$$f'' < -2S + \frac{5\beta^2}{2}(\eta - \eta_1) + \frac{\beta^2}{2}(\eta - \eta_1), \quad \forall \eta \in (\eta_1, \eta_2]. \tag{3.33}$$

Next, consider the line through the points $(\eta_1, -\beta/4)$ and (η_1+1, β), which has slope $m = 5\beta/4$. Choosing $S > 0$ sufficiently large in (3.33) we can ensure that $f'' < 5\beta/4$ on the interval $(\eta_1, \eta_1+1]$, thus forcing the minimum of f' to occur at $\eta_2 > \eta_1 + 1$. However, this then implies that

$$f'' < -2S + 3\beta^2 \leq \frac{5\beta}{4}, \quad \forall \eta \in (\eta_1, \eta_1+1], \tag{3.34}$$

which, on integration from η_1 to $\eta > \eta_1$, then implies that

$$f' < f'(\eta_1) + \frac{5\beta}{4}(\eta - \eta_1) \leq \frac{5\beta}{4}(\eta - \eta_1) - \frac{\beta}{4}, \quad \forall \eta \in (\eta_1, \eta_1+1], \tag{3.35}$$

since $f'(\eta_1) \leq -\beta/4$. Evaluating at $\eta_1 + 1$ we have

$$f'(\eta_1 + 1) < \beta, \tag{3.36}$$

at which point f' can no longer attain a minimum and therefore cannot be a solution to the BVP.

Inequality (3.34) is satisfied if

$$S \geq \frac{3\beta^2}{2} - \frac{5\beta}{8}. \tag{3.37}$$

Thus if $-4 \leq \beta < 0$ and (3.37) is satisfied, then no value of $f''(0) = \alpha \leq 0$ gives a solution to the BVP.

Next consider the case that a value $f''(0) = \alpha > 0$ gives a solution to the BVP. Then f' is initially increasing and must at some first point, η_0, attain a maximum above $f'(0) = 1$ where $f''(\eta_0) = 0$. Again let η_A be the first subsequent point where f' vanishes, $f'(\eta_A) = 0$. Integrating the ODE (2.1) from η_0 to η_A we have

$$f''(\eta_A) = -\left(2f(\eta_0) + \frac{\beta\eta_0}{2}\right) f'(\eta_0) + \frac{\beta}{2}(f(\eta_A) - f(\eta_0)) - 3\int_{\eta_0}^{\eta_A} f'(t)^2\, dt. \tag{3.38}$$

Note that all of $u = 2f + \beta\eta/2$, f', and f are positive and increasing on $(0, \eta_0)$. Thus $(2f(\eta_0) + \beta\eta_0/2)f'(\eta_0) > 2S$. Thus from (3.38) we have

$$f''(\eta_A) < -2S, \tag{3.39}$$

and the proof proceeds as in the case $f''(0) = \alpha \leq 0$. Thus if $-4 \leq \beta < 0$ and $S \geq 3\beta^2/2 - 5\beta/8$, then the BVP has no solution. □

4 Discussion and numerical results

In this section we discuss the mathematical and physical implications of the results obtained in Sections 2 and 3. These results are summarized graphically in Figure 4.1. The region shaded in green indicates parameter values for which at least one solution to the BVP exists. This solution satisfies $f' > 0$, $f'' < 0$, $f''' > 0$, $f^{(4)} < 0$, and $f^{(5)} > 0$ for all $\eta > 0$. The area shaded in red denotes parameter values for which no solution exists. Finally, the region shaded in orange indicates parameter values for which no solution exists that satisfies $f' > 0$ for all $\eta > 0$. Thus, if a solution to the BVP exists for parameter values in this range, f' must at some point become negative. Physically, this represents a region of flow reversal in the radial direction, and such solutions are unlikely to be stable in the full governing PDEs, and thus are not physically realizable.

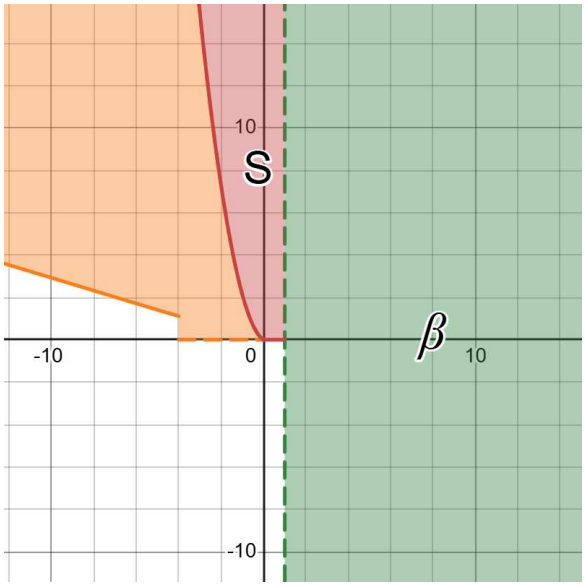


Figure 4.1: β-S parameter space showing regions of existence (green) and nonexistence (red) of solutions to the BVP (2.1–2.2). The region shaded in orange shows parameter values for which no solution exists that satisfies $f' > 0$ for all $\eta > 0$.

To study the problem numerically, we began by replicating the numerical results given in Mehmood *et al.* [8]. Using the same parameter values we found that we could not improve on the results given in [8]. These values include various levels of deceleration ($\beta < 0$) and fluid injection ($S < 0$). A comparison of our results and the results of [8] are given in Tables 4.1 and 4.2.

S	β	α [8]	α
-1.35	-1.14000950	-1.0990	-1.0990
-1.36	-1.66710229	-0.8947	-0.89475
-1.38	-3.17491743	-0.3142	-0.31428
-1.40	-5.57621580	0.5940	-0.5944

Table 4.1: Comparison of numerical values of $f''(0) = \alpha$ between [8] and current results.

Next, based on numerical integration of the problem, it was conjectured in [8] that solutions to the BVP (2.1–2.2) do not exist in the case of fluid injection ($S > 0$). The results given here indicate that, at least for a sufficiently accelerated disk ($\beta > 1$), this conjecture is incorrect. Theorem 2.1 establishes existence of a solution for all $\beta > 1$ and any $S \in \mathbb{R}$.

However, for $\beta > 1$ and $S > 0$, we had difficulty finding these solutions numerically. We employed the numerical integration software XPP-Aut [4]. Using a fourth order Runge–Kutta scheme with a stepsize $d\eta = .0001$ and an interval length $\eta_{\max} = 5$, the automated boundary value solver was unable to converge to a solution.

Consequently, we attempted to pinpoint the solution with manual numerical shooting. We used the bounds on the shooting parameter α given in Theorem 2.2 as a guide. In Figure 4.2,

β	S	α [8]	α
-1	-1.34688680700	-1.1532	-1.1532
-2	-1.36528410900	-0.7659	-0.76596
-3	-1.37810278220	-0.3812	-0.38124
-4	-1.38796975330	0	0
-5	-1.39598577367	0.3779	0.37796

Table 4.2: Comparison of numerical values of $f''(0) = \alpha$ between [8] and current results.

a graph of these bounds on α are given as a function of S for $\beta = 8$. Such bounds greatly narrowed the search for the appropriate value of α, but we were still unable to obtain convergence to a solution.

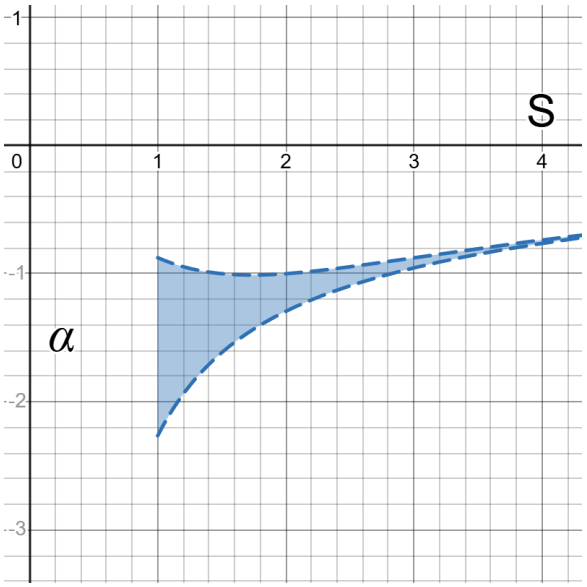


Figure 4.2: Bounds from Theorem 2.2 on $f''(0) = \alpha$ as a function of S for $\beta = 8$. A solution to the BVP (2.1–2.2) must have $f''(0) = \alpha$, where α lies is in this range.

As an example, consider parameter values $\beta = 2$ and $S = 2$. The bounds given in Theorem 2.2 then indicate that a value of α giving a solution to the boundary value problem must lie in the range $-.22 \approx -\frac{11}{50} < \alpha < -\frac{4}{19} \approx -.2105263$. We numerically integrated the differential equation using fourth order Runge–Kutta with a stepsize of $d\eta = .0001$ and an integration length of $\eta_{\max} = 5$. Our manual search resulted in a value of $\alpha \approx -0.217830769818888$ which does lie in the range given by Theorem 2.2. However, the convergence of the solution to the boundary condition at infinity is very slow, and our numerical software lost track of the solution even before $\eta = 4$. Figure 4.3 graphs two numerically integrated solutions to the initial value problem for values of α near the suspected true value. Note the quick divergence at a relatively small value of the independent variable η. Reducing the stepsize from $d\eta = .0001$ to $d\eta = .00001$ did not materially effect the results.

This numerical difficulty can arise from several sources. First, for $\beta > 0$ and $S > 0$,

the coefficient $-(2f + \beta\eta/2)$ of f'' in the ODE (2.1) can lead to rapid growth of the solution of the IVP (2.1), $(2.2)_1$, $(2.2)_2$, and (2.3) away from the solution of the BVP (2.1–2.2) during numerical shooting. Also, for positive β and S, the convergence of the solution is algebraic, not exponential, as η tends to infinity, adding further difficulty.

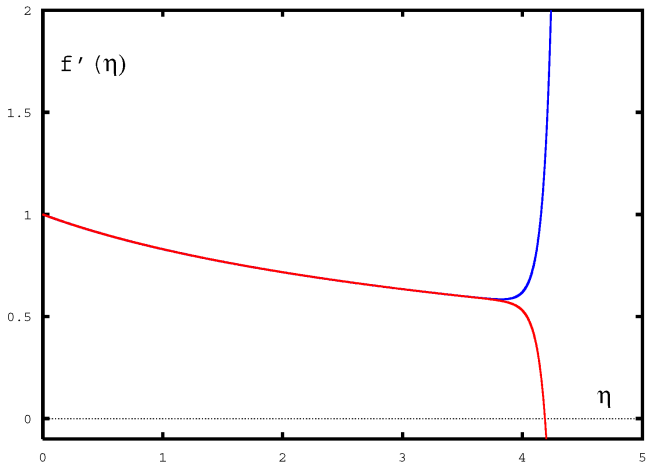


Figure 4.3: Two numerically integrated solutions to the IVP for $\beta = 2$ and $S = 2$ using fourth order Runge–Kutta with $d\eta = .0001$. Upper curve: $\alpha = -.2178307698188879$; lower curve: $\alpha = -.217830769818888$. Theorem 2.2 indicates that the value of α must satisfy $-.22 < \alpha < -.2105263$.

Finally, we note that for $\beta < 0$ and $S < 0$ we were unable to attain any analytical results. In this parameter range, Mehmood *et al.* [8] report dual solutions. However, we conjecture that for $\beta < 0$ and $S < 0$ there exists a continuum of infinitely many solutions. In Figure 4.4 we graph several solutions to the boundary value problem for $\beta = -2$ and $S = -2$. For these parameter values the solutions converge to the boundary condition at infinity very quickly, thus we increased the stepsize to $d\eta = .001$ and increased $\eta_{\max} = 10$. In Figure 4.4, the values of α ranged from 0 up to 5 in increments of 0.5. All solutions still converged to zero as $\eta_{\max}$ was increased. The graph shows the solutions on the range $0 \le \eta \le 10$, however, the interval of integration was increased and $f'(\eta)$ continued to show convergence to zero on any interval examined.

A similar graph can be produced using values of α ranging from -5 to 0. The maximal range of α which produces solutions to the BVP could not be determined exactly since for larger values of $|\alpha|$, the convergence of the solution became very slow.

Using a combination of direct analysis of the BVP (2.1–2.2) and numerical integration of the problem, we were able to extend and refine the results initiated in [8]. However, questions remain regarding the solution set of (2.1–2.2). Specifically, we end this note with two open conjectures:

Conjecture 1. For $\beta < 0$ and $S \ge 0$, no solution to the BVP (2.1–2.2) exists. (Theorems 3.1 and 3.2 above give partial results regarding this conjecture.)

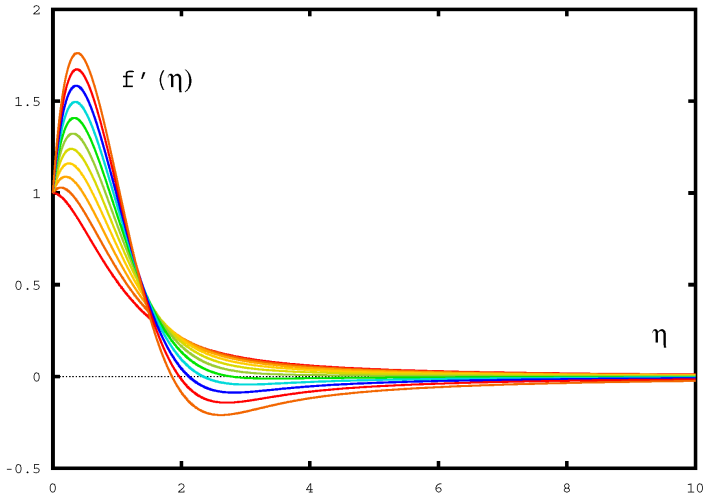


Figure 4.4: Graph of continuum of solutions to the BVP (2.1–2.2) for $\beta = -2$ and $S = -2$. Initial conditions on $f''(0) = \alpha$ range from 0 to 5 in increments of 0.5.

This would imply that for a decelerating disk ($\beta < 0$) and for any amount of fluid injection ($S > 0$), or even an impermeable disk ($S = 0$), no solution exists. However, numerically, it does appear that for a decelerating disk ($\beta < 0$) solutions do exist for sufficiently strong suction (S sufficiently negative). This implies that there exists a critical value for suction, $S_0 < 0$, below which solutions will exist. This critical value is likely to depend on β. Thus we are led to the following conjecture:

Conjecture 2. For $\beta < 0$, there exists an $S_0(\beta) < 0$ such that for all $S < S_0(\beta)$, the BVP (2.1–2.2) possesses a continuum of infinitely many solutions.

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Existence of positive solutions of a semipositone problem with Minkowski-curvature operator

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Abstract. We are concerned with the semipositone problem with Minkowski-curvature operator

$$\begin{cases} -\operatorname{div}\left(\dfrac{\nabla u}{\sqrt{1-|\nabla u|^2}}\right) = \lambda a(|x|)(u^\gamma - \epsilon), & x \in B(R), \\ u = 0, & x \in \partial B(R), \end{cases}$$

where $B(R) = \{x \in \mathbb{R}^N : |x| < R\}$, $N \geq 2$, $R > 0$, $\epsilon > 0$, $\gamma > 1$ and $a : [0,\infty) \to (0,\infty)$ is a continuous function. We show that there exist the constants $\Lambda > 2N/(\max_{r\in[0,R]} a(r)R^{\gamma+1})$ and $\epsilon_0(\Lambda) > 0$ such that, if $\epsilon < \epsilon_0(\Lambda)$, then the problem has least two positive radial solution for any $\lambda \in (\Lambda, \Lambda^*)$, where Λ^* is a constant that depends on ϵ. The proof of our main result is based upon degree theory.

Keywords: mean curvature operator, semipositone, positive solutions, degree theory.

2020 Mathematics Subject Classification: 35B09, 35A16, 35J93.

1 Introduction

In this paper, we are concerned with the problem with Minkowski-curvature operator

$$\begin{cases} -\operatorname{div}\left(\dfrac{\nabla u}{\sqrt{1-|\nabla u|^2}}\right) = \lambda a(|x|)(u^\gamma - \epsilon), & x \in B(R), \\ u = 0, & x \in \partial B(R), \end{cases} \tag{1.1}$$

where $B(R) = \{x \in \mathbb{R}^N : |x| < R\}$, $N \geq 2$, $R > 0$, $\epsilon > 0$, $\gamma > 1$ and $a : [0,\infty) \to (0,\infty)$ is a continuous function.

Boundary value problem of differential equation

$$\begin{cases} -\Delta u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.2}$$

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is called a semipositone problem if $f(\cdot,0)<0$; and it is called a positone problem if $f(\cdot,0)\geq 0$, where Ω is a strictly convex bounded domain in $\mathbb{R}^N$, $N\geq 1$, with C^2 boundary $\partial\Omega$. Semipositone problem has been largely investigated for semilinear elliptic equations (1.2) since the late 1980s, with increasing interest after the paper Castro *et al.* [11] and Brown *et al.* [9], see [4,16,20] and the references therein.

It is well documented (see [8,21]) that, if compared with the positone problems, the study of positive solutions of the semipositone problem is mathematically challenging. For example, the absence of maximum principle, the difficulty in constructing sub and super solutions, and the trivial solution $u\equiv 0$ is not a solution of problem (1.2) with $f(\cdot,0)<0$. These three factors make it unclear how to apply the methods based on the theory of positive operators for cones in Banach spaces, the method of sub and super solutions and bifurcation from the trivial solution.

For quasilinear semipositone problems, there have been some recent papers on the problems with the p-Laplacian operator, see Alves, de Holanda and Santos [2, 3], Castro, de Figueredo and Lopera [10], Chhetri, Drábek and Shivaji [12], Morris, Shivaji and Sim [23]. In particular, Chhetri, Drábek and Shivaji [12] considered a quasilinear elliptic problem of the form

$$\begin{cases} -\Delta_p u=\lambda g(u) & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where $\Delta_p u:=\operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator with $1<p<2$, and $\lambda>0$ is a parameter. The nonlinearity $g:[0,\infty)\to\mathbb{R}$ is a continuous function that satisfies $g(0)<0, g(s)>0$ for $s\gg 1$. They said that λ_∞ is a bifurcation point from infinity if the solution set $S:=\{(\lambda,u)\in\mathbb{R}\times W_0^{1,p}(\Omega): \lambda \text{ and } u \text{ solves (1.3)}\}$ contains a sequence $\{(\lambda_n,u_n)\}$ such that

$$\lambda_n\to\lambda_\infty \quad \text{and} \quad \|u_n\|_\infty\to\infty. \tag{1.4}$$

And they proved the following result.

Theorem A. *Assume $1<p<2$ and $N>p$. Let $g:[0,\infty)\to\mathbb{R}$ be a continuous function satisfying the following conditions:*

(i) $g(0)<0$;

(ii) there exist $b>0$ and $q\in(p-1,N(p-1)/(N-p)]$ such that

$$\lim_{s\to\infty}\frac{g(s)}{s^q}=b.$$

Then (1.1) has a positive solution (λ,u) for $\lambda>0$ small.

The key points of the proof of Theorem A are that rescale the parameter and the solution variable using $\gamma=\lambda^{\frac{1}{q-p+1}}$ and $w=\gamma u$, respectively, then w formally satisfies

$$-\Delta_p w=-\operatorname{div}(|\nabla\gamma u|^{p-2}\nabla\gamma u)=-\gamma^{p-1}\operatorname{div}(|\nabla u|^{p-2}\nabla u)=\gamma^q g(w/\gamma). \tag{1.5}$$

The theory of positive operators for (1.5) can be applied when $\gamma>0$ is small (i.e., $\lambda>0$ is small). This is the standard technique of dealing with semipositone elliptic problems.

However, the mean curvature operator in semipositone problem (1.1) prevents the application of the above standard technique, since $\varphi(ks)\neq k\varphi(s)$ (where $\varphi(s)=\frac{s}{\sqrt{1-s^2}}$) for any

constant $k \in \mathbb{R}\backslash\{0,1\}$ and any solution u of (1.1) is bounded (see Remark 2.1 below). So although there have been significant paper on the problems with mean curvature operator (see Azzollini [5], Bereanu and Mawhin [7], Bereanu, Jebelean and Torres [6], Ma, Gao and Lu [22] and Dai [15] and the references therein) in the last decade, they have had to require that the nonlinearity f be non-negative at 0. In particular, Bereanu, Jebelean and Torres [6] studied the problem (1.1) with $\epsilon = 0$ and proved that there exists $\Lambda > 2N/(\max_{r\in[0,R]} a(r)R^{\gamma+1})$ such that the problem has at least two positive radial solutions according to $\lambda > \Lambda$.

Thus, this paper appears to be the first contribution where consider the problem with mean curvature operator is faced in a semipositone setting. And the aim of this brief paper is to provide a result about the existence of positive solution of the semipositone problem (1.1).

To state our results precisely we recall the following hypothesis,

(H0) $\epsilon > 0$, $\gamma > 1$ and $a : [0,\infty) \to (0,\infty)$ is a continuous function.

Theorem 1.1. *Let (H0) hold. There exist the constants $\Lambda > 2N/(\max_{r\in[0,R]} a(r)R^{\gamma+1})$ and $\epsilon_0(\Lambda) > 0$ such that, for any $\epsilon < \epsilon_0(\Lambda)$ there is $\Lambda^*(\epsilon)$ such that problem (1.1) has at least two positive radial solutions for any $\lambda \in (\Lambda, \Lambda^*)$.*

Remark 1.2. The nonlinearity in (1.1) is related to *pure-power* model $g(s) = s^\gamma$, see W. Allegretto, P. Nistri, P. Zecca [1], D. Costa, H. Ramos Quoirin, H. Tehrani [14], Q. Morris, R. Shivaji, I. Sim [23]. In recent years, for the existence of positive solutions to the semipositone elliptic equation, see, D. D. Hai, A. Muthunayake, R. Shivaji [17], A. Joseph, L. Sankar [18], R. Kajikiya, E. Ko [19] and their references.

2 Preliminary results

Setting, as usual, $r = |x|$ and $u(x) = u(r)$, we reduce the Dirichlet problem (1.1) to the mixed boundary value problem

$$\begin{cases} -(r^{N-1}\varphi(u'(r)))' = \lambda r^{N-1}a(r)(u^\gamma - \epsilon), \quad r \in (0,R), \\ u'(0) = u(R) = 0, \end{cases} \tag{2.1}$$

where $\varphi(s) = \frac{s}{\sqrt{1-s^2}}$. The space $C := C[0,R]$ will be endowed with the usual sup-norm $\|\cdot\|_\infty$ and $C^1 := C^1[0,R]$ will be considered with the norm $\|u\| = \|u\|_\infty + \|u'\|_\infty$. Also, we shall use the closed subspace of C^1 defined by

$$C^1_M = \{u \in C^1 : u'(0) = 0 = u(R)\}.$$

Let

$$W := \left\{w \in C^1_M : \max_{[0,R]} |w'| < 1 \text{ and } \varphi(w') \in C^1[0,R]\right\}.$$

A solution of (2.1) is a function $u \in W$ satisfying (2.1). Function $\Psi = \varphi^{-1}$ is smooth, $w' \in C^1$ can be obtained immediately.

Remark 2.1. Combined with the boundary condition $u(R) = 0$, it is easy to find that the solution $u \in W$ of (2.1) is bounded.

In the sequel, let us consider an auxiliary problem

$$\begin{cases} -(r^{N-1}\varphi(u'(r)))' = \lambda r^{N-1}a(r)(u(r))^{\gamma}, & r \in (0,R), \\ u'(0) = u(R) = 0. \end{cases} \tag{2.2}$$

We introduce the linear operators

$$S : C \to C, \qquad Su(r) = \frac{1}{r^{N-1}}\int_0^r t^{N-1}u(t)dt \qquad (r \in (0,R]), \quad Su(0) = 0;$$

$$K : C \to C^1, \qquad Ku(r) = \int_r^R u(t)dt \qquad (r \in [0,R]).$$

It is easy to see that K is bounded and standard arguments, invoking the Arzelà–Ascoli theorem, show that S is compact. This implies that the nonlinear operator $K \circ \varphi^{-1} \circ S : C \to C^1$ is compact. Let N_λ be the Nemytskii operator associated to $\lambda a(r)u^{\gamma}$, i.e,

$$N_\lambda : C \to C, \qquad N_\lambda(u) = \lambda a(\cdot)u^{\gamma}(\cdot).$$

From [6, Lemma 1], a function $u \in C^1_M$ is a solution of (2.2) if and only if it is a fixed point of the compact nonlinear operator

$$\mathcal{N}_\lambda : C^1_M \to C^1_M, \qquad \mathcal{N}_\lambda = K \circ \varphi^{-1} \circ S \circ N_\lambda. \tag{2.3}$$

A lower solution of (2.2) is a function $\alpha \in C^1$ such that $\|\alpha'\|_\infty < 1$, $r^{N-1}\varphi(\alpha') \in C^1$ and

$$(r^{N-1}\varphi(\alpha'(r)))' + \lambda r^{N-1}a(r)\alpha(r)^{\gamma} \geq 0, \qquad r \in [0,R], \quad \alpha(R) \leq 0.$$

Further, we say that a lower solution α of (2.2) is strict if every solution u of (2.2) with $u \geq \alpha$ in $[0,R]$ satisfies $u \gg \alpha$ in $[0,R)$ (i.e., $u(r) > \alpha(r)$ for any $r \in [0,R)$).

Similarly, a upper solution of (2.2) is a function $\beta \in C^1$ such that $\|\beta'\|_\infty < 1$, $r^{N-1}\varphi(\beta') \in C^1$ and

$$(r^{N-1}\varphi(\beta'(r)))' + \lambda r^{N-1}a(r)\beta(r)^{\gamma} \leq 0, \qquad r \in [0,R], \quad \beta(R) \geq 0.$$

Further, we say that a upper solution β of (2.2) is strict if every solution u of (2.2) with $u \leq \beta$ in $[0,R]$ satisfies $u \ll \beta$ in $[0,R)$ (i.e., $u(r) < \beta(r)$ for any $r \in [0,R)$).

Lemma 2.2 ([13]). *For any fixed $\lambda > 0$, suppose that there exist a lower solution α and an upper solution β of (2.2) with $\alpha \leq \beta$ in $[0,R]$. Then problem (2.2) has solutions v, w with $\alpha \leq v \leq w \leq \beta$ in $[0,R]$ such that every solution u of (2.2) with $\alpha \leq u \leq \beta$ in $[0,R]$ satisfies $v \leq u \leq w$ in $[0,R]$. Further, if α and β are strict, then*

$$\deg(I - \mathcal{N}_\lambda, U^\beta_\alpha, 0) = 1,$$

where $\mathcal{N}_\lambda$ is defined by (2.3) and

$$U^\beta_\alpha = \{z \in C^1_M : \alpha \ll z \ll \beta \text{ in } [0,R) \text{ and } \|z'\|_\infty < 1\}.$$

From [6, Theorem 1.1], there exists

$$\Lambda > 2N/(a_M R^{\gamma+1}),$$

where $a_M := \max_{r\in[0,R]} a(r)$, such that problem (2.2) has at least one positive solutions according to $\lambda = \Lambda$. Let $\lambda_0 > \Lambda$ be arbitrarily chosen. Let $\widetilde{\alpha}$ be a positive solution for (2.2) corresponding to $\lambda = \Lambda$. It is easy to see that $\widetilde{\alpha}$ is a lower solution for (2.2) with $\lambda = \lambda_0$. For fixed $\lambda_2 > \lambda_0$, let $\widetilde{\beta}$ be the unique (positive) solution of

$$\begin{cases} (r^{N-1}\varphi(u'(r)))' + \lambda_2 r^{N-1} a_M \widetilde{R}^\gamma = 0, \quad r \in (0, \widetilde{R}), \\ u'(0) = u(\widetilde{R}) = 0, \end{cases}$$

where $\widetilde{R} > R$. Using that $\widetilde{\beta}(R) > 0$ and

$$\lambda_0 a(r) u^\gamma(r) < \lambda_2 a_M \widetilde{R}^\gamma, \qquad r \in [0, R],$$

it follows that $\widetilde{\beta}$ is an upper solution for (2.2) with $\lambda = \lambda_0$. We can find $\widetilde{R}$ sufficiently large, such that $\widetilde{\alpha}(0) < \widetilde{\beta}(R)$. For any $r \in [0, R)$,

$$\widetilde{\alpha}'(r) = -\varphi^{-1}\Big(\frac{1}{r^{N-1}} \int_0^r t^{N-1}[\Lambda a(t)(\widetilde{\alpha}(t))^\gamma]dt\Big) < 0$$

and

$$\widetilde{\beta}'(r) = -\varphi^{-1}\Big(\frac{1}{r^{N-1}} \int_0^r t^{N-1}[\lambda_2 a_M \widetilde{R}^\gamma]dt\Big) < 0.$$

Then, taking into account that $\widetilde{\alpha}, \widetilde{\beta}$ are strictly decreasing, we infer that $\widetilde{\alpha} < \widetilde{\beta}$ on $[0, R]$.

From Lemma 2.2, problem (2.2) has a solution u_0 satisfies $\widetilde{\alpha} \le u_0 \le \widetilde{\beta}$ in $[0, R]$, i.e., $u_0 \in U_{\widetilde{\alpha}}^{\widetilde{\beta}}$. We will show that $\widetilde{\alpha}$ and $\widetilde{\beta}$ are strict. In fact, for all $r \in [0, R)$, one has

$$\begin{aligned} \widetilde{\beta}(r) &= \int_r^{\widetilde{R}} \varphi^{-1}\Big(\frac{1}{t^{N-1}} \int_0^t s^{N-1}[\lambda_2 a_M \widetilde{R}^\gamma]ds\Big)dt \\ &> \int_r^R \varphi^{-1}\Big(\frac{1}{t^{N-1}} \int_0^t s^{N-1}[\lambda_0 a(s)(u_0(s))^\gamma]ds\Big)dt = u_0(r). \end{aligned}$$

Similarly, for all $r \in [0, R)$,

$$\begin{aligned} \widetilde{\alpha}(r) &= \int_r^R \varphi^{-1}\Big(\frac{1}{t^{N-1}} \int_0^t s^{N-1}[\Lambda a(s)(\widetilde{\alpha}(s))^\gamma]ds\Big)dt \\ &< \int_r^R \varphi^{-1}\Big(\frac{1}{t^{N-1}} \int_0^t s^{N-1}[\lambda_0 a(s)(u_0(s))^\gamma]ds\Big)dt = u_0(r). \end{aligned}$$

This suggests that

$$\deg\Big(I - \mathcal{N}_{\lambda_0}, U_{\widetilde{\alpha}}^{\widetilde{\beta}}, 0\Big) = 1.$$

For any $\rho > 0$, set $B_\rho := \{u \in C_M^1 : \|u\| < \rho\}$. Consider $\rho_1 > 0$ sufficiently small and $\rho_2 \ge R + 1$ such that $U_{\widetilde{\alpha}}^{\widetilde{\beta}} \cap B_{\rho_1} = \emptyset$ and $(U_{\widetilde{\alpha}}^{\widetilde{\beta}} \cup B_{\rho_1}) \subset B_{\rho_2}$. By [6, Lemma 1 and Lemma 4], one has

$$\deg(I - \mathcal{N}_{\lambda_0}, B_{\rho_1}, 0) = 1$$

and

$$\deg(I - \mathcal{N}_{\lambda_0}, B_{\rho_2}, 0) = 1.$$

Then, from the additivity-excision property of the Leray–Schauder degree it follows that

$$\deg(I - \mathcal{N}_{\lambda_0}, B_{\rho_2} \setminus [\overline{U}_{u_1}^{u_2} \cup \overline{B}_{\rho_1}], 0) = -1.$$

3 The proof of main result

Lemma 3.1. *For any fixed $\lambda_0 > \Lambda$, there exists $\delta > 0$ such that if $F_1 : C^1 \to C^1_M$ is a compact map with $\|F_1(v)\| < \delta$ for v in $U^{\widetilde{\beta}}_{\widetilde{\alpha}}$ and $\|F_1(w)\| < \delta$ for w in $B_{\rho_2} \setminus [\overline{U}^{\widetilde{\beta}}_{\widetilde{\alpha}} \cup \overline{B}_{\rho_1}]$, $v, w > 0$, then there exist $u_1 \in U^{\widetilde{\beta}}_{\widetilde{\alpha}}$, $u_2 \in B_{\rho_2} \setminus [\overline{U}^{\widetilde{\beta}}_{\widetilde{\alpha}} \cup \overline{B}_{\rho_1}]$ and $u_1, u_2 > 0$ such that $u_i = \mathcal{N}_{\lambda_0}(u_i) + F_1(u_i)$, $i = 1, 2$.*

Proof. Denote

$$S_1 := \left\{ u \in U^{\widetilde{\beta}}_{\widetilde{\alpha}} : u \text{ is a solution of (2.2) with } \lambda = \lambda_0 \right\}.$$

Choose $\varepsilon_1 > 0$ small and set $\overline{N}_{1,\varepsilon_1} = \bigcup_{u \in S_1} B_{\varepsilon_1}(u)$, where $B_{\varepsilon_1}(u)$ denotes the open ball of radius ε_1 in C^1_M centered at u. Obviously, $\overline{N}_{1,\varepsilon_1}$ is open. Since $\mathcal{N}_{\lambda_0} : C^1_M \to C^1_M$ is compact, there exists $\delta > 0$ such that $\|(I - \mathcal{N}_{\lambda_0})(v)\|_{C^1} > \delta$ if $v \in \partial \overline{N}_{1,\varepsilon_1}$.

Claim. if $v \in \overline{N}_{1,\varepsilon_1}$ and ε_1 is small enough, then $v > 0$ in the interval $[0, R)$.

Suppose on the contrary that there exists $v_\varepsilon \in C^1_M$, $u_\varepsilon \in S_1$ such that $\|v_\varepsilon - u_\varepsilon\| \to 0$ and $v_\varepsilon(r_0) \le 0$ for some $r_0 \in [0, R)$. Since S_1 is compact, without loss of generality, we can assume that $u_\varepsilon \to u$ (as $\varepsilon \to 0$) for some $u \in S_1$, (i.e. $v_\varepsilon \to u$). From $u \in S_1$ and the monotonicity of u, we obtain that $u > 0$ in $[0, R)$ and $u'(R) < 0$. Since $v'_\varepsilon(0) = v_\varepsilon(R) = 0$, it follows that $v_\varepsilon > 0$ in $[0, R)$ for ε small enough. This contradiction ends the proof of Claim.

If $\|F_1(v)\| < \delta$ for v in $U^{\widetilde{\beta}}_{\widetilde{\alpha}}$, we conclude that

$$\deg\left((I - \mathcal{N}_{\lambda_0} - F_1), \overline{N}_{1,\varepsilon_1}, 0\right) = \deg(I - \mathcal{N}_{\lambda_0}, \overline{N}_{1,\varepsilon_1}, 0) = \deg(I - \mathcal{N}_{\lambda_0}, U^{\widetilde{\beta}}_{\widetilde{\alpha}}, 0) = 1.$$

Similarly, let

$$S_2 := \left\{ u \in B_{\rho_2} \setminus [\overline{U}^{\widetilde{\beta}}_{\widetilde{\alpha}} \cup \overline{B}_{\rho_1}] : u \text{ is a solution of (2.2) with } \lambda = \lambda_0 \right\}.$$

Choose $\varepsilon_2 > 0$ small and set $\overline{N}_{2,\varepsilon_2} = \bigcup_{u \in S_2} B_{\varepsilon_2}(u)$. If $\|F_1(v)\| < \delta$ for v in $B_{\rho_2} \setminus [\overline{U}^{\widetilde{\beta}}_{\widetilde{\alpha}} \cup \overline{B}_{\rho_1}]$, we conclude that

$$\deg\left((I - \mathcal{N}_{\lambda_0} - F_1), \overline{N}_{2,\varepsilon_2}, 0\right) = \deg(I - \mathcal{N}_{\lambda_0}, \overline{N}_{2,\varepsilon_2}, 0) = \deg(I - \mathcal{N}_{\lambda_0}, B_{\rho_2} \setminus [\overline{U}^{\widetilde{\beta}}_{\widetilde{\alpha}} \cup \overline{B}_{\rho_1}], 0) = -1.$$

□

The proof of Theorem 1.1. Select δ as in Lemma 3.1. Define

$$\Lambda^* := \frac{N\varphi(\delta/(R+1))}{a_M \epsilon R},$$

where $\epsilon > 0$ is defined in (1.1) and $a_M = \max_{r \in [0,R]} a(r)$. There exist a constant $\epsilon_0(\Lambda) > 0$ such that $\Lambda < \Lambda^*$ for any $\epsilon < \epsilon_0(\Lambda)$, where $\Lambda > 2N/(a_M R^{\gamma+1})$ is the constant that appears in Theorem 1.1 of [6]. Let $\lambda_0 \in [\Lambda, \Lambda^*]$ be arbitrarily chosen.

Let

$$F_1(u)(r) = -\int_r^R \varphi^{-1}\left(\frac{\lambda_0 \epsilon}{t^{N-1}} \int_0^t \tau^{N-1} a(\tau) d\tau\right) dt.$$

With these choices, we obtain that

$$\begin{aligned}\|F_1(u)\| &= \max_{r\in[0,R]}\left|\int_r^R \varphi^{-1}\left(\frac{\lambda_0\epsilon}{t^{N-1}}\int_0^t \tau^{N-1}a(\tau)d\tau\right)dt\right| + \max_{r\in[0,R]}\left|\varphi^{-1}\left(\frac{\lambda_0\epsilon}{r^{N-1}}\int_0^r \tau^{N-1}a(\tau)d\tau\right)\right| \\ &\le \max_{r\in[0,R]}\left|\int_r^R \varphi^{-1}\left(\frac{\lambda_0 a_M\epsilon R}{N}\right)dt\right| + \varphi^{-1}\left(\frac{\lambda_0 a_M\epsilon R}{N}\right) \\ &\le \varphi^{-1}\left(\frac{\lambda_0 a_M\epsilon R}{N}\right)(R+1) < \delta.\end{aligned}$$

By Lemma 3.1, we can conclude the existence of the $\widetilde{u}_1, \widetilde{u}_2 > 0$ such that

$$\widetilde{u}_i = \int_r^R \varphi^{-1}\left(\frac{\lambda_0}{t^{N-1}}\int_0^t \tau^{N-1}a(\tau)(\widetilde{u}_i(\tau)-\epsilon)d\tau\right)dt, \qquad i = 1,2.$$

This suggests that $\widetilde{u}_1, \widetilde{u}_2 > 0$ are the solutions of (2.1). The proof of Theorem 1.1 is complete. □

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Differential inclusion systems with fractional competing operator and multi-valued fractional convection

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Abstract. The existence of solutions to a family of inclusion systems with fractional, possibly competing, elliptic operators, fractional convection, and homogeneous Dirichlet boundary conditions is established. The proof uses Galerkin's method, a surjectivity result for multifunctions in finite dimensional spaces, and approximation techniques.

Keywords: differential inclusion system, fractional competing operator, fractional convection, Dirichlet boundary condition, Galerkin's method.

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1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geq 3$, with a smooth boundary $\partial\Omega$, let $\mu_1, \mu_2 \in \mathbb{R}$, and let $F_1, F_2 : \Omega \times \mathbb{R}^2 \times \mathbb{R}^{2N} \to 2^{\mathbb{R}}$ be two compact convex-valued multifunctions. Consider the differential inclusion system

$$\begin{cases} (-\Delta)_{p_1}^{s_1} u_1 + \mu_1 (-\Delta)_{q_1}^{t_1} u_1 \in F_1(x, u_1, u_2, D^{r_1} u_1, D^{r_2} u_2) & \text{in } \Omega, \\ (-\Delta)_{p_2}^{s_2} u_2 + \mu_2 (-\Delta)_{q_2}^{t_2} u_2 \in F_2(x, u_1, u_2, D^{r_1} u_1, D^{r_2} u_2) & \text{in } \Omega, \\ u_1 = u_2 = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.1}$$

where

(H_1) $0 < t_i < r_i < s_i \leq 1$ and $1 < q_i < p_i < \frac{N}{s_i}$ for each $i = 1, 2$.

The symbol $(-\Delta)_p^s$, with $p > 1$ and $0 < s < 1$, denotes the fractional p-Laplacian, defined by setting, provided u is smooth enough,

$$(-\Delta)_p^s u(x) := 2 \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{N+ps}} \, \mathrm{d}y, \quad x \in \mathbb{R}^N,$$

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with $B_\varepsilon(x) := \{z \in \mathbb{R}^N : \|z-x\|_{\mathbb{R}^N} < \varepsilon\}$. When $s = 1$ it becomes the classical p-Laplacian, namely

$$-\Delta_p u := -\operatorname{div}(|\nabla u|^{p-2}\nabla u).$$

Moreover, $D^s u$ indicates the *distributional Riesz fractional gradient* of u in the sense of [19,20]. If u appropriately decays and is sufficiently smooth then, setting

$$c_{N,s} := -\frac{2^s\,\Gamma\left(\frac{N+s+1}{2}\right)}{\pi^{\frac{N}{2}}\,\Gamma\left(\frac{1-s}{2}\right)},$$

one has [20, pp. 289 and 298]

$$D^s u(x) := c_{N,s}\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^N\setminus B_\varepsilon(x)}\frac{u(x)-u(y)}{|x-y|^{N+s}}\,\frac{x-y}{|x-y|}\mathrm{d}y, \quad x \in \mathbb{R}^N.$$

The right-hand sides F_1 and F_2 satisfy the conditions below, where, to avoid cumbersome formulae, we shall write

$$y := (y_1, y_2), \quad z := (z_1, z_2), \quad p_i^* := \frac{Np_i}{N - s_i p_i}, \quad i = 1, 2. \tag{1.2}$$

(H_2) $x \mapsto F_i(x,y,z)$ is measurable on Ω for all $(y,z) \in \mathbb{R}^2 \times \mathbb{R}^{2N}$ and $(y,z) \mapsto F_i(x,y,z)$ is upper semi-continuous for almost every $x \in \Omega$.

(H_3) There exist $m_i > 0$, $\delta_i \in L^{(p_i^*)'}(\Omega)$, $i = 1,2$, such that

$$\sup_{w_i \in F_i(x,y,z)} |w_i| \le m_i\left(|y_1|^{\frac{p_1^*}{(p_i^*)'}} + |y_2|^{\frac{p_2^*}{(p_i^*)'}} + |z_1|^{\frac{p_1}{(p_i^*)'}} + |z_2|^{\frac{p_2}{(p_i^*)'}}\right) + \delta_i(x)$$

a.e. in Ω and for all $(y,z) \in \mathbb{R}^2 \times \mathbb{R}^{2N}$.

(H_4) There are $M_i, M_i' > 0$, $\sigma_i \in L^1(\Omega)_+$, $i = 1,2$, fulfilling

$$w_i y_i \le M_i(|y_1|^{p_1} + |y_2|^{p_2}) + M_i'(|z_1|^{p_1} + |z_2|^{p_2}) + \sigma_i(x)$$

a.e. in Ω and for all $(y,z) \in \mathbb{R}^2 \times \mathbb{R}^{2N}$, $w_i \in F_i(x,y,z)$.

The involved differential operators are of the type

$$A_\mu(u) := (-\Delta)_p^s u + \mu(-\Delta)_q^t u, \quad u \in W_0^{s,p}(\Omega),$$

where $\mu \in \mathbb{R}$, $0 < t \le r \le s \le 1$, $1 < q < p < \frac{N}{s}$, while convection comes from the presence of fractional gradients $D^r u$ at right-hand sides. A_μ exhibits different behaviors depending on the values of $t, s \in (0,1]$. Precisely, if $t = 1$, then $t = r = s = 1$. Problem (1.1) falls inside the local framework, which has already been investigated in some recent works; see, e.g., [9] for single-valued reactions and [4,18] as regards multi-valued ones. Moreover, the nature of A_μ drastically changes depending on μ. When $\mu > 0$, the operator A_μ is basically patterned after the (possibly) fractional (p,q)-Laplacian, which is non-homogeneous because $p \ne q$. If $\mu = 0$ it coincides with the fractional p-Laplacian. Both cases have been widely studied, and meaningful results are by now available in the literature. On the other hand, for $\mu < 0$ the operator A_μ contains the *difference* between the fractional p- and q-Laplacians. It is usually called competitive and, as already pointed out in [14,17], does not satisfy any ellipticity or

monotonicity condition. In fact, given $u_0 \in W_0^{s,p}(\Omega) \setminus \{0\}$ and chosen $u := \tau u_0$, $\tau > 0$, the expression

$$\langle A_\mu(u), u\rangle = \tau^p \langle (-\Delta)_p^s u_0, u_0\rangle + \mu\, \tau^q \langle (-\Delta)_q^t u_0, u_0\rangle$$

turns out negative for τ small and positive when τ is large, because

$$u_0 \neq 0 \quad \Longrightarrow \quad \langle (-\Delta)_p^s u_0, u_0\rangle > 0,\ \langle (-\Delta)_q^t u_0, u_0\rangle > 0;$$

cf. Section 2. Hence, nonlinear regularity theory, comparison principles, as well as existence theorems for pseudo-monotone maps cannot be employed. Moreover, since the reactions are multi-valued and contain the fractional gradient of the solutions, also variational techniques are no longer directly usable. To overcome these difficulties we first exploit Galerkin's method, thus working with a sequence $\{E_n\}$ of finite dimensional functional spaces. For each integer $n \geq 1$, an *approximate solution* $(u_{1,n}, u_{2,n}) \in E_n$ to (1.1) is obtained via a suitable version (see Proposition 2.3) of a classical surjectivity result. Next, letting $n \to +\infty$ yields a solution in a generalized sense (cf. Definitions 3.3 and 3.5), which turns out weak sense once $\min\{\mu_1, \mu_2\} \geq 0$.

Fractional gradients were first introduced more than sixty years ago by Horváth [12], but they garnered significant interest especially after the works of Shieh and Spector [5,19,20]. The operator $D^s u$ appears as a natural non-local version of ∇u, to which $D^s u$ formally converges when $s \to 1^-$. It possesses favorable geometric and physical properties [2,21], like invariance under translations or rotations, homogeneity of order s, continuity, etc.

Section 2 contains some auxiliary results and the functional framework needed for handling both fractional gradients and the fractional p-Laplacian. The existence of (generalized, strong generalized, or weak) solutions to (1.1) is established in Section 3.

2 Preliminaries

Let X, Y be two nonempty sets. A multifunction $\Phi : X \to 2^Y$ is a map from X into the family of all nonempty subsets of Y. A function $\varphi : X \to Y$ is called a selection of Φ when $\varphi(x) \in \Phi(x)$ for every $x \in X$. Given $B \subseteq Y$, put $\Phi^-(B) := \{x \in X \mid \Phi(x) \cap B \neq \emptyset\}$. If X, Y are topological spaces and $\Phi^-(B)$ turns out closed in X for all closed sets $B \subseteq Y$ then we say that Φ is upper semi-continuous. Suppose $(X, \mathcal{F})$ is a measurable space and Y is a topological space. The multifunction Φ is called measurable when $\Phi^-(B) \in \mathcal{F}$ for every open set $B \subseteq Y$. The result below, stated in [1, p. 215], will be repeatedly useful.

Proposition 2.1. *Let $F : \Omega \times \mathbb{R}^h \to 2^{\mathbb{R}}$ be a closed-valued multifunction such that:*

- *$x \mapsto F(x, \xi)$ is measurable for all $\xi \in \mathbb{R}^h$;*
- *$\xi \mapsto F(x, \xi)$ is upper semi-continuous for almost every $x \in \Omega$.*

Let $w : \Omega \to \mathbb{R}^h$ be measurable. Then the multifunction $x \mapsto F(x, w(x))$ admits a measurable selection.

Let $(X, \|\cdot\|)$ be a real normed space with topological dual X^* and duality brackets $\langle \cdot, \cdot \rangle$. Given a nonempty set $A \subseteq X$, define $|A| := \sup_{x \in A} \|x\|$. We say that $\varphi : X \to X^*$ is *monotone* when

$$\langle \varphi(x) - \varphi(z), x - z\rangle \geq 0 \quad \forall\, x, z \in X,$$

and *of type* $(\mathrm{S})_+$ provided

$$x_n \rightharpoonup x \quad \text{in } X, \quad \limsup_{n\to+\infty} \langle \varphi(x_n), x_n - x\rangle \le 0 \implies x_n \to x \quad \text{in } X.$$

The next elementary result [8, Proposition 2.1] will ensure that condition $(\mathrm{S})_+$ holds true for the differential operators we deal with.

Proposition 2.2. *Let $\varphi : X \to X^*$ be of type $(\mathrm{S})_+$ and let $\psi : X \to X^*$ be monotone. Then $\varphi + \psi$ satisfies condition $(\mathrm{S})_+$.*

A multifunction $\Phi : X \to 2^{X^*}$ is called coercive provided

$$\lim_{\|x\|\to\infty} \frac{\inf\{\langle x^*, x\rangle \mid x \in X,\ x^* \in \Phi(x)\}}{\|x\|} = +\infty\,.$$

The following result is a direct consequence of [11, Proposition 3.2.33].

Theorem 2.3. *Let X be a finite-dimensional normed space and let $\Phi : X \to 2^{X^*}$ be a convex compact-valued multifunction. Suppose Φ is upper semi-continuous and coercive. Then there exists $\hat{x} \in X$ satisfying $0 \in \Phi(\hat{x})$.*

Hereafter, if X and Y are two topological spaces, the symbol $X \hookrightarrow Y$ means that X continuously embeds in Y. Given $p > 1$, put $p' := \frac{p}{p-1}$, denote by $\|\cdot\|_p$ the usual norm of $L^p(\Omega)$, and indicate with $\|\cdot\|_{1,p}$ the norm on $W_0^{1,p}(\Omega)$ arising from Poincaré's inequality, namely

$$\|u\|_{1,p} := \|\nabla u\|_p\,, \quad u \in W_0^{1,p}(\Omega).$$

If $u \in W_0^{1,p}(\Omega)$, we set $u(x) = 0$ on $\mathbb{R}^N \setminus \Omega$; cf. [6, Section 5]. Fix $s \in (0,1)$. The Gagliardo seminorm of a measurable function $u : \mathbb{R}^N \to \mathbb{R}$ is

$$[u]_{s,p} := \left(\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{p}},$$

while $W^{s,p}(\mathbb{R}^N)$ denotes the fractional Sobolev space

$$W^{s,p}(\mathbb{R}^N) := \left\{u \in L^p(\mathbb{R}^N) :\ [u]_{s,p} < \infty\right\},$$

endowed with the norm

$$\|u\|_{W^{s,p}(\mathbb{R}^N)} := \left(\|u\|^p_{L^p(\mathbb{R}^N)} + [u]^p_{s,p}\right)^{\frac{1}{p}}.$$

As usual, on the space

$$W_0^{s,p}(\Omega) := \{u \in W^{s,p}(\mathbb{R}^N) : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\}$$

we will consider the equivalent norm

$$\|u\|_{s,p} := [u]_{s,p}, \quad u \in W_0^{s,p}(\Omega).$$

Let $W^{-s,p'}(\Omega) := (W_0^{s,p}(\Omega))^*$ and let p_s^* be the fractional Sobolev critical exponent, i.e., $p_s^* = \frac{Np}{N-sp}$ when $sp < N$, $p_s^* = +\infty$ otherwise. Thanks to Propositions 2.1–2.2, Theorem 6.7, and Corollary 7.2 of [6] one has

Proposition 2.4. *If* $1 \le p < +\infty$ *then:*

(a) $0 < s' \le s'' \le 1 \implies W_0^{s'',p}(\Omega) \hookrightarrow W_0^{s',p}(\Omega)$.

(b) $sp < N \implies W_0^{s,p}(\Omega) \hookrightarrow L^r(\Omega)$ *for all* $r \in [1, p_s^*]$.

(c) *The embedding in* (b) *is also compact once* $r \in [1, p_s^*)$.

However, contrary to the non-fractional case, we know [15] that

$$1 \le q < p \le +\infty \quad \not\Longrightarrow \quad W_0^{s,p}(\Omega) \subseteq W_0^{s,q}(\Omega).$$

Define, for every $u, v \in W_0^{s,p}(\Omega)$,

$$\langle (-\Delta)_p^s u, v\rangle := \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+ps}} \mathrm{d}x\mathrm{d}y\,.$$

The operator $(-\Delta)_p^s$ is called (negative) s-fractional p-Laplacian. It possesses the following properties.

(p_1) $(-\Delta)_p^s : W_0^{s,p}(\Omega) \to W^{-s,p'}(\Omega)$ turns out monotone, continuous, and of type $(\mathrm{S})_+$; vide, e.g., [7, Lemma 2.1].

(p_2) One has

$$\|(-\Delta)_p^s u\|_{W^{-s,p'}(\Omega)} \le \|u\|_{s,p}^{p-1} \quad \forall\, u \in W_0^{s,p}(\Omega).$$

Hence, $(-\Delta)_p^s$ maps bounded sets into bounded sets.

(p_3) The first eigenvalue $\lambda_{1,p,s}$ of $(-\Delta)_p^s$ is given by (cf. [13])

$$\lambda_{1,p,s} = \inf_{u \in W_0^{s,p}(\Omega), u \neq 0} \frac{\|u\|_{s,p}^p}{\|u\|_p^p}\,.$$

To deal with distributional fractional gradients, we first introduce the Bessel potential spaces $L^{\alpha,p}(\mathbb{R}^N)$, where $\alpha > 0$. Set, for every $x \in \mathbb{R}^N$,

$$g_\alpha(x) := \frac{1}{(4\pi)^{\frac{\alpha}{2}}\Gamma\left(\frac{\alpha}{2}\right)} \int_0^{+\infty} e^{\frac{-\pi|x|^2}{\delta}} e^{\frac{-\delta}{4\pi}} \delta^{\frac{\alpha-N}{2}} \frac{\mathrm{d}\delta}{\delta}\,.$$

On account of [16, Section 7.1] one can assert that:

1) $g_\alpha \in L^1(\mathbb{R}^N)$ and $\|g_\alpha\|_{L^1(\mathbb{R}^N)} = 1$.

2) g_α enjoys the semi-group property, i.e., $g_\alpha * g_\beta = g_{\alpha+\beta}$ for any $\alpha, \beta > 0$, with $*$ being the convolution operator.

We consider

$$L^{\alpha,p}(\mathbb{R}^N) := \{u : u = g_\alpha * \tilde{u} \text{ for some } \tilde{u} \in L^p(\mathbb{R}^N)\}.$$

If $u = g_\alpha * \tilde{u} = g_\alpha * \overline{u}$, with $\tilde{u}, \overline{u} \in L^p(\mathbb{R}^N)$, then a standard argument ensures that $\tilde{u} = \overline{u}$ because $g_\alpha > 0$. So, we can define

$$\|u\|_{L^{\alpha,p}(\mathbb{R}^N)} = \|\tilde{u}\|_{L^p(\mathbb{R}^N)} \quad \text{whenever} \quad u = g_\alpha * \tilde{u}.$$

Using 1) and 2) easily entails

$$0 < \alpha < \beta \implies L^{\beta,p}(\mathbb{R}^N) \subseteq L^{\alpha,p}(\mathbb{R}^N) \subseteq L^p(\mathbb{R}^N).$$

Moreover, by [19, Theorem 2.2], one has

Theorem 2.5. *If $1 < p < +\infty$ and $0 < \varepsilon < \alpha$ then*

$$L^{\alpha+\varepsilon,p}(\mathbb{R}^N) \hookrightarrow W^{\alpha,p}(\mathbb{R}^N) \hookrightarrow L^{\alpha-\varepsilon,p}(\mathbb{R}^N).$$

Finally, given $s \in (0,1)$, set

$$L_0^{s,p}(\Omega) := \{u \in L^{s,p}(\mathbb{R}^N) : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\}$$

and thanks to Theorem 2.5 we infer

$$L_0^{s+\varepsilon,p}(\Omega) \hookrightarrow W_0^{s,p}(\Omega) \hookrightarrow L_0^{s-\varepsilon,p}(\Omega) \quad \forall\, \varepsilon \in (0,s). \tag{2.1}$$

The next basic notion is taken from [19]. For $0 < \alpha < N$, let

$$\gamma(N,\alpha) := \frac{\Gamma\left(\frac{N-\alpha}{2}\right)}{\pi^{\frac{N}{2}} 2^\alpha \Gamma\left(\frac{\alpha}{2}\right)}, \quad I_\alpha(x) := \frac{\gamma(N,\alpha)}{|x|^{N-\alpha}}, \quad x \in \mathbb{R}^N \setminus \{0\}.$$

If $u \in L^p(\mathbb{R}^N)$ and $I_{1-s} * u$ makes sense then the vector

$$D^s u := \left(\frac{\partial}{\partial x_1}(I_{1-s} * u), \dots, \frac{\partial}{\partial x_N}(I_{1-s} * u)\right),$$

where partial derivatives are understood in a distributional sense, is called distributional Riesz s-fractional gradient of u. Theorem 1.2 in [19] ensures that

$$D^s u = I_{1-s} * Du \quad \forall\, u \in C_c^\infty(\mathbb{R}^N).$$

Further, $D^s u$ looks like the natural extension of ∇u to the fractional framework, Indeed, it exhibits analogous properties and, roughly speaking, $D^s u \to \nabla u$ when $s \to 1^-$; see, e.g., [10, Section 2].

According to [19, Definition 1.5], $X^{s,p}(\mathbb{R}^N)$ denotes the completion of $C_c^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_{X^{s,p}(\mathbb{R}^N)} := \left(\|u\|^p_{L^p(\mathbb{R}^N)} + \|D^s u\|^p_{L^p(\mathbb{R}^N)}\right)^{\frac{1}{p}}.$$

Since, by [19, Theorem 1.7], $X^{s,p}(\mathbb{R}^N) = L^{s,p}(\mathbb{R}^N)$ we can deduce many facts about $X^{s,p}(\mathbb{R}^N)$ from the existing literature on $L^{s,p}(\mathbb{R}^N)$. Moreover, if

$$X_0^{s,p}(\Omega) := \{u \in X^{s,p}(\mathbb{R}^N) : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\},$$

then $X_0^{s,p}(\Omega) = L_0^{s,p}(\Omega)$.

3 Existence results

To shorten notation, for $i = 1,2$, we set $U_i := W_0^{s_i,p_i}(\Omega)$ and denote by $\langle \cdot,\cdot \rangle_i$ the duality brackets of U_i. Lemma 2.6 in [3] guarantees that

$$U_i \hookrightarrow W_0^{t_i,q_i}(\Omega). \tag{3.1}$$

Hence, the differential operator $u \mapsto (-\Delta)^{s_i}_{p_i} u + \mu_i (-\Delta)^{t_i}_{q_i} u$ turns out well-defined on U_i. Let $A_i : U_i \to U_i^*$ be given by

$$\begin{aligned}\langle A_i(u), v\rangle_i := &\int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x)-u(y)|^{p_i-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+p_i s_i}} \mathrm{d}x\,\mathrm{d}y \\ &+ \mu_i \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x)-u(y)|^{q_i-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+q_i t_i}} \mathrm{d}x\,\mathrm{d}y\end{aligned}$$

for every $u, v \in U_i$. Thanks to properties (p_1)–(p_2) stated in Section 2, A_i is bounded and continuous. Consequently,

Lemma 3.1. *Under* (H_1)*, the operator* $A : U_1 \times U_2 \to U_1^* \times U_2^*$ *defined by*

$$A(u_1, u_2) := (A_1(u_1), A_2(u_2)) \quad \forall\, (u_1, u_2) \in U_1 \times U_2$$

maps bounded sets into bounded sets and is continuous.

Next, put, provided $(u_1, u_2) \in U_1 \times U_2$,

$$\mathcal{S}_{F_1,F_2}(u_1, u_2) := \{(w_1, w_2) \in L^{(p_1^*)'}(\Omega) \times L^{(p_2^*)'}(\Omega) : \\ w_i(\cdot) \in F_i(\cdot, u_1, u_2, D^{r_1}u_1, D^{r_2}u_2) \text{ a.e. in } \Omega,\ i = 1, 2\},$$

with p_i^* as in (1.2).

Lemma 3.2. *Let* (H_1)–(H_3) *be satisfied. Then:*

- (a_1) $\mathcal{S}_{F_1,F_2}(u_1, u_2)$ *turns out nonempty, convex, closed for all* $(u_1, u_2) \in U_1 \times U_2$.
- (a_2) *The multifunction* $\mathcal{S}_{F_1,F_2} : U_1 \times U_2 \to 2^{L^{(p_1^*)'}(\Omega) \times L^{(p_2^*)'}(\Omega)}$ *is bounded and strongly-weakly upper semi-continuous.*

Proof. Since $r_i < s_i$, if $\varepsilon \in (0, s_i - r_i)$, combining Proposition 2.4 with (2.1) yields

$$U_i \hookrightarrow W_0^{r_i+\varepsilon, p_i}(\Omega) \hookrightarrow L_0^{r_i, p_i}(\Omega).$$

Thus,

$$(u_1, u_2) \in U_1 \times U_2 \implies (D^{r_1}u_1, D^{r_2}u_2) \in L^{p_1}(\Omega) \times L^{p_2}(\Omega).$$

Now, pick $(u_1, u_2) \in U_1 \times U_2$. Via (H_2) and Proposition 2.1 we see that $F_i(\cdot, u_1, u_2, D^{r_1}u_1, D^{r_2}u_2)$ admits a measurable selection $w_i : \Omega \to \mathbb{R}$. By (H_3) one has

$$\begin{aligned} \|w_1\|_{(p_1^*)'}^{(p_1^*)'} &\le \int_\Omega \left[m_1\Big(|u_1|^{p_1^*-1} + |u_2|^{\frac{p_2^*}{(p_1^*)'}} + |D^{r_1}u_1|^{\frac{p_1}{(p_1^*)'}} + |D^{r_2}u_2|^{\frac{p_2}{(p_1^*)'}} \Big) + \delta_1 \right]^{(p_1^*)'} \mathrm{d}x \\ &\le c\Big(\|\delta_1\|_{(p_1^*)'}^{(p_1^*)'} + \|u_1\|_{p_1^*}^{p_1^*} + \|u_2\|_{p_2^*}^{p_2^*} + \|D^{r_1}u_1\|_{p_1}^{p_1} + \|D^{r_2}u_2\|_{p_2}^{p_2} \Big) \end{aligned}$$

for some $c > 0$, whence $\|w_1\|_{(p_1^*)'} < \infty$. Similarly, $\|w_2\|_{(p_2^*)'} < \infty$. So, $\mathcal{S}_{F_1,F_2}(u_1, u_2) \neq \emptyset$. This proves (a_1), because convexity and closedness follow at once from the analogous properties of F_i. Let us next verify (a_2). The above inequalities also guarantee that $\mathcal{S}_{F_1,F_2}$ maps bounded sets into bounded sets. If B is a nonempty weakly closed subset of $L^{(p_1^*)'}(\Omega) \times L^{(p_2^*)'}(\Omega)$ while $\{(u_{1,n}, u_{2,n})\} \subseteq \mathcal{S}^-_{F_1,F_2}(B)$ converges to (u_1, u_2) in $U_1 \times U_2$, then $\{(u_{1,n}, u_{2,n})\} \subseteq U_1 \times U_2$ turns out bounded. The same holds true concerning the set

$$\bigcup_{n \in \mathbb{N}} \mathcal{S}_{F_1,F_2}(u_{1,n}, u_{2,n}) \subseteq L^{(p_1^*)'}(\Omega) \times L^{(p_2^*)'}(\Omega).$$

Thus, up to sub-sequences, there exists $(w_{1,n}, w_{2,n}) \in \mathcal{S}_{F_1,F_2}(u_{1,n}, u_{2,n}) \cap B$, $n \in \mathbb{N}$, such that

$$(w_{1,n}, w_{2,n}) \rightharpoonup (w_1, w_2) \quad \text{in} \quad L^{(p_1^*)'}(\Omega) \times L^{(p_2^*)'}(\Omega).$$

One evidently has $(w_1, w_2) \in B$, because B is weakly closed. Mazur's principle provides a sequence $\{(\tilde{w}_{1,n}, \tilde{w}_{2,n})\}$ of convex combinations of $\{(w_{1,n}, w_{2,n})\}$ satisfying

$$(\tilde{w}_{1,n}, \tilde{w}_{2,n}) \to (w_1, w_2) \quad \text{in } L^{(p_1^*)'}(\Omega) \times L^{(p_2^*)'}(\Omega).$$

By (H_2), after passing to a sub-sequence which converges a.e., this easily entails

$$w_i(x) \in F_i(x, u_1(x), u_2(x), D^{r_1}u_1(x), D^{r_2}u_2(x)) \quad \text{for almost every} \quad x \in \Omega,\ i = 1, 2.$$

Consequently, $(w_1, w_2) \in \mathcal{S}_{F_1,F_2}(u_1, u_2) \cap B$, i.e., $(u_1, u_2) \in \mathcal{S}^-_{F_1,F_2}(B)$, as desired. □

Our existence result can be established after introducing some suitable constants and the notion of generalized solution to (1.1). Since $r_i < s_i$, $i = 1, 2$, embeddings (2.1) produce

$$\|D^{r_1} u_1\|_{p_1}^{p_1} \leq \hat{c}_1 \|u_1\|_{s_1,p_1}^{p_1} \quad \forall\, u_1 \in U_1, \qquad \|D^{r_2} u_2\|_{p_2}^{p_2} \leq \hat{c}_2 \|u_2\|_{s_2,p_2}^{p_2} \quad \forall\, u_2 \in U_2, \tag{3.2}$$

with appropriate $\hat{c}_i > 0$. Via (3.1) and its analogue for couples (s_2, p_2)–(t_2, q_2) we next have

$$\|u_1\|_{t_1,q_1}^{q_1} \leq \tilde{c}_1 \|u_1\|_{s_1,p_1}^{p_1} \quad \forall\, u_1 \in U_1, \qquad \|u_2\|_{t_2,q_2}^{q_2} \leq \tilde{c}_2 \|u_2\|_{s_2,p_2}^{p_2} \quad \forall\, u_2 \in U_2, \tag{3.3}$$

where $\tilde{c}_i > 0$. Finally, given $(T_1, T_2) \in U_1^* \times U_2^*$, set

$$\langle (T_1, T_2), (u_1, u_2) \rangle := \langle T_1, u_1 \rangle_1 + \langle T_2, u_2 \rangle_2, \quad (u_1, u_2) \in U_1 \times U_2.$$

Observe also that, by (b) in Proposition 2.4,

$$L^{(p_1^*)'}(\Omega) \times L^{(p_2^*)'}(\Omega) \hookrightarrow U_1^* \times U_2^*.$$

Hence, every $w \in L^{(p_1^*)'}(\Omega) \times L^{(p_2^*)'}(\Omega)$, $w = (w_1, w_2)$, defines a functional $T_w \in U_1^* \times U_2^*$ through

$$T_w(u_1, u_2) := \int_\Omega (u_1 w_1 + u_2 w_2) \mathrm{d}x \quad \forall\, (u_1, u_2) \in U_1 \times U_2. \tag{3.4}$$

Definition 3.3. We say that $(u_1, u_2) \in U_1 \times U_2$ is a generalized solution of (1.1) if there exist two sequences $(u_{1,n}, u_{2,n}) \in U_1 \times U_2$ and $w_n \in \mathcal{S}_{F_1,F_2}(u_{1,n}, u_{2,n})$, $w_n = (w_{1,n}, w_{2,n})$, fulfilling:

(i) $(u_{1,n}, u_{2,n}) \rightharpoonup (u_1, u_2)$ in $U_1 \times U_2$;

(ii) $A(u_{1,n}, u_{2,n}) - T_{w_n} \rightharpoonup 0$ in $U_1^* \times U_2^*$;

(iii) $\langle A(u_{1,n}, u_{2,n}), (u_{1,n} - u_1, u_{2,n} - u_2) \rangle - T_{w_n}(u_{1,n} - u_1, u_{2,n} - u_2) \to 0$.

Theorem 3.4. *Suppose* (H_1)–(H_4) *are satisfied and, moreover,*

$$\frac{M_1 + M_2}{\lambda_{1,p_i,s_i}} + \hat{c}_i (M_1' + M_2') + \tilde{c}_i |\mu_i| < 1, \quad i = 1, 2. \tag{3.5}$$

Then Problem (1.1) *admits a generalized solution.*

Proof. The space $U_1 \times U_2$ is separable, therefore it possesses a Galerkin's basis, namely a sequence $\{E_n\}$ of linear sub-spaces of $U_1 \times U_2$ such that:

(i_1) $\dim(E_n) < \infty \quad \forall\, n \in \mathbb{N}$;

(i_2) $E_n \subseteq E_{n+1} \quad \forall\, n \in \mathbb{N}$;

(i_3) $\overline{\cup_{n=1}^{\infty} E_n} = U_1 \times U_2$.

Pick $n \in \mathbb{N}$. Consider the following problem: Find $(u_1, u_2) \in E_n$, $(w_1, w_2) \in \mathcal{S}_{F_1,F_2}(u_1, u_2)$ fulfilling

$$\langle A(u_1, u_2), (v_1, v_2) \rangle - T_w(v_1, v_2) = 0 \quad \forall\, (v_1, v_2) \in E_n^*, \tag{3.6}$$

with T_w as in (3.4). Thanks to Lemma 3.2 the multifunction $\Phi : E_n \to 2^{E_n^*}$ defined by

$$\Phi(u_1, u_2) := \{ (A(u_1, u_2) - T_w) \lfloor_{E_n} : w \in \mathcal{S}_{F_1,F_2}(u_1, u_2), w = (w_1, w_2) \}, \quad (u_1, u_2) \in E_n,$$

takes nonempty, convex, closed values and maps bounded sets into bounded sets. We claim that Φ is upper semi-continuous. In fact, let $B \subseteq E_n^*$ closed. If $\{(u_{1,k}, u_{2,k})\} \subseteq \Phi^-(B)$ and $(u_{1,k}, u_{2,k}) \to (u_1, u_2)$ in E_n then there exists a sequence $w_k \in \mathcal{S}_{F_1,F_2}(u_{1,k}, u_{2,k})$, $w_k = (w_{1,k}, w_{2,k})$, such that

$$(A(u_{1,k}, u_{2,k}) - T_{w_k})\lfloor_{E_n} \in B \quad \forall\, k \in \mathbb{N}. \tag{3.7}$$

The same argument used in the proof of Lemma 3.2 gives $w \in \mathcal{S}_{F_1,F_2}(u_1, u_2)$, $w = (w_1, w_2)$, satisfying $(w_{1,k}, w_{2,k}) \rightharpoonup (w_1, w_2)$ in $L^{(p_1^*)'}(\Omega) \times L^{(p_2^*)'}(\Omega)$. Since $\dim(E_n) < \infty$, one has $T_{w_k}\lfloor_{E_n} \to T_w\lfloor_{E_n}$ in E_n^*. Thus, from Lemma 3.1 and (3.7) it follows

$$(A(u_1, u_2) - T_w)\lfloor_{E_n} \in B, \quad \text{i.e.,} \quad (u_1, u_2) \in \Phi^-(B).$$

This shows that $\Phi^-(B)$ is closed. As B was arbitrary, the multifunction Φ turns out upper semi-continuous. Next, if $(u_1, u_2) \in U_1 \times U_2$ and $w \in \mathcal{S}_{F_1,F_2}(u_1, u_2)$ then, thanks to (H_4), we have

$$\begin{aligned}\langle A(u_1, u_2), (u_1, u_2)\rangle - T_w(u_1, u_2) &\geq \|u_1\|_{s_1,p_1}^{p_1} + \|u_2\|_{s_2,p_2}^{p_2} - |\mu_1|\|u_1\|_{t_1,q_1}^{q_1} - |\mu_2|\|u_2\|_{t_2,q_2}^{q_2} \\ &\quad - \int_\Omega \left[M_1(|u_1|^{p_1} + |u_2|^{p_2}) + M_1'(|D^{r_1}u_1|^{p_1} + |D^{r_2}u_2|^{p_2}) + \sigma_1\right] \mathrm{d}x \\ &\quad - \int_\Omega \left[M_2(|u_1|^{p_1} + |u_2|^{p_2}) + M_2'(|D^{r_1}u_1|^{p_1} + |D^{r_2}u_2|^{p_2}) + \sigma_2\right] \mathrm{d}x.\end{aligned}$$

Exploiting (p_3) yields

$$\begin{aligned}&\langle A(u_1, u_2), (u_1, u_2)\rangle - T_w(u_1, u_2) \\ &\quad \geq \left(1 - \frac{M_1 + M_2}{\lambda_{1,p_1,s_1}}\right)\|u_1\|_{s_1,p_1}^{p_1} + \left(1 - \frac{M_1 + M_2}{\lambda_{1,p_2,s_2}}\right)\|u_2\|_{s_2,p_2}^{p_2} - |\mu_1|\|u_1\|_{t_1,q_1}^{q_1} - |\mu_2|\|u_2\|_{t_2,q_2}^{q_2} \\ &\qquad - \int_\Omega (M_1' + M_2')|D^{r_1}u_1|^{p_1}\mathrm{d}x - \int_\Omega (M_1' + M_2')|D^{r_2}u_2|^{p_2}\mathrm{d}x - \|\sigma_1\|_1 - \|\sigma_2\|_1,\end{aligned}$$

whence, on account of (3.2),

$$\begin{aligned}&\langle A(u_1, u_2), (u_1, u_2)\rangle - T_w(u_1, u_2) \\ &\quad \geq \left[1 - \frac{M_1 + M_2}{\lambda_{1,p_1,s_1}} - \hat{c}_1(M_1' + M_2')\right]\|u_1\|_{s_1,p_1}^{p_1} + \left[1 - \frac{M_1 + M_2}{\lambda_{1,p_2,s_2}} - \hat{c}_2(M_1' + M_2')\right]\|u_2\|_{s_2,p_2}^{p_2} \\ &\qquad - |\mu_1|\|u_1\|_{t_1,q_1}^{q_1} - |\mu_2|\|u_2\|_{t_2,q_2}^{q_2} - \|\sigma_1\|_1 - \|\sigma_2\|_1.\end{aligned}$$

Finally, through (3.3) we obtain

$$\begin{aligned}&\langle A(u_1, u_2), (u_1, u_2)\rangle - T_w(u_1, u_2) \\ &\quad \geq \left[1 - \frac{M_1 + M_2}{\lambda_{1,p_1,s_1}} - \hat{c}_1(M_1' + M_2') - |\mu_1|\tilde{c}_1\right]\|u_1\|_{s_1,p_1}^{p_1} \\ &\qquad + \left[1 - \frac{M_1 + M_2}{\lambda_{1,p_2,s_2}} - \hat{c}_2(M_1' + M_2') - |\mu_2|\tilde{c}_2\right]\|u_2\|_{s_2,p_2}^{p_2} - \|\sigma_1\|_1 - \|\sigma_2\|_1 \\ &\quad \geq \min_{i=1,2}\left[1 - \frac{M_1 + M_2}{\lambda_{1,p_i,s_i}} - \hat{c}_i(M_1' + M_2') - |\mu_i|\tilde{c}_i\right]\left(\|u_1\|_{s_1,p_1}^{p_1} + \|u_2\|_{s_2,p_2}^{p_2}\right) - \|\sigma_1\|_1 - \|\sigma_2\|_1,\end{aligned}$$

namely

$$\langle A(u_1, u_2), (u_1, u_2)\rangle - T_w(u_1, u_2) \geq \alpha\left(\|u_1\|_{s_1,p_1}^{p_1} + \|u_2\|_{s_2,p_2}^{p_2}\right) - \beta, \tag{3.8}$$

where

$$\alpha := \min_{i=1,2}\left[1 - \frac{M_1+M_2}{\lambda_{1,p_i,s_i}} - \hat{c}_i(M_1'+M_2') - |\mu_i|\tilde{c}_i\right], \quad \beta := \|\sigma_1\|_1 + \|\sigma_2\|_1.$$

Since (3.5) holds, the multifunction Φ turns out coercive. Now, Theorem 2.3 can be applied, and there exists a solution $(u_{1,n}, u_{2,n}) \in E_n$, $w_n \in \mathcal{S}_{F_1,F_2}(u_{1,n}, u_{2,n})$, $w_n = (w_{1,n}, w_{2,n})$, to Problem (3.6), i.e.,

$$\langle A(u_{1,n}, u_{2,n}), (v_1, v_2)\rangle - T_{w_n}(v_1, v_2) = 0, \quad (v_1, v_2) \in E_n^*. \tag{3.9}$$

From (3.8), written with $(u_1, u_2) := (u_{1,n}, u_{2,n})$, and (3.9) it follows

$$0 \geq \alpha\left(\|u_{1,n}\|_{s_1,p_1}^{p_1} + \|u_{2,n}\|_{s_2,p_2}^{p_2}\right) - \beta \quad \forall\, n \in \mathbb{N}.$$

Thus, $\{(u_{1,n}, u_{2,n})\} \subseteq U_1 \times U_2$ is bounded. By reflexivity one has $(u_{1,n}, u_{2,n}) \rightharpoonup (u_1, u_2)$ in $U_1 \times U_2$, taking a sub-sequence when necessary. Consequently, (i) of Definition 3.3 holds. Through Lemma 3.2 we next infer that $\{(w_{1,n}, w_{2,n})\} \subseteq L^{(p_1^*)'}(\Omega) \times L^{(p_3^*)'}(\Omega)$ turns out bounded. Therefore, always up to sub-sequences,

$$A(u_{1,n}, u_{2,n}) - T_{w_n} \rightharpoonup T \quad \text{in } U_1^* \times U_2^*. \tag{3.10}$$

Given any $(v_1, v_2) \in \cup_{k=1}^{\infty} E_k$, Property (i_2) and (3.9) yield

$$T(v_1, v_2) = \lim_{n\to\infty}\left(\langle A(u_{1,n}, u_{2,n}), (v_1, v_2)\rangle - T_{w_n}(v_1, v_2)\right) = 0.$$

Because of (i_3) this forces $T = 0$, namely condition (ii) is true. Moreover, using (3.9)–(3.10) entails

$$\begin{aligned}\langle A(u_{1,n}, u_{2,n}), (u_{1,n}-u_1, u_{2,n}-u_2)\rangle - T_{w_n}(u_{1,n}-u_1, u_{2,n}-u_2)\\ = -\langle A(u_{1,n}, u_{2,n}), (u_1, u_2)\rangle + T_{w_n}(u_1, u_2) \to 0,\end{aligned} \tag{3.11}$$

which shows (iii) in Definition 3.3. Summing up, the pair (u_1, u_2) turns out to be a generalized solution to (1.1). □

If we strengthen (H_3) as follows:

$(\mathrm{H}_3)'$ For every $i = 1, 2$ there exist $\rho_i, \sigma_i \in (1, p_i^*)$, $m_i > 0$, and $\delta_i \in L^{\sigma_i'}(\Omega)$ such that

$$|F_i(x, y_1, y_2, z_1, z_2)| \leq m_i\left(|y_1|^{\frac{p_1^*}{\rho_i'}} + |y_2|^{\frac{p_2^*}{\rho_i'}} + |z_1|^{\frac{p_1}{\rho_i'}} + |z_2|^{\frac{p_2}{\rho_i'}}\right) + \delta_i(x)$$

a.e. in Ω and for all $(y_1, y_2, z_1, z_2) \in \mathbb{R}^2 \times \mathbb{R}^{2N}$,

then the next notion of strongly generalized solution can be given. Obviously, $(\mathrm{H}_3)'$ implies (H_3), because $\rho_i < p_i^*$ forces

$$\frac{\kappa}{\rho_i'} < \frac{\kappa}{(p_i^*)'} \quad \forall\, \kappa \in \{p_1, p_2, p_1^*, p_2^*\}.$$

Definition 3.5. We say that $(u_1, u_2) \in U_1 \times U_2$ is a strongly generalized solution to (1.1) if there are two sequences $(u_{1,n}, u_{2,n}) \in U_1 \times U_2$ and $w_n \in \mathcal{S}_{F_1,F_2}(u_{1,n}, u_{2,n})$, $w_n = (w_{1,n}, w_{2,n})$, satisfying (i) and (ii) of Definition 3.3 and, moreover,

$(\mathrm{iii})'$ $\lim_{n\to\infty}\langle A(u_{1,n}, u_{2,n}), (u_{1,n}-u_1, u_{2,n}-u_2)\rangle = 0$.

Theorem 3.6. *Under assumptions* (H_1)–(H_2), $(H_3)'$, (H_4), *and* (3.5), *Problem* (1.1) *admits a strongly generalized solution.*

Proof. Reasoning as in the proof of Theorem 3.4 yields both $(u_1, u_2) \in U_1 \times U_2$ and two sequences $(u_{1,n}, u_{2,n}) \in U_1 \times U_2$, $(w_{1,n}, w_{2,n}) \in \mathcal{S}_{F_1,F_2}(u_{1,n}, u_{2,n})$ that comply with (i)–(ii) in Definition 3.3 as well as (3.11). Thus, it remains to show $(iii)'$. By $(H_3)'$ and Hölder's inequality we have

$$\begin{aligned}
&\left|\int_\Omega w_{i,n}(u_{i,n}-u_i)\mathrm{d}x\right| \\
&\quad\le m_i\int_\Omega\left(|u_{1,n}|^{\frac{p_1^*}{\rho_i'}}+|u_{2,n}|^{\frac{p_2^*}{\rho_i'}}+|\nabla u_{1,n}|^{\frac{p_1}{\rho_i'}}+|\nabla u_{2,n}|^{\frac{p_2}{\rho_i'}}\right)|u_{i,n}-u_i|\mathrm{d}x+\int_\Omega\delta_i|u_{i,n}-u_i|\mathrm{d}x \\
&\quad\le m_i\left(\|u_{1,n}\|_{p_1^*}^{p_1^*/\rho_i'}+\|u_{2,n}\|_{p_2^*}^{p_2^*/\rho_i'}+\|u_{1,n}\|_{1,p_1}^{p_1/\rho_i'}+\|u_{2,n}\|_{1,p_2}^{p_2/\rho_i'}\right)\|u_{i,n}-u_i\|_{\rho_i}+\|\delta_i\|_{\sigma_i'}\|u_{i.n}-u_i\|_{\sigma_i} \\
&\quad\le C\|u_{i,n}-u_i\|_{\rho_i}+\|\delta_i\|_{\sigma_i'}\|u_{i,n}-u_i\|_{\sigma_i}\quad\forall\, n\in\mathbb{N},
\end{aligned}$$

with $C > 0$, because $\{u_{i,n}\} \subseteq U_i$ turns out bounded. The condition $\max\{\rho_i, \sigma_i\} < p_i^*$ forces $u_{i,n} \to u_i$ in $L^{\rho_i}(\Omega) \cap L^{\sigma_i}(\Omega)$, where a sub-sequence is considered if necessary; see Proposition 2.4. Hence,

$$\lim_{n\to\infty}\int_\Omega w_{i,n}(u_{i,n}-u_i)\mathrm{d}x=0,\quad i=1,2. \tag{3.12}$$

Through (3.11)–(3.12), we arrive at

$$\lim_{n\to\infty}\langle A(u_{1,n},u_{2,n}),(u_{1,n}-u_1,u_{2,n}-u_2)\rangle=0,$$

namely $(iii)'$ of Definition 3.5 also holds. □

Finally, recall that $(u_1, u_2) \in U_1 \times U_2$ is called a *weak solution* to (1.1) when there exists $(w_1, w_2) \in \mathcal{S}_{F_1,F_2}(u_1, u_2)$ such that

$$A(u_1,u_2)=(w_1,w_2)\quad\text{in } U_1^*\times U_2^*. \tag{3.13}$$

Corollary 3.7. *Let the hypotheses of Theorem 3.6 be satisfied and let* $\min\{\mu_1, \mu_2\} \ge 0$. *Then Problem* (1.1) *possesses a weak solution.*

Proof. Keep the same notation of the previous proof. Since $\mu_i \ge 0$, gathering (p_1) with Proposition 2.2 together ensures that A_i is of type $(S)_+$. Therefore, from $(iii)'$ it follows $(u_{1,n}, u_{2,n}) \to (u_1, u_2)$ in $U_1 \times U_2$. On the other hand, (a_2) in Lemma 3.2 produces, up to subsequences, $(w_{1,n}, w_{2,n}) \rightharpoonup (w_1, w_2)$ in $U_1^* \times U_2^*$. Now, through (ii) and Lemma 3.1 we easily infer (3.13). □

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On the integrability and dynamics of the Hide, Skeldon and Acheson differential system

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Abstract. The family of systems

$$\dot{x} = x(y-1) - \beta z, \quad \dot{y} = \alpha(1-x^2) - \kappa y, \quad \dot{z} = x - \lambda z,$$

where $(x, y, z) \in \mathbb{R}^3$ and α, β, κ, λ are real parameters, was proposed by Hide, Skeldon and Acheson in 1996 for the study of self-excited dynamo action in which a Faraday disc and coil are arranged in series with either a capacitor or a motor. Since then it has been studied by several authors.

Here for the first time we provide, taking into account all values of the parameters, all its 13 types of invariant algebraic surfaces, its 2 polynomial and its 2 rational simple first integrals having at most degree two, and its 6 Darboux invariants coming for the invariant algebraic surfaces of degree two. Moreover, we show how to describe its dynamics in the Poincaré ball when the differential system has either a given first integral, or a given Darboux invariant. Describing the dynamics of the differential system in the Poincaré ball we control the orbits of the system that escape or come from the infinity.

Keywords: Hide–Skeldon–Acheson model, polynomial first integral, rational first integral, invariant algebraic surface, Darboux invariant

2020 Mathematics Subject Classification: 34C05.

1 Introduction and statements of the main results

In this work, we give some results about the integrability and the global dynamics of the Hide–Skeldon–Acheson dynamo model (HSA dynamo model), given by 4-parameter family of systems

$$\dot{x} = x(y-1) - \beta z, \quad \dot{y} = \alpha(1-x^2) - \kappa y, \quad \dot{z} = x - \lambda z, \tag{1.1}$$

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where $(x, y, z) \in \mathbb{R}^3$ and α, β, κ, λ are real parameters.

The family of systems (1.1) was proposed by Hide, Skeldon and Acheson in 1996 [4] for the study of self-excited dynamo action in which a Faraday disc and coil are arranged in series with either a capacitor or a motor. In this case, $x(t)$ is the current flowing through the dynamo, $y(t)$ is the angular rotation rate of the disc, and $z(t)$ is the angular rotation rate of the motor (or the charge in the capacitor). The parameter α is proportional to the steady applied mechanical couple driving the disc into rotation, β^{-1} measures the moment of inertia of the armature of the motor, and κ and λ are the coefficients of friction in the disc and the motor, respectively.

In [9] the authors study the integrability problem of systems (1.1). They provide some first integrals when the parameter $\alpha \neq 0$. They also prove that for $\alpha, \beta, \kappa \neq 0$ the dynamo model does not admit a polynomial or rational first integral.

In [6] the local and global asymptotical stability of the equilibrium point is discussed. All the orbits of the system are proved to be bounded. The existence of periodic orbits and the Hopf bifurcation is studied, and also the coexistence of three types of attractors: equilibria, hidden periodic attractors and hidden chaotic attractors.

In this paper our goal is to show how invariant algebraic surfaces, first integrals and Darboux invariants help in understanding the global dynamics of polynomial vector fields. Here we do this for the HSA dynamo model. More precisely, the existence of invariant surfaces always simplifies the study of the dynamics of a differential system in $\mathbb{R}^3$ because they separate the phase portrait in invariant pieces by the flow. On the other hand, the existence of a first integral, allows to reduce the study of the dynamics of a differential system in one dimension, and the existence of a Darboux invariant allows to determine the α- and ω-limits of the orbits of the system. In summary, to know one of these three objects helps in understanding the dynamics of a differential system.

One can easily check that systems (1.1) are invariant under the change of variables $(x, y, z) \to (-x, y, -z)$. Consequently if $(x(t), y(t), z(t))$ is a solution to one of the systems (1.1), so is $(-x(t), y(t), -z(t))$ (is symmetric with respect to the y-axis).

Let $\mathbb{R}[x, y, z]$ be the ring of the real polynomials in the variables x, y and z. Let $F = F(x, y, z) \in \mathbb{R}[x, y, z]$. We say that $F = F(x, y, z) = 0$ is an *invariant algebraic surface* of system (1.1) if it satisfies

$$\frac{\partial F}{\partial x}(x(y-1) - \beta z) + \frac{\partial F}{\partial y}(\alpha(1-x^2) - \kappa y) + \frac{\partial F}{\partial z}(x - \lambda z) = KF,$$

where $K = K(x, y, z)$ is a real polynomial of degree at most 1, called the *cofactor* of $F(x, y, z)$. In the next proposition we classify all invariant algebraic surfaces of degree at most two of the HSA dynamo model.

Proposition 1.1. *All the invariant algebraic surfaces of degree one or two that could occur for systems* (1.1) *are given in Table* 1.1 *together with their domains of existence in the parameter space .*

Proposition 1.1 will be proved in Section 3.

Let U be an open and dense subset of $\mathbb{R}^3$. Let $H\colon U \to \mathbb{R}$ a C^1 function non-locally constant. We say that H is a *first integral* of a system (1.1) if H is constant in all the solutions $(x(t), y(t), z(t))$ of that system contained in U. In other words, H must satisfy

$$\frac{\partial H}{\partial x}(x(y-1) - \beta z) + \frac{\partial H}{\partial y}(\alpha(1-x^2) - \kappa y) + \frac{\partial H}{\partial z}(x - \lambda z) = 0,$$

Conditions	$F_k = 0$	**Cofactor**
$\alpha = \beta = \lambda = \kappa = 0$	$F_1 = a - bx + cy - bz + dy^2 + byz$	$K_1 = 0$
$\alpha = \kappa = 0$	$F_2 = a + by + cy^2$	$K_2 = 0$
$\alpha = 0, \beta = \kappa = -\lambda$	$F_3 = a - x - z + yz$	$K_3 = 0$
$\beta = 0$	$F_4 = x$	$K_4 = y - 1$
$\alpha = 0$	$F_5 = y$	$K_5 = -\kappa$
$\alpha = \lambda = 0, \beta = \kappa(1-\kappa),$	$F_6 = -x + ay + (\kappa - 1)z + yz$	$K_6 = -\kappa$
$\alpha = 0, \beta = \kappa(1+\kappa), \lambda = -2\kappa$	$F_7 = -x - (\kappa + 1)z + yz$	$K_7 = \kappa$
$\alpha = 0, \beta = \kappa(1-2\kappa), \lambda = \kappa$	$F_8 = -x + ay^2 + (2\kappa - 1)z + yz$	$K_8 = -2\kappa$
$\alpha = \kappa(1-\kappa) < 0, \beta = 0$	$F_9 = \kappa - 1 - \sqrt{\kappa(\kappa-1)}x + y$	$K_9 = -\kappa - \sqrt{\kappa(\kappa-1)}x$
$\alpha = \kappa(1-\kappa) < 0, \beta = 0$	$F_{10} = \kappa - 1 + \sqrt{\kappa(\kappa-1)}x + y$	$K_{10} = -\kappa + \sqrt{\kappa(\kappa-1)}x$
$\alpha = \kappa(1-\kappa) > 0, \beta = 0$	$F_{11} = (y - (1-\kappa))^2 + \kappa(1-\kappa)x^2$	$K_{11} = -2\kappa$
$\alpha = \kappa(1-\kappa), \lambda = \kappa$	$F_{12} = (y - (1-\kappa))^2 + \kappa(1-\kappa)x^2 + \beta\kappa(1-\kappa)z^2$	$K_{12} = -2\kappa$
$\alpha = 0, \beta = \kappa\lambda - \kappa(1+\kappa)$	$F_{13} = \beta x + \kappa xy - \beta\kappa z$	$K_{13} = y - (1+\kappa)$

Table 1.1: Invariant algebraic surfaces of system (1.1), $F_k = F_k(x, y, z) = 0$, for $1 \leq k \leq 13$. Here $a, b, c, d \in \mathbb{R}$.

on the solutions $(x(t), y(t), z(t))$ of the system (1.1), with fixed $\alpha, \beta, \kappa, \lambda \in \mathbb{R}$, contained in U. A first integral is called *simple* if it is not a function of other first integrals, and the first integral H is called *polynomial* (respectively *rational*) if H is a polynomial (respectively a rational) function. The degree of a polynomial first integral is the *degree* of the polynomial, and the *degree* of a rational function, say $G = P_1/P_2$ being $P_1, P_2 \in \mathbb{R}[x, y, z]$ is the maximum of the degrees of the polynomials P_1 and P_2.

In the next two results we characterize, first all polynomial first integrals of degree at most two of systems (1.1), and second all rational first integrals of degree at most two of systems (1.1).

Proposition 1.2. *All simple polynomial first integrals of degree one or two of systems* (1.1)*, together with the corresponding parameters of existence, are listed in Table* 1.2*.*

The proof of Propositions 1.2 and 1.3 are given in Section 3.

Proposition 1.3. *All simple rational first integrals of degree one or two of systems* (1.1) *that are not polynomial, together with the corresponding parameters of existence, are listed in Table* 1.3*.*

Conditions	**Polynomial first integral**
$\alpha = \kappa = 0$	$H_1 = y$
$\alpha = 0, \beta = \kappa = -\lambda$	$H_2 = x + z - yz$

Table 1.2: Simple polynomial first integrals of systems (1.1), $H_k = H_k(x, y, z)$ for $1 \leq k \leq 2$.

Conditions	Rational first integrals
$\alpha=0$, $\beta=\kappa(1-2\kappa)$, $\lambda=\kappa$	$G_1=(-x+(2\kappa-1)z+yz)/y^2$
$\alpha=0$, $\beta=\kappa(1-\kappa)$, $\lambda=0$	$G_2=(-x+(\kappa-1)z+yz)/y$

Table 1.3: Simple rational first integrals of systems (1.1), $G_k=G_k(x,y,z)$ for $1\leq k\leq 2$.

We say that a differential system (1.1) is *completely integrable* if there are two first integrals functionally independent, i.e. the gradients of both first integrals are linear independent except perhaps in a zero Lebesgue measure set.

Corollary 1.4. *If $\alpha=\beta=\kappa=\lambda=0$, the differential system (1.1) is completely integrable with the two first integrals H_1 and H_2 of Table 1.2.*

Corollary 1.4 is given in Section 3. Note that the first integrals H_1 and H_2 are all the rational first integrals of degree one or two, but the systems can have other rational first integrals of higher degree.

An *invariant* of a differential system (1.1) on an open and dense subset U of $\mathbb{R}^3$ is a non-constant C^1 function $I=I(x,y,z,t)$, such that $I(x(t),y(t),z(t),t)$ is constant on all solutions $(x(t),y(t),z(t))$ of that system (1.1) contained in U, i.e.

$$\frac{\partial I}{\partial x}(x(y-1)-\beta z)+\frac{\partial I}{\partial y}(\alpha(1-x^2)-\kappa y)+\frac{\partial I}{\partial z}(x-\lambda z)+\frac{\partial I}{\partial t}=0$$

for all $(x(t),y(t),z(t))\in U$. Note that if an invariant is independent of t it is a first integral.

If the invariant is of the form $I=f(x,y,z)e^{st}$ with f an analytic function in U and $s\in\mathbb{R}\setminus\{0\}$ we say that I is a *Darboux invariant*. A Darboux invariant is *simple* if it is not a function of other Darboux invariants.

Proposition 1.5. *All Darboux invariants $I=f(x,y,z)e^{st}$ with $f(x,y,z)$ formed by the polynomials F_k with $k=1,\ldots,13$ of Table 1.1 are given in Table 1.4.*

Conditions	Darboux invariant
$\alpha=0$	$I_1=ye^{\kappa t}$
$\alpha=0$, $\beta=\kappa(1-\kappa)$, $\lambda=0$	$I_2=(-x+(\kappa-1)z+yz)e^{\kappa t}$
$\alpha=0$, $\beta=\kappa(1+\kappa)$, $\lambda=-2\kappa$	$I_3=(-x-(\kappa+1)z+yz)e^{-\kappa t}$,
$\alpha=0$, $\beta=\kappa(1-2\kappa)$, $\lambda=\kappa$	$I_4=(-x+(2\kappa-1)z+yz)e^{2\kappa t}$
$\alpha=\kappa(1-\kappa)$, $\beta=0$	$I_5=((y-(1-\kappa))^2+\kappa(1-\kappa)x^2)e^{2\kappa t}$
$\alpha=\kappa(1-\kappa)$, $\lambda=\kappa$	$I_6=((y-(1-\kappa))^2+\kappa(1-\kappa)x^2+\beta\kappa(1-\kappa)z^2)e^{2\kappa t}$

Table 1.4: Darboux invariants of systems (1.1), $I_k=I_k(x,y,z)$ for $1\leq k\leq 6$.

Proposition 1.5 is also proved in Section 3.

Let $\phi_p(t)$ be the solution of a system (1.1) such that $\phi_p(0)=p$, defined on its maximal interval (α_p,ω_p). If $\omega_p=+\infty$, we define the ω-limit set of p as

$$\omega(p)=\left\{q\in\mathbb{R}^3:\ \exists\{t_n\}\subset\mathbb{R}\text{ with }\lim_{n\to\infty}t_n=\infty\text{ and }\lim_{n\to\infty}\phi_p(t_n)=q\right\}.$$

In the same way, if $\alpha_p = -\infty$, we define the α-limit set of p as

$$\alpha(p) = \left\{ q \in \mathbb{R}^3 : \exists \{t_n\} \subset \mathbb{R} \text{ with } \lim_{n\to\infty} t_n = -\infty \text{ and } \lim_{n\to\infty} \phi_p(t_n) = q \right\}.$$

While the existence of a first integral allows to reduce in one dimension the study of the dynamics of a differential system in $\mathbb{R}^3$, the existence of a Darboux invariant provides information on the α- and ω-limits of the orbits of the differential system (see Proposition 5 of [8], or Proposition 2.4 in Section 2).

In order to illustrate how to use the existence of a first integral or a Darboux invariant in the study of the global dynamics of the 4-parameter family of systems (1.1) we take the first integral H_2 and the Darboux invariant I_5. This is the content of the next two main results.

Systems (1.1) under the conditions $\alpha = 0$ and $\beta = \kappa = -\lambda$ have the first integral $H_2 = x + z - yz$. In the next theorem we characterize their phase portraits on each level $H_2 = h \in \mathbb{R}$.

Theorem 1.6. *The phase portraits in the Poincaré disc of the differential systems* (1.1) *restricted to the invariant surface* $H_2(x, y, z) = h \in \mathbb{R}$ *under the conditions* $\alpha = 0$ *and* $\beta = \kappa = -\lambda$ *are topologically equivalent to one of the six phase portraits of Figure 1.1. The phase portrait G1 occurs for* $\kappa = 0, h \neq 0$*; G2 for* $\kappa = 0, h = 0$*; G3 for* $\kappa \in (0, 1)$*; G4 for* $k = 1, h \neq 0$*; G5* $\kappa = 1, h = 0$ *and G6 for* $\kappa > 1$ *or* $\kappa = 0$.

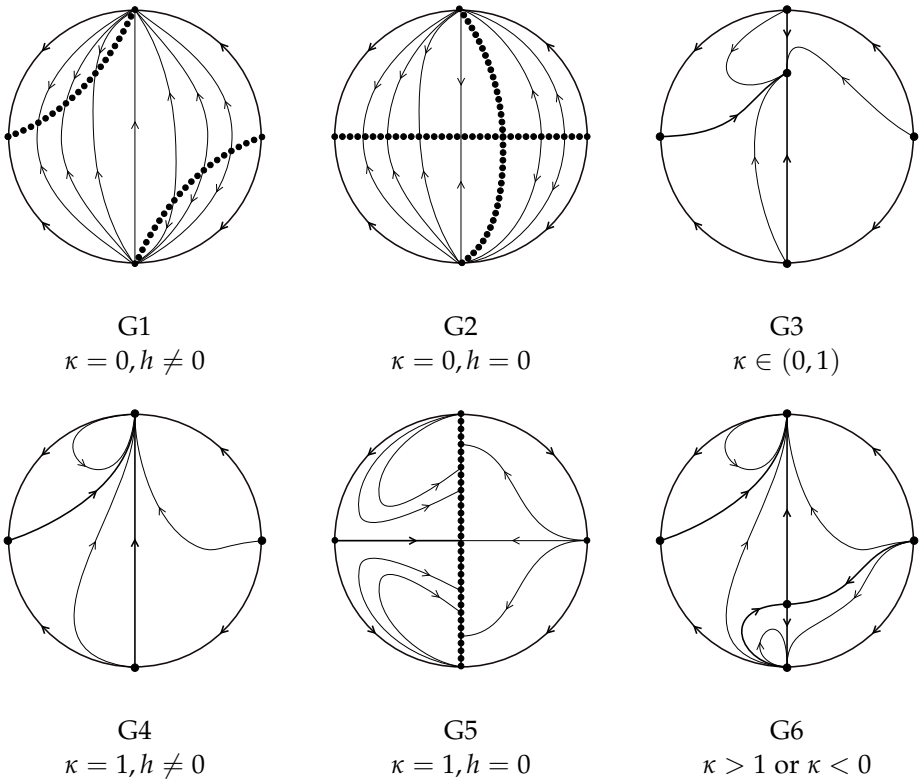


Figure 1.1: Phase portraits of systems (1.1) with $\alpha = 0$ and $\beta = \kappa = -\lambda$ at any level $H_2 = h \in \mathbb{R}$ in the Poincaré disc.

Theorem 1.6 is proved in Section 4.

Systems (1.1) under the conditions $\alpha = \kappa(1 - \kappa) \neq 0$ and $\beta = 0$, have the Darboux invariant $I_5 = ((\kappa - 1 + y)^2 + \kappa(1 - \kappa)x^2)e^{2\kappa t}$. In the next theorem we characterize their dynamics in the Poincaré ball.

We define the infinite equilibrium $S_1 = (\sqrt{\kappa(\kappa-1)}, 0, 0)$ and $S_4 = (-\sqrt{\kappa(\kappa-1)}, 0, 0)$ in the local chart U_1 and $S_3 = (-\sqrt{\kappa(\kappa-1)}, 0, 0)$ and $S_2 = (\sqrt{\kappa(\kappa-1)}, 0, 0)$ in the local chart V_1, and the finite equilibrium $P_1 = (0, 1-\kappa, 0)$.

Theorem 1.7. *The dynamics of the differential systems* (1.1) *under the conditions* $\alpha = \kappa(1-\kappa) \neq 0$ *and* $\beta = 0$ *are described below.*

(a) When $\kappa(\kappa-1) > 0$ *and* $\beta = 0$ *they have two invariant algebraic surfaces* $F_9 = 0$ *and* $F_{10} = 0$ *(see Table 1.1). The boundary of these surfaces are the boundary at infinity of the planes* $y - \sqrt{\kappa(\kappa-1)}x = 0$ *and* $y + \sqrt{\kappa(\kappa-1)}x = 0$*, respectively. Up to topological equivalence there are only two phase portraits on the Poincaré ball for both invariant algebraic surfaces (see Figure 1.2). When* $\kappa(\kappa-1) < 0$ *and* $\beta = 0$ *the invariant surface* $F_{11} = 0$ *corresponds to the straight line* $(0, 1-\kappa, z)$ *with* $z \in \mathbb{R}$*. The boundary of this straight line are the endpoints of the z-axis. The dynamics on this straight line is described by the differential equation* $\dot{z} = -\lambda z$*.*

(b) The phase portraits of systems (1.1) *under the conditions* $\alpha = \kappa(1-\kappa) \neq 0$ *and* $\beta = 0$ *on the infinite sphere is shown in Figures 1.3.*

(b1) When $\kappa(\kappa-1) < 0$ *the boundary at infinity of the plane* $x = 0$ *is filled with equilibria. All the orbits that are not equilibria are heteroclinic orbits starting at an infinite equilibrium with its y-coordinate positive and ending at an infinite equilibrium with its y-coordinate negative.*

(b2) When $\kappa(\kappa-1) > 0$ *the boundary at infinity of the plane* $x = 0$ *is filled with equilibria and there are four additional isolated equilibria* S_k *for* $k = 1, 2, 3, 4$*. On the infinite sphere there are eight separatrices. The equilibria* S_3 *and* S_4 *are the* α*-limit of two separatrices one having its* ω*-limit at the endpoint of the positive z-axis and the other at the endpoint of the negative z-axis. The equilibria* S_1 *and* S_2 *are the* ω*-limits of two separatrices one having its* α*-limit at the endpoint of the positive z-axis and the other at the endpoint of the negative z-axis. The orbits inside the regions delimited by these separatrices and that do not contain equilibria have either* S_4 *as* α*-limit and* S_1 *as* ω*-limit, or* S_3 *as* α*-limit and* S_2 *as* ω*-limit. For all the other orbits: the ones with* $x, y > 0$ *have their* α*-limit at an infinite equilibrium of the plane* $x = 0$ *with* $y > 0$ *and their* ω*-limit at* S_1*; the ones with* $x > 0$*,* $y < 0$ *have their* ω*-limit at one infinite equilibrium of the plane* $x = 0$ *with* $y < 0$ *and their* α*-limit at* S_4*; the ones with* $x < 0$*,* $y > 0$ *have their* α*-limit at one infinite equilibrium of the plane* $x = 0$ *with* $y > 0$ *and their* ω*-limit at* S_2*; the ones with* $x, y < 0$ *have their* ω*-limit at one infinite equilibrium of the plane* $x = 0$ *with* $y < 0$ *and their* α*-limit at* S_3*.*

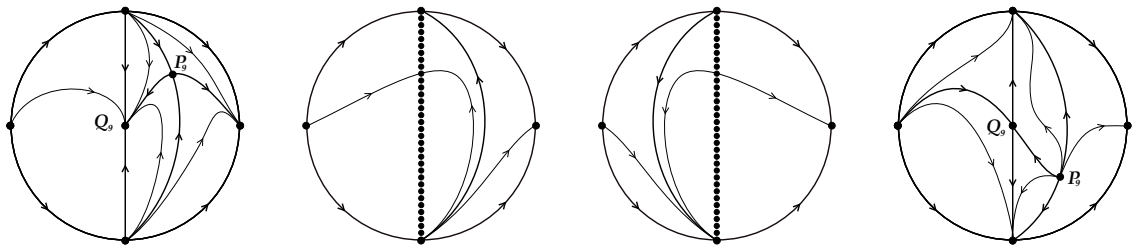
(c) All the α*- and* ω*-limits for the orbits inside the Poincaré ball, i.e. in* $\mathbb{R}^3$*, are the following:*

(c1) Assume $\kappa(\kappa-1) < 0$*. If* $\lambda > 0$ *all the orbits in* $\mathbb{R}^3$ *have their* ω*-limit at* P_1 *and their* α*-limit at an infinite equilibrium. If* $\lambda < 0$*, with the exception of the equilibria on the straight line* $(0, 1-k, z)$ *with* $z \in \mathbb{R}$*, all the orbits in* $\mathbb{R}^3$ *have their* ω*-limit at the infinite equilibria* $(0, 0, 1)$ *or* $(0, 0, -1)$ *of the Poincaré ball, and their* α*-limit is an infinite equilibrium. If* $\lambda = 0$ *all the orbits in* $\mathbb{R}^3$ *have their* ω*-limit in one of the equilibria of the straight line* $(0, 1-k, z)$ *with* $z \in \mathbb{R}$*, and their* α*-limit is an infinite equilibrium.*

(c2) Assume $\kappa(\kappa-1) > 0$*. For any value of the parameters the* α*-limits of the orbits in* $\mathbb{R}^3$ *are the infinite equilibria* S_3 *and* S_4 *and also the positive and negative endpoints of the z-axis. If* $\lambda = 0$ *and* $\kappa < 0$ *all the equilibria of the straight line* $(0, 1-\kappa, z)$ *with* $z \in \mathbb{R}$ *can be also* α*-limit. If* $\lambda < 0$ *and* $\kappa < 0$*, then the finite equilibrium point* P_1 *can be also an* α*-limit.*

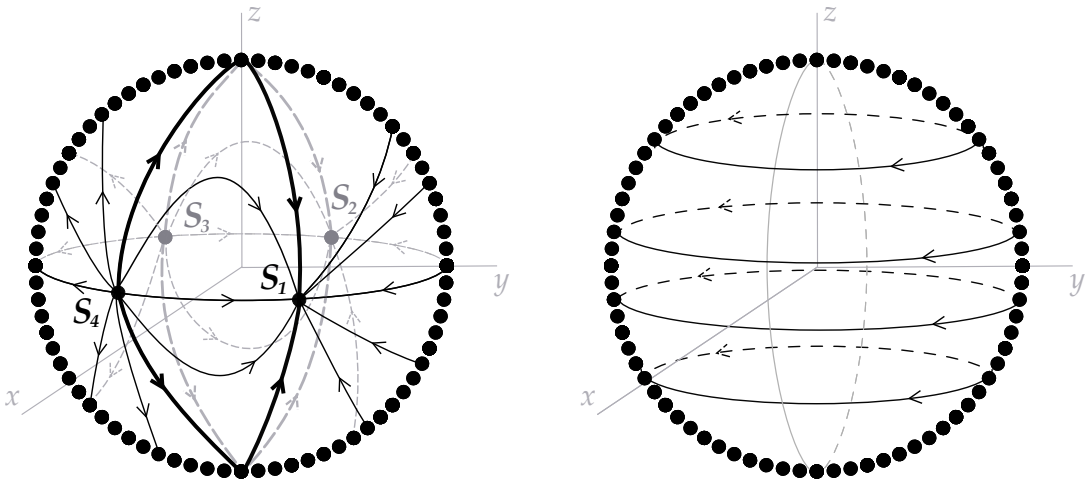
For any values of the parameters the ω-limits are the infinite points S_1 and S_2 and also the positive and negative endpoints of the z-axis. If $\lambda = 0$ and $\kappa > 0$, all the finite equilibria of the straight line $(0, 1-\kappa, z)$ with $z \in \mathbb{R}$ can be also ω-limits. If $\lambda > 0$ and $\kappa > 0$, then the finite equilibrium point P_1 can be also an ω-limit.

Theorem 1.7 is proved in Section 5.



$\lambda > 0$ $\lambda = 0, k > 0$ $\lambda = 0, k < 0$ $\lambda < 0$

Figure 1.2: Phase portraits of systems (1.1) on the invariant surface $F_9 = 0$ and $F_{10} = 0$ with $\alpha = \kappa(1-\kappa) < 0$ and $\beta = 0$.



$\kappa(\kappa-1) > 0$ $\kappa(\kappa-1) < 0$

Figure 1.3: Phase portraits of systems (5.1) on the sphere of the infinity.

2 Preliminaries

2.1 Poincaré compactification

2.1.1 Poincaré compactification in dimension 2

Here we introduce the Poincaré compactification, a technique that allows to study the behavior of the orbits of a polynomial differential system near the infinity. This will be used in Sections 4 and 5.

First, we consider the sphere $\mathbb{S}^2 = \{y \in \mathbb{R}^3 : y_1^2 + y_2^2 + y_3^2 = 1\}$, and we call it the *Poincaré sphere*. We identify its tangent plane at the point $(0,0,1)$ with $\mathbb{R}^2$. There, in $\mathbb{R}^2$, we consider a polynomial system of degree d:

$$\dot{x_1} = P_1(x_1, x_2), \quad \dot{x_2} = P_2(x_1, x_2).$$

We get an induced vector field in the northern and southern hemispheres of the sphere with the differential Df^+ and Df^- of the central projections $f^+ : \mathbb{R}^2 \to \mathbb{S}^2$ and $f^- : \mathbb{R}^2 \to \mathbb{S}^2$, defined as $f^{\pm}(x) = \pm(x_1, x_2, 1)/\Delta(x)$, with $\Delta(x) = \sqrt{x_1^2 + x_2^2 + 1}$.

Multiplying the field by y_3^d, we can extend analytically the vector field to the equator, which correspond to the points at infinity of $\mathbb{R}^2$.

This extension is called the *Poincaré compactification* of the original vector field, and the study of this new field near $\mathbb{S}^1$ allows to determine the dynamics of the original field near the infinity.

To make computations we work in the local charts (U_i, ϕ_i) and (V_i, ψ_i) of the sphere $\mathbb{S}^2$, where $U_i = \{y \in \mathbb{S}^2 : y_i > 0\}$, $V_i = \{y \in \mathbb{S}^2 : y_i < 0\}$, $\phi_i : U_i \longrightarrow \mathbb{R}^2$ and $\psi_i : V_i \longrightarrow \mathbb{R}^2$ for $i = 1, 2$ with $\phi_i(y) = \psi_i(y) = (y_m/y_i, y_n/y_i)$ for $m < n$ and $m, n \neq i$.

The expression of the Poincaré compactification in the local chart (U_1, ϕ_1) is

$$(\dot{u}, \dot{v}) = v^d(-uP_1 + P_2, -vP_1), \quad P_i = P_i(1/v, u/v),$$

in the local chart (U_2, ϕ_2) is

$$(\dot{u}, \dot{v}) = v^d(-uP_2 + P_1, -vP_2), \quad P_i = P_i(u/v, 1/v).$$

The expression for the Poincaré compactification in the local charts (V_i, ψ_i), with $i = 1, 2, 3$ is the same as in the local charts (U_i, ϕ_i) multiplied by $(-1)^{d-1}$.

As we want to study the behavior near the infinity, we study the *infinite equilibrium points*, i.e., the equilibrium points of the Poincaré compactification over the equator $\mathbb{S}^1$. If $y \in \mathbb{S}^1$ is an infinite equilibrium point, then $-y$ is also an infinite equilibrium point and they have the same or opposite stability depending on the degree of the system. Then, it is enough to study the infinite equilibrium points in the local chart U_1 and the origin of the local chart U_2.

More information about the Poincaré compactification in dimension 2 can be found in Chapter 5 of [3].

2.1.2 Poincaré compactification in dimension 3

The technique of the Poincaré compactification can be extended to higher dimensions. In particular, we use this technique in dimension 3 in Section 5. As the ideas are the same explained before, we just include here the expressions obtained in the local charts U_1, U_2 and U_3 for a polynomial system in $\mathbb{R}^3$ of degree d

$$\dot{x} = P_1(x, y, z), \quad \dot{y} = P_2(x, y, z), \quad \dot{z} = P_3(x, y, z).$$

More details about the Poincaré compactification in dimension n can be found in [1].

The phase portrait in the local chart U_1 is given by the system

$$(\dot{u}, \dot{v}, \dot{w}) = w^d(-uP_1 + P_2, -vP_1 + P_3, -wP_1), \quad P_i = P_i(1/w, u/w, v/w),$$

where u and v are the coordinates of the tangent plane to the sphere $\mathbb{S}^2$ at the point $(1,0,0)$.

The phase portrait in the local chart U_2 is given by the system

$$(\dot{u},\dot{v},\dot{w}) = w^d(-uP_2 + P_1, -vP_2 + P_3, -wP_2), \quad P_i = P_i(u/w, 1/w, v/w),$$

with the coordinates u, v of the tangent plane to the sphere at $(0,1,0)$.

In the local chart U_3 the phase portrait is given by

$$(\dot{u},\dot{v},\dot{w}) = w^d(-uP_3 + P_1, -vP_3 + P_2, -wP_3), \quad P_i = P_i(u/w, v/w, 1/w),$$

and now u and v are the coordinates of the tangent plane to the sphere $\mathbb{S}^2$ at the point $(0,0,1)$.

The following result is proved in [7].

Lemma 2.1. *Let $f(x,y,z) = 0$ be an algebraic surface of degree m in $\mathbb{R}^3$. The extension of this surface to the boundary of the Poincaré ball is contained in the surface defined by*

$$w^m f\left(\frac{x}{w}, \frac{y}{w}, \frac{z}{w}\right) = 0, \quad w = 0.$$

2.2 Invariant algebraic surfaces and Darboux invariants

We will use the following results (see Theorem 8.7 and Proposition 8.4 of [3], respectively, for a proof):

Proposition 2.2. *Assume that a system (1.1) admit p irreducible invariant algebraic surfaces $f_i = 0$ with cofactors K_i, for $i = 1, \cdots, p$, not all zero such that $\sum_{i=1}^{p} \lambda_i K_i = -s$ for some $s \in \mathbb{R} \setminus 0$, then the function*

$$f_1^{\lambda_1} \cdots f_p^{\lambda_p} e^{st}$$

is a Darboux invariant of that system.

Proposition 2.3. *Suppose $f \in \mathbb{R}[x,y,z]$ and let $f = f_1^{n_1} \cdots f_r^{n_r}$ be its factorization into irreducible factors over $\mathbb{R}[x,y,z]$. Then, for a polynomial system (1.1), $f = 0$ is an invariant algebraic curve with cofactor K_f if and only if $f_i = 0$ is an invariant algebraic curve for each $i = 1 \cdots, r$ with cofactor K_{f_i}. Moreover $K_f = n_1 K_{f_1} + \cdots + n_r K_{f_r}$.*

The following proposition is Proposition 5 of [8].

Proposition 2.4. *Let $\mathbb{S}^2$ be the infinity of the Poincaré ball and $I(x,y,z,t) = f(x,y,z)\ e^{st}$ be a Darboux invariant of a system (1.1). Let also $p \in \mathbb{R}^3$ and $\phi_p(t)$ be the solution of that system (1.1) with maximal interval (α_p, ω_p) such that $\phi_p(0) = p$. Then*

1. *Assume that $s > 0$. If $\omega_p = +\infty$, then $\omega(p)$ is contained in the closure $\overline{\{f(x,y,z) = 0\}}$ in the Poincaré ball, and if $\alpha_p = -\infty$ then $\alpha(p) \subset \overline{\{f(x,y,z) = 0\}} \cap \mathbb{S}^2$, being $\mathbb{S}^2$ the boundary of the Poincaré ball.*

2. *Assume that $s < 0$. If $\alpha_p = -\infty$, then $\alpha(p)$ is contained in $\overline{\{f(x,y,z) = 0\}}$, and if $\omega_p = +\infty$, then $\omega(p) \subset \overline{\{f(x,y,z) = 0\}} \cap \mathbb{S}^2$.*

The following definition and theorem is Definition 3.1 and Theorem 3.2 in [2] (based on the work in [5]).

Let ϕ_t be a smooth flow on a manifold M and suppose C is a submanifold of M consisting entirely of equilibrium points for the flow. C is said to be *normally hyperbolic* if the tangent bundle to M over C splits into three subbundles TC, E^s, and E^u that are invariant under $d\phi_t$

and satisfy: $d\phi_t$ contracts E^s exponentially, $d\phi_t$ expands E^u exponentially and TC is the tangent bundle of C.

For normally hyperbolic submanifolds one has the usual existence of smooth stable and unstable manifolds together with the persistence of these invariant manifolds under small perturbations. More precisely, we have the following theorem.

Theorem 2.5. *Let C be a normally hyperbolic submanifold of equilibrium points for ϕ_t. Then there exist smooth stable and unstable manifolds tangent along C to $E^s \oplus TC$ and $E^u \oplus TC$ respectively. Furthermore, both C and the stable and unstable manifolds are persistent under small perturbations of the flow.*

3 Proof of Propositions 1.1, 1.2, 1.3, 1.5 and Corollary 1.4

3.1 Proof of Proposition 1.1

Let $F(x,y,z) = a_0 + a_1x + a_2y + a_3z + a_4x^2 + a_5xy + a_6xz + a_7y^2 + a_8yz + a_9z^2$ with $a_i \in \mathbb{R}$ and $K = K(x,y,z) = k_0 + k_1x + k_2y + k_3z$ with $k_i \in \mathbb{R}$. If $F(x,y,z) = 0$ is an invariant algebraic surface of a system (1.1), with fixed $\alpha, \beta, \kappa, \lambda \in \mathbb{R}$, with cofactor K it must verify

$$\frac{\partial F}{\partial x}(x(y-1) - \beta z) + \frac{\partial F}{\partial y}(\alpha(1-x^2) - \kappa y) + \frac{\partial F}{\partial z}(x - \lambda z) = (k_0 + k_1x + k_2y + k_3z)F. \tag{3.1}$$

Equating the coefficients of the left-hand side polynomial of (3.1) with the right-hand side polynomial of (3.1) we get a system of 20 equations with 18 unknowns which are a_j for $j = 0, \ldots, 9$, k_i for $i = 0, \ldots, 3$ and α, β, λ and κ. Solving this system of equations with the help of an algebraic manipulator as Mathematica or Maple we obtain the 13 invariant algebraic surfaces F_k given in Table 1.1. This completes the proof of the proposition.

3.2 Proof of Proposition 1.2

We note that if H is a first integral then $H +$ constant is also a first integral. Without loss of generality we can assume that H has no constant term. Moreover, if H is a first integral then CH, where C is a constant, is also a first integral, so the first integrals in Table 1.2 are determined up to multiplication by a constant.

Since a polynomial first integral is an invariant algebraic surface with zero cofactor, from Proposition 1.1, Table 1.1 and Proposition 2.3, Proposition 1.2 is proved. In particular we need to take among the invariant algebraic surfaces in Table 1.1 with zero cofactor also all possible combinations of the invariant algebraic surfaces in Table 1.1 with the respectively non-zero cofactors such that the sum of the cofactors is zero. Thus we obtain the polynomial first integrals F_1, F_2, F_3 and F_6F_7 when $\alpha = \beta = \lambda = \kappa = 0$. Since the constants a, b, c are arbitrary and we look for simple polynomial first integrals, the first integral F_2 becomes the simple first integral $H_1 = y$, and the first integral F_1 is function of the simple first integrals H_1 and $H_2 = x + z - yz$. Moreover F_6F_7 when $\alpha = \beta = \lambda = \kappa = 0$ can be expressed as a function of H_1 and H_2. This completes the proof of the proposition.

3.3 Proof of Proposition 1.3

Since a rational first integral is the quotient of two invariant algebraic surfaces with the same cofactor, Table 1.3 follows from Proposition 1.1, Table 1.1 and Proposition 2.3, taking all possible combinations of the invariant algebraic surfaces in Table 1.1 with the respectively cofactors

and the resulting function is a non-polynomial rational function. To that end, it is taken into account that the sum of invariant surfaces is an invariant surface if the cofactors of both are the same, and also that the product of invariant surfaces is always an invariant surface, with its cofactor being the sum of the corresponding cofactors associated with each of them. Doing so, all possible rational first integrals are: F_{13}/F_4, F_6/F_5, F_{12}/F_8, F_{12}/F_{11}, F_9F_{10}/F_{12}, F_5F_6/F_8, F_5F_6/F_{12}, F_8/F_5^2, F_{12}/F_5^2, F_9F_{10}/F_5^2, F_8/F_6^2, F_{12}/F_6^2. However the unique real simple rational first integral of degree one or two are G_1 (which is obtained from F_8/F_5^2) and G_2 (which is obtained from F_6/F_5) because for the others, they are either polynomial first integrals or can be expressed as a function of G_1 and G_2.

3.4 Proof of Corollary 1.4

From Table 1.2 it follows that system (1.1) under the conditions $\alpha = \beta = \kappa = \lambda = 0$ have the two polynomial first integrals H_1 and H_2. They clearly are independent because H_1 does not depend on x or z while H_2 does depend on these variables. This proves Corollary 1.4.

3.5 Proof of Proposition 1.5

It follows from Propositions 2.2 and 2.3 that to obtain Table 1.4 we need to take all the invariant algebraic surfaces in Table 1.1 such that their cofactors are non-zero constants, or different invariant algebraic surfaces with nonzero constant cofactor such that the product of them to some power (either positive or negative) provide a polynomial or a rational function $f(x,y,z)$ with constant cofactor $-s$, and consequently a Darboux invariant $f(x,y,z)e^{st}$.

The Darboux invariants I_1, I_2, I_3, I_4 and I_6, of Table 1.4 are simple and are obtained from the invariant algebraic surfaces of Table 1.1 as $I_1 = F_5e^{\kappa t}$, $I_1 + I_2 = F_6e^{\kappa t}$, $I_3 = F_7e^{-\kappa t}$, $I_4 + I_1^2 = F_8e^{2\kappa t}$ and $I_6 = F_{12}e^{-\kappa t}$. Moreover, I_5 is a simple Darboux invariant: it is $I_5 = F_{11}e^{2\kappa t}$ if $\kappa(1-\kappa) > 0$, and it is $I_5 = F_9F_{10}e^{2\kappa t}$ if $\kappa(1-\kappa) < 0$.

4 Proof of Theorem 1.6

The family of systems (1.1) under conditions $\alpha = 0$, $\beta = \kappa = -\lambda$ has the first integral $H_2 = x + z - yz$, so on any level $H_2 = h$ we can reduce these systems to the planar polynomial family of systems

$$\dot{y} = -\kappa y, \quad \dot{z} = h + (\kappa - 1)z + yz. \tag{4.1}$$

In this section we study the dynamics of systems (4.1) in the Poincaré disc for any value of $h \in \mathbb{R}$.

Finite equilibrium points

If $\kappa \neq 1$ and $\kappa \neq 0$ systems (4.1) have the unique equilibrium point $(0, h/(1-\kappa))$ which is a stable hyperbolic node if $\kappa \in (0,1)$, an a hyperbolic saddle if $\kappa > 1$ or $\kappa < 0$.

If $\kappa = 1$ and $h = 0$ then system (4.1) has the straight line $y = 0$ filled of equilibrium points. If $\kappa = 1$ and $h \neq 0$ there are no finite equilibrium points on systems (4.1).

If $\kappa = 0$ and $h \neq 0$ then systems (4.1) have the curve $z = h/(1-y)$ filled with equilibrium points. If $\kappa = 0$ and $h = 0$ then system (4.1) has two curves filled with equilibrium points: $z = 0$ and $y = 1$.

When $\kappa = 1$ and $h = 0$ system (4.1) becomes

$$\dot{y} = -y, \quad \dot{z} = yz,$$

and the orbits behave as in Figure 4.1, so the equilibrium points are all stable.

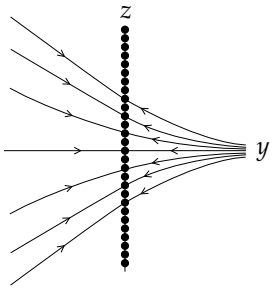
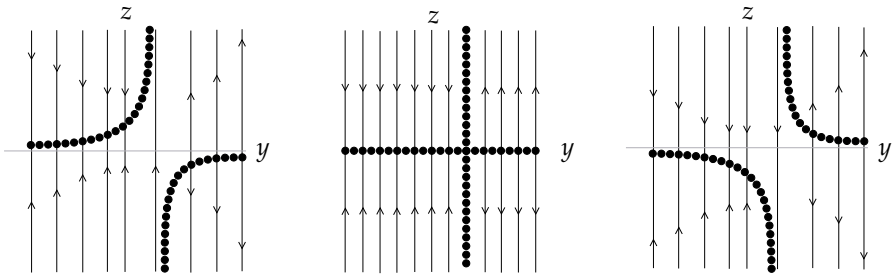


Figure 4.1: Phase portrait around the line $y = 0$ consisting of equilibrium points.

When $\kappa = 0$ systems (4.1) become

$$\dot{y} = 0, \quad \dot{z} = h - z + yz,$$

so all the orbits live in vertical straight lines: if $h > 0$ the orbits are as in Figure 4.2(a), if $h = 0$ then the phase portrait becomes as in Figure 4.2(b), and finally if $h < 0$ the orbits are as in Figure 4.2(c).



(a) $h > 0$ (b) $h = 0$ (c) $h < 0$

Figure 4.2: Finite phase portraits of systems (4.1) with $\kappa = 0$.

Infinite equilibrium points

In order to study the dynamics of systems (4.1) at infinity, we will use the Poincaré compactification introduced in subsection 2.1.1. We only consider the case $\kappa \neq 0$, as the phase portrait in the case $\kappa = 0$ can be totally determined with the finite phase portrait.

In the local chart U_1 systems (4.1) have the expression

$$\dot{u} = u + (2\kappa - 1)uv + hv^2, \quad \dot{v} = \kappa v^2. \tag{4.2}$$

If $\kappa \neq 0$, the only equilibrium point at infinity is the origin of the systems. This point is a semi-hyperbolic equilibrium point. Applying Theorem 2.19 of [3] it is a saddle-node for any $\kappa \neq 0$.

In the local chart U_2 systems(4.1) have the expression

$$\dot{u} = -u^2 - (2\kappa - 1)uv - huv^2, \quad \dot{v} = -uv - (\kappa - 1)v^2 - hv^3. \tag{4.3}$$

The only point we need to study in the local chart U_2 is the origin, and it is linearly zero.

We will study the local phase portrait at the origin of the chart U_2 doing vertical blow-ups. Since the characteristic directions at the origin of system (4.3) are given by the factors of the polynomial $\kappa u(v-2)v^2 = 0$ when $\kappa \neq 0$, we need to do a twist $(u, v) = (u_1 - v_1, v_1)$ because $u = 0$ is a characteristic direction. When $\kappa = 0$ all the directions are characteristic. In the new coordinates (u_1, v_1) system (4.3) becomes

$$\dot{u_1} = -u_1^2 + 2(1-\kappa)u_1v_1 + \kappa v_1^2 - hu_1v_1^2, \quad \dot{v_1} = -v_1(u_1 + (\kappa - 2)v_1 + hv_1^2). \tag{4.4}$$

Now we do the vertical blow up $(u_1, v_1) = (u_2, u_2v_2)$ and system (4.4) becomes

$$\dot{u_2} = -u_2^2(1 + 2(\kappa - 1)v_2 - \kappa v_2^2 + hu_2v_2^2), \quad \dot{v_2} = -\kappa u_2(v_2 - 1)v_2^2. \tag{4.5}$$

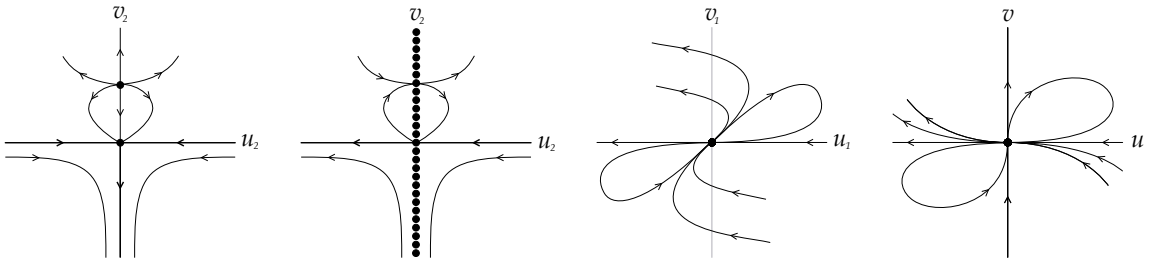
Since u_2 is a common factor of $\dot{u_2}$ and $\dot{v_2}$ we rescale the time and we get the differential system

$$\dot{u_2} = -u_2(1 + 2(\kappa - 1)v_2 - \kappa v_2^2 + hu_2v_2^2), \quad \dot{v_2} = -\kappa(v_2 - 1)v_2^2. \tag{4.6}$$

This system has on the straight line $u_2 = 0$ two equilibrium points: $(0, 0)$ and $(0, 1)$. As $\kappa \neq 0$ by Theorem 2.19 of [3] the point $(0, 0)$ is a saddle-node.

If $\kappa < 0$ the point $(0, 1)$ is a hyperbolic unstable node, and if $\kappa \in (0, 1)$ it is a hyperbolic saddle, if $\kappa = 1$ it is a semihyperbolic saddle node if $h \neq 0$ and it is not isolated if $h = 0$, and if $\kappa > 1$ it is a hyperbolic stable node.

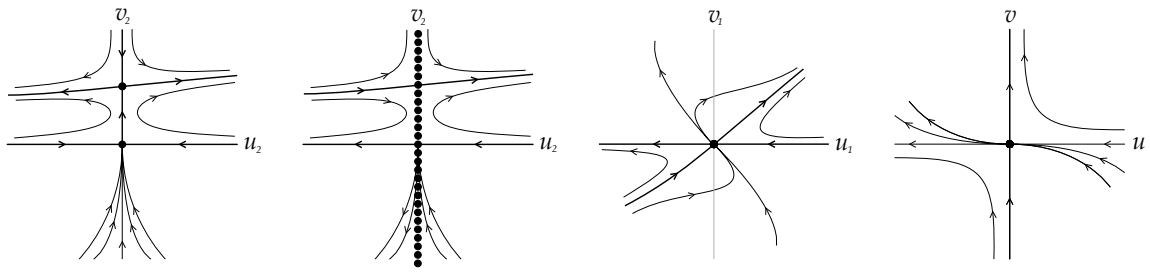
Assume $\kappa < 0$. Then for system (4.6) the origin is a saddle-node and the point $(0, 1)$ is an unstable node, and in Figure 4.3(a) we include the local phase portrait of this system in a neighborhood of the straight line $u_2 = 0$. If we multiply by u_2 to return to system (4.5), then all points on the u_2-axis become equilibrium points, and the orientation of the orbits in the second and third quadrants are reversed. Thus we obtain the phase portrait of Figure 4.3(b) for system (4.5) around the straight line $u_2 = 0$. Now we undo the blow up to obtain the local phase portrait at the origin of system (4.4). We must take into account that for this system the u_1-axis is invariant and as $\dot{u_1}|_{u_1=0} = -u_1^2$, then over this axis the flow points in the negative direction. The v_1-axis is not invariant, and we have that $\dot{u_1}|_{u_1=0} = kv_1^2$ and $\dot{v_1}|_{u_1=0} = (2-k)v_1^2 - hv_1^3$, so in a neighborhood of the origin, over the v_1 axis, the flow points in the negative u_1-direction and in the positive v_1-direction. Then we obtain the local phase portrait in Figure 4.3(c) for system (4.4) . Finally, if we undo the twist, we obtain the local phase portrait at the origin of the local chart U_2, which is given in Figure 4.3(d).



(a) (b) (c) (d)

Figure 4.3: Blowing down of the origin of system (4.3) with $\kappa < 0$.

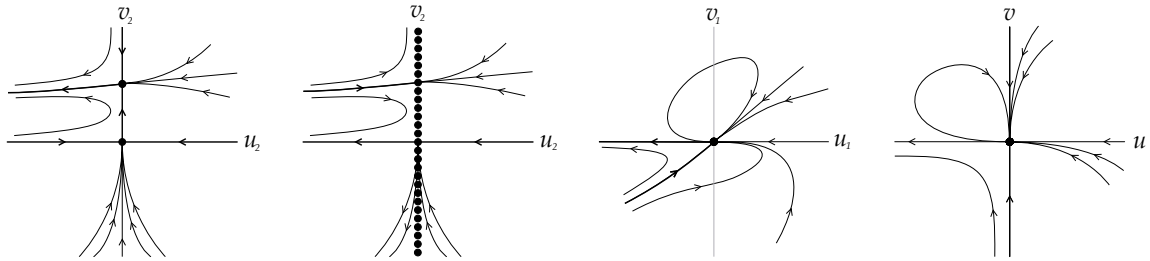
Assume now that $\kappa \in (0,1)$. In Figure 4.4(a) we plotted the local phase portrait of system (4.6) in a neighborhood of the straight line $u_2 = 0$. In Figure 4.4(b) there is the phase portrait of system (4.5) around the straight line $u_2 = 0$ and in Figure 4.4(c) at the origin of system (4.4). Finally in Figure 4.4(d) there is the local phase portrait at the origin of the local chart U_2.



(a) (b) (c) (d)

Figure 4.4: Blowing down of the origin of the system (4.3) with $\kappa \in (0,1)$.

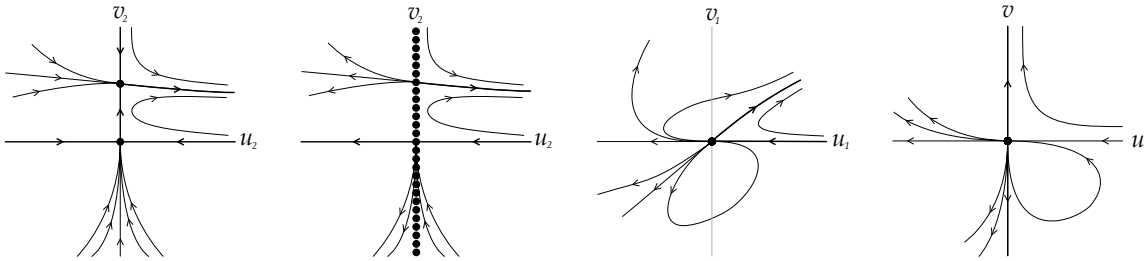
For $\kappa = 1$ and $h > 0$, in Figure 4.5(a) we plotted the local phase portrait of system (4.6) in a neighborhood of the straight line $u_2 = 0$. In Figure 4.5(b) there is the phase portrait of system (4.5) around the straight line $u_2 = 0$ and in Figure 4.5(c) at the origin of system (4.4). Finally in Figure 4.5(d) there is the local phase portrait at the origin of the local chart U_2.



(a) (b) (c) (d)

Figure 4.5: Blowing down of the origin of the system (4.3) with $\kappa = 1$ and $h > 0$.

Assume that $\kappa = 1$ and $h < 0$. In Figure 4.6(a) we plotted the local phase portrait of system (4.6) in a neighborhood of the straight line $u_2 = 0$. In Figure 4.6(b) there is the phase portrait of system (4.5) around the straight line $u_2 = 0$ and in Figure 4.6(c) at the origin of system (4.4). Finally in Figure 4.6(d) there is the local phase portrait at the origin of the local chart U_2.



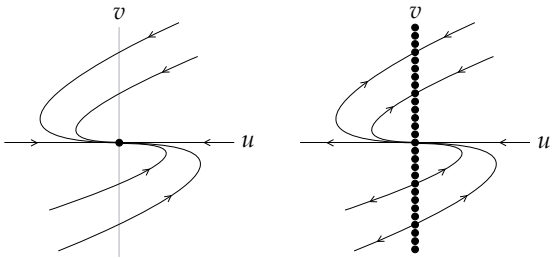
(a) (b) (c) (d)

Figure 4.6: Blowing down of the origin of the system (4.3) with $\kappa = 1$ and $h < 0$.

If $\kappa = 1$ and $h = 0$, the line $u = 0$ is filled of equilibrium points. We eliminate the common factor u rescaling the time and we get the differential system

$$\dot{u} = -u - v, \qquad \dot{v} = -v. \tag{4.7}$$

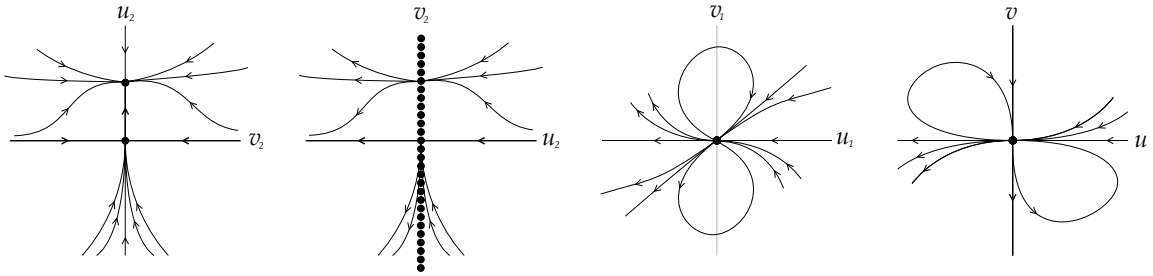
The local phase portrait of system (4.7) is shown in Figure 4.7(a), and the local phase portrait at the origin of U_2 is given in Figure 4.7(b).



(a) (b)

Figure 4.7: Phase portrait at the origin of the system (4.3) with $\kappa = 1$ and $h = 0$.

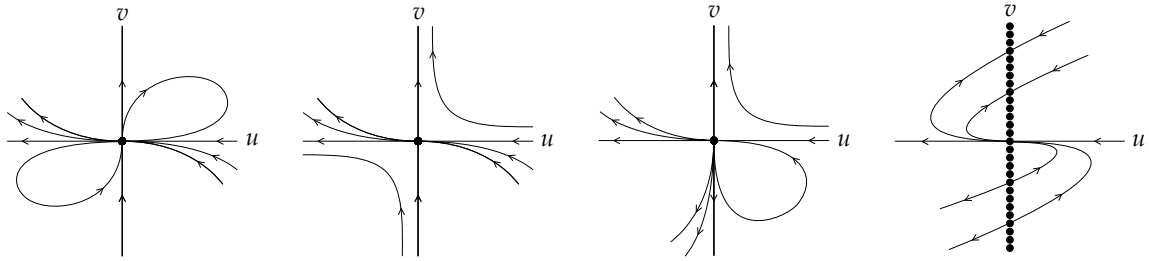
Finally, for $\kappa > 1$, in Figure 4.8(a) we plotted the local phase portrait of system (4.6) in a neighborhood of the straight line $u_2 = 0$. In Figure 4.8(b) there is the phase portrait of system (4.5) around the straight line $u_2 = 0$ and in Figure 4.8(c) at the origin of system (4.4). Finally in Figure 4.8(d) there is the local phase portrait at the origin of the local chart U_2.



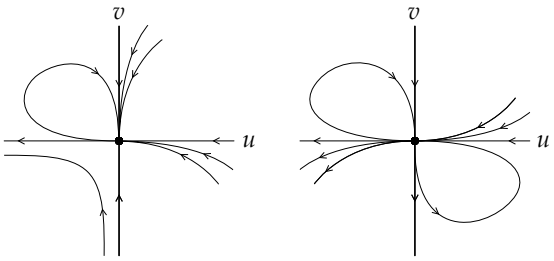
(a) (b) (c) (d)

Figure 4.8: Blowing down of the origin of the system (4.3) with $\kappa \in (1, 2)$.

Summarizing, all the possible phase portraits at the origin of the local chart U_2 for $k \neq 0$, are given in Figure 4.9.



$L_1\ (\kappa < 0)$ $L_2\ (\kappa \in (0,1))$ $L_3\ (\kappa = 1,\ h < 0)$ $L_4\ (\kappa = 1,\ h = 0)$



$L_5\ (\kappa = 1,\ h > 0)$ $L_6\ (\kappa > 1)$

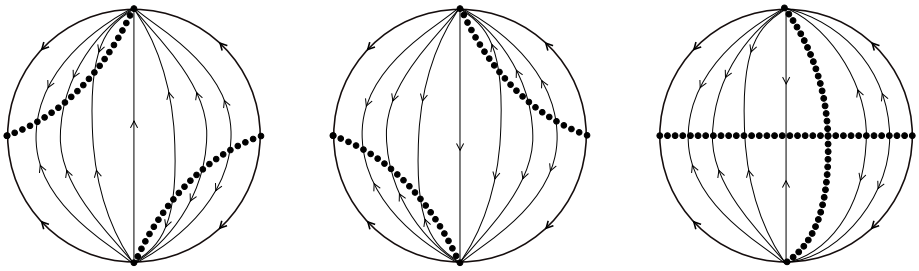
Figure 4.9: Phase portraits of the origin of U_2.

4.1 Phase portraits in the Poincaré disc

In this subsection we give the topological classification of all phase portraits of systems (4.1) stated in Theorem 1.6.

We bring together the local information previously obtained. In some cases the local information gives rise to only one phase portrait in the Poincaré disc, this occurs when the separatrices can be connected in only one way, but in other cases several phase portraits in the Poincaré disc are possible, and we shall prove which of them are feasible.

Case $\kappa = 0$. In this case there exists a continuum of equilibrium points given by the curve $z = h/(1-y)$, and the finite phase portrait completely defines the phase portrait in the Poincaré disc. The infinite equilibrium points are also contained in this continuum. All the orbits in the finite region are contained in vertical straight lines. We will have three different phase portraits depending on whether $h > 0$, $h = 0$ or $h < 0$, because the sign of h determines the position of the continuum of equilibria and the orientation of the orbits. In the case in which h is zero, the curve of equilibrium points previously mentioned becomes the y-axis, and we have another straight line filled of equilibrium points, which is $y = 1$. The three possibilities are given in Figure 4.10. Cases with $h > 0$ and $h < 0$ are topologically equivalent by doing a symmetry with respect to the y-axis, so in the classification it is enough to consider one of them, which is G1 in Figure 1.1. The case with $h = 0$ is G2 in Figure 1.1.



$h > 0$ $h < 0$ $h = 0$

Figure 4.10: Possible phase portraits of systems (4.1) with $\kappa = 0$.

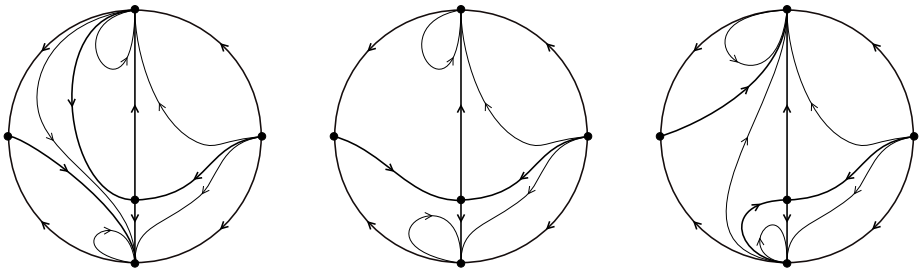
Case $\kappa \in (0,1)$. In this case there exist the finite equilibrium point P which is a stable node, the origin of U_1 is a saddle-node and the origin of U_2 has the local phase portrait L_2. We recall that the origin of the local charts V_1 and V_2 have the same phase portraits as the origin of the local charts U_1 and U_2, respectively, but reversing the orientation of the orbits. Then, we obtain only one possible phase portrait, which is G3 in Figure 1.1. Note that in the figure the point P is represented in the positive part of the z-axis, which corresponds to $h > 0$. For the case $h < 0$ the point is over the negative part of the z-axis, but these phase portraits are topologically equivalent.

Case $\kappa = 1$, $h \neq 0$. There are not finite equilibrium points, the origin of U_1 has the local phase portrait L_{11} and the origin of U_2 has the local phase portrait L_{24} if $h > 0$, and L_{25} if $h < 0$. In both cases there are only one way to connect the separatrices. In the case $h > 0$ we obtain the global phase portrait G4 in Figure 1.1, and in the case $h < 0$ we obtain a topologically equivalent phase portrait, obtained by a symmetry with respect to the y-axis.

Case $\kappa = 1$, $h = 0$. The z-axis is a continuum of equilibrium points, and then the origin of charts U_2 and V_2 are contained in this continuum, and the finite orbits are as in Figure 4.1. The origin of chart U_1 has the local phase portrait L_{11}. There is only one possible global configuration in this case, which is G5 in Figure 1.1.

Case $\kappa > 1$. In this case the finite equilibrium point P is a saddle, the origin of U_1 is a saddle-node and the origin of U_2 has the local phase portrait L_6.

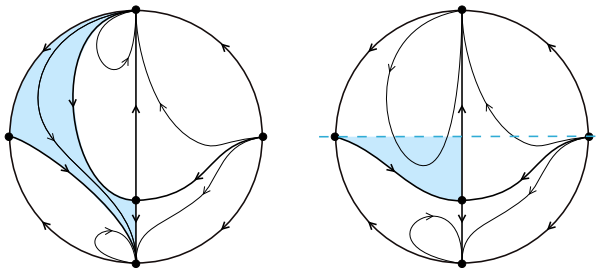
We first consider $h > 0$. The separatrices can be connected in three ways, giving rise to the three phase portraits given in Figure 4.11.



Case 1 Case 2 Case 3

Figure 4.11: Possible phase portraits of systems (4.1) with $\kappa > 1$, $h > 0$.

We can prove that the two first cases of Figure 4.11 are not possible. The flow over the horizontal axis points upward as $\dot{z}|_{z=0} = h > 0$. In Figure 4.12 we can see two shaded regions. In the shaded region of case 1 the flow points downwards over the horizontal axis, so that configuration is not possible. If we take any point in the shaded region of case 2, the orbit passing through that point starts and ends in the origin of the local chart U_2, so at some point of the z-axis the flow points downwards, so this case is also not possible. The only possible phase portrait is the one in case 3, which corresponds to G6 in Figure 1.1.



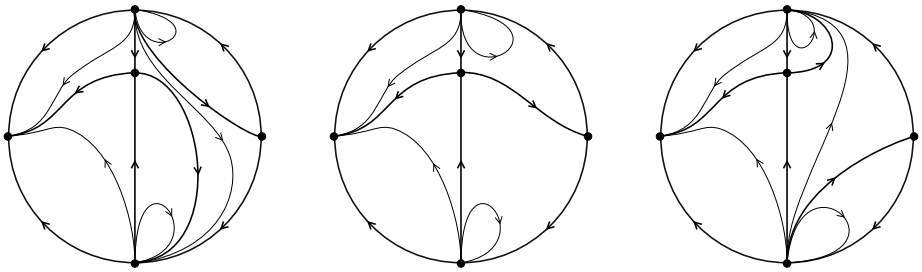
Case 1 Case 2

Figure 4.12: Regions that allow us to prove that the flow points downwards over the z-axis.

For the case with $h < 0$ an analogous reasoning can be done. The obtained phase portrait is topologically equivalent to G6 by doing a symmetry with respect to the y-axis.

Case $\kappa < 0$. As in the previous case, the finite equilibrium point P is a saddle, but the origin of U_1 is a saddle-node, and the origin of U_2 has the local phase portrait L_1.

We consider $h > 0$. We can connect the separatrices in three forms, giving rise to the three phase portraits given in Figure 4.13.



Case 1 Case 2 Case 3

Figure 4.13: Possible phase portraits of systems (4.1) with $\kappa < 0$.

Cases 1 and 2 are not possible and an analogous reasoning to the previous case works, as we find that the flow point downwards at some points of the y-axis, which contradicts the fact that $\dot{z}|_{z=0} = h > 0$. The only feasible phase portrait is case 3, which is topologically equivalent to G6 in Figure 1.1, by doing a symmetry with respect to the x-axis, a symmetry with respect to the z-axis and reversing the orientation.

For the case with $h < 0$ we obtain analogous results. The obtained phase portrait is topologically equivalent to case 3 by doing a symmetry with respect to the y-axis.

Summarizing the information in this section, we can conclude that systems (4.1) can present six topologically different phase portraits, given in Figure 1.1 and according to the classification given in Table 4.1.

Conditions	**Phase portrait**
$\kappa = 0,\ h \neq 0$	G1
$\kappa = 0,\ h = 0$	G2
$\kappa \in (0,1)$	G3
$\kappa = 1,\ h \neq 0$	G4
$\kappa = 1,\ h = 0$	G5
$\kappa > 1$	G6
$\kappa < 0$	

Table 4.1: Classification of phase portraits of systems (4.1).

5 Proof of Theorem 1.7

We consider systems (1.1) under conditions $\alpha = \kappa(1-\kappa) \neq 0$, $\beta = 0$, i.e.,

$$\dot{x} = x(y-1), \quad \dot{y} = \kappa(1-\kappa)(1-x^2) - \kappa y, \quad \dot{z} = x - \lambda z. \tag{5.1}$$

These systems have the Darboux invariant $I_5 = ((\kappa - 1 + y)^2 + \kappa(1-\kappa)x^2)e^{2\kappa t}$.

Finite equilibrium points

We study the finite equilibrium points of systems (5.1).

We first consider the case $\kappa(1-\kappa) > 0$ (i.e. $\kappa \in (0,1)$). If $\lambda \neq 0$, the unique equilibrium point is $P_1 = (0, 1-\kappa, 0)$ contained in $F_{11} = 0$. The Jacobian matrix of systems (5.1) at P_1 has eigenvalues $-\kappa$ with multiplicity two and $-\lambda$. Hence, if $\lambda > 0$ it is a local attractor, and if $\lambda < 0$ it has a 2-dimensional stable manifold and a 1-dimensional unstable manifold. If $\lambda = 0$ then the straight line $(0, 1-\kappa, z)$ with $z \in \mathbb{R}$ is filled of equilibria. The eigenvalues of the Jacobian matrix at any of these points are $-\kappa$ with multiplicity two and 0. All equilibrium points are normally hyperbolic and in view of Theorem 2.5 taking into account that in this case we have $\kappa \in (0,1)$ we get that to each equilibrium point arrives a 2-dimensional stable manifold.

Consider now the case $\kappa(1-\kappa) < 0$ (i.e. $\kappa < 0$ or $\kappa > 1$). If $\lambda \neq 0$ there are three equilibrium points which are

$$P_1 = (0, 1-\kappa, 0), \quad P_2 = \Big(\sqrt{\frac{\kappa}{\kappa-1}}, 1, \frac{1}{\lambda}\sqrt{\frac{\kappa}{\kappa-1}}\Big), \quad P_3 = -\Big(\sqrt{\frac{\kappa}{\kappa-1}}, -1, \frac{1}{\lambda}\sqrt{\frac{\kappa}{\kappa-1}}\Big).$$

All three equilibrium points are contained in $F_9 F_{10} = 0$. The eigenvalues of the Jacobian matrix at the equilibrium point P_1 are $-\kappa$ with multiplicity two and $-\lambda$. Hence, P_1 is a local attractor if $\kappa > 1$ and $\lambda > 0$; has a 2-dimensional stable manifold and a 1-dimensional unstable manifold if $\kappa > 1$ and $\lambda < 0$; it is a local repeller if $\kappa, \lambda < 0$, and it has a 2-dimensional

unstable manifold and a 1-dimensional stable manifold if $\kappa < 0$ and $\lambda > 0$. On the other hand the eigenvalues of the Jacobian matrix at the equilibrium points P_2 and P_3 are $-\lambda$, -2κ and κ. Therefore, both equilibrium points have a 2-dimensional stable manifold and a 1-dimensional unstable manifold if either $\kappa > 1$ and $\lambda > 0$, or $\kappa < 0$ and $\lambda > 0$; and they have a 2-dimensional unstable manifold and a 1-dimensional stable manifold if either $\kappa, \lambda < 0$, or $\kappa > 1$ and $\lambda < 0$.

Conditions	**Finite equilibrium points**
$\kappa \in (0,1)$, $\lambda > 0$	P_1 stable node
$\kappa \in (0,1)$, $\lambda < 0$	P_1 saddle (stable manifold of dimension 2)
$\kappa > 1$, $\lambda > 0$	P_1 stable node, P_2 and P_3 saddles (stable manifold of dimension 2)
$\kappa > 1$, $\lambda < 0$	P_1 saddle (stable manifold of dimension 2), P_2 and P_3 saddles (unstable manifold of dimension 2)
$\kappa < 0$, $\lambda < 0$	P_1 unstable node, P_2 and P_3 saddles (unstable manifold of dimension 2)
$\kappa < 0$, $\lambda > 0$	P_1 saddle (unstable manifold of dimension 2), P_2 and P_3 saddles (stable manifold of dimension 2)

Table 5.1: Local behavior of the finite equilibrium points of systems (5.1) with $\lambda \neq 0$.

If $\lambda = 0$ the straight line $(0, 1-\kappa, z)$ with $z \in \mathbb{R}$ is filled of equilibria. The eigenvalues of the Jacobian matrix at any of these points are $-\kappa$ with multiplicity two and 0. All equilibrium points are normally hyperbolic and in view of Theorem 2.5 we have that if $\kappa > 1$ then to each equilibrium point arrives a 2-dimensional stable manifold, and if $\kappa < 0$ at each equilibrium point starts a 2-dimensional unstable manifold.

Dynamics on the invariant surfaces

We consider two different cases: $\kappa(1-\kappa) > 0$ and $\kappa(1-\kappa) < 0$.

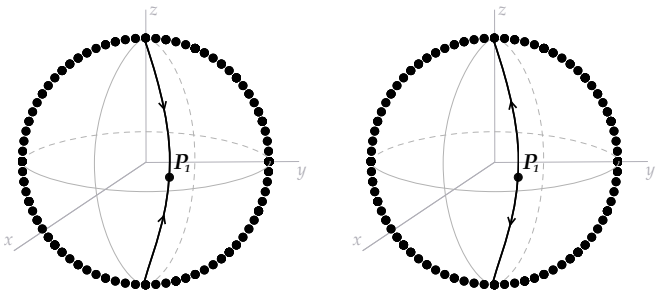
Case $\kappa(1-\kappa) > 0$

In this case $F_{11} = 0$ corresponds to the straight line $(0, 1-\kappa, z)$ with $z \in \mathbb{R}$. The dynamics on this straight line is given by $\dot{z} = -\lambda z$.

If $\lambda > 0$ the points over this straight line have their α-limit at the endpoints of the z-axis in the sphere, and their ω-limit in P_1. If $\lambda < 0$ the points over this straight line have their α-limit at P_1 and their ω-limit at the endpoints of the z-axis in the sphere. This is represented in Figure 5.1. Note that if $\lambda = 0$ the straight line is filled with equilibria.

Case $\kappa(1-\kappa) < 0$

We will study the dynamics on $F_9 = 0$ and on $F_{10} = 0$.



$\lambda > 0$ $\lambda < 0$

Figure 5.1: Phase portraits in the Poincaré disc for systems (5.2) and (5.3).

Dynamics on $F_9 = 0$

We note that systems (5.1) restricted to $F_9 = \kappa - 1 - \sqrt{\kappa(\kappa-1)}x + y = 0$ by setting $y = 1 - \kappa + \sqrt{\kappa(\kappa-1)}x$ are

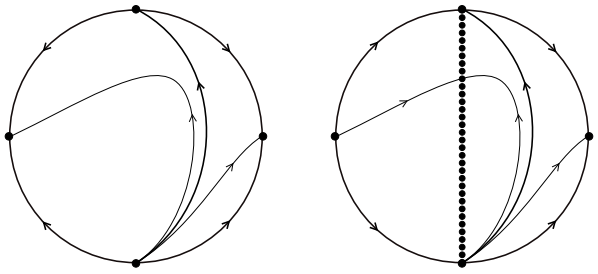
$$\dot{x} = x\left(\sqrt{\kappa(\kappa-1)}x - \kappa\right), \quad \dot{z} = x - \lambda z. \tag{5.2}$$

When $\lambda \neq 0$, there are two equilibrium points $Q_1 = (0,0)$ and $Q_2 = \frac{\sqrt{\kappa(\kappa-1)}}{\kappa-1}(1, 1/\lambda)$. The eigenvalues of the Jacobian matrix at Q_1 are $-\kappa$, $-\lambda$ and the eigenvalues of the Jacobian matrix at Q_2 are κ and $-\lambda$. Hence, Q_1 is a stable node when $\kappa, \lambda > 0$; an unstable node when $\kappa, \lambda < 0$, and a saddle when $\kappa\lambda < 0$. On the other hand Q_2 is a stable node when $\kappa < 0$ and $\lambda > 0$; an unstable node when $\kappa > 0$ and $\lambda < 0$, and a saddle when $\kappa\lambda > 0$.

If $\lambda = 0$ then $x = 0$ is filled with equilibrium points. Removing the common factor x between $\dot{x}$ and $\dot{z}$ with a rescaling of the time we get that system (5.2) when $\lambda = 0$ becomes

$$\dot{x} = \sqrt{\kappa(\kappa-1)}x - \kappa, \quad \dot{z} = 1. \tag{5.3}$$

This linear system has the phase portrait in Figure 5.2(a) for $k > 0$, and so system (5.2) with $\lambda = 0$ has the phase portrait in Figure 5.2(b) for $k > 0$. In the cases with $k < 0$ the phase portrait is the same but doing a symmetry with respect to the z-axis and changing the time t into $-t$.



(a) (b)

Figure 5.2: Phase portraits in the Poincaré disc for systems (5.2) and (5.3).

From now on we focus on the study of system (5.2) with $\lambda \neq 0$. Note that system (5.2) has the invariant straight lines $x = 0$ and $\sqrt{\kappa(\kappa-1)}x - \kappa = 0$ that contain the equilibrium points. Therefore there cannot be periodic orbits.

Now we study system (5.2) in the Poincaré disc. We first study it on the local chart U_1. On the local chart U_1 system (5.2) takes the form

$$\dot{u} = -\sqrt{\kappa(\kappa-1)}u + v + (\kappa-\lambda)uv, \quad \dot{v} = v(-\sqrt{\kappa(\kappa-1)} + \kappa v).$$

The unique equilibrium point on $v = 0$ is the origin $(0,0)$. The Jacobian matrix at that point has the eigenvalue $-\sqrt{\kappa(\kappa-1)}$ with multiplicity two and so the origin of the local chart U_1 is a stable node.

On the local chart U_2 system (5.2) takes the form

$$\dot{u} = u\left(\sqrt{\kappa(\kappa-1)}u + (\lambda-\kappa)v - uv\right), \quad \dot{v} = v^2(\lambda-u). \tag{5.4}$$

The origin is an equilibrium point. The Jacobian matrix at that point is the zero matrix, so in order to study its local phase portrait we need to do blow-ups. Since the characteristic directions at the origin of system (5.4) are given by the factors of the polynomial $uv(\sqrt{\kappa(\kappa-1)}u - \kappa v) = 0$, we need to do a twist $(u,v) = (u_1 - v_1, v_1)$ because $u = 0$ is a characteristic direction. In the new coordinates (u_1, v_1) system (5.4) becomes

$$\begin{aligned} \dot{u_1} &= \sqrt{\kappa(\kappa-1)}u_1^2 - \left(\kappa - \lambda + 2\sqrt{\kappa(\kappa-1)}\right)u_1v_1 + \left(\kappa + \sqrt{\kappa(\kappa-1)}\right)v_1^2 - u_1^2v_1 + u_1v_1^2, \\ \dot{v_1} &= v_1^2(\lambda - u_1 + v_1). \end{aligned} \tag{5.5}$$

Now we do the vertical blow up $(u_1, v_1) = (u_2, u_2v_2)$ and system (5.5) becomes

$$\begin{aligned} \dot{u_2} &= u_2^2\left(\sqrt{\kappa(\kappa-1)} - \left(\kappa - \lambda + 2\sqrt{\kappa(\kappa-1)}\right)v_2 - u_2v_2 + \left(\kappa + \sqrt{\kappa(\kappa-1)}\right)v_2^2 + u_2v_2^2\right), \\ \dot{v_2} &= u_2v_2(v_2-1)\left(\sqrt{\kappa(\kappa-1)} - \left(\kappa + \sqrt{\kappa(\kappa-1)}\right)v_2\right). \end{aligned} \tag{5.6}$$

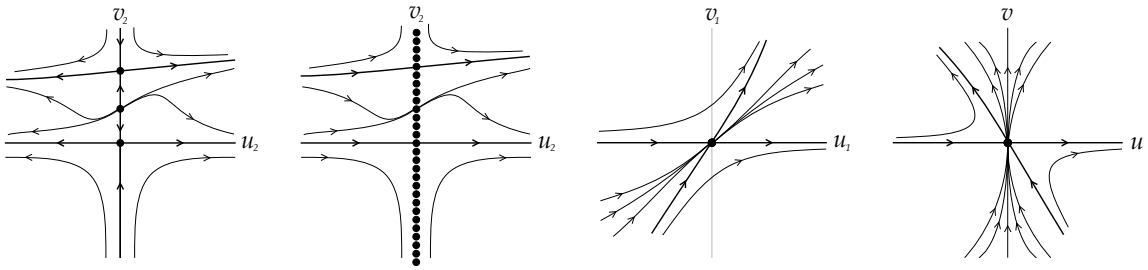
Since u_2 is a common factor of $\dot{u_2}$ and $\dot{v_2}$ we rescale the time and we get the differential system

$$\begin{aligned} \dot{u_2} &= u_2\left(\sqrt{\kappa(\kappa-1)} - \left(\kappa - \lambda + 2\sqrt{\kappa(\kappa-1)}\right)v_2 - u_2v_2 + \left(\kappa + \sqrt{\kappa(\kappa-1)}\right)v_2^2 + u_2v_2^2\right), \\ \dot{v_2} &= v_2(v_2-1)(\sqrt{\kappa(\kappa-1)} - (\kappa + \sqrt{\kappa(\kappa-1)})v_2). \end{aligned} \tag{5.7}$$

This system has on the straight line $u_2 = 0$ three equilibrium points: $(0,0)$, $(0,1)$ and $\left(0, \frac{\sqrt{\kappa(\kappa-1)}}{\kappa+\sqrt{\kappa(\kappa-1)}}\right)$. The equilibrium point $(0,0)$ is a hyperbolic saddle. The equilibrium point $(0,1)$ is a hyperbolic stable node if $\lambda < 0$ and $\kappa > 0$; a hyperbolic unstable node if $\lambda > 0$ and $\kappa < 0$, and a hyperbolic saddle if $\lambda\kappa > 0$. Finally the equilibrium point $\left(0, \frac{\sqrt{\kappa(\kappa-1)}}{\kappa+\sqrt{\kappa(\kappa-1)}}\right)$ is a hyperbolic saddle if $\lambda\kappa < 0$, a hyperbolic stable node if $\lambda, \kappa < 0$ and a hyperbolic unstable node if $\lambda, \kappa > 0$. The equilibrium point $\left(0, \frac{\sqrt{\kappa(\kappa-1)}}{\kappa+\sqrt{\kappa(\kappa-1)}}\right)$ is greater than the point $(0,1)$ if $\kappa < 0$ and is between the points $(0,0)$ and $(0,1)$ if $\kappa > 1$.

Assume $\kappa < 0$ and $\lambda > 0$, in Figure 5.3(a) we plotted the local phase portrait of system (5.7) in a neighborhood of the straight line $u_2 = 0$. In Figure 5.3(b) there is the phase portrait

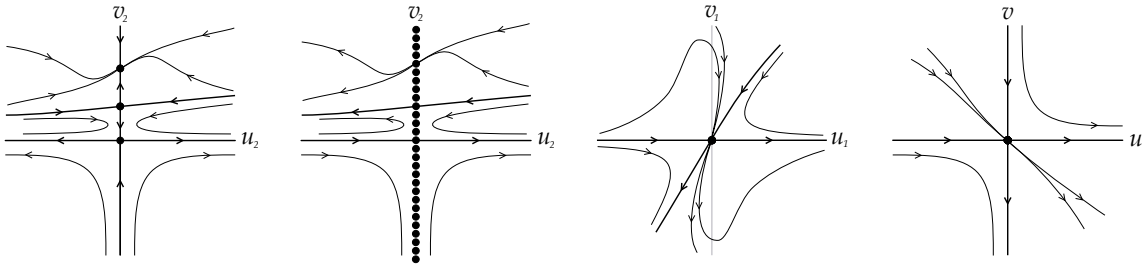
of system (5.6) around the straight line $u_2 = 0$ and in Figure 5.3(c) at the origin of system (5.5). Finally in Figure 5.3(d) there is the local phase portrait at the origin of the local chart U_2.



(a) (b) (c) (d)

Figure 5.3: Blowing down of the origin of the system (5.4) with $\kappa < 0$ and $\lambda > 0$.

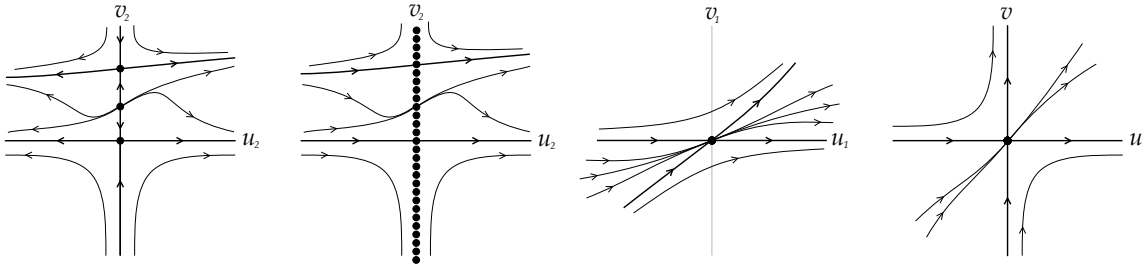
Now consider $\kappa < 0$ and $\lambda < 0$. In Figure 5.4(a) we plotted the local phase portrait of system (5.7) in a neighborhood of the straight line $u_2 = 0$. In Figure 5.4(b) there is the phase portrait of system (5.6) around the straight line $u_2 = 0$ and in Figure 5.4(c) at the origin of system (5.5). Finally in Figure 5.4(d) there is the local phase portrait at the origin of the local chart U_2.



(a) (b) (c) (d)

Figure 5.4: Blowing down of the origin of the system (5.4) with $\kappa < 0$ and $\lambda < 0$.

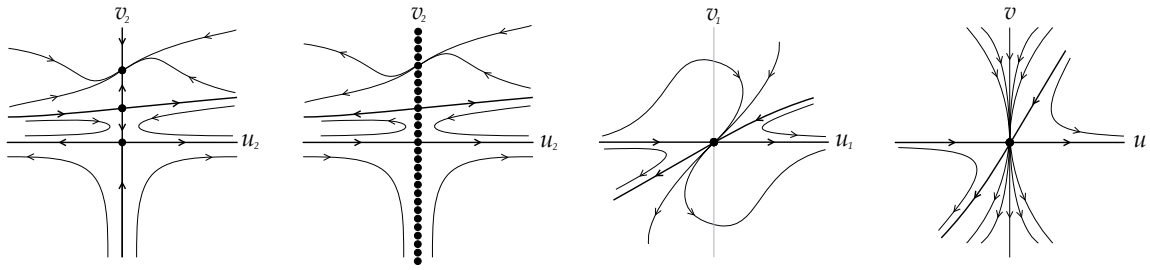
For $\kappa > 1$ and $\lambda > 0$, in Figure 5.5(a) we plotted the local phase portrait of system (5.7) in a neighborhood of the straight line $u_2 = 0$. In Figure 5.5(b) there is the phase portrait of system (5.6) around the straight line $u_2 = 0$ and in Figure 5.5(c) at the origin of system (5.5). Finally in Figure 5.5(d) there is the local phase portrait at the origin of the local chart U_2.



(a) (b) (c) (d)

Figure 5.5: Blowing down of the origin of the system (5.4) with $\kappa > 1$ and $\lambda > 0$.

Consider $\kappa > 1$ and $\lambda < 0$. In Figure 5.6(a) we plotted the local phase portrait of system (5.7) in a neighborhood of the straight line $u_2 = 0$. In Figure 5.6(b) there is the phase portrait of system (5.6) around the straight line $u_2 = 0$ and in Figure 5.6(c) at the origin of system (5.5). Finally in Figure 5.6(d) there is the local phase portrait at the origin of the local chart U_2.



(a) (b) (c) (d)

Figure 5.6: Blowing down of the origin of the system (5.4) with $\kappa > 1$ and $\lambda < 0$.

Phase portraits in the Poincaré disc

Taking into account the local information obtained for the finite and infinite equilibrium points, now we study the local phase portraits in the Poincaré disc. In all the cases the local information gives rise to only one phase portrait, as the separatrices can be connected only in one way.

Case $\lambda > 0$, $k > 0$. In this case the origin is a stable node and the equilibrium point Q_2 is a saddle and it is on the first quadrant. The origin of the local chart U_1 is a stable node and the origin of U_2 has the phase portrait in Figure 5.5(d). The equilibrium point Q_2 is contained in the invariant straight line $\sqrt{\kappa(\kappa-1)}x - \kappa = 0$, and over this line are the two stable separatrices of the saddle Q_2. The unstable separatrices have only one possible ω-limit for each of them: one goes to the origin and the other to the origin of the local chart U_2. Then we have the phase portrait given in Figure 1.2(a).

Case $\lambda > 0$, $k < 0$. In this case the origin is a saddle and the equilibrium point Q_2 is a stable node and it is on the third quadrant. The origin of the local chart U_1 is a stable node and the origin of U_2 has the phase portrait in Figure 5.3(d). The separatrix that leaves the origin in the half of the disc corresponding to the positive x-axis, goes to the origin of U_2. There are three separatrices in the half of the disc corresponding to the negative x-axis, and for the three of them the only possible ω-limit is the equilibrium point Q_2. Then we have a phase portrait which is topologically equivalent to the one in Figure 1.2(a) by moving the equilibrium point Q_2 into the origin and thus the origin into the first quadrant.

Case $\lambda < 0$, $k > 0$. In this case the origin is a saddle and the equilibrium point Q_2 is an unstable node and it is on the fourth quadrant. The origin of the local chart U_1 is a stable node and the origin of U_2 has the phase portrait in Figure 5.6(d). Similar to the previous case, there are three separatrices in the half of the disc corresponding to the positive x-axis whose only possible α-limit is the origin of U_2. The α-limit of the other stable separatrix of the origin is the origin of the local chart V_2. The phase portrait obtained is the one of Figure 1.2(b). It is topologically equivalent to the one of Figure 1.2(a) doing a symmetry with respect to the z-axis and changing t into $-t$, but we include it in order to study the $\alpha-$ and ω-limits.

Case $\lambda < 0$, $k < 0$. In this case the origin is an unstable node and the equilibrium point Q_2

is a saddle and it is on the second quadrant. The origin of the local chart U_1 is a stable node and the origin of U_2 has the phase portrait of Figure 5.4(d). The two unstable separatrices of Q_2 are over the invariant line $\sqrt{\kappa(\kappa-1)}x-\kappa=0$ and the stable separatrices have as α-limit the origin and the origin of the local chart V_2. The phase portrait in this case is topologically equivalent to the one of Figure 1.2(b) by moving the equilibrium point Q_2 into the origin.

Summarizing, we have included four phase portraits in Figure 1.2, attending to the orientations of the orbits, but note that phase portraits (a) and (b) of Figure 1.2 are topologically equivalent, and also phase portraits (c) and (d) are topologically equivalent.

Taking into account these phase portraits we analyze the α- and ω-limits of the orbits in the invariant plane $F_9=0$.

If $\lambda>0$ all the points in the invariant plane have their ω-limit at the point Q_9 or at a point at infinity, and their α-limit at an infinite point, except for the points of the two unstable separatrices of P_9 for which their α-limit is P_9.

If $\lambda<0$ all the points in the invariant plane have their ω-limit at an infinite point, except for the points of the two stable separatrices of Q_9 for which their ω-limit is Q_9. Moreover they have their α-limit at the point P_9 or in a point at infinity.

If $\lambda=0$ and $k>0$ all the points in the invariant plane have their ω-limit at an infinite point or at any point of the straight line filled of equilibria and have their α-limit in a point at infinity.

If $\lambda=0$ and $k<0$ all the points in the invariant plane have their ω-limit in a point at infinity and their α-limit at an infinite point or at any point of the straight line filled with equilibria.

Dynamics on $F_{10}=0$

We note that systems (5.1) restricted to $F_{10}=\kappa-1+\sqrt{\kappa(\kappa-1)}x+y=0$ by setting $y=1-\kappa-\sqrt{\kappa(\kappa-1)}x$ are

$$\dot{x}=-x\left(\sqrt{\kappa(\kappa-1)}x+\kappa\right),\quad \dot{z}=x-\lambda z. \tag{5.8}$$

Doing a study completely analogous to the one carried out to study $F_9=0$, we get that the phase portrait are those included in Figure 1.2.

Dynamics at infinity

Study of the infinity in the local charts U_1 and V_1

The expression of the Poincaré compactification of systems (5.1) in the local chart U_1 is given by

$$\begin{aligned}\dot{u}&=\kappa(\kappa-1)-u^2+(1-\kappa)uw+\kappa(1-\kappa)w^2,\\ \dot{v}&=w-uv+(1-\lambda)vw,\\ \dot{w}&=-w(u-w).\end{aligned}$$

At infinity, i.e. when $w=0$ this system writes as

$$\dot{u}=\kappa(\kappa-1)-u^2,\quad \dot{v}=-uv.$$

We have two different cases.

If $\kappa(1-\kappa) > 0$, there are no infinite equilibrium points in the local chart U_1.

If $\kappa(1-\kappa) < 0$, there are two infinite equilibrium points in the local chart U_1, which are $s_1 = (\sqrt{\kappa(\kappa-1)}, 0)$ and $s_4 = (-\sqrt{\kappa(\kappa-1)}, 0)$. The eigenvalues of the Jacobian matrix at these points are $-2\sqrt{\kappa(\kappa-1)}$ and $-\sqrt{\kappa(\kappa-1)}$ for s_1 and $2\sqrt{\kappa(\kappa-1)}$ and $\sqrt{\kappa(\kappa-1)}$ for s_4. Hence s_1 is a stable node and s_4 is an unstable node, as in Figure 5.7

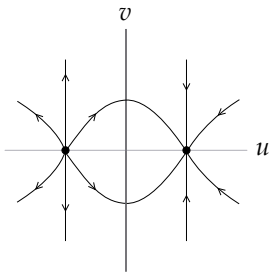


Figure 5.7: Phase portrait at the origin of the local chart U_1 with $\kappa(1-\kappa) < 0$.

Study of the infinity in the local charts U_2 and V_2

The expression of the Poincaré compactification of systems (5.1) in the local chart U_2 is given by

$$\begin{aligned}
\dot{u} &= u(1+\kappa(\kappa-1)w + \kappa(1-\kappa)u^2 + \kappa(\kappa-1)w^2),\\
\dot{v} &= uw + (\kappa-\lambda)vw + \kappa(1-\kappa)u^2v + \kappa(\kappa-1)vw^2,\\
\dot{w} &= \kappa w(w - (\kappa-1)u^2 - (1-\kappa)w^2).
\end{aligned}$$

At infinity, i.e. when $w = 0$ this system writes as

$$\dot{u} = u(1+\kappa(1-\kappa)u^2), \quad \dot{v} = \kappa(1-\kappa)u^3.$$

The line $u = 0$ if filled with equilibria. With a change of time we remove the common factor between $\dot{u}$ and $\dot{v}$ and we get

$$\dot{u} = 1+\kappa(1-\kappa)u^2, \quad \dot{v} = \kappa(1-\kappa)u^2.$$

If $\kappa(1-\kappa) > 0$ there are no equilibrium points and if $\kappa(1-\kappa) < 0$ there are two equilibrium points which are $(\pm 1/\sqrt{\kappa(1-\kappa)}, 0)$. Since we are interested in the equilibrium points with $u = 0$ (the ones with $u \neq 0$ are the same as the ones obtained in the local chart U_1) we do not consider these equilibrium points.

Study of the infinity in the local charts U_3 and V_3

The expression of the Poincaré compactification of systems (5.1) in the local chart U_2 is given by

$$\begin{aligned}
\dot{u} &= u(v + (\lambda-1)w - uw),\\
\dot{v} &= \kappa(\kappa-1)u^2 + (\lambda-\kappa)vw + \kappa(1-\kappa)w^2 - uvw,\\
\dot{w} &= w^2(\lambda-u).
\end{aligned}$$

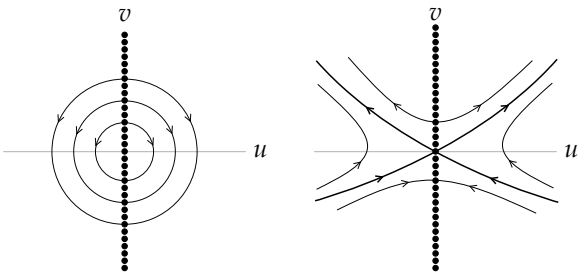
At infinity, i.e. when $w = 0$ this system writes as

$$\dot{u} = uv, \quad \dot{v} = \kappa(\kappa - 1)u^2.$$

Again the line $u = 0$ if filled with equilibria. With a change of time we remove the common factor between $\dot{u}$ and $\dot{v}$ and we get

$$\dot{u} = v, \quad \dot{v} = \kappa(\kappa - 1)u.$$

We are only interested in the point $(0, 0)$. It is indeed an equilibrium point. Again we consider two cases. If $\kappa(1 - \kappa) > 0$ the origin is a center and if $\kappa(1 - \kappa) < 0$ the origin is a saddle. Going back with the rescaling of time we see that the local phase portraits at the origin of the local chart U_3 are the ones given in Figure 5.8.



$\kappa(\kappa - 1) < 0$ $\kappa(\kappa - 1) > 0$

Figure 5.8: Phase portraits at the origin of the local chart U_3.

The phase portrait of systems (1.1) under the conditions $\alpha = \kappa(1 - \kappa) \neq 0$ on the sphere at infinity is as shown in Figure 1.3, because the dynamics depend on the sign of $\kappa(1 - \kappa)$.

When $\kappa(\kappa - 1) < 0$ the infinity of the plane $x = 0$ is filled with equilibria. All the orbits are heteroclinic having their α-limit in one of the equilibria with $y > 0$, and their ω-limit in their symmetric equilibria with respect to the origin of coordinates.

When $\kappa(\kappa - 1) > 0$ the infinity of the plane $x = 0$ is again filled with equilibria. There are eight separatrices. The equilibrium points S_3 and S_4 are the α-limits of the two separatrices whose ω-limit is the endpoint of the positive z-axis and also of the two separatrices whose ω-limit is the endpoint of the negative z-axis. On the other hand, the points S_1 and S_2 are the ω-limits of the two separatrices whose α-limit is the endpoint of the positive z-axis and also of the two separatrices whose α-limit is the endpoint of the negative z-axis. The orbits inside the regions delimited by these separatrices have S_4 as α-limit and S_1 as ω-limit (respectively S_3 as α-limit and S_2 as ω-limit). For all the other orbits: the ones with $x, y > 0$ have their α-limit on the infinity of the plane $x = 0$ with $y > 0$ and their ω-limit is S_1; the ones with $x > 0$, $y < 0$ have their ω-limit on the infinity of the plane $x = 0$ with $y < 0$ and their α-limit is S_4; the ones with $x < 0$, $y > 0$ have their α-limit on the infinity of the plane $x = 0$ with $y > 0$ and their ω-limit is S_2; the ones with $x, y < 0$ have their ω-limit on the infinity of the plane $x = 0$ with $y < 0$ and their α-limit is S_3.

α- and ω-limits for the points in $\mathbb{R}^3$

Case $\kappa(\kappa - 1) < 0$

If $\lambda > 0$, all the points in $\mathbb{R}^3$ have their ω-limit in P_1 and their α-limit in a point at infinity.

If $\lambda < 0$ all the points in $\mathbb{R}^3$ have their ω-limit in the endpoints of the z-axis, and their α-limit is an infinite point except for the points at the straight line $(0, 1-k, z)$ with $z \in \mathbb{R}$, for which their α-limit is P_1.

If $\lambda = 0$ all the points in $\mathbb{R}^3$ have their ω-limit in one of the points of the straight line $(0, 1-k, z)$ with $z \in \mathbb{R}$, which is filled with equilibria. Moreover the α-limit is a point at infinity.

Case $\kappa(\kappa - 1) > 0$

If $\lambda > 0$ and $\kappa > 1$ there are three finite equilibrium points, P_1 which is a stable node and P_2 and P_3 which are saddles with a stable manifold of dimension two, one of them in $F_9 = 0$ and the other in $F_{10} = 0$. Taking into account the phase portrait over these invariant surfaces (see Figure 1.2), there are two stable manifolds of dimension one that enter in the invariant surface $F_9 = 0$ and another two that enter into $F_{10} = 0$. The possible α-limits are the negative endpoint of the x-axis in $F_9 = 0$ and the positive endpoint of the x-axis in $F_{10} = 0$, which correspond to the infinite points S_3 and S_4; the positive and negative endpoints of the z-axis in $\mathbb{R}^3$ (which correspond to the positive and negative endpoints of the z-axis in $F_9 = 0$ and $F_{10} = 0$).

The possible ω-limits are the points Q_9 on $F_9 = 0$, and the equivalent in $F_{10} = 0$, which both correspond with the equilibrium point P_1 in $\mathbb{R}^3$; the positive endpoint of the x-axis in $F_9 = 0$ and the negative endpoint of the x-axis in $F_{10} = 0$, which correspond to the infinite points S_1 and S_2; and also the positive and negative endpoints of the z-axis, but only for the orbits over the infinity in $F_9 = 0$ and $F_{10} = 0$, which are the separatrices in the region with $y < 0$ in $\mathbb{R}^3$.

With a similar reasoning we can analyze the following cases.

If $\lambda > 0$ and $\kappa < 0$ the possible α-limits for any point $p \in \mathbb{R}^3$ are the infinite points S_3 and S_4, and also the positive and negative endpoints of the z-axis. The possible ω-limits are the infinite points S_1 and S_2, and also the positive and negative endpoints of the z-axis in the case of the four separatrices in the region with $y < 0$.

If $\lambda = 0$ and $\kappa > 0$ the possible α-limits for any point $p \in \mathbb{R}^3$ are the infinite points S_3 and S_4, the positive endpoints of the z-axis and also the negative endpoint of the z-axis in the case of the four separatrices in the region with $y > 0$ and $z > 0$. The possible ω-limits are the finite points over the straight line $(0, 1-\kappa, z)$ with $z \in \mathbb{R}$, the infinite points S_1, S_2, the positive endpoints of the z-axis, and also the negative endpoints of the z-axis in the case of the separatrices in the region with $y < 0$ and $z < 0$.

If $\lambda = 0$ and $\kappa < 0$ the possible α-limits for any point $p \in \mathbb{R}^3$ are the finite points over the straight line $(0, 1-\kappa, z)$ with $z \in \mathbb{R}$, the infinite points S_3, S_4, the positive endpoint of the z-axis, and also the negative endpoint of the z-axis for the separatrices in the region with $y > 0$ and $z < 0$. The possible ω-limits are the points S_1, S_2, the negative endpoints of the z-axis, and also the positive endpoints of the z-axis for the separatrices in the region with $y < 0$ and $z > 0$.

If $\lambda < 0$ and $\kappa > 1$ the possible α-limits for any point $p \in \mathbb{R}^3$ are the infinite points S_3 and S_4, and also the positive and negative endpoints of the z-axis in the case of the four separatrices in the region with $y > 0$. The possible ω-limits are the infinite points S_1, S_2, and the positive and negative endpoints of the z-axis.

If $\lambda < 0$ and $\kappa < 0$ the possible α-limits for any point $p \in \mathbb{R}^3$ are the finite point P_1 and the infinite points S_3 and S_4, also the positive and negative endpoints of the z-axis in the case of the four separatrices in the region with $y > 0$. The possible ω-limits are the points S_1, S_2, and the positive and negative endpoints of the z-axis.

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Multiplicity of solutions for nonautonomous (p, q)-equations

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Abstract. We consider a nonlinear Dirichlet problem driven by a nonautonomous (p, q)-Laplacian and a Carathéodory reaction which is $(p-1)$-linear near $\pm\infty$. Using variational tools, comparison principles and critical groups, we prove two multiplicity theorems producing six nontrivial smooth solutions, all with sign information and ordered.

Keywords: comparison principles, nonlinear regularity theory, tangency principle, constant sign solutions, nodal solutions, critical groups.

2020 Mathematics Subject Classification: 35J20, 35J60

1 Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper we study the following nonhomogeneous Dirichlet (p, q)-equation:

$$\begin{cases} -\Delta_p^{a_1} u(z) - \Delta_q^{a_2} u(z) = f(z, u(z)) \text{ in } \Omega, \\ u|_{\partial\Omega} = 0, \ 1 < q < p. \end{cases} \tag{1.1}$$

If $a \in L^\infty(\Omega)$, $0 < \hat{c} \leq \operatorname{essinf}_\Omega a$ and $r \in (1, \infty)$, then by Δ_r^a we denote the weighted r-Laplace differential operator defined by

$$\Delta_r^a u = \operatorname{div}(a(z)|Du|^{r-2}|Du), \quad \text{for all } u \in W_0^{1,r}(\Omega).$$

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In (1.1) the equation is driven by the sum of two such operators with different exponents $(1 < q < p)$ and in general different weights $(a_1(\cdot), a_2(\cdot))$. So in (1.1) the differential operator is nonautonomous and nonhomogeneous. The reaction (right-hand side) $f(z,x)$ is a Carathéodory function (that is, for all $x \in \mathbb{R}$, $z \mapsto f(z,x)$ is measurable, and for a.a. $z \in \Omega$, $x \mapsto f(z,x)$ is continuous). Such a function is jointly measurable and so superpositionally measurable (that is, for every measurable function $u : \Omega \to \mathbb{R}$, the function $z \mapsto f(z,u(z))$ is measurable). We assume that $f(z,\cdot)$ exhibits $(p-1)$-linear growth as $x \to \pm\infty$.

Using variational tools from the critical point theory and the spectral properties of the nonhomogeneous p and q Laplacians (see Liu–Papageorgiou [9]), we prove the existence of at least six nontrivial smooth solutions, all with sign information and ordered. Our work here extends that of Liu–Papageorgiou [8,9] where the authors obtain three nontrivial smooth solutions. In the process of the proof, we prove two strong comparison theorems, which provide powerful tools for the analysis of nonautonomous equations.

2 Mathematical background – hypotheses

The main spaces in the analysis of problem (1.1) are the Sobolev spaces $W_0^{1,p}(\Omega)$ and the Banach space $C_0^1(\overline{\Omega}) = \{u \in C^1(\overline{\Omega}) : u|_{\partial\Omega} = 0\}$. On account of the Poincaré inequality, on $W_0^{1,p}(\Omega)$ we can use the norm

$$\|u\| = \|Du\|_p \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

The Banach space $C_0^1(\overline{\Omega})$ is an ordered Banach space with positive (order) cone

$$C_+ = \left\{u \in C_0^1(\overline{\Omega}) : u(z) \geq 0 \text{ for all } z \in \overline{\Omega}\right\}.$$

This cone has a nonempty interior given by

$$\operatorname{int} C_+ = \left\{u \in C_+ : u(z) > 0 \text{ for all } z \in \Omega,\ \frac{\partial u}{\partial n}\Big|_{\partial\Omega} < 0\right\},$$

where $\frac{\partial u}{\partial n} = (Du, n)_{\mathbb{R}^N}$ with $n(\cdot)$ being the outward unit normal on $\partial\Omega$.

Let $C^{0,1}(\overline{\Omega})$ be the space of all Lipschitz continuous functions on $\overline{\Omega}$. Suppose $a \in C^{0,1}(\overline{\Omega})$, $0 < \hat{c} \leq a(z)$ for all $z \in \overline{\Omega}$ and $1 < r < \infty$. We consider the following nonlinear eigenvalue problem

$$-\Delta_r^a u(z) = \hat{\lambda}|u(z)|^{r-2}u(z) \quad \text{in } \Omega,\ u|_{\partial\Omega} = 0. \tag{2.1}$$

From Liu–Papageorgiou [9], we know that

(a) Problem (2.1) has a smallest eigenvalue $\hat{\lambda}_1^a(r) > 0$ which has the following variational characterization

$$\hat{\lambda}_1^a(r) = \inf\left\{\frac{\int_\Omega a(z)|Du|^r\,dz}{\|u\|_r^r} : u \in W_0^{1,r}(\Omega), u \neq 0\right\}. \tag{2.2}$$

(b) $\hat{\lambda}_1^a(r) > 0$ is isolated, that is, if $\hat{\sigma}(r) > 0$ denotes the spectrum of (2.1), then there exists $\varepsilon > 0$ such that

$$\hat{\sigma}(r) \cap (\hat{\lambda}_1^a(r), \hat{\lambda}_1^a(r) + \varepsilon) = \emptyset.$$

(c) $\hat{\lambda}_1^a(r) > 0$ is simple, that is, if $\hat{u}, \hat{v} \in W_0^{1,p}(\Omega)$ are eigenfunctions corresponding to $\hat{\lambda}_1^a(r)$, then $\hat{u} = \theta\hat{v}$ for some $\theta \in \mathbb{R}\backslash\{0\}$.

(d) All eigenfunctions of (2.1) belong in $C_0^1(\bar{\Omega})$ (nonlinear regularity theory of Lieberman [7]), if $\lambda > \hat{\lambda}_1^a(r)$ is an eigenvalue of (2.1), then the eigenfunctions corresponding to λ are nodal (sign-changing) and the eigenfunctions of $\hat{\lambda}_1^a(r)$ have fixed sign (see (2.2)) and by $\hat{u}_1^a(r)$ we denote the positive, r-normalized (that is, $\|\hat{u}_1^a(r)\|_r = 1$) eigenfunction for $\hat{\lambda}_1^a(r)$. Using the nonlinear maximum principle (see Pacci–Serrin [16], p. 120), we have

$$\hat{u}_1^a(r) \in \operatorname{int} C_+.$$

(e) We do not have a full knowledge of $\hat{\sigma}(r)$, $r \neq 2$. Using the Ljusternik–Schnirelmann minimax scheme, we know that (2.1) has a sequence $\{\hat{\lambda}_k^a(r)\}_{k\in\mathbb{N}}$ of eigenvalues (known as variational eigenvalues) such that $\hat{\lambda}_k^a(r) \to +\infty$ as $k \to \infty$. We do not know if this sequence exhausts $\hat{\sigma}(r)$.

If $r = 2$ (linear eigenvalue problem), then using the spectral theorem for compact operators, we have a full description of the spectrum $\hat{\sigma}(2)$ which consists of a sequence $\{\hat{\lambda}_k(2)\}_{k\in\mathbb{N}}$ of distinct eigenvalues with $\hat{\lambda}_k(2) \to +\infty$. The eigenspaces $E(\hat{\lambda}_k(2))$ are finite dimensional linear spaces and we have $E(\hat{\lambda}_k(2)) \subseteq C_0^1(\bar{\Omega})$ for all $k \in \mathbb{N}$. We have variational characterizations for all the eigenvalues:

$$\hat{\lambda}_1^a(2) = \inf\left\{\frac{\int_\Omega a(z)|Du|^2dz}{\|u\|_2^2} : u \in H_0^1(\Omega), u \neq 0\right\}, \tag{2.3}$$

$$\begin{aligned} m \geq 2 \quad \hat{\lambda}_m^a(2) &= \inf\left\{\frac{\int_\Omega a(z)|Du|^2dz}{\|u\|_2^2} : u \in V = \overline{\bigoplus_{k\geq m} E(\hat{\lambda}_k(2))}, u \neq 0\right\} \\ &= \sup\left\{\frac{\int_\Omega a(z)|Du|^2dz}{\|u\|_2^2} : u \in V = \bigoplus_{k=1}^m E(\hat{\lambda}_k(2)), u \neq 0\right\}. \end{aligned} \tag{2.4}$$

The extrema in (2.3) and (2.4) are realized on the corresponding eigenspaces.

The next inequalities are easy consequences of the aforementioned properties (see Hu–Papageorgiou [5]).

Proposition 2.1. *If $\hat{\eta} \in L^\infty(\Omega)$, $\hat{\eta}(z) \leq \hat{\lambda}_1^a(r)$ for a.a. $z \in \Omega$ and $\hat{\eta} \not\equiv \hat{\lambda}_1^a(r)$, then there exists $c_1 > 0$ such that*

$$c_1\|u\|^r \leq \|Du\|_r^r - \int_\Omega \hat{\eta}(z)|u|^r dz \quad \text{for all } u \in W_0^{1,r}(\Omega).$$

Proposition 2.2.

(a) If $\hat{\eta} \in L^\infty(\Omega)$, $\hat{\eta}(z) \leq \hat{\lambda}_k^a(2)$ for a.a. $z \in \Omega$ and $\hat{\eta} \not\equiv \hat{\lambda}_k^a(2)$, then there exists $c_2 > 0$ such that

$$c_2\|u\|^2 \leq \|Du\|_2^2 - \int_\Omega \hat{\eta}(z)u^2dz \quad \text{for all } u \in V = \overline{\bigoplus_{m\geq k} E(\hat{\lambda}_m(2))}.$$

(b) If $\hat{\eta} \in L^\infty(\Omega)$, $\hat{\eta}(z) \geq \hat{\lambda}_k^a(2)$ for a.a. $z \in \Omega$ and $\hat{\eta} \not\equiv \hat{\lambda}_k^a(2)$, then there exists $c_3 > 0$ such that

$$\|Du\|_2^2 - \int_\Omega \hat{\eta}(z)u^2\,dz \leq -c_3\|u\|_2^2 \quad \text{for all } u \in V = \bigoplus_{m=1}^k E(\hat{\lambda}_m(2)).$$

Let $A_p^{a_1} : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega) = W_0^{1,p}(\Omega)^* \left(p' = \frac{p}{p-1}\right)$ and $A_q^{a_2} : W_0^{1,q}(\Omega) \to W^{-1,q'}(\Omega) = W_0^{1,q}(\Omega)^*$ be defined by

$$\langle A_p^{a_1}(u), h\rangle = \int_\Omega a_1(z)|Du|^{p-2}(Du, Dh)_{\mathbb{R}^N}\, dz \quad \text{for all } u, h \in W_0^{1,p}(\Omega),$$
$$\langle A_q^{a_2}(u), h\rangle = \int_\Omega a_2(z)|Du|^{q-2}(Du, Dh)_{\mathbb{R}^N}\, dz \quad \text{for all } u, h \in W_0^{1,q}(\Omega).$$

We set $V = A_p^{a_1} + A_q^{a_2}$ (recall $W_0^{1,p}(\Omega) \hookrightarrow W_0^{1,q}(\Omega)$).

Proposition 2.3. *The operators $A_p^{a_1}$, $A_q^{a_2}$, V are bounded (map bounded sets to bounded ones), continuous, strictly monotone (thus maximal monotone too), coercive and of type $(S)_+$, that is,*

$$\text{if } u_n \to u \text{ in } X \quad (\text{with } X = W_0^{1,p}(\Omega) \text{ or } X = W_0^{1,q}(\Omega))$$

and $\limsup_{n\to\infty}\langle G(u_n), u_n - u\rangle \le 0$ *(with $G = A_p^{a_1}$ or $G = A_q^{a_2}$ or $G = V$), then $u_n \to u$ in X.*

Let X be a Banach space and $\varphi \in C^1(X)$. We say that $\varphi(\cdot)$ satisfies the "C-condition" if it has the following property:

"every sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq X$ such that $|\varphi(u_n)| \le M$ for some $M > 0$, all $n \in \mathbb{N}$, $(1 + \|u_n\|_X)\varphi'(u_n) \to 0$ in X^* as $n \to \infty$, admits a strongly convergent subsequence".

We introduce the following sets:

$$K_\varphi = \{u \in X : \varphi'(u) = 0\} \quad \text{(critical set of } \varphi(\cdot)),$$
$$\varphi^c = \{u \in X : \varphi(u) \le c\} \quad \text{for every } c \in \mathbb{R}.$$

If $Y_2 \subseteq Y_1 \subseteq X$ and $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, then by $H_k(Y_1, Y_2)$ we denote the kth-singular homology group with real coefficients (so the critical groups are vector spaces). Let $u \in K_\varphi$ be isolated and $c = \varphi(u)$. The critical groups of $\varphi(\cdot)$ at u are defined by

$$C_k(\varphi, u) = H_k(\varphi^c \cap U, \varphi^c \cap U \setminus \{u\}) \quad \text{for all } k \in \mathbb{N}_0,$$

where U is a neighborhood of u such that $K_\varphi \cap U \cap \varphi^c = \{u\}$. The excision property of singular homology implies that this definition is independent of the isolating neighborhood U.

If $u, v : \Omega \to \mathbb{R}$ are measurable functions such that $u(z) \le v(z)$ for a.a. $z \in \Omega$". We define

$$[u, v] = \{h \in W_0^{1,p}(\Omega) : u(z) \le h(z) \le v(z) \text{ for a.a. } z \in \Omega\}$$
$$\text{int}_{C_0^1(\overline{\Omega})}[u, v] = \text{ interior in } C_0^1(\overline{\Omega}) \text{ of } [u, v] \cap C_0^1(\overline{\Omega}).$$

Also given $u : \Omega \to \mathbb{R}$ measurable, we write $0 \prec u$ if for every $K \subseteq \Omega$ compact we have $0 < c_K \le u(z)$ for a.a $z \in K$. Also, we set $u^\pm = \max\{\pm u, 0\}$ and we have $u = u^+ - u^-$, $|u| = u^+ + u^-$ and if $u \in W_0^{1,p}(\Omega)$, then $u^\pm \in W_0^{1,p}(\Omega)$. By $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$.

The hypotheses on the data of (1.1), are the following:

H_0 $a_1, a_2 \in C^{0,1}(\overline{\Omega})$, $0 < \hat{c} \le a_1(z), a_2(z)$ for all $z \in \overline{\Omega}$, $1 < q < p < N$.

H_1 $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that

(i) $|f(z, x)| \le \hat{a}(z)[1 + |x|^{p-1}]$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$, with $\hat{a} \in L^\infty(\Omega)$;

(ii) there exists $\hat{\eta} \in L^\infty(\Omega)$ such that

$$\hat{\lambda}_1^{a_1}(p) \leq \hat{\eta}(z) \text{ for a.a. } z \in \Omega, \quad \hat{\eta} \not\equiv \hat{\lambda}_1^{a_1}(p),$$

$$\hat{\eta}(z) \leq \liminf_{x \to \pm\infty} \frac{f(z,x)}{|x|^{p-2}x} \text{ uniformly for a.a. } z \in \Omega;$$

(iii) there exists $\eta \in L^\infty(\Omega)$ such that

$$\hat{\lambda}_1^{a_2}(q) \leq \eta(z) \text{ for a.a. } z \in \Omega, \quad \eta \not\equiv \hat{\lambda}_1^{a_2}(q),$$

$$\eta(z) \leq \liminf_{x \to 0} \frac{f(z,x)}{|x|^{q-2}x} \text{ uniformly for a.a. } z \in \Omega;$$

(iv) there exist real numbers $\theta_- < 0 < \theta_+$ such that

$$f(z,\theta_+) \leq -l < 0 < l \leq f(z,\theta_-) \text{ for a.a. } z \in \Omega;$$

(v) for every $\rho > 0$, there exists $\hat{\xi}_\rho > 0$ such that for a.a. $z \in \Omega$, the function $x \to f(z,x) + \hat{\xi}_\rho |x|^{p-2}x$ is nondecreasing on $[-\rho, \rho]$.

Remark 2.4. Hypotheses H_1(i),(ii) imply that $f(z,\cdot)$ is $(p-1)$-linear as $x \to \pm\infty$. We can have partial interaction with $\hat{\lambda}_1^{a_1}(p) > 0$ from the right (nonuniform nonresonance). Similarly as $x \to 0$ with respect to $\hat{\lambda}_1^{a_2}(q) > 0$, Hypothesis H_1(i),(iii) imply that $f(z,0) = 0$ for a.a. $z \in \Omega$. Hypothesis H_1(iv) implies that $f(z,\cdot)$ is necessarily sign changing (it exhibits an oscillatory behavior near zero). Later (see hypotheses H_2 in Section 4), we will see that by requiring that $q = 2$, we can relax hypothesis H_1(iii) and incorporate in our framework reactions which satisfy the sign condition (that is, $f(z,x)x \geq 0$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$).

In what follows, for notational economy, we write

$$\rho_{a_1,p}(Du) = \int_\Omega a_1(z)|Du|^p \, dz,$$
$$\rho_{a_2,q}(Du) = \int_\Omega a_2(z)|Du|^q \, dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Let $\varphi : W_0^{1,p}(\Omega) \to \mathbb{R}$ be the energy functional for problem (1.1) defined by

$$\varphi(u) = \frac{1}{p}\rho_{a_1,p}(Du) + \frac{1}{q}\rho_{a_2,q}(Du) - \int_\Omega F(z,u) \, dz \quad \text{for all } u \in W_0^{1,p}(\Omega),$$

where $F(z,x) = \int_0^x f(z,s)ds$. Evidently $\varphi \in C^1(W_0^{1,p}(\Omega))$.

To produce solutions of constant sign, we introduce the positive and negative truncations of $\varphi(\cdot)$, namely the C^1-functionals $\varphi_\pm : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\varphi_\pm(u) = \frac{1}{p}\rho_{a_1,p}(Du) + \frac{1}{q}\rho_{a_2,q}(Du) - \int_\Omega F(z,\pm u^\pm)dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

3 Comparison theorems

In this section, we prove two strong comparison theorems which we will use in the sequel and which are of independent interest.

Given two measurable functions $u, v : \Omega \to \mathbb{R}$, we write

$$u \prec v \quad \text{if } 0 \prec u - v \quad \text{(see Section 2)}.$$

Proposition 3.1. *If hypotheses H_0 hold, $\hat{\xi} \geq 0$, $h_1, h_2 \in L^\infty(\Omega)$ and $h_1 \prec h_2$ and $u \in C_0^1(\overline{\Omega}) \setminus \{0\}$, $v \in \operatorname{int} C_+$ satisfy the equations*

$$\begin{aligned} -\Delta_p^{a_1} u - \Delta_q^{a_2} u + \hat{\xi}|u|^{p-2}u &= h_1 \quad \text{in } \Omega, \\ -\Delta_p^{a_1} v - \Delta_q^{a_2} v + \hat{\xi} v^{p-1} &= h_2 \quad \text{in } \Omega, \end{aligned}$$

then $v - u \in \operatorname{int} C_+$.

Proof. From Theorem 3.4.1, p. 61 of Pucci–Serrin [16] (the weak comparison principle), we have

$$u(z) \leq v(z) \quad \text{for all } z \in \overline{\Omega}. \tag{3.1}$$

We introduce the following closed subsets of Ω:

$$E_0 = \{z \in \Omega : u(z) = v(z)\}, \quad E_1 = \{z \in \Omega : Du(z) = Dv(z) = 0\}.$$

First we show that $E_0 \subseteq E_1$. To this end let $\hat{z} \in E_0$ and set $y = v - u$. Then $y \in C_+ \setminus \{0\}$ and $y(\hat{z}) = \min_{\overline{\Omega}} y$. Hence

$$\begin{aligned} &Dy(\hat{z}) = 0, \\ \Rightarrow\ &Dv(\hat{z}) = Du(\hat{z}). \end{aligned}$$

If $\hat{z} \notin E_1$, then $Dv(\hat{z}) = Du(\hat{z}) \neq 0$ and so we can find an open ball B centered at $\hat{z}$ such that $Dv(z) \neq 0$, $Du(z) \neq 0$ for all $z \in B$. As in the proof of Proposition 2.1 of Guedda–Véron [3], we show that there exists $K \in W^{1,\infty}(\Omega, \mathbb{R}^{N\times N})$ such that

$$-\operatorname{div}(K(z)Dy) = h_2(z) - h_1(z) - \hat{\xi}(v^{p-1} - |u|^{p-2}u) \text{ in } B. \tag{3.2}$$

By taking B even smaller if necessary, we can apply the strong maximum principle of Pucci–Serrin [16] (p. 111) and obtain

$$\begin{aligned} &y(z) > 0 \quad \text{for all } z \in B, \\ \Rightarrow\ &u(z) < v(z) \quad \text{for all } z \in B, \end{aligned}$$

a contradiction since $\hat{z} \in E_0 \cap B$. This proves that

$$E_0 \subseteq E_1. \tag{3.3}$$

Since $v \in \operatorname{int} C_+$, it follows that $E_1 \subseteq \Omega$ is compact. Hence E_0 is compact too (see (3.3)). So, we can find $\Omega' \subset \Omega$ open such that

$$E_0 \subseteq \Omega' \subseteq \overline{\Omega'} \subseteq \Omega. \tag{3.4}$$

From (3.4) and since $\overline{\Omega'}$ is compact, we can find $\varepsilon > 0$ such that

$$u(z) + \varepsilon \leq v(z) \quad \text{for all } z \in \partial\Omega', \tag{3.5}$$

$$h_1(z) + \varepsilon \leq h_2(z) \quad \text{for all } z \in \overline{\Omega'} \quad (\text{recall } h_1 \prec h_2). \tag{3.6}$$

We choose $\delta \in (0, \varepsilon)$ small such that

$$\hat{\xi}\,\big||s|^{p-2}s - |t|^{p-2}t\big| \leq \varepsilon \quad \text{for all } s, t \in [\min_{\overline{\Omega}} u, \|v\|_\infty],\ |t-s| \leq \delta. \tag{3.7}$$

Then we have

$$\begin{aligned}
&-\Delta_p^{a_1}(u+\delta) - \Delta_q^{a_2}(u+\delta) + \hat{\xi}|u+\delta|^{p-2}(u+\delta)\\
&\quad = -\Delta_p^{a_1}u - \Delta_q^{a_2}u + \hat{\xi}|u+\delta|^{p-2}(u+\delta)\\
&\quad = h_1 - \hat{\xi}|u|^{p-2}u + \hat{\xi}|u+\delta|^{p-2}(u+\delta)\\
&\quad \leq h_2 - \varepsilon + \varepsilon \quad (\text{see (3.6), (3.7)})\\
&\quad = -\Delta_p^{a_1}v - \Delta_q^{a_2}v + \hat{\xi}v^{p-1} \quad \text{in } \Omega',\\
&\Rightarrow u + \delta \leq v \quad \text{in } \Omega' \quad (\text{by the weak comparison principle, see [16], p. 61})\\
&\Rightarrow E_0 = \emptyset \quad (\text{see (2.4)}),\\
&\Rightarrow u(z) < v(z) \quad \text{for all } z \in \Omega.
\end{aligned}$$

Now let $z_0 \in \partial\Omega$ and let $z_1 = z_0 - 2rn$ with $n = n(z_0)$ the outward unit normal at $z_0 \in \partial\Omega$. For $r \in (0,1)$ small, we consider the annulus

$$R = \{z \in \Omega : r \leq |z - z_1| \leq 2r\}$$

and let $m = \min\{y(z) : z \in \partial B_r(z_1)\} > 0$. From the proof of Proposition 2.3 of Papageorgiou–Vetro–Vetro [13], we can find $\widetilde{w} \in C^1(\overline{R}) \cap C^2(R)$ satisfying

$$-\Delta_p^{a_1}\widetilde{w} - \Delta_q^{a_2}\widetilde{w} + \hat{\xi}|\widetilde{w}|^{p-2}\widetilde{w} \leq 0 \quad \text{in } R$$

with $\widetilde{w} \leq u$ on ∂R. Then by the weak comparison principle, we have

$$\widetilde{w} \leq u \quad \text{in } R.$$

Moreover, it follows $\widetilde{w}(z_0) = 0$, $\frac{\partial y}{\partial n}(z_0) \leq \frac{\partial \widetilde{w}}{\partial n}(z_0)$ and so $y \in \operatorname{int} C_+$,

$$\Longrightarrow\ v - u \in \operatorname{int} C_+. \qquad \square$$

In the second comparison result, we strengthen the relation between the two forcing terms h_1, h_2 but relax the conditions on u, v.

Proposition 3.2. *If hypothesis H_0 holds, $\hat{\xi} \geq 0$, $h_1, h_2 \in L^1(\Omega)$ satisfy*

$$\alpha \leq \hat{c}_0 \leq h_2(z) - h_1(z) \quad \textit{for a.a. } z \in \Omega$$

and $u, v \in C^1(\overline{\Omega}) \setminus \{0\}$ with $0 < v(z)$ for all $z \in \Omega$ satisfy

$$\begin{aligned}
&-\Delta_p^{a_1}u - \Delta_q^{a_2}u + \hat{\xi}|u|^{p-2}u = h_1 \quad \textit{in } \Omega,\\
&-\Delta_p^{a_1}v - \Delta_q^{a_2}v + \hat{\xi}|v|^{p-1} = h_2 \quad \textit{in } \Omega, \quad u \leq v \textit{ on } \partial\Omega,
\end{aligned}$$

then $u(z) < v(z)$ for all $z \in \Omega$.

Proof. Let $\eta(z,y) = a_1(z)|y|^{p-2}y + a_2(z)|y|^{q-2}y$ for all $z \in \Omega$, all $y \in \mathbb{R}^N$. We have

$$-\operatorname{div}(\eta(z,Dv) - \eta(z,Du)) = h_2 - h_1 - \hat{\xi}(v^{p-1} - |u|^{p-2}u) \quad \text{in } \Omega. \tag{3.8}$$

As before the weak comparison principle implies $u \leq v$. Let $y = v - u$. Then $y \in C^1(\overline{\Omega})$, $y(z) \geq 0$ for all $z \in \Omega$ and as in Guedda–Véron [3], we have

$$-\operatorname{div}(K(z)Dy(z)) = h_2 - h_1 - \hat{\xi}(v^{p-1} - |u|^{p-2}u) \quad \text{in } \Omega,$$

with $K \in W^{1,\infty}(\Omega, \mathbb{R}^{N\times N})$.

Suppose that for some $\hat{z} \in \Omega$, we have $y(\hat{z}) = 0$, hence $v(\hat{z}) = u(\hat{z})$. Then from (3.8) and the hypothesis on h_1, h_2 we can find $r \in (0,1)$ small such that

$$-\operatorname{div}(K(z)Dy(z)) \geq \hat{c}_1 > 0 \quad \text{in } B_r(\hat{z}) = \{z \in \mathbb{R}^N : |z - \hat{z}| \leq r\} \subseteq \Omega.$$

Invoking Theorem 4 of Vázquez [18], we obtain

$$0 < y(z) \quad \text{for all } z \in B_r(\hat{z}),$$

a contradiction since $y(\hat{z}) = 0$. It follows that

$$u(z) < v(z) \quad \text{for all } z \in \Omega. \qquad \square$$

In the next section we will use these comparison results to produce multiple constant sign solutions of (1.1).

4 Constant sign solutions

We start by obtaining constant sign solutions which are local minimizers of the energy functional $\varphi(\cdot)$.

Proposition 4.1. *If hypotheses H_0, H_1 (i), (iii), (iv), (v) hold, then problem (1.1) has two constant sign solutions*

$$u_0 \in \operatorname{int} C_+ \quad \textit{with} \quad u(z) < \theta_+ \quad \textit{for all } z \in \overline{\Omega},$$

$$v_0 \in -\operatorname{int} C_+ \quad \textit{with} \quad \theta_- < v(z) \quad \textit{for all } z \in \overline{\Omega}$$

and u_0 is a local minimizer of the functionals φ, φ_+, v_0 is a local minimizer of the functionals φ, φ_-.

Proof. First we produce the positive solution. To this end we introduce the Carathéodory function $\hat{f}_+ : \Omega \times \mathbb{R} \to \mathbb{R}$ defined by

$$\hat{f}_+(z,x) = \begin{cases} f(z,x^+) & \text{if } x \leq \theta_+, \\ f(z,\theta_+) & \text{if } \theta_+ < x. \end{cases} \tag{4.1}$$

We set $\hat{F}_+(z,x) = \int_0^x \hat{f}_+(z,s)ds$ and consider the C^1-functional $\hat{\varphi}_+ : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\hat{\varphi}_+(u) = \frac{1}{p}\rho_{a_1,p}(Du) + \frac{1}{q}\rho_{a_2,q}(Du) - \int_\Omega \hat{F}_+(z,u)\,dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

From (4.1) and hypotheses H_0, we see that $\hat{\varphi}_+(\cdot)$ is coercive. Also, using the Sobolev embedding theorem, we have that $\hat{\varphi}_+(\cdot)$ is sequentially weakly lower semicontinuous, so by the Weierstrass–Tonelli theorem, we can find $u_0 \in W_0^{1,p}(\Omega)$ such that

$$\hat{\varphi}_+(u_0) = \inf\left\{\hat{\varphi}_+(u) : u \in W_0^{1,p}(\Omega)\right\}. \tag{4.2}$$

On account of hypothesis H_1(iii), given $\varepsilon > 0$, we can find $\delta \in (0, \theta_+)$ such that

$$\frac{1}{q}[\eta(z) - \varepsilon]x^q \leq F(z,x) \quad \text{for a.a. } z \in \Omega, \text{ all } 0 \leq x \leq \delta. \tag{4.3}$$

From Section 2, we know that $\hat{u}_1^{a_2}(q) \in \operatorname{int} C_+$. So, we can find $t \in (0,1)$ small such that

$$0 \leq t\hat{u}_1^{a_2}(q)(z) \leq \delta \quad \text{for all } z \in \overline{\Omega}. \tag{4.4}$$

Using (4.3), (4.4) and (4.1), we have

$$\begin{aligned}\hat{\varphi}_+(t\hat{u}_1^{a_2}(q)) \leq{}& \frac{t^p}{p}\rho_{a_1,p}(D\hat{u}_1^{a_2}(q)) \\ &+ \frac{t^q}{q}\left[\int_\Omega (\hat{\lambda}_1^{a_2}(q) - \eta(z))(\hat{u}_1^{a_2}(q)^q)\,dz + \varepsilon\right] \text{ (recall that } \|\hat{u}_1^{a_2}(q)\|_q = 1).\end{aligned} \tag{4.5}$$

Since by hypothesis the inequality $\hat{\lambda}_1^{a_2}(q) \leq \eta(z)$ is strict on a set of positive Lebesgue measure, we have

$$\beta = \int_\Omega (\eta(z) - \hat{\lambda}_1^{a_2}(q))\hat{u}_1^{a_2}(q)^q\,dz > 0.$$

So, choosing $\varepsilon \in (0, \beta)$ small, from (4.5), we obtain

$$\hat{\varphi}_+(t\hat{u}_1^{a_2}(q)) \leq c_4 t^p - c_5 t^q \quad \text{for some } c_4, c_5 > 0.$$

Since $q < p$, choosing $t \in (0,1)$ even smaller if necessary, we have

$$\begin{aligned}&\hat{\varphi}_+(t\hat{u}_1^{a_2}(q)) < 0, \\ \Rightarrow\ &\hat{\varphi}_+(u_0) < 0 = \hat{\varphi}_+(0) \quad \text{(see (4.2))}, \\ \Rightarrow\ &u_0 \neq 0.\end{aligned}$$

From (4.2) we have

$$\begin{aligned}&\langle \hat{\varphi}_+'(u_0), h\rangle = 0 \quad \text{for all } h \in W_0^{1,p}(\Omega), \\ \Rightarrow\ &\langle V(u_0), h\rangle = \int_\Omega \hat{f}_+(z, u_0)h\,dz \quad \text{for all } h \in W_0^{1,p}(\Omega).\end{aligned} \tag{4.6}$$

In (4.6) first we choose the test function $h = -u_0^- \in W_0^{1,p}(\Omega)$. Then from (4.1) we have

$$\hat{c}\|Du_0^-\|_p^p \leq 0 \Rightarrow u_0 \geq 0, \quad u_0 \neq 0.$$

Next in (4.6) we choose the test function $(u_0 - \theta_+)^+ \in W_0^{1,p}(\Omega)$. Then

$$\begin{aligned}\langle V(u_0), (u_0 - \theta_+)^+\rangle &= \int_\Omega f(z, \theta_+)(u_0 - \theta_+)^+\,dz \quad \text{(see (4.1))} \\ &\leq 0 \quad \text{(see hypothesis } H_1\text{(iv))} \\ &= \langle V(\theta_+), (u_0 - \theta_+)^+\rangle \\ &\Rightarrow u_0 \leq \theta_+.\end{aligned}$$

So, we have proved that

$$u_0 \in [0, \theta_+], \quad u_0 \neq 0. \tag{4.7}$$

From (4.7), (4.1) and (4.6), we infer that u_0 is a nontrivial nonnegative solution of (1.1). Evidently $u_0 \in L^\infty(\Omega)$ and then the nonlinear regularity theory of Lieberman [7] implies $u_0 \in C_+ \setminus \{0\}$. Let $\rho = \|u_0\|_\infty$ and let $\hat{\xi}_\rho > 0$ be as postulated by hypothesis H_1(v). We have

$$\begin{aligned} &-\Delta_p^{a_1} u_0 - \Delta_q^{a_2} u_0 + \hat{\xi}_\rho u_0^{p-1} \geq 0 \quad \text{in } \Omega, \\ &\Rightarrow u_0 \in \operatorname{int} C_+ \quad \text{(see Pucci–Serrin [16], pp. 111, 120).} \end{aligned} \tag{4.8}$$

We have

$$\begin{aligned} -\Delta_p^{a_1} u_0 - \Delta_q^{a_2} u_0 + \hat{\xi}_\rho u_0^{p-1} &= f(z, u_0) + \hat{\xi}_\rho u_0^{p-1} \quad \text{(see (4.7), (4.1))} \\ &\leq f(z, \theta_+) + \hat{\xi}_\rho \theta_+^{p-1} \quad \text{(see (4.7) and hypothesis } H_1\text{(v))} \\ &\leq -l + \hat{\xi}_\rho \theta_+^{p-1} \quad \text{(see hypothesis } H_1\text{(iv))} \\ &\leq -\Delta_p^{a_1} \theta_+ - \Delta_q^{a_2} \theta_+ + \hat{\xi}_\rho \theta_+^{p-1} \quad \text{in } \Omega. \end{aligned}$$

Invoking Proposition 3.2, we infer that

$$u_0(z) < \theta_+ \quad \text{for all } z \in \overline{\Omega}. \tag{4.9}$$

Then (4.8) and (4.9) imply that

$$u_0 \in \operatorname{int}_{C_0^1(\overline{\Omega})} [0, \theta_+]. \tag{4.10}$$

Note that

$$\hat{\varphi}_+\big|_{[0,\theta_+]} = \varphi_+\big|_{[0,\theta_+]} \quad \text{(see (4.1))}. \tag{4.11}$$

Recall that u_0 is a minimizer of $\hat{\varphi}_+(\cdot)$. So, from (4.10) and (4.11) it follows that

$$\begin{aligned} &u_0 \text{ is a local } C_0^1(\overline{\Omega})\text{-minimizer of } \varphi_+(\cdot), \\ &\Rightarrow u_0 \text{ is a local } W_0^{1,p}(\Omega)\text{-minimizer of } \varphi_+(\cdot). \\ &\qquad \text{(see Hu–Papageorgiou [5, p. 339]).} \end{aligned} \tag{4.12}$$

Let $W_+ = \{u \in W_0^{1,p}(\Omega) : u(z) \geq 0 \quad \text{for a.a. } z \in \Omega\}$. We have

$$\varphi\big|_{W_+} = \varphi_+\big|_{W_+}, \quad u_0 \in \operatorname{int} W_+. \tag{4.13}$$

So, from (4.12) and (4.13) it follows that

$$u_0 \text{ is a local } W_0^{1,p}(\Omega)\text{-minimizer of } \varphi(\cdot).$$

For the negative solution, we start with the Carathéodory function $\hat{f}_-(z, x)$ defined by

$$\hat{f}_-(z, x) = \begin{cases} f(z, \theta_-) & \text{if } x < \theta_-, \\ f(z, -x^-) & \text{if } \theta_- \leq x. \end{cases}$$

We set $\hat{F}_-(z, x) = \int_0^x \hat{f}_-(z, s)\, ds$ and consider the C^1-functional $\hat{\varphi}_- : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\hat{\varphi}_-(u) = \frac{1}{p}\rho_{a_1,p}(Du) + \frac{1}{q}\rho_{a_2,q}(Du) - \int_\Omega \hat{F}_-(z, u)\, dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Working as above, this time with the functional $\hat{\varphi}_-(\cdot)$, we produce a negative solution $v_0 \in -\operatorname{int} C_+$, such that

$$v_0 \in \operatorname{int}_{C_0^1(\overline{\Omega})}[\theta_-, 0],$$

and v_0 is a local $W_0^{1,p}(\Omega)$ minimizer of both $\varphi_-(\cdot)$, $\varphi(\cdot)$. □

As we already mentioned, hypothesis H_1(iv) forces $f(z,\cdot)$ to be sign-changing. We can modify hypothesis H_1(iv) so as to include in our framework reactions which satisfy the sign condition. We can do this if we restrict to $q = 2$. The new hypotheses on $f(z,x)$ are the following:

$H_2 : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that

(i) same as hypothesis H_1(i);

(ii) same as hypothesis H_1(ii);

(iii) same as hypothesis H_1(iii) with $q = 2$;

(iv) there exist $\theta_- < 0 < \theta_+$ such that $f(z,\theta_+) \leq 0 \leq f(z,\theta_-)$ for a.a. $z \in \Omega$;

(v) same as hypothesis H_1(v).

Remark 4.2. We see that this set of hypotheses includes also reactions which satisfy the sign condition, that is, $f(z,x)x \geq 0$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$. Such functions were excluded under hypotheses H_1.

Proposition 4.3. *If hypotheses H_0, H_2(i), (iii), (iv), (v) hold, then problem (1.1) has at least two constant sign solutions*

$$u_0 \in \operatorname{int} C_+ \quad \text{with} \quad u_0(z) < \theta_+ \quad \text{for all } z \in \overline{\Omega},$$

$$v_0 \in -\operatorname{int} C_+ \quad \text{with} \quad \theta_- < v_0(z) \quad \text{for all } z \in \overline{\Omega},$$

u_0 is a local minimizer of the functionals φ, φ_+, v_0 is a local minimizer of the functionals φ, φ_-.

Proof. The proof is the same as that of Proposition 4.1 and it differs only at the point where we show that $u_0(z) < \theta_+$ (resp. $\theta_- < v_0(z)$) for all $z \in \overline{\Omega}$. In this case we argue as follows: let $\eta(z,y) = a_1(z)|y|^{p-2}y + a_2(z)|y|^{q-2}y$ for all $z \in \Omega$, all $y \in \mathbb{R}^N$. Then

$$\operatorname{div} \eta(z, Du) = \Delta_p^{a_1} u + \Delta_q^{a_2} u \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Since $2 < p$ (see hypothesis H_2(iii)), we see that $\eta(z,\cdot) \in C^1(\mathbb{R}^N, \mathbb{R}^N)$ and we have

$$\nabla_y \eta(z,y) = a_1(z)|y|^{p-2}\left[\mathrm{id} + (p-2)\frac{y \otimes y}{|y|^2}\right] + a_2(z)\mathrm{id}.$$

Therefore for any $h \in \mathbb{R}^N \setminus \{0\}$, we have

$$\begin{aligned} &(\nabla_y \eta(z,y)h, h)_{\mathbb{R}^N} \geq \hat{c}|h|^2, \\ \Rightarrow\ &\nabla_y \eta(z,\cdot) \text{ is positive definite.} \end{aligned} \tag{4.14}$$

Note that

$$\operatorname{div} \eta(z, Du_0) + f(z,u_0) \geq -\operatorname{div} \eta(z,\theta_+) + f(z,\theta_+) \quad \text{in } \Omega. \tag{4.15}$$

(see hypothesis H_2(iv)).

From (4.14), (4.15) and the tangency principle of Serrin [17] (see also Pucci–Serrin [16], p. 35), we have

$$u_0(z) < \theta_+ \quad \text{for all } z \in \overline{\Omega}.$$

Similarly, we show that $\theta_- < v_0(z)$ for all $z \in \overline{\Omega}$. □

Next we will show the existence of extremal constant sign solutions, that is, a minimal positive solution and a maximal negative solution. In Section 4 we will use these extremal constant sign solutions in order to produce a nodal (sign-changing) solutions.

On account of hypotheses H_1(i), (iii) (they are also valid when H_2 hold, with $q = 2$), given $\varepsilon > 0$, we can find $c_6 = c_6(\varepsilon) \geq 0$ such that

$$f(z,x)x \geq [\eta(z) - \varepsilon]|x|^q - c_6|x|^p \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}. \tag{4.16}$$

This unilateral growth condition suggests that we consider the following auxiliary Dirichlet problem

$$\begin{cases} -\Delta_p^{a_1} u(z) - \Delta_q^{a_2} u(z) = [\eta(z) - \varepsilon]|u(z)|^{q-2}u(z) - c_6|u(z)|^{p-2}u(z) & \text{in } \Omega, \\ u|_{\partial\Omega} = 0. \end{cases} \tag{4.17}$$

For this problem we have the following existence and uniqueness result.

Proposition 4.4. *If hypothesis H_0 holds, then for all $\varepsilon \in (0,1)$ small problem has a unique positive solution*

$$\bar{u} \in \operatorname{int} C_+,$$

and since the equation is odd $\bar{v} = -\bar{u} \in -\operatorname{int} C_+$ is the unique negative solution of (4.17).

Proof. Let $\sigma_+ : W_0^{1,p}(\Omega) \to \mathbb{R}$ be the C^1-functional defined by

$$\sigma_+(u) = \frac{1}{p}\rho_{a_1,p}(Du) + \frac{1}{q}\rho_{a_2,q}(Du) - \frac{1}{q}\int_\Omega [\eta(z) - \varepsilon](u^+)^q \, dz + \frac{c_6}{p}\|u^+\|_p^p$$

for all $u \in W_0^{1,p}(\Omega)$.

Since $q < p$, we see that $\sigma_+(\cdot)$ is coercive. Also using the Sobolev embedding theorem, we show that $\sigma_+(\cdot)$ is sequentially weakly lower semicontinuous. So, we can find $\bar{u} \in W_0^{1,p}(\Omega)$ such that

$$\sigma_+(\bar{u}) = \inf\left\{\sigma_+(u) : u \in W_0^{1,p}(\Omega)\right\}. \tag{4.18}$$

As in the proof of Proposition 4.1, for $t \in (0,1)$ small, we have

$$\begin{aligned} &\sigma_+(t\hat{u}_1^{a_2}(q)) < 0, \\ \Rightarrow\ &\sigma_+(\bar{u}) < 0 = \sigma_+(0) \quad \text{(see (4.18))}, \\ \Rightarrow\ &\bar{u} \neq 0. \end{aligned}$$

From (4.18) we have

$$\begin{aligned} &\langle \sigma_+'(\bar{u}), h\rangle = 0 \quad \text{for all } h \in W_0^{1,p}(\Omega), \\ \Rightarrow\ &\langle V(\bar{u}), h\rangle = \int_\Omega [\eta(z) - \varepsilon](\bar{u}^+)^{q-1}h \, dz - c_6 \int_\Omega (\bar{u}^+)^{p-1}h \, dz \end{aligned} \tag{4.19}$$

for all $h \in W_0^{1,p}(\Omega)$.

In (4.19) we choose the test function $h = \bar{u}^- \in W_0^{1,p}(\Omega)$ and obtain

$$\hat{c}\|D\bar{u}\|_p^p \leq 0, \quad \Rightarrow \quad \bar{u} \geq 0, \quad \bar{u} \neq 0.$$

Therefore $\bar{u}$ is a nontrivial positive solution of (4.17). Using Theorem 7.1, p. 286, of Ladyzhenskaya–Uraltseva [6], we have that $\bar{u} \in L^\infty(\Omega)$ and then the nonlinear regularity theory of Lieberman [7] implies that $\bar{u} \in C_+ \setminus \{0\}$. We have

$$\Delta_p^{a_1}\bar{u} + \Delta_q^{a_2}\bar{u} \leq c_6\bar{u}^{p-1} \quad \text{in } \Omega,$$

$$\Rightarrow \bar{u} \in \operatorname{int} C_+ \quad \text{(see Pucci–Serrin [16], p. 120)}.$$

Now we show the uniqueness of this positive solution of (4.17). To this end, we introduce the integral functional $j : L^1(\Omega) \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ defined by

$$j(u) = \begin{cases} \frac{1}{p}\rho_{a_1,p}(Du^{1/q}) + \frac{1}{q}\rho_{a_2,q}(Du^{1/q}) & \text{if } u \geq 0,\ u^{1/q} \in W_0^{1,p}(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

From Diaz-Saa [2], we know that $j(\cdot)$ is convex. Suppose that $\bar{y}$ is another positive solution of (4.17). Again we have $\bar{y} \in \operatorname{int} C_+$. On account of Proposition 4.1.22, p. 274, of Papageorgiou–Rădulescu–Repovš [12], implies that

$$\frac{\bar{u}}{\bar{y}} \in L^\infty(\Omega), \quad \frac{\bar{y}}{\bar{u}} \in L^\infty(\Omega). \tag{4.20}$$

Set $h = \bar{u}^q - \bar{y}^q \in C_0^1(\overline{\Omega})$. Then (4.20) implies that for $t > 0$ small, we have

$$\bar{u} + th \in \operatorname{dom} j, \quad \bar{y} + th \in \operatorname{dom} j,$$

where $\operatorname{dom} j = \{u \in L^1(\Omega) : j(u) < \infty\}$ (effective domain of $j(\cdot)$). Then we can compute the directional derivatives of $j(\cdot)$ at $\bar{u}^q$ and at $\bar{y}^q$ in the direction h. A simple computation using the convexity of $j(\cdot)$ and the nonlinear Green's theorem (see [12], p. 35), we obtain

$$\begin{aligned} j'(\bar{u}^q)(h) &= \frac{1}{q}\int_\Omega \frac{-\Delta_p^{a_1}\bar{u} - \Delta_q^{a_2}\bar{u}}{\bar{u}^{q-1}}(\bar{u}^q - \bar{y}^q)\,dz \\ &= \frac{1}{q}\int_\Omega \left[(\eta(z) - \varepsilon) - c_4\bar{u}^{p-q}\right](\bar{u}^q - \bar{y}^q)\,dz, \\ j'(\bar{y}^q)(h) &= \frac{1}{q}\int_\Omega \frac{-\Delta_p^{a_1}\bar{y} - \Delta_q^{a_2}\bar{y}}{\bar{y}^{q-1}}(\bar{u}^q - \bar{y}^q)\,dz \\ &= \frac{1}{q}\int_\Omega \left[(\eta(z) - \varepsilon) - c_4\bar{y}^{p-q}\right](\bar{u}^q - \bar{y}^q)\,dz. \end{aligned}$$

The convexity of $j(\cdot)$ implies the monotonicity of $j'(\cdot)$. So, we have

$$0 \leq \frac{c_6}{q}\int_\Omega \left(\bar{y}^{p-q} - \bar{u}^{p-q}\right)(\bar{u}^q - \bar{y}^q)\,dz,$$

$$\Rightarrow \bar{u} = \bar{y} \quad (\text{since } q < p).$$

Therefore $\bar{u} \in \operatorname{int} C_+$ is the unique positive solution of (4.17). The equation is odd, so $\bar{v} = -\bar{u} \in -\operatorname{int} C_+$ is the unique negative solution of (4.17). □

Let S_+ (resp. S_-) be the set of positive (resp. negative) solutions of (1.1). From Propositions 4.1 and 4.3 and the nonlinear regularity theory, we have

$$\emptyset \neq S_+ \subseteq \operatorname{int} C_+ \quad \text{and} \quad \emptyset \neq S_- \subseteq -\operatorname{int} C_+.$$

Next we show that the solutions $\bar{u} \in \operatorname{int} C_+$ (resp. $\bar{v} \in -\operatorname{int} C_+$) from Proposition 4.4, is a lower bound (resp. upper bound) for the elements of S_+ (resp. of S_-).

Proposition 4.5. *If hypotheses H_0 and H_1 or H_2 ($q = 2$) hold, then $\bar{u} \leq u$ for all $u \in S_+$, and $v \leq \bar{v}$ for all $v \in S_-$.*

Proof. For $u \in S_+ \subseteq \operatorname{int} C_+$ and consider the Carathéodory function $k_+ : \Omega \times \mathbb{R} \to \mathbb{R}$ defined by

$$k_+(z,x) = \begin{cases} (\eta(z) - \varepsilon)(x^+)^{q-1} - c_6(x^+)^{p-1} & \text{if } x \leq u(z), \\ (\eta(z) - \varepsilon)(u(z))^{q-1} - c_6(u(z))^{p-1} & \text{if } u(z) < x. \end{cases} \tag{4.21}$$

We set $K_+(z,x) = \int_0^x k_+(z,s)\,ds$ and consider the C^1-functional $\gamma_+ : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\gamma_+(u) = \frac{1}{p}\rho_{a_1,p}(Du) + \frac{1}{q}\rho_{a_2,q}(Du) - \int_\Omega K_+(z,u)\,dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Clearly $\gamma_+(\cdot)$ is coercive (see (4.21)). Also using the Sobolev embedding theorem, we see that $\gamma_+(\cdot)$ is sequentially weakly lower semicontinuous. So, we can find $\tilde{u} \in W_0^{1,p}(\Omega)$ such that

$$\gamma_+(\tilde{u}) = \inf\left\{\gamma_+(u) : u \in W_0^{1,p}(\Omega)\right\}. \tag{4.22}$$

As in the proof of Proposition 4.1, we choose $t \in (0,1)$ small so that $t\hat{u}_1^{a_2}(q) \leq u$ (recall $u \in \operatorname{int} C_+$ and use Proposition 4.1.22, p. 274 of [12]) and using (4.21) and since $q < p$, we have

$$\begin{aligned} &\gamma_+(t\hat{u}_1^{a_2}(q)) < 0 \ \text{ (choosing } t \in (0,1) \text{ even smaller if necessary)}, \\ &\Rightarrow \gamma_+(\tilde{u}) < 0 = \gamma_+(0) \quad \text{(see (4.22))}, \\ &\Rightarrow \tilde{u} \neq 0. \end{aligned}$$

From (4.22) we have

$$\begin{aligned} &\langle \gamma_+'(\tilde{u}), h\rangle = 0 \quad \text{for all } h \in W_0^{1,p}(\Omega), \\ &\Rightarrow \langle V(\tilde{u}), h\rangle = \int_\Omega k_+(z,\tilde{u})h\,dz \quad \text{for all } h \in W_0^{1,p}(\Omega). \end{aligned} \tag{4.23}$$

In (4.23) we choose the test function $h = -\tilde{u}^- \in W_0^{1,p}(\Omega)$ and obtain

$$\begin{aligned} &\hat{c}\|D\tilde{u}^-\|_p^p \leq 0 \quad \text{(see (4.21))}, \\ &\Rightarrow \tilde{u} \geq 0, \quad \tilde{u} \neq 0. \end{aligned}$$

Next in (4.23), we choose the test function $h = (\tilde{u} - u)^+ \in W_0^{1,p}(\Omega)$. Then

$$\begin{aligned} \langle V(\tilde{u}), (\tilde{u} - u)^+\rangle &= \int_\Omega \left[(\eta(z) - \varepsilon)u^{q-1} - c_6 u^{p-1}\right](\tilde{u} - u)^+\,dz \quad \text{(see (4.21))} \\ &\leq \int_\Omega f(z,u)(\tilde{u} - u)^+\,dz \quad \text{(see (4.16))} \\ &= \langle V(u), (\tilde{u} - u)^+\rangle \quad \text{(since } u \in S_+\text{)}. \\ &\Rightarrow \tilde{u} \leq u. \end{aligned}$$

So, we have proved that

$$\tilde{u} \in [0, u], \quad \tilde{u} \neq 0. \tag{4.24}$$

From (4.24), (4.21), (4.23) and Proposition 4.5, it follows that

$$\tilde{u} = \bar{u} \leq u \quad \text{for all } u \in S_+ \ (u \in S_+ \text{ was arbitrary}).$$

Similarly, we show that

$$v \leq \bar{v} \quad \text{for all } v \in S_-. \qquad \square$$

Using these bounds, we can produce the extremal constant sign solutions for problem (1.1).

Proposition 4.6. *If hypotheses H_0 and H_1 or H_2 $(q = 2)$ hold, then there exist $u^* \in S_+$ and $v^* \in S_-$ such that $u^* \leq u$ for all $u \in S_+$, $v \leq v^*$ for all $v \in S_-$.*

Proof. We know that S_+ is downward directed, that is, if $u_1, u_2 \in S_+$, then there exists $u \in S_+$ such that $u \leq u_1$, $u \leq u_2$ (see Papageorgiou–Rădulescu–Repovš [11]). Then using Theorem 5.109, p. 308, of Papageorgiou–Rădulescu [4], we can find $\{u_n\}_{n\in\mathbb{N}} \subseteq S_+$ decreasing such that

$$\inf S_+ = \inf_{n\in\mathbb{N}} u_n.$$

From Proposition 4.5, we have

$$\bar{u} \leq u_n \leq u_1 \quad \text{for all } n \in \mathbb{N}. \tag{4.25}$$

Also, since $u_n \in S_+$ for all $n \in \mathbb{N}$, we have

$$\langle V(u_n), h\rangle = \int_\Omega f(z, u_n)h\,dz \quad \text{for all } h \in W_0^{1,p}(\Omega). \tag{4.26}$$

In (4.26) we choose the test function $h = u_n \in W_0^{1,p}(\Omega)$. Then using (4.25) and hypothesis $H_1(\mathrm{i}) = H_2(\mathrm{i})$, we infer that

$$\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega) \quad \text{is bounded.}$$

So, we may assume that

$$u_n \xrightarrow{w} u^* \quad \text{in } W_0^{1,p}(\Omega), \quad u_n \to u^* \quad \text{in } L^p(\Omega). \tag{4.27}$$

In (4.26) we choose the test function $h = u_n - u^* \in W_0^{1,p}(\Omega)$, pass to the limit as $n \to \infty$ and use (4.27). Then

$$\lim_{n\to\infty} \langle V(u_n), u_n - u^*\rangle = 0,$$

$$\Rightarrow u_n \to u^* \quad \text{in } W_0^{1,p}(\Omega) \quad (\text{see Proposition 2.3}).$$

Passing to the limit as $n \to \infty$ in (4.26), we obtain

$$\langle V(u^*), h\rangle = \int_\Omega f(z, u^*)h\,dz \quad \text{for all } h \in W_0^{1,p}(\Omega),$$
$$\bar{u} \leq u^* \ (\text{see (4.25)}).$$

Therefore, $u^* \in S_+$, $u^* = \inf S_+$.

To produce the maximal negative solution, we note that S_- is upward directed (that is, if $v_1, v_2 \in S_-$, there exists $v \in S_-$ such that $v_1 \leq v,\ v_2 \leq v$). Therefore, we can find $\{v_n\}_{n\in\mathbb{N}}$ increasing such that

$$\sup S_- = \sup_{n\in\mathbb{N}} v_n.$$

Reasoning as above, we produce $v^* \in S_-$ such that

$$v \leq v^* \quad \text{for all } v \in S_-. \qquad \square$$

In Proposition 4.1, to obtain the solutions $u_0 \in \operatorname{int} C_+$ and $v_0 \in -\operatorname{int} C_+$, we did not use the asymptotic condition as $x \to \pm\infty$ (hypothesis $H_1(\text{ii}) = H_2(\text{ii})$). If we bring this condition in the picture and strengthen hypotheses $H_1(\text{iii})$, $H_2(\text{iii})$, we can generate additional constant sign solutions.

The new hypotheses on $f(z,x)$ are the following:

H_1' : $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $H_1'(\text{i})$, (ii), (iv), (v) are the same as the corresponding hypotheses $H_1(\text{i})$, (ii), (iv), (v) and

(iii) there exist $\tau \in (1,q)$, $\delta > 0$ and $c_0^* > 0$ such that

$$c_0^*|x|^\tau \leq f(z,x)x \leq \tau F(z,x) \quad \text{for a.a. } z \in \Omega, \text{ all } |x| \leq \delta.$$

H_2' : $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $H_2'(\text{i})$, (ii), (iv), (v) are the same as the corresponding hypotheses $H_2(\text{i})$, (ii), (iv), (v) and

(iii) $q = 2$ and there exist $\eta_1, \eta_2 \in L^\infty(\Omega)$ and $k \geq 2$ such that

$$\begin{aligned}
&\hat{\lambda}_k^{a_2}(2) \leq \eta_1(z) \leq \eta_2(z) \leq \hat{\lambda}_{k+1}^{a_2}(2) \quad \text{for a.a. } z \in \Omega,\\
&\eta_1 \not\equiv \hat{\lambda}_k^{a_2}(2), \quad \eta_2 \not\equiv \hat{\lambda}_{k+1}^{a_2}(2),\\
&\eta_1(z) \leq \liminf_{x\to 0} \frac{f(z,x)}{x} \leq \limsup_{x\to 0} \frac{f(z,x)}{x} \leq \eta_2(z)\\
&\qquad\qquad \text{uniformly for a.a. } z \in \Omega.
\end{aligned}$$

Proposition 4.7. *If hypotheses H_0 and H_1' or H_2' ($q = 2$) hold, then we can find two more constant sign solutions:*

$$\hat{u} \in \operatorname{int} C_+, \quad u_0 \leq \hat{u}, \quad \hat{u} \neq u_0,$$
$$\hat{v} \in -\operatorname{int} C_+, \quad \hat{v} \leq v_0, \quad \hat{v} \neq v_0.$$

Proof. From Proposition 4.1, we know that there is a positive solution $u_0 \in \operatorname{int} C_+$ such that

$$u_0 \in \operatorname{int}_{C_0^1(\overline{\Omega})}[0, \theta_+]$$

and u_0 is a local minimizer of $\varphi_+(\cdot)$. We assume that K_{φ_+} is finite, otherwise we already have an infinite number of positive smooth solutions and so we are done.

Claim: $\varphi_+(\cdot)$ satisfies the C-condition.

Consider a sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega)$ such that $\{\varphi_+(u_n)\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ is bounded and

$$\left| \langle V(u_n), h\rangle - \int_\Omega f(z, u_n^+)h\, dz \right| \leq \frac{\varepsilon_n \|h\|}{1 + \|u_n\|} \tag{4.28}$$

for all $h \in W_0^{1,p}(\Omega)$, all $n \in \mathbb{N}$ with $\varepsilon_n \to 0$.

In (4.28), we choose the test function $h = -u_n^- \in W_0^{1,p}(\Omega)$ and obtain

$$\hat{c}\|Du_n^-\|_p^p \leq \varepsilon_n \quad \text{for all } n \in \mathbb{N},$$

$$\Rightarrow u_n^- \to 0 \quad \text{in } W_0^{1,p}(\Omega). \tag{4.29}$$

Suppose that $\|u_n^+\| \to \infty$ and let $y_n = \frac{u_n^+}{\|u_n^+\|}$, $n \in \mathbb{N}$. Then

$$\|y_n\| = 1, \ y_n \geq 0 \quad \text{for all } n \in \mathbb{N}.$$

So, we may assume that

$$y_n \xrightarrow{w} y \quad \text{in } W_0^{1,p}(\Omega), \quad y_n \to y \quad \text{in } L^p(\Omega), \quad y \geq 0. \tag{4.30}$$

From (4.28) and (4.29) it follows that

$$|\langle V(u_n^+), h\rangle| \leq \varepsilon_n'\|h\| + \int_\Omega f(z, u_n^+)h\,dz \tag{4.31}$$

$$\text{for all } h \in W_0^{1,p}(\Omega), \text{ all } n \in \mathbb{N}, \text{ with } \varepsilon_n' \to 0^+.$$

$$\Rightarrow \int_\Omega a_1(z)|Dy_n|^{p-2}(Dy_n, Dh)_{\mathbb{R}^N}dz + \frac{1}{\|u_n\|^{p-q}}\int_\Omega a_2|Dy_n|^{q-2}(Dy_n, Dh)_{\mathbb{R}^N}\,dz$$

$$\leq \frac{\varepsilon_n'}{\|u_n^+\|^{p-1}}\|h\| + \int_\Omega \frac{f(z, u_n^+)}{\|u_n^+\|^{p-1}}h\,dz \quad \text{for all } h \in W_0^{1,p}(\Omega), \text{ all } n \in \mathbb{N}. \tag{4.32}$$

On account of hypothesis H_1(i)=H_2(i), we have that

$$\left\{\frac{f(\cdot, u_n^+(\cdot))}{\|u_n^+\|^{p-1}}\right\}_{n\in\mathbb{N}} \subseteq L^{p'}(\Omega) \text{ is bounded.} \tag{4.33}$$

So using hypothesis $H_1(\text{ii}) = H_2(\text{ii})$, we may assume that

$$\frac{f(\cdot, u_n^+(\cdot))}{\|u_n^+\|^{p-1}} \xrightarrow{w} \hat{\eta}_0(\cdot)y^{p-1} \tag{4.34}$$

with $\hat{\eta}_0 \in L^\infty(\Omega)$, $\hat{\eta}(z) \leq \hat{\eta}_0(z)$ for a.a. $z \in \Omega$. (see Aizicovici–Papageorgiou–Staicu [1], proof of Proposition 16). In (4.32), we choose the test function $h = y_n - y \in W_0^{1,p}(\Omega)$, pass to the limit as $n \to \infty$ and use (4.30) and (4.33). Then

$$\limsup_{n\to\infty}\int_\Omega a_1(z)|Dy_n|^{p-2}(Dy_n, Dy_n - Dy)_{\mathbb{R}^N}\,dz \leq 0,$$

$$\Rightarrow y_n \to y \quad \text{in } W_0^{1,p}(\Omega), \quad \|y\| = 1, \quad y \geq 0 \tag{4.35}$$

(see Papageorgiou–Winkert [14], p. 665).

If in (4.32), we pass to the limit as $n \to \infty$ and use (4.34), (4.35), we obtain

$$\int_\Omega a_1(z)|Dy|^{p-2}(Dy, Dh)_{\mathbb{R}^N}\,dz = \int_\Omega \hat{\eta}_0(z)y^{p-1}h\,dz \quad \text{for all } h \in W_0^{1,p}(\Omega),$$

$$\Rightarrow -\Delta_p^{a_1}y = \hat{\eta}_0(z)y^{p-1} \quad \text{in } \Omega, \quad y|_{\partial\Omega} = 0. \tag{4.36}$$

Recall that $\hat{\lambda}_1^{a_1}(p) \leq \hat{\eta}(z) \leq \hat{\eta}_0(z)$ for a.a. $z \in \Omega$. Let $\tilde{\lambda}_1^{a_1}(p, \hat{\eta}_0)$ be the principal eigenvalue of $-\Delta_p^{a_1} y = \tilde{\lambda}\hat{\eta}_0(z)|y|^{p-2}y$ in Ω, $y|_{\partial\Omega} = 0$ (weighted version of the eigenvalue problem (2.1). From Proposition 4.133, p. 271, of Hu–Papageorgiou [5] we have

$$\tilde{\lambda}_1^{a_1}(p, \hat{\eta}_0) < \tilde{\lambda}_1^{a_1}(p, \hat{\lambda}_1^{a_1}(p)) = 1.$$

Then from (4.36) it follows that

$$y(\cdot) \text{ is nodal or zero (see Hu–Papageorgiou [5, Proposition 4.127, p. 268]).}$$

Both possibilities contradict (4.35). Therefore

$$\begin{aligned} &\{u_n^+\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega) \text{ is bounded,} \\ \Rightarrow\ &\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega) \text{ is bounded (see (4.29)).} \end{aligned}$$

So, we may assume that

$$u_n \xrightarrow{w} u \quad \text{in } W_0^{1,p}(\Omega), \quad u_n \to u \quad \text{in } L^p(\Omega). \tag{4.37}$$

In (4.28) we choose the test function $h = u_n - u \in W_0^{1,p}(\Omega)$, pass to the limit as $n \to \infty$ and use (4.37). Then

$$\begin{aligned} &\lim_{n\to\infty} \langle V(u_n), u_n - u\rangle = 0 \\ \Rightarrow\ &u_n \to u \quad \text{in } W_0^{1,p}(\Omega) \quad \text{(see Proposition 2.3)}, \\ \Rightarrow\ &\varphi_+(\cdot) \text{ satisfies the C-condition.} \end{aligned}$$

This proves the claim.

Recall that u_0 is a local minimizer of $\varphi_+(\cdot)$ and that K_{φ_+} is finite (see the beginning of the proof). These facts and Proposition 3.132, p. 179, of Hu–Papageorgiou [5], imply that we can find $\rho \in (0,1)$ small such that

$$\varphi_+(u_0) < \inf\{\varphi_+(u) : \|u - u_0\| = \rho\} = m_+. \tag{4.38}$$

Hypotheses H_1(i),(ii) $= H_2$(i),(ii) imply that given $\varepsilon > 0$, we can find $\delta = \delta(\varepsilon) > 0$ such that

$$F(z,x) \geq \frac{1}{p}[\hat{\eta}(z) - \varepsilon]|x|^p \quad \text{for a.a. } z \in \Omega, \text{ all } |x| \leq \delta.$$

For $t \in (0,1)$, we have

$$\begin{aligned} \varphi_+(t\hat{u}_1^{a_1}(p)) &\leq \frac{t^p}{p}\int_\Omega [\hat{\lambda}_1^{a_1}(p) - \eta(z)]\hat{u}_1^{a_1}(p)^p\, dz + \frac{\varepsilon t^p}{p} + \frac{t^q}{t^q}\rho_{a_2,q}(D\hat{u}_1^{a_1}(p)) \\ &\quad \text{(since } \rho_{a_1,p}(D\hat{u}_1^{a_1}(p)) = \hat{\lambda}_1^{a_1}(p)\|\hat{u}_1^{a_1}(p)\|_p^p,\ \|\hat{u}_1^{a_1}(p)\|_p = 1). \end{aligned}$$

Since $\hat{u}_1^{a_1} \in \operatorname{int} C_+$ and the inequality $\hat{\lambda}_1^{a_1}(p) \leq \eta(z)$ for a.a. $z \in \Omega$ is strict on a set of positive Lebesgue measure, it follows that

$$\hat{\beta} = \int_\Omega [\hat{\eta}(z) - \hat{\lambda}_1^{a_1}(p)]\hat{u}_1(p)^p\, dz > 0.$$

Choosing $\varepsilon \in (0, \hat{\beta})$ small, we obtain

$$\varphi_+(t\hat{u}_1^{a_1}(p)) \leq c_7 t^q - c_8 t^p \quad \text{for some } c_7, c_8 > 0.$$

Since $q < p$, we infer that

$$\varphi_+(t\hat{u}_1(p)) \to -\infty \quad \text{as } t \to +\infty. \tag{4.39}$$

Then (4.38), (4.39) and the Claim, permit the use of the mountain pass theorem and obtain $\hat{u} \in W_0^{1,p}(\Omega)$ such that

$$\begin{aligned} &\hat{u} \in K_{\varphi_+} \subseteq \operatorname{int} C_+ \cup \{0\}, \quad m_+ \leq \varphi_+(\hat{u}), \\ \Rightarrow\ &\hat{u} \neq u_0. \end{aligned}$$

If we show that $\hat{u} \neq 0$, then $\hat{u} \in \operatorname{int} C_+$ is the second positive solution of (1.1). First assume that hypotheses H_1' hold. On account of hypothesis H_1'(iii) and of Lemma 5.125, p. 459, of Hu–Papageorgiou [5] we have

$$C_k(\varphi_+, 0) = 0 \quad \text{for all } k \in \mathbb{N}_0. \tag{4.40}$$

On the other hand since $\hat{u} \in K_{\varphi_+}$ is a critical point of mountain pass type, using Corollary 3.125, p. 178, of Hu–Papageorgiou [5], we have

$$C_1(\varphi_+, \hat{u}) \neq 0. \tag{4.41}$$

Comparing (4.40) and (4.41), we conclude that $\hat{u} \neq 0$. So $\hat{u} \in \operatorname{int} C_+$ is the second positive solution of (1.1).

Similarly working this time with the functional $\varphi_-(\cdot)$, we produce a second negative solution $\hat{v} \in -\operatorname{int} C_+$, $\hat{v} \neq v_0$. □

5 Nodal solutions

In this section we prove the existence of nodal solutions (sign-changing solutions) for problem (1.1) and state the complete multiplicity theorem.

First we consider the case where hypotheses H_1' hold.

Proposition 5.1. *If hypotheses H_0, H_1' hold, then problem (1.1) admits at least two nodal solutions*

$$y_0 \in \operatorname{int}_{C_0^1(\overline{\Omega})}[v_0, u_0], \ \hat{y} \in [v_0, u_0] \cap C_0^1(\overline{\Omega}), \ \hat{y} \neq y_0.$$

Proof. Let $u_* \in \operatorname{int} C_+$, $v_* \in -\operatorname{int} C_+$ be the two extremal constant sign solutions produced in Proposition 4.6. We introduce the Carathéodory function defined by

$$\gamma(z,x) = \begin{cases} f(z, v^*(z)) & \text{if } x < v^*(z), \\ f(z,x) & \text{if } v^*(z) \leq x \leq u^*(z), \\ f(z, u^*(z)) & \text{if } u^*(z) < x. \end{cases} \tag{5.1}$$

In addition, we introduce the positive and negative truncations of $\gamma_*(z,\cdot)$, namely the Carathéodory functions

$$\gamma_\pm(z,x) = \gamma(z, \pm x^\pm) \quad \text{for all } z \in \Omega, \text{ all } x \in \mathbb{R}. \tag{5.2}$$

We set $\Gamma(z,x) = \int_0^x \gamma(z,s)\,ds$, $\Gamma_\pm(z,x) = \int_0^x \gamma_\pm(z,s)\,ds$ and consider the C^1-functionals $\mu, \mu_\pm : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\mu(u) = \frac{1}{p}\rho_{a_1,p}(Du) + \frac{1}{q}\rho_{a_2,q}(Du) - \int_\Omega \Gamma(z,u)\,dz,$$
$$\mu_\pm(u) = \frac{1}{p}\rho_{a_1,p}(Du) + \frac{1}{q}\rho_{a_2,q}(Du) - \int_\Omega \Gamma_\pm(z,u)\,dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Using (5.1) and (5.2), we can check easily that

$$K_\mu \subseteq [v_*, u_*] \cap C_0^1(\overline{\Omega}), \qquad K_{\mu_+} \subseteq [0, u_*] \cap C_+, \qquad K_{\mu_-} \subseteq [v_*, 0] \cap (-C_+).$$

The extremality of u^*, v^* implies that

$$K_\mu \subseteq [v_*, u_*] \cap C_0^1(\overline{\Omega}), \qquad K_{\mu_+} = \{0, u^*\}, \qquad K_{\mu_-} = \{0, v_*\}. \tag{5.3}$$

From (5.1) and (5.2) it is clear that $\mu_+(\cdot)$ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $\hat{u}^* \in W_0^{1,p}(\Omega)$ such that

$$\mu_+(\hat{u}^*) = \inf\left\{\mu_+(u) : u \in W_0^{1,p}(\Omega)\right\} < 0 = \mu_+(0),$$
$$\text{(since } \tau < q \text{ and } u^* \in \operatorname{int} C_+)$$
$$\Rightarrow \hat{u}^* \neq 0 \quad \text{and so } \hat{u}^* = u^* \quad \text{(see (5.3))}.$$

From (5.1) and (5.2) it is clear that

$$\mu_+\big|_{C_+} = \mu\big|_{C_+}.$$

Therefore, we have that

$$u^* \in \operatorname{int} C_+ \text{ is a local } C_0^1(\overline{\Omega})\text{-minimizer of } \varphi(\cdot),$$
$$\Rightarrow u^* \in \operatorname{int} C_+ \text{ is a local } W_0^{1,p}(\Omega)\text{-minimizer of } \mu(\cdot).$$

In a similar fashion, using this time the functional $\mu_-(\cdot)$, we show that

$$v^* \in -\operatorname{int} C_+ \text{ is a local } W_0^{1,p}(\Omega)\text{-minimizer of } \mu(\cdot).$$

We may assume that

$$\mu(v^*) \leq \mu(u^*) \quad \text{and} \quad K_\mu \text{ is finite} \tag{5.4}$$

(see (5.3) and note that any solution $y \notin \{0, u^*, v^*\}$ of (1.1) is necessarily nodal). The functional $\mu(\cdot)$ being coercive, satisfies the C-condition (see Hu–Papageorgiou [5], Proposition 3.19, p. 123). So, using Proposition 3.132, p. 179, of Hu–Papageorgiou [5], we can find $\rho \in (0,1)$ small such that

$$\rho < \|u^* - v^*\|, \ \mu(u^*) < \inf\{\mu(u) : \|u - u^*\| = \rho\} = m^*. \tag{5.5}$$

Then (5.4),(5.5) and the C-condition of $\mu(\cdot)$, permit the use of the mountain pass theorem. So, we can find $y_0 \in W_0^{1,p}(\Omega)$ such that

$$y_0 \in K_\mu \in [v^*, u^*] \cap C_0^1(\overline{\Omega}) \text{ (see(5.3))}, \quad m^* \leq \mu(y_0) \text{ (see (5.5))}. \tag{5.6}$$

We know that

$$C_1(\mu, y_0) \neq 0 \quad \text{(see [5, Corollary 3.123, p. 176])}. \tag{5.7}$$

Using the homotopy invariance property of critical groups (see [5, p. 179]) and the nonlinear regularity theory, we obtain

$$C_k(\mu, 0) = C_k(\varphi, 0) \quad \text{for all } k \in \mathbb{N}_0. \tag{5.8}$$

From hypothesis H_1'(iii) and Hu–Papageorgiou [5, Lemma 5.126, p. 462], we have

$$\begin{aligned} & C_k(\varphi, 0) = 0 \quad \text{for all } k \in \mathbb{N}_0, \\ \Rightarrow\ & C_k(\mu, 0) = 0 \quad \text{for all } k \in \mathbb{N}_0 \quad \text{(see (5.8))}. \end{aligned} \tag{5.9}$$

From (5.7) and (5.9), we infer that

$$y_0 \neq 0.$$

Also, $y_0 \notin \{u^*, v^*\}$ (see (5.5), (5.6)). Therefore

$$y_0 \in [v^*, u^*] \cap C_0^1(\overline{\Omega}) \text{ is a nodal solution of (1.1)}.$$

By Theorem 1.3 of Lucia–Prashanth [10] (see also Pucci–Serrin [15, p. 6]), we have

$$y_0(z) < u^*(z) \quad \text{for all } z \in \Omega. \tag{5.10}$$

Let $\rho = \max\{\|v^*\|_\infty, \|u^*\|_\infty\}$ and let $\hat{\xi}_\rho > 0$ be as postulated by hypothesis H_1(v). We take $\hat{\xi}^* > \hat{\xi}_\rho$ and have

$$\begin{aligned} -\Delta_p^{a_1} y_0 - \Delta_q^{a_2} y_0 + \hat{\xi}^* |y_0|^{p-2} y_0 &= f(z, y_0) + \hat{\xi}^* |y_0|^{p-2} y_0 \\ &\leq f(z, u^*) + \hat{\xi}^* (u^*)^{p-1} \\ &= -\Delta_p^{a_1} u^* - \Delta_q^{a_2} u^* + \hat{\xi}^* (u^*)^{p-1} \quad \text{in } \Omega. \end{aligned}$$

Note that

$$\begin{aligned} f(z, u^*) + \hat{\xi}^* (u^*)^{p-1} &= f(z, u^*) + \hat{\xi}_\rho (u^*) + (\hat{\xi}^* - \hat{\xi}_\rho)(u^*)^{p-1} \\ &\geq f(z, y_0) + \hat{\xi}_\rho |y_0|^{p-2} y_0 + (\hat{\xi}^* - \hat{\xi}_\rho)|y_0|^{p-2} y_0 \\ &= f(z, y_0) + \hat{\xi}^* |y_0|^{p-2} y_0 \quad \text{in } \Omega. \end{aligned} \tag{5.11}$$

From hypothesis H_1(v) and (5.10), we see that

$$f(\cdot, y_0(\cdot)) + \hat{\xi}^* |y_0(\cdot)|^{p-2} y_0(\cdot) \prec f(\cdot, u^*(\cdot)) + \hat{\xi}^* (u^*(\cdot))^{p-1}.$$

So, using Preccosition 3.1, from (5.11) we infer that

$$u^* - y_0 \in \operatorname{int} C_+.$$

Similarly we show that

$$y_0 - v^* \in \operatorname{int} C_+.$$

It follows that

$$y_0 \in \operatorname{int}_{C_0^1(\overline{\Omega})} [v^*, u^*]. \tag{5.12}$$

Moreover, using the flow invariance argument from the proof of Proposition 5.1 of Liu–Papageorgiou [8], we generate another solution

$$\hat{y} \in [v^*, u^*] \cap C_0^1(\overline{\Omega}) \setminus \operatorname{int}_{C_0^1(\overline{\Omega})} [v^*, u^*]. \tag{5.13}$$

From (5.12) and (5.13) it follows that

$$\hat{y} \in C_0^1(\overline{\Omega}) \text{ is a nodal solution of (1.1)}, \quad \hat{y} \neq y_0. \qquad \square$$

When hypotheses $H_2'(q=2)$ hold, since the behavior of $f(z,\cdot)$ near zero is different, we need to strengthen the hypotheses on the reaction $f(z,x)$ in order to have nodal solutions.

H_2'': $f:\Omega\times\mathbb{R}\to\mathbb{R}$ is a measurable function such that $f(z,\cdot)\in C^1(\mathbb{R})$ for a.a. $z\in\Omega$, hypotheses H_2''(i), (ii), (iv), (v) are the same as the corresponding hypotheses H_2(i), (ii), (iv), (v) and

(iii) there exists $k\geq 2$ such that

$$f_x'(z,0)\in\left[\hat{\lambda}_k^{a_2}(2),\hat{\lambda}_{k+1}^{a_2}(2)\right] \quad \text{for a.a. } z\in\Omega,$$
$$f_x'(\cdot,0)\not\equiv\hat{\lambda}_k^{a_2}(2),\ f_x'(\cdot,0)\not\equiv\hat{\lambda}_{k+1}^{a_2}(2).$$

Proposition 5.2. *If hypotheses H_0 and $H_2''(q\neq 2)$ hold, then problem (1.1) $(q=2)$ admits at least two nodal solutions*

$$y_0\in \operatorname{int}_{C_0^1(\overline{\Omega})}[v^*,u^*]$$
$$\hat{y}\in[v^*,u^*]\cap C_0^1(\overline{\Omega}),\quad \hat{y}\neq y_0.$$

Proof. The proof is similar to that of Proposition 5.1.

In this case (5.10) follows from the tangency principle (see Pucci–Serrin [16, p. 35]). Since $f(z,\cdot)$ exhibits a linear behavior near zero (see hypothesis H_2''(iii)) and so there is no resonance, we can not claim that $C_k(\mu,0)=0$ for all $k\in\mathbb{N}_0$. So, to prove the nontriviality of y_0, we have to use a different argument.

Let $\hat{l}:H_0^1(\Omega)\to\mathbb{R}$ be the C^2-functional defined by

$$\hat{l}(u)=\frac{1}{2}\rho_{a_2,2}(Du)-\int_\Omega F(z,u)\,dz \quad \text{for all } u\in H_0^1(\Omega).$$

We consider the following orthogonal direct sum decomposition

$$H_0^1(\Omega)=Y\oplus V$$

with $Y=\bigoplus_{i=1}^k E(\hat{\lambda}_i^{a_2}(2))$, $V=\overline{\bigoplus_{i\geq k}E(\hat{\lambda}_i^{a_2}(2))}$.

Hypotheses H_2''(i), (iii), imply that given $\varepsilon>0$, we can find $c_{10}=c_{10}(\varepsilon)>0$ such that

$$F(z,x)\leq\frac{1}{2}[f_x'(z,0)+\varepsilon]x^2+c_{10}|x|^p \tag{5.14}$$

If $u\in V$, then using (5.14), we have

$$\begin{aligned}\hat{l}(u)&\geq\frac{1}{2}\rho_{a_2,2}(Du)-\frac{1}{2}\int_\Omega f_x'(z,0)u^2\,dz-\frac{\varepsilon}{2\hat{\lambda}_1^{a_2}(2)}\rho_{a_2,2}(Du)-c_{11}\|u\|^p \quad \text{for some } c_{11}>0\\ &\geq\frac{1}{2}\left[c_{12}-\varepsilon\right]\|u\|^2-c_{11}\|u\|^p \quad \text{for some } c_{12}>0 \text{ (see Proposition 2.2)}\end{aligned}$$

Choosing $\varepsilon\in(0,c_{12})$, we obtain

$$\hat{l}(u)\geq c_{13}\|u\|^2-c_{11}\|u\|^p \quad \text{for some } c_{13}>0.$$

Since $2<p$, for $\rho\in(0,1)$ small we can say that

$$\hat{l}(u)\geq 0 \quad \text{for all } u\in V, \text{ with } \|u\|\leq\rho. \tag{5.15}$$

On the other hand, again from hypothesis H_2''(iii), we see that given $\varepsilon > 0$, we can find $\delta = \delta(\varepsilon) > 0$ such that

$$F(z,x) \geq \frac{1}{2}[f_x'(z,0) - \varepsilon]x^2 \quad \text{for a.a. } z \in \Omega, \text{ all } |x| \leq \delta. \tag{5.16}$$

The space Y is finite dimensional and so all norms are equivalent (see Papageorgiou–Winkert [14], p. 183). Therefore we can find $\hat{\rho} > 0$ such that

$$u \in Y,\ \|u\| \leq \hat{\rho} \Rightarrow |u(z)| \leq \delta \quad \text{for all } z \in \Omega. \tag{5.17}$$

Therefore if $u \in Y$ with $\|u\| \leq \hat{\rho}$, then

$$\begin{aligned}\hat{l}(u) &\leq \frac{1}{2}\rho_{a_2,2}(Du) - \frac{1}{2}\int_\Omega [f_x'(z,0) - \varepsilon]u^2\,dz \quad \text{(see (5.16), (5.17))}\\ &\leq \frac{1}{2}\left[\varepsilon - c_{14}\right]\|u\|^2 \quad \text{for some } c_{14} > 0 \quad \text{(see Proposition 2.2)}\end{aligned}$$

Choosing $\varepsilon \in (0, c_{14})$, we conclude that

$$\hat{l}(u) \leq 0 \quad \text{for all } u \in Y,\ \|u\| \leq \hat{\rho}. \tag{5.18}$$

From (5.15) and (5.18), we infer that $\hat{l}(\cdot)$ has a local linking at the origin with respect to the orthogonal decomposition (Y, V) (see [5, p. 145]). If $d_k = \dim Y \geq 2$ (recall $k \geq 2$), then

$$C_m(\hat{l}, 0) = \delta_{m,d_k}\mathbb{R} \quad \text{for all } m \in \mathbb{N}_0 \quad \text{(see [12], p. 539)}. \tag{5.19}$$

Let $l = \hat{l}|_{W_0^{1,p}(\Omega)}$ (recall $W_0^{1,p}(\Omega) \hookrightarrow H_0^1(\Omega)$ densely since $p > 2$). So by Proposition 3.128, p. 178, of Hu–Papageorgiou [5], we have

$$\begin{aligned}&C_m(l,0) = C_m(\hat{l},0) \quad \text{for all } m \in \mathbb{N}_0,\\ \Rightarrow\ &C_m(l,0) = \delta_{m,d_k}\mathbb{R} \quad \text{for all } m \in \mathbb{N}_0 \quad \text{(see (5.17))}.\end{aligned} \tag{5.20}$$

From the C^1-continuity property of critical groups (see Theorem 3.129, p. 179, of Hu–Papageorgiou [5]), we have

$$\begin{aligned}&C_m(\psi,0) = C_m(l,0) \quad \text{for all } m \in \mathbb{N}_0.\\ \Rightarrow\ &C_m(\psi,0) = \delta_{m,d_k}\mathbb{R} \quad \text{for all } m \in \mathbb{N}_0 \quad \text{(see (5.20))}.\end{aligned}$$

Recall via the homotopy invariance property of critical groups (see [5, p. 179]), we have

$$\begin{aligned}&C_m(\mu,0) = C_m(\psi,0)\mathbb{R} \quad \text{for all } m \in \mathbb{N}_0,\\ \Rightarrow\ &C_m(\mu,0) = \delta_{m,d_k}\mathbb{R} \quad \text{for all } m \in \mathbb{N}_0\ (d_k \geq 2).\end{aligned} \tag{5.21}$$

We know that $C_1(\mu, y_0) \neq 0$. Comparing with (5.21) we deduce that

$$\begin{aligned}&y_0 \neq 0,\\ \Rightarrow\ &y_0 \in C_0^1(\overline{\Omega}) \text{ is a nodal solution of (1.1) } (q = 2).\end{aligned}$$

Since $y_0(z) < u^*(z)$ for all $z \in \Omega$, as in the proof of Proposition 5.1, we show that

$$y_0 \in \operatorname{int}_{C_0^1(\overline{\Omega})}[v^*, u^*].$$

Finally, the flow invariance argument of Liu–Papageorgiou [8], generates $\hat{y} \in [v^*, u^*] \cap C_0^1(\overline{\Omega})$ another nodal solution of problem (1.1) ($q = 2$), $\hat{y} \neq y_0$. □

We conclude with the following two multiplicity theorems for

Theorem 5.3. *If hypotheses H_0 and H_1' hold, then problem (1.1) admits at least six nontrivial smooth solutions:*

$$\begin{aligned} &u_0, \hat{u} \in \operatorname{int} C_+, \quad u_0 \leq \hat{u}, \quad u_0(z) < \theta_+ \quad \text{for all } z \in \overline{\Omega}, \\ &v_0, \hat{v} \in -\operatorname{int} C_+, \quad \hat{v} \leq v_0, \quad \theta_- < v_0(z) \quad \text{for all } z \in \overline{\Omega}, \\ &y_0 \in \operatorname{int}_{C_0^1(\overline{\Omega})}[v_0, u_0], \quad \hat{y} \in [v_0, u_0] \cap C_0^1(\overline{\Omega}), \quad y_0 \neq \hat{y} \text{ both nodal.} \end{aligned}$$

Theorem 5.4. *If hypotheses H_0 and H_2' ($q = 2$) hold, then problem (1.1) ($q = 2$) admits at least six nontrivial smooth solutions*

$$\begin{aligned} &u_0, \hat{u} \in \operatorname{int} C_+, \quad u_0 \leq \hat{u}, \quad u_0(z) < \theta_+ \quad \text{for all } z \in \overline{\Omega}, \\ &v_0, \hat{v} \in -\operatorname{int} C_+, \quad \hat{v} \leq v_0, \quad \theta_- < v_0(z) \quad \text{for all } z \in \overline{\Omega}, \\ &y_0 \in \operatorname{int}_{C_0^1(\overline{\Omega})}[v_0, u_0], \quad \hat{y} \in [v_0, u_0] \cap C_0^1(\overline{\Omega}), \quad y_0 \neq \hat{y} \text{ both nodal.} \end{aligned}$$

Remark 5.5. We point out that in the above multiplicity theorems, we provide sign information for all the solutions produced. Moreover, the solutions $\{u_0, \hat{u}, v_0, \hat{v}, y_0\}$ and $\{u_0, \hat{u}, v_0, \hat{v}, y_0\}$ are ordered.

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On a critical Choquard-type equation with vanishing potential in the Heisenberg group

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Abstract. In this article, we prove the existence of a weak solution to the sub-elliptic problem

$$-\Delta_{\mathbb{H}} u + V(\xi)u = \left(\int_{\mathbb{H}^N} \frac{|u(\eta)|^{Q_\alpha^*} + K(\eta)|u(\eta)|^p}{|\eta^{-1}\circ\xi|^\alpha} d\eta\right)\left(Q_\alpha^*|u|^{Q_\alpha^*-2}u + pK(\xi)|u|^{p-2}u\right)$$

in the Heisenberg group $\mathbb{H}^N$, where $0<\alpha<4$, $Q=2N+2$, $Q_\alpha^*=\frac{2Q-\alpha}{Q-2}$, $p\in(2,Q_\alpha^*)$, V is a non-negative continuous function that can vanish at infinity and K is continuous non-negative bounded function. We establish the existence of a weak solution by employing the penalization method in conjunction with the mountain pass theorem.

Keywords: vanishing potential, critical Choquard type non-linearity, Heisenberg group, penalization method.

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1 Introduction

In this paper, we study the existence of a weak solution to the problem:

$$-\Delta_{\mathbb{H}} u + V(\xi)u = \left(\int_{\mathbb{H}^N} \frac{|u(\eta)|^{Q_\alpha^*} + K(\eta)|u(\eta)|^p}{|\eta^{-1}\circ\xi|^\alpha} d\eta\right)\left(Q_\alpha^*|u|^{Q_\alpha^*-2}u + pK(\xi)|u|^{p-2}u\right) \tag{1.1}$$

in the Heisenberg group $\mathbb{H}^N$, where $0<\alpha<4$, $Q=2N+2$, $Q_\alpha^*=\frac{2Q-\alpha}{Q-2}$, $p\in(2,Q_\alpha^*)$, V is a non-negative continuous function that can vanish at infinity and K is continuous non-negative bounded function.

The Heisenberg group has found significant attraction due to its rich geometry, leading to various applications in the field of partial differential equations. In one of the pioneering works, Garofalo and Lanconelli [20] obtained the existence of a solution for a semilinear

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subelliptic PDE in the Heisenberg group via a variational technique. Since then, a wide range of results have been obtained, see [7,8,13,21,25,36,39] and references therein.

The equation (1.1) is analogous to the well-known steady state Schrödinger equation

$$\Delta\Psi + V(x)\Psi = (Z(x) * |\Psi|^q)|\Psi|^{q-2}\Psi \quad \text{in } \mathbb{R}^N, \tag{1.2}$$

where V is the potential, and Z is the response function describing the mutual interaction between the bosons. When $Z(x) = |x|^{-1}$, (1.2) reduces to the Choquard equation. A typical model, corresponding to $q = 2$, is

$$-\Delta\Psi + \Psi = \left(\frac{1}{|x|} * |\Psi|^2\right)\Psi \quad \text{in } \mathbb{R}^N,$$

which was originally introduced in the context of polaron models [19,33]. For a comprehensive overview of Choquard-type problems in $\mathbb{R}^N$, see [31].

On setting the response function $Z(x)$ to be the Dirac-delta function in (1.2), we obtain the Schrödinger equation in the following form:

$$-\Delta u + V(x)u = f(x,u) \quad \text{in } \mathbb{R}^N, \tag{1.3}$$

where $u > 0, V$ is a continuous function and f is a non-linear function. The function V is commonly referred to as the potential function. Equation (1.3) has been extensively studied under various assumptions on V and f. For bounded, coercive potentials V, see [15,30,34,36]. For vanishing potentials ($V(x) \to 0$ as $|x| \to \infty$), notable contributions include [3–5,24,29]. Other situations such as constant, periodic, asymptotically constant, asymptotically periodic, and radial potentials have also been addressed (see [1,2,10,32,40]). For the vanishing potential case Alves–Souto [4], Alves–Figueiredo-Yang [3] employed a penalization technique inspired by Del Pino–Felmer [16].

In recent years, considerable progress has been made on critical Choquard-type problems in the Heisenberg group. Goel and Sreenadh [21] established a Brezis–Nirenberg type result for Choquard equations in $\Omega \subset \mathbb{H}^N$. Sun et al. [36] developed a concentration-compactness lemma for the Choquard equation and applied it to obtain weak solutions of a Kirchhoff–Choquard problem with the critical Hardy–Littlewood–Sobolev exponent. Subsequent works extended and refined these results: Bai et al. [7] proved existence of weak solutions to Kirchhoff–Choquard equations with the same critical exponent in bounded domains, recovering compactness through suitable conditions on the Kirchhoff term, while Sun et al. [36] investigated Kirchhoff-–Choquard systems in bounded domains of $\mathbb{H}^N$. Yang et al. [38] studied (p,q)-type Choquard equations in $\mathbb{H}^N$, focusing on cases where the Choquard term is subcritical. Liang et al. [26] extended the concentration-compactness approach of [36] to the p-sub-Laplacian framework. Further contributions include Bai et al. [6], who considered critical Choquard-type problems with lower-order perturbation terms that may be sublinear or superlinear. Bai et al. [9], who established results for fractional p-sub-Laplacian equations with critical Choquard non-linearities. Most recently, Liang et al. [27] proved the existence of normalized solutions for critical Choquard-type equations with logarithmic perturbations in bounded domains of $\mathbb{H}^N$.

From the above survey, we conclude, to the best of our knowledge, that there are no existing works addressing problems involving vanishing potentials in the Heisenberg group. Moreover, we are not aware of any research in the Heisenberg group that employs the penalization technique. Although Li et al. [24] studied a problem similar to (1.1) in $\mathbb{R}^N$ for the

Laplacian, their approach did not involve the penalization method. Furthermore, to the best of our knowledge, Choquard-type problems have not been investigated via the penalization technique either in $\mathbb{R}^N$ or in $\mathbb{H}^N$. In this article, we obtain an existence result for (1.1) by employing the penalization method, inspired by the works of Alves et al. [3,4].

Before stating our main result, we specify our assumptions on V and K :

(Σ_1) $V : \mathbb{H}^N \to \mathbb{R}$, is continuous, non-negative, $V \in L^\infty(\mathbb{H}^N) \cap L^{\frac{Q}{2}}(\mathbb{H}^N)$ and there exists $\mu > 0, \Lambda > 0$ and $R > 0$ such that

$$\inf_{|\xi| \geq R} |\xi|^\mu V(\xi) \geq \Lambda.$$

(Σ_2) $K : \mathbb{H}^N \to \mathbb{R}$ is continuous, non-negative, bounded and $K \in L^{\frac{Q^*}{Q^*_\alpha - p}}$, where $Q^* = \frac{2Q}{Q-2}$.

From (Σ_1), we have $V \in L^{\frac{2Q}{4-\alpha}}(\mathbb{H}^N)$ by interpolation of the Lebesgue spaces $L^\infty(\mathbb{H}^N)$ and $L^{\frac{Q}{2}}(\mathbb{H}^N)$.

Next, we present and example of V and K satisfying (Σ_1) and (Σ_2), respectively.

Example 1.1. Let $R = 1$. Define

$$V_1(\xi) = \begin{cases} \dfrac{1}{|\xi|^a}, & \xi \in \mathbb{H}^N \setminus B_1(0),\ a > 2, \\ 1, & \xi \in B_1(0). \end{cases}$$

Then V_1 satisfies (Σ_1) for any $\mu > a$ with $\Lambda = 1$.

Examples for K.

(i) Define

$$K_1(\xi) = \begin{cases} \dfrac{1}{|\xi|^a}, & \xi \in \mathbb{H}^N \setminus B_1(0),\ a > \dfrac{Q(Q^*_\alpha - p)}{Q^*}, \\ 1, & \xi \in B_1(0), \end{cases}$$

where

$$p \in \left(2, \frac{2Q-\alpha}{Q-2}\right).$$

Then, K_1 satisfies (Σ_2).

(ii) As a different example, consider

$$K_2(\xi) = e^{-|\xi|^4}.$$

Then K_2 also satisfies (Σ_2).

Now, we state the main result of this paper:

Theorem 1.2. *Let the conditions (Σ_1), (Σ_2) hold. Then, there exists a constant γ such that (1.1) admits a positive weak solution $u \in E$ (see Section 2 for the definition of E) for any $\Lambda \geq \gamma R^\mu$.*

We have organized the paper as follows. Section 2 gives the preliminaries for the Heisenberg group and some primary results on the Heisenberg group. Further, Section 3 converts the main problem into an auxiliary problem and describes its variational framework. In the last section, we prove Theorem 1.2.

2 Preliminaries

We recall some terminologies and definitions related to the Heisenberg group $\mathbb{H}^N = \left(\mathbb{R}^{2N+1}, \circ\right)$, where '$\circ$' denotes the group operation defined as

$$\xi \circ \xi' = (x + x', y + y', t + t' + 2(x' \cdot y - y' \cdot x)), \quad \text{for every } \xi = (x, y, t), \xi' = (x', y', t') \in \mathbb{H}^N,$$

where $x, y, x', y' \in \mathbb{R}^N, t, t' \in \mathbb{R}$. $\xi^{-1} = -\xi$ is the inverse, and therefore $(\xi')^{-1} \circ \xi^{-1} = (\xi \circ \xi')^{-1}$.

The natural group of dilations on $\mathbb{H}^N$ is defined as $\delta_s(\xi) = (sx, sy, s^2 t)$, for every $s > 0$. Hence, $\delta_s(\xi' \circ \xi) = \delta_s(\xi') \circ \delta_s(\xi)$ and $\delta_s(\delta_{s'}(\xi)) = \delta_{ss'}(\xi)$, for $s, s' > 0$. It can be easily proved that the Jacobian determinant of dilations $\delta_s : \mathbb{H}^N \to \mathbb{H}^N$ is constant and equal to s^Q, for every $\xi = (x, y, t) \in \mathbb{H}^N$. The natural number $Q = 2N + 2$ is called the homogeneous dimension of $\mathbb{H}^N$. The homogeneous norm on $\mathbb{H}^N$ is defined as follows

$$|\xi| = |\xi|_{\mathbb{H}} = \left[(|x|^2 + |y|^2)^2 + t^2\right]^{\frac{1}{4}}, \quad \text{for every } \xi \in \mathbb{H}^N.$$

By definition, the homogeneous degree of the norm is 1, in terms of dilations. The following vector fields

$$T = \frac{\partial}{\partial t}, \quad X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}, \quad j = 1, \ldots, N,$$

generate the real Lie algebra of left-invariant vector fields. The vector fields, T, X_j, Y_j satisfy the following relations:

$$[X_j, Y_k] = -4\delta_{jk} T, \qquad [Y_j, Y_k] = [X_j, X_k] = [Y_j, T] = [X_j, T] = 0.$$

The Heisenberg gradient on $\mathbb{H}^N$ is given by

$$\nabla_{\mathbb{H}} = (X_1, X_2, \cdots, X_N, Y_1, Y_2, \cdots, Y_N),$$

and the Kohn-Laplacian on $\mathbb{H}^N$ is given by

$$\Delta_{\mathbb{H}} = \sum_{j=1}^{N} X_j^2 + Y_j^2 = \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2} + 4y_j \frac{\partial^2}{\partial x_j \partial t} - 4x_j \frac{\partial^2}{\partial x_j \partial t} + 4(x_j^2 + y_j^2) \frac{\partial^2}{\partial t^2}.$$

The Haar measure on $\mathbb{H}^N$ coincides with the Lebesgue measure and is Q-homogeneous with respect to dilations. More precisely, it is consistent with the $(2N+1)$-dimensional Lebesgue measure. Consequently, the topological dimension of $\mathbb{H}^N$ is $2N + 1$, which is strictly smaller than its Hausdorff dimension $Q = 2N + 2$.

For a measurable set $\Omega \subseteq \mathbb{H}^N$, we denote by $|\Omega|$ its $(2N+1)$-dimensional Lebesgue measure. Then, for every $s > 0$, we have

$$|\delta_s(\Omega)| = s^Q |\Omega|, \quad d(\delta_s(\xi)) = s^Q \, d\xi, \quad \text{and} \quad |B_r(\xi)| = \alpha_Q r^Q,$$

where $\alpha_Q = |B_1(0)|$. Here, $B_r(\xi)$ denotes the ball in $\mathbb{H}^N$ centered at ξ with radius r.

The horizontal Sobolev space is defined as

$$HW^{1,2}(\mathbb{H}^N) = \left\{ u \in L^2(\mathbb{H}^N) \ : \ X_j u, \, Y_j u \in L^2(\mathbb{H}^N), \ j = 1, \ldots, N \right\}.$$

It is a Hilbert space equipped with the inner product

$$\langle u, v \rangle = \int_{\mathbb{H}^N} \nabla_{\mathbb{H}} u(\xi) \cdot \nabla_{\mathbb{H}} v(\xi) \, d\xi + \int_{\mathbb{H}^N} u(\xi) \, v(\xi) \, d\xi,$$

which induces the norm

$$\|u\|_{HW^{1,2}(\mathbb{H}^N)} = \left(\int_{\mathbb{H}^N} \left| \nabla_{\mathbb{H}} u(\xi) \right|^2 d\xi + \int_{\mathbb{H}^N} |u(\xi)|^2 \, d\xi \right)^{\frac{1}{2}}.$$

We define

$$E = \left\{ u \in S^{1,2}(\mathbb{H}^N) : \int_{\mathbb{H}^N} V(\xi) \, |u(\xi)|^2 \, d\xi < \infty \right\},$$

where $S^{1,2}(\mathbb{H}^N)$ denotes the completion of $C_0^\infty(\mathbb{H}^N)$ with respect to the norm

$$\|u\|_{S^{1,2}(\mathbb{H}^N)} = \left(\int_{\mathbb{H}^N} |\nabla_{\mathbb{H}} u(\xi)|^2 \, d\xi \right)^{\frac{1}{2}}.$$

The space E is a Hilbert space endowed with the inner product

$$\langle u, v \rangle = \int_{\mathbb{H}^N} \left(\nabla_{\mathbb{H}} u(\xi) \cdot \nabla_{\mathbb{H}} v(\xi) + V(\xi) \, u(\xi) \, v(\xi) \right) d\xi,$$

which induces the norm

$$\|u\| = \left(\int_{\mathbb{H}^N} \left(|\nabla_{\mathbb{H}} u(\xi)|^2 + V(\xi) |u(\xi)|^2 \right) d\xi \right)^{1/2}.$$

The continuous embedding

$$E \subset S^{1,2}(\mathbb{H}^N) \hookrightarrow L^{Q^*}(\mathbb{H}^N)$$

holds (see [22]); that is,

$$\|u\|_{L^{Q^*}(\mathbb{H}^N)} \leq C \|u\|, \quad u \in E.$$

Since $S^{1,2}(\mathbb{H}^N) \hookrightarrow L^{Q^*}(\mathbb{H}^N)$, we have

$$S \, \|u\|^2_{L^{Q^*}(\mathbb{H}^N)} \leq \int_{\mathbb{H}^N} |\nabla_{\mathbb{H}} u(\xi)|^2 \, d\xi,$$

where S is the best Sobolev constant, defined by

$$S = \inf_{S^{1,2}(\mathbb{H}^N) \setminus \{0\}} \frac{\int_{\mathbb{H}^N} |\nabla_{\mathbb{H}} u(\xi)|^2 \, d\xi}{\left(\int_{\mathbb{H}^N} |u|^{Q^*} \, d\xi \right)^{\frac{2}{Q^*}}}.$$

It was proved by Jerison and Lee [23] that S is attained by the function

$$U(\xi) = U(x, y, t) = \frac{C}{\left(t^2 + \left(1 + |x|^2 + |y|^2 \right)^2 \right)^{\frac{Q-2}{4}}}, \tag{2.1}$$

(up to dilation and translation), where $C > 0$ is a suitable constant.

Now, we state the Hardy–Littlewood–Sobolev inequality.

Lemma 2.1. *Let $r, s > 1$ and $0 < \alpha < Q$ satisfy*

$$\frac{1}{r} + \frac{\alpha}{Q} + \frac{1}{s} = 2.$$

If $f \in L^r(\mathbb{H}^N)$ and $h \in L^s(\mathbb{H}^N)$, then there exists a sharp constant $C(r,s,\alpha,Q) > 0$, independent of f and h, such that

$$\int_{\mathbb{H}^N}\int_{\mathbb{H}^N} \frac{f(\xi)\,h(\eta)}{|\eta^{-1}\circ\xi|^{\alpha}}\,d\eta\,d\xi \;\leq\; C(r,s,\alpha,Q)\,\|f\|_{L^r(\mathbb{H}^N)}\,\|h\|_{L^s(\mathbb{H}^N)}. \tag{2.2}$$

In particular, if $r = s = \frac{2Q}{2Q-\alpha}$, then the sharp constant is given by

$$C(r,s,\alpha,Q) = C(Q,\alpha) = \left(\frac{\pi^{N+1}}{2^{N-1}N!}\right)^{\frac{\alpha}{Q}} \frac{N!\,\Gamma\left(\frac{Q-\alpha}{2}\right)}{\Gamma^2\left(\frac{2Q-\alpha}{2}\right)}, \tag{2.3}$$

where Γ denotes the Gamma function.

Frank and Lieb [18] proved that the equality holds in (2.2) if and only if $f, h \in L^{\frac{2Q}{2Q-\alpha}}(\mathbb{H}^N)$ defined as $f(\xi) = cW\left(\delta_\theta\left(\eta^{-1}\circ\xi\right)\right)$, $h(\xi) = c'W\left(\delta_\theta\left(\eta^{-1}\circ\xi\right)\right)$, where $c \in \mathbb{C}, \theta > 0$, and $\eta \in \mathbb{H}^N$(unless $f \equiv 0$ or $g \equiv 0$) and

$$W(\xi) = W(x,y,t) = \left(t^2 + \left(1+|x|^2+|y|^2\right)^2\right)^{-\frac{2Q-\alpha}{4}}, \quad \text{for all } \xi = (x,y,t) \in \mathbb{H}^N.$$

S_{HG} is defined as

$$S_{HG} = \inf_{S^{1,2}(\mathbb{H}^N)\setminus\{0\}} \frac{\int_{\mathbb{H}^N} |\nabla_{\mathbb{H}} u(\xi)|^2 d\xi}{\left(\int_{\mathbb{H}^N}\int_{\mathbb{H}^N} \frac{|u(\eta)|^{Q_\alpha^*}|u(\xi)|^{Q_\alpha^*}}{|\eta^{-1}\circ\xi|^{\alpha}} d\eta d\xi\right)^{\frac{1}{Q_\alpha^*}}}.$$

Goel and Sreenadh [21] proved that S_{HG} is attained by the function U (defined in (2.1)), up to the translation and dilation.(cf [21, Lemma 2.1]) and

$$S_{HG} = S\left(C(Q,\alpha)^{-\frac{1}{Q_\alpha^*}}\right),$$

where $C(Q,\alpha)$ is defined in (2.3).

To deal with the lack of compactness caused by the term with critical exponent, we use the Concentration-Compactness Principle, which was given by Lions [28] in the Euclidean framework and by Ivanov et al. [22] in the Carnot groups setting. We state the lemma in the Heisenberg group setting.

Lemma 2.2 ([22, Lemma 1.4.5]). *Let $\{u_n\}$ be a bounded sequence in $S^{1,2}(\mathbb{H}^N)$ converging weakly and a.e. to some $u \in S^{1,2}(\mathbb{H}^N)$. $|\nabla_{\mathbb{H}} u_n|^2 \rightharpoonup \omega$, $|u_n|^{Q^*} \rightharpoonup \zeta$ weakly in the sense of measures where ω and ζ are bounded non-negative measures on $\mathbb{H}^N$. Then we have:*

(1) there exists some at most countable set I, a family $\{z_i : i \in I\}$ of distinct points in $\mathbb{H}^N$, and a family $\{\zeta_i : i \in I\}$ of positive numbers such that

$$\zeta = |u|^{Q^*} + \sum_{i\in I} \zeta_i \delta_{z_i},$$

where $\delta_{\tilde{\xi}}$ is the Dirac-mass of mass 1 concentrated at $\xi \in \mathbb{H}^N$.

(2) In addition, we have

$$\omega \geq |\nabla_{\mathbb{H}} u|^2 + \sum_{i\in I} \omega_i \delta_{z_i}$$

for some family $\{\omega_i : i \in I\}$, $\omega_i > 0$ *satisfying*

$$S\zeta_i^{\frac{2}{Q^*}} \leq \omega_i, \quad \text{for all } i \in I.$$

In particular, $\sum_{i\in I} \zeta_i^{\frac{2}{Q^*}} < \infty$.

We next recall the Concentration–Compactness Principle to handle Choquard-type problems in the Heisenberg group:

Lemma 2.3 ([36, Theorem 3.1]). *Let* $\{u_n\}$ *be a bounded sequence in* $S^{1,2}(\mathbb{H}^N)$ *converging weakly and a.e. to some* u *and* ω, ζ *be the bounded nonnegative measures in Lemma 2.2. Assume that*

$$\int_{\mathbb{H}^N} \left(\frac{|u_n(\eta)|^{Q_\alpha^*}}{|\eta^{-1}\circ\xi|^\alpha} d\eta \right) |u_n(\xi)|^{Q_\alpha^*} \rightharpoonup \nu$$

weakly in the sense of measure where ν *is a bounded positive measure on* $\mathbb{H}^N$. *Then, there exists a countable sequence of points* $\{z_i\}_{i\in I} \subset \mathbb{H}^N$ *and families of positive numbers* $\{\nu_i : i \in I\}$, $\{\zeta_i : i \in I\}$ *and* $\{\omega_i : i \in I\}$ *such that*

$$\nu = \left(\int_{\mathbb{R}^N} \frac{|u(\eta)|^{Q_\alpha^*}}{|\eta^{-1}\circ\xi|^\alpha} d\eta \right) |u(\xi)|^{Q_\alpha^*} + \sum_{i\in I} \nu_i \delta_{z_i}, \qquad \sum_{i\in I} \nu_i^{\frac{1}{Q_\alpha^*}} < \infty,$$

$$\omega \geq |\nabla_{\mathbb{H}} u|^2 + \sum_{i\in I} \omega_i \delta_{z_i},$$

$$\zeta \geq |u|^{Q^*} + \sum_{i\in I} \zeta_i \delta_{z_i},$$

and

$$S_{HG}\nu_i^{\frac{1}{Q_\alpha^*}} \leq \omega_i, \quad \nu_i^{\frac{Q}{2Q-\alpha}} \leq C(Q,\alpha)^{\frac{Q}{2Q-\alpha}} \zeta_i,$$

where $\delta_{\bar{\xi}}$ *is the Dirac-mass of mass 1 concentrated at* $\bar{\xi} \in \mathbb{H}^N$.

Lemma 2.4 ([21, Lemma 2.5]). *Let* $0 < \alpha < Q$. *If* $\{u_n\}$ *is a bounded sequence in* $L^{\frac{2Q}{Q-2}}(\mathbb{H}^N)$ *such that* $u_n \to u$ *a.e. in* $\mathbb{H}^N$ *as* $n \to \infty$. *Then as* $n \to \infty$, *the following holds:*

$$\int_{\mathbb{H}^N} \left(|\xi|^{-\alpha} * |u_n|^{Q_\alpha^*} \right) |u_n|^{Q_\alpha^*} d\xi - \int_{\mathbb{H}^N} \left(|\xi|^{-\alpha} * |u_n - u|^{Q_\alpha^*} \right) |u_n - u|^{Q_\alpha^*} d\xi \\ \to \int_{\mathbb{H}^N} \left(|\xi|^{-\alpha} * |u|^{Q_\alpha^*} \right) |u|^{Q_\alpha^*} d\xi.$$

By following the proof of [11, Proposition 2.2], we get the following embedding result:

Lemma 2.5. *Let* $g \in L^m(\mathbb{H}^N)$, *where* $m = \frac{Q^*}{Q^*-q}, q \in [1, Q^*)$. *Then* $S^{1,2}(\mathbb{H}^N) \hookrightarrow L^q(\mathbb{H}^N, |g|)$ *compactly, where* $L^q(\mathbb{H}^N, |g|) = \{u \text{ is a measurable function} \mid \int_{\mathbb{H}^N} g|u(\xi)|^q d\xi < \infty\}$.

3 Auxiliary problem and variational framework

In this section, adapting the arguments in [3, 29], we study the existence of a weak solution to an auxiliary problem associated with (1.1). Throughout this and the following sections, we use C to denote a generic constant, whose value may change from line to line.

Let $f(\xi,u) = Q_\alpha^*|u(\xi)|^{Q_\alpha^*-2}u(\xi) + pK(\xi)|u(\xi)|^{p-2}u(\xi)$. We penalize the left hand side of (1.1)

Define, for $\ell > 1$ and $R > 1$,

$$\tilde{f}(\xi,t) = \begin{cases} f(\xi,t) & \text{if } \ell f(\xi,t) \le V(\xi)t, \\ \frac{V(\xi)}{\ell}t & \text{if } \ell f(\xi,t) > V(\xi)t, \end{cases} \tag{3.1}$$

and

$$g(\xi,t) = \begin{cases} f(\xi,t) & \text{if } |\xi| \le R, \\ \tilde{f}(\xi,t) & \text{if } |\xi| > R. \end{cases} \tag{3.2}$$

From (3.1) and (3.2), it follows that

$$\begin{aligned} &\tilde{f}(\xi,t) \le f(\xi,t), && \forall\, \xi \in \mathbb{H}^N, \\ &g(\xi,t) \le \frac{V(\xi)}{\ell}\,t, && |\xi| \ge R, \\ &G(\xi,t) = F(\xi,t), && |\xi| \le R, \\ &G(\xi,t) \le \frac{V(\xi)}{2\ell}\,t^2, && |\xi| \ge R, \end{aligned} \tag{3.3}$$

where $F(\xi,t) = \int_0^t f(\xi,s)ds$, and $G(\xi,t) = \int_0^t g(\xi,s)ds$.

We consider the following auxiliary problem associated with (1.1):

$$-\Delta_{\mathbb{H}}u(\xi) + V(\xi)u(\xi) = \left(\int_{\mathbb{H}^N} \frac{G(\eta,u(\eta))}{|\eta^{-1}\circ\xi|^\alpha}d\eta\right) g(\xi,u(\xi)) \quad \text{in } \mathbb{H}^N. \tag{3.4}$$

The energy functional $J : E \to \mathbb{R}$ associated with (3.4) is given by

$$J(u) = \frac{1}{2}\|u\|^2 - \frac{1}{2}\int_{\mathbb{H}^N}\left(\int_{\mathbb{H}^N} \frac{G(\eta,u(\eta))}{|\eta^{-1}\circ\xi|^\alpha}d\eta\right) G(\xi,u(\xi))d\xi. \tag{3.5}$$

One can verify that J is a C^1-functional and its derivative is given by

$$\begin{aligned} \langle J'(u), v\rangle = &\int_{\mathbb{H}^N} \left(\nabla_{\mathbb{H}}u(\xi)\cdot\nabla_{\mathbb{H}}v(\xi) + V(\xi)u(\xi)v(\xi)\right) d\xi \\ &- \int_{\mathbb{H}^N}\left(\int_{\mathbb{H}^N} \frac{G(\eta,u(\eta))}{|\eta^{-1}\circ\xi|^\alpha}d\eta\right) g(\xi,u(\xi))v(\xi)d\xi. \end{aligned}$$

We now present the result that constitutes the core of this paper.

Theorem 3.1. *Let the conditions (Σ_1), (Σ_2) hold. Then, (3.4) admits a positive weak solution $u \in E$.*

To prove Theorem 3.1, we first prove a series of lemmas.

Lemma 3.2. *The functional J satisfies the following properties:*

(i) there exist $\beta, r > 0$ such that $J(u) \geq \beta$ whenever $\|u\| = r$;

(ii) there exists $e \in E$ with $\|e\| > \rho$ (for some $\rho > 0$) such that $J(e) < \beta$.

Proof. Consider,

$$\begin{aligned}
J(u) &= \frac{1}{2}\|u\|^2 - \frac{1}{2}\int_{\mathbb{H}^N}\left(\int_{\mathbb{H}^N}\frac{G(\eta,u(\eta))}{|\eta^{-1}\circ\xi|^{\alpha}}d\eta\right)G(\xi,u(\xi))d\xi \\
&\geq \frac{1}{2}\|u\|^2 - \frac{1}{2}\int_{\mathbb{H}^N}\left(\int_{\mathbb{H}^N}\frac{F(\eta,u)}{|\eta^{-1}\circ\xi|^{\alpha}}d\eta\right)F(\xi,u)d\xi \\
&\geq \frac{1}{2}\|u\|^2 - \frac{1}{2}\int_{\mathbb{H}^N}\left(\int_{\mathbb{H}^N}\frac{|u(\eta)|^{Q^*_\alpha}+K(\eta)|u(\eta)|^p}{|\eta^{-1}\circ\xi|^{\alpha}}d\eta\right)\left(|u(\xi)|^{Q^*_\alpha}+K(\xi)|u(\xi)|^p\right)d\xi \\
&= \frac{1}{2}\|u\|^2 - \frac{1}{2}\int_{\mathbb{H}^N}\int_{\mathbb{H}^N}\frac{|u(\eta)|^{Q^*_\alpha}}{|\eta^{-1}\circ\xi|^{\alpha}}|u(\xi)|^{Q^*_\alpha}d\eta d\xi - \frac{1}{2}\int_{\mathbb{H}^N}\int_{\mathbb{H}^N}\frac{|u(\eta)|^{Q^*_\alpha}}{|\eta^{-1}\circ\xi|^{\alpha}}K(\xi)|u(\xi)|^p d\eta d\xi \\
&- \frac{1}{2}\int_{\mathbb{H}^N}\int_{\mathbb{H}^N}\frac{K(\eta)|u(\eta)|^p}{|\eta^{-1}\circ\xi|^{\alpha}}K(\xi)|u(\xi)|^p d\eta d\xi - \frac{1}{2}\int_{\mathbb{H}^N}\int_{\mathbb{H}^N}\frac{K(\eta)|u(\eta)|^p}{|\eta^{-1}\circ\xi|^{\alpha}}|u(\xi)|^{Q^*_\alpha}d\eta d\xi.
\end{aligned}$$

Using (2.2), we get

$$\int_{\mathbb{H}^N}\int_{\mathbb{H}^N}\frac{K(\eta)|u(\eta)|^p}{|\eta^{-1}\circ\xi|^{\alpha}}K(\xi)|u(\xi)|^p d\eta d\xi \leq C\left(\int_{\mathbb{H}^N}|K(\xi)|^{\frac{2Q}{2Q-\alpha}}|u(\xi)|^{\frac{2Qp}{2Q-\alpha}}d\xi\right)^{\frac{2Q-\alpha}{Q}}. \tag{3.6}$$

By using Hölder's inequality and (Σ_2), we get

$$\begin{aligned}
\int_{\mathbb{H}^N}|K(\xi)|^{\frac{2Q}{2Q-\alpha}}|u(\xi)|^{\frac{2Qp}{2Q-\alpha}}d\xi &\leq \left(\int_{\mathbb{H}^N}|K(\xi)|^{\frac{Q^*}{Q^*_\alpha-p}}d\xi\right)^{\frac{Q^*_\alpha-p}{Q^*_\alpha}}\left(\int_{\mathbb{H}^N}|u(\xi)|^{Q^*}d\xi\right)^{\frac{p}{Q^*_\alpha}} \\
&\leq C\left(\int_{\mathbb{H}^N}|u(\xi)|^{Q^*}d\xi\right)^{\frac{p}{Q^*_\alpha}}.
\end{aligned}$$

On using this in (3.6), we get

$$\begin{aligned}
\int_{\mathbb{H}^N}\int_{\mathbb{H}^N}\frac{K(\eta)|u(\eta)|^p}{|\eta^{-1}\circ\xi|^{\alpha}}K(\xi)|u(\xi)|^p d\eta d\xi &\leq C\left(\int_{\mathbb{H}^N}|u(\xi)|^{Q*}d\xi\right)^{\frac{p}{Q^*}}\left(\int_{\mathbb{H}^N}|u(\xi)|^{Q^*}d\xi\right)^{\frac{p}{Q^*}} \\
&= C\|u\|^{2p}_{L^{Q^*}}(\mathbb{H}^N).
\end{aligned}$$

Similarly, we have

$$\int_{\mathbb{H}^N}\int_{\mathbb{H}^N}\frac{|u(\eta)|^{Q^*_\alpha}}{|\eta^{-1}\circ\xi|^{\alpha}}K(\xi)|u(\xi)|^p d\eta d\xi \leq C\|u\|^{p+Q^*_\alpha}_{L^{Q^*}}$$

and

$$\int_{\mathbb{H}^N}\int_{\mathbb{H}^N}\frac{|u(\eta)|^{Q^*_\alpha}}{|\eta^{-1}\circ\xi|^{\alpha}}|u(\xi)|^{Q^*_\alpha}d\eta d\xi \leq C\|u\|^{2Q^*_\alpha}_{L^{Q^*}}.$$

Hence,

$$\begin{aligned}
J(u) &\geq \frac{1}{2}\|u\|^2 - \frac{1}{2}C\left(\|u\|^{p+Q^*_\alpha}_{L^{Q^*}} + \|u\|^{2Q^*_\alpha}_{L^{Q^*}} + \|u\|^{2p}_{L^{Q^*}}\right) \\
&\geq \frac{1}{2}\|u\|^2 - \frac{1}{2}C\left(\|u\|^{p+Q^*_\alpha} + \|u\|^{2Q^*_\alpha} + \|u\|^{2p}\right).
\end{aligned}$$

Thus for sufficiently small $\|u\| = r$, $J(u) \geq \beta$.

(ii) For $u \in C_0^\infty(B_R \setminus \{0\})$,

$$J(tu) \leq \frac{t^2}{2}\int_{|\xi|\leq R} |\nabla_{\mathbb{H}} u|^2 + V(\xi)u^2 d\xi - \frac{t^{2p}}{2}\int_{|\eta|\leq R}\left(\int_{|\xi|\leq R}\frac{|u(\eta)|^{Q_\alpha^*} + K(\eta)|u(\eta)|^p}{|\eta^{-1}\circ\xi|^\alpha}d\eta\right)\left(|u(\xi)|^{Q_\alpha^*} + K(\xi)|u(\xi)|^p d\xi\right).$$

Thus, we conclude that $J(tu) \to -\infty$ as $t \to \infty$. Hence for some $e \in E$ with $\|e\| > \rho$ such that $J(e) < \beta$. This completes the proof.

□

Lemma 3.3. *Let $\{u_n\}$ be a $(PS)_c$ sequence associated to the functional J in $E \subset S^{1,2}(\mathbb{H}^N)$. Then $\{u_n\}$ is bounded.*

Proof. Let $\{u_n\}$ be a $(PS)_c$ sequence associated to the functional J in $E \subset S^{1,2}(\mathbb{H}^N)$. Then

$$J(u_n) \to c,\ J'(u_n) \to 0. \tag{3.7}$$

Let $\Psi(u) = \frac{1}{2}\int_{\mathbb{H}^N}\left(\int_{\mathbb{H}^N}\frac{G(\eta,u(\eta))}{|\eta^{-1}\circ\xi|^\alpha}d\eta\right)G(\xi,u(\xi))d\xi$.
We prove $\langle\Psi'(u),u\rangle \geq 2\theta\Psi(u) > 0$, for some $\theta \in (1,2)$.
Consider,

$$\begin{aligned}\frac{1}{2\theta}\langle\Psi'(u),u\rangle - \Psi(u) &= \int_{\mathbb{H}^N}\left(\int_{\mathbb{H}^N}\frac{G(\eta,u(\eta))}{|\eta^{-1}\circ\xi|^\alpha}d\eta\right)\left(\frac{1}{2\theta}g(\xi,u(\xi))u - \frac{1}{2}G(\xi,u(\xi))\right)d\xi\\ &= \int_{|\xi|\leq R}\left(\int_{\mathbb{H}^N}\frac{G(\eta,u(\eta))}{|\eta^{-1}\circ\xi|^\alpha}d\eta\right)\left(\frac{1}{2\theta}f(\xi,u)u - \frac{1}{2}F(\xi,u)\right)d\xi\\ &\quad + \int_{|\xi|>R}\left(\int_{\mathbb{H}^N}\frac{G(\eta,u(\eta))}{|\eta^{-1}\circ\xi|^\alpha}d\eta\right)\left(\frac{1}{2\theta}g(\xi,u(\xi))u - \frac{1}{2}G(\xi,u(\xi))\right)d\xi\end{aligned}$$

Let,

$$\begin{aligned}A &= \{\sigma \in \mathbb{H}^N : |\sigma| > R \text{ and } \ell f(\sigma,u) \leq V(\sigma)u(\sigma)\},\\ B &= \{\sigma \in \mathbb{H}^N : |\sigma| > R \text{ and } \ell f(\sigma,u) > V(\sigma)u(\sigma)\}.\end{aligned} \tag{3.8}$$

We have, $uf(\xi,u) \geq \theta F(\xi,u)$. Now for $|\xi| > R$,

$$\begin{aligned}\frac{1}{2\theta}\langle\Psi'(u),u\rangle - \Psi(u) &= \int_{|\xi|>R}\left(\int_{\mathbb{H}^N}\frac{G(\eta,u(\eta))}{|\eta^{-1}\circ\xi|^\alpha}d\eta\right)\left(\frac{1}{2\theta}g(\xi,u(\xi))u - \frac{1}{2}G(\xi,u(\xi))\right)d\xi\\ &\geq \int_A\left(\int_{\mathbb{H}^N}\frac{G(\eta,u(\eta))}{|\eta^{-1}\circ\xi|^\alpha}d\eta\right)\left(\frac{1}{2\theta}g(\xi,u(\xi))u - \frac{1}{2}G(\xi,u(\xi))\right)d\xi\\ &\quad + \int_B\left(\int_{\mathbb{H}^N}\frac{G(\eta,u(\eta))}{|\eta^{-1}\circ\xi|^\alpha}d\eta\right)\left(\frac{1}{2\theta}g(\xi,u(\xi))u - \frac{1}{2}G(\xi,u(\xi))\right)d\xi\\ &\geq \int_A\left(\int_{\mathbb{H}^N}\frac{G(\eta,u(\eta))}{|\eta^{-1}\circ\xi|^\alpha}d\eta\right)\left(\frac{1}{2\theta}f(\xi,u)u - \frac{1}{2}F(\xi,u)\right)d\xi\\ &\quad + \int_B\left(\int_{\mathbb{H}^N}\frac{G(\eta,u(\eta))}{|\eta^{-1}\circ\xi|^\alpha}d\eta\right)\left(\frac{1}{2\theta\ell} - \frac{1}{4\ell}\right)V(\xi)u^2 d\xi\\ &\geq 0.\end{aligned}$$

Consider,

$$\begin{aligned} c + \|u_n\| o(1) &= J(u_n) - \frac{1}{2\theta}\langle J'(u_n), u_n\rangle \\ &= \left(\frac{1}{2} - \frac{1}{2\theta}\right)\|u_n\|^2 + \frac{1}{2\theta}\langle \Psi'(u_n), u_n\rangle - \Psi(u_n) \\ &\geq \left(\frac{1}{2} - \frac{1}{2\theta}\right)\|u_n\|^2. \end{aligned}$$

Thus, $\{u_n\}$ is bounded. □

Lemma 3.4. *Let V and K satisfy (Σ_1), (Σ_2) and let $\{u_n\}$ be as $(PS)_c$ sequence of J. Then, there exists a positive number $c_1 > 0$ such that for $c < c_1$,*

$$\lim_{n\to\infty} \int_{\overline{B_R}} \int_{\mathbb{H}^N} \frac{|(u_n - u)(\xi)|^{Q^*_\alpha} |(u_n - u)(\eta)|^{Q^*_\alpha}}{|\eta^{-1} \circ \xi|^\alpha} \, d\eta \, d\xi = 0$$

Proof. Let $\{u_n\}$ be a $(PS)_c$ sequence. By Lemma 3.3, the sequence $\{u_n\}$ is bounded in E. Since E is reflexive, there exists $u \in E$ such that, up to a subsequence, $u_n \rightharpoonup u$ weakly in E. Consequently, $u_n \to u$ strongly in $L^q_{\text{loc}}(\mathbb{H}^N)$ for every $q \in [2, Q^*)$, and $u_n(\xi) \to u(\xi)$ a.e. in $\mathbb{H}^N$. Moreover, $u_n \rightharpoonup u$ in $S^{1,2}(\mathbb{H}^N)$, and hence, by Sobolev embedding, $u_n \rightharpoonup u$ in $L^{Q^*}(\mathbb{H}^N)$. Therefore, there exist bounded nonnegative measures ω, ζ, and ν such that, as $n \to \infty$,

$$|\nabla_{\mathbb{H}} u_n|^2 \rightharpoonup \omega, \qquad |u_n|^{Q^*} \rightharpoonup \zeta, \qquad \int_{\mathbb{H}^N} \frac{|u_n(\eta)|^{Q^*_\alpha}}{|\eta^{-1} \circ \xi|^\alpha} \, d\eta \, |u_n(\xi)|^{Q^*_\alpha} \rightharpoonup \nu.$$

Therefore by Lemma 2.3, there exists a countable sequence of points $\{z_i\}_{i\in I} \subset \mathbb{H}^N$ and families of positive numbers $\{\nu_i : i \in I\}$, $\{\zeta_i : i \in I\}$ and $\{\omega_i : i \in I\}$ such that

$$\begin{gathered} \nu = \left(\int_{\mathbb{R}^N} \frac{|u(\eta)|^{Q^*_\alpha}}{|\eta^{-1} \circ \xi|^\alpha} dy \right) |u(x)|^{Q^*_\alpha} + \Sigma_{i\in I} \nu_i \delta_{z_i}, \qquad \Sigma_{i\in I} \nu_i^{\frac{1}{Q^*_\alpha}} < \infty, \\ \omega \geq |\nabla_{\mathbb{H}} u|^2 + \sum_{i\in I} \omega_i \delta_{z_i}, \\ \zeta \geq |u|^{Q^*} + \sum_{i\in I} \zeta_i \delta_{z_i}, \end{gathered}$$

and

$$S_{HG} \nu_i^{\frac{1}{Q^*_\alpha}} \leq \omega_i, \qquad \nu_i^{\frac{Q}{2Q-\alpha}} \leq C(Q, \alpha)^{\frac{Q}{2Q-\alpha}} \zeta_i,$$

where $\delta_{\bar{\xi}}$ is the Dirac-mass of mass 1 concentrated at $\bar{\xi} \in \mathbb{H}^N$.

Let $\phi \in C_0^\infty(\mathbb{H}^n)$ be such that $0 \leq \phi \leq 1$,

$$\phi(\xi) = \begin{cases} 1 & \text{if } \xi \in B_1(0), \\ 0 & \text{if } \xi \in \mathbb{H}^N \setminus B_2(0). \end{cases} \tag{3.9}$$

Let $\varepsilon > 0$. For fixed $i \in I$, define

$$\phi_{\varepsilon,i}(\xi) = \phi\left(\delta_{\frac{2}{\varepsilon}}(z_i^{-1} \circ \xi)\right).$$

From (3.7), we have $J'(u_n) \to 0$ as $n \to \infty$. Therefore, $\langle J'(u_n), \phi_{\varepsilon,i} u_n\rangle \to 0$ as $n \to \infty$. Hence,

$$\begin{aligned} \lim_{n\to\infty} \int_{\mathbb{H}^N} &\left(\nabla_{\mathbb{H}} u_n \cdot \nabla_{\mathbb{H}} \left(u_n \cdot \phi_{\varepsilon,i}(\xi) u_n + V(\xi)|u_n|^2 \phi_{\varepsilon,i}\right) d\xi \right. \\ &\left. - \int_{\mathbb{H}^N} \left(\int_{\mathbb{H}^N} \frac{G(\eta, u_n)}{|\eta^{-1} \circ \xi|^\alpha} d\eta \right) g(\xi, u_n) u_n(\xi) \phi_{\varepsilon,i}(\xi) d\xi = 0. \right. \end{aligned} \tag{3.10}$$

Next, we claim that

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_{\mathbb{H}^N} u_n\nabla_{\mathbb{H}}u_n\nabla_{\mathbb{H}}\phi_{\varepsilon,i}=0. \tag{3.11}$$

Indeed, the boundedness of $\{u_n\}$, combined with Hölder's inequality and the Sobolev embedding theorem, yields

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\lim_{n\to\infty}\left|\int_{\mathbb{H}^N} u_n(\xi)\nabla_{\mathbb{H}}u_n(\xi)\cdot\nabla_{\mathbb{H}}\phi_{\varepsilon,i}(\xi)\,d\xi\right|\\
&\quad\le \lim_{\varepsilon\to 0}\lim_{n\to\infty}\left(\int_{\mathbb{H}^N}|\nabla_{\mathbb{H}}u_n(\xi)|^2\,d\xi\right)^{\frac{1}{2}}\left(\int_{\mathbb{H}^N}|u_n(\xi)\nabla_{\mathbb{H}}\phi_{\varepsilon,i}(\xi)|^2\,d\xi\right)^{\frac{1}{2}}\\
&\quad\le \lim_{\varepsilon\to 0}\lim_{n\to\infty} C\left(\int_{B_\varepsilon(z_i)}|u_n(\xi)|^2|\nabla_{\mathbb{H}}\phi_{\varepsilon,i}(\xi)|^2\,d\xi\right)^{\frac{1}{2}}\\
&\quad\le \lim_{\varepsilon\to 0} C\left(\int_{B_\varepsilon(z_i)}|u(\xi)|^2|\nabla_{\mathbb{H}}\phi_{\varepsilon,i}(\xi)|^2\,d\xi\right)^{\frac{1}{2}}\\
&\quad\le \lim_{\varepsilon\to 0} C\left(\int_{B_\varepsilon(z_i)}|\nabla_{\mathbb{H}}\phi_{\varepsilon,i}(\xi)|^Q\,d\xi\right)^{\frac{1}{Q}}\left(\int_{B_\varepsilon(z_i)}|u(\xi)|^{Q^*}\,d\xi\right)^{\frac{1}{Q^*}}.
\end{aligned}$$

One can see that using change of variables, $\int_{B_\varepsilon(z_i)}|\nabla_{\mathbb{H}}\phi_{\varepsilon,i}(\xi)|^Q\,d\xi$ is bounded as $\varepsilon\to 0$. Thus, (3.11) holds. Further, by using (Σ_2) and Lemma 2.5, with $m=2$, we have

$$\lim_{n\to\infty}\int_{\mathbb{H}^N}V(\xi)|u_n(\xi)|^2d\xi=\lim_{n\to\infty}\int_{\mathbb{H}^N}V(\xi)|u(\xi)|^2d\xi.$$

So by Hölder's inequality,

$$\begin{aligned}
\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_{\mathbb{H}^N}V(\xi)|u_n(\xi)|^2\phi_{\varepsilon,i}(\xi)d\xi&=\lim_{\varepsilon\to 0}\int_{\mathbb{H}^N}V(\xi)|u(\xi)|^2\phi_{\varepsilon,i}(\xi)d\xi\\
&\le\lim_{\varepsilon\to 0}\left(\int_{B_\varepsilon(z_i)}|V(\xi)|^{\frac{Q}{2}}\right)\left(\int_{B_\varepsilon(z_i)}|u(\xi)|^{Q^*}d\xi\right)^{\frac{1}{Q^*}}.
\end{aligned}$$

Hence,

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_{\mathbb{H}^N}V(\xi)|u_n(\xi)|^2\phi_{\varepsilon,i}(\xi)d\xi=0. \tag{3.12}$$

Therefore, using Lemma 2.3, (3.11) and (3.12), we get

$$\omega_i\le\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_{\mathbb{H}^N}|\nabla_{\mathbb{H}}u_n|^2\phi_{\varepsilon,i}(\xi)d\xi=\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_{\mathbb{H}^N}\nabla_{\mathbb{H}}u_n\cdot\nabla_{\mathbb{H}}\left(\phi_{\varepsilon,i}(\xi)u_n(\xi)\right)d\xi. \tag{3.13}$$

Suppose that there exists $i\in I$ such that $z_i\in\overline{B_R(0)}$. Hence for small $\varepsilon>0$,

$$\begin{aligned}
&\int_{\mathbb{H}^N}\left(\int_{\mathbb{H}^N}\frac{G(\eta,u_n)}{|\eta^{-1}\circ\xi|^\alpha}d\eta\right)g(\xi,u_n)u_n(\xi)\phi_{\varepsilon,i}(\xi)d\xi\\
&\quad=\int_{|\xi|\le R}\left(\int_{|\eta|\le R}\frac{|u_n(\eta)|^{Q^*_\alpha}+K(\eta)|u_n(\eta)|^p}{|\eta^{-1}\circ\xi|^\alpha}d\eta\right)(Q^*_\alpha|u_n(\xi)|^{Q^*_\alpha}+pK(\xi)|u_n(\xi)|^p)\phi_{\varepsilon,i}(\xi)d\xi\\
&\qquad+\int_{|\xi|\le R}\left(\int_{A}\frac{|u_n(\eta)|^{Q^*_\alpha}+K(\eta)|u_n(\eta)|^p}{|\eta^{-1}\circ\xi|^\alpha}d\eta\right)(Q^*_\alpha|u_n(\xi)|^{Q^*_\alpha}+pK(\xi)|u_n(\xi)|^p)\phi_{\varepsilon,i}(\xi)d\xi\\
&\qquad+\int_{|\xi|\le R}\left(\int_{B}\frac{V(\eta)|u_n(\eta)|^2}{2\ell|\eta^{-1}\circ\xi|^\alpha}d\eta\right)(Q^*_\alpha|u_n(\xi)|^{Q^*_\alpha}+pK(\xi)|u_n(\xi)|^p)\phi_{\varepsilon,i}(\xi)d\xi.
\end{aligned} \tag{3.14}$$

We now claim

$$\lim_{n\to\infty}\int_{|\xi|\le R}\int_{|\eta|\ge R}\frac{V(\eta)|u_n(\eta)|^2|u_n(\xi)|^p}{|\eta^{-1}\circ\xi|^\alpha}\phi_{\varepsilon,i}(\xi)d\eta d\xi\to 0\quad\text{as }\varepsilon\to 0. \tag{3.15}$$

By the Hardy–Littlewood–Sobolev inequality,

$$\int_{|\xi|\le R}\int_{|\eta|\ge R}\frac{V(\eta)|u_n(\eta)|^2|u_n(\xi)|^p}{|\eta^{-1}\circ\xi|^\alpha}\phi_{\varepsilon,i}(\xi)d\eta d\xi\le C\|V|u_n|^2\|_{L^{\frac{2Q}{2Q-\alpha}}(\mathbb{H}^N)}\|u_n\|^p_{L^{\frac{2Qp}{2Q-\alpha}}(B_\varepsilon(z_i))}.$$

On using Hölder's inequality, Sobolev embedding and (Σ_1), we get

$$\int_{|\xi|\le R}\int_{|\eta|\ge R}\frac{V(\eta)|u_n(\eta)|^2|u_n(\xi)|^p}{|\eta^{-1}\circ\xi|^\alpha}\phi_{\varepsilon,i}(\xi)d\eta d\xi\le C\|u\|^p_{L^{\frac{2Qp}{2Q-\alpha}}(B_\varepsilon(z_i))}.$$

Letting $\varepsilon\to 0$, we prove the claim. Similarly, by the Hardy–Littlewood–Sobolev inequality and (Σ_2), we have

$$\lim_{n\to\infty}\int_{|\xi|\le R}\int_{|\eta|\le R}\frac{K(\eta)|u_n(\eta)|^pK(\xi)|u_n(\xi)|^p}{|\eta^{-1}\circ\xi|^\alpha}\phi_{\varepsilon,i}(\xi)d\eta d\xi\to 0\quad\text{as }\varepsilon\to 0, \tag{3.16}$$

$$\lim_{n\to\infty}\int_{|\xi|\le R}\int_{|\eta|\le R}\frac{|u_n(\eta)|^{Q^*_\alpha}K(\xi)|u_n(\xi)|^p}{|\eta^{-1}\circ\xi|^\alpha}\phi_{\varepsilon,i}(\xi)d\eta d\xi\to 0,\quad\text{as }\varepsilon\to 0. \tag{3.17}$$

Next, we claim that

$$\lim_{n\to\infty}\int_{|\xi|\le R}\int_{A}\frac{K(\eta)|u_n(\eta)|^p|u_n(\xi)|^{Q^*_\alpha}}{|\eta^{-1}\circ\xi|^\alpha}\phi_{\varepsilon,i}(\xi)d\eta d\xi\to 0\quad\text{as }\varepsilon\to 0.$$

We have $p\in\left(2,\frac{2Q-\alpha}{Q-2}\right)$, $K\in L^{\frac{Q^*}{Q^*_\alpha-p}}(\mathbb{H}^N)$. Hence, $K^{\frac{2Q}{2Q-\alpha}}\in L^{\frac{Q^*}{Q^*-\frac{2Qp}{2Q-\alpha}}}(\mathbb{H}^N)$. By Lemma 2.5,

$$\int_A(K(\eta)|u_n(\eta)-u(\eta)|^p)^{\frac{2Q}{2Q-\alpha}}\to 0$$

as $n\to\infty$, i.e $K|u_n|^p\to K|u|^p$ in $L^{\frac{2Q}{2Q-\alpha}}(\mathbb{H}^N)$. By the Hardy–Littlewood–Sobolev inequality, we have

$$\int_A\frac{K(\eta)|u_n(\eta)|^p}{|\eta^{-1}\circ\xi|^\alpha}d\eta\to\int_A\frac{K(\eta)|u(\eta)|^p}{|\eta^{-1}\circ\xi|^\alpha}d\eta$$

in $L^{\frac{2Q}{\alpha}}(\mathbb{H}^N)$. Now since u_n is a bounded sequence in E, we have $\int_{|\xi|\le R}(|u_n(\xi)|^{Q^*_\alpha}\phi_{\varepsilon,i})^{\frac{2Q}{2Q-\alpha}}d\xi\le C$. Since $u_n\rightharpoonup u$ in E, as $n\to\infty$, $|u_n(\xi)|^{Q^*_\alpha}\phi_{\varepsilon,i}(\xi)\rightharpoonup|u(\xi)|^{Q^*_\alpha}\phi_{\varepsilon,i}(\xi)$ in $L^{\frac{2Q}{2Q-\alpha}}(|\xi|\le R)$. Thus,

$$\int_{|\xi|\le R}\int_A\frac{K(\eta)|u_n(\eta)|^p}{|\eta^{-1}\circ\xi|^\alpha}d\eta|u_n(\xi)|^{Q^*_\alpha}\phi_{\varepsilon,i}(\xi)d\xi\to\int_{|\xi|\le R}\int_A\frac{K(\eta)|u(\eta)|^p}{|\eta^{-1}\circ\xi|^\alpha}d\eta|u(\xi)|^{Q^*_\alpha}\phi_{\varepsilon,i}(\xi)d\xi,$$

as $n\to\infty$. Now using the Hardy–Littlewood–Sobolev inequality,

$$\int_{|\xi|<R}\int_A\frac{K(\eta)|u(\eta)|^p}{|\eta^{-1}\circ\xi|^\alpha}d\eta|u(\xi)|^{Q^*_\alpha}\phi_{\varepsilon,i}(\xi)d\xi\le C\|K|u|^p\|_{L^{\frac{2Q}{2Q-\alpha}}(|\xi|\le R)}\|u\|^{Q^*_\alpha}_{L^{Q^*}(B_\varepsilon(z_i))}$$

and taking limit as $\varepsilon\to 0$, we prove the claim. Thus, we have

$$\lim_{n\to\infty}\int_{|\xi|}\int_A\frac{K(\eta)|u_n(\eta)|^p|u_n(\xi)|^{Q^*_\alpha}}{|\eta^{-1}\circ\xi|^\alpha}\phi_{\varepsilon,i}(\xi)d\eta d\xi\to 0\quad\text{as }\varepsilon\to 0.$$

Similarly, we can prove

$$\lim_{n\to\infty}\int_{|\xi|\le R}\int_B \frac{V(\eta)|u_n(\eta)|^2|u_n(\xi)|^{Q_\alpha^*}}{|\eta^{-1}\circ\xi|^\alpha}\phi_{\varepsilon,i}(\xi)d\eta d\xi \to 0 \quad \text{as } \varepsilon\to 0.$$

Hence, taking limit as $\varepsilon\to 0$ and $n\to\infty$ in (3.14) and using Lemma 2.3 we get

$$\begin{aligned}
&\lim_{\varepsilon\to\infty}\lim_{n\to\infty}\int_{\mathbb{H}^N}\left(\int_{\mathbb{H}^N}\frac{G(\eta,u_n)}{|\eta^{-1}\circ\xi|^\alpha}d\eta\right)g(\xi,u_n)u_n(\xi)\phi_{\varepsilon,i}(\xi)d\xi\\
&\quad\le \lim_{\varepsilon\to\infty}\lim_{n\to\infty} Q_\alpha^*\int_{|\xi|\le R}\left(\int_{|\eta|\le R}\frac{|u_n(\eta)|^{Q_\alpha^*}}{|\eta^{-1}\circ\xi|^\alpha}d\eta\right)|u_n(\xi)|^{Q_\alpha^*}\phi_{\varepsilon,i}(\xi)d\xi\\
&\qquad + Q_\alpha^*\int_{|\xi|\le R}\left(\int_A\frac{|u_n(\eta)|^{Q_\alpha^*}}{|\eta^{-1}\circ\xi|^\alpha}d\eta\right)|u_n(\xi)|^{Q_\alpha^*}\phi_{\varepsilon,i}(\xi)d\xi\\
&\quad\le Q_\alpha^*\nu_i
\end{aligned}$$

where we have used (2.2) and Hölder's inequality. Hence, from (3.10) and (3.12)

$$\begin{aligned}
\omega_i &\le Q_\alpha^*\nu_i\,,\\
S_{HG}\nu_i^{\frac{Q-2}{2Q-\alpha}} &\le Q_\alpha^*\nu_i\,,\\
S_{HG} &\le Q_\alpha^*\nu_i^{\frac{Q-\alpha+2}{2Q-\alpha}}.
\end{aligned}$$

Hence, we conclude that $\nu_i > \delta > 0$, for some fixed δ. Now by Lemma 2.3, we have $\sum_{i\in I}\nu_i^{\frac{2}{Q_\alpha^*}} < \infty$. Hence, I is finite.

Consider,

$$\begin{aligned}
c+o(1) &= J(u_n) - \frac{1}{2}\langle J'(u_n),u_n\rangle\\
&\ge \frac{1}{2}\int_{\mathbb{H}^N}\left(\int_{\mathbb{H}^N}\frac{G(\eta,u_n)}{|\eta^{-1}\circ\xi|^\alpha}d\eta\right)(g(\xi,u_n)u_n - 2G(\xi,u_n))d\xi\\
&= \frac{1}{2}\int_{\mathbb{H}^N}\left(\int_{|\eta|\le R}\frac{|u_n|^{Q_\alpha^*}+K(\eta)|u_n|^p}{|\eta^{-1}\circ\xi|^\alpha}d\eta\right)((Q_\alpha^*-2)|u_n|^{Q_\alpha^*}+(p-2)K(\xi)|u_n|^p)d\xi\\
&\quad + \frac{1}{2}\int_{\mathbb{H}^N}\left(\int_A\frac{|u_n|^{Q_\alpha^*}+K(\eta)|u_n|^p}{|\eta^{-1}\circ\xi|^\alpha}d\eta\right)((Q_\alpha^*-2)|u_n|^{Q_\alpha^*}+(p-2)K(\xi)|u_n|^p)d\xi\\
&\quad + \frac{1}{2}\int_{\mathbb{H}^N}\left(\int_B\frac{V(\eta)|u_n|^2}{2\ell|\eta^{-1}\circ\xi|^\alpha}d\eta\right)((Q_\alpha^*-2)|u_n|^{Q_\alpha^*}+(p-2)K(\xi)|u_n|^p)d\xi\\
&\ge \left(\frac{Q_\alpha^*-2}{2}\right)\int_{\mathbb{H}^N}\left(\int_{|\eta|\le R}\frac{|u_n(\eta)|^{Q_\alpha^*}|u_n(\xi)|^{Q_\alpha^*}}{|\eta^{-1}\circ\xi|^\alpha}d\eta\right)\phi_{\varepsilon,i}(\xi)d\xi\\
&\quad + \left(\frac{Q_\alpha^*-2}{2}\right)\int_{\mathbb{H}^N}\left(\int_A\frac{|u_n(\eta)|^{Q_\alpha^*}|u_n(\xi)|^{Q_\alpha^*}}{|\eta^{-1}\circ\xi|^\alpha}d\eta\right)\phi_{\varepsilon,i}(\xi)d\xi\\
&\ge \left(\frac{Q_\alpha^*-2}{2Q_\alpha^*}\right)\omega_i\\
&\ge \left(\frac{Q_\alpha^*-2}{2Q_\alpha^*}\right)S_{HG}\nu_i^{\frac{Q-2}{2Q-\alpha}}\\
&\ge \left(\frac{Q_\alpha^*-2}{2}\right)\left(\frac{S_{HG}}{Q_\alpha^*}\right)^{\frac{2Q-\alpha}{Q-\alpha+2}}
\end{aligned}$$

as $\varepsilon \to 0$. Therefore,

$$c \geq \left(\frac{Q_\alpha^* - 2}{2}\right)\left(\frac{S_{HG}}{Q_\alpha^*}\right)^{\frac{2Q-\alpha}{Q-2+\alpha}}.$$

Hence for level less than $c_1 = \left(\frac{Q_\alpha^*-2}{2}\right)\left(\frac{S_{HG}}{Q_\alpha^*}\right)^{\frac{2Q-\alpha}{Q-2+\alpha}}$, I is empty or $\nu_i = 0$ for all $z_i \in B_R(0)$ and using Lemma 2.4 we get the desired result. □

Lemma 3.5. *Let V and K satisfy (Σ_1), (Σ_2) and $\{u_n\}$ be a $(PS)_c$ sequence for J. Then, at least for a subsequence and for any $\varepsilon > 0$ there exists $r \geq R$ such that, for all r*

$$\limsup_{n\to\infty} \int_{|\xi|\geq 2r} (|\nabla_{\mathbb{H}} u_n(\xi)|^2 + V(\xi)u_n(\xi)^2)d\xi < \varepsilon.$$

Proof. For $r > R$, let $\psi_r \in C^\infty(\mathbb{H}^n)$ be such that $0 \leq \psi_r \leq 1$,

$$\psi_r(\xi) = \begin{cases} 0 & \text{if } \xi \in B_r(0), \\ 1 & \text{if } \xi \in \mathbb{H}^N \setminus B_{2r}(0), \end{cases} \tag{3.18}$$

and $|\nabla_{\mathbb{H}}\psi_r(\xi)| \leq \frac{2}{r}$. Since $J'(u_n) \to 0$ as $n \to \infty$, $\langle J'(u_n), \psi_r u_n\rangle \to 0$ as $n \to \infty$. Thus,

$$\begin{aligned} &\lim_{n\to\infty} \int_{\mathbb{H}^N} \nabla_{\mathbb{H}} u_n(\xi) \cdot \nabla_{\mathbb{H}}(u_n(\xi)\psi_r(\xi))d\xi + \lim_{n\to\infty} \int_{\mathbb{H}^N} V(\xi)|u_n(\xi)|^2\psi_r(\xi)d\xi \\ &\quad = \lim_{n\to\infty} \int_{\mathbb{H}^N} \left(\int_{\mathbb{H}^N} \frac{G(\eta, u_n)}{|\eta^{-1}\circ\xi|^\alpha} d\eta\right) g(\xi, u_n)u_n\psi_r(\xi)d\xi, \end{aligned}$$

$$\begin{aligned} &\lim_{n\to\infty} \int_{\mathbb{H}^N} |\nabla_{\mathbb{H}} u_n(\xi)|^2\psi_r(\xi)d\xi + \lim_{n\to\infty} \int_{\mathbb{H}^N} V(\xi)|u_n(\xi)|^2\psi_r(\xi)d\xi \\ &\qquad + \lim_{n\to\infty} \int_{\mathbb{H}^N} u_n(\xi)\nabla_{\mathbb{H}} u_n(\xi) \cdot \nabla_{\mathbb{H}}\psi_r(\xi)d\xi \\ &\quad = \lim_{n\to\infty} \int_{\mathbb{H}^N} \left(\int_{\mathbb{H}^N} \frac{G(\eta, u_n)}{|\eta^{-1}\circ\xi|^\alpha} d\eta\right) g(\xi, u_n)u_n\psi_r(\xi)d\xi. \end{aligned}$$

Next, we claim that

$$\lim_{r\to\infty}\lim_{n\to\infty} \int_{\mathbb{H}^N} u_n(\xi)\nabla_{\mathbb{H}} u_n(\xi) \cdot \nabla_{\mathbb{H}}\psi_r(\xi)d\xi = 0. \tag{3.19}$$

Since $\nabla_{\mathbb{H}}\psi_r(\xi) = 0$ for $|\xi| \leq r$ and $|\xi| \geq 2r$, it follows from Hölder's inequality, boundedness of $\{u_n\}$ and compact embedding $S^{1,2}(\mathbb{H}^N) \hookrightarrow L^2_{\text{loc}}(\mathbb{H}^N)$ that

$$\begin{aligned} \int_{\mathbb{H}^N} u_n(\xi)\nabla_{\mathbb{H}} u_n(\xi) \cdot \nabla_{\mathbb{H}}\psi_r(\xi)d\xi &\leq \frac{2}{r}\int_{r\leq|\xi|\leq 2r} |u_n(\xi)||\nabla_{\mathbb{H}} u_n(\xi)|d\xi \\ &\leq \frac{2}{r}\left(\int_{r\leq|\xi|\leq 2r} |u_n|^2 d\xi\right)^{\frac{1}{2}}\left(\int_{r\leq|\xi|\leq 2r} |\nabla u_n|^2 d\xi\right)^{\frac{1}{2}} \\ &\leq \frac{C}{r}\left(\int_{r\leq|\xi|\leq 2r} |u_n|^2 d\xi\right)^{\frac{1}{2}}. \\ &\to \frac{C}{r}\left(\int_{r\leq|\xi|\leq 2r} |u(\xi)|^2 d\xi\right)^{\frac{1}{2}} \quad \text{as } n \to \infty. \end{aligned}$$

Thus, we have

$$\begin{aligned}\lim_{n\to\infty}\int_{\mathbb{H}^N} u_n(\xi)\nabla_{\mathbb{H}}u_n(\xi)\cdot\nabla_{\mathbb{H}}\psi_r(\xi)d\xi &\le \frac{C}{r}\left(\int_{r\le|\xi|\le 2r}|u(\xi)|^2 d\xi\right)^{\frac{1}{2}}\\ &\le \frac{C}{r}\left(\int_{r\le|\xi|\le 2r} d\xi\right)^{\frac{1}{Q}}\left(\int_{r\le|\xi|\le 2r}|u(\xi)|^{Q^*}d\xi\right)^{\frac{1}{Q^*}}\\ &\le C\left(\int_{r\le|\xi|}|u(\xi)|^{Q^*}d\xi\right)^{\frac{1}{Q^*}},\end{aligned}$$

which shows that

$$\lim_{r\to\infty}\lim_{n\to\infty}\int_{\mathbb{H}^N} u_n(\xi)\nabla_{\mathbb{H}}u_n(\xi)\cdot\nabla_{\mathbb{H}}\psi_r(\xi)d\xi = 0.$$

This proves (3.19). Next, we see that

$$\int_{\mathbb{H}^N}\left(\int_{\mathbb{H}^N}\frac{G(\eta,u_n)}{|\eta^{-1}\circ\xi|^{\alpha}}d\eta\right)g(\xi,u_n)u_n\psi_r(\xi)d\xi \le \int_{|\xi|\ge r}\left(\int_{\mathbb{H}^N}\frac{G(\eta,u_n)}{|\eta^{-1}\circ\xi|^{\alpha}}d\eta\right)g(\xi,u_n)u_n\psi_r(\xi)d\xi.$$

Since $r > R$, using (3.3), boundedness of $\{u_n\}$, (Σ_1), and (Σ_2), we have

$$\begin{aligned}&\int_{\mathbb{H}^N}\left(\int_{\mathbb{H}^N}\frac{G(\eta,u_n)}{|\eta^{-1}\circ\xi|^{\alpha}}d\eta\right)g(\xi,u_n)u_n\psi_r(\xi)d\xi\\ &\le \int_{|\xi|\ge r}\left(\int_{\mathbb{H}^N}\frac{G(\eta,u_n)}{|\eta^{-1}\circ\xi|^{\alpha}}d\eta\right)\frac{V(\xi)|u_n(\xi)|^2}{\ell}d\xi\\ &\le \int_{|\xi|\ge r}\left(\int_{\mathbb{H}^N}\frac{|u_n(\eta)|^{Q^*_\alpha}+K(\eta)|u_n(\eta)|^p}{|\eta^{-1}\circ\xi|^{\alpha}}d\eta\right)\frac{V(\xi)|u_n(\xi)|^2}{\ell}d\xi\\ &\quad+\int_{|\xi|\ge r}\left(\int_{\mathbb{H}^N}\frac{V(\eta)|u_n(\eta)|^2}{2\ell|\eta^{-1}\circ\xi|^{\alpha}}d\eta\right)\frac{V(\xi)|u_n(\xi)|^2}{\ell}d\xi\\ &\le C\left(\int_{|\xi|\ge r}|V(\xi)|^{\frac{2Q}{2Q-\alpha}}|u_n(\xi)|^{\frac{4Q}{2Q-\alpha}}d\xi\right)^{\frac{2Q-\alpha}{2Q}}\left(\int_{\mathbb{H}^N}\left(|u_n(\eta)|^{Q^*}+|K(\eta)|^{\frac{2Q}{2Q-\alpha}}|u_n(\eta)|^{\frac{2Qp}{2Q-\alpha}}\right)d\eta\right)^{\frac{2Q-\alpha}{2Q}}\\ &\quad+C\left(\int_{|\xi|\ge r}|V(\xi)|^{\frac{2Q}{2Q-\alpha}}|u_n(\xi)|^{\frac{4Q}{2Q-\alpha}}d\xi\right)^{\frac{2Q-\alpha}{2Q}}\left(\int_{\mathbb{H}^N}|V(\eta)|^{\frac{2Q}{2Q-\alpha}}|u_n(\eta)|^{\frac{4Q}{2Q-\alpha}}d\eta\right)^{\frac{2Q-\alpha}{2Q}}\\ &\le C\left(\int_{|\xi|\ge r}|V(\xi)|^{\frac{2Q}{2Q-\alpha}}|u_n(\xi)|^{\frac{4Q}{2Q-\alpha}}d\xi\right)^{\frac{2Q-\alpha}{2Q}}\left(\int_{\mathbb{H}^N}\left(|u_n(\eta)|^{Q^*}+|K(\eta)|^{\frac{2Q}{2Q-\alpha}}|u_n(\eta)|^{\frac{2Qp}{2Q-\alpha}}\right)d\eta\right)^{\frac{2Q-\alpha}{2Q}}\\ &\quad+C\left(\int_{|\xi|\ge r}|V(\xi)|^{\frac{2Q}{2Q-\alpha}}|u_n(\xi)|^{\frac{4Q}{2Q-\alpha}}d\xi\right)^{\frac{2Q-\alpha}{2Q}}\left(\int_{\mathbb{H}^N}|V(\eta)|^{\frac{2Q}{4-\alpha}}d\eta\right)^{\frac{4-\alpha}{2Q}}\left(\int_{\mathbb{H}^N}|u_n(\eta)|^{Q^*}d\eta\right)^{\frac{2Q-4}{2Q}}\\ &\le C\left(\int_{|\xi|\ge r}|V(\xi)|^{\frac{2Q}{2Q-\alpha}}|u_n(\xi)|^{\frac{4Q}{2Q-\alpha}}d\xi\right)^{\frac{2Q-\alpha}{2Q}}.\end{aligned}\tag{3.20}$$

Since $\alpha\in(0,4)$, it follows that $\frac{4Q}{2Q-\alpha}\in(2,Q^*)$. Moreover, as $V\in L^{\frac{2Q}{4-\alpha}}(\mathbb{H}^N)$, we have

$$V^{\frac{2Q}{2Q-\alpha}}\in L^{\frac{Q^*}{Q^*-\frac{4Q}{2Q-\alpha}}}(\mathbb{H}^N).$$

Therefore, by applying Lemma 2.5, we deduce that

$$\lim_{n\to\infty}\int_{|\xi|\ge r}|V(\xi)|^{\frac{2Q}{2Q-\alpha}}|u_n(\xi)|^{\frac{4Q}{2Q-\alpha}}d\xi = \int_{|\xi|\ge r}|V(\xi)|^{\frac{2Q}{2Q-\alpha}}|u(\xi)|^{\frac{4Q}{2Q-\alpha}}d\xi.$$

Hence,

$$\lim_{r\to\infty}\lim_{n\to\infty}\int_{|\xi|\geq r}|V(\xi)|^{\frac{2Q}{2Q-\alpha}}|u_n(\xi)|^{\frac{4Q}{2Q-\alpha}}d\xi=0.$$

Substituting this into (3.20), we obtain

$$\lim_{r\to\infty}\lim_{n\to\infty}\int_{\mathbb{H}^N}\left(\int_{\mathbb{H}^N}\frac{G(\eta,u_n)}{|\eta^{-1}\circ\xi|^{\alpha}}d\eta\right)g(\xi,u_n)u_n\psi_r(\xi)d\xi=0.$$

This completes the proof. □

We now find the upper bound for the functional J. Before that, we state some asymptotic estimates which can be proved similar to [21, Lemma 3.1] and [14, Lemma 1.1]. The minimizer for the Sobolev inequality

$$S\,\|u\|^2_{L^{Q^*}(\mathbb{H}^N)}\leq\int_{\mathbb{H}^N}|\nabla_{\mathbb{H}}u(\xi)|^2\,d\xi,$$

is attained by

$$U(\xi)=U(x,y,t)=\frac{C}{((t^2+(1+|x|^2+|y^2|)^2))^{\frac{Q-2}{4}}},$$

where C is a suitable positive constant. Let,

$$U_\varepsilon(\xi)=\frac{C}{((t^2+(\varepsilon+|x|^2+|y^2|)^2))^{\frac{Q-2}{4}}},\tag{3.21}$$

where C is a suitable constant. Hence

$$U_\varepsilon(\xi)=\varepsilon^{-\frac{Q-2}{2}}U(\delta_{\frac{1}{\sqrt{\varepsilon}}}(\xi)).$$

Define,

$$\phi(\xi)=\begin{cases}1 & \text{if } \xi\in B_{\frac{R}{2}}(0),\\ 0 & \text{if } \xi\in B^c_R(0),\end{cases}$$

$$u_\varepsilon(\xi)=U_\varepsilon(\xi)\phi(\xi),$$

$$v_\varepsilon(\xi)=\frac{u_\varepsilon(\xi)}{\left(\int_{\mathbb{H}^N}\int_{\mathbb{H}^N}\frac{|u_\varepsilon(\eta)|^{Q^*_\alpha}|u_\varepsilon(\xi)|^{Q^*_\alpha}}{|\eta^{-1}\circ\xi|^{\alpha}}d\eta d\xi\right)^{\frac{1}{2Q^*_\alpha}}}.\tag{3.22}$$

Since $S^{1,2}(\mathbb{H}^N)\hookrightarrow L^2_{\mathrm{loc}}(\mathbb{H}^N)$ and $\phi\in C^\infty_c(\mathbb{H}^N)$, we have $u_\varepsilon\in L^2_{\mathrm{loc}}(\mathbb{H}^N)$. Consequently, $v_\varepsilon\in E$. Further, we have

$$\int_{\mathbb{H}^N}|\nabla_{\mathbb{H}}u_\varepsilon(\xi)|^2d\xi=\varepsilon^{-\frac{Q-2}{2}}\int_{\mathbb{H}^N}|\nabla_{\mathbb{H}}U(\xi)|^2d\xi+O(1).\tag{3.23}$$

$$\int_{\mathbb{H}^N}|u_\varepsilon(\xi)|^{Q^*}d\xi=\varepsilon^{-\frac{Q}{2}}\int_{\mathbb{H}^N}|U(\xi)|^{Q^*}d\xi+O(1).\tag{3.24}$$

Using the Hardy–Littlewood–Sobolev inequality, we get

$$\int_{\mathbb{H}^N}\int_{\mathbb{H}^N}\frac{|u_\varepsilon(\eta)|^{Q^*_\alpha}|u_\varepsilon(\xi)|^{Q^*_\alpha}}{|\eta^{-1}\circ\xi|^{\alpha}}d\eta d\xi\leq C(Q,\alpha)\varepsilon^{-\frac{2Q-\alpha}{2}}\left(\int_{\mathbb{H}^N}|U(\xi)|^{Q^*}d\xi\right)^{\frac{2Q-\alpha}{Q}}+O(1)\tag{3.25}$$

Following the calculations in [21, Lemma 3.1], we obtain

$$\int_{\mathbb{H}^N}\int_{\mathbb{H}^N}\frac{|u_\varepsilon(\eta)|^{Q^*_\alpha}|u_\varepsilon(\xi)|^{Q^*_\alpha}}{|\eta^{-1}\circ\xi|^{\alpha}}d\eta d\xi\geq\varepsilon^{-\frac{2Q-\alpha}{2}}\int_{\mathbb{H}^N}\int_{\mathbb{H}^N}\frac{|U(\eta)|^{Q^*_\alpha}|U(\xi)|^{Q^*_\alpha}}{|\eta^{-1}\circ\xi|^{\alpha}}d\eta d\xi-O(1).\tag{3.26}$$

Lemma 3.6.

$$\int_{\mathbb{H}^N} |u_\varepsilon(\xi)|^q d\xi \leq \begin{cases} C\varepsilon^{\frac{Q}{2}-\frac{Q-2}{2}q} + O(1), & q \in (2, Q^*), \\ 1 + \log \frac{R}{\sqrt{\varepsilon}}, & q = 2, Q = 4, \\ C\varepsilon^{-\frac{Q-4}{2}} + O(1), & q = 2, Q > 4. \end{cases}$$

Proof. If $q \in (2, Q^*)$, then

$$\begin{aligned} \int_{\mathbb{H}^N} |u_\varepsilon(\xi)|^q d\xi &\leq \int_{|\xi|\leq R} |U_\varepsilon(\xi)|^q d\xi \\ &\leq \varepsilon^{\frac{Q}{2}-\frac{Q-2}{2}q} \int_{|\xi|\leq \frac{R}{\sqrt{\varepsilon}}} |U(\xi)|^q d\xi \\ &= C\varepsilon^{\frac{Q}{2}-\frac{Q-2}{2}q} \int_{|\xi|\leq \frac{R}{\sqrt{\varepsilon}}} \frac{1}{(t^2 + (1+|x|^2+|y|^2)^2)^{\frac{Q-2}{4}q}} d\xi \\ &\leq C\varepsilon^{\frac{Q}{2}-\frac{Q-2}{2}q} \int_{|\xi|\leq \frac{R}{\sqrt{\varepsilon}}} \frac{1}{(1+|\xi|^4)^{\frac{Q-2}{4}q}} d\xi \\ &\leq C\varepsilon^{\frac{Q}{2}-\frac{Q-2}{2}q} \left(\int_{1\leq|\xi|\leq \frac{R}{\sqrt{\varepsilon}}} \frac{1}{|\xi|^{(Q-2)q}} d\xi + \int_{|\xi|\leq 1} 1 d\xi \right) \\ &\leq C\varepsilon^{\frac{Q}{2}-\frac{Q-2}{2}q} \left(1 + \int_1^{\frac{R}{\sqrt{\varepsilon}}} \frac{1}{s^{(Q-2)q-Q+1}} ds \right) \\ &\leq C\varepsilon^{\frac{Q}{2}-\frac{Q-2}{2}q} \left(1 + \frac{1}{\varepsilon^{\frac{Q}{2}-\frac{Q-2}{2}q}} \right), \end{aligned}$$

that is,

$$\int_{\mathbb{H}^N} |u_\varepsilon(\xi)|^q d\xi \leq C\varepsilon^{\frac{Q}{2}-\frac{Q-2}{2}q} + O(1).$$

For $q = 2$, $Q = 4$, we have

$$\begin{aligned} \int_{\mathbb{H}^N} |u_\varepsilon(\xi)|^2 d\xi &\leq \int_{|\xi|\leq R} |U_\varepsilon(\xi)|^2 d\xi \\ &\leq C \int_{|\xi|\leq \frac{R}{\sqrt{\varepsilon}}} \frac{C}{1+|\xi|^4} d\xi \\ &\leq C \left(1 + \int_1^{\frac{R}{\sqrt{\varepsilon}}} \frac{1}{s} ds \right) \\ &\leq 1 + \log \frac{R}{\sqrt{\varepsilon}}. \end{aligned}$$

For $q = 2$, $Q > 4$, we have

$$\begin{aligned} \int_{\mathbb{H}^N} |u_\varepsilon(\xi)|^2 d\xi &\leq \varepsilon^{-\frac{Q-4}{2}} \int_{|\xi|\leq R} |U_\varepsilon(\xi)|^2 d\xi \\ &\leq C\varepsilon^{-\frac{Q-4}{2}} \left(1 + \int_1^{\frac{R}{\sqrt{\varepsilon}}} \frac{1}{s^{Q-3}} ds \right) \\ &\leq C\varepsilon^{-\frac{Q-4}{2}} + O(1). \end{aligned}$$

□

Using these estimates and following the calculations as in [14, Lemma 1.1] we obtain the following: Using (3.23), (3.25) and (3.26) we get,

$$\int_{\mathbb{H}^N} |\nabla_{\mathbb{H}} v_\varepsilon(\xi)|^2 d\xi = S_{HG} + O(\varepsilon^{\frac{Q-2}{2}}), \tag{3.27}$$

Using Lemma 3.6, (3.26) and the Hardy–Littlewood–Sobolev inequality,

$$\int_{\mathbb{H}^N}\int_{\mathbb{H}^N} \frac{|v_\varepsilon(\eta)|^p |v_\varepsilon(\xi)|^p}{|\eta^{-1}\circ\xi|^\alpha} d\eta d\xi \le O\left(\varepsilon^{\frac{2Q-\alpha}{2}-\frac{Q-2}{2}p}\right) + O\left(\varepsilon^{\frac{Q-2}{2}p}\right). \tag{3.28}$$

Using Lemma 3.6 and (3.26),

$$\int_{\mathbb{H}^N} |v_\varepsilon(\xi)|^2 d\xi \le \begin{cases} C\varepsilon\left(1+\log\left(\frac{R}{\sqrt{\varepsilon}}\right)\right), \text{ if } Q=4, \\ O(\varepsilon^2), \text{ if } Q>4. \end{cases} \tag{3.29}$$

Thus, from (3.28) and (3.29), we conclude

$$\lim_{\varepsilon\to 0}\int_{\mathbb{H}^N}\int_{\mathbb{H}^N} \frac{|v_\varepsilon(\eta)|^p |v_\varepsilon(\xi)|^p}{|\eta^{-1}\circ\xi|^\alpha} d\eta d\xi = 0$$

and

$$\lim_{\varepsilon\to 0}\int_{\mathbb{H}^N} |v_\varepsilon(\xi)|^2 d\xi = 0.$$

Now, we prove a bound for the energy functional J.

Lemma 3.7. *There exists $v \in E, v > 0$, such that* $\sup_{t>0} J(tv) < \frac{1}{2}\left(\frac{1}{Q_\alpha^*}\right)^{\frac{1}{Q_\alpha^*-1}}\left(1-\left(\frac{1}{Q_\alpha^*}\right)^{Q_\alpha^*}\right) S_{HG}^{\frac{Q_\alpha^*}{Q_\alpha^*-1}}$.

Proof. Let $v_\varepsilon \in E$ be defined as in (3.22). Define

$$h(t) = \frac{t^2}{2}\|v_\varepsilon\|^2 - \frac{t^{2p}}{2}\int_{|\xi|\le R}\int_{|\eta|\le R} \frac{K(\eta)|v_\varepsilon(\eta)|^p K(\xi)|v_\varepsilon(\xi)|^p}{|\eta^{-1}\circ\xi|^\alpha} d\eta d\xi - \frac{t^{2Q_\alpha^*}}{2}. \tag{3.30}$$

We observe that $h(t) \ge J(tv_\varepsilon)$, with $h(t) \to 0$ as $t \to 0$ and $h(t) \to -\infty$ as $t \to \infty$. Since $2 < 2p < 2Q_\alpha^*$, we have $h(t) > 0$ for small $t > 0$. Therefore, h attains a global maximum. Let t_ε denote a point at which h achieves its global maximum. Then

$$\|v_\varepsilon\|^2 = p t_\varepsilon^{2(p-1)} \int_{|\xi|\le R}\int_{|\eta|\le R} \frac{K(\eta)|v_\varepsilon(\eta)|^p K(\xi)|v_\varepsilon(\xi)|^p}{|\eta^{-1}\circ\xi|^\alpha} d\eta d\xi + Q_\alpha^* t_\varepsilon^{2(Q_\alpha^*-1)}. \tag{3.31}$$

Hence,

$$Q_\alpha^* t_\varepsilon^{2(Q_\alpha^*-1)} \le \|v_\varepsilon\|^2,$$
$$t_\varepsilon \le \left(\frac{1}{Q_\alpha^*}\|v_\varepsilon\|^2\right)^{\frac{1}{2(Q_\alpha^*-1)}}.$$

Therefore, using this preceding inequality in (3.30),

$$\int_{|\xi|\le R} |\nabla_{\mathbb{H}} v_\varepsilon(\xi)|^2 d\xi \le p\left(\frac{1}{Q_\alpha^*}\|v_\varepsilon\|^2\right)^{\frac{2(p-1)}{2(Q_\alpha^*-1)}} \int_{|\xi|\le R}\int_{|\eta|\le R} \frac{K(\eta)|v_\varepsilon(\eta)|^p K(\xi)|v_\varepsilon(\xi)|^p}{|\eta^{-1}\circ\xi|^\alpha} d\eta d\xi + Q_\alpha^* t_\varepsilon^{2(Q_\alpha^*-1)}.$$

From (3.27) for sufficiently small ε,

$$Q_\alpha^* t_\varepsilon^{2(Q_\alpha^*-1)} \geq \frac{S_{HG}}{2}.$$

Hence,

$$h(t_\varepsilon) \leq \frac{t_\varepsilon^2}{2}\int_{|\xi|\leq R}|\nabla_{\mathbb{H}} v_\varepsilon(\xi)|^2 d\xi + C\int_{|\xi|\leq R}|v_\varepsilon(\xi)|^2 d\xi$$
$$- C\int_{|\xi|\leq R}\int_{|\eta|\leq R}\frac{K(\eta)|v_\varepsilon(\eta)|^p K(\xi)|v_\varepsilon(\xi)|^p}{|\eta^{-1}\circ\xi|^\alpha} d\eta d\xi - \frac{t_\varepsilon^{2Q_\alpha^*}}{2}.$$

Let $\rho(\tau) = \frac{\tau^2}{2}\|v_\varepsilon\|^2 - \frac{\tau^{2Q_\alpha^*}}{2}$. By an argument similar to that used for h, the function ρ attains a global maximum. Let $\tau' = \left(\frac{1}{Q_\alpha^*}\right)^{\frac{1}{Q_\alpha^*-1}}\|v_\varepsilon\|^{\frac{1}{Q_\alpha^*-1}}$ be the point at which ρ attains its maximum. Hence,

$$\rho(\tau') = \frac{1}{2}\left(\frac{1}{Q_\alpha^*}\right)^{\frac{1}{Q_\alpha^*-1}}\|v_\varepsilon\|^{\frac{2Q_\alpha^*}{Q_\alpha^*-1}} - \frac{1}{2}\left(\frac{1}{Q_\alpha^*}\right)^{\frac{Q_\alpha^*}{Q_\alpha^*-1}}\|v_\varepsilon\|^{\frac{2Q_\alpha^*}{Q_\alpha^*-1}}$$
$$= \frac{1}{2}\|v_\varepsilon\|^{\frac{2Q_\alpha^*}{Q_\alpha^*-1}}\left(\left(\frac{1}{Q_\alpha^*}\right)^{\frac{1}{Q_\alpha^*-1}} - \left(\frac{1}{Q_\alpha^*}\right)^{\frac{Q_\alpha^*}{Q_\alpha^*-1}}\right).$$

Therefore using (3.27), (3.28) and (3.29),

$$h(t) \leq \frac{1}{2}\left(\frac{1}{Q_\alpha^*}\right)^{\frac{1}{Q_\alpha^*-1}}\|v_\varepsilon\|^{\frac{2Q_\alpha^*}{Q_\alpha^*-1}}\left(1-\left(\frac{1}{Q_\alpha^*}\right)^{Q_\alpha^*}\right) + C\int_{|\xi|\leq R}|v_\varepsilon(\xi)|^2 d\xi$$
$$- C\int_{|\xi|\leq R}\int_{|\eta|\leq R}\frac{K(\eta)|v_\varepsilon(\eta)|^p K(\xi)|v_\varepsilon(\xi)|^p}{|\eta^{-1}\circ\xi|^\alpha} d\eta d\xi$$
$$\leq \frac{1}{2}\left(\left(\frac{1}{Q_\alpha^*}\right)^{\frac{1}{Q_\alpha^*-1}} - \left(\frac{1}{Q_\alpha^*}\right)^{\frac{Q_\alpha^*}{Q_\alpha^*-1}}\right) S_{HG}^{\frac{Q_\alpha^*}{Q_\alpha^*-1}}.$$

□

Proof of Theorem 3.1. Let,

$$c_0 = \min\left\{\left(\frac{Q_\alpha^*-2}{2}\right)\left(\frac{S_{HG}}{Q_\alpha^*}\right)^{\frac{2Q-\alpha}{Q-2+\alpha}}, \frac{1}{2}\left(\left(\frac{1}{Q_\alpha^*}\right)^{\frac{1}{Q_\alpha^*-1}} - \left(\frac{1}{Q_\alpha^*}\right)^{\frac{Q_\alpha^*}{Q_\alpha^*-1}}\right) S_{HG}^{\frac{Q_\alpha^*}{Q_\alpha^*-1}}\right\}.$$

$c_0 > 0$. We conclude that from Lemma 3.4, Lemma 3.5 and Lemma 3.6, if $c \leq c_0$, then $I = \emptyset$. To conclude the result, it is enough to prove that the $(PS)_c-$sequence $\{u_n\}$ is convergent in E. From (3.7) we have $\langle J'(u_n), u_n - u\rangle \to 0$ and so $\langle J'(u_n - u), u_n - u\rangle \to 0$. Hence,

$$\int_{\mathbb{H}^N}|\nabla_{\mathbb{H}}(u_n-u)(\xi)|^2 + V(\xi)|(u_n-u)(\xi)|^2 d\xi$$
$$- \int_{\mathbb{H}^N}\left(\int_{\mathbb{H}^N}\frac{G(\eta, u_n-u)}{|\eta^{-1}\circ\xi|^\alpha} d\eta\right) g(\xi, u_n-u)(u_n-u) d\xi \to 0.$$

We claim that $\int_{\mathbb{H}^N}\left(\int_{\mathbb{H}^N}\frac{G(\eta,u_n-u)}{|\eta^{-1}\circ\xi|^\alpha} d\eta\right) g(\xi, u_n-u)(u_n-u) \to 0$ as $n \to \infty$.

Using Lemma 3.5, for given $\varepsilon > 0$, if $r > R$ is sufficiently large then for all $n \geq n_0$,

$$\int_{|\xi|\geq 2r}\left(\int_{\mathbb{H}^N}\frac{G(\eta, u_n-u)}{|\eta^{-1}\circ\xi|^\alpha} d\eta\right) g(\xi, u_n-u)(u_n-u) d\xi \leq \frac{\varepsilon}{2}.$$

Consider

$$\begin{aligned}
&\int_{|\xi|\le R}\left(\int_{\mathbb{H}^N}\frac{G(\eta,u_n-u)}{|\eta^{-1}\circ\xi|^{\alpha}}d\eta\right)g(\xi,u_n-u)(u_n-u)d\xi\\
&=\int_{|\xi|\le R}\left(\int_{|\eta|\le R}\frac{|(u_n-u)(\eta)|^{Q_\alpha^*}+K(\eta)|(u_n-u)(\eta)|^p}{|\eta^{-1}\circ\xi|^{\alpha}}d\eta\right)(Q_\alpha^*|(u_n-u)(\xi)|^{Q_\alpha^*}+pK(\xi)|(u_n-u)(\xi)|^p)d\xi\\
&\quad+\int_{|\xi|\le R}\left(\int_{A}\frac{|(u_n-u)(\eta)|^{Q_\alpha^*}+K(\eta)|(u_n-u)(\eta)|^p}{|\eta^{-1}\circ\xi|^{\alpha}}d\eta\right)(Q_\alpha^*|(u_n-u)(\xi)|^{Q_\alpha^*}+pK(\xi)|(u_n-u)(\xi)|^p)d\xi\\
&\quad+\int_{|\xi|\le R}\left(\int_{B}\frac{V(\eta)|(u_n-u)(\eta)|^2}{|\eta^{-1}\circ\xi|^{\alpha}}d\eta\right)(Q_\alpha^*|(u_n-u)(\xi)|^{Q_\alpha^*}+pK(\xi)|(u_n-u)(\xi)|^p)d\xi.
\end{aligned}\tag{3.32}$$

Now by virtue of Lemma 3.4, Lemma 2.5 and compact embedding in bounded domain,

$$\int_{|\xi|\le 2r}\left(\int_{\mathbb{H}^N}\frac{G(\eta,u_n-u)}{|\eta^{-1}\circ\xi|^{\alpha}}d\eta\right)g(\xi,u_n-u)(u_n-u)d\xi\le\varepsilon.$$

Hence combining the results, we get $\|u_n-u\|\to 0$ in E, as $n\to\infty$. Therefore, by the Mountain Pass Theorem [37, Theorem 1.17], equation (3.4) admits a weak solution. Moreover, the positivity of the solution follows from the strong maximum principle [12]. □

4 Proof of Theorem 1.2

In this section, we prove the main result of this paper. The following theorem provides the L^∞-regularity of the weak solution to (3.4). The proof is based on an argument of Brezis–Kato type, as carried out in $\mathbb{R}^N$ in [4, Lemma 2.10]. For the adaptation of the Brezis–Kato argument to the Heisenberg group, we refer to [20, Theorem 4.2]. We omit the details.

Theorem 4.1. *Let u be the solution obtained for the equation (3.4). Then, there exists a constant M_0, such that*

$$\|u\|_{L^\infty(\mathbb{H}^N)}\le M_0.$$

Before presenting the proof of Theorem 1.2, we prove the following lemma:

Lemma 4.2. *Let u be a solution of (3.4). Then, there exists $\ell_0>0$ such that*

$$\sup_{u\in E}\frac{H(u)(\xi)}{\ell_0}\le\frac{1}{2},$$

where $H(u)(\xi)=\int_{\mathbb{H}^N}\frac{G(\eta,u(\eta))}{|\eta^{-1}\circ\xi|^{\alpha}}d\eta$

Proof. Consider,

$$-\Delta_{\mathbb{H}}u+V(\xi)u=\left(\int_{\mathbb{H}^N}\frac{G(\eta,u(\eta))}{|\eta^{-1}\circ\xi|^{\alpha}}d\eta\right)g(\xi,u(\xi))$$

and

$$-\Delta_{\mathbb{H}}u=\left(-V(\xi)+\left(\int_{\mathbb{H}^N}\frac{G(\eta,u(\eta))}{|\eta^{-1}\circ\xi|^{\alpha}}d\eta\right)\frac{g(\xi,u(\xi))}{u}\right)u.$$

One can verify that $\int_{\mathbb{H}^N}\left(\frac{G(\eta,u(\eta))}{|\eta^{-1}\circ\xi|^{\alpha}}d\eta\right)\frac{g(\xi,u(\xi))}{u}\in L^{\frac{Q}{2}}(\mathbb{H}^N)$. Hence by Brezis–Kato argument [35, Lemma B.3], $u\in L^t_{\mathrm{loc}}(\mathbb{H}^N)$ for all $t<\infty$. Consider,

$$\begin{aligned}
|H(u)(\xi)| &= \left|\int_{\mathbb{H}^N}\frac{G(\eta,u(\eta))}{|\eta^{-1}\circ\xi|^{\alpha}}d\eta\right| \\
&\le \int_{\mathbb{H}^N}\frac{|u(\eta)|^{Q^*_\alpha}+K(\eta)|u(\eta)|^p}{|\eta^{-1}\circ\xi|^{\alpha}}d\eta \\
&\le \int_{|\eta^{-1}\circ\xi|^{\alpha}\le 1}\frac{|u(\eta)|^{Q^*_\alpha}+K(\eta)|u(\eta)|^p}{|\eta^{-1}\circ\xi|^{\alpha}}d\eta+\int_{|\eta^{-1}\circ\xi|^{\alpha}>1}\frac{|u(\eta)|^{Q^*_\alpha}+K(\eta)|u(\eta)|^p}{|\eta^{-1}\circ\xi|^{\alpha}}d\eta \\
&\le \int_{|\eta^{-1}\circ\xi|^{\alpha}\le 1}\frac{|u(\eta)|^{Q^*_\alpha}+K(\eta)|u(\eta)|^p}{|\eta^{-1}\circ\xi|^{\alpha}}d\eta+C \\
&\le \int_{|\eta^{-1}\circ\xi|^{\alpha}\le 1}\frac{|u(\eta)|^{Q^*_\alpha}}{|\eta^{-1}\circ\xi|^{\alpha}}d\eta+C\int_{|\eta^{-1}\circ\xi|^{\alpha}\le 1}\frac{|u(\eta)|^p}{|\eta^{-1}\circ\xi|^{\alpha}}d\eta+C \\
&\le \left(\int_{|\eta^{-1}\circ\xi|^{\alpha}\le 1}|u(\eta)|^{Q^*_\alpha t}d\eta\right)^{\frac{1}{t}}\left(\int_{|\eta^{-1}\circ\xi|^{\alpha}\le 1}\frac{1}{|\eta^{-1}\circ\xi|^{\alpha t'}}d\eta\right)^{\frac{1}{t'}} \\
&\quad + C\left(\int_{|\eta^{-1}\circ\xi|^{\alpha}\le 1}|u(\eta)|^{ps}d\eta\right)^{\frac{1}{s}}\left(\int_{|\eta^{-1}\circ\xi|^{\alpha}\le 1}\frac{1}{|\eta^{-1}\circ\xi|^{\alpha s'}}d\eta\right)^{\frac{1}{s'}}+C
\end{aligned}$$

where $s',t'\in\left[1,\frac{Q}{\alpha}\right)$ and $\frac{1}{s'}+\frac{1}{s}=1$, $\frac{1}{t'}+\frac{1}{t}=1$. Hence, by using the Hardy–Littlewood–Sobolev inequality, we have

$$|H(u)(\xi)|\le C.$$

Thus, we may choose $\ell_0>0$ such that

$$\sup_{u\in E}\frac{H(u)(\xi)}{\ell_0}\le\frac{1}{2}.$$ □

We take $\ell\ge\ell_0>0$ and consider the penalized problem.

In order to prove the existence of solution for (1.1), we show that there exists $R>1$ such that u satisfies the inequality

$$f(\xi,u)\le\frac{V(\xi)}{\ell}u$$

for $|\xi|\ge R$.

Proof of Theorem 1.2. Let

$$v(\xi)=\frac{R^{Q-2}\|u\|_{L^\infty(\mathbb{H}^N)}}{|\xi|^{Q-2}}.$$

One can verify that $v\in C^\infty(\mathbb{H}^N)\setminus\{0\}$. Observe that $u\le v$ for $|\xi|=R$. Define

$$w=\begin{cases}(u-v)_+ & \text{if } |\xi|\ge R,\\ 0 & \text{if } |\xi|<R.\end{cases}$$

We see that $w=0$ on $\partial B_R(0)$, $w\ge 0$ and $\Delta_{\mathbb{H}}v=0$ in $\mathbb{H}^N\setminus B_R(0)$, since $\frac{C}{|\xi|^{Q-2}}$ is a fundamental

solution where C is suitable constant[17, Theorem 2.1]. Therefore,

$$\begin{aligned}\int_{\mathbb{H}^N} |\nabla_{\mathbb{H}} w(\xi)|^2 d\xi &= \int_{\mathbb{H}^N} \nabla_{\mathbb{H}}(u-v)(\xi)\cdot\nabla_{\mathbb{H}} w(\xi) d\xi \\ &= \int_{|\xi|\geq R} H(\xi)g(\xi,u(\xi))w(\xi) - V(\xi)u(\xi)w(\xi)d\xi \\ &\leq \int_{|\xi|\geq R}\left(\frac{H(\xi)}{\ell_0}-1\right)V(\xi)u(\xi)w(\xi)d\xi \\ &\leq 0.\end{aligned}$$

Thus $w \equiv 0$ and hence, $u \leq v \leq \frac{R^{Q-2}M}{|\xi|^{Q-2}}$ if $|\xi| \geq R$. Consider,

$$\begin{aligned}\frac{f(\xi,u)}{u}p &\leq K(\xi)|u|^{p-2} + Q_\alpha^*|u|^{Q_\alpha^*-2} \\ &\leq pM^{p-2}K(\xi)\left(\frac{R}{|\xi|}\right)^{(p-2)(Q-2)} + Q_\alpha^* M^{Q_\alpha^*-2}\left(\frac{R}{|\xi|}\right)^{(Q_\alpha^*-2)(Q-2)} \\ &\leq C\left(\frac{R}{|\xi|}\right)^{(p-2)(Q-2)} \\ &\leq \frac{C\ell R^{(p-2)(Q-2)}}{\ell|\xi|^{(p-2)(Q-2)}}.\end{aligned}$$

Let $\gamma = C\ell, \Lambda \geq \gamma R^\mu$ and $\mu = (p-2)(Q-2)$. Then, from (Σ_1), we have

$$\frac{f(\xi,u)}{u} \leq \frac{1}{\ell}\frac{\Lambda}{|\xi|^\mu} \leq \frac{V(\xi)}{\ell}.$$

This completes the proof. □

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On the uniqueness of mild solutions and an averaging principle results for partial functional differential equations with infinite delay using the α-norm

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Abstract. In this paper, we study a class of partial functional differential equations with infinite delay within an abstract phase space framework. By employing the α-norm, we establish the existence and uniqueness of mild solutions under a more general condition on the nonlinear term, which is weaker than the classical Lipschitz condition. The linear part is governed by an unbounded operator that generates an analytic semigroup. Furthermore, an averaging principle is established in this case. To illustrate the applicability of the theoretical results, a concrete example is provided.

Keywords: analytic semigroup, infinite delay, alpha-norm, non-Lipschitz condition.

2020 Mathematics Subject Classification: 35A01, 35A02, 35B40.

1 Introduction

This work is concerned with the uniqueness existence of mild solutions and an averaging principle in the case of finite interval, especially $t \in [0, a]$, of the following partial functional differential equations with infinite delay of the form,

$$\begin{cases} x'(t) = -Ax(t) + f\left(\frac{t}{\varepsilon}, x_t\right), \quad t \geq 0, \\ x_0 = \varphi \in \mathcal{B}_\alpha, \end{cases} \tag{P_ε}$$

where $\varepsilon > 0$ and $-A$ generates an analytic semigroup $(T(t))_{t\geq 0}$ on a Banach space $(X, \|\cdot\|)$. The space $\mathcal{B}$ is a Banach space of functions mapping $(-\infty, 0]$ to X defined axiomatically. For $0 < \alpha < 1$, A^α denotes the fractional power of A. We assume that f is defined on $\mathbb{R}^+ \times \mathcal{B}_\alpha$ with values in X, where $\mathcal{B}_\alpha$ is defined by

$$\mathcal{B}_\alpha = \{\varphi \in \mathcal{B} : \varphi(\theta) \in D(A^\alpha) \text{ for } \theta \leq 0 \text{ and } A^\alpha \varphi \in \mathcal{B}\},$$

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and the function $A^{\alpha}\varphi$ is defined by

$$(A^{\alpha}\varphi)(\theta) = A^{\alpha}(\varphi(\theta)) \quad \text{for } \theta \leq 0.$$

For for every $t \geq 0$, the history function $x_t \in \mathcal{B}_{\alpha}$ is defined by

$$x_t(\theta) = x(t+\theta) \quad \text{for } \theta \leq 0.$$

The delay phenomenon is ubiquitous and extremely important in various engineering and dynamic systems, playing a significant role in their evolution. Mathematical models with delays are typically more suitable for practical problems than those without, and they have a stronger application background. The development was initiated for equations with finite delay by Travis and Webb in [23] and [24] and a book of Wu [25]. Concerning the case of infinite delay, an extensive theory has been developed by many authors, including Hino et al. [11], Hale and Kato [9], and Ezzinbi et al. [1] and [2].

However in many practical cases, the source function f may implicitly include spatial derivatives, then the above established results become invalid. In this case, the problem becomes more complicated, and to tackle this difficulty, Ezzinbi et al. in [6] examined the following equation:

$$\begin{cases} x'(t) = -Ax(t) + f(t, x_t), & t \geq 0, \\ x_0 = \varphi \in \mathcal{B}_{\alpha}. \end{cases}$$

In their work, they proved the existence of a mild solution without proving uniqueness, under the assumption that $-A$ generates an analytic semigroup and that the function f is continuous with respect to both its time variable and its history function. As far as we know, no research has been published on the unique existence of mild solutions and the averaging principle for the problem (P_{ε}), when the source function involves implicit spatial derivatives, and the the function f satisfies a condition weaker than the ordinary Lipschitz one as we can see in the sequel.

The initial studies on the averaging principle originate from the problem of celestial mechanic, we refer the reader to [4,12,17], for more details about this topic. The authors provide a justification of the averaging principle in the context of finite-dimensional differential equations. Subsequently, in [5,7,8,10,13–15,20,21], the authors extended the averaging principle to both finite- and infinite-dimensional equations as well as to functional differential equations with finite delay.

The approach here is to consider the following partial functional differential equations with infinite delay in X and a small positive parameter ε of the (normal) form

$$\begin{cases} z'(\tau) = \varepsilon[-Az(\tau) + f(\tau, z_{\tau})], & \tau \geq 0, \\ z_0 = \psi. \end{cases} \tag{1.1}$$

Under the same hypotheses on the operator $-A$ and the function f, and adding the following condition on the phase space $\mathcal{B}$,

$$\psi\left(\frac{\cdot}{\varepsilon}\right) \in \mathcal{B}, \quad \text{for all } \varepsilon > 0, \tag{C_1}$$

we can prove that problem (1.1) can be equivalently rewritten as (P_{ε}).

The averaging principle is based on the replacement of the nonautonomous part of a partial differential equation with small parameter $\varepsilon > 0$ by its time average $f_0 : \mathcal{B} \to E$ defined by

$$f_0(v) = \lim_{t \to +\infty} \frac{1}{t} \int_0^t f(s, v)ds, \quad \text{for } v \in \mathcal{B},$$

and which figure in the parallel problem of (1.1) given by

$$\begin{cases} z'(\tau) = \varepsilon[-Az(\tau) + f_0(z_\tau)], \quad \tau \geq 0, \\ z_0 = \psi. \end{cases} \tag{1.2}$$

Therefore, we conclude that under the additional condition (C_1) on the phase space $\mathcal{B}$ and in order to establish the averaging principle for problem (1.1) in the case of finite time interval, it is necessary to show that, for any fixed time $a > 0$, the (unique) solution of the problem (P_ε) can be approximated (in a certain sense) by the (unique) solution of the parallel problem

$$\begin{cases} x'(t) = -Ax(t) + f_0(x_t), \quad t \in [0, a], \\ x_0 = \varphi, \end{cases} \tag{P_0}$$

as $\varepsilon \to 0^+$.

The paper is organized as follows: In Section 2, we recall some necessary preliminaries. In Section 3, we establish a uniqueness existence result of mild solutions in a Banach space with infinite delay in an abstract phase space $\mathcal{B}$, where the linear part $-A$ generates an analytic semigroup which will be denoted by $(T(t))_{t\geq 0}$ on a Banach space X, and the nonlinear part f satisfies a condition with respect to the second argument, which is weaker than the ordinary Lipschitz condition. In Section 4, we consider the problem (P_ε) and its corresponding parallel problem (P_0). we establish the averaging principle result for the problem (P_ε). Finally we illustrate our results in the third and the last section by showing how they can be applied to a parabolic partial differential equation with infinite delay.

2 Preliminaries

Consider the following partial functional differential equation in X of the form:

$$\begin{cases} x'(t) = -Ax(t) + f(t, x_t), \quad t \in [0, a], \\ x_0 = \varphi \in \mathcal{B}_\alpha, \end{cases} \tag{2.1}$$

In the whole of this work, we assume that

($\mathbf{H}_0$) $-A$ is the infinitesimal generator of a compact analytic semigroup $(T(t))_{t\geq 0}$ on a Banach space X and $0 \in \rho(A)$, where $\rho(A)$ is the resolvent set of A.

Then, there exist constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\|T(t)\| \leq Me^{\omega t}$ for $t \geq 0$. Without loss of generality, we assume that $\omega > 0$. If the assumption $0 \in \rho(A)$ is not satisfied, one can substitute the operator A by the operator $(A - \sigma I)$ with σ large enough such that $0 \in \rho(A - \sigma I)$ and so we can always assume that $0 \in \rho(A)$.

For the fractional power $(A^\alpha, D(A^\alpha))$, for $0 < \alpha < 1$, and its inverse $A^{-\alpha}$. One has the following known result.

Theorem 2.1 ([16, Theorems 6.8–6.13, pp. 72–74]]). *Let $0 < \alpha < 1$ and assume that ($\mathbf{H}_0$) holds. Then*

(i) $D(A^\alpha)$ is a Banach space with the norm $|x|_\alpha = |A^\alpha x|$ for $x \in D(A^\alpha)$,

(ii) $T(t) : X \to D(A^\alpha)$ for $t > 0$,

(iii) $A^{\alpha}T(t)x = T(t)A^{\alpha}x$ *for* $x \in D(A^{\alpha})$ *and* $t \geq 0$,

(iv) for every $t > 0$, $A^{\alpha}T(t)$ *is bounded on* X *and there exists* $M_{\alpha} > 0$ *such that*

$$|A^{\alpha}T(t)| \leq M_{\alpha}\frac{e^{\omega t}}{t^{\alpha}} \text{ for } t > 0, \tag{2.2}$$

(v) $A^{-\alpha}$ *is a bounded linear operator on* X *with* $D(A^{\alpha}) = \mathrm{Im}(A^{-\alpha})$,

(vi) if $0 < \alpha < \beta < 1$, *then* $D(A^{\beta}) \hookrightarrow D(A^{\alpha})$,

(vii) there exists $N_{\alpha} > 0$ *such that*

$$\left|(T(t) - I)A^{-\alpha}\right| \leq N_{\alpha}t^{\alpha} \quad \text{for } t > 0. \tag{2.3}$$

In the sequel, we denote by X_{α} *the Banach space* $(D(A^{\alpha}), |\cdot|_{\alpha})$. *Recall that* $A^{-\alpha}$ *is given by the following formulas*

$$A^{-\alpha} = \frac{\sin(\pi\alpha)}{\alpha}\int_0^{\infty} t^{-\alpha}(t + A)^{-1}dt,$$

or

$$A^{-\alpha} = \frac{1}{\Gamma(\alpha)}\int_0^{\infty} t^{\alpha-1}T(t)\mathrm{d}t.$$

We denote by $C_{\alpha} := C_B((-\infty, 0]; X_{\alpha})$ the space of continuous bounded functions from $(-\infty, 0]$ into X_{α} endowed with the norm

$$\|\varphi\|_{\alpha} := \sup_{\theta \leq 0} |\varphi(\theta)|_{\alpha}, \quad \varphi \in C_{\alpha}.$$

Remark that $(C_{\alpha}, \|\cdot\|_{\alpha})$ is also a Banach space.

From now on, we adopt the axiomatic definition of the phase space $\mathcal{B}$, originally introduced by Hale and Kato in [9]. We denote its norm by $\|\cdot\|_{\mathcal{B}}$. The space $\mathcal{B}$ consists of all functions mapping $(-\infty, 0]$ into X and satisfying the following fundamental axioms:

(A) there exist a positive constant N, a locally bounded function $M(\cdot)$ on $[0, +\infty)$ and a continuous function $K(\cdot)$ on $[0, +\infty)$, such that if $x : (-\infty, a] \to X$ is continuous on $[\sigma, a]$ with $x_{\sigma} \in \mathcal{B}$, for some $\sigma < a$, then for all $t \in [\sigma, a]$,

(i) $x_t \in \mathcal{B}$,

(ii) $t \mapsto x_t$ is continuous with respect to $\|\cdot\|$ on $[\sigma, a]$,

(iii) $N|x(t)| \leq \|x_t\| \leq K(t - \sigma)\sup_{\sigma \leq s \leq t}|x(s)| + M(t - \sigma)\|x_{\sigma}\|$.

(B) $\mathcal{B}$ is a Banach space.

We suppose that

$(\mathbf{H}_1)$ $A^{-\alpha}\varphi \in \mathcal{B}$ for $\varphi \in \mathcal{B}$, where the function $A^{-\alpha}\varphi$ is defined by

$$\left(A^{-\alpha}\varphi\right)(\theta) = A^{-\alpha}(\varphi(\theta)) \text{ for } \theta \leq 0.$$

Consequently, we get the following result.

Proposition 2.2 ([6]). *Assume that* $(\mathbf{H}_0)$ *and* $(\mathbf{H}_1)$ *hold. If* $\mathcal{B}$ *satisfies axioms* (**A**) *and* (**B**). *Then* $\mathcal{B}_{\alpha}$ *satisfies axioms* (**A**) *and* (**B**).

(**A**) *if* $x : (-\infty, a] \to X_\alpha$ *is continuous on* $[\sigma, a]$ *with* $x_\sigma \in \mathcal{B}$, *for some* $\sigma < a$, *then for all* $t \in [\sigma, a]$,

(i) $x_t \in \mathcal{B}$,

(ii) $t \mapsto x_t$ *is continuous with respect to* $\|\cdot\|_\alpha$ *on* $[\sigma, a]$,

(iii) $N|x(t)|_\alpha \leq \|x_t\|_\alpha \leq K(t-\sigma)\sup_{\sigma\leq s\leq t}|x(s)|_\alpha + M(t-\sigma)\|x_\sigma\|_\alpha$.

(**B**) $\mathcal{B}_\alpha$ *is a Banach space.*

We have the following definition.

Definition 2.3. Let $\varphi \in \mathcal{B}_\oslash$. We say that $x \in \mathcal{C}(]-\infty, a]; X_\alpha)$ is a mild solution to problem (2.1) if $x_0 = \varphi$ and,

$$x(t) = T(t)\varphi(0) + \int_0^t T(t-s)f(s, x_s)ds \quad \text{for } 0 \leq t \leq a.$$

3 Existence results

In the whole of this work we suppose that $\mathcal{B}$ satisfies (**A**) and (**B**).

For the existence result we suppose that the function $f : (t, u) \mapsto f(t, u)$ acts from $[0, a] \times \mathcal{B}_\alpha$ into X and continuous with respect to t and u. Then we have the following theorem,

Theorem 3.1 ([6]). *Assume that* $(\mathbf{H}_0)$ *and* $(\mathbf{H}_1)$ *hold. Then for* $\varphi \in \mathcal{B}_\alpha$ *problem* (2.1) *has a mild solution on* $[0, a]$, *for a small enough.*

In order to prove a uniqueness result, we suppose that the function f satisfies a condition weaker than the ordinary Lipschitz condition, which is given as follows

$(\mathbf{H}_2)$ For all $(u, v) \in \mathcal{B}_\alpha \times \mathcal{B}_\alpha$, and $t \geq 0$

$$\|f(t, u) - f(t, v)\| \leq L(t, \|u - v\|_\alpha),$$

where $L : [0, a] \times [0, +\infty[\to [0, +\infty[$ satisfies:

(i) continuous, nondecreasing with respect to the first and the second argument, and $L(t, 0) = 0$, for all $t \in [0, a]$,

(ii) for any $\zeta > 0$ and for all continuous mapping $h : [0, a] \to [0, +\infty[$, such that:

$$h(t) \leq \zeta \int_0^t (t-s)^{-\alpha} L(s, h(s))ds, \quad \text{for all } t \in [0, a],$$

then $h(t) = 0$ for all $t \in [0, a]$.

We have the following theorem.

Theorem 3.2. *Assume that* $(\mathbf{H}_0)$–$(\mathbf{H}_2)$ *are satisfied, then problem* (2.1) *has a unique mild solution.*

Proof. The existence is ensured by Theorem 3.1. To attend for a uniqueness result we suppose that problem (2.1) has two solutions x_1 and x_2 in X_α. Let $s \in [0, t]$, by using the fact that the mapping L is nondecreasing with respect to the second argument and **A**-(iii), we get

$$\begin{aligned}
|x_1(s) - x_2(s)|_\alpha &= \left| \int_0^s T(s-\tau)\Big(f\Big(\tau, x_\tau^1\Big) - f\big(\tau, x_\tau^2\big)\Big)d\tau \right|_\alpha \\
&\leq M_\alpha e^{\omega a} \int_0^s (s-\tau)^{-\alpha} \Big\| f\Big(\tau, x_\tau^1\Big) - f\big(\tau, x_\tau^2\big) \Big\| d\tau \\
&\leq M_\alpha e^{\omega a} \int_0^s (s-\tau)^{-\alpha} L\left(\tau, K_a \sup_{0\leq\theta\leq\tau} |x_1(\theta) - x_2(\theta)|_\alpha\right) d\tau.
\end{aligned}$$

We claim that the function $s \mapsto \int_0^s (s-\tau)^{-\alpha} L(\tau, K_a \sup_{0\le\theta\le\tau} |x_1(\theta) - x_2(\theta)|_\alpha) d\tau$ is nondecreasing on $[0,t]$. In fact let us denote $g(\tau) = L(\tau, K_a \sup_{0\le\theta\le\tau} |x_1(\theta) - x_2(\theta)|_\alpha)$, which is a nondecreasing function by $(\mathbf{H}_2)$-(i). Then for $s, s' \in [0,t]$ be such that $s < s'$ we have

$$\int_0^s (s-\tau)^{-\alpha} g(\tau) d\tau = \int_0^s \tau^{-\alpha} g(s-\tau) d\tau \le \int_0^{s'} \tau^{-\alpha} g(s'-\tau) d\tau = \int_0^{s'} (s'-\tau)^{-\alpha} g(\tau) d\tau.$$

Then $\sup_{s\in[0,t]} \int_0^s (s-\tau)^{-\alpha} g(\tau) d\tau = \int_0^t (t-\tau)^{-\alpha} g(\tau) d\tau$. Thus,

$$\sup_{0\le s\le t} |x_1(s) - x_2(s)|_\alpha \le M_\alpha e^{\omega a} \int_0^t (t-\tau)^{-\alpha} L\left(\tau, K_a \sup_{0\le\theta\le\tau} |x_1(\theta) - x_2(\theta)|_\alpha\right) d\tau.$$

Consequently,

$$K_a \sup_{0\le s\le t} |x_1(s) - x_2(s)|_\alpha \le \zeta_{\alpha,a} \int_0^t (t-\tau)^{-\alpha} L\left(\tau, K_a \sup_{0\le\theta\le\tau} |x_1(\theta) - x_2(\theta)|_\alpha\right) d\tau.$$

Using $(\mathbf{H}_2)$-(ii), we get

$$\sup_{0\le s\le t} |x_1(s) - x_2(s)|_\alpha = 0 \quad \text{for all } t \in [0,a].$$

Thus, problem (2.1) has a unique mild solution. □

Corollary 3.3. Let k a positive constant. If the mapping $L(t,u) = ku$ for all $t \ge 0$, then our assumption $(\mathbf{H}_2)$ becomes the ordinary Lipschitz condition and $(\mathbf{H}_2)$-(ii) still hold.

Proof. Let $h : [0,a] \to [0,+\infty[$ a continuous function then by taking $L(t,u) = ku$, for all $t \ge 0$, we can see that

$$h(t) \le \zeta \int_0^t (t-s)^{-\alpha} L(s, h(s)) ds, \quad \text{for all } t \in [0,a],$$

becomes

$$h(t) \le \zeta' \int_0^t \frac{h(s)}{(t-s)^\alpha} ds, \quad \text{for all } t \in [0,a], \tag{3.1}$$

for $\zeta' > 0$. To complete to proof we need the following lemma.

Lemma 3.4 ([16, Lemma 6.7, p. 159])**.** *Let $\varphi : [0,a] \to [0,\infty[$ be continuous. If there are positive constants A, B and $0 < \alpha < 1$ such that*

$$\varphi(t) \le A + B \int_0^t \frac{\varphi(s)}{(t-s)^\alpha} ds, \quad \text{for } 0 \le t \le a,$$

then there exist $A' > 0, B' > 0$, such that

$$\varphi(t) \le A' + B' \int_0^t \varphi(s) ds,$$

where $A' = A \sum_{j=0}^{n-1} \left(\frac{Ba^{1-\alpha}}{1-\alpha}\right)^j$, and $B' = \frac{(B\Gamma(1-\alpha))^n}{\Gamma(n(1-\alpha))} a^{n(1-\alpha)-1}$. In particular there exists $C' \ge 0$ such that $\varphi(t) \le C'$, for all $t \ge 0$.

Proof. By taking $A = 0$, $B = \zeta'$ in (3.1), we can deduce that

$$h(t) \le B' \int_0^t h(s) ds.$$

Gronwall's inequality leads to $h(t) = 0$ for all $t \ge 0$. □

Another example of a mapping with the Assumption $(\mathbf{H}_2)$, we give the following:

Example 3.5. [22] We take $L(t,u) = q(t)r(u)$, for all $t \in [0,a]$ and $u \in [0,+\infty[$, where $q : [0,a] \to [0,+\infty[$ is a continuous function, and $r : [0,+\infty[\to [0,+\infty[$, is a continuous nondecreasing function with $r(0) = 0$, such that

$$\int_{0^+} \frac{1}{r(u)} du = \infty.$$

Remark 3.6. To establish the existence of global solutions, we assume the same conditions on f as in Theorem 3.2, and we introduce the following additional assumption:

$(\mathbf{H}_3)$ There exists a continuous function g defined on $\mathbb{R}^+$ such that,

$$\|f(t,u)\| \leq g(t)(1 + \|u\|_\alpha) \quad \text{for all } t \in \mathbb{R}^+, \ \ u \in \mathcal{B}_\alpha.$$

The proof is similar to that of Corollary 6 in [6].

4 Averaging result

In order to prove an averaging principle result, we choose the phase space $\mathcal{B} = \mathcal{C}_\infty$ (see [11]). The space $\mathcal{C}_\infty$ is formed by all the continuous functions $\varphi : \mathbb{R}^- \to X$, such that $\lim_{\theta \to -\infty} \varphi(\theta)$ exists in X. Endowed with the following norm

$$\|\varphi\|_{\mathcal{B}} = \sup_{\theta \in]-\infty,0]} |\varphi(\theta)|.$$

Consider the partial functional differential equation in X with a small positive parameter ε of the form

$$\begin{cases} x'(t) = Ax(t) + f\left(\frac{t}{\varepsilon}, x_t\right), \quad t \in [0,a], \\ x_0 = \varphi. \end{cases} \tag{P_ε}$$

Further, parallel to the problem (P_ε), $\varepsilon > 0$, we consider the averaged problem

$$\begin{cases} x'(t) = Ax(t) + f_0(x_t), \\ x_0 = \varphi. \end{cases} \tag{P_0}$$

Remark 4.1. When the delay is infinite the choice of the phase space is crucial to establish the averaging principle result associate to problem (P_ε). To guarantee the equivalence between

$$\begin{cases} w'(\tau) = \varepsilon[Aw(\tau) + f(\tau, w_\tau)], \quad \tau \in \left[0, \frac{a}{\varepsilon}\right], \\ w_0 = \psi \end{cases} \tag{4.1}$$

and (P_ε). Also between

$$\begin{cases} w'(t) = \varepsilon[Aw(t) + f_0(w_t)] \\ w_0 = \psi \end{cases} \tag{4.2}$$

and (P_0), we have to suppose that the condition (C_1) is satisfied. Clearly, this condition is not satisfied for any choice of the space $\mathcal{B}$ introduced by Hale and Kato [9], as shown by the following phase space examples:

Let $E = \mathcal{C}([0,1];\mathbb{R})$ be the space of continuous functions endowed with the uniform norm topology. Let $\gamma > 0$ and $\mathcal{L}_\gamma$ be the space of measurable functions $\varphi : (-\infty, 0] \to X$ such that $e^{\gamma\theta}\varphi(\theta)$ is integrable on $(-\infty, 0\ ]$. We consider the phase space $\mathcal{B} = E \times \mathcal{L}_\gamma$. The space of functions $\varphi \in \mathcal{L}_\gamma$ which is uniquely defined at 0 and endowed with the following norm

$$\|\varphi\| = |\varphi(0)| + \int_{-\infty}^{0} e^{\gamma\theta}|\varphi(\theta)|d\theta \quad \text{for } \varphi \in \mathcal{B}.$$

Let the function φ be defined on $(-\infty, 0]$ by $\varphi(\theta) = e^{-\frac{\gamma}{2}\theta}$.
Since, for $\theta = 0$, we have $\varphi(0) = 1$, and

$$\int_{-\infty}^{0} e^{\gamma\theta}|\varphi(\theta)|d\theta = \int_{-\infty}^{0} e^{\gamma\theta}e^{-\frac{\gamma}{2}\theta}d\theta = \int_{-\infty}^{0} e^{\frac{\gamma}{2}\theta}d\theta = \frac{2}{\gamma},$$

it follows, $\varphi \in \mathcal{B}$.
Now, for $0 < \varepsilon < 1$, we consider the scaled function $\varphi\left(\frac{\cdot}{\varepsilon}\right)$, given by:

$$\varphi\left(\frac{\theta}{\varepsilon}\right) = e^{-\frac{\gamma}{2\varepsilon}\theta} \text{ for all } \theta \leq 0.$$

We can easily check that the integral condition is not satisfied.

In fact, we have

$$\int_{-\infty}^{0} e^{\gamma\theta}\left|\varphi\left(\frac{\theta}{\varepsilon}\right)\right|d\theta = \int_{-\infty}^{0} e^{\gamma\theta}e^{-\frac{\gamma}{2\varepsilon}\theta}d\theta = \int_{-\infty}^{0} e^{\left(1-\frac{1}{2\varepsilon}\right)\gamma\theta}d\theta.$$

For $\varepsilon \leq \frac{1}{2}$, the integral diverges, and $\varphi\left(\frac{\cdot}{\varepsilon}\right)$ does not belong to $\mathcal{B}$.

Another example of phase space given in [9], where the condition (C_1) fails, is the phase space $\mathcal{B} = C_\gamma$, where C_γ is the space of continuous functions $\psi : \mathbb{R}^- \to \mathbb{R}$ such $e^{\gamma\theta}\psi(\theta)$ has the limit in $\mathbb{R}$ as $\theta \to -\infty$ endowed with the following norm:

$$\|\psi\|_{\mathcal{B}} = \sup\left\{e^{\gamma\theta}|\psi(\theta)| : \theta \in \right] -\infty, 0\left]\right\}.$$

It is clear that the function $\psi(\theta) = e^{-\gamma\theta}$ belongs to $\mathcal{B}$ but, for every $0 < \varepsilon < 1$,

$$\left\|\psi\left(\frac{\cdot}{\varepsilon}\right)\right\|_{\mathcal{B}} = \sup_{\theta\in]-\infty,0]} e^{\gamma\theta}e^{-\gamma(\theta/\varepsilon)} = \sup_{\theta\in]-\infty,0]} e^{\gamma\theta(1-1/\varepsilon)} = +\infty.$$

Now, in problem (4.1), we consider the change of variables

$$\begin{cases} \theta = \dfrac{\theta'}{\varepsilon}, & \tau = \dfrac{t}{\varepsilon}, \quad \theta \leq 0, \quad \tau \geq 0, \\ \psi\left(\dfrac{\theta'}{\varepsilon}\right) = \varphi(\theta), & w\left(\dfrac{t}{\varepsilon}\right) = x(t). \end{cases}$$

Then we have the following proposition.

Proposition 4.2 ([7]). *For the choice $\mathcal{B} = C_\infty$, the condition (C_1) is satisfied and the problems (4.1) and (4.2) can be equivalently rewritten as (P_ε) and (P_0), respectively.*

We suppose the following hypotheses:
Hypotheses. Suppose that the unbounded linear operator A satisfies the condition $(\mathbf{H}_0)$, $(\mathbf{H}_1)$ and the function f acts from $[0, a] \times \mathcal{B}_\alpha$ into X. We consider the following hypotheses:

(f_1) f is continuous with respect to the first and the second argument,

(f_2) for all $(u,v) \in \mathcal{B}_\alpha \times \mathcal{B}_\alpha$,

$$\|f(t,u) - f(t,v)\| \leq \mathbb{L}(\|u-v\|_\alpha) \quad \text{for all} \quad t \in [0,a],$$

where $\mathbb{L} : [0,+\infty[\to [0,+\infty[$ is such that:
(i) $\mathbb{L}(\cdot)$ is continuous, nondecreasing, and $\mathbb{L}(0) = 0$,
(ii) for every nonnegative continuous mapping $h : [0,a] \to [0,+\infty[$ and every constant ζ, we have,

$$h(t) \leq \zeta \int_0^t (t-s)^{-\alpha} \mathbb{L}(h(s))ds \quad \text{for all } t \in [0,a],$$

then $h(t) = 0$.

(f_3) there exists a constant $C > 0$ such that,

$$\|f(t,u)\| \leq C(1 + \|u\|_\alpha) \quad \text{for all } t \in \mathbb{R}^+, \text{ and } u \in \mathcal{B}_\alpha$$

(Af) there exist $\Delta_0 > 0$ and a function $f_0 : \mathcal{B}_\alpha \to X$ satisfying (f_2) and (f_3) and, for all $u \in \mathcal{B}_\alpha$ and $t_1, t_2 \in [0,a]$ with $0 \leq t_1 \leq t_2 \leq t_1 + \Delta_0 \leq a$, we have

$$\lim_{\varepsilon \to 0^+} \int_{t_1}^{t_2} T(t_2 - \theta) \left[f\left(\frac{\theta}{\varepsilon}, u\right) - f_0(u) \right] d\theta = 0.$$

The function $f_0 : u \mapsto \lim_{T\to\infty} \frac{1}{T} \int_0^T f(t,u)dt$ satisfies a condition similar to (f_2) and (f_3), which is

(f_1') for all $(u,v) \in \mathcal{B}_\alpha \times \mathcal{B}_\alpha$,

$$\|f_0(u) - f_0(v)\| \leq \mathbb{L}(\|u-v\|_\alpha),$$

(f_2') there exists a constant $C > 0$ such that, for all $u \in \mathcal{B}_\alpha$,

$$\|f_0(u)\| \leq C(1 + \|u\|_\alpha).$$

In fact, for $(u,v) \in \mathcal{B}^2$ we have

$$\begin{aligned}
\|f_0(u)\| &= \left\| \lim_{T\to\infty} \frac{1}{T} \int_0^T f(t,u)dt \right\| \\
&\leq \lim_{T\to\infty} \frac{1}{T} \int_0^T \|f(t,u)\|dt \\
&\leq \lim_{T\to\infty} \frac{1}{T} \int_0^T C(|1 + \|u\|_\alpha)dt \\
&\leq C(\|1 + \|u\|_\alpha)
\end{aligned}$$

and

$$\begin{aligned}
\|f_0(x) - f_0(y)\| &= \left\| \lim_{T\to\infty} \frac{1}{T} \int_0^T f(t,u) - f(t,v)dt \right\| \\
&\leq \lim_{T\to\infty} \frac{1}{T} \int_0^T \|f(t,u) - f(t,v)\|dt \\
&\leq \lim_{T\to\infty} \frac{1}{T} \int_0^T \mathbb{L}(\|u-v\|_\alpha)dt \\
&\leq \mathbb{L}(\|u-v\|_\alpha).
\end{aligned}$$

By (f_1') we can easily deduce the continuity of the function f_0.

Theorem 4.3. *Assume that the hypotheses* $(\mathbf{H}_0)$, $(\mathbf{H}_1)$, (f_1)–(f_3), *and* (Af) *are satisfied. For* $\varphi \in \mathcal{B}_\alpha$, *the unique mild solution of the problem* (P_ε), *converges uniformly in each set* $[0,a]$ *to the mild solution* x^∞ *of the problem* (P_0), *as* $\varepsilon \to 0^+$.

Proof. According to Theorem 3.2 , for each $\varepsilon > 0$ there exists a unique mild solution x^ε of the problem (P_ε) and there exists a unique mild solution x^∞ of the problem (P_0). Moreover, for each $\varepsilon > 0$, x^ε is defined by

$$x^\varepsilon(t) = T(t)\varphi(0) + \int_0^t T(t-s) f\Big(\frac{s}{\varepsilon}, x_s^\varepsilon\Big) ds, \quad 0 \le t \le a,$$

and x^∞ is defined by

$$x^\infty(t) = T(t)\varphi(0) + \int_0^t T(t-s) f_0(x_s^\infty) ds, \quad 0 \le t \le a.$$

By axiom $(\mathbf{A})$-(iii) if $x^\varepsilon \to x^\infty$ in $\mathcal{C}([0,a]; X_\alpha)$, then $x_t^\varepsilon \to x_t^\infty$ in $\mathcal{C}(]-\infty, a]; \mathcal{B}_\alpha)$ as $\varepsilon \to 0^+$. Therefore, in order to prove Theorem 4.3, it remains to show that

$$\lim_{\varepsilon \to 0^+} \sup_{t \in [0,a]} |x^\varepsilon(t) - x^\infty(t)|_\alpha = 0.$$

First, we prove that the sequence $(x^\varepsilon)_{\varepsilon>0}$ is bounded. Let $\varepsilon > 0$, by using (f_3) and Axiom $(\mathbf{A})$-(iii) and let $M := \sup_{0 \le t \le a} \|T(t)\|$. Then for every $t \in [0,a]$, we have

$$\begin{aligned}
|x^\varepsilon(t)|_\alpha &= \left| T(t)\varphi(0) + \int_0^t T(t-s) f\Big(\frac{s}{\varepsilon}, x_s\Big) ds \right|_\alpha \\
&\le \|T(t)A^\alpha \varphi(0)\| + \left\| \int_0^t A^\alpha T(t-s) f\Big(\frac{s}{\varepsilon}, x_s^\varepsilon\Big) ds \right\| \\
&\le M|\varphi(0)|_\alpha + M_\alpha \int_0^t \frac{e^{\omega(t-s)}}{(t-s)^\alpha} C(1 + \|x_s^\varepsilon\|_\alpha) ds. \\
&\le MH\|\varphi\|_\alpha + M_\alpha \int_0^t \frac{e^{\omega(t-s)}}{(t-s)^\alpha} C\left(1 + K_a \sup_{0 \le \theta \le s} |x^\varepsilon(\theta)|_\alpha + N_a \|\varphi\|_\alpha\right) ds.
\end{aligned}$$

Therefore,

$$|x^\varepsilon(t)|_\alpha \le A + B \int_0^t \frac{1}{(t-s)^\alpha} \sup_{0 \le \theta \le s} |x^\varepsilon(\theta)|_\alpha ds, \tag{4.3}$$

where

$$A = MH\|\varphi\|_\alpha + M_\alpha [C(1 + N_a \|\varphi\|_\alpha)] \int_0^a \frac{e^{\omega s}}{s^\alpha} ds,$$

and

$$B = M_\alpha C K_a e^{\omega a}.$$

Let $z(\sigma) = \sup_{0 \le \tau \le \sigma} |x^\varepsilon(\tau)|_\alpha$. We claim that the function $s \mapsto \int_0^s \frac{1}{(s-\sigma)^\alpha} z(\sigma) d\sigma$ is nondecreasing on $[0,t]$.

In fact let $s, s' \in [0,t]$ be such that $s < s'$. Then

$$\int_0^s \frac{1}{(s-\sigma)^\alpha} z(\sigma) d\sigma = \int_0^s \frac{1}{\sigma^\alpha} z(s-\sigma) d\sigma \le \int_0^{s'} \frac{1}{\sigma^\alpha} z(s'-\sigma) d\sigma \le \int_0^{s'} \frac{1}{(s'-\sigma)^\alpha} z(\sigma) d\sigma.$$

Then we obtain

$$\sup_{0 \le s \le t} |x^\varepsilon(s)|_\alpha \le A + B \int_0^t \frac{1}{(t-s)^\alpha} \sup_{0 \le \sigma \le s} |x^\varepsilon(\sigma)|_\alpha ds.$$

Thus, by Lemma 3.4 there is a constant C' such that

$$\sup_{0\le s\le t}|x^\varepsilon(s)|_\alpha \le C', \quad \text{for all } \varepsilon>0,$$

Which proves that the sequence $(x^\varepsilon)_{\varepsilon>0}$ is bounded.

Now by Axiom $(\mathbf{A})$-(iii), the function $t\mapsto x_t^\infty$ is continuous on $[0,a]$. Let $\delta>0$, then we can find a partition $0=t_0<t_1<\ldots<t_m=a$ of $[0,a]$ such that, if $\max_{1\le i\le m}(t_i-t_{i-1})\le \min\{\delta,\Delta_0\}$, then

$$\left\|x_t^\infty - x_{t_i}^\infty\right\| \le \delta \quad \text{for all } t\in[t_{i-1},t_i], \quad i=1,\ldots,m, \tag{4.4}$$

where Δ_0 is from (Af).

We define a function $t\mapsto \bar{x}_t^\infty$by

$$\bar{x}_t^\infty = x_{t_{i-1}}^\infty \quad \text{for } t\in[t_{i-1},t_i[, \quad i=1,\ldots,m. \tag{4.5}$$

Setting $\tau(t)=\max\{i,t_i\le t\}$ and $t'=t_{\o(t)}$, for every $t\in[0,a]$, we get

$$\begin{aligned}
x^\varepsilon(t)-x^\infty(t) &= \int_{t'}^t T(t-\theta)\left[f\left(\frac{\theta}{\varepsilon},x_\theta^\varepsilon\right)-f_0(x_\theta^\infty)\right]d\theta\\
&\quad+\int_0^{t'} T(t-\theta)\left[f\left(\frac{\theta}{\varepsilon},x_\theta^\varepsilon\right)-f\left(\frac{\theta}{\varepsilon},x_\theta^\infty\right)\right]d\theta\\
&\quad+\int_0^{t'} T(t-\theta)\left[f\left(\frac{\theta}{\varepsilon},x_\theta^\infty\right)-f\left(\frac{\theta}{\varepsilon},\bar{x}_\theta^\infty\right)\right]d\theta\\
&\quad+\int_0^{t'} T(t-\theta)\left[f\left(\frac{\theta}{\varepsilon},\bar{x}_\theta^\infty\right)-f_0(\bar{x}_\theta^\infty)\right]d\theta\\
&\quad+\int_0^{t'} T(t-\theta)[f_0(\bar{x}_\theta^\infty)-f_0(x_\theta^\infty)]d\theta\\
&= I_1+I_2+I_3+I_4+I_5.
\end{aligned}$$

Since the sequence $(x^\varepsilon)_{\varepsilon>0}$ is bounded, by Axiom $(\mathbf{A})$-(iii), the sequence $(x_t^\varepsilon)_{\varepsilon>0}$ is also bounded in $\mathcal{C}([0,a];\mathcal{B}_\alpha)$, for all $t\in[0,a]$. Thus,

$$\begin{aligned}
|I_1|_\alpha &= \left|\int_{t'}^t T(t-\theta)\left[f\left(\frac{\theta}{\varepsilon},x_\theta^\varepsilon\right)-f_0(x_\theta^\infty)\right]d\theta\right|_\alpha\\
&\le M_\alpha\int_{t'}^t \frac{e^{\omega(t-\theta)}}{(t-\theta)^\alpha}\left\|f\left(\frac{\theta}{\varepsilon},x_\theta^\varepsilon\right)-f_0(x_\theta^\infty)\right\|d\theta\\
&\le M_\alpha\int_{t'}^t \frac{e^{\omega(t-\theta)}}{(t-\theta)^\alpha}\left[C\left(1+\sup_{\varepsilon>0}\left\|x_{(.)}^\varepsilon\right\|_{\mathcal{C}([0,a];\mathcal{B}_\alpha)}\right)+C\left(1+\max_{t\in[0,a]}\|x_t^\infty\|_\alpha\right)\right]d\theta\\
&\le M_\alpha\left[C\left(1+\sup_{\varepsilon>0}\left\|x_{(.)}^\varepsilon\right\|_{\mathcal{C}([0,a];\mathcal{B}_\alpha)}\right)+C\left(1+\max_{t\in[0,a]}\|x_t^\infty\|_\alpha\right)\right]e^{\omega a}\int_{t'}^t\frac{1}{(t-\theta)^\alpha}d\theta.
\end{aligned}$$

Then by a simple change of variable, we get to

$$\int_{t'}^t\frac{1}{(t-\theta)^\alpha}d\theta=\frac{1}{1-\alpha}(t-t')^{1-\alpha}\le\frac{1}{1-\alpha}\delta^{1-\alpha}.$$

This leads to

$$|I_1|_\alpha\le\tilde{\mu}\delta^{1-\alpha},$$

where

$$\tilde{\mu} = M_\alpha C \frac{e^{\omega a}}{1-\alpha}\left[\left(1+\sup_{\varepsilon>0}\left\|x^\varepsilon_{(.)}\right\|_{\mathcal{C}([0,a];\mathcal{B}_\alpha)}\right)+\left(1+\max_{t\in[0,a]}\|x^\infty_t\|_\alpha\right)\right].$$

By axiom $(\mathbf{A})$, and hypothesis (f_2), we get

$$\begin{aligned}|I_2|_\alpha &\leq \left|\int_0^t T(t-\theta)\left[f\left(\frac{\theta}{\varepsilon},x^\varepsilon_\theta\right)-f\left(\frac{\theta}{\varepsilon},x^\infty_\theta\right)\right]d\theta\right|_\alpha\\ &\leq M_\alpha e^{\omega a}\int_0^t (t-\theta)^{-\alpha}\mathbb{L}\left(K(\theta)\sup_{0\leq s\leq\theta}|x^\varepsilon(s)-x^\infty(s)|_\alpha\right)d\theta.\end{aligned}$$

Further, let $q=\frac{2-\alpha}{1-\alpha}$, $p=2-\alpha$. By using (f_2), (4.4), (4.5) and Hölder's inequality we obtain

$$\begin{aligned}|I_3|_\alpha &\leq M_\alpha\int_0^t \frac{e^{\omega(t-\theta)}}{(t-\theta)^\alpha}\left\|f\left(\frac{\theta}{\varepsilon},x^\infty_\theta\right)-f\left(\frac{\theta}{\varepsilon},\bar{x}^\infty_\theta\right)\right\|d\theta\\ &\leq M_\alpha\left(\int_0^t\frac{e^{\omega p\theta)}}{\theta^{p\alpha}}d\theta\right)^{\frac{1}{p}}\left(\int_0^t\left\|f\left(\frac{\theta}{\varepsilon},x^\infty_\theta\right)-f\left(\frac{\theta}{\varepsilon},\bar{x}^\infty_\theta\right)\right\|^q d\theta\right)^{\frac{1}{q}}\\ &\leq C''\left(\sum_{i=1}^m\int_{t_{i-1}}^{t_i}\left\|f\left(\frac{\theta}{\varepsilon},x^\infty_\theta\right)-f\left(\frac{\theta}{\varepsilon},\bar{x}^\infty_\theta\right)\right\|^q d\theta\right)^{\frac{1}{q}}\\ &\leq C''\left(\sum_{i=1}^m\int_{t_{i-1}}^{t_i}\left\|f\left(\frac{\theta}{\varepsilon},x^\infty_\theta\right)-f\left(\frac{\theta}{\varepsilon},x^\infty_{t_{i-1}}\right)\right\|^q d\theta\right)^{\frac{1}{q}}\\ &\leq C''\mathbb{L}(\delta)a^{\frac{1}{q}},\end{aligned}$$

where $C''=M_\alpha\left(\int_0^t\frac{e^{\omega p\theta)}}{\theta^{p\alpha}}d\theta\right)^{\frac{1}{p}}$.

For the term I_4, we have

$$\begin{aligned}|I_4|_\alpha &= \left|\int_0^{t'}T(t-\theta)\left[f\left(\frac{\theta}{\varepsilon},\bar{x}^\infty_\theta\right)-f_0(\bar{x}^\infty_\theta)\right]d\theta\right|_\alpha\\ &= \left|\sum_{i=1}^{\tau(t)}\int_{t_{i-1}}^{t_i}T(t-t_i)T(t_i-\theta)\left[f\left(\frac{\theta}{\varepsilon},x^\infty_{t_{i-1}}\right)-f_0\left(x^\infty_{t_{i-1}}\right)\right]d\theta\right|_\alpha\\ &\leq M\sum_{i=1}^{\tau(t)}\left|\int_{t_{i-1}}^{t_i}T(t_i-\theta)\left[f\left(\frac{\theta}{\varepsilon},x^\infty_{t_{i-1}}\right)-f_0\left(x^\infty_{t_{i-1}}\right)\right]d\theta\right|_\alpha,\end{aligned}$$

where $M=\sup_{t\in[0,a]}\|T(t)\|$.

Bearing in mind hypothesis (Af), we can see

$$\max_{1\leq i\leq m}\lim_{\varepsilon\to0^+}\int_{t_{i-1}}^{t_i}T(t_i-\theta)\left[f\left(\frac{\theta}{\varepsilon},x^\infty_{t_{i-1}}\right)-f_0\left(x^\infty_{t_{i-1}}\right)\right]d\theta=0.$$

Thus,

$$\max_{1\leq i\leq m}\left|\int_{t_{i-1}}^{t_i}T(t_i-\theta)\left[f\left(\frac{\theta}{\varepsilon},x^\infty_{t_i-1}\right)-f_0\left(x^\infty_{t_{i-1}}\right)\right]d\theta\right|_\alpha\leq\frac{\delta}{m}\quad\text{as }\varepsilon\to0^+.$$

Therefore,

$$|I_4|_\alpha\leq M\frac{\tau(t)}{m}\delta\leq M\delta\quad\text{as }\varepsilon\to0^+.$$

Since the function f_0 satisfies a condition similar to (f_2), the term I_5 can be estimated as I_3, thus,

$$|I_5|_\alpha \leq C''\mathbb{L}(\delta)a^{\frac{1}{q}}.$$

From the estimates obtained for the terms I_1 to I_5, for every $t \in [0,a]$, we get

$$\begin{aligned}|x^\varepsilon(t) - x^\infty(t)|_\alpha &\leq \tilde{\mu}\delta^{1-\alpha} + M\delta + 2C''\mathbb{L}(\delta)a^{\frac{1}{q}} \\ &\quad + M_\alpha e^{\omega a}\int_0^t (t-\theta)^\alpha \mathbb{L}\left(K(\theta)\sup_{0\leq s\leq\theta}|x^\varepsilon(s) - x^\infty(s)|_\alpha\right)d\theta, \quad \text{as } \varepsilon \to 0^+.\end{aligned}$$

By the same way as in the proof of Theorem 3.2, the last expression does not decrease, and if we set

$$r(\delta) = \tilde{\mu}\delta^{1-\alpha} + M\delta + 2C''\mathbb{L}(\delta)a^{\frac{1}{q}}, \quad \text{for every } t \in [0,a],$$

we get

$$\begin{aligned}&K_a \sup_{0\leq s\leq t}|x^\varepsilon(s) - x^\infty(s)|_\alpha \\ &\quad \leq K_a r(\delta) + K_a M_\alpha e^{\omega a}\int_0^t (t-\theta)^\alpha \mathbb{L}\left(K_a \sup_{0\leq s\leq\theta}|x^\varepsilon(s) - x^\infty(s)|_\alpha\right)d\theta \quad \text{as } \varepsilon \to 0^+.\end{aligned}$$

In view of (f_2), the function $\delta \to K_a r(\delta)$ is continuous on $[0,+\infty[$ and $K_a r(0) = 0$. Thus, in view of the arbitrariness of δ, we necessarily conclude that

$$K_a \sup_{0\leq s\leq t}|x^\varepsilon(s) - x^\infty(s)|_\alpha \leq M_\alpha e^{\omega a}\int_0^t (t-\theta)^\alpha \mathbb{L}\left(K_a \sup_{0\leq s\leq\theta}|x^\varepsilon(s) - x^\infty(s)|_\alpha\right)d\theta$$

as $\varepsilon \to 0^+$.

By using (f_2) once again, for every $t \in [0,a]$, we can write

$$K_a \sup_{0\leq s\leq t}|x^\varepsilon(s) - x^\infty(s)|_\alpha = 0 \quad \text{as } \varepsilon \to 0^+.$$

Since K_a is a positive constant, we conclude

$$\sup_{0\leq s\leq t}|x^\varepsilon(s) - x^\infty(s)|_\alpha = 0 \quad \text{as } \varepsilon \to 0^+.$$

This completes the proof. □

Now we give the averaging principle with a more general form.

We denote by $(Af)'$ the following hypothesis :

$$\lim_{T\to+\infty}\frac{1}{T}\int_0^T f(s,u)ds = f_0(u), \quad \text{for all } u \in \mathcal{B}_\alpha.$$

We have the following lemma.

Lemma 4.4 ([19]). *Assume that the hypotheses* ($\mathbf{H}_0$), (f_1)-(f_3) *and* $(Af)'$ *are satisfied. Moreover, suppose that* X *is reflexive. Then*

$$\lim_{\varepsilon\to0^+}\int_{t_1}^{t_2} T(t_2-\theta)\left[f\left(\frac{\theta}{\varepsilon},u\right) - f_0(u)\right]d\theta = 0.$$

Corollary 4.5. *Assume that the hypotheses* $(\mathbf{H}_0)$, $(\mathbf{H}_1)$, (f_1)–(f_3) *and* $(Af)'$ *are satisfied. Moreover, suppose that* X *is reflexive. Then the unique mild solution* x^ε *of the problem* (P_ε) *converges to the unique mild solution* x^∞ *of the problem* (P_0)*, i.e.,*

$$\lim_{\varepsilon\to 0^+} \|x^\varepsilon - x^\infty\|_{\mathcal{C}(]-\infty,a],X_\alpha)} = 0.$$

Proof. According to Lemma 4.4 all the hypotheses of Theorem 4.3 are satisfieded then the result. □

5 Application

Example 5.1 (Existence result)**.** We consider the following partial differential equation:

$$\begin{cases} \dfrac{\partial}{\partial t}v(t,\xi) = \dfrac{\partial^2}{\partial \xi^2}v(t,\xi) + F\left(t, \displaystyle\int_{-\infty}^{0} a_1(\theta)\int_0^\xi \frac{\partial}{\partial\sigma}v(t-r,\sigma)d\sigma d\theta, \int_{-\infty}^{t}\int_0^\xi a_2(\theta-t)v(\theta,\sigma)d\sigma d\theta\right) \\ v(t,0) = v(t,1) = 0, \quad t \geq 0, \\ v(\theta,\xi) = v_0(\theta,\xi), \quad \theta \leq 0, \quad \xi \in [0,1], \end{cases} \tag{5.1}$$

where $F : \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. The function $v_0 : (-\infty, 0] \times [0,1] \to \mathbb{R}$ is continuous and it will be described below.

In order to rewrite equation (5.1) in the abstract form, we choose $X = L^2([0,1];\mathbb{R})$ and we define the operator A on X by

$$\begin{cases} D(A) = H^2(0,1) \cap H_0^1(0,1), \\ Ay = -y''. \end{cases}$$

Then, by Theorem 2.7, page 211 in [16], we get that $-A$ generates an analytic compact semigroup $(T(t))_{t\geq 0}$ on X. From [24], the semigroup $T(t)$ is explicitly given by

$$T(t)y = \sum_{n=1}^{\infty} e^{-n^2 t}\langle y, e_n\rangle e_n \quad \text{for } y \in X.$$

If we choose $\alpha = \frac{1}{2}$, then

$$\begin{cases} A^{\frac{1}{2}}T(t)y = \sum_{n=1}^{\infty} n e^{-n^2 t}\langle y, e_n\rangle e_n & \text{for } y \in X, \\ A^{-\frac{1}{2}}y = \sum_{n=1}^{\infty} \frac{1}{n}\langle y, e_n\rangle e_n & \text{for } y \in X, \\ A^{\frac{1}{2}}y = \sum_{n=1}^{\infty} n\langle y, e_n\rangle e_n & \text{for } y \in D\left(A^{\frac{1}{2}}\right). \end{cases} \tag{5.2}$$

Lemma 5.2 ([24])**.** *If* $y \in D\left(A^{\frac{1}{2}}\right)$*, then* y *is absolutely continuous,* $y' \in X$ *and*

$$\|y'\| = \left\|A^{\frac{1}{2}}y\right\|.$$

Let $\gamma > 0$, we consider the following phase space introduced in [9], which satisfies all the axioms given in the first section

$$\mathcal{B} = \left\{\varphi \in C((-\infty,0];X) : \lim_{\theta\to-\infty} e^{\gamma\theta}\varphi(\theta) \text{ exists in } X\right\}.$$

We equip it with the following norm

$$\|\varphi\|_{\mathcal{B}} = \sup_{\theta\leq 0} e^{\gamma\theta}\|\varphi(\theta)\| \quad \text{for } \varphi \in \mathcal{B}.$$

The norm in $\mathcal{B}_{\frac{1}{2}}$ is given by

$$\|\varphi\|_{\frac{1}{2}} = \sup_{\theta\leq 0} e^{\gamma\theta}\left\|A^{\frac{1}{2}}\varphi(\theta)\right\| = \sup_{\theta\leq 0} e^{\gamma\theta}\sqrt{\int_0^\pi \left(\frac{\partial}{\partial\xi}(\varphi)(\theta)(\xi)\right)^2 d\xi}.$$

Assume that

$(\mathbf{H}_4)$ $e^{-2\gamma\cdot}a_i(\cdot) \in L^2(\mathbb{R}^-)$, for $i \in \{1,2\}$.

Since

$$\begin{aligned}\int_{-\infty}^t a_2(\theta-t)\int_0^\xi v(\theta,\sigma)\mathrm{d}\sigma\mathrm{d}\theta &= \int_{-\infty}^0 a_2(\theta)\int_0^\xi v(t+\theta,\sigma)\mathrm{d}\sigma\mathrm{d}\theta\\ &= \int_{-\infty}^0 a_2(\theta)\int_0^\xi v_t(\theta,\sigma)\mathrm{d}\sigma\mathrm{d}\theta.\end{aligned}$$

We introduce

$$x(t)(\xi) = v(t,\xi), \quad t \geq 0, 0 \leq \xi \leq 1,$$

$$\varphi(\theta)(\xi) = v_0(\theta,\xi), \quad -\infty < \theta \leq 0, 0 \leq \xi \leq 1,$$

$$f(t,\varphi)(\xi) = F\left(t, \int_{-\infty}^0 a_1(\theta)\int_0^\xi \frac{\partial}{\partial\sigma}(\varphi)(-r)(\sigma)d\sigma d\theta, \int_{-\infty}^0 a_2(\theta)\int_0^\xi \varphi(\theta)(\sigma)d\sigma d\theta\right).$$

Then the problem (5.1) can be rewritten as follows

$$\begin{cases} x'(t) = Ax(t) + f(t,x_t) & \text{for } t \geq 0,\\ x_0 = \varphi \in \mathcal{B}_\alpha. \end{cases} \tag{5.3}$$

Let us consider the following hypotheses on the function F.

(a) $F : [0,1]\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is continuous.

(b) There exists a constant $C > 0$, such that $|F(t,\mu,y) - F(t,\mu',y')| \leq C\lambda(|\mu-\mu'| + |y-y'|)$, where $\lambda(\cdot) : [0,+\infty[\to [0,+\infty[$ is a continuous, nondecreasing function with $\lambda(0) = 0$.

As example of such function $\lambda, +\infty[\to [0,+\infty[$, one can write

$$\lambda(z) = \begin{cases} 0, & z = 0\\ -z\ln z, & 0 < z < 1/e,\\ 1/e, & z > 1/e. \end{cases}$$

The function λ is continuous nondecreasing. For more example one can see [22].

Theorem 5.3. *Assume that the hypotheses (a), (b) and* $(\mathbf{H}_4)$ *are satisfied. Then the problem* (5.3) *has a unique mild solution in X.*

Proof. According to Theorem 3.2, we have to prove that f satisfies the hypotheses $(\mathbf{H}_2)$. Define in the space $\mathcal{B}_{\frac{1}{2}}$, an equivalent norm $|||\cdot|||_{\frac{1}{2}}$ by

$$|||\varphi|||_{\frac{1}{2}} = 2v\|\psi\|_{\frac{1}{2}},$$

with v is a positive constant chosen, such that

$$v \geq \frac{1}{\sqrt{2\gamma}} \max\Big(\|e^{-2\gamma(\cdot)}a_1(\cdot)\|_{L^2(\mathbb{R}^-)}; \|e^{-2\gamma(\cdot)}a_2(\cdot)\|_{L^2(\mathbb{R}^-)}\Big).$$

Using hypothesis (b) and the fact that the function $\lambda(\cdot)$ is nondecreasing, for $\psi_1, \psi_2 \in \mathcal{B}_{\frac{1}{2}}$, and for $\xi \in [0,1]$, we have

$$\begin{aligned}
&|\, f(t,\psi_1)(\xi) - f(t,\psi_2)(\xi)|^2 \\
&\leq C^2\Bigg[\lambda\Bigg(\Big(\int_{-\infty}^0 e^{-4\gamma\theta}a_1(\theta)^2 d\theta\Big)^{\frac{1}{2}}\Big(\int_{-\infty}^0 e^{4\gamma\theta}\int_0^\xi \frac{\partial}{\partial\sigma}(\psi_1(r)-\psi_2)^2(-r)(\sigma)d\sigma d\theta\Big)^{\frac{1}{2}} \\
&\quad +\Big(\int_{-\infty}^0 e^{-4\gamma\theta}a_2(\theta)^2 d\theta\Big)^{\frac{1}{2}}\Big(\int_{-\infty}^0 e^{4\gamma\theta}\Big(\int_0^\xi \Big(\psi_1(\theta)(\sigma)-\psi_2(\theta)(\sigma)\Big)\Big)^2 \mathrm{d}\sigma \mathrm{d}\theta\Big)^{\frac{1}{2}}\Bigg)\Bigg]^2 \\
&\leq C^2\Bigg[\lambda\Bigg(\Big(\int_{-\infty}^0 e^{-4\gamma\theta}a_1(\theta)^2 d\theta\Big)^{\frac{1}{2}}\frac{1}{\sqrt{2\gamma}}\sup_{\theta\leq 0}e^{\gamma\theta}\Big(\int_0^1 \frac{\partial}{\partial\sigma}(\psi_1-\psi_2)^2(-r)(\sigma)d\sigma\Big)^{\frac{1}{2}} \\
&\quad +\Big(\int_{-\infty}^0 e^{-4\gamma\theta}a_2(\theta)^2 d\theta\Big)^{\frac{1}{2}}\frac{1}{\sqrt{2\gamma}}\sup_{\theta\leq 0}e^{\gamma\theta}\Big(\int_0^1 \Big(\psi_1(\theta)(\sigma)-\psi_2(\theta)(\sigma)\Big)\Big)^2 \mathrm{d}\sigma\Big)^{\frac{1}{2}}\Bigg)\Bigg]^2 \\
&\leq C^2\Bigg[\lambda\Bigg(v\sup_{\theta\leq 0}e^{\gamma\theta}\Big(\|A^{\frac{1}{2}}(\psi_1-\psi_2)(\theta)\|+\Big(\int_0^1 \big(\psi_1(\theta)(\sigma)-\psi_2(\theta)(\sigma)\big)^2 \mathrm{d}\sigma\Big)^{\frac{1}{2}}\Big)\Bigg)\Bigg]^2.
\end{aligned}$$

Using formulas (5.2), one has

$$\begin{aligned}
\int_0^1 \Big(\psi_1(\theta)(\sigma)-\psi_2(\theta)(\sigma)\Big)^2 d\xi &\leq \int_0^1 \Big(\frac{\partial}{\partial\sigma}\Big(\psi_1(\theta)(\sigma)-\psi_2(\theta)(\sigma)\Big)\Big)^2 d\sigma \\
&= \Big\|A^{\frac{1}{2}}\Big(\psi_1(\theta)-\psi_2(\theta)\Big)\Big\|^2.
\end{aligned}$$

Then we get

$$|\, f(t,\psi_1)(\xi) - f(t,\psi_2)(\xi)|^2 \leq C^2\Bigg[\lambda\Big(2v\sup_{\theta\leq 0}e^{\gamma\theta}\|\psi_1-\psi_2\|_{\frac{1}{2}}\Big)\Bigg]^2,$$

which implies that for all $\psi_1,\ \psi_2 \in \mathcal{B}_{\frac{1}{2}}$, we have

$$\|f(t,\psi_1)-f(t,\psi_2)\|_{L^2(0,1)} \leq C\lambda(|||\psi_1-\psi_2|||_{\frac{1}{2}}).$$ □

Example 5.4 (Averaging principle)**.** Now, we present an example about the averaging principle result.

Consider the following partial differential equation

$$\begin{cases} x'(t) = -Ax(t) + f\big(\frac{t}{\varepsilon}, x_t\big), \\ x_0 = \varphi \in \mathcal{B}_\alpha, \end{cases} \tag{5.4}$$

such that, $-A$ generates an analytic semigroup compact on a reflexive space X (as example we can take A and X given in Example 5.1.

Concerning the phase space $\mathcal{B}$ is identically the phase space $\mathcal{C}_\infty$ described in Section 4, and which satisfies the condition (C_1) (for more examples we refer the reader to a book by Hino, Murakami and Naito [11]).

The function $f : \mathbb{R}^+ \times \mathcal{B}_\alpha \to \mathbb{R}$ is defined as follows

$$f\left(\frac{t}{\varepsilon}, u\right) = \frac{\sin\left(\frac{t}{\varepsilon} + u\right)}{e^{\frac{t}{\varepsilon}}} + 1, \quad \text{for all } t > 0,\ \varepsilon > 0 \ \text{ and } u \in \mathcal{B}_\alpha.$$

Since $-A$ generates an analytic semigroup compact, then $(\mathbf{H}_0)$ is satisfied.

Due to the boundedness of f we can easily show that f meets all the conditions (f_1)–(f_3).

Taking $f_0(u) = 1$, then we can verify that the averaging result condition is satisfied, in fact:

$$\begin{aligned}
\left|\frac{1}{T}\int_0^T f(t,u)dt - f_0(u)\right| &= \left|\frac{1}{T}\int_0^T \frac{\sin(t+u)}{e^t} + 1dt\ - 1\right| \\
&= \left|\frac{1}{T}\int_0^T \frac{\sin(t+u)}{e^t} + 1dt\ - \frac{1}{T}\int_0^T dt\right| \\
&\leq \frac{1}{T}\int_0^T \left|\frac{\sin(t+u)}{e^t}\right| dt \\
&\leq \frac{1}{T}\int_0^T e^{-t}dt \\
&= \frac{1-e^{-T}}{T} \to 0, \quad \text{as } T \to \infty.
\end{aligned}$$

Based on the above discussion and the result in Corollary 4.5, we can deduce that the solution of Equation (5.4) can be approximated as $\varepsilon \to 0$, by the solution of the following autonomous partial differential equation,

$$\begin{cases} x'(t) = -Ax(t) + f_0(x_t), \\ x_0 = \varphi \in \mathcal{B}_\alpha. \end{cases} \tag{5.5}$$

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Nonexistence and existence of solutions for a Choquard–Kirchhoff type equation involving mixed local and nonlocal operators

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Abstract. In this paper, we study the following Choquard–Kirchhoff type equation

$$\left(a+b\|u\|^{p(\theta-1)}\right)\left(-\Delta_p u+(-\Delta)_p^s u\right)=\left(\int_{\mathbb{R}^N}\frac{|u(y)|^{p_\mu^*}}{|x-y|^\mu}dy\right)|u|^{p_\mu^*-2}u+\lambda f(x)\ \text{ in } \mathbb{R}^N,$$

where $a,b>0$, $\theta>1$, $\lambda\geq 0$, $0<s<1<p<N$, and $0<\mu<N$. Here $p_\mu^*=\frac{p(2N-\mu)}{2(N-p)}$ denotes the critical exponent associated with the Hardy–Littlewood–Sobolev inequality, and the norm

$$\|u\|=\left(\int_{\mathbb{R}^N}|\nabla u|^p\,dx+\iint_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\right)^{\frac{1}{p}}.$$

The function $f\in L^{\frac{p^*}{p^*-1}}(\mathbb{R}^N)\setminus\{0\}$ is nonnegative, with $p^*=\frac{Np}{N-p}$ being the Sobolev critical exponent. For $\lambda=0$, we establish the nonexistence of nontrivial solutions. For $\lambda>0$, by developing a concentration compactness principle suited to the local-nonlocal setting and combining it with Ekeland's variational principle and the mountain pass theorem, we obtain the existence of multiple nonnegative solutions.

Keywords: Kirchhoff-type equation, local-nonlocal operators, critical equation, Choquard nonlinearity, concentration compactness principle.

2020 Mathematics Subject Classification: 35A15, 35M10, 35J92, 35R11.

1 Introduction

In this paper, we investigate the following Choquard–Kirchhoff type problem that combines mixed local and non-local operators

$$\begin{cases}\left(a+b\|u\|^{p(\theta-1)}\right)\left(-\Delta_p u+(-\Delta)_p^s u\right)=\left(\int_{\mathbb{R}^N}\frac{|u(y)|^{p_\mu^*}}{|x-y|^\mu}\,dy\right)|u|^{p_\mu^*-2}u+\lambda f(x)\ \text{ in } \mathbb{R}^N,\\ u\in X^{1,p}(\mathbb{R}^N),\end{cases}\tag{1.1}$$

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where $a > 0$, $b > 0$, $\theta > 1$, $\lambda \geq 0$ is a parameter. Here, $0 < s < 1 < p < N$, $0 < \mu < N$, and $p_\mu^* = \frac{p(2N-\mu)}{2(N-p)}$ is the critical exponent in the sense of the Hardy–Littlewood–Sobolev inequality. The left hand side of (1.1) involves the classical p-Laplace operator $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, and $(-\Delta)_p^s$ is the fractional p-Laplacian operator. Up to normalization factors, for any $x \in \mathbb{R}^N$ and $u \in C_0^\infty(\mathbb{R}^N)$, it can be defined as

$$(-\Delta)_p^s u(x) = \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|u(y) - u(x)|^{p-2}(u(y) - u(x))}{|x-y|^{N+ps}}\, dy,$$

where $B_\varepsilon(x)$ denotes the ball in $\mathbb{R}^N$ centered at x with radius ε. For $s \in (0,1)$ and a measurable function $u : \mathbb{R}^{\mathbb{N}} \longrightarrow \mathbb{R}$, we define the Gagliardo seminorm as

$$[u]_{s,p} = \left(\iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x-y|^{N+ps}}\, dx\, dy \right)^{\frac{1}{p}}.$$

Moreover, the space $X^{1,p}(\mathbb{R}^N)$ is defined as the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\| = \left(\int_{\mathbb{R}^N} |\nabla u|^p\, dx + \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x-y|^{N+ps}}\, dx\, dy \right)^{\frac{1}{p}}.$$

In recent years, the research on partial differential equations incorporating a blend of local and non-local operators has witnessed substantial advancements in both theoretical exploration and practical implementations. These hybrid operators, constructed through the superposition of the p-Laplacian $(-\Delta)_p$ and the fractional p-Laplacian $(-\Delta)_p^s$, have emerged as significant mathematical models in diverse disciplines. For instance, they play a crucial role in population dynamics, where they can describe the movement and distribution of species, in biomathematics for modeling biological processes, and in optimal foraging strategies to understand how organisms search for food (see [19, 20, 37] and the references therein). Concurrently, these operators have spurred novel developments within the realm of mathematical analysis. This includes the generalization of classical inequalities such as the Hong–Krahn–Szegő inequality [10] and the Faber–Krahn inequality [9], as well as the in-depth investigation of non-local evolution equations, exemplified by the Cahn–Hilliard equation [14] and the porous medium equation [18].

The interplay between local and non-local diffusions has become a focal point of interest for numerous mathematicians. In the specific case of the fractional Laplacian ($p = 2$), variational techniques were employed in [5, 39, 40] to prove the existence and multiplicity of solutions for elliptic equations. When $p \neq 2$, the Brézis–Nirenberg problem associated with fractional p-Laplacian operators was explored in [35]. Moreover, in [30], Morse theory was applied to study the existence of solutions under general reaction terms. Fiscella and Pucci [23] delved into stationary Kirchhoff problems involving fractional operators, Hardy potentials, and critical nonlinearities, deriving results regarding the asymptotic behavior of solutions. The ground state solutions for fractional Choquard equations with sub-critical nonlinearities were also investigated in [7].

Kirchhoff-type equations that combine fractional Laplacians and critical Sobolev exponents have attracted considerable attention from the academic community. These studies cover both bounded domains [22, 36, 42] and the entire space [24, 29, 33], involving Kirchhoff models with critical-growth nonlinearities. Recent contributions in this area include fractional p-Laplacian Choquard–Kirchhoff equations [38], fractional Kirchhoff problems with a

combination of Choquard and singular terms [41], and Choquard–Kirchhoff equations featuring Hardy–Littlewood–Sobolev critical exponents [31]. In [43], nonhomogeneous p-Kirchhoff problems with convex-concave nonlinearities were analyzed. Additionally, Xiang et al. [44] studied the following fractional p-Kirchhoff problem:

$$\begin{cases} \left(a + b[u]_{s,p}^{p(\theta-1)}\right)(-\Delta)_p^s u = |u|^{p_s^*-2}u + \lambda f(x) \ \text{ in } \mathbb{R}^N, \\ u \in D^{s,p}(\mathbb{R}^N), \end{cases} \tag{1.2}$$

where $a \geq 0, b > 0, \theta > 1, 1 < p < N/s$, $p_s^* = \frac{Np}{N-ps}$, $\lambda \geq 0$ is a parameter, $f \in L^{\frac{p_s^*}{p_s^*-1}}(\mathbb{R}^N) \setminus \{0\}$, and $D^{s,p}(\mathbb{R}^N)$ is defined as the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the Gagliardo seminorm $[u]_{s,p}$. For the case of $\lambda = 0$, Xiang et al. demonstrated that the non - existence of non-trivial solutions to (1.2) depends on the parameters N, θ, s, p, a and b. When $\lambda > 0$, they utilized Ekeland's variational principle and the mountain pass theorem to prove the existence of at least two non-negative solutions.

Driven by the extensive applications of mixed local-nonlocal operators, considerable research efforts have been devoted to the existence and qualitative analysis of solutions to equations of the form

$$-\Delta_p u + (-\Delta)_p^s u = f(x,u) \quad \text{in } \Omega,$$

where $\Omega \subset \mathbb{R}^N$ is a domain, $p \in (1,\infty)$, $s \in (0,1)$ and f is a given function. For $p = 2$, symmetry properties of solutions to semilinear equations involving such hybrid operators were established in [11]. Fundamental contributions to existence theory, local boundedness, the strong maximum principle, Lipschitz regularity, and interior Sobolev regularity can be found in [1,6,8]. By combining variational techniques with a Pohožaev identity, Anthal et al. [4] derived existence and nonexistence results for Choquard-type equations with subcritical perturbations.

When $p \neq 2$, Garain and Kinnunen [27] investigated regularity properties-including local boundedness, Harnack estimates, local Hölder continuity, and semicontinuity-for weak solutions in the homogeneous case $f = 0$. In the nonhomogeneous case $f \neq 0$, Da Silva et al. [17] recently proved the existence and multiplicity of nontrivial solutions by integrating variational methods with topological tools such as the Krasnoselskii genus and Lusternik–Schnirelman category theories.

Define the optimal constants

$$S_{H,C} = \inf_{u \in C_0^\infty(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^p \, dx}{\left(\iint_{\mathbb{R}^{2N}} \frac{|u(x)|^{p_\mu^*} |u(y)|^{p_\mu^*}}{|x-y|^\mu} \, dx \, dy \right)^{\frac{p}{2p_\mu^*}}}, \tag{1.3}$$

and

$$S_{H,M} = \inf_{u \in C_0^\infty(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^p \, dx + \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} \, dx \, dy}{\left(\iint_{\mathbb{R}^{2N}} \frac{|u(x)|^{p_\mu^*} |u(y)|^{p_\mu^*}}{|x-y|^\mu} \, dx \, dy \right)^{\frac{p}{2p_\mu^*}}}, \tag{1.4}$$

well-posed via the classical Hardy–Littlewood–Sobolev inequality (cf. Proposition 2.1). Leveraging these definitions, we establish the following nonexistence result for trivial solutions when $\lambda = 0$.

Theorem 1.1. *Assume $\lambda = 0$ and $\theta > 1$. Then problem (1.1) admits no nontrivial solution.*

For the case $\lambda \neq 0$, we analyze problem (1.1). Owing to the lack of explicit expressions and asymptotic estimates for minimizers of $S_{H,M}$, we cannot directly characterize the threshold for nontrivial solutions as in [12, 26, 35]. To address the compactness deficit, we develop a specialized concentration-compactness principle for mixed local-nonlocal operators (see Section 4).

Henceforth, unless stated otherwise, we assume f satisfies

(f) $f \geq 0$, $f \not\equiv 0$ and $f \in L^{\frac{p^*}{p^*-1}}(\mathbb{R}^N)$.

Theorem 1.2. *Suppose $a > 0$, $b > 0$, and either $\theta = \frac{3N-\mu-p}{2(N-p)}$ or $a > 0$, $0 < b < S_{H,M}^{-\frac{2p_\mu^*}{p}}$, $\theta = \frac{2N-\mu}{N-p}$. Under assumption (f), there exists $\lambda^* > 0$ such that for all $\lambda \in (0, \lambda^*)$, problem (1.1) possesses a nonnegative weak solution $u_1 \in X^{1,p}(\mathbb{R}^N)$ satisfying $I_\lambda(u_1) < 0$. Moreover, there exists $\lambda^{**} \in (0, \lambda^*]$ such that for all $\lambda \in (0, \lambda^{**})$, problem (1.1) admits another nonnegative weak solution $u_2 \in X^{1,p}(\mathbb{R}^N)$ with $I_\lambda(u_2) > 0$.*

The paper is organized as follows. Section 2 reviews essential preliminaries. In Section 3, we analyze the optimal constant $S_{H,M}$ and prove Theorem 1.1. Section 4 extends Lions' concentration-compactness principle [32] to the mixed operator framework. Finally, Section 5 employs Ekeland's variational principle and the mountain pass theorem to establish the existence of two nonnegative solutions for (1.1) within a suitable parameter range of λ.

Notations

- The standard norm on $L^q(\mathbb{R}^N)$ denotes by $|\cdot|_q$ for $q \in [1, \infty)$.
- We shall also use the notation $\|u\|_{HL}^p = \big(\iint_{\mathbb{R}^{2N}} \frac{|u(x)|^{p_\mu^*}|u(y)|^{p_\mu^*}}{|x-y|^\mu}\,dx\,dy\big)^{\frac{p}{2p_\mu^*}}$ for $u \in X^{1,p}(\mathbb{R}^N)$.
- C denotes positive constant possibly different from line to line.

2 Preliminaries

Proposition 2.1 (Hardy–Littlewood–Sobolev inequality)**.** *Let $r, q > 1$ and $0 < \mu < N$ satisfy $\frac{1}{r} + \frac{1}{q} + \frac{\mu}{N} = 2$. For $g \in L^r(\mathbb{R}^N)$ and $h \in L^q(\mathbb{R}^N)$, there exists a constant $C(r, q, n, \mu)$ independent of g and h such that*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{g(x)h(x)}{|x-y|^\mu}\,dx\,dy \leq C(r, q, N, \mu)|g|_r|h|_q.$$

In particular, setting $g = h = |u|^t$, the Hardy–Littlewood–Sobolev inequality ensures the integral

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^t|u(y)|^t}{|x-y|^\mu}\,dx\,dy$$

is well defined provided $|u|^t \in L^v(\mathbb{R}^N)$ with $v = \frac{2N}{2N-\mu}$. By the Sobolev embedding $X^{1,p}(\mathbb{R}^N) \hookrightarrow L^{p^*}(\mathbb{R}^N)$, we deduce $t = \frac{p(2N-\mu)}{2(N-p)} := p_\mu^*$. Moreover, for $u \in X^{1,p}(\mathbb{R}^N)$, it holds that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^{p_\mu^*}|u(y)|^{p_\mu^*}}{|x-y|^\mu}\,dx\,dy \leq C(N, \mu)|u|_{p^*}^{2p_\mu^*}.$$

Next, we introduce the concept of weak solutions for problem (1.1).

Definition 2.2. A function $u \in X^{1,p}(\mathbb{R}^N)$ is a weak solution of (1.1) if for all $\varphi \in X^{1,p}(\mathbb{R}^N)$, it satisfies

$$\begin{aligned}
&\Big(a + b\|u\|^{(\theta-1)p}\Big) \\
&\quad \times \left(\int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u \nabla \varphi \, dx + \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+ps}} \, dx\, dy \right) \\
&= \iint_{\mathbb{R}^{2N}} \frac{|u(y)|^{p_\mu^*}|u(x)|^{p_\mu^*-2}u(x)\varphi(x)}{|x-y|^{\mu}} \, dx\, dy + \lambda \int_{\mathbb{R}^N} f(x)\varphi(x)\, dx
\end{aligned} \tag{2.1}$$

We define the energy functional

$$I_\lambda(u) = \frac{a}{p}\|u\|^p + \frac{b}{\theta p}\|u\|^{\theta p} - \frac{1}{2p_\mu^*}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u^+(y)|^{p_\mu^*}|u^+(x)|^{p_\mu^*}}{|x-y|^{\mu}} \, dx\, dy - \lambda \int_{\mathbb{R}^N} f(x)u \, dx \tag{2.2}$$

for $u \in X^{1,p}(\mathbb{R}^N)$, where $u^+ = \max\{u, 0\}$. Since $f \in (X^{1,p}(\mathbb{R}^N))'$, the functional I_λ belongs to $C^1(X^{1,p}(\mathbb{R}^N))$. For all $u, \varphi \in X^{1,p}(\mathbb{R}^N)$, its derivative is given by

$$\begin{aligned}
&\langle I_\lambda'(u), \varphi\rangle \\
&\quad = \Big(a + b\|u\|^{(\theta-1)p}\Big) \\
&\qquad \times \left(\int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u \nabla \varphi \, dx + \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+ps}} \, dx\, dy \right) \\
&\qquad - \iint_{\mathbb{R}^{2N}} \frac{|u^+(y)|^{p_\mu^*}|u^+(x)|^{p_\mu^*-2}u^+(x)\varphi(x)}{|x-y|^{\mu}} \, dx\, dy - \lambda \int_{\mathbb{R}^N} f(x)\varphi(x)\, dx.
\end{aligned}$$

Consequently, u is a weak solution of (1.1) if and only if u is a critical point of the functional I_λ.

Remark 2.3. To verify the well-definedness of (2.2), we confirm the finiteness of all integrals in (2.1). For $u, \varphi \in X^{1,p}(\mathbb{R}^N)$, Hölder's inequality yields

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u \nabla \varphi \, dx + \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+ps}} \, dx\, dy \\
&\qquad \le |\nabla u|_p^{p-1}|\nabla \varphi|_p + [u]_{s,p}^{p-1}[\varphi]_{s,p} < \infty.
\end{aligned}$$

Additionally, leveraging the Sobolev embedding $X^{1,p}(\mathbb{R}^N) \hookrightarrow L^{p^*}(\mathbb{R}^N)$ and the Hardy–Littlewood–Sobolev inequality, we estimate

$$\begin{aligned}
&\iint_{\mathbb{R}^{2N}} \frac{|u(y)|^{p_\mu^*}|u(x)|^{p_\mu^*-2}u(x)\varphi(x)}{|x-y|^{\mu}} \, dx\, dy + \lambda \int_{\mathbb{R}^N} f(x)\varphi(x)\, dx \\
&\qquad \le C(N,\mu)|u|_{p^*}^{\frac{p(2N-\mu)+2Np}{2(N-p)}-\frac{2N}{2N-\mu}}|\varphi|_{p^*}^{\frac{2N}{2N-\mu}} + \lambda|f|_{(p^*)'}|\varphi|_{p^*} < \infty.
\end{aligned}$$

3 Proof of Theorem 1.1

We dedicate this section to prove Theorem 1.1 in the case $\lambda = 0$.

Lemma 3.1. *Let $s \in (0,1)$. For $S_{H,C}$ and $S_{H,M}$ defined in (1.3) and (1.4), it holds that $S_{H,M} = S_{H,C}$.*

Proof. From the definitions of $S_{H,C}$ and $S_{H,M}$, the inequality $S_{H,M} \geq S_{H,C}$ is immediate. To prove the reverse inequality, consider any $u \in C_0^\infty(\mathbb{R}^N)$. There exists $k_0(u) \in \mathbb{N}$ such that

$$\text{supp}(u) \subseteq B_{kr}(0), \quad \text{for all } k \geq k_0.$$

Define the scaled function $u_k := k^{\frac{N-p}{p}} u(kx)$ for $k \geq k_0$, which satisfies $supp(u_k) \subseteq B_r(0)$. By the definition of $S_{H,M}$, we have for every $k \geq k_0$,

$$S_{H,M} \leq \frac{\|u_k\|^p}{\|u_k\|_{HL}^p} = \frac{|\nabla u|_p^p}{\|u\|_{HL}^p} + k^{ps-p} \frac{[u]_{s,p}^p}{\|u\|_{HL}^p}.$$

Letting $k \to \infty$ and using $0 < s < 1$, the second term vanishes, yielding

$$S_{H,M} \leq \frac{|\nabla u|_p^p}{\|u\|_{HL}^p}.$$

By the arbitrariness of $u \in C_0^\infty(\mathbb{R}^N)$, we conclude

$$S_{H,M} \leq S_{H,C}.$$

Thus $S_{H,M} = S_{H,C}$. □

Lemma 3.2. *The optimal constant $S_{H,M}$ is never achieved.*

Proof. Assume for contradiction that there exists a nonzero function $v \in X^{1,p}(\mathbb{R}^N)$ such that $\|v\|_{HL} = 1$ and $\|v\|^p = S_{H,M} = S_{H,L}$, then

$$S_{H,L} \leq |\nabla v|_p^p \leq |\nabla v|_p^p + [v]_{s,p}^p = S_{H,L}.$$

This implies $[v]_{s,p} = 0$, meaning v is a constant in $\mathbb{R}^N$. However, a constant function v satisfies $\|v\| = 0$, contradicting $\|v\|_{HL} = 1$. Therefore, $S_{H,M}$ can not be achieved. □

Lemma 3.3. *Let $\alpha > 0$. The problem*

$$\begin{cases} -\Delta_p u + (-\Delta)_p^s u = \alpha \left(\displaystyle\int_{\mathbb{R}^N} \frac{|u(y)|^{p_\mu^*}}{|x-y|^\mu} \, dy \right) |u|^{p_\mu^* - 2} u \quad \text{in } \mathbb{R}^N, \\ u \in X^{1,p}(\mathbb{R}^N), \end{cases} \tag{3.1}$$

admits no nontrivial solutions.

Proof. Suppose u is a solution to (3.1). Then u satisfies

$$\|u\|^p = \alpha \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^{p_\mu^*} |u(y)|^{p_\mu^*}}{|x-y|^\mu} \, dx \, dy \right).$$

Define $w = lu$ for some $l > 0$. Substituting into the equation gives

$$\frac{\|w\|^p}{l^p} = \alpha \cdot \frac{\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|w(x)|^{p_\mu^*} |w(y)|^{p_\mu^*}}{|x-y|^\mu} \, dx \, dy}{l^{2p_\mu^*}}.$$

Setting $S_{H,M} = \alpha l^{p - 2p_\mu^*}$, we obtain

$$\|w\|^p = S_{H,M} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|w(x)|^{p_\mu^*} |w(y)|^{p_\mu^*}}{|x-y|^\mu} \, dx \, dy.$$

This implies w achieves $S_{H,M}$, contradicting Lemma 3.2. Hence, no nontrivial solutions exist. □

Proof of Theorem 1.1. Assume for contradiction that there exists a nonzero solution v to (1.1) with $\lambda = 0$. Then v satisfies

$$\left(a + b\|v\|^{p(\theta-1)}\right)\left(-\Delta_p v + (-\Delta)_p^s v\right) = \left(\int_{\mathbb{R}^N} \frac{|v(y)|^{p_\mu^*}}{|x-y|^\mu} dy\right)|v|^{p_\mu^*-2}v.$$

For any $\psi \in X^{1,p}(\mathbb{R}^N)$, testing the equation with ψ yields

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla v|^{p-2}\nabla v \nabla \psi \, dx + \iint_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^{p-2}(v(x)-v(y))(\psi(x)-\psi(y))}{|x-y|^{N+ps}} \, dx\, dy \\ = \frac{1}{a + b\|v\|^{(\theta-1)p}} \iint_{\mathbb{R}^{2N}} \frac{|v(y)|^{p_\mu^*}|v(x)|^{p_\mu^*-2}v(x)\psi(x)}{|x-y|^\mu} \, dx\, dy.\end{aligned}$$

Let $\alpha = \frac{1}{a+b\|v\|^{(\theta-1)p}} > 0$. The equation then reduces to

$$-\Delta_p v + (-\Delta)_p^s v = \alpha\left(\int_{\mathbb{R}^N} \frac{|v(y)|^{p_\mu^*}}{|x-y|^\mu} \, dy\right)|v|^{p_\mu^*-2}v,$$

which contradicts Lemma 3.3. Therefore, no nontrivial solutions exist when $\lambda = 0$. □

4 The principle of concentration compactness

In [32], Lions established the celebrated concentration compactness principle in classical Sobolev spaces, which has since been widely applied to solve elliptic problems involving critical exponents. Xiang et al. [44] later extended this principle to fractional Sobolev spaces $D^{s,p}(\mathbb{R}^N)$, while Gao et al. [25] developed a concentration compactness framework for convolution nonlinearities to study critical Choquard equations. However, to the best of our knowledge, no existing results have formulated the concentration compactness principle for mixed local-nonlocal operators with Choquard-type nonlinearities. Motivated by these contributions, we establish such a principle in $X^{1,p}(\mathbb{R}^N)$, which will be pivotal in Section 5. Define

$$C_c(\mathbb{R}^N) = \{u \in C(\mathbb{R}^N) : \operatorname{supp}(u) \text{ is a compact subset of } \mathbb{R}^N\}$$

and denote by $C_0(\mathbb{R}^N)$ the closure of $C_c(\mathbb{R}^N)$ with respect to the supremum norm $|\eta|_\infty = \sup_{x\in\mathbb{R}^N} |\eta(x)|$. Recall that a finite measure on $\mathbb{R}^N$ corresponds to a continuous linear functional on $C_0(\mathbb{R}^N)$. For a measure ξ, its total variation norm is given by:

$$\|\xi\| = \sup_{\eta\in C_0(\mathbb{R}^N), |\eta|_\infty = 1} |(\xi, \eta)|,$$

where $(\xi, \eta) = \int_{\mathbb{R}^N} \eta \, d\xi$.

Definition 4.1. Let $\mathcal{M}(\mathbb{R}^N)$ denote the space of finite nonnegative Borel measure space on $\mathbb{R}^N$. For $\xi \in \mathcal{M}(\mathbb{R}^N)$, we have $\xi(\mathbb{R}^N) = \|\xi\|$. A sequence $\{\xi_n\} \subset \mathcal{M}(\mathbb{R}^N)$ converges weak-$*$ to $\xi \in \mathcal{M}(\mathbb{R}^N)$ (denoted $\xi_n \rightharpoonup \xi$) if $(\xi_n, \eta) \to (\xi, \eta)$ for all $\eta \in C_0(\mathbb{R}^N)$ as $n \to \infty$.

Theorem 4.2. *Let $\{u_n\}_n \subset X^{1,p}(\mathbb{R}^N)$ be a sequence with $\|u_n\| \le C$ for some constant $C > 0$, and suppose*

$$\begin{cases} u_n \rightharpoonup u & \text{weakly in } X^{1,p}(\mathbb{R}^N), \\ |\nabla u_n|^p + \displaystyle\int_{\mathbb{R}^N} \frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}} \, dy \rightharpoonup \xi & \text{weak-}* \text{ in } \mathcal{M}(\mathbb{R}^N), \\ \displaystyle\int_{\mathbb{R}^N} \frac{|u_n(y)|^{p_\mu^*}}{|x-y|^\mu} \, dy |u_n(x)|^{p_\mu^*} \rightharpoonup \nu & \text{weak-}* \text{ in } \mathcal{M}(\mathbb{R}^N). \end{cases}$$

Then the following hold

$$\xi(\mathbb{R}^N) \le C^p, \qquad \nu(\mathbb{R}^N) \le S_{H,M}^{-\frac{2p_\mu^*}{p}} C^{2p_\mu^*}, \tag{4.1}$$

$$\xi = |\nabla u|^p + \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\, dy + \sum_{j\in J} \xi_j \delta_{x_j} + \tilde{\xi}, \tag{4.2}$$

$$\nu = \int_{\mathbb{R}^N} \frac{|u(y)|^{p_\mu^*}}{|x-y|^\mu}\, dy |u(x)|^{p_\mu^*} + \sum_{j\in J} \nu_j \delta_{x_j}. \tag{4.3}$$

where J is at most countable, sequence $\{\xi_j\}_j$, $\{\nu_j\}_j \subset \mathbb{R}_0^+$, $\{x_j\}_j \subset \mathbb{R}^N$, δ_{x_j} is the Dirac mass centered at x_j, $\tilde{\xi}$ is a nonnegative non-atomic measure. Moreover, we have

$$\nu(\mathbb{R}^N) \le S_{H,M}^{-\frac{2p_\mu^*}{p}} \xi(\mathbb{R}^N)^{\frac{2p_\mu^*}{p}}, \qquad \nu_j \le S_{H,M}^{-\frac{2p_\mu^*}{p}} \xi_j^{\frac{2p_\mu^*}{p}}, \quad \text{for all } j \in J, \tag{4.4}$$

where $S_{H,M}$ is the optimal constant defined in (1.4).

Lemma 4.3. *Let $\{u_n\}_n \subset X^{1,p}(\mathbb{R}^N)$ be the sequence given by Theorem 4.2. Fix $x_0 \in \mathbb{R}^N$ and let $\varphi \in C_0^\infty(\mathbb{R}^N)$ satisfy $0 \le \varphi \le 1$, $\varphi \equiv 1$ on $B_1(0)$, $\varphi \equiv 0$ in $\mathbb{R}^N \setminus B_2(0)$, and $|\nabla \varphi| \le 2$. Define $\varphi_\varepsilon = \varphi((x-x_0)/\varepsilon)$ for all $x \in \mathbb{R}^N$. Then the following hold*

$$\lim_{\varepsilon\to 0} \lim_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^p |\nabla \varphi_\varepsilon|^p\, dx = 0, \tag{4.5}$$

$$\lim_{\varepsilon\to 0} \limsup_{n\to\infty} \iint_{\mathbb{R}^{2N}} \frac{|u_n(y)|^p |\varphi_\varepsilon(x)-\varphi_\varepsilon(y)|^p}{|x-y|^{N+ps}}\, dx\, dy = 0. \tag{4.6}$$

Proof. We first prove (4.5). By the definition of φ_ε, we have:

$$\int_{\mathbb{R}^N} |u_n|^p |\nabla \varphi_\varepsilon|^p\, dx \le \int_{B_{2\varepsilon}(x_0)\setminus B_\varepsilon(x_0)} \frac{2^p}{\varepsilon^p} |u_n|^p\, dx.$$

Scaling the variable $z = \frac{x-x_0}{\varepsilon}$, this becomes

$$\int_{B_2(0)\setminus B_1(0)} 2^p \varepsilon^{N-p} |u_n(x_0+\varepsilon z)|^p\, dz.$$

For the bounded domain $\Omega = B_2(0) \setminus B_1(0)$, the Sobolev embedding $X^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ holds (cf. [13, Lemma 2.1]), where $X^{1,p}(\Omega)$ is the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{X^{1,p}(\Omega)} = \left(\int_\Omega |\nabla u|^p\, dx + \iint_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\, dx\, dy \right)^{\frac{1}{p}}.$$

Using the boundedness of $\{u_n\}$ in $X^{1,p}(\mathbb{R}^N)$ and $N > p$, we estimate

$$\int_{\mathbb{R}^N} |u_n|^p |\nabla \varphi_\varepsilon|^p\, dx \le C\varepsilon^{N-p} \|u_n\|_{X^{1,p}(\Omega)}^p \le C\varepsilon^{N-p} \|u_n\|^p \to 0$$

as $n \to \infty$ and $\varepsilon \to 0$, establishing (4.5). For (4.6), the proof follows similarly to [44, Lemma 2.3] by leveraging the fractional Sobolev seminorm properties and the decay of u_n at infinity. Details are omitted here for brevity. □

Proof of Theorem 4.2. We divide the proof into four steps.

Step 1. Proof of (4.1).
Fix $R > 0$ and let $\eta \in C_0^\infty(B_{2R}(0))$ satisfy $0 \le \eta \le 1$ and $\eta \equiv 1$ on $B_R(0)$. By weak-$*$ convergence,

$$\int_{\mathbb{R}^N} \left(|\nabla u_n|^p + \int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^p}{|x-y|^{N+ps}}\, dy \right) \eta(x)\, dx \to \int_{\mathbb{R}^N} \eta(x)\, d\xi$$

as $n \to \infty$. Since $\|u_n\| \le C$, we have

$$\begin{aligned} &\int_{\mathbb{R}^N} \left(|\nabla u_n|^p + \int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^p}{|x-y|^{N+ps}}\, dy \right) \eta(x)\, dx \\ &\quad \le \int_{\mathbb{R}^N} |\nabla u_n|^p\, dx + \iint_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^p}{|x-y|^{N+ps}}\, dy\, dx \le C^p. \end{aligned}$$

Taking $R \to \infty$, we conclude $\xi(\mathbb{R}^N) \le C^p$. For $\nu(\mathbb{R}^N)$, by the definition of $S_{H,M}$ and $\|u_n\| \le C$,

$$\iint_{\mathbb{R}^{2N}} \frac{|u_n(x)|^{p_\mu^*} |u_n(y)|^{p_\mu^*}}{|x-y|^\mu}\, dx\, dy \le S_{H,M}^{-\frac{2p_\mu^*}{p}} \|u_n\|^{2p_\mu^*} \le S_{H,M}^{-\frac{2p_\mu^*}{p}} C^{2p_\mu^*},$$

yielding $\nu(\mathbb{R}^N) \le S_{H,M}^{-\frac{2p_\mu^*}{p}} C^{2p_\mu^*}$.

Step 2. Proof of (4.2).
Define the functional

$$K(u) = \int_{\mathbb{R}^N} \left(|\nabla u|^p + \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x-y|^{N+ps}}\, dy \right) \eta(x)\, dx$$

for $\eta \in C_0(\mathbb{R}^N)$. Since K is continuously differentiable and convex on $X^{1,p}(\mathbb{R}^N)$, it is weakly lower semicontinuous. Thus,

$$\begin{aligned} &\liminf_{n\to\infty} \int_{\mathbb{R}^N} \left(|\nabla u_n|^p + \int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^p}{|x-y|^{N+ps}}\, dy \right) \eta(x)\, dx \\ &\quad \ge \int_{\mathbb{R}^N} \left(|\nabla u|^p + \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x-y|^{N+ps}}\, dy \right) \eta(x)\, dx. \end{aligned}$$

By weak-$*$ convergence, the left-hand side equals $\int_{\mathbb{R}^N} \eta\, d\xi$. Therefore,

$$\int_{\mathbb{R}^N} \eta\, d\xi \ge \int_{\mathbb{R}^N} \left(|\nabla u|^p + \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x-y|^{N+ps}}\, dy \right) \eta(x)\, dx.$$

By the arbitrariness of $\eta \ge 0$ in $C_0(\mathbb{R}^N)$, we deduce

$$\xi \ge |\nabla u|^p + \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x-y|^{N+ps}}\, dy.$$

The decomposition

$$\xi = |\nabla u|^p + \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x-y|^{N+ps}}\, dy + \sum_{j\in J} \xi_j \delta_{x_j} + \tilde{\xi}$$

follows from the structure of nonnegative Radon measures.

Step 3. Proof of (4.3).

For $\eta \in C_0(\mathbb{R}^N)$, using [34, Equation (3.6)], we have

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|u_n(y)|^{p_\mu^*}}{|x-y|^\mu}\,dy\,|u_n(x)|^{p_\mu^*}-\int_{\mathbb{R}^N}\frac{|\bar{u}_n(y)|^{p_\mu^*}}{|x-y|^\mu}\,dy\,|\bar{u}_n(x)|^{p_\mu^*}\right)\eta(x)\,dx$$
$$=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(y)|^{p_\mu^*}}{|x-y|^\mu}\,dy\,|u(x)|^{p_\mu^*}\eta(x)\,dx,$$

where $\bar{u}_n = u_n - u$. Define $\bar{\nu} = \nu - \int_{\mathbb{R}^N}\frac{|u(y)|^{p_\mu^*}}{|x-y|^\mu}\,dy\,|u(x)|^{p_\mu^*}$. By weak-$*$ convergence,

$$\int_{\mathbb{R}^N}\eta\,d\bar{\nu}=\lim_{n\to\infty}\iint_{\mathbb{R}^{2N}}\frac{|\bar{u}_n(y)|^{p_\mu^*}}{|x-y|^\mu}\,dy\,|\bar{u}_n(x)|^{p_\mu^*}\eta(x)\,dx.$$

To analyze the atomic part, fix $x_0 \in \mathbb{R}^N$ and let $\varphi \in C_0^\infty(\mathbb{R}^N)$ satisfy $0 \le \varphi \le 1$, $\varphi \equiv 1$ on $B_1(0)$, $\varphi \equiv 0$ outside $B_2(0)$, and $|\nabla\varphi| \le 2$. Define $\varphi_\varepsilon(x) = \varphi((x-x_0)/\varepsilon)$. Using [25, Equation (2.8)], we derive

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\iint_{\mathbb{R}^{2N}}\frac{|\varphi_\varepsilon u_n(y)|^{p_\mu^*}|\varphi_\varepsilon u_n(x)|^{p_\mu^*}}{|x-y|^\mu}\,dy\,dx=\nu(\{x_0\})$$

and

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\left(\int_{\mathbb{R}^N}|\nabla u_n|^p\varphi_\varepsilon^p\,dx+\iint_{\mathbb{R}^{2N}}\frac{|\varphi_\varepsilon u_n(x)-\varphi_\varepsilon u_n(y)|^p}{|x-y|^{N+ps}}\,dy\,dx\right)=\xi(\{x_0\}).$$

Recall the following Young inequality

$$|\zeta_1+\zeta_2|^p \le (1+\beta)^{p-1}|\zeta_1|^p+(1+1/\beta)^{p-1}|\zeta_2|^p, \tag{4.7}$$

where $\zeta_1, \zeta_2 \in \mathbb{R}$ and $\beta > 0$. Applying (4.7) and Lemma 4.3, we find

$$\nu(\{x_0\}) \le S_{H,M}^{-\frac{2p_\mu^*}{p}}\xi(\{x_0\})^{\frac{2p_\mu^*}{p}}.$$

Thus, the atoms of ν are contained in those of ξ. For the non-atomic part, we use the Radon–Nikodym theorem and Lebesgue's differentiation theorem to show $\tilde{\nu} = 0$.

Step 4. Proof of (4.4).

Fix $\eta \in C_0^\infty(B_{2R}(0))$ with $0 \le \eta \le 1$, $\eta \equiv 1$ on $B_R(0)$, and $|\nabla\eta| \le 2/R$. By Young's inequality and Lemma 4.3,

$$\iint_{\mathbb{R}^{2N}}\frac{|\eta u_n(y)|^{p_\mu^*}|\eta u_n(x)|^{p_\mu^*}}{|x-y|^\mu}\,dy\,dx$$
$$\le S_{H,M}^{-\frac{2p_\mu^*}{p}}\left((1+\beta)^{p-1}\int_{\mathbb{R}^N}|u_n\nabla\eta|^p\,dx+(1+\beta^{-1})^{p-1}\int_{\mathbb{R}^N}\eta^p\,d\xi\right)^{\frac{2p_\mu^*}{p}}.$$

Letting $R \to \infty$ and $\beta \to \infty$, we obtain

$$\int_{\mathbb{R}^N}\eta^{2p_\mu^*}\,d\nu \le S_{H,M}^{-\frac{2p_\mu^*}{p}}\left(\int_{\mathbb{R}^N}\eta^p\,d\xi\right)^{\frac{2p_\mu^*}{p}}.$$

Taking $R \to \infty$ and using the atomic decomposition, we conclude

$$\nu(\mathbb{R}^N) \le S_{H,M}^{-\frac{2p_\mu^*}{p}}\xi(\mathbb{R}^N)^{\frac{2p_\mu^*}{p}} \quad\text{and}\quad \nu_j \le S_{H,M}^{-\frac{2p_\mu^*}{p}}\xi_j^{\frac{2p_\mu^*}{p}} \quad \text{for all } j \in J.$$

This completes the proof. □

Theorem 4.4. *Let $\{u_n\}_n \subset X^{1,p}(\mathbb{R}^N)$ be a bounded sequence such that*

$$\begin{cases} |\nabla u_n|^p + \displaystyle\int_{\mathbb{R}^N} \frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}\,dy \rightharpoonup \xi & \text{weak-}* \text{ in } \mathcal{M}(\mathbb{R}^N), \\ \displaystyle\int_{\mathbb{R}^N} \frac{|u_n(y)|^{p_\mu^*}}{|x-y|^\mu}\,dy |u_n(x)|^{p_\mu^*} \rightharpoonup \nu & \text{weak-}* \text{ in } \mathcal{M}(\mathbb{R}^N), \end{cases}$$

and set

$$\xi_\infty = \lim_{R\to\infty} \limsup_{n\to\infty} \int_{\{x\in\mathbb{R}^N:|x|>R\}} \left(|\nabla u_n|^p + \int_{\mathbb{R}^N} \frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}} dy \right) dx,$$

$$\nu_\infty = \lim_{R\to\infty} \limsup_{n\to\infty} \int_{\{x\in\mathbb{R}^N:|x|>R\}} \int_{\mathbb{R}^N} \frac{|u_n(y)|^{p_\mu^*}}{|x-y|^\mu}\,dy\,|u_n(x)|^{p_\mu^*}\,dx,$$

where the quantities ξ_∞ and ν_∞ are well defined. Then the following hold

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} \left(|\nabla u_n|^p + \int_{\mathbb{R}^N} \frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}} dy \right) dx = \xi(\mathbb{R}^N) + \xi_\infty,$$

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(y)|^{p_\mu^*}}{|x-y|^\mu}\,dy\,|u_n(x)|^{p_\mu^*}\,dx = \nu(\mathbb{R}^N) + \nu_\infty,$$

$$\nu_\infty \le S_{H,M}^{-\frac{2p_\mu^*}{p}} \xi_\infty^{\frac{2p_\mu^*}{p}}.$$

Proof. Fix $\chi \in C^\infty(\mathbb{R}^N)$ with $0 \le \chi \le 1$, $\chi = 0$ in $B_1(0)$, $\chi = 1$ in $\mathbb{R}^N \setminus B_2(0)$, and $|\nabla\chi| \le 2$. Define $\chi_R(x) = \chi(x/R)$ for all $x \in \mathbb{R}^N$ and $R > 0$. Then

$$\begin{aligned} &\int_{\{x\in\mathbb{R}^N:|x|>2R\}} \left(|\nabla u_n|^p + \int_{\mathbb{R}^N} \frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}\,dy \right) dx \\ &\quad\le \int_{\mathbb{R}^N} \left(|\nabla u_n|^p + \int_{\mathbb{R}^N} \frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}\,dy \right) \chi_R^p\,dx \\ &\quad\le \int_{\{x\in\mathbb{R}^N:|x|>R\}} \left(|\nabla u_n|^p + \int_{\mathbb{R}^N} \frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}\,dy \right) dx, \end{aligned}$$

which means

$$\xi_\infty = \lim_{R\to\infty} \limsup_{n\to\infty} \int_{\mathbb{R}^N} \left(|\nabla u_n|^p + \int_{\mathbb{R}^N} \frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}} dy \right) \chi_R^p\,dx.$$

Similarly,

$$\nu_\infty = \lim_{R\to\infty} \limsup_{n\to\infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(y)|^{p_\mu^*}}{|x-y|^\mu}\,dy\,|u_n(x)|^{p_\mu^*} \chi_R^{2p_\mu^*}\,dx.$$

By the definition of weak-$*$ convergence, we have

$$\int_{\mathbb{R}^N} \left(|\nabla u_n|^p + \int_{\mathbb{R}^N} \frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}} dy \right) (1-\chi_R^p)\,dx \to \int_{\mathbb{R}^N} (1-\chi_R^p)\,d\xi$$

as $n \to \infty$. So we get

$$\xi(\mathbb{R}^N) = \lim_{R\to\infty} \lim_{n\to\infty} \int_{\mathbb{R}^N} \left(|\nabla u_n|^p + \int_{\mathbb{R}^N} \frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}} dy \right) (1-\chi_R^p)\,dx.$$

Then the following decomposition

$$\begin{aligned}
&\int_{\mathbb{R}^N}\left(|\nabla u_n|^p+\int_{\mathbb{R}^N}\frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}dy\right)dx\\
&\quad=\int_{\mathbb{R}^N}\left(|\nabla u_n|^p+\int_{\mathbb{R}^N}\frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}dy\right)\chi_R^p\,dx\\
&\quad\quad+\int_{\mathbb{R}^N}\left(|\nabla u_n|^p+\int_{\mathbb{R}^N}\frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}dy\right)(1-\chi_R^p)\,dx
\end{aligned}$$

gives that

$$\limsup_{n\to\infty}\int_{\mathbb{R}^N}\left(|\nabla u_n|^p+\int_{\mathbb{R}^N}\frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}dy\right)dx=\xi_\infty+\xi(\mathbb{R}^N).$$

Similarly, we can conclude that

$$\limsup_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_n(y)|^{p_\mu^*}}{|x-y|^\mu}\,dy\,|u_n(x)|^{p_\mu^*}\,dx=\nu(\mathbb{R}^N)+\nu_\infty.$$

To prove the last inequality

$$\nu_\infty\le S_{H,M}^{-\frac{2p_\mu^*}{p}}\xi_\infty^{\frac{2p_\mu^*}{p}},$$

we first claim that

$$\lim_{R\to\infty}\lim_{n\to\infty}\int_{\mathbb{R}^N}|u_n|^p|\nabla\chi_R|^p\,dx=0, \tag{4.8}$$

$$\lim_{R\to\infty}\limsup_{n\to\infty}\iint_{\mathbb{R}^{2N}}\frac{|u_n(y)|^p|\chi_R(x)-\chi_R(y)|^p}{|x-y|^{N+ps}}\,dx\,dy=0. \tag{4.9}$$

By the same argument as equation (4.5) in Lemma 4.3, we can derive (4.8). For (4.9), the proof follows similarly to [44, Equation (2.15)]. Details are omitted here for brevity.

By Young's inequality, (4.8) and (4.9), we deduce

$$\begin{aligned}
&\lim_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_n(y)|^{p_\mu^*}}{|x-y|^\mu}\,dy\,|u_n(x)|^{p_\mu^*}|\chi_R|^{2p_\mu^*}\,dx\\
&=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_n(y)\chi_R|^{p_\mu^*}}{|x-y|^\mu}dy\,|u_n(x)\chi_R|^{p_\mu^*}\,dx\\
&\le S_{H,M}^{-\frac{2p_\mu^*}{p}}\left((1+\beta^{-1})^{p-1}\int_{\mathbb{R}^N}|\nabla u_n\chi_R|^p\,dx+(1+\beta^{-1})^{p-1}\iint_{\mathbb{R}^{2N}}\frac{|u_n(x)-u_n(y)|^p|\chi_R(x)|^p}{|x-y|^{N+ps}}\,dx\,dy\right)^{\frac{2p_\mu^*}{p}}
\end{aligned}$$

Letting $R\to\infty$ and $\beta\to\infty$, we conclude

$$\nu_\infty\le S_{H,M}^{-\frac{2p_\mu^*}{p}}\xi_\infty^{\frac{2p_\mu^*}{p}}.$$

Therefore, the proof is complete. □

5 Proof of Theorem 1.2

Throughout this section, we consistently assume that $1<\theta\le\frac{2N-\mu}{N-p}$, without additional mention.

Definition 5.1. Let $c \in \mathbb{R}$, suppose $X^{1,p}(\mathbb{R}^N)$ is a Banach space and $I_\lambda \in C^1(X^{1,p}(\mathbb{R}^N), \mathbb{R})$. We say $\{u_n\}_n$ is a $(PS)_c$ sequence in $X^{1,p}(\mathbb{R}^N)$ if $I_\lambda(u_n) \to c$ and $I'_\lambda(u_n) \to 0$.

Lemma 5.2. *If $\{u_n\}_n$ is a $(PS)_c$ sequence, then $\{u_n\}_n$ is bounded in $X^{1,p}(\mathbb{R}^N)$ and $\{u_n^+\}_n$ is also a $(PS)_c$ sequence.*

Proof. Since $\{u_n\}_n$ is a $(PS)_c$ sequence, there exists $C > 0$ such that

$$|I_\lambda(u_n)| \le C, \ \left| \left\langle I'_\lambda(u_n), \frac{u_n}{\|u_n\|} \right\rangle \right| \le C.$$

Then we obtain

$$I_\lambda(u_n) - \frac{1}{2p_\mu^*}\langle I'_\lambda(u_n), u_n\rangle \le C + C\|u_n\|. \tag{5.1}$$

When $\theta < \frac{2N-\mu}{N-p}$ and $a > 0, \ b > 0$, we have

$$\begin{aligned} I_\lambda(u_n) - \frac{1}{2p_\mu^*}\langle I'_\lambda(u_n), u_n\rangle &= a\left(\frac{1}{p} - \frac{1}{2p_\mu^*}\right)\|u_n\|^p + b\left(\frac{1}{\theta p} - \frac{1}{2p_\mu^*}\right)\|u_n\|^{\theta p} \\ &\quad + \lambda\left(\frac{1}{2p_\mu^*} - 1\right)\int_{\mathbb{R}^N} f(x)u_n\,dx \\ &\ge b\left(\frac{1}{\theta p} - \frac{1}{2p_\mu^*}\right)\|u_n\|^{\theta p} - \lambda\left(1 - \frac{1}{2p_\mu^*}\right)S^{-\frac{1}{p}}|f|_{(p^*)'}\|u_n\|. \end{aligned}$$

When $\theta = \frac{2N-\mu}{N-p}$ and $a > 0, \ b > 0$, we have

$$\begin{aligned} I_\lambda(u_n) - \frac{1}{2p_\mu^*}\langle I'_\lambda(u_n), u_n\rangle &= a\left(\frac{1}{p} - \frac{1}{2p_\mu^*}\right)\|u_n\|^p + b\left(\frac{1}{\theta p} - \frac{1}{2p_\mu^*}\right)\|u_n\|^{\theta p} \\ &\quad + \lambda\left(\frac{1}{2p_\mu^*} - 1\right)\int_{\mathbb{R}^N} f(x)u_n\,dx \\ &\ge a\left(\frac{1}{p} - \frac{1}{2p_\mu^*}\right)\|u_n\|^p - \lambda\left(1 - \frac{1}{2p_\mu^*}\right)S^{-\frac{1}{p}}|f|_{(p^*)'}\|u_n\|. \end{aligned}$$

By invoking $1 < p < \theta p$ and (5.1), we conclude that $\{u_n\}_n$ is bounded in $X^{1,p}(\mathbb{R}^N)$.
Next, we aim to demonstrate that $\{u_n^+\}_n$ is also a $(PS)_c$ sequence, the proof is divided into two steps.

Step 1. We prove $I_\lambda(u_n^+) \to c$.
Since $\{u_n\}_n$ is a $(PS)_c$ sequence, we have $I_\lambda(u_n) \to c$ and $\langle I'_\lambda(u_n), u_n\rangle \to 0$. So $\langle I'_\lambda(u_n), u_n^+\rangle \to 0$. From the definition of I_λ, we have

$$\begin{aligned} &\langle I'_\lambda(u_n), u_n\rangle - \langle I'_\lambda(u_n), u_n^+\rangle \\ &= \left(a + b\|u_n\|^{p(\theta-1)}\right)\left(\int_{\mathbb{R}^N} |\nabla u_n|^{p-2}\nabla u_n(-\nabla u_n^-)\,dx\right. \\ &\quad \left. + \iint\limits_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y))(-u_n^-(x) + u_n^-(y))}{|x-y|^{N+ps}}\,dx\,dy\right) \\ &\quad - \lambda\int_{\mathbb{R}^N} f(x)u_n\,dx + \lambda\int_{\mathbb{R}^N} f(x)u_n^+\,dx \to 0 \end{aligned} \tag{5.2}$$

as $n \to \infty$. Based on the fact that

$$|\xi^- - \eta^-|^p \le |\xi - \eta|^{p-2}(\xi - \eta)(-\xi^- + \eta^-),$$

we obtain

$$|u_n(x) - u_n(y)|^{p-2}\,(u_n(x) - u_n(y))\,\left(-u_n^-(x) + u_n^-(y)\right) \ge |u_n^-(x) - u_n^-(y)|^p \ge 0. \tag{5.3}$$

Besides,

$$\begin{aligned}&\int_{\mathbb{R}^N} |\nabla u_n|^{p-2}\nabla u_n\left(-\nabla u_n^-\right)\,dx\\ &\quad= \int_{\mathbb{R}^N} |\nabla u_n|^{p-2}\left(\nabla u_n^+ - \nabla u_n^-\right)\left(-\nabla u_n^-\right)\,dx = \int_{\mathbb{R}^N} |\nabla u_n^-|^p\,dx \ge 0.\end{aligned} \tag{5.4}$$

Moreover, $f \ge 0$ and $\lambda \ge 0$ implies

$$\lambda \int_{\mathbb{R}^N} f(x)\left(u_n^+ - u_n\right)\,dx = \lambda \int_{\mathbb{R}^N} f(x)u_n^-\,dx \ge 0. \tag{5.5}$$

By (5.2)–(5.5), we derive

$$[u_n^-]_{s,p}^p \to 0, \qquad \int_{\mathbb{R}^N} |\nabla u_n^-|^p\,dx \to 0, \qquad \int_{\mathbb{R}^N} f u_n^-\,dx \to 0 \tag{5.6}$$

as $n \to \infty$. By

$$[u_n^+]_{s,p} - [u_n^-]_{s,p} \le [u_n]_{s,p} = [u_n^+ - u_n^-]_{s,p} \le [u_n^+]_{s,p} + [u_n^-]_{s,p},$$

we eventually obtain $[u_n^+]_{s,p}^p \to [u_n]_{s,p}^p$. Similarly, we can get $\int_{\mathbb{R}^N} |\nabla u_n^+|^p\,dx \to \int_{\mathbb{R}^N} |\nabla u_n|^p\,dx$. It is also easy to deduce $\int_{\mathbb{R}^N} f u_n^+\,dx \to \int_{\mathbb{R}^N} f u_n\,dx$. Hence we conclude $I_\lambda(u_n^+) \to c$ since $I_\lambda(u_n) \to c$.

Step 2. We prove $\langle I_\lambda'(u_n^+), v\rangle \to 0$ for all $v \in X^{1,p}(\mathbb{R}^N)$.

Note that

$$\langle I_\lambda'(u_n), v\rangle - \langle I_\lambda'(u_n^+), v\rangle = \left(a + b\|u_n\|^{p(\theta-1)}\right)(\mathrm{I}_1 + \mathrm{I}_2) + o(1), \tag{5.7}$$

where

$$\mathrm{I}_1 = \int_{\mathbb{R}^N} \left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_n^+|^{p-2}\nabla u_n^+\right) v\,dx$$

and

$$\mathrm{I}_2 = \iint_{\mathbb{R}^N} \frac{\left(|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y)) - |u_n^+(x)-u_n^+(y)|^{p-2}(u_n^+(x)-u_n^+(y))\right)(v(x)-v(y))}{|x-y|^{N+ps}}\,dx\,dy.$$

Next we are going to estimate I_1 and I_2.

For I_1:

Case 1. When $u_n \ge 0$, we have $\mathrm{I}_1 = 0$.

Case 2. When $u_n < 0$,

$$\mathrm{I}_1 = \int_{\{x:\ u_n(x)<0\}} |\nabla u_n|^{p-2}\nabla u_n v\,dx \le \int_{\mathbb{R}^N} |\nabla u_n^-|^p\,dx \int_{\mathbb{R}^N} |v|^p\,dx \to 0$$

as $n \to \infty$ by (5.6).

For I_2:

Case 1. When $u_n(x) \geq 0,\ u_n(y) \geq 0$, it's obvious that $\mathrm{I}_2 = 0$.

Case 2. When $u_n(x) \geq 0,\ u_n(y) < 0$,

$$\begin{aligned}
\mathrm{I}_2 &= \int_{\{x:\ u_n(x)\geq 0\}} \int_{\{y:\ u_n(y)<0\}} \frac{\left[(u_n(x)-u_n(y))^{p-1} - (u_n(x))^{p-1}\right](v(x)-v(y))}{|x-y|^{N+ps}}\,dx\,dy. \\
&= \int_{\{x:\ u_n(x)\geq 0\}} \int_{\{y:\ u_n(y)<0\}} \frac{\left(\sum_{i,j\in Z^+,\ i+j=p-1,\ j\geq 1} (u_n(x))^i(-u_n(y))^j\right)(v(x)-v(y))}{|x-y|^{N+ps}}\,dx\,dy \\
&= \sum_{\substack{i,j\in Z^+\\ i+j=p-1\\ j\geq 1}} \int_{\{x:\ u_n(x)\geq 0\}} \int_{\{y:\ u_n(y)<0\}} \frac{(u_n(x))^i (v(x)-v(y))^{\frac{i}{p-1}} (-u_n(y))^j (v(x)-v(y))^{\frac{j}{p-1}}}{|x-y|^{N+ps}}\,dx\,dy \\
&\leq \sum_{\substack{i,j\in Z^+\\ i+j=p-1\\ j\geq 1}} \iint_{\mathbb{R}^{2N}} \frac{|u_n^+(x)-u_n^+(y)|^i |v(x)-v(y)|^{\frac{i}{p-1}}}{|x-y|^{(N+ps)\cdot\frac{i}{p-1}}} \cdot \frac{|u_n^-(x)-u_n^-(y)|^j |v(x)-v(y)|^{\frac{j}{p-1}}}{|x-y|^{(N+ps)\cdot\frac{j}{p-1}}}\,dx\,dy \\
&\leq \sum_{\substack{i,j\in Z^+\\ i+j=p-1\\ j\geq 1}} \left(\iint_{\mathbb{R}^{2N}} \frac{|u_n^+(x)-u_n^+(y)|^{p-1}|v(x)-v(y)|}{|x-y|^{N+ps}}\,dx\,dy\right)^{\frac{i}{p-1}} \\
&\quad \left(\iint_{\mathbb{R}^{2N}} \frac{|u_n^-(x)-u_n^-(y)|^{p-1}|v(x)-v(y)|}{|x-y|^{N+ps}}\,dx\,dy\right)^{\frac{j}{p-1}} \\
&\leq \sum_{\substack{i,j\in Z^+\\ i+j=p-1\\ j\geq 1}} \left(\iint_{\mathbb{R}^{2N}} \frac{|u_n^+(x)-u_n^+(y)|^{p}}{|x-y|^{N+ps}}\,dx\,dy\right)^{\frac{p-1}{p}} \left(\iint_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^{p}}{|x-y|^{N+ps}}\,dx\,dy\right)^{\frac{1}{p}} \\
&\quad \left(\iint_{\mathbb{R}^{2N}} \frac{|u_n^-(x)-u_n^-(y)|^{p}}{|x-y|^{N+ps}}\,dx\,dy\right)^{\frac{p-1}{p}} \left(\iint_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^{p}}{|x-y|^{N+ps}}\,dx\,dy\right)^{\frac{1}{p}} \\
&\leq \sum_{\substack{j\in Z^+\\ 1\leq j\leq p-1}} C\|u_n^-\|^{\frac{j}{p}} \to 0
\end{aligned}$$

as $n \to \infty$ by (5.6).

Case 3. When $u_n(x) < 0,\ u_n(y) \geq 0$, we can obtain $\mathrm{I}_2 \to 0$ as $n \to \infty$ through a discussion similar to Case 2.

Case 4. When $u_n(x) < 0,\ u_n(y) < 0$, we have

$$\begin{aligned}
\mathrm{I}_2 &= \int_{\{x:\ u_n(x)<0\}} \int_{\{y:\ u_n(y)<0\}} \frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))(v(x)-v(y))}{|x-y|^{N+ps}}\,dx\,dy \\
&\leq \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy \iint_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^p}{|x-y|^{N+ps}}\,dx\,dy \\
&\leq C\|u_n^-\| \to 0
\end{aligned}$$

as $n \to \infty$ by (5.6).

By virtue of (5.7) and the fact that $\langle I'_\lambda(u_n), v\rangle \to 0$, we derive that $\langle I'_\lambda(u_n^+), v\rangle \to 0$ for all $v \in X^{1,p}(\mathbb{R}^N)$. Consequently, we conclude that $\{u_n^+\}_n$ is a $(PS)_c$ sequence. □

Lemma 5.3. *Suppose $a > 0,\ b > 0$ and either $\theta < \frac{2N-\mu}{N-p}$, or $a > 0,\ 0 < b < S_{H,M}^{-\frac{2p_\mu^*}{p}}$, $\theta = \frac{2N-\mu}{N-p}$. Then there exist $\lambda_0 > 0$, $\rho > 0$ and $\alpha > 0$ such that for any $\lambda \in (0, \lambda_0)$, $I_\lambda(u) \geq \alpha$ holds for every $u \in X^{1,p}(\mathbb{R}^N)$ satisfying $\|u\| = \rho$.*

Proof. For any $u \in X^{1,p}(\mathbb{R}^N)$, by the Sobolev inequality, we have

$$\begin{aligned} I_\lambda(u) &= \frac{a}{p}\|u\|^p + \frac{b}{\theta p}\|u\|^{\theta p} - \frac{1}{2p_\mu^*}\iint_{\mathbb{R}^{2N}} \frac{|u^+(x)|^{p_\mu^*}|u^+(y)|^{p_\mu^*}}{|x-y|^\mu}\,dx\,dy - \lambda\int_{\mathbb{R}^N} f(x)u\,dx \\ &\geq \frac{a}{p}\|u\|^p + \frac{b}{\theta p}\|u\|^{\theta p} - \frac{1}{2p_\mu^*}S_{H,M}^{-\frac{2p_\mu^*}{p}}\|u\|^{2p_\mu^*} - \lambda S^{-\frac{1}{p}}|f|_{(p^*)'}\|u\|, \end{aligned}$$

where S is the optimal constant for the Sobolev embedding $X^{1,p}(\mathbb{R}^N) \hookrightarrow L^{p^*}(\mathbb{R}^N)$.

Case 1. $a > 0,\ b > 0$ and $\theta < \frac{2N-\mu}{N-p}$. We have

$$I_\lambda(u) \geq \left(\frac{b}{\theta p}\|u\|^{\theta p-1} - \frac{1}{2p_\mu^*}S_{H,M}^{-\frac{2p_\mu^*}{p}}\|u\|^{2p_\mu^*-1} - \lambda S^{-\frac{1}{p}}|f|_{(p^*)'}\right)\|u\|.$$

Define

$$k(t) := \frac{b}{\theta p}t^{\theta p-1} - \frac{1}{2p_\mu^*}S_{H,M}^{-\frac{2p_\mu^*}{p}}t^{2p_\mu^*-1}$$

for all $t \geq 0$. Since $2p_\mu^* > \theta p$ by $\theta < \frac{2N-\mu}{N-p}$, we obtain that $\max_{t\geq 0} k(t) = k(\rho) > 0$, where

$$\rho = \left(\frac{2bp_\mu^*(\theta p-1)S_{H,M}^{\frac{2p_\mu^*}{p}}}{\theta p(2p_\mu^*-1)}\right)^{\frac{1}{2p_\mu^*-\theta p}}.$$

Set

$$\lambda_0 = \frac{k(\rho)}{S^{-\frac{1}{p}}|f|_{(p^*)'}},$$

then for any $\lambda \in (0, \lambda_0)$ we have

$$I_\lambda(u) \geq \left(k(\rho) - \lambda S^{-\frac{1}{p}}|f|_{(p^*)'}\right)\rho =: \alpha > 0 \quad \text{for all } \|u\| = \rho.$$

Case 2. $a > 0,\ 0 < b < S_{H,M}^{-\frac{2p_\mu^*}{p}}$ and $\theta = \frac{2N-\mu}{N-p}$. We have

$$I_\lambda(u) \geq \left[\frac{a}{p}\|u\|^{p-1} - \left(\frac{1}{2p_\mu^*}S_{H,M}^{-\frac{2p_\mu^*}{p}} - \frac{b}{\theta p}\right)\|u\|^{\theta p-1} - \lambda S^{-\frac{1}{p}}|f|_{(p^*)'}\right]\|u\|$$

with $2p_\mu^* = \theta p$. Define

$$l(t) := \frac{a}{p}t^{p-1} - \left(\frac{1}{2p_\mu^*}S_{H,M}^{-\frac{2p_\mu^*}{p}} - \frac{b}{\theta p}\right)t^{\theta p-1}$$

for all $t \geq 0$. Then $\max_{t\geq 0} l(t) = l(\rho) > 0$, where

$$\rho = \left(\frac{2a(p-1)p_\mu^*}{p(2p_\mu^*-1)(S_{H,M}^{-\frac{2p_\mu^*}{p}} - b)}\right)^{\frac{1}{2p_\mu^*-p}}.$$

Set

$$\lambda_0 = \frac{l(\rho)}{S^{-\frac{1}{p}}|f|_{(p^*)'}},$$

then for all $\lambda \in (0, \lambda_0)$ we deduce

$$I_\lambda(u) \geq \left(l(\rho) - \lambda S^{-\frac{1}{p}}|f|_{(p^*)'}\right)\rho =: \alpha > 0 \text{ for all } \|u\| = \rho.$$

Thus, the proof is completed. □

Lemma 5.4. *Suppose $a > 0$, $b > 0$, and either $\theta < \frac{2N-\mu}{N-p}$ or $a > 0$, $0 < b < S_{H,M}^{-\frac{2p_\mu^*}{p}}$, $\theta = \frac{2N-\mu}{N-p}$. Then for any $\lambda > 0$, there exists $e \in X^{1,p}(\mathbb{R}^N)$ such that $\|e\| \geq \rho$ and $I_\lambda(e) < 0$, where ρ is the number given in Lemma 5.3.*

Proof. By the definition of $S_{H,M}$ as given in (1.4), for all $\varepsilon > 0$, there exists $u \in X^{1,p}(\mathbb{R}^N) \setminus \{0\}$ such that

$$\frac{\|u\|^p}{\left(\iint_{\mathbb{R}^{2N}} \frac{|u(x)|^{p_\mu^*}|u(y)|^{p_\mu^*}}{|x-y|^\mu}\,dx\,dy\right)^{\frac{p}{2p_\mu^*}}} \leq S_{H,M} + \varepsilon.$$

Then there exists a positive U satisfying $\|U\|_{HL} = 1$ such that

$$\|U\|^p = S_{H,M} + h(\varepsilon), \tag{5.8}$$

where $h(\varepsilon)$ is a nonnegative constant associated with ε and $h(\varepsilon) \leq \varepsilon$. Then for any $t > 0$, we have

$$\begin{aligned} I_\lambda(tU) &= \frac{a\|U\|^p}{p}t^p + \frac{b\|U\|^{\theta p}}{\theta p}t^{\theta p} - \frac{t^{2p_\mu^*}}{2p_\mu^*} - \lambda t\int_{\mathbb{R}^N} f(x)U\,dx \\ &\leq \frac{a\|U\|^p}{p}t^p + \frac{b\|U\|^{\theta p}}{\theta p}t^{\theta p} - \frac{t^{2p_\mu^*}}{2p_\mu^*}. \end{aligned}$$

When $\theta < \frac{2N-\mu}{N-p}$(i.e. $\theta p < 2p_\mu^*$), there exists a sufficiently large $t > 0$ such that $\|tU\| \geq \rho$ and $I_\lambda(tU) < 0$.

When $\theta = \frac{2N-\mu}{N-p}$(i.e. $\theta p = 2p_\mu^*$), we derive

$$I_\lambda(tU) \leq \frac{a\|U\|^p}{p}t^p - \left(\frac{1}{2p_\mu^*} - \frac{b\|U\|^{\theta p}}{\theta p}\right)t^{2p_\mu^*}.$$

Given $0 < b < S_{H,M}^{-\theta}$, there exists $t > 0$ such that $\|tU\| \geq \rho$ and $I_\lambda(tU) < 0$. Setting $e = tU$ completes the proof. □

From Lemma 5.2, we have that any $(PS)_c$ sequence can be regarded as a sequence of nonnegative functions. Thus, we may assume without loss of generality that all $(PS)_c$ sequences are nonnegative in the subsequent analysis.

Define

$$\Lambda = \begin{cases} \left(\frac{1}{p} - \frac{1}{\theta p}\right)(aS_{H,M})^{\frac{\theta}{\theta-1}} \left(\frac{1}{1-bS_{H,M}^{\theta}}\right)^{\frac{1}{\theta-1}}, & \text{if } \theta = \frac{2N-\mu}{N-p},\ a>0,\ 0<b<S_{H,M}^{-\theta}, \\ \left(\frac{a}{p} - \frac{a}{\theta p}\right) S_{H,M} \left(\frac{bS_{H,M}^{\theta} + \sqrt{b^2 S_{H,M}^{2\theta} + 4aS_{H,M}}}{2}\right)^{\frac{2\theta-1}{\theta-1}\cdot\frac{p}{2p_{\mu}^*}} \\ \quad + \left(\frac{1}{\theta p} - \frac{1}{2p_{\mu}^*}\right) \left(\frac{bS_{H,M}^{\theta} + \sqrt{b^2 S_{H,M}^{2\theta} + 4aS_{H,M}}}{2}\right)^{\frac{2\theta-1}{\theta-1}}, & \text{if } \theta = \frac{3N-\mu-p}{2(N-p)},\ a>0,\ b>0, \end{cases}$$

and

$$\Theta(\lambda) = \frac{p-1}{p}\left[\left(a - \frac{a}{\theta}\right)S\right]^{-\frac{1}{p-1}} \left(1 - \frac{1}{\theta p}\right)^{\frac{p}{p-1}} \lambda^{\frac{p}{p-1}} |f|_{(p^*)'}^{\frac{p}{p-1}},$$

where $(p^*)' = \frac{p^*}{p^*-1}$.

Lemma 5.5. *Assume that $a>0,\ b>0$ and $\theta = \frac{3N-\mu-p}{2(N-p)}$, or $a>0,\ 0<b<S_{H,M}^{-\frac{2\cdot p_{\mu}^*}{p}}$ and $\theta = \frac{2N-\mu}{N-p}$, if $\{u_n\}_n \in X^{1,p}(\mathbb{R}^N)$ is a bounded and nonnegative sequence satisfying $I_\lambda(u_n) \to c < \Lambda - \Theta(\lambda)$ and $I'_\lambda(u_n) \to 0$ in $(X^{1,p}(\mathbb{R}^N))'$ as $n \to \infty$, then there exists a nonnegative function $u \in X^{1,p}(\mathbb{R}^N)$ such that, up to a subsequence, $u_n \to u$ in $X^{1,p}(\mathbb{R}^N)$ as $n \to \infty$.*

Proof. As $\{u_n\}_n$ is bounded and nonnegative, then there exists a subsequence, still noted $\{u_n\}_n$, and a nonnegative function $u \in X^{1,p}(\mathbb{R}^N)$ such that $u_n \rightharpoonup u$ in $X^{1,p}(\mathbb{R}^N)$, $u_n \to u$ in L^r_{loc} for $r \in [1, p^*)$ and $u_n \to u$ *a.e* in $\mathbb{R}^N$. Applying Theorem 4.2, up to a further subsequence, there exists a (at most) countable set J, a non-atomic measure $\tilde{\xi}$ and points $\{x_j\}_{j\in J} \subset \mathbb{R}^N$ and $\{\xi_j\}_{j\in J},\ \{\nu_j\}_{j\in J} \subset \mathbb{R}^+$ such that as $n \to \infty$ we have

$$|\nabla u_n|^p + \int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^p}{|x-y|^{N+ps}}\, dy \rightharpoonup \xi = |\nabla u|^p + \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\, dy + \sum_{j\in J} \xi_j \delta_{x_j} + \tilde{\xi}, \quad (5.9)$$

$$\int_{\mathbb{R}^N} \frac{|u_n(y)|^{p_{\mu}^*}}{|x-y|^{\mu}}\, dy\, |u_n(x)|^{p_{\mu}^*} \rightharpoonup \nu = \int_{\mathbb{R}^N} \frac{|u(y)|^{p_{\mu}^*}}{|x-y|^{\mu}}\, dy\, |u(x)|^{p_{\mu}^*} + \sum_{j\in J} \nu_j \delta_{x_j} \quad (5.10)$$

in the measure sense, where δ_{x_j} is the Dirac measure concentrated at x_j. Moreover,

$$\nu_j \le S_{H,M}^{-\frac{2p_{\mu}^*}{p}} \xi_j^{\frac{2p_{\mu}^*}{p}} \quad (5.11)$$

for all $j \in J$, where $S_{H,M}$ is the optimal constant defined in (1.4). We now proceed with the proof in three steps.

Step 1. We claim that $J = \emptyset$. Suppose by contradiction that $J \neq \emptyset$. Fix $j \in J$. For any $\varepsilon > 0$, choose $\varphi_{\varepsilon,j} \in C_0^\infty(\mathbb{R}^N)$ such that

$$\varphi_{\varepsilon,j} = 1 \quad \text{in } B_\varepsilon(x_j), \qquad \varphi_{\varepsilon,j} = 0 \quad \text{in } \mathbb{R}^N \setminus B_{2\varepsilon}(x_j), \qquad |\nabla \varphi_{\varepsilon,j}| \le 2/\varepsilon.$$

Evidently, $\varphi_{\varepsilon,j}u_n \in X^{1,p}(\mathbb{R}^N)$. It follows from $\langle I'_\lambda(u_n), \varphi_{\varepsilon,j}u_n\rangle \to 0$ that

$$\begin{aligned}
&\Big(a + b\|u_n\|^{(\theta-1)p}\Big)\left(\int_{\mathbb{R}^N} |\nabla u_n|^{p-2}\nabla u_n \nabla(\varphi_{\varepsilon,j}u_n)\,dx\right.\\
&\quad\left.+\iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))(\varphi_{\varepsilon,j}(x)u_n(x)-\varphi_{\varepsilon,j}(y)u_n(y))}{|x-y|^{N+ps}}\,dx\,dy\right)\\
&= \iint_{\mathbb{R}^{2N}} \frac{|u_n(y)|^{p^*_\mu}|u_n(x)|^{p^*_\mu-2}u_n(x)u_n(x)\varphi_{\varepsilon,j}(x)}{|x-y|^{\mu}}\,dx\,dy + \lambda\int_{\mathbb{R}^N} f(x)\varphi_{\varepsilon,j}(x)u_n\,dx + o(1).
\end{aligned} \tag{5.12}$$

For the left-hand side of (5.12), applying Hölder's inequality and Lemma 4.3 yields

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_{\mathbb{R}^N} |\nabla u_n|^{p-2}\nabla u_n\nabla\varphi_{\varepsilon,j}u_n\,dx\\
&\quad\le \lim_{\varepsilon\to 0}\lim_{n\to\infty}\left(\int_{\mathbb{R}^N}|\nabla u_n|^p\,dx\right)^{\frac{p-1}{p}}\left(\int_{\mathbb{R}^N}|\nabla\varphi_{\varepsilon,j}u_n|^p\,dx\right)^{\frac{1}{p}}\\
&\quad\le C\lim_{\varepsilon\to 0}\lim_{n\to\infty}\left(\int_{\mathbb{R}^N}|\nabla\varphi_{\varepsilon,j}u_n|^p\,dx\right)^{\frac{1}{p}} = 0,
\end{aligned} \tag{5.13}$$

and

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\lim_{n\to\infty}\left|\iint_{\mathbb{R}^{2N}}\frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))(\varphi_{\varepsilon,j}(x)-\varphi_{\varepsilon,j}(y))u_n(y)}{|x-y|^{N+ps}}\,dx\,dy\right|\\
&\le C\lim_{\varepsilon\to 0}\lim_{n\to\infty}\left(\iint_{\mathbb{R}^{2N}}\frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\right)^{\frac{p-1}{p}}\left(\iint_{\mathbb{R}^{2N}}\frac{|(\varphi_{\varepsilon,j}(x)-\varphi_{\varepsilon,j}(y))u_n(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\right)^{\frac{1}{p}}\\
&\le C\lim_{\varepsilon\to 0}\lim_{n\to\infty}\left(\frac{|(\varphi_{\varepsilon,j}(x)-\varphi_{\varepsilon,j}(y))u_n(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\right)^{\frac{1}{p}} = 0.
\end{aligned} \tag{5.14}$$

This together with equations (5.9) leads to

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\lim_{n\to\infty}\Big(a+b\|u_n\|^{(\theta-1)p}\Big)\int_{\mathbb{R}^N}\left(|\nabla u_n|^p+\int_{\mathbb{R}^N}\frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}dy\right)\varphi_{\varepsilon,j}(x)\,dx\\
&\quad\ge \lim_{\varepsilon\to 0}\lim_{n\to\infty} a\int_{\mathbb{R}^N}\left(|\nabla u_n|^p+\int_{\mathbb{R}^N}\frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}dy\right)\varphi_{\varepsilon,j}(x)\,dx\\
&\qquad+\lim_{\varepsilon\to 0}\lim_{n\to\infty} b\left(\int_{\mathbb{R}^N}\left(|\nabla u_n|^p+\int_{\mathbb{R}^N}\frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}\,dy\right)\varphi_{\varepsilon,j}(x)\,dx\right)^\theta\\
&\quad= a\xi_j + b\xi_j^\theta.
\end{aligned} \tag{5.15}$$

For the first term on the right-hand side of (5.12), it follows from (5.10) that

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\lim_{n\to\infty}\iint_{\mathbb{R}^{2N}}\frac{|u_n(y)|^{p^*_\mu}}{|x-y|^\mu}\,dy\,|u_n(x)|^{p^*_\mu}\varphi_{\varepsilon,j}\,dx\\
&\quad=\lim_{\varepsilon\to 0}\left(\iint_{\mathbb{R}^{2N}}\frac{|u(y)|^{p^*_\mu}}{|x-y|^\mu}\,dy\,|u(x)|^{p^*_\mu}\varphi_{\varepsilon,j}\,dx+\sum_{j\in J}\nu_j\delta_{x_j}\right)=\nu_j.
\end{aligned} \tag{5.16}$$

Moreover,

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_{\mathbb{R}^N} f\varphi_{\varepsilon,j}u_n\,dx = \lim_{\varepsilon\to 0}\int_{\mathbb{R}^N} f\varphi_{\varepsilon,j}u\,dx = 0. \tag{5.17}$$

Therefore, by substituting (5.13)–(5.17) into (5.12), we obtain

$$\nu_j \geq a\xi_j + b\xi_j^{\theta}. \tag{5.18}$$

Combining (5.11) with (5.18) implies

$$\nu_j \geq \begin{cases} \left(\dfrac{aS_{H,M}}{1-bS_{H,M}^{\theta}}\right)^{\frac{\theta}{\theta-1}} & \text{if } \theta = \frac{2N-\mu}{N-p},\ a>0,\ 0<b<S_{H,M}^{-\theta}, \\ \left(\dfrac{bS_{H,M}^{\theta}+\sqrt{b^2S_{H,M}^{2\theta}+4aS_{H,M}}}{2}\right)^{\frac{2\theta-1}{\theta-1}} & \text{if } \theta = \frac{3N-\mu-p}{2(N-p)},\ a>0,\ b>0. \end{cases} \tag{5.19}$$

For $R>0$, assume that $\psi_R \in C_0^{\infty}(\mathbb{R}^N)$ satisfies $\psi_R \in [0,1]$, $\psi_R(x)=1$ for $|x-x_j|\leq R$, $\psi_R(x)=0$ for $|x-x_j|>2R$, and $|\nabla\psi_R|\leq \frac{2}{R}$. By (5.9) and (5.10), we have

$$\begin{aligned}
c &= \lim_{n\to\infty}\left(I_\lambda(u_n)-\frac{1}{\theta p}\langle I_\lambda'(u_n),u_n\rangle\right) \\
&= \lim_{R\to\infty}\lim_{n\to\infty}\left(\left(\frac{a}{p}-\frac{a}{\theta p}\right)\left(\int_{\mathbb{R}^N}|\nabla u_n|^p\psi_R\,dx+\iint_{\mathbb{R}^{2N}}\frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}\psi_R(x)\,dy\,dx\right)\right. \\
&\quad \left.+\left(\frac{1}{\theta p}-\frac{1}{2p_\mu^*}\right)\iint_{\mathbb{R}^{2N}}\frac{|u_n(y)|^{p_\mu^*}|u_n(x)|^{p_\mu^*}\psi_R(x)}{|x-y|^{\mu}}\,dy\,dx-\left(1-\frac{1}{\theta p}\right)\lambda\int_{\mathbb{R}^N}fu_n\,dx\right) \\
&= \lim_{R\to\infty}\left(\left(\frac{a}{p}-\frac{a}{\theta p}\right)\left(\int_{\mathbb{R}^N}|\nabla u|^p\psi_R\,dx+\iint_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\psi_R(x)\,dy\,dx\right)+\left(\frac{a}{p}-\frac{a}{\theta p}\right)\xi_j\right. \\
&\quad +\left(\frac{1}{\theta p}-\frac{1}{2p_\mu^*}\right)\iint_{\mathbb{R}^{2N}}\frac{|u(y)|^{p_\mu^*}|u(x)|^{p_\mu^*}\psi_R(x)}{|x-y|^{\mu}}\,dy\,dx \\
&\quad \left.+\left(\frac{1}{\theta p}-\frac{1}{2p_\mu^*}\right)\nu_j-\left(1-\frac{1}{\theta p}\right)\lambda\int_{\mathbb{R}^N}fu\,dx\right) \\
&\geq \left(\frac{a}{p}-\frac{a}{\theta p}\right)\left(\int_{\mathbb{R}^N}|\nabla u|^p dx+\iint_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dy\,dx\right)+\left(\frac{a}{p}-\frac{a}{\theta p}\right)\xi_j \\
&\quad +\left(\frac{1}{\theta p}-\frac{1}{2p_\mu^*}\right)\iint_{\mathbb{R}^{2N}}\frac{|u(y)|^{p_\mu^*}|u(x)|^{p_\mu^*}}{|x-y|^{\mu}}\,dy\,dx+\left(\frac{1}{\theta p}-\frac{1}{2p_\mu^*}\right)\nu_j-\left(1-\frac{1}{\theta p}\right)\lambda\int_{\mathbb{R}^N}fu\,dx \\
&\geq \left(\frac{a}{p}-\frac{a}{\theta p}\right)S|u|_{p^*}^p+\left(\frac{a}{p}-\frac{a}{\theta p}\right)\xi_j+\left(\frac{1}{\theta p}-\frac{1}{2p_\mu^*}\right)\iint_{\mathbb{R}^{2N}}\frac{|u(y)|^{p_\mu^*}|u(x)|^{p_\mu^*}}{|x-y|^{\mu}}\,dy\,dx \\
&\quad +\left(\frac{1}{\theta p}-\frac{1}{2p_\mu^*}\right)\nu_j-\left(1-\frac{1}{\theta p}\right)\lambda\int_{\mathbb{R}^N}fu\,dx,
\end{aligned}$$

which implies that

$$c \geq \begin{cases} \left(\dfrac{a}{p}-\dfrac{a}{\theta p}\right)S|u|_{p^*}^p+\left(\dfrac{a}{p}-\dfrac{a}{\theta p}\right)\xi_j-\left(1-\dfrac{1}{\theta p}\right)\lambda\displaystyle\int_{\mathbb{R}^N}fu\,dx, \\ \qquad\qquad \text{if } \theta=\frac{2N-\mu}{N-p},\ a>0,\ b>0, \\ \left(\dfrac{a}{p}-\dfrac{a}{\theta p}\right)S|u|_{p^*}^p+\left(\dfrac{a}{p}-\dfrac{a}{\theta p}\right)\xi_j+\left(\dfrac{1}{\theta p}-\dfrac{1}{2p_\mu^*}\right)\nu_j-\left(1-\dfrac{1}{\theta p}\right)\lambda\displaystyle\int_{\mathbb{R}^N}fu\,dx, \\ \qquad\qquad \text{if } \theta=\frac{3N-\mu-p}{2(N-p)},\ a>0,\ b>0. \end{cases}$$

Then we apply the Hölder inequality, the Sobolev embedding, and Young inequality to derive

$$\begin{aligned}\left(1-\frac{1}{\theta p}\right)\lambda\int_{\mathbb{R}^N} fu\,dx &= \left(1-\frac{1}{\theta p}\right)\lambda|f|_{(p^*)'}|u|_{p^*}\\ &\le \left(\frac{a}{p}-\frac{a}{\theta p}\right)S|u|_{p^*}^p + \frac{p-1}{p}\left[\left(a-\frac{a}{\theta}\right)S\right]^{-\frac{1}{p-1}}\left(1-\frac{1}{\theta p}\right)^{\frac{p}{p-1}}\lambda^{\frac{p}{p-1}}|f|_{(p^*)'}^{\frac{p}{p-1}}.\end{aligned} \tag{5.20}$$

By (5.20) we obtain

$$c \ge \begin{cases} \left(\frac{a}{p}-\frac{a}{\theta p}\right)\xi_j - \frac{p-1}{p}\left[\left(a-\frac{a}{\theta}\right)S\right]^{-\frac{1}{p-1}}\left(1-\frac{1}{\theta p}\right)^{\frac{p}{p-1}}\lambda^{\frac{p}{p-1}}|f|_{(p^*)'}^{\frac{p}{p-1}}, \\ \qquad\qquad \text{if } \theta = \frac{2N-\mu}{N-p},\ a>0,\ b>0, \\ \left(\frac{a}{p}-\frac{a}{\theta p}\right)\xi_j + \left(\frac{1}{\theta p}-\frac{1}{2p_\mu^*}\right)\nu_j - \frac{p-1}{p}\left[\left(a-\frac{a}{\theta}\right)S\right]^{-\frac{1}{p-1}}\left(1-\frac{1}{\theta p}\right)^{\frac{p}{p-1}}\lambda^{\frac{p}{p-1}}|f|_{(p^*)'}^{\frac{p}{p-1}}, \\ \qquad\qquad \text{if } \theta = \frac{3N-\mu-p}{2(N-p)},\ a>0,\ b>0. \end{cases}$$

Combining (5.11) with (5.19), we arrive at $c \ge \Lambda - \Theta(\lambda)$, which contradicts our initial assumption. Thus the claim $J = \emptyset$ holds.

Step 2. We claim $\nu_\infty = 0$. Let $R > 0$. Set

$$\xi_\infty = \lim_{R\to\infty}\limsup_{n\to\infty}\int_{\{x\in\mathbb{R}^N:|x|>R\}}\left(|\nabla u_n|^p + \int_{\mathbb{R}^N}\frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}\,dy\right)dx,$$

$$\nu_\infty = \lim_{R\to\infty}\limsup_{n\to\infty}\int_{\{x\in\mathbb{R}^N:|x|>R\}}\int_{\mathbb{R}^N}\frac{|u_n(y)|^{p_\mu^*}}{|x-y|^\mu}\,dy\,|u_n(x)|^{p_\mu^*}\,dx.$$

According to Theorem 4.4, we know that ξ_∞ and ν_∞ are well defined and we have

$$\limsup_{n\to\infty}\int_{\mathbb{R}^N}\left(|\nabla u_n|^p + \int_{\mathbb{R}^N}\frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}\,dy\right)dx = \xi(\mathbb{R}^N) + \xi_\infty, \tag{5.21}$$

$$\limsup_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_n(y)|^{p_\mu^*}}{|x-y|^\mu}\,dy\,|u_n(x)|^{p_\mu^*}\,dx = \nu(\mathbb{R}^N) + \nu_\infty. \tag{5.22}$$

To rule out the possibility of concentration for mass at infinity, we take a suitable cutoff function $\chi_R \in C^\infty(\mathbb{R}^N)$ satisfying $\chi_R \in [0,1]$, $\chi_R(x) = 0$ for $|x| < R$ and $\chi_R(x) = 1$ for $|x| > 2R$, $|\nabla\chi_R| < \frac{2}{R}$. Following a discussion analogous to that of Theorem 4.4, we derive

$$\xi_\infty = \lim_{R\to\infty}\limsup_{n\to\infty}\int_{\mathbb{R}^N}|\nabla u_n|^p\chi_R(x)\,dx + \int_{\mathbb{R}^N}\frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}\chi_R(x)\,dy\,dx, \tag{5.23}$$

and

$$\nu_\infty = \lim_{R\to\infty}\limsup_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_n(y)\chi_R(y)|^{p_\mu^*}}{|x-y|^\mu}\,dy\,|u_n(x)\chi_R(x)|^{p_\mu^*}\,dx. \tag{5.24}$$

Moreover, we have

$$S_{H,M}\nu_\infty^{\frac{p}{2p_\mu^*}} < \xi_\infty. \tag{5.25}$$

Given that $\|u_n\|^p$ and $\|u_n\|_{HL}^{2p_\mu^*}$ are bounded, up to subsequence, we can assume that $\|u_n\|^p$ and $\|u_n\|_{HL}^{2p_\mu^*}$ are both convergent. Then by (5.21) and (5.22) we obtain

$$\lim_{n\to\infty} \|u_n\|^p = \int_{\mathbb{R}^N} d\xi + \xi_\infty, \tag{5.26}$$

$$\iint_{\mathbb{R}^{2N}} \frac{|u_n(y)|^{p_\mu^*}}{|x-y|^\mu} dy |u_n(x)|^{p_\mu^*} dx = \int_{\mathbb{R}^N} d\nu + \nu_\infty. \tag{5.27}$$

The fact $\langle I'_\lambda(u_n), \chi_R u_n\rangle \to 0$ as $n \to \infty$ entails that

$$\begin{aligned}
&\left(a + b\|u_n\|^{(\theta-1)p}\right)\left(\int_{\mathbb{R}^N} |\nabla u_n|^{p-2}\nabla u_n \nabla(\chi_R u_n)\, dx\right.\\
&\quad\left. + \iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))(\chi_R(x)u_n(x)-\chi_R(y)u_n(y))}{|x-y|^{N+ps}}\, dx\, dy\right)\\
&= \iint_{\mathbb{R}^{2N}} \frac{|u_n(y)|^{p_\mu^*}|u_n(x)|^{p_\mu^*-2}u_n(x)u_n(x)\chi_R(x)}{|x-y|^\mu}\, dx\, dy + \lambda\int_{\mathbb{R}^N} f(x)\chi_R u_n\, dx + o(1).
\end{aligned} \tag{5.28}$$

By (4.8), (4.9) and the Hölder inequality, we get

$$\lim_{R\to\infty}\lim_{n\to\infty}\int_{\mathbb{R}^N} |\nabla u_n|^{p-2}\nabla u_n \nabla\chi_R u_n\, dx = 0, \tag{5.29}$$

$$\lim_{R\to\infty}\limsup_{n\to\infty}\iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))u_n(y)(\chi_R(x)-\chi_R(y))}{|x-y|^{N+ps}}\, dx\, dy = 0. \tag{5.30}$$

Hence, it follows from $\theta > 1$, (5.23), (5.26) and (5.28)–(5.30) that

$$\begin{aligned}
&\lim_{R\to\infty}\limsup_{n\to\infty}\left(a + b\|u_n\|^{(\theta-1)p}\right)\left(\int_{\mathbb{R}^N}|\nabla u_n|^p\chi_R\, dx + \iint_{\mathbb{R}^{2N}}\frac{|u_n(x)-u_n(y)|^p\chi_R(x)}{|x-y|^{N+ps}}\, dy\, dx\right)\\
&= \left[a + \left(\int_{\mathbb{R}^N} d\xi + \xi_\infty\right)^{\theta-1}\right]\\
&\quad\times\lim_{R\to\infty}\limsup_{n\to\infty}\left(\int_{\{x\in\mathbb{R}^N:|x|>R\}}|\nabla u_n|^p\, dx + \int_{\{x\in\mathbb{R}^N:|x|>R\}}\int_{\mathbb{R}^N}\frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}\, dy\, dx\right)\\
&\geq \left(a + b\xi_\infty^{\theta-1}\right)\xi_\infty = a\xi_\infty + b\xi_\infty^\theta.
\end{aligned} \tag{5.31}$$

Besides, it is straightforward to observe that

$$\lim_{R\to\infty}\limsup_{n\to\infty}\int_{\mathbb{R}^N} f(x)\chi_R u_n\, dx = \lim_{R\to\infty}\int_{\mathbb{R}^N} f(x)\chi_R u\, dx = 0. \tag{5.32}$$

Therefore, we combine (5.28)-(5.32) with (5.24) and obtain that

$$a\xi_\infty + b\xi_\infty^\theta \leq \nu_\infty.$$

In conjunction with (5.25), this leads to

$$aS_{H,M}\nu_\infty^{\frac{p}{2p_\mu^*}} + bS_{H,M}^\theta\nu_\infty^{\frac{\theta p}{2p_\mu^*}} \leq \nu_\infty,$$

indicating that $\nu_\infty = 0$ or

$$\nu_\infty \geq \begin{cases} \left(\dfrac{aS_{H,M}}{1-bS_{H,M}^{\theta}}\right)^{\frac{\theta}{\theta-1}} & \text{if } \theta = \frac{2N-\mu}{N-p},\ a>0,\ 0<b<S_{H,M}^{-\theta}, \\ \left(\dfrac{bS_{H,M}^{\theta}+\sqrt{b^2S_{H,M}^{2\theta}+4aS_{H,M}}}{2}\right)^{\frac{2\theta-1}{\theta-1}} & \text{if } \theta = \frac{3N-\mu-p}{2(N-p)},\ a>0,\ b>0. \end{cases} \tag{5.33}$$

Suppose that (5.33) holds, then we have

$$\begin{aligned} c &= \lim_{n\to\infty}\left(I_\lambda(u_n) - \frac{1}{\theta p}\langle I'_\lambda(u_n), u_n\rangle\right) \\ &= \lim_{n\to\infty}\left(\left(\frac{a}{p}-\frac{a}{\theta p}\right)\|u_n\|^p + \left(\frac{1}{\theta p}-\frac{1}{2p_\mu^*}\right)\iint_{\mathbb{R}^{2N}}\frac{|u_n(y)|^{p_\mu^*}|u_n(x)|^{p_\mu^*}}{|x-y|^\mu}\,dy\,dx\right. \\ &\quad \left. -\left(1-\frac{1}{\theta p}\right)\lambda\int_{\mathbb{R}^N} f u_n\,dx\right) \\ &= \left(\frac{a}{p}-\frac{a}{\theta p}\right)\int_{\mathbb{R}^N} d\xi + \left(\frac{a}{p}-\frac{a}{\theta p}\right)\xi_\infty + \left(\frac{1}{\theta p}-\frac{1}{2p_\mu^*}\right)\int_{\mathbb{R}^N} d\nu + \left(\frac{1}{\theta p}-\frac{1}{2p_\mu^*}\right)\nu_\infty \\ &\quad -\left(1-\frac{1}{\theta p}\right)\lambda\int_{\mathbb{R}^N} f u\,dx \\ &\geq \left(\frac{a}{p}-\frac{a}{\theta p}\right)\|u\|^p + \left(\frac{a}{p}-\frac{a}{\theta p}\right)\xi_\infty + \left(\frac{1}{\theta p}-\frac{1}{2p_\mu^*}\right)\iint_{\mathbb{R}^{2N}}\frac{|u(y)|^{p_\mu^*}|u(x)|^{p_\mu^*}}{|x-y|^\mu}\,dy\,dx \\ &\quad + \left(\frac{1}{\theta p}-\frac{1}{2p_\mu^*}\right)\nu_\infty - \left(1-\frac{1}{\theta p}\right)\lambda\int_{\mathbb{R}^N} f u\,dx. \end{aligned}$$

Ultimately, from equations (5.20) and (5.33), we infer that $c \geq \Lambda - \Theta(\lambda)$, leading to a contradiction. Hence, $\nu_\infty = 0$. In view of $J = \emptyset$ and (5.27), we conclude that

$$\lim_{n\to\infty}\iint_{\mathbb{R}^{2N}}\frac{|u_n(y)|^{p_\mu^*}}{|x-y|^\mu}\,dy\,|u_n(x)|^{p_\mu^*}\,dx = \iint_{\mathbb{R}^{2N}}\frac{|u(y)|^{p_\mu^*}}{|x-y|^\mu}\,dy\,|u(x)|^{p_\mu^*}\,dx. \tag{5.34}$$

Step 3. We are now in a position to prove that $u_n \to u$ in $X^{1,p}(\mathbb{R}^N)$. Suppose $d := \inf_{n\geq 1}\|u_n\| > 0$. For any $w, v \in X^{1,p}(\mathbb{R}^N)$, we define

$$\begin{aligned} \langle L(w), v\rangle = &\int_{\mathbb{R}^N}|\nabla w_n|^{p-2}\nabla w_n \nabla v\,dx \\ &+ \iint_{\mathbb{R}^{2N}}\frac{|w_n(x)-w_n(y)|^{p-2}(w_n(x)-w_n(y))(v_n(x)-v_n(y))}{|x-y|^{N+ps}}\,dx\,dy. \end{aligned}$$

Clearly, $L(w)$ is a continuous linear functional in $X^{1,p}(\mathbb{R}^N)$. Then by the boundedness of $\{u_n\}_n$ and $u_n \rightharpoonup u$ in $X^{1,p}(\mathbb{R}^N)$, we deduce

$$\lim_{n\to\infty}\left(a+b\|u_n\|^{(\theta-1)p}\right)\langle L(u), u_n-u\rangle = 0,$$

and

$$\lim_{n\to\infty}\left(a+b\|u\|^{(\theta-1)p}\right)\langle L(u), u_n-u\rangle = 0.$$

Therefore, as $n \to \infty$, there holds

$$\begin{aligned}&\lim_{n\to\infty}\left(a+b\|u_n\|^{(\theta-1)p}\right)\langle L(u_n),u_n-u\rangle-\left(a+b\|u\|^{(\theta-1)p}\right)\langle L(u),u_n-u\rangle\\&\quad=\lim_{n\to\infty}\left(a+b\|u_n\|^{(\theta-1)p}\right)\left(\langle L(u_n),u_n-u\rangle-\langle L(u),u_n-u\rangle\right).\end{aligned}$$

It follows from $\langle I'_\lambda(u_n)-I'_\lambda(u),u_n-u\rangle\to 0$ that

$$\begin{aligned}&\left(a+b\|u_n\|^{(\theta-1)p}\right)\langle L(u_n),u_n-u\rangle-\left(a+b\|u\|^{(\theta-1)p}\right)\langle L(u),u_n-u\rangle\\&\quad=\iint_{\mathbb{R}^{2N}}\left(\frac{|u_n(y)|^{p^*_\mu}}{|x-y|^\mu}dy\,|u_n(x)|^{p^*_\mu-2}u_n(x)-\frac{|u(y)|^{p^*_\mu}}{|x-y|^\mu}dy\,|u(x)|^{p^*_\mu-2}u(x)\right)(u_n(x)-u(x))\,dx\\&\qquad+o(1).\end{aligned}$$

Hence, we conclude from (5.34) that

$$\lim_{n\to\infty}\left(a+b\|u_n\|^{(\theta-1)p}\right)\left(\langle L(u_n),u_n-u\rangle-\langle L(u),u_n-u\rangle\right)=0.$$

Since $d:=\inf_{n\geq 1}\|u_n\|>0$ and $b>0$, we have

$$\lim_{n\to\infty}\left(\langle L(u_n),u_n-u\rangle-\langle L(u),u_n-u\rangle\right)=0.$$

Now recall the famous Simon inequalities:

$$|\alpha-\beta|^p\leq\begin{cases}C'_p\left(|\alpha|^{p-2}\alpha-|\beta|^{p-2}\beta\right)\cdot(\alpha-\beta) & \text{if } p\geq 2,\\ C''_p\left[\left(|\alpha|^{p-2}\alpha-|\beta|^{p-2}\beta\right)\cdot(\alpha-\beta)\right]^{\frac{p}{2}}\left(|\alpha|^p+|\beta|^p\right)^{\frac{2-p}{2}} & \text{if } 1<p<2,\end{cases}\tag{5.35}$$

for all $\alpha,\beta\in\mathbb{R}^N$, where C'_p and C''_p are positive constants depending only on p.

If $p\geq 2$, from equation (5.35), we derive that

$$\begin{aligned}&\int_{\mathbb{R}^N}|\nabla(u_n-u)|^p\,dx+\iint_{\mathbb{R}^{2N}}\frac{|u_n(x)-u_n(y)-u(x)+u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\\&\quad\leq C'_p\left(\langle L(u_n),u_n-u\rangle-\langle L(u),u_n-u\rangle\right)\to 0\end{aligned}$$

as $n\to\infty$. Hence $u_n\to u$ in $X^{1,p}(\mathbb{R}^N)$.

For the case $1<p<2$, let $A=u_n(x)-u_n(y)$, $B=u(x)-u(y)$. Using (5.35) and concavity we have

$$\begin{aligned}&\int_{\mathbb{R}^N}|\nabla(u_n-u)|^p\,dx+\iint_{\mathbb{R}^{2N}}\frac{|u_n(x)-u_n(y)-u(x)+u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\\&\quad\leq C''_p\int_{\mathbb{R}^N}\left[\left(|\nabla u_n|^{p-2}\nabla u_n-|\nabla u|^{p-2}\nabla u\right)\cdot(\nabla u_n-\nabla u)\right]^{\frac{p}{2}}\left(|\nabla u_n|^p+|\nabla u|^p\right)^{\frac{2-p}{2}}dx\\&\qquad+C''_p\iint\limits_{\mathbb{R}^{2N}}\frac{\left[\left(|A|^{p-2}A-|B|^{p-2}B\right)\cdot(A-B)\right]^{\frac{p}{2}}\left(|A|^p+|B|^p\right)^{\frac{2-p}{2}}}{|x-y|^{N+ps}}\,dx\,dy\\&\quad\leq C\left(\langle L(u_n),u_n-u\rangle-\langle L(u),u_n-u\rangle\right)^{\frac{p}{2}}\to 0\end{aligned}$$

as $n\to\infty$. Hence $u_n\to u$ in $X^{1,p}(\mathbb{R}^N)$. In conclusion, we get $u_n\to u$ strongly in $X^{1,p}(\mathbb{R}^N)$ as $n\to\infty$.

Finally, we consider $\inf_n \|u_n\|_{X^{1,p}(\mathbb{R}^N)} = 0$. Then either 0 is an accumulation point of the sequence $\{u_n\}_n$ and so there exists a subsequence of $\{u_n\}_n$ strongly converging to $u = 0$, or 0 is an isolated point of the sequence $\{u_n\}_n$ and so there exists a subsequence, still denoted by $\{u_n\}_n$, such that $\inf_n \|u_n\| > 0$. In the first case we are done, while in the latter case we can process as above. □

Set $B_\rho(0) := \{u \in X^{1,p}(\mathbb{R}^N) : \|u\| < \rho\}$.

Theorem 5.6. *There exists $\lambda^* > 0$ such that for all $\lambda \in (0, \lambda^*)$, problem (1.1) admits a nonnegative weak solution u_1 in $X^{1,p}(\mathbb{R}^N)$ with $I_\lambda(u_1) < 0$.*

Proof. By Lemma 5.3, there exist $\lambda_0 > 0$ and $\gamma > 0$ such that

$$I_\lambda(u) \geq \gamma > 0 \quad \text{for } \lambda \in (0, \lambda_0) \quad \text{and} \quad u \in X^{1,p}(\mathbb{R}^N) \quad \text{with } \|u\| = \rho.$$

In the complete metric space $\overline{B}_\rho(0) := \{u \in X^{1,p}(\mathbb{R}^N) : \|u\| < \rho\}$, we apply Ekeland's variational principle [21] to prove that there exists a $(PS)_{c_0}$ sequence $\{u_n\}_n \subset B_\rho(0)$ with

$$c_0 = \inf\{I_\lambda(u) : u \in \overline{B}_\rho(0)\}.$$

Next, we aim to show that $-\infty < c_0 < 0$. Choose $\varphi_0 \in X^{1,p}(\mathbb{R}^N)$ such that $\int_{\mathbb{R}^N} f\varphi_0 \, dx > 0$ and fix $\lambda \in (0, \lambda^*)$. Then for any $t > 0$, we have

$$I_\lambda(t\varphi_0) = \frac{at^p}{p}\|\varphi_0\|^p + \frac{bt^{\theta p}}{\theta p}\|\varphi_0\|^{\theta p} - \frac{t^{2p_\mu^*}}{2p_\mu^*}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\varphi_0{}^+(y)|^{p_\mu^*}}{|x-y|^\mu}\, dy\, |\varphi_0{}^+(x)|^{p_\mu^*}\, dx - \lambda t \int_{\mathbb{R}^N} f(x)\varphi_0 \, dx.$$

Then we can deduce that there exists a sufficiently small $t > 0$ such that $\|t\varphi_0\| \leq \rho$ and $I_\lambda(t\varphi_0) < 0$. Hence, according to the definition of c_0, it follows that $c_0 < I_\lambda(t\varphi_0) < 0$ and it's evident that $c_0 > -\infty$.

Selecting $\lambda^* \in (0, \lambda_0]$ such that $0 < \Lambda - \Theta(\lambda)$ for all $\lambda \in (0, \lambda^*)$. From Lemma 5.5 and $c_0 < 0$, we can obtain that there exist a subsequence of $\{u_n\}_n$ and $u_1 \in X^{1,p}(\mathbb{R}^N)$ such that $u_n \to u_1$ strongly in $X^{1,p}(\mathbb{R}^N)$. Therefore, $I'_\lambda(u_1) = 0$ and $I_\lambda(u_1) = c_0 < 0$. Hence we complete the proof of Theorem 1.2. □

Drawing on the ideas introduced by Chabrowski [16], Alves [2] and Goncalves and Alves [28], we are going to explore that equation (1.1) has anther solution. Firstly we know that I_λ satisfies the mountain pass geometry by Lemmas 5.3 and 5.4. Hence applying the mountain pass theorem [3], there exists a $(PS)_c$ sequence $\{v_n\}_n$ with

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I_\lambda(\gamma(t)),$$

$$\Gamma = \{\gamma \in C([0,1], X^{1,p}(\mathbb{R}^N)) : \gamma(0) = 0, \gamma(1) = e\}.$$

Lemma 5.7. *Suppose $a > 0$, $b > 0$, and either $\theta = \frac{3N-\mu-p}{2(N-p)}$ or $a > 0$, $0 < b < S_{H,M}^{-\theta}$, $\theta = \frac{2N-\mu}{N-p}$. Under the assumption (f), there exists $\lambda^{**} \in (0, \lambda^*)$, such that for any $\lambda \in (0, \lambda^{**})$, one has*

$$c \leq \sup_{t \geq 0} I_\lambda(tU) < \Lambda - \Theta(\lambda).$$

Proof. Employing the function U in Lemma 5.4, we consider the funtions

$$g(t) := I_\lambda(tU) = \frac{a\|U\|^p}{p}t^p + \frac{b\|U\|^{\theta p}}{\theta p}t^{\theta p} - \frac{t^{2p^*_\mu}}{2p^*_\mu} - \lambda t\int_{\mathbb{R}^N} fU\,dx$$

and

$$\tilde{g}(t) := \frac{a\|U\|^p}{p}t^p + \frac{b\|U\|^{\theta p}}{\theta p}t^{\theta p} - \frac{t^{2p^*_\mu}}{2p^*_\mu}$$

for all $t \geq 0$. There exists $t_1 > 0$ such that $\tilde{g}'(t_1) = 0$ and $\max_{t\geq 0}\tilde{g}(t) = \tilde{g}(t_1) = \Lambda + h(\varepsilon) > 0$, where t_1 satisfies

$$t_1 = \begin{cases} \left(\dfrac{a\|U\|^p}{1-b\|U\|^{\theta p}}\right)^{\frac{1}{\theta p-p}} & \text{if } \theta = \frac{2N-\mu}{N-p},\ a>0,\ 0<b<S^{-\theta}_{H,M}, \\ \left(\dfrac{b\|U\|^{\theta p}+\sqrt{b^2\|U\|^{2\theta p}+4a\|U\|^p}}{2}\right)^{\frac{1}{\theta p-p}} & \text{if } \theta = \frac{3N-\mu-p}{2(N-p)},\ a>0,\ b>0. \end{cases}$$

When $c = 0$, the proof is similar to Theorem 1.2. When $c > 0$, we can choose $\lambda_1 \in (0, \lambda^*]$ such that $\Lambda - \Theta(\lambda_1) > 0$. It is easy to observe that

$$\lim_{t\to 0^+}\left(\frac{a\|U\|^p}{p}t^p + \frac{b\|U\|^{\theta p}}{\theta p}t^{\theta p}\right) = 0.$$

So there exists $t_2 \in (0, t_1)$ such that for any $\lambda \in (0, \lambda_1)$, we have

$$\max_{0\leq t\leq t_2} g(t) \leq \max_{0\leq t\leq t_2}\left(\frac{a\|U\|^p}{p}t^p + \frac{b\|U\|^{\theta p}}{\theta p}t^{\theta p}\right) \leq \Lambda - \Theta(\lambda_1) < \Lambda - \Theta(\lambda).$$

Choosing $\lambda^{**} \in (0, \lambda_1]$ such that for any $\lambda \in (0, \lambda^{**})$, there holds

$$\lambda t_2\int_{\mathbb{R}^N} fU^+\,dx > \frac{p-1}{p}\left[\left(a-\frac{a}{\theta}\right)S\right]^{-\frac{1}{p-1}}\lambda^{\frac{p}{p-1}}|f|^{\frac{p}{p-1}}_{(p^*)'} + h(\varepsilon).$$

Then, for any $\lambda \in (0, \lambda^{**})$, one can obtain

$$\begin{aligned}\sup_{t\geq t_2} g(t) &\leq \sup_{t\geq t_2}\tilde{g}(t) - \lambda t_2\int_{\mathbb{R}^N} fU^+\,dx \\ &< \tilde{g}(t_1) - \frac{p-1}{p}\left[\left(a-\frac{a}{\theta}\right)S\right]^{-\frac{1}{p-1}}\lambda^{\frac{p}{p-1}}|f|^{\frac{p}{p-1}}_{(p^*)'} - h(\varepsilon) \\ &= \Lambda - \Theta(\lambda).\end{aligned}$$

Therefore, for any $\lambda \in (0, \lambda^{**})$ we have

$$c \leq \sup_{t\geq 0} I_\lambda(tU) = \sup_{t\geq 0} g(t) < \Lambda - \Theta(\lambda).$$

Hence, the proof is complete. □

Theorem 5.8. *There exists $\lambda^{**} \in (0, \lambda^*]$ such that for all $\lambda \in (0, \lambda^{**})$, problem (1.1) has another nonnegative weak solution u_2 in $X^{1,p}(\mathbb{R}^N)$ with $I_\lambda(u_2) > 0$.*

Proof. Using Lemmas 5.5 and 5.7, there exist a subsequence of $\{v_n\}_n$, still denoted by $\{v_n\}_n$, and a function $u_2 \in X^{1,p}(\mathbb{R}^N)$ such that $v_n \to u_2$ strongly in $X^{1,p}(\mathbb{R}^N)$. Therefore,

$$I'_\lambda(u_2) = 0 \quad \text{and} \quad I_\lambda(u_2) = c > 0 = I_\lambda(0),$$

indicating that u_2 is a nontrivial and nonnegative solution of (1.1). □

Proof of Theorem 1.2. Combining the proof of Theorem 5.6 with Theorem 5.8, we can obtain Theorem 1.2. □

Statements and declarations

The authors declare that they have no conflict of interest.

Data availability

Date sharing is not applicable to this article as no new data were created analyzed in this study.

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