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CONTENTS

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CONTENTS

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VOLUME 27 (2026), NUMBER 1

CONTENTS

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CONTENTS

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HERMITE-HADAMARD TYPE INEQUALITIES FOR (M_ϕ, M_ψ)-CONVEX FUNCTIONS

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Abstract. In this paper, the authors introduce the generalization of Hermite-Hadamard inequality by using (M_φ, M_ψ)-convex functions and getting some other theorems with (M_φ, M_ψ)-convex functions. Some natural applications to special means of real numbers are also given.

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1. Introduction and preliminaries

It is well known in mathematical analysis that a function $\mathfrak{f}\colon I \subseteq \mathbb{R} \to \mathbb{R}$, $I \neq \emptyset$ is said to be convex on I if the inequality

$$\mathfrak{f}(\lambda x + (1-\lambda)y) \leq \lambda \mathfrak{f}(x) + (1-\lambda)\mathfrak{f}(y) \tag{1.1}$$

holds for all $x, y \in I$ and $\lambda \in [0,1]$ [25] .

Definition 1 ([4, 12])**.** A function $\mathfrak{f}\colon I \to [0,\infty)$ is said to be AG-convex or log-convex or multiplicatively convex if $\log \mathfrak{f}$ is convex, or equivalently if for all $x, y \in I$ and $\lambda \in [0,1]$ one has the inequality:

$$\mathfrak{f}(\lambda x + (1-\lambda)y) \leq \mathfrak{f}(x)^{\lambda} \mathfrak{f}(y)^{1-\lambda}. \tag{1.2}$$

Definition 2 ([16])**.** Let $\mathbb{I} \subseteq \mathbb{R} \setminus \{0\}$ be an interval. Then a real-valued function $\mathfrak{f}\colon I \to \mathbb{R}$ is said to be harmonically convex if

$$\mathfrak{f}\left(\frac{xy}{\lambda x + (1-\lambda)y}\right) \leq \lambda \mathfrak{f}(y) + (1-\lambda)\mathfrak{f}(x) \tag{1.3}$$

holds for all $x, y \in \mathbb{I} = [c, d]$ and $\lambda \in [0,1]$.

Definition 3 ([4, 12])**.** A function $\mathfrak{f}\colon \mathbb{I}\subseteq\mathbb{R}\to\mathbb{J}\subseteq\mathbb{R}\setminus\{0\}$ is called AH-convex on the convex set C if the following inequality holds

$$\mathfrak{f}(\lambda x+(1-\lambda)y)\leq\frac{\mathfrak{f}(x)\mathfrak{f}(y)}{(1-\lambda)\mathfrak{f}(y)+\lambda\mathfrak{f}(x)} \tag{1.4}$$

for any $x,y\in C$ and $\lambda\in[0,1]$.

Definition 4 ([4])**.** Let $I\subset(0,\infty)$ be an interval; a real-valued function $\mathfrak{f}\colon I\to\mathbb{R}$ is said to be GH-convex on I if

$$\mathfrak{f}(x^{1-\lambda}y^{\lambda})\leq\frac{\mathfrak{f}(x)\mathfrak{f}(y)}{(1-\lambda)\mathfrak{f}(y)+\lambda\mathfrak{f}(x)} \tag{1.5}$$

for all $x,y\in I$ and $\lambda\in[0,1]$.

Definition 5 ([4, 12])**.** Let $I\subset(0,\infty)$ be an interval; a real-valued function $\mathfrak{f}\colon I\to\mathbb{J}\subseteq\mathbb{R}\setminus\{0\}$ is said to be GA-convex on I if

$$\mathfrak{f}(x^{1-\lambda}y^{\lambda})\leq(1-\lambda)\mathfrak{f}(x)+\lambda\mathfrak{f}(y) \tag{1.6}$$

for all $x,y\in I$ and $\lambda\in[0,1]$.

Definition 6 ([4, 12])**.** The function $\mathfrak{f}\colon I\subset(0,\infty)\to(0,\infty)$ is called GG-convex on the interval I of real numbers $\mathbb{R}$ if

$$\mathfrak{f}(x^{1-\lambda}y^{\lambda})\leq\mathfrak{f}(x)^{(1-\lambda)}\cdot\mathfrak{f}(y)^{\lambda} \tag{1.7}$$

for all $x,y\in I$ and $\lambda\in[0,1]$.

Definition 7 ([4, 12])**.** We say that the function $\mathfrak{f}\colon I\subset\mathbb{R}\setminus\{0\}\to(0,\infty)$ is HG-convex or harmonically convex if

$$\mathfrak{f}\left(\frac{xy}{(1-\lambda)y+\lambda x}\right)\leq\mathfrak{f}(x)^{(1-\lambda)}\mathfrak{f}(y)^{\lambda} \tag{1.8}$$

for all $x,y\in I$ and $\lambda\in[0,1]$.

Definition 8 ([4, 12])**.** We say that the function $\mathfrak{f}\colon I\subset\mathbb{R}\setminus\{0\}\to(0,\infty)$ is HH-convex or harmonically convex if

$$\mathfrak{f}\left(\frac{xy}{(1-\lambda)y+\lambda x}\right)\leq\frac{\mathfrak{f}(x)\mathfrak{f}(y)}{(1-\lambda)\mathfrak{f}(y)+\lambda\mathfrak{f}(x)} \tag{1.9}$$

for all $x,y\in I$ and $\lambda\in[0,1]$.

Definition 9 ([18,28])**.** Let $I\subset(0,\infty)$ be a real interval and $p\in\mathbb{R}\setminus\{0\}$. A function $\mathfrak{f}\colon I\to\mathbb{R}$ is said to be p-convex, if

$$\mathfrak{f}\left([\lambda x^{p}+(1-\lambda)y^{p}]^{\frac{1}{p}}\right)\leq\lambda\mathfrak{f}(x)+(1-\lambda)\mathfrak{f}(y) \tag{1.10}$$

for all $x,y\in I$ and $\lambda\in[0,1]$.

Definition 10 ([27]). Let $I \subset (0,\infty)$ be a interval, $\varphi\colon I \to \mathbb{R}$ be a continuous and strictly monotonic function. $\mathfrak{f}\colon I \to \mathbb{R}$ is said to be $M_\varphi A$ convex, if

$$\mathfrak{f}\left(\varphi^{-1}(\lambda\varphi(x)+(1-\lambda)\varphi(y))\right) \leq \lambda\mathfrak{f}(x)+(1-\lambda)\mathfrak{f}(y) \tag{1.11}$$

for all $x, y \in I$ and $\lambda \in [0,1]$.

A number of inequalities have been written for convex functions but the most famous is the Hermite-Hadamard inequality which is stated as follows (see, e.g., [22]):

If $\mathfrak{f}\colon [c,d] \to \mathbb{R}$ is a convex function, then the following inequality is known as Hermite-Hadamard inequality:

$$\mathfrak{f}\left(\frac{c+d}{2}\right) \leq \frac{1}{d-c}\int_c^d \mathfrak{f}(x)\mathrm{d}x \leq \frac{\mathfrak{f}(c)+\mathfrak{f}(d)}{2}. \tag{1.12}$$

Note that some of classical inequalities for means can be derived from (1.12) for appropriate particular selections of the mapping $\mathfrak{f}$. Both inequalities hold in the reversed direction if $\mathfrak{f}$ is concave. For some results which generalize, improve and extend the inequality (1.12) we refer the reader to the recent papers (see [2, 3, 5–7, 14, 15, 17, 19–21, 23, 24]).

A convex (or concave) function $\mathfrak{f}$ is bounded on every compact subinterval $[u,v]$ of its interval of definition. If $\mathfrak{f}\colon I \subset R \to R$ is convex, then $\mathfrak{f}$ is continuous on interior I^o of I [26].

Let $A(c,d;\lambda) = \lambda c + (1-\lambda)d$, $G(c,d;\lambda) = c^\lambda d^{1-\lambda}$, $H(c,d;\lambda) = cd/(\lambda c + (1-\lambda)d)$ and $M_p(c,d;\lambda) = (\lambda c^p + (1-\lambda)d^p)^{1/p}$ be the weighted arithmetic, geometric, harmonic, power of order p means of two positive real numbers c and d with $c \neq d$ for $\lambda \in [0,1]$, respectively. The most used class of means is quasi-arithmetic mean, which is associated to a continuous and strictly monotonic function $\varphi\colon I \to R$ by the formula

$$M_\varphi(x,y) = \varphi^{-1}\left(\frac{\varphi(x)+\varphi(y)}{2}\right), \text{ for } x, y \in I.$$

Weighted quasi-arithmetic mean is given by the formula

$$M_\varphi(x,y;\lambda) = \varphi^{-1}\left(\lambda\varphi(x)+(1-\lambda)\varphi(y)\right), \text{ for } x, y \in I,\ \lambda \in [0,1].$$

Here $\lambda \in (0,1)$ and $x < y$ always implies $x < M_\varphi(x,y;\lambda) < y$. The function is called Kolmogoroff-Naguma function of M. Of special interest are the power means M_p on $\mathbb{R}_+$, defined by

$$\varphi_p(x) := \begin{cases} x^p, & p \neq 0, \\ \ln x, & p = 0. \end{cases}$$

For $p = 1$, we get the arithmetic mean $A = M_1$, for $p = 0$, we get the geometric mean $G = M_0$ and for $p = -1$, we get the harmonic mean $H = M_{-1}$.

Definition 11 ([1]). For any two quasi-arithmetic means M, N (with Kolmogoroff-Naguma function φ, ψ defined on I, J, respectively), a function $\mathfrak{f}\colon I \to J$ can be called (M_φ, M_ψ)-convex if it satisfies

$$\mathfrak{f}(M_\varphi(x,y;\lambda)) \leq M_\psi(\mathfrak{f}(x),\mathfrak{f}(y);\lambda) \tag{1.13}$$

for all $x, y \in I$ and $\lambda \in [0,1]$. If the inequality in (1.13) is reversed, then $\mathfrak{f}$ said to be (M_φ, M_ψ)-concave.

If $\psi\colon \mathbb{R} \to \mathbb{R}$, $\psi(x) = x$, (i.e., $M_\psi(\mathfrak{f}(x),\mathfrak{f}(y);\lambda) = A(c,d;\lambda)$), then we just say that $\mathfrak{f}$ is $M_\varphi A$-convex. Let $\mathfrak{f}$ be $M_\varphi A$-convex.

(1) If we take $\varphi\colon I \subset \mathbb{R} \to \mathbb{R}$, $\varphi(x) = x$, then $M_\varphi A$-convexity deduces usual convexity.
(2) If we take $\varphi\colon I \subset (0,\infty) \to \mathbb{R}$, $\varphi(x) = \ln x$, then $M_\varphi A$-convexity deduces GA-convexity.
(3) If we take $\varphi\colon I \subset (0,\infty) \to \mathbb{R}$, $\varphi(x) = x^{-1}$, then $M_\varphi A$-convexity deduces harmonically convexity.
(4) If we take $\varphi\colon I \subset (0,\infty) \to \mathbb{R}$, $\varphi(x) = x^p$, then $M_\varphi A$-convexity deduces p-convexity.

If $\psi\colon (0,\infty) \to \mathbb{R}$, $\psi(x) = \ln x$, (i.e., $M_\psi(\mathfrak{f}(x),\mathfrak{f}(y);\lambda) = G(c,d;\lambda)$), then we just say that $\mathfrak{f}$ is $M_\varphi G$-convex. Let $\mathfrak{f}$ be $M_\varphi G$-convex.

(1) If we take $\varphi\colon I \subset \mathbb{R} \to \mathbb{R}$, $\varphi(x) = x$, then $M_\varphi G$-convexity deduces logarithmic convexity.
(2) If we take $\varphi\colon I \subset (0,\infty) \to \mathbb{R}$, $\varphi(x) = \ln x$, then $M_\varphi G$-convexity deduces GG-convexity.
(3) If we take $\varphi\colon I \subset (0,\infty) \to \mathbb{R}$, $\varphi(x) = x^{-1}$, then $M_\varphi G$-convexity deduces harmonically G-convexity.
(4) If we take $\varphi\colon I \subset (0,\infty) \to \mathbb{R}$, $\varphi(x) = x^p$, then $M_\varphi G$-convexity deduces pG-convexity.

If $\psi\colon I \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $\psi(x) = x^{-1}$, (i.e., $M_\psi(\mathfrak{f}(x),\mathfrak{f}(y);\lambda) = H(c,d;\lambda)$), then we just say that $\mathfrak{f}$ is $M_\varphi H$-convex. Let $\mathfrak{f}$ be $M_\varphi H$-convex.

(1) If we take $\varphi\colon I \subset \mathbb{R} \to \mathbb{R}$, $\varphi(x) = x$, then $M_\varphi H$-convexity deduces AH-convexity.
(2) If we take $\varphi\colon I \subset (0,\infty) \to \mathbb{R}$, $\varphi(x) = \ln x$, then $M_\varphi H$-convexity deduces GH-convexity.
(3) If we take $\varphi\colon I \subset (0,\infty) \to \mathbb{R}$, $\varphi(x) = x^{-1}$, then $M_\varphi H$-convexity deduces HH-convexity.
(4) If we take $\varphi\colon I \subset (0,\infty) \to \mathbb{R}$, $\varphi(x) = x^p$, then $M_\varphi H$-convexity deduces pH-convexity.

If $\psi\colon (0,\infty) \to \mathbb{R}$, $\psi(x) = x^p$, then we just say that $\mathfrak{f}$ is $M_\varphi p$-convex. Let $\mathfrak{f}$ be $M_\varphi p$-convex.

(1) If we take $\varphi\colon I \subset \mathbb{R} \to \mathbb{R}$, $\varphi(x) = x$, then $M_\varphi p$-convexity deduces p-convexity.

(2) If we take $\varphi\colon I \subset (0,\infty) \to \mathbb{R},\ \varphi(x) = \ln x$, then $M_\varphi p$-convexity deduces Gp-convexity.
(3) If we take $\varphi\colon I \subset (0,\infty) \to \mathbb{R},\ \varphi(x) = x^{-1}$, then $M_\varphi p$-convexity deduces harmonically p-convexity.
(4) If we take $\varphi\colon I \subset (0,\infty) \to \mathbb{R},\ \varphi(x) = x^p$, then $M_\varphi p$-convexity deduces pp-convexity.

Lemma 1 ([1]). *Let φ and ψ be two continuous and strictly monotonic functions on intervals I and J respectively and let $\mathfrak{f}\colon I \to J$ is a function.*

If ψ is strictly increasing , then $\mathfrak{f}$ is (M_φ, M_ψ)-convex (concave) if and only if $\psi \circ \mathfrak{f} \circ \varphi^{-1}$ is convex (concave) on $\varphi(I)$ in the usual sense.

If ψ is strictly decreasing , then $\mathfrak{f}$ is (M_φ, M_ψ)-convex (concave) if and only if $\psi \circ \mathfrak{f} \circ \varphi^{-1}$ is concave (convex) on $\varphi(I)$ in the usual sense.

The main purpose of this paper is to introduce the generalization of Hermite-Hadamard inequality by using (M_φ, M_ψ)-convex functions and getting some other theorems with (M_φ, M_ψ)-convex functions.

2. Main results

Theorem 1. *Let φ and ψ are two continuous and strictly monotonic functions on intervals I and J respectively and let $\mathfrak{f}, \mathfrak{g}\colon I \to J$ are two functions. If $\mathfrak{f}$ and $\mathfrak{g}$ (M_φ, M_ψ)-convex functions, then $\mathfrak{f} \oplus_\psi \mathfrak{g}$ is a (M_φ, M_ψ)-convex function, where $(\mathfrak{f} \oplus_\psi \mathfrak{g})(x) := f(x) \oplus_\psi g(x) := \psi^{-1}(\psi(\mathfrak{f}(x)) + \psi(\mathfrak{g}(x)))$, $x \in I$.*

Proof. Firstly, let ψ be strictly increasing, then ψ^{-1} is also strictly increasing. Since $\mathfrak{f}$ and $\mathfrak{g}$ are (M_φ, M_ψ)-convex functions, we have

$$\mathfrak{f}(M_\varphi(x,y;\lambda)) \le M_\psi(\mathfrak{f}(x), \mathfrak{f}(y); \lambda) \tag{2.1}$$

and

$$\mathfrak{g}(M_\varphi(x,y;\lambda)) \le M_\psi(\mathfrak{g}(x), \mathfrak{g}(y); \lambda), \tag{2.2}$$

for all $x, y \in I$ and $\lambda \in [0,1]$. Then, since ψ and ψ^{-1} are strictly increasing, with (2.1) and (2.2), we have

$$\begin{aligned}
(\mathfrak{f} \oplus_\psi \mathfrak{g})(M_\varphi(x,y;\lambda)) &= \psi^{-1}(\psi(\mathfrak{f}(M_\varphi(x,y;\lambda))) + \psi(\mathfrak{g}(M_\varphi(x,y;\lambda)))) \\
&\le \psi^{-1}(\psi(M_\psi(\mathfrak{f}(x), \mathfrak{f}(y);\lambda)) + \psi(M_\psi(\mathfrak{g}(x), \mathfrak{g}(y);\lambda))) \\
&= \psi^{-1}(\lambda(\psi(\mathfrak{f}(x)) + \psi(\mathfrak{g}(x))) + (1-\lambda)(\psi(\mathfrak{f}(y)) + \psi(\mathfrak{g}(y)))) \\
&= \psi^{-1}(\lambda\psi((\mathfrak{f} \oplus_\psi \mathfrak{g})(x)) + (1-\lambda)\psi((\mathfrak{f} \oplus_\psi \mathfrak{g})(y))) \\
&= M_\psi((\mathfrak{f} \oplus_\psi \mathfrak{g})(x), (\mathfrak{f} \oplus_\psi \mathfrak{g})(y);\lambda),
\end{aligned}$$

which completes the proof. □

Remark 1. In the above theorem, it reduces to geometric and arithmetic inequalities when special choices are made.

Theorem 2. *Let φ and ψ be two continuous and strictly monotonic functions on intervals I and J respectively. If $\mathfrak{f}$ (M_φ, M_ψ)-convex function on $[u,v] \subset I$, then $\mathfrak{f}$ is bounded on $[u,v]$.*

Proof. Since $\mathfrak{f}$ is (M_φ, M_ψ)-convex function on $[u,v]$, if φ is strictly increasing (φ is strictly decreasing), then $\psi \circ \mathfrak{f} \circ \varphi^{-1}$ is convex (concave) on $\varphi([u,v])$. Thus $\psi \circ \mathfrak{f} \circ \varphi^{-1}$ is bounded on $\varphi([u,v])$. Therefore , with the continuity of ψ^{-1}, $\mathfrak{f} \circ \varphi^{-1}$ becomes bounded on $\varphi([u,v])$ which gives us that $\mathfrak{f}$ is bounded on $[u,v]$. This completes the proof. □

Theorem 3. *Let φ and ψ be two continuous and strictly monotonic functions on intervals I and J respectively. If $\mathfrak{f}$ (M_φ, M_ψ)-convex function on I, then $\mathfrak{f}$ is a continuous function on I^o.*

Proof. Since $\mathfrak{f}$ is (M_φ, M_ψ)-convex function, if ψ is strictly increasing (ψ is strictly decreasing), then $\psi \circ \mathfrak{f} \circ \varphi^{-1}$ is convex(concave) on $\varphi(I)$. Thus $\psi \circ \mathfrak{f} \circ \varphi^{-1}$ is continuous on $\varphi(I^o) = (\varphi(I))^o$ (this equality is satisfied because φ is a homeomorphism). Therefore, with continuity of ψ^{-1}, $\mathfrak{f} \circ \varphi^{-1}$ becomes continuous on $\varphi(I^o)$ which gives us that $\mathfrak{f}$ is continuous on I^o. This completes the proof. □

Theorem 4. *Let φ and ψ be two continuous and strictly monotonic functions on $(0,\infty)$ and let $\mathfrak{f}\colon (0,\infty) \to \mathbb{R}$ is a function. If $c,d \in (0,\infty)$ with $c<d$ and $\mathfrak{f}$ is (M_φ, M_ψ)-convex then the following inequalities hold:*

$$\mathfrak{f}\left(\varphi^{-1}\left(\frac{\varphi(c)+\varphi(d)}{2}\right)\right) \le \psi^{-1}\left[\frac{1}{\varphi(d)-\varphi(c)}\int_{\varphi(c)}^{\varphi(d)} (\psi \circ \mathfrak{f} \circ \varphi^{-1})(x)\mathrm{d}x\right]$$
$$\le \psi^{-1}\left(\frac{\psi(\mathfrak{f}(c))+\psi(\mathfrak{f}(d))}{2}\right). \tag{2.3}$$

The above inequalities are sharp.

Proof. Firstly, let ψ be strictly increasing. Since $\mathfrak{f}\colon (0,\infty) \to \mathbb{R}$ is a (M_φ, M_ψ)-convex function, $\psi \circ \mathfrak{f} \circ \varphi^{-1}$ is convex on $\varphi((0,\infty))$. So, by the inequalities (1.12) we have

$$(\psi \circ \mathfrak{f} \circ \varphi^{-1})\left(\frac{\varphi(c)+\varphi(d)}{2}\right) \le \frac{1}{\varphi(d)-\varphi(c)}\int_{\varphi(c)}^{\varphi(d)} (\psi \circ \mathfrak{f} \circ \varphi^{-1})(x)\mathrm{d}x$$
$$\le \frac{(\psi \circ \mathfrak{f} \circ \varphi^{-1})(\varphi(c)) + (\psi \circ \mathfrak{f} \circ \varphi^{-1})(\varphi(d))}{2},$$

i.e.

$$(\psi \circ \mathfrak{f} \circ \varphi^{-1})\left(\frac{\varphi(c)+\varphi(d)}{2}\right) \le \frac{1}{\varphi(d)-\varphi(c)}\int_{\varphi(c)}^{\varphi(d)} (\psi \circ \mathfrak{f} \circ \varphi^{-1})(x)\mathrm{d}x$$
$$\le \frac{(\psi \circ \mathfrak{f})(c) + (\psi \circ \mathfrak{f})(d)}{2}.$$

Since ψ^{-1} is strictly increasing, we have (2.3).

Secondly, let ψ be strictly decreasing. Since $\mathfrak{f}\colon (0,\infty)\to\mathbb{R}$ is a (M_φ,M_ψ)-convex function, $\psi\circ\mathfrak{f}\circ\varphi^{-1}$ is concave on $\varphi((0,\infty))$. So, by the inequalities (1.12) we have

$$(\psi\circ\mathfrak{f}\circ\varphi^{-1})\left(\frac{\varphi(c)+\varphi(d)}{2}\right)\geq\frac{1}{\varphi(d)-\varphi(c)}\int_{\varphi(c)}^{\varphi(d)}(\psi\circ\mathfrak{f}\circ\varphi^{-1})(x)\mathrm{d}x$$
$$\geq\frac{(\psi\circ\mathfrak{f}\circ\varphi^{-1})(\varphi(c))+(\psi\circ\mathfrak{f}\circ\varphi^{-1})(\varphi(d))}{2},$$

i.e.

$$(\psi\circ\mathfrak{f}\circ\varphi^{-1})\left(\frac{\varphi(c)+\varphi(d)}{2}\right)\geq\frac{1}{\varphi(d)-\varphi(c)}\int_{\varphi(c)}^{\varphi(d)}(\psi\circ\mathfrak{f}\circ\varphi^{-1})(x)\mathrm{d}x$$
$$\geq\frac{(\psi\circ\mathfrak{f})(c)+(\psi\circ\mathfrak{f})(d)}{2}.$$

Since ψ^{-1} is strictly decreasing, we have (2.3). □

Remark 2. (1) If we choose $\varphi(x)=x$ and $\psi(x)=x$ in Theorem 4, our result deduces to (1.12).
(2) If we choose $\varphi(x)=\ln x$ and $\psi(x)=x$ in Theorem 4, our result deduces to Hermite-Hadamard inequality for GA-convex functions in [9].
(3) If we choose $\varphi(x)=x^{-1}$ and $\psi(x)=x$ in Theorem 4, our result deduces to Hermite-Hadamard inequality for Harmonic functions in [16].
(4) If we choose $\varphi(x)=x^p$ and $\psi(x)=x$ in Theorem 4, our result deduces to Hermite-Hadamard inequality for p-convex functions in [13, 18].
(5) If we choose $\varphi(x)=x$ and $\psi(x)=\ln x$ in Theorem 4, our result deduces to Hermite-Hadamard inequality for Logaritmic convex functions in [12].
(6) If we choose $\varphi(x)=\ln x$ and $\psi(x)=\ln x$ in Theorem 4, our result deduces to Hermite-Hadamard inequality for GG-convex functions in [10].
(7) If we choose $\varphi(x)=x^{-1}$ and $\psi(x)=\ln x$ in Theorem 4, our result deduces to Hermite-Hadamard inequality for HG-convex functions in [8].
(8) If we choose $\varphi(x)=\ln x$ and $\psi(x)=x^{-1}$ in Theorem 4, our result deduces to Hermite-Hadamard inequality for GH-convex functions in [11].

3. Conclusion

The aim of this study is to make a generalized proof with this function that can reach more specific results for the algebraic properties of many convexities in the literature or for the Hermite-Hadamard inequality. This function will give us more general results and its special cases will be reduced to the literature.

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STUDY ON A SUBCLASS OF HOLOMORPHIC FUNCTIONS ASSOCIATED TO THE q-ANALOGUE MULTIPLIER TRANSFORMATION DEFINED IN A JANOWSKI DOMAIN

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Abstract. The present study uses differential subordination in conjunction with Janowski-type functions to establish the particular class $\mathcal{F}(m,n,\lambda,q,D,E)$ of holomorphic functions in the open unit disk. This class is associated with the q-analogue multiplier transformation. Using both the Keogh-Merkes and Ma-Minda's inequalities and the well-known Carathéodory's inequality for functions with positive real parts, an upper bound for the first two initial coefficients of the Taylor-Maclaurin power series expansion is derived. Also, for the functions in this family, an upper bound on the Fekete-Szegö functional is provided. Furthermore, for the function $\mathcal{G}^{-1}$, a similar conclusion is derived for the Fekete-Szegö inequality and the first two coefficients when $\mathcal{G} \in \mathcal{F}(m,n,\lambda,q,D,E)$. Properties regarding partial sums, necessary and sufficient conditions for functions to be part of $\mathcal{F}(m,n,\lambda,q,D,E)$, radii of close-to-convexity and starlikeness for this class, as well as distortion bounds are also established. The novelty of the results consists in the investigation of the basic properties of the new class of functions using simple methods, and the fact that the class is connected with the new above-mentioned q-operator and the Janowski functions.

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1. INTRODUCTION

The set of functions $\mathcal{G}$ of the type

$$\mathcal{G}(\eta) = \eta + \sum_{n=2}^{\infty} a_n \eta^n, \tag{1.1}$$

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That are holomorphic on the open unit disk $\mathcal{U} := \{\eta \in \mathbb{C} : |\eta| < 1\}$ of the complex plane are denoted by $\mathcal{A}$, and the subclass $\mathcal{S}$ of $\mathcal{A}$ refers to functions that are *univalent* in $\mathcal{U}$.

Given $\mathcal{G}, \mathcal{F} \in \mathcal{A}$, the function $\mathcal{G}$ is referred to as subordinate to $\mathcal{F}$ if there is a function $w(\eta) \in \mathcal{U}$ satisfying $w(0) = 0$, $|w(\eta)| < 1$, $\eta \in \mathcal{U}$, known as *Schwarz function*, and $\mathcal{G}(\eta) = \mathcal{F}(w(\eta))$ for all $\eta \in \mathcal{U}$. The mathematical sign used for subordination is:

$$\mathcal{G} \prec \mathcal{F} \quad \text{or} \quad \mathcal{G}(\eta) \prec \mathcal{F}(\eta).$$

The following inclusion equivalency holds if the function $\mathcal{F} \in \mathcal{S}$:

$$\mathcal{G}(\eta) \prec \mathcal{F}(\eta) \Leftrightarrow \mathcal{G}(0) = \mathcal{F}(0) \quad \text{and} \quad \mathcal{G}(\mathcal{U}) \subset \mathcal{F}(\mathcal{U}).$$

The *starlike* and *convex* functions in $\mathcal{U}$, respectively, are the subfamilies of $\mathcal{S}$:

$$\mathcal{S}^* := \left\{ \mathcal{G} \in \mathcal{A} : \mathrm{Re}\left\{ \frac{\eta \mathcal{G}'(\eta)}{\mathcal{G}(\eta)} \right\} > 0,\ \eta \in \mathcal{U} \right\}$$

and

$$\mathcal{K} := \left\{ \mathcal{G} \in \mathcal{A} : \mathrm{Re}\left\{ \frac{(\eta \mathcal{G}'(\eta))'}{\mathcal{G}'(\eta)} \right\} > 0,\ \eta \in \mathcal{U} \right\},$$

respectively. Similarly,

$$\mathcal{S}^*(\varphi) = \left\{ \mathcal{G} \in \mathcal{A} : \frac{\eta \mathcal{G}'(\eta)}{\mathcal{G}(\eta)} \prec \varphi(\eta) \right\}, \quad \mathcal{K}(\varphi) = \left\{ \mathcal{G} \in \mathcal{A} : \frac{(\eta \mathcal{G}'(\eta))'}{\mathcal{G}'(\eta)} \prec \varphi(\eta) \right\},$$

where

$$\varphi(\eta) = \frac{1+\eta}{1-\eta}.$$

Janowski defined in [19] the *Janwoski class of functions* $\mathfrak{S}^*[\mathcal{D}, \mathcal{E}]$, an extended function family of starlike functions. A function $\mathcal{G} \in \mathcal{A}$ is said to be in the family $\mathfrak{S}^*[\mathcal{D}, \mathcal{E}]$ if

$$\frac{\eta \mathcal{G}'(\eta)}{\mathcal{G}(\eta)} \prec \frac{1+\mathcal{D}\eta}{1-\mathcal{E}\eta} \quad (-1 \le \mathcal{E} < \mathcal{D} \le 1).$$

We mention that the above subordination could be written as:

$$\frac{\eta \mathcal{G}'(\eta)}{\mathcal{G}(\eta)} = \frac{(\mathcal{D}+1)\,p(\eta) - (\mathcal{D}-1)}{(\mathcal{E}+1)\,p(\eta) - (\mathcal{E}-1)} \quad (-1 \le \mathcal{E} < \mathcal{D} \le 1),$$

where $p(\eta)$ is an analytical function with a positive real part in $\mathcal{U}$.

The Janowski convex and Janowski starlike functions are obtained by reducing the above-described classes to the requirement $-1 \le \mathcal{E} < \mathcal{D} \le 1$. The starlike and convex functions of order ϑ $(0 \le \vartheta < 1)$ are obtained for the special cases when

$\mathcal{D} := 1-2\vartheta$ and $\mathcal{E} := -1$, where $0 \leq \vartheta < 1$. These functions were earlier defined by Robertson in [26], and were considered as:

$$\mathcal{S}^*(\vartheta) := \left\{ \mathcal{G} \in \mathcal{A} : \operatorname{Re}\left\{ \frac{\eta \mathcal{G}'(\eta)}{\mathcal{G}(\eta)} \right\} > \vartheta,\ \eta \in \mathcal{U} \right\},$$

$$\mathcal{K}(\vartheta) := \left\{ \mathcal{G} \in \mathcal{A} : \operatorname{Re}\left\{ \frac{(\eta \mathcal{G}'(\eta))'}{\mathcal{G}'(\eta)} \right\} > \vartheta,\ \eta \in \mathcal{U} \right\}.$$

Considering the well-known inclusions $\mathcal{S}^*(\vartheta) \subset \mathcal{S}$ and $\mathcal{K}(\vartheta) \subset \mathcal{S}$, it follows from the familiar Alexander's duality relation that $\mathcal{G} \in \mathcal{K}(\vartheta)$ if and only if $\eta \mathcal{G}'(\eta) \in \mathcal{S}^*(\vartheta)$ for each $0 \leq \vartheta < 1$. Geometric Function Theory (GFT) has been developed substantially based on the aforementioned families, and several key properties of $\mathcal{S}$ have been examined considering various perspectives.

When $0 < q < 1$, $[n]_q!$ represents the q-factorial described as (see [18]):

$$[n]_q! = \begin{cases} [n]_q[n-1]_q \cdots [2]_q[1]_q, & \text{if } n = 1,2,3,\ldots, \\ 1, & \text{if } n = 0, \end{cases}$$

where $[n]_q$, named the q-analogue of $n \in \mathbb{N}$, is given by:

$$[n]_q = \frac{1-q^n}{1-q} \quad \text{for } n \in \mathbb{N},$$

The applications of q-calculus across various mathematical fields, as well as in physics and engineering, are widely recognized.

Jackson [18] proposed the q-derivative operator $\mathfrak{D}_q$ of a function $\mathcal{G}$ as follows:

$$\mathfrak{D}_q \mathcal{G}(\eta) = \frac{\mathcal{G}(\eta) - \mathcal{G}(q\eta)}{(1-q)\eta} \quad (0 < q < 1;\ \eta \neq 0).$$

It is clear that

$$\lim_{q \to 1-} \mathfrak{D}_q \mathcal{G}(\eta) = \mathcal{G}'(\eta) \quad \text{and} \quad \mathfrak{D}_q \mathcal{G}(0) = \mathcal{G}'(0).$$

For additional information on the q-derivative operator's theory $\mathfrak{D}_q$, one can refer to [11–13].

The relationship between q-calculus and the theory of univalent functions was made clear by Ismail et al. [17] who introduced and investigated a specific class of q-stalike functions. It was Srivastava who established the foundational principles for the applications of q-calculus within the realm of geometric function theory, as presented in the book chapter that appeared in 1989 [29]. A recent study [1] highlights some aspects of the application of quantum calculus in geometric function theory, while Srivastava's review from 2020 [30] highlights other breakthroughs.

The introduction of new q-analogue operators led to a multitude of applications of q-calculus on univalent functions. Convolution was used by Kanas and Răducanu [20] to establish the q-analogue of the Ruscheweyh differential operator. Further research on the use of this differential operator was conducted by Mahmood and Sokół [24] and Mohammed and Darus [6]. The same pattern led to the emergence of the q-analogue of the Sălăgean differential operator [14], which sparked a number of applications among very recent ones being [16, 22]. Other interesting very recent studies involving q-analogue operators can be seen in [4, 5, 7].

Recently, Shah and Noor [27] defined the q-analogue multiplier transformation $I_q^{m,\lambda}\mathcal{G} : \mathcal{A} \to \mathcal{A}$ by

$$I_q^{m,\lambda}\mathcal{G}(\eta) := \eta + \sum_{n=2}^{\infty} [\Phi_n(\lambda, q)]^m a_n \eta^n, \quad \eta \in \mathcal{U}, \tag{1.2}$$

where

$$\Phi_n(\lambda, q) := \frac{[n+\lambda]_q}{[1+\lambda]_q}, \qquad \lambda > -1,\ q \in (0,1),\ m \in \mathbb{R},\ \eta \in \mathcal{U}. \tag{1.3}$$

Motivated by the recent new studies involving q-analogue operators like [2, 15, 32, 33], in this article, using the q-analogue multiplier transformation defined in (1.2), we define a new subclass of $\mathcal{A}$ given by:

$$\mathcal{F}(m,n,\lambda,q,\mathcal{D},\mathcal{E}) = \left\{ \mathcal{G} \in \mathcal{A} : \frac{I_q^{m,\lambda}\mathcal{G}(\eta)}{I_q^{n,\lambda}\mathcal{G}(\eta)} \prec \frac{1+\mathcal{D}\eta}{1+\mathcal{E}\eta} \right\}, \tag{1.4}$$

where $-1 \le E < D \le 1$, $\lambda > -1$, $q \in (0,1)$ and $m \in \mathbb{R}$.

Specializing the parameters $\mathcal{D}$ and $\mathcal{E}$, one can obtain the particular cases

$$\mathcal{F}(m,n,\lambda,q,1-2\alpha,-1) =: \mathcal{F}(m,n,\lambda,q,\alpha),$$

and

$$\mathcal{F}(m,n,\lambda,q,1,-1) =: \mathcal{F}(m,n,\lambda,q,\varphi).$$

The study described in this paper focuses on investigating several coefficient properties of this subclass. The study starts in the next section with the assessment of the Fekete-Szegö problem. In addition, the following sections establish the outcomes of partial sums, specific properties, and coefficient estimates.

2. The Fekete-Szegö functional bounds for the class $\mathcal{F}(m,n,\lambda,q,\mathcal{D},\mathcal{E})$

In order to assess the Fekete-Szegö type inequality for $\mathcal{G} \in \mathcal{F}(m,n,\lambda,q,\mathcal{D},\mathcal{E})$ the next already established results will be employed (the first part is due to Carathéodory [9]):

Lemma 1 ([21, 23]). *If* $P(\eta) = 1 + p_1\eta + p_2\eta^2 + \cdots \in \mathcal{P}$ *where* $\mathcal{P}$ *the class of functions* $P \in \mathcal{A}$ *with* $\operatorname{Re} P(\eta) > 0$ *and* $P(0) = 1$*, then*

$$|p_n| \leq 1,\ n \geq 1, \tag{2.1}$$

and for $\hbar \in \mathbb{C}$ *we have*

$$\left|p_2 - \hbar p_1^2\right| \leq 2\max\{1; |1 - 2\hbar|\}. \tag{2.2}$$

If $\hbar \in \mathbb{R}$*, then*

$$\left|p_2 - \hbar p_1^2\right| \leq \begin{cases} -4\hbar + 2, & \text{if } \hbar \leq 0, \\ 2, & \text{if } 0 \leq \hbar \leq 1, \\ 4\hbar - 2, & \text{if } \hbar \geq 1. \end{cases} \tag{2.3}$$

If $\hbar > 1$ *or* $\hbar < 0$*, (2.3) remains valid if and only if* $P_1(\eta) = \dfrac{1+\eta}{1-\eta}$ *or one of its rotations.*

When $0 < \hbar < 1$*, then the equality (2.3) remains valid if and only if* $P_2(\eta) = \dfrac{1+\eta^2}{1-\eta^2}$ *or one of its rotations.*

When $\hbar = 0$*, equality equality (2.3) remains valid if and only if*

$$P_3(\eta) = \left(\frac{1+c}{2}\right)\frac{1+\eta}{-\eta+1} + \left(\frac{1-c}{2}\right)\frac{-\eta+1}{1+\eta} \quad (0 \leq c \leq 1)$$

or one of its rotations.

Theorem 1. *If* $\mathcal{G} \in \mathcal{A}$ *has the form (1.1) and* $\mathcal{G} \in \mathcal{F}(m, n, \lambda, q, \mathcal{D}, \mathcal{E})$*, then*

$$|a_2| \leq \frac{|\mathcal{D} - \mathcal{E}|}{[\Phi_2(\lambda, q)]^m - [\Phi_2(\lambda, q)]^n}, \tag{2.4}$$

$$|a_3| \leq \frac{|\mathcal{D} - \mathcal{E}|}{2\left([\Phi_3(\lambda, q)]^m - [\Phi_3(\lambda, q)]^n\right)} \times \max\left\{1; \left|\frac{1 + 2\mathcal{E} - \mathcal{D}}{\mathcal{D} - \mathcal{E}} - \frac{(\mathcal{D} - \mathcal{E})\left([\Phi_2(\lambda, q)]^{n+m} - [\Phi_2(\lambda, q)]^{2n}\right)}{\left([\Phi_2(\lambda, q)]^m - [\Phi_2(\lambda, q)]^n\right)^2}\right|\right\}, \tag{2.5}$$

and for a complex number τ*, we have*

$$\left|a_3 - \tau a_2^2\right| \leq \frac{2(\mathcal{D} - \mathcal{E})}{[\Phi_3(\lambda, q)]^m - [\Phi_3(\lambda, q)]^n} \max\{1; |\Theta(\tau, \mathcal{D}, \mathcal{E})|\}, \tag{2.6}$$

where

$$\Theta(\tau, \mathcal{D}, \mathcal{E}) := \frac{1 + 2\mathcal{E} - \mathcal{D}}{\mathcal{D} - \mathcal{E}} - \frac{(\mathcal{D} - \mathcal{E})\left([\Phi_2(\lambda, q)]^{n+m} - [\Phi_2(\lambda, q)]^{2n}\right)}{\left([\Phi_2(\lambda, q)]^m - [\Phi_2(\lambda, q)]^n\right)^2} + \frac{2\tau(\mathcal{D} - \mathcal{E})\left([\Phi_3(\lambda, q)]^m - [\Phi_3(\lambda, q)]^n\right)}{\left([\Phi_2(\lambda, q)]^m - [\Phi_2(\lambda, q)]^n\right)^2},$$

and $\Phi_n(\lambda, q)$ *is given by* (1.3).

Proof. Our aim is to demonstrate that the relations (2.4), (2.5) and (2.6) remain valid for $\mathcal{G} \in \mathcal{F}(m,n,\lambda,q,\mathcal{D},\mathcal{E})$. Considering $\mathcal{G} \in \mathcal{F}(m,n,\lambda,q,\mathcal{D},\mathcal{E})$, we have:

$$\frac{I_q^{m,\lambda}\mathcal{G}(\eta)}{I_q^{n,\lambda}\mathcal{G}(\eta)} \prec \frac{1+\mathcal{D}\eta}{1+\mathcal{E}\eta},$$

which yields

$$\frac{I_q^{m,\lambda}\mathcal{G}(\eta)}{I_q^{n,\lambda}\mathcal{G}(\eta)} = \frac{1+\mathcal{D}w(\eta)}{1+\mathcal{E}w(\eta)} = G(w(\eta)), \quad (-1 \le \mathcal{E} < \mathcal{D} \le 1).$$

Since $w(\eta)$ can be written as:

$$w(\eta) = \frac{1-h(\eta)}{1+h(\eta)} = \frac{p_1\eta + p_2\eta^2 + p_3\eta^3 + \cdots}{2 + p_1\eta + p_2\eta^2 + p_3\eta^3 + \cdots},$$

we get:

$$G(w(\eta)) = 1 + \frac{1}{2}(\mathcal{D}-\mathcal{E})p_1\eta + \frac{1}{4}\left(2(\mathcal{D}-\mathcal{E})p_2 - (1+\mathcal{E})p_1^2\right)\eta^2 + \cdots, \tag{2.7}$$

and therefore

$$\begin{aligned}\frac{I_q^{m,\lambda}\mathcal{G}(\eta)}{I_q^{n,\lambda}\mathcal{G}(\eta)} &= 1 + \left([\Phi_2(\lambda,q)]^m - [\Phi_2(\lambda,q)]^n\right)a_2\eta \\ &\quad + \Big(\left([\Phi_3(\lambda,q)]^m - [\Phi_3(\lambda,q)]^n\right)a_3 \\ &\quad - \left([\Phi_2(\lambda,q)]^{n+m} - [\Phi_2(\lambda,q)]^{2n}\right)a_2^2\Big)\eta^2 + \cdots. \end{aligned} \tag{2.8}$$

If we compare the first coefficients of (2.7) and (2.8) we have:

$$a_2 = \frac{\mathcal{D}-\mathcal{E}}{2\left([\Phi_2(\lambda,q)]^m - [\Phi_2(\lambda,q)]^n\right)}p_1, \tag{2.9}$$

$$\begin{aligned} a_3 &= \frac{\mathcal{D}-\mathcal{E}}{2\left([\Phi_3(\lambda,q)]^m - [\Phi_3(\lambda,q)]^n\right)} \\ &\times \left(p_2 - \frac{p_1^2}{2}\left[\frac{1+\mathcal{E}}{\mathcal{D}-\mathcal{E}} - \left(\frac{(\mathcal{D}-\mathcal{E})\left([\Phi_2(\lambda,q)]^{n+m} - [\Phi_2(\lambda,q)]^{2n}\right)}{\left([\Phi_2(\lambda,q)]^m - [\Phi_2(\lambda,q)]^n\right)^2}\right)\right]\right), \end{aligned} \tag{2.10}$$

and by using (2.1) in (2.9) and (2.2) in (2.10) we get:

$$|a_2| \le \frac{|\mathcal{D}-\mathcal{E}|}{2[\Phi_2(\lambda,q)]^m - [\Phi_2(\lambda,q)]^n},$$

$$|a_3| \le \frac{|\mathcal{D}-\mathcal{E}|}{2\left([\Phi_3(\lambda,q)]^m - [\Phi_3(\lambda,q)]^n\right)}$$

$$\times \max\left\{1;\left|\frac{1+2\mathcal{E}-\mathcal{D}}{\mathcal{D}-\mathcal{E}}-\frac{(\mathcal{D}-\mathcal{E})\left([\Phi_2(\lambda,q)]^{n+m}-[\Phi_2(\lambda,q)]^{2n}\right)}{\left([\Phi_2(\lambda,q)]^m-[\Phi_2(\lambda,q)]^n\right)^2}\right|\right\}.$$

In addition, from (2.9) and (2.10), we have

$$\left|a_3-\tau a_2^2\right|=\frac{|\mathcal{D}-\mathcal{E}|}{2\left([\Phi_3(\lambda,q)]^m-[\Phi_3(\lambda,q)]^n\right)}\left|p_2-p_1^2\mathcal{K}(\tau,\mathcal{D},\mathcal{E})\right|, \tag{2.11}$$

where

$$\mathcal{K}(\tau,\mathcal{D},\mathcal{E}):=\frac{1+\mathcal{E}}{\mathcal{D}-\mathcal{E}}-\frac{(\mathcal{D}-\mathcal{E})\left([\Phi_2(\lambda,q)]^{n+m}-[\Phi_2(\lambda,q)]^{2n}\right)}{\left([\Phi_2(\lambda,q)]^m-[\Phi_2(\lambda,q)]^n\right)^2}+\frac{\tau(\mathcal{D}-\mathcal{E})\left([\Phi_3(\lambda,q)]^m-[\Phi_3(\lambda,q)]^n\right)}{\left([\Phi_2(\lambda,q)]^m-[\Phi_2(\lambda,q)]^n\right)^2}. \tag{2.12}$$

The necessary results are now obtained if we apply (2.2) in (2.11). Furthermore, we obtain our inequality for real τ by applying (2.3) to the previously mentioned (2.11). □

Theorem 2. *If $\mathcal{G}\in\mathcal{A}$ is given by (1.1) and $\mathcal{G}\in\mathcal{F}(m,n,\lambda,q,\mathcal{D},\mathcal{E})$, then for any $\tau\in\mathbb{R}$ we obtain:*

$$\left|a_3-\tau a_2^2\right|\leq\frac{|\mathcal{D}-\mathcal{E}|}{\left|[\Phi_3(\lambda,q)]^m-[\Phi_3(\lambda,q)]^n\right|}\begin{cases}1-2\mathcal{K}(\tau,\mathcal{D},\mathcal{E}), & \text{if } \tau\leq\sigma_1,\\ 1, & \text{if } \sigma_1\leq\tau\leq\sigma_2,\\ 2\mathcal{K}(\tau,\mathcal{D},\mathcal{E})-1, & \text{if } \tau\geq\sigma_2,\end{cases}$$

where $\mathcal{K}(\tau,\mathcal{D},\mathcal{E})$ defined by (2.12) and

$$\sigma_1=\frac{\left([\Phi_2(\lambda,q)]^m-[\Phi_2(\lambda,q)]^n\right)^2}{(\mathcal{D}-\mathcal{E})\left([\Phi_3(\lambda,q)]^m-[\Phi_3(\lambda,q)]^n\right)}\times\left(\frac{(\mathcal{D}-\mathcal{E})\left([\Phi_2(\lambda,q)]^{n+m}-[\Phi_2(\lambda,q)]^{2n}\right)}{\left([\Phi_2(\lambda,q)]^m-[\Phi_2(\lambda,q)]^n\right)^2}-\frac{1+\mathcal{E}}{\mathcal{D}-\mathcal{E}}\right),$$

and

$$\sigma_2=\frac{\left([\Phi_2(\lambda,q)]^m-[\Phi_2(\lambda,q)]^n\right)^2}{(\mathcal{D}-\mathcal{E})\left([\Phi_3(\lambda,q)]^m-[\Phi_3(\lambda,q)]^n\right)}\times\left(\frac{(\mathcal{D}-\mathcal{E})\left([\Phi_2(\lambda,q)]^{n+m}-[\Phi_2(\lambda,q)]^{2n}\right)}{\left([\Phi_2(\lambda,q)]^m-[\Phi_2(\lambda,q)]^n\right)^2}+\frac{\mathcal{D}-2\mathcal{E}-1}{\mathcal{D}-\mathcal{E}}\right).$$

Proof. For real τ, using Lemma 1 and equation (2.12), we get:

$$\left|a_3-\tau a_2^2\right|=\frac{|\mathcal{D}-\mathcal{E}|}{2\left|[\Phi_3(\lambda,q)]^m-[\Phi_3(\lambda,q)]^n\right|}\left|p_2-p_1^2\mathcal{K}(\tau,\mathcal{D},\mathcal{E})\right|$$

$$\leq \frac{|\mathcal{D}-\mathcal{E}|}{2\left|[\Phi_3(\lambda,q)]^m-[\Phi_3(\lambda,q)]^n\right|}\begin{cases}-4\mathcal{K}(\tau,\mathcal{D},\mathcal{E})+2, & \text{if } \mathcal{K}(\tau,\mathcal{D},\mathcal{E})\leq 0,\\ 2 & \text{if } 0\leq \mathcal{K}(\tau,\mathcal{D},\mathcal{E})\leq 1,\\ 4\mathcal{K}(\tau,\mathcal{D},\mathcal{E})-2 & \text{if } \mathcal{K}(\tau,\mathcal{D},\mathcal{E})\geq 1.\end{cases}$$

$$\leq \frac{|\mathcal{D}-\mathcal{E}|}{\left|[\Phi_3(\lambda,q)]^m-[\Phi_3(\lambda,q)]^n\right|}\begin{cases}-2\mathcal{K}(\tau,\mathcal{D},\mathcal{E})+1, & \text{if } \mathcal{K}(\tau,\mathcal{D},\mathcal{E})\leq 0,\\ 1 & \text{if } 0\leq \mathcal{K}(\tau,\mathcal{D},\mathcal{E})\leq 1,\\ 2\mathcal{K}(\tau,\mathcal{D},\mathcal{E})-1 & \text{if } \hbar\geq \mathcal{K}(\tau,\mathcal{D},\mathcal{E}).\end{cases}$$

where $\mathcal{K}(\tau,\mathcal{D},\mathcal{E})$ defined by (2.12).

If

$$\mathcal{K}(\tau,\mathcal{D},\mathcal{E})\leq 0,$$

then

$$\mathcal{K}(\tau,\mathcal{D},\mathcal{E}):=\frac{1+\mathcal{E}}{\mathcal{D}-\mathcal{E}}-\frac{(\mathcal{D}-\mathcal{E})\left([\Phi_2(\lambda,q)]^{n+m}-[\Phi_2(\lambda,q)]^{2n}\right)}{\left([\Phi_2(\lambda,q)]^m-[\Phi_2(\lambda,q)]^n\right)^2}+\frac{\tau(\mathcal{D}-\mathcal{E})\left([\Phi_3(\lambda,q)]^m-[\Phi_3(\lambda,q)]^n\right)}{\left([\Phi_2(\lambda,q)]^m-[\Phi_2(\lambda,q)]^n\right)^2}\leq 0,$$

and we have

$$\tau\leq \frac{\left([\Phi_2(\lambda,q)]^m-[\Phi_2(\lambda,q)]^n\right)^2}{(\mathcal{D}-\mathcal{E})\left([\Phi_3(\lambda,q)]^m-[\Phi_3(\lambda,q)]^n\right)}\times\left(\frac{(\mathcal{D}-\mathcal{E})\left([\Phi_2(\lambda,q)]^{n+m}-[\Phi_2(\lambda,q)]^{2n}\right)}{\left([\Phi_2(\lambda,q)]^m-[\Phi_2(\lambda,q)]^n\right)^2}-\frac{1+\mathcal{E}}{\mathcal{D}-\mathcal{E}}\right)=\sigma_1.$$

If

$$\mathcal{K}(\tau,\mathcal{D},\mathcal{E})\geq 1,$$

then

$$\mathcal{K}(\tau,\mathcal{D},\mathcal{E}):=\frac{1+\mathcal{E}}{\mathcal{D}-\mathcal{E}}-\frac{(\mathcal{D}-\mathcal{E})\left([\Phi_2(\lambda,q)]^{n+m}-[\Phi_2(\lambda,q)]^{2n}\right)}{\left([\Phi_2(\lambda,q)]^m-[\Phi_2(\lambda,q)]^n\right)^2}+\frac{\tau(\mathcal{D}-\mathcal{E})\left([\Phi_3(\lambda,q)]^m-[\Phi_3(\lambda,q)]^n\right)}{\left([\Phi_2(\lambda,q)]^m-[\Phi_2(\lambda,q)]^n\right)^2}\geq 1,$$

and we get

$$\tau\geq \frac{\left([\Phi_2(\lambda,q)]^m-[\Phi_2(\lambda,q)]^n\right)^2}{(\mathcal{D}-\mathcal{E})\left([\Phi_3(\lambda,q)]^m-[\Phi_3(\lambda,q)]^n\right)}\times\left(\frac{(\mathcal{D}-\mathcal{E})\left([\Phi_2(\lambda,q)]^{n+m}-[\Phi_2(\lambda,q)]^{2n}\right)}{\left([\Phi_2(\lambda,q)]^m-[\Phi_2(\lambda,q)]^n\right)^2}+\frac{\mathcal{D}-2\mathcal{E}-1}{\mathcal{D}-\mathcal{E}}\right)=\sigma_2.$$

□

3. Characterization properties

We will introduce some characteristic properties of the functions $\mathcal{G} \in \mathcal{F}(m,n,\lambda,q,\mathcal{D},\mathcal{E})$ by using the techniques that Silverman introduced in [28]. These properties include partial sums results, necessary and sufficient conditions for functions to be part of $\mathcal{F}(m,n,\lambda,q,\mathcal{D},\mathcal{E})$, radii of close-to-convexity and starlikeness for this class as well as distortion bounds.

Theorem 3. *If $\mathcal{G} \in \mathcal{A}$ has the form (1.1) and $\mathcal{G} \in \mathcal{F}(m,n,\lambda,q,\mathcal{D},\mathcal{E})$, then*

$$\sum_{j=2}^{\infty} \left((1-\mathcal{E})\left[\Phi_j(\lambda,q)\right]^m + (\mathcal{D}-1)\left[\Phi_j(\lambda,q)\right]^n\right)|a_n| \leq |\mathcal{D}-\mathcal{E}|, \tag{3.1}$$

where $\Phi_j(\lambda,q)$ is given by (1.3).

Proof. Letting $\mathcal{G} \in \mathcal{F}(m,n,\lambda,q,\mathcal{D},\mathcal{E})$, by (1.4) we deduce that

$$\frac{I_q^{m,\lambda}\mathcal{G}(\eta)}{I_q^{n,\lambda}\mathcal{G}(\eta)} = \frac{1+\mathcal{D}w(\eta)}{\mathcal{E}w(\eta)+1}, \quad \eta \in \mathcal{U},$$

with $w(\eta)$ a Schwarz function, meaning that:

$$\left|\frac{I_q^{m,\lambda}\mathcal{G}(\eta) - I_q^{n,\lambda}\mathcal{G}(\eta)}{\mathcal{D}I_q^{n,\lambda}\mathcal{G}(\eta) - \mathcal{E}I_q^{m,\lambda}\mathcal{G}(\eta)}\right| < 1, \quad \eta \in \mathcal{U}.$$

Thus, the above relation leads us to

$$\begin{aligned}
&\left|\frac{I_q^{m,\lambda}\mathcal{G}(\eta) - I_q^{n,\lambda}\mathcal{G}(\eta)}{\mathcal{D}I_q^{n,\lambda}\mathcal{G}(\eta) - \mathcal{E}I_q^{m,\lambda}\mathcal{G}(\eta)}\right| \\
&= \left|\frac{\sum_{j=2}^{\infty}\left(\left[\Phi_j(\lambda,q)\right]^m - \left[\Phi_j(\lambda,q)\right]^n\right)a_j\eta^j}{(\mathcal{D}-\mathcal{E})\eta + \sum_{j=2}^{\infty}\left(\mathcal{D}\left[\Phi_j(\lambda,q)\right]^n - \mathcal{E}\left[\Phi_j(\lambda,q)\right]^m\right)a_j\eta^j}\right| \\
&\leq \frac{\sum_{j=2}^{\infty}\left(\left[\Phi_j(\lambda,q)\right]^m - \left[\Phi_j(\lambda,q)\right]^n\right)|a_j|k^{j-1}}{|\mathcal{D}-\mathcal{E}| - \sum_{j=2}^{\infty}\left(\mathcal{D}\left[\Phi_j(\lambda,q)\right]^n - \mathcal{E}\left[\Phi_j(\lambda,q)\right]^m\right)|a_j|k^{j-1}} < 1,
\end{aligned}$$

and taking $k \to 1^-$, a simple computation yields (3.1). □

Example 1. For

$$\mathcal{G}(\eta) = \eta + \sum_{j=2}^{\infty} \frac{|\mathcal{D}-\mathcal{E}|}{(1-\mathcal{E})\left[\Phi_j(\lambda,q)\right]^m + (\mathcal{D}-1)\left[\Phi_j(\lambda,q)\right]^n}\ell_j\eta^j, \ \eta \in \Omega,$$

such that $\sum\limits_{j=2}^{\infty} \ell_j = 1$, we get

$$\sum_{j=2}^{\infty} \left((1-\mathcal{E})\left[\Phi_j(\lambda,q)\right]^m + (\mathcal{D}-1)\left[\Phi_j(\lambda,q)\right]^n\right)|a_j|$$

$$= \sum_{j=2}^{\infty} \left((1-\mathcal{E}) [\Phi_j(\lambda,q)]^m + (\mathcal{D}-1) [\Phi_j(\lambda,q)]^n \right)$$
$$\times \frac{|\mathcal{D}-\mathcal{E}|}{\left((1-\mathcal{E}) [\Phi_j(\lambda,q)]^m + (\mathcal{D}-1) [\Phi_j(\lambda,q)]^n \right)} \ell_j = |\mathcal{D}-\mathcal{E}| \sum_{j=2}^{\infty} \ell_j = |\mathcal{D}-\mathcal{E}|.$$

Then $\mathcal{G}(\eta) \in \mathcal{F}(m,n,\lambda,q,\mathcal{D},\mathcal{E})$.

Corollary 1. *Consider $\mathcal{G} \in \mathcal{F}(m,n,\lambda,q,\mathcal{D},\mathcal{E})$ described by (1.1). We have:*

$$|a_j| \le \frac{|\mathcal{D}-\mathcal{E}|}{(1-\mathcal{E}) [\Phi_j(\lambda,q)]^m + (\mathcal{D}-1) [\Phi_j(\lambda,q)]^n}, \quad \text{for } j \ge 2,$$

where $\Phi_j(\lambda,q)$ is defined by (1.3).

Theorem 4. *For $\mathcal{G} \in \mathcal{F}(m,n,\lambda,q,\mathcal{D},\mathcal{E})$, then*

$$r - \frac{|\mathcal{D}-\mathcal{E}|}{(1-\mathcal{E}) [\Phi_2(\lambda,q)]^m + (\mathcal{D}-1) [\Phi_2(\lambda,q)]^n} r^2 \le |\mathcal{G}(\eta)|$$
$$\le r + \frac{|\mathcal{D}-\mathcal{E}|}{(1-\mathcal{E}) [\Phi_2(\lambda,q)]^m + (\mathcal{D}-1) [\Phi_2(\lambda,q)]^n} r^2.$$

For the function defined by

$$\widehat{\mathcal{G}}(\eta) := \eta - \frac{|\mathcal{D}-\mathcal{E}|}{(1-\mathcal{E}) [\Phi_2(\lambda,q)]^m + (\mathcal{D}-1) [\Phi_2(\lambda,q)]^n} \eta^2, \quad |\eta| = r < 1, \tag{3.2}$$

the approximation is sharp.

Proof. For $|\eta| = r < 1$ we have

$$|\mathcal{G}(\eta)| = \left| \eta + \sum_{j=2}^{\infty} a_j \eta^j \right| \le |\eta| + \sum_{j=2}^{\infty} a_j |\eta|^j = r + \sum_{j=2}^{\infty} a_j |r|^j.$$

Moreover, since for $|\eta| = r < 1$ we get $r^j < r^2$ for all $j \ge 2$, the above relation implies that

$$|\mathcal{G}(\eta)| \le r + r^2 \sum_{j=2}^{\infty} |a_j|. \tag{3.3}$$

Similarly, we get

$$|\mathcal{G}(\eta)| \ge r - r^2 \sum_{j=2}^{\infty} |a_j|. \tag{3.4}$$

From the relation (3.1) we have

$$\sum_{j=2}^{\infty} \left((1-\mathcal{E}) [\Phi_j(\lambda,q)]^m + (\mathcal{D}-1) [\Phi_j(\lambda,q)]^n \right) |a_j| \le |\mathcal{D}-\mathcal{E}|,$$

but

$$\left((1-\mathcal{E})\left[\Phi_2(\lambda,q)\right]^m+(\mathcal{D}-1)\left[\Phi_2(\lambda,q)\right]^n\right)\sum_{j=2}^{\infty}\left|a_j\right|$$

$$\leq\sum_{j=2}^{\infty}\left((1-\mathcal{E})\left[\Phi_j(\lambda,q)\right]^m+(\mathcal{D}-1)\left[\Phi_j(\lambda,q)\right]^n\right)\left|a_n\right|\leq\left|\mathcal{D}-\mathcal{E}\right|.$$

Therefore,

$$\sum_{j=2}^{\infty}a_j\leq\frac{\left|\mathcal{D}-\mathcal{E}\right|}{(1-\mathcal{E})\left[\Phi_2(\lambda,q)\right]^m+(\mathcal{D}-1)\left[\Phi_2(\lambda,q)\right]^n},\tag{3.5}$$

and using (3.5) in (3.3) and (3.4) we get the desired result. □

The next distortion theorem for the family $\mathcal{F}(m,n,\lambda,q,\mathcal{D},\mathcal{E})$ could be similarly obtained:

Theorem 5. *If $\mathcal{G}\in\mathcal{F}(m,n,\lambda,q,\mathcal{D},\mathcal{E})$, we have:*

$$1-\frac{2\left|\mathcal{D}-\mathcal{E}\right|}{(1-\mathcal{E})\left[\Phi_2(\lambda,q)\right]^m+(\mathcal{D}-1)\left[\Phi_2(\lambda,q)\right]^n}k\leq\left|\mathcal{G}'(\eta)\right|$$

$$\leq1+\frac{2\left|\mathcal{D}-\mathcal{E}\right|}{(1-\mathcal{E})\left[\Phi_2(\lambda,q)\right]^m+(\mathcal{D}-1)\left[\Phi_2(\lambda,q)\right]^n}k.$$

The equality holds if the function is $\widehat{\mathcal{G}}$ given by (3.2).

Proof. We shall skip the proof since it closely resembles the arguments presented in Theorem 4. □

The next result deals with the fact that a convex combination of functions from the class $\mathcal{F}(m,n,\lambda,q,\mathcal{D},\mathcal{E})$ belongs to the same class, as follows:

Theorem 6. *Consider $\mathcal{G}_i\in\mathcal{F}(m,n,\lambda,q,\mathcal{D},\mathcal{E})$ having the form:*

$$\mathcal{G}_i(\eta)=\eta+\sum_{j=2}^{\infty}a_{i,j}\eta^j,\quad i=1,2,3,\ldots,m.\tag{3.6}$$

Then $H\in\mathcal{F}(m,n,\lambda,q,\mathcal{D},\mathcal{E})$, where

$$H(\eta):=\sum_{i=1}^{m}c_i\mathcal{G}_i(\eta),\quad\textit{and}\quad\sum_{i=1}^{m}c_i=1.$$

Proof. Applying the outcome of Theorem 3 we write:

$$\sum_{j=2}^{\infty}\left((1-\mathcal{E})\left[\Phi_j(\lambda,q)\right]^m+(\mathcal{D}-1)\left[\Phi_j(\lambda,q)\right]^n\right)\left|a_j\right|\leq\left|\mathcal{D}-\mathcal{E}\right|,$$

and, in addition,

$$H(\eta)=\sum_{i=1}^{m}c_i\left(\eta+\sum_{j=2}^{\infty}a_{i,j}\eta^j\right)=\eta+\sum_{j=2}^{\infty}\left(\sum_{i=1}^{m}c_ia_{i,j}\right)\eta^j.$$

Therefore

$$\begin{aligned}&\sum_{j=2}^{\infty}\left((1-\mathcal{E})\left[\Phi_j(\lambda,q)\right]^m+(\mathcal{D}-1)\left[\Phi_j(\lambda,q)\right]^n\right)\left|\sum_{i=1}^{m}c_i a_{i,j}\right|\\&\leq\sum_{i=1}^{m}\left[\sum_{j=2}^{\infty}\left((1-\mathcal{E})\left[\Phi_j(\lambda,q)\right]^m+(\mathcal{D}-1)\left[\Phi_j(\lambda,q)\right]^n\right)\left|a_{i,j}\right|\right]c_i\\&=\sum_{i=1}^{m}|\mathcal{D}-\mathcal{E}|c_i=|\mathcal{D}-\mathcal{E}|\sum_{i=1}^{m}c_i=|\mathcal{D}-\mathcal{E}|,\end{aligned}$$

thus $H(\eta)\in\mathcal{F}(m,n,\lambda,q,\mathcal{D},\mathcal{E})$. □

Regarding the arithmetic means of the functions of the family $\mathcal{F}(m,n,\lambda,q,\mathcal{D},\mathcal{E})$ the next result holds:

Theorem 7. *When $\mathcal{G}_i\in\mathcal{F}(m,n,\lambda,q,\mathcal{D},\mathcal{E})$ have the form seen in (3.6), we obtain:*

$$\mathcal{G}(\eta):=\eta+\frac{1}{m}\sum_{j=2}^{\infty}\left(\sum_{i=1}^{m}a_{i,j}\eta^j\right)\in\mathcal{F}(m,n,\lambda,q,\mathcal{D},\mathcal{E}),\tag{3.7}$$

with function $\mathcal{G}$ being the arithmetic mean of $\mathcal{G}_i$, $i=1,2,3,\ldots,m$.

Proof. Using (3.7) we get

$$\mathcal{G}(\eta)=\frac{1}{m}\sum_{i=1}^{m}f_i(\eta)\,\frac{1}{m}\sum_{i=1}^{m}\left(\eta+\sum_{j=2}^{\infty}a_{i,j}\eta^j\right)\eta+\sum_{j=2}^{\infty}\left(\frac{1}{m}\sum_{i=1}^{m}a_{i,j}\right)\eta^j,$$

and to prove that $\mathcal{G}(\eta)\in\mathcal{F}(m,n,\lambda,q,\mathcal{D},\mathcal{E})$, according to the Theorem 3, it suffices to demonstrate that:

$$\sum_{j=2}^{\infty}\left((1-\mathcal{E})\left[\Phi_j(\lambda,q)\right]^m+(\mathcal{D}-1)\left[\Phi_j(\lambda,q)\right]^n\right)\left(\frac{1}{m}\sum_{i=1}^{m}\left|a_{i,j}\right|\right)\leq|\mathcal{D}-\mathcal{E}|.$$

A quick calculation reveals that:

$$\begin{aligned}&\sum_{j=2}^{\infty}\left((1-\mathcal{E})\left[\Phi_j(\lambda,q)\right]^m+(\mathcal{D}-1)\left[\Phi_j(\lambda,q)\right]^n\right)\left(\frac{1}{m}\sum_{i=1}^{m}\left|a_{i,j}\right|\right)\\&=\frac{1}{m}\sum_{i=1}^{m}\left(\sum_{j=2}^{\infty}\left((1-\mathcal{E})\left[\Phi_j(\lambda,q)\right]^m+(\mathcal{D}-1)\left[\Phi_j(\lambda,q)\right]^n\right)\left|a_{i,j}\right|\right)\\&\leq\frac{1}{m}\sum_{i=1}^{m}|\mathcal{D}-\mathcal{E}|=|\mathcal{D}-\mathcal{E}|,\end{aligned}$$

therefore $\mathcal{G}\in\mathcal{F}(m,n,\lambda,q,\mathcal{D},\mathcal{E})$. □

Theorem 8. *When $\mathcal{G} \in \mathcal{F}(m,n,\lambda,q,\mathcal{D},\mathcal{E})$, we have that $\mathcal{G} \in \mathcal{S}^*(\vartheta)$ $(0 \le \vartheta < 1)$, $|\eta| < k_1^*$,*

$$k_1^* = \inf_{j \ge 2} \left(\frac{(1-\vartheta)\left((1-\mathcal{E})\left[\Phi_j(\lambda,q)\right]^m + (\mathcal{D}-1)\left[\Phi_j(\lambda,q)\right]^n\right]}{(j-\vartheta)\,|\mathcal{D}-\mathcal{E}|} \right)^{\frac{1}{j-1}}.$$

Proof. Consider $\mathcal{G} \in \mathcal{F}(m,n,\lambda,q,\mathcal{D},\mathcal{E})$. Then $\mathcal{G} \in \mathcal{S}^*(\vartheta)$ if:

$$\left| \frac{\eta \mathcal{G}'(\eta)}{\mathcal{G}(\eta)} - 1 \right| < 1-\vartheta.$$

By applying a simple calculation, we deduce:

$$\sum_{j=2}^{\infty} \left(\frac{j-\vartheta}{1-\vartheta} \right) \left|a_j\right| |\eta|^{j-1} < 1. \tag{3.8}$$

Since $\mathcal{G} \in \mathcal{F}(m,n,\lambda,q,\mathcal{D},\mathcal{E})$, considering (3.1) we have:

$$\sum_{j=2}^{\infty} \frac{(1-\mathcal{E})\left[\Phi_j(\lambda,q)\right]^m + (\mathcal{D}-1)\left[\Phi_j(\lambda,q)\right]^n}{|\mathcal{D}-\mathcal{E}|} \left|a_j\right| < 1.$$

Inequality (3.8) is true when:

$$\sum_{j=2}^{\infty} \left(\frac{j-\vartheta}{1-\vartheta} \right) \left|a_j\right| |\eta|^{j-1} < \sum_{j=2}^{\infty} \frac{(1-\mathcal{E})\left[\Phi_j(\lambda,q)\right]^m + (\mathcal{D}-1)\left[\Phi_j(\lambda,q)\right]^n}{|\mathcal{D}-\mathcal{E}|} \left|a_j\right|,$$

which implies that

$$|\eta|^{j-1} < \left(\frac{(1-\vartheta)\left((1-\mathcal{E})\left[\Phi_j(\lambda,q)\right]^m + (\mathcal{D}-1)\left[\Phi_j(\lambda,q)\right]^n\right)}{|\mathcal{D}-\mathcal{E}|\,(j-\vartheta)} \right),$$

or, equivalently

$$|\eta| < \left(\frac{(1-\vartheta)\left((1-\mathcal{E})\left[\Phi_j(\lambda,q)\right]^m + (\mathcal{D}-1)\left[\Phi_j(\lambda,q)\right]^n\right)}{|\mathcal{D}-\mathcal{E}|\,(j-\vartheta)} \right)^{\frac{1}{j-1}},$$

hence, the family is starlike. □

Theorem 9. *Any function $\mathcal{G} \in \mathcal{F}(m,n,\lambda,q,\mathcal{D},\mathcal{E})$, is a close-to-convex function of order ϑ $(0 \le \vartheta < 1)$, $|\eta| < k_2^*$,*

$$k_2^* = \inf_{j \ge 2} \left(\frac{(1-\vartheta)\left((1-\mathcal{E})\left[\Phi_j(\lambda,q)\right]^m + (\mathcal{D}-1)\left[\Phi_j(\lambda,q)\right]^n\right)}{j|\mathcal{D}-\mathcal{E}|} \right)^{\frac{1}{j-1}}.$$

Proof. Let $\mathcal{G} \in \mathcal{F}(m,n,\lambda,q,\mathcal{D},\mathcal{E})$. If $\mathcal{G}$ is a close-to-convex function of order ϑ, then we can write:

$$\left| \mathcal{G}'(\eta) - 1 \right| < 1-\vartheta,$$

equivalently written,

$$\sum_{j=2}^{\infty} \frac{j}{1-\vartheta} \left|a_j\right| |\eta|^{j-1} < 1.$$

Since $\mathcal{G} \in \mathcal{F}(m,n,\lambda,q,\mathcal{D},\mathcal{E})$, using (3.1) we obtain:

$$\sum_{j=2}^{\infty} \frac{(1-\mathcal{E})\left[\Phi_j(\lambda,q)\right]^m + (\mathcal{D}-1)\left[\Phi_j(\lambda,q)\right]^n}{|\mathcal{D}-\mathcal{E}|} \left|a_j\right| < 1.$$

Inequality (3.8) is true when:

$$\sum_{j=2}^{\infty} \frac{j}{1-\vartheta} \left|a_j\right| |\eta|^{j-1} < \sum_{j=2}^{\infty} \frac{(1-\mathcal{E})\left[\Phi_j(\lambda,q)\right]^m + (\mathcal{D}-1)\left[\Phi_j(\lambda,q)\right]^n}{|\mathcal{D}-\mathcal{E}|} \left|a_j\right|,$$

which implies that

$$|\eta|^{j-1} < \left(\frac{(1-\vartheta)\left[(1-\mathcal{E})\left[\Phi_j(\lambda,q)\right]^m + (\mathcal{D}-1)\left[\Phi_j(\lambda,q)\right]^n\right]}{j|\mathcal{D}-\mathcal{E}|}\right),$$

or, equivalently

$$|\eta| < \left(\frac{(1-\vartheta)\left[(1-\mathcal{E})\left[\Phi_j(\lambda,q)\right]^m + (\mathcal{D}-1)\left[\Phi_j(\lambda,q)\right]^n\right]}{j|\mathcal{D}-\mathcal{E}|}\right)^{\frac{1}{j-1}},$$

which yields the desired result. □

4. THE COEFFICIENT INEQUALITIES FOR $\mathcal{G}^{-1} \in \mathcal{F}(m,n,\lambda,q,\mathcal{D},\mathcal{E})$

According to the "Koebe one quarter theorem" [10], there is a disk with radius $\frac{1}{4}$ in the image of $\mathcal{U}$ through any function $\mathcal{G} \in \mathcal{S}$. Consequently, an inverse function $\mathcal{G}^{-1}$ for each $\mathcal{G} \in \mathcal{S}$ exists and satisfies:

$$\mathcal{G}^{-1}(\mathcal{G}(\eta)) = \eta,\ (\eta \in \Omega) \quad \text{and} \quad \mathcal{G}\left(\mathcal{G}^{-1}(w)\right) = w,\ \left(|w| < r_0(\mathcal{G}),\ r_0(\mathcal{G}) \geq \frac{1}{4}\right).$$

When both $\mathcal{G}$ and $\mathcal{G}^{-1}$ are univalent in $\mathcal{U}$, a function $\mathcal{G} \in \mathcal{A}$ is referred to as bi-univalent in $\mathcal{U}$. It is important to remember that the set of bi-univalent functions is not empty. The bi-univalent function family includes, for instance, the functions η, $\frac{\eta}{1-\eta}$, $-\log(1-\eta)$, and $\frac{1}{2}\log\frac{1+\eta}{1-\eta}$ but the Koebe function is not included.

Theorem 10. *Considering $\mathcal{G} \in \mathcal{F}(m,n,\lambda,q,\mathcal{D},\mathcal{E})$ and $\mathcal{G}^{-1}(w) = w + \sum_{n=2}^{\infty} d_n w^n$, we have:*

$$|d_2| = \frac{|\mathcal{D}-\mathcal{E}|}{\left[\Phi_2(\lambda,q)\right]^m - \left[\Phi_2(\lambda,q)\right]^n},$$

$$|d_3| = \frac{|\mathcal{D}-\mathcal{E}|}{\left[\Phi_3(\lambda,q)\right]^m - \left[\Phi_3(\lambda,q)\right]^n} \max\{1; |\mathcal{K}(2,\mathcal{D},\mathcal{E})-1|\},$$

and for any $\hbar \in \mathbb{C}$, *obtain:*

$$\left|d_3 - \hbar d_2^2\right| \leq \frac{|\mathcal{D}-\mathcal{E}|}{[\Phi_3(\lambda,q)]^m - [\Phi_3(\lambda,q)]^n} \times \max\left[1; \left|\mathcal{K}(2,\mathcal{D},\mathcal{E}) + \hbar\frac{(\mathcal{D}-\mathcal{E})\left([\Phi_3(\lambda,q)]^m - [\Phi_3(\lambda,q)]^n\right)}{\left([\Phi_2(\lambda,q)]^m - [\Phi_2(\lambda,q)]^n\right)^2} - 1\right|\right],$$

where

$$\mathcal{K}(2,\mathcal{D},\mathcal{E}) := \frac{1+\mathcal{E}}{\mathcal{D}-\mathcal{E}} - \frac{(\mathcal{D}-\mathcal{E})\left([\Phi_2(\lambda,q)]^{n+m} - [\Phi_2(\lambda,q)]^{2n}\right)}{\left([\Phi_2(\lambda,q)]^m - [\Phi_2(\lambda,q)]^n\right)^2} + \frac{2(\mathcal{D}-\mathcal{E})\left([\Phi_3(\lambda,q)]^m - [\Phi_3(\lambda,q)]^n\right)}{\left([\Phi_2(\lambda,q)]^m - [\Phi_2(\lambda,q)]^n\right)^2}.$$

Proof. Since

$$\mathcal{G}^{-1}(w) = w + \sum_{n=2}^{\infty} d_n w^n,$$

is the inverse of the function $\mathcal{G}$, it can be seen that

$$\eta = \mathcal{G}^{-1}\left(\mathcal{G}(\eta)\right) = \mathcal{G}\left(\mathcal{G}^{-1}(\eta)\right), \quad |\eta| < r_0(\mathcal{G}). \tag{4.1}$$

From (1.1) and (4.1), we obtain that

$$\eta = \mathcal{G}^{-1}\left(\eta + \sum_{n=2}^{\infty} a_n \eta^n\right), \quad |\eta| < r_0(\mathcal{G}), \tag{4.2}$$

therefore from (4.1) and (4.2) we get:

$$\eta + (a_2 + d_2)\eta^2 + (a_3 + 2a_2 d_2 + d_3)\eta^3 + \cdots = \eta, \quad |\eta| < r_0(\mathcal{G}). \tag{4.3}$$

Equating the corresponding coefficients of the relation (4.3), we conclude that

$$d_2 = -a_2, \tag{4.4}$$

$$d_3 = 2a_2^2 - a_3. \tag{4.5}$$

First, from the relations (2.9) and (4.4) we have

$$d_2 = -\frac{\mathcal{D}-\mathcal{E}}{2\left([\Phi_2(\lambda,q)]^m - [\Phi_2(\lambda,q)]^n\right)} p_1.$$

To find $|d_3|$, from (4.5) we write:

$$|d_3| = |a_3 - 2a_2^2|,$$

hence by using (2.11) for real $\tau = 2$ we deduce that:

$$|d_3| = \left|a_3 - 2a_2^2\right| = \frac{|\mathcal{D}-\mathcal{E}|}{2\left([\Phi_3(\lambda,q)]^m - [\Phi_3(\lambda,q)]^n\right)} \left|p_2 - \frac{p_1^2}{2}\mathcal{K}(2,\mathcal{D},\mathcal{E})\right|$$

$$= \frac{|\mathcal{D}-\mathcal{E}|}{2\left([\Phi_3(\lambda,q)]^m-[\Phi_3(\lambda,q)]^n\right)} \max\left\{1; |\mathcal{K}(2,\mathcal{D},\mathcal{E})-1|\right\},$$

where

$$\mathcal{K}(2,\mathcal{D},\mathcal{E}) := \frac{1+\mathcal{E}}{\mathcal{D}-\mathcal{E}} - \frac{(\mathcal{D}-\mathcal{E})\left([\Phi_2(\lambda,q)]^{n+m}-[\Phi_2(\lambda,q)]^{2n}\right)}{\left([\Phi_2(\lambda,q)]^m-[\Phi_2(\lambda,q)]^n\right)^2} + \frac{2(\mathcal{D}-\mathcal{E})\left([\Phi_3(\lambda,q)]^m-[\Phi_3(\lambda,q)]^n\right)}{\left([\Phi_2(\lambda,q)]^m-[\Phi_2(\lambda,q)]^n\right)^2}.$$

For any complex number $\hbar$, a simple computation gives us that:

$$\begin{aligned} d_3-\hbar d_2^2 &= \frac{\mathcal{D}-\mathcal{E}}{2\left([\Phi_3(\lambda,q)]^m-[\Phi_3(\lambda,q)]^n\right)}\left(p_2-\frac{p_1^2}{2}\mathcal{K}(2,\mathcal{D},\mathcal{E})\right) \\ &\quad -\hbar\frac{(\mathcal{D}-\mathcal{E})^2}{4\left([\Phi_2(\lambda,q)]^m-[\Phi_2(\lambda,q)]^n\right)^2}p_1^2 \\ &= \frac{\mathcal{D}-\mathcal{E}}{2\left([\Phi_3(\lambda,q)]^m-[\Phi_3(\lambda,q)]^n\right)} \\ &\quad \times\left(p_2-\frac{p_1^2}{2}\left[\mathcal{K}(2,\mathcal{D},\mathcal{E})+\hbar\frac{(\mathcal{D}-\mathcal{E})\left([\Phi_3(\lambda,q)]^m-[\Phi_3(\lambda,q)]^n\right)}{\left([\Phi_2(\lambda,q)]^m-[\Phi_2(\lambda,q)]^n\right)^2}\right]\right). \end{aligned} \tag{4.6}$$

After using Lemma 1 and (2.1), and considering modulus on both sides of (4.6), we determine:

$$\begin{aligned} \left|d_3-\hbar d_2^2\right| &\le \frac{|\mathcal{D}-\mathcal{E}|}{[\Phi_3(\lambda,q)]^m-[\Phi_3(\lambda,q)]^n} \\ &\quad \times\max\left\{1; \left|\mathcal{K}(2,\mathcal{D},\mathcal{E})+\hbar\frac{(\mathcal{D}-\mathcal{E})\left([\Phi_3(\lambda,q)]^m-[\Phi_3(\lambda,q)]^n\right)}{\left([\Phi_2(\lambda,q)]^m-[\Phi_2(\lambda,q)]^n\right)^2}-1\right|\right\} \end{aligned}$$

and this completes our proof. □

5. Conclusions

The study on the generalized differential operator $I_q^{m,\lambda}$ given by (1.2), used for introducing the new subclass of $\mathcal{A}$ given by (1.4), contains information that may serve as a basis for future research efforts on introducing other new classes of analytic functions.

Furthermore, we hope that this study will inspire other researchers to develop this concept further for new families that can be obtained by applying the concept of subordination with connection to specific probability distribution series [3, 31] or with generalized telephone numbers[8, 25].

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SYMMETRIC AND GENERATING FUNCTIONS FOR CERTAIN PRODUCTS OF NUMBERS AND POLYNOMIALS WITH APPLICATION TO BIFURCATION ANALYSIS

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Abstract. In this paper, we establish a novel theorem pertaining to symmetric and generating functions. Utilizing this theorem, we derive new generating functions for products of k-Fibonacci numbers and Fibonacci polynomials with certain (p,q)-numbers, such as (p,q)-Fibonacci and (p,q)-Jacobsthal-Lucas numbers. Furthermore, we examine the bifurcation and chaotic behavior of the generating functions associated with (p,q)-Jacobsthal numbers for specific parameter values of p and q.

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1. Introduction and preliminaries

The Fibonacci sequence, denoted by $(F_n)_{n\geqslant 0}$, is one of the most well-known and intriguing numerical sequences due to its numerous properties and connections to various fields [15]. It is defined by the recurrence relation $F_{n+2} = F_{n+1} + F_n$ for every $n \geqslant 2$, with initial values $F_0 = 0$ and $F_1 = 1$. Many generalizations of this sequence have been proposed, some by modifying the initial conditions and others by preserving the recurrence relation.

The k-Fibonacci sequences, denoted by $\{F_{k,n}\}$, are defined recurrently for any positive real number k by the relation $F_{k,n} = kF_{k,n-1} + F_{k,n-2}$, with initial conditions $F_{k,0} = 0$ and $F_{k,1} = 1$ [7, 10]. Using this recurrence, it is possible to calculate the k-Fibonacci sequence both forward and backward. Substituting n with $-n$ yields the relation $F_{k,-n} = F_{k,-(n-1)} + F_{k,-(n-2)}$. The characteristic equation $x^2 - kx - 1 = 0$ of this sequence has two roots, denoted by α and β. Consequently, the Binet formula leads to

$$F_{k,-n} = (-1)^{n+1} F_{k,n}, \quad \text{for all } n \geq 0, \quad (\text{see [19]}), \tag{1.1}$$

demonstrating a close relationship between the positive and negative indices of the Fibonacci numbers.

On the other hand, the generalized Fibonacci sequence $\{F_{p,q,n}\}_{n\in\mathbb{N}}$, referred to as the (p,q)-Fibonacci sequence, is defined in [25] by

$$F_{p,q,0}=0,\quad F_{p,q,1}=1,\quad \text{and}\quad F_{p,q,n}=pF_{p,q,n-1}+qF_{p,q,n-2},\quad \text{for } n\geq 2.$$

Each term of the (p,q)-Fibonacci sequence is called a (p,q)-Fibonacci number.

It is well known that the (p,q)-Lucas, (p,q)-Pell, (p,q)-Pell-Lucas, (p,q)-Jacobsthal, and (p,q)-Jacobsthal-Lucas numbers, denoted respectively by $\{L_{p,q,n}\}_{n\in\mathbb{N}}$, $\{P_{p,q,n}\}_{n\in\mathbb{N}}$, $\{Q_{p,q,n}\}_{n\in\mathbb{N}}$, $\{J_{p,q,n}\}_{n\in\mathbb{N}}$, and $\{j_{p,q,n}\}_{n\in\mathbb{N}}$, are defined by the following recurrence relations for any positive real numbers p and q (see [5, 14, 22, 24, 26]):

$$\begin{aligned}
&L_{p,q,0}=2, && L_{p,q,1}=p, && \text{and} && L_{p,q,n}=pL_{p,q,n-1}+qL_{p,q,n-2}, && \text{for } n\geq 2,\\
&P_{p,q,0}=0, && P_{p,q,1}=1, && \text{and} && P_{p,q,n}=2pP_{p,q,n-1}+qP_{p,q,n-2}, && \text{for } n\geq 2,\\
&Q_{p,q,0}=2, && Q_{p,q,1}=2p, && \text{and} && Q_{p,q,n}=2pQ_{p,q,n-1}+qQ_{p,q,n-2}, && \text{for } n\geq 2,\\
&J_{p,q,0}=0, && J_{p,q,1}=1, && \text{and} && J_{p,q,n}=pJ_{p,q,n-1}+2qJ_{p,q,n-2}, && \text{for } n\geq 2,\\
&j_{p,q,0}=2, && j_{p,q,1}=p, && \text{and} && j_{p,q,n}=pj_{p,q,n-1}+2qj_{p,q,n-2}, && \text{for } n\geq 2.
\end{aligned}$$

The generating function for the (p,q)-Jacobsthal numbers is given by

$$J_{p,q}(z)=\frac{z}{1-pz-2qz^2}. \tag{1.2}$$

Large classes of polynomials can be defined by Fibonacci-like recurrence relations. One such class, known as Fibonacci polynomials, was studied in 1883 by Catalan and Jacobsthal. These polynomials, denoted by $F_n(x)$, are defined by the recurrence relation

$$\begin{cases} F_n(x)=xF_{n-1}(x)+F_{n-2}(x), & \text{for } n\geq 2,\\ F_0(x)=1, \quad F_1(x)=x. \end{cases}$$

For more details about this sequence, the reader is referred to [8].

In their seminal work [4], Berry, Lewis, and Nye pioneered the study of fractal structures by establishing the self-similarity of the Weierstrass–Mandelbrot function. Building upon this foundation, Benbernou et al. [3] derived regularity conditions for the three-dimensional magnetohydrodynamic equations, while Guariglia and Silvestrov [13] extended wavelet theory through the introduction of fractional wavelets. Subsequent research by Guariglia [11] further demonstrated the significance of fractality in primality theory and image analysis. More recent advancements include the development of Chebyshev wavelet techniques by Guariglia and Guido [12] and the establishment of novel integral inequalities for generalized convex functions by Akdemir et al. [2]. Etemad et al. [9] contributed to this evolving landscape by proving existence results for solutions to multi-order q-difference fractional boundary value problems. Collectively, these studies underscore the growing interplay between fractals, wavelet theory, and analytic inequalities, highlighting their unifying role across pure and applied mathematics.

Now, we present certain essential information and outcomes concerning symmetric functions.

Definition 1 ([1, Definition 2.1])**.** Consider A and B as two alphabets. We define $S_n(A-B)$ as

$$\frac{\prod_{b\in B}(1-bz)}{\prod_{a\in A}(1-az)}=\sum_{n=0}^{\infty}S_n(A-B)z^n,$$

with the condition $S_n(A-B)=0$ for $n<0$.

Definition 2 ([23, Definition 1.5])**.** Let k be a positive integer and $A=\{a_1,a_2\}$ be a set of given variables. The k^{th} symmetric function $S_k(A)=S_k(a_1+a_2)$ is defined by

$$S_k(A)=S_k(a_1+a_2)=\frac{a_1^{k+1}-a_2^{k+1}}{a_1-a_2},$$

with

$$\begin{aligned}
S_0(A)&=S_0(a_1+a_2)=1,\\
S_1(A)&=S_1(a_1+a_2)=a_1+a_2,\\
S_2(A)&=S_2(a_1+a_2)=a_1^2+a_1a_2+a_2^2,\\
&\vdots
\end{aligned}$$

Now, we give some definitions from [17] that will be useful in the sequel.

Definition 3. Assume that $\{a_1,a_2,\dots,a_n\}$ is an alphabet and k and n are two positive integers. The k^{th} elementary symmetric function, denoted as $e_k(a_1,a_2,\dots,a_n)$, is the sum of the products of k distinct elements selected from the set $\{a_1,a_2,\dots,a_n\}$, i.e.,

$$e_k^{(n)}=e_k(a_1,a_2,\dots,a_n)=\sum_{i_1+i_2+\dots+i_n=k}a_1^{i_1}a_2^{i_2}\dots a_n^{i_n}\quad(0\le k\le n),$$

where $i_1,i_2,\dots,i_n\in\{0,1\}$.

Remark 1. By convention, $e_0(a_1,a_2,\dots,a_n)=1$, and we set $e_k(a_1,a_2,\dots,a_n)=0$ for $k<0$ or $k>n$.

Definition 4. Assume that $\{a_1,a_2,\dots,a_n\}$ is an alphabet and k and n are two positive integers. The k^{th} complete homogeneous symmetric function, denoted as $h_k(a_1,a_2,\dots,a_n)$, is given by

$$h_k^{(n)}=h_k(a_1,a_2,\dots,a_n)=\sum_{i_1+i_2+\dots+i_n=k}a_1^{i_1}a_2^{i_2}\dots a_n^{i_n}\quad(k\ge 0),\tag{1.3}$$

where $i_1,i_2,\dots,i_n\ge 0$.

Remark 2. By convention, $h_0(a_1,a_2,\dots,a_n)=1$, and we set $h_k(a_1,a_2,\dots,a_n)=0$ for $k<0$.

For $n=2$, the k^{th} complete homogeneous symmetric function (1.3) is given by

$$h_k^{(2)}=h_k(a_1,a_2)=S_k(a_1+a_2)=\frac{a_1^{k+1}-a_2^{k+1}}{a_1-a_2},\quad \text{for all } k\in\mathbb{N}_0.$$

Proposition 1 ([19, Proposition 2.1])**.** *The generating function for the elementary symmetric function based on the alphabet $A=\{a_1,a_2,\dots,a_n\}$ is given by*

$$\sum_{k=0}^{\infty}e_k(a_1,a_2,\dots,a_n)z^k=\prod_{a\in A}(1+az).$$

Proposition 2 ([19, Proposition 2.2])**.** *The generating function for the complete homogeneous symmetric function based on the alphabet $A=\{a_1,a_2,\dots,a_n\}$ is given by*

$$\sum_{k=0}^{\infty}h_k(a_1,a_2,\dots,a_n)z^k=\frac{1}{\prod_{a\in A}(1-az)}.$$

A fundamental relationship exists between elementary symmetric functions and complete homogeneous symmetric functions:

$$\sum_{j=0}^{k}(-1)^j e_j(a_1,a_2,\dots,a_n)h_{k-j}(a_1,a_2,\dots,a_n)=0,\quad \forall k>0.$$

Definition 5 ([6, Definition 2])**.** Given an alphabet $A=\{a_1,a_2\}$, the symmetrizing operator $\delta_{a_1a_2}^k$ is defined by

$$\delta_{a_1a_2}^k(f)=\frac{a_1^kf(a_1)-a_2^kf(a_2)}{a_1-a_2},\quad \text{for all } k\in\mathbb{N}_0:=\mathbb{N}\cup\{0\}=\{0,1,2,\dots\}.$$

In this paper, we apply the operator $\delta_{b_1b_2}^{4-l}$ to derive generating functions for certain generalized products of numbers and polynomials. Additionally, we explore the bifurcation and chaotic behavior exhibited by the generating function for (p,q)-Jacobsthal numbers.

This paper is structured as follows: In Section 2, we present our main theorem, which establishes a connection between the symmetric functions defined in the preceding section and the symmetrizing operator. This theorem unifies various previously established generating function results into a single framework. It is used in Section 3 to find the generating functions of the products of (p,q)-numbers with Pell polynomials and k-Fibonacci numbers at positive and negative indices. This section is divided into two parts: Part 1 focuses on calculating some ordinary generating functions of the products of (p,q)-numbers with k-Fibonacci polynomials, while Part 2 focuses on calculating some ordinary generating functions of the products of (p,q)-numbers with Fibonacci polynomials. In Section 4, we investigate the behavior of the family of maps (1.2) corresponding to different values of the parameters p and q. These maps serve as generating functions for sequences of generalized Jacobsthal numbers. It is observed that as the parameters vary, the behavior of these maps evolves from periodicity, through bifurcation, to chaos. Studies on bifurcation and chaotic behavior in nonlinear dynamical systems have become increasingly significant. Numerous studies have focused on chaos in various sequences and polynomials. In [18], the authors studied the bifurcation of Fibonacci generating functions associated with the golden mean. A novel approach involving a chaos-based generating function for Chebyshev polynomials has been explored in [16]. We have analytically determined a new generating function for the Jacobsthal numbers. Furthermore, the chaotic behavior of this generating function is verified through the examination of the bifurcation diagram and the Lyapunov exponent.

2. Main results

In this section, we establish the main theorem of this paper. This result provides a unified framework that encapsulates all previously established results, enabling them to be interpreted as special cases, such as those found in [20].

Theorem 1. *Assume that A and B are two alphabets, denoted by* $\{a_1,a_2,\dots,a_k\}$ *and* $\{b_1,b_2\}$ *respectively, then we have*

$$\begin{aligned}
&\sum_{n=0}^{\infty} h_n(a_1,a_2,\dots,a_k)\,h_{n-l+3}(b_1,b_2)\,z^n \\
&= \frac{h_{3-l}(b_1,b_2) - e_1(a_1,a_2,\dots,a_k)\,b_1b_2h_{2-l}(b_1,b_2)\,z}{\left(\sum\limits_{n=0}^{\infty}(-1)^n e_n(a_1,a_2,\dots,a_k)\,b_1^nz^n\right)\left(\sum\limits_{n=0}^{\infty}(-1)^n e_n(a_1,a_2,\dots,a_k)\,b_2^nz^n\right)} \\
&\quad + \frac{e_2(a_1,a_2,\dots,a_k)\,b_1^2b_2^2h_{1-l}(b_1,b_2)z^2}{\left(\sum\limits_{n=0}^{\infty}(-1)^n e_n(a_1,a_2,\dots,a_k)\,b_1^nz^n\right)\left(\sum\limits_{n=0}^{\infty}(-1)^n e_n(a_1,a_2,\dots,a_k)\,b_2^nz^n\right)} \\
&\quad - \frac{-e_3(a_1,a_2,\dots,a_k)\,b_1^3b_2^3h_{-l}(b_1,b_2)z^3}{\left(\sum\limits_{n=0}^{\infty}(-1)^n e_n(a_1,a_2,\dots,a_k)\,b_1^nz^n\right)\left(\sum\limits_{n=0}^{\infty}(-1)^n e_n(a_1,a_2,\dots,a_k)\,b_2^nz^n\right)} \qquad (2.1) \\
&\quad - \frac{b_1^{4-l}b_2^{4-l}z^{5-l}\sum\limits_{n=0}^{+\infty}(-1)^{n-l+5}\,e_{n-l+5}(a_1,a_2,\dots,a_k)\,h_n(b_1,b_2)\,z^n}{\left(\sum\limits_{n=0}^{\infty}(-1)^n e_n(a_1,a_2,\dots,a_k)\,b_1^nz^n\right)\left(\sum\limits_{n=0}^{\infty}(-1)^n e_n(a_1,a_2,\dots,a_k)\,b_2^nz^n\right)},
\end{aligned}$$

for all $n,\ k \in \mathbb{N}_0$ *and* $l \in \{0,1,2,3,4\}$.

Proof. By applying the operator $\delta_{b_1 b_2}^{4-l}$ to the series

$$f(b_1 z) = \sum_{n=0}^{\infty} h_n(a_1, a_2, \ldots, a_k) b_1^n z^n,$$

we have

$$\begin{aligned}
\delta_{b_1 b_2}^{4-l} f(b_1 z) &= \frac{b_1^{4-l} \sum\limits_{n=0}^{\infty} h_n(a_1, a_2, \ldots, a_k) b_1^n z^n - b_2^{4-l} \sum\limits_{n=0}^{\infty} h_n(a_1, a_2, \ldots, a_k) b_2^n z^n}{b_1 - b_2} \\
&= \sum_{n=0}^{\infty} h_n(a_1, a_2, \ldots, a_k) \left(\frac{b_1^{n-l+4} - b_2^{n-l+4}}{b_1 - b_2} \right) z^n \\
&= \sum_{n=0}^{\infty} h_n(a_1, a_2, \ldots, a_k) h_{n-l+3}(b_1, b_2) z^n.
\end{aligned}$$

On the other hand, by applying the operator $\delta_{b_1 b_2}^{4-l}$ to the series

$$f(b_1 z) = \frac{1}{\sum\limits_{n=0}^{\infty} (-1)^n e_n(a_1, a_2, \ldots, a_k) b_1^n z^n},$$

we obtain

$$\begin{aligned}
&\delta_{b_1 b_2}^{4-l} f(b_1 z) = \delta_{b_1 b_2}^{4-l} \left(\frac{1}{\sum\limits_{n=0}^{\infty} (-1)^n e_n(a_1, a_2, \ldots, a_k) b_1^n z^n} \right) \\
&= \frac{\frac{b_1^{4-l}}{\sum\limits_{n=0}^{\infty} (-1)^n e_n(a_1, a_2, \ldots, a_k) b_1^n z^n} - \frac{b_2^{4-l}}{\sum\limits_{n=0}^{\infty} (-1)^n e_n(a_1, a_2, \ldots, a_k) b_2^n z^n}}{b_1 - b_2} \\
&= \frac{b_1^{4-l} \sum\limits_{n=0}^{\infty} (-1)^n e_n(a_1, a_2, \ldots, a_k) b_2^n z^n - b_2^{4-l} \sum\limits_{n=0}^{\infty} (-1)^n e_n(a_1, a_2, \ldots, a_k) b_1^n z^n}{(b_1 - b_2) \left(\sum\limits_{n=0}^{\infty} (-1)^n e_n(a_1, a_2, \ldots, a_k) b_1^n z^n \right) \left(\sum\limits_{n=0}^{\infty} (-1)^n e_n(a_1, a_2, \ldots, a_k) b_2^n z^n \right)} \\
&= \frac{\sum\limits_{n=0}^{\infty} (-1)^n e_n(a_1, a_2, \ldots, a_k) b_1^n b_2^n \frac{b_1^{4-l-n} - b_2^{4-l-n}}{b_1 - b_2} z^n}{\left(\sum\limits_{n=0}^{\infty} (-1)^n e_n(a_1, a_2, \ldots, a_k) b_1^n z^n \right) \left(\sum\limits_{n=0}^{\infty} (-1)^n e_n(a_1, a_2, \ldots, a_k) b_2^n z^n \right)}
\end{aligned}$$

$$= \frac{\sum_{n=0}^{+\infty}(-1)^n e_n(a_1,a_2,\ldots,a_k)\, b_1^n b_2^n h_{3-n-l}(b_1,b_2)\, z^n}{\left(\sum_{n=0}^{+\infty}(-1)^n e_n(a_1,a_2,\ldots,a_k)\, b_1^n z^n\right)\left(\sum_{n=0}^{\infty}(-1)^n e_n(a_1,a_2,\ldots,a_k)\, b_2^n z^n\right)}$$

$$= \frac{\sum_{n=0}^{3-l}(-1)^n e_n(a_1,a_2,\ldots,a_k)\, b_1^n b_2^n h_{3-n-l}(b_1,b_2)\, z^n}{\left(\sum_{n=0}^{\infty}(-1)^n e_n(a_1,a_2,\ldots,a_k)\, b_1^n z^n\right)\left(\sum_{n=0}^{\infty}(-1)^n e_n(a_1,a_2,\ldots,a_k)\, b_2^n z^n\right)}$$

$$+ \frac{\sum_{n=5-l}^{\infty}(-1)^n e_n(a_1,a_2,\ldots,a_k)\, b_1^n b_2^n h_{3-n-l}(b_1,b_2)\, z^n}{\left(\sum_{n=0}^{\infty}(-1)^n e_n(a_1,a_2,\ldots,a_k)\, b_1^n z^n\right)\left(\sum_{n=0}^{\infty}(-1)^n e_n(a_1,a_2,\ldots,a_k)\, b_2^n z^n\right)}$$

$$= \frac{\sum_{n=0}^{3-l}(-1)^n e_n(a_1,a_2,\ldots,a_k)\, b_1^n b_2^n h_{3-n-l}(b_1,b_2)\, z^n}{\left(\sum_{n=0}^{\infty}(-1)^n e_n(a_1,a_2,\ldots,a_k)\, b_1^n z^n\right)\left(\sum_{n=0}^{\infty}(-1)^n e_n(a_1,a_2,\ldots,a_k)\, b_2^n z^n\right)}$$

$$- \frac{\sum_{n=5-l}^{\infty}(-1)^n e_n(a_1,a_2,\ldots,a_k)\, b_1^{4-l} b_2^{4-l}\left(\frac{b_1^{n+l-4}-b_2^{n+l-4}}{b_1-b_2}\right) z^n}{\left(\sum_{n=0}^{\infty}(-1)^n e_n(a_1,a_2,\ldots,a_n)\, b_1^n z^n\right)\left(\sum_{n=0}^{\infty}(-1)^n e_n(a_1,a_2,\ldots,a_n)\, b_2^n z^n\right)},$$

accordingly,

$$\delta_{b_1 b_2}^{4-l} f(b_1 z)$$

$$= \frac{h_{3-l}(b_1,b_2) - e_1(a_1,a_2,\ldots,a_k)\, b_1 b_2 h_{2-l}(b_1,b_2)\, z}{\left(\sum_{n=0}^{\infty}(-1)^n e_n(a_1,a_2,\ldots,a_k)\, b_1^n z^n\right)\left(\sum_{n=0}^{\infty}(-1)^n e_n(a_1,a_2,\ldots,a_k)\, b_2^n z^n\right)}$$

$$+ \frac{e_2(a_1,a_2,\ldots,a_k)\, b_1^2 b_2^2 h_{1-l}(b_1,b_2) z^2}{\left(\sum_{n=0}^{\infty}(-1)^n e_n(a_1,a_2,\ldots,a_k)\, b_1^n z^n\right)\left(\sum_{n=0}^{\infty}(-1)^n e_n(a_1,a_2,\ldots,a_k)\, b_2^n z^n\right)}$$

$$- \frac{-e_3(a_1,a_2,\ldots,a_k)\, b_1^3 b_2^3 h_{-l}(b_1,b_2) z^3}{\left(\sum_{n=0}^{\infty}(-1)^n e_n(a_1,a_2,\ldots,a_k)\, b_1^n z^n\right)\left(\sum_{n=0}^{\infty}(-1)^n e_n(a_1,a_2,\ldots,a_k)\, b_2^n z^n\right)}$$

$$- \frac{b_1^{4-l} b_2^{4-l} \sum_{n=5-l}^{\infty}(-1)^n e_n(a_1,a_2,\ldots,a_k)\, h_{n+l-5}(b_1,b_2)\, z^n}{\left(\sum_{n=0}^{\infty}(-1)^n e_n(a_1,a_2,\ldots,a_k)\, b_1^n z^n\right)\left(\sum_{n=0}^{\infty}(-1)^n e_n(a_1,a_2,\ldots,a_k)\, b_2^n z^n\right)}$$

$$= \frac{h_{3-l}(b_1,b_2) - e_1(a_1,a_2,\dots,a_k)\, b_1 b_2 h_{2-l}(b_1,b_2)\, z}{\left(\sum_{n=0}^{\infty}(-1)^n e_n(a_1,a_2,\dots,a_k)\, b_1^n z^n\right)\left(\sum_{n=0}^{\infty}(-1)^n e_n(a_1,a_2,\dots,a_k)\, b_2^n z^n\right)}$$
$$+ \frac{e_2(a_1,a_2,\dots,a_k)\, b_1^2 b_2^2 h_{1-l}(b_1,b_2) z^2}{\left(\sum_{n=0}^{\infty}(-1)^n e_n(a_1,a_2,\dots,a_k)\, b_1^n z^n\right)\left(\sum_{n=0}^{\infty}(-1)^n e_n(a_1,a_2,\dots,a_k)\, b_2^n z^n\right)}$$
$$- \frac{e_3(a_1,a_2,\dots,a_k)\, b_1^3 b_2^3 h_{-l}(b_1,b_2) z^3}{\left(\sum_{n=0}^{\infty}(-1)^n e_n(a_1,a_2,\dots,a_k)\, b_1^n z^n\right)\left(\sum_{n=0}^{\infty}(-1)^n e_n(a_1,a_2,\dots,a_k)\, b_2^n z^n\right)}$$
$$- \frac{b_1^{4-l} b_2^{4-l} z^{5-l} \sum_{n=0}^{\infty}(-1)^{n-l+5} e_{n-l+5}(a_1,a_2,\dots,a_k)\, h_n(b_1,b_2)\, z^n}{\left(\sum_{n=0}^{\infty}(-1)^n e_n(a_1,a_2,\dots,a_k)\, b_1^n z^n\right)\left(\sum_{n=0}^{\infty}(-1)^n e_n(a_1,a_2,\dots,a_k)\, b_2^n z^n\right)}.$$

Therefore,

$$\sum_{n=0}^{\infty} h_n(a_1,a_2,\dots,a_k)\, h_{n-l+3}(b_1,b_2)\, z^n$$
$$= \frac{h_{3-l}(b_1,b_2) - e_1(a_1,a_2,\dots,a_k)\, b_1 b_2 h_{2-l}(b_1,b_2)\, z}{\left(\sum_{n=0}^{\infty}(-1)^n e_n(a_1,a_2,\dots,a_k)\, b_1^n z^n\right)\left(\sum_{n=0}^{\infty}(-1)^n e_n(a_1,a_2,\dots,a_k)\, b_2^n z^n\right)}$$
$$+ \frac{e_2(a_1,a_2,\dots,a_k)\, b_1^2 b_2^2 h_{1-l}(b_1,b_2) z^2}{\left(\sum_{n=0}^{\infty}(-1)^n e_n(a_1,a_2,\dots,a_k)\, b_1^n z^n\right)\left(\sum_{n=0}^{\infty}(-1)^n e_n(a_1,a_2,\dots,a_k)\, b_2^n z^n\right)}$$
$$- \frac{e_3(a_1,a_2,\dots,a_k)\, b_1^3 b_2^3 h_{-l}(b_1,b_2) z^3}{\left(\sum_{n=0}^{\infty}(-1)^n e_n(a_1,a_2,\dots,a_k)\, b_1^n z^n\right)\left(\sum_{n=0}^{\infty}(-1)^n e_n(a_1,a_2,\dots,a_k)\, b_2^n z^n\right)}$$
$$- \frac{b_1^{4-l} b_2^{4-l} z^{5-l} \sum_{n=0}^{\infty}(-1)^{n-l+5} e_{n-l+5}(a_1,a_2,\dots,a_k)\, h_n(b_1,b_2)\, z^n}{\left(\sum_{n=0}^{\infty}(-1)^n e_n(a_1,a_2,\dots,a_k)\, b_1^n z^n\right)\left(\sum_{n=0}^{\infty}(-1)^n e_n(a_1,a_2,\dots,a_k)\, b_2^n z^n\right)}.$$

Thus, this completes the proof. □

For $A = \{a_1, a_2\}$, $B = \{b_1, b_2\}$, $l = 3$ and $l = 4$ in Theorem 1, we deduce the following lemmas.

Lemma 1. *Given two alphabets $A = \{a_1, a_2\}$ and $B = \{b_1, b_2\}$, then we have*

$$\sum_{n=0}^{\infty} h_n(a_1,a_2) h_n(b_1,b_2) z^n$$

$$= \frac{1 - a_1 a_2 b_1 b_2 z^2}{\left(\sum\limits_{n=0}^{\infty} (-1)^n e_n(a_1, a_2) b_1^n z^n\right)\left(\sum\limits_{n=0}^{\infty} (-1)^n e_n(a_1, a_2) b_2^n z^n\right)}. \quad (2.2)$$

Based on relationship (2.2), we get

$$\sum_{n=0}^{\infty} h_{n-1}(a_1, a_2) h_{n-1}(b_1, b_2) z^n$$
$$= \frac{z - a_1 a_2 b_1 b_2 z^3}{\left(\sum\limits_{n=0}^{\infty} (-1)^n e_n(a_1, a_2) b_1^n z^n\right)\left(\sum\limits_{n=0}^{\infty} (-1)^n e_n(a_1, a_2) b_2^n z^n\right)}. \quad (2.3)$$

Lemma 2. *Given two alphabets $B = \{b_1, b_2\}$ and $A = \{a_1, a_2\}$, then we have*

$$\sum_{n=0}^{\infty} h_n(a_1, a_2) h_{n-1}(b_1, b_2) z^n$$
$$= \frac{(a_1 + a_2) z - a_1 a_2 (b_1 + b_2) z^2}{\left(\sum\limits_{n=0}^{\infty} (-1)^n e_n(a_1, a_2) b_1^n z^n\right)\left(\sum\limits_{n=0}^{\infty} (-1)^n e_n(a_1, a_2) b_2^n z^n\right)}. \quad (2.4)$$

From (2.4), we get

$$\sum_{n=0}^{\infty} h_{n-1}(a_1, a_2) h_n(b_1, b_2) z^n$$
$$= \frac{(b_1 + b_2) z - b_1 b_2 (a_1 + a_2) z^2}{\left(\sum\limits_{n=0}^{\infty} (-1)^n e_n(a_1, a_2) b_1^n z^n\right)\left(\sum\limits_{n=0}^{\infty} (-1)^n e_n(a_1, a_2) b_2^n z^n\right)}. \quad (2.5)$$

3. Generating functions for products of (p,q)-numbers with Fibonacci polynomials and k-Fibonacci numbers at positive and negative indices

In this section, we derive new generating functions for products of (p,q)-Fibonacci numbers, (p,q)-Lucas numbers, (p,q)-Pell numbers, (p,q)-Pell–Lucas numbers, (p,q)-Jacobsthal numbers, and (p,q)-Jacobsthal–Lucas numbers with k-Fibonacci numbers and Fibonacci polynomials.

For the case where $A = \{a_1, -a_2\}$ and $B = \{b_1, -b_2\}$, substituting a_2 with $(-a_2)$ and b_2 with $(-b_2)$ into Eqs. (2.2), (2.3), (2.4), and (2.5) yields

$$\sum_{n=0}^{\infty} h_n(a_1, [-a_2]) h_n(b_1, [-b_2]) z^n = \frac{1 - a_1 a_2 b_1 b_2 z^2}{(1 - a_1 b_1 z)(1 + a_2 b_1 z)(1 + a_1 b_2 z)(1 - a_2 b_2 z)}.$$

$$\sum_{n=0}^{\infty} h_{n-1}(a_1,[-a_2])h_{n-1}(b_1,[-b_2])z^n = \frac{z-a_1a_2b_1b_2z^3}{(1-a_1b_1z)(1+a_2b_1z)(1+a_1b_2z)(1-a_2b_2z)}. \quad (3.1)$$

$$\sum_{n=0}^{\infty} h_n(a_1,[-a_2])h_{n-1}(b_1,[-b_2])z^n = \frac{(a_1-a_2)z+a_1a_2(b_1-b_2)z^2}{(1-a_1b_1z)(1+a_2b_1z)(1+a_1b_2z)(1-a_2b_2z)}. \quad (3.2)$$

$$\sum_{n=0}^{\infty} h_{n-1}(a_1,[-a_2])h_n(b_1,[-b_2])z^n = \frac{(b_1-b_2)z+b_1b_2(a_1-a_2)z^2}{(1-a_1b_1z)(1+a_2b_1z)(1+a_1b_2z)(1-a_2b_2z)}.$$

3.1. *Ordinary generating functions of the products of (p,q)-numbers with k-Fibonacci numbers*

This case consists of three related parts.

First: By making the substitutions

$$\begin{cases} a_1-a_2=p \\ a_1a_2=q \end{cases} \text{and} \begin{cases} b_1-b_2=k \\ b_1b_2=1 \end{cases},$$

in Eqs. (3.1) and (3.2), we obtain

$$\sum_{n=0}^{\infty} h_{n-1}(a_1,[-a_2])h_{n-1}(b_1,[-b_2])z^n = \frac{z-qz^3}{1-pkz-(qk^2+2q+p^2)z^2-pqkz^3+q^2z^4}, \quad (3.3)$$

$$\sum_{n=0}^{\infty} h_n(a_1,[-a_2])h_{n-1}(b_1,[-b_2])z^n = \frac{pz+qkz^2}{1-pkz-(qk^2+2q+p^2)z^2-pqkz^3+q^2z^4}, \quad (3.4)$$

respectively, and we have the following results.

Proposition 3. *For $n \in \mathbb{N}$, the new generating function of the product of (p,q)-Fibonacci numbers with k-Fibonacci numbers is given by*

$$\sum_{n=0}^{\infty} F_{p,q,n}F_{k,n}z^n = \frac{z-qz^3}{1-pkz-(qk^2+2q+p^2)z^2-pqkz^3+q^2z^4}, \quad (3.5)$$

with $F_{p,q,n}F_{k,n} = h_{n-1}(a_1,[-a_2])h_{n-1}(b_1,[-b_2])$.

Theorem 2. *For $n \in \mathbb{N}$, the new generating function of the product of (p,q)-Lucas numbers with k-Fibonacci numbers is given by*

$$\sum_{n=0}^{\infty} L_{p,q,n} F_{k,n} z^n = \frac{pz + 2qkz^2 + pqz^3}{1 - pkz - (qk^2 + 2q + p^2)z^2 - pqkz^3 + q^2z^4}. \tag{3.6}$$

Proof. By [21], we have $L_{p,q,n} = 2h_n(a_1,[-a_2]) - ph_{n-1}(a_1,[-a_2])$. Then, we can see that:

$$\begin{aligned}\sum_{n=0}^{\infty} L_{p,q,n} F_{k,n} z^n &= \sum_{n=0}^{\infty} \left(2h_n(a_1,[-a_2]) - ph_{n-1}(a_1,[-a_2])\right) h_{n-1}(b_1,[-b_2]) z^n \\ &= 2\sum_{n=0}^{\infty} h_n(a_1,[-a_2]) h_{n-1}(b_1,[-b_2]) z^n \\ &\quad - p\sum_{n=0}^{\infty} h_{n-1}(a_1,[-a_2]) h_{n-1}(b_1,[-b_2]) z^n,\end{aligned}$$

by using the relationships (3.3) and (3.4), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} L_{p,q,n} F_{k,n} z^n &= \frac{2\left(pz + qkz^2\right)}{1 - pkz - (qk^2 + 2q + p^2)z^2 - pqkz^3 + q^2z^4} \\ &\quad - \frac{p\left(z - qz^3\right)}{1 - pkz - (qk^2 + 2q + p^2)z^2 - pqkz^3 + q^2z^4} \\ &= \frac{pz + 2qkz^2 + pqz^3}{1 - pkz - (qk^2 + 2q + p^2)z^2 - pqkz^3 + q^2z^4}.\end{aligned}$$

This completes the proof. □

Proposition 4. *By using the change of variable $z = -z$ in Eqs. (3.5) and (3.6) and according to relation (1.1), we give the following new generating functions*

$$\sum_{n=0}^{\infty} F_{p,q,n} F_{k,-n} z^n = \frac{z - qz^3}{1 + pkz - (qk^2 + p^2 + 2q)z^2 + pqkz^3 + q^2z^4}. \tag{3.7}$$

$$\sum_{n=0}^{\infty} L_{p,q,n} F_{k,-n} z^n = \frac{pz - 2qkz^2 + pqz^3}{1 + pkz - (qk^2 + p^2 + 2q)z^2 + pqkz^3 + q^2z^4}. \tag{3.8}$$

By putting $k = 1$ in relationships (3.5), (3.6), (3.7) and (3.8), we obtain the following new generating functions. The calculation results are indicated in Tab.1.

Second: By making the substitutions

$$\begin{cases} a_1 - a_2 = p \\ a_1 a_2 = 2q \end{cases} \text{and} \begin{cases} b_1 - b_2 = k \\ b_1 b_2 = 1 \end{cases},$$

in Eqs. (3.1) and (3.2), we obtain

TABLE 1. New generating functions for the products of some sequences.

Coefficient of z^n	Generating function
$F_{p,q,n}F_n$	$\frac{z-qz^3}{1-pz-(3q+p^2)z^2-pqz^3+q^2z^4}$
$L_{p,q,n}F_n$	$\frac{pz+2qz^2+pqz^3}{1-pz-(3q+p^2)z^2-pqz^3+q^2z^4}$
$F_{p,q,n}F_{-n}$	$\frac{z-qz^3}{1+pz-(3q+p^2)z^2+pqz^3+q^2z^4}$
$L_{p,q,n}F_{-n}$	$\frac{pz-2qz^2+pqz^3}{1+pz-(3q+p^2)z^2+pqz^3+q^2z^4}$

$$\sum_{n=0}^{\infty} h_{n-1}(a_1,[-a_2])\,h_{n-1}(b_1,[-b_2])\,z^n = \frac{z-2qz^3}{1-pkz-(2qk^2+p^2+4q)z^2-2pqkz^3+4q^2z^4},$$

$$\sum_{n=0}^{\infty} h_{n}(a_1,[-a_2])\,h_{n-1}(b_1,[-b_2])\,z^n = \frac{pz+2qkz^2}{1-pkz-(2qk^2+p^2+4q)z^2-2pqkz^3+4q^2z^4},$$

respectively, and we have the following results.

Proposition 5. *For $n \in \mathbb{N}$, the new generating function of the product of (p,q)-Jacobsthal numbers with k-Fibonacci numbers is given by*

$$\sum_{n=0}^{\infty} J_{p,q,n}F_{k,n}z^n = \frac{z-2qz^3}{1-pkz-(2qk^2+p^2+4q)z^2-2pqkz^3+4q^2z^4}, \tag{3.9}$$

with $J_{p,q,n}F_{k,n} = h_{n-1}(a_1,[-a_2])\,h_{n-1}(b_1,[-b_2])$.

Theorem 3. *For $n \in \mathbb{N}$, the new generating function of the product of (p,q)-Jacobsthal Lucas numbers with k-Fibonacci numbers is given by*

$$\sum_{n=0}^{\infty} j_{p,q,n}F_{k,n}z^n = \frac{pz+4qkz^2+2pqz^3}{1-pkz-(2qk^2+p^2+4q)z^2-2pqkz^3+4q^2z^4}. \tag{3.10}$$

Proof. We know that

$$j_{p,q,n} = 2h_n(a_1,[-a_2]) - ph_{n-1}(a_1,[-a_2]),\ \text{(see [21])}.$$

So

$$\begin{aligned}\sum_{n=0}^{\infty} j_{p,q,n}F_{k,n}z^n &= \sum_{n=0}^{\infty}\left(2h_n(a_1,[-a_2]) - ph_{n-1}(a_1,[-a_2])\right)h_{n-1}(b_1,[-b_2])z^n \\ &= 2\sum_{n=0}^{\infty} h_n(a_1,[-a_2])\,h_{n-1}(b_1,[-b_2])z^n\end{aligned}$$

$$- p\sum_{n=0}^{\infty} h_{n-1}(a_1,[-a_2])\,h_{n-1}(b_1,[-b_2])z^n$$

$$= \frac{2\left(pz+2qkz^2\right)}{1-pkz-(2qk^2+p^2+4q)z^2-2pqkz^3+4q^2z^4}$$

$$- \frac{p\left(z-2qz^3\right)}{1-pkz-(2qk^2+p^2+4q)z^2-2pqkz^3+4q^2z^4}$$

$$= \frac{pz+4qkz^2+2pqz^3}{1-pkz-(2qk^2+p^2+4q)z^2-2pqkz^3+4q^2z^4}.$$

This completes the proof. □

Proposition 6. *By using the change of variable $z=-z$ in Eqs. (3.9) and (3.10) and according to relation (1.1), we derive the following new generating functions*

$$\sum_{n=0}^{\infty} J_{p,q,n}F_{k,-n}z^n = \frac{z-2qz^3}{1+pkz-(2qk^2+p^2+4q)z^2+2pqkz^3+4q^2z^4}. \tag{3.11}$$

$$\sum_{n=0}^{\infty} j_{p,q,n}F_{k,-n}z^n = \frac{pz-4qkz^2+2pqz^3}{1+pkz-(2qk^2+p^2+4q)z^2+2pqkz^3+4q^2z^4}. \tag{3.12}$$

By setting $k=1$ in relationships (3.9), (3.10), (3.11) and (3.12), we obtain the following new generating functions. The calculation results are indicated in Tab. 2.

TABLE 2. New generating functions for the products of some sequences.

Coefficient of z^n	Generating function
$J_{p,q,n}F_n$	$\frac{z-2qz^3}{1-pz-(6q+p^2)z^2-2pqz^3+4q^2z^4}$
$j_{p,q,n}F_n$	$\frac{pz+4qz^2+2pqz^3}{1-pz-(6q+p^2)z^2-2pqz^3+4q^2z^4}$
$J_{p,q,n}F_{-n}$	$\frac{z-2qz^3}{1+pz-(6q+p^2)z^2+2pqz^3+4q^2z^4}$
$j_{p,q,n}F_{-n}$	$\frac{pz-4qz^2+2pqz^3}{1+pz-(6q+p^2)z^2+2pqz^3+4q^2z^4}$

Third: the substitutions of

$$\begin{cases} a_1-a_2=2p \\ a_1a_2=q \end{cases} \text{and} \begin{cases} b_1-b_2=k \\ b_1b_2=1 \end{cases},$$

in Eqs. (3.1) and (3.2) we obtain

$$\sum_{n=0}^{\infty} h_{n-1}(a_1,[-a_2])\,h_{n-1}(b_1,[-b_2])\,z^n$$

$$= \frac{z - qz^3}{1 - 2pkz - (qk^2 + 4p^2 + 2q)z^2 - 2pqkz^3 + q^2z^4}, \quad (3.13)$$

$$\sum_{n=0}^{\infty} h_n(a_1, [-a_2]) h_{n-1}(b_1, [-b_2]) z^n$$
$$= \frac{2pz + qkz^2}{1 - 2pkz - (qk^2 + 4p^2 + 2q)z^2 - 2pqkz^3 + q^2z^4}, \quad (3.14)$$

respectively, and we have the following results.

Proposition 7. *For $n \in \mathbb{N}$, the new generating function of the product of (p,q)-Pell numbers with k-Fibonacci numbers is given by*

$$\sum_{n=0}^{\infty} P_{p,q,n} F_{k,n} z^n = \frac{z - qz^3}{1 - 2pkz - (qk^2 + 4p^2 + 2q)z^2 - 2pqkz^3 + q^2z^4}, \quad (3.15)$$

with $P_{p,q,n} F_{k,n} = h_{n-1}(a_1, [-a_2]) h_{n-1}(b_1, [-b_2])$.

We have the following theorem.

Theorem 4. *For $n \in \mathbb{N}$, the new generating function of the product of (p,q)-Pell Lucas numbers with k-Fibonacci numbers is given by*

$$\sum_{n=0}^{\infty} Q_{p,q,n} F_{k,n} z^n = \frac{2pz + 2qkz^2 + 2pqz^3}{1 - 2pkz - (qk^2 + 4p^2 + 2q)z^2 - 2pqkz^3 + q^2z^4}. \quad (3.16)$$

Proof. By referred to [21], we have

$$Q_{p,q,n} = 2h_n(a_1, [-a_2]) - 2ph_{n-1}(a_1, [-a_2]).$$

We see that

$$\sum_{n=0}^{\infty} Q_{p,q,n} F_{k,n} z^n = \sum_{n=0}^{\infty} \left(2h_n(a_1, [-a_2]) - 2ph_{n-1}(a_1, [-a_2])\right) h_{n-1}(b_1, [-b_2]) z^n$$
$$= 2\sum_{n=0}^{\infty} h_n(a_1, [-a_2]) h_{n-1}(b_1, [-b_2]) z^n$$
$$- 2p\sum_{n=0}^{\infty} h_{n-1}(a_1, [-a_2]) h_{n-1}(b_1, [-b_2]) z^n.$$

Using relationships (3.13) and (3.14), we obtain

$$\sum_{n=0}^{\infty} Q_{p,q,n} F_{k,n} z^n = \frac{2\left(2pz + qkz^2\right)}{1 - 2pkz - (qk^2 + 4p^2 + 2q)z^2 - 2pqkz^3 + q^2z^4}$$
$$- \frac{2p\left(z - qz^3\right)}{1 - 2pkz - (qk^2 + 4p^2 + 2q)z^2 - 2pqkz^3 + q^2z^4}$$

$$= \frac{2pz + 2qkz^2 + 2pqz^3}{1 - 2pkz - (qk^2 + 4p^2 + 2q)z^2 - 2pqkz^3 + q^2z^4}.$$

This completes the proof. □

Proposition 8. *By using the change of variable $z = -z$ in Eqs. (3.15) and (3.16) and according to relation (1.1), we give the following new generating functions:*

$$\sum_{n=0}^{\infty} P_{p,q,n} F_{k,-n} z^n = \frac{z - qz^3}{1 + 2pkz - (qk^2 + 4p^2 + 2q)z^2 + 2pqkz^3 + q^2z^4}. \tag{3.17}$$

$$\sum_{n=0}^{\infty} Q_{p,q,n} F_{k,-n} z^n = \frac{2pz - 2qkz^2 + 2pqz^3}{1 + 2pkz - (qk^2 + 4p^2 + 2q)z^2 + 2pqkz^3 + q^2z^4}. \tag{3.18}$$

By taking $k = 1$ in relationships (3.15), (3.16), (3.17) and (3.18), we obtain the following new generating functions. The calculation results are indicated in Tab.3

TABLE 3. New generating functions for the products of some sequences.

Coefficient of z^n	Generating function
$P_{p,q,n}F_n$	$\frac{z-qz^3}{1-2pz-(3q+4p^2)z^2-2pqz^3+q^2z^4}$
$Q_{p,q,n}F_n$	$\frac{2pz+2qz^2+2pqz^3}{1-2pz-(3q+4p^2)z^2-2pqz^3+q^2z^4}$
$P_{p,q,n}F_{-n}$	$\frac{z-qz^3}{1+2pz-(3q+4p^2)z^2+2pqz^3+q^2z^4}$
$Q_{p,q,n}F_{-n}$	$\frac{2pz-2qz^2+2pqz^3}{1+2pz-(3q+4p^2)z^2+2pqz^3+q^2z^4}$

3.2. *Ordinary generating functions of the products of (p,q)-numbers with Fibonacci polynomials*

This part consists of three cases.

Case 1: The substitutions of $\begin{cases} a_1 - a_2 = p \\ a_1 a_2 = q \end{cases}$ and $\begin{cases} b_1 - b_2 = x \\ b_1 b_2 = 1 \end{cases}$ in Eqs. (2.1) and (2.4), gives

$$\sum_{n=0}^{\infty} h_n(a_1, [-a_2]) h_n(b_1, [-b_2]) z^n = \frac{1 - qz^2}{1 - pxz - (qx^2 + 2q + p^2)z^2 - pqxz^3 + q^2z^4},$$

$$\sum_{n=0}^{\infty} h_{n-1}(a_1, [-a_2]) h_n(b_1, [-b_2]) z^n = \frac{xz + pz^2}{1 - pxz - (qx^2 + 2q + p^2)z^2 - pqxz^3 + q^2z^4},$$

respectively, and we deduce the following results.

Proposition 9. *For $n \in \mathbb{N}$, the new generating function of the product of (p,q)-Fibonacci numbers with Fibonacci polynomials is given by:*

$$\sum_{n=0}^{\infty} F_{p,q,n}F_n(x)z^n = \frac{xz+pz^2}{1-pxz-(qx^2+2q+p^2)z^2-pqxz^3+q^2z^4},$$

with $F_{p,q,n}F_n(x) = h_{n-1}(a_1,[-a_2])\,h_n(b_1,[-b_2])$.

Theorem 5. *Let n be a natural number. Then we have the following new generating function for the product of (p,q)-Lucas numbers with Fibonacci polynomials*

$$\sum_{n=0}^{\infty} L_{p,q,n}F_n(x)z^n = \frac{2-pxz-\left(2q+p^2\right)z^2}{1-pxz-(qx^2+2q+p^2)z^2-pqxz^3+q^2z^4}.$$

Proof. By [21], we have $L_{p,q,n} = 2h_n(a_1,[-a_2]) - ph_{n-1}(a_1,[-a_2])$. Then, we can see that

$$\begin{aligned}
\sum_{n=0}^{\infty} L_{p,q,n}F_n(x)z^n &= \sum_{n=0}^{\infty}\left(2h_n(a_1,[-a_2]) - ph_{n-1}(a_1,[-a_2])\right)h_n(b_1,[-b_2])z^n \\
&= 2\sum_{n=0}^{\infty} h_n(a_1,[-a_2])h_n(b_1,[-b_2])z^n \\
&\quad - p\sum_{n=0}^{\infty} h_{n-1}(a_1,[-a_2])h_n(b_1,[-b_2])z^n \\
&= \frac{2\left(1-qz^2\right)}{1-pxz-(qx^2+2q+p^2)z^2-pqxz^3+q^2z^4} \\
&\quad - \frac{p\left(xz+pz^2\right)}{1-pxz-(qx^2+2q+p^2)z^2-pqxz^3+q^2z^4},
\end{aligned}$$

after simple calculations, we obtain

$$\sum_{n=0}^{\infty} L_{p,q,n}F_n(x)z^n = \frac{2-pxz-\left(2q+p^2\right)z^2}{1-pxz-(qx^2+2q+p^2)z^2-pqxz^3+q^2z^4}.$$

So, the desired result is achieved. □

Case 2: The substitutions of $\begin{cases} a_1-a_2=p \\ a_1a_2=2q \end{cases}$ and $\begin{cases} b_1-b_2=x \\ b_1b_2=1 \end{cases}$ in Eqs. (2.1) and (2.4), gives:

$$\begin{aligned}
&\sum_{n=0}^{\infty} h_n(a_1,[-a_2])\,h_n(b_1,[-b_2])\,z^n \\
&\qquad = \frac{1-2qz^2}{1-pxz-(2qx^2+4q+p^2)z^2-2pqxz^3+4q^2z^4},
\end{aligned}$$

$$\sum_{n=0}^{\infty} h_{n-1}(a_1,[-a_2])\,h_n(b_1,[-b_2])\,z^n$$
$$= \frac{xz+pz^2}{1-pxz-(2qx^2+4q+p^2)z^2-2pqxz^3+4q^2z^4},$$

respectively, and we deduce the following results.

Proposition 10. *For $n\in\mathbb{N}$, the new generating function of the product of (p,q)-Jacobsthal numbers with Fibonacci polynomials is given by*

$$\sum_{n=0}^{\infty} J_{p,q,n}F_n(x)\,z^n = \frac{xz+pz^2}{1-pxz-(2qx^2+4q+p^2)z^2-2pqxz^3+4q^2z^4},$$

with $J_{p,q,n}F_n(x) = h_{n-1}(a_1,[-a_2])\,h_n(b_1,[-b_2])$.

Theorem 6. *Let n be a natural number. Then, we have the following new generating function for the product of (p,q)-Jacobsthal Lucas numbers with Fibonacci polynomials*

$$\sum_{n=0}^{\infty} j_{p,q,n}F_n(x)\,z^n = \frac{2-pxz-\left(4q+p^2\right)z^2}{1-pxz-(2qx^2+4q+p^2)z^2-2pqxz^3+4q^2z^4}.$$

Proof. By [21], we have $j_{p,q,n} = 2h_n(a_1,[-a_2]) - ph_{n-1}(a_1,[-a_2])$. Then, we can see that

$$\begin{aligned}\sum_{n=0}^{\infty} j_{p,q,n}F_n(x)\,z^n &= \sum_{n=0}^{\infty}\left(2h_n(a_1,[-a_2]) - ph_{n-1}(a_1,[-a_2])\right)h_n(b_1,[-b_2])z^n\\ &= 2\sum_{n=0}^{\infty} h_n(a_1,[-a_2])\,h_n(b_1,[-b_2])z^n\\ &\quad - p\sum_{n=0}^{\infty} h_{n-1}(a_1,[-a_2])\,h_n(b_1,[-b_2])z^n\\ &= \frac{2\left(1-2qz^2\right)}{1-pxz-(2qx^2+4q+p^2)z^2-2pqxz^3+4q^2z^4}\\ &\quad - \frac{p\left(xz+pz^2\right)}{1-pxz-(2qx^2+4q+p^2)z^2-2pqxz^3+4q^2z^4},\end{aligned}$$

after simple calculations, we obtain

$$\sum_{n=0}^{\infty} j_{p,q,n}F_n(x)\,z^n = \frac{2-pxz-\left(4q+p^2\right)z^2}{1-pxz-(2qx^2+4q+p^2)z^2-2pqxz^3+4q^2z^4}.$$

As required. □

Case 3: The substitutions of $\begin{cases} a_1-a_2=2p \\ a_1a_2=q \end{cases}$ and $\begin{cases} b_1-b_2=x \\ b_1b_2=1 \end{cases}$ in Eqs. (2.1) and (2.4), gives

$$\sum_{n=0}^{\infty} h_n(a_1,[-a_2])\,h_n(b_1,[-b_2])\,z^n$$
$$= \frac{1-qz^2}{1-2pxz-(qx^2+2q+4p^2)z^2-2pqxz^3+q^2z^4},$$

$$\sum_{n=0}^{\infty} h_{n-1}(a_1,[-a_2])\,h_n(b_1,[-b_2])\,z^n$$
$$= \frac{xz+2pz^2}{1-2pxz-(qx^2+2q+4p^2)z^2-2pqxz^3+q^2z^4},$$

respectively, and we deduce the following results.

Proposition 11. *For $n \in \mathbb{N}$, the new generating function of the product of (p,q)-Pell numbers with Fibonacci polynomials is given by:*

$$\sum_{n=0}^{\infty} P_{p,q,n}F_n(x)\,z^n = \frac{xz+2pz^2}{1-2pxz-(qx^2+2q+4p^2)z^2-2pqxz^3+q^2z^4},$$

with $P_{p,q,n}F_n(x) = h_{n-1}(a_1,[-a_2])\,h_n(b_1,[-b_2])$.

Theorem 7. *For $n \in \mathbb{N}$, the new generating function of the product of (p,q)-Pell Lucas numbers with Fibonacci polynomials is given by*

$$\sum_{n=0}^{\infty} Q_{p,q,n}F_n(x)\,z^n = \frac{2-2pxz-\left(2q+4p^2\right)z^2}{1-2pxz-(qx^2+2q+4p^2)z^2-2pqxz^3+q^2z^4}.$$

Proof. By referred to [21], we have

$$Q_{p,q,n} = 2h_n(a_1,[-a_2]) - 2ph_{n-1}(a_1,[-a_2]).$$

Then, we see that

$$\begin{aligned}
\sum_{n=0}^{\infty} Q_{p,q,n}F_n(x)\,z^n &= \sum_{n=0}^{\infty}\left(2h_n(a_1,[-a_2]) - 2ph_{n-1}(a_1,[-a_2])\right)h_n(b_1,[-b_2])z^n \\
&= 2\sum_{n=0}^{\infty} h_n(a_1,[-a_2])\,h_n(b_1,[-b_2])z^n \\
&\quad - 2p\sum_{n=0}^{\infty} h_{n-1}(a_1,[-a_2])\,h_n(b_1,[-b_2])z^n \\
&= \frac{2\left(1-qz^2\right)}{1-2pxz-(qx^2+2q+4p^2)z^2-2pqxz^3+q^2z^4} \\
&\quad - \frac{2p\left(xz+2pz^2\right)}{1-2pxz-(qx^2+2q+4p^2)z^2-2pqxz^3+q^2z^4} \\
&= \frac{2-2pxz-\left(2q+4p^2\right)z^2}{1-2pxz-(qx^2+2q+4p^2)z^2-2pqxz^3+q^2z^4}.
\end{aligned}$$

This completes the proof. □

4. Chaotic Behavior and Bifurcation Analysis of Generalized Jacobsthal Numbers

In this section, we examine the dynamical behavior of the proposed generating function and investigate the chaotic nature of the recurrent form of the generalized (p,q)-Jacobsthal numbers (1.2)

$$x_n = \frac{x_{n-1}}{1 - px_{n-1} - 2qx_{n-1}^2}. \quad (4.1)$$

The suggested generating function (4.1) is a discrete-time dynamical system that exhibits sensitive dependence on initial conditions and non-periodic behavior. Through numerical simulations and analysis of the map's iterates, we explore the parameter space where chaos emerges and examine key properties such as bifurcations and period-doubling cascades. These two parameters, p and q, make the study important and unusual. When giving some values to this two parameters, the sequence of iterates generating from the function change the behavior and give transition from periodic to chaotic behavior of the parameter.

4.1. *Bifurcation diagram*

Our study identifies regions of stability and observes the emergence of bifurcation points. The numerical bifurcation diagram provides valuable insights into the transition from stable periodic orbits to chaotic behavior, as shown in Figure.1. For $p = 2.7$ and $-2 \leq q \leq -1$, the bifurcation diagram shows that the system undergoes a series of bifurcation.

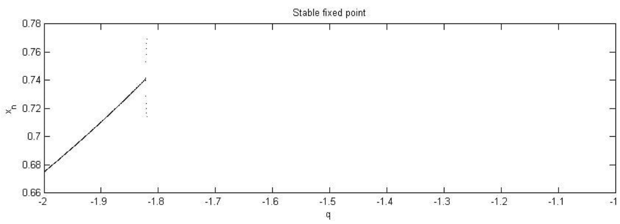

FIGURE 1.

For q between -2 and approximately -1.8, the system exhibits a stable fixed point, which corresponds to a single vertical line in the diagram. At $q = -1.8$, this fixed point undergoes a period-doubling bifurcation, splitting into two stable fixed points see Figure.2. As q increased further, the system exposes additional period-doubling bifurcation, producing periodic orbits of periods $4, 8, 16$ and so on.

Finally, for $-1.3 \leq q \leq -1$, the system enters a region of chaotic behavior where the attractor is a strange attractor with a fractal structure, as depicted in Figure.3.

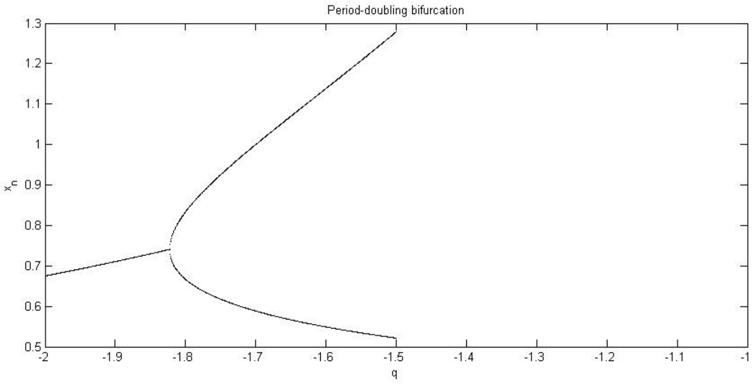

FIGURE 2.

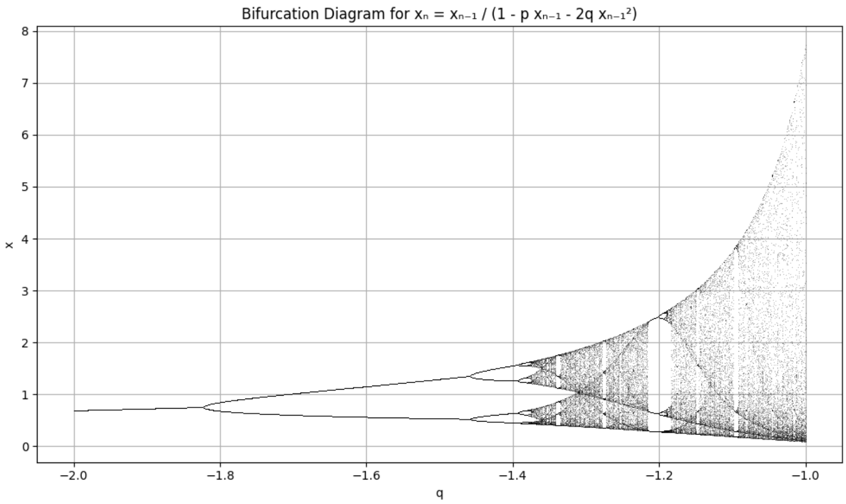

FIGURE 3.

4.2. *Lyapunov Exponent*

To verify the presence of chaos in the proposed generating function, we analyze the Lyapunov exponent. This quantity measures the rate of exponential divergence or convergence of nearby trajectories in a dynamical system. A positive Lyapunov exponent indicates exponential divergence of trajectories, which is a hallmark of chaotic behavior.

The Lyapunov exponent is computed using the formula:

$$\lambda = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \ln \left(\left| \frac{dx_{n+1}}{dx_n} \right| \right), \tag{4.2}$$

which represents the limit, as N approaches infinity, of the average logarithmic derivative of the map over a trajectory of N points.

To evaluate (4.2), we first generate a trajectory of N points using the generating function (4.1). The expression then becomes:

$$\lambda = \lim_{n \longrightarrow \infty} \frac{1}{N} \sum_{n=1}^{N} \ln \left(\left| \frac{1 + 2qx_n^2}{(1 - px_n - 2qx_n^2)^2} \right| \right).$$

To demonstrate that the Lyapunov exponent is positive in certain parameter regions, we compute its values numerically. We used MATLAB to compute both the bifurcation diagram and the Lyapunov exponent; the results are presented in Figure .4.

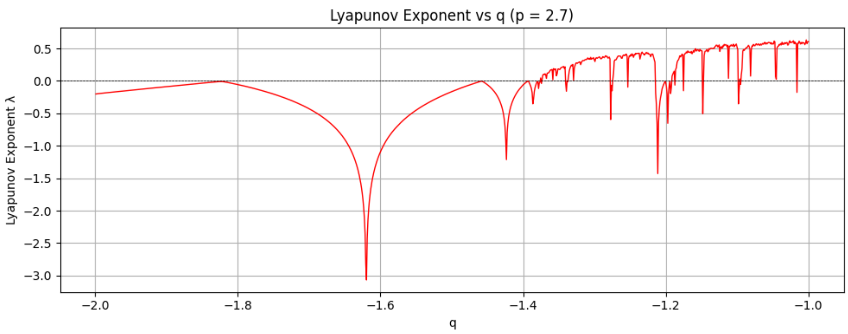

FIGURE 4.

So, the corresponding plot of the Lyapunov exponent provide clear visual evidence of chaotic behavior in the proposed generating map and the chaotic regimes corresponding to the regimes of the bifurcation diagram.

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NONLOCAL INTEGRAL BOUNDARY VALUE PROBLEMS FOR SEQUENTIAL DIFFERENTIAL EQUATION INVOLVING A FRACTIONAL MIXED DERIVATIVES

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Abstract. The study of fractional differential equations occupies an important place in various fields of science. In this paper, we investigate the existence result for a nonlocal integral boundary value problems for a sequential differential equation involving a fractional mixed derivatives. Our method consists to define an extended space on which we can apply the Mönch fixed point theorem via the noncompactness measure. In addition, the compactness of the solution set is studied using the sequential method. Finally, an example is given to illustrate the results obtained.

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1. Introduction

The aim of this paper is to study the existence and the compactness of the solution set for a sequential fractional differential equation with nonlocal integral boundary value conditions. More precisely, we consider the following problem:

$$ {}^{H}\mathcal{D}^{\rho,\sigma,\chi}_{\underline{\xi}^+}\Big(D^{\chi}y(\xi)-\kappa y(\xi)\Big)=\hbar\Big(\xi,y(\xi),{}^{C}\mathcal{D}^{\gamma,\chi}_{\underline{\xi}^+}y(\xi)\Big),\ \ \xi\in(\underline{\xi},\overline{\xi}], \tag{1.1}$$

$$ I^{1-\gamma,\chi}_{\underline{\xi}^+}D^{\chi}y(\underline{\xi}^+)=\sum_{i=1}^{n}\zeta_i D^{\chi}y(\xi_i),\quad y(\underline{\xi})=0, \tag{1.2}$$

where:

- κ is a real number,
- ${}^{H}\mathcal{D}^{\rho,\sigma,\chi}_{\underline{\xi}^+}$ denotes the χ-Hilfer fractional derivative of order ρ and parameter σ such that $0<\rho<1$ and $0\le\sigma\le 1$,
- ${}^{C}\mathcal{D}^{\gamma,\chi}_{\underline{\xi}^+}$ is the χ-Caputo fractional derivative of order $\gamma=\rho+\sigma-\sigma\rho$,

- E is a Banach space and $\hbar\colon (\underline{\xi},\overline{\xi}] \times E^2 \to E$ is a function that satisfies certain conditions (see Section 3),
- $\chi \in \mathcal{C}^1([\underline{\xi},\overline{\xi}],\mathbb{R})$ such that $\chi'(\xi) > 0$ for all $\xi \in [\underline{\xi},\overline{\xi}]$,
- $D^\chi = \frac{1}{\chi'(\xi)}\frac{\mathrm{d}}{\mathrm{d}\xi}$, $\underline{\xi}$, $\overline{\xi} \in \mathbb{R}_+^*$ with $\underline{\xi} < \overline{\xi}$ and $\xi_i \in (\underline{\xi},\overline{\xi}), i = 1,\dots,n$ such that

$$\Gamma(\gamma) \neq \sum_{i=1}^{n} \zeta_i(\chi(\xi_i) - \chi(\underline{\xi}))^{\gamma-1},$$

  where Γ is the gamma function defined by $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\mathrm{d}t \quad (x > 0)$.

The domain of fractional differential equations becomes a very important tool for understanding many physical phenomena. Moreover, their contributions to mathematical analysis help us to obtain certain appreciable results in the economic and engineering fields. For details, we refer the reader to [3, 4, 6, 14, 17, 20, 22, 23].

On the other hand, fractional differential equations with nonlocal conditions are of great importance in several branches of applied analysis. For example, in [13], the author claimed that the nonlocal conditions can be more effective than others to describe some physical situations. Furthermore, there is an extensive literature that focused on the study of the existence, uniqueness and stability for nonlocal fractional differential equations involving Riemann and Hilfer derivatives [7, 8, 26]. In the references [5, 10, 11], the authors studied the topological properties of some fractional differential equations, especially the compactness and the stability.

Recently, in [24], the authors studied the following nonlocal boundary value problems of sequential $\psi-$Hilfer-type fractional differential equations:

$$\left({}^H\mathcal{D}^{\alpha,\beta,\psi} + k\,{}^H\mathcal{D}^{\alpha-1,\beta,\psi}\right)x(t) = f\left(t,x(t)\right), \quad t \in [a,b],$$

$$x(a) = 0 \text{ and } x(b) = \sum_{i=1}^{n}\mu_i\int_a^{\varsigma_i}\psi'(s)x(s)\mathrm{d}s + \sum_{j=1}^{m}\theta_j x(\xi_j),$$

where ${}^H\mathcal{D}^{\alpha,\beta,\psi}$ is the ψ-Hilfer fractional derivative of order α, $1 < \alpha < 2$ and parameter β, $0 \leq \beta \leq 1$, $k \in \mathbb{R}$, $f\colon [a,b] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $a > 0$, μ_i, $\theta_j \in \mathbb{R}$, ς_i, $\xi_j \in (a,b]$ and ψ is a positive increasing function on $(a,b]$, which has a continuous derivative $\psi'(t)$ on (a,b). See also the work discussed by Ragusa [21], on the inclusion of the commutators of fractional integral operators to vanishing Morrey spaces. For other interesting papers which consider fractional differential problems, we mention [1, 12, 15, 16, 19].

The present work is organized as follows: In Section 2, we give some general results and preliminaries. The Section 3 presents two important results concerning the existence of solutions and compactness of (1.1)-(1.2) applying the fixed point theorem. An example to reinforce our work in Section 4.

2. Basic results and background

In this section, we will give some concepts and notations about the functional spaces, fractional calculus, noncompactness measure which are used throughout this paper. we denote by $\mathcal{C}([\underline{\xi},\overline{\xi}])$ (resp. by $L^1([\underline{\xi},\overline{\xi}])$) the space of E-valued continuous functions (resp. the space of E-Bochner's integrable functions) with the following norm

$$\|u\|_\infty = \sup\Big\{\|u(\xi)\|,\ \xi\in[\underline{\xi},\overline{\xi}]\Big\}\ \Big(\text{resp. } \|u\|_{L^1} = \int_{\underline{\xi}}^{\overline{\xi}}\|u(\xi)\|\mathrm{d}\xi\Big).$$

Let $\mathcal{C}_{1-\gamma}([\underline{\xi},\overline{\xi}])$ be the Banach spaces of functions from $(\underline{\xi},\overline{\xi}]$ into E which is defined as:

$$\mathcal{C}_{1-\gamma;\chi}([\underline{\xi},\overline{\xi}]) = \Big\{u\in\mathcal{C}((\underline{\xi},\overline{\xi}]):\ (\chi(.)-\chi(\underline{\xi}))^{1-\gamma}u(.)\in\mathcal{C}([\underline{\xi},\overline{\xi}],E)\Big\}.$$

with his norm $\|u\|_{\gamma,\chi}$, that is given by

$$\|u\|_{\gamma;\chi} = \sup_{\xi\in(\underline{\xi},\overline{\xi}]}(\chi(\xi)-\chi(\underline{\xi}))^{1-\gamma}\|u(\xi)\|.$$

Next, we denote by $\mathcal{C}^1_{1-\gamma;\chi}((\underline{\xi},\overline{\xi}])$ the space of functions (γ,χ) - continuously differentiable defined as follows

$$\mathcal{C}^1_{1-\gamma;\chi}([\underline{\xi},\overline{\xi}]) = \Big\{u\colon(\underline{\xi},\overline{\xi}]\to E:\ u(.)\in\mathcal{C}([\underline{\xi},\overline{\xi}]) \text{ and } D^\chi u(.)\in\mathcal{C}_{1-\gamma;\chi}([\underline{\xi},\overline{\xi}])\Big\}.$$

We note that the space $\mathcal{C}^1_{1-\gamma;\chi}((\underline{\xi},\overline{\xi}])$ with the norm $\|u\|^1_{\gamma;\chi} = \|u\|_\infty + \|D^\chi u\|_{\gamma;\chi}$ is a Banach space.

In the following, for all $\eta > -1$, we put $\Psi_\eta(r,s) = (\chi(r)-\chi(s))^\eta$, for all $s,r\in[\underline{\xi},\overline{\xi}]$ with $r>s$ and $\Psi^*_\eta = (\chi(\overline{\xi})-\chi(\underline{\xi}))^\eta$.

First, we introduce the notions of χ-fractional derivative according to the Riemann-Liouville and Hilfer concept and their properties.

Definition 1 ([17, 25])**.** Let $\ell\in L^1([\underline{\xi},\overline{\xi}])$ and $\chi\in\mathcal{C}^1([\underline{\xi},\overline{\xi}])$ such that $\chi'(\xi)>0$, for all $\xi\in[\underline{\xi},\overline{\xi}]$,

(i) the χ-Riemann- Liouville fractional integral of order $\rho>0$ of the function ℓ is defined by

$$\mathfrak{I}^{\rho;\chi}_{\underline{\xi}^+}\ell(\xi) = \frac{1}{\Gamma(\rho)}\int_{\underline{\xi}}^{\xi}\chi'(s)\Psi_{\rho-1}(\xi,s)\ell(s)\mathrm{d}s,$$

(ii) the χ-Riemann- Liouville fractional derivative of order $\rho>0$ of the function ℓ is defined by

$${}^{RL}\mathcal{D}^{\rho;\chi}_{\underline{\xi}^+}\ell(\xi) = \frac{1}{\Gamma(n-\rho)}\left(\frac{1}{\chi'(\xi)}\frac{\mathrm{d}}{\mathrm{d}\xi}\right)^n\left(\int_{\underline{\xi}}^{\xi}\chi'(s)\Psi_{n-\rho-1}(\xi,s)\ell(s)\mathrm{d}s\right),$$

where $n=[\rho]+1$ such that $[\rho]$ represents the integer part of the real number ρ.

Definition 2 ([17,25]). Let $\chi\in\mathcal{C}^1([\underline{\xi},\overline{\xi}],\mathbb{R})$ be a function satisfying $\chi'(\xi)>0$, for all $\xi\in[\underline{\xi},\overline{\xi}]$. The χ-Hilfer fractional derivative of a function ℓ of order $0<\rho<1$ and type $0\leq\sigma\leq1$ is given by

$$^{H}\mathcal{D}^{\rho,\sigma,\chi}_{\underline{\xi}^+}\ell(\xi)=\mathfrak{I}^{\sigma(1-\rho),\chi}_{\underline{\xi}^+}\left(\frac{1}{\chi'(\xi)}\frac{\mathrm{d}}{\mathrm{d}\xi}\right)\mathfrak{I}^{(1-\sigma)(1-\rho),\chi}_{\underline{\xi}^+}\ell(\xi)=\mathfrak{I}^{1-\gamma,\chi}_{\underline{\xi}^+}{}^{RL}D^{\gamma,\chi}_{\underline{\xi}^+}\ell(\xi),$$

where $\gamma=\rho+\sigma(1-\rho)$.

Lemma 1 ([17]). *Let $\rho,\mu\in\mathbb{R}^*_+$ and $\xi>\underline{\xi}$, then*

(i_1) $\mathfrak{I}^{\rho,\chi}_{\underline{\xi}^+}\Psi_{\mu-1}(\xi,\underline{\xi})=\frac{\Gamma(\mu)}{\Gamma(\rho+\mu)}\Psi_{\rho+\mu-1}(\xi,\underline{\xi})$.

(i_2) ${}^{RL}\mathcal{D}^{\rho,\chi}_{\underline{\xi}^+}\Psi_{\mu-1}(\xi,\underline{\xi})=\frac{\Gamma(\mu)}{\Gamma(\mu-\rho)}\Psi_{\mu-\rho-1}(\xi,\underline{\xi}),0<\rho<1,\ \mu>1$, *in the case when $\rho=\mu$, we get* ${}^{RL}\mathcal{D}^{\rho,\chi}_{\underline{\xi}^+}\Psi_{\mu-1}(\xi,\underline{\xi})=0$.

We consider the following auxiliary spaces

$$\mathcal{C}^{\gamma}_{1-\gamma,\chi}([\underline{\xi},\overline{\xi}])=\left\{u\colon(\underline{\xi},\overline{\xi}]\to E/u\in\mathcal{C}_{1-\gamma,\chi}([\underline{\xi},\overline{\xi}]),\ {}^{RL}\mathcal{D}^{\gamma,\chi}_{\underline{\xi}^+}u\in\mathcal{C}_{1-\gamma,\chi}([\underline{\xi},\overline{\xi}])\right\},$$

$$\mathcal{C}^{1,\gamma}_{1-\gamma,\chi}([\underline{\xi},\overline{\xi}])=\left\{u\colon(\underline{\xi},\overline{\xi}]\to E/u\in\mathcal{C}([\underline{\xi},\overline{\xi}]),\ D^{\chi}u\in\mathcal{C}^{\gamma}_{1-\gamma,\chi}([\underline{\xi},\overline{\xi}])\right\}\ \text{and}$$

$$\mathcal{C}^{1,\rho,\sigma}_{1-\gamma,\chi}([\underline{\xi},\overline{\xi}])=\left\{u\colon(\underline{\xi},\overline{\xi}]\to E/u\in\mathcal{C}([\underline{\xi},\overline{\xi}]),\ D^{\chi}u,\ {}^{H}\mathcal{D}^{\rho,\sigma,\chi}_{\underline{\xi}^+}D^{\chi}u\in\mathcal{C}_{1-\gamma,\chi}([\underline{\xi},\overline{\xi}])\right\},$$

it is clear to see that $\mathcal{C}^{1,\gamma}_{1-\gamma,\chi}([\underline{\xi},\overline{\xi}]\subseteq\mathcal{C}^{1,\rho,\sigma}_{1-\gamma,\chi}([\underline{\xi},\overline{\xi}])$.

Lemma 2 ([18]). *Let $0<\rho<1$, $0\leq\sigma\leq1$ and $\gamma=\rho+\sigma-\rho\sigma$. If $\omega(.)\in\mathcal{C}^{\gamma}_{1-\gamma}([\underline{\xi},\overline{\xi}])$, then*

$$\mathfrak{I}^{\gamma,\chi}_{\underline{\xi}^+}\mathcal{D}^{\gamma,\chi}_{\underline{\xi}^+}\omega=\mathfrak{I}^{\rho,\chi}_{\underline{\xi}^+}\mathcal{D}^{\rho,\sigma,\chi}_{\underline{\xi}^+}\omega$$

and

$$\mathcal{D}^{\gamma,\chi}_{\underline{\xi}^+}\mathfrak{I}^{\rho,\chi}_{\underline{\xi}^+}\omega=\mathcal{D}^{\sigma(1-\rho)}_{\underline{\xi}^+}\omega.$$

Lemma 3 ([18]). *Suppose that $f(.,y(.))\in\mathcal{C}_{1-\gamma,\chi}([\underline{\xi},\overline{\xi}])$ for all $y(.)\in\mathcal{C}^{\gamma}_{1-\gamma,\chi}([\underline{\xi},\overline{\xi}])$. If $y(.)\in\mathcal{C}^{\gamma}_{1-\gamma,\chi}([\underline{\xi},\overline{\xi}])$, then, $y(.)$ is a solution of the fractional differential problem:*

$$\begin{cases}{}^{H}\mathcal{D}^{\rho,\sigma,\chi}_{\underline{\xi}^+}y(\xi)=f(\xi,y(\xi)), & 0<\rho<1,\ 0\leq\sigma\leq1;\\ \mathfrak{I}^{1-\gamma,\chi}_{\underline{\xi}^+}y(\underline{\xi}^+)=\omega_0, & \gamma=\rho+\sigma-\rho\sigma,\end{cases}$$

if and only if y satisfies the following integral equation:

$$y(\xi)=\frac{\omega_0\Psi_{\gamma-1}(\xi,\underline{\xi})}{\Gamma(\gamma)}+\frac{1}{\Gamma(\rho)}\int_{\underline{\xi}}^{\xi}\chi'(s)\Psi_{\rho-1}(\xi,s)f(s,y(s))\mathrm{d}s.$$

Next we give the notion of the noncompactness measure in the sense of Kuratowski and its properties which will be used in the next section, for this purpose, we denote by $\mathrm{Set}_b(E)$ the set of all bounded subsets of Banach space E.

Definition 3 ([9]). Let $D \in \mathrm{Set}_b(E)$. The Kuratowski noncompactness measure ϑ of the subset D is defined as follows:

$$\vartheta(\Omega) = \inf\{e > 0 \colon \Omega \text{ admits a finite cover by sets of diameter} \leq e\}.$$

Lemma 4 ([9]). *Let $A, B \in \mathrm{Set}_b(E)$, we have the following properties*

(i_1) $\vartheta(A) = 0$ *if and only if A is relatively compact,*
(i_2) $\vartheta(A) = \vartheta(\overline{A})$, *where $\overline{A}$ denotes the closure of A,*
(i_3) $\vartheta(A+B) \leq \vartheta(A) + \vartheta(B)$,
(i_4) $A \subset B$ *implies* $\vartheta(A) \leq \vartheta(B)$,
(i_5) $\vartheta(a.A) = |a|.\vartheta(A)$ *for all* $a \in \mathbb{R}$,
(i_6) $\vartheta(\{a\} \cup A) = \vartheta(A)$ *for all* $a \in E$,
(i_7) $\vartheta(A) = \vartheta(Conv(A))$, *where $Conv(A)$ is the smallest convex that contains A.*

Lemma 5 ([9]). *If D is a equicontinuous and bounded subset of $\mathcal{C}([\underline{\xi}, \overline{\xi}])$, then $\vartheta(D(.)) \in \mathcal{C}([\underline{\xi}, \overline{\xi}], \mathbb{R}_+)$*

$$\vartheta_{\mathcal{C}}(D) = \max_{\xi \in [\underline{\xi}, \overline{\xi}]} \vartheta(D(\xi)), \ \vartheta\left(\left\{ \int_{\underline{\xi}}^{\overline{\xi}} w(\xi)\mathrm{d}\xi : w \in D \right\}\right) \leq \int_{\underline{\xi}}^{\overline{\xi}} \vartheta(D(\xi))\mathrm{d}r,$$

where $D(\xi) = \{w(\xi) \colon w \in D\}$ and $\vartheta_{\mathcal{C}}$ is the noncompactness measure on the space $\mathcal{C}([\underline{\xi}, \overline{\xi}])$.

Theorem 1 ([2]). *Let E be a Banach space and D a closed and convex subset of E such that D is bounded and contains 0, and let $N \colon D \longrightarrow D$ be a continuous mapping. If the following implication:*

$$V = N(V) \cup \{0\} \ \text{ or } \ V = \overline{conv} N(V) \implies \gamma(V) = 0,$$

is satisfied for every subset V of D, then N has at least one fixed point.

3. Main results

3.1. *Integral equation*

In the content of Lemma below, we will illustrate the equivalence between the problem at hand (1.1)-(1.2) and the following integral equation

$$\begin{aligned} y(\xi) = & \frac{\sum_{i=1}^{n} \zeta_i \left[\kappa y(\xi_i) + I^{\rho} \hbar\left(\xi_i, y(\xi_i), {}^{C}\mathcal{D}_{\underline{\xi}^+}^{\gamma,\chi} y(\xi_i)\right)\right]}{\Gamma(\gamma+1) - \gamma \sum_{i=1}^{n} \zeta_i \Psi_{\gamma-1}(\xi_i, \underline{\xi})} \Psi_{\gamma}(\xi, \underline{\xi}) + \kappa \int_{\underline{\xi}}^{\xi} \chi'(s) y(s)\mathrm{d}s \\ & + \frac{1}{\Gamma(\rho+1)} \int_{\underline{\xi}}^{\xi} \chi'(s) \Psi_{\rho}(\xi, s) \hbar\left(s, y(s), {}^{C}\mathcal{D}_{\underline{\xi}^+}^{\gamma,\chi} y(s)\right)\mathrm{d}s. \end{aligned} \tag{3.1}$$

Lemma 6. *Let* $\gamma = \rho + \sigma - \rho\sigma$ *with* $0 < \rho < 1$ *and* $0 \leq \sigma \leq 1$, *we assume that the function* $\hbar \colon (\underline{\xi}, \overline{\xi}] \times E^2 \to E$ *satisfies* $\hbar\Big(.,y(.),{}^{C}\mathcal{D}^{\gamma,\chi}_{\underline{\xi}^+}y(.)\Big) \in \mathcal{C}_{1-\gamma,\chi}([\underline{\xi},\overline{\xi}])$, *for all* $y(.) \in \mathcal{C}^{1}_{1-\gamma,\chi}([\underline{\xi},\overline{\xi}])$. *If* $y \in \mathcal{C}^{1,\gamma}_{1-\gamma,\chi}([\underline{\xi},\overline{\xi}])$. *Then, y is a solution of the problem* (1.1)-(1.2) *if and only if y satisfies the integral equation* (3.1).

Proof. Let $y \in \mathcal{C}^{1,\gamma}_{1-\gamma,\chi}([\underline{\xi},\overline{\xi}])$ be a solution of the problem (1.1)-(1.2), since $\hbar\Big(.,y(.),{}^{C}\mathcal{D}^{\gamma,\chi}_{\underline{\xi}^+}y(.)\Big) \in \mathcal{C}_{1-\gamma,\chi}([\underline{\xi},\overline{\xi}])$, from Lemma 3 we have

$$D^{\chi}y(\xi) = \frac{I^{1-\gamma,\chi}_{\underline{\xi}^+}D^{\chi}y(\underline{\xi}^+)}{\Gamma(\gamma)}\Psi_{\gamma-1}(\xi,\underline{\xi}) + \kappa y(\xi) + I^{\rho,\chi}_{\underline{\xi}^+}\hbar\Big(t,y(\xi),{}^{C}\mathcal{D}^{\gamma,\chi}_{\underline{\xi}^+}y(\xi)\Big). \tag{3.2}$$

Next, we substitute ξ by ξ_i into the above equation, we get

$$D^{\chi}y(\xi_i) = \frac{I^{1-\gamma,\chi}_{\underline{\xi}^+}D^{\chi}y(\underline{\xi}^+)}{\Gamma(\gamma)}\Psi_{\gamma-1}(\xi_i,\underline{\xi}) + \kappa y(\xi_i) + I^{\rho,\chi}_{\underline{\xi}^+}\hbar\Big(\xi_i,y(\xi_i),{}^{C}\mathcal{D}^{\gamma,\chi}_{\underline{\xi}^+}y(\xi_i)\Big).$$

By utilizing the second condition (1.2), we obtain

$$\begin{aligned} I^{1-\gamma,\chi}_{\underline{\xi}^+}D^{\chi}y(\underline{\xi}^+) = {} & \frac{I^{1-\gamma,\chi}_{\underline{\xi}^+}D^{\chi}y(\underline{\xi}^+)}{\Gamma(\gamma)}\sum_{i=1}^{n}\zeta_i\Psi_{\gamma-1}(\xi_i,\underline{\xi}) \\ & + \sum_{i=1}^{n}\zeta_i\Big[\kappa y(\xi_i) + I^{\rho,\chi}_{\underline{\xi}^+}\hbar\Big(\xi_i,y(\xi_i),{}^{C}\mathcal{D}^{\gamma,\chi}_{\underline{\xi}^+}y(\xi_i)\Big)\Big], \end{aligned}$$

this implies

$$I^{1-\gamma,\chi}_{\underline{\xi}^+}D^{\chi}y(\underline{\xi}^+) = \frac{\Gamma(\gamma)\sum_{i=1}^{n}\zeta_i\Big[\kappa y(\xi_i) + I^{\rho,\chi}_{\underline{\xi}^+}\hbar\Big(\xi_i,y(\xi_i),{}^{C}\mathcal{D}^{\gamma,\chi}_{\underline{\xi}^+}y(\xi_i)\Big)\Big]}{\Gamma(\gamma) - \sum_{i=1}^{n}\zeta_i\Psi_{\gamma-1}(\xi_i,\underline{\xi})}. \tag{3.3}$$

By substituting (3.3) to (3.2), we deduce that

$$\begin{aligned} D^{\chi}y(\xi) = {} & \frac{\sum_{i=1}^{n}\zeta_i\Big[\kappa y(\xi_i) + I^{\rho,\chi}_{\underline{\xi}^+}\hbar\Big(\xi_i,y(\xi_i),{}^{C}\mathcal{D}^{\gamma,\chi}_{\underline{\xi}^+}y(\xi_i)\Big)\Big]}{\Gamma(\gamma) - \sum_{i=1}^{n}\zeta_i\Psi_{\gamma-1}(\xi_i,\underline{\xi})}\Psi_{\gamma-1}(\xi,\underline{\xi}) + \kappa y(\xi) \\ & + \frac{1}{\Gamma(\rho)}\int_{\underline{\xi}}^{\xi}\chi'(s)\Psi_{\rho-1}(\xi,s)\hbar\Big(s,y(s),{}^{C}\mathcal{D}^{\gamma,\chi}_{\underline{\xi}^+}y(s)\Big)\mathrm{d}s. \end{aligned} \tag{3.4}$$

Next, applying $I^{\chi}_{\underline{\xi}^+}$ to both sides of (3.4), we obtain

$$y(\xi) = \frac{\sum_{i=1}^{n}\zeta_i\Big[\kappa y(\xi_i) + I^{\rho,\chi}_{\underline{\xi}^+}\hbar\Big(\xi_i,y(\xi_i),{}^{C}\mathcal{D}^{\gamma,\chi}_{\underline{\xi}^+}y(\xi_i)\Big)\Big]}{\Gamma(\gamma+1) - \gamma\sum_{i=1}^{n}\zeta_i\Psi_{\gamma-1}(\xi_i,\underline{\xi})}\Psi_{\gamma}(\xi,\underline{\xi}) + \kappa\int_{\underline{\xi}}^{\xi}\chi'(s)y(s)\mathrm{d}s$$

$$+\frac{1}{\Gamma(\rho+1)}\int_{\underline{\xi}}^{\xi}\chi'(s)\Psi_\rho(\xi,s)\hbar\Big(s,y(s),{}^{C}\mathcal{D}_{\underline{\xi}^+}^{\gamma,\chi}y(s)\Big)\mathrm{d}s.$$

Conversely, let $y\in C_{1-\gamma,\chi}^{1,\gamma}([\underline{\xi},\overline{\xi}])$ be a function verifies equation (3.1), it is clear that $y(0)=0$. By applying D^{χ} to both sides of (3.1), we obtain equation (3.4), using Lemma 3, we can easily establish that the function y satisfies the second condition (1.2). □

3.2. *Existence and compactness*

In this subsection, we will prove that the solution set (denoted **SS**) of the problem (1.1)-(1.2) is nonempty and compact, we necessarily assume the following hypotheses

($\mathbf{H_1}$) Suppose that the function $\hbar\colon(\underline{\xi},\overline{\xi}]\times E^2\to E$ verifies $\hbar(.,u(.),v(.))\in C_{1-\gamma,\chi}^{\sigma(1-\rho)}([\underline{\xi},\overline{\xi}])$, for all $u(.),v(.)\in C([\underline{\xi},\overline{\xi}])$, $\hbar(.,0,0)\in C([\underline{\xi},\overline{\xi}],E)$ and there exists $\alpha,\beta\in\mathbb{R}_+$ such that

($\mathbf{H_{1-1}}$) For all $u,v,\overline{u},\overline{v}\in E$:

$$\|\hbar(\xi,u,v)-\hbar(\xi,\overline{u},\overline{v})\|\le\alpha\|u-\overline{u}\|+\beta\|v-\overline{v}\|.$$

($\mathbf{H_{1-2}}$) For each nonempty, bounded set $\Omega\subset C_{1-\gamma,\chi}^{1}([\underline{\xi},\overline{\xi}])$, for all $\xi\in(\underline{\xi},\overline{\xi}]$, we have

$$\vartheta\Big(\hbar(\xi,\Omega(\xi),{}^{C}\mathcal{D}_{\underline{\xi}^+}^{\gamma,\chi}\Omega(\xi)\Big)\le\alpha\vartheta\Big(\Omega(\xi))\Big)+\beta\vartheta\Big({}^{C}\mathcal{D}_{\underline{\xi}^+}^{\gamma,\chi}\Omega(\xi)\Big),$$

where

$$\Omega(\xi)=\Big\{y(\xi),\,y\in C_{1-\gamma,\chi}^{1}([\underline{\xi},\overline{\xi}])\Big\}\text{ and }{}^{C}\mathcal{D}_{\underline{\xi}^+}^{\gamma,\chi}\Omega(\xi)=\Big\{{}^{C}\mathcal{D}_{\underline{\xi}^+}^{\gamma,\chi}y(\xi),\,y\in C_{1-\gamma,\chi}^{1}([\underline{\xi},\overline{\xi}])\Big\}.$$

($\mathbf{H_2}$)

$$\Big(\kappa\Gamma(\rho+2)+(\rho+1)A_0\Big)\Big(|\mathcal{T}|\zeta^{*}n(\Psi_\gamma^{*}+\gamma)+\Psi_{1-\gamma}^{*}\Big)+\Big(A_0+\kappa\Gamma(\rho+2)\Big)\Psi_1^{*}<\frac{\Gamma(\rho+2)}{2},$$

where

$$\mathcal{T}=\frac{1}{\Gamma(\gamma+1)-\gamma\sum_{i=1}^{n}\zeta_i\Psi_{\gamma-1}(\xi_i,\underline{\xi})}\quad\text{and}\quad A_0=\Big(\alpha+\beta\Gamma(\gamma)\Big)\psi_\rho^{*}.$$

Define the operator $\Xi\colon C_{1-\gamma,\chi}^{1}([\underline{\xi},\overline{\xi}])\}\to C_{1-\gamma,\chi}^{1}([\underline{\xi},\overline{\xi}])\}$ by

$$\Xi y(\xi)=\mathcal{T}\sum_{i=1}^{n}\zeta_i\Big[\kappa y(\xi_i)+I_{\underline{\xi}^+}^{\rho,\chi}\hbar\Big(\xi_i,y(\xi_i),{}^{C}\mathcal{D}_{\underline{\xi}^+}^{\gamma,\chi}y(\xi_i)\Big)\Big]\Psi_\gamma(\xi,\underline{\xi})+\kappa\int_{\underline{\xi}}^{\xi}\chi'(s)y(s)\mathrm{d}s$$
$$+\frac{1}{\Gamma(\rho+1)}\int_{\underline{\xi}}^{\xi}\chi'(s)\Psi_\rho(\xi,s)\hbar\Big(s,y(s),{}^{C}\mathcal{D}_{\underline{\xi}^+}^{\gamma,\chi}y(s)\Big)\mathrm{d}s.$$

and the operator $D^{\chi}\Xi\colon\ \mathcal{C}_{1-\gamma;\chi}([\underline{\xi},\overline{\xi}])\}\to\mathcal{C}_{1-\gamma;\chi}([\underline{\xi},\overline{\xi}])\}$ by

$$D^{\chi}\Xi y(\xi)=\gamma\mathcal{T}\sum_{i=1}^{n}\zeta_i\Big[\kappa y(\xi_i)+I_{\underline{\xi}^+}^{\rho;\chi}\hbar\Big(\xi_i,y(\xi_i),{}^{C}\mathcal{D}_{\underline{\xi}^+}^{\gamma;\chi}y(\xi_i)\Big)\Big]\Psi_{\gamma-1}(\xi,\underline{\xi})+\kappa y(\xi)$$
$$+\frac{1}{\Gamma(\rho)}\int_{\underline{\xi}}^{\xi}\chi'(s)\Psi_{\rho-1}(\xi,s)\hbar\Big(s,y(s),{}^{C}\mathcal{D}_{\underline{\xi}^+}^{\gamma;\chi}y(s)\Big)\mathrm{d}s.$$

In this part, we will present the result concerning the existence of solutions of the problem (1.1)-(1.2). First, we will give some useful lemmas to demonstrate this result.

Lemma 7. *We assume the hypotheses* $(\mathbf{H_1})$ *and* $(\mathbf{H_{1-1}})$ *hold. Then*

(1) Ξ *is bounded and continuous.*
(2) $\Xi(B)$ *is equicontinuous for all bounded subset* B *of* $\mathcal{C}^{1}_{1-\gamma;\chi}([\underline{\xi},\overline{\xi}])$.

Proof. Let us show condition (1); we begin to prove that Ξ is a bounded operator. Let $y\in\mathcal{C}^{1}_{1-\gamma;\chi}([\underline{\xi},\overline{\xi}])$, it is clear to see that $\Xi y\in\mathcal{C}^{1}_{1-\gamma;\chi}([\underline{\xi},\overline{\xi}])$. Using $(\mathbf{H_1})$ and $(\mathbf{H_{1-1}})$, for all $y\in B_r=\{y\in C^{1}_{1-\gamma;\chi}([\underline{\xi},\overline{\xi}])\colon\ \|y\|^{1}_{1-\gamma;\chi}<r\}$ and $\xi\in(\underline{\xi},\overline{\xi}]$, we have

$$\|\Xi y(\xi)\|\le|\mathcal{T}|\sum_{i=1}^{n}|\zeta_i|\Big[\kappa\|y(\xi_i)\|+I_{\underline{\xi}^+}^{\rho;\chi}\|\hbar\Big(\xi_i,y(\xi_i),{}^{C}\mathcal{D}_{\underline{\xi}^+}^{\gamma;\chi}y(\xi_i)\Big)\|\Big]\Psi_{\gamma}(\xi,\underline{\xi})$$
$$+\kappa\int_{\underline{\xi}}^{\xi}\chi'(s)\|y(s)\|\mathrm{d}s+\frac{1}{\Gamma(\rho+1)}\int_{\underline{\xi}}^{\xi}\chi'(s)\Psi_{\rho}(\xi,s)\|\hbar\Big(s,y(s),{}^{C}\mathcal{D}_{\underline{\xi}^+}^{\gamma;\chi}y(s)\Big)\|\mathrm{d}s,$$
$$\le|\mathcal{T}|\zeta^{*}n\Psi_{\gamma}^{*}\Big[\kappa r+\frac{\hbar^{*}\Psi_{\rho}^{*}}{\Gamma(\rho+1)}+\frac{r\alpha\Psi_{\rho}^{*}}{\Gamma(\rho+1)}+\frac{r\beta\Gamma(\gamma)\Psi_{\rho}^{*}}{\Gamma(\rho+1)}\Big]+\kappa r\Psi_{1}^{*}$$
$$+\frac{\hbar^{*}\Psi_{\rho+1}^{*}}{\Gamma(\rho+2)}+\frac{r\alpha\Psi_{\rho+1}^{*}}{\Gamma(\rho+2)}+\frac{r\beta\Gamma(\gamma)\Psi_{\rho+1}^{*}}{\Gamma(\rho+2)}.$$

We also have, for each $\xi\in(\underline{\xi},\overline{\xi}]$

$$\|\Psi_{1-\gamma}(\xi,\underline{\xi})D^{\chi}\Xi y(\xi)\|\le\gamma|\mathcal{T}|\sum_{i=1}^{n}\zeta_i\Big[\kappa y(\xi_i)+I_{\underline{\xi}^+}^{\rho;\chi}\hbar\Big(\xi_i,y(\xi_i),{}^{C}\mathcal{D}_{\underline{\xi}^+}^{\gamma;\chi}y(\xi_i)\Big)\Big]$$
$$+\kappa\Psi_{1-\gamma}(\xi,\underline{\xi})y(\xi)+\frac{\Psi_{1-\gamma}(\xi,\underline{\xi})}{\Gamma(\rho)}\int_{\underline{\xi}}^{\xi}\chi'(s)\Psi_{\rho-1}(\xi,s)\hbar\Big(s,y(s),{}^{C}\mathcal{D}_{\underline{\xi}^+}^{\gamma;\chi}y(s)\Big)\mathrm{d}s$$
$$\le(\gamma|\mathcal{T}|\zeta^{*}n+\Psi_{1-\gamma}^{*})\Big[\kappa r+\frac{\hbar^{*}\Psi_{\rho}^{*}}{\Gamma(\rho+1)}+\frac{r\alpha\Psi_{\rho}^{*}}{\Gamma(\rho+1)}+\frac{r\beta\Gamma(\gamma)\Psi_{\rho}^{*}}{\Gamma(\rho+1)}\Big].$$

So,

$$\|\Xi y\|_{\infty}+\|D^{\chi}\Xi y\|_{\gamma;\chi}\ \le\Big(|\mathcal{T}|\zeta^{*}n(\gamma+\Psi_{\gamma}^{*})+\Psi_{1-\gamma}^{*}\Big)\Big[\kappa r+\frac{\hbar^{*}\Psi_{\rho}^{*}}{\Gamma(\rho+1)}+\frac{r\alpha\Psi_{\rho}^{*}}{\Gamma(\rho+1)}$$

$$+\frac{r\beta\Gamma(\gamma)\Psi^*_\rho}{\Gamma(\rho+1)}\Big]+\kappa r\Psi^*_1+\frac{\hbar^*\Psi^*_{\rho+1}}{\Gamma(\rho+2)}+\frac{r\alpha\Psi^*_{\rho+1}}{\Gamma(\rho+2)}+\frac{r\beta\Gamma(\gamma)\Psi^*_{\rho+1}}{\Gamma(\rho+2)}.$$

Now we will show that Ξ is continuous. Let $\{y_n\}_{n\in\mathbb{N}}\to y$ in $\mathcal{C}^1_{1-\gamma,\chi}([\underline{\xi},\overline{\xi}])$, from $(\mathbf{H_{1-1}})$ and Lemma 1 we can easily prove that $\Xi y_n(.)\to\Xi y(.)$ in $\mathcal{C}([\underline{\xi},\overline{\xi}])$ and $D^\chi\Xi y_n(.)\to D^\chi\Xi y(.)$ in $\mathcal{C}_{1-\gamma,\chi}([\underline{\xi},\overline{\xi}])$, that implies $\Xi y_n(.)\to\Xi y(.)$ in $\mathcal{C}^1_{1-\gamma,\chi}([\underline{\xi},\overline{\xi}])$, then Ξ is continuous.

Let us show the second condition (2), it is enough to show that $\Xi(B_r)$ (resp. $D^\chi\Xi(B_r)$) is equicontinuous in $\mathcal{C}([\underline{\xi},\overline{\xi}])$ (resp. in $\mathcal{C}_{1-\gamma,\chi}([\underline{\xi},\overline{\xi}])$). Let $y\in B_r$ and $\xi_1,\xi_2\in(\underline{\xi},\overline{\xi}]$ with $\xi_1<\xi_2$, from $(\mathbf{H_{1-1}})$, we have

$$\begin{aligned}
&\|\Xi y(\xi_2)-\Xi y(\xi_1)\|\\
&\leq\Big[\kappa r+\frac{\hbar^*\Psi^*_\rho}{\Gamma(\rho+1)}+\frac{r\alpha\Psi^*_\rho}{\Gamma(\rho+1)}+\frac{r\beta\Gamma(\gamma)\Psi^*_\rho}{\Gamma(\rho+1)}\Big]\times\Big(\Psi_\gamma(\xi_2,\underline{\xi})-\Psi_\gamma(\xi_1,\underline{\xi})\Big)\\
&\quad+\frac{1}{\Gamma(\rho+1)}\int_{\underline{\xi}}^{\xi_1}\chi'(s)[\Psi_\rho(\xi_2,s)-\Psi_\rho(\xi_1,s)]\hbar\Big(s,y(s),{}^C\mathcal{D}^{\gamma;\chi}_{\underline{\xi}^+}y(s)\Big)\mathrm{d}s\\
&\quad+\frac{1}{\Gamma(\rho+1)}\int_{\xi_1}^{\xi_2}\chi'(s)\Psi_\rho(\xi_2,s)\hbar\Big(s,y(s),{}^C\mathcal{D}^{\gamma;\chi}_{\underline{\xi}^+}y(s)\Big)\mathrm{d}s+\kappa\int_{\xi_1}^{\xi_2}\chi'(s)y(s)\mathrm{d}s\\
&\leq\Big[\kappa r+\frac{\hbar^*\Psi^*_\rho}{\Gamma(\rho+1)}+\frac{r\alpha\Psi^*_\rho}{\Gamma(\rho+1)}+\frac{r\beta\Gamma(\gamma)\Psi^*_\rho}{\Gamma(\rho+1)}\Big]\Big(\Psi_\gamma(\xi_2,\underline{\xi})-\Psi_\gamma(\xi_1,\underline{\xi})\Big)\\
&\quad+\frac{\hbar^*+r(\alpha+\beta\Gamma(\gamma))}{\Gamma(\rho+2)}\Big[\Psi_{\rho+1}(\xi_2,\underline{\xi})-\Psi_{\rho+1}(\xi_1,\underline{\xi})+\Psi_{\rho+1}(\xi_2,\xi_1)\Big]\\
&\quad+\frac{\hbar^*+r(\alpha+\beta\Gamma(\gamma))}{\Gamma(\rho+2)}\Psi_{\rho+1}(\xi_2,\xi_1)+\kappa r\Psi_1(\xi_2,\xi_1).
\end{aligned}$$

As ξ_2 tends to ξ_1, the right-hand side of the last inequality tends to 0. Therefore $\Xi(B_r)$ is equicontinuous in $\mathcal{C}([\underline{\xi},\overline{\xi}])$.

And, we also have

$$\begin{aligned}
&\|\Psi_{1-\gamma}(\xi_2,\underline{\xi})D^\chi\Xi y(\xi_2)-\Psi_{1-\gamma}(\xi_1,\underline{\xi})D^\chi\Xi y(\xi_1)\|\\
&\leq\kappa\|\Psi_{1-\gamma}(\xi_2,\underline{\xi})y(\xi_2)-\Psi_{1-\gamma}(\xi_1,\underline{\xi})y(\xi_1)\|\\
&\quad+\Bigg\|\frac{\Psi_{1-\gamma}(\xi_2,\underline{\xi})}{\Gamma(\rho)}\int_{\underline{\xi}}^{\xi_2}\chi'(s)\Psi_{\rho-1}(\xi_2,s)\hbar\Big(s,y(s),{}^C\mathcal{D}^{\gamma;\chi}_{\underline{\xi}^+}y(s)\Big)\mathrm{d}s\\
&\qquad-\frac{\Psi_{1-\gamma}(\xi_1,\underline{\xi})}{\Gamma(\rho-1)}\int_{\underline{\xi}}^{\xi_1}\chi'(s)\Psi_\rho(\xi_1,s)\hbar\Big(s,y(s),{}^C\mathcal{D}^{\gamma;\chi}_{\underline{\xi}^+}y(s)\Big)\mathrm{d}s\Bigg\|\\
&\leq\kappa\Big(\Psi_{1-\gamma}(\xi_2,\underline{\xi})-\Psi_{1-\gamma}(\xi_1,\underline{\xi})\Big)\|y(\xi_2)\|+\kappa\Psi_{1-\gamma}(\xi_1,\underline{\xi})\|y(\xi_2)-y(\xi_1)\|
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\Psi_{1-\gamma}(\xi_1,\underline{\xi})}{\Gamma(\rho)}\int_{\underline{\xi}}^{\xi_1}\chi'(s)\Big[\Psi_{\rho-1}(\xi_1,s)-\Psi_{\rho-1}(\xi_2,s)\Big]\|\hbar\Big(s,y(s),{}^{C}\mathcal{D}^{\gamma,\chi}_{\underline{\xi}^+}y(s)\Big)\|\mathrm{d}s\\
&+\frac{\Psi_{1-\gamma}(\xi_2,\underline{\xi})-\Psi_{1-\gamma}(\xi_1,\underline{\xi})}{\Gamma(\alpha)}\int_{\underline{\xi}}^{\xi_1}\chi'(s)\Psi_{\rho-1}(\xi_2,s)\|\hbar\Big(s,y(s),{}^{C}\mathcal{D}^{\gamma,\chi}_{\underline{\xi}^+}y(s)\Big)\|\mathrm{d}s\\
&+\frac{\Psi_{1-\gamma}(\xi_2,\underline{\xi})}{\Gamma(\rho)}\int_{\xi_1}^{\xi_2}\chi'(s)\Psi_{\rho-1}(\xi_2,s)\|\hbar\Big(s,y(s),{}^{C}\mathcal{D}^{\gamma,\chi}_{\underline{\xi}^+}y(s)\Big)\|\mathrm{d}s\\
\leq{}&\kappa\Big(\Psi_{1-\gamma}(\xi_2,\underline{\xi})-\Psi_{1-\gamma}(\xi_1,\underline{\xi})\Big)r+\frac{r\kappa\Psi^*_{1-\gamma}}{\gamma}\Psi_\gamma(\xi_2,\xi_1)\\
&+\frac{\Big(\hbar^*+r[\alpha+\beta\Gamma(\gamma)]\Big)\Psi_{1-\gamma}(\xi_1,\underline{\xi})}{\Gamma(\rho)}\int_{\underline{\xi}}^{\xi_1}\chi'(s)\Big[\Psi_{\rho-1}(\xi_1,s)-\Psi_{\rho-1}(\xi_2,s)\Big]\mathrm{d}s\\
&+\frac{\Big(\hbar^*+r[\alpha+\beta\Gamma(\gamma)]\Big)\Big(\Psi_{1-\gamma}(\xi_2,\underline{\xi})-\Psi_{1-\gamma}(\xi_1,\underline{\xi})\Big)}{\Gamma(\rho)}\int_{\underline{\xi}}^{\xi_1}\chi'(s)\Psi_{\rho-1}(\xi_2,s)\mathrm{d}s\\
&+\frac{\Big(\hbar^*+r[\alpha+\beta\Gamma(\gamma)]\Big)\Psi_{1-\gamma}(\xi_2,\underline{\xi})}{\Gamma(\rho)}\int_{\xi_1}^{\xi_2}\chi'(s)\Psi_{\rho-1}(\xi_2,s)\mathrm{d}s\\
\leq{}&\kappa\Big(\Psi_{1-\gamma}(\xi_2,\underline{\xi})-\Psi_{1-\gamma}(\xi_1,\underline{\xi})\Big)r+\frac{r\kappa\Psi^*_{1-\gamma}}{\gamma}\Psi_\gamma(\xi_2,\xi_1)\\
&+\frac{\Big(\hbar^*+r[\alpha+\beta\Gamma(\gamma)]\Big)\Psi^*_{1-\gamma}}{\Gamma(\rho+1)}\Big[\Psi_\rho(\xi_2,\underline{\xi})-\Psi_\rho(\xi_1,\underline{\xi})+2\Psi_\rho(\xi_2,\xi_1)\Big]\\
&+\frac{\Big(\hbar^*+r[\alpha+\beta\Gamma(\gamma)]\Big)\Psi^*_\rho}{\Gamma(\rho)}\Big[\Psi_{1-\gamma}(\xi_2,\underline{\xi})-\Psi_{1-\gamma}(\xi_1,\underline{\xi})\Big].
\end{aligned}
$$

By taking ξ_2 tends to ξ_1, the right-hand side of the last inequality tends to 0, and hence $D^\chi\Xi(B_r)$ is equicontinuous in $\mathcal{C}_{1-\gamma;\chi}([\underline{\xi},\overline{\xi}])$, thus, $\Xi(B_r)$ is equicontinuous in $\mathcal{C}^1_{1-\gamma;\chi}([\underline{\xi},\overline{\xi}])$. □

We denote by $\vartheta_{\mathcal{C}},\vartheta_\gamma$ and ϑ^1_γ the Kuratowski noncompactness measure defined respectively on $\mathcal{C}([\underline{\xi},\overline{\xi}]))$, $\mathcal{C}_{1-\gamma;\chi}([\underline{\xi},\overline{\xi}]))$ and $\mathcal{C}^1_{1-\gamma;\chi}([\underline{\xi},\overline{\xi}]))$.

Lemma 8. *Let B be a bounded subset of $\mathcal{C}^1_{1-\gamma;\chi}([\underline{\xi},\overline{\xi}]))$, we have*

$$\vartheta^1_\gamma(B)\leq\vartheta(B)+\vartheta_\gamma(D^\chi B)\leq 2\vartheta^1_\gamma(B). \tag{3.5}$$

Proof. Let B be a bounded subset of $\mathcal{C}^1_{1-\gamma;\chi}([\underline{\xi},\overline{\xi}]))$ and let ε be a strictly positive real number. So, there exists a finite partition B_i, $i=1,\dots m$, such that

$$\mathrm{Diam}^1_\gamma(B_i)\leq\varepsilon+\vartheta^1_\gamma(B),\ \ i=1,\dots m.$$

Then for all y_1, y_2 in B_i and $\xi \in (\underline{\xi}, \overline{\xi}]$, we have

$$\|y_2(\xi) - y_1(\xi)\| \leq \varepsilon + \vartheta^1_\gamma(B) \text{ and } \|D^\chi y_2(\xi) - D^\chi y_1(\xi)\| \leq \varepsilon + \vartheta^1_\gamma(B),\ i = 1, \ldots m.$$

So,

$$\mathrm{Diam}(B_i) \leq \varepsilon + \vartheta^1_\gamma(B) \text{ and } \mathrm{Diam}_\gamma(D^\chi B_i) \leq \varepsilon + \vartheta^1_\gamma(B),\ i = 1, \ldots m.$$

Thus,

$$\vartheta(B) + \vartheta_\gamma(D^\chi B) \leq 2\varepsilon + 2\vartheta^1_\gamma(B).$$

Since ε is arbitrary, this means that we arrive at

$$\vartheta(B) + \vartheta_\gamma(D^\chi B) \leq 2\vartheta^1_\gamma(B). \tag{3.6}$$

Conversely, we want to prove that $\vartheta^1_\gamma(B) \leq \vartheta(B) + \vartheta_\gamma(D^\chi B)$, from the definition of Kuratowski noncompactness measure, we have, for each $\varepsilon > 0$, there are a finite partitions $\{B_i\}_{i=1,\ldots,m_1}$ of B and $\{D_j\}_{j=1,\ldots,m_2}$ of $D^\chi B$ such that

$$\mathrm{Diam}(B_i) \leq \varepsilon + \vartheta(B), \text{ and } \mathrm{Diam}_\gamma(D_j) \leq \varepsilon + \vartheta_\gamma(D^\chi B).$$

It is clear that the partition $\{B_i \cap I^\chi_{\underline{\xi}^+} D_j\}_{i,j}$ belongs to $\mathcal{C}^1_{1-\gamma,\chi}([\underline{\xi}, \overline{\xi}]))$ and verifies the following inequality:

$$\mathrm{Diam}(B_i \cap I^\chi_{\underline{\xi}^+} D_j) + \mathrm{Diam}_\gamma(D^\chi(B_i \cap I^\chi_{\underline{\xi}^+} D_j)) \leq 2\varepsilon + \vartheta(B) + \vartheta_\gamma(D^\chi B).$$

As ε is arbitrary, we obtain

$$\vartheta^1_\gamma(B) \leq \vartheta(B) + \vartheta_\gamma(D^\chi B). \tag{3.7}$$

From (3.6)-(3.7), we get

$$\vartheta^1_\gamma(B) \leq \vartheta(B) + \vartheta_\gamma(D^\chi B) \leq 2\vartheta^1_\gamma(B).$$

□

From Lemma 5 and Lemma 8, we easily show the following inequality

$$\vartheta^1_\gamma(D) \leq \sup_{\xi \in [\underline{\xi}, \overline{\xi}]} \vartheta(D(\xi)) + \sup_{\xi \in [\underline{\xi}, \overline{\xi}]} \vartheta(\Psi_{1-\gamma}(\xi, \underline{\xi}) D^\chi D(\xi)) \leq 2\vartheta^1_\gamma(D), \tag{3.8}$$

where D is a bounded and equicontinuous subset of $\mathcal{C}^1_{1-\gamma,\chi}([\underline{\xi}, \overline{\xi}]))$,

$$D(\xi) = \{y(\xi) : y \in D\} \quad \text{and} \quad D^\chi D(\xi) = \{D^\chi y(\xi) : y \in D\}.$$

Let

$$B_R = \left\{ y \in \mathcal{C}_{1-\gamma,\chi}([\underline{\xi}, \overline{\xi}])) : \|y\|^1_{\gamma,\chi} \leq R \right\}.$$

We are about to present our main result which is as follows.

Theorem 2. *Assume that the hypotheses* $(\mathbf{H_1})-(\mathbf{H_2})$ *are satisfied and that* R *verifies the following inequality*

$$\frac{1}{R}<\frac{\Gamma(\rho+2)-\Big(\kappa\Gamma(\rho+2)+(\rho+1)A_0\Big)\Big(|\mathcal{T}|\zeta^* n(\Psi_\gamma^*+\gamma)+\Psi_{1-\gamma}^*\Big)}{(\rho+1)\Big(|\mathcal{T}|\zeta^* n(\Psi_\gamma^*+\gamma)+\Psi_{1-\gamma}^*\Big)\Psi_\rho^*\hbar^*+\Psi_{\rho+1}^*\hbar^*} \qquad (3.9)$$
$$-\frac{\Big(A_0+\kappa\Gamma(\rho+2)\Big)\Psi_1^*}{(\rho+1)\Big(|\mathcal{T}|\zeta^* n(\Psi_\gamma^*+\gamma)+\Psi_{1-\gamma}^*\Big)\Psi_\rho^*\hbar^*+\Psi_{\rho+1}^*\hbar^*}.$$

Then, the problem (1.1)-(1.2) *has at least one solution in* $\mathcal{C}^{1,\gamma}_{1-\gamma,\chi}([\underline{\xi},\overline{\xi}])$. *In addition, the solution set* **SS** *of the problem* (1.1)-(1.2) *is compact in* $\mathcal{C}^{1}_{1-\gamma,\chi}([\underline{\xi},\overline{\xi}])$.

Proof. From the definition of Ξ and Lemma 6, it is clear that the solutions of (1.1)-(1.2) is equivalent to the fixed point of Ξ. For this reason, we want to verify that Ξ satisfies the assumptions of Mönch fixed point theorem. First, we will prove that Ξ is well defined from B_R to B_R, indeed, let $y\in B_R$. By using the condition $(\mathbf{H_{1-1}})$ and after some calculations, for each $\xi\in(\underline{\xi},\overline{\xi}]$ and $y\in B_R$, we get

$$\begin{aligned}
&\|\Xi y(\xi)\|+\|\Psi_{1-\gamma}(\xi,\underline{\xi})D^\chi\Xi y(\xi)\|\\
&\leq|\mathcal{T}|\sum_{i=1}^n|\zeta_i|\Big[\kappa\|y(\xi_i)\|+I^{\rho,\chi}_{\underline{\xi}^+}\|\hbar\Big(\xi_i,y(\xi_i),{}^C\mathcal{D}^{\gamma,\chi}_{\underline{\xi}^+}y(\xi_i)\Big)\|\Big]\Psi_\gamma^*\\
&\quad+\kappa\int_{\underline{\xi}}^{\xi}\chi'(s)\|y(s)\|\mathrm{d}s+\frac{1}{\Gamma(\rho+1)}\int_{\underline{\xi}}^{\xi}\chi'(s)\Psi_\rho(\xi,s)\|\hbar\Big(s,y(s),{}^C\mathcal{D}^{\gamma,\chi}_{\underline{\xi}^+}y(s)\Big)\|\mathrm{d}s\\
&\quad+\gamma|\mathcal{T}|\sum_{i=1}^n\zeta_i\Big[\kappa y(\xi_i)+I^{\rho,\chi}\hbar\Big(\xi_i,y(\xi_i),{}^C\mathcal{D}^{\gamma,\chi}_{\underline{\xi}^+}y(\xi_i)\Big)\Big]+\kappa\Psi_{1-\gamma}^*y(\xi)\\
&\quad+\frac{\Psi_{1-\gamma}^*}{\Gamma(\rho)}\int_{\underline{\xi}}^{\xi}\chi'(s)\Psi_{\rho-1}(\xi,s)\hbar\Big(s,y(s),{}^C\mathcal{D}^{\gamma,\chi}_{\underline{\xi}^+}y(s)\Big)\mathrm{d}s\\
&\leq\frac{(\rho+1)\Big(|\mathcal{T}|\zeta^* n(\Psi_\gamma^*+\gamma)+\Psi_{1-\gamma}^*\Big)\Psi_\rho^*\hbar^*+\Psi_{\rho+1}^*\hbar^*}{\Gamma(\rho+2)}\\
&\quad+\frac{\Big(\kappa\Gamma(\rho+2)+(\rho+1)A_0\Big)\Big(|\mathcal{T}|\zeta^* n(\Psi_\gamma^*+\gamma)+\Psi_{1-\gamma}^*\Big)}{\Gamma(\rho+2)}R+\frac{\Big(A_0+\kappa\Gamma(\rho+2)\Big)\Psi_1^*}{\Gamma(\rho+2)}R.
\end{aligned}$$

From $(\mathbf{H_2})$ and the inequality (3.9), we obtain

$$\forall y\in B_R:\|\Xi y\|^1_{\gamma,\chi}<R.$$

Note that B_R is bounded, convex and closed subset of $\mathcal{C}^{1}_{1-\gamma,\chi}([\underline{\xi},\overline{\xi}]))$ and Ξ is continuous on B_R. Next, it is enough to show the following implication:

$$V\subset\overline{conv}\{N(V)\cup\{0\}\}\Longrightarrow\vartheta^1_\gamma(V)=0,\ \text{for any } V\subset B_R.$$

Let V be a subset of B_R such that $V \subset \overline{conv}\{N(V) \cup \{0\}\}$. By using Lemmas 4 and 5, we obtain

$$\vartheta(\Xi V(\xi)) + \vartheta\Big(\psi_{1-\gamma}(\xi,\underline{\xi})D^{\chi}\Xi(V(\xi))\Big)$$
$$\leq |\mathcal{T}|\sum_{i=1}^{n}\zeta^*\left[\kappa\vartheta\Big(V(\xi_i)\Big) + + I_{\underline{\xi}^+}^{\rho;\chi}\vartheta\Big(\hbar\Big(\xi_i, V(\xi_i), {}^{C}\mathcal{D}_{\underline{\xi}^+}^{\gamma;\chi}V(\xi_i)\Big)\Big)\right]\Psi_{\gamma}^*$$
$$+\kappa\int_{\underline{\xi}}^{\xi}\chi'(s)\vartheta\Big(V(s)\Big)\mathrm{d}s + \frac{1}{\Gamma(\rho+1)}\int_{\underline{\xi}}^{\xi}\chi'(s)\Psi_{\rho}(\xi,s)\vartheta\Big(\hbar\Big(s, V(s), {}^{C}\mathcal{D}_{\underline{\xi}^+}^{\gamma;\chi}V(s)\Big)\Big)\mathrm{d}s$$
$$+\gamma|\mathcal{T}|\sum_{i=1}^{n}\zeta^*\left[\kappa\vartheta\Big(V(\xi_i)\Big) + I_{\underline{\xi}^+}^{\rho;\chi}\vartheta\Big(\hbar\Big(\xi_i, V(\xi_i), {}^{C}\mathcal{D}_{\underline{\xi}^+}^{\gamma;\chi}V(\xi_i)\Big)\Big)\right] + \kappa\Psi_{1-\gamma}^*\vartheta\Big(V(\xi)\Big)$$
$$+\frac{\Psi_{1-\gamma}^*}{\Gamma(\rho)}\int_{\underline{\xi}}^{\xi}\chi'(s)\Psi_{\rho-1}(\xi,s)\vartheta\Big(\hbar\Big(s, V(s), {}^{C}\mathcal{D}_{\underline{\xi}^+}^{\gamma;\chi}V(s)\Big)\mathrm{d}s\Big).$$

From Lemmas 5, 7 and 8 and the hypotheses $(\mathbf{H_{1-2}}) - (\mathbf{H_2})$ and inequality (3.8), we arrive at

$$\begin{aligned}\vartheta_{\gamma}^{1}(\Xi V) &\leq \sup_{\xi\in[\underline{\xi},\overline{\xi}]}\vartheta(\Xi V(\xi)) + \sup_{\xi\in[\underline{\xi},\overline{\xi}]}\vartheta\Big(\psi_{1_{\gamma(\xi,\underline{\xi})}}D^{\chi}\Xi(V(\xi))\Big)\\ &\leq \frac{2\Big(\kappa\Gamma(\rho+2)+(\rho+1)A_0\Big)\Big(|\mathcal{T}|\zeta^* n(\Psi_{\gamma}^*+\gamma)+\Psi_{1-\gamma}^*\Big)}{\Gamma(\rho+2)}\vartheta_{\gamma}^{1}(\Xi V)\\ &\quad+\frac{2\Big(A_0+\kappa\Gamma(\rho+2)\Big)\Psi_1^*}{\Gamma(\rho+2)}\vartheta_{\gamma}^{1}(\Xi V).\end{aligned}$$

By the condition $(\mathbf{H_2})$, we get $\vartheta_{\gamma}^{1}(\Xi V) = 0$, that means $\vartheta_{\gamma}^{1}(V) = 0$. From Theorem 1, the operator Ξ has at least one fixed point $y \in B_R$. By using Lemma 6, we conclude that the problem (1.1)-(1.2) has at least one solution. Let us prove that solution set **SS** of (1.1)-(1.2) is included in $\mathcal{C}_{1-\gamma;\chi}^{1,\gamma}([\underline{\xi},\overline{\xi}])$, Let $w \in \{u \in \mathcal{C}_{1-\gamma;\chi}^{1}([\underline{\xi},\overline{\xi}]) : \Xi u = u \text{ and } D^{\chi}\Xi u = D^{\chi}u\}$, we need to show that $D^{\chi}w \in \mathcal{C}_{1-\gamma;\chi}^{\gamma}([\underline{\xi},\overline{\xi}])$, so, for all $\xi \in (\underline{\xi},\overline{\xi}]$, we have

$$D^{\chi}w(\xi) = \gamma\mathcal{T}\sum_{i=1}^{n}\zeta_i\left[\kappa w(\xi_i) + I_{\underline{\xi}^+}^{\rho;\chi}\hbar\Big(\xi_i, w(\xi_i), {}^{C}\mathcal{D}_{\underline{\xi}^+}^{\gamma;\chi}w(\xi_i)\Big)\right]\Psi_{\gamma-1}(\xi,\underline{\xi}) + \kappa w(\xi)$$
$$+\frac{1}{\Gamma(\rho)}\int_{\underline{\xi}}^{\xi}\chi'(s)\Psi_{\rho-1}(\xi,s)\hbar\Big(s, w(s), {}^{C}\mathcal{D}_{\underline{\xi}^+}^{\gamma;\chi}w(s)\Big)\mathrm{d}s.$$

By using ${}^{RL}\mathcal{D}_{\underline{\xi}^+}^{\gamma}$ on both sides the last inequality, from Lemmas 1, 2 we obtain

$$(1-\kappa)^{RL}\mathcal{D}_{\underline{\xi}^+}^{\gamma}D^{\chi}w(t) = {}^{RL}\mathcal{D}_{\underline{\xi}^+}^{\gamma}I_{\underline{\xi}^+}^{\rho;\chi}\hbar\Big(s, w(s), {}^{C}\mathcal{D}_{\underline{\xi}^+}^{\gamma;\chi}w(s)\Big)$$

$$={}^{RL}\mathcal{D}^{\sigma(1-\rho)}_{\underline{\xi}^+}\hbar\Big(\xi, w(\xi), {}^{C}\mathcal{D}^{\gamma,\chi}_{\underline{\xi}^+}w(\xi)\Big).$$

So, from $(\mathbf{H_1})$, we have ${}^{RL}\mathcal{D}^{\gamma}_{\underline{\xi}^+}D^{\chi}w(t)\in\mathcal{C}^{\gamma}_{1-\gamma}([\underline{\xi},\overline{\xi}])$, that means $w\in\mathcal{C}^{1,\gamma}_{1-\gamma,\chi}([\underline{\xi},\overline{\xi}])$. Finally, the solution set **SS** of problem (1.1)-(1.2) is included in $\mathcal{C}^{1,\gamma}_{1-\gamma,\chi}([\underline{\xi},\overline{\xi}])$.

We show now that the solution set **SS** of the problem (1.1)-(1.2) is compact subset o $\mathcal{C}^{1}_{1-\gamma,\chi}([\underline{\xi},\overline{\xi}])$. Let $\{y_n\}_{n\in\mathbb{N}}$ be a sequence of the solution set, as $\mathcal{C}^{1}_{1-\gamma,\chi}([\underline{\xi},\overline{\xi}])$ is compact space, there exists a subsequence of $\{y_n\}_{n\in\mathbb{N}}$ (still denoted $\{y_n\}_{n\in\mathbb{N}}$) converges to y^*, it is enough to demonstrate that y^* is a solution of (1.1)-(1.2), for each $\xi\in(\underline{\xi},\overline{\xi}]$, we have

$$y_n(\xi)=\mathcal{T}\sum_{i=1}^{n}\zeta_i\Big[\kappa y_n(\xi_i)+I^{\rho,\chi}_{\underline{\xi}^+}\hbar\Big(\xi_i,y_n(\xi_i),{}^{C}\mathcal{D}^{\gamma,\chi}_{\underline{\xi}^+}y_n(\xi_i)\Big)\Big]\Psi_\gamma(\xi,\underline{\xi})$$
$$+\kappa\int_{\underline{\xi}}^{\xi}\chi'(s)y_n(s)\mathrm{d}s+\frac{1}{\Gamma(\rho+1)}\int_{\underline{\xi}}^{\xi}\chi'(s)\Psi_\rho(\xi,s)\hbar\Big(s,y_n(s),{}^{C}\mathcal{D}^{\gamma,\chi}_{\underline{\xi}^+}y_n(s)\Big)\mathrm{d}s$$

and

$$D^{\chi}y_n(\xi)=\gamma\mathcal{T}\sum_{i=1}^{n}\zeta_i\Big[\kappa y_n(\xi_i)+I^{\rho,\chi}_{\underline{\xi}^+}\hbar\Big(\xi_i,y_n(\xi_i),{}^{C}\mathcal{D}^{\gamma,\chi}_{\underline{\xi}^+}y_n(\xi_i)\Big)\Big]\Psi_{\gamma-1}(\xi,\underline{\xi})+\kappa y_n(\xi)$$
$$+\frac{1}{\Gamma(\rho)}\int_{\underline{\xi}}^{\xi}\chi'(s)\Psi_{\rho-1}(\xi,s)\hbar\Big(s,y_n(s),{}^{C}\mathcal{D}^{\gamma,\chi}_{\underline{\xi}^+}y_n(s)\Big)\mathrm{d}s.$$

From $(\mathbf{H_1})$, we have $\hbar(.,y_n(.),{}^{C}\mathcal{D}^{\gamma,\chi}_{\underline{\xi}^+}y_n(.))$ converges to $\hbar(.,y^*(.),{}^{C}\mathcal{D}^{\gamma,\chi}_{\underline{\xi}^+}y^*(.))$ as $n\to+\infty$, let $\xi\in(\underline{\xi},\overline{\xi}]$, form $(\mathbf{H_{1-1}})$, for all $n\in\mathbb{N}$, we have

$\chi'(s)\Psi_\rho(\xi,s)\|\hbar\Big(s,y_n(s),{}^{C}\mathcal{D}^{\gamma,\chi}_{\underline{\xi}^+}y_n(s)\Big)\|\le\Big(\hbar^*+(\alpha+\beta\Gamma(\gamma))M\Big)\chi'(s)\Psi_\rho(\xi,s)$ and

$\chi'(s)\Psi_{\rho-1}(\xi,s)\|\hbar\Big(s,y_n(s),{}^{C}\mathcal{D}^{\gamma,\chi}_{\underline{\xi}^+}y_n(s)\Big)\|\le\Big(\hbar^*+(\alpha+\beta\Gamma(\gamma))M\Big)\chi'(s)\Psi_{\rho-1}(\xi,s)$.

Using Lebesgue's dominated convergence theorem, for each $\xi\in(\underline{\xi},\overline{\xi}]$, we obtain

$$y^*(\xi)=\mathcal{T}\sum_{i=1}^{n}\zeta_i\Big[\kappa y^*(\xi_i)+I^{\rho,\chi}_{\underline{\xi}^+}\hbar\Big(\xi_i,y^*(\xi_i),{}^{C}\mathcal{D}^{\gamma,\chi}_{\underline{\xi}^+}y^*(\xi_i)\Big)\Big]\Psi_\gamma(\xi,\underline{\xi})$$
$$+\kappa\int_{\underline{\xi}}^{\xi}\chi'(s)y^*(s)\mathrm{d}s+\frac{1}{\Gamma(\rho+1)}\int_{\underline{\xi}}^{\xi}\chi'(s)\Psi_\rho(\xi,s)\hbar\Big(s,y^*(s),{}^{C}\mathcal{D}^{\gamma,\chi}_{\underline{\xi}^+}y^*(s)\Big)\mathrm{d}s$$

and

$$D^{\chi}y^*(\xi)=\gamma\mathcal{T}\sum_{i=1}^{n}\zeta_i\Big[\kappa y^*(\xi_i)+I^{\rho,\chi}_{\underline{\xi}^+}\hbar\Big(\xi_i,y^*(\xi_i),{}^{C}\mathcal{D}^{\gamma,\chi}_{\underline{\xi}^+}y^*(\xi_i)\Big)\Big]\Psi_{\gamma-1}(\xi,\underline{\xi})+\kappa y^*(\xi)$$
$$+\frac{1}{\Gamma(\rho)}\int_{\underline{\xi}}^{\xi}\chi'(s)\Psi_{\rho-1}(\xi,s)\hbar\Big(s,y^*(s),{}^{C}\mathcal{D}^{\gamma,\chi}_{\underline{\xi}^+}y^*(s)\Big)\mathrm{d}s.$$

So, the solution set of Problem (1.1)-(1.2) is a compact subset of $\mathcal{C}^1_{1-\gamma;\chi}([\underline{\xi},\overline{\xi}])$. $\square$

4. Example

We take $\psi(t)=\frac{4\arctan t}{10\pi}$, $\underline{\xi}=0, \xi_1=0.5$, $\overline{\xi}=1$, $\sigma=\rho=0.25$, $\kappa=\frac{1}{40}$, E the Banach space defined by

$$E=\Big\{(y_1,y_2,\ldots,y_n,\ldots)\colon \sup_n |y_n|<\infty\Big\},$$

with the norm $\|y\|=\sup_n|y_n|$, we define the function $\hbar\colon (0,1]\times E^2\to E$ by

$$\hbar\Big(\xi,y(\xi),{}^{C}\mathcal{D}^{\gamma;\chi}_{\underline{\xi}^+}y(\xi)\Big)=\Big(\hbar_1\Big(\xi,y_1(\xi),{}^{C}\mathcal{D}^{\gamma;\chi}_{\underline{\xi}^+}y_1(\xi)\Big),\ldots,\hbar_n\Big(\xi,y_n(\xi),{}^{C}\mathcal{D}^{\gamma;\chi}_{\underline{\xi}^+}y_n(\xi)\Big),\ldots\Big),$$

where

$$\hbar_n\Big(\xi,y_n(\xi),{}^{C}\mathcal{D}^{\gamma;\chi}_{\underline{\xi}^+}y_n(\xi)\Big)=\frac{{}^{C}\mathcal{D}^{\gamma;\chi}_{\underline{\xi}^+}y_n(\xi)}{40+nt^2}+\frac{y_n(\xi)}{40+t^n},\ \xi\in(0,1].$$

We easily see that $\hbar\colon (0,1]\times E^2\to E$ is continuous and

$$\|\hbar(\xi,u,v)-\hbar(\xi,\overline{u},\overline{v})\|\le\frac{1}{40}\|u-\overline{u}\|+\frac{1}{40}\|v-\overline{v}\|,\ \text{for all } \xi\in(0,1] \text{ and } u,v,\overline{u},\overline{v}\in E.$$

Next, for all Ω a bounded subset of $\mathcal{C}^1_{1-\gamma;\chi}([0,1])$, we have

$$\vartheta\Big(\hbar\Big(\xi,\Omega(\xi),{}^{C}\mathcal{D}^{\gamma;\chi}_{\underline{\xi}^+}\Omega(\xi)\Big)\Big)\le\frac{1}{40}\Big(\vartheta(\Omega(\xi))+\vartheta\big({}^{C}\mathcal{D}^{\gamma;\chi}_{\underline{\xi}^+}\Omega(\xi)\big)\Big),\ \xi\in(0,1].$$

So, $(\mathbf{H_1})$, $(\mathbf{H_{1-1}})$ and $(\mathbf{H_{1-2}})$ are satisfied. A quick calculation gives us

$$\Big(\kappa\Gamma(\rho+2)+(\rho+1)A_0\Big)\Big(|\mathcal{T}|\zeta^* n(\Psi^*_\gamma+\gamma)+\Psi^*_{1-\gamma}\Big)+\Big(A_0+\kappa\Gamma(\rho+2)\Big)\Psi^*_1<\frac{\Gamma(\rho+2)}{2}.$$

So, $(\mathbf{H_2})$ holds. Therefore, Theorem 2 ensures that the solution set of Problem (1.1)-(1.2) is nonempty and compact.

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DEMI VERSION OF QUASI LEVI AND LEVI OPERATORS ON BANACH LATTICES

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Abstract. Demi versions of different classes of operators have been recently investigated by many researchers. In this paper we study the demi version of quasi Levi and Levi operators. We investigate their properties. Moreover, their relations with other classes of operators are also discussed.

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1. Introduction

Many papers have been devoted to studying the properties of different classes of operators in Banach lattice theory. Some of them introduced some operators involving demi criteria, for instance in [10, 11, 15, 18], the notion of demi KB operators using the definition of KB operators from [9] is defined. They also showed that demi KB and b-weakly demicompact operators are the same.

In the present paper we consider the classes of (quasi)-KB and (quasi)-Levi operators as they were introduced in [4, 16]. For alternative versions of the definitions we refer to [6, 9, 11, 20]. The main motivation of this paper is to explore the demi version of operators using the study from [12–14]. We then expand our approach by defining the concepts of demi quasi Levi operators and demi Levi operators. The goal of this paper is to examine these newly defined operators.

The paper is organized as follows. In Section 2, we will address demi quasi Levi operators and study their basic properties depending on order convergence. We characterize Banach lattices on which all operators are demi quasi Levi operators. Section 3 is devoted to demi Levi operators. In the last section, the demi version of KB and quasi KB operators are studied depending on the definition given by [13].

Throughout this paper, E denotes real Banach lattices, B_E is the closed unit ball of E, I is the identity operator on E. The set of all bounded linear operators on E

is denoted by $L(E)$. The positive cone of E is denoted by $E_+ := \{x \in E \ 0 \leq x\}$. A Banach lattice E is called a KB space if every increasing, norm bounded sequence in E_+ converges in norm. A Banach lattice is called an abstract L_1-space (AL-space), whenever its norm is additive in the sense that $\|x+y\| = \|x\| + \|y\|$ holds for all $x, y \in E_+$ with $x \wedge y = 0$. A normed Riesz space is called a Levi lattice if each increasing norm bounded sequence in E_+ has a supremum in E. For more information, see [1, 2, 17].

According to [11], an operator $T : E \to E$ is a demi KB operator if for every positive increasing sequence $x_n \in B_E$ such that $(x_n - Tx_n)$ is convergent to some $x \in E$, there is a convergent subsequence of (x_n). Moreover, $T : E \to E$ is called a weak demi KB operator if for every positive increasing sequence $x_n \in B_E$ such that $(x_n - Tx_n)$ is weakly convergent to $x \in E$, there is a weakly convergent subsequence of (x_n). Recall that a subset A of a Banach lattice E is said b-order bounded if there exists some $0 \leq x^{''} \in E^{''}$ such that $|x| \leq x^{''}$ for all $x \in A$. An operator T from a Banach lattice E to a Banach space X is said to be b-weakly compact if the image of every b-order bounded subset of E under T is relatively weakly compact. An operator $T : E \to E$ is said to be b-weakly demicompact if for every b-order bounded sequence in E_+ such that $x_n \to 0$ in $\sigma(E, E')$ and $\|x_n - Tx_n\| \to 0$, we have $\|x_n\| \to 0$. The definiton of demicompact operators was firstly given by [18]. $T : D(T) \subseteq E \to E$, where $D(T)$ is a subspace of E, is said to be demicompact if, for every bounded sequence (x_n) in the domain $D(T)$ such that $(x_n - Tx_n)$ converges to $x \in E$, there is a convergent subsequence of (x_n). For the other necessary definitions, see [3, 5–8].

We provide four fundamental definitions given in [4, 13, 14], which form the basis of this paper.

Let E be a normed Riesz space. An operator $T : E \to E$ is called a quasi Levi (σ-quasi Levi) operator if T takes increasing norm bounded net (sequence) in E_+ to an order Cauchy net (sequence). The collection of the quasi Levi (σ-quasi Levi) operators is denoted by $L_{ql}(E)$ ($L^{\sigma}_{ql}(E)$). An operator $T : E \to E$ is called Levi (σ-Levi) operator if for every norm bounded increasing net (sequence) in E_+ such that $Tx_\alpha \xrightarrow{o} Tx$, $x \in E$. The collection of the Levi (σ-Levi) operators is denoted by $L_l(E)$ ($L^{\sigma}_l(E)$).

Let T be an operator from a Banach lattice E to Banach space Y. T is a quasi KB (σ-quasi KB) operator if (Tx_α) converges in the norm for every increasing norm bounded net (sequence) in E_+. The collection of quasi KB (σ-quasi KB) operators is denoted by $L_{qKB}(E, Y)$ ($L^{\sigma}_{qKB}(E, Y)$). T is a KB (σ-KB) operator if for every increasing norm bounded net (sequence) in E_+, $\|Tx_\alpha - Tx\| \to 0$ for some $x \in E$. The collection of the KB (σ-KB) operators is denoted by $L_{KB}(E, Y)$ ($L^{\sigma}_{KB}(E, Y)$). In [4], Proposition 1.2 shows that $L^{\sigma}_{qKB}(E, Y) = L_{qKB}(E, Y)$. In the following we study the demi version of such operators.

2. DEMI QUASI LEVI OPERATORS

In this section, demi quasi Levi operators are introduced and their properties are studied. Since the ideas of the general proofs in most cases are similar to the σ-case, we only restrict to sequences.

Proposition 1. *Let E be a normed Riesz space. $L_{ql}(E)$ is a vector space. Moreover every quasi Levi operator $T: E \to E$ is σ-order bounded.*

Proof. The proof is directly from [20, Theorem 2.2]. □

Definition 1. Let E be a normed Riesz space. An operator $T: E \to E$ is called a demi quasi Levi (σ-demi quasi Levi) operator if for every increasing and norm bounded net (sequence) in E_+ such that $(x_\alpha - Tx_\alpha)$ is order Cauchy, there exists a subnet (subsequence) of (x_α) that is order convergent. The collection of the demi quasi Levi (σ-demi quasi Levi) operators is denoted by $L_{Dql}(E)$ ($L^{\sigma}_{Dql}(E)$).

Lemma 1. *Let E be a Dedekind complete normed Riesz space. Then λI is a demi quasi Levi operator for every $\lambda \neq 1$.*

Proof. Let $0 \le x_n \uparrow$, $x_n \in B_E$ and $(x_n - \lambda I(x_n))$ be order Cauchy. We have

$$\big(x_n - \lambda I(x_n) - (x_m - \lambda I(x_m))\big) \xrightarrow{o} 0.$$

Then there exists $y_k \downarrow 0$ such that $|x_n - \lambda I(x_n) - x_m + \lambda I(x_m)| \le y_k$. Therefore we get

$$|x_n - x_m| \le \frac{y_k}{|1-\lambda|},\ \lambda \neq 1,$$

and so (x_n) is an order Cauchy sequence. Hence (x_n) is order convergent. □

Proposition 2. *Let E be a Dedekind complete normed Riesz space. Every quasi Levi operator $T: E \to E$ is a demi quasi Levi operator.*

Proof. Let T be a quasi Levi operator and $0 \le x_n \uparrow$, $x_n \in B_E$ such that $(x_n - Tx_n)$ is an order Cauchy sequence. Therefore $(I - T)$ is a quasi Levi operator. Since $L_{ql}(E)$ is a vector space, we have $I = (I - T) + T$ that the identity operator is a quasi Levi operator. It follows that (x_n) is order convergent. □

Demi quasi Levi operator need not to be quasi Levi as the following example shows.

Example 1. Let $E = c_{00}$ and consider the operator $2I: E \to E$. $2I$ is not a quasi Levi operator but from Lemma 1, it is a demi quasi Levi operator.

Proposition 3. *Let E be a normed Riesz space and $T, S: E \to E$ be two operators. If T is a demi quasi Levi operator and S is a quasi Levi operator, then $T + S$ is a demi quasi Levi operator.*

Proof. Let $0 \leq x_n \uparrow$, $x_n \in B_E$ be such that $(x_n - (T+S)x_n)$ is an order Cauchy sequence. Since S is a quasi Levi operator, (Sx_n) is an order Cauchy sequence. We can write

$$\begin{aligned}(x_n - Tx_n) - (x_m - Tx_m) &= (x_n - Tx_n \pm Sx_n) - (x_m - Tx_m \pm Sx_m) \\ &= (x_n - (T+S)x_n) - (x_m - (T+S)x_m) + (Sx_n - Sx_m).\end{aligned}$$

Since $(x_n - (T+S)x_n)$ and (Sx_n) are order Cauchy sequences, then $(x_n - Tx_n)$ is order Cauchy. Hence there exists an order convergent subsequence (x_{n_k}) as T is demi quasi Levi. Therefore $T+S$ is a demi quasi Levi operator. □

Theorem 1. *Let E be a Banach lattice with σ-order continuous norm. Then the following statements are equivalent.*

(i) *E is a KB space.*
(ii) *Every σ-order continuous operator $T: E \to E$ is a σ-quasi Levi operator.*
(iii) *Every σ-order continuous operator $T: E \to E$ is a σ-demi quasi Levi operator.*
(iv) *The identity operator of E is a σ-demi quasi Levi operator.*

Proof. $(i) \Rightarrow (ii)$ Let E be a KB space and $0 \leq x_n \uparrow$, $x_n \in B_E$. Since E is a KB space, (x_n) is norm convergent. There exists a subsequence (x_{n_k}) of (x_n) such that $x_{n_k} \xrightarrow{o} x$. If $x_{n_k} \uparrow$ and $x_{n_k} \xrightarrow{o} x$, then $x_n \xrightarrow{o} x$. Therefore $Tx_n \xrightarrow{o} Tx$ as T is a σ-order continuous operator.
$(ii) \Rightarrow (iii)$ and $(iii) \Rightarrow (iv)$ From Proposition 2.
$(iv) \Rightarrow (i)$ Let I be a σ-demi quasi Levi operator and $0 \leq x_n \uparrow$, $x_n \in B_E$. $(x_n - I(x_n))$ is an order Cauchy sequence. Since I is a σ-demi quasi Levi operator then there exists order convergent subsequence (x_{n_k}). Since E has a σ-order continuous norm, (x_{n_k}) is norm convergent and (x_n) is increasing, then (x_n) is a norm convergent sequence which means that E is a KB space. □

In Theorem 1, it is important that the space E must be a KB space. For example, let E not be a KB space. The operators $T = 2I$ and $S = -I$ are demi quasi Levi operators by Lemma 1. However the operator $T+S = I$ is not a demi quasi Levi operator by Theorem 1. Additionally, the collection of operators $L_{Dql}(E)$ is not closed under multiplication and does not form a vector space structure. For instance, let $E = c_{00}$ and consider the operator $(-I) \in L_{Dql}(E)$. Since $E = c_{00}$ is not a KB space, by Theorem 1, the operator $(-1)(-I) = I$ is not a demi quasi Levi operator.

Moreover, on AL space, more generally on KB space, every operator $T: E \to E$ is demi quasi Levi.

The following theorem demonstrates that the domination property holds for demi quasi Levi operators. In the following proofs σ version is only considered. However net version is the same.

Theorem 2. *Let E be a normed Riesz space and T be a positive demi quasi Levi operator. Then every operator S satisfying $0 \leq S \leq T \leq I$ is a demi quasi Levi operator.*

Proof. Let (x_n) be a sequence such that $(x_n - Sx_n)$ is a Cauchy sequence, where $0 \leq x_n \uparrow$ and $x_n \in B_E$. Since S is a central operator, we can write $|x_n - Sx_n| = |(I-S)x_n| = |I-S||x_n|$. Hence we have,

$$|(I-S)(x_n - x_m)| = |I-S||x_n - x_m| = (I-S)|x_n - x_m| \xrightarrow{o} 0.$$

On the other hand from the inequality $0 \leq S \leq T \leq I$, we get $(I-T)|x_n| \leq (I-S)|x_n|$ and $(I-T)|x_n - x_m| \xrightarrow{o} 0$. By the assumption that $T\colon E \to E$ is a demi quasi Levi operator, there exists a subsequence (x_{n_k}) such that $x_{n_k} \xrightarrow{o} y$. Consequently $S\colon E \to E$ is also a demi quasi Levi operator. □

The collection of $L_{Dql}(E)$ is closed under order convergence.

Proposition 4. *Let E be a Banach lattice and $T_\gamma \in L_{Dql}(E)$. If $T_\gamma \xrightarrow{o} T$, then $T \in L_{Dql}(E)$.*

Proof. Let $0 \leq x_n \uparrow$, $x_n \in B_E$ be such that $(x_n - Tx_n)$ is an order Cauchy sequence. From assumption when $T_\gamma \xrightarrow{o} T$, we have $T_\gamma - T \xrightarrow{o} 0$. This implies that the sequence $(T_\gamma - T)$ is order Cauchy. For each (x_n) in B_E, the sequence $(T_\gamma x_n - Tx_n)$ is also order Cauchy. We can write $x_n - T_\gamma x_n = x_n - Tx_n + Tx_n - T_\gamma x_n$. Since both $(x_n - Tx_n)$ and $(T_\gamma x_n - Tx_n)$ are order Cauchy, it follows that $(x_n - T_\gamma x_n)$ is an order Cauchy sequence. Given that $T_\gamma \in L_{Dql}(E)$, there exists a subsequence (x_{n_k}) such that $x_{n_k} \xrightarrow{o} y$. Therefore $T\colon E \to E$ is a demi quasi Levi operator. □

The other important question is whether demi quasi Levi operators satisfy the modul property. Even E is a Dedekind complete Banach lattice and T is order continuous, $|T|$ is not necessarily to be demi quasi Levi. For instance $-I\colon c_0 \to c_0$ is demi quasi Levi but $|-I| = I$ is not demi quasi Levi by Theorem 1.

3. Demi Levi operators

In this section the new operator class called demi Levi operators are introduced. Since the ideas of the general proofs in most cases are similar to the σ-case, we only restrict to sequences.

Every Levi operator is quasi Levi. For the converse, necessary conditions are given in [14, Theorem 3.3] as the following.

Theorem 3 ([14]). *Let E be a Banach lattice with order continuous norm. The followings are equivalent.*

(*i*) *E is a KB space.*
(*ii*) $L_+(E) = L_{l+}(E)$.
(*iii*) $L_{ql+}(E) = L_{l+}(E)$.

Definition 2. Let E be a normed Riesz space. An operator $T\colon E \to E$ is called a demi Levi operator if for every increasing and norm bounded net (sequence) in E_+ such that $x_\alpha - Tx_\alpha \xrightarrow{o} Tx$ and $x \in E$, there exists an order convergent subnet (subsequence) of (x_α). The collection of the demi Levi operators is denoted by $L_{Dl}(E)$ ($L^{\sigma}_{Dl}(E)$).

Lemma 2. *Let E be a normed Riesz space, λI is a demi Levi operator for every $\lambda \neq 1$.*

Proof. Let $0 \leq x_n \uparrow, x_n \in B_E$ and $x_n - \lambda I(x_n) \xrightarrow{o} \lambda I(x), x \in E$. Since $(1-\lambda)x_n \xrightarrow{o} \lambda x$, $x \in E$, so there exists order convergent subsequence x_{n_k} of x_n. □

In case $\lambda = 1$, the identify operator is not necessarily demi Levi. Consider the operator $I\colon c_0 \to c_0$ and the sequence

$$x_n = (x_m)_n = \begin{cases} 1, & m \leq n, \\ 0, & m > n. \end{cases}$$

It is clear that $0 \leq x_n \uparrow$ ve $\|x\|_\infty = 1$. Hence $(x_n - I(x_n))$ is an order convergent but (x_n) is not.

The natural question is the following. Is $L_{Dl}(E)$ a vector space? The answer is negative. Consider, $T = -I$ and $S = 2I$ on c_0. From Lemma 2, T and S are demi Levi operators. But $T+S$ is not a demi Levi operator. Therefore $L_{Dl}(E)$ is not closed with respect to addition. Also, consider $(-I) \in L_{Dl}(E)$. $(-1)(-I) = I \notin L_{Dl}(E)$. Hence $L_{Dl}(E)$ is not a vector space.

Proposition 5. *Let E be a normed Riesz space. Every Levi operators $T\colon E \to E$ is a demi Levi operator.*

Proof. Let $0 \leq x_n \uparrow$, $x_n \in B_E$ and consider $x_n - Tx_n \xrightarrow{o} Tx_1,\ x_1 \in E$. Since T is a Levi operator, there exists $x_2 \in E$ such that $Tx_n \xrightarrow{o} Tx_2$. We can write

$$x_{n_k} = Tx_{n_k} - Tx_{n_k} + x_{n_k} \xrightarrow{o} Tx_1 + Tx_2,$$

and so operator T is a demi Levi operator. □

Lemma 3. *Let E be a normed Riesz space. If $T\colon E \to E$ is a demi quasi Levi operator, then T is a demi Levi operator.*

Proof. Let $0 \leq x_n \uparrow$, $x_n \in B_E$ such that $x_n - Tx_n \xrightarrow{o} Tx$, $x \in E$. Since an order convergent sequence is order Cauchy, $(x_n - Tx_n)$ is order Cauchy. So there exists order convergent subsequence (x_{n_k}) of (x_n). □

[14, Example 3.4] is the example for the strict inclusion of $L_l(E) \subset L_{Dl}(E)$. This example is quasi Levi but it is not Levi. So it is demi quasi Levi by Proposition 2. Therefore by Lemma 3, it is demi Levi.

Before the following theorem, the definition of a Levi lattice should be given. A normed Riesz space E is called a Levi lattice if every increasing norm bounded positive sequence (x_n) in E has a supremum. Every Levi lattice is Dedekind complete.

Theorem 4. *Let E be a normed Riesz space. The followings are equivalent:*

(i) *E is a Levi lattice.*
(ii) *Every order continuous operator $T: E \to E$ is a Levi operator.*
(iii) *Every order continuous operator $T: E \to E$ is a demi Levi operator.*
(iv) *The identity operator of E is a demi Levi operator.*

Proof. $(i) \Rightarrow (ii)$ Let $0 \leq x_n \uparrow, x_n \in B_E$. Since E is Levi lattice, $\sup x_n = x$. (x_n) is order convergent to $x \in E$ as $x_n \uparrow$ and $\sup x_n = x$. It is clear that $Tx_n \xrightarrow{o} Tx,\ x \in E$ as T is order continuous. Hence T is Levi.
$(ii) \Rightarrow (iii)$ From Proposition 5.
$(iii) \Rightarrow (iv)$ Obvious.
$(iv) \Rightarrow (i)$ Let $0 \leq x_n \uparrow,\ x_n \in B_E$. We have to show that (x_n) has a supremum in E. It is obvious that $(x_n - I(x_n))$ order convergent. Since I is a demi Levi operator, there exists an order convergent subsequence of (x_n) such that $x_{n_k} \xrightarrow{o} y,\ y \in E$. $x_n \uparrow$ and $x_{n_k} \xrightarrow{o} y$ implies $x_n \xrightarrow{o} y$, hence $\sup x_n = y,\ y \in E$ is satisfied and so E is a Levi lattice. □

Since every KB space is a Levi lattice with an order continuous complete norm, it follows from Theorem 4 that every order continuous operator is a demi quasi Levi operator if and only if it is demi Levi.

It is an important topic to examine whether the same properties hold for the modul of operators. The modulus of a demi Levi operator is not necessarily a demi Levi operator. For example, the operator $-I: c_0 \to c_0$ is a demi Levi operator, but the operator $|-I| = I$ is not a demi Levi operator, as c_0 is not a Levi lattice by Theorem 4.

4. DEMI KB AND DEMI QUASI KB OPERATORS

In this section, demi KB and demi quasi KB operators are introduced and their relations are discussed. We restrict ourselves to the sequence case.

Definition 3. Let T be an operator from a Banach lattice E to E. We say that:

(i) T is a demi quasi KB (σ-demi quasi KB) operator if for every increasing norm bounded net (sequence) in E_+ such that $(x_\alpha - Tx_\alpha)$ is norm Cauchy net (sequence), there is a norm convergent subnet (subsequence) of (x_α). The collection of demi quasi KB (σ-demi quasi KB) operators is denoted by $L_{qKB}(E)$ ($L^{\sigma}_{qKB}(E)$).

(ii) T is a demi KB (σ-demi KB)operator if for every increasing norm bounded net (sequence) in E_+ such that $x_\alpha - Tx_\alpha \xrightarrow{\|\cdot\|} Tx$ and $x \in E$, there exists a norm convergent subnet (subsequence) of (x_α). The collection of the demi KB (σ-demi KB)operators is denoted by $L_{DKB}(E)$ $(L^{\sigma}_{DKB}(E))$.

Lemma 4. *Let E be a Banach lattice and $T : E \to E$ be an operator. Every KB operator is demi KB, and every quasi KB operator is demi quasi KB.*

Lemma 5. *Let E be a Banach lattice. If $T : E \to E$ is a demi quasi KB operator, then T is a demi KB operator.*

Proof. Let $0 \leq x_n \uparrow$, $x_n \in B_E$ and for $x \in E$, $x_n - Tx_n \xrightarrow{\|\cdot\|} Tx$. Since the norm convergent sequence is norm Cauchy, $(x_n - Tx_n)$ is norm Cauchy. So there exists norm convergent subsequence (x_{n_k}) of (x_n). □

Example 1 in [13] is the example of an operator that is quasi KB but not KB. By Lemma 4, it is demi quasi KB. This operator is also demi KB by Lemma 5. Therefore it is considered for demi KB operator not necessarly to be KB.

Lemma 6. *Let E be a Banach lattice. λI is a demi KB operator for every $\lambda \neq 1$. Hence $\lambda I \in L_{DqKB}(E)$ for every $\lambda \neq 1$.*

Proof. Let $0 \leq x_n \uparrow, x_n \in B_E$ and $x_n - \lambda I(x_n) \xrightarrow{\|\cdot\|} I(x)$. Since $(1-\lambda)x_n \xrightarrow{\|\cdot\|} x$, there exists a norm convergent subsequence of (x_n). □

The following corollary gives an important fact on Banach lattices that b-weakly demicompact operators, demi KB operators and weak demi KB operators of [11] and the demi quasi KB operators coincide. For the proof, see [19].

Corollary 1. *Let E be a Banach lattice and $T : E \to E$ be an operator. The following statements are equivalent.*

- (i) *For every b-order bounded sequence (x_n) in E_+ such that $x_n \to 0$ in $\sigma(E,E')$ and $\|x_n - Tx_n\| \to 0$ as $n \to \infty$, we have $\|x_n\| \to 0$ as $n \to \infty$.*
- (ii) *For every positive, increasing sequence (x_n) in B_E such that $(x_n - Tx_n)$ is convergent to some $x \in E$, there is a norm convergent subsequence of (x_n).*
- (iii) *For every positive, increasing sequence (x_n) in B_E such that $(x_n - Tx_n)$ is a norm Cauchy sequence, there is a norm convergent subsequence of (x_n).*
- (iv) *For every positive, increasing sequence (x_n) in B_E such that $(x_n - Tx_n)$ is weakly convergent to some $x \in E$, there is a weakly convergent subsequence of (x_n).*

Therefore, if a b-weakly demi-compact operator fulfills certain properties, then any demi quasi-KB operator inherently satisfy those same properties. Consequently, we can restate the following theorem.

Theorem 5. *Let E be a Banach lattice. Then, the following statements are equivalent.*

- (i) *E is a KB space.*

(ii) *Every operator $T: E \to E$ is a quasi KB operator.*
(iii) *The identity operator of E is a demi quasi KB operator.*

Proof. $(i) \Rightarrow (ii)$ from Proposition 2.13 [9]. By Lemma 4, (ii) implies (iii). (iii) implies (i) is obtained directly by KB space definition. □

The following two propositions examine the relationships between different operators.

Proposition 6. *Let E be a Dedekind complete Banach lattice.*
(i) *Every quasi KB operator $T: E \to E$ is demi quasi Levi.*
(ii) *Every quasi KB operator $T: E \to E$ is demi Levi.*

Proof. (i) Let $0 \le x_n \uparrow$, $x_n \in B_E$ and $(x_n - Tx_n)$ be order Cauchy sequence. Since E is Dedekind complete, then $x_n - Tx_n \xrightarrow{o} x$, $x \in E$. Since $T \in L_{qKB}(E)$, (Tx_n) is norm Cauchy. Hence, $Tx_n \xrightarrow{\|\cdot\|} y$, $y \in E$. Then there exists subsequence (x_{n_k}) of (x_n) such that $Tx_{n_k} \xrightarrow{o} y$. We can write

$$x_{n_k} = Tx_{n_k} - Tx_{n_k} + x_{n_k} \xrightarrow{o} x + y,\ x + y \in E,$$

and so T is demi quasi Levi.
(ii) Since every demi quasi Levi operator is demi Levi by Lemma 3, the proof is completed. □

Proposition 7. *Let E be a Banach lattice with order continuous norm. Then the followings are valid.*
(i) *Every demi quasi KB operator $T: E \to E$ is demi quasi Levi.*
(ii) *$T: E \to E$ is a demi KB operator iff $T: E \to E$ is demi Levi.*
(iii) *Every quasi Levi operator $T: E \to E$ is quasi KB.*
(iv) *Every Levi operator $T: E \to E$ is quasi KB.*

Proof. (i) Let $0 \le x_n \uparrow$, $x_n \in B_E$ and $(x_n - Tx_n)$ be order Cauchy. We aim to show that $x_{n_k} \xrightarrow{o} y$, $y \in E$. Since E has an order continuous norm, $(x_n - Tx_n)$ is norm Cauchy. Since T is a demi quasi KB operator, there exists subsequence (x_{n_k}) of (x_n) such that (x_{n_k}) is norm convergent. Then there exists an order convergent subsequence $(x_{n_{k_l}})$. As $(x_{n_{k_l}})$ is increasing and order convergent, we can write $x_{n_k} \xrightarrow{o} y$, $y \in E$. Therefore, T is demi quasi Levi.
(ii) $(\Rightarrow)$ Let $0 \le x_n \uparrow$, $x_n \in B_E$ and $x_n - Tx_n \xrightarrow{o} Tx$, $x \in E$. Since E has an order continuous norm, $x_n - Tx_n \xrightarrow{\|\cdot\|} Tx$, $x \in E$. From assumption T is a demi KB operator, we obtain $x_{n_k} \xrightarrow{\|\cdot\|} y$, so $x_n \xrightarrow{\|\cdot\|} y$. Therefore $x_{n_k} \xrightarrow{o} y$. It implies that T is a demi Levi operator.
(ii) $(\Leftarrow)$ Let $0 \le x_n \uparrow$, $x_n \in B_E$ and $x_n - Tx_n \xrightarrow{\|\cdot\|} Tx$, $x \in E$. We know that, $x_{n_k} - Tx_{n_k} \xrightarrow{o} Tx$. Since T is a demi Levi operator, a subsequence $(x_{n_{k_t}})$ exists and

it converges to $y \in E$ in order. As (x_{n_k}) is increasing and $x_{n_{k_t}} \xrightarrow{o} y$, then we obtaing $x_{n_k} \xrightarrow{o} y$. On the other hand, since E has an order continuous norm, then $x_{n_k} \xrightarrow{\|\cdot\|} y$. Therefore, T is a demi KB operator.
(iii) Let $0 \leq x_n \uparrow$, $x_n \in B_E$. From the assumption (Tx_n) is order Cauchy. Since E is Dedekind complete, $Tx_n \xrightarrow{o} y$ and E has order continuous norm, we obtain $Tx_n \xrightarrow{\|\cdot\|} y$. So (Tx_n) is norm Cauchy. Therefore, T is a quasi KB operator.
(iv) Since every Levi operator is quasi Levi, the proof is completed. □

Items (iii) and (iv) of the above proposition are also mentioned in [4, 16].

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SOME PROPERTIES OF r-SMALL ELEMENTS IN LATTICES

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Abstract. In this paper, all lattices are complete modular lattices with the greatest element 1 and the smallest element 0. Let L be a lattice, $a \in L$ and $a \leq r(L)$. If $a \ll r(L)/0$, then a is called an r-small (or r-superfluous) element of L and denoted by $a \ll_r L$. In this work, some properties of these elements are investigated. This concept is a generalization of an r-small submodule of any module. It is clear that every r-small element is small. But the converse of this statement is not true in general. Let L be a lattice, $a \in r(L)/0$ and $r(L)$ be a supplement element in L. Then $a \ll L$ if and only if $a \ll_r L$. Let L be a lattice, $a \in L$ and $b \ll_r L$. Then $a \ll_r L$ if and only if $a \vee b \ll_r 1/b$.

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1. INTRODUCTION

In this paper, all lattices are complete modular lattices with the greatest element 1 and the smallest element 0. Let L be a lattice and $a, b \in L$ with $a \leq b$. A sublattice $\{x \in L | a \leq x \leq b\}$ is called a *quotient sublattice* and denoted by b/a. Let L be a lattice and $a, b \in L$. If $a \vee b = 1$ and $a \wedge b = 0$, then a is called a *complement* of b in L and denoted by $1 = a \oplus b$ (here we also call a and b are *direct summands* of L). L is called a *complemented* lattice if every element of L has at least one complement in L. Let L be a lattice and $a \in L$. a is called a *compact* element of L if every subset X of L with $a \leq \vee X$ there exists a finite subset F of X such that $a \leq \vee F$. L is said to be *compact* if 1 is compact in L. L is said to be *compactly generated* if each element of L is a join of compact elements in L. Let L be a lattice and $a \in L$. If $b = 1$ for every $b \in L$ with $a \vee b = 1$, then a is called a *small* (or *superfluous*) element of L and denoted by $a \ll L$. The meet of all maximal $(\neq 1)$ elements of a lattice L is called the *radical* of L and denoted by $r(L)$. If L have no maximal $(\neq 1)$ elements, then the radical of L is defined by $r(L) = 1$. Let L be a lattice and $a, b \in L$. If a is minimal for $1 = b \vee a$, then a is called a *supplement* of b in L. a is a supplement of b in a lattice L if and only if $1 = b \vee a$ and $b \wedge a \ll a/0$. A lattice L is said to be *supplemented* if every element of L has at least one supplement in L. L is said to be *hollow* if every element with distinct from 1 is small in L, and L is said to be *local* if L has the greatest element $(\neq 1)$. We say an element $a \in L$ has *ample supplements* in L if for every $b \in L$ with $a \vee b = 1$, a has a supplement x in L with $x \leq b$. L is said to be *amply supplemented* if every element of L has ample supplements in L. Let L be a lattice. It is defined β_* relation on the elements of L by $x\beta_* y$ with $x, y \in L$ if and only if for each $t \in L$ such that $x \vee t = 1$ then $y \vee t = 1$ and for each $k \in L$ such that $y \vee k = 1$ then $x \vee k = 1$. We say that an element y of L *lies above* an element x of L if $x \leq y$ and $y \ll 1/x$.

More information about (amply) supplemented lattices are in [1,5,7]. More results about (amply) supplemented modules are in [4, 6, 12]. The definition of β_* relation on lattices and some properties of this relation are in [8]. This relation is a generalization of β^* relation on modules. The definition of β^* relation on modules and some properties of this relation are in [3].

Lemma 1. *Let L be a lattice and $a, b, c, d \in L$. Then the followings hold.*

(i) *If $a \leq b$ and $b \ll L$, then $a \ll L$.*
(ii) *Let $a \leq b$. If $a \ll L$ and $b \ll 1/a$, then $b \ll L$.*
(iii) *If $a \ll b/0$, then $a \ll t/0$ for every $t \in L$ with $b \leq t$.*
(iv) *Let $a \leq b$ and b be a supplement element in L. Then $a \ll b/0$ if and only if $a \ll L$.*
(v) *If $a \ll b/0$, then $a \vee c \ll (b \vee c)/c$.*
(vi) *If $a \ll L$, then $a \vee b \ll 1/b$.*
(vii) *If $a \ll b/0$ and $c \ll d/0$, then $a \vee c \ll (b \vee d)/0$.*
$(viii)$ *If $a \ll L$ and $b \ll L$, then $a \vee b \ll L$.*
(ix) *If $a \ll L$, then $a \leq r(L)$.*
(x) *If a is compact in L and $a \leq r(L)$, then $a \ll L$.*
(xi) *If L is compactly generated, then $r(L) = \bigvee_{x \ll L} x$.*
(xii) *If L is compact, then $r(L) \ll L$.*

Proof. See [5]. □

2. r-SMALL ELEMENTS IN LATTICES

Definition 1. Let L be a lattice, $a \in L$ and $a \leq r(L)$. If $a \ll r(L)/0$, then a is called an r-small (or r-superfluous) element of L and denoted by $a \ll_r L$. (See also [2]).

This concept is a generalization of an r-small submodule of any module. The definition of r-small submodules and some properties of these submodules are in [9–11].

Proposition 1. *Let L be a lattice, $a \in L$ and $r(L) = 1$. Then $a \ll_r L$ if and only if $a \ll L$.*

Proof. Clear from definitions. □

Lemma 2. *Let L be a lattice and $a \in L$. If $a \ll_r L$, then $a \ll L$.*

Proof. Since $a \ll_r L$, $a \ll r(L)/0$. Then by Lemma 1, $a \ll L$, as desired. □

The converse of this lemma is not true in general. (See Example 1, Example 2 and Example 3).

Lemma 3. *Let L be a lattice and $a \in L$. If $a \ll L$ and $r(L)$ is a supplement element in L, then $a \ll_r L$.*

Proof. Since $a \ll L$, by Lemma 1, $a \leq r(L)$. Since $a \leq r(L)$ and $r(L)$ is a supplement element in L, by Lemma 1, $a \ll r(L)/0$ and $a \ll_r L$, as desired. □

Corollary 1. *Let L be a lattice, $a \in r(L)/0$ and $r(L)$ be a supplement element in L. Then $a \ll L$ if and only if $a \ll_r L$.*

Proof. ($\Longrightarrow$) Clear from Lemma 3.
($\Longleftarrow$) Clear from Lemma 2. □

Corollary 2. *Let L be a lattice, $a \in r(L)/0$ and $r(L)$ be a direct summand of L. Then $a \ll L$ if and only if $a \ll_r L$.*

Proof. Clear from Corollary 1. □

Proposition 2. *Let L be a lattice and $a\beta_* b$ in $r(L)/0$. If $b \ll_r L$, then $a \ll_r L$.*

Proof. Since $b \ll_r L$, then $b \ll r(L)/0$. Since $a\beta_* b$ in $r(L)/0$, by [8, Theorem 2], $a \ll r(L)/0$. Hence $a \ll_r L$, as desired. □

Lemma 4. *Let L be a lattice, $a \ll L$ and $r(L)$ be a supplement element in L. If $b \in r(L)/0$ and $a\beta_* b$ in L, then $b \ll_r L$.*

Proof. Since $a\beta_* b$ in L and $a \ll L$, by [8, Theorem 2], $b \ll L$. Since $b \leq r(L)$ and $r(L)$ is a supplement element in L, by Corollary 1, $b \ll_r L$. □

Corollary 3. *Let L be a lattice, $a \ll L$ and $r(L)$ be a direct summand of L. If $b \in r(L)/0$ and $a\beta_* b$ in L, then $b \ll_r L$.*

Proof. Clear from Lemma 4. □

Proposition 3. *Let L be a lattice and $a \ll_r L$. Then $a \ll c/0$ for every maximal $(\neq 1)$ element c of L.*

Proof. Let c be a maximal $(\neq 1)$ element of L. Since $a \ll_r L$, $a \ll r(L)/0$ and since $r(L) \leq c$, by Lemma 1, $a \ll c/0$, as desired. □

Lemma 5. *Let L be a lattice, $a, b \in L$ and $a \leq b$. If $a \ll_r b/0$, then $a \ll_r L$.*

Proof. Since $a \ll_r b/0$, $a \ll r(b/0)/0$. Then because of $r(b/0) \leq r(L)$, by Lemma 1, $a \ll r(L)/0$. Hence $a \ll_r L$, as desired. □

Proposition 4. *Let L be a lattice, $a, b \in L$ and $a \leq b$. If $b \ll_r L$, then $a \ll_r L$.*

Proof. Since $b \ll_r L$, $b \ll r(L)/0$. Since $a \leq b$, by Lemma 1, $a \ll r(L)/0$. Hence $a \ll_r L$, as desired. □

Lemma 6. *Let L be a lattice and $a, b \in L$. If $a \ll_r L$, then $a \vee b \ll_r 1/b$.*

Proof. Since $a \ll_r L$, $a \ll r(L)/0$ and by Lemma 1, $a \vee b \ll (r(L) \vee b)/b$. Then by $r(L) \vee b \leq r(1/b)$ and Lemma 1, $a \vee b \ll r(1/b)/b$. Hence $a \vee b \ll_r 1/b$, as desired. □

Lemma 7. *Let L be a lattice and $a, b \in L$. If $b \ll_r L$ and $a \vee b \ll_r 1/b$, then $a \ll_r L$.*

Proof. Since $b \ll_r L$, $b \ll r(L)/0$. Then $b \leq r(L)$. Here we can see that $r(1/b) = r(L)$. Since $a \vee b \ll_r 1/b$, $a \vee b \ll r(1/b)/b = r(L)/b$. Let $a \vee t = r(L)$ for $t \in r(L)/0$. Here $(a \vee b) \vee (b \vee t) = r(L)$ and since $a \vee b \ll r(L)/b$, $b \vee t = r(L)$. Since $b \ll r(L)/0$, $t = r(L)$. Hence $a \ll r(L)/0$ and $a \ll_r L$. □

Corollary 4. *Let L be a lattice, $a \in L$ and $b \ll_r L$. Then $a \ll_r L$ if and only if $a \vee b \ll_r 1/b$.*

Proof. Clear from Lemma 6 and Lemma 7. □

Lemma 8. *Let L be a lattice and $a, b, c, d \in L$. If $a \ll_r b/0$ and $c \ll_r d/0$, then $a \vee c \ll_r (b \vee d)/0$.*

Proof. Since $a \ll_r b/0$ and $b \leq b \vee d$, by Lemma 5, $a \ll_r (b \vee d)/0$ and $a \ll r((b \vee d)/0)/0$. Similarly we can see that $c \ll r((b \vee d)/0)/0$. Since $a \ll r((b \vee d)/0)/0$ and $c \ll r((b \vee d)/0)/0$, by Lemma 1, $a \vee c \ll r((b \vee d)/0)/0$ and $a \vee c \ll_r (b \vee d)/0$, as desired. □

Corollary 5. *Let L be a lattice and $a, b \in L$. If $a \ll_r L$ and $b \ll_r L$, then $a \vee b \ll_r L$.*

Proof. Clear from Lemma 8. □

Corollary 6. *Let L be a lattice and $a_1, a_2, \ldots, a_n \in L$. If $a_i \ll_r L$ for every $i = 1, 2, \ldots, n$, then $a_1 \vee a_2 \vee \cdots \vee a_n \ll_r L$.*

Proof. Clear from Corollary 5. □

Proposition 5. *Let L be a lattice and $a \ll_r L$. Then $a \leq r(r(L)/0)$.*

Proof. Since $a \ll_r L$, $a \ll r(L)/0$. Then by Lemma 1, $a \leq r(r(L)/0)$, as desired. □

Lemma 9. *Let L be a lattice and $a \leq r(r(L)/0)$. If a is compact in $r(L)/0$, then $a \ll_r L$.*

Proof. Since a is compact in $r(L)/0$ and $a \leq r(r(L)/0)$, by Lemma 1, $a \ll r(L)/0$. Then $a \ll_r L$, as desired. □

Corollary 7. *Let L be a lattice and $a \leq r(r(L)/0)$. If a is compact in L, then $a \ll_r L$.*

Proof. Clear from Lemma 9. □

Lemma 10. *Let L be a lattice and $r(L)/0$ is compactly generated. Then*

$$r(r(L)/0) = \bigvee_{a \ll_r L} a.$$

Proof. Since $r(L)/0$ is compactly generated, by Lemma 1(xi),

$$r(r(L)/0) = \bigvee_{a \ll r(L)/0} a.$$

Hence $r(r(L)/0) = \bigvee_{a \ll_r L} a$, as desired. □

Corollary 8. *Let L be a compactly generated lattice. Then $r(r(L)/0) = \bigvee_{a \ll_r L} a$.*

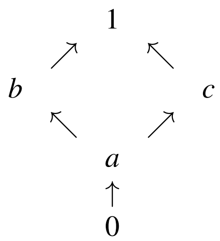
Proof. Clear from Lemma 10. □

Proposition 6. *Let L be a lattice. If $r(L)$ is compact, then $r(r(L)/0) \ll_r L$.*

Proof. Clear from Lemma 1(xii). □

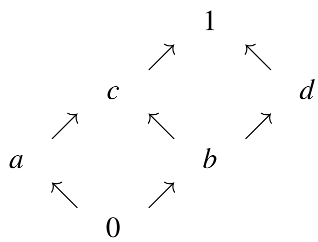
Remark 1. Let L be a compact lattice and $r(L) \neq 0$. Then $r(L) \ll L$, but not $r(L) \ll_r L$.

Example 1. Consider the lattice $L = \{0, a, b, c, 1\}$ given by the following diagram.



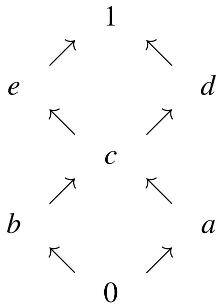
Here $a \ll L$, but not $a \ll_r L$.

Example 2. Consider the lattice $L = \{0, a, b, c, d, 1\}$ given by the following diagram.



Here $r(L) = b \ll L$, but not $b \ll_r L$.

Example 3. Consider the lattice $L = \{0, a, b, c, d, e, 1\}$ given by the following diagram.



In this lattice $r(L) = c$. Here $a \ll L$, but not $a \ll_r L$. Here also $b \ll L$, but not $b \ll_r L$.

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SOME APPROXIMATIONS ON A SYSTEM OF MULTI-QUADRATIC-QUARTIC FUNCTIONAL EQUATIONS

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Abstract. In the current work, we define multi-quadratic-quartic mappings as a system of k quadratic and $n-k$ quartic functional equations and then present them as an equation. In continuation, we establish the (Hyers-Ulam, Rassias and Găvruţa) stability and hyperstability of the mentioned mappings, by applying the direct (Hyers) method in the setting of Banach spaces. Using a characterization result, we illustrate an example for the case that a multi-quadratic-quartic mapping in the singularity case can not be stable.

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Keywords: Banach space, Hyers-Ulam stability, multi-quadratic mapping, multi-quartic mapping, multi-quadratic-quartic mapping

1. INTRODUCTION

Let $(\mathcal{G},+)$ be a commutative group, W be a linear space, and $n \geq 2$ be a natural number. Throughout this paper, for a set X, we denote $\overbrace{X \times X \times \cdots \times X}^{n-\text{times}}$ by X^n. A multivariable mapping $f\colon \mathcal{G}^n \longrightarrow W$ is called *multi-quadratic* [17] if f satisfies the quadratic functional equation

$$Q(g_1+g_2)+Q(g_1-g_2)=2Q(g_1)+2Q(g_2) \tag{1.1}$$

in each component (see also [11]). Moreover, f is defined as *multi-quartic* if it satisfies the quartic functional equation

$$\begin{aligned}\mathfrak{Q}(2g_1+g_2)&+\mathfrak{Q}(2g_1-g_2)\\ &=4\mathfrak{Q}(g_1+g_2)+4\mathfrak{Q}(g_1-g_2)+24\mathfrak{Q}(g_1)-6\mathfrak{Q}(g_2).\end{aligned} \tag{1.2}$$

in each variable. For more details we refer to [1] and [16].

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Zhao et al. [21] proved that a mapping $f\colon \mathcal{G}^n \longrightarrow W$ is multi-quadratic if and only if the relation

$$\sum_{s\in\{-1,1\}^n} f(\vartheta_1+s\vartheta_2)=2^n \sum_{j_1,\ldots,j_n\in\{1,2\}} f(\vartheta_{j_1 1},\ldots,\vartheta_{j_n n})$$

is valid, where $\vartheta_j=(\vartheta_{j1},\ldots,\vartheta_{jn})\in\mathcal{G}^n$ with $j\in\{1,2\}$. Then, various versions of multi-quadratic mappings (with unification each of them as an equation) were studied in [5, 8, 12, 19]. Moreover, Abbasbeygi et al. [1] presented a characterization of multi-quartic mappings. In fact, they illustrated that every multi-quartic mapping can be described as a equation and vice versa; see also [7] for a different class.

Let us recall that the stability theory has been pioneered by Ulam [20] concerning a question stability of of homomorphisms on groups. Hyers [15] reacted positively to the mentioned query for more groups, assuming that Banach spaces are the groups and homomorphisms are the linear mappings. Recall that a functional equation $\mathfrak{F}$ is said to be *stable* if any approximate solution φ of $\mathfrak{F}$ is near to an exact solution. If φ is an exact solution of $\mathfrak{F}$, then $\mathfrak{F}$ is *hyperstable* [9]. Next, Aoki [3] (resp. Th. M. Rassias [18]) solved the Ulam problem for additive mappings (resp. linear mappings) by an unbounded Cauchy difference. After that, many Hyers-Ulam stability problems for various functional equations and mappings were introduced and investigated by authors; see for instance [11, 13, 14] and other resources. Some stability results for multi-quadratic and multi-quartic mappings in various spaces are available in [7, 10, 17, 19].

Motivated by the discussion above, in this study, we define multi-quadratic-quartic mappings and investigate their structure. In other words, we provide a characterization of such mappings. In fact, we show that every multi-quadratic-quartic mapping can be presented as an equation. we bring some Hyers-Ulam, Rassias and Găvruţa stability results for multi-quadratic-quartic mappings in Banach spaces through the direct method. Finally, we bring an example to show that a multi-quadratic mapping is non-stable in the case of singularity.

2. Presentation of multi-quadratic-quartic mappings

Throughout this paper, for any $t\in\mathbb{N}_0=\mathbb{N}\bigcup\{0\}, n\in\mathbb{N}$, $s=(s_1,\cdots,s_n)\in\{-1,1\}^n$ and $v=(v_1,\cdots,v_n)\in V^n$, where V is a linear space. We write $tv:=(tv_1,\ldots,tv_n)$ and $sv:=(s_1v_1,\ldots,s_nv_n)$.

Definition 1. Given $n\in\mathbb{N}$ and $k\in\{0,\cdots,n\}$. Suppose that V and W are linear spaces. A multivariable mapping $f\colon V^n\longrightarrow W$ is called k-quadratic and $n-k$-quartic (briefly, multi-quadratic-quartic) if $v\mapsto f^{\#}_j(v)$ satisfies (1.1) for all $j\in\{1,\ldots,k\}$ and fulfills (1.2) for all $j\in\{k+1,\ldots,n\}$, where $f^{\#}(v)=f(u_1,\ldots,u_{j-1},v,u_{j+1},\ldots,u_n)$, in which $u_1,\ldots,u_{j-1},u_{j+1},\ldots,u_n$ are fixed and arbitrary elements in V.

It is clear that for $k=n$ (resp. $k=0$), we arrive at the multi-quadratic (resp. the multi-quartic) mappings. It is easily verified that the mapping $f\colon \mathbb{R}^n \longrightarrow \mathbb{R}$ defined through $f(u_1,\cdots,u_n)=\prod_{j=1}^{k}\prod_{i=k+1}^{n}u_j^2u_i^4$ is multi-quadratic-quartic.

In this section, we assume that V and W are vector spaces over the field $\mathbb{Q}$. Moreover, we consider $v^{[n]}=(v_1,\ldots,v_n)\in V^n$ with $\left(v^{[k]},v^{[n-k]}\right)\in V^k\times V^{n-k}$, where $v^{[k]}:=(v_1,\ldots,v_k)$ and $v^{[n-k]}:=(v_{k+1},\ldots,v_n)$. Put $v_i^{[k]}=(v_{i1},\ldots,v_{ik})\in V^k$ and $v_i^{[n-k]}$ $=(v_{i,k+1}\ldots,v_{in})\in V^{n-k}$ where $i\in\{1,2\}$. Furthermore, for $v_1^{[n]},v_2^{[n]}\in V^n$, we set

$$\mathcal{A}=\left\{\mathfrak{A}_n=(A_{k+1},\ldots,A_n)|\, A_j\in\{v_{1j}\pm v_{2j},v_{1j},v_{2j}\}\right\},$$

where $j\in\{k+1,\ldots,n\}$. Let $p_i\in\mathbb{N}_0$ with $0\le p_i\le n$ and $i\in\{1,2\}$. Consider $\mathcal{A}_{(p_1,p_2)}^{n-k}$ of $\mathcal{A}$ as follows:

$$\mathcal{A}_{(p_1,p_2)}^{n-k}:=\left\{\mathfrak{A}_n\in\mathcal{A}|\;\mathrm{Card}\{A_j:A_j=v_{ij}\}=p_i,\; i\in\{1,2\},j\in\{k+1,\ldots,n\}\right\}.$$

where Card$X$ is the cardinality of the set X.

From now on, we use the following notations:

$$f\left(\mathcal{A}_{(p_1,p_2)}^{n-k}\right):=\sum_{\mathfrak{A}_n\in\mathcal{A}_{(p_1,p_2)}^{n-k}}f(\mathfrak{A}_n)$$

and

$$f\left(u_1,\ldots,u_k,\mathcal{A}_{(p_1,p_2)}^{n-k}\right):=\sum_{\mathfrak{A}_n\in\mathcal{A}_{(p_1,p_2)}^{n-k}}f\left(u_1,\ldots,u_k,\mathfrak{A}_n\right). \tag{2.1}$$

Proposition 1. *If a mapping $f\colon V^n\longrightarrow W$ is k-quadratic and $n-k$-quartic mapping, then f satisfies the equation*

$$\sum_{s\in\{-1,1\}^k}\sum_{t\in\{-1,1\}^{n-k}}f\left(v_1^{[k]}+sv_2^{[k]},2v_1^{[n-k]}+tv_2^{[n-k]}\right)$$
$$=2^k\sum_{p_1=0}^{n-k}\sum_{p_2=0}^{n-k-p_1}\sum_{j_1,\ldots,j_k\in\{1,2\}}4^{n-k-p_1-p_2}24^{p_1}(-6)^{p_2}f\left(v_{j_11},\ldots,v_{j_kk},\mathcal{A}_{(p_1,p_2)}^{n-k}\right), \tag{2.2}$$

for all $v_i^{[k]}=(v_{i1},\ldots,v_{ik})\in V^k$, $v_i^{[n-k]}=(v_{i,k+1}\ldots,v_{in})\in V^{n-k}$ with $i\in\{1,2\}$, where $f\left(v_{j_11},\ldots,v_{j_kk},\mathcal{A}_{(p_1,p_2)}^{n-k}\right)$ is defined in (2.1).

Proof. It is known that for $k\in\{0,n\}$, our result concludes from [21, Theorem 3] and [1, Proposition 2.2] and hence it is assumed that $k\in\{1,\ldots,n-1\}$. Let $v^{[n-k]}\in V^{n-k}$ be a fixed and arbitrary element. Define the mapping $\Phi_{v^{[n-k]}}: V^k\longrightarrow W$ through $\Phi_{v^{[n-k]}}\left(v^{[k]}\right):=f\left(v^{[k]},v^{[n-k]}\right)$ for $v^{[k]}\in V^k$. By hypothesis, $\Phi_{v^{[n-k]}}$ is k-quadratic, and [21, Theorem 3] necessitates that

$$\sum_{s\in\{-1,1\}^k}\Phi_{v^{[n-k]}}\left(v_1^{[k]}+sv_2^{[k]}\right)$$

$$= 2^k \sum_{j_1,\ldots,j_k\in\{1,2\}} \Phi_{v^{[n-k]}}\left(v_{j_1 1},\cdots,v_{j_k k}\right), \qquad \left(v_1^{[k]},v_2^{[k]}\in V^k\right).$$

By the definition of Φ, the above equality implies that

$$\sum_{s\in\{-1,1\}^k} f\left(v_1^{[k]}+sv_2^k, v^{[n-k]}\right) = 2^k \sum_{j_1,\ldots,j_k\in\{1,2\}} f\left(v_{j_1 1},\cdots,v_{j_k k},v^{[n-k]}\right) \tag{2.3}$$

for all $v_1^{[k]},v_2^{[k]}\in V^k$ and $v^{[n-k]}\in V^{n-k}$. Similar to the above, for any $v^{[k]}\in V^k$, define the mapping $h_{v^{[k]}}\colon V^{n-k}\longrightarrow W$ via $\Psi_{v^{[k]}}\left(v^{[n-k]}\right):=f\left(v^{[k]},v^{[n-k]}\right)$, $v^{[n-k]}\in V^{n-k}$ which is $n-k$-quartic and so we figure out from [1, Proposition 2.2] that

$$\sum_{t\in\{-1,1\}^{n-k}} \Psi_{v^{[k]}}\left(2v_1^{[n-k]}+tv_2^{[n-k]}\right) = \sum_{p_1=0}^{n-k}\sum_{p_2=0}^{n-k-p_1} 4^{n-k-p_1-p_2}24^{p_1}(-6)^{p_2}\Psi_{v^{[k]}}\left(\mathcal{A}^{n-k}_{(p_1,p_2)}\right) \tag{2.4}$$

for all $v_1^{[n-k]},v_2^{[n-k]}\in V^{n-k}$. It follows from (2.4) that

$$\sum_{t\in\{-1,1\}^{n-k}} f\left(v^{[k]},2v_1^{[n-k]}+tv_2^{[n-k]}\right) = \sum_{p_1=0}^{n-k}\sum_{p_2=0}^{n-k-p_1} 4^{n-k-p_1-p_2}24^{p_1}(-6)^{p_2} f\left(v^{[k]},\mathcal{A}^{n-k}_{(p_1,p_2)}\right) \tag{2.5}$$

for all $v_1^{[n-k]},v_2^{[n-k]}\in V^{n-k}$ and $v^{[k]}\in V^k$. Plugging (2.3) and (2.5), we get

$$\begin{aligned}
&\sum_{s\in\{-1,1\}^k}\sum_{t\in\{-1,1\}^{n-k}} f\left(v_1^{[k]}+sv_2^{[k]},2v_1^{[n-k]}+tv_2^{[n-k]}\right)\\
&=\sum_{p_1=0}^{n-k}\sum_{p_2=0}^{n-k-p_1} 4^{n-k-p_1-p_2}24^{p_1}(-6)^{p_2} f\left(v_1^{[k]}+sv_2^{[k]},\mathcal{A}^{n-k}_{(p_1,p_2)}\right)\\
&=2^k\sum_{p_1=0}^{n-k}\sum_{p_2=0}^{n-k-p_1} 4^{n-k-p_1-p_2}24^{p_1}(-6)^{p_2}\sum_{j_1,\ldots,j_k\in\{1,2\}} f\left(v_{j_1 1},\ldots,v_{j_k k},\mathcal{A}^{n-k}_{(p_1,p_2)}\right)
\end{aligned}$$

for all $v_i^{[k]}=(v_{i1},\ldots,v_{ik})\in V^k$ and $v_i^{[n-k]}=(v_{ik+1}\ldots,v_{in})\in V^{n-k}$. $\square$

Definition 2. We say a mapping $f\colon V^n\longrightarrow W$ satisfies (has) the *quartic condition* in the jth variable if the mapping $f_j^\bullet\colon V\longrightarrow W$ defined by

$$f_j^\bullet(u)=f(u_1,\ldots,u_{j-1},u,u_{j+1},\ldots,u_n)$$

that has the property $f_j^\bullet(2u)=2^4 f_j^\bullet(u)$ for all $j\in\{1,\ldots,n\}$, where $u_1,\ldots,u_{j-1},u_{j+1},\ldots,u_n$ are fixed and arbitrary elements in V.

In the following result, we show that if every mapping f satisfies (2.2), then it is a multi-quadratic-quartic under the condition which is given in Definition 2.

Proposition 2. *Given a mapping $f\colon V^n \longrightarrow W$. Suppose that f satisfies equation (2.2) and moreover has the assumption $(H1)$ as follows:*

$(H1)$**:** *f has the quartic condition in the last $n-k$ variables.*

Then, f is a multi-quadratic-quartic mapping.

Proof. We first note that similar to proof of [4, Lemma 2.5], one can show that $(H2)$ is true, where

$(H2)$**:** $f\left(v^{[n]}\right)=0$ for any $v^{[n]}\in V^n$ provided that at least one component of $v^{[n]}$ is zero.

By putting $v_2^{[n-k]}=(\overbrace{0,\dots,0}^{n-k-\text{times}})$ in (2.2) and using the hypothesis, the left side of (2.2) will be

$$\begin{aligned}2^{n-k}\times 2^{4(n-k)}&\sum_{s\in\{-1,1\}^k} f\left(v_1^{[k]}+sv_2^{[k]},v_1^{[n-k]}\right)\\&=2^{5(n-k)}\sum_{s\in\{-1,1\}^k} f\left(v_1^{[k]}+sv_2^{[k]},v_1^{[n-k]}\right),\end{aligned}\tag{2.6}$$

for all $v_1^{[k]},v_2^{[k]}\in V^k$ and $v_1^{[n-k]}\in V^{n-k}$. Moreover, under the above replacement, the right side of (2.2) converts to

$$\begin{aligned}&2^k\sum_{p_1=0}^{n-k}2^{n-k-p_1}4^{n-k-p_1}24^{p_1}\sum_{j_1,\dots,j_k\in\{1,2\}} f\left(v_{j_11},\dots,v_{j_kk},v_1^{[n-k]}\right)\\&=2^k\sum_{p_1=0}^{n-k}8^{n-k-p_1}24^{p_1}\sum_{j_1,\dots,j_k\in\{1,2\}} f\left(v_{j_11},\dots,v_{j_kk},v_1^{[n-k]}\right)\\&=2^{5n-4k}\sum_{j_1,\dots,j_k\in\{1,2\}} f\left(v_{j_11},\dots,v_{j_kk},v_1^{[n-k]}\right),\end{aligned}\tag{2.7}$$

for all $v_1^{[k]},v_2^{[k]}\in V^k$ and $x_1^{[n-k]}\in V^{n-k}$. It follows from (2.6) and (2.7) that

$$\sum_{s\in\{-1,1\}^k} f\left(v_1^{[k]}+sv_2^{[k]},v_1^{[n-k]}\right)=2^k\sum_{j_1,\dots,j_k\in\{1,2\}} f\left(v_{j_11},\dots,v_{j_kk},v_1^{[n-k]}\right),$$

for all $v_1^{[k]},v_2^{[k]}\in V^k$ and $v_1^{[n-k]}\in V^{n-k}$. By [21, Theorem 3], we find that f is quadratic in each of the k first components. In addition, replacing $v_2^{[k]}$ by $(0,\dots,0)$ in (2.2), we get

$$2^k\sum_{t\in\{-1,1\}^{n-k}} f\left(v_1^{[k]},2v_1^{[n-k]}+tv_2^{[n-k]}\right)$$

$$= 2^k \sum_{p_1=0}^{n-k} \sum_{p_2=0}^{n-k-p_1} 4^{n-k-p_1-p_2} 24^{p_1} (-6)^{p_2} f\left(v_1^{[k]}, \mathcal{A}_{(p_1,p_2)}^{n-k}\right),$$

for all $v_1^{[k]} \in V^k$ and $v_1^{[n-k]}, v_2^{[n-k]} \in V^{n-k}$. It follows from [1, Proposition 2.2] that f is multi-quartic in the $n-k$ last variables and now it completes the proof. □

3. STABILITY OF (2.2)

For a given mapping $f\colon V^n \longrightarrow W$, for simplicity, we use the difference notation

$$\mathfrak{D}_q f\left(v_1^{[n]}, v_2^{[n]}\right) := \sum_{s\in\{-1,1\}^k} \sum_{t\in\{-1,1\}^{n-k}} f\left(v_1^{[k]} + s v_2^{[k]}, 2v_1^{[n-k]} + t v_2^{[n-k]}\right)$$

$$- 2^k \sum_{p_1=0}^{n-k} \sum_{p_2=0}^{n-k-p_1} \sum_{j_1,\ldots,j_k\in\{1,2\}} 4^{n-k-p_1-p_2} 24^{p_1} (-6)^{p_2} f\left(v_{j_1 1}, \ldots, v_{j_k k}, \mathcal{A}_{(p_1,p_2)}^{n-k}\right),$$

for all $v_i^{[k]} = (v_{i1}, \ldots, v_{ik}) \in V^k$, $v_i^{[n-k]} = (v_{i,k+1} \ldots, v_{in}) \in V^{n-k}$ and $i \in \{1,2\}$, where $f\left(v_{j_1 1}, \ldots, v_{j_k k}, \mathcal{A}_{(p_1,p_2)}^{n-k}\right)$ is defined in (2.1). From now on, we assume that all multivariable mappings f have the condition $(H2)$.

Theorem 1. *Let V and W be a linear space and a Banach space, respectively. Let also $f\colon V^n \longrightarrow W$ be a mapping in which there exists a function $\psi\colon V^n \times V^n \longrightarrow [-\alpha, \infty)$ so that*

$$\widetilde{\psi}\left(v_1^{[n]}, v_2^{[n]}\right) := \sum_{j=0}^{\infty} \frac{1}{2^{(5n-3k)\beta j}} \psi\left(2^{\frac{\beta-1}{2}+\beta j} v_1^{[n]}, 2^{\frac{\beta-1}{2}+\beta j} v_2^{[n]}\right) < \infty, \tag{3.1}$$

$$\left\|\mathfrak{D}_q f(v_1, v_2)\right\| \leq \alpha\left(\frac{\beta+1}{2}\right) + \psi\left(v_1^{[n]}, v_2^{[n]}\right), \tag{3.2}$$

for all $v_1^{[n]}, v_2^{[n]} \in V^n$, where $\alpha \in [0,\infty)$ and $\beta \in \{-1,1\}$. Then, there exists a solution $Q\colon V^n \longrightarrow W$ of (2.2) such that

$$\left\|f\left(v^{[n]}\right) - Q\left(v^{[n]}\right)\right\|$$
$$\leq \frac{1}{2^{\frac{\beta+1}{2}(5n-3k)}} \left[\frac{a^{(5n-3k)\beta}\alpha}{(a^{(5n-3k)\beta} - 1)}\left(\frac{\beta+1}{2}\right) + \widetilde{\psi}\left(v_1^{[n]}, v_1^{[k]}, \mathbf{0}\right)\right], \tag{3.3}$$

for all $v_1^{[n]} := v^{[n]} \in V^n$, where $\mathbf{0} = (\overbrace{0,\ldots,0}^{n-k-times})$. Moreover, if Q has assumption $(H1)$, then it is a unique multi-quadratic-quartic mapping satisfying (3.3).

Proof. Putting $v_1^{[k]} = v_2^{[k]}$ and $v_2^{[n-k]} = \mathbf{0}$ in (3.2), we have

$$\left\|f\left(2v_1^{[n]}\right) - Sf\left(v_1^{[n]}\right)\right\| \leq \alpha\left(\frac{\beta+1}{2}\right) + \psi\left(v_1^{[n]}, v_1^{[k]}, \mathbf{0}\right), \tag{3.4}$$

for all $v_1^{[n]} \in V^n$ in which

$$S = 2^{2k} \sum_{p_1=0}^{n-k} \binom{n}{p_1} 2^{n-k-p_1} 4^{n-k-p_1} 24^{p_1} f\left(v_1^{[n]}\right).$$

It is not hard to show that $S = 2^{5(n-k)}$. For the rest of proof, we set $v_1^{[n]}$ by $v^{[n]}$ unless otherwise stated explicitly. It concludes from relation (3.4) that

$$\left\| f\left(2v^{[n]}\right) - 2^{5n-3k} f\left(v^{[n]}\right) \right\| \leq \alpha\left(\frac{\beta+1}{2}\right) + \psi\left(v^{[n]}, v^{[k]}, \mathbf{0}\right),$$

and so the equation above can be rewritten as

$$\left\| \frac{f\left(a^{\beta} v^{[n]}\right)}{2^{(5n-3k)\beta}} - f\left(v^{[n]}\right) \right\| \leq \frac{1}{2^{\frac{\beta+1}{2}(5n-3k)}} \left[\alpha\left(\frac{\beta+1}{2}\right) + \phi\left(2^{\frac{\beta-1}{2}} v^{[n]}, 2^{\frac{\beta-1}{2}} v^{[k]}, \mathbf{0}\right)\right], \tag{3.5}$$

for all $v^{[n]} \in V^n$. Replacing v by $2^{\beta} v$ in (3.5), one can obtain

$$\left\| \frac{f\left(2^{\beta m} v^{[n]}\right)}{2^{(5n-3k)m\beta}} - f\left(v^{[n]}\right) \right\| \leq \frac{1}{2^{\frac{\beta+1}{2}(5n-3k)}} \left[\frac{\beta+1}{2} \sum_{j=0}^{m-1} \frac{\alpha}{2^{(5n-3k)\beta j}} + \sum_{j=0}^{m-1} \frac{\psi\left(2^{\frac{\beta-1}{2}+j\beta} v^{[n]}, 2^{\frac{\beta-1}{2}+j\beta} v^{[k]}, \mathbf{0}\right)}{2^{(5n-3k)\beta j}}\right], \tag{3.6}$$

for all $v^{[n]} \in V^n$. On the other hand, we can use induction to find

$$\left\| \frac{f\left(2^{\beta m} v^{[n]}\right)}{2^{(5n-3k)m\beta}} - \frac{f\left(2^{\beta l} v^{[n]}\right)}{2^{(5n-3k)l\beta}} \right\| \leq \frac{1}{2^{\frac{\beta+1}{2}(5n-3k)}} \left[\frac{\beta+1}{2} \sum_{j=l}^{m-1} \frac{\alpha}{2^{(5n-3k)\beta j}} + \sum_{j=l}^{m-1} \frac{\psi\left(2^{\frac{\beta-1}{2}+j\beta} v^{[n]}, 2^{\frac{\beta-1}{2}+j\beta} v^{[k]}, \mathbf{0}\right)}{2^{(5n-3k)\beta j}}\right], \tag{3.7}$$

for all $v^{[n]} \in V^n$, and $m > l \geq 0$. By applying (3.1) and (3.7), we deduce that the sequence $\left\{\frac{f\left(2^{\beta m} v^{[n]}\right)}{2^{(5n-3k)m\beta}}\right\}$ is Cauchy. The completeness of W necessitates that there exists a mapping $Q\colon V^n \longrightarrow W$ so that

$$\lim_{m\to\infty} \frac{f\left(2^{\beta m} v^{[n]}\right)}{2^{(5n-3k)m\beta}} = Q\left(v^{[n]}\right), \qquad \left(v^{[n]} \in V^n\right). \tag{3.8}$$

By letting $m \to \infty$ in (3.6) and using (3.8), the validity of inequality (3.3) is now shown. By switching $v_1^{[n]}, v_2^{[n]}$ into $2^m v_1^{[n]}, 2^m v_2^{[n]}$, respectively in (3.2) and dividing to

$2^{(5n-3k)m\beta}$, we get

$$\frac{1}{2^{(5n-3k)m\beta}}\left\|\mathfrak{D}_q f\left(2^{\beta m}v_1^{[n]},2^{\beta m}v_2^{[n]}\right)\right\| \leq \frac{\alpha}{2^{(5n-3k)m\beta}}\left(\frac{\beta+1}{2}\right)+\frac{\psi\left(2^{\beta m}v_1^{[n]},2^{\beta m}v_2^{[n]}\right)}{2^{(5n-3k)m\beta}}.$$

Taking the limit as $m\to\infty$ in the last relation, and applying (3.1) and (3.8) we obtain

$$\mathfrak{D}_q Q\left(v_1^{[n]},v_2^{[n]}\right)=0, \qquad \left(v_1^{[n]},v_2^{[n]}\in V^n\right)$$

and so Q is a solution of (2.2). Assume now that Q has $(H1)$, then it is a multi-quadratic-quartic mapping by Proposition 2. Let now $Q'\colon V^n\longrightarrow W$ be another multi-quartic-quadratic mapping with (3.3). Then, we find

$$\begin{aligned}
&\left\|Q\left(v^{[n]}\right)-Q'\left(v^{[n]}\right)\right\|\\
&=\frac{1}{2^{(5n-3k)m\beta}}\left\|C\left(2^{\beta m}v^{[n]}\right)-Q'\left(2^{\beta m}v^{[n]}\right)\right\|\\
&\leq\frac{1}{2^{(5n-3k)m\beta}}\left(\left\|Q\left(2^{\beta m}v^{[n]}\right)-f\left(2^{\beta m}v^{[n]}\right)\right\|+\left\|f\left(2^{\beta m}v^{[n]}\right)-Q'\left(2^{\beta m}v^{[n]}\right)\right\|\right)\\
&\leq\frac{2}{2^{(5n-3k)m\beta}}\left[\frac{2^{(5n-3k)\beta}\alpha}{2^{(5n-3k)\beta}-1}\left(\frac{\beta+1}{2}\right)+\widetilde{\psi}\left(2^{\beta m}v^{[n]},2^{\beta m}v^{[k]},\mathbf{0}\right)\right]\\
&=\frac{2}{2^{(5n-3k)m\beta}}\left[\frac{2^{(5n-3k)\beta}\alpha}{2^{(5n-3k)\beta}-1}\left(\frac{\beta+1}{2}\right)\right.\\
&\qquad\left.+\sum_{j=0}^{\infty}\frac{1}{2^{(5n-3k)\beta j}}\psi\left(2^{\frac{\beta-1}{2}+(j+m)\beta}v^{[n]},2^{\frac{\beta-1}{2}+(j+m)\beta}v^{[k]},\mathbf{0}\right)\right]\\
&=\frac{2}{2^{(5n-3k)m\beta}}\left[\frac{2^{(5n-3k)\beta}\alpha}{2^{(5n-3k)\beta}-1}\left(\frac{\beta+1}{2}\right)\right.\\
&\qquad\left.+2^{(5n-3k)m\beta}\sum_{j=m}^{\infty}\frac{1}{2^{(5n-3k)\beta j}}\psi\left(2^{\frac{\beta-1}{2}+j\beta}v^{[n]},2^{\frac{\beta-1}{2}+j\beta}v^{[k]},\mathbf{0}\right)\right],
\end{aligned}$$

for all $v^{[n]}\in V^n$. Taking $m\to\infty$ in the above inequality, we have $Q=Q'$ and hence it is shown the uniqueness of solution. □

In the next results, it is assumed that V is a normed space and W is a Banach space. The upcoming (Rassias) corollary is a consequence of Theorem 1, which investigates the Hyers-Ulam stability of (2.2).

Corollary 1. *Given $\alpha,\delta,r\in\mathbb{R}$ with $r\neq 5n-3k$ and $\delta,\alpha\in[0,\infty)$. Moreover, $r_{ij}>0$ with $\sum_{i=1}^{2}\sum_{j=1}^{n} r_{ij}\neq 5n-3k$. Let a mapping $f\colon V^n\longrightarrow W$ satisfies*

$$\|\mathfrak{D}_q f\left(v_1^{[n]},v_2^{[n]}\right)\|\leq \alpha+\delta\sum_{i=1}^{2}\sum_{j=1}^{n}\|v_{ij}\|^r+\prod_{i=1}^{2}\prod_{j=1}^{n}\|v_{ij}\|^{r_{ij}},$$

for all $v_1^{[n]},v_2^{[n]}\in V^n$. Then, there exists a mapping $Q\colon V^n\longrightarrow W$ as a solution of (2.2) such that

$$\left\|f\left(v^{[n]}\right)-Q\left(v^{[n]}\right)\right\|$$
$$\leq\begin{cases}\frac{\alpha}{2^{5n-3k}-1}+\frac{\delta}{2^{5n-3k}-2^r}\left(\sum_{j=1}^{n}\|v_{1j}\|^r+\sum_{j=1}^{k}\|v_{1j}\|^r\right) & r\in(0,5n-3k),\\ \frac{\delta}{2^r-2^{5n-3k}}\left(\sum_{j=1}^{n}\|v_{1j}\|^r+\sum_{j=1}^{k}\|v_{1j}\|^r\right) & r\in(5n-3k,\infty),\end{cases}\tag{3.9}$$

for all $v^{[n]}:=v_1^{[n]}\in V^n$. Furthermore, if Q has $(H1)$, then it is a unique multi-quadratic-quartic mapping fulfilling (3.9).

Proof. Setting

$$\psi\left(v_1^{[n]},v_2^{[n]}\right)=\delta\sum_{i=1}^{2}\sum_{j=1}^{n}\|v_{ij}\|^r+\prod_{i=1}^{2}\prod_{j=1}^{n}\|v_{ij}\|^r$$

in Theorem 1, we reach to the desired results. □

Remark 1. Considering $\beta=1$ and putting $\psi:=0$ in Theorem 1, we conclude that there exists a mapping $Q\colon V^n\longrightarrow W$ as a solution of (2.2) such that

$$\left\|f\left(v^{[n]}\right)-Q\left(v^{[n]}\right)\right\|\leq\frac{\alpha}{2^{5n-3k}-1},$$

for all $v^{[n]}\in V^n$. In addition, if Q has $(H1)$, then it is a unique multi-quadrati-quartic mapping satisfies the last inequality. In the case that $n=k$, with the above assumptions, there exists a multi-quadratic mapping $Q\colon V^n\longrightarrow W$ such that

$$\left\|f\left(v^{[n]}\right)-Q\left(v^{[n]}\right)\right\|\leq\frac{\alpha}{2^{2n}-1},$$

for all $v^{[n]}\in V^n$. Furthermore, in the case that $k=0$ and Q has the quartic condition in each component, there exists a multi-quartic mapping $Q\colon V^n\longrightarrow W$ such that

$$\left\|f\left(v^{[n]}\right)-Q\left(v^{[n]}\right)\right\|\leq\frac{\alpha}{2^{5n}-1},$$

for all $v^{[n]}\in V^n$. Furthermore, if $\alpha=\delta=0$ in Theorem 1, then f satisfies (2.2). Furthermore, if it has assumption $(H1)$, then it is a multi-quadratic-quartic mapping.

The next merged proposition was proved in [2] and [6]. We use it to constract a counterexample.

Proposition 3. *Given a function* $\rho\colon \mathbb{R}^n \longrightarrow \mathbb{R}$. *If it is continuous n-quadratic (resp. n-quartic), then* ρ *has the form*

$$\rho(r_1,\ldots,r_n) = c\prod_{j=1}^{n} r_j^2 \quad \left(\text{resp. } \rho(r_1,\ldots,r_n) = c\prod_{j=1}^{n} r_j^4\right),$$

where c is a constant in $\mathbb{R}$.

Applying the above proposition, we have the following characterization.

Theorem 2. *Let* $\rho\colon \mathbb{R}^n \longrightarrow \mathbb{R}$ *be a continuous k-quadratic and* $n-k$*-quartic function. Then, it has the form*

$$\rho(r_1,\ldots,r_n) = c\prod_{j=1}^{k}\prod_{l=k+1}^{n} r_j^2 r_l^4,$$

for all $r_1,\ldots,r_n \in \mathbb{R}$, *where* $c \in \mathbb{R}$ *is a constant.*

Proof. We firstly identify $r = (r_1,\ldots,r_n) \in \mathbb{R}^n$ with $\left(r^{[k]}, r^{[n-k]}\right) \in \mathbb{R}^k \times \mathbb{R}^{n-k}$, where $r^{[k]} := (r_1,\ldots,r_k)$ and $r^{[n-k]} := (r_{k+1},\ldots,r_n)$. For any $r^{[n-k]} \in \mathbb{R}^{n-k}$, consider the mapping $\mathcal{T}_{r^{[n-k]}} : \mathbb{R}^k \longrightarrow \mathbb{R}$ defined by

$$\mathcal{T}_{r^{[n-k]}}(r_1,\ldots,r_k) := \rho\left(r_1,\ldots,r_k, r^{[n-k]}\right).$$

By assumption, $\mathcal{T}_{r^{[n-k]}}$ is k-quadratic. It follows from Proposition 3 that there exists a constant $c_1 \in \mathbb{R}$ such that

$$\mathcal{T}_{r^{[n-k]}}(r_1,\ldots,r_k) = \rho\left(r_1,\ldots,r_k, r^{[n-k]}\right) = c_1\prod_{j=1}^{k} r_j^2. \tag{3.10}$$

Note that c_1 depends on $r^{[n-k]}$. In fact,

$$c_1 = T\left(r^{[n-k]}\right). \tag{3.11}$$

Putting $r_1 = \cdots = r_k = 1$ in (3.10) and applying (3.11), we get

$$c_1 = T(r^{[n-k]}) = \rho\left(1,\ldots,1, r^{[n-k]}\right). \tag{3.12}$$

Once again, for any $r^{[k]} \in \mathbb{R}^k$, define the mapping $\mathcal{S}_{r^{[k]}} : \mathbb{R}^{n-k} \longrightarrow \mathbb{R}$ through

$$\mathcal{S}_{r^{[k]}}(r_{k+1},\ldots,r_n) := \rho(1,\ldots,1,r_{k+1},\ldots,r_n).$$

Since $\mathcal{S}_{r^{[k]}}$ is $n-k$-quartic, by Proposition 3 there exists a constant $c_2 \in \mathbb{R}$ such that

$$\mathcal{S}_{r^{[k]}}(r_{k+1},\ldots,r_n) = \rho\left(1,\ldots,1, r^{[n-k]}\right) = c_2\prod_{l=k+1}^{n} r_l^4. \tag{3.13}$$

It is obvious that c_2 depends on $r^{[k]}$ and hence

$$c_2 = S\left(r^{[k]}\right). \tag{3.14}$$

Letting $r_{k+1}=\cdots=r_n=1$ in (3.13) and using (3.14), we get

$$c_2=\rho(\overbrace{1,\ldots,1}^{k-\text{times}},\overbrace{1,\ldots,1}^{n-k-\text{times}}). \tag{3.15}$$

The result now follows from (3.10), (3.12), (3.13) and (3.15). □

In the following, we show the assumption $r\neq 5n-3k$ is necessary and can not be eliminated in Corollary 1 when $\alpha=0$. This means that a multi-quadratic-quartic mapping can be non-stable example; see [6, Example 1] and [13]). In view of the proof of Theorem 2, we observe that by removing the continuity condition of ρ, the result remains valid when the domain of ρ is replaced by $\mathbb{Q}^n$, where $c=\rho(\overbrace{1,\ldots,1}^{n-\text{times}})$.

Example 1. Let $\delta>0$ and $n\in\mathbb{N}$. Consider the function $\mathbf{1}\colon\mathbb{Q}^n\longrightarrow\mathbb{R}$ whose range is the constant 1. Put $\mu=\frac{2^{5n-3k}-1}{2^{2(5n-3k)}S}\delta$, where S is grater than or equal to

$$\sum_{s\in\{-1,1\}^k}\sum_{t\in\{-1,1\}^{n-k}}\mathbf{1}\left(v_1^{[k]}+sv_2^{[k]},2v_1^{[n-k]}+tv_2^{[n-k]}\right)$$
$$+2^k\sum_{p_1=0}^{n-k}\sum_{p_2=0}^{n-k-p_1}\sum_{j_1,\ldots,j_k\in\{1,2\}}4^{n-k-p_1-p_2}24^{p_1}6^{p_2}\mathbf{1}\left(v_{j_11},\ldots,v_{j_kk},\mathcal{A}_{(p_1,p_2)}^{n-k}\right);$$

see the definition of $\mathfrak{D}_qf\left(v_1^{[n]},v_2^{[n]}\right)$. Consider the function $\psi\colon\mathbb{Q}^n\longrightarrow\mathbb{R}$ defined via

$$\psi(r_1,\ldots,r_n)=\begin{cases}\mu\prod_{j=1}^k\prod_{t=k+1}^n r_j^2r_t^4 & r_j \text{ with } |r_j|<1,\\ \mu & \text{otherwise.}\end{cases}$$

According the function above, consider the function $\rho\colon\mathbb{Q}^n\longrightarrow\mathbb{R}$ defined by

$$\rho(r_1,\ldots,r_n)=\sum_{l=0}^{\infty}\frac{\psi(2^lr_1,\ldots,2^lr_n)}{2^{(5n-3k)l}},\qquad (r_j\in\mathbb{R}).$$

It is clear that ρ is a non-negative and even function in all components. Moreover, ψ is bounded by μ, and so ρ is bounded by $\frac{2^{5n-3k}}{2^{5n-3k}-1}\mu$, and additionally

$$\left|\mathfrak{D}_q\rho\left(v_1^{[n]},v_2^{[n]}\right)\right|\leq\frac{2^{5n-3k}}{2^{5n-3k}-1}\mu S, \tag{3.16}$$

where $v_i^{[n]}=(v_{i1},\ldots,v_{in})\in\mathbb{Q}^n$ for $i\in\{1,2\}$. We show that

$$\left|\mathfrak{D}_q\rho(v_1,v_2)\right|\leq\delta\sum_{i=1}^2\sum_{j=1}^n|v_{ij}|^{5n-3k}, \tag{3.17}$$

for all $v_1^{[n]}, v_2^{[n]} \in \mathbb{Q}^n$. Clearly, for $v_1^{[n]} = v_2^{[n]} = \mathbf{0} := (\overbrace{0,\ldots,0}^{n-\text{times}})$, inequality (3.17) is true. Let $v_1^{[n]}, v_2^{[n]} \in \mathbb{Q}^n$ with

$$\sum_{i=1}^{2}\sum_{j=1}^{n}|v_{ij}|^{5n-3k} < \frac{1}{2^{5n-3k}}. \tag{3.18}$$

It concludes from (3.18) that there is $M \in \mathbb{N}$ such that

$$\frac{1}{2^{(5n-3k)(M+1)}} < \sum_{i=1}^{2}\sum_{j=1}^{n}|v_{ij}|^{5n-3k} < \frac{1}{2^{(5n-3k)M}}, \tag{3.19}$$

and hence $|v_{ij}|^{5n-3k} < \sum_{i=1}^{2}\sum_{j=1}^{n}|v_{ij}|^{5n-3k} < \frac{1}{2^{(5n-3k)M}}$. Now, the last relation necessitates that $2^M|v_{ij}| < 1$ for all $i = 1,2$ and $j = 1,\ldots,n$. Hence, $2^{M-1}|v_{ij}| < 1$. Let $u_1, u_2 \in \{v_{ij} | i = 1,2,\ j = 1,\ldots,n\}$. Then

$$\{2^{M-1}|u_1-u_2|, 2^{M-1}|u_1+2u_2|\} \subseteq (-1,1)$$

for all $l \in \{0,1,2,\ldots,M-1\}$. Since ψ is multi-quadratic-quartic function on $(-1,1)^n$, we find $\mathfrak{D}_q\psi\left(2^l v_1^{[n]}, 2^l v_2^{[n]}\right) = 0$ for all $l \in \{0,1,2,\ldots,M-1\}$. It follows from the last equality and (3.19) that

$$\begin{aligned}
\frac{\left|\mathfrak{D}_q\rho\left(2^l v_1^{[n]}, 2^l v_2^{[n]}\right)\right|}{\sum_{i=1}^{2}\sum_{j=1}^{n}|v_{ij}|^{5n-3k}} &\le \sum_{l=M}^{\infty}\frac{\left|\mathfrak{D}_q\psi\left(2^l v_1^{[n]}, 2^l v_2^{[n]}\right)\right|}{2^{(5n-3k)l}\sum_{i=1}^{2}\sum_{j=1}^{n}|v_{ij}|^{5n-3k}}\\
&\le \sum_{l=0}^{\infty}\frac{\mu S}{2^{(5n-3k)(l+N)}\sum_{i=1}^{2}\sum_{j=1}^{n}|v_{ij}|^{5n-3k}}\\
&\le \mu S\sum_{l=0}^{\infty}\frac{1}{2^{(5n-3k)l}} \le \mu S 2^{5n-3k}\frac{2^{5n-3k}}{2^{5n-3k}-1} = \mu S\frac{2^{2(5n-3k)}}{2^{5n-3k}-1} = \delta,
\end{aligned}$$

for all $v_1^{[n]}, v_2^{[n]} \in \mathbb{Q}^n$. Hence, (3.17) holds for the case (3.18). Assume that $\sum_{i=1}^{2}\sum_{j=1}^{n}|v_{ij}|^{5n-3k} \ge \frac{1}{2^{5n-3k}}$. A direct consequence of (3.16) shows that

$$\frac{\left|\mathfrak{D}_q\rho\left(2^l v_1^{[n]}, 2^l v_2^{[n]}\right)\right|}{\sum_{i=1}^{2}\sum_{j=1}^{n}|v_{ij}|^{5n-3k}} \le 2^{2n}\frac{2^{5n-3k}}{2^{5n-3k}-1}\mu S = \delta.$$

Thus, ρ satisfies in (3.17) for all $v_1^{[n]}, v_2^{[n]} \in \mathbb{Q}^n$. Now, suppose the assertion is false, that there exist a multi-quadratic-quartic mapping $\mathcal{Q}\colon \mathbb{Q}^n \longrightarrow \mathbb{R}$ and $\eta > 0$ such that

$$|\rho(r_1,\ldots,r_n) - \mathcal{Q}(r_1,\ldots,r_n)| \le \eta\left(\sum_{j=1}^{n}|r_j|^{5n-3k} + \sum_{l=1}^{k}|r_l|^{5n-3k}\right).$$

Since $5n-3k$ is a fixed number, one can find a number $\lambda \in [0,\infty)$ as large enough such that

$$\eta\left(\sum_{j=1}^{n} r_j^{5n-3k}+\sum_{l=1}^{k} r_l^{5n-3k}\right)\leq \lambda \prod_{j=1}^{k}\prod_{l=k+1}^{n} r_j^2 r_l^4,$$

and so

$$|\rho(r_1,\ldots,r_n)-Q(r_1,\ldots,r_n)|<\lambda \prod_{j=1}^{k}\prod_{l=k+1}^{n} r_j^2 r_l^4.$$

On the other hand, Theorem 2 and the paragraph before this example imply that there is a constant $\alpha \in \mathbb{R}$ such that $Q(r_1,\ldots,r_n)=\alpha\prod_{j=1}^{k}\prod_{l=k+1}^{n} r_j^2 r_l^4$ and hence

$$\rho(r_1,\ldots,r_n)\leq (|\alpha|+\lambda)\prod_{j=1}^{k}\prod_{l=k+1}^{n} r_j^2 r_l^4 \tag{3.20}$$

holds. In addition, take $M\in\mathbb{N}$ such that $M\mu>|\alpha|+\lambda$. Take $r=(r_1,\ldots,r_n)$ in $\mathbb{Q}^n$ in which $r_j\in\left(0,\frac{1}{2^{M-1}}\right)$ for all $j\in\{1,\ldots,n\}$, then $2^l r_j\in(0,1)$ for all $l=0,1,\ldots,M-1$. Therefore

$$\begin{aligned}\rho(r_1,\ldots,r_n)&=\sum_{l=0}^{\infty}\frac{\psi\left(2^l r_1,\ldots,2^l r_2\right)}{2^{(5n-3k)l}}\geq \mu\sum_{l=0}^{M-1}\frac{2^{(5n-3k)l}\prod_{j=1}^{k}\prod_{l=k+1}^{n} r_j^2 r_l^4}{2^{(5n-3k)l}}\\&=M\mu\prod_{j=1}^{k}\prod_{l=k+1}^{n} r_j^2 r_l^4>(|\alpha|+\lambda)\prod_{j=1}^{k}\prod_{l=k+1}^{n} r_j^2 r_l^4.\end{aligned}$$

The relation above contradicts inequality (3.20).

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EXISTENCE AND STABILITY OF MILD SOLUTIONS FOR HYBRID FRACTIONAL SEMI-LINEAR EVOLUTION EQUATIONS

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Abstract. This paper examines the existence and stability of mild solutions for initial value problems associated with hybrid fractional semi-linear evolution equations. The existence of mild solutions is established using Dhage's fixed point theorem. Additionally, we investigate four distinct types of Mittag-Leffler-Ulam-Hyers stability to analyze the behavior of solutions under perturbations. To demonstrate the applicability of our theoretical findings, we provide a concrete example. These results contribute to the advancement of fractional evolution equations and their stability theory, with potential applications in various fields of applied mathematics and engineering.

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1. INTRODUCTION

Fractional differential equations (FDEs) have garnered significant attention across various scientific fields, including economics, engineering, chemistry, aerodynamics, and the control of dynamical systems. The primary appeal of fractional calculus lies in its ability to capture the complex behaviors of systems that traditional integer-order derivatives fail to represent. By incorporating memory effects and non-local interactions, fractional models provide a more accurate depiction of phenomena, offering deeper insights into the dynamics of systems such as spring pendulums, particle motion in circular cavities, and the spread of epidemics.

The interest in fractional calculus is driven by its capacity to model real-world processes more effectively. Researchers have increasingly recognized its advantages, including the ability to uncover hidden dynamics that are not observable using integer-order derivatives. In this context, Almeida [2] extended the classical Caputo fractional derivative by introducing dependencies on an auxiliary function, which improved model adaptability and accuracy. Recent advancements have also introduced new techniques for solving FDEs, such as the generalized Laplace transform developed by Jarad and Abdeljawad [10], which simplifies the resolution of fractional differential equations in the framework of generalized Caputo derivatives. In parallel, significant progress has been made in understanding the mathematical structure of fractional differential equations. For instance, the study of commutators of fractional integral operators on vanishing-Morrey spaces has revealed new insights into the behavior of these operators [13]. Additionally, the application of fractional calculus to the Lerch zeta function has provided a deeper understanding of the connections between fractional calculus and special functions [9]. The study of hybrid fractional differential equations with fractional proportional derivatives, which combine fractional derivatives with proportional terms, has further enriched the field [1]. Furthermore, investigations into fractional calculus, zeta functions, and Shannon entropy have opened new avenues for research in applied mathematics and information theory [8]. Numerical methods also play a crucial role in solving fractional differential equations. High-order numerical techniques, such as those developed for solving two-dimensional Riesz space fractional advection-dispersion equations, have proven effective in addressing complex dynamical systems [3]. Additionally, the study of fractional derivatives in complex planes has contributed to the advancement of both theoretical and computational aspects of fractional calculus [8]. This growing body of research not only enhances our theoretical understanding of fractional calculus but also expands its applicability to a wide range of real-world problems. As the field continues to evolve, fractional differential equations are poised to provide more precise models for complex systems across multiple disciplines, offering promising solutions to long-standing scientific and engineering challenges.

Inspired by the research presented in [6], we delve into the analysis of the existence and different classifications of Ulam-Hyers stability outcomes for mild solutions for the semi-linear fractional hybrid differential equations that include the $\rho-$Caputo fractional derivative of order $0 \leq \beta \leq 1$.

$$\begin{cases} {}^{C}D_{0+}^{\beta,\rho}\left(\frac{u(t)}{g(t,u(t))}\right) = \mathcal{A}\left(\frac{u(t)}{g(t,u(t))}\right) + h(t,u(t)) \qquad t \in [0,T] \quad 0 < \beta < 1, \\ u(0) = 0. \end{cases} \tag{1.1}$$

Where $T > 0$, $\mathcal{A}$ is the infinitesimal generator of $\mathcal{C}_0-$semigroup $(T(t))_{t\geq 0}$ on a Banach space $\mathcal{X}$, $g \in \mathcal{C}([0,T] \times \mathcal{X}, \mathcal{X} \setminus \{0\})$ and $h \in \mathcal{C}_c([0,T] \times \mathcal{X}, \mathcal{X})$.

2. Preliminaries

We provide initial context that will be referenced throughout this paper.

Let $\mathcal{X}$ be a Banach space and $\mathcal{C}([0,T],\mathcal{X})$ be the Banach space of continuous functions from $[0,T]$ to $\mathcal{X}$ with the norm $||u|| = \sup_{t\in[0,T]} ||u(t)||$.

Definition 1 (ρ−Riemann-Liouville fractional integral [4, Definition 2.1])**.** Let $\beta > 0$, f be an integrable function defined on $[a,b]$ and $\rho\colon [a,b] \to \mathbb{R}$ that is an increasing differentiable function such that $\rho'(t) \neq 0$, for all $t \in [a,b]$.

The ρ-Riemann-Liouville fractional integral operator of order β of a function f is defined by

$$I_{a+}^{\beta,\rho} f(t) = \frac{1}{\Gamma(\beta)} \int_a^t \rho'(s)(\rho(t)-\rho(s))^{\beta-1} f(s)\mathrm{d}s.$$

Definition 2 (ρ-Riemann-Liouville fractional derivative [4, Definition 2.2])**.** Let $n \in \mathbb{N}$, $k,\rho \in \mathcal{C}^n([a,b])$ be two functions such that ρ is increasing with $\rho'(t) \neq 0$, for all $t \in [a,b]$.

ρ-Riemann-Liouville fractional derivative of order β of a function f is defined by

$$\mathcal{D}_{a+}^{\beta,\rho} f(t) = \frac{1}{\Gamma(n-\beta)} \left(\frac{1}{\rho'(t)}\frac{d}{dt}\right)^n \int_a^t \rho'(s)(\rho(t)-\rho(s))^{n-\beta-1} f(s)\mathrm{d}s,$$

where $n = [\beta]+1$ and $[\beta]$ denotes the integer part of β.

Definition 3 (ρ-Caputo fractional derivative [4, Definition 2.3])**.** Let $n \in \mathbb{N}$, $f,\rho \in \mathcal{C}^n([a,b])$ be two functions such that ρ is increasing with $\rho'(t) \neq 0$, for all $t \in [a,b]$.

ρ-Caputo fractional derivative of order β of a function f is defined by

$${}^C D_{a+}^{\beta,\rho} f(t) = \frac{1}{\Gamma(n-\beta)} \int_a^t \rho'(s)(\rho(t)-\rho(s))^{n-\beta-1} f_\rho^{[n]}(s)\mathrm{d}s,$$

where $n = [\beta]+1$, for $\beta \notin \mathbb{N}$. And $f_\beta^{[n]}(t) = \left(\frac{1}{\rho'(t)}\frac{\mathrm{d}}{\mathrm{d}t}\right)^n f(t)$ on $[a,b]$.

Remark 1.

(1) It is clear that when $\beta = n \in \mathbb{N}$, we have

$${}^C\mathcal{D}_{a+}^{\beta,\rho} f(t) = f_\rho^{[n]}(t).$$

(2) If $f \in \mathcal{C}^n([a,b])$ and $\beta > 0$. The relation between the two types of fractional derivatives is given by

$${}^C\mathcal{D}_{a+}^{\beta,\rho} f(t) = \mathcal{D}_{a+}^{\beta,\rho}\left(f(t) - \sum_{k=0}^{n-1} \frac{f_\rho^{[k]}(a)}{k!}(\rho(t)-\rho(a))^k\right).$$

Theorem 1. *Given $f \in \mathcal{C}^n([a,b])$ and $\beta > 0$. Then we have*

$$I_{a+}^{\beta,\rho}\, {}^C\mathcal{D}_{a+}^{\beta,\rho} f(t) = f(t) - \sum_{k=0}^{n-1} \frac{f_\rho^{[k]}(a)}{k!}(\rho(t)-\rho(a))^k.$$

In particular, if $\beta \in (0,1)$ we have:

$$I_{a+}^{\beta,\rho}\, {}^C\mathcal{D}_{a+}^{\beta,\rho} f(t) = f(t) - f(a).$$

Now, we present a generalized integral transform introduced by Jarad and Abdeljawwad [10] which can be used to solve linear FDEs involving $\rho-$Riemann-Liouville and $\rho-$Caputo fractional derivatives.

Definition 4. Let $v, \rho\colon [a,\infty) \to \mathbb{R}$ be e a real-valued function and ρ be a non-negative increasing function such that $\rho'(0) > 0$. Then the Laplace transform of v with respect to ρ is defined by

$$\mathcal{L}_\rho\{v(t)\} = \mathfrak{V}(\lambda) = \int_0^\infty \exp\{-\lambda(\rho(t) - \rho(0)\}\rho'(t)v(t)\mathrm{d}t,$$

for all $\lambda \in \mathbb{C}$ such that this integral converges.

Here $\mathcal{L}_\rho$ denotes the Laplace transform with respect to ρ, which we call a *generalized Laplace transform*.

Theorem 2. *Let $\beta > 0$ and let v be a function of $\rho-$exponential order $c > 0$, piecewise continuous over each finite interval $[a,T]$. Then*

$$\mathcal{L}_\rho\{(I^{\beta,\rho}_{a+}v)(t)\} = \frac{\mathcal{L}_\rho\{v(t)\}}{\lambda^\beta}.$$

Next, we present some information regarding the semigroups of linear operators. These findings are documented in references [7, 12].

For a strongly continuous semigroup, often denoted as $\mathcal{C}_0-$semigroup, represented by $(T(t))_{t\geq 0}$, the infinitesimal generator of $(T(t))_{t\geq 0}$ is defined as follows:

$$\mathcal{A}x = \lim_{t\to 0^+} \frac{T(t)x - x}{x} \qquad x \in \mathcal{X}.$$

The domain of $\mathcal{A}$, denoted as $\mathcal{D}(\mathcal{A})$, is such that

$$\mathcal{D}(\mathcal{A}) = \left\{x \in \mathcal{X} : \lim_{t\to 0^+} \frac{T(t)x - x}{x} \text{ exists}\right\}.$$

Theorem 3 ([14, Lemma 2.24])**.** *Let be $(T(t))_{t\geq 0}$ be a $\mathcal{C}_0-$semigroup then there exist constants $w \in \mathbb{R}$ and $M \geq 1$, such that*

$$||T(t)|| \leq M \exp\{wt\} \qquad 0 \leq t < +\infty.$$

Theorem 4 ([14, Lemma 2.25] (Hille-Yosida))**.** *A linear operator $\mathcal{A}$ is the infinitesimal generator $\mathcal{C}_0-$semigroup of contraction (it means $||T(t)|| \leq 1 \qquad t \geq 0$) if and only if:*

(1) *$\mathcal{A}$ is closed and $\overline{\mathcal{D}(\mathcal{A})} = \mathcal{X}$.*
(2) *The resolvent set of $\mathcal{A}$, $\rho(A)$ contains $\mathbb{R}^+_*$ and for every $\lambda > 0$*

$$\mathcal{R}(\lambda, \mathcal{A}) \leq \frac{1}{\lambda},$$

where $\mathcal{R}(\lambda, \mathcal{A}) := (\lambda^\beta I - \mathcal{A})^{-1} = \int_0^\infty \exp\{-\lambda^\beta s\}T(s)x\mathrm{d}s$.

Throughout this paper, let $\mathcal{A}$ denote the infinitesimal generator of a C_0-semigroup of uniformly bounded linear operators $(T(t))_{t\geq 0}$ on the Banach space $\mathcal{X}$. Consequently, there exists $M \geq 1$ such that $M = \sup_{t\in[0,\infty)} ||T(t)||$.

Definition 5 ([14, Definition 2.13]). The Wright type function is defined by

$$\phi_\beta(s) = \sum_{k=0}^{\infty} \frac{(-s)^k}{k!\Gamma(-\beta k + 1 - \beta)} = \sum_{k=0}^{\infty} \frac{(-s)^k \Gamma(\beta(k+1)) \sin(\pi(k+1)\beta)}{k!},$$

for $0 < \beta < 1$ and $s \in \mathbb{C}$.

Proposition 1 ([14, Proposition 2.14]). *The Wright function ϕ_β is an entire function and has the following properties:*

(1) $\phi_\beta(\delta) \geq 0$ *for* $\delta \geq 0$ *and* $\int_0^\infty \phi_\beta(\delta)\mathrm{d}\delta = 1$;
(2) $\int_0^\infty \phi_\beta(\delta)\delta^p \mathrm{d}\delta = \frac{\Gamma(1+p)}{\Gamma(1+\beta p)}$ *for* $p > -1$;
(3) $\int_0^\infty \phi_\beta(\delta)\exp\{-s\delta\}\mathrm{d}\delta = E_\beta(-s)$, $s \in \mathbb{C}$,
where $E_\beta(s) = \sum_{k=0}^\infty \frac{s^k}{\Gamma(k\beta+1)}$ *is Mittag-Leffler function with one parameter for* $s \in \mathbb{C}$ *and* $\beta > 0$.

Theorem 5 (Gronwall's inequality [14, Theorem 2.11]). *Let μ, ν be two integrable functions and κ be a continuous function on $[a,b]$. Let $\rho \in \mathcal{C}^1([a,b])$ be an increasing function such that $\rho'(t) \neq 0$ for all $t \in [a,b]$. Assume that*

(i) *μ and ν are nonnegative;*
(ii) *κ is nonnegative and nondecreasing.*

If

$$\mu(t) \leq \nu(t) + \kappa(t) \int_a^t (\rho(t) - \rho(s))^{\beta-1} \mu(s)\rho'(s)\mathrm{d}s,$$

then

$$\mu(t) \leq \nu(t) + \int_a^t \sum_{k=1}^{\infty} \frac{\{\kappa(t)\Gamma(\beta)\}^k}{\Gamma(k\beta)} (\rho(t) - \rho(s))^{k\beta-1} \nu(s)\rho'(s)\mathrm{d}s \qquad t \in [a,b].$$

Corollary 1. *Under the hypotheses of Theorem 5, let ν be a nondecreasing function on $[a,b]$. Then we have*

$$\mu(t) \leq \nu(t) E_\beta(\kappa(t)\Gamma(\beta)[\rho(t) - \rho(a)]^\beta),$$

for all $t \in [a,b]$.

Theorem 6 ([5, Corollary 2.1]). *Let $\mathcal{S}$ be a closed, bounded and convex subset of the Banach algebra $\mathcal{X}$. We consider the two operators $I \colon \mathcal{X} \to \mathcal{X}$ and $\mathcal{J} \colon \mathcal{S} \to \mathcal{X}$ such that*

(i) *I is Lipschitzian with a lipschitz constant α;*
(ii) *$\mathcal{J}$ is completely continuous;*
(iii) $u = Iu\mathcal{J}v \Rightarrow u \in \mathcal{S} \qquad v \in \mathcal{S}$;

(iv) $\alpha\mathcal{K} < 1$, *where* $\mathcal{K} = ||\mathcal{I}(\mathcal{S})||$.

Then the operator function $u = IuJu$ *has a solution on* $\mathcal{S}$.

3. Existence results

According to Definition 3 and Theorem 1 it is suitable to rewrite the Cauchy problem in the equivalent integral equation

$$u(t) = g(t,u(t))\left\{\frac{1}{\Gamma(\beta)}\int_0^t \rho'(s)(\rho(t)-\rho(s))^{\beta-1}\mathcal{A}\left(\frac{u(s)}{g(t,u(s))}\right)\mathrm{d}s \right. \\ \left. + \frac{1}{\Gamma(\beta)}\int_0^t \rho'(s)(\rho(t)-\rho(s))^{\beta-1}h(s,u(s))\mathrm{d}s\right\}. \tag{3.1}$$

Proof.

$$I_{0+}^{\beta,\rho}\,{}^{C}\mathcal{D}_{0+}^{\beta,\rho}\left(\frac{u(t)}{g(t,u(t))}\right) = I_{0+}^{\beta,\rho}\left\{\mathcal{A}\left(\frac{u(t)}{g(t,u(t))}\right) + h(t,u(t))\right\},$$

then

$$\frac{u(t)}{g(t,u(t))} - \frac{u(0)}{g(0,u(0))} = I_{0+}^{\beta,\rho}\mathcal{A}\left(\frac{u(t)}{g(t,u(t))}\right) + I_{0+}^{\beta,\rho}h(t,u(t)).$$

□

Lemma 1. *If* (3.1) *holds, then we have*

$$u(t) = g(t,u(t))\left\{\beta\int_0^t\int_0^\infty \theta\Phi_\beta(\theta)(\rho(t)-\rho(s))^{\beta-1}T(\rho(t)-\rho(s))\theta) \right. \\ \left. \times h(s,u(s))\rho'(s)d\theta\mathrm{d}s\right\}.$$

Proof. Let $\lambda > 0$. Applying the generalized Laplace transforms to (3.1), we have

$$\mathfrak{L}(\lambda) = \frac{1}{\lambda^\beta}(\mathcal{A}\mathfrak{U}(\lambda) + \mathfrak{H}(\lambda)),$$

where

$$\mathfrak{U}(\lambda) = \int_0^\infty \exp\{-\lambda(\rho(\tau)-\rho(0))\}\frac{u(\tau)}{g(\tau,u(\tau))}\rho'(\tau)\mathrm{d}\tau,$$

$$\mathfrak{H}(\lambda) = \int_0^\infty \exp\{-\lambda(\rho(\tau)-\rho(0))\}h(\tau,u(\tau))\rho'(\tau)\mathrm{d}\tau.$$

It follows that

$$\mathfrak{L}(\lambda) = \int_0^\infty \exp\{-\lambda^\beta s\}T(s)\mathfrak{H}(\lambda)\mathrm{d}s = \int_0^\infty \beta w^{\beta-1}\exp\{-(\lambda w)^\beta\}T(w^\beta)\mathfrak{H}(\lambda)\mathrm{d}w.$$

Taking $w = \rho(t) - \rho(0)$, we get

$$\mathfrak{L}(\lambda) = \int_0^\infty \beta(\rho(t)-\rho(0))^{\beta-1}\exp\{-\lambda(\rho(t)-\rho(0))^\beta\}T((\rho(t)-\rho(0))^\beta)\mathfrak{H}(\lambda)\rho'(t)\mathrm{d}t \\ = \int_0^\infty\int_0^\infty \beta(\rho(t)-\rho(0))^{\beta-1}\exp\{-\lambda(\rho(t)-\rho(0))^\beta\}T((\rho(t)-\rho(0))^\beta)$$

$$\times \exp\{-\lambda(\rho(s)-\rho(0))\}h(s,u(s))\rho'(s)\rho'(t)\mathrm{d}s\mathrm{d}t.$$

We consider the following over-sided stable probability density in [11]

$$\rho_\beta(\theta)=\frac{1}{\pi}\sum_{i=1}^{\infty}(-1)^{i-1}\theta^{-\beta i-1}\frac{\Gamma(\beta i+1)}{i!}\sin(i\pi\beta) \qquad \theta\in(0,\infty),$$

whose integration, is given by

$$\int_0^\infty \exp\{-\lambda\theta\}\rho_\beta(\theta)\mathrm{d}\theta=\exp\{-\lambda^\beta\} \qquad \beta\in(0,1). \tag{3.2}$$

Using (3.2), we get

$$\begin{aligned}
\mathfrak{U}(\lambda)&=\int_0^\infty\int_0^\infty\int_0^\infty \beta\rho_\beta(\theta)(\rho(t)-\rho(0))^{\beta-1}\exp\{-\lambda(\rho(t)-\rho(0))\theta\}\\
&\quad\times T((\rho(t)-\rho(0))^\beta)\exp\{-\lambda(\rho(s)-\rho(0))\}h(s,u(s))\rho'(s)\rho'(t)\mathrm{d}\theta\mathrm{d}s\mathrm{d}t\\
&=\int_0^\infty\int_0^\infty\int_0^\infty \beta\rho_\beta(\theta)\frac{(\rho(t)-\rho(0))^{\beta-1}}{\theta^\beta}\exp\{-\lambda(\rho(t)+\rho(s)-2\rho(0))\}\\
&\quad\times T\left(\frac{(\rho(t)-\rho(0))^\beta}{\theta^\beta}\right)h(s,u(s))\rho'(s)\rho'(t)\mathrm{d}\theta\mathrm{d}s\mathrm{d}t\\
&=\int_0^\infty\int_t^\infty\int_0^\infty \beta\rho_\beta(\theta)\frac{(\rho(t)-\rho(0))^{\beta-1}}{\theta^\beta}\exp\{-\lambda(\rho(\tau)-\rho(0))\}\\
&\quad\times T\left(\frac{(\rho(t)-\rho(0))^\beta}{\theta^\beta}\right)\\
&\qquad\times h(\rho^{-1}(\rho(\tau)-\rho(t)+\rho(0)),u(\rho^{-1}(\rho(\tau)-\rho(t)+\rho(0))))\\
&\qquad\times\rho'(t)\rho'(\tau)\mathrm{d}\theta\mathrm{d}\tau\mathrm{d}t,
\end{aligned}$$

by Fubini's theorem, we have

$$\begin{aligned}
\mathfrak{U}(\lambda)&=\int_0^\infty\exp\{-\lambda(\rho(\tau)-\rho(0))\}\\
&\times\left\{\int_0^\tau\int_0^\infty\beta\rho_\beta(\theta)\frac{(\rho(t)-\rho(0))^{\beta-1}}{\theta^\beta}T\left(\frac{(\rho(t)-\rho(0))^\beta}{\theta^\beta}\right)\right.\\
&\times \left. h(\rho^{-1}(\rho(\tau)-\rho(t)+\rho(0)),u(\rho^{-1}(\rho(\tau)-\rho(t)+\rho(0))))\rho'(t)\mathrm{d}\theta\mathrm{d}t\right\}\rho'(\tau)\mathrm{d}\tau\\
&=\int_0^\infty\exp\{-\lambda(\rho(\tau)-\rho(0))\}\\
&\quad\times\left\{\int_0^\tau\int_0^\infty\beta\rho_\beta(\theta)\frac{(\rho(\tau)-\rho(s))^{\beta-1}}{\theta^\beta}\right.\\
&\quad\times\left.T\left(\frac{(\rho(\tau)-\rho(s))^\beta}{\theta^\beta}\right)h(s,u(s))\rho'(s)\mathrm{d}\theta\mathrm{d}s\right\}\rho'(\tau)\mathrm{d}\tau.
\end{aligned}$$

Now, we can invert the Laplace transform to get

$$\frac{u(t)}{g(t,u(t))} = \beta \int_0^t \int_0^\infty \theta\phi_\beta(\theta)(\rho(t)-\rho(s))^{\beta-1} \times T((\rho(t)-\rho(s))^\beta\theta)h(s,u(s))\rho'(s)\mathrm{d}\theta\mathrm{d}s,$$

where $\phi_\beta(\theta) = \frac{1}{\beta}\theta^{-1-\frac{1}{\beta}}\rho_\beta(\theta^{-\frac{1}{\beta}})$ is the probability density function defined in $(0,\infty)$. Thus

$$u(t) = g(t,u(t))\left\{\beta \int_0^t \int_0^\infty \theta\phi_\beta(\theta)(\rho(t)-\rho(s))^{\beta-1} \times T((\rho(t)-\rho(s))^\beta\theta)h(s,u(s))\rho'(s)\mathrm{d}\theta\mathrm{d}s\right\}.$$

□

For any $u \in \mathcal{X}$, define the operator $\omega_\rho^\beta(t,s)$ by

$$\omega_\rho^\beta(t,s)u = \beta \int_0^\infty \theta\phi_\beta(\theta)T((\rho(t)-\rho(s))^\beta\theta)\, u\,\mathrm{d}\theta \qquad 0 \le s \le t \le T.$$

Lemma 2. *The operator ω_ρ^β has the following properties:*

(1) *For any fixed $t \ge s \ge 0$ $\omega_\rho^\beta(t,s)$ is bounded linear operator with*

$$\|\omega_\rho{}^\beta(t,s)(\omega)\| \le \frac{\beta.M}{\Gamma(1+\beta)}||u|| = \frac{M}{\Gamma(\beta)}\|u\| \qquad u \in \mathcal{X}.$$

(2) *The operator $\omega_\rho^\beta(t,s)$ is strongly continuous for all $t \ge s \ge 0$, that is for every $u \in \mathcal{X}$ and $0 \le s \le t_1 \le t_2 \le T$ we have*

$$\|\omega_\rho^\beta(t_2,s)u - \omega_\rho^\beta(t_1,s)\| \longrightarrow 0, \qquad \text{as } t_1 \longrightarrow t_2.$$

Proof. The proof of this lemma is similar to the one given in [15]. □

Definition 6. A function $u \in \mathcal{C}([0,T],\mathcal{X})$ is called a mild solution of (1.1) if satisfies

$$u(t) = g(t,u(t))\left\{\int_0^t (\rho(t)-\rho(s))^{\beta-1}\omega_\rho^\beta(t,s)h(s,u(s))\rho'(s)\mathrm{d}s\right\}.$$

Before starting and proving the main result, we introduce the following hypothesis

(C_1) $T(t)$ is compact operator for every $t > 0$.

(C_2) For any $r > 0$, there exists a function $h_r \in L^\infty([0,T],\mathcal{X})$ such that

$$\sup_{||u||\le r} \|h(t,u)\| \le h_r(t) \qquad t \in [0,T],$$

and there is a constant ζ >0 such that

$$\limsup_{r\to\infty} \frac{||h_r(t)||_{L^\infty}}{r} = \zeta.$$

(C_3) The function $g \in \mathcal{C}_c([0,T]\times\mathcal{X},\mathcal{X}\setminus\{0\})$ is bounded and there exist constants $\mu > 0$ and $L > 0$ such that for all $u,v \in \mathcal{X}$ and $t \in [0,T]$, we have

$$|g(t,u)-g(t,v)| \le \mu|u-v| \quad |g(t,u)| < L.$$

Theorem 7. *Assume that condition $(C_1)-(C_3)$ hold. Then the problem (1.1) has at least mild solution provided that*

$$\frac{\mu M||h_r||_{L^\infty}}{\Gamma(\beta+1)}(\rho(T)-\rho(0))^\beta < 1.$$

Proof. Let $\mathcal{V}=\{u\in\mathcal{X}, ||u||\le b\}$ $\qquad b=\frac{LM||h_r||_{L^\infty}}{\Gamma(\beta+1)}(\rho(t)-\rho(0))^\beta$.
We have

$$u(t)=g(t,u(t))\left\{\int_0^t(\rho(t)-\rho(s))^{\beta-1}\omega_\rho{}^\beta(t,s)h(s,u(s))\rho'(s)\mathrm{d}s\right\}.$$

Then we can transform into $u(t)=Iu(t)\mathcal{J}u(t) \qquad t\in[0,T]$.
Now we prove that all conditions of Theorem 6 are satisfied.

Step 1: Let $u,v\in\mathcal{X}$ then

$$\begin{aligned} |\ Iu(t)-Iv(t)\ | &= |\ g(t,u(t))-g(t,v(t))\ | \\ &\le \mu\ |\ u(t)-v(t) \qquad t\in[0,T]. \end{aligned}$$

Step 2: Firstly, we prove that $\mathcal{J}$ is completely continuous.
Let $u_n,u\in\mathcal{V}$ with $\lim\limits_{n\to+\infty}\|u_n-u\|=0$. Then

$$h(s,u_n(s))\to h(s,u(s)), \text{ as } n\to\infty.$$

Therefore

$$\|\mathcal{J}u_n(t)-\mathcal{J}u(t)\| \le \int_0^t(\rho(t)-\rho(s))^{\beta-1}\frac{M}{\Gamma(\beta)}||h(s,u_n(s))-h(s,u(s))||\rho'(s)\mathrm{d}s.$$

Via Lebesgue dominated convergence theorem, we get

$$\|\mathcal{J}u_n(t)-\mathcal{J}u(t)\| \longrightarrow 0 \qquad \text{as} \quad n\to\infty.$$

• $\mathcal{J}(\mathcal{V})$ is uniformly bounded.
Let $u\in\mathcal{V}$,

$$\begin{aligned} \|\mathcal{J}u(t)\| &\le \int_0^t(\rho(t)-\rho(s))^{\beta-1}\frac{M}{\Gamma(\beta)}\|h_r\|_{L^\infty}\rho'(s)\mathrm{d}s \\ &= \frac{M||h_r||_{L^\infty}}{\Gamma(\beta+1)}(\rho(t)-\rho(0))^\beta \qquad 0\le t\le T \\ &\le \frac{M||h_r||_{L^\infty}}{\Gamma(\beta+1)}(\rho(T)-\rho(0))^\beta. \end{aligned}$$

• $\mathcal{J}(\mathcal{V})$ is equicontinuous.

Let $t_1, t_2 \in [0,T]$ with $t_1 < t_2$ and $u \in \mathcal{V}$.

$$||\mathcal{J}u(t_2) - \mathcal{J}u(t_1)|| \leq \int_{t_1}^{t_2} (\rho(t_1) - \rho(s))^{\beta-1} \frac{M}{\Gamma(\beta)} ||h_r||_{L^\infty} \rho'(s)\mathrm{d}s$$
$$+ \int_0^{t_1} ||\omega_\rho^\beta(t_2,s)|| \{(\rho(t_2) - \rho(s))^{\beta-1} - (\rho(t_1) - \rho(s))^{\beta-1}\} \|h(s,u(s))\| \rho'(s)\mathrm{d}s$$
$$+ \int_0^{t_1} (\rho(t_1) - \rho(s))^{\beta-1} ||(\omega_\rho^\beta(t_2,s) - \omega_\rho^\beta(t_1,s))|| \times ||h(s,u(s))|| \rho'(s)\mathrm{d}s$$
$$\leq \frac{-M\|h_r\|_{L^\infty}}{\Gamma(\beta+1)} (\rho(t_1) - \rho(t_2))^\beta + \frac{M\|h_r\|_{L^\infty}}{\Gamma(\beta+1)} \Big\{ (\rho(t_1) - \rho(0))^\beta - (\rho(t_2) - \rho(t_1))^\beta$$
$$- (\rho(t_1) - \rho(0))^\beta \} + \frac{\|h_r\|_{L^\infty}}{\beta} (\rho(t_1) - \rho(0))^\beta ||\omega_\rho^\beta(t_2,s) - \omega_\rho^\beta(t_1,s)||.$$

Thus

$$\|\mathcal{J}u(t_2) - \mathcal{J}u(t_1)\| \to 0 \qquad \text{as } t_2 \to t_1.$$

Step 3: Let $u \in \mathcal{X}$ and $v \in \mathcal{V}$ such that $Iu\mathcal{J}v = u$, prove that $u \in \mathcal{V}$.

$$|u(t)| = |Iu\mathcal{J}v| = |g(t,u(t))| \times |\mathcal{J}v(t)|$$
$$\leq L \times \frac{M||h_r||_{L^\infty}}{\Gamma(\beta+1)} (\rho(T) - \rho(0))^\beta.$$

Step 4: Suppose that $\mathcal{S} = \sup\{||\mathcal{J}u||, u \in \mathcal{V}\} \leq b$. Then

$$\mu \times \mathcal{S} \leq \mu \times b < 1.$$

□

4. MITTAG-LEFFLER-ULAM-HYERS STABILITY

For $g \in \mathcal{C}([0,T] \times \mathcal{X}, \mathcal{X} \setminus \{0\})$, $h \in \mathcal{C}_c([0,T] \times \mathcal{X}, \mathcal{X})$, $\psi \in \mathcal{C}([0,T], \mathbb{R}^+)$ and $\varepsilon > 0$. We consider the equation

$${}^C D_{0+}^{\beta,\rho} \left(\frac{u(t)}{g(t,u(t))} \right) = \mathcal{A} \left(\frac{u(t)}{g(t,u(t))} \right) + h(t,u(t)) \qquad t \in [0,T] \quad 0 < \beta < 1. \tag{4.1}$$

And the inequalities

$$\left| {}^C D_{0+}^{\beta,\rho} \left(\frac{u(t)}{g(t,u(t))} \right) - \mathcal{A} \left(\frac{u(t)}{g(t,u(t))} \right) - h(t,u(t)) \right| \leq \varepsilon \qquad t \in [0,T]; \tag{4.2}$$

$$\left| {}^C D_{0+}^{\beta,\rho} \left(\frac{u(t)}{g(t,u(t))} \right) - \mathcal{A} \left(\frac{u(t)}{g(t,u(t))} \right) - h(t,u(t)) \right| \leq \psi \qquad t \in [0,T]; \tag{4.3}$$

$$\left| {}^C D_{0+}^{\beta,\rho} \left(\frac{u(t)}{g(t,u(t))} \right) - \mathcal{A} \left(\frac{u(t)}{g(t,u(t))} \right) - h(t,u(t)) \right| \leq \varepsilon\psi \qquad t \in [0,T]. \tag{4.4}$$

Definition 7. We said that the equation (4.1) is Mittag-Leffler-Ulam-Hyers stable with respect to E_β, if there exists a real number $\delta > 0$ such that for each $\varepsilon > 0$ and for each solution $\nu \in \mathcal{C}^1([0,T],\mathcal{X})$ of inequality (4.2) there exists a mild solution $u \in \mathcal{C}([0,T],\mathcal{X})$ of equation (4.1) with $|\nu(t) - u(t)| \leq \delta\varepsilon E_\beta(t)$, $t \in [0,T]$.

Definition 8. We said that the equation (4.1) is generalized Mittag-Leffler-Ulam-Hyers stable with respect to E_β, if there exists $\mu \in \mathcal{C}(\mathbb{R}^+,\mathbb{R}^+)$ with $\mu(0) = 0$ such that for each solution $\nu \in \mathcal{C}^1([0,T],\mathcal{X})$ of inequality (4.2) there exists a mild solution $u \in \mathcal{C}([0,T],\mathcal{X})$ of equation (4.1) with $|\nu(t) - u(t)| \leq C\mu(t)E_\beta(t)$, $C > 0$ and $t \in [0,T]$.

Definition 9. Equation (4.1) is Mittag-Leffler-Ulam-Hyers-Rassias stable with respect to ψE_β, if there exists a real number $C_\psi > 0$ such that for each $\varepsilon > 0$ and for each solution $\nu \in \mathcal{C}^1([0,T],\mathcal{X})$ of inequality (4.4) there exists a mild solution $u \in \mathcal{C}([0,T],\mathcal{X})$ of equation (4.1) with $|\nu(t) - u(t)| \leq C_\psi \varepsilon E_\beta(t)$, $t \in [0,T]$.

Definition 10. Equation (4.1) is generalized Mittag-Leffler-Ulam-Hyers-Rassias stable with respect to ψE_β, if there exists a real number $C_\psi > 0$ such that for each solution $\nu \in \mathcal{C}^1([0,T],\mathcal{X})$ of inequality (4.3) there exists a mild solution $u \in \mathcal{C}([0,T],\mathcal{X})$ of equation (4.1) with $|\nu(t) - u(t)| \leq C_\psi \psi(t) E_\beta(t)$, $t \in [0,T]$.

Remark 2. A function $u \in \mathcal{C}^1([0,T],\mathcal{X})$ is a solution of inequality (4.2) if and only if there exists a function $\varphi \in \mathcal{C}^1([0,T],\mathcal{X})$ (which depend on u) such that

(1) $|\varphi(t)| \leq \varepsilon \qquad t \in [0,T]$.
(2) ${}^C D^{\beta,\rho}_{0+}\left(\frac{u(t)}{g(t,u(t))}\right) = \mathcal{A}\left(\frac{u(t)}{g(t,u(t))}\right) + h(t,u(t)) + \varphi(t) \qquad t \in [0,T]$.

Lemma 3. *If* $\nu \in \mathcal{C}^1([0,T],\mathcal{X})$ *is a solution of* (4.2), ν *is a solution of the following integral inequality*

$$\left|\nu(t) - g(t,\nu(t))\left\{\int_0^t (\rho(t)-\rho(s))^{\beta-1}\omega_\rho^\beta(t,s)h(s,\nu(s))\rho'(s)\mathrm{d}s\right\}\right| \leq \frac{LM\varepsilon}{\Gamma(\beta+1)}(\rho(T)-\rho(0))^\beta.$$

Proof. By the previous remark, we have

$${}^C D^{\beta,\rho}_{0+}\left(\frac{u(t)}{g(t,u(t))}\right) = \mathcal{A}\left(\frac{u(t)}{g(t,u(t))}\right) + h(t,u(t)) + \varphi(t) \qquad t \in [0,T].$$

And from Theorem 7, we get

$$\left|\nu(t) - g(t,\nu(t))\left\{\int_0^t (\rho(t)-\rho(s))^{\beta-1}\mathcal{W}_\rho^\beta(t,s)h(s,\nu(s))\rho'(s)\mathrm{d}s\right\}\right| \leq L \times \varepsilon \times \frac{M}{\Gamma(\beta)}\{\rho(T)-\rho(0)\}^\beta.$$

□

Theorem 8. *Assume that $h \in \mathcal{C}([0,T]\times \mathcal{X},\mathcal{X})$ and there exists $L_h > 0$ such that $|h(t,u_1)-h(t,u_2)| \leq L_h|u_1-u_2|$, for all $t\in[0,T]$ and $u_1,u_2\in\mathcal{X}$.*
Then equation (4.1) is Mittag-Leffler-Ulam-Hyers stable.

Proof. Let $\nu \in \mathcal{C}^1([0,T],\mathcal{X})$ be a solution of inequality (4.2) and let us denote by $u\in\mathcal{C}([0,T],\mathcal{X})$ the unique mild solution of the cauchy problem

$$\begin{cases} {}^{C}D^{\beta,\rho}_{0+}\left(\frac{u(t)}{g(t,u(t))}\right) = \mathcal{A}\left(\frac{u(t)}{g(t,u(t))}\right) + h(t,u(t)) & t\in[0,T],\\ u(0)=\nu(0)=0. \end{cases}$$

We have

$$\begin{aligned}
|\nu(t)-u(t)| &\leq \left|\nu(t) - g(t,\nu(t))\left\{\int_0^t(\rho(t)-\rho(s))^{\beta-1}\omega_\rho^\beta(t,s)h(s,\nu(s))\rho'(s)\mathrm{d}s\right\}\right|\\
&+\left|g(t,\nu(t))\int_0^t(\rho(t)-\rho(s))^{\beta-1}\omega_\rho^\beta(t,s)h(s,\nu(s))\rho'(s)\mathrm{d}s\right.\\
&\left.-g(t,\nu(t))\int_0^t(\rho(t)-\rho(s))^{\beta-1}\omega_\rho^\beta(t,s)h(s,u(s))\rho'(s)\mathrm{d}s\right|\\
&+\left|g(t,\nu(t))\int_0^t(\rho(t)-\rho(s))^{\beta-1}\omega_\rho^\beta(t,s)h(s,u(s))\rho'(s)\mathrm{d}s\right.\\
&\left.-g(t,u(t))\int_0^t(\rho(t)-\rho(s))^{\beta-1}\omega_\rho^\beta(t,s)h(s,u(s))\rho'(s)\mathrm{d}s\right|\\
&\leq \frac{LM\varepsilon}{\Gamma(\beta+1)}(\rho(T)-\rho(0))^\beta + \frac{LL_hM}{\Gamma(\beta)}\int_0^t(\rho(t)-\rho(s))^{\beta-1}|\nu(s)-u(s)|\rho'(s)\mathrm{d}s\\
&\quad+\frac{M\mu||h_r||_{L^\infty}}{\Gamma(\beta+1)}(\rho(T)-\rho(0))^\beta|\nu(t)-u(t)|\\
&\leq \frac{LM\varepsilon}{\Gamma(\beta+1)}(\rho(T)-\rho(0))^\beta\times\frac{1}{d}+\frac{LL_hM}{d\Gamma(\beta)}\int_0^t(\rho(t)-\rho(s))^{\beta-1}|\nu(s)-u(s)|\rho'(s)\mathrm{d}s,
\end{aligned}$$

with

$$d = 1-\frac{M\mu||h_r||_{L^\infty}}{\Gamma(\beta+1)}(\rho(T)-\rho(0))^\beta > 0 \text{ (under the hypothesis of Theorem 7).}$$

By Gronwall's inequality, we get

$$|\nu(t)-u(t)| \leq \frac{LM\varepsilon}{\Gamma(\beta+1)}\times\frac{1}{d}(\rho(T)-\rho(0))^\beta E_\beta\left(\frac{LL_hM}{d}(\rho(t)-\rho(0))^\beta\right)\ (d>0).$$

□

Theorem 9. *Assume that the following conditions hold:*

(i) $h\in\mathcal{C}([0,\infty)\times\mathcal{X},\mathcal{X})$;

(ii) *the function* $\psi \in \mathcal{C}([0,\infty],\mathbb{R}^+)$ *is increasing and there exists* $\lambda > 0$ *such that*

$$\frac{LM\varepsilon}{\Gamma(\beta+1)}(\rho(T)-\rho(0))^{\beta} \leq \lambda\psi(t) \qquad t \in [0,\infty);$$

(iii) $\mu(t)$ *is nonnegative, nondecreasing continuous function defined on* $t \in [0,\infty)$

$$\text{and } |h(t,u_1)-h(t,u_2)| \leq \mu(t)|u_1-u_2|, \quad \text{for all } t \geq 0 \text{ and } u_1,u_2 \in \mathcal{X}.$$

Then, equation (4.1) *is generalized Mittag-Leffler-Ulam-Hyers-Rassias stable with respect to* ψE_{β}.

Proof. Let $\nu \in \mathcal{C}^1([0,T],\infty)$ be a solution of equation (4.3). Then, we get

$$\left|\nu(t)-g(t,\nu(t))\left\{\int_0^t(\rho(t)-\rho(s))^{\beta-1}\omega_{\rho}^{\beta}(t,s)h(s,\nu(s))\rho'(s)\mathrm{d}s\right\}\right| \leq \frac{LM\varepsilon}{\Gamma(\beta+1)}(\rho(T)-\rho(0))^{\beta} \leq \lambda\psi(t) \qquad t \in [0,\infty).$$

Let us denote by $u \in \mathcal{C}([0,T],\infty)$ the unique mild solution of the cauchy problem

$$\begin{cases} {}^{C}D_{0+}^{\beta,\rho}\left(\frac{u(t)}{g(t,u(t))}\right) = \mathcal{A}\left(\frac{u(t)}{g(t,u(t))}\right) + h(t,u(t)) & \in t[0,\infty), \\ u(0) = \nu(0) = 0. \end{cases}$$

We have

$$u(t) = g(t,u(t))\left\{\int_0^t(\rho(t)-\rho(s))^{\beta-1}\omega_{\rho}^{\beta}(t,s)h(s,u(s))\rho'(s)\mathrm{d}s\right\} \qquad t \in [0,\infty).$$

It follows that

$$\begin{aligned}
|\nu(t)-u(t)| &\leq \left|\nu(t)-g(t,\nu(t))\left\{\int_0^t(\rho(t)-\rho(s))^{\beta-1}\omega_{\rho}^{\beta}(t,s)h(s,\nu(s))\rho'(s)\mathrm{d}s\right\}\right| \\
&+ \left|g(t,\nu(t))\int_0^t(\rho(t)-\rho(s))^{\beta-1}\omega_{\rho}^{\beta}(t,s)h(s,\nu(s))\rho'(s)\mathrm{d}s\right. \\
&\quad \left. - g(t,u(t))\int_0^t(\rho(t)-\rho(s))^{\beta-1}\omega_{\rho}^{\beta}(t,s)h(s,u(s))\rho'(s)\mathrm{d}s\right| \\
&\leq \lambda\psi(t) + |g(t,\nu(t))|\int_0^t(\rho(t)-\rho(s))^{\beta-1}\omega_{\rho}^{\beta}(t,s)|h(s,\nu(s))-h(s,u(s))|\rho'(s)\mathrm{d}s \\
&\quad + |g(t,\nu(t))-g(t,u(t))|\,||h_r||_{L^{\infty}}\frac{M}{\Gamma(\beta+1)}(\rho(T)-\rho(s))^{\beta-1} \\
&\leq \lambda\psi(t) + L\times\mu(t)\frac{M}{\Gamma(\beta)}\int_0^t(\rho(t)-\rho(s))^{\beta-1}\times|\nu(s)-u(s)|\rho'(s)\mathrm{d}s \\
&\quad + \mu|\nu(t)-u(t)|\,||h_r||_{L^{\infty}}\frac{M}{\Gamma(\beta+1)}(\rho(T)-\rho(0))^{\beta}
\end{aligned}$$

$$\leq \frac{\lambda}{d}\psi(t) + \frac{LM\mu(t)}{\Gamma(\beta)}\frac{1}{d}\int_0^t (\rho(t)-\rho(s))^{\beta-1} \times |\nu(s)-u(s)|\rho'(s)\mathrm{d}s, \qquad d>0.$$

By Gronwall's inequality, we get

$$|\nu(t)-u(t)| \leq \frac{\lambda}{d}\psi(t)E_\beta\left(\frac{LM\mu(t)}{d}(\rho(t)-\rho(0))^\beta\right),$$

with $d = 1-\mu||h_r||_{L^\infty}\frac{M}{\Gamma(\beta+1)}(\rho(T)-\rho(0))^\beta$. □

5. An illustrative example

Let $\mathcal{X} = L^2([0,\pi])$ equipped with the norm and inner product defined respectively, for all $u,\nu \in L^2([0,\pi])$ by

$$||u|| = \left(\int_0^\pi |u(x)|^2\mathrm{d}x\right)^{\frac{1}{2}} \qquad <u,\nu> = \int_0^\pi u(x)\,\overline{\nu(x)}\,\mathrm{d}x.$$

Consider the following initial-boundary value problem of tire fractional parabolic partial differential equation with nonlinear source term

$$\begin{cases} {}^C\mathcal{D}_{0^+}^{\beta,\rho}\left(\frac{u(x,t)}{e^{-t}\,u(x,t)}\right) = \frac{\partial^2}{\partial x^2}\left(\frac{u(x,t)}{e^{-t}\,u(x,t)}\right) + \frac{1}{2}e^{-t}\,u(x,t) & (t,x)\in[0,1]\times[0,\pi] \\ u(0,t) = u(\pi,t) = 0 & t\in[0,1] \\ u(x,0) = 0 & x\in[0,\pi]. \end{cases}$$

Where $\beta = \frac{2}{3}$ $\quad T = 1 \quad \rho = t$.

We define an operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathcal{X} \to \mathcal{X}$ by

$$\mathcal{D}(\mathcal{A}) := \{\nu \in \mathcal{E}; \nu,\nu^{'} \text{are absolutely continuous and } \nu^{''},\ \nu(0)=\nu(\pi)=0\},$$

and

$$\mathcal{A}u = \frac{\partial^2}{\partial x^2}u.$$

It is well known that $\mathcal{A}$ has a discrete spectrum, the eigenvalue are $-j^2$, $j\in\mathbb{N}$, with corresponding normalized eigenvectors $e_j(z) = \sqrt{\frac{2}{\pi}}\sin(jz)$. Then

$$\mathcal{A}x = \sum_{j=1}^{\infty} -j^2 <x,e_j> e_j \qquad x\in\mathcal{D}(A).$$

Thus, $\mathcal{A}$ generates a uniformly bounded analytic semigroup $\{T(t)\}_{t\geq 0}$ in $\mathcal{X}$ and it is given by

$$T(t)x = \sum_{j=1}^{\infty} e^{-j^2 t} <x,e_j> e_j \qquad x\in\mathcal{X}$$

with

$$||T(t)|| \leq e^{-t} \qquad \forall t\geq 0.$$

Hence, we take $M=1$ which implies that $\sup\limits_{t\in[0,\infty)} ||T(t)|| = 1$ and $(\mathcal{C}_1)$ is satisfied.

Then for all $t \in [0,1]$, we have

$$\begin{cases} ||h(t,u)|| = \frac{1}{2} e^{-t} ||u||, \\ \sup_{||u|| \leq r} ||h(t,u)|| \leq \frac{1}{2} e^{-t} r := h_r(t), \\ \limsup_{r \to \infty} \frac{||h_r(t)||_{L^\infty}}{r} = \frac{1}{2} := L. \end{cases}$$

And

$$||g(t,u_1) - g(t,u_2)|| \leq |u_1 - u_2| \qquad u_1, u_2 \in \mathcal{X}.$$

Therefore $(\mathcal{C}_2)$ and $(\mathcal{C}_3)$ are satisfied, which is given us

$$\frac{M\mu ||h_r||_{L^\infty}}{\Gamma(\beta+1)} (\rho(T) - \rho(0))^\beta = \frac{1}{2\Gamma\left(\frac{5}{3}\right)} \simeq 0.45137 < 1.$$

According to Theorem 7. The problem (5) has a unique mold solution on $[0,1]$.

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ALGEBRAS AND VARIETIES WHERE SASAKI OPERATIONS FORM AN ADJOINT PAIR

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Abstract. The so-called Sasaki projection was introduced by U. Sasaki on the lattice $\mathbf{L}(\mathbf{H})$ of closed linear subspaces of a Hilbert space $\mathbf{H}$ as a projection of $\mathbf{L}(\mathbf{H})$ onto a certain sublattice of $\mathbf{L}(\mathbf{H})$. Since $\mathbf{L}(\mathbf{H})$ is an orthomodular lattice, the Sasaki projection and its dual can serve as the logical connectives conjunction and implication within the logic of quantum mechanics. It was shown by the authors in their previous paper [5] that these operations form a so-called adjoint pair. The natural question arises if this result can be extended also to lattices with a unary operation which need not be orthomodular or to other algebras with two binary and one unary operation. To show that this is possible is the aim of the present paper. We determine a variety of lattices with a unary operation where the Sasaki operations form an adjoint pair and we continue with so-called λ-lattices and certain classes of semirings. We show that the Sasaki operations have a deeper sense than originally assumed by their author and can be applied also outside the lattices of closed linear subspaces of a Hilbert space.

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1. PRELIMINARIES

Consider a bounded complemented lattice $\mathbf{L} = (L, \vee, \wedge, ', 0, 1)$ where the unary operation $'$ is a *complementation*, i.e. $\mathbf{L}$ satisfies the identities $x \vee x' \approx 1$ and $x \wedge x' \approx 0$. $\mathbf{L}$ is called *orthomodular* (see [1]) if the complementation $'$ is an *antitone involution* and $\mathbf{L}$ satisfies the *orthomodular law*, i.e. the identity

(OM) $x \vee ((x \vee y) \wedge x') \approx x \vee y.$

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Apparently, the class of orthomodular lattices forms a variety.

A *projection* is a mapping f from a set M to M satisfying $f \circ f = f$. In such a case f is called a projection from M onto $f(M)$. Let $\mathbf{P} = (P, \leq)$ be a poset, $'$ an antitone involution on $\mathbf{P}$ and $f\colon P \to P$. Then the *dual* of f is the mapping $\overline{f}\colon P \to P$ defined by $\overline{f}(x) := (f(x'))'$ for all $x \in P$. If f is a projection or a monotone mapping then $\overline{f}$ has the same property, respectively. Now let $(L, \vee, \wedge, ', 0, 1)$ be an orthomodular lattice and $a \in L$. The following mapping $p_a\colon L \to L$ was introduced by U. Sasaki [9], see also [1]:

$$p_a(x) := (x \vee a') \wedge a$$

for all $x \in L$. This mapping is a monotone projection from L onto $[0, a]$ and is usually called the *Sasaki projection* from L onto $[0, a]$. The dual $\overline{p_a}$ of p_a is defined by

$$\overline{p_a}(x) := \big(p_a(x')\big)' = \big((x' \vee a') \wedge a\big)' = (x \wedge a) \vee a'$$

for all $x \in L$ and it is a monotone projection from L onto $[a', 1]$. For more information on Sasaki projections cf. [7]. In what follows we will call binary operations defined in a similar way *Sasaki operations*.

Let $(P, \leq)$ be a poset and f, g binary operations on P. We introduce the following statements:

(A1) If $f(x, y) \leq z$ then $x \leq g(y, z)$,
(A2) if $x \leq g(y, z)$ then $f(x, y) \leq z$

for all $x, y, z \in P$. Recall that f and g are said to form an *adjoint pair* if they satisfy both conditions (A1) and (A2). In such a case we say that f and g are connected via *adjointness*. If f and g form an adjoint pair then each of the two operations f and g determines the other one. Namely, for every $x, y \in P$, $f(x, y)$ is the smallest element z of P satisfying the inequality $x \leq g(y, z)$, and for every $y, z \in P$, $g(y, z)$ is the greatest element x of P satisfying the inequality $f(x, y) \leq z$.

It is easy to prove that if the binary operations f and g form an adjoint pair on a given poset $(P, \leq)$ then f is monotone in the first variable and g in the second one.

Lemma 1. *Let $(P, \leq)$ be a poset, $a, b, c \in P$ with $a \leq b$ and f, g binary operations on P forming an adjoint pair. Then $f(a, c) \leq f(b, c)$ and $g(c, a) \leq g(c, b)$.*

Proof. Any of the following assertions implies the next one:

$$\begin{aligned} f(b,c) &\leq f(b,c), \\ b &\leq g\big(c, f(b,c)\big), \\ a &\leq g\big(c, f(b,c)\big), \\ f(a,c) &\leq f(b,c). \end{aligned}$$

Moreover, any of the following assertions implies the next one:

$$\begin{aligned} g(c,a) &\leq g(c,a), \\ f\big(g(c,a), c\big) &\leq a, \end{aligned}$$

$$f(g(c,a),c) \le b,$$
$$g(c,a) \le g(c,b).$$

□

The classical example of an adjoint pair are the operations $\wedge$ and $\to$ on a Boolean algebra $(B,\vee,\wedge,{}', 0,1)$ where $x \to y := x' \vee y$ for all $x,y \in B$ or, more general, the operations $\wedge$ and $\to$ on a relatively pseudocomplemented meet-semilattice $(S,\wedge,*)$ where $x \to y := x * y$ for all $x,y \in S$ and $x * y$ denotes the relative pseudocomplement of x with respect to y. Recall that for two elements x and y of a meet-semilattice $(S,\wedge)$ the *relative pseudocomplement* of x with respect to y is the greatest element z of S satisfying $x \wedge z \le y$. The *meet-semilattice* is called *relatively pseudocomplemented* if any two of its elements have a relative pseudocomplement, see [2] for details.

It was shown by the authors in [5] that if $\mathbf{L} = (L,\vee,\wedge,{}',0,1)$ is an orthomodular lattice then the Sasaki operations on L defined by the aforementioned projections, i.e.

(S1) $x \odot y := (x \vee y') \wedge y$ and $x \to y := x' \vee (x \wedge y)$

for all $x,y \in L$, form an adjoint pair. Note that for the Sasaki operations defined by (S1) we have $x \odot y = p_y(x)$ and $x \to y = \overline{p_x}(y)$ for all $x,y \in L$. In case of (S1), conditions (A1) and (A2) read as follows:

(A1) If $x \odot y \le z$ then $x \le y \to z$,
(A2) if $x \le y \to z$ then $x \odot y \le z$

for all $x,y,z \in L$. However, such conditions hold also in the case when $\mathbf{L}$ is not an orthomodular lattice. Namely, in order to prove adjointness we only used (OM), but not the fact that $'$ is an antitone involution. In fact, in a modular lattice with complementation, the choice of $'$ even determines whether $\mathbf{L}$ is orthomodular or not. For example, consider the complemented modular lattice $\mathbf{L} = (L,\vee,\wedge,{}',0,1)$ depicted in Fig. 1: If we choose $'$ as follows:

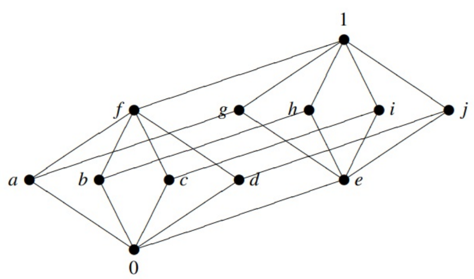

Fig. 1

Complemented modular lattice

x	0	a	b	c	d	e	f	g	h	i	j	1
x'	1	h	i	j	g	f	e	b	c	d	a	0

then $\mathbf{L}$ is not an orthomodular lattice since $'$ is not an involution. However, also in this case one can introduce $\odot$ and $\rightarrow$ by Sasaki operations on L in such a way that these operations form an adjoint pair (cf. Proposition 2 (iii)).

Hence the natural question arises when two binary operations $\odot$ and $\rightarrow$ on a set form an adjoint pair. In general, we need not consider even a complemented lattice, we ask only an algebra with two binary operations and one unary operation, for example a semiring $(S,+,\cdot,0,')$ with an additional unary operation $'$. We need not assume $\cdot$ to be distributive with respect to $+$, i.e.

$$(x+y)z \approx xz+yz \text{ or } z(x+y) \approx zx+zy,$$

but we need that a partial order relation $\leq$ is defined on our algebra. An example of such an algebra may e.g. be a so-called λ-lattice, see [4]. However, if the Sasaki operations on a bounded lattice with a unary operation $'$ form an adjoint pair then $'$ must be a complementation, see the following result.

Lemma 2. *Let $\mathbf{L}=(L,\vee,\wedge,')$ be a lattice with a unary operation $'$ and $\odot$ and $\rightarrow$ denote the Sasaki operations on L defined by* (S1). *Then the following holds:*

- (i) *If $\mathbf{L}$ has a top element 1 and $\odot$ and $\rightarrow$ satisfy condition* (A1) *then $\mathbf{L}$ satisfies the identity $x\vee x' \approx 1$,*
- (ii) *if $\mathbf{L}$ has a bottom element 0 and $\odot$ and $\rightarrow$ satisfy condition* (A2) *then $\mathbf{L}$ satisfies the identity $x\wedge x' \approx 0$,*
- (iii) *if $\mathbf{L}$ is bounded and $\odot$ and $\rightarrow$ form an adjoint pair then $'$ is a complementation on $\mathbf{L}$.*

Proof. Let $a\in L$.

- (i) Because of $1\odot a\leq 1$ we have $1\leq a\rightarrow 1 = a'\vee(a\wedge 1) = a'\vee a$ and hence $a\vee a' = 1$.
- (ii) Because of $0\leq a\rightarrow 0$ we have $a'\wedge a = (0\vee a')\wedge a = 0\odot a\leq 0$ and hence $a\wedge a' = 0$.
- (iii) This follows from (i) and (ii).

□

2. LATTICES

In this section we investigate the Sasaki operations on lattices with a unary operation $'$. We are going to present some classes of lattices, in fact varieties, where the Sasaki operations form an adjoint pair.

For lattices $(L,\vee,\wedge,')$ with a unary operation $'$ we define the following identities:

(B1) $y'\vee((x\vee y')\wedge y) \approx x\vee y'$,

(B2) $(x'\vee(x\wedge y))\wedge x \approx x\wedge y$.

We study the Sasaki operations in the variety of lattices satisfying identities (B1) and (B2). Observe that if $'$ is an antitone involution then any of the identities (B1) and (B2) implies the other one.

Proposition 1. *Identities* (B1) *and* (B2) *are independent.*

Proof. Let $(L,\vee,\wedge,0,1)$ be a non-trivial bounded lattice. If we define a unary operation $'$ on L by $x' := 1$ for all $x \in L$ then $(L,\vee,\wedge,')$ satisfies identity (B1) since

$$y' \vee \big((x \vee y') \wedge y\big) \approx 1 \vee ((x \vee 1) \wedge y) \approx 1 \approx x \vee 1 \approx x \vee y',$$

but does not satisfy identity (B2) since

$$\big(1' \vee (1 \wedge 0)\big) \wedge 1 = 1 \vee 0 = 1 \neq 0 = 1 \wedge 0.$$

If we define a unary operation $'$ on L by $x' := 0$ for all $x \in L$ then $(L,\vee,\wedge,')$ satisfies identity (B2) since

$$\big(x' \vee (x \wedge y)\big) \wedge x \approx (0 \vee (x \wedge y)) \wedge x \approx (x \wedge y) \wedge x \approx x \wedge y,$$

but does not satisfy identity (B1) since

$$0' \vee \big((1 \vee 0') \wedge 0\big) = 0 \vee 0 = 0 \neq 1 = 1 \vee 0 = 1 \vee 0'.$$

□

The following theorem shows when the Sasaki operations $\odot$ and $\to$ satisfy condition (A1) or condition (A2), respectively, depending on the aforementioned identities.

Theorem 1. *Let* $\mathbf{L} = (L,\vee,\wedge,')$ *be a lattice with a unary operation* $'$ *and* $\odot$ *and* $\to$ *denote the Sasaki operations on* L *defined by* (S1)*. Then the following holds:*

(i) *If* $\mathbf{L}$ *satisfies identity* (B1) *then* $\odot$ *and* $\to$ *satisfy condition* (A1)*,*
(ii) *if* $\mathbf{L}$ *satisfies identity* (B2) *then* $\odot$ *and* $\to$ *satisfy condition* (A2)*,*
(iii) *if* $\mathbf{L}$ *satisfies identities* (B1) *and* (B2) *then* $\odot$ *and* $\to$ *form an adjoint pair.*

Proof. Let $a,b,c \in A$.

(i) If $a \odot b \le c$ then using identity (B1) we obtain

$$\begin{aligned} a \le a \vee b' &= b' \vee \big((a \vee b') \wedge b\big) = b' \vee \big(b \wedge \big((a \vee b') \wedge b\big)\big) \\ &= b' \vee (b \wedge (a \odot b)) \le b' \vee (b \wedge c) = b \to c. \end{aligned}$$

(ii) If $a \le b \to c$ then using identity (B2) we obtain

$$\begin{aligned} a \odot b &= (a \vee b') \wedge b \le \big((b \to c) \vee b'\big) \wedge b = \big(\big(b' \vee (b \wedge c)\big) \vee b'\big) \wedge b \\ &= \big(b' \vee (b \wedge c)\big) \wedge b = b \wedge c \le c. \end{aligned}$$

(iii) This follows from (i) and (ii).

□

If the lattice with a unary operation is even modular, we can simplify our assumptions essentially, i.e. we need not assume identities (B1) and (B2) a priori, see the following result.

Proposition 2. *Let* $\mathbf{L} = (L, \vee, \wedge, ')$ *be a modular lattice with a unary operation* $'$ *and* $\odot$ *and* $\to$ *denote the Sasaki operations on* L *defined by* (S1)*. Then the following holds:*

(i) *If* $\mathbf{L}$ *has a top element* 1 *and satisfies the identity* $x \vee x' \approx 1$ *then* $\mathbf{L}$ *satisfies identity* (B1) *and hence* $\odot$ *and* $\to$ *satisfy condition* (A1)*,*

(ii) *if* $\mathbf{L}$ *has a bottom element* 0 *and satisfies the identity* $x \wedge x' \approx 0$ *then* $\mathbf{L}$ *satisfies identity* (B2) *and hence* $\odot$ *and* $\to$ *satisfy condition* (A2)*,*

(iii) *if* $\mathbf{L}$ *is complemented then* $\odot$ *and* $\to$ *form an adjoint pair.*

Proof.

(i) Assume $\mathbf{L}$ to have a top element 1 and to satisfy the identity $x \vee x' \approx 1$. Then

$$y' \vee \big((x \vee y') \wedge y\big) \approx y' \vee \big(y \wedge (x \vee y')\big) \approx (y' \vee y) \wedge (x \vee y') \approx 1 \wedge (x \vee y') \approx x \vee y'$$

and hence $\mathbf{L}$ satisfies identity (B1) and therefore $\odot$ and $\to$ satisfy condition (A1) according to Theorem 1 (i).

(ii) Assume $\mathbf{L}$ to have a bottom element 0 and to satisfy the identity $x \wedge x' \approx 0$. Then

$$\big(x' \vee (x \wedge y)\big) \wedge x \approx \big((x \wedge y) \vee x'\big) \wedge x \approx (x \wedge y) \vee (x' \wedge x) \approx (x \wedge y) \vee 0 \approx x \wedge y$$

and hence $\mathbf{L}$ satisfies identity (B2) and therefore $\odot$ and $\to$ satisfy condition (A2) according to Theorem 1 (ii).

(iii) This follows from (i) and (ii).

□

Concerning Proposition 2 we make the following remark.

Remark 1. Recall that a *meet-semilattice* $(S, \wedge, 0)$ with bottom element 0 is called *pseudocomplemented* if for every $x \in S$ there exists a greatest element x^* of S satisfying $x \wedge x^* = 0$. It is clear that every finite distributive lattice is pseudocomplemented. Hence we can apply Proposition 2 (ii) to finite distributive lattices in order to see that the Sasaki operations defined by (S1) satisfy condition (A2). Recall that a *join-semilattice* $(S, \vee, 1)$ with top element 1 is called *dually pseudocomplemented* if for every $x \in S$ there exists a smallest element x^d of S satisfying $x \vee x^d = 1$. It is clear that every finite distributive lattice is dually pseudocomplemented. Hence we can apply Proposition 2 (i) to finite distributive lattices in order to see that the Sasaki operations defined by (S1) satisfy condition (A1).

For the next result, recall the following concepts.

A *lattice* $(L, \vee, \wedge, ')$ with a unary operation $'$ is called *weakly orthomodular* respectively *dually weakly orthomodular* (cf. [6]) if it satisfies the identity

$$x \approx (x \wedge y) \vee \big(x \wedge (x \wedge y)'\big) \text{ or}$$
$$x \approx (x \vee y) \wedge \big(x \vee (x \vee y)'\big),$$

respectively. Hence, weakly orthomodular as well as dually weakly orthomodular lattices form a variety. Let us note that the unary operation $'$ need neither be a complementation nor an antitone involution.

Proposition 3. *Let $\mathbf{L} = (L, \vee, \wedge, ')$ be a lattice with a unary operation $'$ and $\odot$ and $\rightarrow$ denote the Sasaki operations on L defined by* (S1). *Then the following holds:*

(i) *If $\mathbf{L}$ is weakly orthomodular and $'$ is an involution then $\mathbf{L}$ satisfies identity* (B1) *and hence $\odot$ and $\rightarrow$ satisfy condition* (A1),

(ii) *if $\mathbf{L}$ is dually weakly orthomodular then $\mathbf{L}$ satisfies identity* (B2) *and hence $\odot$ and $\rightarrow$ satisfy condition* (A2),

(iii) *if $\mathbf{L}$ is orthomodular then $\odot$ and $\rightarrow$ form an adjoint pair.*

Proof.

(i) Assume $\mathbf{L}$ to be weakly orthomodular and $'$ to be an involution. Then

$$y' \vee \big((x \vee y') \wedge y\big) \approx y' \vee \big((x \vee y') \wedge y''\big) \approx x \vee y'$$

and hence $\mathbf{L}$ satisfies identity (B1) and therefore $\odot$ and $\rightarrow$ satisfy condition (A2) according to Theorem 1 (i).

(ii) Assume $\mathbf{L}$ to be dually weakly orthomodular. Then

$$\big(x' \vee (x \wedge y)\big) \wedge x \approx x \wedge \big((x \wedge y) \vee x'\big) \approx x \wedge y$$

and hence $\mathbf{L}$ satisfies identity (B2) and therefore $\odot$ and $\rightarrow$ satisfy condition (A2) according to Theorem 1 (ii).

(iii) This follows from (i) and (ii).

□

However, a lattice satisfying identity (B1) need not be modular, see the following example.

Example 1. Consider the non-modular lattice $\mathbf{N}_5 = (N_5, \vee, \wedge)$ visualized in Fig. 2: where the complementation $'$ is defined as follows:

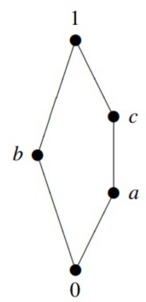

Fig. 2

Non-modular lattice $\mathbf{N}_5$

x	0	a	b	c	1
x'	1	b	b'	b	0

with $b' \in \{a,c\}$. Then $'$ is not an involution since $c'' = b' = a \neq c$ in case $b' = a$ and $a'' = b' = c \neq a$ in case $b' = c$. Abbreviate $(N_5, \vee, \wedge, ')$ by $\mathbf{N}'_5$ and let $\odot$ and $\to$ denote the Sasaki operations on N_5 defined by (S1). In case $b' = c$ the algebra $\mathbf{N}'_5$ satisfies identity (B1) and hence also condition (A1). In case $b' = a$ the algebra $\mathbf{N}'_5$ does not satisfy condition (A1) since

$$c \odot b = (c \vee b') \wedge b = (c \vee a) \wedge b = c \wedge b = 0,$$

but

$$c \not\leq a = a \vee 0 = b' \vee (b \wedge 0) = b \to 0.$$

and hence $\mathbf{N}'_5$ does not satisfy identity (B1). In any case, $\mathbf{N}'_5$ does not satisfy condition (A2) since

$$a \leq 1 = b \vee a = c' \vee (c \wedge a) = c \to a,$$

but

$$a \odot c = (a \vee c') \wedge c = (a \vee b) \wedge c = 1 \wedge c = c \not\leq a$$

and hence $\mathbf{N}'_5$ does not satisfy identity (B2).

3. λ-LATTICES

Other ordered algebras with two binary and one unary operation where the Sasaki operations can be studied are the so-called λ-lattices.

For every poset $(P, \leq)$ and any $a, b \in P$ we define the upper cone $U(a,b)$ of a and b by

$$U(a,b) := \{x \in A \mid a \leq x \text{ and } b \leq x\}$$

and the lower cone $L(a,b)$ of a and b by

$$L(a,b) := \{x \in A \mid x \leq a \text{ and } x \leq b\}.$$

Let us recall the concept of a λ-lattice introduced by V. Snášel [10], see also [4]. A *λ-lattice* is an algebra $(A, \sqcup, \sqcap)$ of type $(2,2)$ satisfying the following identities:

$$\begin{array}{ll} x \sqcup y \approx y \sqcup x, & x \sqcap y \approx y \sqcap x, \\ x \sqcup ((x \sqcup y) \sqcup z) \approx (x \sqcup y) \sqcup z, & x \sqcap ((x \sqcap y) \sqcap z) \approx (x \sqcap y) \sqcap z, \\ x \sqcup (x \sqcap y) \approx x, & x \sqcap (x \sqcup y) \approx x. \end{array}$$

It is evident that the class of λ-lattices forms a variety. It is immediate to check that it satisfies the idempotent laws

$$x \sqcup x \approx x \text{ and } x \sqcap x \approx x.$$

In a λ-lattice a partial order relation $\leq$, the so-called *induced order*, can be introduced by

$$x \leq y \text{ if and only if } x \sqcup y = y \text{ if and only if } x \sqcap y = x$$

$(x,y \in A)$, see [4] for details. Every poset $(A, \leq)$ having the property that any two elements have at least one lower bound and at least one upper bound can be converted into a λ-lattice by defining binary operations $\sqcup$ and $\sqcap$ as follows:

If $a \leq b$ then $a \sqcup b = b \sqcup a := b$ and $a \sqcap b = b \sqcap a := a$.

If $a \parallel b$ then $a \sqcup b = b \sqcup a$ is an arbitrary element of $U(a,b)$, and $a \sqcap b = b \sqcap a$ is an arbitrary element of $L(a,b)$. It is elementary to verify the identities of a λ-lattice. Of course, every lattice is a λ-lattice, but not vice versa. Fig. 3 shows a λ-lattice that is not a lattice:

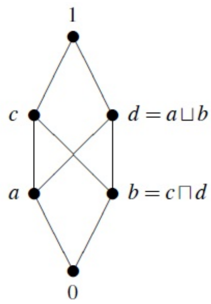

Fig. 3

A λ-lattice

For λ-lattices $(A, \sqcup, \sqcap, ')$ with a unary operation $'$ we introduce the following identities and conditions (which could be rewritten in the form of identities) being variants of the identities (B1) and (B2) from the previous section:

(C1) $y' \sqcup ((x \sqcup y') \sqcap y) \approx x \sqcup y'$,

(C2) $(x' \sqcup (x \sqcap y)) \sqcap x \approx x \sqcap y$

for all $x, y \in A$, or, in the form of inequalities,

(D1) $x \sqcup y' \leq y' \sqcup ((x \sqcup y') \sqcap y)$,

(D2) $(x' \sqcup (x \sqcap y)) \sqcap x \leq x \sqcap y$

for all $x, y \in A$. Obviously, identity (C1) implies condition (D1) and identity (C2) implies condition (D2). In λ-lattices $(A, \sqcup, \sqcap, ')$ with a unary operation, the Sasaki operations can be defined by

(S2) $x \odot y := (x \sqcup y') \sqcap y$ and $x \to y := x' \sqcup (x \sqcap y)$

for all $x, y \in A$.

Similarly as in the case of lattices, we can prove the following result.

Lemma 3. *Let* $\mathbf{A} = (A, \sqcup, \sqcap, ')$ *be a* λ*-lattice with a unary operation* $'$ *and* $\odot$ *and* $\to$ *denote the Sasaki operations on* A *defined by* (S2). *Then the following holds:*

(i) *If* $\mathbf{A}$ *has a top element* 1 *and* $\odot$ *and* $\to$ *satisfy condition* (A1) *then* $\mathbf{A}$ *satisfies the identity* $x \sqcup x' \approx 1$,

(ii) *if* **A** *has a bottom element* 0 *and* $\odot$ *and* $\to$ *satisfy condition* (A2) *then* **A** *satisfies the identity* $x \sqcap x' \approx 0$,
(iii) *if* **A** *is bounded and* $\odot$ *and* $\to$ *form an adjoint pair then* $'$ *is a complementation on* **A**.

Proof. Let $a \in A$.

(i) Because of $1 \odot a \leq 1$ we have $1 \leq a \to 1 = a' \sqcup (a \sqcap 1) = a' \sqcup a$ and hence $a \sqcup a' = 1$.
(ii) Because of $0 \leq a \to 0$ we have $a' \sqcap a = (0 \sqcup a') \sqcap a = 0 \odot a \leq 0$ and hence $a \sqcap a' = 0$.
(iii) This follows from (i) and (ii).

□

Consider the following bounded poset $\mathbf{P} = (A, \leq, ', 0, 1)$ with involution $'$:

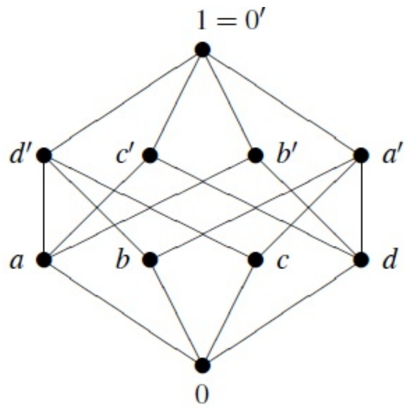

Fig. 4

A bounded poset with involution

This poset **P** can be converted into a λ-lattice in several ways. The converse of Lemma 3 (iii) does not hold, see the following λ-lattice. If $(A, \sqcup, \sqcap)$ is a λ-lattice with involution corresponding to **P** then the Sasaki operations $\odot$ and $\to$ on A defined by (S2) do not form an adjoint pair, independent from the fact how $\sqcup$ and $\sqcap$ are defined within this λ-lattice. Suppose $\odot$ and $\to$ form an adjoint pair. Then we have

$b \leq a \sqcup b = a \sqcup (a' \sqcap b) = a' \to b$ and hence $b \odot a' \leq b$,
$b \leq a \sqcup c = a \sqcup (a' \sqcap c) = a' \to c$ and hence $b \odot a' \leq c$,
$b \not\leq a = a \sqcup 0 = a \sqcup (a' \sqcap 0) = a' \to 0$ and hence $b \odot a' \not\leq 0$, i.e. $b \odot a' \neq 0$

which is a contradiction.

But there is another essential difference from the case of lattices. It is known (see e.g. [4]) that the λ-lattice operations $\sqcup$ and $\sqcap$ need not be compatible with the induced order. For example we have $a \leq c$ in the λ-lattice depicted in Fig. 3, but $a \sqcup b = d \not\leq c = c \sqcup b$. Moreover, a λ-lattice is a lattice if and only if $\sqcup$ and $\sqcap$ are compatible with the induced order, see e.g. [10] or Theorem 2.14 in [4]. We consider a weaker version of compatibility expressed by the following conditions (E1) and (E2). These conditions are not trivial, they do not imply that the λ-lattice in question is a lattice.

For λ-lattices $(A, \sqcup, \sqcap, ')$ with a unary operation we define the following conditions (which could be rewritten in the form of identities):

(E1) $x \leq y$ implies $z' \sqcup (z \sqcap x) \leq z' \sqcup (z \sqcap y)$,
(E2) $x \leq y$ implies $(x \sqcup z') \sqcap z \leq (y \sqcup z') \sqcap z$.

Moreover, we consider also a weaker version of these conditions, namely

(F1) $x \odot y \leq z$ implies $y' \sqcup (y \sqcap (x \odot y)) \leq y' \sqcup (y \sqcap z)$,
(F2) $x \leq y \to z$ implies $(x \sqcup y') \sqcap y \leq ((y \to z) \sqcup y') \sqcap y$

for all $x, y, z \in A$. Obviously, condition (E1) implies condition (F1) and condition (E2) implies condition (F2).

Proposition 4. *Let* $\mathbf{A} = (A, \sqcup, \sqcap, ')$ *be a λ-lattice with a unary operation* $'$ *and* $\odot$ *and* $\to$ *denote the Sasaki operations on* A *defined by* (S2)*. Then the following holds:*

(i) *If* $\mathbf{A}$ *satisfies conditions* (D1) *and* (F1) *then* $\odot$ *and* $\to$ *satisfy condition* (A1)*,*
(ii) *if* $\mathbf{A}$ *satisfies conditions* (D2) *and* (F2) *then* $\odot$ *and* $\to$ *satisfy condition* (A2).

Proof. Let $a, b, c \in A$.

(i) If $\mathbf{A}$ satisfies conditions (D1) and (F1) and $a \odot b \leq c$ then we obtain

$$\begin{aligned} a &\leq a \sqcup b' \leq b' \sqcup \big((a \sqcup b') \sqcap b\big) = b' \sqcup \big(b \sqcap \big((a \sqcup b') \sqcap b\big)\big) = b' \sqcup (b \sqcap (a \odot b)) \\ &\leq b' \sqcup (b \sqcap c) = b \to c. \end{aligned}$$

(ii) If $\mathbf{A}$ satisfies conditions (D2) and (F2) and $a \leq b \to c$ then we obtain

$$\begin{aligned} a \odot b &= (a \sqcup b') \sqcap b \leq \big((b \to c) \sqcup b'\big) \sqcap b = \big(\big(b' \sqcup (b \sqcap c)\big) \sqcup b'\big) \sqcap b \\ &= \big(b' \sqcup (b \sqcap c)\big) \sqcap b \leq b \sqcap c \leq c. \end{aligned}$$

□

In the next example we present a λ-lattice whose Sasaki operations defined by (S2) satisfy condition (A1), but not condition (A2).

Example 2. Let $\mathbf{A} = (A, \sqcup, \sqcap, ')$ denote the λ-lattice from Fig. 3 with the unary operation $'$ defined by

x	0	a	b	c	d	1
x'	1	1	1	d	c	0

and $\odot$ and $\rightarrow$ denote the Sasaki operations on A defined by (S2). If $y \neq c, d$ then condition (C1) clearly holds. If $y = c$ then condition (C1) reads $d \sqcup ((x \sqcup d) \sqcap c) = x \sqcup d$ which holds since $x \sqcup d \geq d$. If, finally, $y = d$ then condition (C1) reads $c \sqcup ((x \sqcup c) \sqcap d) = x \sqcup c$ which holds since $x \sqcup c \geq c$. Now assume $x, y, z \in A$ and $x \leq y$. If $z \neq c, d$ then clearly

(0) $z' \sqcup (z \sqcap x) \leq z' \sqcup (z \sqcap y)$.

If $z = c$ then (0) holds since

$$z' \sqcup (z \sqcap x) = d \sqcup (c \sqcap x) = \begin{cases} d & \text{if } x \leq d, \\ 1 & \text{otherwise.} \end{cases}$$

If, finally, $z = d$ then (0) holds since

$$z' \sqcup (z \sqcap x) = c \sqcup (d \sqcap x) = \begin{cases} c & \text{if } x \leq c, \\ 1 & \text{otherwise.} \end{cases}$$

Hence $\mathbf{A}$ satisfies also condition (E1) and by Proposition 4 (i) $\odot$ and $\rightarrow$ satisfy condition (A1). But $\odot$ and $\rightarrow$ do not satisfy condition (A2) since

$$0 \leq d = d \sqcup 0 = c' \sqcup (c \sqcap 0) = c \rightarrow 0,$$

but

$$0 \odot c = (0 \sqcup c') \sqcap c = d \sqcap c = b \not\leq 0.$$

However, the unary operation $'$ on A cannot be defined in such a way that both condition (D2) and condition (E2) are satisfied. This can be seen as follows: Suppose there exists some unary operation $'$ on A satisfying both condition (D2) and condition (E2). Putting $x = c$ and $y = 0$ in condition (D2) yields

$$c' \sqcap c = (c' \sqcup 0) \sqcap c = \big(c' \sqcup (c \sqcap 0)\big) \sqcap c \leq c \sqcap 0 = 0$$

whence $c' = 0$. Because of $a \leq d$ we have according to condition (E2)

$$a = (a \sqcup c') \sqcap c \leq (d \sqcup c') \sqcap c = d \sqcap c = b,$$

a contradiction.

We now present an example of a λ-lattice whose Sasaki operations defined by (S2) satisfy condition (A2), but not condition (A1).

Example 3. Consider the following bounded poset $\mathbf{P} = (A, \leq, ', 0, 1)$ with involution $'$:

Define a bounded λ-lattice $\mathbf{A} = (A, \sqcup, \sqcap, ', 0, 1)$ with involution corresponding to $\mathbf{P}$ in the following way: Put $B := \{a, b, c, d\}$ and $B' := \{a', b', c', d'\}$ and for different $x, y \in B$ assume $x \sqcup y \in B' \setminus \{x', y'\}$ and put $x' \sqcap y' := 0$. Let $\odot$ and $\rightarrow$ denote the Sasaki operations on A defined by (S2). Evidently, $'$ is a complementation as well as an antitone involution. We show that for all $x, y, z \in A$ we have

(A2) $x \leq y \rightarrow z = y' \sqcup (y \sqcap z)$ implies $x \odot y = (x \sqcup y') \sqcap y \leq z$.

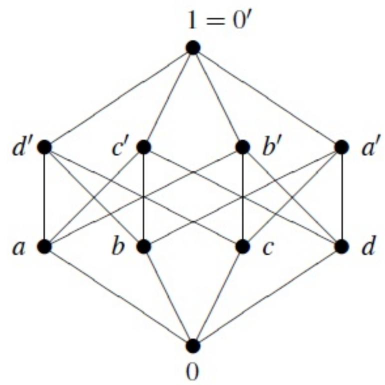

Fig. 5

A bounded poset with involution

First observe that **A** satisfies the identities $x \sqcup x' \approx 1$ and $x \sqcap x' \approx 0$. If $x = 0$ or $y \in \{0,1\}$ then condition (A2) holds. If $x = 1$ and $x \leq y' \sqcup (y \sqcap z)$ then $y' \sqcup (y \sqcap z) = 1$ and hence $y \sqcap z = y$, i.e. $y \leq z$ showing $(x \sqcup y') \sqcap y = y \leq z$. Now let e, f be different elements of B.

If $(x,y) = (e,f)$ then $(x \sqcup y') \sqcap y = (e \sqcup f') \sqcap f = 0 \leq z$,
if $(x,y) = (e,e')$ then $(x \sqcup y') \sqcap y = (e \sqcup e) \sqcap e' = 0 \leq z$,
if $(x,y) = (e,f')$ then $(x \sqcup y') \sqcap y = (e \sqcup f) \sqcap f' = 0 \leq z$,
if $(x,y) = (e',e)$ then $(x \sqcup y') \sqcap y = (e' \sqcup e') \sqcap e = 0 \leq z$,
if $(x,y) = (e',f')$ then $(x \sqcup y') \sqcap y = (e' \sqcup f) \sqcap f' = 0 \leq z$,
if $(x,y) = (e,e)$ and $x \leq y' \sqcup (y \sqcap z)$ then $e \leq e' \sqcup (e \sqcap z)$ and hence $e \sqcap z = e$, i.e. $e \leq z$ showing $(x \sqcup y') \sqcap y = (e \sqcup e') \sqcap e = e \leq z$,
if $(x,y) = (e',f)$ and $x \leq y' \sqcup (y \sqcap z)$ then $e' \leq f' \sqcup (f \sqcap z)$ and hence $f \sqcap z = f$, i.e. $f \leq z$ showing $(x \sqcup y') \sqcap y = (e' \sqcup f') \sqcap f = f \leq z$,
if, finally, $(x,y) = (e',e')$ and $x \leq y' \sqcup (y \sqcap z)$ then $e' \leq e \sqcup (e' \sqcap z)$ and hence $e' \sqcap z = e'$, i.e. $e' \leq z$ showing $(x \sqcup y') \sqcap y = (e' \sqcup e) \sqcap e' = e' \leq z$. This shows that $\odot$ and $\to$ satisfy condition (A2). However, $\odot$ and $\to$ do not satisfy condition (A1) since

$$a \odot c' = (a \sqcup c) \sqcap c' \in \{b' \sqcap c', d' \sqcap c'\} = \{0\}$$

and hence $a \odot c' \leq 0$, but

$$a \not\leq c = c \sqcup 0 = c \sqcup (c' \sqcap 0) = c' \to 0.$$

Moreover, **A** does not satisfy identity (C2) since by putting $(x,y) = (a',b)$ we obtain

$$\big(x' \sqcup (x \sqcap y)\big) \sqcap x = \big(a \sqcup (a' \sqcap b)\big) \sqcap a' = (a \sqcup b) \sqcap a' = 0 \neq b = a' \sqcap b = x \sqcap y.$$

Remark 2. As shown in Example 3, identity (C2) is not a necessary condition for Sasaki operations in a λ-lattice to satisfy condition (A2).

We are going to derive a characterization of λ-lattices satisfying conditions (D1) and (D2) in which the Sasaki operations defined by (S2) form an adjoint pair.

Theorem 2. *Let* $\mathbf{A} = (A, \sqcup, \sqcap, ')$ *be a* λ*-lattice with a unary operation* $'$ *satisfying conditions* (D1) *and* (D2) *and* $\odot$ *and* $\to$ *denote the Sasaki operations on* A *defined by* (S2). *Then the following are equivalent:*

(i) *The operations* $\odot$ *and* $\to$ *form an adjoint pair,*
(ii) *the* λ*-lattice* $\mathbf{A}$ *satisfies conditions* (E1) *and* (E2),
(iii) *the* λ*-lattice* $\mathbf{A}$ *satisfies conditions* (F1) *and* (F2).

Proof. Let $a, b, c \in A$.
(i) $\Rightarrow$ (ii):
If $a \le b$ then according to Lemma 1 we have

$$c' \sqcup (c \sqcap a) = c \to a \le c \to b = c' \sqcup (c \sqcap b),$$
$$(a \sqcup c') \sqcap c = a \odot c \le b \odot c = (b \sqcup c) \sqcap c.$$

(ii) $\Rightarrow$ (iii):
This is clear.
(iii) $\Rightarrow$ (i):
This follows from Proposition 4. □

A λ-lattice as described in Theorem 2 which is not a lattice is presented in Example 5 below.

It is worth noticing that we do not know an example of a λ-lattice (with a unary operation $'$) not being a lattice, but satisfying identities (C1) and (C2) whose Sasaki operations $\odot$ and $\to$ defined by (S2) form an adjoint pair. This indicates that the identities (C1) and (C2) are too strong for λ-lattices despite the fact that their lattice versions work well for lattices, see the previous section. This can be explained by the next theorem showing that a λ-lattice satisfying identities (C1) and (C2) where $\odot$ and $\to$ form an adjoint pair is very close to a lattice.

Theorem 3. *Let* $\mathbf{A} = (A, \sqcup, \sqcap, ')$ *be a* λ*-lattice with a surjective unary operation* $'$ *satisfying identities* (C1) *and* (C2) *and assume that the Sasaki operations on* A *defined by* (S2) *form an adjoint pair. Then* $\mathbf{A}$ *is a lattice.*

Proof. Let $a, b, c \in A$ with $a \le b$. Since $'$ is surjective there exists some $d \in A$ with $d' = b$. Using identities (C1) and (C2) as well as Lemma 1 we obtain

$$\begin{aligned} a \sqcup c &= a \sqcup d' = d' \sqcup \big((a \sqcup d') \sqcap d\big) = d' \sqcup \big(d \sqcap \big((a \sqcup d') \sqcap d\big)\big) \\ &= d \to (a \odot d) \le d \to (b \odot d) = d' \sqcup \big(d \sqcap \big((b \sqcup d') \sqcap d\big)\big) = d' \sqcup \big((b \sqcup d') \sqcap d\big) \\ &= b \sqcup d' = b \sqcup c, \\ c \sqcap a &= \big(c' \sqcup (c \sqcap a)\big) \sqcap c = \big(\big(c' \sqcup (c \sqcap a)\big) \sqcup c'\big) \sqcap c = (c \to a) \odot c \le (c \to b) \odot c \\ &= \big(\big(c' \sqcup (c \sqcap b)\big) \sqcup c'\big) \sqcap c = \big(c' \sqcup (c \sqcap b)\big) \sqcap c = c \sqcap b. \end{aligned}$$

This means that $\sqcup$ and $\sqcap$ are monotone and according to Theorem 2.14 in [4], **A** is a lattice. ☐

We now present a λ-lattice whose Sasaki operations satisfy identities (C1) and (C2), but neither condition (A1), nor condition (A2).

Example 4. Put $B := \{a,b,c,d,e,f,g\}$ and $B' := \{x' \mid x \in B\}$, assume B, B' and $\{0,1\}$ to be pairwise disjoint and put

$$C := \{\{a,b,c\},\{a,d,f\},\{a,e,g\},\{b,d,g\},\{b,e,f\},\{c,d,e\},\{c,f,g\}\}.$$

Then to any two different elements x and y of B there exists exactly one element $F(x,y)$ of B satisfying $\{x,y,F(x,y)\} \in C$. (C is the set of all "lines" of the Fano plane with the set B of points.) We illustrate this construction by the following diagram: Put $A := B \cup B' \cup \{0,1\}$ and let $x,y \in A$. We define $x \leq y$ if $x = 0$ or $y = 1$ or $x = y$ or

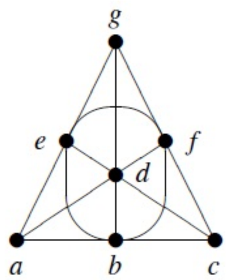

Fig. 6

The Fano plane

if $x \in B$ and $y \in B' \setminus \{x'\}$. Then $(A,\leq)$ is a poset. If x and y are different elements of B then we define $x \sqcup y := (F(x,y))'$. In all the other cases we define $x \sqcup y := \max(x,y)$ provided x and y are comparable with each other and $x \sqcup y := 1$ otherwise. If x and y are different elements of B then we define $x' \sqcap y' := F(x,y)$. In all the other cases we define $x \sqcap y := \min(x,y)$ provided x and y are comparable with each other and $x \sqcap y := 0$ otherwise. Then $(A,\sqcup,\sqcap)$ is a λ-lattice that is not a lattice. We extend the mapping $'$ from B to B' to a unary operation $'$ on A by the following table:

x	0	a'	b'	c'	d'	e'	f'	g'	1
x'	1	a	b	c	d	e	f	g	0

Put $\mathbf{A} := (A,\sqcup,\sqcap,')$. It is easy to see that $'$ is an antitone involution and a complementation and that **A** satisfies the identities $(x \sqcup y)' \approx x' \sqcap y'$ and $(x \sqcap y)' \approx x' \sqcup y'$. Let h and i be different elements of B and put $j := F(h,i)$. We prove that **A** satisfies the identity

(1) $(x \sqcup y) \sqcap y' \approx x \sqcap y'$.

If $x,y\in\{0,1\}$ or $y\in\{x,x'\}$ then (1) holds.

If $(x,y)=(h,i)$ then $(x\sqcup y)\sqcap y'=(h\sqcup i)\sqcap i'=j'\sqcap i'=h=h\sqcap i'=x\sqcap y'$.

If $(x,y)=(h,i')$ then $(x\sqcup y)\sqcap y'=(h\sqcup i')\sqcap i=i'\sqcap i=0=h\sqcap i=x\sqcap y'$.

If $(x,y)=(h',i)$ then $(x\sqcup y)\sqcap y'=(h'\sqcup i)\sqcap i'=h'\sqcap i'=x\sqcap y'$.

If $(x,y)=(h',i')$ then $(x\sqcup y)\sqcap y'=(h'\sqcup i')\sqcap i=1\sqcap i=i=h'\sqcap i=x\sqcap y'$.

By duality we obtain that **A** satisfies the identity

(2) $(x\sqcap y)\sqcup y'\approx x\sqcup y'$.

Now we have

$$y'\sqcup\big((x\sqcup y')\sqcap y\big)\approx y'\sqcup(x\sqcap y)\approx x\sqcup y',$$
$$\big(x'\sqcup(x\sqcap y)\big)\sqcap x\approx(x'\sqcup y)\sqcap x\approx x\sqcap y$$

showing that **A** satisfies the identities (C1) and (C2). Now let $k\in B\setminus\{h,i,j\}$ and put $l:=F(i,k)$ and $m:=F(h,k)$. Then $m=i$ would imply $j=F(h,i)=F(h,m)=k$, a contradiction. Hence $m\neq i$ and $h\leq i'$, but

$$k\sqcup(k'\sqcap h)=k\sqcup h=m'\not\leq i'=k\sqcup l=k\sqcup(k'\sqcap i')$$

showing that **A** does not satisfy condition (E1). By duality, **A** does not satisfy condition (E2). Let $\odot$ and $\to$ denote the Sasaki operations on A defined by (S2). According to Theorem 2, $\odot$ and $\to$ do not form an adjoint pair. Because of (1) and (2) we have

$$x\odot y\approx(x\sqcup y')\sqcap y\approx x\sqcap y,$$
$$x\to y\approx x'\sqcup(x\sqcap y)\approx x'\sqcup y.$$

Now $h'\odot i'=h'\sqcap i'=j\leq k'$, but $h'\not\leq k'=i\sqcup k'=i'\to k'$ showing directly that $\odot$ and $\to$ do not satisfy condition (A1). But $\odot$ and $\to$ do not satisfy condition (A2), too, since

$$k\leq j'=h\sqcup i=h\sqcup(h'\sqcap i)=h'\to i,$$

but

$$k\odot h'=(k\sqcup h)\sqcap h'=m'\sqcap h'=k\not\leq i.$$

Although the λ-lattices visualized in Figures 4 and 5 are complemented and the complementation is an antitone involution, the corresponding Sasaki operations defined by (S2) do not form an adjoint pair without regard how the $\sqcup$ and $\sqcap$ are defined.

However, there exist λ-lattices with a unary operation not being lattices, but whose Sasaki operations defined by (S2) form an adjoint pair. At first, consider the following example.

Example 5. Consider the λ-lattice $\mathbf{A}=(A,\sqcup,\sqcap,')$ with the unary operation $'$ defined by the following table:

x	0	a	b	c	d	1
x'	1	b	a	d	c	0

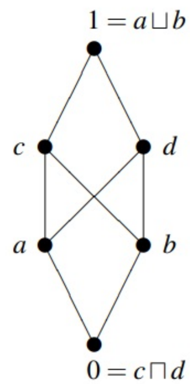

Fig. 7

A λ-lattice

Then the operation tables of the Sasaki operations $\odot$ and $\to$ on A defined by (S2) look as follows:

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	0	0	0	a
b	0	0	b	0	0	b
c	0	a	b	c	0	c
d	0	a	b	0	d	d
1	0	a	b	c	d	1

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	b	1	b	1	1	1
b	a	a	1	1	1	1
c	d	d	d	1	d	1
d	c	c	c	c	1	1
1	0	a	b	c	d	1

It can be easily verified that for all $x, y \in A$, $x \odot y$ is the smallest element z of A satisfying $x \leq y \to z$. Hence $\odot$ and $\to$ form an adjoint pair. It is interesting and not hard to prove that **A** satisfies conditions (D1) and (D2), but according to Theorem 3 cannot satisfy both identities (C1) and (C2). Indeed, **A** satisfies neither identity (C1) nor identity (C2) since

$$a' \sqcup \big((c \sqcup a') \sqcap a\big) = b \sqcup ((c \sqcup b) \sqcap a) = b \sqcup (c \sqcap a) = b \sqcup a = 1 \neq c = c \sqcup b = c \sqcup a',$$
$$\big(c' \sqcup (c \sqcap a)\big) \sqcap c = (d \sqcup a) \sqcap c = d \sqcap c = 0 \neq a = c \sqcap a.$$

There exist infinitely many of λ-lattices the Sasaki operations of which defined by (S2) form an adjoint pair. Namely, for every positive integer n consider the direct power $\mathbf{A}^n$ of the λ-lattice **A** (with unary operation) from Example 5. Since the operations on $\mathbf{A}^n$ are defined componentwise, every such $\mathbf{A}^n$ satisfies both conditions (A1) and (A2) and is not a lattice. Moreover, **A** is subdirectly irreducible. Consider the variety $\mathcal{V}(\mathbf{A})$ of λ-lattices (with unary operation) generated by **A**. According to Theorem 4.15 in [4] this variety is congruence distributive. Hence the only finite subdirectly irreducible members of this variety are homomorphic images of subalgebras of **A**. Because **A** is simple, all finite subdirectly irreducible algebras in $\mathcal{V}(\mathbf{A})$ are subalgebras of **A**. Up to **A** itself, these are only the subalgebras with universes $\{0,1\}$, $\{0,a,b,1\}$ and $\{0,c,d,1\}$ which are lattices, in fact Boolean algebras. Because every algebra in $\mathcal{V}(\mathbf{A})$ is a λ-lattice (with unary operation) which is a subdirect product of subdirectly irreducible members, i.e. a subalgebra of a direct product of an arbitrary number of these four algebras, their Sasaki operations form an adjoint pair again.

Of course, the λ-lattices (with unary operation) mentioned before are not the only λ-lattices (with unary operations) the Sasaki operations of which form an adjoint pair, other such examples are e.g. direct products of **A** with orthomodular lattices (considered as λ-lattices with a unary operation).

4. Ordered semirings and ring-like structures

In this section we investigate so-called *ordered semirings with a unary operation*, i.e. ordered sixtuples $(S,+,\cdot,0,',\leq)$ where $(S,+,\cdot,0)$ is a commutative semiring (see e.g. [8]), $'$ a unary operation and $\leq$ a partial order relation on S satisfying the identity $xx' \approx 0$. Recall from [8] that a *commutative semiring* is an algebra $(S,+,\cdot,0)$ of type $(2,2,0)$ such that the following holds:

$(S,+,0)$ is a commutative monoid,

$(S,\cdot)$ is a commutative semigroup,

$x0 \approx 0$,

the operation $\cdot$ is distributive with respect to $+$.

We can transform the Sasaki operations from (S1) by replacing $\vee$ and $\wedge$ with $+$ and $\cdot$, respectively. In this way we obtain the Sasaki operations on S defined by

(S3) $x \odot y := (x+y')y \quad$ and $\quad x \to y := x' + xy$

for all $x,y \in S$. Observe that in our case

$$x \odot y \approx (x+y')y \approx xy + y'y \approx xy + 0 \approx xy.$$

We investigate when the Sasaki operations on S defined by (S3) form an adjoint pair.

We introduce the following conditions:

(3) $x \leq y' + xyy$,

(4) $x \leq y$ implies $z' + zx \leq z' + zy$,

(5) $x \leq y$ implies $xz \leq yz$,

(6) $xy \leq x$.

Using of these conditions, we can state and prove the following result.

Theorem 4. *Let* $\mathbf{S} = (S,+,\cdot,0,',\leq)$ *be an ordered semiring with a unary operation and* $\odot$ *and* $\to$ *denote the Sasaki operations on* S *defined by* (S3). *Then the following holds:*

(i) *If* **S** *satisfies conditions* (3) *and* (4) *then* $\odot$ *and* $\to$ *satisfy condition* (A1)*,*

(ii) *if* **S** *satisfies conditions* (5) *and* (6) *then* $\odot$ *and* $\to$ *satisfy condition* (A2)*,*

(iii) *if* $\mathbf{S}$ *satisfies conditions* (3) – (6) *then* $\odot$ *and* $\rightarrow$ *form an adjoint pair.*

Proof. Let $a,b,c \in S$.

(i) If $\mathbf{S}$ satisfies conditions (3) and (4) and $a \odot b \leq c$ then we obtain

$$a \leq b' + abb = b' + b(ab) = b' + b(a \odot b) \leq b' + bc = b \rightarrow c.$$

(ii) If $\mathbf{S}$ satisfies conditions (5) and (6) and $a \leq b \rightarrow c$ then we obtain

$$a \odot b = ab \leq (b \rightarrow c)b = (b' + bc)b = b'b + bcb = 0 + c(bb) = c(bb) \leq c.$$

(iii) This follows from (i) and (ii).

□

Example 6. Consider a unital Boolean ring $\mathbf{R} = (R,+,\cdot,0,1)$ and define a unary operation $'$ and a binary relation $\leq$ on R by $x' := x+1$ and $x \leq y$ whenever $xy = x$ for all $x,y \in R$, respectively. Then $\mathbf{R} := (R,+,\cdot,0,',\leq)$ is an ordered semiring with a unary operation satisfying conditions (3) – (6). Namely, let $a,b,c \in R$. Then we have:

$aa' = a \wedge a' = 0,$

$a \leq b' \vee a = (b \wedge a')' = (ba')' = b(a+1)+1 = b+1+ab = b'+abb,$

$a \leq b$ implies $c' + ca = c+1+ca = c(a+1)+1 = (c \wedge a')' \leq (c \wedge b')' = c(b+1)+1 =$

$= c+1+cb = c'+cb,$

$a \leq b$ implies $ac = a \wedge c \leq b \wedge c = bc,$

$ab = a \wedge b \leq a.$

According to Theorem 4 (iii) the Sasaki operations on a Boolean ring $\mathbf{R}$ defined by (S3) form an adjoint pair.

Finally, we are interested in algebras similar to semirings in which also Sasaki operations forming an adjoint pair can be defined.

In [3] the first author introduced the following notion:

Definition 1. An *orthomodular pseudoring* is an algebra $(R,+,\cdot,0,1)$ of type $(2,2,0,0)$ such that $(A,+,0)$ is a commutative groupoid with neutral element 0, $(A,\cdot,1)$ is a semilattice with neutral element 1 and the following identities are satisfied:

$$\begin{aligned} x+x &\approx 0, \\ x0 &\approx 0, \\ (x+1)+y &\approx x+(1+y), \\ (1+xy)x &\approx x+xyx, \\ (1+x)(1+xy) &\approx 1+x, \\ (1+x(1+y))(1+y(1+x)) &\approx 1+(x+y), \end{aligned}$$

$$(x+xy)+xy \approx x.$$

Orthomodular pseudorings are closely related to orthomodular lattices in a similar way as Boolean rings are related to Boolean algebras.

Theorem 5. (*cf.* [3]) *If* $(L,\vee,\wedge,',0,1)$ *is an orthomodular lattice and*

$$x+y := (x\wedge y')\vee(x'\wedge y),$$
$$xy := x\wedge y$$

for all $x,y\in L$ *then* $(L,+,\cdot,0,1)$ *is an orthomodular pseudoring. If, conversely,* $(R,+,\cdot,0,1)$ *is an orthomodular pseudoring and*

$$x\vee y := 1+(1+x)(1+y),$$
$$x\wedge y := xy,$$
$$x' := 1+x$$

for all $x,y\in R$ *then* $(R,\vee,\wedge,',0,1)$ *is an orthomodular lattice. This correspondence between orthomodular lattices and orthomodular pseudorings is one-to-one.*

We can translate the Sasaki operations defined by (S1) for orthomodular lattices into the operations $+$ and $\cdot$ of the corresponding orthomodular pseudoring $(R,+,\cdot,0,1)$ as follows:

(S4) $x\odot y := (1+(1+x)y)y$ and $x\to y := 1+x(1+xy)$

for all $x,y\in R$.

Example 7. Consider the following orthomodular pseudoring $(R,+,\cdot,',0,1)$ with $R=\{0,a,b,c,d,1\}$ and

$+$	0	a	b	c	d	1
0	0	a	b	c	d	1
a	a	0	0	1	0	c
b	b	0	0	0	1	d
c	c	1	0	0	0	a
d	d	0	1	0	0	b
1	1	c	d	a	b	0

$\cdot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	0	0	0	a
b	0	0	b	0	0	b
c	0	0	0	c	0	c
d	0	0	0	0	d	d
1	0	a	b	c	d	1

x	x'
0	1
a	c
b	d
c	a
d	b
1	0

The Sasaki operations on R defined by (S4) read as follows:

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	b	0	d	a
b	0	a	b	c	0	b
c	0	0	b	c	d	c
d	0	a	0	c	d	d
1	0	a	b	c	d	1

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	c	1	c	c	c	1
b	d	d	1	d	d	1
c	a	a	a	1	a	1
d	b	b	b	b	1	1
1	0	a	b	c	d	1

Using Theorem 5 and the result in [5] on orthomodular lattices mentioned in the beginning we immediately obtain

Theorem 6. *If* $(R,+,\cdot,0,1)$ *is an orthomodular pseudoring then the Sasaki operations on R defined by* (S4) *form an adjoint pair.*

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SEMISIMPLE-CONTINUOUS MODULES

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Abstract. An R-module M is said to be a semisimple-continuous, if it is weak CS (i.e., every semisimple submodule of M is essential in a direct summand of M) and semisimple-direct injective (i.e., if A and B are isomorphic semisimple submodules of M such that A is a direct summand of M, then B is a direct summand of M). It is proved that any semisimple-continuous module is decomposed as a direct sum of a semisimple module and a module with square-free socle. We investigate when the finite exchange property implies full exchange property for the former class of modules. Moreover, we explore the notion of the semisimple-continuity for Abelian groups. We also characterize right Noetherian right V-rings in terms of semisimple-continuous modules. Examples are delimit our results.

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1. Introduction

In this paper, all rings are associative with unity and all modules are unital right R-modules. We use M to denote such a module. Recall that a module is *extending* (or CS), if every submodule is essential in a direct summand, and a module is a $C2$-module, if every submodule isomorphic to a direct summand is a direct summand, in addition a module is a $C3$-module, if the sum of any two direct summands with zero intersection is again a direct summand.

In [5], the "simple" versions of $C2$ and $C3$-modules are introduced and analyzed. The authors in [5] call these modules simple-direct injective. Then M is *simple-direct injective*, if every simple submodule isomorphic to a direct summand is itself a direct summand, or equivalently if the sum of any two simple direct summands with zero intersection is again a direct summand. In [1], the authors explore the "semisimple" version of $C2$ and $C3$-modules. They call a module M *semisimple-direct injective*, if whenever S and T are semisimple submodules of M with $S \cong T$ and T is a direct summand of M, then S is a direct summand of M, or equivalently, for any semisimple direct summands S and T of M with $S \cap T = 0$, $S \oplus T$ is a direct summand of M. The class of simple-direct injective modules properly contains semisimple-direct injective modules. Extending modules play an important role in rings and categories of modules, their generalizations and related modules have been studied extensively. In [19], the concept of simple-continuous modules is defined by a kind of extending condition with simple-direct injectivity. A module is said to be *simple-continuous*, if every simple submodule of M is essential in a direct summand of M, and M is simple-direct injective. This module class has already been known in the literature as min-continuous modules in [16].

The motivation of this paper comes from the studies in [1, 5, 19]. Our goal is to introduce and investigate semisimple-continuous modules. We call a module M *semisimple-continuous*, if it is both weak CS (i.e., every semisimple submodule of M is essential in a direct summand of M [17]) and semisimple-direct injective. The class of semisimple-continuous modules is a proper subclass of simple-continuous modules. We start by presenting some results which are related to the notions of weak CS and semisimple-direct injectivity in Section 2.

In Section 3, we introduce semisimple-continuous modules and provide several examples. Observe that it is unknown whether the direct summand of a weak CS-module is weak CS or not. We examine when the semisimple-continuous module property is inherited by direct summands. As a result of this fact, we underline when the direct summand of weak CS-modules enjoys this property. It is shown that semisimple-continuous modules are not closed under direct sums. Hence, we also deal with when the direct sums of semisimple-continuous modules are semisimple-continuous.

We obtain decomposition results in Section 4. We show that if M is semisimple-continuous, then M can be decomposed as $M = A \oplus B \oplus K$ where $A \cong B$, $A \oplus B$ is semisimple, and K, which is $(A \oplus B)$-injective, has a square-free socle. As a consequence, we show that if M is semisimple-continuous with the finite exchange, then M has the full exchange. Moreover, we characterize right Noetherian right V-rings in terms of semisimple-continuous modules.

For a nonempty subset X of M, $X \leq M$, $X \leq^{ess} M$ and $X \leq^{\oplus} M$ denote X is a right R-submodule of M, X is an essential right R-submodule of M and X is a direct summand of M, respectively. For notation we use $Soc(M)$, $Rad(M)$, $End(M)$ and $E(M)$ the socle, the radical, the endomorphism ring and the injective hull of a module M, respectively. Note that $M_n(R)$, $T_n(R)$ and $l_R(M)$ stand for the n-by-n matrix ring over R, the n-by-n upper triangular matrix ring over R, and left annihilators of M in R, respectively. Other terminology and notation can be found in [13] and [14].

2. More properties of weak CS modules and semisimple-direct injective modules

Recall that a module M is called *weak CS* [17], if every semisimple submodule of M is essential in a direct summand of M. A ring R is right *weak CS* provided that R_R is a right weak CS-module. In this section, we investigate the behavior of weak CS rings with respect to the generalized upper triangular matrix rings. Let R and S be rings with unity and M be a (R,S)-bimodule. Then $T = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ denotes the generalized upper triangular matrix ring. Recall from [12, Proposition 1.17] that $\begin{pmatrix} J_1 & \\ & J_2 \\ 0 & \end{pmatrix}$ is the right ideal of T such that J_1 is a right ideal of R and J_2 is a right S-submodule of $M \oplus S$ with $J_1M \subseteq J_2$. For this notation, we refer to [15].

Lemma 1 ([10, Exercise 3])**.** *Let T be a generalized triangular matrix ring. Then*

$$Soc(T_T) = \begin{pmatrix} Soc(l_R(M)) & Soc(M_S) \\ 0 & Soc(S_S) \end{pmatrix}.$$

Proposition 1. *Let T be a generalized triangular matrix ring. Then T_T is weak CS if and only if for any semisimple right ideal $\begin{pmatrix} J_1 & \\ & J_2 \\ 0 & \end{pmatrix}$ of T, there exists an idempotent $\begin{pmatrix} e & m \\ 0 & f \end{pmatrix}$ of T such that $J_1 \leq^{ess} [eR \cap l_R(M)]$ and $J_2 \leq^{ess} (eM + V)$, where $V = \left\{ \begin{pmatrix} ms \\ fs \end{pmatrix} : \forall s \in S \right\}$.*

Proof. Let T be a right weak CS-ring and $\begin{pmatrix} J_1 & \\ & J_2 \\ 0 & \end{pmatrix} \leq Soc(T_T)$. Then $\begin{pmatrix} J_1 & \\ & J_2 \\ 0 & \end{pmatrix}$ $\leq^{ess} \begin{pmatrix} e & m \\ 0 & f \end{pmatrix} T$ for some idempotent $\begin{pmatrix} e & m \\ 0 & f \end{pmatrix}$ of T. Now by [15, Lemma 2.2], $J_2 \leq^{ess} (eM + V)$ as right S-modules, where $V = \left\{ \begin{pmatrix} ms \\ fs \end{pmatrix} : \forall s \in S \right\}$ and $[J_1 \cap l_R(M)]$ $\leq^{ess} [eR \cap l_R(M)]$ as right R-modules. However, by Lemma 1, $[J_1 \cap l_R(M)] = J_1$, so as desired. The converse is clear. □

Corollary 1. *Let T be a generalized triangular matrix ring and ${}_RM$ is faithful. Then T_T is a weak CS-ring if and only if for any right S-module*

$$\begin{pmatrix} 0 & \\ & J_2 \\ 0 & \end{pmatrix} \leq \begin{pmatrix} 0 & Soc(M_S) \\ 0 & Soc(S_S) \end{pmatrix},$$

there exists an idempotent $\begin{pmatrix} 0 & m \\ 0 & f \end{pmatrix}$ *of* T *such that* $J_2 \leq^{ess} V$*, where*

$$V = \left\{ \begin{pmatrix} ms \\ fs \end{pmatrix} : \forall s \in S \right\}.$$

Let M and N be right R-modules. Recall from [2] that M is called *socle-N-injective*, denoted soc-N-injective, if any R-homomorphism $f\colon Soc(N) \to M$ extends to N. M is called *soc-injective*, if M is soc-R-injective. In case, M is soc-N-injective and N is soc-M-injective, then M and N are called *relatively soc-injective*.

Proposition 2. *Let T be a generalized triangular matrix ring. If T is a right weak CS-ring, then the followings hold:*

- (i) *S_S is weak CS, and if $l_R(M) = R$, then R_R is weak CS.*
- (ii) *M_S is a soc-injective module.*
- (iii) *$l_R(M)$ (without identity) is a weak CS-module.*

Proof. (i) If $l_R(M) = R$, then by Proposition 1, R is a right weak CS-ring. To see that S is right weak CS, let $X \leq Soc(S_S)$. Then $\begin{pmatrix} 0 & 0 \\ 0 & X \end{pmatrix} \leq Soc(T_T)$, so there exist idempotent $\alpha = \begin{pmatrix} r & m \\ 0 & s \end{pmatrix}$ of T such that $(0 \oplus X) \leq^{ess} \alpha T$. Then for $x \in X$, write $\bar{x} = \begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix}$, so we have $\alpha\bar{x} = \bar{x}$, this implies that $x = sx$. Thus $X \leq sS$, where $s^2 = s$. Let $0 \neq ss_1 \in sS$. Then $\begin{pmatrix} r & m \\ 0 & s \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & s_1 \end{pmatrix} = \begin{pmatrix} 0 & ms_1 \\ 0 & ss_1 \end{pmatrix} \neq 0$. Thus $(0 \oplus X)$ is essential in αT, so there exists $\begin{pmatrix} r_2 & m_2 \\ 0 & s_2 \end{pmatrix} \in T$ such that $0 \neq \begin{pmatrix} 0 & ms_1 \\ 0 & ss_1 \end{pmatrix}\begin{pmatrix} r_2 & m_2 \\ 0 & s_2 \end{pmatrix} \in (0 \oplus X)$. Hence, $\begin{pmatrix} 0 & ms_1s_2 \\ 0 & ss_1s_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & x' \end{pmatrix}$ for some $x' \in X$, and so $0 \neq ss_1s_2 = x' \in X$. It shows that $X_S \leq^{ess} sS_S$.

(ii) Let $X \leq Soc(S_S)$. Consider the S-linear map $\phi\colon X \to M$. Write

$$F = \left\{ \begin{pmatrix} 0 & \phi(x) \\ 0 & x \end{pmatrix} : x \in X \right\}.$$

Then $F \leq Soc(T_T)$. Therefore, $F \leq^{ess} tT$ for some $t^2 = t = \begin{pmatrix} 0 & m \\ 0 & s \end{pmatrix}$ of T by [18, Lemma 3.87]. Hence, for each $\alpha \in F$, $\alpha = \begin{pmatrix} 0 & \phi(x) \\ 0 & x \end{pmatrix}$ for some $x \in X$. Thus $\alpha = t\alpha$, so $\phi(x) = mx$, as required.

(iii) Let $B = l_R(M)$ and $X \leq Soc(B_B)$. Then $\begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix} \leq Soc(T_T)$. So there exists an element $\alpha^2 = \alpha = \begin{pmatrix} e & k \\ 0 & f \end{pmatrix} \in T$ such that $X \leq^{ess} \begin{pmatrix} e & k \\ 0 & f \end{pmatrix} T_T$. Observe that $f = 0$. Assume $0 \neq k \in M$. Then $0 \neq \begin{pmatrix} e & k \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & k \\ 0 & 0 \end{pmatrix} \in \begin{pmatrix} e & k \\ 0 & f \end{pmatrix} T$. So there is $\begin{pmatrix} r_1 & m_1 \\ 0 & s_1 \end{pmatrix} \in T$ such that $0 \neq \begin{pmatrix} 0 & k \\ 0 & 0 \end{pmatrix} \begin{pmatrix} r_1 & m_1 \\ 0 & s_1 \end{pmatrix} = \begin{pmatrix} 0 & ks_1 \\ 0 & 0 \end{pmatrix} \in \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix}$, a contradiction. So, $k = 0$. Then $X \leq^{ess} \begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix} T_T$. By Proposition 1, $X \leq^{ess} (eR \cap B)$ and $eM = 0$. It follows that $e \in B$, and hence $eR \cap B = eB$, so $X_B \leq^{ess} eB_B$. Therefore B_B is weak CS. □

Proposition 3. *If T is a right weak CS-ring, then $(M \oplus S)_S$ is a weak CS-module.*

Proof. Let $X \leq Soc(M \oplus S)_S$. Then $\begin{pmatrix} 0 & \\ & X \\ 0 & \end{pmatrix} \leq Soc(T_T)$. Hence there exists $\alpha^2 = \alpha = \begin{pmatrix} e & m \\ 0 & f \end{pmatrix} \in T$ such that $\begin{pmatrix} 0 & \\ & X \\ 0 & \end{pmatrix} \leq^{ess} \alpha T$. Note that $T = \begin{pmatrix} e & m \\ 0 & f \end{pmatrix} T \oplus \begin{pmatrix} (1-e) & -m \\ 0 & (1-f) \end{pmatrix} T$ and $\alpha T = \begin{bmatrix} eR & eM \\ 0 & 0 \end{bmatrix} + K$, where $K = \left\{ \begin{pmatrix} 0 & ms \\ 0 & fs \end{pmatrix} : \forall s \in S \right\}$. Now by [15, Lemma 2.2], $X \leq^{ess} (eM + V)$ where $V = \left\{ \begin{pmatrix} ms \\ fs \end{pmatrix} : \forall s \in S \right\}$. Hence $(eM + V) \oplus [(1-e)M \oplus U] = (M \oplus S)_S$, where $U = \left\{ \begin{pmatrix} -ms \\ (1-f)s \end{pmatrix} : \forall s \in S \right\}$. Thus $(M \oplus S)_S$ is weak CS. □

Recall from [9] that a ring extension T of R is said to be *right intrinsic over* R, if $X \cap R \neq 0$ for each nonzero right ideal X of T, denoted by $R \leq_r^{int} T$.

Proposition 4. *If $R \leq_r^{int} S$ and R_R is a weak CS-module, then S_S is a weak CS-module.*

Proof. Let Y be a semisimple submodule of S and $X = R \cap Y$. Then X is a semisimple submodule R_R. Thus there is $e = e^2 \in R$ such that $X_R \leq^{ess} eR_R$. Let $y \in Y$. Then $y = ey + (1-e)y$. If $(1-e)y \neq 0$, then there exists $r \in R$ such that $0 \neq (1-e)yr \in R \cap Y = X \subseteq eR$, a contradiction. Therefore $(1-e)y = 0$. So $Y \leq eS$. Let $0 \neq es \in eS$. Then $0 \neq esr_1 \in R \cap eS \leq eR$, for some $r_1 \in R$. Hence $R \cap Y \leq R \cap eS \leq eR$. It follows that $0 \neq es(r_1r_2) \in R \cap Y \leq Y$, for some $r_2 \in R$. Thus $Y_S \leq^{ess} eS_S$ which gives that S_S is a weak CS-module. □

Corollary 2. *Let S be an essential overring of a ring R. If R_R is weak CS, then so is S_R.*

Proof. It is a consequence of Proposition 4. □

Theorem 1. (i) *Let R be a ring such that $R = ReR$ and $S = eRe$ for some $e^2 = e \in R$. Then M_R is weak CS if and only if the right S-module Me is weak CS.*

(ii) *Let R be a ring such that $R = ReR$ for some $e^2 = e \in R$. Then R_R is weak CS if and only if the right eRe-module Re is weak CS.*

Proof. It is clear from [18, Propositions 2.77, 2.78]. □

Corollary 3. *$M_n(R)$ is a right weak CS-ring if and only if the free right R-module R^n is weak CS.*

Proof. Note that $M_n(R) = M_n(R)eM_n(R)$, where e is the matrix unit with 1 in the (1,1)th position and zero elsewhere. Then the result follows from Theorem 1. □

Observe that for any ring R, the polynomial ring $R[x]$ has zero socle. Thus $M_n(R[x])$ is also a weak CS-module by Corollary 3.

Corollary 4. (i) *If $T_n(R)$ is weak CS, then so is $M_n(R)$.*

(ii) *If $T_n(R)$ is weak CS, then the free right R-module R^n is weak CS.*

Proof. (i) Note that $T_n(R)$ is an essential overring of $M_n(R)$. Thus Corollary 2 yields the result.

(ii) It follows from part (i) and Corollary 3. □

In the rest of this section, we deal with some structural properties of semisimple-direct injective modules.

Proposition 5. *Let $M = \bigoplus_{i \in I} M_i$ for some $M_i \leq M$, and consider one of the following conditions are satisfied:*

(i) *For each semisimple direct summand D of M, $D \subseteq M_i$ for some $i \in I$.*

(ii) *For each direct summand D of M, $D = \bigoplus_{i \in I}(D \cap M_i)$.*

(iii) *Each direct summand D of M is fully invariant.*

Then M is semisimple-direct injective if and only if M_i is semisimple-direct injective for all $i \in I$.

Proof. Assume M is semisimple-direct injective and let U and V be semisimple submodules of M_i such that $U \cong V \leq^{\oplus} M_i$. Then we have $V \leq^{\oplus} M$ and hence $U \leq^{\oplus} M$. Then $U \leq^{\oplus} M_i$, so M_i is semisimple-direct injective. Conversely, suppose M_i is semisimple-direct injective for all $i \in I$. Observe that (iii) implies (ii), and (ii) implies (i). Thus it is enough to complete the proof for condition (i). Now, let A and B be semisimple direct summands of M with $A \cap B = 0$. By (i), $A \subseteq M_i$ and $B \subseteq M_j$ for some $j, k \in I$. Note that $A \leq^{\oplus} M_j$ and $B \leq^{\oplus} M_k$. Assume $j \neq k$. Then $A \oplus B \leq^{\oplus} M_j \oplus M_k \leq^{\oplus} M$. If $j = k$, then $A \oplus B \leq^{\oplus} M_j$, as M_j is semisimple-direct injective. Hence $A \oplus B \leq^{\oplus} M$. Therefore M is semisimple-direct injective by [1, Proposition 2.1]. □

The conditions (i), (ii) and (iii) in Proposition 5 are not superfluous. Consider the $\mathbb{Z}$-module $M = \mathbb{Z}_p \oplus \mathbb{Z}(p^\infty)$ for a prime p. Since M is not simple-direct injective by [11, Lemma 2.5], M is not semisimple-direct injective. Let $X = (1,a)\mathbb{Z}$, where $0 \neq a \in \mathbb{Z}(p^\infty)$ such that $pa = 0$. Then $X \oplus \mathbb{Z}(p^\infty) = M$. But $X \nsubseteq \mathbb{Z}_p \oplus 0$ and $X \nsubseteq 0 \oplus \mathbb{Z}(p^\infty)$. Note that $T_p(M) = \{x \in M \mid xp^k = 0$ for some non-negative integer $k\}$ is a submodule of M which is called the *p-primary component* of M. It is well-known that every torsion R-module is a direct sum of its p-primary components.

Corollary 5. *Let R be a Dedekind domain and T a torsion module over R. Then T is semisimple-direct injective if and only if the $T_p(M)$ is semisimple-direct injective for each prime p.*

Proof. Note that $T_p(M)$ is fully invariant. Now, it follows from Proposition 5. □

Proposition 6. *Let M be an R-module. If $M \oplus M$ is a semisimple-direct injective module, then M is a semisimple-direct injective module.*

Proof. Let $M \oplus M$ be a semisimple-direct injective module and S be a semisimple submodule of M such that $S \cong S' \leq^{\oplus} M$. Clearly, S' is a semisimple submodule of M. We need to show that $S \leq^{\oplus} M$. Write $M = S' \oplus T$ for some $T \leq M$. Since $M \oplus M = (S' \oplus T) \oplus M = S' \oplus (M \oplus T)$ is a semisimple-direct injective module, and if we take $\tau \colon S \to S'$ as the preceding isomorphism, $\tau^{-1} \colon S' \to M = (S' \oplus T)$ splits by [1, Proposition 2.1(4)]. Hence $S \leq^{\oplus} M$. □

Theorem 2. *Let $M = A \oplus B$, where A and B are relatively soc-injective. Then M is a semisimple-direct injective module.*

Proof. Let X and Y be any two semisimple direct summands of M with $X \cap Y = 0$. We will show that $X \oplus Y \leq^{\oplus} M$. We have submodules X', Y' of M such that $M = X \oplus X'$ and $M = Y \oplus Y'$. By the hypothesis, X and X' are relatively soc-injective and so also Y and Y'. Hence by [2, Theorem 2.2 (4)], X (respectively, Y) is soc-$(X \oplus X')$-injective (respectively, soc-$(Y \oplus Y')$-injective). Then the semisimple module $X \oplus Y$ is soc-M-injective by [2, Theorem 2.2 (1)]. Therefore $X \oplus Y \leq^{\oplus} M$. □

3. SEMISIMPLE-CONTINUOUS MODULES

In this section, the class of semisimple-continuous modules is introduced and investigated. Module theoretical properties such as direct summands and direct sums of the former class of modules are examined and examples are given to illustrate the results.

Definition 1. A module M is called *semisimple-continuous*, if M is both weak CS and semisimple-direct injective. A ring R is called *right semisimple-continuous* if the module R_R is semisimple-continuous.

Example 1. (i) Semisimple, uniform and (quasi-)continuous modules are semisimple-continuous modules.

(ii) Semisimple-continuous modules are simple-continuous, but not vice versa:

(1) Consider $R = < \oplus_{i=1}^{\infty} F_i, 1_{\prod_{i=1}^{\infty} F_i} >$ as a subring of $\prod_{i=1}^{\infty} F_i$ generated by $\oplus_{i=1}^{\infty} F_i$ and $1_{\prod_{i=1}^{\infty} F_i}$, where $F_i = \mathbb{Z}_2$ for any $i \in \mathbb{N}$. Then R is a commutative, non self-injective V-ring and $Soc(R)$ is essential in R. It implies that R is not Noetherian. Therefore we can infer from Theorem 6 or [19, Theorem 3.1] that there exists a simple-continuous module over R which is not semisimple-continuous.

(2) Let V be an infinite-dimensional vector space over a field F. Let $Q = End_F(V)$, $J = \{x \in Q : dim_F(xV) < \infty\}$ and $R = F + J$. Then R is a right V-ring and R is not right Noetherian. By Theorem 6 and [19, Theorem 3.1], there is a simple-continuous right R-module which is not semisimple-continuous.

(3) Let $L = End(V_F)$ be the full right linear ring of an infinite dimensional right vector space V over a field F, let S be the ideal consisting of linear transformations of finite rank, and let $R = S + F$ be the subring generated by S and the subring F consisting of scalar transformations (sending every $v \longrightarrow va$ for some $a \in F$). Then R is a right V-ring such that $R/Soc(R)$ is a field, hence a simple right R-module. Now it is easy to see that R_R is not weak CS. For this write $Soc(R) = A \oplus B$, where A and B are infinitely generated semisimple right ideals. If R_R were weak CS, A and B would be essential in some direct summands A' and B', respectively. It is clear that A and B are proper submodules of A' and B'. But then, $\frac{A' \oplus B'}{A \oplus B} \subseteq \frac{R}{Soc(R)}$, a contradiction since $\frac{A' \oplus B'}{A \oplus B}$ is not simple. Note that R_R is simple-continuous by [19, Theorem 3.1].

As an application of Proposition 1, we construct the following example.

Example 2. Let $T = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ be a generalized triangular matrix ring, where $R = M = \mathbb{Z}_2$ and $S = \mathbb{Z}$. Then ${}_RM$ is a faithful module and $Soc(T_T) = \begin{pmatrix} 0 & \mathbb{Z}_2 \\ 0 & 0 \end{pmatrix}$. Clearly, $Soc(T_T)$ essential in αT, where $\alpha^2 = \alpha = \begin{pmatrix} \bar{1} & \bar{0} \\ 0 & 0 \end{pmatrix} \in T$. Hence T_T is right weak CS by Corollary 1. Moreover, T_T is right semisimple-direct injective by [1, Theorem 2.21], so T_T is right semisimple-continuous.

Proposition 7. *M is a semisimple-continuous module if and only if the followings hold:*

(i) Let $X \leq Soc(M)$. Then there exists a closed submodule K in M such that $X \leq^{ess} K$ and any homomorphism $\phi \colon K \oplus L \to M$ can be extended to M for some complement L of K.

(ii) For $X, Y \leq Soc(M)$ with $X \cong Y \leq^{\oplus} M$, every homomorphism $\theta \colon X \to M$ can be extended to M.

Proof. Let M be a semisimple-continuous module. Then (i) and (ii) hold clearly. For converse, condition (i) implies that K is a direct summand of M by [18, Lemma 3.97]. Hence $X \leq^{ess} K \leq^{\oplus} M$ for all $X \leq Soc(M)$. Thus M is weak CS. By condition

(ii), X is a direct summand of M by virtue of [18, Proposition 2.85]. Hence M is semisimple-direct injective, so M is semisimple-continuous. □

Proposition 8. *Let R be a Dedekind domain. If the component $T_p(M)$ is semisimple-direct injective for every prime p, then every finitely generated R-module M is a semisimple-continuous module.*

Proof. Let M be a finitely generated module, then by [17, Theorem 1.16] M is a weak CS-module. To see that M is semisimple-direct injective, note that M has a decomposition $M = M_1 \oplus M_2$ for some torsion module M_1 and torsion-free module M_2. Note that, if S is any semisimple submodule of M, then $S \subseteq M_1$. Hence M is a semisimple-direct injective module by Corollary 5. □

Consider the $\mathbb{Z}$-module $M = \mathbb{Z}_8 \oplus \mathbb{Z}_2$. It is clear that M is finitely generated module over Dedekind domain. But it is not a semisimple-continuous module by Example 3(1). The following results are related to the direct summand of semisimple-continuous modules.

Proposition 9. *Let M be a semisimple-continuous module module and X a fully invariant direct summand of M. Then X is semisimple-continuous.*

Proof. Let X be a fully invariant direct summand of a semisimple-continuous module M. Then X is semisimple-direct injective by Proposition 5. Assume A is a semisimple submodule of X. Then by hypothesis, $A \leq^{ess} Y \leq^{\oplus} M$ for some $Y \leq M$. Then A is essential in $Y \cap X$ and clearly, $Y \cap X$ is a direct summand of X. Thus X is weak CS. Therefore X is semisimple-continuous. □

Recall that M is called a *UC-module*, if every submodule of M has a unique essential closure.

Proposition 10. (i) *Let M be a semisimple-continuous and UC module. Then any direct summand of M is semisimple-continuous.*

(ii) *Let M be a semisimple-continuous module satisfying $C3$. Then any direct summand of M is semisimple-continuous.*

(iii) *Any direct summand of a nonsingular semisimple-continuous module is semisimple-continuous.*

Proof. The proof follows from Proposition 5 and [18, Propositions 4.7-4.9, Corollary 4.8]. □

Corollary 6. *Let M be a semisimple-continuous module. If M is a duo, or M has an Abelian endomorphism ring, then every direct summand of M is semisimple-continuous.*

Proof. It follows from [7, Theorem 4.4] and Proposition 10. □

Proposition 11. *Let R be an Artian serial ring with $J(R)^2 = 0$. Then any direct summand of semisimple-continuous R-module is semisimple-continuous.*

Proof. By Proposition 5, any direct summand of semisimple-direct injective R-module is semisimple-direct injective. Hence [1, Theorem 2.10] implies that every semisimple-continuous R-module has $C3$. Thus the proof follows from Proposition 10. □

Proposition 12. *Let $M = M_1 \oplus M_2$ be a semisimple-continuous module and M_1 be a semisimple fully invariant submodule of M. Then M_1 and M_2 are semisimple-continuous.*

Proof. M_1 is a semisimple-continuous module by Proposition 9. Let X be a semisimple submodule of M_2. Then $M_1 \oplus X$ is a semisimple submodule of M. Then there exists a direct summand N of M such that $M_1 \oplus X \leq^{ess} N$. Note that $M_1 \subseteq M_1 \oplus X \subseteq N$, so by modular law, $N = N \cap (M_1 \oplus M_2) = M_1 \oplus (N \cap M_2)$. Since $M_1 \oplus X \subseteq N$, we have $M_2 \cap (M_1 \oplus X) \leq^{ess} N \cap M_2$. Notice that $X = M_2 \cap (M_1 \oplus X)$ by modular law. Thus, $X \leq^{ess} N \cap M_2$. It is enough to show that $N \cap M_2$ is a direct summand of M_2. Observe that $M = N \oplus W = M_1 \oplus (N \cap M_2) \oplus W$, and hence $M_2 = M_2 \cap M = M_2 \cap (M_1 \oplus (N \cap M_2) \oplus W) = (N \cap M_2) \oplus [M_2 \cap (M_1 \oplus W]$. Therefore $N \cap M_2$ is a direct summand of M_2. Thus M_2 is a weak CS-module. Note that M_2 is semisimple-direct injective by Proposition 5. Therefore M_2 is semisimple-continuous. □

Recall that a module M has the *summand intersection property*, *SIP*, in case the intersection of any two direct summands is again a direct summand of M.

Proposition 13. *Assume M is semisimple-continuous with SIP. Then any direct summand of M is semisimple-continuous.*

Proof. Assume $M = D \oplus D'$ for some $D, D' \leq M$. To see D is weak CS, let $A \leq Soc(D)$. Then $A \oplus Soc(D') \leq Soc(M)$. By hypothesis, $A \oplus Soc(D') \leq^{ess} K$, for some direct summand K of M. Now clearly, $D \cap (A \oplus Soc(D')) \leq^{ess} D \cap K$. By modular law, $D \cap (A \oplus Soc(D')) = A$. Hence $A \leq^{ess} D \cap K$. But then M has SIP and so $D \cap K$ a direct summand of M. Observe that $D \cap K$ is a direct summand of D. Thus D is weak CS. Note that SIP condition implies semisimple-direct injectivity, hence M is semisimple-continuous. □

Proposition 14. *Let M be an R-module. If $M \oplus M$ is a semisimple-continuous module, then M is semisimple-continuous.*

Proof. In view of Proposition 6, we only show that M is a weak CS-module. Let S be a semisimple submodule of M. Then $S' = S \oplus 0$ is semisimple in $M \oplus M$. So by hypothesis, $S' \leq^{ess} D$, where $D \leq^{\oplus} (M \oplus M)$. Consider $T = \pi_1(D)$ where $\pi_1 : M \oplus M \to M$ is the first projection on M. Then T is a summand of M and clearly, S is essential in T. □

The next example shows that the direct sum of any two semisimple-continuous modules may not be semisimple-continuous.

Example 3. (1) Let $M = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$ be a $\mathbb{Z}$-module. Clearly, $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/8\mathbb{Z}$ are semisimple-continuous $\mathbb{Z}$-modules. However, M is not a semisimple continuous $\mathbb{Z}$-module, because the simple non-summand $0 \oplus \mathbb{Z}(4+8\mathbb{Z})$ is isomorphic to the simple summand $\mathbb{Z}/2\mathbb{Z} \oplus 0$.

(2) Let R be the trivial extension of $\mathbb{Z}_4$ with the $\mathbb{Z}_4$-module $2\mathbb{Z}_4$, such that

$$R = \left\{ \begin{pmatrix} a & 2b \\ 0 & a \end{pmatrix} : a, b \in \mathbb{Z}_4 \right\}.$$

Let $X = 2\mathbb{Z}_4 \oplus 0$, $Y = 0 \oplus 2\mathbb{Z}_4$, and $A = R/X$ and $B = R/Y$. Then A and B are uniform, and hence semisimple-continuous. Since $A \oplus B$ is not weak CS from [18, Example 4.3], $A \oplus B$ is not semisimple-continuous.

Proposition 15. *Let $M = A \oplus B$, where A is weak CS-module such that $Soc(A) \leq Rad(A)$ and B is fully invariant semisimple submodule of M. Then M is a semisimple-continuous module.*

Proof. M is a weak CS-module, by [17, Lemma 1.10]. We show that M is a semisimple-direct injective module. Consider semisimple submodules X and Y of M such that $X \cong Y$ and $Y \leq^{\oplus} M$. Then Y is not contained in A because each simple module of A is small in A. This shows that $\pi_2(Y) \neq 0$, where $\pi_2\colon M \to B$ is the projection. Since B is semisimple, $\pi_2(Y)$ is a direct summand of B. So $X \cong Y \cong \pi_2(Y) \leq^{\oplus} B$ which implies that there exists $f \in End(M_R)$ such that $f(\pi_2(Y)) = X$. Since $Soc(B)$ is fully invariant in M, then $X \subseteq Soc(B)$. As above X is not small in B, $X \leq^{\oplus} B$. Hence $X \leq^{\oplus} M$. □

The fully invariant condition in Proposition 15 is not superflous. For example, let $M_{\mathbb{Z}} = A \oplus B$, where $A = \mathbb{Z}/8\mathbb{Z}$ and $B = \mathbb{Z}/2\mathbb{Z}$ as $\mathbb{Z}$-modules. Then A is a weak CS-module such that $4\mathbb{Z}/8\mathbb{Z} = Soc(A) \leq Rad(B) = 2\mathbb{Z}/8\mathbb{Z}$ and B is a semisimple module but B is not fully invariant in M. Indeed, take $\theta = i \circ f \circ \pi$, where $i\colon (4\mathbb{Z}/8\mathbb{Z} \oplus 0) \to M$ is the inclusion map, $f\colon (0 \oplus \mathbb{Z}/2\mathbb{Z}) \to (4\mathbb{Z}/8\mathbb{Z} \oplus 0)$ is the isomorphism and $\pi\colon M \to (0 \oplus \mathbb{Z}/2\mathbb{Z})$ is the natural projection map. Then clearly $\theta \in End(M)$ and $\theta(B) = (4\mathbb{Z}/8\mathbb{Z} \oplus 0) \nleq B$. By Example 3, M is not a semisimple-continuous module.

Theorem 3. *Let $M = \bigoplus_{i \in I} X_i$, where each X_i is a fully invariant submodule of M for all $i \in I$, and all semisimple submodules of M are fully invariant in M. Then X_i is semisimple-continuous for all $i \in I$ if and only if M is a semisimple-continuous module.*

Proof. $(\Leftarrow)$ It is clear from Proposition 9.

$(\Rightarrow)$ By Proposition 5, M is semisimple-direct injective. Now we show that M is a weak CS-module. Let S be a semisimple submodule of M. By hypothesis, S is fully invariant. Then $S = \oplus_{i \in I}(S \cap X_i)$. As X_i is weak CS, there is $D_i \leq^{\oplus} X_i$ such that $(S \cap X_i) \leq^{ess} D_i$ for all $i \in I$. Hence $S = \oplus_{i \in I}(S \cap X_i) \leq^{ess} (\oplus_{i \in I} D_i) \leq^{\oplus} (\oplus_{i \in I} X_i) = M$. □

Theorem 4. *Let $M = A \oplus B$ be a UC-module such that $Soc(A) \leq^{ess} A$ and $Soc(B) = 0$. Then M is semisimple-continuous if and only if A and B are semisimple-continuous.*

Proof. $(\Rightarrow)$ Assume M is semisimple-continuous then both A and B are semisimple-continuous by Proposition 10.

$(\Leftarrow)$ Let X, Y be semisimple submodules of M with $X \cong Y \leq^{\oplus} M$. Then Y is not contained in B because $Soc(B) = 0$. Then $Y \leq A$, so $Y \leq^{\oplus} A$ but then A is semisimple-direct injective and so $X \leq^{\oplus} A$ which in turn implies that $X \leq^{\oplus} M$. Thus M is semisimple-direct injective. [6, Corollary 1.1] yields the result. □

4. Decompositions

Recall that any submodules A and B of M, A is *superspective* to B, if for any submodule $X \leq M$, $M = A \oplus X$ if and only if $M = B \oplus X$. Two modules are *orthogonal*, if they have no non-zero isomorphic submodules. For any class $\mathcal{K}$ of modules, $\mathcal{K}^{\perp}$ denotes the class of modules orthogonal to all members of $\mathcal{K}$. A pair of classes $\mathcal{A}$ and $\mathcal{B}$ are called an *orthogonal pair*, if $\mathcal{A}^{\perp} = \mathcal{B}$ and $\mathcal{B}^{\perp} = \mathcal{A}$.

Proposition 16. *Let $\mathcal{A}$ and $\mathcal{B}$ be an orthogonal pair of classes of modules:*

(1) *If M is weak CS which is closed under direct summand, then $M = A \oplus B$ with semisimple $A \in \mathcal{A}$ and $B \in \mathcal{B}$. In fact, in this case, M is necessarily semisimple.*
(2) *Every semisimple-continuous module M which is closed under direct summand, has a decomposition such that $M = A \oplus B$ with semisimple $A \in \mathcal{A}$ and $B \in \mathcal{B}$ unique up to superspectivity.*

Proof. (1) By Zorn's Lemma, M has a semisimple submodule A maximal w.r.t. $A \in \mathcal{A}$. Since $\mathcal{A}$ is closed under essential extensions, A is a closed submodule of M; hence $A \leq^{\oplus} M$, as M is weak CS. Then $M = A \oplus B$ for some $B \leq M$. Applying the same argument to B, we get $B = C \oplus D$ where C is maximal semisimple submodule such that $C \in \mathcal{B}$. Assume that $D \neq 0$. Since $D \notin \mathcal{B}$, D contains a non-zero semisimple submodule $Z \in \mathcal{A}$; which is a contradiction to the maximality of $A \in \mathcal{A}$. Hence $D = 0$ and so $M = A \oplus B$ with $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

(2) Let $M = A_1 \oplus B_1 = A_2 \oplus B_2$ with $A_i \in \mathcal{A}$ and $B_i \in \mathcal{B}$ for all $i = 1, 2$. Assume that $M = A_1 \oplus X$. Then $X \cong B_1$, hence $X \in \mathcal{B}$ and therefore $A_2 \cap X = 0$. By hypothesis, $A_2 \oplus X \leq^{\oplus} M$, and so $M = A_2 \oplus X \oplus Y$. Then $A_2 \oplus Y \cong A_1$ and $X \oplus Y \cong B_2$. Consequently $Y \in \mathcal{A} \cap \mathcal{B} = 0$, so $M = A_2 \oplus X$. This proves that A_1 and A_2 are superspective. Similarly one can prove that B_1 and B_2 are superspective. □

We provide the decomposition theorem for the semisimple-continuous modules.

Theorem 5. *If M is a semisimple-continuous module, then $M = A \oplus B \oplus K$, where*

(1) *$A \cong B$,*
(2) *$A \oplus B$ is semisimple,*
(3) *$Soc(K)$ is a square-free module, and*

(4) *K is $(A \oplus B)$-injective.*

Proof. Let $F = \{(A, B, f) \mid A, B$ are semisimple closed submodules in M such that $A \cap B = 0,$ and $A \cong^f B\}$. Define an order on F as follows:

$$(A, B, f) \leq (A_1, B_1, f_1) \Leftrightarrow A \leq A_1, B \leq B_1, \text{ and } f_1 \text{ extends } f.$$

Clearly, F is a non-empty partially ordered set and every chain of elements of F has an upper bound in F. Then by Zorn's Lemma, F has a maximal member, say, (A, B, f). Since M is weak CS, there exist $A_1, B_1 \leq M$ such that $A \leq^{ess} A_1 \leq^{\oplus} M$, $B \leq^{ess} B_1 \leq^{\oplus} M$. Note that $A_1 \cap B_1 = 0$. But A and B are closed in M, so $(A, B, f) = (A_1, B_1, f)$. By semisimple-direct injectivity, we have $A \oplus B \leq^{\oplus} M$. Write $M = (A \oplus B) \oplus K$ for some $K \leq M$. Since $A \oplus B$ is semisimple with $A \cong B$, we prove $(1), (2)$ and (4). To show that condition (3) holds, let X and Y are submodules of $Soc(K)$ such that $X \cong^{\sigma} Y$ and $X \cap Y = 0$. Then $(A, B, f) \leq (A \oplus X, B \oplus Y, f \oplus \sigma)$. By maximality of (A, B, f), we have $(A, B, f) = (A \oplus X, B \oplus Y, f \oplus \sigma)$. Hence $A = A \oplus X$ and $B = B \oplus Y$ which yield $X = Y = 0$. Thus $Soc(K)$ is a square-free module. □

Proposition 17. *Let G be an Abelian group. Then G is semisimple-continuous if and only if $G = T \oplus F$, where F is torsion-free and T is a torsion Abelian group with each p-component, $T_p(M)$, a direct sum of a bounded Abelian group and a divisible Abelian group. Moreover, $T_p(M)$ is semisimple or $Soc(T_p(M)) \subseteq Rad(T_p(M))$, for each prime p.*

Proof. $(\Rightarrow)$ Assume G is semisimple-continuous. Then $G = T \oplus F$, where $Soc(G) \leq^{ess} T$ such that F is torsion-free and T is a torsion Abelian group with each p-component, T_p, a direct sum of a bounded Abelian group and a divisible Abelian group by [18, Corollary 5.98]. Note that $Soc(G) = Soc(T)$ and $Soc(F) = 0$. Now T is semisimple-direct injective and so T is simple-direct injective, then by [3, Theorem 2 (iv)], for each prime p, T_p is semisimple, or $Soc(T_p) \subseteq Rad(T_p(M))$.

$(\Leftarrow)$ Suppose $G = T \oplus F$, where F is torsion-free and T is a torsion Abelian group with each p-component, T_p, a direct sum of a bounded Abelian group and a divisible Abelian group. Then G is weak CS by [18, Corollary 5.98]. Now we will show that G is semisimple-direct injective. Firstly, if for each prime p, T_p is semisimple, then T is semisimple-direct injective by Corollary 5. Also, $Soc(F) = 0$ so it is straightforward to see that $T \oplus F$ is semisimple-direct injective. Now if $Soc(T_p) \leq Rad(T_p(M))$, then $Soc(T) \leq Rad(T)$ and $Soc(F) \cap Rad(F) = 0$. Let A, B be semisimple submodules of G with $A \cong B \leq^{\oplus} G$. Then B is not contained in T because each simple module of T is small in T. This shows that $\pi(B) \neq 0$, where $\pi \colon T \oplus F \to F$ is the projection map. But $Soc(F) = 0$, so $\pi(B) = 0$, a contradiction and hence $A = B = 0$. This shows that G is semisimple-direct injective. □

Recall that an R-module M is said to be a *C4-module* [8], if $B \cong A \leq^{\oplus} M$, $B \leq M$ and $A \cap B = 0$, then $B \leq^{\oplus} M$. Clearly, $C4$ condition implies semisimple-direct injective, but not vice versa. For instance, let $M_{\mathbb{Z}} = \mathbb{Z} \oplus 2\mathbb{Z}$. Since $Soc(M) = 0$, M is semisimple-direct injective. However it is not a $C4$-module. Note that a module M is called *pseudo- N-injective* if every monomorphism $f\colon K \to M$, where $K \leq N$, can be extended to a homomorphism from N into M.

Proposition 18. *Let $M = A \oplus B$ be a weak CS and C4-module, then A is pseudo-Soc(B)-injective and B is pseudo-Soc(A)-injective.*

Proof. We only show that B is pseudo- Soc(A)-injective. In a similar manner, it can be shown that A is pseudo- Soc(B)-injective. Let $A' \leq Soc(A)$ and consider a monomorphism $\phi\colon A' \to B$. Then $K = \{a' - \phi(a')\colon a' \in A'\} \leq Soc(M)$. Also $K \cap A = K \cap B = 0$. Hence $K \leq^{ess} D \leq^{\oplus} M$ by weak-CS condition. Now consider $\pi\colon M \to A$. Then $B \oplus D = B \oplus \pi(D)$. Hence we have $D \cong \pi(D)$ and $D \cap \pi(D) = 0$. Since M has C4, $\pi(D) \leq^{\oplus} M$. So $\pi(D) \leq^{\oplus} A$. Write $A = \pi(D) \oplus D'$ for some $D' \leq A$. Then $M = B \oplus A = B \oplus \pi(D) \oplus D' = B \oplus (D \oplus D')$. Now take the canonical projection $\pi\colon B \oplus (D \oplus D') \to B$. Then we have the restriction map $\pi|_A\colon A \to B$. Hence $\pi(a') = \pi(a' - \phi(a')) + \pi(\phi(a')) = \phi(a')$. This shows that $\pi|_A$ extends ϕ. □

Corollary 7. (i) *Let $M = A \oplus B$ be a weak CS and C4-module, then A is min-B-injective and B is min-A-injective.*

(ii) *Let R be a commutative ring. If $R \oplus R$ is semisimple-continuous as an R-module, then R is mininjective.*

Proof. It follows from Proposition 18. □

Proposition 19. *If M is weak CS and a C4-module and $A \leq^{\oplus} M$, then A is pseudo-Soc(B)-injective for any submodule B of M with $A \cap B = 0$.*

Proof. Write $M = A \oplus A'$ for a submodule $A' \leq M$ and consider the natural projection $\gamma\colon M \to A'$. Clearly, the restriction of γ on B is a monomorphism and so $B \cong \gamma(B) \leq A'$. Since M is weak CS and C4, A is pseudo-soc-A'-injective by Proposition 18. Consequently, A is pseudo-soc-$\gamma(B)$- injective which in turn gives that A is pseudo-soc-B-injective. □

In the following result, we examine when the finite exchange property implies full exchange property for the related module classes.

Proposition 20. (i) *Let M be a weak CS module and every submodule of M is a C4-module. If M has the finite exchange property, then M has the full exchange property.*

(ii) *Let R be a semiartinian ring and M be an R-module. If M is a semisimple-continuous module with the finite exchange property, then M has the full exchange property.*

Proof. (i) Let M be a weak CS module and every submodule of M be a C4-module. Clearly, M is semisimple-continuous. Then, by Proposition 5, $M = (A \oplus B) \oplus K$, where $A \oplus B$ is semisimple and K has square-free socle. [8, Proposition 2.20] yields that K is square-free. Since direct summands and finite direct sums of modules with the (finite) exchange property also have the (finite) exchange property, M has the full exchange property.

(ii) By Proposition 5, we have $M = (A \oplus B) \oplus K$, where $A \oplus B$ is semisimple and K has square-free socle. Now by [8, Proposition 2.21], K is square-free. Hence the proof follows from similar arguments in the proof of part (i). □

Lemma 2. *If M is semisimple and $M \oplus E(M)$ is semisimple-continuous, then M is injective.*

Proof. Let M be semisimple and $i\colon M \to E(M)$ be the inclusion map. Hence $i(M) = M \leq^{\oplus} E(M)$ by [1, Proposition 2.1(4)]. Therefore $M = E(M)$ and M is injective. □

Proposition 21. *The following assertions are equivalent for a ring R:*

(i) *Every projective semisimple right R-module is injective.*
(ii) *Every nonsingular right R-module is semisimple-continuous.*

Proof. $(i) \Rightarrow (ii)$ It is known from [10, Corollary 1.25] that any nonsingular semisimple right R-module is projective. Let M be a nonsingular module with semisimple direct summands $A, B \leq M$ such that $A \cap B = 0$. Then A and B are injective by (i). Hence $A \oplus B \leq^{\oplus} M$. Thus M is semisimple-direct injective. To show that M is weak CS, let S be a semisimple submodule of M. Then S is also injective by hypothesis. Therefore $S \leq^{\oplus} M$, so clearly M is weak CS. Thus, (ii) holds.

$(ii) \Rightarrow (i)$ Let P be a projective semisimple right R-module. By [13, p. 269], P is nonsingular and hence $E(P)$ and $P \oplus E(P)$ both are nonsingular. By assertion (ii), we have $P \oplus E(P)$ is semisimple-continuous. Thus, P is injective by Lemma 2. □

Theorem 6. *The following conditions are equivalent for a ring R.*

(i) *R is a right Noetherian right V-ring.*
(ii) *Every right R-module is a semisimple-continuous module.*
(iii) *The direct sum of any two semisimple-continuous right R-modules is semisimple-continuous.*
(iv) *Every semisimple-continuous module is strongly soc-injective.*

Proof. $(i) \Rightarrow (ii)$ It is clear because R is right Noetherian right V-ring if and only if every semisimple R-module is injective by [4, Proposition 1].

$(ii) \Rightarrow (iii)$ It is clear.

$(iii) \Rightarrow (i)$ Let M be a semisimple module. By the hypothesis, $M \oplus E(M)$ is a semisimple-continuous module. By Lemma 2, M is injective. Therefore, R is a right Noetherian right V-ring.

$(i) \Leftrightarrow (iv)$ This part follows from [1, Proposition 2.15]. □

Corollary 8. *A ring R is semisimple Artinian if and only if all semisimple-continuous right R-modules are injective.*

Corollary 9. *Let R be a right V-ring. Then R is right Noetherian if and only if every simple-continuous right R-module is semisimple-continuous.*

Proposition 22. *The followings are equivalent for a regular ring R:*

(i) *R is a right V-ring.*
(ii) *Every cyclic right R-module is semisimple-continuous.*
(iii) *Every cyclic right R-module is simple-continuous.*

Proof. It follows from Example 1(ii) and [19, Proposition 3.3]. □

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VALLÉE-POUSSIN THEOREM FOR KATUGAMPOLA FRACTIONAL FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. We propose Vallée-Poussin theorem in form of three equivalent assertions for Katugampola fractional functional differential equation. Choosing corresponding function, we obtain explicit test of negativity of Green's function in form of algebraic inequality. We discuss particular cases of functional equation such as equations with deviation to illustrate application of our technique. Further, we demonstrate applications of Katugampola derivatives as it generalizes previous inequalities available in literature for Riemann–Liouville fractional boundary value problem and Hadamard fractional boundary value problem.

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1. INTRODUCTION

In this paper, we consider the following fractional functional differential equation

$$(D_{a+}^{\alpha,\rho}x)(t)+(Tx)(t)=f(t), \quad t\in[a,b], \tag{1.1}$$

with boundary condition

$$x(a)=x(b)=0, \tag{1.2}$$

where $1<\alpha\leq 2$ and $D_{a+}^{\alpha,\rho}$ is Katugampola derivative which depends on extra parameter ρ and generalizes the Riemann–Liouville and Hadamard fractional derivative. If $\rho=1$, then it reduces to Riemann–Liouville fractional derivative, and if $\rho\to 0_+$, it becomes Hadamard fractional derivative [14, 15]. The operator $T:C\to L_\infty$ are linear continuous operators acting from the space of the continuous functions C to the space of essentially bounded functions L_∞ and $f\in L_\infty$. The operator T can be of the forms $(Tx)(t)=q(t)x(t-\tau(t))$, $x(\xi)=0$, if $\xi\neq[a,b]$, $(Tx)(t)=\int_a^b Q(t,s)x(\theta(s))\mathrm{d}s$, or $(Tx)(t)=\int_a^b x(s)\mathrm{d}_sQ(t,s)$. The conditions describing assumptions about all their coefficients are explained in [5], which allows acting of these operators $T:C\to L_\infty$.

For the application of fractional differential equations in various fields of science and engineering one can refer to the known monographs [16, 23]. In the last few decades, analysis of positive solutions and investigation of various inequalities for fractional differential equations has been an active area of research. Qualitative theory for fractional differential equations such as oscillation theory, zeros of solutions, disconjugacy and comparison theory for fractional differential equations can be studied and many results were obatined on the basis of various inequalities, for example, Lyapunov-type inequalities, De La Vallée-Poussin inequalities and Hartman–Wintner-type inequalities (see, for example [1, 20, 25]). There are various methods such as different fixed point theorems, topological methods, coincidence degree theory, upper and lower solution method and different numerical methods which are used to study fractional differential equations with different fractional derivatives, to mention some of them one can see articles for Riemann–Liouville [13,24], Caputo [25], Hadamard [11], Caputo-Hadamard [3], ψ-operators [18], along with relevant references therein.

Let us note some of the recent work using Katugampola fractional derivatives. In [20], Lupinska and Odzijewicz obtained Lyapunov inequality for Katugampola fractional differential equation. In [9], authors used Guo-Krasnoselskii and Banach fixed point theorems to study the existence and uniqueness of solutions for nonlinear Katugampola fractional differential equation

$$\begin{cases} -D_{a+}^{\alpha,\rho}x(t) = \beta f(t,x(t)), \quad 1 < \alpha \leq 2, \quad t \in [0,T], \\ x(0) = x(T) = 0, \end{cases}$$

where $\beta \in \mathbb{R}$ and $f : [0,T] \times [0,\infty) \to [h,\infty)$ is a continuous function with finite positive constants h, T. In [22], Łupinska and Schmeidel by proving the Lyapunov-type inequality deduced the conditions for the existence, and non-existence of the solutions for fractional differential equations under fractional boundary conditions with the Katugampola derivative.

$$\begin{cases} D_{a+}^{\alpha,\rho}x(t) + g(t)x(t) = 0, \\ x(a) = D_{a+}^{\alpha,\rho}x(b) = 0. \end{cases}$$

In [4], authors studied existence and uniqueness theorem for a fractional equation with Caputo–Katugampola derivative. Some other analysis work for equations with Katugampola derivatives can be found in [19, 21].

Investigation of equations with Katugampola fractional derivatives is seldom in literature. It looks very natural in mathematical modeling to consider memory effects not only in the left-hand (i.e., in the “derivative part”), but also in another term. This leads us to equation (1.1). Another motivation presents a corresponding new step in the studying system: using representation of solution for one of the components of the solution vectors of the system to come to a scalar functional differential equation for another component of the solution-vector. For functional differential equation with classical derivatives this idea was formulated in [2]. It should be stressed that even in the case of ordinary differential system, an equation for a corresponding component is an integro-differential (i.e., the functional differential equations). This motivates us to study problem (1.1)–(1.2). In this article, we apply Vallée-Poussin theorem about differential inequality to study problem (1.1)–(1.2). We obtain explicit tests of negativity of Green's function in the form of algebraic inequalities. For n-th order functional differential equations, an analog of the Vallée-Poussin theorem and results on sign-constancy of Green's functions on its basis were obtained in [6–8,10]. Recent work for fractional functional differential equations with operator T of the general form using Vallée-Poussin theorem can be found in [11–13,24].

Theorem 1 ([20]). *Let $0 < a < b < \infty$, $1 < \alpha \le 2$, $\rho > 0$ and $q : [a,b] \to \mathbb{R}$ be a continuous function and x be a solution of the boundary value problem*

$$\begin{cases} (D_{a+}^{\alpha,\rho}x)(t) + q(t)x(t) = 0, \\ x(a) = x(b) = 0. \end{cases} \tag{1.3}$$

If $x(t) \neq 0$ for all $t \in (a,b)$, then we have the inequality

$$\int_a^b |q(s)|ds \ge \frac{\Gamma(\alpha)}{\max\{a^{\rho-1},\ b^{\rho-1}\}} \left(\frac{4\rho}{b^\rho - a^\rho}\right). \tag{1.4}$$

Note that in [20], it was not assumed that $x(t) \neq 0$ for $t \in (a,b)$. We have (1.4) in the form

$$\min_{t\in[a,b]} q(t) \ge \frac{\Gamma(\alpha)}{\max\{a^{\rho-1},\ b^{\rho-1}\}} \left(\frac{4\rho}{b^\rho - a^\rho}\right)\left(\frac{1}{b-a}\right). \tag{1.5}$$

From Corollary 3, obtained below in Section 4 (see Remark 1), we get that the inequality

$$q(t) < \frac{\alpha^\alpha}{(\alpha-1)^{\alpha-1}} \frac{\Gamma(\alpha+1)}{\rho^{2-\alpha}(b^\rho - a^\rho)}, \tag{1.6}$$

guarantees that the problem (1.3) has only the trivial solution and its Green's function is negative for $(t,s) \in (a,b)\times(a,b)$. Note that inequality (1.6) we have constructed for a more general problem (4.1) in which we have $x(h(t))$ instead of $x(t)$ compared to problem (1.3). Inequality (1.6) means that in the case of zeros of solution $x(t)$ at the points a and b, we obtain that

$$\min_{t\in[a,b]} q(t) \ge \frac{\alpha^\alpha}{(\alpha-1)^{\alpha-1}} \frac{\Gamma(\alpha+1)}{\rho^{2-\alpha}(b^\rho - a^\rho)}, \tag{1.7}$$

since in the case of the coefficient q satisfying inequality (1.6) we exclude the existence of zero at the points a and b, i.e., one does not have the $x(a) = x(b) = 0$. Let us compare (1.5) and (1.7), computing the right-hand sides (RHS) in them, we obtain values estimating $\min\limits_{t\in[a,b]} q(t)$ in Table 1 and graphical representation in Figure 1. We see that our estimate of q in right-hand side of inequality (1.5) gives sharper values of $\min_{t\in[a,b]} q(t)$ in compared to previous Lyapunov inequality (1.7).

α	RHS in inequality (1.5)	RHS in inequality (1.7)
2	4	8
1.9888	3.91993	7.85644
1.95	3.65704	7.37800
1.9	3.34906	6.80198
1.88	3.23840	6.58362
1.85	3.07229	6.62682
1.8	2.82342	5.773238

TABLE 1. Comparing our results with the known ones.

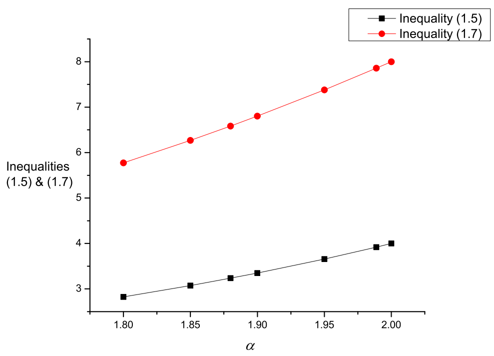

FIGURE 1. Describing that our inequality (1.7) is more exact than (1.5)

2. PRELIMINARIES

In [14], to define the generalized fractional derivative Katugampola considered the space $X_c^p(a,b)$ (where $c \in \mathbb{R}$, $1 \le p \le \infty$) of those Lebesgue measurable functions f on $[a,b]$ for which $\|f\|_{X_c^p} < \infty$, where the norm is defined by

$$\|f\|_{X_c^p} = \left(\int_a^b |t^c f(t)|^p \frac{dt}{t} \right)^{\frac{1}{p}} < \infty, \quad (c \in \mathbb{R},\ 1 \le p \le \infty)$$

and for the case $p = \infty$

$$\|f\|_{X_c^\infty} = \text{ess sup}_{\mathrm{a \le t \le b}} \left[t^c |f(t)| \right], \quad c \in \mathbb{R}.$$

In particular, when $c = \frac{1}{p}$, $(1 \le p \le \infty)$ the space $X_c^p(a,b)$ coincides with the classical $L_p(a,b)$-space. In this article, we consider the case when $p \to \infty$, which brings us to the classical L_∞ space.

Definition 1 (see [9, 20]). Let $\alpha > 0$, $\rho > 0$, $0 < a < b \leq \infty$, $f(t) \in L_\infty$. The operator

$$I_{a+}^{\alpha,\rho} = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t \frac{\tau^{\rho-1}}{(t^\rho - \tau^\rho)^{1-\alpha}} f(\tau) d\tau,$$

for $t \in (a,b)$ is called left-side Katugampola integral of fractional order α.

Definition 2 (see [20]). Let $\alpha > 0$, $\rho > 0$, $n = [\alpha] + 1$, $0 < a < t < b \leq \infty$, $f \in L_\infty$. The operator

$$\begin{aligned} D_{a+}^{\alpha,\rho} f(t) &= \left(t^{1-\rho} \frac{d}{dt}\right)^n I_{a+}^{n-\alpha,\rho} f(t) \\ &= \frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)} \left(t^{1-\rho} \frac{d}{dt}\right)^n \int_a^t \frac{\tau^{\rho-1}}{(t^\rho - \tau^\rho)^{\alpha-n+1}} f(\tau) d\tau \end{aligned}$$

for $t \in (a,b)$ is called left-side Katugampola derivatives of fractional order α.

Lemma 1 (see [20]). *Let $\alpha, \rho > 0$, $n = [\alpha] + 1$, where $[\alpha]$ is the integer part of α, $f \in L_\infty$. The fractional differential equation*

$$(D_{a+}^{\alpha,\rho} x)(t) = f(t)$$

has the general solution of the form

$$x(t) = \sum_{i=1}^n c_i \left(\frac{t^\rho - a^\rho}{\rho}\right)^{i-n+\alpha} - I_{a+}^{\alpha,\rho} f(t),$$

where c_i are real constants.

Lemma 2 (see [20]). *Assume $1 < \alpha \leq 2$ and $f \in L_\infty$. Then the unique solution to the problem*

$$(D_{a+}^{\alpha,\rho} x)(t) = f(t) \tag{2.1}$$

with boundary condition (1.2) is given by

$$x(t) = \int_a^b G(t,s) f(s) ds \tag{2.2}$$

where $G(t,s)$ is the Green's function given by

$$G(t,s) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \begin{cases} -\frac{s^{\rho-1}}{(b^\rho - s^\rho)^{1-\alpha}} \left(\frac{t^\rho - a^\rho}{b^\rho - a^\rho}\right)^{\alpha-1} + \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\alpha}}, & a \leq s \leq t \leq b \\ -\frac{s^{\rho-1}}{(b^\rho - s^\rho)^{1-\alpha}} \left(\frac{t^\rho - a^\rho}{b^\rho - a^\rho}\right)^{\alpha-1}, & a \leq t \leq s \leq b. \end{cases} \tag{2.3}$$

Now, in the following lemma we use the technique of [20], where authors proved the sign-constancy of Green's function.

Lemma 3. *Green's function represented by (2.3) is negative for $s, t \in (a,b)$.*

Proof. For $a<t<s<b$, it is clear that $G(t,s)<0$. For $a<s<t<b$,

$$G(t,s) = -\frac{s^\rho \rho^{1-\alpha}}{\Gamma(\alpha)}\left(\frac{t^\rho - a^\rho}{b^\rho - a^\rho}\right)^{\alpha-1}\left[(b^\rho - s^\rho)^{\alpha-1} + \left((t^\rho - s^\rho)\left(\frac{b^\rho - a^\rho}{t^\rho - a^\rho}\right)\right)^{\alpha-1}\right]$$

$$= -\frac{s^\rho \rho^{1-\alpha}}{\Gamma(\alpha)}\left(\frac{t^\rho - a^\rho}{b^\rho - a^\rho}\right)^{\alpha-1}\left[(b^\rho - s^\rho)^{\alpha-1} + \left(b^\rho - \left(a^\rho + \frac{(s^\rho - a^\rho)(b^\rho) - a^\rho}{t^\rho - a^\rho}\right)\right)\right].$$

Observing that

$$s^\rho \leq a^\rho + \frac{(s^\rho - a^\rho)(b^\rho - a^\rho)}{t^\rho - a^\rho} \leq b^\rho,$$

because of the fact that

$$\frac{(s^\rho - a^\rho)(b^\rho) - a^\rho}{t^\rho - a^\rho} \geq 0 \quad \text{and} \quad \frac{(s^\rho - a^\rho)(b^\rho) - a^\rho}{t^\rho - a^\rho} \leq 0,$$

we get $G(t,s)<0$ also for $s<t$. Hence, $G(t,s)<0$ for $t,s\in(1,e)$. □

3. Main Results

Let us define the operator $K: L_\infty \to L_\infty$ by the equality

$$(Kz)(t) = -T\left[\int_a^b G(\cdot,s)z(s)ds\right](t). \tag{3.1}$$

We use here and below the notation $T[\gamma(t)]$ meaning that the operator T acts on the continuous function γ, i.e., $T[\gamma(t)] = (T\gamma)(t)$. We assume in this paper the positivity of operators in the standard sense, i.e., the operator K is positive if $(Kz)(t)\geq 0$ for $t\in[a,b]$ for every nonnegative $z\in L_\infty$.

The following assertion can be considered as an analog of the Vallée-Poussin theorem on differential inequality [6].

Theorem 2. *Let $T: C\to L_\infty$, be positive operator, $1<\alpha\leq 2$. Then the following assertions are equivalent:*

1) *there exist a positive number ε and a function $v\in X_c^p(a,b)\cap C$ such that $v(t)>0$ for $t\in(a,b)$, $v(a)=0$, $v(b)=0$ and*

$$(D_{a+}^{\alpha,\rho}v)(t) + (Tv)(t) \equiv \psi(t) \leq -\varepsilon < 0 \;\; \text{for} \;\; t\in(a,b); \tag{3.2}$$

2) *the spectral radius $r(K)$ of the operator $K: L_\infty\to L_\infty$ is less than 1;*
3) *problem (1.1)–(1.2), is uniquely solvable for any $f\in L_\infty$ and its Green's function $G(t,s)$ is negative for $(t,s)\in(a,b)\times(a,b)$.*

Proof. 1) $\Rightarrow$ 2). The function v in condition 1) satisfies the boundary value problem

$$\begin{cases} (D_{a+}^{\alpha,\rho}x)(t) = z(t), \\ x(a) = 0,\ x(b) = v(b), \end{cases} \tag{3.3}$$

where z from L_∞ is $z(t) = \psi(t) - (Tv(t))$. Thus, there exists $\delta > 0$ such that $z(t) \leq -\delta$ for $t \in (a,b)$. It is clear that

$$x(t) = \int_a^b G(t,s)z(s)ds + u(t), \tag{3.4}$$

where u is a solution of the homogeneous equation

$$\begin{cases} (D_{a+}^{\alpha,\rho}u)(t) = 0, \quad t \in [a,b], \\ u(a) = 0, \quad u(b) = 0. \end{cases} \tag{3.5}$$

Substituting this representation in the place of v into (3.2) yields

$$z(t) + T\left[\int_a^b G(t,s)z(s)ds\right] + (Tu)(t) = \psi(t), \tag{3.6}$$

and

$$z(t) - (Kz)(t) = \psi(t) - (Tu)(t),\ \ t \in (a,b). \tag{3.7}$$

Let us prove that $u(t) \equiv 0$ for $t \in (a,b)$. From condition 1) of Theorem 2 we have $v(a) = 0$ and $v(b) = 0$. It is clear that $u(t) \equiv 0$ for $t \in (a,b]$ according to Lemma 2. Thus, $\Psi(t) = \psi(t) - (Tu)(t) \leq -\varepsilon < 0$. The function $w = -z$ satisfies the inequalities

$$w(t) - (Kw)(t) = -\Psi(t) > 0,$$

and

$$w(t) > (Kw)(t).$$

According to [17, Theorem 5.8 on p. 84], we obtain $r(K) < 1$.

This completes the proof of the implication 1) $\Rightarrow$ 2).

Let us prove now the implication 2) $\Rightarrow$ 3). Consider the boundary value problem (1.1). Let us use the substitution

$$x(t) = \int_a^b G_0(t,s)z(s)ds, \tag{3.8}$$

where $G_0(t,s)$ is Green's function of the problem consisting of the equation

$$(D_{a+}^{\alpha,\rho}x)(t) = z(t) \tag{3.9}$$

with the boundary conditions (1.2). Substituting representation (3.8) into (1.1), we get (3.7), where $\psi(t) = f(t)$ and $u(t) \equiv 0$ for $t \in [a,b]$. If $r(K) < 1$, then (3.7) is uniquely solvable and its solution is

$$z(t) = (I-K)^{-1}\psi(t) = ((I + K + K^2 + K^3 + \ldots)\psi)(t). \tag{3.10}$$

We obtain that the solution x defined by (3.8) exists and is unique, and this proves that the problem (1.1) is uniquely solvable. We see also that $(I-K)^{-1}$ is a positive operator if K is positive. The assumption about positivity of the operator $T: C \to L_\infty$ and Lemma 2 imply that $K: L_\infty \to L_\infty$ is positive. Then from $\psi(t) \leq 0$, it follows that $z(t) \leq 0$ for $t \in [a,b]$. Thus, if $f(t) \leq 0$, then $z(t) \leq 0$ for $t \in [a,b]$. If $z(t) \leq 0$, then from the fact of nonpositivity of Green's function $G(t,s)$ in the formula (3.8), we get $x(t) \geq 0$ for $t \in [a,b]$. This is possible only in the case when Green's function $G(t,s)$ of problem (1.1)–(1.2) satisfies the inequality $G(t,s) \leq 0$. From the inequalities

$$\begin{aligned} x(t) &= \int_a^b G_0(t,s)z(s)ds = \int_a^b G_0(t,s)(I-K)^{-1}\psi(s)\mathrm{d}s \\ &= \int_a^b G_0(t,s)[I+K+K^2+K^3+\ldots]\psi(s)ds \end{aligned}$$

and the fact that $G_0(t,s) < 0$, it follows that $G(t,s) \leq G_0(t,s) < 0$ for $t,s \in (a,b)$. This completes the proof of the implication 2) $\Rightarrow$ 3).

In order to prove the implication 3) $\Rightarrow$ 1), we set

$$v(t) = -\int_a^b G(t,s)\mathrm{d}s.$$

Since $G(t,s) < 0$ for $(t,s) \in (a,b)$, we get $v(t) > 0$ for $t \in (a,b)$. It is clear that $v(a) = v(b) = 0$, $\psi(t) = -1$. This completes the proof of the implication 3)$\Rightarrow$1).

The proof is now complete. □

Corollary 1. *If $1 < \alpha \leq 2$ and the following inequality is fulfilled*

$$T\left[\rho^{2-\alpha}(t^\rho - a^\rho)^{\alpha-1}[(b^\rho - a^\rho) - (t^\rho - a^\rho)]\right] < \Gamma(\alpha+1) \tag{3.11}$$

then problem (1.1)–(1.2) is uniquely solvable for any $f \in L_\infty$ and its Green's function $G(t,s)$ is negative for $(t,s) \in (a,b)$.

Proof. For $\delta > 0$, consider the auxiliary equation

$$\begin{cases} (D_{a+}^{\alpha,\rho} v)(t) = -\delta, \\ v(a) = v(b) = 0, \end{cases}$$

Using Lemma 1, we write

$$v(t) = c_1\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha-1} + c_2\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha-2} + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^t \frac{\tau^{\rho-1}}{(t^\rho - \tau^\rho)^{1-\alpha}}(-\delta)\mathrm{d}\tau$$

for some real constants c_1 and c_2. Applying boundary condition, we obtain $c_2 = 0$ and so,

$$v(t) = c_1\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha-1} + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^t \frac{\tau^{\rho-1}}{(t^\rho - \tau^\rho)^{1-\alpha}}(-\delta)\mathrm{d}\tau$$

We have to "connect" c_1 with δ to guarantee the inequality

$$v(t) > 0 \tag{3.12}$$

In order to continue the proof, let us describe in more detail an idea of this "connection". It is clear that condition (3.12) is fulfilled for sufficiently large c_1, but to achieve inequality (3.2), we need sufficiently small c_1. Thus, we have to choose a minimal possible c_1 such that inequality (3.12) is fulfilled.

$$\begin{aligned} v(t) &= c_1\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha-1} - \frac{\delta\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^t \frac{\tau^{\rho-1}}{(t^\rho-\tau^\rho)^{1-\alpha}}d\tau \\ &= c_1\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha-1} - \frac{\delta\rho^{2-\alpha}}{\Gamma(\alpha+1)}(t^\rho-a^\rho)^\alpha \end{aligned}$$

Using the boundary condition $v(b) = 0$, we obtain

$$c_1 = \frac{\delta\rho}{\Gamma(\alpha+1)}(b^\rho - a^\rho)$$

putting this c_1 back in $v(t)$, we get

$$\begin{aligned} v(t) &= \frac{\delta\rho}{\Gamma(\alpha+1)}(b^\rho-a^\rho)\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha-1} - \frac{\delta\rho^{2-\alpha}}{\Gamma(\alpha+1)}(t^\rho-a^\rho)^\alpha \\ &= \frac{\delta\rho^{2-\alpha}}{\Gamma(\alpha+1)}(t^\rho-a^\rho)^{\alpha-1}\left[(b^\rho-a^\rho)-(t^\rho-a^\rho)\right] \end{aligned}$$

Now, considering $(D^{\alpha,\rho}_{a+}v)(t) = -\delta$ and obtained $v(t)$, we get (3.11). Using now the equivalence of the assertions 1) and 3) of Theorem 2, we obtain negativity of Green's function $G(t,s)$ of problem (1.1), (1.2). □

Let us consider the equation with deviations

$$(D^{\alpha,\rho}_{a+}x)(t) + \sum_{j=1}^{m} q_j(t)x(h_j(t)) = f(t), \quad t\in(a,b), \tag{3.13}$$

where

$$x(\xi) = 0, \quad \xi \notin (a,b), \tag{3.14}$$

$q_j, f \in L_\infty$, h is a measurable function. We obtain the following assertion.

Corollary 2. *Let $q_j(t)\ge 0$, $1<\alpha<2$ and assume*

$$\sum_{j=1}^{m}\chi(a<h_j(t)<b)q_j(t)\rho^{2-\alpha}(h_j(t)^\rho-a^\rho)^{\alpha-1}\left[(b^\rho-a^\rho)-(h_j(t)^\rho-a^\rho)\right] < \Gamma(\alpha+1),\ t\in(a,b)$$

where

$$\chi(a<h_j(t)<b) = \begin{cases} 1, & h_j(t)\in(a,b), \\ 0, & h_j(t)\notin(a,b). \end{cases}$$

Then the problem consisting of equation (3.13) and the boundary conditions (1.2) is uniquely solvable for any $f \in L_\infty$ and its Green's function $G(t,s)$ is negative for $t,s \in (a,b)$.

4. Remarks and Examples

Consider the particular case of (1.1) given as

$$\begin{cases} (D^{\alpha,\rho}x)(t) + q(t)x(h(t)) = f(t), \quad t \in (a,b), \\ x(a) = x(b) = 0. \end{cases} \tag{4.1}$$

Corollary 3. *If $1 < \alpha < 2$, $q(t) \geq 0$ for $t \in (a,b)$ and*

$$\{\text{ess sup}_{t\in[a,b]} q(t)\}\rho^{2-\alpha}((h(t))^\rho - a^\rho)^{\alpha-1}[(b^\rho - a^\rho) - ((h(t))^\rho - a^\rho)] < \Gamma(\alpha+1), \quad t \in (a,b) \tag{4.2}$$

is fulfilled, then problem (4.1) is uniquely solvable for any $f \in L_\infty$ and its Green's function is negative for $(t,s) \in (a,b)\times(a,b)$.

Remark 1. Taking into account

$$\max_{a\leq t\leq b} \rho^{2-\alpha}(t^\rho - a^\rho)^{\alpha-1}[(b^\rho - a^\rho) - (t^\rho - a^\rho)] = \rho^{2-\alpha}(b^\rho - a^\rho)^\alpha \frac{(\alpha-1)^{\alpha-1}}{\alpha^\alpha}, \tag{4.3}$$

which is achieved at the point $t^\rho = a^\rho + \frac{(\alpha-1)}{\alpha}(b^\rho - a^\rho)$, we get

$$q(t) < \frac{\alpha^\alpha}{(\alpha-1)^{\alpha-1}} \frac{\Gamma(\alpha+1)}{\rho^{2-\alpha}(b^\rho - a^\rho)}. \tag{4.4}$$

Remark 2. Inequality (4.4) cannot be improved. Actually, assume that

$$(h(t))^\rho = a^\rho + \frac{(\alpha-1)}{\alpha}(b^\rho - a^\rho)$$

in (4.1) and consider the equation

$$(D^{\alpha,\rho}_{a+}x)(t) + \frac{\alpha^\alpha}{(\alpha-1)^{\alpha-1}} \frac{\Gamma(\alpha+1)}{\rho^{2-\alpha}(b^\rho - a^\rho)} x\left(\left(a^\rho + \frac{(\alpha-1)}{\alpha}(b^\rho - a^\rho)\right)^{\frac{1}{\rho}}\right) = 0, \quad t \in (a,b), \quad 1 < \alpha < 2. \tag{4.5}$$

The function $x(t) = \rho^{2-\alpha}(t^\rho - a^\rho)^{\alpha-1}[(b^\rho - a^\rho) - (t^\rho - a^\rho)]$ satisfies (4.5) and indeed (4.5) has an infinite number of solutions of the form $c(\rho^{2-\alpha}(t^\rho - a^\rho)^{\alpha-1}[(b^\rho - a^\rho) - (t^\rho - a^\rho)])$ for every real number c.

Corollary 4. *If $1 < \alpha \leq 2$, $h(t) < \varepsilon$ for $a = 0$, $b = 1$, for $t \in (0,1)$, then the inequality (4.4) becomes*

$$q(t) < \frac{\Gamma(\alpha+1)}{(1-\varepsilon)\varepsilon^{\alpha-1}} \tag{4.6}$$

and implies the unique solvability for any $f \in L_\infty$ of the problem (4.1).

Example 1. If we take $\varepsilon = 0.1,\ 0.001,\ 0.0001$, $\alpha = 1.5$ in inequality (4.6) of Corollary 4, then particular bounds of inequalities calculated in Table 1 and represented in Figure 2.

ε	$\frac{\Gamma(\alpha+1)}{(1-\varepsilon)\varepsilon^{\alpha-1}}$
0.09	4.869379
0.05	6.257885
0.009	14.139736
0.005	18.8941829
0.0009	44.351262
0.0005	59.47964
0.0001	[illegible]

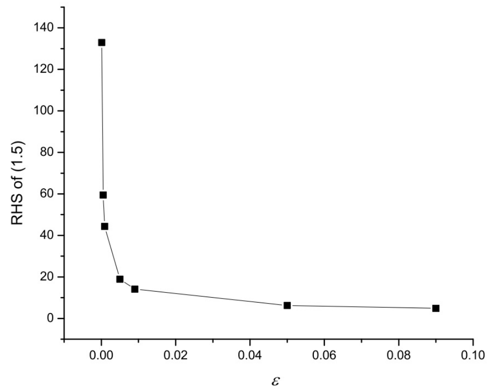

FIGURE 2.

Corollary 5. *In the case of superposition of integral and deviation operators*

$$(Tx)(t) = \int_a^b k(t,s)x(h(s))ds,$$

we get to

$$\int_a^b k(t,s)[\rho^{2-\alpha}(s-a^{\rho})[(b^{\rho}-a^{\rho})-(t^{\rho}-a^{\rho})]]ds < \Gamma(\alpha+1).$$

Using estimate of (4.3)*, we get to*

$$\int_a^b k(t,s)ds < \frac{1}{\max\left\{\rho^{2-\alpha}(b^{\rho}-a^{\rho})^{\alpha}\frac{(\alpha-1)^{\alpha-1}}{\alpha^{\alpha}}\right\}},$$

which implies that problem

$$\begin{cases} (D^{\alpha,\rho}x)(t) + \int_a^b k(t,s)x(h(s))ds = f(t), & t \in (a,b), \\ x(a) = x(b) = 0. \end{cases}$$

is uniquely solvable for any $f \in L_\infty$ and its Green's function is negative for $(t,s) \in (a,b)\times(a,b)$.

5. Applications

If we take $\rho = 1$ in Definition 2, then we obtain the Riemann–Liouville fractional derivative [16, 23]

$$D_{a+}^{\alpha,1} f(t) = {}^{RL}D_{a+}^{\alpha} f(t) = \left(\frac{d}{dt}\right)^n \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau.$$

Now, for $\rho \to 1$, $a = 0$ and $b = 1$, problem (1.1)–(1.2) becomes a Riemann–Liouville fractional boundary value problem which coincides with the problem studied in [13] for $k = 0$,

$$\begin{cases} ({}^{RL}D_{0+}^{\alpha} x)(t) + (Tx)(t) = f(t), \\ x(0) = x(1) = 0, \end{cases} \tag{5.1}$$

where $({}^{RL}D_{0+}^{\alpha})$ is Riemann–Liouville fractional derivative.

Corollary 6. *For $\rho \to 1$ and $a = 0$, $b = 1$, we get inequality (3.11) as [13, Corollary 6]*

$$T\left[t^{\alpha-1}(1-t)\right] < \Gamma(\alpha+1)$$

Similarly, if we take $\rho \to 0_+$ in Definition 2, then we get the Hadamard fractional derivative [16, 23]

$$\lim_{\rho\to 0^+} D_{1+}^{\alpha,\rho} f(t) = {}^{H}D_{1+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(t\frac{d}{dt}\right)^n \int_a^t \left(\log\frac{t}{\tau}\right)^{n-\alpha-1} f(\tau)\frac{d\tau}{\tau}.$$

Next, for $\rho \to 0_+$, $a = 1$ and $b = e$, problem (1.1)–(1.2) becomes a Hadamard fractional boundary value problem which coincides with the problem studied in [11]

$$\begin{cases} ({}^{H}D_{1+}^{\alpha} x)(t) + (Tx)(t) = f(t), \\ x(1) = x(e) = 0, \end{cases} \tag{5.2}$$

where $({}^{H}D_{1+}^{\alpha})$ is Hadamard fractional derivative.

Corollary 7. *For $\rho \to 0_+$ and $a = 1$, $b = e$, we get inequality (3.11) as [11, Corollary 3.2]*

$$T\left[(\ln t)^{\alpha-1}(1-\ln t)\right] < \Gamma(\alpha+1)$$

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EXTENDING A THEOREM OF DATKO FOR EVOLUTIONARY FAMILIES

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Abstract. In this note, we extend Datko's result in the paper [4, 1972]. In particular, the exponential stability of evolutionary families is characterized by its pointwise trajectories in which the norm mapping of each pointwise trajectory lies in a Banach function space.

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1. INTRODUCTION

The concept of an evolutionary family arises naturally from the well-posed theory of non-autonomous abstract differential equations on $\mathbb{R}$ or $\mathbb{R}_+$, it is also a quite natural generalization of strongly continuous semigroups. Readers can refer to Engel and Nagel [6], Chicone and Latushkin [2], Pazy [15], Daleckii and Krein [3] for this topic. In this note, X is a real or complex Banach space with a norm $\|\cdot\|$ and I is either $\mathbb{R}$ or $\mathbb{R}_+$.

Definition 1. A family of bounded linear operators $(U(t,s))_{t\geq s}$ on a Banach space X is an (strongly continuous and exponentially bounded) evolutionary family on I (means $t,s\in I$) if

(i) $U(t,t) = Id$ and $U(t,r)U(r,s) = U(t,s)$ for all $t\geq r\geq s$ and $t,r,s\in I$.

(ii) The map $(t,s)\mapsto U(t,s)x$ is continuous for every $x\in X$.

(iii) There are constants $K,c>0$ such that $\|U(t,s)x\|\leq Ke^{c(t-s)}\|x\|$ for all $t\geq s$ and $x\in X$.

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Notice that a strongly continuous semigroup $(T(t))_{t\geq 0}$ naturally gives rise to the evolutionary family $U(t,s) = T(t-s)$ for $t \geq s$ and $t,s \in I$. Among the many interesting types of stability of evolutionary families, of interest in this note is the exponentially stable evolutionary families because such families are very useful when we study nonlinear problems associated with those families.

Definition 2. An evolutionary family $(U(t,s))_{t\geq s}$ on I will be called *exponentially stable* if there exist constants $K_1, \alpha > 0$ such that $\|U(t,s)x\| \leq K_1 e^{-\alpha(t-s)}\|x\|$ for all $x \in X$ and $t \geq s$.

By properties of (i) and (iii) from Definition 1, the exponential stability of evolutionary families is equivalent to the uniformly asymptotic stability of those families, see [4, Lemma 1]. To study the exponential stability of evolutionary families, there are two basic approaches. One is based on Perron's method, for instance [14]. The other is based on Datko's result, for example [18]. In [4], Datko pointed out that *an evolutionary family* $(U(t,s))_{t\geq s}$ *is exponentially stable on* $\mathbb{R}_+$ *if and only if for each* $x \in X$ *there exists a constant* $M(x) > 0$ *such that*

$$\int_{t_0}^{\infty} \|U(t,t_0)x\|^2 \mathrm{d}t \leq M(x) \quad \text{for all} \quad t_0 \geq 0.$$

In case the evolutionary family $(U(t,s))_{t\geq s\geq 0}$ is a strongly continuous semigroup that means $U(t,s) = U(t-s,0)$ for $t \geq s \geq 0$ or $U(t,s) = T(t-s)$ with $(T(t))_{t\geq 0}$ is a strongly continuous semigroup, this result was extended by Pazy in [15] for the Lebesgue spaces $L^p(\mathbb{R}_+)$ with $p \in [1,\infty)$ and by Neerven in [18]. Neerven's result covers a wide class of function spaces. However, it only holds for strongly continuous semigroups. In addition, another limitation in Neerven's result is that it only offers a sufficient condition for exponential stability of strongly continuous semigroups, and his result is valid only for complex Banach spaces. In this case, we mention [19, Theorem 1.1] as an interesting application for Datko's result and Pazy's result.

Using the idea of replacing the squared function in the integral with a function of two variables that satisfies some certain conditions, Rolewicz [16, Theorem 2] had generalized Datko's result. With that same idea, Megan et al. extended Datko's result for linear skew-product semiflows and skew-evolution semiflows on Banach spaces in [11, Theorem 3.4] and [17, Theorem 1], respectively. In [13, Theorem 4.1 and Theorem 4.2], Megan et al. characterized the uniform exponential stability of periodic evolution families by the belonging of the associated trajectories to Banach sequence spaces and Banach function spaces having unbounded fundamental functions. In [12], Megan et al. characterized the uniform exponential stability of general evolution families on the half-line by the ownership of their associated trajectories to Banach function spaces that belong to a broad class of function spaces.

In [1, Theorem 0.1], Buse used normed function spaces, which were the same as in [18], to give a more general result for the evolutionary family on the half-line. More precisely, this result improves Theorem 4.2 in the paper [18]. Furthermore, Datko's result was also extended to nonuniform behavior of evolutionary family in papers [5] and [9].

Follow the second approach in studying the exponential stability of an evolutionary family, our purpose is the generalization of Datko's result for an evolutionary family on the line or the half-line and a broader class of function spaces. To accomplish this task, we introduce the concept of Banach function space in Subsection 2.1. Using the concept of Banach function space, we get the expected results which are stated in Subsection 2.2. Our results hold for both Banach spaces over complex and real fields. Because we provide both necessary and sufficient conditions for the exponential stability of the evolutionary family, the class of Banach function spaces in this note is smaller than the class of function spaces in [18]. However, it is still very wide and most of the known function spaces belong to the class of Banach function spaces that we define in this note.

2. Banach function spaces and main results

2.1. *Banach function spaces*

In the monograph book [10, Chapter 2], Massera and Schäffer introduced several classes of function spaces that play a fundamental role in studying differential equations. Based on the concepts of function spaces given by Massera and Schäffer, we have collected some basic properties to give the notion of Banach function space. This definition method is also similar to that in [7, Section 2].

Definition 3. Let $\mathcal{B}$ be the Borel σ-algebra and let μ be the Lebesgue measure on $\mathbb{R}$. A vector space E of real-valued measurable functions on $\mathbb{R}$ is called a Banach function space if

i. $(E,\|\cdot\|_E)$ is a Banach space and $\|\cdot\|_E$ guarantees the property: if $\varphi_2\in E$ and φ_1 is real-valued measurable function such that $|\varphi_1(t)|\le|\varphi_2(t)|$ almost everywhere on $\mathbb{R}$ then $\varphi_1\in E$ and $\|\varphi_1\|_E\le\|\varphi_2\|_E$;

ii. the characteristic function $\chi_{[a,b]}\in E$ for all $[a,b]\subset\mathbb{R}$ and $\inf_{t\in\mathbb{R}}\|\chi_{[t,t+1]}\|_E>0$;

iii. there is a constant $M\ge 1$ such that
$$\frac{1}{b-a}\int_a^b|\varphi(t)|\mathrm{d}t\le\frac{M\|\varphi\|_E}{\|\chi_{[a,b]}\|_E}\ \text{ for all } \varphi\in E \text{ and } [a,b]\subset\mathbb{R}; \tag{2.1}$$

iv. the function $\Lambda_1\varphi$ defined by $(\Lambda_1\varphi)(t)=\int_t^{t+1}\varphi(\tau)\mathrm{d}\tau$ belongs to E for each $\varphi\in E$;

v. E is T_τ-invariant and there exists a constant $N>0$ such that $\|T_\tau\|\le N$ for all $\tau\in\mathbb{R}$, where T_τ is shift operator on E defined by $(T_\tau\varphi)(t)=\varphi(t+\tau)$ for $t\in\mathbb{R}$.

Denote by $L_{1,\text{loc}}(\mathbb{R})$ space of real-valued locally integrable functions on $\mathbb{R}$. A family of seminorms defining the topology of $L_{1,\text{loc}}(\mathbb{R})$ is given by

$$\left\{ p_n\colon n \in \mathbb{Z} \text{ and } p_n(\varphi) = \int_n^{n+1} |\varphi(t)| \mathrm{d}t \right\}.$$

Then, $L_{1,\text{loc}}(\mathbb{R})$ is a Fréchet space. Therefore, by (2.1) then $E \hookrightarrow L_{1,\text{loc}}(\mathbb{R})$.

By direct inspection, it can be easily seen that class of Banach function spaces includes the Lebesgue spaces $L^p(\mathbb{R})$ with $p \in [1,\infty]$, the Lorentz spaces $L^{p,q}(\mathbb{R})$ with $p,q \in [1,\infty]$, the Orlicz spaces, etc, and the space

$$\mathbf{M}(\mathbb{R}) = \left\{ \varphi \in L_{1,\text{loc}}(\mathbb{R})\colon \sup_{t\in\mathbb{R}} \int_t^{t+1} |\varphi(\tau)| \mathrm{d}\tau < \infty \right\}$$

with the norm $\|\varphi\|_{\mathbf{M}} := \sup_{t\in\mathbb{R}} \int_t^{t+1} |\varphi(\tau)| \mathrm{d}\tau$. By (2.1) and the item ii. in Definition 3, we have the estimate

$$\|\varphi\|_{\mathbf{M}} \le \frac{M}{\inf_{t\in\mathbb{R}} \|\chi_{[t,t+1]}\|_E} \|\varphi\|_E, \tag{2.2}$$

for all $\varphi \in E$. Thus, $E \hookrightarrow \mathbf{M}(\mathbb{R})$.

Throughout this note, function $\chi_{D(\varphi)}\varphi$ has the domain $\mathbb{R}$ and defines as follows:

$$(\chi_{D(\varphi)}\varphi)(t) = \begin{cases} \varphi(t), & \text{if } t \in D(\varphi), \\ 0, & \text{otherwise}, \end{cases}$$

where $D(\varphi) \subset \mathbb{R}$ is the domain of the function φ.

2.2. *Main results*

Let $(U(t,s))_{t\ge s}$ be an evolutionary family on I. For $t_0 \in I$ and $x \in X$, put

$$g_{t_0,x}(t) = \|U(t,t_0)x\| \quad \text{for} \quad t \ge t_0.$$

Because the evolutionary family is strongly continuous, $\chi_{[t_0,\infty)}g_{t_0,x}$ is measurable function. By the properties of the Banach function space, the exponential stability of the evolutionary family will now be characterized by its pointwise trajectories. First, we give the necessary condition.

Theorem 1. *Let E be any Banach function space. If $(U(t,s))_{t\ge s}$ is exponentially stable on I, then for each $x \in X$ the function $\chi_{[t_0,\infty)}g_{t_0,x}$ belongs to E and there is a constant $M(x) > 0$ such that*

$$\|\chi_{[t_0,\infty)}g_{t_0,x}\|_E \le M(x),$$

for all $t_0 \in I$.

Proof. First we show $e^{-\alpha|t|} \in E$. Put

$$v(t) = \int_{-\infty}^{t} e^{-\alpha(t-\tau)}\chi_{[0,1]}(\tau)\mathrm{d}\tau + \int_{t}^{\infty} e^{-\alpha(\tau-t)}\chi_{[0,1]}(\tau)\mathrm{d}\tau.$$

Then,

$$v(t) = \begin{cases} \frac{e^{-\alpha t}(e^{\alpha}-1)}{\alpha}, & \text{if } t \geq 1, \\ \frac{e^{\alpha t}(1-e^{-\alpha})}{\alpha}, & \text{if } t \leq 0, \\ \frac{1-e^{-\alpha t}}{\alpha} + \frac{1-e^{-\alpha(1-t)}}{\alpha}, & \text{if } t \in (0,1). \end{cases}$$

Therefore, $e^{\alpha|t|}v(t) \geq \frac{1-e^{-\alpha}}{\alpha}$ for all $t \in \mathbb{R}$. On the other hand,

$$\begin{aligned} v(t) &= \sum_{k=0}^{\infty} \int_{t-(k+1)}^{t-k} e^{-\alpha(t-\tau)}\chi_{[0,1]}(\tau)\mathrm{d}\tau + \sum_{k=0}^{\infty} \int_{t+k}^{t+k+1} e^{-\alpha(\tau-t)}\chi_{[0,1]}(\tau)\mathrm{d}\tau \\ &\leq \sum_{k=0}^{\infty} e^{-\alpha k} \int_{t-(k+1)}^{t-k} \chi_{[0,1]}(\tau)\mathrm{d}\tau + \sum_{k=0}^{\infty} e^{-\alpha k} \int_{t+k}^{t+k+1} \chi_{[0,1]}(\tau)\mathrm{d}\tau \\ &= \sum_{k=0}^{\infty} e^{-\alpha k}(T_{-k-1}(\Lambda_1\chi_{[0,1]}))(t) + \sum_{k=0}^{\infty} e^{-\alpha k}(T_k(\Lambda_1\chi_{[0,1]}))(t) =: \varphi(t). \end{aligned}$$

This implies that

$$e^{-\alpha|t|} \leq \frac{\alpha}{1-e^{-\alpha}}v(t) \leq \frac{\alpha}{1-e^{-\alpha}}\varphi(t) \quad \text{for all } t \in \mathbb{R}.$$

We also have the following estimates.

$$\begin{aligned} \sum_{k=0}^{\infty} e^{-\alpha k}\|T_{-k-1}(\Lambda_1\chi_{[0,1]})\|_E + \sum_{k=0}^{\infty} e^{-\alpha k}\|T_k(\Lambda_1\chi_{[0,1]})\|_E &\leq \sum_{k=0}^{\infty} e^{-\alpha k}2N\|\Lambda_1\chi_{[0,1]}\|_E \\ &= \frac{2N}{1-e^{-\alpha}}\|\Lambda_1\chi_{[0,1]}\|_E. \end{aligned}$$

So, function series φ is absolutely convergent in the Banach function space E. Therefore, $\varphi \in E$ and

$$\|\varphi\|_E \leq \frac{2N\|\Lambda_1\chi_{[0,1]}\|_E}{1-e^{-\alpha}}.$$

By the item i. in Definition 3, we get $e^{-\alpha|t|} \in E$ and

$$\|e^{-\alpha|\cdot|}\|_E \leq \frac{2N\alpha\|\Lambda_1\chi_{[0,1]}\|_E}{(1-e^{-\alpha})^2}.$$

Because $(U(t,s))_{t\geq s}$ is exponentially stable on I,

$$\begin{aligned} (\chi_{[t_0,\infty)}g_{t_0,x})(t) &\leq K_1\chi_{[t_0,\infty)}(t)e^{-\alpha(t-t_0)}\|x\| \\ &\leq K_1\|x\|(T_{-t_0}e^{-\alpha|\cdot|})(t), \end{aligned} \tag{2.3}$$

for all $t \in \mathbb{R}$. By the items i., v. in Definition 3 and (2.3), we obtain $\chi_{[t_0,\infty)} g_{t_0,x} \in E$ and

$$\|\chi_{[t_0,\infty)} g_{t_0,x}\|_E \le \frac{2N^2 K_1 \alpha \|\Lambda_1 \chi_{[0,1]}\|_E}{(1-e^{-\alpha})^2} \|x\|,$$

for all $t_0 \in I$. □

In sufficient condition, we need to add a constraint of Banach function space.

Theorem 2. *Let E be a Banach function space such that $\lim_{t\to+\infty} \|\chi_{[0,t]}\|_E = \infty$. If for each $x \in X$, the function $\chi_{[t_0,\infty)} g_{t_0,x}$ belongs to E and there exists a constant $M(x) > 0$ such that*

$$\|\chi_{[t_0,\infty)} g_{t_0,x}\|_E \le M(x),$$

for all $t_0 \in I$, then the evolutionary family $(U(t,s))_{t\ge s}$ is exponentially stable on I.

Remark 1. It is easy to see that the Lebesgue spaces $L^p(\mathbb{R})$ with $p \in [1,\infty)$ and the Lorentz spaces $L^{p,q}(\mathbb{R})$ with $p \in [1,\infty), q \in [1,\infty]$ satisfy the constraint above. Thus, Theorem 1 and Theorem 2 are an extension for Datko's result in the paper [4, Theorem 1]. This constraint is not to be missed; moreover, it also appears naturally in the proof of the theorem. On the other hand, we can give a simple example below to see that the condition $\lim_{t\to+\infty} \|\chi_{[0,t]}\|_E = \infty$ can not be omitted.

In $\mathbb{R}^2$, consider $E = L^\infty(\mathbb{R})$ and the evolutionary family $(U(t,s))_{t\ge s}$ on I, in which

$$U(t,s) = \begin{pmatrix} \cos(t-s) & -\sin(t-s) \\ \sin(t-s) & \cos(t-s) \end{pmatrix}.$$

Obviously, the evolutionary family $(U(t,s))_{t\ge s}$ is not exponentially stable on I. For each $x \in \mathbb{R}^2$ and $t_0 \in I$, we have

$$g_{t_0,x}(t) = \|U(t,t_0)x\|_{\mathbb{R}^2} = \|x\|_{\mathbb{R}^2}, \quad \text{for} \quad t \ge t_0.$$

Therefore, $\|\chi_{[t_0,\infty)} g_{t_0,x}\|_{L^\infty(\mathbb{R})} = \|x\|_{\mathbb{R}^2}$ for all $t_0 \in I$.

Proof. For $t_0 \in I$ and $t > t_0$, for each $x \in X$ then mapping $\varphi_{t,t_0,x}$ is defined as follows:

$$\varphi_{t,t_0,x}(\xi) = \begin{cases} \|U(\xi,t_0)x\|, & \text{if } \xi \in [t_0,t], \\ 0, & \text{if } \xi \notin [t_0,t]. \end{cases}$$

Then, $\varphi_{t,t_0,x} \in E$ and

$$\|\varphi_{t,t_0,x}\|_E \le \|\chi_{[t_0,\infty)} g_{t_0,x}\|_E \le M(x). \tag{2.4}$$

We now construct a family of functions $\{\Phi_{t,t_0}\}$ determining by

$$\Phi_{t,t_0} : X \to \mathbb{R} \quad \text{with} \quad \Phi_{t,t_0}(x) = \|\varphi_{t,t_0,x}\|_E.$$

It is easy to see that Φ_{t,t_0} is a seminorm on X. On the other hand, we have

$$\varphi_{t,t_0,x}(\xi) \le K e^{c(t-t_0)} \chi_{[t_0,t]}(\xi) \|x\|,$$

for all $\xi \in \mathbb{R}$. Thus, $\Phi_{t,t_0}(x) \le Ke^{c(t-t_0)}\|\chi_{[t_0,t]}\|_E\|x\|$. So, Φ_{t,t_0} is a continuous seminorm on X. Moreover, by (2.4), the family of continuous seminorms $\{\Phi_{t,t_0} : t_0 \in I, t > t_0\}$ is pointwise bounded. Applying uniform boundedness principle (see Appendix), there exists a constant $C > 0$ such that

$$\Phi_{t,t_0}(x) \le C\|x\|, \tag{2.5}$$

for all $x \in X$ and $t_0 \in I, t > t_0$.

For $\xi \in [t_0,t]$, we have

$$e^{-c(t-\xi)}\|U(t,t_0)x\| = e^{-c(t-\xi)}\|U(t,\xi)U(\xi,t_0)x\| \le K\|U(\xi,t_0)x\|.$$

Therefore,

$$\chi_{[t_0,t]}(\xi)e^{-c(t-\xi)}\|U(t,t_0)x\| \le K\varphi_{t,t_0,x}(\xi),$$

for all $\xi \in \mathbb{R}$. So,

$$\|\chi_{[t_0,t]}e^{-c(t-\cdot)}\|_E\|U(t,t_0)x\| \le KC\|x\|.$$

By (2.2),

$$\|\chi_{[t_0,t]}e^{-c(t-\cdot)}\|_M\|U(t,t_0)x\| \le \frac{KCM}{\inf_{\tau\in\mathbb{R}}\|\chi_{[\tau,\tau+1]}\|_E}\|x\|,$$

for all $t_0 \in I$ and $t > t_0$.

For $t \ge t_0 + 1$, we have

$$\|\chi_{[t_0,t]}e^{-c(t-\cdot)}\|_M \ge \int_{t-1}^{t} e^{-c(t-\xi)}\mathrm{d}\xi = \frac{1-e^{-c}}{c}.$$

Thus,

$$\|U(t,t_0)x\| \le \frac{KCMc}{(1-e^{-c})\inf_{\tau\in\mathbb{R}}\|\chi_{[\tau,\tau+1]}\|_E}\|x\|,$$

for all $t_0 \in I$ and $t \ge t_0 + 1$. Because the evolutionary family $(U(t,s))_{t\ge s}$ is exponentially bounded, there exists a constant $C_1 > 0$ such that

$$\|U(t,t_0)x\| \le C_1\|x\|,$$

for all $x \in X$ and $t_0 \in I, t \ge t_0$.

For $\xi \in [t_0,t]$, we have $\|U(t,t_0)x\| \le C_1\|U(\xi,t_0)x\|$. Therefore,

$$\chi_{[t_0,t]}(\xi)\|U(t,t_0)x\| \le C_1\varphi_{t,t_0,x}(\xi),$$

for all $\xi \in \mathbb{R}$. So,

$$\|\chi_{[t_0,t]}\|_E\|U(t,t_0)x\| \le C_1C\|x\|,$$

for all $t_0 \in I$ and $t > t_0$. On the other hand,

$$\chi_{[0,t-t_0]}(\xi) = \chi_{[t_0,t]}(\xi + t_0) = (T_{t_0}\chi_{[t_0,t]})(\xi).$$

Therefore, $\|\chi_{[0,t-t_0]}\|_E \le N\|\chi_{[t_0,t]}\|_E$. So,

$$\|U(t,t_0)x\| \le \frac{NC_1C}{\|\chi_{[0,t-t_0]}\|_E}\|x\|,$$

for all $t_0 \in I$ and $t > t_0$. Because of $\lim_{t\to+\infty} \|\chi_{[0,t]}\|_E = \infty$, the evolutionary family $(U(t,s))_{t\geq s}$ is exponentially stable on I. □

The following corollaries are a minor weakening of Theorem 2 for special evolutionary families.

Corollary 1. *Let E be a Banach function space such that $\lim_{t\to+\infty} \|\chi_{[0,t]}\|_E = \infty$ and $(T(t))_{t\geq 0}$ be a strongly continuous semigroup. If function $\chi_{[0,\infty)} g_{0,x}$ belongs to E for each $x \in X$ then $(T(t))_{t\geq 0}$ is exponentially stable, where $g_{0,x}(t) = \|T(t)x\|$ for $t \geq 0$.*

Remark 2. This corollary is more general than Pazy's result in [15, Chapter 4, Theorem 4.1].

Proof. For $t_0 \in \mathbb{R}$ and $x \in X$, we have $g_{t_0,x}(t) = \|T(t-t_0)x\|$ for $t \geq t_0$. Therefore,

$$(\chi_{[t_0,\infty)} g_{t_0,x})(t) = (\chi_{[0,\infty)} g_{0,x})(t-t_0) = (T_{-t_0}(\chi_{[0,\infty)} g_{0,x}))(t),$$

for $t \in \mathbb{R}$. By the item v. in Definition 3, we get $\chi_{[t_0,\infty)} g_{t_0,x} \in E$ and

$$\|\chi_{[t_0,\infty)} g_{t_0,x}\|_E \leq N \|\chi_{[0,\infty)} g_{0,x}\|_E, \quad \text{for all } t_0 \in \mathbb{R}.$$

By Theorem 2, $(T(t))_{t\geq 0}$ is an exponentially stable semigroup. □

Next, we will give the exponentially stable characterization for a periodic evolutionary family.

Definition 4. An evolutionary family $(U(t,s))_{t\geq s}$ is said to be periodic with a period $T > 0$ if $U(t+T, s+T) = U(t,s)$ for all $t \geq s$ and $t,s \in I$.

Corollary 2. *Let E be a Banach function space such that $\lim_{t\to+\infty} \|\chi_{[0,t]}\|_E = \infty$ and $(U(t,s))_{t\geq s}$ be a periodic evolutionary family with period $T > 0$. If the function $\chi_{[T,\infty)} g_{T,x}$ belongs to E for each $x \in X$, then the evolutionary family $(U(t,s))_{t\geq s}$ is exponentially stable on I.*

Proof. By the periodicity of the evolutionary family $(U(t,s))_{t\geq s}$ on I so we just need to consider $t_0 \geq 0$. We will repeat manner of the proof in Theorem 2 to obtain (2.5) as follows.

Replace t_0 with T and discuss the same as in the first paragraph in the proof of Theorem 2, there exists a constant $D > 0$ such that

$$\Phi_{t,T}(x) \leq D\|x\|,$$

for all $x \in X$ and $t > T$. For $t_0 \in [0,T)$ and $t \in (t_0, T]$, we have

$$\varphi_{t,t_0,x}(\xi) \leq \chi_{[0,T]}(\xi) K e^{cT} \|x\|,$$

for all $\xi \in \mathbb{R}$. Therefore, $\varphi_{t,t_0,x} \in E$ and $\|\varphi_{t,t_0,x}\|_E \leq K e^{cT} \|\chi_{[0,T]}\|_E \|x\|$. For $t > T$,

$$\varphi_{t,t_0,x}(\xi) \leq \chi_{[0,T]}(\xi) K e^{cT} \|x\| + \chi_{[T,t]}(\xi) \|U(\xi,T) U(T,t_0) x\|,$$

for all $\xi \in \mathbb{R}$. Thus, $\varphi_{t_0,x} \in E$ and

$$\|\varphi_{t,t_0,x}\|_E \leq Ke^{cT}\|\chi_{[0,T]}\|_E\|x\| + \Phi_{t,T}(U(T,t_0)x) \leq Ke^{cT}(\|\chi_{[0,T]}\|_E + D)\|x\|.$$

For $t_0 \geq T$ and $t > t_0$, we can write $t_0 = nT + \tau$ with $n \in \mathbb{N}$ and $\tau \in [0,T)$. Then,

$$\|U(\xi,t_0)x\| = \|U(\xi,nT+\tau)x\| = \|U(\xi-nT,\tau)x\| = \|U(\xi-t_0+\tau,\tau)x\|,$$

for $\xi \in [t_0,t]$. Put

$$\psi_{\tau,x}(\xi) = \begin{cases} \|U(\xi,\tau)x\|, & \text{if } \xi \in [\tau, t-t_0+\tau], \\ 0, & \text{if } \xi \notin [\tau, t-t_0+\tau]. \end{cases}$$

Then, $\varphi_{t,t_0,x}(\xi) = (T_{-t_0+\tau}\psi_{\tau,x})(\xi)$ for all $\xi \in \mathbb{R}$. Because of $\tau \in [0,T)$ so $\psi_{\tau,x} \in E$, hence $\varphi_{t,t_0,x} \in E$ and

$$\|\varphi_{t,t_0,x}\|_E \leq NKe^{cT}(\|\chi_{[0,T]}\|_E + D)\|x\|.$$

So, for all $t_0 \geq 0$ and $t > t_0$ then $\varphi_{t,t_0,x} \in E$ and

$$\|\varphi_{t,t_0,x}\|_E \leq NKe^{cT}(\|\chi_{[0,T]}\|_E + D)\|x\|.$$

Therefore,

$$\Phi_{t,t_0}(x) \leq NKe^{cT}(\|\chi_{[0,T]}\|_E + D)\|x\|,$$

for all $x \in X$ and $t_0 \geq 0, t > t_0$. The next step, by the same discussions as in the proof of Theorem 2 we deduce that the evolutionary family $(U(t,s))_{t\geq s}$ is exponentially stable on I. □

Appendix

For completeness, we restate here the boundedness principle for a family of continuous seminorms.

Let X be a Banach space over the field K ($K = \mathbb{R}$ or $\mathbb{C}$). A mapping $p\colon X \to \mathbb{R}_+$ is a seminorm on X if $p(\theta x) = |\theta|p(x)$ and $p(x+y) \leq p(x) + p(y)$ for all $x, y \in X$ and $\theta \in K$. As is known, a seminorm p is continuous on X if and only if p is continuous at 0. Moreover, if A is a bounded linear operator on X then $p_A(x) = \|Ax\|$ for $x \in X$ is a continuous seminorm on X.

Let Λ be an index set. A family of continuous seminorms $\{p_\lambda\colon \lambda \in \Lambda\}$ on X is called

- pointwise bounded if for each $x \in X$ there exists a constant $M(x) > 0$ such that $p_\lambda(x) \leq M(x)$ for all $\lambda \in \Lambda$;
- uniformly bounded if there is a constant $M > 0$ such that $p_\lambda(x) \leq M\|x\|$ for all $\lambda \in \Lambda$ and $x \in X$.

A similar proof to the uniform boundedness principle for a family of bounded linear operators (see Kreyszig [8]), we get a version of the uniform boundedness principle for a family of continuous seminorms.

Uniform boundedness principle. *Let* $\{p_\lambda : \lambda \in \Lambda\}$ *be a family of continuous seminorms on* X. *If this family is pointwise bounded then it is also uniformly bounded.*

Obviously, this version is more general than the old version in functional analysis.

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BOURGAIN-LEBESGUE SPACES

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Abstract. Bourgain initially introduced a specific instance of Bourgain-Morrey spaces to investigate the restriction and multiplier problems in $\mathbb{R}^3$. Following this, the concept of Bourgain-type function spaces garnered considerable attention among researchers. In the paper, we aim to introduce Bourgain-Lebesgue spaces, and delve into the embedding properties, the Young inequality, dilation properties in the spaces. Additionally, we explore the boundedness properties within Bourgain-Lebesgue spaces concerning local Hardy-Littlewood maximal operators and their vector-valued counterparts.

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1. Introduction

Let $\nu \in \mathbb{Z}$ and $\vec{m} = (m_1, m_2, \dots, m_n) \in \mathbb{Z}^n$. A dyadic cube $Q_{\nu\vec{m}}$ is defined by

$$Q_{\nu\vec{m}} := \prod_{i=1}^{n} \left[\frac{m_i}{2^\nu}, \frac{m_i+1}{2^\nu}\right),$$

denote $\mathcal{D}_\nu := \{Q_{\nu\vec{m}} : \vec{m} \in \mathbb{Z}^n\}$ and $\mathcal{D} := \bigcup_{\nu\in\mathbb{Z}} \mathcal{D}_\nu$. Let $0 < p \leqslant q < \infty$ and $0 < r \leqslant \infty$. The Bourgain-Morrey space $\mathcal{M}^q_{p,r}(\mathcal{D})$ is the set of all $f \in L^p_{\text{loc}}(\mathbb{R}^n)$ satisfying

$$\|f\|_{\mathcal{M}^q_{p,r}(\mathcal{D})} = \left\| \left\{ |Q_{\nu\vec{m}}|^{\frac{1}{q}-\frac{1}{p}} \left(\int_{Q_{\nu\vec{m}}} |f(y)|^p \, \mathrm{d}y \right)^{\frac{1}{p}} \right\}_{\nu\in\mathbb{Z}, \vec{m}\in\mathbb{Z}^n} \right\|_{\ell^r} < \infty.$$

Here $\|\cdot\|_{\ell^r}$ is the norm of discrete Lebesgue space ℓ^r.

In [2], Bourgain introduced a function space to study the restriction and multiplier problems in $\mathbb{R}^3$. The function space can now be viewed as a special case of Bourgain-Morrey spaces. In [4], Hatano et al studied the Bourgain-Morrey space from the perspectives of harmonic analysis and functional analysis. They obtain some classical

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results related to the spaces, such as approximation properties in the spaces, interpolation properties between the spaces and the boundedness of operators in the spaces, as well as the dual of the spaces. Besides, there are some general spaces related to Bourgain-Morrey spaces, such as Triebel-Lizorkin-Bourgain-Morrey spaces [5] and Besov-Bourgain-Morrey spaces [11]. The Bourgain-Morrey spaces are important in partial differential equations, for more details we refer the reader to references [1, 6–9] and the references therein.

Our interests are successfully attracted by Example 2.9 in [4]. The example shows that the Bourgain-Morrey space $\mathcal{M}^q_{p,r}(\mathcal{D})$ is not trivial if and only if $0 < p < q < r < \infty$ or $0 < p \leqslant q < r = \infty$. In the case of $0 < p \leqslant q < r = \infty$, the Bourgain-Morrey space $\mathcal{M}^q_{p,\infty}(\mathcal{D})$ coincides with the Morrey space $\mathcal{M}^q_p(\mathbb{R}^n)$ which is defined by

$$\mathcal{M}^q_p(\mathbb{R}^n) := \left\{ f \in L^p_{\mathrm{loc}}(\mathbb{R}^n) : \ \|f\|_{\mathcal{M}^q_p(\mathbb{R}^n)} < \infty \right\},$$

and

$$\|f\|_{\mathcal{M}^q_p(\mathbb{R}^n)} := \sup_{Q \in \mathcal{D}} |Q|^{\frac{1}{q}-\frac{1}{p}} \left(\int_Q |f(y)|^p \, \mathrm{d}y \right)^{\frac{1}{p}}.$$

It is well know that $\mathcal{M}^q_p(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ when $p = q$, so the Bourgain-Morrey space $\mathcal{M}^p_{p,\infty}(\mathcal{D})$ is identical with $L^p(\mathbb{R}^n)$.

While in the case of $0 < p < q < r < \infty$, the Bourgain-Morrey space $\mathcal{M}^q_{p,r}(\mathcal{D})$ can not be reduced to the Lebesgue space. In the setting, an interesting question is that how to introduce a corresponding space in the case $0 < p = q < r < \infty$. We try to answer this question in the paper.

Inspired by the above works, in Section 2, we aim to introduce the Bourgain-Lebesgue space and delve into the embedding properties of the spaces. Besides, there are some examples show that Lebesgue spaces and the Bourgain-Lebesgue space are not contained within each other. In Section 3, the Young inequality and dilation properties are obtained in the Bourgain-Lebesgue space. Section 4 contains the boundedness and weak boundedness of local Hardy-Littlewood maximal operators in Bourgain-Lebesgue spaces. In Section 5, we obtain the boundedness of vector-valued local Hardy-Littlewood maximal operators in Bourgain-Lebesgue spaces.

At the end of the section, we need to explain some conventions on notations. The C is a positive constant and independent of the main parameters in the formula, but may vary from line to line. The symbol $f \lesssim g$ means $f \leqslant Cg$. The symbol $f \gtrsim g$ means $f \geq Cg$.

2. The Bourgain-Lebesgue space and its basic properties

In order to introduce the Bourgain-Lebesgue space, we have to throw the large cubes in $\mathcal{D}$ away (for more details see Example 1). Let $\mathbb{N}$ be the set of nonnegative integers, $\mathcal{B} := \bigcup_{\nu \in \mathbb{N}} \mathcal{D}_\nu$. So $\mathcal{B}$ contains all the cubes in $\mathcal{D}$ with the length less than or equal to 1.

Definition 1. Let $0 < p < r \leqslant \infty$, the Bourgain-Lebesgue space is defined by

$$\mathcal{B}^r L^p(\mathbb{R}^n) := \left\{ f \in L^p_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{\mathcal{B}^r L^p(\mathbb{R}^n)} < \infty \right\},$$

where

$$\|f\|_{\mathcal{B}^r L^p(\mathbb{R}^n)} := \begin{cases} \left(\sum_{Q \in \mathcal{B}} \|f\|^r_{L^p(Q)} \right)^{\frac{1}{r}}, & r \in (0, \infty), \\ \sup_{Q \in \mathcal{B}} \|f\|_{L^p(Q)}, & r = \infty. \end{cases}$$

It is easy to know that $L^p(\mathbb{R}^n)$ and $L^\infty(\mathbb{R}^n)$ are subspaces of $\mathcal{B}^\infty L^p(\mathbb{R}^n)$. $\mathcal{B}^\infty L^p(\mathbb{R}^n)$ is equivalent to the Wiener amalgam space $W(L^p, \ell^\infty)(\mathbb{R}^n)$ which is important to consider the problems related to multidimensional summation [10]. We also mention that the space $\mathcal{B}^\infty L^\infty(\mathbb{R}^n)$ is not contained in the Definition 1, because the norm of $\mathcal{B}^\infty L^\infty(\mathbb{R}^n)$ is equal to the norm of $L^\infty(\mathbb{R}^n)$.

Next example means that the condition "$p < r$" is needed in Definition 1.

Example 1. Let $0 < p < \infty$ and $0 < r < \infty$. Then $\chi_{[0,1)^n} \in \mathcal{B}^r L^p(\mathbb{R}^n)$ if and only if $0 < p < r < \infty$.

Next two embedding properties are obvious of Bourgain-Lebesgue spaces.

Proposition 1. *Let* $0 < p < r_1 \leqslant r_2 \leqslant \infty$. *Then* $\mathcal{B}^{r_1} L^p(\mathbb{R}^n) \subseteq \mathcal{B}^{r_2} L^p(\mathbb{R}^n)$ *with continuous embedding.*

Proof. The proposition is due to $l^{r_1} \subseteq l^{r_2}$ with $r_1 \leqslant r_2$. □

Proposition 2. *Let* $0 < p_1 \leqslant p_2 < r \leqslant \infty$. *Then* $\mathcal{B}^r L^{p_2}(\mathbb{R}^n) \subseteq \mathcal{B}^r L^{p_1}(\mathbb{R}^n)$ *with continuous embedding.*

Proof. The proposition is obtained by $\|f\|_{L^{p_1}(Q)} \leqslant \|f\|_{L^{p_2}(Q)}$ for $Q \in \mathcal{B}$. □

The following three examples show that Lebesgue spaces $L^p(\mathbb{R}^n)$ and Bourgain-Lebesgue spaces $\mathcal{B}^r L^p(\mathbb{R}^n)$ do not contain each other for $0 < p < r < \infty$.

Example 2. Let $0 < p < r < \infty$, $a \in (n, \infty)$ and $f(x) = |x|^{-\frac{a}{p}} \cdot \chi_{\mathbb{R}^n \setminus B(\vec{0},1)}(x)$, $x \in \mathbb{R}^n$. Then $f \in L^p(\mathbb{R}^n)$, but $f \notin \mathcal{B}^r L^p(\mathbb{R}^n)$.

Let $x \in \mathbb{R}^n$, we use $x_i \in \mathbb{R}$ to denote the i-th coordinate component of x, then $x = (x_1, x_2, \ldots, x_n)$.

Example 3. Let $0 < p < r < \infty$, $f(x) = x_1^{-\frac{1}{p}} \left(\sum_{k \in \mathbb{Z}^+} \chi_{[k,k+1)^n}(x) \right)$, $x \in \mathbb{R}^n$. Then $f \in \mathcal{B}^r L^p(\mathbb{R}^n)$, but $f \notin L^p(\mathbb{R}^n)$.

Example 4. Let $0 < p < r < \infty$, $a \in (0,1)$ and

$$f(x) := \left[(1-a)^n \prod_{i=1}^n x_i^{-a} \right]^{\frac{1}{p}} \cdot \sum_{k \in \mathbb{Z}^+} \chi_{[k,k+1)^n}(x), \qquad x \in \mathbb{R}^n.$$

Then

(1) $f \notin L^p(\mathbb{R}^n)$ and $f \notin \mathcal{B}^r L^p(\mathbb{R}^n)$, $a \in \left(0, \frac{p}{nr}\right]$;
(2) $f \notin L^p(\mathbb{R}^n)$ but $f \in \mathcal{B}^r L^p(\mathbb{R}^n)$, $a \in \left(\frac{p}{nr}, \frac{1}{n}\right]$;
(3) $f \in L^p(\mathbb{R}^n)$ and $f \in \mathcal{B}^r L^p(\mathbb{R}^n)$, $a \in \left(\frac{1}{n}, 1\right)$.

At the end of the section, we show Bourgain-Lebesgue spaces are complete.

Theorem 1. *Let $1 \leqslant p < r \leqslant \infty$. Then $\mathcal{B}^r L^p(\mathbb{R}^n)$ is a Banach space.*

Proof. We only prove the theorem for $1 \leqslant p < r < \infty$ because the proof is similar to the theorem for $1 \leqslant p < r = \infty$. The fact is obvious that $\|\cdot\|_{\mathcal{B}^r L^p(\mathbb{R}^n)}$ is a norm. Let $\{f_m\}_{m\in\mathbb{Z}^+} \subseteq \mathcal{B}^r L^p(\mathbb{R}^n)$ be a Cauchy sequence, it can deduce that $\{f_m\}_{m\in\mathbb{Z}^+}$ is a Cauchy sequence in $L^p(Q)$ independent of $Q \in \mathcal{B}$. So, there exist $\{g_Q\}_{Q\in\mathcal{B}} \subset L^p(\mathbb{R}^n)$ such that

$$\lim_{m\to\infty} \|f_m - g_Q\|_{L^p(Q)} = 0, \text{ uniformly for } Q \in \mathcal{B},$$

and

$$g_Q(x) = g_{Q_{(k)}}(x) \qquad \text{a.e., } x \in Q_k,$$

where $Q_{(k)}$ is a k-th subgeneration of Q.

Let $Q \in \mathcal{D}_0$ and

$$f_Q(x) := \begin{cases} g_Q(x), & x \in Q, \\ 0, & x \notin Q, \end{cases}$$

we denote f by

$$f(x) := \sum_{Q\in\mathcal{D}_0} f_Q(x),\ x \in \mathbb{R}^n.$$

Now we prove that $f \in \mathcal{B}^r L^p(\mathbb{R}^n)$. Due to the fact

$$\begin{aligned} \|f\|_{\mathcal{B}^r L^p(\mathbb{R}^n)} &= \left(\sum_{Q\in\mathcal{B}} \|g_Q\|_{L^p(Q)}^r\right)^{\frac{1}{r}} = \left[\sum_{Q\in\mathcal{B}} \left(\lim_{m\to\infty} \|f_m\|_{L^p(Q)}\right)^r\right]^{\frac{1}{r}} \\ &= \lim_{m\to\infty} \|f_m\|_{\mathcal{B}^r L^p(\mathbb{R}^n)}, \end{aligned}$$

there exist a constant C such that

$$\|f\|_{\mathcal{B}^r L^p(\mathbb{R}^n)} \leqslant C.$$

It means $f \in \mathcal{B}^r L^p(\mathbb{R}^n)$. Besides,

$$\lim_{m\to\infty} \|f_m - f\|_{\mathcal{B}^r L^p(\mathbb{R}^n)} = \left[\sum_{Q\in\mathcal{B}} \left(\lim_{m\to\infty} \|f_m - f\|_{L^p(Q)}\right)^r\right]^{\frac{1}{r}} = 0.$$

To sum up, $\mathcal{B}^r L^p(\mathbb{R}^n)$ is a Banach space. □

3. THE CONVOLUTION OPERATION IN BOURGAIN-LEBESGUE SPACES

In the section, some convolution inequalities are obtained in Bourgain-Lebesgue spaces. We begin with the translation properties in the spaces.

Lemma 1. *Let* $1 \leqslant p < r < \infty$*. Then*

$$\frac{1}{2^n}\|f\|_{\mathcal{B}^r L^p(\mathbb{R}^n)} \leqslant \|f(\cdot - y)\|_{\mathcal{B}^r L^p(\mathbb{R}^n)} \leqslant 2^n \|f\|_{\mathcal{B}^r L^p(\mathbb{R}^n)}, \qquad y \in \mathbb{R}^n.$$

Proof. Let $\nu \in \mathbb{N}$ and $y \in \mathbb{R}^n$, there exist $\vec{m}_1(\nu, y), \vec{m}_2(\nu, y), \ldots, \vec{m}_{2^n}(\nu, y) \in \mathbb{Z}^n$ such that

$$Q_{\nu\vec{m}} - y \subset \bigcup_{k=1}^{2^n} Q_{\nu(\vec{m}+\vec{m}_k(\nu,y))},$$

for each $\vec{m} \in \mathbb{Z}^n$. Let $z = x - y$, then

$$\|f(\cdot - y)\|_{\mathcal{B}^r L^p(\mathbb{R}^n)} \leqslant \left[\sum_{\nu\in\mathbb{N},\vec{m}\in\mathbb{Z}^n}\left(\sum_{k=1}^{2^n}\int_{Q_{\nu(\vec{m}+\vec{m}_k(\nu,y))}}|f(z)|^p\,\mathrm{d}z\right)^{\frac{r}{p}}\right]^{\frac{1}{r}} \leqslant 2^n\|f\|_{\mathcal{B}^r L^p(\mathbb{R}^n)}.$$

On the other hand, by a similar way,

$$\|f\|_{\mathcal{B}^r L^p(\mathbb{R}^n)} = \left[\sum_{\nu\in\mathbb{N},\vec{m}\in\mathbb{Z}^n}\left(\int_{Q_{\nu\vec{m}}+y}|f(x-y)|^p\,\mathrm{d}x\right)^{\frac{r}{p}}\right]^{\frac{1}{r}} \leqslant 2^n\|f(\cdot - y)\|_{\mathcal{B}^r L^p(\mathbb{R}^n)}. \square$$

The following theorem states that convolution operation is well defined in Bourgain-Lebesgue spaces.

Theorem 2. *Let* $1 < p < r < \infty$*,* $f \in \mathcal{B}^r L^p(\mathbb{R}^n)$ *and* $g \in L^1(\mathbb{R}^n)$*. Then*

$$\|g * f\|_{\mathcal{B}^r L^p(\mathbb{R}^n)} \leqslant 2^n\|f\|_{\mathcal{B}^r L^p(\mathbb{R}^n)}\|g\|_{L^1(\mathbb{R}^n)}.$$

Proof. It is easy to know that

$$\begin{aligned}\|g * f\|_{\mathcal{B}^r L^p(\mathbb{R}^n)} &\leqslant \left[\sum_{Q\in\mathcal{B}}\left(\int_{\mathbb{R}^n}\|f(\cdot - y)\|_{L^p(Q)}|g(y)|\,\mathrm{d}y\right)^r\right]^{\frac{1}{r}} \\ &\leqslant \int_{\mathbb{R}^n}\left(\sum_{Q\in\mathcal{B}}\|f(\cdot - y)\|_{L^p(Q)}^r\right)^{\frac{1}{r}}|g(y)|\,\mathrm{d}y,\end{aligned}$$

where the penultimate inequality is due to the Minkowski inequality for the L^p-norm and the last inequality is because of the Minkowski inequality for the ℓ^r-norm. By Lemma 1, we have

$$\|g * f\|_{\mathcal{B}^r L^p(\mathbb{R}^n)} \leqslant \int_{\mathbb{R}^n}\|f(\cdot - y)\|_{\mathcal{B}^r L^p(\mathbb{R}^n)}|g(y)|\,\mathrm{d}y \leqslant 2^n\|f\|_{\mathcal{B}^r L^p(\mathbb{R}^n)}\|g\|_{L^1(\mathbb{R}^n)}. \quad \square$$

By the next lemma, we will show that the Young inequality is true in Bourgain-Lebesgue spaces. Let $a>0$, Q be a cube, denote aQ by the dilation of Q around its centre by a.

Lemma 2. *Let $0<p<r<\infty$ and $a>0$. There exists a positive constant $C_{a,n}$ related to a and n such that*

$$\left(\sum_{\nu\in\mathbb{N},\vec{m}\in\mathbb{Z}^n}\|f\|^r_{L^p(aQ_{\nu\vec{m}})}\right)^{\frac{1}{r}}\leqslant C_{a,n}\|f\|_{\mathcal{B}^rL^p(\mathbb{R}^n)}.$$

Proof. Let $a\in(0,1]$, the lemma is obviously correct because $aQ_{\nu\vec{m}}\subset Q_{\nu\vec{m}}$ for $\nu\in\mathbb{N}$ and $\vec{m}\in\mathbb{Z}^n$. Let $a\in(1,\infty)$, there exists $\beta_a\in\mathbb{Z}^+$ such that $2^{\beta_a-1}\leqslant a<2^{\beta_a}$. For any $Q_{\nu\vec{m}}\in\mathcal{B}$ with $\nu\in\{0,1,\ldots,\beta_a-1\}$ and $\vec{m}\in\mathbb{Z}^n$, there are at most $\tau(a):=\left(2^{\beta_a}+1\right)^n$ cubes in $\mathcal{D}_0$, for example $Q_{0\vec{m}_1},Q_{0\vec{m}_2},\ldots,Q_{0\vec{m}_{\tau(a)}}$, such that

$$aQ_{\nu\vec{m}}\subset\bigcup_{i=1}^{\tau(a)}Q_{0\vec{m}_i}.$$

So

$$\begin{aligned}
&\left(\sum_{\nu\in\mathbb{N},\vec{m}\in\mathbb{Z}^n}\|f\|^r_{L^p(aQ_{\nu\vec{m}})}\right)^{\frac{1}{r}}\\
&\leqslant\left[\sum_{\substack{\nu\in\{0,1,\ldots,\beta_a-1\}\\ \vec{m}\in\mathbb{Z}^n}}\left(\int_{aQ_{\nu\vec{m}}}|f(x)|^p\,\mathrm{d}x\right)^{\frac{r}{p}}\right]^{\frac{1}{r}}\\
&\quad+\left[\sum_{\substack{\nu\in\mathbb{N}\setminus\{0,1,\ldots,\beta_a-1\}\\ \vec{m}\in\mathbb{Z}^n}}\left(\int_{aQ_{\nu\vec{m}}}|f(x)|^p\,\mathrm{d}x\right)^{\frac{r}{p}}\right]^{\frac{1}{r}}\\
&\leqslant\sum_{i=1}^{\tau(a)}\left[\sum_{\substack{\nu\in\{0,1,\ldots,\beta_a-1\}\\ \vec{m}\in\mathbb{Z}^n}}\left(\int_{Q_{0\vec{m}_i}}|f(x)|^p\,\mathrm{d}x\right]^{\frac{r}{p}}\right)^{\frac{1}{r}}+2^n\|f\|_{\mathcal{B}^rL^p(\mathbb{R}^n)}\\
&\leqslant\left(\tau(a)\beta_a+2^n\right)\|f\|_{\mathcal{B}^rL^p(\mathbb{R}^n)}.
\end{aligned}$$

□

Now, we state Young's inequality for Bourgain-Lebesgue spaces.

Theorem 3 (The Young inequality)**.** *Let $1\leqslant p<r<\infty$, $1\leqslant p_0<r_0<\infty$, $1\leqslant p_1<r_1<\infty$ and*

$$\frac{1}{p}+1=\frac{1}{p_0}+\frac{1}{p_1},\qquad\frac{1}{r}+1=\frac{1}{r_0}+\frac{1}{r_1}.$$

Then, for $f \in \mathcal{B}^{r_0}L^{p_0}(\mathbb{R}^n)$ and $g \in \mathcal{B}^{r_1}L^{p_1}(\mathbb{R}^n)$, there exists

$$\|f * g\|_{\mathcal{B}^rL^p(\mathbb{R}^n)} \lesssim \|f\|_{\mathcal{B}^{r_0}L^{p_0}(\mathbb{R}^n)}\|g\|_{\mathcal{B}^{r_1}L^{p_1}(\mathbb{R}^n)}.$$

Proof. Let $Q_{\nu\vec{m}} \in \mathcal{B}$, by the Minkowski inequality and the Young inequality for the L^p-norm,

$$\begin{aligned}\|f * g\|_{L^p(Q_{\nu\vec{m}})} &\leqslant \sum_{\vec{m}'\in\mathbb{Z}^n} \left\|\left(f\chi_{Q_{\nu\vec{m}'}}\right) * \left(g\chi_{Q_{\nu\vec{m}}-Q_{\nu\vec{m}'}}\right)\right\|_{L^p(\mathbb{R}^n)} \\ &\lesssim \sum_{\vec{m}'\in\mathbb{Z}^n} \left\|f\chi_{Q_{\nu\vec{m}'}}\right\|_{L^{p_0}(\mathbb{R}^n)} \left\|g\chi_{Q_{\nu\vec{m}}-Q_{\nu\vec{m}'}}\right\|_{L^{p_1}(\mathbb{R}^n)},\end{aligned}$$

where $Q_{\nu\vec{m}} - Q_{\nu\vec{m}'} = \{x-x' :\ x \in Q_{\nu\vec{m}}, x' \in Q_{\nu\vec{m}'}\}$.

Due to the fact that $Q_{\nu\vec{m}} - Q_{\nu\vec{m}'} \subset 3Q_{\nu(\vec{m}-\vec{m}')}$, we have

$$\begin{aligned}\left(\sum_{\vec{m}\in\mathbb{Z}^n} \|f * g\|^r_{L^p(Q_{\nu\vec{m}})}\right)^{\frac{1}{r}} &\lesssim \left[\sum_{\vec{m}\in\mathbb{Z}^n}\left(\sum_{\vec{m}'\in\mathbb{Z}^n} \left\|f\chi_{Q_{\nu\vec{m}'}}\right\|_{L^{p_0}(\mathbb{R}^n)} \cdot \left\|g\chi_{3Q_{\nu(\vec{m}-\vec{m}')}}\right\|_{L^{p_1}(\mathbb{R}^n)}\right)^r\right]^{\frac{1}{r}} \\ &\leqslant \left(\sum_{\vec{m}\in\mathbb{Z}^n} \|f\chi_{Q_{\nu\vec{m}}}\|^{r_0}_{L^{p_0}(\mathbb{R}^n)}\right)^{\frac{1}{r_0}} \cdot \left(\sum_{\vec{m}\in\mathbb{Z}^n} \|g\chi_{3Q_{\nu\vec{m}}}\|^{r_1}_{L^{p_1}(\mathbb{R}^n)}\right)^{\frac{1}{r_1}} \\ &= \left(\sum_{\vec{m}\in\mathbb{Z}^n} \|f\|^{r_0}_{L^{p_0}(Q_{\nu\vec{m}})}\right)^{\frac{1}{r_0}} \cdot \left(\sum_{\vec{m}\in\mathbb{Z}^n} \|g\|^{r_1}_{L^{p_1}(3Q_{\nu\vec{m}})}\right)^{\frac{1}{r_1}},\end{aligned}$$

where the second inequality is due to the Young inequality for the discrete Lebesgue space ℓ^r. Then

$$\begin{aligned}&\left\|\left\{\left(\sum_{\vec{m}\in\mathbb{Z}^n} \|f * g\|^r_{L^p(Q_{\nu\vec{m}})}\right)^{\frac{1}{r}}\right\}_{\nu\in\mathbb{N}}\right\|_{\ell^r} \\ &\leqslant \left\|\left\{\left(\sum_{\vec{m}\in\mathbb{Z}^n} \|f\chi_{Q_{\nu\vec{m}}}\|^{r_0}_{L^{p_0}(\mathbb{R}^n)}\right)^{\frac{1}{r_0}}\right\}_{\nu\in\mathbb{N}}\right\|_{\ell^r} \cdot \left\|\left\{\left(\sum_{\vec{m}\in\mathbb{Z}^n} \|g\|^{r_1}_{L^{p_1}(3Q_{\nu\vec{m}})}\right)^{\frac{1}{r_1}}\right\}_{\nu\in\mathbb{N}}\right\|_{\ell^\infty} \\ &\leqslant \left\|\left\{\left(\sum_{\vec{m}\in\mathbb{Z}^n} \|f\chi_{Q_{\nu\vec{m}}}\|^{r_0}_{L^{p_0}(\mathbb{R}^n)}\right)^{\frac{1}{r_0}}\right\}_{\nu\in\mathbb{N}}\right\|_{\ell^{r_0}} \cdot \left\|\left\{\left(\sum_{\vec{m}\in\mathbb{Z}^n} \|g\|^{r_1}_{L^{p_1}(3Q_{\nu\vec{m}})}\right)^{\frac{1}{r_1}}\right\}_{\nu\in\mathbb{N}}\right\|_{\ell^{r_1}},\end{aligned}$$

where the last inequality is due to the embedding properties of the discrete Lebesgue space ℓ^r. By Lemma 2, we obtain

$$\|f * g\|_{\mathcal{B}^rL^p(\mathbb{R}^n)} \lesssim \|f\|_{\mathcal{B}^{r_0}L^{p_0}(\mathbb{R}^n)}\|g\|_{\mathcal{B}^{r_1}L^{p_1}(\mathbb{R}^n)}.$$ □

Finally, we consider the dilation properties in Bourgain-Lebesgue spaces.

Theorem 4. *Let* $0 < p < r < \infty$ *and* $t \in (0,1]$. *Then*

$$\|f(t\cdot)\|_{\mathcal{B}^r L^p(\mathbb{R}^n)} \leqslant 2^n t^{-\frac{n}{p}} \|f\|_{\mathcal{B}^r L^p(\mathbb{R}^n)}.$$

Proof. The theorem is correct when $t = 1$, so we only need to prove the case when $t \in (0,1)$. For such t, there exists $\nu_t \in \mathbb{N}$ such that $2^{-\nu_t - 1} \leqslant t < 2^{-\nu_t}$. So there are $\vec{m}_1(\nu+\nu_t,t), \vec{m}_2(\nu+\nu_t,t), \ldots, \vec{m}_{2^n}(\nu+\nu_t,t) \in \mathbb{Z}^n$ such that

$$\prod_{i=1}^{n}\left[t\frac{m_i}{2^\nu}, t\frac{m_i+1}{2^\nu}\right) \subset \bigcup_{k=1}^{2^n} Q_{\nu+\nu_t, \vec{m}+\vec{m}_k(\nu+\nu_t,t)},$$

for each $\vec{m} \in \mathbb{Z}^n$. Let $x = ty$, $y \in Q_{\nu\vec{m}}$, then

$$\begin{aligned}\|f(t\cdot)\|_{\mathcal{B}^r L^p(\mathbb{R}^n)} &\leqslant t^{-\frac{n}{p}}\left[\sum_{\nu\in\mathbb{N}, \vec{m}\in\mathbb{Z}^n}\left(\int_{\bigcup_{k=1}^{2^n} Q_{\nu+\nu_t,\vec{m}+\vec{m}_k(\nu+\nu_t,t)}} |f(x)|^p \,\mathrm{d}x\right)^{\frac{r}{p}}\right]^{\frac{1}{r}} \\ &\leqslant 2^n t^{-\frac{n}{p}} \|f\|_{\mathcal{B}^r L^p(\mathbb{R}^n)}.\end{aligned}$$

□

The dilation properties become complicated in the spaces when $t \in (1,\infty)$.

Theorem 5. *Let* $0 < p < r < \infty$ *and* $t \in (1,\infty)$. *Then there exists a positive constant* $C_{n,t}$ *related to* n *and* t *such that*

$$\|f(t\cdot)\|_{\mathcal{B}^r L^p(\mathbb{R}^n)} \leqslant C_{n,t} t^{-\frac{n}{p}} \|f\|_{\mathcal{B}^r L^p(\mathbb{R}^n)}.$$

Proof. For $t \in (1,\infty)$, there exists $\nu_t \in \mathbb{Z}^+$ such that $2^{\nu_t - 1} \leqslant t < 2^{\nu_t}$. Let $x = ty$, $y \in Q_{\nu\vec{m}}$, then

$$\begin{aligned}\|f(t\cdot)\|_{\mathcal{B}^r L^p(\mathbb{R}^n)} \leqslant &\left[\sum_{\nu=0}^{\nu_t-1}\sum_{\vec{m}\in\mathbb{Z}^n}\left(\int_{Q_{\nu\vec{m}}} |f(ty)|^p \,\mathrm{d}y\right)^{\frac{r}{p}}\right]^{\frac{1}{r}} \\ &+\left[\sum_{\nu=\nu_t}^{\infty}\sum_{\vec{m}\in\mathbb{Z}^n}\left(\int_{Q_{\nu\vec{m}}} |f(ty)|^p \,\mathrm{d}y\right)^{\frac{r}{p}}\right]^{\frac{1}{r}}.\end{aligned}$$

When $\nu \in \mathbb{N}\backslash\{0,\ldots,\nu_t - 1\}$, similar to the proof of Theorem 4, there are $\vec{m}_1(\nu - \nu_t,t), \vec{m}_2(\nu-\nu_t,t), \ldots, \vec{m}_{2^n}(\nu-\nu_t,t) \in \mathbb{Z}^n$ such that

$$\prod_{i=1}^{n}\left[t\frac{m_i}{2^\nu}, t\frac{m_i+1}{2^\nu}\right) \subset \bigcup_{k=1}^{2^n} Q_{\nu-\nu_t, \vec{m}+\vec{m}_k(\nu-\nu_t,t)},$$

for each $\vec{m} \in \mathbb{Z}^n$. Then

$$\begin{aligned}\left[\sum_{\nu=\nu_t}^{\infty}\sum_{\vec{m}\in\mathbb{Z}^n}\left(\int_{Q_{\nu\vec{m}}} |f(ty)|^p \,\mathrm{d}y\right)^{\frac{r}{p}}\right]^{\frac{1}{r}} &\leqslant 2^n t^{-\frac{n}{p}}\left[\sum_{\nu=0}^{\infty}\sum_{\vec{m}\in\mathbb{Z}^n}\left(\int_{Q_{\nu\vec{m}}} |f(x)|^p \,\mathrm{d}x\right)^{\frac{r}{p}}\right]^{\frac{1}{r}} \\ &\leqslant 2^n t^{-\frac{n}{p}} \|f\|_{\mathcal{B}^r L^p(\mathbb{R}^n)}.\end{aligned}$$

When $\nu \in \{0, \dots, \nu_t - 1\}$, we know that

$$2^{\nu_t-\nu-1} \leqslant \ell\left(\prod_{i=1}^{n}\left[t\frac{m_i}{2^\nu}, t\frac{m_i+1}{2^\nu}\right)\right) < 2^{\nu_t-\nu}.$$

So, there are at most $\gamma(t) := (2^{\nu_t}+1)^n$ cubes in $\mathcal{D}_0$, for example $Q_{0\vec{m}_1}, Q_{0\vec{m}_2}, \dots, Q_{0\vec{m}_{\gamma(t)}}$, such that

$$\prod_{i=1}^{n}\left[t\frac{m_i}{2^\nu}, t\frac{m_i+1}{2^\nu}\right) \subset \bigcup_{i=1}^{\gamma(t)} Q_{0\vec{m}_i}.$$

Then

$$\left[\sum_{\nu=0}^{\nu_t-1}\sum_{\vec{m}\in\mathbb{Z}^n}\left(\int_{Q_{\nu\vec{m}}}|f(ty)|^p\,\mathrm{d}y\right)^{\frac{r}{p}}\right]^{\frac{1}{r}} \leqslant t^{-\frac{n}{p}}\sum_{i=1}^{\gamma(t)}\left[\sum_{\nu=0}^{\nu_t-1}\sum_{\vec{m}\in\mathbb{Z}^n}\left(\int_{Q_{0\vec{m}_i}}|f(x)|^p\,\mathrm{d}x\right)^{\frac{r}{p}}\right]^{\frac{1}{r}}$$
$$\leqslant \gamma(t)\nu_t t^{-\frac{n}{p}}\|f\|_{\mathcal{B}^r L^p(\mathbb{R}^n)}.$$

Thus

$$\|f(t\cdot)\|_{\mathcal{B}^r L^p(\mathbb{R}^n)} \leqslant (\gamma(t)\nu_t + 2^n)t^{-\frac{n}{p}}\|f\|_{\mathcal{B}^r L^p(\mathbb{R}^n)}. \qquad \square$$

4. The boundedness of local Hardy-Littlewood maximal operators in Bourgain-Lebesgue spaces

The Hardy-Littlewood maximal operator and its local versions are important in the theory of function spaces. In the section, we investigate the boundedness and weak boundedness of local Hardy-Littlewood maximal operators in Bourgain-Lebesgue spaces. Let $f \in L_{\mathrm{loc}}(\mathbb{R}^n)$ and $Q \subset \mathbb{R}^n$ be a cube, the Hardy-Littlewood maximal operator $\mathcal{M}$ is as follows

$$(\mathcal{M}f)(x) := \sup_{Q\subset\mathbb{R}^n, x\in Q}\frac{1}{|Q|}\int_Q |f(y)|\,\mathrm{d}y, \qquad x \in \mathbb{R}^n.$$

Let $\ell(Q)$ be the length of Q, we restrict $\ell(Q)$ to be less than or equal to one, then the local Hardy-Littlewood maximal operator $\mathcal{M}_{\mathrm{loc}}$ is defined by

$$(\mathcal{M}_{\mathrm{loc}}f)(x) := \sup_{\substack{Q\subset\mathbb{R}^n, x\in Q\\ \ell(Q)\leqslant 1}}\frac{1}{|Q|}\int_Q |f(y)|\,\mathrm{d}y, \qquad x \in \mathbb{R}^n.$$

In addition, the definition of the Hardy-Littlewood maximal operator associated with $\mathcal{B}$, denoted by $\mathcal{M}_{\mathcal{B}}$, is

$$(\mathcal{M}_{\mathcal{B}}f)(x) := \sup_{Q\in\mathcal{B}, x\in Q}\frac{1}{|Q|}\int_Q |f(y)|\,\mathrm{d}y, \qquad x \in \mathbb{R}^n.$$

It is easy to know that

$$(\mathcal{M}_{\mathcal{B}} f)(x) \leqslant (\mathcal{M}_{\mathrm{loc}} f)(x) \leqslant (\mathcal{M} f)(x), \qquad \text{a.e. } x \in \mathbb{R}^n,$$

so $\mathcal{M}_{\mathcal{B}}$ is a bounded sublinear operator in $L^p(\mathbb{R}^n), 1 < p \leqslant \infty$.

Next, we prove that $\mathcal{M}_{\mathcal{B}}$ is bounded in $\mathcal{B}^r L^p(\mathbb{R}^n)$ for $1 < p < r < \infty$.

Theorem 6. *Let $1 < p < r < \infty$. The Hardy-Littlewood maximal operator $\mathcal{M}_{\mathcal{B}}$ is bounded in $\mathcal{B}^r L^p(\mathbb{R}^n)$.*

Proof. Let $Q \in \mathcal{B}$,

$$f_1 := f\chi_Q, \; f_2 := f\chi_{\mathbb{R}^n \setminus Q}.$$

Then $f = f_1 + f_2$ and

$$\left\|\mathcal{M}_{\mathcal{B}} f_1\right\|_{L^p(Q)} \lesssim \|f_1\|_{L^p(\mathbb{R}^n)} = \|f\|_{L^p(Q)}.$$

It means

$$\left\|\mathcal{M}_{\mathcal{B}} f_1\right\|_{\mathcal{B}^r L^p(\mathbb{R}^n)} \lesssim \left(\sum_{Q \in \mathcal{B}} \|f\|_{L^p(Q)}^r\right)^{\frac{1}{r}} = \|f\|_{\mathcal{B}^r L^p(\mathbb{R}^n)}.$$

Let $x \in Q$, $k \in \mathbb{Z}^+$, Q_k be the k-th dyadic parent of Q, then $|Q_k| = 2^{nk}|Q|$ and

$$(\mathcal{M}_{\mathcal{B}} f_2)(x) \leqslant \sup_{Q_k \in \mathcal{B}} \frac{1}{|Q_k|} \int_{Q_k} |f(y)| \, \mathrm{d}y \leqslant \sum_{k=1}^{\infty} I_{Q_k} \left(\frac{1}{|Q_k|}\right)^{\frac{1}{p}} \|f\|_{L^p(Q_k)},$$

where $I_{Q_k} = 1$ if $Q_k \in \mathcal{B}$, $I_{Q_k} = 0$ if $Q_k \in \mathcal{D} \setminus \mathcal{B}$. So we have

$$\left\|\mathcal{M}_{\mathcal{B}} f_2\right\|_{L^p(Q)} \leqslant \sum_{k=1}^{\infty} 2^{-\frac{nk}{p}} I_{Q_k} \|f\|_{L^p(Q_k)}.$$

Let $\mathcal{Q} := \{Q_k \in \mathcal{B} : \; Q \in \mathcal{B}\}$, then

$$\left\|\mathcal{M}_{\mathcal{B}} f_2\right\|_{\mathcal{B}^r L^p(\mathbb{R}^n)} \leqslant \sum_{k=1}^{\infty} 2^{\frac{nk}{r} - \frac{nk}{p}} \left(\sum_{Q_k \in \mathcal{Q}} \|f\|_{L^p(Q_k)}^r\right)^{\frac{1}{r}} \lesssim \|f\|_{\mathcal{B}^r L^p(\mathbb{R}^n)},$$

where the penultimate inequality is due to the fact that there are at most 2^{nk} dyadic cubes in $\mathcal{B}$ such that their k-th dyadic parent is Q_k.

To sum up, we obtain

$$\left\|\mathcal{M}_{\mathcal{B}} f\right\|_{\mathcal{B}^r L^p(\mathbb{R}^n)} \lesssim \left\|\mathcal{M}_{\mathcal{B}} f_1\right\|_{\mathcal{B}^r L^p(\mathbb{R}^n)} + \left\|\mathcal{M}_{\mathcal{B}} f_2\right\|_{\mathcal{B}^r L^p(\mathbb{R}^n)} \lesssim \|f\|_{\mathcal{B}^r L^p(\mathbb{R}^n)}.$$

Thus we finish the proof. □

Denote $\alpha \in \{0,1,2\}^n$ by $\alpha := (\alpha_1, \alpha_2, \ldots, \alpha_n), \; \alpha_j \in \{0,1,2\}, j = 1,2,\ldots,n$. Let

$$Q_{\nu\vec{m}}^{\alpha} := \prod_{j=1}^{n} \left[\frac{m_j + \frac{\alpha_j}{3}}{2^\nu}, \frac{m_j + 1 + \frac{\alpha_j}{3}}{2^\nu}\right),$$

$\mathcal{D}_\nu^\alpha := \{Q_{\nu\vec{m}}^\alpha : \vec{m} \in \mathbb{Z}^n\}$, and $\mathcal{B}_\alpha := \bigcup_{\nu\in\mathbb{N}} \mathcal{D}_\nu^\alpha$, as well as $\mathcal{D}_\alpha := \bigcup_{\nu\in\mathbb{Z}} \mathcal{D}_\nu^\alpha$. By the notations we know that $\mathcal{B}_{(0,0,\dots,0)} = \mathcal{B}$ and $\mathcal{D}_{(0,0,\dots,0)} = \mathcal{D}$.

The notation $\mathcal{M}_{\mathcal{B}_\alpha}$ means the Hardy-Littlewood maximal operator associated with $\mathcal{B}_\alpha$ and is defined by

$$(\mathcal{M}_{\mathcal{B}_\alpha} f)(x) := \sup_{Q\in\mathcal{B}_\alpha, x\in Q} \frac{1}{|Q|}\int_Q |f(y)|\,dy, \qquad x \in \mathbb{R}^n.$$

By a similar methods of Theorem 6, we have

Corollary 1. *Let $1 < p < r < \infty$. The Hardy-Littlewood maximal operator $\mathcal{M}_{\mathcal{B}_\alpha}$ is bounded on $\mathcal{B}_\alpha^r L^p(\mathbb{R}^n)$.*

To establish the boundedness of the Hardy-Littlewood maximal operator $\mathcal{M}_{\mathcal{B}_\alpha}$ in the space $\mathcal{B}^r L^p(\mathbb{R}^n)$, it is essential to demonstrate the norm equivalence between the spaces $\mathcal{B}^r L^p(\mathbb{R}^n)$ and $\mathcal{B}_\alpha^r L^p(\mathbb{R}^n)$.

Lemma 3. *Let $1 < p < r < \infty$ and $\alpha \in \{0,1,2\}^n$. The norms of $\mathcal{B}^r L^p(\mathbb{R}^n)$ and $\mathcal{B}_\alpha^r L^p(\mathbb{R}^n)$ are equivalent.*

Proof. Let $Q_{\nu\vec{m}}^\alpha \in \mathcal{B}_\alpha$, there exists $\vec{m}_1(\nu,\alpha), \vec{m}_2(\nu,\alpha), \dots, \vec{m}_{2^n}(\nu,\alpha) \in \mathbb{Z}^n$ such that

$$Q_{\nu\vec{m}}^\alpha \subset \bigcup_{k=1}^{2^n} Q_{\nu(\vec{m}+\vec{m}_k(\nu,\alpha))}.$$

Then we have

$$\|f\|_{\mathcal{B}_\alpha^r L^p(\mathbb{R}^n)} \leqslant \sum_{k=1}^{2^n}\left(\sum_{\nu\in\mathbb{N},\vec{m}\in\mathbb{Z}^n} \|f\|^r_{L^p\left(Q_{\nu(\vec{m}+\vec{m}_k(\nu,\alpha))}\right)}\right)^{\frac{1}{r}} \leqslant 2^n \|f\|_{\mathcal{B}^r L^p(\mathbb{R}^n)}.$$

By a similar way,

$$\|f\|_{\mathcal{B}^r L^p(\mathbb{R}^n)} \leqslant 2^n \|f\|_{\mathcal{B}_\alpha^r L^p(\mathbb{R}^n)}.$$

Thus

$$\frac{1}{2^n}\|f\|_{\mathcal{B}_\alpha^r L^p(\mathbb{R}^n)} \leqslant \|f\|_{\mathcal{B}^r L^p(\mathbb{R}^n)} \leqslant 2^n \|f\|_{\mathcal{B}_\alpha^r L^p(\mathbb{R}^n)}.$$

Thus we finish the proof. □

Based on Lemma 3, we state the following theorem.

Theorem 7. *Let $1 < p < r < \infty$. The Hardy-Littlewood maximal operator $\mathcal{M}_{\mathcal{B}_\alpha}$ is bounded in $\mathcal{B}^r L^p(\mathbb{R}^n)$.*

Local Hardy-Littlewood maximal operators can be dominated by a sum of a sequence of Hardy-Littlewood maximal operators $\{\mathcal{M}_{\mathcal{B}_\alpha}\}_{\alpha\in\{0,1,2\}^n}$ as follows

$$\mathcal{M}_{\mathrm{loc}} f \lesssim \sum_{\alpha\in\{0,1,2\}^n} \mathcal{M}_{\mathcal{B}_\alpha} f.$$

This fact leads to the conclusion that the local Hardy-Littlewood maximal operator is bounded in $\mathcal{B}^r L^p(\mathbb{R}^n)$.

Theorem 8. *Let $1 < p < r < \infty$. The local Hardy-Littlewood maximal operator $\mathcal{M}_{\mathrm{loc}}$ is bounded in $\mathcal{B}^r L^p(\mathbb{R}^n)$.*

We now aim to prove the local Hardy-Littlewood maximal operator $\mathcal{M}_{\mathrm{loc}}$ is bounded from Bourgain-Lebesgue spaces to weak Bourgain-Lebesgue spaces.

Definition 2 (The weak Bourgain-Lebesgue space)**.** Let $0 < p < r < \infty$, the weak Bourgain-Lebesgue space $\mathcal{B}^r L^{p,\infty}(\mathbb{R}^n)$ is defined as

$$\mathcal{B}^r L^{p,\infty}(\mathbb{R}^n) := \left\{ f \in L^p_{\mathrm{loc}}(\mathbb{R}^n) : \ \|f\|_{\mathcal{B}^r L^{p,\infty}(\mathbb{R}^n)} < \infty \right\},$$

where

$$\|f\|_{\mathcal{B}^r L^{p,\infty}(\mathbb{R}^n)} := \left(\sum_{Q \in \mathcal{B}} \|f\|^r_{L^{p,\infty}(Q)} \right)^{\frac{1}{r}},$$

$$\|f\|_{L^{p,\infty}(Q)} = \inf\{C > 0 : \lambda^p |\{x \in Q : |f(x)| > \lambda\}| \leqslant C^p\}.$$

It is easy to see that $\|f\|_{\mathcal{B}^r L^{p,\infty}(\mathbb{R}^n)} \leqslant \|f\|_{\mathcal{B}^r L^p(\mathbb{R}^n)}$. By Theorem 8 we can state the following theorem.

Theorem 9. *Let $1 < p < r < \infty$. The local Hardy-Littlewood maximal operator $\mathcal{M}_{\mathrm{loc}}$ is bounded from $\mathcal{B}^r L^p(\mathbb{R}^n)$ to $\mathcal{B}^r L^{p,\infty}(\mathbb{R}^n)$.*

We proceed to prove the Hardy-Littlewood maximal operator $\mathcal{M}$ is bounded from $\mathcal{B}^r L^1(\mathbb{R}^n)$ to $\mathcal{B}^r L^{1,\infty}(\mathbb{R}^n)$ for $1 < r < \infty$.

Lemma 4 (see Lemma 2.1.5 in [3])**.** *Let $\{Q_1, Q_2, \ldots, Q_k\}$ be a finite collection of open cubes in $\mathbb{R}^n$. Then there exists a finite subcollection $\{Q_{j_1}, \ldots, Q_{x_{j_l}}\}$ of pairwise disjoint cubes such that*

$$\left| \bigcup_{i=1}^{k} Q_i \right| \leqslant 3^n \sum_{r=1}^{l} |Q_{j_r}|.$$

Inspired by Theorem 2.1.6 in [3] and with the help of Lemma 4, we obtain the following lemma.

Lemma 5. *Let $f \in L^1(\mathbb{R}^n)$ and $Q \subset \mathbb{R}^n$, then*

$$\left| \left\{ x \in Q : \ (\mathcal{M} f)(x) > \alpha \right\} \right| \leqslant \frac{3^n}{\alpha} \int_{\left\{ x \in Q : \ (\mathcal{M} f)(x) > \alpha \right\}} |f(y)| \, dy.$$

Proof. The proof is similar to Theorem 2.1.6 in [3], so we omit it here . □

By Lemma 5 we know that the Hardy-Littlewood maximal operator $\mathcal{M}$ is bounded from $\mathcal{B}^r L^1(\mathbb{R}^n)$ to $\mathcal{B}^r L^{1,\infty}(\mathbb{R}^n)$.

Theorem 10. *Let $1 < r < \infty$. The Hardy-Littlewood maximal operator $\mathcal{M}$ is bounded from $\mathcal{B}^r L^1(\mathbb{R}^n)$ to $\mathcal{B}^r L^{1,\infty}(\mathbb{R}^n)$.*

Proof. Let $Q \in \mathcal{B}$, by Lemma 5, we have

$$\left|\left\{x \in Q:\ (\mathcal{M}f)(x) > \alpha\right\}\right| \leqslant \frac{3^n}{\alpha} \int_{\left\{x \in Q:\ (\mathcal{M}f)(x) > \alpha\right\}} |f(y)|\,\mathrm{d}y \leqslant \frac{3^n}{\alpha} \|f\|_{L^1(Q)}.$$

Thus

$$\left\|\mathcal{M}f\right\|_{L^{1,\infty}(Q)} \leqslant 3^n \|f\|_{L^1(Q)},$$

which means

$$\left\|\mathcal{M}f\right\|_{\mathcal{B}^r L^{1,\infty}(\mathbb{R}^n)} \leqslant 3^n \|f\|_{\mathcal{B}^r L^1(\mathbb{R}^n)}.$$

Thus we finish the proof. □

By the fact that $(\mathcal{M}_{\text{loc}} f)(x) \leqslant (\mathcal{M}f)(x)$ for a.e. $x \in \mathbb{R}^n$ and Theorem 10, we have the following corollary.

Corollary 2. *Let $1 < r < \infty$. The local Hardy-Littlewood maximal operator $\mathcal{M}_{\text{loc}}$ is bounded from $\mathcal{B}^r L^1(\mathbb{R}^n)$ to $\mathcal{B}^r L^{1,\infty}(\mathbb{R}^n)$.*

By the way, the constant 3^n in Lemma 5 can be replaced by 1 if we only focus on the Hardy-Littlewood maximal operator $\mathcal{M}_{\mathcal{B}}$.

Corollary 3. *Let $f \in L^1(\mathbb{R}^n)$ and $Q \in \mathcal{B}$, then*

$$\left|\left\{x \in Q:\ (\mathcal{M}_{\mathcal{B}} f)(x) > \alpha\right\}\right| \leqslant \frac{1}{\alpha} \int_{\left\{x \in Q:\ (\mathcal{M}_{\mathcal{B}} f)(x) > \alpha\right\}} |f(y)|\,\mathrm{d}y.$$

Remark 1. The method used in Theorem 6 fails to prove the boundedness of the Hardy-Littlewood maximal operator $\mathcal{M}$ in $\mathcal{B}^r L^p(\mathbb{R}^n)$. We do not know whether the Hardy-Littlewood maximal operator $\mathcal{M}$ is bounded on $\mathcal{B}^r L^p(\mathbb{R}^n)$.

5. The boundedness of vector-valued local Hardy-Littlewood maximal operators in Bourgain-Lebesgue spaces

In the last section, we focus on the boundedness of vector-valued local Hardy-Littlewood maximal operators in Bourgain-Lebesgue spaces.

Theorem 11. *Let $1 < p < r < \infty$, $0 < q < \infty$ and $\{f_i\}_{i \in \mathbb{Z}^+}$ be an sequence of functions contained in $L^p_{\text{loc}}(\mathbb{R}^n)$. Then*

$$\left\|\left(\sum_{i=1}^{\infty} |\mathcal{M}_{\mathcal{B}} f_i|^q\right)^{\frac{1}{q}}\right\|_{\mathcal{B}^r L^p(\mathbb{R}^n)} \lesssim \left\|\left(\sum_{i=1}^{\infty} |f_i|^q\right)^{\frac{1}{q}}\right\|_{\mathcal{B}^r L^p(\mathbb{R}^n)}.$$

Proof. Let $Q \in \mathcal{B}$, $i \in \mathbb{Z}^+$,

$$f_{i,1} := f_i \chi_Q,\ f_{i,2} := f_i \chi_{\mathbb{R}^n \setminus Q}.$$

Then $f = f_{i,1} + f_{i,2}$. By the Fefferman-Stein vector valued inequality, for example see (5.6.25) in [3], we have

$$\left\| \left(\sum_{i=1}^{\infty} \left| \mathcal{M}_{\mathcal{B}} f_{i,1} \right|^q \right)^{\frac{1}{q}} \right\|_{L^p(Q)} \lesssim \left\| \left(\sum_{i=1}^{\infty} |f_{i,1}|^q \right)^{\frac{1}{q}} \right\|_{L^p(\mathbb{R}^n)} = \left\| \left(\sum_{i=1}^{\infty} |f_i|^q \right)^{\frac{1}{q}} \right\|_{L^p(Q)}.$$

So

$$\left\| \left(\sum_{i=1}^{\infty} \left| \mathcal{M}_{\mathcal{B}} f_{i,1} \right|^q \right)^{\frac{1}{q}} \right\|_{\mathcal{B}^r L^p(\mathbb{R}^n)} \lesssim \left\| \left(\sum_{i=1}^{\infty} |f_i|^q \right)^{\frac{1}{q}} \right\|_{\mathcal{B}^r L^p(\mathbb{R}^n)}.$$

Let $k \in \mathbb{Z}^+$, Q_k be the k-th dyadic parent of Q, then $|Q_k| = 2^{nk}|Q|$. For all $x \in Q$,

$$\left(\mathcal{M}_{\mathcal{B}} f_{i,2} \right)(x) = \sup_{Q_k \in \mathcal{B}} \frac{1}{|Q_k|} \int_{Q_k \setminus Q} |f_i(y)|\, \mathrm{d}y \leqslant \sum_{k=1}^{\infty} \frac{I_{Q_k}}{|Q_k|} \int_{Q_k} |f_i(y)|\, \mathrm{d}y,$$

where $I_{Q_k} = 1$ if $Q_k \in \mathcal{B}$, $I_{Q_k} = 0$ if $Q_k \in \mathcal{D} \setminus \mathcal{B}$. By the Minkowski inequality and the Hölder inequality, we have

$$\left(\sum_{i=1}^{\infty} \left| \left(\mathcal{M}_{\mathcal{B}} f_{i,2} \right)(x) \right|^q \right)^{\frac{1}{q}} \lesssim \sum_{k=1}^{\infty} \left(\frac{I_{Q_k}}{|Q_k|} \right)^{\frac{1}{p}} \left\| \left(\sum_{i=1}^{\infty} |f_i|^q \right)^{\frac{1}{q}} \right\|_{L^p(Q_k)}.$$

So

$$\left\| \left(\sum_{i=1}^{\infty} \left| \mathcal{M}_{\mathcal{B}} f_{i,2} \right|^q \right)^{\frac{1}{q}} \right\|_{L^p(Q)} \lesssim \sum_{k=1}^{\infty} \left(\frac{|Q| I_{Q_k}}{|Q_k|} \right)^{\frac{1}{p}} \left\| \left(\sum_{i=1}^{\infty} |f_i|^q \right)^{\frac{1}{q}} \right\|_{L^p(Q_k)}.$$

It means

$$\begin{aligned} \left\| \left(\sum_{i=1}^{\infty} \left| \mathcal{M}_{\mathcal{B}} f_{i,2} \right|^q \right)^{\frac{1}{q}} \right\|_{\mathcal{B}^r L^p(\mathbb{R}^n)} &\lesssim \sum_{k=1}^{\infty} 2^{\frac{nk}{r} - \frac{nk}{p}} \left[\sum_{Q_k \in \mathcal{Q}} I_{Q_k} \left\| \left(\sum_{i=1}^{\infty} |f_i|^q \right)^{\frac{1}{q}} \right\|_{L^p(Q_k)}^r \right]^{\frac{1}{r}} \\ &\lesssim \left\| \left(\sum_{i=1}^{\infty} |f_i|^q \right)^{\frac{1}{q}} \right\|_{B^r L^p(\mathbb{R}^n)}, \end{aligned}$$

where the inequalities are due to $\mathcal{Q} := \{Q_k \in \mathcal{B} : Q \in \mathcal{B}\}$ and the fact that there are at most 2^{nk} dyadic cubes in $\mathcal{B}$ such that their k-th dyadic parent is Q_k.

By the fact $|\mathcal{M}_{\mathcal{B}} f_i| \leqslant |\mathcal{M}_{\mathcal{B}} f_{i,1}| + |\mathcal{M}_{\mathcal{B}} f_{i,2}|$, we have

$$\left\| \left(\sum_{i=1}^{\infty} |\mathcal{M}_{\mathcal{B}} f_i|^q \right)^{\frac{1}{q}} \right\|_{\mathcal{B}^r L^p(\mathbb{R}^n)} \lesssim \left\| \left(\sum_{i=1}^{\infty} |f_i|^q \right)^{\frac{1}{q}} \right\|_{\mathcal{B}^r L^p(\mathbb{R}^n)} . \qquad \square$$

By the similar argument in the proof of Theorem 11 and the equivalent norms of $\mathcal{B}^r L^p(\mathbb{R}^n)$ and $\mathcal{B}^r_\alpha L^p(\mathbb{R}^n)$, we have the following corollary.

Corollary 4. *Let* $1 < p < r < \infty$, $0 < q < \infty$ *and* $\{f_i\}_{i \in \mathbb{Z}^+}$ *be an sequence of functions contained in* $L^p_{\mathrm{loc}}(\mathbb{R}^n)$. *Then*

$$\left\| \left(\sum_{i=1}^{\infty} |\mathcal{M}_{\mathcal{B}_\alpha} f_i|^q \right)^{\frac{1}{q}} \right\|_{\mathcal{B}^r L^p(\mathbb{R}^n)} \lesssim \left\| \left(\sum_{i=1}^{\infty} |f_i|^q \right)^{\frac{1}{q}} \right\|_{\mathcal{B}^r L^p(\mathbb{R}^n)} .$$

Then we have the boundedness of vector-valued local Hardy-Littlewood maximal operators on $\mathcal{B}^r L^p(\mathbb{R}^n)$.

Theorem 12. *Let* $1 < p < r < \infty$, $0 < q \leqslant \infty$ *and* $\{f_i\}_{i \in \mathbb{Z}^+}$ *be an sequence of functions contained in* $L^p_{\mathrm{loc}}(\mathbb{R}^n)$. *Then*

$$\left\| \left(\sum_{i=1}^{\infty} |\mathcal{M}_{\mathrm{loc}} f_i|^q \right)^{\frac{1}{q}} \right\|_{\mathcal{B}^r L^p(\mathbb{R}^n)} \lesssim \left\| \left(\sum_{i=1}^{\infty} |f_i|^q \right)^{\frac{1}{q}} \right\|_{\mathcal{B}^r L^p(\mathbb{R}^n)} .$$

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EXISTENCE OF SUPER-SOLUTIONS OF A DISCRETE EQUATION OF WOLFF TYPE

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Abstract. In this paper, we are concerned with the Wolff-type equation

$$u_i = \int_0^\infty \left(\frac{\sum_{j\in\mathbb{Z}^n, |j-i|<t} u_j^p}{t^{n-\beta\gamma}} \right)^{\frac{1}{\gamma-1}} \frac{dt}{t}, \quad u_i > 0 \quad \text{for } i \in \mathbb{Z}^n,$$

where $n \geq 1$, $\min\{\beta, p\} > 0$, $\gamma > 1$ and $\beta\gamma < n$. Such an equation is related to the study in the theory of nonlinear PDEs and mathematical physics. Here we study the existence of positive super-solutions and obtain the critical exponent of the Serrin type.

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1. INTRODUCTION

Let $u = (u_i)_{i\in\mathbb{Z}^n}$ be a nonnegative sequence. The Wolff potential of u is (cf. [4])

$$W_{\beta,\gamma}(u)(i) = \int_0^\infty \left(\frac{\sum_{j\in\mathbb{Z}^n, |j-i|<t} u_j}{t^{n-\beta\gamma}} \right)^{\frac{1}{\gamma-1}} \frac{dt}{t}, \tag{1.1}$$

where $n \geq 1$, $\beta > 0$, and $\gamma > 1$. In this paper, we study the existence of positive super-solutions of the equation

$$u_i = W_{\beta,\gamma}(u^p)(i), \quad i \in \mathbb{Z}^n, \tag{1.2}$$

where $n \geq 1$, $\min\{\beta, p\} > 0$, $\gamma > 1$ and $\beta\gamma < n$. A sequence $u = (u_i)_{i\in\mathbb{Z}^n}$ is called a positive super-solution of (1.2), if u satisfies

$$\begin{cases} u_i > 0 \;\; for \;\; all \;\; i \in \mathbb{Z}^n, \\ u_i < \infty \;\; when \;\; |i| < R \;\; for \;\; any \;\; R > 0, \end{cases} \tag{1.3}$$

and (1.2) with '$=$' replaced by '$\geq$' holds.

Recently, the authors in [11] obtained an optimal summability of positive solutions of the equation (1.2) by means of regularity lifting lemma and a Wolff-type inequality. Then, they also derived the decay rate of u_i when $|i| \to \infty$.

It is easy to see that (1.1) is one of the discrete forms of the Wolff potential of a locally integrable nonnegative function f

$$W_{\beta,\gamma}(f)(x) := \int_0^\infty \left[\frac{\int_{B_t(x)} f(y)dy}{t^{n-\beta\gamma}} \right]^{\frac{1}{\gamma-1}} \frac{dt}{t}.$$

This potential can be used to study nonlinear PDEs, such as $\mathcal{A}$-harmonic equations and k-Hessian equations. According to results in [6,7,15], if $\inf_{\mathbb{R}^n} u = 0$, there exists $C > 1$ such that some positive solution u of the Lane-Emden equation

$$Lu = u^p, \quad u > 0 \ \ in \ \ \mathbb{R}^n$$

satisfy

$$C^{-1} W_{\beta,\gamma}(u^p)(x) \le u(x) \le C W_{\beta,\gamma}(u^p)(x), \quad x \in \mathbb{R}^n. \tag{1.4}$$

Here $\beta = 1$ and $\gamma = q$ when $Lu = -div(A(x,u,\nabla u))$ or $Lu = -div(|\nabla u|^{q-2}\nabla u)$, and $\beta = 2k/(k+1)$ and $\gamma = k+1$ when $Lu = \sigma_k(D^2(-u))$. In view of (1.4), the following Wolff-type integral equation

$$u(x) = K(x) W_{\beta,\gamma}(u^p(y))(x), \quad u > 0 \ \ on \ \ \mathbb{R}^n \tag{1.5}$$

comes into play in those work. Here, a function $K(x)$ is called *double bounded*, if there exist positive constants c and C such that $c \le K(x) \le C$ for all $x \in \mathbb{R}^n$.

When $K(x) \equiv n - \alpha$, $\gamma = 2$ and $\beta = \alpha/2$, (1.5) is reduced to

$$u(x) = \int_{\mathbb{R}^n} \frac{u^q(y)dy}{|x-y|^{n-\alpha}}, \quad u > 0 \ \ on \ \ \mathbb{R}^n. \tag{1.6}$$

In addition, (1.6) is the Euler-Lagrange equation satisfied by the extremal functions of the Hardy-Littlewood-Sobolev inequality (cf. [3,12,13]).

For the coupling system

$$\begin{cases} u(x) = W_{\beta,\gamma}(v^q)(x) \\ v(x) = W_{\beta,\gamma}(u^p)(x), \end{cases} \tag{1.7}$$

Chen and Li [2] proved the radial symmetry for the integrable solutions. Afterward, Ma, Chen and Li [14] used the regularity lifting lemmas to obtain the optimal integrability and the Lipschitz continuity. Based on these results, [16] obtained the decay rates of the integrable solutions when $|x| \to \infty$.

The main result in this paper is the following theorem.

Theorem 1. *The Wolff-type equation* (1.2) *has a positive super-solution if and only if* $p > \frac{n(\gamma-1)}{n-\beta\gamma}$.

Remark 1. When $\beta = \alpha/2$ and $\gamma = 2$, the discrete form of (1.7) is reduced to

$$\begin{cases} u_i = \sum_{j\in\mathbb{Z}^n, i\neq j} \frac{v_j^q}{|i-j|^{n-\alpha}}, \\ v_i = \sum_{j\in\mathbb{Z}^n, i\neq j} \frac{u_j^p}{|i-j|^{n-\alpha}}, \end{cases}$$

This system is associated with the best constant of the discrete Hardy-Littlewood-Sobolev inequality (cf. [5]). Theorem 1 is consistent with Theorem 1.3 in [10]. Although Corollary 2.1 in [14] provides a Wolff-type inequality, the Euler-Lagrange equation corresponding to this extremum function is not (1.2). In fact, we have not found the Euler-Lagrange equation that satisfies the extremum function of the inequality corresponding to (1.2) at present.

Remark 2. The existence results and the critical exponents of integral equations (1.5), (1.6) and (1.7) can be seen in [1,8,9].

Remark 3. Another discrete Wolff potential of a nonnegative Borel measure ω is (cf. [4])

$$\tilde{W}_{\beta,\gamma}(\omega)(x) = \sum_{Q\in\mathcal{D}} \left[\frac{\omega(Q)}{|Q|^{1-\beta\gamma/n}}\right]^{\frac{1}{\gamma-1}} \chi_Q(x),$$

where $\mathcal{D} = \{Q\}$ and $Q = 2^i(k+[0,1)^n)$, $i \in \mathbb{Z}$, $k \in \mathbb{Z}^n$. The corresponding discrete equation of the Wolff type is

$$u(x) = \tilde{W}_{\beta,\gamma}(u^p)(x) + f(x),$$

where $f \in L^p_{loc}(\mathbb{R}^n)$ is a nonnegative function. The existence results and the critical exponents can be seen in [15].

Clearly, the discrete equation of (1.5) is

$$u_i = c_i \int_0^\infty \left(\sum_{j\in\mathbb{Z}^n, |j-i|<t} u_j^p\right)^{\frac{1}{\gamma-1}} t^{\frac{\beta\gamma-n}{\gamma-1}} \frac{dt}{t}, \quad i \in \mathbb{Z}^n, \tag{1.8}$$

where c_i is a double bounded sequence. Namely, there is a constant $C > 1$ such that $C^{-1} \le c_i \le C$ for all $i \in \mathbb{Z}^n$.

Theorem 1 is a corollary of the following two lemmas.

Lemma 1. *If* $p > \frac{n(\gamma-1)}{n-\beta\gamma}$, (1.8) *has positive solutions for some double bounded* c_i.

Lemma 2. *If* $0 < p \le \frac{n(\gamma-1)}{n-\beta\gamma}$, (1.8) *has no positive solution satisfying* (1.3) *for any double bounded* c_i.

2. Proof of Theorem 1

Proof of Lemma 1.

Let θ be a positive constant which will be determined later. Inserting

$$v_i = (1+|i|^2)^{-\theta} \tag{2.1}$$

into $W_{\beta,\gamma}(v^p)(i)$, we obtain

$$\begin{aligned} W_{\beta,\gamma}(v^p)(i) = &\int_0^{|i|/2} [\sum_{|j-i|<t} (1+|j|^2)^{-p\theta} t^{\beta\gamma-n}]^{\frac{1}{\gamma-1}} \frac{dt}{t} \\ &+ \int_{|i|/2}^{\infty} [\sum_{|j-i|<t} (1+|j|^2)^{-p\theta} t^{\beta\gamma-n}]^{\frac{1}{\gamma-1}} \frac{dt}{t} \\ := &I_1(i) + I_2(i). \end{aligned}$$

When $|i| \leq R$ for some $R > 0$, then u_i is proportional to $W_{\beta,\gamma}(u^p)(i)$. So we also only consider suitably large $|i|$.

Clearly, when $t \in (0, |i|/2)$, $|j-i| < t$ implies $|i|/2 < |j| < 3|i|/2$. Therefore, we can find positive constants c and C such that

$$\frac{c}{(1+|i|^2)^{\frac{p\theta}{\gamma-1}}} \int_0^{|i|/2} t^{\frac{\beta\gamma}{\gamma-1}} \frac{dt}{t} \leq I_1(i) \leq \frac{C}{(1+|i|^2)^{\frac{p\theta}{\gamma-1}}} \int_0^{|i|/2} t^{\frac{\beta\gamma}{\gamma-1}} \frac{dt}{t}.$$

Namely,

$$c(1+|i|^2)^{\frac{\beta\gamma-2p\theta}{2(\gamma-1)}} \leq I_1(i) \leq C(1+|i|^2)^{\frac{\beta\gamma-2p\theta}{2(\gamma-1)}}. \tag{2.2}$$

Take the slow rate

$$2\theta = \frac{\beta\gamma}{p-\gamma+1}. \tag{2.3}$$

There holds $\beta\gamma < 2p\theta < n$ by $p > \frac{n(\gamma-1)}{n-\beta\gamma}$. In addition, when $t \geq |i|/2$,

$$\{j \in \mathbb{Z}^n; |j-i| < t\} \subset \{j \in \mathbb{Z}^n; |j| < 3t\}.$$

Therefore,

$$I_2(i) \leq C \int_{|i|/2}^{\infty} \left(\frac{t^{n-2p\theta}}{t^{n-\beta\gamma}}\right)^{\frac{1}{\gamma-1}} \frac{dt}{t} \leq C(1+|i|^2)^{\frac{\beta\gamma-2p\theta}{2(\gamma-1)}}.$$

Thus,

$$c(1+|i|^2)^{\frac{\beta\gamma-2p\theta}{2(\gamma-1)}} \leq I_1(i) + I_2(i) \leq C(1+|i|^2)^{\frac{\beta\gamma-2p\theta}{2(\gamma-1)}}. \tag{2.4}$$

Set

$$c_i := (1+|i|^2)^{\frac{2p\theta-\beta\gamma}{2(\gamma-1)}} [I_1(i) + I_2(i)]$$

Then $I_1(i) + I_2(i) = c_i v_i$ in view of (2.1) with (2.3). In addition, (2.4) implies that c_i is double bounded. Thus, v_i is a solution of (1.8).

Similarly, we also find a fast decaying solution. In fact, taking

$$2\theta = \frac{n-\beta\gamma}{\gamma-1}, \tag{2.5}$$

from $p > \frac{n(\gamma-1)}{n-\beta\gamma}$, we also have

$$2p\theta > n, \tag{2.6}$$

and

$$\frac{\beta\gamma-n}{\gamma-1} > \frac{\beta\gamma-2p\theta}{\gamma-1}. \tag{2.7}$$

When $t > 2|i|$, there holds

$$\{j \in Z^n; |j-i| < t\} \supset \{j \in Z^n; |j| \le 1\}.$$

Therefore,

$$I_2(i) \ge \int_{2|i|}^{\infty} [\sum_{|j|\le 1} (1+|j|^2)^{-p\theta}]^{\frac{1}{\gamma-1}} t^{\frac{\beta\gamma-n}{\gamma-1}} \frac{dt}{t} \ge c\int_{2|i|}^{\infty} t^{\frac{\beta\gamma-n}{\gamma-1}} \frac{dt}{t} \ge c(1+|i|^2)^{-\frac{n-\beta\gamma}{2(\gamma-1)}}. \tag{2.8}$$

On the other hand,

$$\sum_{|j|\le 1, |j-i|<t} (1+|j|^2)^{-p\theta} \le \sum_{|j|\le 1} 1,$$

and by (2.6), there holds

$$\sum_{|j|>1, |j-i|<t} (1+|j|^2)^{-p\theta} \le \sum_{|j|>1} (1+|j|^2)^{-p\theta} \le C.$$

Therefore,

$$I_2(i) = \int_{|i|/2}^{\infty} \left(\frac{\Sigma_{|j|\le 1, |j-i|<t} (1+|j|^2)^{-p\theta} + \Sigma_{|j|>1, |j-i|<t} (1+|j|^2)^{-p\theta}}{t^{n-\beta\gamma}} \right)^{\frac{1}{\gamma-1}} \frac{dt}{t}$$

$$\le C\int_{|i|/2}^{\infty} t^{-\frac{n-\beta\gamma}{\gamma-1}} \frac{dt}{t} \le C(1+|i|^2)^{-\frac{n-\beta\gamma}{2(\gamma-1)}}.$$

Combining with (2.8), we get

$$c(1+|i|^2)^{-\frac{n-\beta\gamma}{2(\gamma-1)}} \le I_2(i) \le C(1+|i|^2)^{-\frac{n-\beta\gamma}{2(\gamma-1)}}.$$

This result, together with (2.2) and (2.7), implies

$$c(1+|i|^2)^{-\frac{n-\beta\gamma}{2(\gamma-1)}} \le I_1(i) + I_2(i) \le C(1+|i|^2)^{-\frac{n-\beta\gamma}{2(\gamma-1)}}. \tag{2.9}$$

Set

$$c_i = (1+|i|^2)^{\frac{n-\beta\gamma}{2(\gamma-1)}} [I_1(i) + I_2(i)].$$

Then, there also holds

$$I_1(i) + I_2(i) = c_i (1+|i|^2)^{-\frac{n-\beta\gamma}{2(\gamma-1)}} = c_i v_i,$$

in view of (2.1) with (2.5), and c_i is bounded in view of (2.9). Thus, v_i is also a solution of (1.8).

Proof of Lemma 2.

Suppose that u solves (1.8). We will deduce a contradiction.
Step 1. Let

$$0 < p < \frac{n(\gamma-1)}{n-\beta\gamma}. \tag{2.10}$$

From (1.8) it follows

$$u_i \geq c\int_{2|i|}^{\infty} t^{\frac{\beta\gamma-n}{\gamma-1}}\frac{dt}{t} = \frac{c}{|i|^{a_0}},$$

since $\Sigma_{|j|<1}u_j^p \geq c$, where $a_0 = \frac{n-\beta\gamma}{\gamma-1}$. By this estimate, we have

$$u_i \geq c\int_{2|i|}^{\infty}\left(\frac{\sum_{|j|<t-|i|}|j|^{-pa_0}}{t^{n-\beta\gamma}}\right)^{\frac{1}{\gamma-1}}\frac{dt}{t} \geq c\int_{2|i|}^{\infty} t^{\frac{\beta\gamma-pa_0}{\gamma-1}}\frac{dt}{t}. \tag{2.11}$$

When $\frac{p}{\gamma-1} \in (0, \frac{\beta\gamma}{n-\beta\gamma}]$, we have $\beta\gamma - pa_0 \geq 0$. Eq. (2.11) implies $u_i = \infty$. This contradicts with (1.3).

Next, we consider the case $\frac{p}{\gamma-1} \in (\frac{\beta\gamma}{n-\beta\gamma}, \frac{n}{n-\beta\gamma})$. Now (2.11) leads to

$$u_i \geq \frac{c}{|i|^{a_1}},$$

where $a_1 = \frac{p}{\gamma-1}a_0 - \frac{\beta\gamma}{\gamma-1}$. Write

$$a_k = \frac{p}{\gamma-1}a_{k-1} - \frac{\beta\gamma}{\gamma-1},\ k = 1, 2, \cdots. \tag{2.12}$$

We claim that there must be $k_0 > 0$ such that $a_{k_0} \leq 0$. This leads to $u_i = \infty$, which is impossible.

In fact, by (2.12) we get

$$a_k = \left(\frac{p}{\gamma-1}\right)^k a_0 - \left[1 + \frac{p}{\gamma-1} + \cdots + \left(\frac{p}{\gamma-1}\right)^{k-1}\right]\frac{\beta\gamma}{\gamma-1}.$$

If $\frac{p}{\gamma-1} = 1$, then we can find a large k_0 such that

$$a_{k_0} = a_0 - k_0\frac{\beta\gamma}{\gamma-1} \leq 0.$$

If $\frac{p}{\gamma-1} \in (1, \frac{n}{n-\beta\gamma})$, then using $a_0 - \frac{\beta\gamma}{p-\gamma+1} < 0$ which is implied by (2.10), we can find a large k_0 such that

$$a_{k_0} = \left(\frac{p}{\gamma-1}\right)^{k_0} a_0 - \frac{(\frac{p}{\gamma-1})^{k_0} - 1}{\frac{p}{\gamma-1} - 1}\frac{\beta\gamma}{\gamma-1}$$

$$= \left(\frac{p}{\gamma-1}\right)^{k_0}\left(a_0 - \frac{\beta\gamma}{p-\gamma+1}\right) + \frac{\beta\gamma}{p-\gamma+1} \le 0.$$

If $\frac{p}{\gamma-1} \in (0,1)$, letting $k \to \infty$, we get

$$a_k = \left(\frac{p}{\gamma-1}\right)^k a_0 - \frac{1-(\frac{p}{\gamma-1})^k}{1-\frac{p}{\gamma-1}}\frac{\beta\gamma}{\gamma-1} \to \frac{\beta\gamma}{p-\gamma+1} < 0.$$

Thus, there must be k_0 such that $a_{k_0} \le 0$.

Step 2. Let $p = \frac{n(\gamma-1)}{n-\beta\gamma}$. We deduce the contradiction if u is a positive solution of (1.8).

Let $R > 0$. Clearly, when $t \in (R/2, R)$ and $|i| < R/4$, there holds

$$\{j \in \mathbb{Z}^n; |j-i| < t\} \supset \{j \in \mathbb{Z}^n; |j| < R/4\}.$$

From (1.8), it follows that

$$u_i \ge c\int_{R/2}^{R}(\sum_{|j-i|<t} u_j^p)^{\frac{1}{\gamma-1}} t^{\frac{\beta\gamma-n}{\gamma-1}}\frac{dt}{t}$$

$$\ge c\int_{R/2}^{R}(\sum_{|j|<R/4} u_j^p)^{\frac{1}{\gamma-1}} t^{\frac{\beta\gamma-n}{\gamma-1}}\frac{dt}{t} \ge cR^{-\frac{n-\beta\gamma}{\gamma-1}}\left(\sum_{|j|<R/4} u_j^p\right)^{\frac{1}{\gamma-1}}.$$

Therefore, we get

$$u_i^p \ge cR^{p\frac{\beta\gamma-n}{\gamma-1}}(\sum_{|j|<R/4} u_j^p)^{\frac{p}{\gamma-1}}. \tag{2.13}$$

Summing for $|i| < R/4$ and noting $p = \frac{n(\gamma-1)}{n-\beta\gamma}$, we get

$$\sum_{|j|<R/4} u_i^p \ge cR^{p\frac{\beta\gamma-n}{\gamma-1}}(\sum_{|j|<R/4} 1)(\sum_{|j|<R/4} u_j^p)^{\frac{p}{\gamma-1}} \ge c(\sum_{|j|<R/4} u_j^p)^{\frac{p}{\gamma-1}}.$$

Here $c > 0$ is independent of R. Letting $R \to \infty$ and noting $p > \gamma - 1$, we have

$$\sum_{j\in\mathbb{Z}^n} u_j^p < \infty. \tag{2.14}$$

Summing (2.13) for $R/8 < |i| < R/4$ yields

$$\sum_{R/8<|i|<R/4} u_i^p \ge cR^{p\frac{\beta\gamma-n}{\gamma-1}}(\sum_{R/8<|i|<R/4} 1)(\sum_{|j|<R/4} u_j^p)^{\frac{p}{\gamma-1}}.$$

By $p = \frac{n(\gamma-1)}{n-\beta\gamma}$, it follows

$$\sum_{R/8<|i|<R/4} u_i^p \ge c(\sum_{|j|<R/4} u_j^p)^{\frac{p}{\gamma-1}},$$

where $c > 0$ is independent of R. Letting $R \to \infty$, and noting (2.14), we obtain

$$\sum_{j \in \mathbb{Z}^n} u_j^p = 0,$$

which contradicts with (1.3).

Proof of Theorem 1.

Necessity. Replacing (1.8) by (1.2) in the proof of Lemma 2, we easily see that (1.2) has no super-solution when $p \in (0, \frac{n(\gamma-1)}{n-\beta\gamma}]$.

Sufficiency. By Lemma 1, we assume that v_i solves (1.8) for some c_i. Since c_i is double bounded, we can find a constant $b_0 > 0$ such that $c_i \geq b_0$ for all $i \in \mathbb{Z}^n$.

Set $u_i = \lambda v_i$, where $\lambda > 0$ will be determined later. Thus, by (1.8),

$$u_i \geq \lambda b_0 W_{\beta,\gamma}(\lambda^{-p} u^p)(i) = b_0 \lambda^{1-\frac{p}{\gamma-1}} W_{\beta,\gamma}(u^p)(i).$$

Take $\lambda = b_0^{1/[p/(\gamma-1)-1]}$. Then $b_0 \lambda^{1-\frac{p}{\gamma-1}} = 1$ and hence u_i is a super-solution of (1.2).

The proof is complete.

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EXISTENCE OF SOLUTIONS FOR A GENERAL CLASS OF NONLINEAR FRACTIONAL INTEGRAL OPERATORS

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Abstract. In this study, we aim to investigate the existence results for a general class of fractional nonlinear integral equations with order $\alpha \in (0,1)$ in a continuous function space $\big(C[a,b], \|\cdot\|\big)$. We use the Schauder fixed point theorem as a tool with providing a compact integral operator from a subset of the Banach space $\big(C[a,b], \|\cdot\|\big)$ into itself. Furthermore, we present an example to illustrate and support the work.

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1. Introduction

In many applications, nonlinear fractional integral equations play a considerable role in mathematical physics and engineering due to their ability to model complex phenomena in real-world problems, particularly in fields such as fluid mechanics, engineering, and chemical reactions [1, 2, 7, 9]. Nonlinear fractional and weakly singular integral equations naturally arise in a wide range of applications across many other fields, including physics, biology (involving population dynamics), chemical, mechanical engineering, thermal explosions, nuclear physics, chemical kinetics, and control theory [1–3, 7]. We consider the following general nonlinear fractional integral equation:

$$u(\xi) = g(\xi) + \int_{t_0}^{\xi} (\xi - \rho)^{\alpha-1} \Psi\Big(\rho, u(\rho), \mathrm{D}^{\gamma} u(\rho)\Big) \mathrm{d}\rho, \tag{1.1}$$

where $\alpha \in (0,1)$, g is a given continuous function on $[t_0, T]$, $\Psi\colon [t_0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a function satisfying specified regularity conditions, and u is unknown solution of (1.1), while D^{γ} is the Riemann-Liouville fractional integral operator of order $\gamma > 0$. It is important to emphasize that the operator D^{γ} in (1.1) can be switched to other differential or integral operators with changing the relative continuous space $\big(C[a,b], \|\cdot\|\big)$.

The kernels in numerous real modeled integral equations are not smooth so this makes challenging to determine the exact and numerical solutions in such type of problems. Existence and uniqueness of solutions for nonlinear modeling for example nonlinear singular integral equations are crucial because it is impossible to accomplish analytic solutions to almost mathematical models of real-world problems. Existence and uniqueness results are the theoretical principle for effective numerical methods to establish approximate solutions.

Our goal in this work is to use the Schauder fixed point theorem as a main tool to achieve an existence of solution u to the proposed problem (1.1) in a subset of the Banach space $\big(C[t_0,T], \|\cdot\|\big)$.

In regard to previous study on existence and uniqueness of solutions to different type of singular integral equations, we recommend [2,4–7,9] as references related to the subject in this work.

In 2023, existence and properties of travelling-wave solutions for a family of singular Volterra integral equations:

$$\mathbb{X}(t) = \mathbb{F}(t) + \int_0^t \frac{\mathbb{G}(\tau)}{\mathbb{X}^{\beta}(\tau)} \mathrm{d}\tau, \quad \text{where } \beta \in (0,\infty) \text{ and } t > 0.$$

have been studied by Garriz [4]. Matoog and Abdou [7] have worked on the existence and uniqueness of solutions for the nonlinear integral equation of the following problem:

$$\frac{\partial}{\partial s}(\omega\Theta(s,u) - f(s,u)) = \lambda\xi(s) \int_0^1 k(u,v)\Theta(s,v,\Theta(t,v))\mathrm{d}v$$

in the Banach space $L^2[0,1] \times C[0,T]$. Aghajani, Banaś, and Jalilian [1] demonstrated the existence of solutions for the following singular integral equations using the fixed point theorem of the Darbo type:

$$\mathbb{F}(x) = \mathbb{F}_1(x,\mathbb{F}(x),\mathbb{F}(a(x))) + G\mathbb{F}(x) \int_0^x k(x,s)/(x-s)^{1-\alpha} Q\mathbb{F}(s)\mathrm{d}s.$$

In 2023, Alhazmi, Mahdy, and Mohamed [2] studied stability and error analysis for the symmetric singular kernel mixed integral equations with discussing existence and uniqueness of solution for the following problem:

$$\Phi(\tau,\xi)\Theta(\tau,\xi) = f(\tau,\xi) + \lambda \int_0^{\xi} \int_{-1}^1 (\tau - y)^{-2} F(\xi,s)\Theta(y,s)\mathrm{d}y\mathrm{d}s,$$

subject to the boundary conditions; $\Theta(1,\xi) = \Theta(-1,\xi) = 0$, for all $\xi \in [0,a]$ and $a < 1$.

2. Preliminary notation

In this section, we recall some fundamental concepts and definitions, which are used as supplementary in this paper.

Definition 1. Let $\gamma > 0$ and let R be a continuous function on $[a_0, \infty)$. Then

$$\mathrm{D}^{\gamma} R(\zeta) = \frac{1}{\Gamma(\gamma)} \int_{a_0}^{\zeta} (\zeta - \xi)^{\gamma-1} R(\xi) \mathrm{d}\xi, \tag{2.1}$$

is said to be the Riemann-Liouville fractional integral operator of order γ.

Definition 2. [8]] Let $-\infty < t_0 < b < \infty$, and let $\left(X[t_0, b], \|\cdot\|\right)$ be a Banach space. Let W be a subset of $X[t_0, b]$ with the following conditions:

- W is bounded set,
- W is equicontinuous. That is, for every positive ε, there corresponds a positive δ such that

$$\left|w(t) - w(\tau)\right| < \varepsilon, \text{when } |t - \tau| < \delta, \text{ and for all } w \in W. \tag{2.2}$$

Then W is a relatively compact subset of $X[t_0, b]$.

Definition 3 ([8, Definition 11])**.** Let $\left(X_1, \|\cdot\|\right)$ and $\left(X_2, \|\cdot\|\right)$ be Banach spaces over $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ *or* $\mathbb{K} = \mathbb{C}$). The operator

$$\mathrm{A}\colon W \subseteq X_1 \to X_2,$$

is said to be compact if and only if

1) A is continuous;
2) A maps bounded sets into relatively compact sets.

Theorem 1 ([8])**.** *Let $-\infty < t_0 < b < \infty$, and let $\left(X[t_0, b], \|\cdot\|\right)$ be a Banach space. Let $W \neq \varnothing$ be closed, bounded, and convex subset of $X[t_0, b]$. Then any compact operator $\mathrm{B}\colon W \to W$ has at least one fixed point. This is Schauder's fixed point theorem.*

3. Existence of solutions

The main findings of the existence of a solution for general nonlinear singular integral operators are discussed and illustrated in this section. Let $I = [t_0, T] \times \mathbb{R} \times \mathbb{R}$ and assume that the following hypotheses are true for Ψ:

(H1) $\Psi\colon I \to \mathbb{R}$ is a measurable function and uniformly bounded by some positive constant M on each compact subset of I;

(H2) For each compact interval $[-\theta, \theta] \subset \mathbb{R}$, there corresponds a positive constant L such that:

$$\left|\Psi(\omega, \phi_1, \varphi_1) - \Psi(\omega, \phi_2, \varphi_2)\right| \le L\left(|\phi_1 - \varphi_1| + |\phi_2 - \varphi_2|\right), \text{ for all } \omega \in [t_0, T],$$

and all $\phi_1, \phi_2, \varphi_1, \varphi_2 \in [-\theta, \theta]$.

Let y be a function in $C[t_0, T]$ and let its supremum norm be denoted and defined by:

$$\|y\|_{\sup} = \sup_{\tau \in [t_0, T]} \left\{|y(\tau)|\right\},$$

for simplicity in this paper, we use $\|\cdot\|$ sometimes for $\|\cdot\|_{\sup}$.

Lemma 1. *The fractional integral operator* $\mathrm{D}^{\gamma}\mathfrak{g}$ *is continuous.*

Proof. Let $\mathfrak{g}_n$ be a sequence of continuous functions on $[t_0,T]$ that converges to a function $\mathfrak{g}$. That is, for each $\varepsilon>0$, there is $N>0$ such that $\|\mathfrak{g}_n-\mathfrak{g}\|<\frac{\varepsilon\gamma\Gamma(\gamma)}{(T-t_0)^{\gamma}}$ for all $n>N$.

$$\begin{aligned}\left\|\mathrm{D}^{\gamma}\mathfrak{g}_n(\xi)-\mathrm{D}^{\gamma}\mathfrak{g}(\xi)\right\| &\le \frac{1}{\Gamma(\gamma)}\sup_{\xi\in[t_0,T]}\int_{t_0}^{\xi}(\xi-\tau)^{\gamma-1}\left|\mathfrak{g}_n(\tau)-\mathfrak{g}(\tau)\right|\mathrm{d}\tau\\ &\le \frac{1}{\Gamma(\gamma)}\left\|\mathfrak{g}_n-\mathfrak{g}\right\|\sup_{\xi\in[t_0,T]}\int_{t_0}^{\xi}(\xi-\tau)^{\gamma-1}\mathrm{d}\tau\\ &= \frac{1}{\Gamma(\gamma)}\left\|\mathfrak{g}_n-\mathfrak{g}\right\|\frac{(T-t_0)^{\gamma}}{\gamma}\\ &<\varepsilon\end{aligned}$$

for all $n>N$. Thus, $\mathrm{D}^{\gamma}\mathfrak{g}$ is continuous on $[t_0,T]$. □

Lemma 2. *Let* $a,b\in[0,\infty)$ *and* $\theta\in(0,1)$. *Then the following inequality holds*

$$b^{\theta}-a^{\theta}\le\left|b-a\right|^{\theta}.$$

Proof. Since, $0<\theta<1$ and for all $\mu,\nu\ge 0$. Then

$$1\le\left(\frac{\mu}{\mu+\nu}\right)^{\theta}+\left(\frac{\nu}{\mu+\nu}\right)^{\theta},$$

implies that

$$(\mu+\nu)^{\theta}\le\mu^{\theta}+\nu^{\theta}.$$

Let $\mu=b-a$ and $\nu=a$ implies that

$$b^{\theta}\le(b-a)^{\theta}+a^{\theta}\quad\Rightarrow\quad b^{\theta}-a^{\theta}\le\left(b-a\right)^{\theta}\le\left|b-a^{\theta}\right|.$$

□

For any fix $f\in C[t_0,T]$ and define an operator Λ as:

$$\Lambda u(\eta)=f(\eta)+\int_{t_0}^{\eta}(\eta-\rho)^{\alpha-1}\Psi(\rho,u(\rho),\mathrm{D}^{\gamma}u(\rho))\mathrm{d}\rho,$$

for all $u\in C[t_0,T]$ and $\eta\in[t_0,T]$.

Theorem 2. *If* $u\in C[t_0,T]$ *and* $\alpha\in(0,1)$. *Then* Λu *belongs to* $C[t_0,T]$.

Proof. Let u_k be a convergent sequence of functions in $C[t_0,T]$ that converges to u. That is, for any $\varepsilon>0$ there is a sufficiently large $N>0$ such that for all $k>N$ then,

$$\left\|u_k - u\right\| = \sup_{\eta\in[t_0,T]} \left|u_k(\eta) - u(\eta)\right| < \frac{\alpha\varepsilon}{2L(T-t_0)^{\alpha}},$$

and it follows from Lemma 1 that

$$\left\|\mathrm{D}^{\gamma}u_k - \mathrm{D}^{\gamma}u\right\| = \sup_{\eta\in[t_0,T]} \left|\mathrm{D}^{\gamma}u_k(\eta) - \mathrm{D}^{\gamma}u(\eta)\right| < \frac{\alpha\varepsilon}{2L(T-t_0)^{\alpha}}.$$

Implies from hypothesis $(H2)$ that

$$\begin{aligned}
&\left\|\Lambda u_k(\eta) - \Lambda u(\eta)\right\| \\
&= \sup_{\eta\in[t_0,T]} \left|\int_{t_0}^{\eta} (\eta-\rho)^{\alpha-1}\Big(\Psi(\rho,u_k(\rho),\mathrm{D}^{\gamma}u_k(\rho)) - \Psi(\rho,u(\rho),\mathrm{D}^{\gamma}u(\rho))\Big)\mathrm{d}\rho\right| \\
&\leq \sup_{\eta\in[t_0,T]} \int_{t_0}^{\eta} (\eta-\rho)^{\alpha-1}\Big|\Psi(\rho,u_k(\rho),\mathrm{D}^{\gamma}u_k(\rho)) - \Psi(\rho,u(\rho),\mathrm{D}^{\gamma}u(\rho))\Big|\mathrm{d}\rho \\
&\leq L \sup_{\eta\in[t_0,T]} \int_{t_0}^{\eta} (\eta-\rho)^{\alpha-1}\Big(\left|u_k(\rho) - u(\rho)\right| + \left|\mathrm{D}^{\gamma}u_k(\rho) - \mathrm{D}^{\gamma}u(\rho)\right|\Big)\mathrm{d}\rho \\
&\leq L\Big(\left\|u_k - u\right\| + \left\|\mathrm{D}^{\gamma}u_k - \mathrm{D}^{\gamma}u\right\|\Big) \sup_{\eta\in[t_0,T]} \int_{t_0}^{\eta} (\eta-\rho)^{\alpha-1}\mathrm{d}\rho \\
&< L\left(\frac{\alpha\varepsilon}{2L(T-t_0)^{\alpha}} + \frac{\alpha\varepsilon}{2L(T-t_0)^{\alpha}}\right) \sup_{\eta\in[t_0,T]} \int_{t_0}^{\eta} (\eta-\rho)^{\alpha-1}\mathrm{d}\rho \\
&= \frac{\alpha\varepsilon}{(T-t_0)^{\alpha}} \sup_{\eta\in[t_0,T]} \int_{t_0}^{\eta} (\eta-\rho)^{\alpha-1}\mathrm{d}\rho \\
&\leq \frac{\alpha\varepsilon}{(T-t_0)^{\alpha}} \frac{(T-t_0)^{\alpha}}{\alpha} = \varepsilon.
\end{aligned}$$

Therefore, Λu is continuous on $[t_0,T]$. □

Let $\alpha \in (0,1)$ and for any fix $v \in C[t_0,T]$, there corresponds a constant $K > 0$ such that $K = \max_{\tau\in[t_0,T]} |v(\tau)|$. We define a subspace $Y[t_0,T] = \{y \in C[t_0,T]$ such that $\|y\| \leq r\}$, where r is some positive constant. It is easy to observe that $Y[t_0,T]$ is closed, bounded, and convex subset of the Banach space $(C[t_0,T],\|\cdot\|)$. For simplicity, in the rest of the paper we use Y sometimes instead of $Y[t_0,T]$.

Now, we want to show that $\Lambda(Y) \subset C[t_0,T]$ is relatively compact. To do this, we state the following Lemmas and Theorem.

Lemma 3. $\Lambda(Y)$ *is a bounded set.*

Proof. For any continuous function f in $[t_0,T]$ and any $u \in Y$. It follows from $(H1)$ that; there is a constant $M > 0$ so that:

$$\begin{aligned}
\|\Lambda u\| &= \sup_{\tau\in[t_0,T]} \left| f(\tau) + \int_{t_0}^{\tau} (\tau-\rho)^{\alpha-1}\Psi(\rho,u(\rho),D^{\gamma}u(\rho))d\rho \right| \\
&\leq \sup_{\tau\in[t_0,T]} |f(\tau)| + \sup_{\tau\in[t_0,T]} \int_{t_0}^{\tau} (\tau-\rho)^{\alpha-1} |\Psi(\rho,u(\rho),D^{\gamma}u(\rho))| d\rho \\
&\leq K + M \sup_{\tau\in[t_0,T]} \int_{t_0}^{\tau} (\tau-\rho)^{\alpha-1} d\rho \\
&\leq K + M \sup_{\tau} \int_{t_0}^{\tau} (\tau-\rho)^{\alpha-1} d\rho \\
&\leq K + M \sup_{\tau\in[t_0,T]} \int_{t_0}^{\tau} (\tau-\rho)^{\alpha-1} d\rho \\
&\leq K + M \frac{(T-t_0)^{\alpha}}{\alpha} \\
&= r,
\end{aligned}$$

where $r = K + \frac{M(T-t_0)^{\alpha}}{\alpha}$ is a positive constant. Hence, $\Lambda(Y)$ is bounded. □

Lemma 4. *If $u \in Y$. Then $\Lambda u \in Y$.*

Proof. The proof is followed by using Theorem 2 and Lemma 3. □

Theorem 3. *The operator $\Lambda\colon Y \to Y$ is equicontinuous.*

Proof. Let $\{u_k\}_{k=1}^{\infty}$ be a sequence of functions in Y and let $\alpha \in (0,1)$. Without loss of generality, let $t_0 \leq \tau < \xi \leq T$, and since from hypothesis $(H1)$; Ψ is uniformly bounded function by a positive number M. Assume f belongs to Y then for any $\varepsilon > 0$, there is $\delta = \left(\frac{\alpha\varepsilon}{2M}\right)^{1/\alpha}$, whenever $|\xi-\tau| < \delta$ implies $|f(\xi)-f(\tau)| < \frac{\varepsilon}{2}$ for all $\tau,\xi \in [t_0,T]$. Then,

$$\begin{aligned}
|\Lambda u_k(\xi) - \Lambda u_k(\tau)| \leq\ & |f(\xi)-f(\tau)| + \left| \int_{t_0}^{\xi} (\xi-\rho)^{\alpha-1}\Psi(\rho,u_k(\rho),D^{\gamma}u_k(\rho))d\rho \right. \\
& \left. - \int_{t_0}^{\tau} (\tau-\rho)^{\alpha-1}\Psi(\rho,u_k(\rho),D^{\gamma}u_k(\rho))d\rho \right| \\
<\ & \frac{\varepsilon}{2} + \left| \int_{t_0}^{\xi} (\xi-\rho)^{\alpha-1}\Psi(\rho,u_k(\rho),D^{\gamma}u_k(\rho))d\rho \right. \\
& \left. - \int_{t_0}^{\tau} (\tau-\rho)^{\alpha-1}\Psi(\rho,u_k(\rho),D^{\gamma}u_k(\rho))d\rho \right| \\
\leq\ & \frac{\varepsilon}{2} + \int_{t_0}^{\tau} \left((\xi-\rho)^{\alpha-1} - (\tau-\rho)^{\alpha-1} \right) |\Psi(\rho,u_k(\rho),D^{\gamma}u_k(\rho))| d\rho
\end{aligned}$$

$$+\int_{\tau}^{\xi}(\xi-\rho)^{\alpha-1}\left|\Psi(\rho,u_k(\rho),D^{\gamma}u_k(\rho))\right|d\rho.$$

It follows from boundedness of Ψ in $(H1)$ that

$$\begin{aligned}
&\left|\Lambda u_k(\xi)-\Lambda u_k(\tau)\right| \\
&\leq \frac{\varepsilon}{2}+M\int_{t_0}^{\tau}\left((\xi-\rho)^{\alpha-1}-(\tau-\rho)^{\alpha-1}\right)d\rho+M\int_{\tau}^{\xi}(\xi-\rho)^{\alpha-1}d\rho. \\
&=\frac{\varepsilon}{2}+\frac{M}{\alpha}\left\{-(\xi-\tau)^{\alpha}+(\xi-t_0)^{\alpha}-(\tau-t_0)^{\alpha}+(\xi-\tau)^{\alpha}\right\} \\
&=\frac{\varepsilon}{2}+\frac{M}{\alpha}\left\{(\xi-t_0)^{\alpha}-(\tau-t_0)^{\alpha}\right\}.
\end{aligned}$$

Since $|\xi-\tau|<\delta$ and applying Lemma 2 yields to

$$\begin{aligned}
\left|\Lambda u_k(\xi)-\Lambda u_k(\tau)\right| &\leq \frac{\varepsilon}{2}+\frac{M}{\alpha}\left|\xi-\tau\right|^{\alpha} \\
&<\frac{\varepsilon}{2}+\frac{M}{\alpha}\delta^{\alpha} \\
&<\frac{\varepsilon}{2}+\frac{\varepsilon}{2} \\
&=\varepsilon,
\end{aligned}$$

for all $\tau,\xi\in[t_0,T]$. Therefore, $\Lambda(Y)$ is equicontinuous. □

Theorem 4. *Let $0\leq t_0<T<\infty$, let Ψ satisfy hypotheses $(H1)-(H2)$, and f belongs to Y. Then there is at least a solution u belongs to Y for problem (1.1).*

Proof. We note that $u\mapsto\Lambda u$ maps Y into itself for all u and all f in Y. Clearly, Y is bounded, closed, and convex subset of the Banach space $\left(C[t_0,T],\|\cdot\|\right)$. In addition, it follows from Theorem 2, Lemma 3, and Theorem 3 that Λ is continuous, $\Lambda(Y)$ is bounded and equicontinuous, respectively. Hence, by the Arzela-Ascoli Theorem, $\Lambda(Y)$ is a relatively compact set in $C[t_0,T]$. As a result, $\Lambda\colon Y\to Y$ is a compact operator because Y is closed. In conclusion, the Schauder fixed point theorem guarantees that problem (1.1) has a fixed point. That is, there is some $u^*\in Y$ so that $u^*=\Lambda(u^*)$. □

4. APPLICATION

In this section, we provide an example to support our work. It is clear that determining the true solution to such kind of nonlinear fractional problem (1.1) especially with non smooth kernel is either quite difficult or impossible unless for a simple case. For this matter, we consider the following example.

Let $\alpha = \gamma = 1/2$ and we consider the space $Y = \{u \in C[0,1] \colon \|u\| \leq r\}$. Then the following problem

$$u(\xi) = g(\xi) + \int_0^{\xi} (\xi - \rho)^{-1/2} \Psi\Big(\rho, u(\rho), \mathrm{D}^{\gamma} u(\rho)\Big)\, \mathrm{d}\rho, \tag{4.1}$$

where $g(\xi) = \xi^{1/2} - \frac{2}{3}\sqrt{\pi}\xi^{3/2} - \frac{1}{2}\pi\xi$ with $\Psi\Big(\rho, u(\rho), \mathrm{D}^{\gamma} u(\rho)\Big) = u(\rho) + \mathrm{D}^{1/2} u(\rho)$. We observe that Ψ satisfy the hypotheses $(H1) - (H2)$ and its easy to see that $g \in Y[0,1]$ where $r = K + M\frac{(T-t_0)^{\alpha}}{\alpha} \approx 5.5$. It follows from Theorem 4 that problem (4.1) has a solution u in $Y[0,1]$. In this example, we used the Adomian decomposition method with noise terms phenomenon to get the exact solution and the true solution is $u(\xi) = \xi^{1/2}$ in $Y[0,1]$.

CONCLUSIONS

In this work, we demonstrated that the integral operator (1.1) is compact and maps a closed, bounded, and convex subset of the Banach space $\big(C[t_0, T], \|\cdot\|\big)$ into itself. Consequently, Schauder's fixed point theorem guarantees the existence of at least one solution. Moreover, the illustrative example further verifies the applicability and effectiveness of the theoretical results. These findings contribute to the broader study of fractional integral equations and provide a framework that can be extended in future work to more complex models arising in applied mathematics and related fields.

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DYNAMICAL BEHAVIOR OF A TWO DIMENSIONAL DIFFERENCE EQUATION SYSTEM WITH QUADRATIC TERM

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Abstract. This article investigates the dynamical behavior of a two-dimensional asymmetric system of fractional difference equations given by

$$\mu_{n+1} = 1 + \hbar \frac{\mu_n}{\nu_{n-1}^2}, \quad \nu_{n+1} = 1 + \hbar \frac{\nu_n}{\mu_{n-1}^2}, \quad n \in \mathbb{N},$$

where $\hbar > 0$ is a parameter and the initial values μ_i, ν_i $(i = -1, 0)$ are positive real numbers. By employing linear stability analysis and eigenvalue localization within the unit disk, we rigorously establish the existence and stability of equilibrium points. For $0 < \hbar \leq \frac{3}{4}$, the system exhibits a unique symmetric equilibrium (ξ, ξ) that is globally asymptotically stable. When $\frac{3}{4} < \hbar < 1$, two distinct asymmetric equilibria emerge, both of which retain local and global asymptotic stability. Furthermore, the boundedness and persistence of the solutions are demonstrated for all $0 < \hbar < 1$ using induction and comparison principles. The convergence rate of solutions toward equilibrium is quantified through error term linearization, revealing the dependence on the spectral radius of the system's Jacobian. Numerical simulations validate the theoretical findings, illustrating bifurcation phenomena and stability transitions as $\hbar$ crosses the critical thresholds. This work extends the existing models by incorporating asymmetric interactions, offering insights into the qualitative behavior of nonlinear discrete dynamical systems with delayed feedback. The results contribute to broader applications in population dynamics, epidemiology, and other fields governed by coupled difference equations.

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1. INTRODUCTION

Difference equations or systems of difference equations have been widely utilized across various disciplines, including physics, economics, ecology, and infectious disease dynamics. These include population growth, HIV/AIDS and tuberculosis transmission, and influenza prevention and control (refer to [1–5,21,22]). Thus, they play a crucial role in applied mathematics. In particular, numerous academics have conducted thorough investigations into the dynamics of difference equations owing to their theoretical and practical relevance. The examination of the dynamic properties of ordinary or partial difference equations and systems of difference equations has emerged as a prominent research focus over the last decade (refer to [6–17]). With the behavior qualitatively examined through various methods and numerically solved using different techniques, the solutions to these equations were determined. Consequently, the study of difference equations has developed rapidly. Subsequently, a detailed discussion of the research background of the equation under investigation in this study was presented.

Gumus [18] focused on the qualitative behavior of discrete difference system

$$x_{n+1}=\alpha+\frac{\sum_{i=1}^{m}x_{n-i}}{y_n},\ \ y_{n+1}=\beta+\frac{\sum_{i=1}^{m}y_{n-i}}{x_n},\ n\in\mathbb{N},$$

where $\alpha,\beta\in(0,\infty)$, m is a positive integer, both x_{-i} and y_{-i} are positive real numbers for $i\in\{0,1,2,\dots,m\}$.

Khan [19] researched the global dynamics of asymmetric difference, which include boundedness and persistence, existence and local dynamics of fixed points, and global dynamics of asymmetric difference systems and convergence rates.

$$x_{n+1}=A+B\frac{y_n}{y_{n-1}^2},\ \ y_{n+1}=C+D\frac{x_n}{x_{n-1}^2},\ n\in\mathbb{N},$$

where A,B,C,D, are positive and $x_i,y_i,(i=-1,0)$ may be positive or negative.

Okumus and Soykan [20] explored asymptotic stability, periodicity, and boundedness of a nonlinear three dimensional system of difference equations

$$x_{n+1}=A+\frac{x_{n-1}}{z_n},y_{n+1}=A+\frac{y_{n-1}}{z_n},z_{n+1}=A+\frac{z_{n-1}}{y_n},\ n\in\mathbb{N},$$

where the parameter $A\in(0,\infty)$ and the initial values $x_i,y_i,z_i\in(0,\infty),(i=-1,0)$.

After examining several equations explored by other researchers, we will now focus on the equation we have studied, aiming to analyze the solutions of a fractional difference system represented by the following equations. At the same time, we will modify Khan's model [19] by incorporating asymmetric terms to obtain the following model:

$$\mu_{n+1}=1+\hbar\frac{\mu_n}{\nu_{n-1}^2},\quad \nu_{n+1}=1+\hbar\frac{\nu_n}{\mu_{n-1}^2},\quad n\in\mathbb{N},\tag{1.1}$$

where the parameter $\hbar>0$ and initial values $\mu_i,\ \nu_i\in(0,\infty)$, $(i=-1,0)$.

2. PRELIMINARIES

In this section, we recall some definitions and theorems that will be used in system (1.1).

Consider the following discrete dynamical system:

$$\begin{aligned} \mu_n &= \rho(\mu_n, \mu_{n-1}, \nu_n, \nu_{n-1}), \quad n \in \mathbb{N}, \\ \nu_n &= \rho(\mu_n, \mu_{n-1}, \nu_n, \nu_{n-1}), \quad n \in \mathbb{N}. \end{aligned} \tag{2.1}$$

Here, ρ and ρ are continuously differentiable functions. The solution (μ_n, ν_n) of system (2.1) is determined by the specified initial values.

Definition 1.

(i) If there exist positive real numbers M and N such that $\operatorname{supp} x_n \in (0, M]$ and $\operatorname{supp} y_n \in (0, N]$, then we say that the sequences $\{x_n\}$ and $\{y_n\}$ are bounded.

(ii) If there exist positive real numbers m and n such that $\operatorname{supp} x_n \in [m, \infty)$ and $\operatorname{supp} y_n \in [n, \infty)$, then we say that the sequences $\{x_n\}$ and $\{y_n\}$ are persistent.

(iii) If there exist positive real numbers M, N, m, n such that $\operatorname{supp} x_n \in [m, M]$ and $\operatorname{supp} y_n \in [n, N]$, then we say that the sequences $\{x_n\}$ and $\{y_n\}$ are both bounded and persistent.

Definition 2 ([13])**.** Assume that ρ and ρ are continuously differentiable at the equilibrium point (ξ, ξ) and that this point is also an equilibrium point of the mapping $\digamma$, such that the linearized system about the equilibrium point (ξ, ξ) is given by

$$\Phi_{n+1} = \digamma(\Phi_n) = \Upsilon \Phi_n,$$

where

$$\Phi_n = \begin{pmatrix} \mu_n \\ \mu_{n-1} \\ \nu_n \\ \nu_{n-1} \end{pmatrix}$$

and the Jacobian matrix of system (2.1) about the equilibrium point (ξ, ξ) is denoted as Υ.

Theorem 1 ([13])**.** *Let*

$$\Phi_{n+1} = \digamma(\Phi_n), \; n \in \mathbb{N}$$

be a difference system. Assume (ξ, ξ) is a fixed point of $\digamma$. An equilibrium point (ξ, ξ) is locally exponentially stable if and only if all eigenvalues of the Jacobian matrix of $\digamma$ evaluated at (ξ, ξ) reside within the open unit disk (i.e., $|\lambda| < 1$. Conversely, if at least one eigenvalue exceeds 1 (i.e., $|\lambda| > 1$), the system is unstable.

This paper presents a solid theoretical foundation for comprehensively analyzing the boundedness, persistence, and stability of the positive solution of system (1.1). These results carry significant implications for comprehending the dynamic characteristics of system (1.1) and addressing relevant practical problems.

3. RESULTS

In this study, we have identified that the point $(\bar{\mu},\bar{\nu}) = (\xi,\xi) = \left(\frac{1+\sqrt{1+4\hbar}}{2}, \frac{1+\sqrt{1+4\hbar}}{2}\right)$ is a fixed point of system (1.1) for $0<\hbar\leq\frac{3}{4}$. For $(\frac{3}{4}<\hbar<1)$, system (1.1) exhibits additional equilibrium points

$$\left(\frac{\hbar+\hbar\sqrt{4\hbar-3}}{2(1-\hbar)}, \frac{\hbar-\hbar\sqrt{4\hbar-3}}{2(1-\hbar)}\right) = (\bar{\mu}_1,\bar{\nu}_1),$$
$$\left(\frac{\hbar-\hbar\sqrt{4\hbar-3}}{2(1-\hbar)}, \frac{\hbar+\hbar\sqrt{4\hbar-3}}{2(1-\hbar)}\right) = (\bar{\nu}_1,\bar{\mu}_1).$$

(1) If $0<\hbar<1$, every positive solution of system (1.1) is persistence and boundedness.
(2) If $0<\hbar\leq\frac{3}{4}$, (ξ,ξ) is globally asymptotically stable.
(3) If $\frac{3}{4}<\hbar<1$,$(\bar{\mu}_1,\bar{\nu}_1)$ and $(\bar{\nu}_1,\bar{\mu}_1)$ is globally asymptotically stable.

3.1. *Boundedness*

Theorem 2. *The following two statements are true:*

(1) *Both $\mu_n>1$ and $\nu_n>1$ for all $n\geq 1$, independently of the initial conditions.*
(2) *If $0<\hbar<1$, then for all values of $k\geq 3$, we obtain the following inequality:*

$$\begin{cases}\mu_k\leq M_1+\frac{1}{1-\hbar}, \\ \nu_k\leq M_2+\frac{1}{1-\hbar}, \end{cases} \quad k\geq 3, \tag{3.1}$$

where $M_1 = 1+\frac{\hbar}{\nu_0^2}+\frac{\hbar^2\mu_0}{\nu_0^2\nu_{-1}^2}$, $M_2 = 1+\frac{\hbar}{\mu_0^2}+\frac{\hbar^2\nu_0}{\mu_0^2\mu_{-1}^2}$.

Proof. Inference (i) is evidently true. We now assess the validity of inference (ii). Utilizing (1.1) and inference (i), we derive the following expressions for $k\geq 3$:

$$\begin{aligned}\mu_k &= 1+\hbar\frac{\mu_{k-1}}{\nu_{k-2}^2}\leq 1+\hbar\mu_{k-1}, \\ \nu_k &= 1+\hbar\frac{\nu_{k-1}}{\mu_{k-2}^2}\leq 1+\hbar\nu_{k-1}.\end{aligned} \tag{3.2}$$

Here, let $\{x_k\}$ and $\{w_k\}$ be the solutions of the following system.

$$\begin{cases}x_k = 1+\hbar x_{k-1}, \\ w_k = 1+\hbar w_{k-1}, \end{cases} \quad k\geq 3, \tag{3.3}$$

such that

$$\mu_2 = x_2, \quad \nu_2 = w_2. \tag{3.4}$$

We prove by induction that

$$\mu_k\leq x_k, \quad \nu_k\leq w_k, \quad k\geq 3. \tag{3.5}$$

Assume $\mu_m \le x_m$ and $\nu_m \le w_m$ for $k = m \ge 3$. Then from (3.2) we get

$$\begin{aligned} \mu_{m+1} &\le 1 + \hbar\mu_m \le 1 + \hbar x_m = x_{m+1}, \\ \nu_{m+1} &\le 1 + \hbar\nu_m \le 1 + \hbar w_m = w_{m+1}. \end{aligned} \tag{3.6}$$

Therefore, $\mu_k \le x_k$ and $\nu_k \le w_k$ for $k \ge 3$. (3.5) holds.

From (3.3), we have the following inequalities:

$$\begin{cases} x_k \le \frac{1-\hbar^{k-2}}{1-\hbar} + \hbar^{k-2}x_2 = \frac{1-\hbar^{k-2}}{1-\hbar} + \hbar^{k-2}\mu_2, \\ w_k \le \frac{1-\hbar^{k-2}}{1-\hbar} + \hbar^{k-2}w_2 = \frac{1-\hbar^{k-2}}{1-\hbar} + \hbar^{k-2}\nu_2. \end{cases} \tag{3.7}$$

From (1.1), we get

$$\mu_2 = 1 + \hbar\frac{\mu_1}{\nu_0^2}, \quad \mu_1 = 1 + \hbar\frac{\mu_0}{\nu_{-1}^2}, \quad \nu_2 = 1 + \hbar\frac{\nu_1}{\mu_0^2}, \quad \nu_1 = 1 + \hbar\frac{\nu_0}{\mu_{-1}^2},$$

where $\mu_{-1}, \mu_0, \nu_{-1}, \nu_0$ are arbitrary positive numbers.

Since $0 < \hbar < 1$, $k \ge 3$ we have $\hbar^{k-2} < 1$, that is

$$\begin{cases} \mu_k \le M_1 + \frac{1}{1-\hbar}, \\ \nu_k \le M_2 + \frac{1}{1-\hbar}, \end{cases} \quad k \ge 3, \tag{3.8}$$

where $M_1 = 1 + \frac{\hbar}{\nu_0^2} + \frac{\hbar^2\mu_0}{\nu_0^2\nu_{-1}^2}$, $M_2 = 1 + \frac{\hbar}{\mu_0^2} + \frac{\hbar^2\nu_0}{\mu_0^2\mu_{-1}^2}$. Afterward, by utilizing (3.2), (3.5), and (3.6), (3.1) can be proven.

According to Definition 1, system (1.1) is bounded and persistent. □

3.2. *Linear stability*

This section will explore the stability of system (1.1), first analyzing its local stability and then extending the discussion to global stability.

Theorem 3. *For $0 < \hbar \le \frac{3}{4}$, the unique positive equilibrium point of system (1.1) is $(\xi, \xi) = \left(\frac{1+\sqrt{1+4\hbar}}{2}, \frac{1+\sqrt{1+4\hbar}}{2}\right)$ and exhibits local asymptotic stability. When $\frac{3}{4} < \hbar < 1$, system (1.1) possesses two distinct positive equilibrium points $(\bar{\mu}_1, \bar{\nu}_1) = \left(\frac{\hbar+\hbar\sqrt{4\hbar-3}}{2(1-\hbar)}, \frac{\hbar-\hbar\sqrt{4\hbar-3}}{2(1-\hbar)}\right)$ and $(\bar{\nu}_1, \bar{\mu}_1) = \left(\frac{\hbar-\hbar\sqrt{4\hbar-3}}{2(1-\hbar)}, \frac{\hbar+\hbar\sqrt{4\hbar-3}}{2(1-\hbar)}\right)$, both of which demonstrate local asymptotic stability.*

Proof of Theorem 3. (i) When $\bar{x} = \bar{y} = \xi$, the steady-state equation for system (3.1) is

$$\xi = 1 + \frac{\hbar}{\xi}.$$

This equation can be solved to yield $\xi = \frac{1\pm\sqrt{1+4\hbar}}{2}$, thus the unique positive equilibrium point of system (3.1) is

$$(\xi, \xi) = \left(\frac{1+\sqrt{1+4\hbar}}{2}, \frac{1+\sqrt{1+4\hbar}}{2}\right).$$

In accordance with Definition 2, the linear equation for system (1.1) about the equilibrium point (ξ,ξ) is

$$\Phi_{n+1} = \Upsilon \Phi_n, \tag{3.9}$$

where

$$\Phi_n = \begin{pmatrix} x_n \\ x_{n-1} \\ y_n \\ y_{n-1} \end{pmatrix}, \quad \Upsilon = \begin{pmatrix} \frac{\hbar}{\xi^2} & 0 & 0 & -\frac{2\hbar}{\xi^2} \\ 1 & 0 & 0 & 0 \\ 0 & -\frac{2\hbar}{\xi^2} & \frac{\hbar}{\xi^2} & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

The characteristic equation of (3.9) is

$$\lambda^4 - 2\frac{\hbar}{\xi^2}\lambda^3 + \left(\frac{\hbar}{\xi^2}\right)^2 \lambda^2 - 4\left(\frac{\hbar}{\xi^2}\right)^2 = 0,$$

then

$$\lambda^2 \left(\lambda - \frac{\hbar}{\xi^2}\right)^2 = 4\left(\frac{\hbar}{\xi^2}\right)^2. \tag{3.10}$$

(a) When $\lambda < \frac{\hbar}{\xi^2}$, (3.10) simplifies to:

$$\frac{\hbar}{\xi^2}\lambda - \lambda^2 = \frac{2\hbar}{\xi^2}.$$

From this, we can calculate:

$$\lambda_{1,2} = \frac{\frac{\hbar}{\xi^2} \pm \sqrt{\left(\frac{\hbar}{\xi^2}\right)^2 - 8\frac{\hbar}{\xi^2}}}{2}. \tag{3.11}$$

According to Theorem 1, for the equilibrium point (ξ,ξ) of system (1.1) to be locally asymptotically stable, we need $|\lambda| < 1$, leading to:

$$\left| \frac{\frac{\hbar}{\xi^2} \pm \sqrt{\left(\frac{\hbar}{\xi^2}\right)^2 - 8\frac{\hbar}{\xi^2}}}{2} \right| < 1. \tag{3.12}$$

Since $\frac{\hbar}{\xi^2}, \sqrt{\left(\frac{\hbar}{\xi^2}\right)^2 - 8\frac{\hbar}{\xi^2}} > 0$, it follows that:

$$\left|\frac{\hbar}{\xi^2} - \sqrt{\left(\frac{\hbar}{\xi^2}\right)^2 - 8\frac{\hbar}{\xi^2}}\right| < \left|\frac{\hbar}{\xi^2} + \sqrt{\left(\frac{\hbar}{\xi^2}\right)^2 - 8\frac{\hbar}{\xi^2}}\right| < 2. \tag{3.13}$$

If $\sqrt{\left(\frac{\hbar}{\xi^2}\right)^2 - 8\frac{\hbar}{\xi^2}} > 0$, then $\frac{\hbar}{\xi^2} \in (-\infty, 0) \cup (8, +\infty)$. However, given that equation (3.13) holds, it follows that $\frac{\hbar}{\xi^2} < 2$, which precludes any solutions in this case.

(b) When $\lambda > \frac{\hbar}{\xi^2}$, equation (3.10) can be rewritten as

$$\lambda^2 - \frac{\hbar}{\xi^2}\lambda = \frac{2\hbar}{\xi^2}.$$

At this point, we can calculate the eigenvalues:

$$\lambda_{3,4} = \frac{\frac{\hbar}{\xi^2} \pm \sqrt{\left(\frac{\hbar}{\xi^2}\right)^2 + 8\frac{\hbar}{\xi^2}}}{2}. \tag{3.14}$$

We set $|\lambda_{3,4}| < 1$:

$$\left| \frac{\frac{\hbar}{\xi^2} \pm \sqrt{\left(\frac{\hbar}{\xi^2}\right)^2 + 8\frac{\hbar}{\xi^2}}}{2} \right| < 1. \tag{3.15}$$

Similarly, based on (3.15), we have

$$\left| \frac{\hbar}{\xi^2} - \sqrt{\left(\frac{\hbar}{\xi^2}\right)^2 + 8\frac{\hbar}{\xi^2}} \right| < \left| \frac{\hbar}{\xi^2} + \sqrt{\left(\frac{\hbar}{\xi^2}\right)^2 + 8\frac{\hbar}{\xi^2}} \right| < 2. \tag{3.16}$$

From (3.16), we deduce that when $\frac{\hbar}{\xi^2} < 2$:

$$\left(\frac{\hbar}{\xi^2}\right)^2 + 8\frac{\hbar}{\xi^2} < \left(2 - \frac{\hbar}{\xi^2}\right)^2. \tag{3.17}$$

Solving (3.17) gives us $\frac{\hbar}{\xi^2} < \frac{1}{3}$. Substituting $\xi = \frac{1 \pm \sqrt{1+4\hbar}}{2}$ yields $0 < \hbar < \frac{3}{4}$. When $\frac{\hbar}{\xi^2} > 2$, (3.16) does not hold. In summary, when $0 < \hbar < \frac{3}{4}$, the unique positive equilibrium point (ξ, ξ) of system (1.1) is locally asymptotically stable.
(ii) When the equilibrium point is $(\bar{\mu}, \bar{\nu})$, system (1.1) can be transformed into

$$\bar{\mu} = 1 + \hbar\bar{\mu}\left(\frac{1}{\hbar} - \frac{1}{\bar{\mu}}\right)^2. \tag{3.18}$$

Solving equation (3.18), we obtain

$$\bar{\mu} = \frac{\hbar \pm \hbar\sqrt{4\hbar - 3}}{2(1-\hbar)}.$$

Similarly, we can find

$$\bar{\nu} = \frac{\hbar \pm \hbar\sqrt{4\hbar - 3}}{2(1-\hbar)}.$$

Since μ_n, ν_n are non-negative real numbers, we have $\frac{3}{4} \le \hbar < 1$. Notably, when $\hbar = \frac{3}{4}$, we find $\bar{\mu} = \bar{\nu} = 1.5 = \xi$. Because $\bar{\mu} \neq \bar{\nu}$, system (1.1) has two different positive equilibrium points:

$$(\bar{\mu}_1, \bar{\nu}_1) = \left(\frac{\hbar + \hbar\sqrt{4\hbar - 3}}{2(1-\hbar)}, \frac{\hbar - \hbar\sqrt{4\hbar - 3}}{2(1-\hbar)} \right),$$

$$(\bar{\nu}_1, \bar{\mu}_1) = \left(\frac{\hbar - \hbar\sqrt{4\hbar - 3}}{2(1-\hbar)}, \frac{\hbar + \hbar\sqrt{4\hbar - 3}}{2(1-\hbar)} \right).$$

In system (1.1), the Jacobian matrix at the equilibrium point $(\bar{\mu}_1, \bar{\nu}_1)$ is given by:

$$\Upsilon = \begin{pmatrix} \frac{\hbar}{\bar{\nu}_1^2} & 0 & 0 & -\frac{2\hbar\bar{\mu}_1}{\bar{\nu}_1^3} \\ 1 & 0 & 0 & 0 \\ 0 & -\frac{2\hbar\bar{\nu}_1}{\bar{\mu}_1^3} & \frac{\hbar}{\bar{\mu}_1^2} & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Let the eigenvalues of matrix Υ be $\lambda_1, \lambda_2, \lambda_3, \lambda_4$. Define Θ as a 4×4 diagonal matrix with elements d_1, d_2, d_3, d_4, where $d_1 = d_3 = 1$ and $d_2 = 1 - 2\varepsilon$ and $d_4 = 1 - 4\varepsilon$. Furthermore, we have

$$0 < \varepsilon < \min\left\{ \frac{1}{4}\left(1 - (\hbar + \sqrt{4\hbar - 3})(1-\hbar)\right), \frac{1}{4}\left(1 - \frac{(1-\hbar)^2}{\hbar}\right) \right\}.$$

It is evident that matrix Θ is invertible. To show this, we can compute $\Theta\Upsilon\Theta^{-1}$:

$$\Theta\Upsilon\Theta^{-1} = \begin{pmatrix} \frac{\hbar}{\bar{\nu}_1^2} & 0 & 0 & -\frac{2\hbar\bar{\mu}_1}{\bar{\nu}_1^3} d_1 d_4^{-1} \\ d_2 d_1^{-1} & 0 & 0 & 0 \\ 0 & -\frac{2\hbar\bar{\nu}_1}{\bar{\mu}_1^3} d_3 d_2^{-1} & \frac{\hbar}{\bar{\mu}_1^2} & 0 \\ 0 & 0 & d_4 d_3^{-1} & 0 \end{pmatrix}.$$

Since $d_1 > d_2 > 0$ and $d_3 > d_4 > 0$, we have

$$d_2 d_1^{-1} < 1, \quad d_4 d_3^{-1} < 1.$$

Additionally, we have

$$\begin{aligned} &\frac{\hbar}{\bar{\nu}_1^2} + \frac{2\hbar\bar{\mu}_1}{\bar{\nu}_1^3} d_1 d_4^{-1} = \frac{\hbar}{\bar{\nu}_1^2}(1 + \frac{2\bar{\mu}_1}{\bar{\nu}_1} d_1 d_4^{-1}) = \frac{\hbar}{\bar{\nu}_1^2}(1 + \frac{2\bar{\mu}_1}{\bar{\nu}_1}\frac{1}{1-4\varepsilon}) \\ &< \frac{\hbar}{\bar{\nu}_1^2}(1 + \frac{2\bar{\mu}_1}{\bar{\nu}_1})\frac{1}{1-4\varepsilon} = \frac{1}{1-4\varepsilon} \cdot \frac{2(\hbar + \sqrt{4\hbar - 3})(1-\hbar)}{2\hbar^2 - \hbar - \sqrt{4\hbar^3 - 3\hbar^2}} < 1, \end{aligned} \tag{3.19}$$

$$\begin{aligned} &\frac{\hbar}{\bar{\mu}_1^2} + \frac{2\hbar\bar{\nu}_1}{\bar{\mu}_1^3} d_3 d_2^{-1} = \frac{\hbar}{\bar{\mu}_1^2}(1 + \frac{2\bar{\nu}_1}{\bar{\mu}_1} d_3 d_2^{-1}) < \frac{\hbar}{\bar{\mu}_1^2}(1 + \frac{2\bar{\nu}_1}{\bar{\mu}_1}) d_3 d_2^{-1} \\ &= \frac{1}{1-4\varepsilon} \cdot \frac{3\hbar - 2\hbar^2 + \sqrt{4\hbar^3 - 3\hbar^2}}{2\hbar^2 - \hbar + \sqrt{4\hbar^3 - 3\hbar^2}} \cdot \frac{2(1-\hbar)^2}{2\hbar^2 - \hbar + \sqrt{4\hbar^3 - 3\hbar^2}} < 1. \end{aligned} \tag{3.20}$$

From (3.19), we have

$$1-4\varepsilon > \frac{2(\hbar+\sqrt{4\hbar-3})(1-\hbar)}{2\hbar^2-\hbar-\sqrt{4\hbar^3-3\hbar^2}}.$$

When $\frac{3}{4} \leq \hbar < 1$, it follows that $2\hbar^2-\hbar-\sqrt{4\hbar^3-3\hbar^2} < \hbar-\sqrt{4\hbar^3-3\hbar^2}$, leading to

$$\begin{aligned} 1-4\varepsilon &> \frac{(\hbar+\sqrt{4\hbar-3})(1-\hbar)}{2\hbar^2-\hbar-\sqrt{4\hbar^3-3\hbar^2}} > \frac{(\hbar+\sqrt{4\hbar-3})(1-\hbar)}{\hbar-\sqrt{4\hbar^3-3\hbar^2}} \\ &> \frac{(\hbar+\sqrt{4\hbar-3})(1-\hbar)}{\hbar} > (\hbar+\sqrt{4\hbar-3})(1-\hbar). \end{aligned} \tag{3.21}$$

Thus, we establish that $\varepsilon < \frac{1}{4}(1-(\hbar+\sqrt{4\hbar-3})(1-\hbar))$.

From (3.20), we obtain

$$1-4\varepsilon > \frac{3\hbar-2\hbar^2+\sqrt{4\hbar^3-3\hbar^2}}{2\hbar^2-\hbar+\sqrt{4\hbar^3-3\hbar^2}} \cdot \frac{2(1-\hbar)^2}{2\hbar^2-\hbar+\sqrt{4\hbar^3-3\hbar^2}}. \tag{3.22}$$

If $\frac{3}{4} \leq \hbar < 1$, then

$$2\hbar^2-\hbar+\sqrt{4\hbar^3-3\hbar^2} < 3\hbar-2\hbar^2+\sqrt{4\hbar^3-3\hbar^2},\ \ 2\hbar^2-\hbar+\sqrt{4\hbar^3-3\hbar^2} < 2\hbar,$$

$$\begin{aligned} 1-4\varepsilon &> \frac{3\hbar-2\hbar^2+\sqrt{4\hbar^3-3\hbar^2}}{2\hbar^2-\hbar+\sqrt{4\hbar^3-3\hbar^2}} \cdot \frac{2(1-\hbar)^2}{2\hbar^2-\hbar+\sqrt{4\hbar^3-3\hbar^2}} \\ &> \frac{3\hbar-2\hbar^2+\sqrt{4\hbar^3-3\hbar^2}}{2\hbar^2-\hbar+\sqrt{4\hbar^3-3\hbar^2}} \cdot \frac{2(1-\hbar)^2}{3\hbar-2\hbar^2+\sqrt{4\hbar^3-3\hbar^2}} \\ &= \frac{2(1-\hbar)^2}{2\hbar^2-\hbar+\sqrt{4\hbar^3-3\hbar^2}} > \frac{2(1-\hbar)^2}{2\hbar} = \frac{(1-\hbar)^2}{\hbar}. \end{aligned} \tag{3.23}$$

From (3.23), we conclude that $\varepsilon < \frac{1}{4}(1-\frac{(1-\hbar)^2}{\hbar})$.
Since Υ and $\Theta\Upsilon\Theta^{-1}$ have the same eigenvalues, we obtain

$$\begin{aligned} &\max_{1\leq i\leq 4}|\lambda_i| \leq \|\Theta\Upsilon\Theta^{-1}\|_\infty \\ &= \max\left\{d_2d_1^{-1}, d_4d_3^{-1}, \frac{\hbar}{\bar{\nu}_1^2}\left(1+\frac{2\bar{\mu}_1}{\bar{\nu}_1}d_1d_4^{-1}\right), \frac{\hbar}{\bar{\mu}_1^2}\left(1+\frac{2\bar{\nu}_1}{x_1}d_3d_2^{-1}\right)\right\} \\ &< 1. \end{aligned}$$

So the equilibrium point $(\bar{\mu}_1,\bar{\nu}_1)$ of system (1.1) is locally asymptotically stable for $\frac{3}{4} \leq \hbar < 1$. The local asymptotic stability of system (1.1) at the equilibrium point $(\bar{\nu}_1,\bar{\mu}_1)$ is similar to that of $(\bar{\mu}_1,\bar{\nu}_1)$, so we omit that proof.

Thus, the proof of Theorem 3 is complete. □

Theorem 4. *The equilibrium point* $(\bar{\mu},\bar{\nu})$ *of system* (1.1) *is globally asymptotically stable if* $\hbar \in (0,1)$.

Proof of Theorem 4. If we take into account that (μ_n, ν_n) serves as a positive solution to system (1.1), we can deduce that

$$\lim_{n\to\infty} \mu_n = \bar{\mu}, \qquad \lim_{n\to\infty} \nu_n = \bar{\nu}. \tag{3.24}$$

By applying Theorem 2, we derive this

$$\begin{aligned} \Gamma_1 &= \lim_{n\to\infty} \sup \mu_n < \infty, & \Gamma_2 &= \lim_{n\to\infty} \sup \nu_n < \infty, \\ \gamma_1 &= \lim_{n\to\infty} \inf \mu_n \geq 1, & \gamma_2 &= \lim_{n\to\infty} \inf \nu_n \geq 1. \end{aligned} \tag{3.25}$$

Then from system (1.1) and (3.25), we get

$$\Gamma_1 \leq 1 + \hbar\frac{\Gamma_1}{\gamma_2^2}, \;\; \Gamma_2 \leq 1 + \hbar\frac{\Gamma_2}{\gamma_1^2}, \;\; \gamma_1 \geq 1 + \hbar\frac{\gamma_1}{\Gamma_2^2}, \;\; \gamma_2 \geq 1 + \hbar\frac{\gamma_2}{\Gamma_1^2}. \tag{3.26}$$

Now we set $\alpha = \frac{\gamma_2}{\Gamma_1}$, $\beta = \frac{\gamma_1}{\Gamma_2}$, then from (3.26) we have

$$\alpha \geq \frac{1 + \hbar\frac{\gamma_2}{\Gamma_1^2}}{1 + \hbar\frac{\Gamma_1}{\gamma_2^2}} = \frac{1 + \hbar\frac{\alpha}{\Gamma_1}}{1 + \hbar\frac{1}{\alpha\gamma_2}} \geq \frac{1}{1 + \frac{\hbar}{\alpha}}. \tag{3.27}$$

Then the following relationship holds:

$$\alpha + \hbar \geq 1. \tag{3.28}$$

If $0 < \hbar < 1$, then it follows that $\alpha \geq 1$, which implies $\gamma_2 \geq \Gamma_1$. Similarly, when $0 < \hbar < 1$, it follows that $\gamma_1 \geq \Gamma_2$. Therefore, when $0 < \hbar < 1$, we have the following relationship:

$$\gamma_2 \geq \Gamma_1 \geq \gamma_1 \geq \Gamma_2. \tag{3.29}$$

Based on (3.25) and (3.29), we obtain the following results:

$$\Gamma_1 = \gamma_1 = \Gamma_2 = \gamma_2. \tag{3.30}$$

Hence from system (1.1) and (3.30), there exist the $\lim \mu_n$, $\lim \nu_n$, as $n \to \infty$,

$$\lim_{n\to\infty} \mu_n = \bar{\mu}, \quad \lim_{n\to\infty} \nu_n = \bar{\nu}.$$

The proof is fully established when the point $(\bar{\mu}, \bar{\nu})$ is identified as a positive equilibrium within system (1.1) if $0 < \hbar < 1$. □

4. RATE OF CONVERGENCE

In this section,we will discuss the rate of convergence result of the solution that tends to $(\bar{\mu}, \bar{\nu})$. Consider the system

$$Ł_{n+1} = (\imath + \kappa(n))Ł_n. \tag{4.1}$$

Here, $Ł_n$ represents an n-dimensional vector, $\imath$ denotes a constant matrix, and κ signifies a constant matrix transformation satisfying the condition:

$$\|\kappa(n)\| \to 0 \quad \text{as} \quad n \to \infty. \tag{4.2}$$

In this context, $\|\cdot\|$ represents any chosen matrix norm associated with a vector norm.

Theorem 5. *Assuming condition (4.2) holds, $0 < \hbar < 1$ and $\{Ł_n\}$ serves as a solution for equation (4.1), then either $Ł_n = 0$ for all large n or $\Lambda = \lim_{n\to\infty} \frac{||Ł_{n+1}||}{||Ł_n||}$ or $\Lambda = \lim_{n\to\infty}(||Ł||)^{\frac{1}{n}}$ exists and Λ is equal to the modulus of an eigenvalue of the matrix ι.*

Theorem 6. *Assume that the solution (μ_n, ν_n) of system (1.1) converges to $(\bar{\mu}, \bar{\nu})$. Then, the error vector*

$$e_n = \begin{pmatrix} e_n^1 \\ e_{n-1}^1 \\ e_n^2 \\ e_{n-1}^2 \end{pmatrix} = \begin{pmatrix} \mu_n - \bar{\mu} \\ \mu_{n-1} - \bar{\mu} \\ \nu_n - \bar{\nu} \\ \nu_{n-1} - \bar{\nu} \end{pmatrix}$$

for all solutions of system (1.1) satisfies the following asymptotic relation:

$$\lim_{n\to\infty} \frac{||Ł_{n+1}||}{||Ł_n||} = \lambda_{1,2,3,4}\Upsilon(\bar{\mu}, \bar{\nu}), \qquad \lim_{n\to\infty}(||Ł||)^{\frac{1}{n}} = \lambda_{1,2,3,4}\Upsilon(\bar{\mu}, \bar{\nu}),$$

where $\lambda_{1,2,3,4}\Upsilon(\bar{\mu}, \bar{\nu})$ are the characteristic roots of $\Upsilon(\bar{\mu}, \bar{\nu})$.

Proof of Theorem 6. The error terms can be expressed as follows:

$$\begin{aligned} \mu_{n+1} - \bar{\mu} &= 1 + \hbar\frac{\mu_n}{\nu_{n-1}^2} - (1 + \hbar\frac{\bar{\mu}}{\bar{\nu}^2}), \\ \nu_{n+1} - \bar{\nu} &= 1 + \hbar\frac{\nu_n}{\mu_{n-1}^2} - (1 + \hbar\frac{\bar{\nu}}{\bar{\mu}^2}). \end{aligned} \tag{4.3}$$

(4.3) can be transformed into

$$\begin{aligned} \mu_{n+1} - \bar{\mu} &= \frac{\hbar}{\nu_{n-1}^2}(\mu_n - \bar{\mu}) - \frac{\hbar\bar{\mu}(\nu_{n-1} + \bar{\nu})}{\bar{\nu}^2\nu_{n-1}^2}(\nu_{n-1} - \bar{\nu}), \\ \nu_{n+1} - \bar{\nu} &= \frac{\hbar}{\mu_{n-1}^2}(\nu_n - \bar{\nu}) - \frac{\hbar\bar{\nu}(\mu_{n-1} + \bar{\mu})}{\bar{\mu}^2\mu_{n-1}^2}(\mu_{n-1} - \bar{\mu}). \end{aligned} \tag{4.4}$$

Let $e_n^1 = \mu_n - \bar{\mu}$, $e_{n-1}^1 = \mu_{n-1} - \bar{\mu}$, $e_n^2 = \nu_n - \bar{\nu}$, $e_{n-1}^2 = \nu_{n-1} - \bar{\nu}$, system (4.4) can be expressed as

$$e_{n+1}^1 = a_n e_n^1 + b_n e_{n-1}^2, \quad e_{n+1}^2 = c_n e_n^2 + d_n e_{n-1}^1, \tag{4.5}$$

where

$$a_n = \frac{\hbar}{\nu_{n-1}^2}, \quad b_n = -\frac{\hbar\bar{\mu}(\nu_{n-1} + \bar{\nu})}{\bar{\nu}^2\nu_{n-1}^2}, \quad c_n = \frac{\hbar}{\mu_{n-1}^2}, \quad d_n = -\frac{\hbar\bar{\nu}(\mu_{n-1} + \bar{\mu})}{\bar{\mu}^2\mu_{n-1}^2}. \tag{4.6}$$

By taking the limits of (4.6) to obtain

$$\lim_{n\to\infty} a_n = \frac{\hbar}{\bar{\nu}^2}, \quad \lim_{n\to\infty} b_n = -\frac{2\hbar\bar{\mu}}{\bar{\nu}^3}, \quad \lim_{n\to\infty} c_n = \frac{\hbar}{\bar{\mu}^2}, \quad \lim_{n\to\infty} d_n = -\frac{2\hbar\bar{\nu}}{\bar{\mu}^3}. \tag{4.7}$$

That is

$$a_n = \frac{\hbar}{\bar{\nu}^2} + \kappa_a, \;\; b_n = -\frac{2\hbar\bar{\mu}}{\bar{\nu}^3} + \kappa_b, \;\; c_n = \frac{\hbar}{\bar{\mu}^2} + \kappa_c, \;\; d_n = -\frac{2\hbar\bar{\nu}}{\bar{\mu}^3} + \kappa_d, \tag{4.8}$$

where $\kappa_a, \kappa_b, \kappa_c, \kappa_d \to 0$ as $n \to \infty$. Thus,we get the system of the form (4.1)

$$Ł_{n+1} = (\iota + \kappa(n))Ł_n, \tag{4.9}$$

where

$$\iota = \begin{pmatrix} \frac{\hbar}{\bar{\nu}^2} & 0 & 0 & -\frac{2\hbar\bar{\mu}}{\bar{\nu}^3} \\ 1 & 0 & 0 & 0 \\ 0 & -\frac{2\hbar\bar{\nu}}{\bar{\mu}^3} & \frac{\hbar}{\bar{\mu}^2} & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \kappa(n) = \begin{pmatrix} \kappa_a & 0 & 0 & \kappa_b \\ 1 & 0 & 0 & 0 \\ 0 & \kappa_d & \kappa_c & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \tag{4.10}$$

and when $n \to \infty$, $||\kappa(n)|| \to 0$. Therefore, the limiting system of error terms can be formulated as follows:

$$\begin{pmatrix} e^1_{n+1} \\ e^1_n \\ e^2_{n+1} \\ e^2_n \end{pmatrix} = \begin{pmatrix} \frac{\hbar}{\bar{\nu}^2} & 0 & 0 & -\frac{2\hbar\bar{\mu}}{\bar{\nu}^3} \\ 1 & 0 & 0 & 0 \\ 0 & -\frac{2\hbar\bar{\nu}}{\bar{\mu}^3} & \frac{\hbar}{\bar{\mu}^2} & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} e^1_n \\ e^1_{n-1} \\ e^2_n \\ e^2_{n-1} \end{pmatrix}. \tag{4.11}$$

□

5. FIGURES

To validate the theoretical findings, we present numerical simulations of system (1.1) under distinct parameter regimes. The initial conditions are chosen as $\mu_{-1} = 0.5$, $\mu_0 = 1.0$, $\nu_{-1} = 0.6$, and $\nu_0 = 1.2$ for all cases unless specified otherwise.

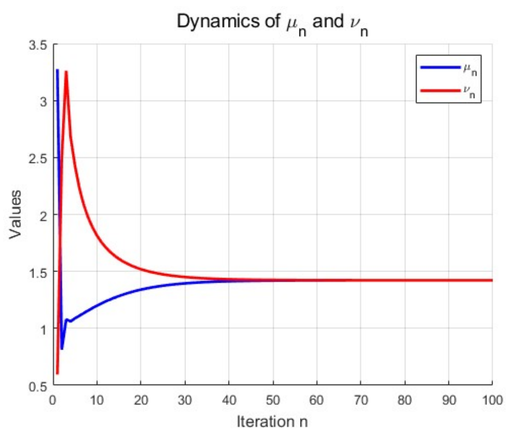

FIGURE 1. Behavior of system (1.1) at $\hbar = 0.6$

Case 1: $\hbar = 0.6$ $(0 < \hbar \leq 3/4)$. Figure 1 illustrates the trajectories of μ_n and ν_n. Both variables converge monotonically to the symmetric equilibrium $(\xi, \xi) = (1.366, 1.366)$, consistent with Theorem 3. This confirms the global asymptotic stability of the unique equilibrium in this parameter range.

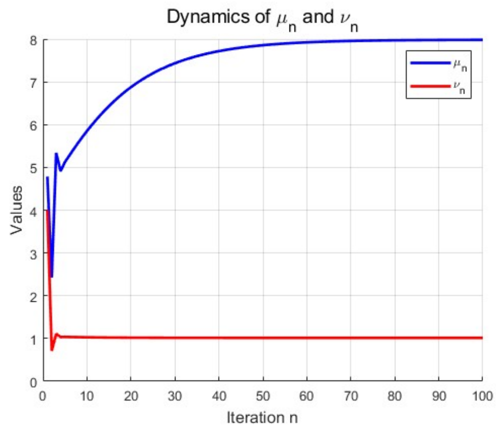

FIGURE 2. Behavior of system (1.1) at $\hbar = 0.9$

Case 2: $\hbar = 0.9$ $(3/4 < \hbar < 1)$. As shown in Figure 2, the system exhibits bistability. Depending on initial perturbations, solutions converge to either $(\bar{\mu}_1, \bar{\nu}_1) = (2.12, 0.48)$ or $(\bar{\nu}_1, \bar{\mu}_1) = (0.48, 2.12)$. This bifurcation aligns with the emergence of asymmetric equilibria predicted in Section 3.

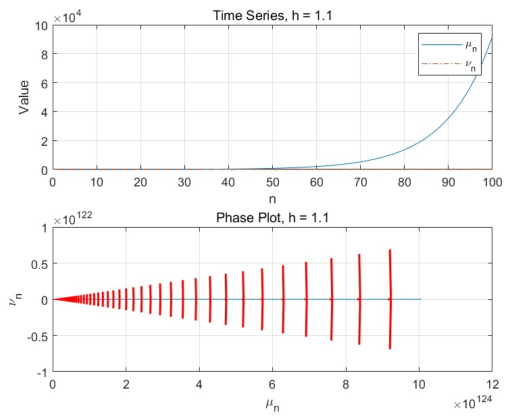

FIGURE 3. Behavior of system (1.1) at $\hbar = 1.1$

Case 3: $\hbar = 1.1$ ($\hbar > 1$). While our theoretical analysis focuses on $0 < \hbar < 1$, Figure 3 demonstrates oscillatory divergence when $\hbar$ exceeds 1. This highlights the critical role of $\hbar$ in maintaining system stability.

6. Conclusion

This paper investigates a two-dimensional asymmetric fractional difference equation system that reveals rich dynamic properties. A unique symmetric equilibrium (ξ, ξ) is globally asymptotically stable for $0 < \hbar \leq \frac{3}{4}$, whereas two asymmetric equilibria emerge, and are stable for $\frac{3}{4} < \hbar < 1$. Solution boundedness and persistence were proven using the induction and comparison principles. Numerical simulations confirmed monotonic convergence for $\hbar = 0.6$, bistability for $\hbar = 0.9$, and oscillatory divergence for $\hbar > 1$. This study highlights the threshold effect of $\hbar$ on stability, offering insights for discrete dynamical modeling in fields such as population dynamics and epidemiology. Future studies may explore higher-dimensional systems or specific applications.

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ANALYTICAL SOLUTIONS TO THE DOUBLE-CHAIN DNA SYSTEM BY TWO COMPUTATIONAL TECHNIQUES

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Abstract. In this article, two algebraic techniques are employed to analyze the double-chain model of deoxyribonucleic acid, a crucial component in the realm of biology. The solutions obtained by these methods include the trigonometric function solution, hyperbolic function solution and rational solution. These methodologies demonstrate significant effectiveness in obtaining exact solutions for many nonlinear differential equations.

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1. INTRODUCTION

Deoxyribonucleic Acid (DNA), is a genetic material that contains all the evolutionary and functional information of living organisms that they need for their reproduction and life. Friedrich Miescher first discovered DNA in the 1800s. But it took a long time for scientists to discover its structure and realize its importance in biology. Structurally, DNA consists of two strands that are parallel and anti-parallel to each other, each filled with numerous nucleotides. These two strands are held together by hydrogen bonds between the bases of each nucleotide pair, keeping them side by side. The physical structure of the two DNA strands as a double helical ladder was first discovered using X-rays in 1953 by James Watson and Francis Crick and was accepted and expanded by scientific communities. This structure allows it to carry biological information. During the last several decades, the structure of DNA has been extensively studied by many scientists. DNA has a complex structure and has many longitudinal, transverse and torsional motions. For this reason, a mathematical model with all its characteristics has not yet been presented. However, to study the structure of DNA, we must provide suitable nonlinear mathematical models. Some scientists have attempted to present some models to describe it, such as the property of the open state in long polynucleotide double helices and possibility of soliton excitations [5], soliton excitations in DNA double helices [22], a coupled base-rotator model for structure and dynamics of DNA [7], two-dimensional discrete model: denaturation and longitudinal wave propagation for DNA dynamics [14], nonlinear dynamics in a new double-chain model of DNA [3], solitary wave solutions for longitudinal and transverse movements of DNA [2, 17], explicit solutions of double-chain DNA dynamical system [12], simulation of the coupled DNA nonlinear dynamical equation bell-shaped [16] and so on.

In this paper, we investigate the double-chain DNA system [18]:

$$\Omega_{tt} - c_1^2\Omega_{xx} = \lambda_1\Omega + \gamma_1\Omega\Gamma + \mu_1\Omega^3 + \beta_1\Omega\Gamma^2,$$
$$\Gamma_{tt} - c_2^2\Gamma_{xx} = \lambda_2\Gamma + \gamma_2\Omega^2 + \mu_2\Omega^2\Gamma + \beta_2\Gamma^3 + c_0,$$

where

$$c_1 = \pm\frac{\chi_1}{\rho},\ c_2 = \pm\frac{\chi_2}{\rho},\ \lambda_1 = \frac{-2\mu}{\rho\sigma h}(h-l_0),\ \lambda_2 = \frac{-2\mu}{\rho\sigma},\ \gamma_1 = 2\gamma_2 = \frac{2\sqrt{2}\mu l_0}{\rho\sigma h^2},$$
$$\mu_1 = \mu_2 = \frac{-2\mu l_0}{\rho\sigma h^3},\ \beta_1 = \beta_2 = \frac{4\mu l_0}{\rho\sigma h^3},\ c_0 = \frac{\sqrt{2}\mu}{\rho\sigma}(h-l_0).$$

Here Ω and Γ indicate the difference in the longitudinal and transverse displacements of the top and bottom strands. This model with two long strands, homogeneous and elastic, demonstrates two polynucleotide chains of the DNA molecule. where χ_1 and χ_2 are the Young's modulus and the tension density of each strand, ρ and σ denote the mass density and the area of transverse cross-section. Also, μ, l_0 and h are the stiffness of the elastic membrane, the height of the membrane in the equilibrium positive and the distance between the two strands, respectively. Exact solutions of nonlinear partial differential equations (NLPDEs) play a critical role in better realizing qualitative features and physical interpretations of many occurrences. Many complicated events can be described by these solutions. For this purpose some techniques have been suggested, such as Kudryashov method [11], $\tan(\phi(\xi)/2)$-expansion method [10], $\exp(-\varphi(\xi))$-expansion method [19], first integral method [8], sine-Gordon method [21], Legendre wavelets [6], $\frac{G'}{G^2}$-expansion method [9], and so on. Many scientists have been applied some methods to study the double-chain DNA system. For examples, Riccati parameterized factorization method [2], $\frac{G'}{G}$-expansion method [12], φ^6-model expansion approach [18], improved generalized Riccati equation mapping method [13], Lie transformation method [20], $\exp(-\varphi(\xi))$-expansion method [1], φ^4-expansion method [4] and so on. Our purpose in this article is to obtain exact solutions to the double-chain DNA system using two algebraic methods.

The remaining parts of this article are constructed as follows. In section 2, we describe the first algebraic method and its application to the double-chain DNA system.

The application of $\frac{G'}{G^2}$-expansion method for the double-chain DNA system is presented in section 2.2. Graphical representations of some solutions is shown in section 4. In the last section, the conclusion is given.

2. The first algebraic method

2.1. *Description of the first method to look exact solutions of NLPDEs*

In this section, we consider the following NLPDE

$$R(\phi,\phi_x,\phi_t,\phi_{xx},\phi_{xt},\phi_{tt},\ldots)=0, \tag{2.1}$$

where $\phi=\phi(x,t)$ is an unknown function in two variables x and t. Substituting the travelling wave transformation

$$\xi=\kappa x+\omega t, \tag{2.2}$$

into (2.1), it can be reduced to the following ODE

$$\tilde{R}(\Phi,\Phi',\Phi'',\Phi''',\ldots)=0. \tag{2.3}$$

Here $\Phi^{(n)}=\frac{d^n\Phi}{d\xi^n}$. The solution of Eq. (2.3) can be written:

$$\Phi(\xi)=\frac{\sum_{j=0}^{\eta_1}A_j\Theta(\xi)}{\sum_{j=0}^{\eta_2}B_j\Theta(\xi)}, \tag{2.4}$$

where the positive constants η_1 and η_2 can be calculated by considering the homogeneous balance between the highest order derivatives and the highest nonlinear terms of $\Phi(\xi)$ in equation (2.3), and $A_j\,(0\le j\le\eta_1)$, $B_j\,(0\le j\le\eta_2)$ are constants to be found later and $A_{\eta_1},B_{\eta_2}\neq 0$. Here $\Theta=\Theta(\xi)$ satisfies the following ODE

$$\Theta'(\xi)=p+\Theta^2(\xi) \tag{2.5}$$

where p is a constant and which has the following special solutions [15].

Case 1: When $p<0$,

$$\Theta_1(\xi)=-\sqrt{-p}\tanh(\sqrt{-p}\,\xi), \tag{2.6}$$

$$\Theta_2(\xi)=-\sqrt{-p}\coth(\sqrt{-p}\,\xi), \tag{2.7}$$

$$\Theta_3(\xi)=-\sqrt{-p}\tanh(2\sqrt{-p}\,\xi)\pm i\sqrt{-p}\,sech(2\sqrt{-p}\,\xi), \tag{2.8}$$

$$\Theta_4(\xi)=-\sqrt{-p}\coth(2\sqrt{-p}\,\xi)\pm\sqrt{-p}\,csch(2\sqrt{-p}\,\xi), \tag{2.9}$$

$$\Theta_5(\xi)=-\frac{1}{2}\left(\sqrt{-p}\tanh(\frac{\sqrt{-p}}{2}\xi)+\sqrt{-p}\coth(\frac{\sqrt{-p}}{2}\xi)\right). \tag{2.10}$$

Case 2: When $p>0$,

$$\Theta_6(\xi)=\sqrt{p}\tan(\sqrt{p}\,\xi), \tag{2.11}$$

$$\Theta_7(\xi)=-\sqrt{p}\cot(\sqrt{p}\,\xi), \tag{2.12}$$

$$\Theta_8(\xi)=-\sqrt{p}\tan(2\sqrt{p}\,\xi)\pm\sqrt{p}\sec(2\sqrt{p}\,\xi), \tag{2.13}$$

$$\Theta_9(\xi) = -\sqrt{p}\cot(2\sqrt{p}\xi) \pm \sqrt{p}\csc(2\sqrt{p}\xi), \tag{2.14}$$

$$\Theta_{10}(\ xi) = \frac{1}{2}\left(\sqrt{p}\tan(\frac{\sqrt{p}}{2}\xi) - \sqrt{p}\cot(\frac{\sqrt{p}}{2}\xi)\right). \tag{2.15}$$

Case 3: When $p = 0$,

$$\Theta_{11}(\xi) = -\frac{1}{\xi + d}. \tag{2.16}$$

Where d is a constant.

The exact solutions for NLPDEs can be obtained by following these steps:

First, We substitute in (2.3) the (2.4) with Eq. (2.5). After making this substitution, we obtain a polynomial that involves $\Theta(\xi)$. We then collect the terms with the same powers of $\Theta(\xi)$ and set all the coefficients of the resulting polynomial to zero. This operation yields a set of algebraic equations in terms of $A_j\,(j = 0,1,2,...,\eta_1), B_j\,(j = 0,1,2,...,\eta_2)$, κ and ω. Solving this system, it gives solutions of equation(2.3).

2.2. *Application*

Consider the double-chain DNA system

$$\Omega_{tt} - c_1^2\Omega_{xx} = \lambda_1\Omega + \gamma_1\Omega\Gamma + \mu_1\Omega^3 + \beta_1\Omega\Gamma^2, \tag{2.17}$$

$$\Gamma_{tt} - c_2^2\Gamma_{xx} = \lambda_2\Gamma + \gamma_2\Omega^2 + \mu_2\Omega^2\Gamma + \beta_2\Gamma^3 + c_0. \tag{2.18}$$

By using the transformation

$$\Gamma = a\Omega + b, \tag{2.19}$$

where a and b are constants, Eqs. (2.17)-(2.18) can be converted into the following form

$$\Omega_{tt} - c_1^2\Omega_{xx} = \Omega^3(\mu_1 + \beta_1 a^2) + \Omega^2(2\beta_1 ab + \gamma_1 a) + \Omega\left(\lambda_1 + b\gamma_1 + \beta_1 b^2\right), \tag{2.20}$$

$$\Omega_{tt} - c_2^2\Omega_{xx} = \Omega^3(\mu_2 + \beta_2 a^2) + \Omega^2\left(3\beta_2 ab + \frac{\gamma_2}{a} + \frac{\mu_2 b}{a}\right) + \Omega\left(\lambda_2 + 3\beta_2 b^2\right)$$
$$+\frac{\lambda_2 b}{a} + \frac{\beta_2 b^3}{a} + \frac{c_0}{a}, \tag{2.21}$$

Eqs. (2.20) and (2.21) are similar for

$$b = \frac{h}{\sqrt{2}},\ \chi_1 = \chi_2. \tag{2.22}$$

Now Eqs. (2.20) and (2.21) can be reduced to a single equation as

$$\Omega_{tt} - c_1^2\Omega_{xx} = K\Omega^3 + L\Omega^2 + M\Omega \tag{2.23}$$

where

$$K = \frac{m(4a^2-2)}{h^3},\ L = \frac{6\sqrt{2}am}{h^2},\ M = \left(-\frac{2m}{l_0} + \frac{6m}{h}\right),\ m = \frac{\mu l_0}{\rho\sigma},\ c_1^2 = \frac{\chi_1}{\rho}. \tag{2.24}$$

For obtaining exact solutions of (2.23), We take the travelling wave transformation

$$\Omega(x,t) = \Psi(\xi), \quad \xi = \kappa x + \omega t, \tag{2.25}$$

where κ and ω are constants that should be determined later. Substituting (2.25) into Eq. (2.23), we have

$$(\omega^2 - \kappa^2 c_1^2)\Psi'' - K\Psi^3 - L\Psi^2 - M\Psi = 0, \quad \omega^2 - \kappa^2 c_1^2 \neq 0. \tag{2.26}$$

By balancing Ψ'' with Ψ^3 in (2.26) along with (2.4), we get the below:

$$\eta_1 - \eta_2 + 2 = 3(\eta_1 - \eta_2) \Longrightarrow \eta_1 = \eta_2 + 1. \tag{2.27}$$

Therefore, the exact solution of Eq. (2.26) can be written in the following forms.

Type 1: $\eta_1 = 1$ and $\eta_2 = 0$,

$$\Psi(\xi) = \frac{A_0 + A_1\Theta(\xi)}{B_0}, \tag{2.28}$$

where Θ is the solution of equation (2.5). Substituting Eq. (2.28) along Eq. (2.5) into Eq. (2.26) and equating all of the same powers $\Theta(\xi)$ to zero, we obtain a system of algebraic equations for A_0, A_1, B_0, p, κ and ω. Solving obtained system using *Mathematica*, we obtain

$$A_0 = A_0,\ A_1 = -\frac{1}{\sqrt{-p}}A_0,\ B_0 = -\frac{3K}{L}A_0,$$
$$M = -4p(\omega^2 - \kappa^2 c_1^2),\ K = \frac{2L^2}{9M},\ p < 0. \tag{2.29}$$

By using (2.28), (2.29) and cases (2.6)-(2.10) respectively, we get

$$\Psi_1(x,t) = -\frac{(1 + \tanh(\sqrt{-p}(\kappa x + \omega t)))L}{3K},$$
$$\Psi_2(x,t) = -\frac{(1 + \coth(\sqrt{-p}(\kappa x + \omega t)))L}{3K},$$
$$\Psi_3(x,t) = -\frac{[1 + \tanh(2\sqrt{-p}(\kappa x + \omega t)) \mp isech(2\sqrt{-p}(\kappa x + \omega t))]L}{3K},$$
$$\Psi_4(x,t) = -\frac{[1 + \coth(2\sqrt{-p}(\kappa x + \omega t)) \mp csch(2\sqrt{-p}(\kappa x + \omega t))]L}{3K},$$
$$\Psi_5(x,t) = \frac{[1 + \frac{1}{2}(\tanh(\frac{\sqrt{-p}}{2}(\kappa x + \omega t)) + \coth(\frac{\sqrt{-p}}{2}(\kappa x + \omega t)))]L}{3K}.$$

Type 2: $\eta_1 = 2$ and $\eta_2 = 1$,

$$\Psi(\xi) = \frac{A_0 + A_1\Theta(\xi) + A_2\Theta^2(\xi)}{B_0 + B_1\Theta(\xi)}, \tag{2.30}$$

Substituting Eq. (2.30) along Eq. (2.5) into Eq. (2.26), same as before, we obtain

- *Set* 1 : $A_0 = A_0,\ A_1 = \frac{2\sqrt{5}}{3\sqrt{-2p}}A_0,\ A_2 = -\frac{1}{3p}A_0,\ B_0 = -\frac{12K}{5L}A_0,$

$$B_1 = -\frac{6\sqrt{5}K}{5\sqrt{-2p}}A_0,\ M = -16p(\omega^2 - \kappa^2 c_1^2),\ K = \frac{2L^2}{9M},\ p < 0. \tag{2.31}$$

By using (2.30), (2.31) and cases (2.6)-(2.10) respectively, we get

$$\Psi_1(x,t) = \frac{1 - [\frac{\sqrt{10}}{3} - \frac{1}{3}\tanh(\sqrt{-p}(\kappa x + \omega t))]\tanh(\sqrt{-p}(\kappa x + \omega t))}{\frac{12K}{5L} + \frac{6}{\sqrt{10}}\tanh(\sqrt{-p}(\kappa x + \omega t))},$$

$$\Psi_2(x,t) = \frac{1 - [\frac{\sqrt{10}}{3} - \frac{1}{3}\coth(\sqrt{-p}(\kappa x + \omega t))]\coth(\sqrt{-p}(\kappa x + \omega t))}{\frac{12K}{5L} + \frac{6}{\sqrt{10}}\coth(\sqrt{-p}(\kappa x + \omega t))},$$

$$\begin{aligned}\Psi_3(x,t) = &\left(\frac{[-\frac{\sqrt{10}}{3} + \frac{1}{3}(-\tanh(2\sqrt{-p}(\kappa x + \omega t)) \pm i\,sech(2\sqrt{-p}(\kappa x + \omega t)))]}{\frac{12K}{5L} + \frac{6}{\sqrt{10}}(-\tanh(2\sqrt{-p}(\kappa x + \omega t)) \pm i\,sech(2\sqrt{-p}(\kappa x + \omega t)))}\right)\\ &\times(-\tanh(2\sqrt{-p}(\kappa x + \omega t)) \pm i\,sech(2\sqrt{-p}(\kappa x + \omega t)))\\ &+ \frac{1}{\frac{12K}{5L} + \frac{6}{\sqrt{10}}(-\tanh(2\sqrt{-p}(\kappa x + \omega t)) \pm i\,sech(2\sqrt{-p}(\kappa x + \omega t)))},\end{aligned}$$

$$\begin{aligned}\Psi_4(x,t) = &\left(\frac{[-\frac{\sqrt{10}}{3} + \frac{1}{3}(-\coth(2\sqrt{-p}(\kappa x + \omega t)) \pm csch(2\sqrt{-p}(\kappa x + \omega t)))]}{\frac{12K}{5L} + \frac{6}{\sqrt{10}}(-\coth(2\sqrt{-p}(\kappa x + \omega t)) \pm csch(2\sqrt{-p}(\kappa x + \omega t)))}\right)\\ &\times(-\coth(2\sqrt{-p}(\kappa x + \omega t)) \pm csch(2\sqrt{-p}(\kappa x + \omega t)))\\ &+ \frac{1}{\frac{12K}{5L} + \frac{6}{\sqrt{10}}(-\coth(2\sqrt{-p}(\kappa x + \omega t)) \pm csch(2\sqrt{-p}(\kappa x + \omega t)))},\end{aligned}$$

$$\begin{aligned}\Psi_5(x,t) = &\left(\frac{[\frac{\sqrt{10}}{3} + \frac{1}{6}(\tanh(\frac{\sqrt{-p}}{2}(\kappa x + \omega t)) + \coth(\frac{\sqrt{-p}}{2}(\kappa x + \omega t)))]}{\frac{24K}{5L} + \frac{12}{\sqrt{10}}(\tanh(\frac{\sqrt{-p}}{2}(\kappa x + \omega t)) + \coth(\frac{\sqrt{-p}}{2}(\kappa x + \omega t)))}\right)\\ &\times(\tanh(\frac{\sqrt{-p}}{2}(\kappa x + \omega t)) + \coth(\frac{\sqrt{-p}}{2}(\kappa x + \omega t)))\\ &+ \frac{1}{\frac{12K}{5L} + \frac{6}{\sqrt{10}}(\tanh(\frac{\sqrt{-p}}{2}(\kappa x + \omega t)) + \coth(\frac{\sqrt{-p}}{2}(\kappa x + \omega t)))}.\end{aligned}$$

• *Set* 2 : $A_0 = A_0,\ A_1 = 0,\ A_2 = \frac{1}{p}A_0,\ B_0 = -\frac{2L}{3M}A_0,\ B_1 = \frac{\sqrt{-6KL}}{\sqrt{pM}}A_0,$

$$M = -4p(\omega^2 - \kappa^2 c_1^2),\ K = K,\ p > 0. \tag{2.32}$$

By using (2.30), (2.32) and cases (2.11)-(2.15) respectively, we get

$$\Psi_6(x,t) = \frac{1 + \tan^2(\sqrt{p}(\kappa x + \omega t))}{-\frac{2L}{3M} + \frac{\sqrt{-6KL}}{\sqrt{M}}\tan(\sqrt{p}(\kappa x + \omega t))},$$

$$\Psi_7(x,t)=\frac{1+\cot^2(\sqrt{p}(\kappa x+\omega t))}{-\frac{2L}{3M}-\frac{\sqrt{-6KL}}{\sqrt{M}}\cot(\sqrt{p}(\kappa x+\omega t))},$$

$$\Psi_8(x,t)=\frac{1+(-\tan(2\sqrt{p}(\kappa x+\omega t))\pm\sec(2\sqrt{p}(\kappa x+\omega t)))^2}{-\frac{2L}{3M}+\frac{\sqrt{-6KL}}{\sqrt{M}}(-\tan(2\sqrt{p}(\kappa x+\omega t))\pm\sec(2\sqrt{p}(\kappa x+\omega t)))},$$

$$\Psi_9(x,t)=\frac{1+(-\cot(2\sqrt{p}(\kappa x+\omega t))\pm\csc(2\sqrt{p}(\kappa x+\omega t)))^2}{-\frac{2L}{3M}+\frac{\sqrt{-6KL}}{\sqrt{M}}(-\cot(2\sqrt{p}(\kappa x+\omega t))\pm\csc(2\sqrt{p}(\kappa x+\omega t)))},$$

$$\Psi_{10}(x,t)=\frac{1+\frac{1}{4}(\tan(\frac{\sqrt{p}}{2}(\kappa x+\omega t))-\cot(\frac{\sqrt{p}}{2}(\kappa x+\omega t)))^2}{-\frac{2L}{3M}+\frac{\sqrt{-6KL}}{2\sqrt{M}}(\tan(\frac{\sqrt{p}}{2}(\kappa x+\omega t))-\cot(\frac{\sqrt{p}}{2}(\kappa x+\omega t)))}.$$

$$\bullet\, Set\,3:\quad A_0=A_0,\ A_1=0,\ A_2=\frac{1}{p}A_0,\ B_0=-\frac{3K}{L}A_0,\ B_1=\frac{\sqrt{3K}}{\sqrt{-pL}}A_0,$$
$$M=-4p(\omega^2-\kappa^2c_1^2),\ K=\frac{2L^2}{9M},\ p<0. \qquad (2.33)$$

By using (2.30), (2.33) and cases (2.6)-(2.10) respectively, we get

$$\Psi_{11}(x,t)=\frac{1-\tanh^2(\sqrt{-p}(\kappa x+\omega t))}{-\frac{3K}{L}-\frac{3K}{\sqrt{L}}\tanh(\sqrt{-p}(\kappa x+\omega t))},$$

$$\Psi_{12}(x,t)=\frac{1-\coth^2(\sqrt{-p}(\kappa x+\omega t))}{-\frac{3K}{L}-\frac{3K}{\sqrt{L}}\coth(\sqrt{-p}(\kappa x+\omega t))},$$

$$\Psi_{13}(x,t)=\frac{1-(-\tanh(2\sqrt{-p}(\kappa x+\omega t))\pm i\,sech(2\sqrt{-p}(\kappa x+\omega t)))^2}{-\frac{3K}{L}-\frac{3K}{\sqrt{L}}(-\tanh(2\sqrt{-p}(\kappa x+\omega t))\pm i\,sech(2\sqrt{-p}(\kappa x+\omega t)))},$$

$$\Psi_{14}(x,t)=\frac{1-(-\coth(2\sqrt{-p}(\kappa x+\omega t))\pm csch(2\sqrt{-p}(\kappa x+\omega t)))^2}{-\frac{3K}{L}-\frac{3K}{\sqrt{L}}(-\coth(2\sqrt{-p}(\kappa x+\omega t))\pm csch(2\sqrt{-p}(\kappa x+\omega t)))},$$

$$\Psi_{15}(x,t)=\frac{1-\frac{1}{4}(\tanh(\frac{\sqrt{-p}}{2}(\kappa x+\omega t))+\coth(\frac{\sqrt{-p}}{2}(\kappa x+\omega t)))^2}{-\frac{3K}{L}-\frac{3K}{2\sqrt{L}}(\tanh(\frac{\sqrt{-p}}{2}(\kappa x+\omega t))+\coth(\frac{\sqrt{-p}}{2}(\kappa x+\omega t)))}.$$

$$\bullet\, Set\,4:\quad A_0=A_0,\ A_1=\frac{3\sqrt{2(\omega^2-\kappa^2c_1^2)}}{\sqrt{M}}A_0,\ A_2=\frac{6(\omega^2-\kappa^2c_1^2)}{M}A_0,$$
$$B_0=-\frac{2K}{L}A_0,\ B_1=-\frac{6K\sqrt{2(\omega^2-\kappa^2c_1^2)}}{L\sqrt{M}}A_0,\ K=\frac{L^2}{4M},\ p=0. \qquad (2.34)$$

By using (2.30), (2.34) and case (2.16), we get

$$\Psi_{16}(x,t)=\frac{1+\left(3\sqrt{2}+\frac{6\sqrt{(\omega^2-\kappa^2c_1^2)}}{\sqrt{M}(\kappa x+\omega t+d)}\right)\frac{\sqrt{(\omega^2-\kappa^2c_1^2)}}{\sqrt{M}(\kappa x+\omega t+d)}}{-\frac{2K}{L}-\frac{6\sqrt{2}K}{L}\frac{\sqrt{(\omega^2-\kappa^2c_1^2)}}{\sqrt{M}(\kappa x+\omega t+d)}}.$$

3. The second algebraic method

3.1. *Description of extended $\frac{G'}{G^2}$-expansion method*

In this section, we employ the $\frac{G'}{G^2}$-expansion method to obtain the exact solutions of NLPDEs.

For the following NLPDE

$$R(\phi,\phi_x,\phi_t,\phi_{xx},\phi_{xt},\phi_{tt},\ldots)=0, \tag{3.1}$$

where $\phi=\phi(x,t)$ is an unknown function in two variables x and t. Substituting the following transformation

$$\xi=\kappa x+\omega t, \tag{3.2}$$

into (3.1), it can be transformed to the following ODE

$$\tilde{R}(\Phi,\Phi',\Phi'',\Phi''',\ldots)=0. \tag{3.3}$$

Here $\Phi^{(n)}=\frac{d^n\Phi}{d\xi^n}$. We assume the exact solution of Eq. (3.3) as follow

$$\Phi(\xi)=\sum_{i=-m}^{m}A_i\left(\frac{G'(\xi)}{G^2(\xi)}\right)^i, \tag{3.4}$$

where $A_i\,(A_m\neq 0)$ are constants and $G(\xi)$ satisfies the following ODE:

$$G''(\xi)\,G^2(\xi)-2G(\xi)\,G'^2(\xi)=p\,G^4(\xi)+q\,G'(\xi)\,G^2(\xi)+r\,G'^2(\xi). \tag{3.5}$$

We know the Eq. (3.5) has the following special solutions:

Case 1. If $pr>0$ and $q=0$,

$$\frac{G'}{G^2}=\frac{\sqrt{pr}}{p}\left(\frac{C_1\cos\sqrt{pr}\xi+C_2\sin\sqrt{pr}\xi}{C_1\sin\sqrt{pr}\xi-C_2\cos\sqrt{pr}\xi}\right). \tag{3.6}$$

Case 2. If $pr<0$ and $q=0$,

$$\frac{G'}{G^2}=-\frac{\sqrt{-pr}}{p}\left(\frac{C_1\sinh 2\sqrt{-pr}\xi+C_2\cosh 2\sqrt{-pr}\xi+C_2}{C_1\cosh 2\sqrt{-pr}\xi+C_2\sinh 2\sqrt{-pr}\xi-C_2}\right). \tag{3.7}$$

Case 3. If $p=q=0$ and $r\neq 0$,

$$\frac{G'}{G^2}=-\frac{C_1}{r(C_1\xi+C_2)}. \tag{3.8}$$

Case 4. If $q \neq 0$ and $q^2 - 4pr \geq 0$,

$$\frac{G'}{G^2} = -\frac{q}{2r} - \left(\frac{\sqrt{q^2-4pr}(C_1 \sinh(\frac{\sqrt{q^2-4pr}}{2})\xi + C_2 \cosh(\frac{\sqrt{q^2-4pr}}{2})\xi)}{2r(C_1 \cosh(\frac{\sqrt{q^2-4pr}}{2})\xi + C_2 \sinh(\frac{\sqrt{q^2-4pr}}{2})\xi)} \right). \quad (3.9)$$

Case 5. If $q \neq 0$ and $q^2 - 4pr < 0$,

$$\frac{G'}{G^2} = -\frac{q}{2r} - \left(\frac{\sqrt{4pr-q^2}(-C_1 \sinh(\frac{\sqrt{4pr-q^2}}{2})\xi + C_2 \cosh(\frac{\sqrt{4pr-q^2}}{2})\xi)}{2r(C_1 \cosh(\frac{\sqrt{4pr-q^2}}{2})\xi + C_2 \sinh(\frac{\sqrt{4pr-q^2}}{2})\xi)} \right). \quad (3.10)$$

Where C_1 and C_2 are arbitrary constants. The following steps can be used to obtain the exact solutions of NLPDEs.

First, substitute (3.4) into (3.3) using (3.5). Next, collect the terms with the same powers of $\frac{G'}{G^2}$ in the resulting polynomial. Then, set all the coefficients of this polynomial to zero to obtain a system of algebraic equations in terms of $A_j(-m \leq j \leq m)$, p, q and r. Finally, solve this system to obtain the solutions of (3.3).

3.2. *Application*

For the double-chain DNA system

$$\Omega_{tt} - c_1^2\Omega_{xx} = \lambda_1\Omega + \gamma_1\Omega\Gamma + \mu_1\Omega^3 + \beta_1\Omega\Gamma^2, \quad (3.11)$$

$$\Gamma_{tt} - c_2^2\Gamma_{xx} = \lambda_2\Gamma + \gamma_2\Omega^2 + \mu_2\Omega^2\Gamma + \beta_2\Gamma^3 + c_0, \quad (3.12)$$

similar to the first method, we can obtain

$$(\omega^2 - \kappa^2 c_1^2)\Psi'' - K\Psi^3 - L\Psi^2 - M\Psi = 0, \quad \omega^2 - \kappa^2 c_1^2 \neq 0. \quad (3.13)$$

Balancing Ψ'' with Ψ^3 in (3.13) gives n=1. Therefore, the exact solution of Eq. (3.13) can be written in the form:

$$\Psi(\xi) = A_0 + A_1\left(\frac{G'}{G^2}\right) + A_{-1}\left(\frac{G'}{G^2}\right)^{-1}, \quad A_1 \neq 0. \quad (3.14)$$

Therefore, we have

$$\bullet\, Set\,1: \quad A_0 = \frac{L}{3K}\left(\frac{q}{\sqrt{q^2-4pr}} - 1\right), A_1 = \frac{2Lr}{3Kq}\sqrt{\frac{4pr}{q^2-4pr}+1}, A_{-1} = 0,$$

$$M = (q^2-4pr)(\omega^2 - \kappa^2 c_1^2), K = \frac{2L^2}{9M}, q^2 - 4pr \neq o, q \neq o, r \neq o. \quad (3.15)$$

By using (3.15), (3.14) and case (3.9), we get the general hyperbolic function solutions

$$\Psi_1(x,t) = \frac{L}{3K}\left(\frac{q}{\sqrt{q^2-4pr}} - 1\right) + \frac{L}{3Kq}\sqrt{\frac{4pr}{q^2-4pr}+1}\Big[-q$$

$$-\left(\frac{\sqrt{q^2-4pr}\left[C_1\sinh(\frac{\sqrt{q^2-4pr}}{2}(\kappa x+\omega t))+C_2\cosh(\frac{\sqrt{q^2-4pr}}{2}(\kappa x+\omega t))\right]}{C_1\cosh(\frac{\sqrt{q^2-4pr}}{2}(\kappa x+\omega t))+C_2\sinh(\frac{\sqrt{q^2-4pr}}{2}(\kappa x+\omega t))}\right)\Bigg],$$

In particular, if we choose $C_2 = 0$, then this solution gives the solitary wave solution

$$\Psi_2(x,t) = \frac{L}{3K}\left(\frac{q}{\sqrt{q^2-4pr}}-1\right)+\frac{L}{3Kq}\sqrt{\frac{4pr}{q^2-4pr}+1}$$
$$\times\left[-q-\sqrt{q^2-4pr}\tanh\left(\frac{\sqrt{q^2-4pr}}{2}(\kappa x+\omega t)\right)\right].$$

By using (3.15), (3.14) and case (3.10), we get the general hyperbolic function solutions

$$\Psi_3(x,t) = \frac{L}{3K}\left(\frac{q}{\sqrt{4pr-q^2}}-1\right)+\frac{L}{3Kq}\sqrt{\frac{4pr}{4pr-q^2}+1}\Bigg[-q+$$
$$\left(\frac{\sqrt{4pr-q^2}\left[C_1\sinh\left(\frac{\sqrt{4pr-q^2}}{2}(\kappa x+\omega t)\right)-C_2\cosh\left(\frac{\sqrt{4pr-q^2}}{2}(\kappa x+\omega t)\right)\right]}{C_1\cosh(\frac{\sqrt{4pr-q^2}}{2}(\kappa x+\omega t))+C_2\sinh(\frac{\sqrt{4pr-q^2}}{2}(\kappa x+\omega t))}\right)\Bigg],$$

In particular, if we choose $C_2 = 0$, then this solution gives the solitary wave solution

$$\Psi_4(x,t) = \frac{L}{3K}\left(\frac{q}{\sqrt{4pr-q^2}}-1\right)+\frac{L}{3Kq}\sqrt{\frac{4pr}{4pr-q^2}+1}$$
$$\times\left[-q+\sqrt{4pr-q^2}\tanh\left(\frac{\sqrt{4pr-q^2}}{2}(\kappa x+\omega t)\right)\right],$$

$$\bullet\, Set\, 2: \quad A_0 = -\frac{L}{3K},\ A_1 = \frac{\sqrt{Mr}}{2\sqrt{Kp}},\ A_{-1} = \frac{\sqrt{Mp}}{2\sqrt{K}},$$
$$M = -4pr(\omega^2-\kappa^2c_1^2),\ K = \frac{2L^2}{9M},\ q = o,\ p \neq o,\ r \neq o. \tag{3.16}$$

By using (3.16), (3.14) and case (3.6), we get the general trigonometric function solutions

$$\Psi_5(x,t) = -\frac{L}{3K}+\frac{\sqrt{Mr}}{2\sqrt{Kp}}\left[\left(\frac{C_1\cos(\sqrt{pr}(\kappa x+\omega t))+C_2\sin(\sqrt{pr}(\kappa x+\omega t))}{C_1\sin(\sqrt{pr}(\kappa x+\omega t))-C_2\cos(\sqrt{pr}(\kappa x+\omega t))}\right)\right]$$
$$+\frac{\sqrt{Mp}}{2\sqrt{Kr}}\left[\left(\frac{C_1\sin(\sqrt{pr}(\kappa x+\omega t))-C_2\cos(\sqrt{pr}(\kappa x+\omega t))}{C_1\cos(\sqrt{pr}(\kappa x+\omega t))+C_2\sin(\sqrt{pr}(\kappa x+\omega t))}\right)\right].$$

In particular, if we choose $C_2 = 0$, then this solution gives the solitary wave solution

$$\Psi_6(x,t) = -\frac{L}{3K} + \frac{\sqrt{Mr}}{2\sqrt{Kp}}\cot(\sqrt{pr}(\kappa x+\omega t)) + \frac{\sqrt{Mp}}{2\sqrt{Kr}}\tan(\sqrt{pr}(\kappa x+\omega t)).$$

By using (3.16), (3.14) and case (3.7), we get the general trigonometric function solutions

$$\begin{aligned}\Psi_7(x,t) = &-\frac{L}{3K}\\ &-\frac{\sqrt{-Mr}}{2\sqrt{Kp}}\left(\frac{C_1\sinh(2\sqrt{-pr}(\kappa x+\omega t))+C_2\cosh(2\sqrt{-pr}(\kappa x+\omega t))+C_2}{C_1\cosh(2\sqrt{-pr}(\kappa x+\omega t))+C_2\sinh(2\sqrt{-pr}(\kappa x+\omega t))-C_2}\right)\\ &+\frac{\sqrt{-Mp}}{2\sqrt{Kr}}\left(\frac{C_1\cosh(2\sqrt{-pr}(\kappa x+\omega t))+C_2\sinh(2\sqrt{-pr}(\kappa x+\omega t))-C_2}{C_1\sinh(2\sqrt{-pr}(\kappa x+\omega t))+C_2\cosh(2\sqrt{-pr}(\kappa x+\omega t))+C_2}\right).\end{aligned}$$

In particular, if we choose $C_2 = 0$, then this solution gives the solitary wave solution

$$\Psi_8(x,t) = -\frac{L}{3K} - \frac{\sqrt{-Mr}}{2\sqrt{Kp}}\tanh(2\sqrt{-pr}(\kappa x+\omega t)) + \frac{\sqrt{-Mp}}{2\sqrt{Kr}}\coth(2\sqrt{-pr}(\kappa x+\omega t)).$$

$$\begin{aligned}\bullet\, Set\, 3: \quad & A_0 = \frac{2L}{7K},\ A_1 = \frac{26Lr}{21Kq},\ A_{-1} = \frac{26Lp}{11Kq},\\ & M = -16pr(\omega^2-\kappa^2c_1^2),\ K = \frac{2L^2}{9M},\ q\neq o,\ p\neq o,\ r\neq o.\end{aligned} \tag{3.17}$$

By using (3.17), (3.14) and case (3.9), we get the general hyperbolic function solutions

$$\begin{aligned}\Psi_9(x,t) = &\frac{2L}{7K} - \frac{13L}{21Kq}\left(q + \frac{\sqrt{Z}[C_1\sinh(\frac{\sqrt{Z}}{2}(\kappa x+\omega t))+C_2\cosh(\frac{\sqrt{Z}}{2}(\kappa x+\omega t))]}{C_1\cosh(\frac{\sqrt{Z}}{2}(\kappa x+\omega t))+C_2\sinh(\frac{\sqrt{Z}}{2}(\kappa x+\omega t))}\right)\\ &-\frac{26Lp}{11Kq}\left(\frac{q}{2r} + \frac{\sqrt{Z}[C_1\sinh(\frac{\sqrt{Z}}{2}(\kappa x+\omega t))+C_2\cosh(\frac{\sqrt{Z}}{2}(\kappa x+\omega t))]}{2r(C_1\cosh(\frac{\sqrt{Z}}{2}(\kappa x+\omega t))+C_2\sinh(\frac{\sqrt{Z}}{2}(\kappa x+\omega t)))}\right)^{-1},\end{aligned}$$

In particular, if we choose $C_2 = 0$, then this solution gives the solitary wave solution

$$\begin{aligned}\Psi_{10}(x,t) = &\frac{2L}{7K} - \frac{13L}{21Kq}\left(q+\sqrt{Z}\tanh(\frac{\sqrt{Z}}{2}(\kappa x+\omega t))\right)\\ &-\frac{26Lp}{11Kq}\left(\frac{q}{2r}+\frac{\sqrt{Z}}{2r}\tanh(\frac{\sqrt{Z}}{2}(\kappa x+\omega t))\right)^{-1}.\end{aligned}$$

Where $Z = q^2 - 4pr$.

By using (3.17), (3.14) and case (3.10), we get the general hyperbolic function solutions

$$\Psi_{11}(x,t)=\frac{2L}{7K}-\frac{13L}{21Kq}\left(q-\frac{\sqrt{-Z}[C_1\sinh(\frac{\sqrt{-Z}}{2}(\kappa x+\omega t))-C_2\cosh(\frac{\sqrt{-Z}}{2}(\kappa x+\omega t))]}{C_1\cosh(\frac{\sqrt{-Z}}{2}(\kappa x+\omega t))+C_2\sinh(\frac{\sqrt{-Z}}{2}(\kappa x+\omega t))}\right)$$
$$-\frac{26Lp}{11Kq}\left(\frac{q}{2r}-\frac{\sqrt{-Z}[C_1\sinh(\frac{\sqrt{-Z}}{2}(\kappa x+\omega t))-C_2\cosh(\frac{\sqrt{Z}}{2}(\kappa x+\omega t))]}{2r(C_1\cosh(\frac{\sqrt{-Z}}{2}(\kappa x+\omega t))+C_2\sinh(\frac{\sqrt{-Z}}{2}(\kappa x+\omega t)))}\right)^{-1},$$

In particular, if we choose $C_2=0$, then this solution gives the solitary wave solution

$$\Psi_{12}(x,t)=\frac{2L}{7K}-\frac{13L}{21Kq}\left(q-\sqrt{-Z}\tanh(\frac{\sqrt{-Z}}{2}(\kappa x+\omega t))\right)$$
$$-\frac{26Lp}{11Kq}\left(\frac{q}{2r}-\frac{\sqrt{-Z}}{2r}\tanh(\frac{\sqrt{-Z}}{2}(\kappa x+\omega t))\right)^{-1}.$$

Where $Z=q^2-4pr$.

4. Graphical presentments of some solutions

In figures 1-4 and 5-7, we plot 2D and 3D graphics of some obtained solutions to Eq. (2.23) in the first and second methods respectively, which denote the dynamics of solutions with appropriate parameters selection.

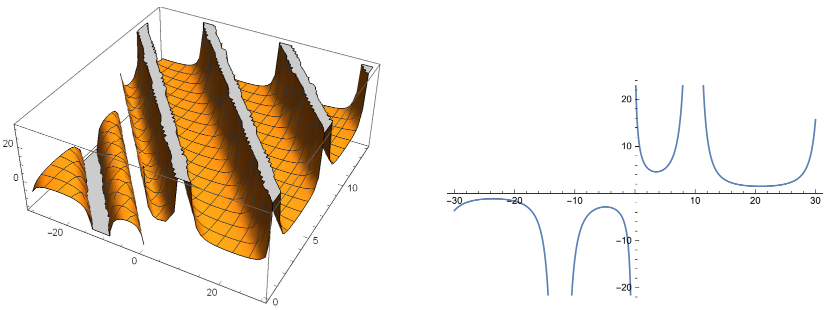

FIGURE 1. The 3D and 2D surfaces of $\Psi_1(x,t)$ (Set 1) for the values $p=-0.5$, $\kappa=0.2$, $\omega=0.7$, $K=-0.05$, $L=-3.9$, and $t=0.4$.

5. Concluding remarks

The purpose of this paper is to study the double-chain DNA. A travelling wave transformation has been utilized on this model to convert it into an ordinary differential equation. In this paper, two algebraic methods were successfully used to study the double-chain DNA system. The exact solutions obtained include the trigonometric function solutions, rational solutions and hyperbolic function solutions. This

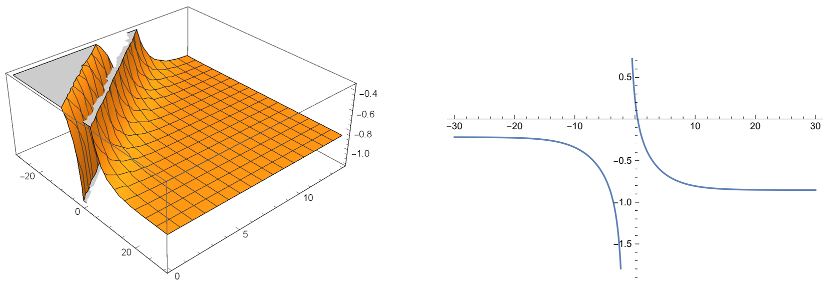

FIGURE 2. The 3D and 2D surfaces of $\Psi_4(x,t)$ (Set 1) for the values $p=-0.5$, $\kappa=0.2$, $\omega=0.7$, $K=-0.05$, $L=-3.9$, and $t=0.4$.

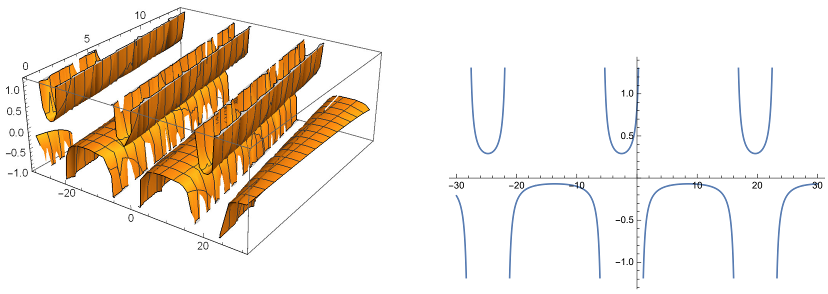

FIGURE 3. The 3D and 2D surfaces of $\Psi_8(x,t)$ (Set 2) for the values $p=0.5$, $\kappa=0.2$, $\omega=0.1$, $K=-2$, $L=1$, $M=0.06$ and $t=0.4$.

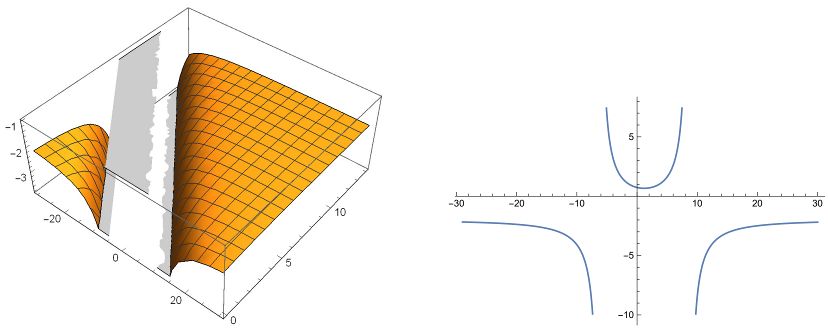

FIGURE 4. The 3D and 2D surfaces of $\Psi_{16}(x,t)$ (Set 4) for the values $p=0$, $\kappa=0.2$, $\omega=0.7$, $K=0.25$, $L=1$, $M=1$, $c_1=0.2$, $d=-2$ and $t=0.4$.

research illustrates the high effectiveness and practical utility of these methods in obtaining exact solutions for various types of nonlinear differential equations. We used Mathematica for computations.

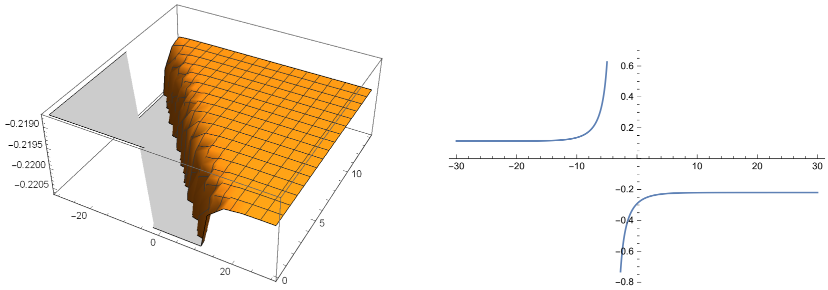

FIGURE 5. The 3D and 2D surfaces of $\Psi_1(x,t)$ (Set 1) for the values $p=-0.5$, $q=3$, $r=-2$, $\kappa=0.2$, $\omega=0.7$, $K=2$, $L=1$, $M=1$, $C_1=1$, $C_2=2$ and $t=0.4$.

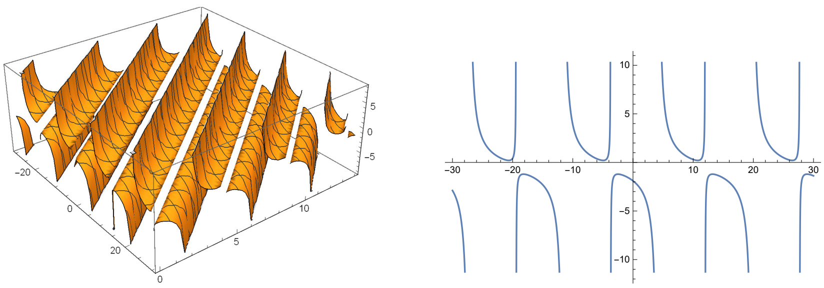

FIGURE 6. The 3D and 2D surfaces of $\Psi_5(x,t)$ (Set 2) for the values $p=0.5$, $q=0$, $r=2$, $\kappa=0.2$, $\omega=0.7$, $K=2$, $L=1$, $M=1$, $C_1=-2$, $C_2=1$ and $t=0.4$.

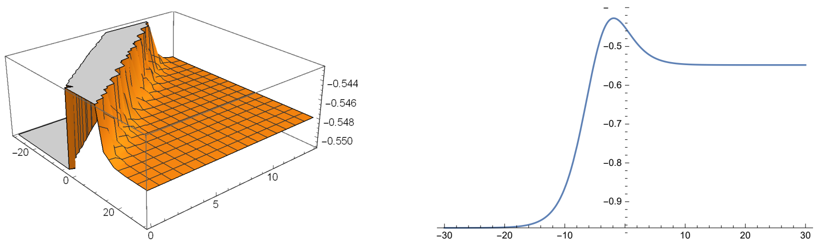

FIGURE 7. The 3D and 2D surfaces of $\Psi_{10}(x,t)$ (Set 3) for the values $p=-0.5$, $q=3$, $r=-2$, $\kappa=0.2$, $\omega=0.7$, $K=2$, $L=1$, $M=1$, $C_1=1$, $C_2=0$ and $t=0.4$.

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SOME GENERALIZATIONS OF MERCER INEQUALITY AND ITS OPERATOR EXTENSIONS

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Abstract. We give a more general form of the Mercer inequality by replacing some constants by positive operators. As some consequences, our results produce a Jensen operator inequality for superquadratic functions. Moreover, we present some Mercer inequalities of Hermite–Hadamard's type.

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Keywords: Mercer inequality, Hermite–Hadamard inequality, superquadratic function, positive operator

1. INTRODUCTION

Mercer [11] proved a variant of the Jensen inequality for convex functions as follows: If $f\colon [m,M] \to \mathbb{R}$ is a convex function, then

$$f\left(M+m-\sum_{j=1}^{n}\lambda_j x_j\right) \le f(M)+f(m)-\sum_{j=1}^{n}\lambda_j f(x_j) \tag{1.1}$$

for all $x_j \in [m,M]$ and all $\lambda_j \in [0,1]$ with $\sum_{j=1}^{n}\lambda_j = 1$. An operator extension of (1.1) has been presented in [10]:

$$f\left(M+m-\sum_{j=1}^{n}\Phi_j(A_j)\right) \le f(M)+f(m)-\sum_{j=1}^{n}\Phi_j(f(A_j)) \tag{1.2}$$

in which $(\Phi_1,\ldots,\Phi_n)$ is a tuple of positive linear maps on $\mathcal{B}(\mathcal{H})$ with $\sum_{i=1}^{n}\Phi_i(I) = I$ and A_j's are self-adjoint operators with spectra in $[m,M]$. Here we denote by $\mathcal{B}(\mathcal{H})$ the algebra of all bounded linear operators on a Hilbert space $\mathcal{H}$ and I is the identity operator. When $f\colon [m,M] \to \mathbb{R}$ is a continuous function and A is a self-adjoint operator with spectrum in $[m,M]$, the operator $f(A)$ is defined by the well-known Gelfand's mapping. This is called the continuous functional calculus, see [4].

Utilising the famous Hermite–Hadamard inequality, the first author and Moslehian [7] presented a variant of the operator Mercer inequality (1.2). Some reverse Mercer operator inequalities have been given in [2]. The authors of [9] introduced logarithmic superquadratic functions. In [12], some sub-additivity inequalities for this class of functions have been presented.

In [14], Moslehian, Mićić, and the first author extended the operator Mercer inequality (1.2) by replacing the scalars m and M by operators and showed that with some conditions on the spectra of operators, the inequality

$$f(\Phi(C))+f(\Phi(B))\leq\Phi(f(A))+\Phi(f(D)) \tag{1.3}$$

holds.

Superquadratic functions are introduced in [1] as modifications of convex functions. Since then, this class of functions has been utilised to improve many results concerning convex functions. A function $f\colon[0,\infty)\to\mathbb{R}$ is called superquadratic if for all $x\geq 0$ there exists a constant $C_x\in\mathbb{R}$ such that

$$f(y)\geq f(x)+C_x(y-x)+f(|y-x|)$$

for all $y\geq 0$. These functions enjoy a Jensen type inequality as

$$\begin{aligned}&f(\lambda x+(1-\lambda)y)\\&\qquad\leq\lambda f(x)+(1-\lambda)f(y)-\lambda f((1-\lambda)|x-y|)-(1-\lambda)f(\lambda|x-y|)\end{aligned} \tag{1.4}$$

for all $x,y\geq 0$ and $\lambda\in[0,1]$, see [1, 9].

The authors of [3] proved a variant of the operator Mercer inequality (1.2) for superquadratic functions: If $f\colon[0,\infty)\to\mathbb{R}$ is a continuous superquadratic function and m,M are positive scalars, then

$$f(M+m-\Phi(A))-\beta(\Phi(A))\leq f(m)+f(M)-\Phi(f(A))-\Phi(\beta(A)) \tag{1.5}$$

holds for every positive operator $A\in\mathcal{B}(\mathcal{H})$ with spectrum in $[m,M]$ and every positive linear map Φ on $\mathcal{B}(\mathcal{H})$, when we set the notation

$$\beta(t)=\frac{t-m}{M-m}f(M-t)+\frac{M-t}{M-m}f(t-m),\qquad(t\in[m,M]). \tag{1.6}$$

Recall that an operator $A\in\mathcal{B}(\mathcal{H})$ is called positive if $\langle A\eta,\eta\rangle\geq 0$ for every $\eta\in\mathcal{H}$.

The importance of (1.2), (1.3) and (1.5) is that they are available without the restrictive condition of being operator convex for the function f. Recall that a function $f\colon[m,M]\subseteq\mathbb{R}\to\mathbb{R}$ is *operator convex*, when

$$f(\lambda A+(1-\lambda)B)\leq\lambda f(A)+(1-\lambda)f(B)$$

holds for all self-adjoint operators A and B with spectra in $[m,M]$ and every $\lambda\in[0,1]$. It is known that if $f\colon[m,M]\to\mathbb{R}$ is operator convex, then the Jensen operator inequality

$$f(\Phi(A))\leq\Phi(f(A)) \tag{1.7}$$

holds for every self-adjoint operator $A \in \mathcal{B}(\mathcal{H})$ with spectrum in $[m,M]$ and every unital positive linear map Φ on $\mathcal{B}(\mathcal{H})$. However, if f is convex, but not operator convex, then (1.7) does not hold in general. However, (1.2) is valid for every convex function ((1.5) is valid for every superquadratic function).

In this paper, we study the Mercer inequality and its operator extension for superquadratic functions. In particular, we extend (1.5) by replacing scalars m,M by operators. As applications, a Jensen operator inequality has been presented for superquadratic functions. Moreover, we present a Mercer inequality of Hemite–Hadamard's type. The Hemite–Hadamard inequality asserts that

$$f\left(\frac{x+y}{2}\right) \leq \frac{1}{y-x}\int_x^y f(t)\mathrm{d}t \leq \frac{f(x)+f(y)}{2}$$

holds for every convex function f on $[x,y]$. The reader can refer to [5, 6, 13] for operator versions of this inequality.

2. Results

We begin by presenting a Mercer inequality of Hemite–Hadamard's type for superquadratic functions. We need the following lemma. For simplicity, we use the notation $x\nabla_\lambda y$ for the λ-weighted arithmetic mean $\lambda x+(1-\lambda)y$ of x and y.

Lemma 1. *Let $0 \leq m < M$ and let $f\colon [0,\infty) \to \mathbb{R}$ be a superquadratic function. Then*

$$\begin{aligned} &f(m+M-x\nabla_\lambda y)+2\beta(x)\nabla_\lambda\beta(y) \\ &\quad \leqslant f(m)+f(M)-f(x)\nabla_\lambda f(y)-f((1-\lambda)|x-y|)\nabla_\lambda f(\lambda|x-y|)) \end{aligned}$$

for all $x,y \in [m,M]$ and every $\lambda \in [0,1]$.

Proof. For any $x \in [m,M]$, we put $y = m+M-x$ so that $y+x = m+M$. There exists $\lambda \in [0,1]$ such that $y = \lambda m+(1-\lambda)M$. Since f is superquadratic, we cal apply (1.4) to write

$$\begin{aligned} f(M+m-x) &= f(\lambda m+(1-\lambda)M) \\ &\leqslant \lambda f(m)+(1-\lambda)f(M)-\lambda f((1-\lambda)|m-M|)-(1-\lambda)f(\lambda|m-M|) \\ &= f(m)+f(M) \qquad (2.1) \\ &\quad -(\lambda f(M)+(1-\lambda)f(m)+\lambda f((1-\lambda)|M-m|)+(1-\lambda)f(\lambda|M-m|)). \end{aligned}$$

On the other hand, $x = m+M-y = \lambda M+(1-\lambda)m$ and so we have

$$\begin{aligned} f(x) &= f(\lambda M+(1-\lambda)m) \\ &\leq \lambda f(M)+(1-\lambda)f(m)-\lambda f((1-\lambda)|M-m|)-(1-\lambda)f(\lambda|M-m|). \quad (2.2) \end{aligned}$$

It follows from (2.1) and (2.2) that

$$f(M+m-x)$$

$$\leq f(m)+f(M)-f(x)-2\left(\lambda f((1-\lambda)|M-m|)+(1-\lambda)f(\lambda|M-m|)\right). \quad (2.3)$$

Applying (2.3) with $\lambda = \dfrac{x-m}{M-m}$ we obtain

$$f(M+m-x) \leqslant f(m)+f(M)-f(x)-2\beta(x), \quad (2.4)$$

in which β is defined as (1.6). Now, for every $x,y \in [m,M]$ and every $\lambda \in [0,1]$, using (1.4) we have

$$\begin{aligned} f(M+m-(\lambda x+(1-\lambda)y)) &= f(\lambda(m+M-x)+(1-\lambda)(m+M-y)) \\ &\leq \lambda f(m+M-x)+(1-\lambda)f(m+M-y) \\ &\quad -\lambda f((1-\lambda)|x-y|)-(1-\lambda)f(\lambda|x-y|) \end{aligned} \quad (2.5)$$

as f is superquadratic. Now, by applying (2.4) we get

$$\begin{aligned} &\lambda f(m+M-x)+(1-\lambda)f(m+M-y) \\ &\quad \leqslant \lambda(f(m)+f(M)-f(x)-2\beta(x))+(1-\lambda)(f(m)+f(M)-f(y)-2\beta(y)) \\ &\quad = f(m)+f(M)-(\lambda f(x)+(1-\lambda)f(y))-2(\lambda\beta(x)+(1-\lambda)\beta(y)). \end{aligned} \quad (2.6)$$

It follows from (2.5), (2.6) that

$$\begin{aligned} &f(m+M-(\lambda x+(1-\lambda)y) \\ &\quad \leq f(m)+f(M)-(\lambda f(x)+(1-\lambda)f(y))-2(\lambda\beta(x)+(1-\lambda)\beta(y)) \\ &\qquad -(\lambda f((1-\lambda)|x-y|)+(1-\lambda)f(\lambda|x-y|)), \end{aligned}$$

as required. □

In the next result, we present a Mercer inequality of Hermite–Hadamard's type for superquadratic functions. The reader may compare it to [7, Theorem 2.1]. Also refer to the paper [12].

Theorem 1. *Let $0 \leq m < M$ and let f be a superquadratic function on $[m,M]$. Then*

$$\begin{aligned} &f\left(m+M-\frac{x+y}{2}\right)+2\int_0^{1/2} f(u|x-y|)\,\mathrm{d}u \\ &\quad \leq \frac{1}{y-x}\int_x^y f(m+M-u)\mathrm{d}u \\ &\quad \leq f(m)+f(M)-\frac{f(x)+f(y)}{2}-(\beta(x)+\beta(y))-2\int_0^1 (1-u)f(u|x-y|)\mathrm{d}u, \end{aligned} \quad (2.7)$$

and

$$\begin{aligned}
&f\left(m+M-\frac{x+y}{2}\right)+2\int_0^{1/2} f(u|x-y|)\mathrm{d}u \\
&\quad\le f(m)+f(M)-\frac{1}{y-x}\int_x^y (f(u)+2\beta(u))\mathrm{d}u \qquad (2.8)\\
&\quad\le f(m)+f(M)-f\left(\frac{x+y}{2}\right)-\frac{2}{y-x}\int_x^y \beta(u)\mathrm{d}u-2\int_0^{1/2} f(u|x-y|)\mathrm{d}u
\end{aligned}$$

for all $x,y\in[m,M]$.

Proof. Assume that $x,y\in[m,M]$ and put $a=M+m-x$ and $b=m+M-y$. Without loss of generality we assume that $x<y$. It follows from (1.4) that

$$\begin{aligned}
f\left(m+M-\frac{x+y}{2}\right)&=f\left(\frac{(ta+(1-t)b)+((1-t)a+tb)}{2}\right)\\
&\le\frac{f(ta+(1-t)b)+f((1-t)a+tb)}{2}-f\left(\left|\frac{2t-1}{2}\right||a-b|\right),
\end{aligned}$$

since f is superquadratic. Integrating both sides of the above inequality with respect to t over $[0,1]$ yields

$$\begin{aligned}
f\left(m+M-\frac{x+y}{2}\right)&\le\int_0^1 f(m+M-((1-t)x+ty))\mathrm{d}t-\int_0^1 f\left(\left|\frac{2t-1}{2}\right||a-b|\right)\mathrm{d}t\\
&=\frac{1}{y-x}\int_x^y f(m+M-u)\mathrm{d}u-2\int_0^{1/2} f\left(u|x-y|\right)\mathrm{d}u, \qquad (2.9)
\end{aligned}$$

where in the las equality, we employ change of variables in both integrals. On the other hand, it follows from Lemma 1 that

$$\begin{aligned}
&f(m+M-(tx+(1-t)y))\\
&\quad\le f(m)+f(M)-(tf(x)+(1-t)f(y)) \qquad (2.10)\\
&\qquad-2(t\beta(x)+(1-t)\beta(y))-(tf((1-t)|x-y|)+(1-t)f(t|x-y|)).
\end{aligned}$$

Noting that

$$\int_0^1 tf((1-t)|x-y|)\mathrm{d}t=\int_0^1(1-t)f(t|x-y|)\mathrm{d}t$$

and integrating both sides of (2.10) with respect to t over $[0,1]$ we get

$$\begin{aligned}
&\int_0^1 f(m+M-(tx+(1-t)y))\mathrm{d}t \qquad (2.11)\\
&\quad\le f(m)+f(M)-\frac{f(x)+f(y)}{2}-(\beta(x)+\beta(y))-2\int_0^1(1-t)f(t|x-y|)\mathrm{d}t.
\end{aligned}$$

Combining (2.9) and (2.11), we reach (2.7).

Next, it follows from Lemma 1 that

$$f\left(m+M-\frac{a+b}{2}\right) \leqslant f(m)+f(M)-\frac{f(a)+f(b)}{2}-(\beta(a)+\beta(b))-f\left(\left|\frac{a-b}{2}\right|\right) \quad (2.12)$$

holds for all $a,b\in[m,M]$. Let $t\in[0,1]$ and $x,y\in[m,M]$. Replacing a and b, respectively, by $tx+(1-t)y$ and $(1-t)x+ty$ in (2.12), we obtain

$$f\left(m+M-\frac{tx+(1-t)y+(1-t)x+ty}{2}\right) \leqslant f(m)+f(M)-\frac{f(tx+(1-t)y)+f((1-t)x+ty)}{2} - (\beta(tx+(1-t)y)+\beta((1-t)x+ty))-f\left(\left|\frac{tx+(1-t)y-(1-t)x+ty}{2}\right|\right),$$

or equivalently, we get

$$f\left(m+M-\frac{x+y}{2}\right) \leqslant f(m)+f(M)-\frac{f(tx+(1-t)y)+f((1-t)x+ty)}{2} - (\beta(tx+(1-t)y)+\beta((1-t)x+ty))-f\left(\frac{|1-2t||x-y|}{2}\right). \quad (2.13)$$

Note that

$$\int_0^1 f(tx+(1-t)y)\mathrm{d}t=\int_0^1 f((1-t)x+ty)\mathrm{d}t=\frac{1}{x-y}\int_x^y f(u)\mathrm{d}u,$$

and

$$\int_0^1 \beta(tx+(1-t)y)\mathrm{d}t=\int_0^1 \beta((1-t)x+ty)\mathrm{d}t=\frac{1}{x-y}\int_x^y \beta(u)\mathrm{d}u.$$

Consequently, the first inequality of (2.8) follows by integrating both sides of (2.13) over $t\in[0,1]$. To obtain the second inequality, we write

$$f\left(\frac{x+y}{2}\right)=f\left(\frac{(tx+(1-t)y)+((1-t)x+ty)}{2}\right) \leq \frac{f(tx+(1-t)y)+f((1-t)x+ty)}{2}-f\left(\left|\frac{1-2t}{2}\right||x-y|\right).$$

Integrating both sides with respect to t over $[0,1]$ we get

$$f\left(\frac{x+y}{2}\right)\leq\frac{1}{y-x}\int_x^y f(u)\,\mathrm{d}u-2\int_0^{1/2} f(u|x-y|)\mathrm{d}u.$$

This completes the proof.

□

In a particular case, the Mercer type inequality presented in Theorem 1 concludes a Hermite–Hadamard inequality for superquadratic functions. The next corollary follows from Theorem 1, when we consider $m = x$ and $M = y$.

Corollary 1. *If $f\colon [0,\infty) \to \mathbb{R}$ is a superquadratic function, then*

$$f\left(\frac{x+y}{2}\right) + 2\int_0^{1/2} f\left(u|x-y|\right)\mathrm{d}u \le \frac{1}{y-x}\int_x^y f(t)\mathrm{d}t$$
$$\le \frac{f(x)+f(y)}{2} - 2f(0) - 2\int_0^1 (1-u)f(u|x-y|)\mathrm{d}u$$

for all $0 \le x < y$.

The power functions $f(t) = t^p$ and $g(t) = -t^q$ are superquadratic, when $p \ge 2$ and $q \in [1,2]$. Hence, the next result follows.

Corollary 2. *Let $0 \le m < M$ and let $x, y \in [m,M]$. If $p \ge 2$, then*

$$\begin{aligned}&\left(m+M-\frac{x+y}{2}\right)^p + \frac{1}{2^p(p+1)}|x-y|^p \\ &\quad\le \frac{(M+m-x)^{p+1}-(M+m-y)^{p+1}}{(p+1)(y-x)} \\ &\quad\le m^p + M^p - \frac{x^p+y^p}{2} - (\beta_p(x)+\beta_p(y)) - 2\frac{|x-y|^p}{(p+1)(p+2)},\end{aligned} \tag{2.14}$$

in which $\beta_p(x) = \frac{(M-x)(x-m)}{M-m}\left((M-x)^{p-1}+(x-m)^{p-1}\right)$. If $p \in [1,2]$, then (2.14) is reversed.

Let $f(t) = t^2$. Then f is superquadratic as well as subquadratic. This fact together Corollary 2 produce an equation as

$$\begin{aligned}&\left(m+M-\frac{x+y}{2}\right)^2 + \frac{1}{12}|x-y|^2 \\ &\quad= \frac{(M+m-x)^3-(M+m-x)^3}{3(y-x)} \\ &\quad= m^2+M^2-\frac{x^2+y^2}{2} - ((M-x)(x-m)+(M-y)(y-m)) - \frac{|x-y|^2}{6}.\end{aligned}$$

The next proposition gives a generalization of [12, Theorem 2.8].

Proposition 1. *Let $f\colon [0,\infty) \to \mathbb{R}$ be a superquadratic function and let $0 \le y_1 \le x_1 \le x_2 \le y_2$. If $x_1 + x_2 = y_1 + y_2$, then*

$$f(x_1)+f(x_2) \le f(y_1)+f(y_2) - 2\frac{y_2-x_1}{y_2-y_1}f(x_1-y_1) - 2\frac{x_1-y_1}{y_2-y_1}f(x_2-y_1). \tag{2.15}$$

Proof. Applying (1.4) with $\lambda = \frac{y_2-x_1}{y_2-y_1}$ we obtain

$$\begin{aligned} f(x_1) &= f\left(\frac{y_2-x_1}{y_2-y_1}y_1 + \frac{x_1-y_1}{y_2-y_1}y_2\right) \\ &\le \frac{y_2-x_1}{y_2-y_1}f(y_1) + \frac{x_1-y_1}{y_2-y_1}f(y_2) - \frac{y_2-x_1}{y_2-y_1}f(x_1-y_1) - \frac{x_1-y_1}{y_2-y_1}f(y_2-x_1). \end{aligned}$$

Similarly with $\lambda = \frac{y_2-x_2}{y_2-y_1}$ we get

$$f(x_2) \le \frac{y_2-x_2}{y_2-y_1}f(y_1) + \frac{x_2-y_1}{y_2-y_1}f(y_2) - \frac{y_2-x_2}{y_2-y_1}f(x_2-y_1) - \frac{x_2-y_1}{y_2-y_1}f(y_2-x_2).$$

The desired inequality now follows from summing two last inequalities. □

Corollary 3 ([12, Theorem 2.8])**.** *Let $f\colon [0,\infty) \to \mathbb{R}$ be a superquadratic function. Then*

$$f(a) + f(b) \le f(a+b) - 2\frac{bf(a)+af(b)}{a+b} \tag{2.16}$$

for all $a,b > 0$. In particular, if f is positive, then f is super-additive. If f is non-positive, then $-f$ is sub-additive.

Proof. Without loss of generality we may assume that $a \le b$. Considering $y_1 = 0$, $x_1 = a$, $x_2 = b$ and $y_2 = a+b$, Proposition 1 concludes the desired result. . □

Now we present our main result. It is an operator extension of (2.15). It also gives a generalization of the operator Mercer inequality for superquadratic functions, see [3].

Theorem 2. *Let $f\colon [0,\infty) \to \mathbb{R}$ be a continuous superquadratic function. Let A,B,C,D be positive operators on a Hilbert space $\mathcal{H}$ such that $A+D = B+C$ and $0 \le A \le mI \le B \le C \le MI \le D$ for some positive scalars m,M. If Φ is a unital positive linear map on $\mathcal{B}(\mathcal{H})$, then*

$$\begin{aligned} &f(\Phi(B)) + f(\Phi(C)) + \beta(\Phi(B)) + \beta(\Phi(C)) \\ &\quad \le \Phi(f(A)) + \Phi(f(D)) - \Phi(f(m-A)) - \Phi(f(D-M)) \\ &\qquad + \frac{\Phi(A-D)+M-m}{M-m}f(M-m) \end{aligned} \tag{2.17}$$

in which $\beta(t)$ is defined by (1.6).

Proof. As f is continuous, the function β is continuous too. Moreover, $\beta(t) = \beta(M+m-t)$ for every $t \in [0,\infty)$. Hence, we can apply the functional calculus to define $\beta(X)$ for every positive operator X.

If $0 \le s \notin (m,M)$, then $s \in (0,m] \cup [M,\infty)$. First we assume that $s \in [M,\infty)$ and we put $\mu = \frac{M-m}{s-m} \in [0,1]$. Applying (1.4) we obtain

$$\begin{aligned} f(M) &= f(\mu s + (1-\mu)m) \\ &\le \frac{M-m}{s-m} f(s) + \frac{s-M}{s-m} f(m) - \frac{M-m}{s-m} f(s-M) - \frac{s-M}{s-m} f(M-m) \end{aligned}$$

or equivalently,

$$f(s) - f(s-M) + \frac{M-s}{M-m} f(M-m) \ge \frac{M-s}{M-m} f(m) + \frac{s-m}{M-m} f(M). \tag{2.18}$$

If $s \in [0,m)$, then a similar argument yields

$$f(s) - f(m-s) + \frac{s-m}{M-m} f(M-m) \ge \frac{M-s}{M-m} f(m) + \frac{s-m}{M-m} f(M). \tag{2.19}$$

As $A \le mI$ and $D \ge MI$, we can apply functional calculus to (2.18) and (2.19), respectively, with $s = D$ and $s = A$ to derive

$$f(D) - f(D-M) + \frac{M-D}{M-m} f(M-m) \ge \frac{M-D}{M-m} f(m) + \frac{D-m}{M-m} f(M)$$

and

$$f(A) - f(m-A) + \frac{A-m}{M-m} f(M-m) \ge \frac{M-A}{M-m} f(m) + \frac{A-m}{M-m} f(M).$$

Applying the positive linear map Φ to both sides of the last two inequalities we reach

$$\begin{aligned} \Phi(f(D)) - \Phi(f(D-M)) + \frac{M-\Phi(D)}{M-m} f(M-m) \\ \ge \frac{M-\Phi(D)}{M-m} f(m) + \frac{\Phi(D)-m}{M-m} f(M) \end{aligned} \tag{2.20}$$

and

$$\begin{aligned} \Phi(f(A)) - \Phi(f(m-A)) + \frac{\Phi(A)-m}{M-m} f(M-m) \\ \ge \frac{M-\Phi(A)}{M-m} f(m) + \frac{\Phi(A)-m}{M-m} f(M). \end{aligned} \tag{2.21}$$

Next let $t \in [m,M]$ and put $\lambda = \frac{M-t}{M-m}$. It follows from (1.4) that

$$f(t) = f(\lambda m + (1-\lambda)M) \le \frac{M-t}{M-m} f(m) + \frac{t-m}{M-m} f(M) - \beta(t), \tag{2.22}$$

where $\beta(t)$ is defined by (1.6). Since Φ is unital and positive, the spectra of operators $\Phi(B)$ and $\Phi(C)$ are contained in $[m,M]$. Accordingly, we can apply the continuous functional calculus to (2.22) with $t = \Phi(B)$ and $t = \Phi(C)$ to get

$$f(\Phi(B)) + \beta(\Phi(B)) \le \frac{M-\Phi(B)}{M-m} f(m) + \frac{\Phi(B)-m}{M-m} f(M) \tag{2.23}$$

and

$$f(\Phi(C))+\beta(\Phi(C)) \leq \frac{M-\Phi(C)}{M-m}f(m)+\frac{\Phi(C)-m}{M-m}f(M). \tag{2.24}$$

Summing (2.23) and (2.24) we get

$$\begin{aligned}
&f(\Phi(B))+f(\Phi(C))+\beta(\Phi(B))+\beta(\Phi(C))\\
&\leq \frac{2M-\Phi(B+C)}{M-m}f(m)+\frac{\Phi(B+C)-2m}{M-m}f(M)\\
&= \frac{2M-\Phi(A+D)}{M-m}f(m)+\frac{\Phi(A+D)-2m}{M-m}f(M) \qquad (\text{by } A+D=B+C)\\
&\leq \Phi(f(A))+\Phi(f(D))-\Phi(f(m-A))-\Phi(f(D-M))\\
&\quad +\frac{\Phi(A-D)+M-m}{M-m}f(M-m)
\end{aligned}$$

where the last inequality follows from summing (2.20) and (2.21). This completes the proof. □

The next corollary gives another variant of (2.17). We omit the proof as it is similar to the proof of Theorem 2.

Corollary 4. *With the hypotheses as in Theorem 2:*

$$\begin{aligned}
&\Phi(f(B))+f(\Phi(C))+\Phi(\beta(B))+\beta(\Phi(C))\\
&\leq \Phi(f(A))+f(\Phi(D))-\Phi(f(m-A))-f(\Phi(D)-M)\\
&\quad +\frac{\Phi(A-D)+M-m}{M-m}f(M-m).
\end{aligned}$$

As a consequence, the Jensen-Mercer operator inequality for superquadratic functions holds:

Corollary 5 ([3, Theorem 1])**.** *Let $f\colon [0,\infty)\to\mathbb{R}$ be a continuous superquadratic function and let $0<m\leq M$. If C is a positive operator, whose spectrum is contained in $[m,M]$, then*

$$f(M+m-\Phi(C))+\beta(\Phi(C))+f(0)\leq f(m)+f(M)-\Phi(\beta(C))-f(0).$$

Proof. Let C be a positive operator with spectrum in $[m,M]$. Apply Corollary 4 with $A=mI$, $B=(M+m)I-C$ and $D=MI$. □

As another consequence, we have the following Jensen operator inequality.

Corollary 6. *Let $f\colon [0,\infty)\to\mathbb{R}$ be a continuous superquadratic function and let $0<m\leq M$. Then*

$$f\left(\frac{A+D}{2}\right)+\beta\left(\frac{A+D}{2}\right)\leq \frac{f(A)+f(D)}{2}-\frac{f(m-A)+f(D-M)}{2}$$

for all positive operators A and D satisfying $A\leq mI\leq \frac{A+D}{2}\leq MI\leq D$.

Remark 1. If the superquadratic function f is positive, then f is convex and Corollary 6 provide an improvement of [14, Corollary 2.7]. For example, if $f(t)=t^p$ with $p\geq 2$, then

$$\left(\frac{A+D}{2}\right)^p+\beta\left(\frac{A+D}{2}\right)\leq\frac{A^p+D^p}{2}-\frac{(m-A)^p+(D-M)^p}{2} \tag{2.25}$$

holds for all positive operators A and D satisfying $A\leq mI\leq\frac{A+D}{2}\leq MI\leq D$. The existence of scalars m,M are necessary in Corollary 6. For example, it is known that the function $f(t)=t^3$ is not operator convex and so one can find positive operators A and D such that the operator

$$\frac{A^3+D^3}{2}-\left(\frac{A+D}{2}\right)^3$$

is not positive. Accordingly, (2.25) does not hold in general, while the function $f(t)=t^3$ is superquadratic.

Moreover, if f is a non-positive superquadratic function, then Corollary 6 gives a reverse of [14, Corollary 2.7].

Remark 2. An operator version of (2.16) also follows from Theorem 1 as follows:

$$f(B)+f(C)+\beta(B)+\beta(C)\leq f(B+C)-f(B+C-M) \tag{2.26}$$

for all positive operators B,C satisfying $0<B,C\leq M\leq B+C$ with $M>0$. To see this apply (2.17) with $A=m=0$ and $D=B+C$ and note that $f(0)\leq 0$ for every superquadratic function f. It is known that (see e.g. [8]) if $f\colon[0,\infty)\to[0,\infty)$ is an increasing convex function with $f(0)=0$, then

$$\|f(B)+f(C)\|\leq\|f(B+C)\| \tag{2.27}$$

for all positive operators B and C and every unitarily invariant norm $\|\cdot\|$. We note that every positive superquadratic function f is convex and satisfies $f(0)=0$. Hence, inequality (2.26) gives an stronger result than (2.27). However, the existence of positive scalar M with $B,C\leq M\leq B+C$ is necessary for (2.26). We give an example of such operators. Let $f(t)=t^3$ and put

$$B=\begin{bmatrix}2&-1\\-1&2\end{bmatrix}\quad\text{and}\quad C=\begin{bmatrix}3&0\\0&2\end{bmatrix}$$

so that $B,C\leq 3I\leq B+C$. We calculate

$$f(B)=\begin{bmatrix}14&-13\\-13&14\end{bmatrix}\quad\text{and}\quad f(C)=\begin{bmatrix}8&0\\0&27\end{bmatrix}\quad\text{and}\quad \beta(B)=5/3\begin{bmatrix}1&1\\1&1\end{bmatrix}$$

and

$$\beta(C)=\frac{1}{3}\begin{bmatrix}10&0\\0&0\end{bmatrix}\text{ and } f(B+C)=\begin{bmatrix}77&-62\\-62&139\end{bmatrix}\text{ and } f(B+C-M)=\begin{bmatrix}5&-8\\-8&13\end{bmatrix}.$$

Accordingly, we have

$$f(B)+f(C)+\beta(B)+\beta(C) \simeq \begin{bmatrix} 27 & -11.33 \\ -11.33 & 42.67 \end{bmatrix} \leq \begin{bmatrix} 72 & -54 \\ -54 & 126 \end{bmatrix}$$
$$= f(B+C) - f(B+C-M).$$

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SOME PROPERTIES OF WEAKLY G-SUPPLEMENTED LATTICES

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Abstract. In this work, all lattices are complete modular lattices with the greatest element 1 and the smallest element 0. Let L be a lattice and $a,b \in L$. If $a \vee b = 1$ and $a \wedge b \ll_g L$, then b is called a weak g-supplement of a in L. If every element of L has a weak g-supplement in L, then L is called a weakly g-supplemented lattice. In this work, some properties of these lattices are investigated. Let L be a lattice and $a,b,c \in L$. If $c \ll L$, then b is a weak g-supplement of a in L if and only if b is a weak g-supplement of $a \vee c$ in L. Let L be a lattice and $1 = a_1 \vee a_2 \vee \cdots \vee a_n$ with $a_i \in L$ $(1 \leq i \leq n)$. If $a_i/0$ is weakly g-supplemented for every $i = 1,2,\ldots,n$, then L is also weakly g-supplemented. Let L be a weakly g-supplemented lattice. Then $1/a$ is weakly g-supplemented for every $a \in L$. If L is a weakly g-supplemented lattice, then $1/r_g(L)$ is complemented. Let L be a lattice. Then L is weakly g-supplemented if and only if for every $a,b \in L$ with $1 = a \vee b$, a has a weak g-supplement c in L with $c \leq b$. Let L be a lattice and $a \beta_* b$ in L. If a and b have weak g-supplements in L, then they have the same weak g-supplements in L.

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1. INTRODUCTION

In this paper, every lattice is complete modular lattice with the smallest element 0 and the greatest element 1. Let L be a lattice, $x,y \in L$ and $x \leq y$. A sublattice $\{a \in L | x \leq a \leq y\}$ is called a *quotient sublattice* and denoted by y/x. An element y of a lattice L is called a *complement* of x in L if $x \wedge y = 0$ and $x \vee y = 1$, this case we denote $1 = x \oplus y$ (in this case we call x and y are *direct summands* of L). L is said to be *complemented* if each element of L has at least one complement in L. An element x of L is said to be *small* or *superfluous* and denoted by $x \ll L$ if $y = 1$ for every $y \in L$ such that $x \vee y = 1$. The meet of all maximal elements of the poset $L - \{1\}$ is called the *radical* of L and denoted by $r(L)$. If $L - \{1\}$ have not any maximal elements, then the radical of L is defined by $r(L) = 1$. An element a of L is called a *supplement* of b in L if it is minimal for $a \vee b = 1$. a is a supplement of b in a lattice L if and only if $a \vee b = 1$ and $a \wedge b \ll a/0$. A lattice L is called a *supplemented* lattice if every element of L has a supplement in L. L is said to be $\oplus-$*supplemented* if every element of L has a supplement that is a direct summand in L. Let L be a lattice and $a,b \in L$. If $a \vee b = 1$ and $a \wedge b \ll L$, then a is called a *weak supplement* of b in L. L is said to be *weakly supplemented* if every element of L has a weak supplement in L. We say that an element y of L *lies above* an element x of L if $x \leq y$ and $y \ll 1/x$. L is said to be *hollow* if every element distinct from 1 is superfluous in L, and L is said to be *local* if $L - \{1\}$ has the greatest element. We say an element $x \in L$ has *ample supplements* in L if for every $y \in L$ with $x \vee y = 1$, x has a supplement z in L with $z \leq y$. L is said to be *amply supplemented* if every element of L has ample supplements in L. It is clear that every amply supplemented lattice is supplemented. Let L be a lattice and $k \in L$. If $t = 0$ for every $t \in L$ with $k \wedge t = 0$, then k is called an *essential* element of L and denoted by $k \trianglelefteq L$. Let L be a lattice and $a \in L$. If $b = 1$ for every $b \trianglelefteq L$ with $a \vee b = 1$, then a is called a *generalized small* (briefly, *g-small*) element of L and denoted by $a \ll_g L$. Let L be a lattice and $a,b \in L$. If $1 = a \vee b$ and $1 = a \vee t$ with $t \trianglelefteq b/0$ implies that $t = b$, then b is called a *g-supplement* of a in L. b is a g-supplement of a in L if and only if $1 = a \vee b$ and $a \wedge b \ll_g b/0$. If every element of L has a g-supplement in L, then L is called a *g-supplemented* lattice. Let L be a lattice and t be a maximal element of $L - \{1\}$. If $t \trianglelefteq L$, then t is called a *g-maximal* element of L. The meet of all g-maximal elements of L is called the *g-radical* of L and denoted by $r_g(L)$. If L have not any g-maximal elements, then we call $r_g(L) = 1$. Let L be a lattice. If every element of L distinct from 1 is g-small in L, then L is called a *g-hollow* lattice.

More details about (amply) supplemented lattices are in [1, 2, 5, 10]. More results about (amply) supplemented modules are in [6, 9, 15]. More details about weakly supplemented lattices are in [1]. More details about g-small elements and g-supplemented lattices are in [14]. More details about g-small submodules and g-supplemented modules are in [7, 8, 12].

Lemma 1. *Let L be a lattice and $a,b,c,d \in L$. Then the followings hold.*

(i) *If $a \leq b$ and $b \ll_g L$, then $a \ll_g L$.*
(ii) *If $a \ll_g b/0$, then $a \ll_g t/0$ for every $t \in L$ with $b \leq t$.*
(iii) *If $a \ll_g L$, then $a \vee b \ll_g 1/b$.*
(iv) *If $a \ll_g b/0$ and $c \ll_g d/0$, then $a \vee c \ll_g (b \vee d)/0$.*
(v) *If $a_i \ll_g b_i/0$ for $a_i, b_i \in L$ $(i = 1,2,\ldots,n)$, then $a_1 \vee a_2 \vee \cdots \vee a_n \ll_g (b_1 \vee b_2 \vee \cdots \vee b_n)/0$.*
(vi) *If $a \leq b$ and $b \ll_g L$, then $b \ll_g 1/a$.*
(vii) *If $a \ll_g L$, then $a \leq r_g(L)$.*
(viii) *$r_g(a/0) \leq r_g(L)$.*

Proof. See [14, Lemma 1, Lemma 6 and Lemma 7]. □

Lemma 2. *Let L be a lattice and $a,b,c \in L$. If $a \vee b = 1$ and $(a \wedge b) \vee c = 1$, then $a \vee (b \wedge c) = b \vee (a \wedge c) = 1$.*

Proof. See [11, Lemma 2]. □

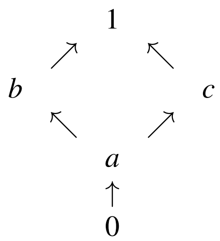
2. Weakly g-supplemented lattices

Definition 1. Let L be a lattice and $a,b \in L$. If $1 = a \vee b$ and $a \wedge b \ll_g L$, then b is called a weak g-supplement of a in L. If every element of L has a weak g-supplement in L, then L is called a weakly g-supplemented lattice. (See also [13])

Every g-supplemented lattice is weakly g-supplemented. But the converse of this statement is not true in general (see Example 1). Hollow, local and g-hollow lattices are weakly g-supplemented. Every weakly supplemented lattice is weakly g-supplemented.

Example 1. Consider the $\mathbb{Z}-$module ${}_{\mathbb{Z}}\mathbb{Q}$. Let L be the set of all submodules of ${}_{\mathbb{Z}}\mathbb{Q}$. Then L is a complete modular lattice with the greatest element $\mathbb{Q}$ and the smallest element 0 by the operation $\subset$. By [6, Example 20.12], ${}_{\mathbb{Z}}\mathbb{Q}$ is weakly supplemented but not supplemented. Hence L is weakly supplemented but not supplemented. Since L is weakly supplemented, it is weakly g-supplemented. Since every nonzero element of L is essential in L and L is not supplemented, L is not g-supplemented.

Example 2. Consider the lattice $L = \{0,a,b,c,1\}$ given by the following diagram.



Then L is weakly g-supplemented but not g-hollow.

Proposition 1. *Let L be a weakly g-supplemented lattice. If every nonzero element of L is essential in L, then L is weakly supplemented.*

Proof. Clear from definitions. □

Proposition 2. *Let L be a lattice and $a,b,c \in L$. If $c \ll L$, then b is a weak g-supplement of a in L if and only if b is a weak g-supplement of $a \vee c$ in L.*

Proof. $(\Longrightarrow)$ Since b is a weak g-supplement of a in L, $1 = a \vee b$ and $a \wedge b \ll_g L$. Here $1 = a \vee b = a \vee c \vee b$. Let $((a \vee c) \wedge b) \vee t = 1$ with $t \trianglelefteq L$. Since $1 = a \vee c \vee b$ and $((a \vee c) \wedge b) \vee t = 1$, by Lemma 2, $a \vee c \vee (b \wedge t) = 1$. Then by $c \ll L$, $a \vee (b \wedge t) = 1$. Since $b \vee t = 1$, by Lemma 2, $(a \wedge b) \vee t = 1$ and since $a \wedge b \ll_g L$ and $t \trianglelefteq L$, $t = 1$. Hence $(a \vee c) \wedge b \ll_g L$ and b is a weak g-supplement of $a \vee c$ in L.

$(\Longleftarrow)$ Since b is a weak g-supplement of $a \vee c$ in L, $1 = a \vee c \vee b$ and $(a \vee c) \wedge b \ll_g L$. Since $1 = a \vee c \vee b$ and $c \ll L$, $1 = a \vee b$. Since $a \wedge b \leq (a \vee c) \wedge b$ and

$(a\vee c)\wedge b\ll_g L$, by Lemma 1(i), $a\wedge b\ll_g L$. Hence b is a weak g-supplement of a in L, as desired. □

Proposition 3. *Let L be a lattice, $a,b,c\in L$ and $b\trianglelefteq L$. If $c\ll_g L$, then b is a weak g-supplement of a in L if and only if b is a weak g-supplement of $a\vee c$ in L.*

Proof. $(\Longrightarrow)$ Since b is a weak g-supplement of a in L, $1=a\vee b$ and $a\wedge b\ll_g L$. Here $1=a\vee b=a\vee c\vee b$. Let $((a\vee c)\wedge b)\vee t=1$ with $t\trianglelefteq L$. Since $1=a\vee c\vee b$ and $((a\vee c)\wedge b)\vee t=1$, by Lemma 2, $a\vee c\vee(b\wedge t)=1$. Since $b\trianglelefteq L$ and $t\trianglelefteq L$, we have $b\wedge t\trianglelefteq L$ and $a\vee(b\wedge t)\trianglelefteq L$. Then by $c\ll_g L$, $a\vee(b\wedge t)=1$. Since $b\vee t=1$, by Lemma 2, $(a\wedge b)\vee t=1$ and since $a\wedge b\ll_g L$ and $t\trianglelefteq L$, $t=1$. Hence $(a\vee c)\wedge b\ll_g L$ and b is a weak g-supplement of $a\vee c$ in L.

$(\Longleftarrow)$ Since b is a weak g-supplement of $a\vee c$ in L, $1=a\vee c\vee b$ and $(a\vee c)\wedge b\ll_g L$. Since $b\trianglelefteq L$, we have $a\vee b\trianglelefteq L$. Then by $1=a\vee c\vee b$ and $c\ll_g L$, $1=a\vee b$. Since $a\wedge b\leq(a\vee c)\wedge b$ and $(a\vee c)\wedge b\ll_g L$, by Lemma 1(i), $a\wedge b\ll_g L$. Hence b is a weak g-supplement of a in L, as desired. □

Lemma 3. *Let L be a lattice and $a,b\in L$. If $a\vee b$ has a weak g-supplement x in L and $(a\vee x)\wedge b$ has a weak g-supplement y in $b/0$, then $x\vee y$ is a weak g-supplement of a in L.*

Proof. Since x is a weak g-supplement of $a\vee b$ in L, $1=a\vee b\vee x$ and $(a\vee b)\wedge x\ll_g L$. Since y is a weak g-supplement of $(a\vee x)\wedge b$ in $b/0$, $b=((a\vee x)\wedge b)\vee y$ and $(a\vee x)\wedge y=(a\vee x)\wedge b\wedge y\ll_g b/0$. Then $1=a\vee b\vee x=a\vee x\vee((a\vee x)\wedge b)\vee y=a\vee x\vee y$ and by Lemma 1(iv), $a\wedge(x\vee y)\leq((a\vee x)\wedge y)\vee((a\vee y)\wedge x)\leq((a\vee x)\wedge y)\vee((a\vee b)\wedge x)\ll_g L$. Hence $x\vee y$ is a weak g-supplement of a in L. □

Corollary 1. *Let L be a lattice and $a,b\in L$. If $a\vee b$ has a weak g-supplement in L and $b/0$ is weakly g-supplemented, then a has a weak g-supplement in L.*

Proof. Clear from Lemma 3. □

Lemma 4. *Let $1=a\vee b$ with $a,b\in L$. If $a/0$ and $b/0$ are weakly g-supplemented, then L is also weakly g-supplemented.*

Proof. Let x be any element of L. Then 0 is a weak g-supplement of $x\vee a\vee b$ in L and since $b/0$ is weakly g-supplemented, by Corollary 1, $x\vee a$ has a weak g-supplement in L. Since $a/0$ is weakly g-supplemented, again by Corollary 1, x has a weak g-supplement in L. Hence L is weakly g-supplemented. □

Corollary 2. *Let $1=a_1\vee a_2\vee\cdots\vee a_n$ with $a_i\in L$ $(1\leq i\leq n)$. If $a_i/0$ is weakly g-supplemented for every $i=1,2,\ldots,n$, then L is also weakly g-supplemented.*

Proof. Clear from Lemma 4. □

Lemma 5. *Let L be a lattice and $a,b,c\in L$ with $c\leq a$. If b is a weak g-supplement of a in L, then $b\vee c$ is a weak g-supplement of a in $1/c$.*

Proof. Since b is a weak g-supplement of a in L, $1 = a \vee b$ and $a \wedge b \ll_g L$. Since $a \wedge b \ll_g L$, by Lemma 1(*iii*), $(a \wedge b) \vee c \ll_g 1/c$. Hence $1 = a \vee b = a \vee b \vee c$ and $a \wedge (b \vee c) = (a \wedge b) \vee c \ll_g 1/c$ and $b \vee c$ is a weak g-supplement of a in $1/c$. □

Corollary 3. *Let L be a weakly g-supplemented lattice. Then $1/a$ is weakly g-supplemented for every $a \in L$.*

Proof. Clear from Lemma 5. □

Lemma 6. *Let L be a weakly g-supplemented lattice. Then $1/r_g(L)$ is complemented.*

Proof. Let x be any element of $1/r_g(L)$. Since L is weakly g-supplemented, x has a weak g-supplement y in L. Here $1 = x \vee y$ and $x \wedge y \ll_g L$. Since $x \wedge y \ll_g L$, by Lemma 1(*vii*), $x \wedge y \leq r_g(L)$. Hence $1 = x \vee y \vee r_g(L)$ and $x \wedge (y \vee r_g(L)) = (x \wedge y) \vee r_g(L) = r_g(L)$. Therefore, $y \vee r_g(L)$ is a complement of x in $1/r_g(L)$ and $1/r_g(L)$ is complemented. □

Corollary 4. *Let L be a weakly g-supplemented lattice. Then $1/r_g(L)$ is $\oplus-$supplemented.*

Proof. Clear from [3, Definition 1] and Lemma 6. □

Lemma 7. *Let L be a lattice and $a, b \in L$. If $1 = a \vee b$ and $a \wedge b$ has a weak g-supplement x in $b/0$, then x is a weak g-supplement of a in L.*

Proof. Since x is a weak g-supplement of $a \wedge b$ in $b/0$, $b = (a \wedge b) \vee x$ and $a \wedge x = a \wedge b \wedge x \ll_g b/0$. Then $1 = a \vee b = a \vee (a \wedge b) \vee x = a \vee x$ and $a \wedge x \ll_g L$. Hence x is a weak g-supplement of a in L. □

Corollary 5. *Let L be a lattice, $a, b \in L$ and $1 = a \vee b$. If $b/0$ is weakly g-supplemented, then a has a weak g-supplement in L.*

Proof. Clear from Lemma 7. □

Lemma 8. *Let L be a lattice, $a, b \in L$ and $1 = a \vee b$. If L is weakly g-supplemented, then a has a weak g-supplement c in L with $c \leq b$.*

Proof. Since L is weakly g-supplemented, $a \wedge b$ has a weak g-supplement x in L. Here $1 = (a \wedge b) \vee x$ and $a \wedge b \wedge x \ll_g L$. Since $1 = (a \wedge b) \vee x$, by modularity, $b = b \wedge 1 = b \wedge ((a \wedge b) \vee x) = (a \wedge b) \vee (b \wedge x)$. Let $c = b \wedge x$. Then $1 = a \vee b = a \vee (a \wedge b) \vee (b \wedge x) = a \vee (b \wedge x) = a \vee c$ and $a \wedge c = a \wedge b \wedge x \ll_g L$. Hence c is a weak g-supplement of a in L. Moreover, $c \leq b$. Hence the desired result is obtained. □

Corollary 6. *Let L be a lattice. Then L is weakly g-supplemented if and only if for every $a, b \in L$ with $1 = a \vee b$, a has a weak g-supplement c in L with $c \leq b$.*

Proof. Clear from Lemma 8. □

Let $x, y \in L$. A relation β_* is defined on the elements of L by $x\beta_* y$ if and only if for every $t \in L$ with $1 = x \vee t$ then $1 = y \vee t$ and for every $k \in L$ with $1 = y \vee k$ then $1 = x \vee k$. (See [11, Definition 1]). More information about β_* relation are in [11]. More information about β^* relation on modules are in [4].

Lemma 9. *Let L be a lattice and $a\beta_* b$ in L. If a and b have weak g-supplements in L, then they have the same weak g-supplements in L.*

Proof. Let x be a weak g-supplement of a in L. Then $1 = a \vee x$ and $a \wedge x \ll_g L$. Since $1 = a \vee x$ and $a\beta_* b$, we have $1 = b \vee x$. Let $1 = (b \wedge x) \vee t$ with $t \trianglelefteq L$. Then by Lemma 2, $1 = b \vee (x \wedge t)$ and since $a\beta_* b$, we have $1 = a \vee (x \wedge t)$. Since $1 = x \vee t$ and $1 = a \vee (x \wedge t)$, by Lemma 2, $1 = (a \wedge x) \vee t$. Since $a \wedge x \ll_g L$ and $t \trianglelefteq L$, $t = 1$. Hence $b \wedge x \ll_g L$ and since $1 = b \vee x$, x is a weak g-supplement of b in L. Similarly, interchanging the roles of a and b we can prove that each weak g-supplement of b in L is also a weak g-supplement of a in L. □

Corollary 7. *Let L be a lattice and let a lie above b in L. If a and b have weak g-supplements in L, then they have the same weak g-supplements in L.*

Proof. By [11, Theorem 3], $a\beta_* b$ and by Lemma 9, the desired result is obtained. □

Lemma 10. *Let L be a lattice. If every element of L is β_*−equivalent to a weak g-supplement element in L, then L is weakly g-supplemented.*

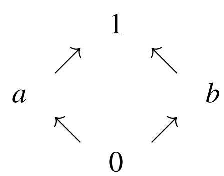
Proof. Let $a \in L$. By hypothesis, there exists a weak g-supplement element x in L such that $a\beta_* x$. Let x be a weak g-supplement of b in L. Then by definition, b is a weak g-supplement of x in L. Since $a\beta_* x$, by Lemma 9, b is a weak g-supplement of a in L. Hence L is weakly g-supplemented. □

Corollary 8. *Let L be a lattice. If every element of L lies above a weak g-supplement element in L, then L is weakly g-supplemented.*

Proof. Clear from [11, Theorem 3] and Lemma 10. □

Example 3. Let L be a nonzero complemented lattice. Here $1 \ll_g L$, but not $1 \ll L$. 1 is a weak g-supplement of 1 in L, but 1 is not a weak supplement of 1 in L.

Example 4. Consider the lattice $L = \{0, a, b, 1\}$ given by the following diagram.



Then L is g-hollow but not hollow. Here $1 \ll_g L$, but not $1 \ll L$. 1 is a weak g-supplement of 1 in L, but 1 is not a weak supplement of 1 in L.

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FURTHER RESULTS ON THE LEBESGUE-NAGELL EQUATION $dx^2 + p^{2m}q^{2n} = 4y^p$

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Abstract. Let d be a fixed positive integer with $d > 3$ is square-free, and let $h(-d)$ denote the class number of the imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$. Further, let p and q be odd primes such that $p > 3$, $p \neq q$ and $p \nmid h(-d)$. In this paper, we give a sufficient and necessary condition for the Lebesgue-Nagell equation $(*)$ $dx^2 + p^{2m}q^{2n} = 4y^p$ to have positive integer solutions (x,y,m,n) with $\gcd(x,y) = 1$. It can be seen from this condition that if $q \not\equiv \pm 1 \pmod{2p}$, then $(*)$ has no positive integer solutions (x,y,m,n) with $\gcd(x,y) = 1$.

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1. INTRODUCTION

Let $\mathbb{Z}, \mathbb{N}, \mathbb{Q}$ be the sets of all integers, positive integers and rational numbers, respectively. Let d be a fixed positive integer with $d > 3$ is square-free, and let $h(-d)$ denote the class number of the imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$. Further, let p and q be distinct odd primes such that $p > 3$ and $p \nmid h(-d)$. As we all know, the Lebesgue-Nagell equation is a class of polynomial-exponential Diophantine equations with a long history and rich content (see [1–5, 8–16, 18, 22–24] and the references of [19]). Recently, K. Chakraborty and A. Hoque [8] discussed in detail a Lebesgue-Nagell equation of the form

$$dx^2 + p^{2m}q^{2n} = 4y^p,\ x,y,m,n \in \mathbb{N},\ \gcd(x,y) = 1. \tag{1.1}$$

They proved that if one of the following conditions is satisfied, then (1.1) has no solutions (x,y,m,n).

(i) $d \equiv 1 \text{ or } 2 \pmod 4$.

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(ii) $d \equiv 3 \pmod 4$ and $q^n \not\equiv \pm 1 \pmod p$.
(iii) $q = p+2$.
(iv) $h(-d) = 1$, $p > 41$, $q = d+p$ and $n = p$.
(v) $h(-d) \in \{1,2,4,8,16,32\}$, $q = 3$ and $n = p$.

Clearly, if (x,y,m,n) is a solution of (1.1), then the general equation

$$X^2 + dY^2 = 4Z^p,\ X,Y \in \mathbb{Z},\ \gcd(X,Y) = 1,\ Z \in \mathbb{N} \tag{1.2}$$

has a solution

$$(X,Y,Z) = (p^m q^n, x, y) \tag{1.3}$$

with $p \mid X$. In this paper, we first start with (1.2) to prove that

Theorem 1. *If (X,Y,Z) is a solution of (1.2) with $p \mid X$, then*

$$X = \lambda_1 a \sum_{i=0}^{(p-1)/2} (-1)^i \begin{bmatrix} p \\ i \end{bmatrix} a^{p-2i-1} \left(\frac{a^2+db^2}{4}\right)^i,$$

$$Y = \lambda_1 \lambda_2 b \sum_{i=0}^{(p-1)/2} \begin{bmatrix} p \\ i \end{bmatrix} (-db^2)^{(p-1)/2-i} \left(\frac{a^2+db^2}{4}\right)^i, \tag{1.4}$$

$$Z = \frac{a^2+db^2}{4},\ a,b \in \mathbb{N},\ \gcd(a,b) = 1,\ 2 \nmid ab,\ p \mid a,\ \lambda_1, \lambda_2 \in \{1,-1\},$$

where

$$\begin{bmatrix} p \\ i \end{bmatrix} = \frac{(p-i-1)!\,p}{(p-2i)!\,i!} \in \mathbb{N},\ i = 0,1,\ldots,\frac{p-1}{2}. \tag{1.5}$$

By Theorem 1, we can obtain the following results from the relation (1.3) for (1.1).

Theorem 2. *A sufficient and necessary condition for (1.1) to have solutions (x,y,m,n) is that there exist positive integers b,r,s which make*

$$pq^s = \left| \sum_{i=0}^{(p-1)/2} (-1)^i \begin{bmatrix} p \\ i \end{bmatrix} p^{r(p-2i-1)} \left(\frac{p^{2r}+db^2}{4}\right)^i \right|, \tag{1.6}$$

where $\begin{bmatrix} p \\ i \end{bmatrix}$ is defined as in (1.5). Moreover, if (1.6) holds, then (1.1) has the solution

$$(x,y,m,n) = \left(b \left| \sum_{i=0}^{(p-1)/2} \begin{bmatrix} p \\ i \end{bmatrix} (-db^2)^{(p-1)/2-i} \left(\frac{p^{2r}+db^2}{4}\right)^i \right|, \frac{p^{2r}+db^2}{4}, r+1, s \right). \tag{1.7}$$

Corollary 1. *If $q \not\equiv \pm 1 \pmod{2p}$, then (1.1) has no solutions (x,y,m,n).*

Obviously, Corollary 1 covers and expands on the results $(i)-(iv)$ in [8].

2. PRELIMINARIES

Lemma 1 ([20, Formula 3.76]). *For any positive integer t and any complex numbers ε and $\bar{\varepsilon}$, we have*

$$\varepsilon^t+\bar{\varepsilon}^t=\sum_{i=0}^{[t/2]}(-1)^i\begin{bmatrix} t \\ i \end{bmatrix}(\varepsilon+\bar{\varepsilon})^{t-2i}(\varepsilon\bar{\varepsilon})^i,$$

where

$$\begin{bmatrix} t \\ i \end{bmatrix}=\frac{(t-i-1)!\,t}{(t-2i)!\,i!}\in\mathbb{N},\ i=0,\dots,\left[\frac{t}{2}\right],$$

$\left[\frac{t}{2}\right]$ *is the integer part of $t/2$.*

Lemma 2 ([17, Theorem 3.3], [21, Chapter 8]). *For any positive integer t, let F_t and L_t denote the t-th Fibonacci and Lucas numbers, respectively. Then we have*

(i) *F_t and L_t are positive integers satisfying $L_t^2-5F_t^2=(-1)^t4$.*

(ii) *The equation*

$$F_k=5z^2,\ k,z\in\mathbb{N} \tag{2.1}$$

has only one solution $(k,z)=(5,1)$.

Let α,β be algebraic integers. If $\alpha+\beta$ and $\alpha\beta$ are nonzero coprime integers, and α/β is not a root of unity, then (α,β) is called a Lucas pair. If $u=\alpha+\beta$ and $w=\alpha\beta$, then we have

$$\alpha=\frac{1}{2}(u+\lambda\sqrt{v}),\ \beta=\frac{1}{2}(u-\lambda\sqrt{v}),\ \lambda\in\{1,-1\}, \tag{2.2}$$

where $v=u^2-4w$.

Lemma 3 ([7]). *Let (α,β) be a Lucas pair with (2.2), and let ℓ be an odd prime. If $\ell\mid u$, then $(\alpha^\ell+\beta^\ell)/\ell u$ is a nonzero integer and its prime divisor q satisfies $q\equiv\pm1$ (mod $2p$).*

Let $\tilde{\alpha}$ and $\tilde{\beta}$ be algebraic numbers. If $(\tilde{\alpha}+\tilde{\beta})^2$ and $\tilde{\alpha}\tilde{\beta}$ are nonzero coprime integers and $\tilde{\alpha}/\tilde{\beta}$ is not root of unity, then $(\tilde{\alpha},\tilde{\beta})$ is called a Lehmer pair. If $\tilde{u}=(\alpha+\beta)^2$ and $\tilde{w}=\tilde{u}\tilde{\beta}$, then we have

$$\tilde{\alpha}=\frac{1}{2}(\sqrt{\tilde{u}}+\tilde{\lambda}\sqrt{\tilde{v}}),\ \tilde{\beta}=\frac{1}{2}(\sqrt{\tilde{u}}-\tilde{\lambda}\sqrt{\tilde{v}}),\ \tilde{\lambda}\in\{1,-1\}, \tag{2.3}$$

where $\tilde{v}=\tilde{u}-4\tilde{w}$. For any positive integer t, let

$$\tilde{L}_t(\tilde{\alpha},\tilde{\beta})=\begin{cases}\dfrac{\tilde{\alpha}^t-\tilde{\beta}^t}{\tilde{\alpha}-\tilde{\beta}}, & \text{if } 2\nmid t,\\[2ex] \dfrac{\tilde{\alpha}^t-\tilde{\beta}^t}{\tilde{\alpha}^2-\tilde{\beta}^2}, & \text{if } 2\mid t.\end{cases} \tag{2.4}$$

Then, $\tilde{L}_t(\tilde{\alpha},\tilde{\beta})$ $(t=1,2,\dots)$ are called the corresponding Lehmer numbers of Lehmer pair $(\tilde{\alpha},\tilde{\beta})$. It is well known that Lehmer numbers are nonzero integers. A prime q is called a primitive divisor of $\tilde{L}_t(\tilde{\alpha},\tilde{\beta})$ $(t>2)$ if $q \mid \tilde{L}_t(\tilde{\alpha},\tilde{\beta})$ and

$$q \nmid \tilde{u}\tilde{v}\prod_{j=1}^{t-1}\tilde{L}_t(\tilde{\alpha},\tilde{\beta}).$$

Lemma 4 ([6]). *For any Lehmer pair $(\tilde{\alpha},\tilde{\beta})$, if $t>30$, then $\tilde{L}_t(\tilde{\alpha},\tilde{\beta})$ has primitive divisors.*

3. Proof of Theorem 1

Let (X,Y,Z) be a solution of (1.2). Since $d>3$ and $p \nmid h(-d)$, by Lemma 1 of [8], we have

$$\frac{X+Y\sqrt{-d}}{2}=\lambda_1\left(\frac{a+\lambda_2 b\sqrt{-d}}{2}\right)^p,\ \lambda_1,\lambda_2\in\{1,-1\}, \tag{3.1}$$

$$Z=\frac{a^2+db^2}{4},\ a,b\in\mathbb{N},\ \gcd(a,b)=1,\ 2\nmid ab. \tag{3.2}$$

Further, let

$$\varepsilon=\lambda_1\left(\frac{a+\lambda_2 b\sqrt{-d}}{2}\right),\ \bar{\varepsilon}=\lambda_1\left(\frac{a-\lambda_2 b\sqrt{-d}}{2}\right). \tag{3.3}$$

Then we have

$$\varepsilon+\bar{\varepsilon}=\lambda_1 a,\ \varepsilon-\bar{\varepsilon}=\lambda_1\lambda_2 b\sqrt{-d},\ \varepsilon\bar{\varepsilon}=\frac{a^2+db^2}{4}. \tag{3.4}$$

Since

$$\frac{X-Y\sqrt{-d}}{2}=\lambda_1\left(\frac{a-\lambda_2 b\sqrt{-d}}{2}\right)^p \tag{3.5}$$

by (3.1), we get from (3.1), (3.3), (3.4) and (3.5) that

$$X=\varepsilon^p+\bar{\varepsilon}^p \tag{3.6}$$

and

$$Y=\frac{\varepsilon^p-\bar{\varepsilon}^p}{\sqrt{-d}}. \tag{3.7}$$

By Lemma 1, we obtain from (3.4), (3.6) and (3.7) that

$$X=\lambda_1 a\sum_{i=0}^{(p-1)/2}(-1)^i\begin{bmatrix}p\\ i\end{bmatrix}a^{p-2i-1}\left(\frac{a^2+db^2}{4}\right)^i, \tag{3.8}$$

and

$$Y=\lambda_1\lambda_2 b\sum_{i=0}^{(p-1)/2}\begin{bmatrix}p\\ i\end{bmatrix}(-db^2)^{(p-1)/2-i}\left(\frac{a^2+db^2}{4}\right)^i, \tag{3.9}$$

where $\begin{bmatrix} p \\ i \end{bmatrix}$ is defined as in (1.5). By (1.5), we have

$$\begin{bmatrix} p \\ 0 \end{bmatrix} = 1, \ p \ \Bigg| \begin{bmatrix} p \\ j \end{bmatrix}, \ j = 1, \dots, \frac{p-1}{2}. \tag{3.10}$$

When $p \mid X$, by (3.8) and (3.10), we have $0 \equiv X \equiv \lambda_1 a^p \pmod{p}$, whence we get $p \mid a$. Thus, by (3.2), (3.8) and (3.9), we obtain (1.4). The theorem is proved.

4. Proof of Theorem 2

The sufficiency of the theorem is obvious, and we will prove its necessity below. Let (x,y,m,n) be a solution of (1.1). Since $\gcd(x,y)=1$ and d is square free, by (1.1), we have

$$p \nmid d, \ p \nmid x, \ p \nmid y. \tag{4.1}$$

Then, it is well known that (1.2) has the solution (1.3) with $p \mid X$. Hence, by Theorem 1, we get from (1.3) and (1.4) that

$$p^m q^n = a \left| \sum_{i=0}^{(p-1)/2} (-1)^i \begin{bmatrix} p \\ i \end{bmatrix} a^{p-2i-1} \left(\frac{a^2+db^2}{4} \right)^i \right|, \tag{4.2}$$

$$x = b \left| \sum_{i=0}^{(p-1)/2} \begin{bmatrix} p \\ i \end{bmatrix} (-db^2)^{(p-1)/2-i} \left(\frac{a^2+db^2}{4} \right)^i \right| \tag{4.3}$$

and

$$y = \frac{a^2+db^2}{4}, \ a,b \in \mathbb{N}, \ \gcd(a,b)=1, \ 2 \nmid ab, \ p \mid a. \tag{4.4}$$

Since $p \mid a$, we have

$$a = p^r f, \ r,f \in \mathbb{N}, \ r \le m, \ p \nmid f. \tag{4.5}$$

Substitute (4.5) into (4.2) yields

$$p^{m-r} q^n = f \left| \sum_{i=0}^{(p-1)/2} (-1)^i \begin{bmatrix} p \\ i \end{bmatrix} (p^r f)^{p-2i-1} \left(\frac{p^{2r}f^2+db^2}{4} \right)^i \right|. \tag{4.6}$$

Further, by (1.5), we have

$$\begin{bmatrix} p \\ (p-1)/2 \end{bmatrix} = p, \tag{4.7}$$

and by (4.1), (4.4) and (4.5), we have $p \nmid p^{2r}f^2+db^2$. Hence, by (4.7), we get

$$p \, \Big\| \sum_{i=0}^{(p-1)/2} (-1)^i \begin{bmatrix} p \\ i \end{bmatrix} (p^r f)^{p-2i-1} \left(\frac{p^{2r}f^2+db^2}{4} \right)^i. \tag{4.8}$$

Therefore, by (4.5), (4.6) and (4.8), we have $m>1$,

$$r = m-1 \tag{4.9}$$

and

$$pq^n = f\left|\sum_{i=0}^{(p-1)/2}(-1)^i\begin{bmatrix}p\\i\end{bmatrix}(p^{m-1}f)^{p-2i-1}\left(\frac{p^{2m-2}f^2+db^2}{4}\right)^i\right|. \tag{4.10}$$

Since $\gcd(a,b)=1$, by (4.1) and (4.5), we have

$$\gcd\left(f,\ \sum_{i=0}^{(p-1)/2}(-1)^i\begin{bmatrix}p\\i\end{bmatrix}(p^{m-1}f)^{p-2i-1}\left(\frac{p^{2m-2}f^2+db^2}{4}\right)^i\right)=1. \tag{4.11}$$

If $f>1$, since $p\nmid f$, then from (4.10) and (4.11) we get $f=q^n$ and

$$p=\left|\sum_{i=0}^{(p-1)/2}(-1)^i\begin{bmatrix}p\\i\end{bmatrix}(p^{m-1}q^n)^{p-2i-1}\left(\frac{p^{2m-2}q^{2n}+db^2}{4}\right)^i\right|. \tag{4.12}$$

Let

$$\tilde{\alpha}=\frac{\sqrt{-db^2}+\sqrt{p^{2m-2}q^{2n}}}{2},\ \tilde{\beta}=\frac{\sqrt{-db^2}-\sqrt{p^{2m-2}q^{2n}}}{2}. \tag{4.13}$$

Then we have

$$(\tilde{\alpha}+\tilde{\beta})^2=-db^2,\ \tilde{\alpha}-\tilde{\beta}=p^{m-1}q^n,\ \tilde{\alpha}\tilde{\beta}=-db^2-p^{2m-2}q^{2n}. \tag{4.14}$$

We see from (2.3), (4.13) and (4.14) that $(\tilde{\alpha},\tilde{\beta})$ is a Lehmer pair. Further, let $\tilde{L}_t(\tilde{\alpha},\tilde{\beta})$ $(t=1,2,\dots)$ denote the corresponding Lehmer numbers. By Lemma 1, we get from (2.4), (4.12), (4.13) and (4.14) that

$$|\tilde{L}_p(\tilde{\alpha},\tilde{\beta})|=p. \tag{4.15}$$

Hence, we find from (4.14) and (4.15) that the Lehmer number $\tilde{L}_t(\tilde{\alpha},\tilde{\beta})$ has no primitive divisor. Therefore, by Lemma 4, we have $p<30$. Based on the results in [25], we can prove that p does not satisfy $6<p\leq 30$. So we have $p=5$, by (4.12), we have

$$\left(\frac{-5^{2m-2}q^{2n}+db^2}{2}\right)^2-5(5^{2m-3}q^{2n})^2=\pm 4. \tag{4.16}$$

However, by Lemma 2, (4.16) is false. Thus, we get

$$f=1, \tag{4.17}$$

and by (4.3), (4.4), (4.5), (4.9), (4.10) and (4.17) we obtain (1.6) and (1.7). The theorem is proved.

5. Proof of Corollary 1

By Theorem 2, if (x,y,m,n) is a solution of (1.1), then it can be expressed as (1.7) with (1.6). Let

$$\alpha=\frac{p^{m-1}+b\sqrt{-d}}{2},\ \beta=\frac{p^{m-1}-b\sqrt{-d}}{2}. \tag{5.1}$$

Then we have

$$\alpha+\beta=p^{m-1},\ \alpha-\beta=b\sqrt{-d},\ \alpha\beta=\frac{p^{2m-2}+db^2}{4}. \tag{5.2}$$

It implies that (α,β) is a Lucas pair. By Lemma 1, we see from (1.6), (1.7), (5.1) and (5.2) that

$$pq^n=\left|\frac{\alpha^p+\beta^p}{\alpha+\beta}\right|. \tag{5.3}$$

Therefore, Lemma 3, we get from (5.2) and (5.3) that $q\equiv\pm1 \pmod{2p}$. Thus, the corollary is proved.

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SÁNDOR'S INEQUALITY FOR RIEMANN-LIOUVILLE FRACTIONAL INTEGRALS

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Abstract. The Sándor inequality is a highly significant result in both pure and applied mathematics, providing an upper bound for the mean square of a positive convex function. This paper presents an extension of the Sándor inequality to the case of fractional integrals in the sense of Riemann-Liouville, as well as a generalization for any positive power r.

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Keywords: Sándor's inequality, Hermite-Hadamard inequality, convex functions, Riemann-Liouville fractional integrals

1. INTRODUCTION

Let I be an interval of real numbers. A function $\mathcal{G}\colon I \to \mathbb{R}$ is said to be convex, if for all $\xi, \varsigma \in I$ and all $t \in [0,1]$, we have

$$\mathcal{G}(t\varsigma + (1-t)\xi) \leq t\,\mathcal{G}(\varsigma) + (1-t)\,\mathcal{G}(\xi).$$

The renowned Hermite-Hadamard inequality (refer to [3,4]) is a fundamental result for the class of convex functions, stated as follows:

Theorem 1. *Let $\mathcal{G}\colon [\varsigma,\xi] \to \mathbb{R}$ be a convex function. Then, we have*

$$\mathcal{G}\left(\frac{\varsigma+\xi}{2}\right) \leq \frac{1}{\xi-\varsigma}\int_{\varsigma}^{\xi} \mathcal{G}(z)\,\mathrm{d}z \leq \frac{\mathcal{G}(\varsigma)+\mathcal{G}(\xi)}{2}. \tag{1.1}$$

Fractional calculus extends classical calculus to non-integer orders, capturing memory and non-local effects. It is widely used to model complex phenomena like anomalous diffusion, viscoelasticity, and heat conduction, offering a more accurate description of systems with hereditary properties. In recent decades, this field has experienced significant development. In particular, in approximation theory, several famous inequalities, already crucial in the classical case, have been extended to the fractional setting (see [1,2,6,9]).

In [7], Sándor proved the following result connected with (1.1).

Theorem 2. *Let $\mathcal{G}\colon [\varsigma,\xi] \to \mathbb{R}$ be a non-negative convex function. Then*

$$\frac{1}{\xi-\varsigma}\int_{\varsigma}^{\xi} \mathcal{G}^2(z)\,\mathrm{d}z \le \frac{1}{3}\left(\mathcal{G}^2(\varsigma)+\mathcal{G}(\varsigma)\,\mathcal{G}(\xi)+\mathcal{G}^2(\xi)\right).$$

The Sándor inequality is a mathematical inequality that connects different special functions, particularly convex functions, within the framework of inequality analysis. It plays a crucial role in both pure and applied mathematics, especially in optimization and inequality theory. This inequality is used to establish precise bounds and estimates, providing powerful tools for solving various problems in number theory, real and complex analysis, and mathematical physics. Its significance lies in its ability to generalize and unify several classical results. In this note, we generalize Sándor's inequality to the case of Riemann-Liouville fractional integrals.

2. Preliminaries

Definition 1 ([5])**.** Let $\mathcal{G} \in L^1[\varsigma,\xi]$. The left- and right-side Riemann-Liouville fractional integrals $\mathcal{J}^{\alpha}_{\varsigma^+}\mathcal{G}$ and $\mathcal{J}^{\alpha}_{\xi^-}\mathcal{G}$ of order $\alpha > 0$ are defined by

$$\mathcal{J}^{\alpha}_{\varsigma^+}\mathcal{G}(x) = \frac{1}{\Gamma(\alpha)}\int_{\varsigma}^{x} (x-t)^{\alpha-1}\,\mathcal{G}(t)\,\mathrm{d}t, \qquad x>\varsigma,$$

$$\mathcal{J}^{\alpha}_{\xi^-}\mathcal{G}(x) = \frac{1}{\Gamma(\alpha)}\int_{x}^{\xi} (t-x)^{\alpha-1}\,\mathcal{G}(t)\,\mathrm{d}t, \qquad \xi>x,$$

respectively, where $\Gamma(\alpha) = \int_0^{\infty} e^{-t}t^{\alpha-1}\,\mathrm{d}t$ is the gamma function and

$$\mathcal{J}^{0}_{\varsigma^+}\mathcal{G}(x) = \mathcal{J}^{0}_{\xi^-}\mathcal{G}(x) = \mathcal{G}(x).$$

Definition 2 ([5])**.** The beta function is defined by

$$B(\varsigma,\xi) = \int_0^1 s^{\varsigma-1}(1-s)^{\xi-1}\,\mathrm{d}s = \frac{\Gamma(\varsigma)\Gamma(\xi)}{\Gamma(\varsigma+\xi)},$$

where $\varsigma,\xi > 0$ and $\Gamma(.)$ is the Euler gamma function.

Definition 3 ([5])**.** The incomplete beta function is given by

$$B_{\delta}(\varsigma,\xi)=\int_{0}^{\delta}s^{\varsigma-1}(1-s)^{\xi-1}\,\mathrm{d}s,$$

where $\varsigma,\xi>0$ and $0<\delta<1$.

Definition 4 ([10])**.**

$$(x+y)^{r}=\sum_{k=0}^{\infty}\frac{(r)_k}{k!}x^{r-k}y^{k},$$

$(r)_k$ is the Pochhammer symbol.

3. Main results

Theorem 3. *Let $\mathcal{G}\colon[\varsigma,\xi]\to\mathbb{R}$ be a non-negative convex mapping. Then, we have*

$$\frac{\Gamma(\alpha+1)}{2(\xi-\varsigma)^{\alpha}}\left(\mathcal{J}^{\alpha}_{\varsigma^{+}}\mathcal{G}^{2}(\xi)+\mathcal{J}^{\alpha}_{\xi^{-}}\mathcal{G}^{2}(\varsigma)\right)$$
$$\le\frac{1}{2(\alpha+2)(\alpha+1)}\left[(2+\alpha(\alpha+1))\left(\mathcal{G}^{2}(\varsigma)+\mathcal{G}^{2}(\xi)\right)+4\alpha\mathcal{G}(\varsigma)\,\mathcal{G}(\xi)\right],$$

where $\alpha>0$.

Proof. Using the convexity of $\mathcal{G}$ on $[\varsigma,\xi]$, we have

$$\mathcal{G}(t)\le\tfrac{\xi-t}{\xi-\varsigma}\mathcal{G}(\varsigma)+\tfrac{t-\varsigma}{\xi-\varsigma}\mathcal{G}(\xi).\tag{3.1}$$

Since $\mathcal{G}$ is non-negative, from (3.1) we deduce

$$\begin{aligned}\mathcal{G}^{2}(t)&\le\left(\tfrac{\xi-t}{\xi-\varsigma}\mathcal{G}(\varsigma)+\tfrac{t-\varsigma}{\xi-\varsigma}\mathcal{G}(\xi)\right)^{2}\\&=\tfrac{(\xi-t)^{2}}{(\xi-\varsigma)^{2}}\mathcal{G}^{2}(\varsigma)+2\tfrac{(\xi-t)(t-\varsigma)}{(\xi-\varsigma)^{2}}\mathcal{G}(\varsigma)\,\mathcal{G}(\xi)+\tfrac{(t-\varsigma)^{2}}{(\xi-\varsigma)^{2}}\mathcal{G}^{2}(\xi).\end{aligned}\tag{3.2}$$

Multiplying both sides of (3.2) by $\frac{1}{\Gamma(\alpha)}(\xi-t)^{\alpha-1}$, then integrating the resulting inequality with respect to t over $[\varsigma,\xi]$, we get

$$\begin{aligned}\frac{1}{\Gamma(\alpha)}\int_{\varsigma}^{\xi}(\xi-t)^{\alpha-1}\mathcal{G}^{2}(t)\,\mathrm{d}t&\le\frac{1}{\Gamma(\alpha)}\int_{\varsigma}^{\xi}\frac{(\xi-t)^{\alpha+1}}{(\xi-\varsigma)^{2}}\mathcal{G}^{2}(\varsigma)\,\mathrm{d}t+\frac{2}{\Gamma(\alpha)}\int_{\varsigma}^{\xi}\frac{(\xi-t)^{\alpha}(t-\varsigma)}{(\xi-\varsigma)^{2}}\mathcal{G}(\varsigma)\,\mathcal{G}(\xi)\,\mathrm{d}t\\&\quad+\frac{1}{\Gamma(\alpha)}\int_{\varsigma}^{\xi}\frac{(t-\varsigma)^{2}(\xi-t)^{\alpha-1}}{(\xi-\varsigma)^{2}}\mathcal{G}^{2}(\xi)\,\mathrm{d}t\\&=\frac{(\xi-\varsigma)^{\alpha}}{(\alpha+2)\Gamma(\alpha)}\left(\mathcal{G}^{2}(\varsigma)+\frac{2}{\alpha+1}\mathcal{G}(\varsigma)\,\mathcal{G}(\xi)+\frac{2}{\alpha(\alpha+1)}\mathcal{G}^{2}(\xi)\right).\end{aligned}\tag{3.3}$$

Similarly, multiplying both sides of (3.2) by $\frac{1}{\Gamma(\alpha)}(t-\varsigma)^{\alpha-1}$, then integrating the resulting inequality with respect to t over $[\varsigma,\xi]$, we obtain

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha)}\int_{\varsigma}^{\xi}(t-\varsigma)^{\alpha-1}\mathcal{G}^{2}(t)\,\mathrm{d}t\\
&\leq \frac{1}{\Gamma(\alpha)}\left[\left(\int_{\varsigma}^{\xi}\frac{(\xi-t)^{2}(t-\varsigma)^{\alpha-1}}{(\xi-\varsigma)^{2}}\mathrm{d}t\right)\mathcal{G}^{2}(\varsigma)+2\left(\int_{\varsigma}^{\xi}\frac{(\xi-t)(t-\varsigma)^{\alpha}}{(\xi-\varsigma)^{2}}\mathrm{d}t\right)\mathcal{G}(\varsigma)\,\mathcal{G}(\xi)\right.\\
&\quad\left.+\left(\int_{\varsigma}^{\xi}\frac{(t-\varsigma)^{\alpha+1}}{(\xi-\varsigma)^{2}}\mathrm{d}t\right)\mathcal{G}^{2}(\xi)\right]\\
&=\frac{(\xi-\varsigma)^{\alpha}}{(\alpha+2)\Gamma(\alpha)}\left(\frac{2}{\alpha(\alpha+1)}\mathcal{G}^{2}(\varsigma)+\frac{2}{\alpha+1}\mathcal{G}(\varsigma)\,\mathcal{G}(\xi)+\mathcal{G}^{2}(\xi)\right),
\end{aligned} \tag{3.4}$$

where we have used

$$\int_{\varsigma}^{\xi}(\xi-t)^{\alpha+1}\mathrm{d}t=\int_{\varsigma}^{\xi}(t-\varsigma)^{\alpha+1}\mathrm{d}t=\frac{(\xi-\varsigma)^{\alpha+2}}{\alpha+2},$$

$$\int_{\varsigma}^{\xi}(\xi-t)^{\alpha}(t-\varsigma)\,\mathrm{d}t=\int_{\varsigma}^{\xi}(t-\varsigma)^{\alpha}(\xi-t)\,\mathrm{d}t=\frac{1}{(\alpha+1)(\alpha+2)}(\xi-\varsigma)^{\alpha+2}$$

and

$$\int_{\varsigma}^{\xi}(t-\varsigma)^{2}(\xi-t)^{\alpha-1}\mathrm{d}t=\int_{\varsigma}^{\xi}(\xi-t)^{2}(t-\varsigma)^{\alpha-1}\mathrm{d}t=\frac{2}{\alpha(\alpha+1)(\alpha+2)}(\xi-\varsigma)^{\alpha+2}.$$

Finally, summing (3.3) and (3.4), then multiplying the resulting inequality by $\frac{1}{2(\xi-\varsigma)}$, we get the desired result. □

Remark 1. In Theorem 3, if we take $\alpha=1$, we obtain Theorem 2.

Theorem 4. *Let $\mathcal{G}\colon[\varsigma,\xi]\to\mathbb{R}$ be a non-negative convex mapping. Then we have*

$$\begin{aligned}
&\frac{\Gamma(\alpha+1)}{2(\xi-\varsigma)^{\alpha}}\left(\mathcal{J}^{\alpha}_{\varsigma^{+}}\mathcal{G}^{r}(\xi)+\mathcal{J}^{\alpha}_{\xi^{-}}\mathcal{G}^{r}(\varsigma)\right)\\
&\leq\frac{\alpha}{2^{2-\frac{r}{[r]}}}\sum_{k=0}^{[r]}\left(C^{k}_{[r]}\right)^{\frac{r}{[r]}}(\mathcal{G}(\varsigma))^{r-\frac{kr}{[r]}}(\mathcal{G}(\xi))^{\frac{kr}{[r]}}\\
&\quad\times\left(B\left(\alpha+r-\frac{kr}{[r]},\frac{kr}{[r]}+1\right)+B\left(\alpha+\frac{kr}{[r]},r-\frac{kr}{[r]}+1\right)\right),
\end{aligned}$$

where $\alpha>0$ and $r\geq1$.

Proof. Since $\mathcal{G}$ is non-negative and convex, from (3.1) we deduce

$$\begin{aligned}
\mathcal{G}^r(t) &\leq \left(\tfrac{\xi-t}{\xi-\varsigma}\mathcal{G}(\varsigma)+\tfrac{t-\varsigma}{\xi-\varsigma}\mathcal{G}(\xi)\right)^r \\
&= \left(\left(\tfrac{\xi-t}{\xi-\varsigma}\mathcal{G}(\varsigma)+\tfrac{t-\varsigma}{\xi-\varsigma}\mathcal{G}(\xi)\right)^{[r]}\right)^{\frac{r}{[r]}} \\
&= \left(\sum_{k=0}^{[r]} C_{[r]}^k\left(\tfrac{\xi-t}{\xi-\varsigma}\mathcal{G}(\varsigma)\right)^{[r]-k}\left(\tfrac{t-\varsigma}{\xi-\varsigma}\mathcal{G}(\xi)\right)^k\right)^{\frac{r}{[r]}} \\
&\leq 2^{\frac{r}{[r]}-1}\sum_{k=0}^{[r]}\left(C_{[r]}^k\right)^{\frac{r}{[r]}}\frac{(\xi-t)^{r-\frac{kr}{[r]}}(t-\varsigma)^{\frac{kr}{[r]}}}{(\xi-\varsigma)^r}(\mathcal{G}(\varsigma))^{r-\frac{kr}{[r]}}(\mathcal{G}(\xi))^{\frac{kr}{[r]}}, \qquad (3.5)
\end{aligned}$$

where we have used the following algebraic inequality $(a+b)^h \leq 2^{h-1}\left(a^h+b^h\right)$ for $a,b>0$ and $h>1$ and $(a+b)^h \leq \left(a^h+b^h\right)$ for $0<h<1$ with $[r]$ denoting the integer part of r.

Multiplying both sides of (3.5) by $\frac{1}{\Gamma(\alpha)}(\xi-t)^{\alpha-1}$, then integrating the resulting inequality with respect to t over $[\varsigma,\xi]$, we get

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha)}\int_{\varsigma}^{\xi}(\xi-t)^{\alpha-1}\mathcal{G}^r(t)\,\mathrm{d}t \\
&\leq \frac{1}{\Gamma(\alpha)}\int_{\varsigma}^{\xi}\left(2^{\frac{r}{[r]}-1}\sum_{k=0}^{[r]}\left(C_{[r]}^k\right)^{\frac{r}{[r]}}\frac{(\xi-t)^{\alpha-1+r-\frac{kr}{[r]}}(t-\varsigma)^{\frac{kr}{[r]}}}{(\xi-\varsigma)^r}(\mathcal{G}(\varsigma))^{r-\frac{kr}{[r]}}(\mathcal{G}(\xi))^{\frac{kr}{[r]}}\right)\mathrm{d}t \\
&= \frac{2^{\frac{r}{[r]}-1}}{(\xi-\varsigma)^r\Gamma(\alpha)}\sum_{k=0}^{[r]}\left(C_{[r]}^k\right)^{\frac{r}{[r]}}(\mathcal{G}(\varsigma))^{r-\frac{kr}{[r]}}(\mathcal{G}(\xi))^{\frac{kr}{[r]}}\int_{\varsigma}^{\xi}(\xi-t)^{\alpha-1+r-\frac{kr}{[r]}}(t-\varsigma)^{\frac{kr}{[r]}}\mathrm{d}t \\
&= \frac{2^{\frac{r}{[r]}-1}(\xi-\varsigma)^{\alpha}}{\Gamma(\alpha)}\sum_{k=0}^{[r]}\left(C_{[r]}^k\right)^{\frac{r}{[r]}}(\mathcal{G}(\varsigma))^{r-\frac{kr}{[r]}}(\mathcal{G}(\xi))^{\frac{kr}{[r]}}\int_0^1(1-u)^{\alpha-1+r-\frac{kr}{[r]}}u^{\frac{kr}{[r]}}\mathrm{d}u \\
&= \frac{2^{\frac{r}{[r]}-1}(\xi-\varsigma)^{\alpha}}{\Gamma(\alpha)}\sum_{k=0}^{[r]}\left(C_{[r]}^k\right)^{\frac{r}{[r]}}(\mathcal{G}(\varsigma))^{r-\frac{kr}{[r]}}(\mathcal{G}(\xi))^{\frac{kr}{[r]}}B\left(\alpha+r-\frac{kr}{[r]},\frac{kr}{[r]}+1\right). \qquad (3.6)
\end{aligned}$$

Similarly, multiplying both sides of (3.5) by $\frac{1}{\Gamma(\alpha)}(t-\varsigma)^{\alpha-1}$, then integrating the resulting inequality with respect to t over $[\varsigma,\xi]$, we obtain

$$\frac{1}{\Gamma(\alpha)}\int_{\varsigma}^{\xi}(t-\varsigma)^{\alpha-1}\mathcal{G}^r(t)\,\mathrm{d}t$$

$$\leq \frac{1}{\Gamma(\alpha)}\int_{\varsigma}^{\xi}\left(2^{\frac{r}{[r]}-1}\sum_{k=0}^{[r]}\left(C_{[r]}^{k}\right)^{\frac{r}{[r]}}\frac{(\xi-t)^{r-\frac{kr}{[r]}}(t-\varsigma)^{\alpha-1+\frac{kr}{[r]}}}{(\xi-\varsigma)^{r}}\left(\mathcal{G}(\varsigma)\right)^{r-\frac{kr}{[r]}}\left(\mathcal{G}(\xi)\right)^{\frac{kr}{[r]}}\right)\mathrm{d}t$$

$$= \frac{2^{\frac{r}{[r]}-1}}{(\xi-\varsigma)^{r}\Gamma(\alpha)}\sum_{k=0}^{[r]}\left(C_{[r]}^{k}\right)^{\frac{r}{[r]}}\left(\mathcal{G}(\varsigma)\right)^{r-\frac{kr}{[r]}}\left(\mathcal{G}(\xi)\right)^{\frac{kr}{[r]}}\int_{\varsigma}^{\xi}(\xi-t)^{r-\frac{kr}{[r]}}(t-\varsigma)^{\alpha-1+\frac{kr}{[r]}}\mathrm{d}t$$

$$= \frac{2^{\frac{r}{[r]}-1}(\xi-\varsigma)^{\alpha}}{\Gamma(\alpha)}\sum_{k=0}^{[r]}\left(C_{[r]}^{k}\right)^{\frac{r}{[r]}}\left(\mathcal{G}(\varsigma)\right)^{r-\frac{kr}{[r]}}\left(\mathcal{G}(\xi)\right)^{\frac{kr}{[r]}}\int_{0}^{1}(1-u)^{r-\frac{kr}{[r]}}u^{\alpha-1+\frac{kr}{[r]}}\mathrm{d}u$$

$$= \frac{2^{\frac{r}{[r]}-1}(\xi-\varsigma)^{\alpha}}{\Gamma(\alpha)}\sum_{k=0}^{[r]}\left(C_{[r]}^{k}\right)^{\frac{r}{[r]}}\left(\mathcal{G}(\varsigma)\right)^{r-\frac{kr}{[r]}}\left(\mathcal{G}(\xi)\right)^{\frac{kr}{[r]}}B\left(\alpha+\frac{kr}{[r]},r-\frac{kr}{[r]}+1\right), \tag{3.7}$$

Now, summing (3.6) and (3.7), we get

$$\frac{1}{\Gamma(\alpha)}\int_{\varsigma}^{\xi}(\xi-t)^{\alpha-1}\mathcal{G}^{r}(t)\,\mathrm{d}t+\frac{1}{\Gamma(\alpha)}\int_{\varsigma}^{\xi}(t-\varsigma)^{\alpha-1}\mathcal{G}^{r}(t)\,\mathrm{d}t$$
$$\leq \frac{2^{\frac{r}{[r]}-1}(\xi-\varsigma)^{\alpha}}{\Gamma(\alpha)}\sum_{k=0}^{[r]}\left(C_{[r]}^{k}\right)^{\frac{r}{[r]}}\left(\mathcal{G}(\varsigma)\right)^{r-\frac{kr}{[r]}}\left(\mathcal{G}(\xi)\right)^{\frac{kr}{[r]}}$$
$$\times\left(B\left(\alpha+r-\frac{kr}{[r]},\frac{kr}{[r]}+1\right)+B\left(\alpha+\frac{kr}{[r]},r-\frac{kr}{[r]}+1\right)\right). \tag{3.8}$$

Multiplying both sides of (3.8) by $\frac{1}{2(\xi-\varsigma)}$, we get the desired result. □

Corollary 1. *For $r=1$, Theorem 4 gives*

$$\frac{\Gamma(\alpha+1)}{2(\xi-\varsigma)^{\alpha}}\left(\mathcal{J}_{\varsigma^{+}}^{\alpha}\mathcal{G}(\xi)+\mathcal{J}_{\xi^{-}}^{\alpha}\mathcal{G}(\varsigma)\right)\leq\frac{\mathcal{G}(\varsigma)+\mathcal{G}(\xi)}{2},$$

which is the fractional version of Hermite-Hadamard inequality obtained by Sarikaya et al. in [8].

Moreover, if we take $\alpha=1$, then we get the second inequality given in Theorem 1.

Corollary 2. *For $r=2$, Theorem 4 gives*

$$\frac{\Gamma(\alpha+1)}{2(\xi-\varsigma)^{\alpha}}\left(\mathcal{J}_{\varsigma^{+}}^{\alpha}\mathcal{G}^{2}(\xi)+\mathcal{J}_{\xi^{-}}^{\alpha}\mathcal{G}^{2}(\varsigma)\right)$$
$$\leq\frac{\alpha}{2}\left((B(1,\alpha+2)+B(3,\alpha))\,\mathcal{G}^{2}(\varsigma)+4B(2,\alpha+1)\,\mathcal{G}(\varsigma)\,\mathcal{G}(\xi)\right)$$
$$+\left(B(3,\alpha)+B(1,\alpha+2)\right)\mathcal{G}^{2}(\xi)\big)$$
$$=\frac{1}{2(\alpha+2)(\alpha+1)}\left[(2+\alpha(\alpha+1))\left(\mathcal{G}^{2}(\varsigma)+\mathcal{G}^{2}(\xi)\right)+4\alpha\mathcal{G}(\varsigma)\,\mathcal{G}(\xi)\right],$$

which is the same result obtained in Theorem 3.

4. CONCLUSION

In conclusion, this paper successfully extends the Sándor inequality to the context of Riemann-Liouville fractional integrals, offering new insights into the application of this important inequality in fractional calculus. By generalizing the inequality for any positive power r, we have expanded its utility in both pure and applied mathematics, particularly in areas such as optimization and inequality theory. The results presented not only build upon classical findings but also provide a broader framework for addressing complex problems in number theory, real and complex analysis, and mathematical physics. This work highlights the continued relevance and versatility of the Sándor inequality in modern mathematical research.

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A GENERALIZATION OF ESSENTIAL SUPPLEMENTED LATTICES

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Abstract. In this work, all lattices are complete modular lattices with the smallest element 0 and the greatest element 1. Let L be a lattice. If every essential element of L has a weak supplement in L, then L is called a weakly essential supplemented (briefly, weakly e-supplemented) lattice. In this work, some properties of these lattices are investigated. The concept of weakly essential supplemented lattice is a generalization of the concept of essential supplemented lattice. Let L be a weakly e-supplemented lattice. Then $1/r(L)$ have no essential elements with distinct from 1. Let L be a lattice, $a_1, a_2, \ldots, a_n \in L$ and $1 = a_1 \vee a_2 \vee \cdots \vee a_n$. If $a_i/0$ is weakly e-supplemented for every $i = 1, 2, \ldots, n$, then L is also weakly e-supplemented. Let L be a weakly e-supplemented lattice and $a \in L$. Then the quotient sublattice $1/a$ is weakly e-supplemented. Let L be a lattice. Then L is weakly e-supplemented if and only if every essential element of L is β_* equivalent to a weak supplement in L.

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Keywords: essential elements, small elements, radical, supplemented lattices

1. INTRODUCTION

Throughout this paper, all lattices are complete modular lattices with the smallest element 0 and the greatest element 1. Let L be a lattice, $a, b \in L$ and $a \leq b$. A sublattice $\{x \in L | a \leq x \leq b\}$ is called a *quotient sublattice*, denoted by b/a. An element a' of a lattice L is called a *complement* of a in L if $a \wedge a' = 0$ and $a \vee a' = 1$. In this case we say a and a' are *direct summands* of L and denoted by $1 = a \oplus a'$. A lattice L is said to be *complemented* if each element of L has at least one complement in L. An element c of L is said to be *compact* if for every subset X of L such that $c \leq \vee X$ there is a finite $F \subset X$ such that $c \leq \vee F$. A lattice L is said to be *compact* if 1 is compact. An element a of L is said to be *small* or *superfluous* in L and denoted by $a \ll L$ if $a \vee b \neq 1$ holds for every $b \neq 1$, or equivalently, $b = 1$ for every $b \in L$ with $a \vee b = 1$. An element a of L is said to be *essential* if $a \wedge b \neq 0$ holds for every $b \neq 0$ and denoted by $a \trianglelefteq L$. This equivalent to $a \wedge b = 0$ implies that $b = 0$. The meet of all maximal elements ($\neq 1$) of a lattice L is called the *radical* of L and denoted by $r(L)$. An element c of L is called a *supplement* of b in L if it is minimal for $b \vee c = 1$. a is a supplement of b in a lattice L if and only if $a \vee b = 1$ and $a \wedge b \ll a/0$. L is called a *supplemented* lattice if every element of L has a supplement in L. If every element of L has a supplement that is a direct summand of L, then L is called a $\oplus-$*supplemented* lattice. L is called an *essential supplemented* (briefly, *e-supplemented*) lattice if every essential element of L has a supplement in L. We say that an element b of L *lies above* an element a of L if $a \leq b$ and $b \ll 1/a$. L is said to be *hollow* if every element ($\neq 1$) is superfluous in L, and L is said to be *local* if L has the greatest element ($\neq 1$). An element a of L is called a *weak supplement* of b in L if $a \vee b = 1$ and $a \wedge b \ll L$. L is called a *weakly supplemented* lattice if every element of L has a weak supplement in L. An element $a \in L$ has *ample supplements* in L if for every $b \in L$ with $a \vee b = 1$, a has a supplement b' in L with $b' \leq b$. L is called an *amply supplemented* lattice if every element of L has ample supplements in L. L is called an *amply essential supplemented* (briefly, *amply e-supplemented*) lattice if every essential element of L has ample supplements in L. It is clear that every supplemented lattice is weakly supplemented and every amply supplemented lattice is supplemented. Let L be a lattice. It is defined β_* relation on the elements of L by $a\beta_* b$ with $a, b \in L$ if and only if for each $t \in L$ such that $a \vee t = 1$ then $b \vee t = 1$ and for each $k \in L$ such that $b \vee k = 1$ then $a \vee k = 1$.

More information about (amply) supplemented lattices are in [5, 11]. More details about weakly supplemented lattices are in [1]. More information about $\oplus-$supplemented lattices are in [2]. More details about (amply) essential supplemented lattices are in [14, 19]. More information about (amply) supplemented modules are in [4, 7–10]. More information about weakly supplemented modules are in [6]. More details about (amply) essential supplemented modules are in [16, 17]. More details about weakly essential supplemented modules are in [12]. The definition of β_* relation on lattices and some properties of this relation are in [13]. This relation is a generalization of β^* relation on modules. The definition of β^* relation on modules and some properties of this relation are in [3, 18]. More details about lying above on lattices are in [5, 11, 13]. More details about lying above on modules are in [9, 10].

Lemma 1. *Let L be a lattice. The following assertions hold.*

(1) *If* $a, b \in L$ *and* $a \leq b$, *then* $a \trianglelefteq L$ *if and only if* $a \trianglelefteq b/0$ *and* $b \trianglelefteq L$.
(2) *Let* $a, b \in L$ *and* $a \leq b$. *If* $b \trianglelefteq 1/a$, *then* $b \trianglelefteq L$.
(3) *Let* $a, b, c, d \in L$, $a \leq c$ *and* $b \leq d$. *If* $a \trianglelefteq c/0$ *and* $b \trianglelefteq d/0$, *then* $a \wedge b \trianglelefteq (c \wedge d)/0$.
(4) *If* $a \trianglelefteq L$ *and* $b \trianglelefteq L$, *then* $a \wedge b \trianglelefteq L$.
(5) *If* $a \trianglelefteq L$, *then* $a \wedge b \trianglelefteq b/0$ *for every* $b \in L$.
(6) *If* $a \trianglelefteq L$, *then* $a \vee b \trianglelefteq L$ *for every* $b \in L$.

Proof. See [5]. □

2. WEAKLY ESSENTIAL SUPPLEMENTED LATTICES

Definition 1. Let L be a lattice. If every essential element of L has a weak supplement in L, then L is called a weakly essential supplemented (briefly, weakly e-supplemented) lattice. (See also [15]).

Clearly we can see that every essential supplemented lattice is weakly essential supplemented, but the converse of this statement is not true in general. (See Example 2). The concept of weakly essential supplemented lattice is a generalization of the concept of essential supplemented lattice.

Definition 2. Let L be a lattice and $x \in L$. If x is a weak supplement of an essential element of L, then x is called a weak e-supplement element in L.

Proposition 1. *Let L be a weakly e-supplemented lattice. If every element of L with distinct from 0 is essential in L, then L is weakly supplemented.*

Proof. Clear from definitions. □

Lemma 2. *Let L be a weakly e-supplemented lattice. Then $1/r(L)$ have no essential elements with distinct from 1.*

Proof. Let k be any essential element of $1/r(L)$. Since $k \trianglelefteq 1/r(L)$, by Lemma 1, $k \trianglelefteq L$ and since L is weakly e-supplemented, k has a weak supplement t in L. Then $1 = k \vee t$ and $k \wedge t \ll L$. Since $1 = k \vee t$, $1 = k \vee (t \vee r(L))$. Since $k \wedge t \ll L$, by [5, Lemma 7.6], $k \wedge t \leq r(L)$. Then $k \wedge (t \vee r(L)) = (k \wedge t) \vee r(L) = r(L)$ and $1 = k \oplus (t \vee r(L))$ in $1/r(L)$. Since $1 = k \oplus (t \vee r(L))$ in $1/r(L)$ and $k \trianglelefteq 1/r(L)$, $k = 1$. Hence $1/r(L)$ have no essential elements with distinct from 1. □

Corollary 1. *Let L be a weakly e-supplemented lattice. If $r(L) = 0$, then L have no essential elements with distinct from 1.*

Proof. Since $r(L) = 0$, $L = 1/0 = 1/r(L)$. Then by Lemma 2, L have no essential elements with distinct from 1. □

Corollary 2. *Let L be a weakly e-supplemented lattice, $k \trianglelefteq L$ and $k \vee r(L) \neq 1$. Then $k \vee r(L)$ is not essential in $1/r(L)$.*

Proof. Clear from Lemma 2. □

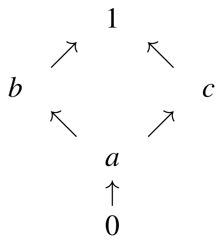
Corollary 3. *Let L be a weakly e-supplemented lattice, $k \trianglelefteq L$ and $r(L) \leq k$. Then k is not essential in $1/r(L)$.*

Proof. Clear from Corollary 2. □

Corollary 4. *Let L be an essential supplemented lattice, $k \trianglelefteq L$ and $r(L) \leq k$. Then k is not essential in $1/r(L)$.*

Proof. Clear from Corollary 3, since every essential supplemented lattice is weakly essential supplemented. □

Example 1. Consider the lattice $L = \{0, a, b, c, 1\}$ given by the following diagram:



Here $r(L) = a$, $b \trianglelefteq L$ but $b \ntrianglelefteq 1/r(L)$. Here also $c \trianglelefteq L$ but $c \ntrianglelefteq 1/r(L)$.

Lemma 3. *Let L be a lattice and $r(L) \ll L$. If $a \vee r(L)$ is a direct summand of $1/r(L)$ for every essential element a of L, then L is weakly e-supplemented.*

Proof. Let $a \trianglelefteq L$. By hypothesis, $a \vee r(L)$ is a direct summand of $1/r(L)$ and there exists $x \in 1/r(L)$ such that $a \vee r(L) \vee x = 1$ and $(a \vee r(L)) \wedge x = r(L)$. Here $1 = a \vee r(L) \vee x = a \vee x$ and $a \wedge x \leq (a \vee r(L)) \wedge x = r(L)$. Since $r(L) \ll L$, $a \wedge x \ll L$. Hence x is a weak supplement of a in L and L is weakly e-supplemented. □

Corollary 5. *Let L be a compact lattice. If $a \vee r(L)$ is a direct summand of $1/r(L)$ for every essential element a of L, then L is weakly e-supplemented.*

Proof. Since L is compact, by [5, Lemma 7.8 (iii)], $r(L) \ll L$. Then by Lemma 3, L is weakly e-supplemented, as desired. □

Corollary 6. *Let L be a lattice and $r(L) \ll L$. If a is a direct summand of $1/r(L)$ for every essential element a of L with $r(L) \leq a$, then L is weakly e-supplemented.*

Proof. Let $x \trianglelefteq L$. By Lemma 1, $x \vee r(L) \trianglelefteq L$ and by hypothesis, $x \vee r(L)$ is a direct summand of $1/r(L)$. Then by Lemma 3, L is weakly e-supplemented, as desired. □

Corollary 7. *Let L be a compact lattice. If a is a direct summand of $1/r(L)$ for every essential element a of L with $r(L) \leq a$, then L is weakly e-supplemented.*

Proof. Since L is compact, by [5, Lemma 7.8 (iii)], $r(L) \ll L$. Then by Corollary 6, L is weakly e-supplemented, as desired. □

Lemma 4. *Let L be a lattice and $r(L) \ll L$. If $a \vee r(L)$ is a direct summand of $1/r(L)$ for every $a \in L$, then L is weakly supplemented.*

Proof. Similar to proof of Lemma 3. □

Corollary 8. *Let L be a compact lattice. If $a \vee r(L)$ is a direct summand of $1/r(L)$ for every $a \in L$, then L is weakly supplemented.*

Proof. Since L is compact, by [5, Lemma 7.8 (iii)], $r(L) \ll L$. Then by Lemma 4, L is weakly supplemented, as desired. □

Corollary 9. *Let L be a lattice and $r(L) \ll L$. If every element of $1/r(L)$ is a direct summand of $1/r(L)$, then L is weakly supplemented.*

Proof. Let $x \in L$. By hypothesis, $x \vee r(L)$ is a direct summand of $1/r(L)$. Then by Lemma 4, L is weakly supplemented, as desired. □

Corollary 10. *Let L be a compact lattice. If every element of $1/r(L)$ is a direct summand of $1/r(L)$, then L is weakly supplemented.*

Proof. Since L is compact, by [5, Lemma 7.8 (iii)], $r(L) \ll L$. Then by Corollary 9, L is weakly supplemented, as desired. □

Lemma 5. *Let L be a lattice, x be an essential element of L and $m \in L$. If $m/0$ is weakly e-supplemented and $x \vee m$ has a weak supplement in L, then x has a weak supplement in L.*

Proof. Let y be a weak supplement of $x \vee m$ in L. Then $1 = x \vee m \vee y$ and $(x \vee m) \wedge y \ll L$. Since $x \trianglelefteq L$, by Lemma 1, $(x \vee y) \trianglelefteq L$ and $(x \vee y) \wedge m \trianglelefteq m/0$. Since $m/0$ is weakly e-supplemented, $(x \vee y) \wedge m$ has a weak supplement z in $m/0$. This case $m = ((x \vee y) \wedge m) \vee z$ and $(x \vee y) \wedge z = (x \vee y) \wedge m \wedge z \ll m/0$. Then $1 = x \vee m \vee y = x \vee ((x \vee y) \wedge m) \vee z \vee y = x \vee y \vee z$ and $x \wedge (y \vee z) \leq ((x \vee y) \wedge z) \vee ((x \vee z) \wedge y) \leq ((x \vee y) \wedge z) \vee ((x \vee m) \wedge y) \ll L$. Hence $y \vee z$ is a weak supplement of x in L. □

Corollary 11. *Let L be a lattice, $x \trianglelefteq L$ and $m_1, m_2, \ldots, m_n \in L$. If $x \vee m_1 \vee m_2 \vee \cdots \vee m_n$ has a weak supplement in L and $m_i/0$ is weakly e-supplemented for every $i = 1, 2, \ldots, n$, then x has a weak supplement in L.*

Proof. Clear from Lemma 5. □

Lemma 6. *Let L be a lattice, $a_1, a_2 \in L$ and $1 = a_1 \vee a_2$. If $a_1/0$ and $a_2/0$ are weakly e-supplemented, then L is also weakly e-supplemented.*

Proof. Let $x \trianglelefteq L$. Then 0 is a weak supplement of $x \vee a_1 \vee a_2$ in L. Since $a_2/0$ is weakly e-supplemented and $x \vee a_1 \trianglelefteq L$, by Lemma 5, $x \vee a_1$ has a weak supplement in L. Since $a_1/0$ is weakly e-supplemented and $x \trianglelefteq L$, by Lemma 5, x has a weak supplement in L. Hence L is weakly e-supplemented. □

Corollary 12. *Let L be a lattice, $a_1, a_2, \ldots, a_n \in L$ and $1 = a_1 \vee a_2 \vee \cdots \vee a_n$. If $a_i/0$ is weakly e-supplemented for every $i = 1, 2, \ldots, n$, then L is also weakly e-supplemented.*

Proof. Clear from Lemma 6. □

Lemma 7. *Let L be a weakly e-supplemented lattice and $a \in L$. Then the quotient sublattice $1/a$ is weakly e-supplemented.*

Proof. Let $x \trianglelefteq 1/a$. Then by Lemma 1, $x \trianglelefteq L$ and since L is weakly e-supplemented, x has a weak supplement y in L. Since $a \leq x$, we can easily see that $a \vee y$ is a weak supplement of x in $1/a$. Hence $1/a$ is weakly e-supplemented. □

Corollary 13. *Let L be a weakly e-supplemented lattice. Then $a/0$ is weakly e-supplemented for every direct summand a of L.*

Proof. Let a be a direct summand of L. Then there exists $b \in L$ such that $a \oplus b = 1$. By Lemma 7, $1/b$ is weakly e-supplemented. Then by $1/b = (a \vee b)/b \cong a/(a \wedge b) = a/0$, $a/0$ is weakly e-supplemented. □

Lemma 8. *Let L be a lattice and $a, b, c \in L$. If $a \vee b = 1$ and $(a \wedge b) \vee c = 1$, then $a \vee (b \wedge c) = 1$ and $b \vee (a \wedge c) = 1$.*

Proof. See [13, Lemma 2]. □

Lemma 9. *Let L be a lattice. Then L is weakly e-supplemented if and only if every essential element of L is β_* equivalent to a weak supplement in L.*

Proof. ($\Longrightarrow$) Let L be a weakly e-supplemented lattice and a be any essential element of L. Since L is weakly e-supplemented, a has a weak supplement b in L. Then a is also a weak supplement of b in L. Since $a\beta_* a$ in L, a is β_* equivalent to a weak supplement in L.

($\Longleftarrow$) Let every essential element of L is β_* equivalent to a weak supplement in L. Let a be any essential element of L. By hypothesis, there exists a weak supplement element b in L with $a\beta_* b$. Let b be a weak supplement of c in L. Then c is a weak supplement of b in L and since $a\beta_* b$, by [13, Theorem 4], c is a weak supplement of a in L. Hence L is weakly e-supplemented. □

Corollary 14. *Let L be a lattice. Then L is weakly e-supplemented if and only if every essential element of L lies above a weak supplement in L.*

Proof. ($\Longrightarrow$) Let L be a weakly e-supplemented lattice and a be any essential element of L. Since L is weakly e-supplemented, a has a weak supplement b in L. Then a is also a weak supplement of b in L. Since a lies above a in L, a lies above a weak supplement in L.

($\Longleftarrow$) Clear from Lemma 9. But we prove this part as follows:

Let every essential element of L lies above a weak supplement in L. Let a be any essential element of L. By hypothesis, a lies above a weak supplement b in L. Let b be a weak supplement of c in L. Since b is a weak supplement of c in L, $b \vee c = 1$ and $b \wedge c \ll L$. Since a lies above b in L, $b \leq a$ and $a \ll 1/b$. Since $b \leq a$ and $b \vee c = 1$, $a \vee c = 1$. Let $(a \wedge c) \vee t = 1$ for $t \in L$. By Lemma 8, $a \vee (c \wedge t) = 1$. Then $a \vee (c \wedge t) \vee b = 1$ and since $a \ll 1/b$, $(c \wedge t) \vee b = 1$. By also Lemma 8, $(b \wedge c) \vee t = 1$. Since $(b \wedge c) \vee t = 1$ and $b \wedge c \ll L$, $t = 1$. Hence c is a weak supplement of a in L. Thus L is weakly e-supplemented. □

Corollary 15. *Let L be a lattice. Then L is weakly e-supplemented if and only if every essential element of L is a weak supplement in L.*

Proof. Clear from Lemma 9. □

Corollary 16. *Let L be a lattice. If every essential element of L is β_* equivalent to a weak e-supplement element in L, then L is weakly e-supplemented.*

Proof. Clear from Lemma 9. □

Corollary 17. *Let L be a lattice. If every essential element of L lies above a weak e-supplement element in L, then L is weakly e-supplemented.*

Proof. Clear from Corollary 16. □

Lemma 10. *Let L be a lattice. If every element of L has a weak supplement that is a supplement element in L, then L is supplemented.*

Proof. Let $a \in L$. By hypothesis, a has weak supplement b that is a supplement element in L. Here $a \vee b = 1$ and $a \wedge b \ll L$. Since $a \wedge b \ll L$ and b is a supplement element in L, by [11, Lemma 10], $a \wedge b \ll b/0$ and b is a supplement of a in L. Hence L is supplemented, as desired. □

Corollary 18. *Let L be a lattice. If every element of L has a weak supplement that is a direct summand of L, then L is $\oplus-$supplemented. (See also [2, Proposition 2]).*

Proof. Clear from Lemma 10. □

Lemma 11. *Let L be a lattice. If every essential element of L has a weak supplement that is a supplement element in L, then L is essential supplemented.*

Proof. Similar to proof of Lemma 10. □

Corollary 19. *Let L be a lattice. If every essential element of L has a weak supplement that is a direct summand of L, then L is essential supplemented.*

Proof. Clear from Lemma 11. □

Example 2. Let Γ be a family all submodules of the $\mathbb{Z}-$module ${}_{\mathbb{Z}}\mathbb{Q}$. It is clear that Γ is a lattice by the operation $\subset$. Here for $A, B \in \Gamma$, $A \vee B = A + B$ and $A \wedge B = A \cap B$. By [4, Example 20.12], ${}_{\mathbb{Z}}\mathbb{Q}$ is weakly supplemented but not supplemented. Hence the lattice Γ is weakly supplemented but not supplemented. Since every nonzero element of Γ is essential in Γ, Γ is weakly essential supplemented but not essential supplemented.

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STRONG ALMOST CONVERGENCE WITH RESPECT TO AN ORLICZ FUNCTION

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Abstract. In this paper, using an Orlicz function we extend the concept of strong almost convergence and show that strong almost convergence, uniform statistical convergence and strong almost convergence with respect to an Orlicz function are all equivalent on bounded sequences. The main tool in proving the result is to consider ideals in bounded sequences.

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1. Introduction

Convergence methods such as statistical and uniform statistical convergences, strong almost convergence are of some interest in mathematical analysis (see, e.g., [1, 5–7, 16–18, 20, 21]).

In the present paper, motivated by those of Demirci [7] and Şahin Bayram [6] we introduce the concept of strong almost convergence with respect to an Orlicz function and show that uniform statistical convergence, strong almost convergence and strong almost convergence with respect to an Orlicz function are all equivalent on bounded sequences.

We first collect some basic concept and notation.

Let $E \subseteq \mathbb{N}$, where $\mathbb{N}$ is the set of all natural numbers. By $E(m,n)$ we denote the cardinality of the set of natural numbers i so that $m \le i \le n$, where $m,n \in \mathbb{N}$. We now consider the numbers

$$\underline{\mathrm{d}}(E) = \liminf_n \frac{E(1,n)}{n}, \quad \bar{\mathrm{d}}(E) = \limsup_n \frac{E(1,n)}{n}.$$

$\underline{\mathrm{d}}(E)$ and $\bar{\mathrm{d}}(E)$ are respectively called the lower and upper densities of the set E. If $\underline{\mathrm{d}}(E) = \bar{\mathrm{d}}(E)$ then the common value $\mathrm{d}(E)$ is called the asymptotic density of E.

The lower and upper uniform densities of $E \subseteq \mathbb{N}$ have been respectively introduced in ([2],[3]) as follows;

$$\underline{\mathrm{u}}(E) = \lim_n \min_{i \geqslant 0} \frac{E(i+1,i+n)}{n}, \quad \bar{\mathrm{u}}(E) = \lim_n \max_{i \geqslant 0} \frac{E(i+1,i+n)}{n}.$$

If $\mathrm{u}(E) := \underline{\mathrm{u}}(E) = \bar{\mathrm{u}}(E)$ then $\mathrm{u}(E)$ is called the uniform density of E. It is known [1] that

$$\underline{\mathrm{u}}(E) \leq \underline{\mathrm{d}}(E) \leq \bar{\mathrm{d}}(E) \leq \bar{\mathrm{u}}(E).$$

Let $x := (x_n)$ be a sequence of real numbers and let

$$K_\varepsilon = \{n \in \mathbb{N} : \ |x_n - L| \geq \varepsilon\}, \quad (\varepsilon > 0).$$

If, for every $\varepsilon > 0$, $\mathrm{d}(K_\varepsilon) = 0$ then x is said to be statistically convergent to L (see, e.g., [8],[11],[22]); and $\mathrm{u}(K_\varepsilon) = 0$ then x is said to be uniformly statistically convergent to L (see, e.g., [1],[21]).

By S and S_u we respectively denote the spaces of all statistically convergent and uniformly statistically convergent sequences. A connection between strong convergence, statistical convergence may be found in [4] (see also [13],[14]) and strong convergence with respect to a modulus function is also made in [5].

There is also a close connection between statistical convergence and almost convergence [19],[23]. Recall that a sequence $x = (x_k)$ is almost convergent to L if and only if

$$\lim_n \frac{1}{n} \sum_{k=i+1}^{i+n} x_k = L, \quad (uniformly\ in\ i)$$

(see [16]). If

$$\lim_n \frac{1}{n} \sum_{k=i+1}^{i+n} |x_k - L| = 0$$

then x is said to be strongly almost convergent to L (see, e.g., [1],[10],[17]).

By c, f, $[f]$, $[f]_0$, l^∞ we respectively denote the spaces of convergent, almost convergent, strongly almost convergent, strongly almost convergent to 0 and bounded sequences then we have the inclusions [17] that

$$c \subset [f] \subset f \subset l^\infty.$$

Combining this inclusion result with Theorem 2 of [1] we have the following

$$[f] = S_u \cap l^\infty. \tag{1.1}$$

Some inclusion relations between strong almost convergence and lacunary statistical convergence is considered in [12]. Later on Pehlivan [21] gave inclusion results between uniform statistical convergence and strong almost convergence with respect to a modulus function.

Motivated by those of Demirci [7] and Şahin Bayram [6] we establish relationships between uniform statistical convergence and strong almost convergence with respect to an Orlicz function.

We collect some basic concepts and notation.

Let $F : [0,\infty) \to [0,\infty)$ be a continuous, nondecreasing and convex function with $F(0) = 0$, $F(x) > 0$ for $x > 0$ and $F(x) \to \infty$ as $x \to \infty$.

Such a function is called Orlicz function [15]. If the convexity condition of F is replaced by $F(x+y) \leq F(x) + F(y)$ then F is called modulus function (see, e.g., [18]).

Throughout the paper let e denote the sequence which is identically 1 and let

$$w := \{all\ real\ valued\ sequences\}.$$

Given an Orlicz function F we introduce the following sequence spaces;

$$[f,F]_0 = \left\{ x \in w : \lim_n \frac{1}{n} \sum_{k=i+1}^{i+n} F(|x_k|) = 0,\ uniformly\ in\ i \right\},$$
$$[f,F] = \{x \in w : x - Le \in [f,F]_0\ for\ some\ L\}.$$

Note that when $F(x) = x$ then we get that

$$[f,F]_0 = [f]_0\ and\ [f,F] = [f].$$

If $x \in [f,F]$ we say that x is strongly almost convergent to L with respect to an Orlicz function F.

Recall that $x = (x_k)$ is uniformly statistically convergent to L if $\chi_{K(x-Le;\varepsilon)}$ is contained in $[f]_0$ for every $\varepsilon > 0$ where $\chi_{K(x;\varepsilon)}$ is the characteristic function of the set

$$K(x;\varepsilon) = \{k \in \mathbb{N} : |x_k| \geqslant \varepsilon\}.$$

Let F be an Orlicz function. If there is a constant $H > 0$ such that

$$F(2u) \leq HF(u),\ (for\ all\ u > 0)$$

we say that the Orlicz function F satisfies Δ_2-condition. This condition is equivalent to the fact that

$$F(tu) \leq HtF(u), \tag{1.2}$$

for all $u \geqslant 0$ and for $t \geqslant 1$ (see, e.g., [15]).

2. Main Results

In this section we are concerned with some inclusion results among the spaces $[f]$, $[f,F]$ and S_u; and show that strong almost convergence, uniform statistical convergence and strong almost convergence with respect to a modulus function F are equivalent on bounded sequences provided that F satisfies Δ_2-condition.

In order to prove our first result we use the idea given by Parashar and Choudhary [20].

Proposition 1. *Let F be an Orlicz function satisfying Δ_2-condition. Then we have the inclusions*

$$[f]_0 \subset [f,F]_0 \text{ and } [f] \subset [f,F].$$

Proof. It is enough to prove that $[f]_0 \subset [f,F]_0$. Let $x=(x_k) \in [f]_0$ and F be an Orlicz function satisfying Δ_2-condition. Since F is right continuous at zero, given $\varepsilon > 0$ there exists δ with $0<\delta<1$ such that $F(t)<\varepsilon$ whenever $0 \le t < \delta$.

Hence we get

$$\frac{1}{n}\sum_{k=i+1}^{i+n} F(|x_k|) = \frac{1}{n}\sum_{\substack{k=i+1\\|x_k|<\delta}}^{i+n} F(|x_k|) + \frac{1}{n}\sum_{\substack{k=i+1\\|x_k|\geqslant\delta}}^{i+n} F(|x_k|) < \frac{1}{n}\varepsilon n + \frac{1}{n}\sum_{\substack{k=i+1\\|x_k|\geqslant\delta}}^{i+n} F(|x_k|). \quad (2.1)$$

Since $0<\delta<1$, we get for every $k\in\mathbb{N}$, that $|x_k| < \frac{1}{\delta}|x_k| < 1+\frac{|x_k|}{\delta}$.

On the other hand F is an Orlicz function satisfying Δ_2-condition, so we have by (1.2) that

$$F(|x_k|) < F\left(1+\frac{|x_k|}{\delta}\right) = F\left(\frac{1}{2}2+\frac{1}{2}\frac{2|x_k|}{\delta}\right) \le \frac{1}{2}F(2)+\frac{H|x_k|}{2\delta}F(2)$$
$$< (1+H)F(2)\frac{|x_k|}{\delta}. \quad (2.2)$$

Combining (2.1) and (2.2) we get

$$\frac{1}{n}\sum_{k=i+1}^{i+n} F(|x_k|) < \varepsilon + \frac{(1+H)}{\delta}F(2)\frac{1}{n}\sum_{k=i+1}^{i+n}|x_k|.$$

Since $x=(x_k)\in[f]_0$ we conclude that $x\in[f,F]_0$. This proves the result. □

The next result is an analog of Lemma 1 of Demirci [7].

Lemma 1. *Let F be an Orlicz function satisfying Δ_2-condition. Then $[f,F]_0$ is an ideal in l^∞.*

Proof. Let $x\in[f,F]_0$ and $y\in l^\infty$. We prove that $xy\in[f,F]_0\cap l^\infty$. Since $y\in l^\infty$, there is $H_1>1$ so that $\|y\|\le H_1$. Since F is nondecreasing and satisfies Δ_2-condition we have

$$F(|x_k y_k|) \le F(H_1|x_k|) \le H(1+H_1)F(|x_k|),\ (H>0).$$

Hence, one can get

$$\frac{1}{n}\sum_{k=i+1}^{i+n} F(|x_k y_k|) \le H(1+H_1)\frac{1}{n}\sum_{k=i+1}^{i+n} F(|x_k|)$$

from which we conclude that

$$\lim_n \frac{1}{n}\sum_{k=i+1}^{i+n} F(|x_k y_k|) = 0, \text{ uniformly in } i.$$

□

We will require the following lemmas:

Lemma 2. *Let M be an ideal in l^∞ and let $x \in l^\infty$.Then x is in the closure of M in l^∞ if and only if $\chi_{K(x;\varepsilon)} \in M$ for all $\varepsilon > 0$ (see, [5]).*

As in Lemma 1 one can see that $[f]_0$ is an ideal in l^∞. On the other hand we can get the following result by using the idea that Freedman and Sember [9] used.

Lemma 3. *$[f]_0$ is closed ideal in l^∞.*

Our next result concerns the equality $[f] = [f,F] \cap l^\infty$.

Theorem 1. *Let F be an Orlicz function which satisfies Δ_2-condition. Then we have*

$$[f] = [f,F] \cap l^\infty.$$

Proof. We just prove that $[f]_0 = [f,F]_0 \cap l^\infty$. By Proposition 1 we have $[f]_0 \subset [f,F]_0$. In order to prove the opposite inclusion we first note that

$$\frac{1}{n}\sum_{k=i+1}^{i+n} F\left(\chi_{K(x;\varepsilon)}(k)\right) = F(1)\frac{1}{n}\sum_{k=i+1}^{i+n} \chi_{K(x;\varepsilon)}(k). \tag{2.3}$$

Let $x \in [f,F]_0 \cap l^\infty$ and $\varepsilon > 0$. Now define a sequence $y = (y_k)$ by $y_k = \dfrac{1}{x_k}$ if $|x_k| \geqslant \varepsilon$ and $y_k = 0$ otherwise. Since $xy = \chi_{K(x;\varepsilon)}$ and $[f,F]_0 \cap l^\infty$ is an ideal in l^∞ we get that $\chi_{K(x;\varepsilon)} \in [f,F]_0 \cap l^\infty$ and therefore

$$\lim_n \frac{1}{n}\sum_{k=i+1}^{i+n} F\left(\chi_{K(x;\varepsilon)}(k)\right) = 0, \text{ uniformly in } i.$$

Now (2.3) implies that

$$\lim_n \frac{1}{n}\sum_{k=i+1}^{i+n} \chi_{K(x;\varepsilon)}(k) = 0.$$

Combining this result with Lemma 2 and Lemma 3 we conclude that $x \in [f]_0$. This proves the theorem. □

By (1.1) and Theorem 1 we get the following

Corollary 1. *If F is an Orlicz function satisfying Δ_2-condition then we have*

$$[f] = [f,F] \cap l^\infty = S_u \cap l^\infty.$$

Our final result shows that $[f,F]$ lies between $[f]$ and S_u.

Theorem 2. *Let F be an Orlicz function satisfying Δ_2-condition. Then we have*

$$[f] \subset [f,F] \subset S_u.$$

Proof. The first inclusion is given in Proposition 1. We now prove that $[f,F] \subset S_u$. Let $x \in [f,F]_0$ and $y \in l^{\infty}$. By Lemma 1 we have that $xy \in [f,F]_0$. Let $\varepsilon > 0$ and $x \in [f,F]_0$. We define a bounded sequence $y = (y_k)$ by $y_k = \frac{1}{x_k}$, if $|x_k| \geqslant \varepsilon$ and $y_k = 0$ otherwise. Hence $xy = \chi_{K(x;\varepsilon)} \in [f,F]_0$. Now Theorem 1 implies that $\chi_{K(x;\varepsilon)} \in [f]_0$. So inclusion (1.1) implies that x is uniformly statistically convergent. □

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ON THE REPRESENTATION OF SOLUTION FOR A CLASS OF PERTURBED CONTROLLED NEUTRAL FUNCTIONAL-DIFFERENTIAL EQUATION WITH THE CONTINUOUS INITIAL CONDITION

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Abstract. The analytic relation between of solutions of the original Cauchy problem and a corresponding perturbed problem is established for the controlled neutral functional-differential equation with the continuous initial condition, whose right-hand side is linear with respect to the prehistory of the phase velocity. In the representation formula of solution the effects of perturbations of the delay parameter containing in the phase coordinates, of the initial and control functions are revealed. Continuity at the initial moment means that at the initial moment values of the initial function and trajectory always coincide. The representation formula of solution plays an important role in proving the necessary conditions of optimality in neutral optimization problems, allows one to get an approximate solution of the perturbed equation and to carry out a sensitivity analysis of mathematical models.

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1. Introduction

The neutral functional-differential equation is a mathematical model of such system whose behavior at a given moment depends on the velocity of the system in the past. Many real processes are described by neutral functional-differential equations [3, 6, 8, 14, 23]. Many works are dedicated to the investigation of neutral functional-differential equations and neutral optimization problems, including [2, 8, 11, 14, 18, 20, 23, 24]. In the paper the quasi-linear neutral functional-differential equation

$$\dot{x}(t) = A(t,x(t))\dot{x}(t-\sigma) + f(t,x(t),x(t-\tau_0),u_0(t)), t \in I = [t_0,t_1] \tag{1.1}$$

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with the continuous initial condition

$$x(t)=\varphi_0(t), t\le t_0 \tag{1.2}$$

is considered. The condition (1.1) is called the continuous initial condition because always $x(t_0)=\varphi_0(t_0)$. Let $x_0(t)$ be solution of the original Cauchy problem (1.1) - (1.2) and let $x(t)$ be solution of the corresponding perturbed (with respect to delay τ_0, initial function $\varphi_0(t)$ and control function $u_0(t)$) problem. In the paper, for the first time the analytic relation between solutions $x_0(t)$ and $x(t)$ is proved on the interval I in the case when the coefficient $A(\cdot)$ depends on the phase coordinate and perturbations do not depends on a small parameter $\varepsilon>0$. Moreover, the essential novelty here is the effects of the continuous initial condition (1.2) and perturbation of the delay τ_0 in the representation formula. We note that such analytic relation plays an important role in proving the necessary conditions of optimality [1, 4, 5, 7, 9, 10, 12, 13, 16, 17, 21, 23]. Besides, such relation allows one to get an approximate solution of the perturbed equation and to carry out a sensitivity analysis of mathematical models. The case when $A(t,x(t))\equiv A(t)$ and perturbations depends on a parameter ε is considered in [11, 18, 24]. The case when $A(t,x(t))\equiv 0$ is considered in [13, 15, 19, 21, 22]. The paper is organized as follows. In Section 2, the main theorem is formulated and some comments are given. In Section 3 the auxiliary lemmas are given and proved. In Section 4 the main theorem is proved.

2. FORMULATION OF THE MAIN RESULT

Let $\mathbb{R}^n$ be the n-dimensional vector space of points $x=(x^1,\dots,x^n)^T$ and let $O\subset\mathbb{R}^n$, $U\subset\mathbb{R}^r$ be convex open sets; let $\sigma>0$ and $\tau_2>\tau_1>0$ be given numbers, with

$$t_0+\max\{\sigma,\tau_2\}<t_1. \tag{2.1}$$

Suppose that the $n\times n$ -dimensional matrix function $A(t,x)$ is continuous on the set $I\times O$ and continuously differentiable with respect to $x^i, i=1,2,\dots,n$; moreover, there exists $M_1>0$ such that

$$|A(t,x)|+\sum_{i=1}^{n}\left|\frac{\partial}{\partial x^i}A(t,x)\right|\le M_1, \forall(t,x)\in I\times O. \tag{2.2}$$

Let the n-dimensional function $f(t,x,y,u)$ be continuous on the set $I\times O^2\times U$ and continuously differentiable with respect to x,y,u; moreover, there exists $M_2>0$ such that

$$|f(t,x,y,u)|+|f_x(\cdot)|+|f_y(\cdot)|+|f_u(\cdot)|\le M_2, \forall(t,x,y,u)\in I\times O^2\times U. \tag{2.3}$$

Further, denote by Φ and Ω the sets of continuous differentiable functions $\varphi(t)\in O, t\in I_1=[\hat{\tau},t_0]$, where $\hat{\tau}=t_0-\max\{\sigma,\tau_2\}$ and measurable functions $u(t)\in U, t\in I$, respectively, with the set $clu(I)$ is compact and $clu(I)\subset U$.

To each element

$$\mu=(\tau,\varphi(t),u(t))\in\Lambda=(\tau_1,\tau_2)\times\Phi\times\Omega$$

we assign the quasi-linear neutral functional-differential equation

$$\dot{x}(t) = A(t,x(t))\dot{x}(t-\sigma) + f(t,x(t),x(t-\tau),u(t)), t \in I \tag{2.4}$$

with the continuous initial condition

$$x(t) = \varphi(t), t \in [\hat{\tau}, t_0]. \tag{2.5}$$

Definition 1. Let $\mu \in \Lambda$, a function $x(t) = x(t;\mu) \in O, t \in [\hat{\tau}, t_1]$ is called a solution of equation (2.4) with the condition (2.5) or a solution corresponding to the element μ and defined on the interval I_1 if it satisfies the condition (2.5) and is absolutely continuous on the interval I and satisfies equation (2.4) almost everywhere (a.e.) on I.

Let us introduce the notations:

$$|\mu| = |\tau| + \|\varphi\|_1 + \|u\|, \quad \Lambda_\varepsilon(\mu_0) = \Big\{\mu \in \Lambda : |\mu - \mu_0| \le \varepsilon\Big\},$$

where

$$\|\varphi\|_1 = \sup\Big\{|\varphi(t)| + |\dot{\varphi}(t)| : t \in I_1\Big\}, \quad \|u\| = \sup\Big\{|u(t)| : t \in I\Big\},$$

$\varepsilon > 0$ is a fixed number and $\mu_0 = (\tau_0, \varphi_0(t), u_0(t)) \in \Lambda$ is a fixed element; furthermore,

$$\delta\tau = \tau - \tau_0, \quad \delta\varphi(t) = \varphi(t) - \varphi_0(t), \quad \delta u(t) = u(t) - u_0(t),$$
$$\delta\mu = \mu - \mu_0 = (\delta\tau, \delta\varphi(t), \delta u(t)),$$

$$\begin{cases} |\delta\mu| = |\delta\tau| + \|\delta\varphi\|_1 + \|\delta u\|, \\ \|\delta\varphi\|_1 = \sup\{|\delta\varphi(t)| + |\dot{\delta\varphi}(t)| : t \in I_1\}. \end{cases} \tag{2.6}$$

Let $x(t;\mu_0)$ be solution corresponding to the element $\mu_0 \in \Lambda$ and defined on the interval I_1. Then there exists a number $\varepsilon_1 > 0$ such that to each element $\mu = \mu_0 + \delta\mu \in \Lambda_{\varepsilon_1}(\mu_0)$ corresponds solution $x(t;\mu)$ i. e. Cauchy's perturbed problem has solution, defined on the interval I_1 (see Lemma 1 in the Section 3).

Theorem 1. *Let $x_0(t) = x(t;\mu_0)$ be solution corresponding to the element $\mu_0 = (\tau_0, \varphi_0(t), u_0(t)) \in \Lambda$ and defined on the interval I_1. Then there exist number $\varepsilon_2 \in (0, \varepsilon_1)$ such that, for arbitrary $\mu \in \Lambda_{\varepsilon_2}(\mu_0)$ on the interval I the following representation holds:*

$$x(t;\mu) = x_0(t) + \delta x(t;\delta\mu) + o(t;\delta\mu), \tag{2.7}$$

where

$$\begin{aligned} \delta x(t;\delta\mu) = {} & \Psi(t_0;t)\delta\varphi(t_0) + \int_{t_0-\sigma}^{t_0} Y(\xi+\sigma;t)A[\xi+\sigma]\dot{\delta\varphi}(\xi)d\xi \\ & + \int_{t_0-\tau_0}^{t_0} Y(\xi+\tau_0;t)f_y[\xi+\tau_0]\delta\varphi(\xi)d\xi + \int_{t_0}^{t} Y(\xi;t)f_u[\xi]\delta u(\xi)d\xi \\ & - \Big\{\int_{t_0}^{t} Y(\xi;t)f_y[\xi]\dot{x}_0(\xi-\tau_0)d\xi\Big\}\delta\tau \end{aligned} \tag{2.8}$$

and

$$\lim_{|\delta\mu|\to 0} o(t;\delta\mu)/|\delta\mu| = 0 \text{ uniformly for } t\in I.$$

Here,

$$A[\xi]=A(\xi,x_0(\xi)), f_y[\xi]=f_y(\xi,x_0(\xi),x_0(\xi-\tau_0),u_0(\xi));$$

for the fixed $t\in(t_0,t_1)$, $n\times n$ *matrix functions* $\Psi(\xi;t)$ *and* $Y(\xi;t)$ *satisfy the linear system*

$$\begin{cases}\Psi_\xi(\xi;t) &= -Y(\xi;t)\Big\{\frac{\partial}{\partial x}\Big[A[\xi]\dot{x}_0(\xi-\sigma)\Big]+f_x[\xi]\Big\}\\ & \quad -Y(\xi+\tau_0;t)f_y[\xi+\tau_0],\\ Y(\xi;t) &= \Psi(\xi;t)+Y(\xi+\sigma;t)A[\xi+\sigma], \xi\in(t_0,t)\end{cases} \tag{2.9}$$

and the condition

$$\Psi(\xi;t)=Y(\xi;t)=\begin{cases}E, \ \xi=t,\\ \Theta, \ \xi>t,\end{cases} \tag{2.10}$$

where E *is the identity matrix and* Θ *is the zero matrix and*

$$\frac{\partial}{\partial x}\Big[A[\xi]\dot{x}_0(\xi-\sigma)\Big]=\frac{\partial}{\partial x}\Big[A(\xi,x)\dot{x}_0(\xi-\sigma)\Big]_{x=x_0(\xi)}.$$

Some Comments. The function $\delta x(t;\delta\mu)$ in (2.7) is called the first variation of solution $x_0(t)$. The expression (2.8) is called the variation formula of solution. The term "variation formula of solution" has been introduced by R. V. Gamkrelidze and proved for the ordinary differential equation in [5].

The expression

$$Y(\xi;t_0)\delta\varphi(t_0)+\int_{t_0-\sigma}^{t_0}Y(\xi+\sigma;t)A[\xi+\sigma]\delta\dot{\varphi}(\xi)d\xi$$
$$+\int_{t_0-\tau_0}^{t_0}Y(\xi+\tau_0;t)f_y[\xi+\tau_0]\delta\varphi(\xi)d\xi$$

in formula (2.8) is the effect of perturbation $\varphi_0(t)$, where $Y(\xi;t_0)\delta\varphi(t_0)$ is the effect of the continuous initial condition.

The addend

$$\int_{t_0}^{t}Y(\xi;t)f_u[\xi]\delta u(\xi)d\xi$$

in formula (2.8) is the effect of perturbation $u_0(t)$.

The expression

$$\Big\{\int_{t_0}^{t}Y(\xi;t)f_y[\xi]\dot{x}_0(\xi-\tau_0)d\xi\Big\}\delta\tau$$

in formula (2.8) is the effect of perturbation τ_0.

Formula (2.7) allows us to obtain an approximate solution of the perturbed equation in the analytical form on the interval I. In fact, for a small $|\delta\mu|$ from (2.7) it follows

$$x(t;\delta\mu)\approx x_0(t)+\delta x(t;\delta\mu),$$

where $\delta x(t;\delta\mu)$ has the form (2.8). We note that in order to construct $\delta x(t;\delta\mu)$ it is sufficient to find a solution to the linear problem (2.9)- (2.10).

3. AUXILIARY ASSERTIONS

Lemma 1. *[23] Let $x_0(t) = x(t;\mu_0)$ be the solution corresponding to the element $\mu_0 = (\tau_0, \varphi_0(t), u_0(t)) \in \Lambda$, defined on the interval I_1. Then there exists a number $\varepsilon_1 > 0$ such that to each element $\mu = (\tau, \varphi(t), u(t)) \in \Lambda_{\varepsilon_1}(\mu_0)$ there corresponds solution $x(t) = x(t;\mu) \in O$ defined on the interval I_1.*

Lemma 2. *There exist numbers $L_i > 0, i = 1,2$ such that the following inequalities hold*

$$|A(t,x) - A(t,y)| \le L_1|x-y|, \forall (t,x,y) \in I \times O^2;$$

$$|f(t,x_1,y_1,u_1) - f(t,x_2,y_2,u_2)| \le L_2\left[|x_1-x_2| + |y_1-y_2| + |u_1-u_2|\right],$$

$$\forall t \in I,\ \forall (x_i,y_i,u_i) \in O^2 \times U, i = 1,2.$$

On the basis of (2.2) and (2.3) the Lemma 2 can be proved analogously to Lemma 2.2 (see [21]).

Lemma 3. *There exist a number $N > 0$ such that the following inequality holds*

$$|\dot{x}_0(t)| \le N,\ a.e.\ t \in I_1. \tag{3.1}$$

Proof. Let $t \in [\hat{\tau}, t_0)$ then

$$|\dot{x}_0(t)| = |\dot{\varphi}_0(t)| < \|\varphi_0\|_1 = N_0.$$

Let $t \in [t_0, t_0+\sigma]$ then

$$|\dot{x}_0(t)| = |A[t]\dot{x}_0(t-\sigma) + f[t]| \le M_1N_0 + M_2 = N_1,$$

where

$$A[t] = A(t,x_0(t)),\ f[t] = f(t,x_0(t),x_0(t-\tau_0),u_0(t))$$

(see (2.2),(2.3)). If $t \in [t_0+\sigma, t_0+2\sigma]$ we get

$$|\dot{x}_0(t)| \le M_1N_1 + M_2 = N_2.$$

Continuing this process to t_1 we get the finite quantity numbers $N_0, N_1, \dots, N_k$, where

$$k = \begin{cases} m & \text{if } t_0 + m\sigma < t_1 < t_0 + (m+1)\sigma, \\ m-1 & \text{if } t_0 + m\sigma = t_1 \end{cases}$$

(see (2.1)). Thus, $N = \max\{N_0, N_1, \dots, N_k\}$. □

Lemma (1) allows one to introduce the increment of the solution $x_0(t)$:

$$\Delta x(t) = \Delta x(t;\delta\mu) = x(t;\mu) - x_0(t), \quad t \in I_1, \tag{3.2}$$

where $\mu = \mu_0 + \delta\mu \in \Lambda_{\varepsilon_1}(\mu_0)$, i.e. $\delta\mu \in \Lambda_{\varepsilon_1}(\mu_0) - \mu_0$.

Lemma 4. *For arbitrary $\delta\mu \in \Lambda_{\varepsilon_1}(\mu_0) - \mu_0$ the following inequality holds*

$$\sup_{t\in I_1} |\Delta x(t)| \le O(\delta\mu), \tag{3.3}$$

where

$$\lim_{|\delta\mu|\to 0} O(\delta\mu)|\delta\mu| < \infty.$$

Proof. Let $t \in [\hat{\tau}, t_0]$ then

$$|\Delta x(t)| = |\delta\varphi(t)| \le |\delta\mu| = O(\delta\mu) \tag{3.4}$$

(see (2.6)). It is not difficult to see that the function $\Delta x(t) = x(t) - x_0(t)$ satisfies the equation

$$\dot{\Delta} x(t) = A(t, x_0(t) + \Delta x(t))\dot{\Delta} x(t-\sigma) + \alpha(t;\delta\mu) + \beta(t;\delta\mu), \ a.e.\ t \in I \tag{3.5}$$

and the initial condition

$$\Delta x(t) = \delta\varphi(t), t \in [\hat{\tau}, t_0], \tag{3.6}$$

where

$$\alpha(t;\delta\mu) = \Big[A(t, x_0(t) + \Delta x(t)) - A[t]\Big]\dot{x}_0(t-\sigma)$$

$$\beta(t;\delta\mu) = f(t, x_0(t) + \Delta x(t), x_0(t-\tau) + \Delta x(t-\tau), u_0(t) + \delta u(t)) - f[t].$$

The solution $\Delta x(t)$ of the problem (3.5)-(3.6) can be represented on the interval I in the following form

$$\Delta x(t) = \delta\varphi(t_0) + \int_{t_0-\sigma}^{t_0} Y_0(\xi;t) A(\xi+\sigma, x_0(\xi+\sigma) + \Delta x(\xi+\sigma))\dot{\delta\varphi}(\xi) d\xi$$
$$+ \int_{t_0}^{t} Y_0(\xi;t)[\alpha(\xi;\delta\mu) + \beta(\xi;\delta\mu)] d\xi$$

(see Theorem 1.7 in [23]), where $Y_0(\xi;t) = Y_0(\xi;t,\delta\mu)$ is $n\times n$ matrix function satisfying the difference equation

$$Y_0(\xi;t) = E + Y_0(\xi+\sigma;t) A(\xi+\sigma, x_0(\xi+\sigma) + \Delta x(\xi+\sigma)), \xi \in [t_0, t]$$

and the condition

$$Y_0(\xi;t) = \begin{cases} E, \ \xi = t, \\ \Theta, \ \xi > t, \end{cases}$$

where E is the identity matrix and Θ is the zero matrix. It is clear that

$$|Y_0(\xi;t,\delta\mu)| < const, \forall(\xi, t, \delta\mu) \in I^2 \times (\Lambda_{\varepsilon_1}(\mu_0) - \mu_0)$$

(see (2.2)). Further,

$$|\Delta x(t)| \le |\delta\varphi(t_0)| + \sigma M_1 \parallel Y_0 \parallel ||\delta\varphi||_1 + ||Y_0|| \int_{t_0}^{t} \Big[L_1 N |\Delta x(\xi)| + L_2 |\Delta x(\xi)|$$
$$+ L_2|\Delta x(\xi-\tau)| + L_2[x_0(\xi-\tau) - x_0(\xi-\tau_0)] + L_2|\delta u(\xi)|\Big] d\xi$$

$$\leq (1+||Y_0||M_1\sigma+||Y_0||L_2(t_1-t_0))|\delta\mu|+||Y_0||(L_1N+L_2)\int_{t_0}^{t}|\Delta x(\xi)|d\xi$$
$$+||Y_0||L_2\int_{t_0}^{t}|\Delta x(\xi-\tau)|d\xi+||Y_0||L_2\int_{t_0}^{t_1}|x_0(\xi-\tau)-x_0(\xi-\tau_0)|d\xi \tag{3.7}$$

(see Lemmas 2 and 3), where

$$||Y_0||=\sup\{|Y_0(\xi;t,\delta\mu)|:(\xi,t)\in I^2,\delta\mu\in(\Lambda_{\varepsilon_1}(\mu_0)-\mu_0)\}.$$

Now we transform the two last addends of (3.7). We have,

$$\int_{t_0}^{t}|\Delta x(\xi-\tau)|d\xi=\int_{t_0-\tau}^{t-\tau}|\Delta x(\xi)|d\xi.$$

Let $t_0+\tau\geq t_0+\sigma$ then if $t\in[t_0,t_0+\sigma]$ we have $t-\tau\leq t_0+\sigma-\tau\leq t_0+\tau-\tau=t_0$, i. e.

$$\int_{t_0-\tau}^{t-\tau}|\Delta x(\xi)|d\xi\leq\int_{t_0-\tau}^{t_0}|\delta\varphi(\xi)|d\xi\leq\tau|\delta\mu|\leq\tau_2|\delta\mu|=O(\delta\mu). \tag{3.8}$$

Let $t_0+\tau<t_0+\sigma$ then if $t\in[t_0,t_0+\tau]$ we have $t-\tau\leq t_0$; if $t\in[t_0+\tau,t_0+\sigma]$ then we have $t-\tau\geq t_0+\tau-\tau=t_0$. Consequently, if $t\in[t_0,t_0+\tau]$ we have (3.8) and if $t\in[t_0+\tau,t_0+\sigma]$ we have

$$\int_{t_0-\tau}^{t-\tau}|\Delta x(\xi)|d\xi=\int_{t_0-\tau}^{t_0}|\Delta x(\xi)|d\xi+\int_{t_0}^{t-\tau}|\Delta x(\xi)|d\xi\leq O(\delta\mu)$$
$$+\int_{t_0}^{t}|\Delta x(\xi)|d\xi.$$

(see (3.4)). Thus,

$$\int_{t_0}^{t}|\Delta x(\xi-\tau)|d\xi\leq O(\delta\mu)+\int_{t_0}^{t}|\Delta x(\xi)|d\xi. \tag{3.9}$$

Further,

$$\int_{t_0}^{t_1}|x_0(\xi-\tau)-x_0(\xi-\tau_0)|d\xi=\int_{t_0}^{t_1}\left|\int_{\xi-\tau}^{\xi-\tau_0}|\dot{x}_0(\varsigma)|d\varsigma\right|d\xi$$
$$\leq(t_1-t_0)N|\delta\mu|=O(\delta\mu). \tag{3.10}$$

Taking into account (3.9), (3.10) from (3.7) it follows

$$|\Delta x(t)|\leq O(\delta\mu)+||Y_0||(L_1N+2L_2)\int_{t_0}^{t}|\Delta x(\xi)|d\xi,\ t\in I.$$

By the Gronwall-Bellman inequality, from the last inequality it follows

$$|\Delta x(t)|\leq O(\delta\mu)\exp[||Y_0||(L_1N+2L_2)(t-t_0)]\leq O(\delta\mu),t\in I. \tag{3.11}$$

On the basis of (3.4) and (3.11) we get (3.3). □

Lemma 5. *The following inequality holds*

$$|\dot{\Delta x}(t)|\leq O(\delta\mu),\ a.e.\ t\in I_1.$$

Using Lemma 2 and equation (3.5) Lemma 5 without principle difficulties can be proved by the step method with respect to σ (see proof of Lemma 3).

4. PROOF OF THEOREM 1

The function $\Delta x(t)$ satisfies the equation (see the previous section)

$$\dot{\Delta} x(t) = A[t]\dot{\Delta} x(t-\sigma) + \Big\{\frac{\partial}{\partial x}\Big[A[t]\dot{x}_0(t-\sigma)\Big] + f_x[t]\Big\}\Delta x(t)$$
$$+ f_y[t]\Delta x(t-\tau_0) + f_u[t]\delta u(t) + a(t;\delta\mu) + b(t;\delta\mu) \qquad (4.1)$$

with the initial condition

$$\Delta x(t) = \delta\varphi(t), t \in [\hat{\tau}, t_0],$$

where

$$a(t;\delta\mu) = A(t, x_0(t) + \Delta x(t))\dot{x}_0(t-\sigma) - A[t]\dot{x}_0(t-\sigma)$$
$$-\frac{\partial}{\partial x}\Big[A[t]\dot{x}_0(t-\sigma)\Big]\Delta x(t) + \Big(A(t, x_0(t) + \Delta x(t)) - A[t]\Big)\dot{\Delta} x(t-\sigma);$$
$$b(t;\delta\mu) = b_0(t;\delta\mu) - f_x[t]\Delta x(t) - f_y[t]\Delta x(t-\tau_0) - f_u[t]\delta u(t),$$
$$b_0(t;\delta\mu) = f(t, x_0(t) + \Delta x(t), x_0(t-\tau) + \Delta x(t-\tau), u_0(t) + \delta u(t)) - f[t].$$

By using the Cauchy formula (see Theorem 1.5, [23]) one can represent the solution of equation (4.1) in the form

$$\Delta x(t) = \Psi(t_0;\xi)\delta\varphi(t_0) + \int_{t_0-\sigma}^{t_0} Y(\xi+\sigma;t)A[\xi+\sigma]\dot{\delta\varphi}(\xi)d\xi$$
$$+ \int_{t_0-\tau_0}^{t_0} Y(\xi+\tau_0;\xi)f_y[\xi+\tau_0]\delta\varphi(\xi)d\xi + a_1(t;\delta\mu) + b_1(t;\delta\mu), \qquad (4.2)$$

where

$$a_1(t;\delta\mu) = \int_{t_0}^{t} Y(\xi;t)a(\xi;\delta\mu)d\xi, \;\; b_1(t;\delta\mu) = \int_{t_0}^{t} Y(\xi;t)b(\xi;\delta\mu)d\xi;$$

$\Psi(\xi;t)$ and $Y(\xi;t)$ are matrix functions satisfying equation (2.9) and condition (2.10). Now we estimate $|a_1(t;\delta\mu)|$, we have

$$|a_1(t;\delta\mu)| = \Big|\int_{t_0}^{t} Y(\xi;t)\Big[\int_0^1 \frac{d}{ds}A(\xi, x_0(\xi) + s\Delta x(\xi))\dot{x}_0(\xi-\sigma)ds\Big]d\xi$$
$$- \int_{t_0}^{t} Y(\xi;t)\frac{\partial}{\partial x}\Big[A[\xi]\dot{x}_0(\xi-\sigma)\Big]\Delta x(\xi)d\xi$$
$$+ \int_{t_0}^{t} Y(\xi;t)\Big(A(\xi, x_0(\xi) + \Delta x(\xi)) - A[\xi]\Big)\dot{\Delta} x(\xi-\sigma)\Big|$$
$$\le \Big|\int_{t_0}^{t} Y(\xi;t)\Big\{\int_0^1 \Big(\frac{\partial}{\partial x}\Big[A(\xi, x_0(\xi) + s\Delta x(\xi))\dot{x}_0(\xi-\sigma)\Big]$$

$$-\frac{\partial}{\partial x}\Big[A[\xi]\dot{x}_0(\xi-\sigma)\Big]\Big)\Delta x(\xi)ds\Big\}d\xi$$
$$+\int_{t_0}^{t} Y(\xi;t)\Big(A(\xi,x_0(\xi)+\Delta x(\xi))-A[\xi]\Big)\dot{\Delta x}(\xi-\sigma)d\xi\Big|$$
$$\le \|Y\|O(\delta\mu)\Big\{\int_{t_0}^{t_1}\rho(\xi;\delta\mu)d\xi+L_1(t_1-t_0)O(\delta\mu)\Big\}$$

(see Lemma 5), where

$$\|Y\|=\sup\{|Y(\xi;t)|:\xi,t\in I\},$$
$$\rho(\xi;\delta\mu)=\int_0^1\Big|\frac{\partial}{\partial x}\Big[A(\xi,x_0(\xi)+s\Delta x(\xi))\dot{x}_0(\xi-\sigma)\Big]-\frac{\partial}{\partial x}\Big[A[\xi]\dot{x}_0(\xi-\sigma)\Big]\Big|ds,$$

By the Lebesgue theorem on the passage to the limit under the integral sign, we have

$$\lim_{|\delta\mu|\to 0}\int_{t_0}^{t_1}\rho(\xi;\delta\mu)d\xi=0.$$

Consequently,

$$a_1(t;\delta\mu)=o(t;\delta\mu). \tag{4.3}$$

We introduce the notations:

$$f[t;\theta,\delta\mu]=f(t,x_0(t)+\theta\Delta x(t),x_0(t-\tau_0)+\theta[x_0(t-\tau)-x_0(t-\tau_0)+\Delta x(t-\tau)],$$
$$u_0(t)+\theta\delta u(t))$$
$$\alpha_x(t;\theta,\delta\mu)=f_x[t;\theta,\delta\mu]-f_x[t].$$

Obviously,

$$\begin{aligned}
b_0(t;\delta\mu)&=\int_0^1\frac{d}{d\theta}f[t;\theta,\delta\mu]d\theta\\
&=\int_0^1\{f_x[t;\theta,\delta\mu]\Delta x(t)+f_y[t;\theta,\delta\mu][x_0(t-\tau)-x_0(t-\tau_0)\\
&\quad+\Delta x(t-\tau)]+f_u[t;\theta,\delta\mu]\delta u(t)\}\,d\theta=\left[\int_0^1\alpha_x(t;\theta,\delta\mu)d\theta\right]\Delta x(t)\\
&\quad+\left[\int_0^1\alpha_y(t;\theta,\delta\mu)d\theta\right](x_0(t-\tau)-x_0(t-\tau_0)+\Delta x(t-\tau))\\
&\quad+\left[\int_0^1\alpha_u(t;\theta,\delta\mu)d\theta\right]\delta u(t)+f_x[t]\Delta x(t)+f_y[t][x_0(t-\tau)\\
&\quad-x_0(t-\tau_0)+\Delta x(t-\tau)]+f_u[t]\delta u(t)\\
&=\alpha_x(t;\delta\mu)\Delta x(t)+\alpha_y(t;\delta\mu)\,(x_0(t-\tau)-x_0(t-\tau_0)+\Delta x(t-\tau))\\
&\quad+\alpha_u(t;\delta\mu)\delta u(t)+f_x[t]\Delta x(t)+f_y[t][x_0(t-\tau)-x_0(t-\tau_0)+\Delta x(t-\tau)]\\
&\quad+f_u[t]\delta u(t),
\end{aligned}$$

where

$$\alpha_x(t;\delta\mu)=\int_0^1 \alpha_x(t;\theta,\delta\mu)d\theta.$$

Taking into account the last relations we have

$$b_1(t;\delta\mu)=\sum_{i=1}^{5} b_{2i}(t;\delta\mu),$$

where

$$b_{21}(t;\delta\mu)=\int_{t_0}^{t} Y(s;t)\alpha_x(s;\delta\mu)\Delta x(s)ds,$$

$$b_{22}(t;\delta\mu)=\int_{t_0}^{t} Y(s;t)\alpha_y(s;\delta\mu)[x_0(s-\tau)-x_0(s-\tau_0)+\Delta x(s-\tau)]ds,$$

$$b_{23}(t;\delta\mu)=\int_{t_0}^{t} Y(s;t)f_y[s][x_0(s-\tau)-x_0(s-\tau_0)]ds,$$

$$b_{24}(t;\delta\mu)=\int_{t_0}^{t} Y(s;t)f_y[s][\Delta x(s-\tau)-\Delta x(s-\tau_0)]ds,$$

$$b_{25}(t;\delta\mu)=\int_{t_0}^{t} Y(s;t)\alpha_u(s;\delta\mu)\delta u(s)ds.$$

It is not difficult to see that

$$|b_{21}(t;\delta\mu)|\le \|Y\|O(\delta\mu)\alpha_x(\delta\mu),\ \alpha_x(\delta\mu)=\int_{t_0}^{t_1}|\alpha_x(s;\delta\mu)|ds$$

(see Lemma 5);

$$|b_{22}(t;\delta\mu)|\le \|Y\|\alpha_y(\delta\mu)\int_{t_0}^{t_1}\Big\{\Big|\int_{s-\tau_0}^{s-\tau}|\dot x_0(\xi)|d\xi\Big|+O(\delta\mu)\Big\}d\xi$$
$$\le \|Y\|O(\delta\mu)\alpha_y(\delta\mu)$$

(see Lemma 4);

$$|b_{25}(t;\delta\mu)|\|Y\|O(\delta\mu)\alpha_u(\delta\mu).$$

Further,

$$b_{23}(t;\delta\mu)=\int_{t_0}^{t} Y(s;t)f_y[s]\Big\{\int_{s-\tau_0}^{s-\tau}\dot x_0(\xi)d\xi\Big\}ds.$$

The function $x_0(\xi),\xi\in I_1$, is absolutely continuous, therefore for each fixed Lebesgue point $s\in(t_0,t_1)$ of the function $\dot x(\varsigma-\tau_0),\quad \varsigma\in(t_0,t_1)$, we get

$$\int_{s-\tau_0}^{s-\tau}\dot x_0(\xi)d\xi=\int_{s}^{s-\delta\tau}\dot x_0(\varsigma-\tau_0)d\varsigma=-\dot x_0(s-\tau_0)\delta\tau$$
$$+\rho(s;\delta\tau), \tag{4.4}$$

where

$$\lim_{|\delta\tau|\to 0}\rho(s;\delta\tau)/|\delta\tau|=0$$

From boundedness of the function $\dot{x}_0(s), s \in I_1$ and (4.4) it follows

$$|\rho(s;\delta\tau)|/|\delta\tau| \leq const$$

a. e. on I. Thus,

$$b_{23}(t;\delta\mu) = -\Big\{\int_{t_0}^{t} Y(s;t)f_y[s]\dot{x}_0(s-\tau_0)ds\Big\}\delta\tau + \rho_1(t;\delta\tau),$$

where

$$\rho_1(t;\delta\tau) = \int_{t_0}^{t} Y(s;t)f_y[s]\rho(s;\delta\tau)ds.$$

It is clear that

$$|\Delta x(s-\tau) - \Delta x(s-\tau)| \leq \Big|\int_{s-\tau_0}^{s-\tau} |\dot{\Delta x}(\xi)|d\xi\Big| \leq O(\delta\mu)|\delta\mu|$$

(see Lemma 5). On the basis above obtained estimates and the Lebesgue theorem can be concluded that

$$\begin{aligned}
&b_{21}(t;\delta\mu) = o(t;\delta\mu),\ b_{22}(t;\delta\mu) = o(t;\delta\mu),\\
&b_{23}(t;\delta\mu) = -\Big\{\int_{t_0}^{t} Y(s;t)f_y[s]\dot{x}_0(s-\tau_0)ds\Big\}\delta\tau + o(t;\delta\mu),\\
&b_{21}(t;\delta\mu) = o(t;\delta\mu),\ b_{22}(t;\delta\mu) = o(t;\delta\mu).
\end{aligned}$$

Consequently, we get

$$b_1(t;\delta\mu) = -\Big\{\int_{t_0}^{t} Y(s;t)f_y[s]\dot{x}_0(s-\tau_0)ds\Big\}\delta\tau + o(t;\delta\mu). \tag{4.5}$$

From (4.2) by virtue of (4.3) and (4.5), we obtain (2.7), where $\delta x(t;\delta\mu)$ has the form (2.8).

5. CONCLUSION

The formula (2.7) plays an important role in proving the necessary conditions of optimality in the optimization problems. Besides, this formula allows one to get an approximate solution of the perturbed equation and to carry out a sensitivity analysis of mathematical models. Future work will consider the case when the initial moment t_0 is non-fixed.

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ON FRACTAL BERNOULLI DIFFERENTIAL EQUATIONS AND THEIR APPLICATIONS

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Abstract. In this paper we study a Bernoulli-type differential equation that replace the usual derivative by a fractal derivative. We show the goodness-of-fit of this model for real data comparing it with classic models.

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Keywords: fractal derivatives, Bernoulli differential equation, real data fitting

1. Introduction

Fractal calculus (also called local fractional calculus) is utilized to handle various nondifferentiable problems that appear in complex systems of the real-world phenomena. This new calculus was first proposed by Kolwankar and Gangal in 1996 through renormalization of Riemann–Liouville definition [23]. Fractal derivatives (or local fractional derivative) are defined as a non-Newtonian generalization of the derivative dealing with the measurement of fractals and play an important role in the study of anomalous diffusion. There are many definitions of fractal derivatives and local fractional integrals (see for instance [3, 4, 6, 8, 21]). In this paper we used a particular version of the definitions given in [12, 13, 31]: for $\alpha, \beta > 0$, let us define the (β, α)-*fractal derivative* of a function f at the point t_0 by

$$\frac{\mathrm{d}^{\beta} f}{\mathrm{d}t^{\alpha}}(t_0) = \lim_{t \to t_0} \frac{f^{\beta}(t) - f^{\beta}(t_0)}{t^{\alpha} - t_0^{\alpha}} \tag{1.1}$$

if there exists this limit and it is finite. In this case, we say that f is (β, α)-*fractal differentiable* at t_0. Here, the function t^{α} is defined for $t \in \mathbb{R}$ as $t^{\alpha} := t|t|^{\alpha-1}$ and so,

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$(t^\alpha)' = \alpha|t|^{\alpha-1}$; also, $(|t|^\alpha)' = \alpha t^{\alpha-1}$. When $\beta = 1$, this derivative (1.1) can be viewed as a fractional derivatives via fractional differences, which are very useful for solving numerical problems of fractional differential equations (see e.g. [2, 16, 20, 22, 29]). However, for $\beta \neq 1$, this fractal derivative is not a linear operator and so, it does not enjoy some desirable properties.

These derivatives engender a new kind of differential equations, referred as fractal differential equations essentially different from the well-known fractional differential equations. As of today, many applications of these fractal derivatives are studied, e.g. the Fokker-Planck equation in modelling phenomena involving fractal time [25] and the time-space fabric underlying anomalous diffusion [12]. Moreover, in fractal cosmology it has proved to be a nice derivative to treat certain universe models considering fractal space-time. Good references on this topic are the books [18, 29–31] and the papers [7, 8, 11, 14, 21, 24, 32].

The classic Bernoulli differential equation is a non-linear differential equation of the form

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = a(t)y(t) + b(t)y^n(t), \quad n \in \mathbb{Z}_+,$$

where $a(t)$ and $b(t)$ are continuous functions. The first discussion of this equation goes back to Jacob Bernoulli in his work from 1695 [9]. Modern physics indeed uses Bernoulli differential equations for modelling the dynamics behind certain circuit elements, known as Bernoulli memristors [17]. The Bernoulli differential equation also show up in some economic utility maximization problems (see for example [26]).

In the present paper we study Bernoulli-type differential equations of the form

$$\frac{\mathrm{d}^\beta y(t)}{\mathrm{d}t^\alpha} = a(t)\big(y(t)\big)^\beta + b(t)\big|y(t)\big|^{\beta\gamma}, \tag{1.2}$$

where $\gamma \in \mathbb{R} \setminus \{1\}$. We solve an initial value problem for the equation (1.2). Finally, it is shown the effectiveness of a fractal Bernoulli differential model to fit real data associated with tuberculosis in Mexico.

2. BASIC PROPERTIES

This section summarizes a series of preliminary notions and tools that are required in the main part of this work (for more details we refer to [5]).

Proposition 1. *Let f be a differentiable function at t_0. Assume that $t_0 \neq 0$ if $\alpha > 1$, and $f(t_0) \neq 0$ if $0 < \beta < 1$. Then f is (β,α)-fractal differentiable at t_0, and the following formula holds*

$$\frac{\mathrm{d}^\beta f}{\mathrm{d}t^\alpha}(t_0) = \frac{\beta}{\alpha}|t_0|^{1-\alpha}|f(t_0)|^{\beta-1}f'(t_0).$$

In particular,

$$\frac{\mathrm{d}^1 f}{\mathrm{d}t^\alpha}(t_0) = \frac{1}{\alpha}|t_0|^{1-\alpha}f'(t_0).$$

Proposition 2. *Let $c \in \mathbb{R}$ and let f be a (β,α)-fractal differentiable function at t_0. Then the following statements hold:*

i. *We have*

$$\frac{\mathrm{d}^1 f}{\mathrm{d}t^1}(t_0) = f'(t_0), \qquad \frac{\mathrm{d}^\beta f}{\mathrm{d}t^\alpha}(t_0) = \frac{\mathrm{d}^1(f^\beta)}{\mathrm{d}t^\alpha}(t_0).$$

ii. *The function c is (β,α)-fractal differentiable at t_0 and*

$$\frac{\mathrm{d}^\beta c}{\mathrm{d}t^\alpha}(t_0) = 0.$$

iii. *The function cf is (β,α)-fractal differentiable at t_0 and*

$$\frac{\mathrm{d}^\beta (cf)}{\mathrm{d}t^\alpha}(t_0) = c^\beta \frac{\mathrm{d}^\beta f}{\mathrm{d}t^\alpha}(t_0).$$

iv. $\mathrm{d}^\beta/\mathrm{d}t^\alpha$ *is a linear operator if and only if* $\beta = 1$.

The following result shows how to compute the fractal derivative of the product and the quotient of two functions.

Proposition 3 (Leibniz rule)**.** *Let f, g be (β,α)-differentiable functions at t_0. Then the following statements hold:*

i. *fg is (β,α)-fractal differentiable at t_0 and*

$$\frac{\mathrm{d}^\beta (fg)}{\mathrm{d}t^\alpha}(t_0) = \frac{\mathrm{d}^\beta f}{\mathrm{d}t^\alpha}(t_0)\, g^\beta(t_0) + f^\beta(t_0)\, \frac{\mathrm{d}^\beta g}{\mathrm{d}t^\alpha}(t_0).$$

ii. *If $g(t_0) \neq 0$, then $1/g$ is (β,α)-fractal differentiable at t_0 and*

$$\frac{\mathrm{d}^\beta}{\mathrm{d}t^\alpha}\left(\frac{1}{g}\right)(t_0) = \frac{-\frac{\mathrm{d}^\beta g}{\mathrm{d}t^\alpha}(t_0)}{|g(t_0)|^{2\beta}}.$$

iii. *If $g(t_0) \neq 0$, then f/g is (β,α)-fractal differentiable at t_0 and*

$$\frac{\mathrm{d}^\beta}{\mathrm{d}t^\alpha}\left(\frac{f}{g}\right)(t_0) = \frac{\frac{\mathrm{d}^\beta f}{\mathrm{d}t^\alpha}(t_0)\, g^\beta(t_0) - f^\beta(t_0)\, \frac{\mathrm{d}^\beta g}{\mathrm{d}t^\alpha}(t_0)}{|g(t_0)|^{2\beta}}.$$

The next result is a kind of chain rule for the fractal derivatives.

Proposition 4 (Chain rule)**.** *Let g be a continuous and (α,α)-differentiable function at t_0 and let f be a (β,α)-differentiable function at $g(t_0)$. Then $f \circ g$ is (β,α)-fractal differentiable at t_0 and*

$$\frac{\mathrm{d}^\beta}{\mathrm{d}t^\alpha}(f \circ g)(t_0) = \frac{\mathrm{d}^\beta f}{\mathrm{d}t^\alpha}(g(t_0))\, \frac{\mathrm{d}^\alpha g}{\mathrm{d}t^\alpha}(t_0).$$

Let I be an interval with $t_0 \in I$ and $\alpha, \beta > 0$. If f is a locally integrable function on I with respect to the measure $|s|^{\alpha-1} \mathrm{d}s$, let us define the operators

$$\begin{aligned} L_{t_0}^{\alpha}(f)(t) &= \int_{t_0}^{t} \alpha |s|^{\alpha-1} f(s)\, \mathrm{d}s, \\ K_{t_0}^{\alpha,\beta}(f)(t) &= \Big(\int_{t_0}^{t} \alpha |s|^{\alpha-1} f(s)\, \mathrm{d}s \Big)^{1/\beta} = \big(L_{t_0}^{\alpha}(f)(t) \big)^{1/\beta}, \end{aligned} \tag{2.1}$$

for $t \in I$. These operators behave as inverse operators of $\mathrm{d}^1/\mathrm{d}t^{\alpha}$ and $\mathrm{d}^{\beta}/\mathrm{d}t^{\alpha}$, respectively,

$$\frac{\mathrm{d}^1}{\mathrm{d}t^{\alpha}} \big(L_{t_0}^{\alpha}(f) \big)(t) = \frac{\mathrm{d}^{\beta}}{\mathrm{d}t^{\alpha}} \big(K_{t_0}^{\alpha,\beta}(f) \big)(t) = f(t).$$

3. Fractal Bernoulli differential equation

The Bernoulli differential equation is a fundamental theoretical element in the solution of problems involving nonlinear differential equations. From the practical point of view, its importance lies in its ability to describe and analyze problems in Physics, Biology and Engineering, fundamentally, it allows the study of the behavior of flows in systems where pressure, velocity and height phenomena intervene. It also serves as a basis for the study of system dynamics, enzyme kinetics and population dynamics (see [10, 15, 28]). This equation has recently been studied in the context of the global fractional operators [27]. The previous research has been complemented by studying this Bernoulli differential equation by means of the local operators (conformable and non-conformable). Likewise, the research leaves open the possibility of comparing the scope of both approaches (global and local) in practice (see for example [20]). One of the advantages of the local differential operators studied, as well as the global differential operators (Caputo, Fabricio, etc.) for $\alpha \in (0,1)$, is that they allow to use, generalize and extend many classic results.

Fractal calculus enjoys opening up new lines of research that also generalize many classic results to distant and multiple areas [19]. The linear operator $L_{t_0}^{\alpha}$ allows to solve Bernoulli-type differential equation. The following theorem describes an initial value problem for (1.2).

Theorem 1. *Let $\gamma \in \mathbb{R} \setminus \{1\}$, $y_0 \in \mathbb{R}$, I an interval with $t_0 \in I$, and let a, b be continuous functions on I. The function*

$$y(t) = \begin{cases} \exp\Big(\frac{1}{\beta} L_{t_0}^{\alpha}(a)(t)\Big) \Big((1-\gamma) L_{t_0}^{\alpha}\Big(b(s) \exp\big((\gamma-1) L_{t_0}^{\alpha}(a)(s)\big)\Big)(t) + y_0^{\beta-\beta\gamma} \Big)^{1/(\beta-\beta\gamma)}, & \text{if } \gamma < 1, \\ y_0 \exp\Big(\frac{1}{\beta} L_{t_0}^{\alpha}(a)(t)\Big) \Big(y_0^{\beta\gamma-\beta} (1-\gamma) L_{t_0}^{\alpha}\Big(b(s) \exp\big((\gamma-1) L_{t_0}^{\alpha}(a)(s)\big)\Big)(t) + 1 \Big)^{1/(\beta-\beta\gamma)}, & \text{if } \gamma > 1, \end{cases} \tag{3.1}$$

will be a solution of the following initial value problem:

$$\frac{\mathrm{d}^\beta y}{\mathrm{d}t^\alpha}(t) = a(t)\big(y(t)\big)^\beta + b(t)\big|y(t)\big|^{\beta\gamma}, \qquad y(t_0) = y_0,$$

with $t \in I$*, and*

$$(1-\gamma)L^\alpha_{t_0}\Big(b(s)\exp\big((\gamma-1)L^\alpha_{t_0}(a)(s)\big)\Big)(t) + y_0^{\beta-\beta\gamma} \neq 0,$$

provided that $\gamma < 0$*, whereas*

$$y_0^{\beta\gamma-\beta}(1-\gamma)L^\alpha_{t_0}\Big(b(s)\exp\big((\gamma-1)L^\alpha_{t_0}(a)(s)\big)\Big)(t) + 1 \neq 0,$$

as long as $\gamma > 1$.

Proof. Firstly, let us collect here the equalities that we will use:

$$\frac{\mathrm{d}^1(fg)}{\mathrm{d}t^\alpha}(t) = \frac{\mathrm{d}^1 f}{\mathrm{d}t^\alpha}(t)\,g(t) + f(t)\,\frac{\mathrm{d}^1 g}{\mathrm{d}t^\alpha}(t), \tag{3.2}$$

$$\frac{\mathrm{d}^1}{\mathrm{d}t^\alpha}\big(f^a\big)(t) = a\,|f(t)|^{a-1}\,\frac{\mathrm{d}^1 f}{\mathrm{d}t^\alpha}(t), \tag{3.3}$$

$$\frac{\mathrm{d}^1}{\mathrm{d}t^\alpha}\big(e^g\big)(t) = e^{g(t)}\,\frac{\mathrm{d}^1 g}{\mathrm{d}t^\alpha}(t), \tag{3.4}$$

$$\frac{\mathrm{d}^1}{\mathrm{d}t^\alpha}\big(L^\alpha_t(f)\big)(t) = f(t). \tag{3.5}$$

Since the case $\gamma > 1$ is similar, we can assume $\gamma < 1$. Let us define

$$\begin{aligned} z(t) &= y(t)^\beta \\ &= \exp\big(L^\alpha_{t_0}(a)(t)\big)\Big((1-\gamma)L^\alpha_{t_0}\Big(b(s)\exp\big((\gamma-1)L^\alpha_{t_0}(a)(s)\big)\Big)(t) + y_0^{\beta-\beta\gamma}\Big)^{1/(1-\gamma)}. \end{aligned}$$

Proposition 2 gives that it suffices to prove that $z(t)$ satisfies the initial value problem

$$\frac{\mathrm{d}^1 z}{\mathrm{d}t^\alpha}(t) = a(t)\,z(t) + b(t)\big|z(t)\big|^\gamma, \qquad z(t_0) = y_0^\beta.$$

We have

$$\begin{aligned} z(t_0) &= \exp\big(L^\alpha_{t_0}(a)(t_0)\big)\Big((1-\gamma)L^\alpha_{t_0}\Big(b(s)\exp\big((\gamma-1)L^\alpha_{t_0}(a)(s)\big)\Big)(t_0) + y_0^{\beta-\beta\gamma}\Big)^{1/(1-\gamma)} \\ &= \exp(0)\Big(0 + y_0^{\beta-\beta\gamma}\Big)^{1/(1-\gamma)} \\ &= y_0^\beta. \end{aligned}$$

Note that $1/(1-\gamma) \geq 0$ and $\gamma/(1-\gamma) \geq 0$ if and only if $\gamma \in [0,1)$; if $\gamma < 0$, we have the hypothesis $(1-\gamma)L^\alpha_{t_0}\Big(b(s)\exp\big((\gamma-1)L^\alpha_{t_0}(a)(s)\big)\Big)(t) + y_0^{\beta-\beta\gamma} \neq 0$.

Since a and b are continuous functions on I, (3.5) implies

$$\frac{\mathrm{d}^1}{\mathrm{d}t^\alpha}\left(L^\alpha_{t_0}(a)\right)(t)=a(t),$$

$$\frac{\mathrm{d}^1}{\mathrm{d}t^\alpha}\left(L^\alpha_{t_0}\left(b(s)\exp\left((\gamma-1)L^\alpha_{t_0}(a)(s)\right)\right)\right)(t)=b(t)\exp\left((\gamma-1)L^\alpha_{t_0}(a)(t)\right).$$

By the above relations, in view of (3.2), (3.3) and (3.4), we obtain

$$\begin{aligned}
\frac{\mathrm{d}^1 z}{\mathrm{d}t^\alpha}(t)=&\frac{\mathrm{d}^1}{\mathrm{d}t^\alpha}\left(\exp\left(L^\alpha_{t_0}(a)(t)\right)\right)\left((1-\gamma)L^\alpha_{t_0}\left(b(s)\exp\left((\gamma-1)L^\alpha_{t_0}(a)(s)\right)\right)(t)+y_0^{\beta-\beta\gamma}\right)^{1/(1-\gamma)}\\
&+\exp\left(L^\alpha_{t_0}(a)(t)\right)\frac{\mathrm{d}^1}{\mathrm{d}t^\alpha}\left(\left((1-\gamma)L^\alpha_{t_0}\left(b(s)\exp\left((\gamma-1)L^\alpha_{t_0}(a)(s)\right)\right)(t)+y_0^{\beta-\beta\gamma}\right)^{1/(1-\gamma)}\right)\\
=&\,a(t)\exp\left(L^\alpha_{t_0}(a)(t)\right)\left((1-\gamma)L^\alpha_{t_0}\left(b(s)\exp\left((\gamma-1)L^\alpha_{t_0}(a)(s)\right)\right)(t)+y_0^{\beta-\beta\gamma}\right)^{1/(1-\gamma)}\\
&+\exp\left(L^\alpha_{t_0}(a)(t)\right)\frac{1}{1-\gamma}\left|(1-\gamma)L^\alpha_{t_0}\left(b(s)\exp\left((\gamma-1)L^\alpha_{t_0}(a)(s)\right)\right)(t)+y_0^{\beta-\beta\gamma}\right|^{1/(1-\gamma)-1}\\
&\times(1-\gamma)\,b(t)\exp\left((\gamma-1)L^\alpha_{t_0}(a)(t)\right)\\
=&\,a(t)\exp\left(L^\alpha_{t_0}(a)(t)\right)\left((1-\gamma)L^\alpha_{t_0}\left(b(s)\exp\left((\gamma-1)L^\alpha_{t_0}(a)(s)\right)\right)(t)+y_0^{\beta-\beta\gamma}\right)^{1/(1-\gamma)}\\
&+b(t)\exp\left(\gamma L^\alpha_{t_0}(a)(t)\right)\left|(1-\gamma)L^\alpha_{t_0}\left(b(s)\exp\left((\gamma-1)L^\alpha_{t_0}(a)(s)\right)\right)(t)+y_0^{\beta-\beta\gamma}\right|^{\gamma/(1-\gamma)}\\
=&\,a(t)\,z(t)+b(t)\left|z(t)\right|^\gamma.
\end{aligned}$$

□

If the function b or a is identically zero, Theorem 1 has the following consequences.

Corollary 1. *Let $y_0\in\mathbb{R}$, I an interval with $t_0\in I$, and let $a(t)$ be a continuous function on I. Then the function*

$$y(t)=y_0\exp\left(\frac{1}{\beta}L^\alpha_{t_0}(a)(t)\right)$$

is a solution of the initial value problem

$$\frac{\mathrm{d}^\beta y}{\mathrm{d}t^\alpha}(t)=a(t)\left(y(t)\right)^\beta,\qquad y(t_0)=y_0.$$

Corollary 2. *Let $\gamma\in\mathbb{R}\setminus\{1\}$, $y_0\in\mathbb{R}$, I an interval with $t_0\in I$, and let $b(t)$ be a continuous function on I. The function*

$$y(t)=\begin{cases}\left((1-\gamma)L^\alpha_{t_0}(b)(t)+y_0^{\beta-\beta\gamma}\right)^{1/(\beta-\beta\gamma)}, & \text{if } \gamma<1,\\ y_0\left(y_0^{\beta\gamma-\beta}(1-\gamma)L^\alpha_{t_0}(b)(t)+1\right)^{1/(\beta-\beta\gamma)}, & \text{if } \gamma>1,\end{cases}$$

is a solution of the initial value problem

$$\frac{\mathrm{d}^\beta y}{\mathrm{d}t^\alpha}(t)=b(t)\left|y(t)\right|^{\beta\gamma},\qquad y(t_0)=y_0,$$

with

$$(1-\gamma)L^{\alpha}_{t_0}(b)(t)+y_0^{\beta-\beta\gamma}\neq 0,$$

provided that $\gamma<0$*, whereas*

$$y_0^{\beta\gamma-\beta}(1-\gamma)L^{\alpha}_{t_0}(b)(t)+1\neq 0,$$

as long as $\gamma>1$.

If the functions $a(t)$ and $b(t)$ are constants, we have the following result:

Proposition 5. *Let* $\gamma\in\mathbb{R}\setminus\{1\}$*,* $y_0,a,b\in\mathbb{R}$ *with* $a\neq 0$*. The function*

$$y(t)=\begin{cases}\left(\left(y_0^{\beta-\beta\gamma}+\frac{b}{a}\right)\exp\left(a(1-\gamma)\left(t^{\alpha}-t_0^{\alpha}\right)\right)-\frac{b}{a}\right)^{1/(\beta-\beta\gamma)}, & \text{if } \gamma<1,\\ y_0\left(\left(1+y_0^{\beta\gamma-\beta}\frac{b}{a}\right)\exp\left(a(1-\gamma)\left(t^{\alpha}-t_0^{\alpha}\right)\right)-y_0^{\beta\gamma-\beta}\frac{b}{a}\right)^{1/(\beta-\beta\gamma)}, & \text{if } \gamma>1,\end{cases}$$

is a solution of the initial value problem

$$\frac{\mathrm{d}^{\beta}y}{\mathrm{d}t^{\alpha}}(t)=a\big(y(t)\big)^{\beta}+b\big|y(t)\big|^{\beta\gamma},\qquad y(t_0)=y_0,$$

when $t\geq t_0$*, and the following assumption are required:*

$$\left(y_0^{\beta-\beta\gamma}+\frac{b}{a}\right)\exp\left(a(1-\gamma)\left(t^{\alpha}-t_0^{\alpha}\right)\right)-\frac{b}{a}\neq 0,$$

if $\gamma<0$*, and*

$$\left(1+y_0^{\beta\gamma-\beta}\frac{b}{a}\right)\exp\left(a(1-\gamma)\left(t^{\alpha}-t_0^{\alpha}\right)\right)-y_0^{\beta\gamma-\beta}\frac{b}{a}\neq 0,$$

if $\gamma>1$.

Proof. Without loss of generality we can assume that $\gamma<1$. The case when $\gamma>1$ is analogous. We have

$$L^{\alpha}_{t_0}(a)(t)=a\int_{t_0}^{t}\alpha|s|^{\alpha-1}\,\mathrm{d}s=a\big(t^{\alpha}-t_0^{\alpha}\big).$$

Theorem 1 yields

$$\begin{aligned}y(t)&=\exp\left(\frac{1}{\beta}L^{\alpha}_{t_0}(a)(t)\right)\left((1-\gamma)L^{\alpha}_{t_0}\Big(b\exp\left((\gamma-1)L^{\alpha}_{t_0}(a)(s)\right)\Big)(t)+y_0^{\beta-\beta\gamma}\right)^{1/(\beta-\beta\gamma)}\\&=\exp\left(\frac{a}{\beta}\left(t^{\alpha}-t_0^{\alpha}\right)\right)\left(b(1-\gamma)L^{\alpha}_{t_0}\Big(\exp\left((\gamma-1)a\left(s^{\alpha}-t_0^{\alpha}\right)\right)\Big)(t)+y_0^{\beta-\beta\gamma}\right)^{1/(\beta-\beta\gamma)}.\end{aligned}$$

Since

$$\begin{aligned}(1-\gamma)L^{\alpha}_{t_0}\Big(\exp\big((\gamma-1)a\big(s^{\alpha}-t_0^{\alpha}\big)\big)\Big)(t) &= (1-\gamma)\int_{t_0}^{t}\alpha\,|s|^{\alpha-1}\exp\big((\gamma-1)a\big(s^{\alpha}-t_0^{\alpha}\big)\big)\,\mathrm{d}t\\ &= \frac{-1}{a}\int_{t_0}^{t}\alpha\,|s|^{\alpha-1}a(\gamma-1)\exp\big(a(\gamma-1)\big(s^{\alpha}-t_0^{\alpha}\big)\big)\,\mathrm{d}t\\ &= \frac{-1}{a}\Big[\exp\big(a(\gamma-1)\big(s^{\alpha}-t_0^{\alpha}\big)\big)\Big]_{s=t_0}^{s=t}\\ &= \frac{1}{a}\Big(1-\exp\big(a(\gamma-1)\big(t^{\alpha}-t_0^{\alpha}\big)\big)\Big),\end{aligned}$$

we conclude that

$$\begin{aligned}y(t) &= \exp\Big(\frac{a}{\beta}\big(t^{\alpha}-t_0^{\alpha}\big)\Big)\Big(b(1-\gamma)L^{\alpha}_{t_0}\Big(\exp\big((\gamma-1)a\big(s^{\alpha}-t_0^{\alpha}\big)\big)\Big)(t)+y_0^{\beta-\beta\gamma}\Big)^{1/(\beta-\beta\gamma)}\\ &= \exp\Big(\frac{a}{\beta}\big(t^{\alpha}-t_0^{\alpha}\big)\Big)\Big(\frac{b}{a}\Big(1-\exp\big(a(\gamma-1)\big(t^{\alpha}-t_0^{\alpha}\big)\big)\Big)+y_0^{\beta-\beta\gamma}\Big)^{1/(\beta-\beta\gamma)}\\ &= \Big(\Big(y_0^{\beta-\beta\gamma}+\frac{b}{a}\Big)\exp\big(a(1-\gamma)\big(t^{\alpha}-t_0^{\alpha}\big)\big)-\frac{b}{a}\Big)^{1/(\beta-\beta\gamma)}.\end{aligned}$$

□

We have the following consequence of Proposition 5 when $\gamma=2$, $a=A$ and $b=-B$, which solves a kind of fractal differential equation.

Corollary 3. *Let $y_0,A,B\in\mathbb{R}$ with $A>0$ and $y_0B\geq 0$. The function given by the expression*

$$y(t)=y_0\Big(\Big(1-y_0^{\beta}\frac{B}{A}\Big)\exp\big(-A\big(t^{\alpha}-t_0^{\alpha}\big)\big)+y_0^{\beta}\frac{B}{A}\Big)^{-1/\beta}$$

is a solution for $t\geq t_0$ of the initial value problem

$$\frac{\mathrm{d}^{\beta}y}{\mathrm{d}t^{\alpha}}(t)=\big(y(t)\big)^{\beta}\big(A-B\big(y(t)\big)^{\beta}\big),\qquad y(t_0)=y_0.$$

Proof. By Proposition 5, we need to prove that the function

$$u(t)=\Big(1-y_0^{\beta}\frac{B}{A}\Big)\exp\big(-A\big(t^{\alpha}-t_0^{\alpha}\big)\big)+y_0^{\beta}\frac{B}{A}$$

is not 0 for every $t\geq t_0$. Since $A>0$ and $y_0B\geq 0$, we have that $y_0^{\beta}B/A\geq 0$ and

$$u(t)=\exp\big(-A\big(t^{\alpha}-t_0^{\alpha}\big)\big)+y_0^{\beta}\frac{B}{A}\Big(1-\exp\big(-A\big(t^{\alpha}-t_0^{\alpha}\big)\big)\Big)>0,\quad t\geq t_0.$$

□

4. A REAL DATA FITTING WITH MATLAB

First we will graph the solutions to the following initial value problem for the fractal Bernoulli differential equation by varying α and β:

$$\frac{d^{\beta}y}{dt^{\alpha}}(t) = t(y(t))^{\beta}, \quad y(0) = 1. \tag{4.1}$$

In Figure 1, black curve is related to the usual derivative ($\beta = 1, \alpha = 1$). The yellow, green, blue, red and brown curves correspond to the following (α, β) values: $(1.1, 1.5)$, $(1.8, 3.0)$, $(1.5, 0.8)$, $(0.2, 0.7)$ and $(0.4, 2.1)$, respectively.

We also plot the solutions to the following equation

$$\frac{d^{\beta}y}{dt^{\alpha}}(t) = 2(y(t))^{\beta} - |y(t)|^{1.8\beta} \tag{4.2}$$

with the same initial condition as the problem (4.1). In Figure 2, black curve is also related to the ordinary derivative ($\beta = 1, \alpha = 1$). However, in this case the yellow, green, blue, red and brown curves correspond to the following (α, β) values: $(0.7, 1.0)$, $(1.5, 1.0)$, $(1.2, 1.4)$, $(0.7, 0.8)$ and $(0.8, 1.5)$, respectively.

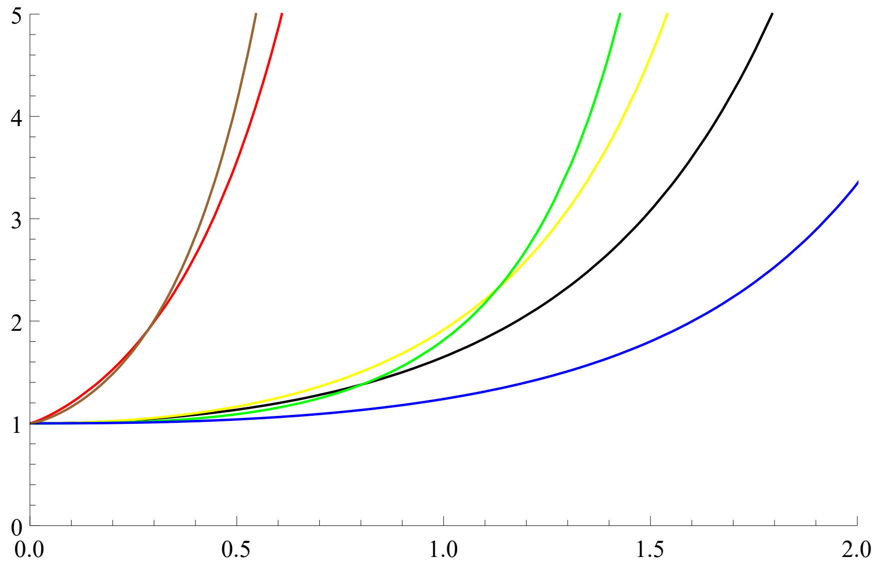

FIGURE 1. Fractal Bernoulli differential model (4.1) for arbitrary values of α and β

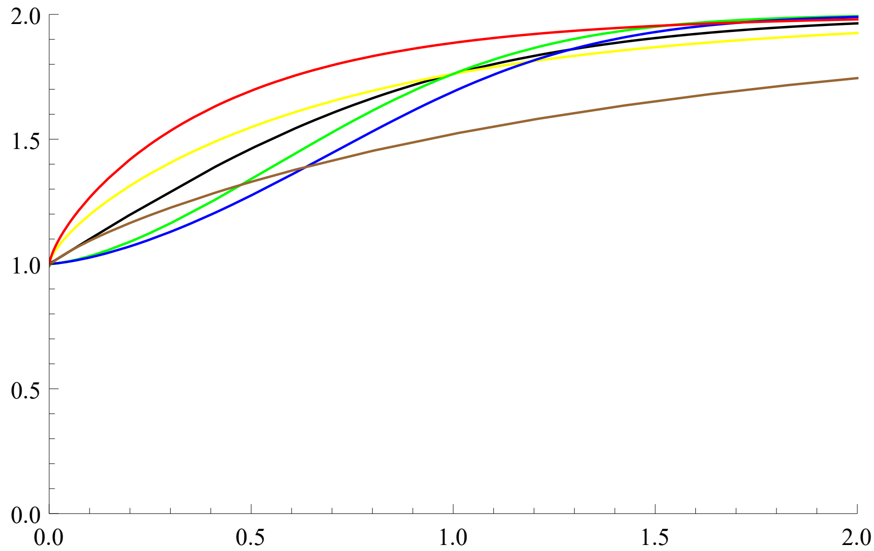

FIGURE 2. Fractal Bernoulli differential model (4.2) for arbitrary values of α and β

In [20], the authors proposed a generalized conformal fractional derivative G_T^α given by the formula

$$(G_T^\alpha f)(t) = \lim_{h\to 0} \frac{f(t) - f(t - he^{(\alpha-1)t})}{h},$$

and applied it to study a Gompertz model. In addition, a real data set on tuberculosis in Mexico was studied and used to solve the inverse problem to estimate the order of the proposed fractional derivative and compare it with the usual derivative and other fractional derivatives (e.g. Khalil and Caputo derivatives). The results they yielded were surprising because the proposed fractional conformable approximation minimizes the error in the adjustments of the parameters associated to the Gompertz model. In what follows we will see that a particular fractal Bernoulli differential equation best fits the data set studied in [20]. For this study we used the same data of the percentage of people with tuberculosis in Mexico between the years 1990 and 2015 [1].

Figure 3 shows the data associated with the percentage of people with tuberculosis in Mexico (black asterisk) and the corresponding fits to the Gompertz model

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = 0.2008 \cdot y(t) \ln\left(\frac{1}{y(t)}\right), \tag{4.3}$$

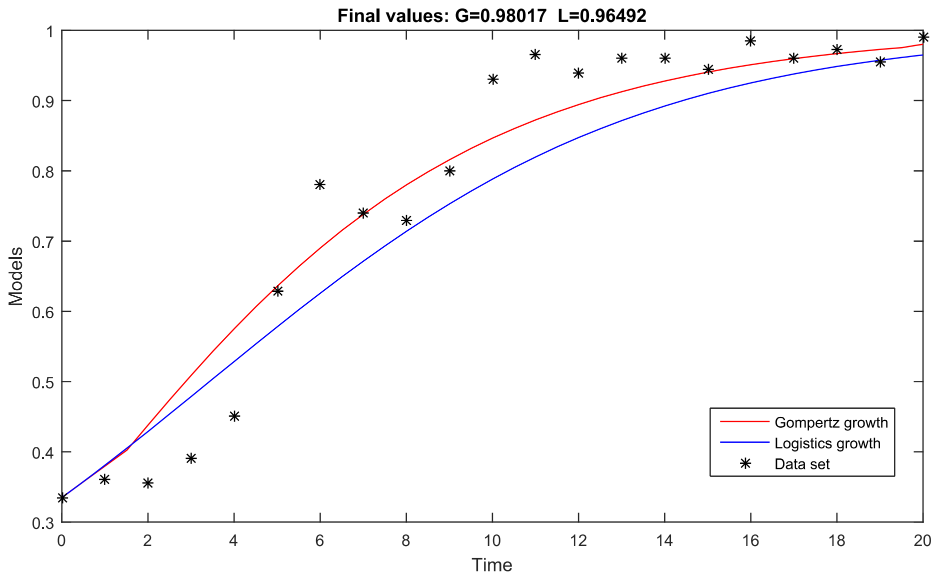

FIGURE 3. Data and estimates of tuberculosis infectious using Gompertz and logistics models

and logistics model

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = 0.2008 \cdot y(t)\,(1 - y(t))\,, \tag{4.4}$$

both with initial condition $y(0) = 0.335$. Using Matlab we have written a script to obtain numerical calculation of the solution of fractal Bernoulli differential equations. For its implementation we used the efficient *ode45* function based on an explicit Runge-Kutta formula, the Dormand-Prince pair. We arrive that the following fractal Bernoulli differential model:

$$\frac{\mathrm{d}^{0.9}y(t)}{\mathrm{d}t^{1.4}} = \left(1 - \frac{1}{0.99}\right) \cdot \left[0.087(y(t))^{0.9} + 0.008(y(t))^{0.63}\right], \tag{4.5}$$
$$y(0) = 0.335,$$

best fits the data of the percentage of people with tuberculosis in Mexico with respect to the previous models (see Figure 4).

Table 1 shows the fitting errors of the studied models: the mean absolute error (MAE) and the mean squared error (MSE). It should also be noted that the fitting

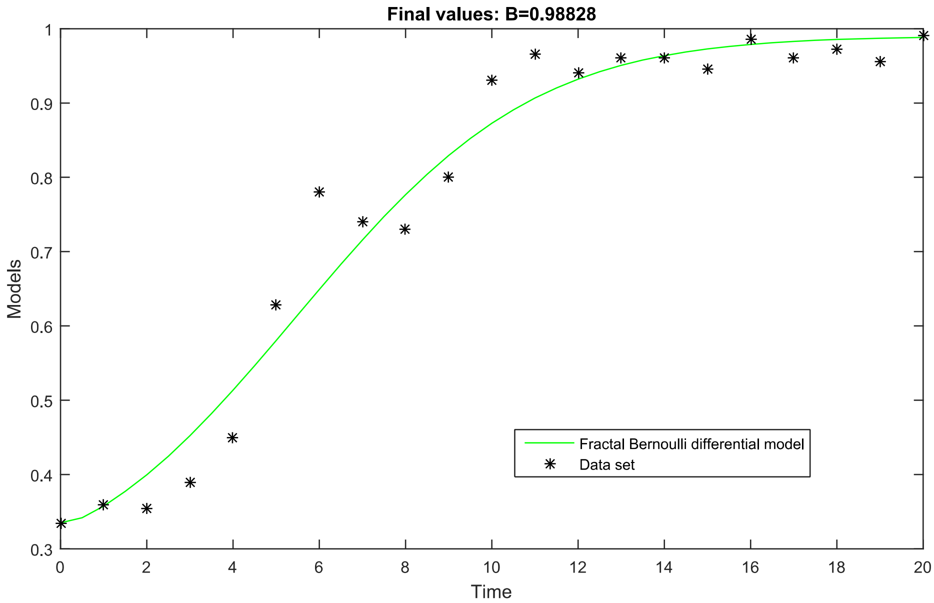

FIGURE 4. Data and estimates of tuberculosis infectious using the fractal Bernoulli differential model (4.5)

errors of the generalized Gompertz model proposed in [20] were relatively larger than those obtained by our model (4.5). The MAE and MSE had values of 0.0416 and 0.0027, respectively.

	Fitting errors	
Models	MAE	MSE
Gompertz growth (4.3)	0.0495	0.0047
Logistics growth (4.4)	0.0614	0.0057
Fractal Bernoulli differential model (4.5)	0.0329	0.0020

TABLE 1. Fitting errors for the studied models

The script is shown below:

```
function fbernoulli
global alpha beta
%%% Set parameters
```

```
alpha=1.4 ; % FILL IN A VALUE FOR ALPHA
beta=0.9; % FILL IN A VALUE FOR BETA
%%% Solve equations
pt = linspace(0,20,100); % Generate t for p
p = (alpha/beta)*pt.^(alpha-1); % Generate p(t)
Tspan = [0 20]; % Solve from t=0 to t=20
IC = 0.335; % y(t=0)=1
[T L] = ode45(@(t,l) myode2(t,l,pt,p),Tspan,IC)
%%%Errors
err2=immse([0.335,0.36,0.355,0.39,0.45,0.628,0.78,0.74,0.73,0.8,0.93,0.965,
0.94,0.96,0.96,0.945,0.985,0.96,0.973,0.955,0.99],[0.335,deval(ode45(@(t,l)
myode2(t,l,pt,p),Tspan,IC),1),deval(ode45(@(t,l) myode2(t,l,pt,p),Tspan,IC),2),
deval(ode45(@(t,l) myode2(t,l,pt,p),Tspan,IC),3),deval(ode45(@(t,l)
myode2(t,l,pt,p),Tspan,IC),4),deval(ode45(@(t,l) myode2(t,l,pt,p),Tspan,IC),5),
deval(ode45(@(t,l) myode2(t,l,pt,p),Tspan,IC),6),deval(ode45(@(t,l)
myode2(t,l,pt,p),Tspan,IC),7),deval(ode45(@(t,l) myode2(t,l,pt,p),Tspan,IC),8),
deval(ode45(@(t,l) myode2(t,l,pt,p),Tspan,IC),9),deval(ode45(@(t,l)
myode2(t,l,pt,p),Tspan,IC),10),deval(ode45(@(t,l) myode2(t,l,pt,p),Tspan,IC),11),
deval(ode45(@(t,l) myode2(t,l,pt,p),Tspan,IC),12),deval(ode45(@(t,l)
myode2(t,l,pt,p),Tspan,IC),13),deval(ode45(@(t,l) myode2(t,l,pt,p),
Tspan,IC),14),deval(ode45(@(t,l) myode2(t,l,pt,p),
Tspan,IC),15),deval(ode45(@(t,l) myode2(t,l,pt,p),
Tspan,IC),16),deval(ode45(@(t,l) myode2(t,l,pt,p),Tspan,IC),17),
deval(ode45(@(t,l) myode2(t,l,pt,p),Tspan,IC),18),deval(ode45(@(t,l)
myode2(t,l,pt,p),Tspan,IC),19),deval(ode45(@(t,l) myode2(t,l,pt,p),
Tspan,IC),20)])
mae2=mae([0.335,0.36,0.355,0.39,0.45,0.628,0.78,0.74,0.73,0.8,0.93,0.965,
0.94,0.96,0.96,0.945,0.985,0.96,0.973,0.955,0.99],[0.335,deval(ode45(@(t,l)
myode2(t,l,pt,p),Tspan,IC),1),deval(ode45(@(t,l) myode2(t,l,pt,p),Tspan,IC),2),
deval(ode45(@(t,l) myode2(t,l,pt,p),Tspan,IC),3),deval(ode45(@(t,l)
myode2(t,l,pt,p),Tspan,IC),4),deval(ode45(@(t,l) myode2(t,l,pt,p),
Tspan,IC),5),deval(ode45(@(t,l) myode2(t,l,pt,p),Tspan,IC),6),
deval(ode45(@(t,l) myode2(t,l,pt,p),Tspan,IC),7),deval(ode45(@(t,l)
myode2(t,l,pt,p),Tspan,IC),8),deval(ode45(@(t,l) myode2(t,l,pt,p),Tspan,IC),9),
deval(ode45(@(t,l) myode2(t,l,pt,p),Tspan,IC),10),deval(ode45(@(t,l)
myode2(t,l,pt,p),Tspan,IC),11),deval(ode45(@(t,l) myode2(t,l,pt,p),
Tspan,IC),12),deval(ode45(@(t,l) myode2(t,l,pt,p),Tspan,IC),13),
deval(ode45(@(t,l) myode2(t,l,pt,p),Tspan,IC),14),deval(ode45(@(t,l)
myode2(t,l,pt,p),Tspan,IC),15),deval(ode45(@(t,l) myode2(t,l,pt,p),
Tspan,IC),16),deval(ode45(@(t,l) myode2(t,l,pt,p),Tspan,IC),17),
deval(ode45(@(t,l) myode2(t,l,pt,p),Tspan,IC),18),deval(ode45(@(t,l)
myode2(t,l,pt,p),Tspan,IC),19),deval(ode45(@(t,l) myode2(t,l,pt,p),
Tspan,IC),20)])
%%% Plot results
figure;
plot(T,L,'green',0,0.335,'k*',1,0.36,'k*',2,0.355,'k*',3,0.39,'k*',4,0.45,
'k*',5,0.628,'k*',6,0.78,'k*',7,0.74,'k*',8,0.73,'k*',9,0.8,'k*',10,0.93,
'k*',11,0.965,'k*',12,0.94,'k*',13,0.96,'k*',14,0.96,'k*',15,0.945,
'k*',16,0.985,'k*',17,0.96,'k*',18,0.973,'k*',19,0.955,'k*',20,0.99,'k*');
```

```
title('Plot of y as a function of time');
xlabel('Time');
ylabel('Models');
legend('Fractal Bernoulli differential model','Data set')
function dl\mathrm{d}t = myode2(t,l,pt,p)
global beta
f = interp1(pt,p,t); % Interpolate the data set (pt,p) at time t
dl\mathrm{d}t =
f.*(abs(l))^(1-beta)*0.1*(0.87*l^beta+0.08*l^(0.7*beta))*(1-(l/0.99));
```

5. Conclusions

The advantage of using fractal differential equations in general is that we can incorporate the variable order of this operator, unlike ordinary differential equations where the simulation and modeling is referred only to the time variable. Thus we have a theoretical-practical panorama that is much broader, more general and richer than that of ordinary differential equations, and which has demonstrated its effectiveness and efficiency in the solution of multiple problems. In this work we studied Bernoulli-type differential equations using fractal derivatives. We were able to show the goodness-of-fit of a particular fractal Bernoulli differential model for the study of real data related to tuberculosis in Mexico. This last fact is remarkable because with such fractal derivatives the curve fit is better than with classic models using usual derivatives. The question of finding a fractal model that further minimizes the fitting error of the data remains open to us. As future work, we plan to use Bayesian statistics to estimate the alpha and beta orders with the aim of finding an optimal model, following the ideas of the paper [16].

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RECONSTRUCTION OF THE STURM-LIOUVILLE DIFFERENTIAL OPERATORS WITH TWO CONSTANT DELAYS

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Abstract. In this manuscript, we study the Sturm–Liouville differential operator with two constant delays. We investigate the properties of the asymptotic form of solutions, eigenvalues, and eigenfunctions of the operator. An inverse spectral problem is studied of recovering the potential functions and delay points from four boundary value problems. Also, we construct the Fourier coefficients. So, we construct the potential functions by using the Fourier series.

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Keywords: Sturm–Liouville problems, inverse problems, constant delay, asymptotic form of solution

1. Introduction

We consider the Sturm–Liouville differential equations

$$\ell_i y := -y''(x) + q_1(x)y(x-a_1) + (-1)^i q_2(x)y(x-a_2) = \lambda y(x), \quad x \in (0,\pi), \tag{1.1}$$

subject to the boundary conditions

$$y(0) = y^{(j)}(\pi) = 0, \tag{1.2}$$

where $q_1(x) \in L(a_1,\pi)$, $q_2(x) \in L(a_2,\pi)$, $q_1(x) = 0$ for $x < a_1$, and $q_2(x) = 0$ for $x < a_2$, are real functions. In what follows, we always take $i, j = 0, 1$. The coefficient $a_1, a_2 \in [0,\pi)$ are real and assumed to be known a priori and fixed and $a_1 < a_2$. For simplicity we use the notation $L_{i,j} := L_{i,j}(q_1(x); q_2(x); a_1; a_2)$, for the problems (1.1)–(1.2).

For the Sturm–Liouville problems, we have three types of problems: *Direct problems*, *Isospectral problems* and *Inverse problems*. In direct problems, the eigenvalues, eigenfunctions and some properties of the problem are estimated from the known coefficients. In isospectral problems, for a given problem, we want to obtain different problems of the same form, which have the same eigenvalues of the initial problem. Isospectral Sturm–Liouville problems are studied in [6, 9, 10]. The third type of problems related to the Sturm–Liouville problems are inverse problem. The inverse spectral Sturm–Liouville problem can be regarded as three aspects: existence, uniqueness and reconstruction of the coefficients with specific properties of eigenvalues and eigenfunctions, (see [4, 8, 12, 16–19, 21, 22, 26] and the references therein).

In the seminal paper [5], G. Freiling and V. A. Yurko motivated by the inverse Sturm–Liouville problem with a constant delays inside the interval. In this paper, they proved that if the spectra of the problems $L_j(q),\ j=0,1,$ coincide with the spectra of $L_j(0),\ j=0,1,$ respectively, then $q(x)=0$ a.e. on $(0,\pi)$. So, they proved a spacial uniqueness theorem in the case of one constant delay. This uniqueness theorem was later extended by me to the cases of two and finite number of constant delays [20, 21]. In [23], M. Shahriari, B. N. Saray, and J. Manafian studied the inverse delay Sturm–Liouville problems with a transmission conditions inside the interval. We constructed delay point and the potential function by using the coefficients of the Fourier series of the Sturm–Liouville differential operator. Moreover, in [11], S. Mosazadeh were studied an inverse Sturm–Liouville problem with a delay and eigenparameter–dependent boundary conditions.

Although other effective methods have been created and some aspects of the direct and inverse problems for operators with a delay can be found in [1–3, 7, 14, 15, 24, 27]. For general background on the delay differential equations we refer (e.g.) to the monographs [13, 25].

In the present paper, we study an inverse problem of Sturm–Liouville differential operators. In this note, we discuss the reconstruction of the potential functions $q_1(x)$, $q_2(x)$ and constant delays a_1, a_2 with four boundary spectral problems (1.1)–(1.2). For this purpose, we study the asymptotic form of eigenvalues and eigenfunctions of the problems. So, we investigate the inverse spectral problem of recovering operators from their four spectra in the Dirichlet–Dirichlet and Dirichlet–Neumann boundary conditions with two constant delays inside the interval. Finally, we construct the potential functions by using the Fourier series.

2. Asymptotic form of solutions and eigenvalues

Let $\varphi^i(x,\lambda)$ be the solution of Eq. (1.1) with $a_1N<\pi\le a_1(N+1)$ and $a_2M<\pi\le a_2(M+1)$ under the initial conditions $\varphi^i(0,\lambda)=0$, $\varphi^{i'}(0,\lambda)=1$. For each fixed x, the functions $\varphi^{i(j)}(x,\lambda)$ are entire in λ of order $1/2$. The function $\varphi^i(x,\lambda)$ is the unique solution of the integral equation

$$\varphi^i(x,\lambda)=\frac{\sin\rho x}{\rho}+\int_0^x\frac{\sin\rho(x-t)}{\rho}\left(q_1(t)\varphi^i(t-a_1,\lambda)+(-1)^iq_2(t)\varphi^i(t-a_2,\lambda)\right)\mathrm{d}t, \tag{2.1}$$

with $\rho^2=\lambda$ and $\rho=\sigma+i\tau$. Solving (2.1) by the method of successive approximations, we get

$$\varphi^i(x,\lambda)=\varphi^i_0(x,\lambda)+\varphi^i_1(x,\lambda)+\cdots+\varphi^i_N(x,\lambda). \tag{2.2}$$

So, we have

$$\varphi_0^i(x,\lambda) = \frac{\sin\rho x}{\rho}, \tag{2.3}$$

$$\varphi_k^i(x,\lambda) = \begin{cases} 0, & x \le ka_1, \\ \int_{ka_1}^x \frac{\sin\rho(x-t)}{\rho} q_1(t)\varphi_{k-1}^i(t-a_1,\lambda)\mathrm{d}t, & ka_1 \le x \le ka_2, \\ \int_{ka_1}^x \frac{\sin\rho(x-t)}{\rho} q_1(t)\varphi_{k-1}^i(t-a_1,\lambda)\mathrm{d}t \\ \quad +(-1)^i \int_{ka_2}^x \frac{\sin\rho(x-t)}{\rho} q_2(t)\varphi_{k-1}^i(t-a_2,\lambda)\mathrm{d}t, & x \ge ka_2, \end{cases} \tag{2.4}$$

$$\varphi_k^{i\,\prime}(x,\lambda) = \begin{cases} 0, & x \le ka_1, \\ \int_{ka_1}^x \cos\rho(x-t) q_1(t)\varphi_{k-1}^i(t-a_1,\lambda)\mathrm{d}t, & ka_1 \le x \le ka_2, \\ \int_{ka_1}^x \cos\rho(x-t) q_1(t)\varphi_{k-1}^i(t-a_1,\lambda)\mathrm{d}t \\ \quad +(-1)^i \int_{ka_2}^x \cos\rho(x-t) q_2(t)\varphi_{k-1}^i(t-a_2,\lambda)\mathrm{d}t, & x \ge ka_2, \end{cases} \tag{2.5}$$

for $k = 1,2,\ldots,M$ and

$$\varphi_k^i(x,\lambda) = \begin{cases} 0, & x \le ka_1, \\ \int_{ka_1}^x \frac{\sin\rho(x-t)}{\rho} q_1(t)\varphi_{k-1}^i(t-a_1,\lambda)\mathrm{d}t, & x \ge ka_1, \end{cases} \tag{2.6}$$

$$\varphi_k^{i\,\prime}(x,\lambda) = \begin{cases} 0, & x \le ka_1, \\ \int_{ka_1}^x \cos\rho(x-t) q_1(t)\varphi_{k-1}^i(t-a_1,\lambda)\mathrm{d}t, & x \ge ka_1, \end{cases} \tag{2.7}$$

for $k = M+1, M+2, \ldots, N$. Using the formulas (2.3)–(2.5) we calculate

$$\varphi_1^i(x,\lambda) = \begin{cases} 0, & x \le a_1, \\ \int_{a_1}^x \frac{\sin\rho(x-t)}{\rho} q_1(t)\varphi_0^i(t-a_1,\lambda)\mathrm{d}t, & a_1 \le x \le a_2, \\ \int_{a_1}^x \frac{\sin\rho(x-t)}{\rho} q_1(t)\varphi_0^i(t-a_1,\lambda)\mathrm{d}t \\ \quad +(-1)^i \int_{a_2}^x \frac{\sin\rho(x-t)}{\rho} q_2(t)\varphi_0^i(t-a_2,\lambda)\mathrm{d}t, & x \ge a_2, \end{cases}$$

$$= \begin{cases} 0, & x \le a_1, \\ \int_{a_1}^{x} \frac{\sin\rho(x-t)}{\rho} q_1(t) \frac{\sin\rho(t-a_1)}{\rho} \mathrm{d}t, & a_1 \le x \le a_2, \\ \int_{a_1}^{x} \frac{\sin\rho(x-t)}{\rho} q_1(t) \frac{\sin\rho(t-a_1)}{\rho} \mathrm{d}t \\ \quad +(-1)^i \int_{a_2}^{x} \frac{\sin\rho(x-t)}{\rho} q_2(t) \frac{\sin\rho(t-a_2)}{\rho} \mathrm{d}t, & x \ge a_2, \end{cases}$$

$$= \begin{cases} 0, & x \le a_1, \\ \frac{1}{2\rho^2} \big(-\cos\rho(x-a_1) \int_{a_1}^{x} q_1(t)\mathrm{d}t \\ \quad + \int_{a_1}^{x} \cos\rho(2t-x-a_1) q_1(t)\mathrm{d}t \big), & a_1 \le x \le a_2, \\ \frac{1}{2\rho^2} \big(-\cos\rho(x-a_1) \int_{a_1}^{x} q_1(t)\mathrm{d}t \\ \quad -(-1)^i \cos\rho(x-a_2) \int_{a_2}^{x} q_2(t)\mathrm{d}t \\ \quad + \int_{a_1}^{x} \cos\rho(2t-x-a_1) q_1(t)\mathrm{d}t \\ \quad +(-1)^i \int_{a_2}^{x} \cos\rho(2t-x-a_2) q_2(t)\mathrm{d}t \big), & x \ge a_2, \end{cases}$$

$$= \begin{cases} 0, & x \le a_1, \\ \frac{1}{2\rho^2} \big(-\cos\rho(x-a_1) \int_{a_1}^{x} q_1(t)\mathrm{d}t \\ \quad + \cos\rho(x+a_1) \int_{a_1}^{x} \cos(2\rho t) q_1(t)\mathrm{d}t \\ \quad + \sin\rho(x+a_1) \int_{a_1}^{x} \cos(2\rho t) q_1(t)\mathrm{d}t \big), & a_1 \le x \le a_2, \\ \frac{1}{2\rho^2} \big(-\cos\rho(x-a_1) \int_{a_1}^{x} q_1(t)\mathrm{d}t \\ \quad -(-1)^i \cos\rho(x-a_2) \int_{a_2}^{x} q_2(t)\mathrm{d}t \\ \quad + \cos\rho(x+a_1) \int_{a_1}^{x} \cos(2\rho t) q_1(t)\mathrm{d}t \\ \quad + \sin\rho(x+a_1) \int_{a_1}^{x} \sin(2\rho t) q_1(t)\mathrm{d}t \\ \quad +(-1)^i \cos\rho(x+a_2) \int_{a_2}^{x} \cos(2\rho t) q_2(t)\mathrm{d}t \\ \quad + \sin\rho(x+a_2) \int_{a_2}^{x} \sin(2\rho t) q_2(t)\mathrm{d}t \big), & x \ge a_2, \end{cases} \tag{2.8}$$

and

$\varphi_1^{i\,\prime}(x,\lambda)$

$$= \begin{cases} 0, & x \le a_1, \\ \frac{1}{2\rho}\left(\sin\rho(x-a_1)\int_{a_1}^{x} q_1(t)\mathrm{d}t + \int_{a_1}^{x}\sin\rho(2t-x-a_1)q_1(t)\mathrm{d}t\right), & a_1 \le x \le a_2, \\ \frac{1}{2\rho}\Big(\sin\rho(x-a_1)\int_{a_1}^{x} q_1(t)\mathrm{d}t + (-1)^i\sin\rho(x-a_2)\int_{a_2}^{x} q_2(t)\mathrm{d}t \\ \quad + \int_{a_1}^{x}\sin\rho(2t-x-a_1)q_1(t)\mathrm{d}t \\ \quad + (-1)^i\int_{a_2}^{x}\sin\rho(2t-x-a_2)q_2(t)\mathrm{d}t\Big), & x \ge a_2, \end{cases}$$

$$= \begin{cases} 0, & x \le a_1, \\ \frac{1}{2\rho^2}\Big(\sin\rho(x-a_1)\int_{a_1}^{x} q_1(t)\mathrm{d}t - \sin\rho(x+a_1)\int_{a_1}^{x}\cos(2\rho t)q_1(t)\mathrm{d}t \\ \quad + \sin\rho(x+a_1)\int_{a_1}^{x}\cos(2\rho t)q_1(t)\mathrm{d}t\Big), & a_1 \le x \le a_2, \\ \frac{1}{2\rho^2}\Big(-\cos\rho(x-a_1)\int_{a_1}^{x} q_1(t)\mathrm{d}t - (-1)^i\cos\rho(x-a_2)\int_{a_2}^{x} q_2(t)\mathrm{d}t \\ \quad + \cos\rho(x+a_1)\int_{a_1}^{x}\sin(2\rho t)q_1(t)\mathrm{d}t \\ \quad - \sin\rho(x+a_1)\int_{a_1}^{x}\cos(2\rho t)q_1(t)\mathrm{d}t \\ \quad + (-1)^i\cos\rho(x+a_2)\int_{a_2}^{x}\sin(2\rho t)q_2(t)\mathrm{d}t \\ \quad - (-1)^i\sin\rho(x+a_2)\int_{a_2}^{x}\cos(2\rho t)q_2(t)\mathrm{d}t\Big), & x \ge a_2. \end{cases} \tag{2.9}$$

For $k = 2$ from (2.5)–(2.9) we get

$$\varphi_2^i(x,\lambda) = \begin{cases} 0, & x \le 2a_1, \\ \int_{2a_1}^{x} \frac{\sin\rho(x-t)}{\rho} q_1(t)\varphi_1(t-a_1,\lambda)\mathrm{d}t, & 2a_1 \le x \le 2a_2, \\ \int_{2a_1}^{x} \frac{\sin\rho(x-t)}{\rho} q_1(t)\varphi_1(t-a_1,\lambda)\mathrm{d}t \\ \quad + (-1)^i\int_{2a_2}^{x} \frac{\sin\rho(x-t)}{\rho} q_2(t)\varphi_1(t-a_2,\lambda)\mathrm{d}t, & x \ge 2a_2, \end{cases} \tag{2.10}$$

and

$$\varphi_2^{i\,\prime}(x,\lambda) = \begin{cases} 0, & x \le 2a_1, \\ \int_{2a_1}^{x} \cos\rho(x-t) q_1(t)\varphi_1(t-a_1,\lambda)\mathrm{d}t, & 2a_1 \le x \le 2a_2, \\ \int_{2a_1}^{x} \cos\rho(x-t) q_1(t)\varphi_1(t-a_1,\lambda)\mathrm{d}t \\ \quad + (-1)^i\int_{2a_2}^{x} \cos\rho(x-t) q_2(t)\varphi_1(t-a_2,\lambda)\mathrm{d}t, & x \ge 2a_2. \end{cases} \tag{2.11}$$

From Eqs. (2.8)–(2.11) with a simple calculation we obtain

$$\varphi_2^{i\,(j)}(x,\lambda) = \begin{cases} 0, & x \le 2a_1, \\ O(\rho^{j-3}\exp(|\tau|(x-2a_1))), & x \ge 2a_1, \end{cases} \qquad |\rho| \to \infty,$$

where $\tau = \mathrm{Im}\rho$. Using Eqs. (2.3)–(2.9) by induction one can easily show that

$$\varphi_k^{i\,(j)}(x,\lambda) = \begin{cases} 0, & x \le ka_1, \\ O(\rho^{j-k-1}\exp(|\tau|(x-ka_1))), & x \ge ka_1, \end{cases} \qquad |\rho| \to \infty.$$

Denote $\Delta_j^i(\lambda) := \varphi^{i\,(j)}(\pi,\lambda)$. The functions $\Delta_j^i(\lambda)$ are entire functions in λ of order $\frac{1}{2}$ and the zeroes of $\Delta_j^i(\lambda)$ coincide with the eigenvalues λ_n^i of $L_{i,0}$ and μ_n^i of $L_{i,1}$. So, the function $\Delta_j^i(\lambda)$ is called the characteristic function for $L_{i,j}$. From Eqs. (2.1)–(2.2), (2.8)–(2.9), and (2.11) we calculate the following asymptotic formula for $|\rho| \to \infty$,

$$\begin{aligned} \Delta_0^i(\lambda) &= \varphi^i(\pi,\lambda) \\ &= \frac{\sin\rho\pi}{\rho} + \frac{1}{2\rho^2}\Bigg[-\cos\rho(\pi-a_1)w_1 - (-1)^i\cos\rho(\pi-a_2)w_2 \\ &\quad + \int_{a_1}^{\pi}\cos\rho(2t-\pi-a_1)q_1(t)\mathrm{d}t + (-1)^i\int_{a_2}^{\pi}\cos\rho(2t-\pi-a_2)q_2(t)\mathrm{d}t \Bigg] \\ &\quad + O\left(\frac{\exp(|\tau|(\pi-a_1))}{\rho^3}\right), \end{aligned} \tag{2.12}$$

and

$$\begin{aligned} \Delta_1^i(\lambda) &= \varphi^{i\prime}(\pi,\lambda) \\ &= \cos\rho\pi + \frac{1}{2\rho}\Bigg[\sin\rho(\pi-a_1)w_1 + (-1)^i\sin\rho(\pi-a_2)w_2 \\ &\quad + \int_{a_1}^{\pi}\sin\rho(2t-\pi-a_1)q_1(t)\mathrm{d}t + (-1)^i\int_{a_2}^{\pi}\sin\rho(2t-\pi-a_2)q_2(t)\mathrm{d}t \Bigg] \\ &\quad + O\left(\frac{\exp(|\tau|(\pi-a_1))}{\rho^2}\right), \end{aligned} \tag{2.13}$$

where $w_1 := \int_{a_1}^{\pi} q_1(t)\mathrm{d}t$ and $w_2 := \int_{a_2}^{\pi} q_2(t)\mathrm{d}t$.

Lemma 1 ([20, Sec. 2]). *The asymptotic formula for the eigenvalues $\lambda_n^i = {\rho_{n0}^i}^2$ and $\mu_n^i = {\rho_{n1}^i}^2$ as $n \to \infty$ are of the following forms:*

$$\rho_{n0}^i = n + \frac{1}{2\pi n}\left[w_1\cos na_1 + (-1)^i w_2\cos na_2\right] + o\left(\frac{1}{n}\right), \tag{2.14}$$

$$\rho_{n1}^i = n - \frac{1}{2} + \frac{1}{2\pi n}\left[w_1\cos\left(n-\frac{1}{2}\right)a_1 + (-1)^i w_2\cos\left(n-\frac{1}{2}\right)a_2\right] + o\left(\frac{1}{n}\right).$$

Lemma 2 ([20, Lemma 2.1], [23, Lemma 2.1]). *The specification of the spectrum* $\{\lambda_n^i\}$, $n \geq 1$ *and* $j = 0, 1$, *uniquely determines the characteristic function* $\Delta_j^i(\lambda)$ *by the formulaes*

$$\Delta_0^i(\lambda) = \pi \prod_{n=1}^{\infty} \frac{\lambda_{n0}^i - \lambda}{n^2}, \qquad \Delta_1^i(\lambda) = \prod_{n=1}^{\infty} \frac{\lambda_{n1}^i - \lambda}{(n - \frac{1}{2})^2}.$$

3. Reconstruction of potential function

In this section, we study the Sturm–Liouville differential operator with two constant delays. We solve the inverse spectral problems of these operators when $a_1 \in (\frac{\pi}{2}, \pi)$ and $a_2 \in (\frac{\pi}{2}, \pi)$. We will first show that the delay points a_1, a_2 and the values of w_1 and w_2 are uniquely determined by the spectrum. So, we prove our main theorem.

Lemma 3. *If* $\{\lambda_n^i\}_{n\geq 1}$ *are the spectrum of* $L_{i,0}$ *then the delay points* a_1 *and* a_2 *are uniquely determined.*

Proof. Let us consider the sequence

$$\tilde{\lambda}_n = \frac{\lambda_n^0 + \lambda_n^1}{2}$$

and

$$\hat{\lambda}_n = \frac{\lambda_n^0 - \lambda_n^1}{2}.$$

From (2.14) we get the asymptotic formulas:

$$\tilde{\lambda}_n = n^2 + \frac{w_1}{\pi} \cos n a_1 + o(1) \tag{3.1}$$

and

$$\hat{\lambda}_n = \frac{w_2}{\pi} \cos n a_2 + o(1). \tag{3.2}$$

There are infinitely many numbers $k, l \in \mathbb{N}$ with $\sin(k a_2) \neq 0$ and $\sin(l a_1) \neq 0$. From (3.1) and (3.2) we get

$$\begin{aligned}
\lim_{k\to\infty} \frac{\hat{\lambda}_{k+2} - \hat{\lambda}_{k-2}}{\hat{\lambda}_{k+1} - \hat{\lambda}_{k-1}} &= \lim_{k\to\infty} \frac{w_2(\cos a_2(k+2) - \cos a_2(k-2)) + o(1)}{w_2(\cos a_2(k+1) - \cos a_2(k-1)) + o(1)} \\
&= \lim_{k\to\infty} \frac{\sin k a_2 \sin 2a_2 + o(1)}{\sin k a_2 \sin a_2 + o(1)} \\
&= \frac{\sin 2a_2}{\sin a_2} = 2\cos a_2
\end{aligned} \tag{3.3}$$

and

$$\lim_{k\to\infty} \frac{\tilde{\lambda}_{l+2} - (l+2)^2 - \tilde{\lambda}_{l-2} + (l-2)^2}{\tilde{\lambda}_{l+1} - (l+1)^2 - \tilde{\lambda}_{l-1} + (l-1)^2}$$

$$
\begin{aligned}
&= \lim_{l\to\infty} \frac{\cos a_1(l+2) - \cos a_1(l-2) + o(1)}{\cos a_1(l+1) - \cos a_1(l-1) + o(1)} \\
&= \lim_{k\to\infty} \frac{\sin la_1 \sin 2a_1 + o(1)}{\sin la_1 \sin a_1 + o(1)} \\
&= \frac{\sin 2a_1}{\sin a_1} \\
&= 2\cos a_1.
\end{aligned} \tag{3.4}
$$

Then, we get the delay points a_1 and a_2. □

Lemma 4. *If $\left\{\lambda_n^i\right\}_{n\geq 1}$ are the spectrum of $L_{i,0}$ then the values of w_1 and w_2 are uniquely determined.*

Proof. There are infinitely many $k \in \mathbb{N}$ satisfying

$$
\det \begin{bmatrix} \cos ka_1 & \cos ka_2 \\ \cos ka_1 & -\cos ka_2 \end{bmatrix} \neq 0. \tag{3.5}
$$

From (2.13), we get

$$
w_1 \cos na_1 + (-1)^i w_2 \cos na_2 = \pi(\lambda_n^i - n^2) + o(1), \tag{3.6}
$$

So, from (3.5) and (3.6), we calculate

$$
\begin{aligned}
w_1 &= \lim_{k\to\infty} \frac{\det \begin{bmatrix} \pi(\lambda_k^0 - k^2) + o(1) & \cos ka_2 \\ \pi(\lambda_k^1 - k^2) + o(1) & -\cos ka_2 \end{bmatrix}}{\det \begin{bmatrix} \cos ka_1 & \cos ka_2 \\ \cos ka_1 & -\cos ka_2 \end{bmatrix}} \\
w_2 &= \lim_{k\to\infty} \frac{\det \begin{bmatrix} \cos ka_1 & \pi(\lambda_k^0 - k^2) + o(1) \\ \cos ka_1 & \pi(\lambda_k^1 - k^2) + o(1) \end{bmatrix}}{\det \begin{bmatrix} \cos ka_1 & \cos ka_2 \\ \cos ka_1 & -\cos ka_2 \end{bmatrix}}
\end{aligned} \tag{3.7}
$$

□

Applying $q_1(x) = 0$ for $x \in [0, a_1)$ and $q_2(x) = 0$ for $x \in [0, a_2)$, we obtain that $\int_0^\pi q_1(t)\mathrm{d}t = \int_{a_1}^\pi q_1(t)\mathrm{d}t$ and $\int_0^\pi q_2(t)\mathrm{d}t = \int_{a_2}^\pi q_2(t)\mathrm{d}t$. By using Lemma 4, we get the first Fourier coefficient of the potential q_1 and q_2 on $[0, \pi]$.

Now, we will obtain the Fourier coefficients of the potential function $q_1(x)$ and $q_2(x)$. So, we will prove that these coefficients are uniquely determined from the spectrum of $L_{i,j}$. Denote the Fourier coefficients of q_1 and q_2 by

$$
a_n = \int_0^\pi q_1(t) \cos 2nt\, \mathrm{d}t, \qquad b_n = \int_0^\pi q_1(t) \sin 2nt\, \mathrm{d}t,
$$
$$
c_n = \int_0^\pi q_2(t) \cos 2nt\, \mathrm{d}t, \qquad d_n = \int_0^\pi q_2(t) \sin 2nt\, \mathrm{d}t.
$$

So, we finally come to our main result.

Theorem 1. *If* $\{\lambda_n^i\}_{n\geq 1}$ *and* $\{\mu_n^i\}_{n\geq 1}$ *be the eigenvalues of the boundary value problems* $L_{i,0}$ *and* $L_{i,1}$*, respectively, then the Fourier coefficient* a_n*,* b_n*,* c_n*, and* d_n *of the potential functions* q_1 *and* q_2 *uniquely determined for all* $n\in\mathbb{N}$*.*

Proof. From the Eq. (2.12), (2.13) and $a_1,a_2\in(\frac{\pi}{2},\pi)$ we obtain

$$
\begin{aligned}
F_{i,0}(\rho) &= \Delta_0^i(\lambda)\\
&= \varphi^i(\pi,\lambda)\\
&= \frac{\sin\rho\pi}{\rho}+\frac{1}{2\rho^2}\Bigg[-\cos\rho(\pi-a_1)w_1-(-1)^i\cos\rho(\pi-a_2)w_2\\
&\quad+\int_{a_1}^{\pi}\cos\rho(2t-\pi-a_1)q_1(t)\mathrm{d}t+(-1)^i\int_{a_2}^{\pi}\cos\rho(2t-\pi-a_2)q_2(t)\mathrm{d}t\Bigg]\\
&= \frac{\sin\rho\pi}{\rho}+\frac{1}{2\rho^2}\Bigg[-\cos\rho(\pi-a_1)w_1-(-1)^i\cos\rho(\pi-a_2)w_2\\
&\quad+\cos\rho(\pi+a_1)\int_{a_1}^{\pi}\cos(2\rho t)q_1(t)\mathrm{d}t+\sin\rho(\pi+a_1)\int_{a_1}^{\pi}\sin(2\rho t)q_1(t)\mathrm{d}t\\
&\quad+(-1)^i\cos\rho(\pi+a_2)\int_{a_2}^{\pi}\cos(2\rho t)q_2(t)\mathrm{d}t\\
&\quad+(-1)^i\sin\rho(\pi+a_2)\int_{a_2}^{\pi}\sin(2\rho t)q_2(t)\mathrm{d}t\Bigg],
\end{aligned}
$$

and

$$
\begin{aligned}
F_{i,1}(\rho) &= \Delta_1^i(\lambda)=\varphi^{i\prime}(\pi,\lambda)\\
&= \cos\rho\pi+\frac{1}{2\rho}\Bigg[\sin\rho(\pi-a_1)w_1+(-1)^i\sin\rho(\pi-a_2)w_2\\
&\quad+\int_{a_1}^{\pi}\sin\rho(2t-\pi-a_1)q_1(t)\mathrm{d}t+(-1)^i\int_{a_2}^{\pi}\sin\rho(2t-\pi-a_2)q_2(t)\mathrm{d}t\Bigg]\\
&= \cos\rho\pi+\frac{1}{2\rho}\Bigg[\sin\rho(\pi-a_1)w_1+(-1)^i\sin\rho(\pi-a_2)w_2\\
&\quad+\cos\rho(\pi+a_1)\int_{a_1}^{\pi}\sin(2\rho t)q_1(t)\mathrm{d}t-\sin\rho(\pi+a_1)\int_{a_1}^{\pi}\cos(2\rho t)q_1(t)\mathrm{d}t\\
&\quad+(-1)^i\cos\rho(\pi+a_2)\int_{a_2}^{\pi}\sin(2\rho t)q_2(t)\mathrm{d}t
\end{aligned}
$$

$$-(-1)^i \sin\rho(\pi+a_2)\int_{a_2}^{\pi}\cos(2\rho t)q_2(t)\mathrm{d}t\Bigg].$$

Now by putting $\rho = n$ (for $n \in \mathbb{N}$), we obtain

$$A_i(n) = A_n + (-1)^i C_n \tag{3.8}$$

$$B_i(n) = B_n + (-1)^i D_n \tag{3.9}$$

where

$$A_i(n) := 2n^2 F_{i,0}(n) + \cos n(\pi - a_1)w_1 + (-1)^i \cos n(\pi - a_2)w_2$$
$$B_i(n) := 2n(F_{i,1}(n) - (-1)^n) - \sin n(\pi - a_1)w_1 - (-1)^i \sin n(\pi - a_2)w_2$$
$$A_n := \int_{a_1}^{\pi}\cos n(2t-\pi-a_1)q_1(t)\mathrm{d}t, \qquad B_n := \int_{a_1}^{\pi}\sin n(2t-\pi-a_1)q_1(t)\mathrm{d}t$$
$$C_n := \int_{a_2}^{\pi}\cos n(2t-\pi-a_2)q_2(t)\mathrm{d}t, \qquad D_n := \int_{a_2}^{\pi}\sin n(2t-\pi-a_2)q_2(t)\mathrm{d}t$$

Applying Lemmas 2, 3, and 4, we can compute $A_i(n)$ and $B_i(n)$. Using the formulas (3.8) and (3.9), we determine the coefficients A_n, B_n, C_n and D_n of the following forms:

$$A_n = \frac{A_0(n)+A_1(n)}{2}, \qquad C_n = \frac{A_0(n)-A_1(n)}{2},$$
$$B_n = \frac{B_0(n)+B_1(n)}{2}, \qquad D_n = \frac{B_0(n)-B_1(n)}{2}.$$

So, with a simple calculation we obtain the coefficients of the Fourier series

$$a_n = A_n \cos n(\pi+a_1) + B_n \sin n(\pi+a_1),$$
$$b_n = A_n \sin n(\pi+a_1) - B_n \cos n(\pi+a_1),$$
$$c_n = D_n \sin n(\pi+a_2) + C_n \cos n(\pi+a_2),$$
$$d_n = C_n \sin n(\pi+a_2) - D_n \cos n(\pi+a_2).$$

Finally, we use the Fourier series for computing the potential functions $q_1(x)$ and $q_2(x)$. □

Algorithm 1.

(i) *Using Eqs.* (3.3) *and* (3.4)*, for recovering the delay point a_1 and a_2.*
(ii) *Using* (3.7) *for constructing the values w_0 and w_1.*
(iii) *Applying Lemma* 2 *for computing $\Delta_0^i(n)$ and $\Delta_1^i(n)$.*
(iv) *Compute the value of a_n, b_n, c_n, and d_n the coefficients of Fourier series by applying Theorem* 1.
(v) *Applying the Fourier series for reconstruction the unknown potentials.*

3.1. *Example*

In this section, an example is presented.

Example 1. Suppose that in Eqs. (1.1)–(1.2), $a_1 = 2$, $a_2 = 2/5$, and the potential functions are

$$q_1(x) = \begin{cases} 7(x-2)(\pi - x), & x \geq 2, \\ 0, & x < 2. \end{cases}$$

and

$$q_2(x) = \begin{cases} 20(x-2.5)\sin(\pi - x), & x \geq 2/5, \\ 0, & x < 2/5, \end{cases}$$

Figure 1 shows that the reconstruction of $q_1(x)$ and $q_2(x)$.

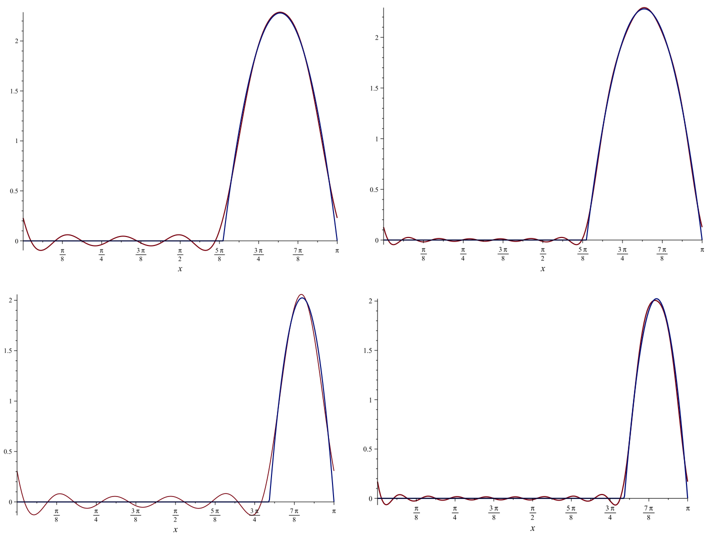

FIGURE 1. Reconstruction of potential functions $q_1(x)$ ((a) $n = 5$, (b) $n = 10$) and $q_2(x)$ ((c) $n = 5$, (d) $n = 10$).

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FURTHER RESULTS ON I-CONVERGENCE OF MULTISET SEQUENCES

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Abstract. The notions of multiset and multiset sequences play an important role in the theory of computation and information sciences. To study convergence properties of multiset sequences, notions of statistical convergence, statistical limit points, and cluster points for multiset sequences were introduced by Debnath and Debnath [5]. Later, in [6], Demir and Gümüş introduced the notions of I-convergence and I^*-convergence for multiset sequences to generalize the results in [5] and investigated the connections between these two notions. In this paper, we extend the results in [6] and introduce and study the notion of I-limit points and I-cluster points for multiset sequences. Further, we introduce and study the notions of I-Cauchy and I^*-Cauchy multiset sequences, and establish relationships with the notions of I-convergence and I^*-convergence of multiset sequences.

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1. INTRODUCTION

According to classical set theory, a set is a well-defined collection of distinct objects, and there is no ordering of the objects in the set. Each object occurs only once in a set. But, in many situations, multiple occurrences of a particular object play an important role in our daily lives. For example, in the case of a cellphone number, one digit can occur multiple times. Also, we know that duplicates could appear at different points in the information retrieval process. In such cases, the notion of multisets comes into play.

Let X be a non-empty set. A multiset M with elements from the set X contains elements $x \in X$ with the multiplicity $C(x)$, where $C : X \to \mathbb{N}$, where $\mathbb{N}$ is the set of all positive integers. The positive integer $C(x)$ represents the multiplicity of the element x. Consider the set $\{3,3,8,8,8,8,7,7\}$. This is a multiset as 3 occurs 2 times, 8 occurs 4 times, and 7 occurs 2 times. In this case, we represent this multiset as

$\{3|2,8|4,7|2\}$. Here, $C(3)=2, C(8)=4, C(7)=2$. Multiset theory can be employed in instances where classical set theory is inadequate. In 1980, Hickman [10] studied algebraic operations on multisets. In 1981, Knuth studied multisets related to the computer programming [11]. Bender [1] investigated the partitions of multisets. In 1976, Lake [14] gave axiomatization of the theory of multisets. In [15], Majumdar introduced the notion of soft multisets, and studied distance, and similarity between two soft multisets. The theory of multiset has many applications in computer and information sciences, interested readers can see [1, 2, 20, 22, 23], and many other references therein. In 2021, Pachilangode and John [17] introduced the notion of distance $d_M(x,y)=d(x,y)+|C(x)-C(y)|$ on a multiset M whose elements are from a metric space (Z,d), and in addition, they introduced and studied the notions of Wijsman convergence, and Hausdorff convergence in the realm of multisets.

On the other hand, the idea of convergence of a sequence of real numbers has been extended to statistical convergence by Fast [8], and Steinhaus [21] independently, and later on re-introduced by Schoenberg [19], and is based on the notion of asymptotic density of the subset of natural numbers. Let $K \subseteq \mathbb{N}$. The asymptotic density of K is defined as $d(K)=\lim_{n\to\infty}\frac{1}{n}|\{k\leq n : k\in K\}|$ if the limit exists. In 1980, Šalát [18] has considered the set of all statistically convergent sequences in l_∞ over the sup norm and showed that the set is dense in l_∞. In 1985, Fridy [9] has defined the notion of statistically Cauchy sequence and investigated the relationships between statistical convergence and statistically Cauchy sequence. In 2000, Kostyrko et al. [12] generalized the notion of statistical convergence of sequences of real numbers by introducing the notion of I-convergence (I is an ideal on the set of natural numbers $\mathbb{N}$) for sequence of elements in a metric space (Z,d). In addition, they introduced and studied the notions of I-limit points and I-cluster points for sequence of elements in a metric space (Z,d). In [7], Dems introduced and studied the notion of I-Cauchy sequences of real numbers and established its relationships with the notion of I-convergence of sequence of real numbers. Later in 2007, Nabiev et al. [16] introduced and studied the notions of I-Cauchy sequence and I^*-Cauchy sequence of elements in a metric space (Z,d). Moreover, they have established relationships between the notions of I-convergence and I-Cauchy. For further studies in the direction, one can see [3, 4, 13], and references therein.

In 2021, Debnath and Debnath [5] introduced the notion of statistical convergence for multiset sequences, and established various properties of this new convergence. Moreover, they defined the notion of statistically boundedness in case of multiset sequences, and established the relation between statistically boundedness and statistical convergence of multiset sequences. Later in 2023, Demir and Gümüş [6] introduced and discussed the notion of ideal convergence for multiset sequences.

The notion of multisets plays an important role in the realm of computer and information sciences. For example, some domain-specific language, such as Structured Query Language (SQL), operates on multisets and displays identical data. To see some applications of multisets, we refer interested readers to [20]. In recent years one can see a surge in studies on the notion of multisets. Naturally, some people desire to investigate further in the area of multisets. In this paper, we plan to do so. The significance of this work lies in the fact that further investigation on multisets may lead to advancements in the realm of computer science and information sciences. We note that this work advances the work done in [6]. This paper is organized as follows: In section 2, we discuss some preliminary notions about ideal and multiset sequence. In section 3, we introduce the notions of I-limits and I-cluster points for multiset sequences and study the concepts. We also investigate the between ideal convergence and I-limit points, and I-cluster points. In section 4, we introduce the notions of I-Cauchy and I^*-Cauchy multiset sequences. We prove that a multiset sequence is I-convergent if and only if it is I-Cauchy. Also, we show that every I^*-Cauchy multiset sequence is I-Cauchy. We provide an example to show that there exists an ideal I for which the notions of I-Cauchy and I^*-Cauchy are different. Furthermore, we establish that the notions of I-Cauchy, and I^*-Cauchy are equivalent if I satisfies weakly additive property.

2. Preliminaries

In this section, we consider some basic notions and results that help us to understand the paper thoroughly.

Definition 1 ([12])**.** Let Z be a non-empty set. A non-void family I of subsets of Z is said to be an ideal on Z if the following conditions hold:

(1) $A,B \in I \Longrightarrow A \cup B \in I$;
(2) $A \in I, B \subseteq A \Longrightarrow B \in I$.

Definition 2 ([12])**.** Let I be an ideal on Z. Then, I is said to be admissible if $\{z\} \in I$ for each $z \in Z$.

Clearly, $\varnothing \in I$, so, I is always non-empty. If $Z \notin I$ and $I \neq \{\varnothing\}$, then I will be called a non-trivial proper ideal on Z. From now on, we always consider an ideal to be non-trivial, proper, and admissible unless otherwise stated.

Definition 3 ([12])**.** Let Z be a non-empty set. A family $\mathcal{F}$ of subsets of Z is said to be a filter on Z if the following conditions hold:

(1) $\varnothing \notin \mathcal{F}$;
(2) $A,B \in \mathcal{F} \Longrightarrow A \cap B \in \mathcal{F}$;
(3) $A \in \mathcal{F}, A \subseteq B \Longrightarrow B \in \mathcal{F}$.

Let I be an ideal on Z. Then, the family $\mathcal{F}(I) = \{A \subset Z : Z \setminus A \in I\}$ is a filter on Z. We say that $\mathcal{F}(I)$ is the filter associated with ideal I.

Definition 4 ([12])**.** admissible ideal I on $\mathbb{N}$ is said to have the property (AP) if for any sequence $\{A_1, A_2, \dots\}$ of mutually disjoint sets in I, there is a sequence

$\{B_1, B_2, \dots\}$ of sets such that for each $i \in \mathbb{N}$, the symmetric difference $A_i \Delta B_i$ is finite and $\bigcup_i A_i = \bigcup_i B_i \in I$.

Definition 5 ([12])**.** Let (Z,d) be a metric space and I be an admissible ideal on $\mathbb{N}$. A sequence (z_n) in Z is said to be I-convergent to $a \in Z$ if for every $\varepsilon > 0$, $\{n \in \mathbb{N} : d(z_n, a) \geq \varepsilon\} \in I$.

Definition 6 ([12])**.** Let (Z,d) be a metric space and I be an admissible ideal on $\mathbb{N}$. A sequence (z_n) in Z is said to be I^*-convergent to $a \in Z$ if a set $M = \{k_1 < k_2 < \dots\} \in \mathcal{F}(I)$ such that (z_{k_n}) is convergent to a.

Definition 7 ([12])**.** Let (Z,d) be a metric space, I be an admissible ideal on $\mathbb{N}$, and (z_n) be a sequence in Z.

(1) An element $a \in Z$ is said to be an I-limit point of (z_n) if there exists a set $M = \{k_1 < k_2 < \dots\} \notin I$ such that (z_{k_n}) is convergent to a.
(2) An element $a \in Z$ is said to be an I-cluster point of (z_n) if for every $\varepsilon > 0$, $\{n \in \mathbb{N} : d(z_n, a) < \varepsilon\} \notin I$.

Definition 8 ([16])**.** Let (Z,d) be a metric space and I be an admissible ideal on $\mathbb{N}$. A sequence (z_n) in Z is said to be I-Cauchy if for each $\varepsilon > 0$ there exists a $k = k(\varepsilon) \in \mathbb{N}$ such that

$$\{n \in \mathbb{N} : d(z_n, z_k) \geq \varepsilon\} \in I.$$

Definition 9 ([16])**.** Let (Z,d) be a metric space and I be an admissible ideal on $\mathbb{N}$. A sequence (z_n) in Z is said to be I^*-Cauchy if there exists a set $M = \{k_1 < k_2 < \dots\} \in \mathcal{F}(I)$ such that (z_{k_n}) is a Cauchy subsequence of (z_n).

Proposition 1 ([16])**.** *Let (Z,d) be a metric space and I be an admissible ideal on $\mathbb{N}$. If (z_n) is I^*-Cauchy, then (z_n) is I-Cauchy. In addition, if we consider the ideal I with property (AP), then the concepts I-Cauchy sequence and I^*-Cauchy sequence coincide.*

Definition 10 ([5])**.** A set of real numbers where repetition of real numbers is allowed, is called a multiset of real numbers, denoted by $m\mathbb{R}$. Thus,

$$m\mathbb{R} = \{x|c : x \in \mathbb{R} \wedge c \in \mathbb{N}_0\}.$$

Definition 11 ([5])**.** A function whose domain is the set $\mathbb{N}$ of natural numbers and co-domain $m\mathbb{R}$ is said to be a multiset sequence of $m\mathbb{R}$ (in short, multisequence). We denote a multiset sequence by $H = \{x_n|c_n\}$, where $x_n \in \mathbb{R}$ and $c_n \in \mathbb{N}$ for each $n \in \mathbb{N}$.

Let (Z,d) be a metric space. On a multiset M whose elements are from Z one can define different types of metric, for more details see [17]. In this paper, we consider the metric

$$d_M(x_n|c_n, y_n|d_n) = \sqrt{(x_n - y_n)^2 + (c_n - d_n)^2}$$

on M.

Definition 12 ([6]). Let I be an ideal on $\mathbb{N}$. A multiset sequence by $H = \{x_n|c_n\}$ is said to be I-convergent to $l|c$ if for each $\varepsilon > 0$,

$$\{n \in \mathbb{N} : d_M(x_n|c_n, l|c) \geq \varepsilon\} \in I.$$

Definition 13 ([6]). Let I be an ideal on $\mathbb{N}$. A multiset sequence by $H = \{x_n|c_n\}$ is said to be I^*-convergent to $l|c$ if there exists a set $M = \{m_1 < m_2 < \dots\} \in \mathcal{F}(I)$ such that the multisubsequence $\{x_{m_i}|c_{m_i}\}$ is convergent to $l|c$.

Throughout the paper, whenever we use the term multiset sequence, we mean multiset sequence of $m\mathbb{R}$ unless otherwise stated.

3. I-LIMIT POINTS AND I-CLUSTER POINTS

We introduce the notions of I-limit points and I-cluster points for a sequence of multisets as follows:

Definition 14. Let $H = \{x_n|c_n\}$ be a multiset sequence. An element $\{x|c\}$ is said to be an I-limit point of H if there exists a set $T = \{h_1 < h_2 < \dots\} \subset \mathbb{N}, T \notin I$ such that the multiset sequence $\{x_{h_i}|c_{h_i}\}$ is convergent to $x|c$; i.e, $\forall \varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $d_M(x_{h_i}|c_{h_i}, x|c) < \varepsilon$ for all $i \geq n_0$. The collection of all I-limit point of H is denoted by $I(\Lambda_H^m)$.

Definition 15. Let $H = \{x_n|c_n\}$ be a multiset sequence. An element $\{x|c\}$ is said to be an I-cluster point of H if for all $\varepsilon > 0$, $\{n \in \mathbb{N} : d_M(x_n|c_n, x|c) < \varepsilon\} \notin I$. The collection of all I-cluster point of H is denoted by $I(\Gamma_H^m)$.

Theorem 1. *Let I be any proper nontrivial admissible ideal and $H = \{x_n|c_n\}$ be a multiset sequence. Then, $I(\Lambda_H^m) \subset I(\Gamma_H^m)$.*

Proof. Let $x|c$ be an I-limit point of H. Then, there exists a set $T = \{h_1 < h_2 < \dots\} \subset \mathbb{N}, T \notin I$ such that $d_M(x_{h_i}|c_{h_i}, x|c) \to 0$ as $n \to \infty$. Let $\varepsilon > 0$. So, there exists $n_0 \in \mathbb{N}$ such that $d_M(x_{h_i}|c_{h_i}, x|c) < \varepsilon$ for all $i > n_0$. We have $T \setminus \{h_1, h_2, \dots, h_{n_0}\} \subseteq \{i \in \mathbb{N} : d_M(x_i|c_i, x|c) < \varepsilon\}$. But, if $\{i \in \mathbb{N} : d_M(x_i|c_i, x|c) < \varepsilon\} \in I$, then this will imply $T \in I$. So, $\{i \in \mathbb{N} : d_M(x_i|c_i, x|c) < \varepsilon\} \notin I$. So, $x|c$ be an I-cluster point of H. □

But, converse of Theorem 1 is not true. Let $\mathfrak{I} = \{J_1, J_2, J_3, \dots\}$ be mutually disjoint partition of $\mathbb{N}$ such that each of J_i is infinite. If we take

$$I = \{A \subset \mathbb{N} : \text{ there exist } l_1, l_2, \dots, l_p \text{ such that } A \cap J_{l_i} \neq \phi \text{ for all } i = 1, 2, \dots, p\},$$

then I becomes a proper nontrivial admissible ideal. Define a multiset sequence H with same multiplicity p by

$$x_n = \begin{cases} \frac{1}{i}, & \text{if } n \in J_i \text{ and } i \text{ is even,} \\ 1, & \text{otherwise.} \end{cases}$$

Now, consider the multiset $\{0|p\}$. Then, for any $\varepsilon > 0$, we have $\{i \in \mathbb{N} : d_M(x_i|p, 0|p) < \varepsilon\} \notin I$. So, $0|p$ is an I-cluster point of H. Also, it can be easily verified that $0|p$ is not an I-limit point of H.

Theorem 2. *Let $H = \{x_n|c_n\}$ be a multiset sequence and I be a proper nontrivial admissible ideal in $\mathbb{N}$. Let H is I-convergent to $x|c$. Then, $x|c$ is an I-cluster point of H.*

Proof. Since H is I-convergent to $x|c$, for $\varepsilon > 0$,

$$\left\{n \in \mathbb{N} : \sqrt{(x_n - x)^2 + (c_n - c)^2} \geq \varepsilon\right\} \in I.$$

Therefore, $\left\{n \in \mathbb{N} : \sqrt{(x_n - x)^2 + (c_n - c)^2} < \varepsilon\right\} \notin I$. Otherwise, $\mathbb{N} \in I$ which is not possible. This shows that $x|c$ is an I-cluster point of H. □

Theorem 3. *Let $H = \{x_n|c_n\}$ be a multiset sequence and I be a proper nontrivial admissible ideal in $\mathbb{N}$. Let H is I^*-convergent to $x|c$. Then, $x|c$ is an I-limit point of H.*

Theorem 4. *Let $H = \{x_n|c_n\}$ be a multiset sequence and $I, \mathcal{J}$ be two ideals in $\mathbb{N}$ with $\mathcal{J} \subset I$. If $\{x|c\}$ is an I-limit point (I-cluster point) of H, then $\{x|c\}$ is an $\mathcal{J}$-limit point ($\mathcal{J}$-cluster point) of H.*

Proof. Let $H = \{x_n|c_n\}$ be a multiset sequence and $I, \mathcal{J}$ be two ideals in $\mathbb{N}$ with $\mathcal{J} \subset I$. Let $\{x|c\}$ is an I-limit point of H. Then, there exists a set $T = \{t_1 < t_2 < \dots\} \notin I$ such that the multisubsequence $\{x_{t_i}|c_{t_i}\}$ is convergent to $\{x|c\}$. Since $T = \{t_1 < t_2 < \dots\} \notin \mathcal{J}$, so, $\{x|c\}$ is also, an $\mathcal{J}$-limit point of H. Similarly, we can show that $\{x|c\}$ is an $\mathcal{J}$-cluster point of H. □

Now, we introduce the notion of $I - \limsup$ and $I - \liminf$ for a multiset sequence H with respect to a proper nontrivial admissible ideal I. The introduced notion will improve the corresponding notion in [5] as the results will become a particular case for the density zero ideal I_d. We start with a multiset sequence $H = \{x_n|c_n\}$. Consider the two sets,

$$B_H = \left\{x|c : \left\{n \in \mathbb{N} : \sqrt{x_n^2 + (c_n - 1)^2} > \sqrt{x^2 + (c-1)^2}\right\} \notin I\right\}$$

and

$$A_H = \left\{x|c : \left\{n \in \mathbb{N} : \sqrt{x_n^2 + (c_n - 1)^2} < \sqrt{x^2 + (c-1)^2}\right\} \notin I\right\}.$$

$\sup B_H$ and $\inf A_H$ is defined in the same way as in [5].

Definition 16. Let $H = \{x_n|c_n\}$ be a multiset sequence and I be a proper nontrivial admissible ideal in $\mathbb{N}$. We define

$$I - \limsup H = \begin{cases} \sup B_H, & \text{if } B_H \neq \phi, \\ -\infty, & \text{if } B_H = \phi. \end{cases}$$

$$I-\liminf H=\begin{cases}\inf A_H, & \text{if } A_H\neq\phi,\\ +\infty, & \text{if } A_H=\phi.\end{cases}$$

Example 1. Consider the ideal $I=I_f$ of all finite subsets of $\mathbb{N}$. Let $H=\{x_n|c_n\}$ be defined by

$$(x_i|c_i)=\begin{cases}0|4, \text{ if } i \text{ is even},\\ 1|3 \text{ if } i \text{ is odd}.\end{cases}$$

Here, it can be shown that

$$B_H=\left\{[-2,2]|1,[-2,2]|2,[-2,2]|3,[-\sqrt{2},\sqrt{2}]|1,[-\sqrt{2},\sqrt{2}]|2\right\}.$$

So, $I-\limsup H=2|3$. Also,

$$A_H=\left\{(-\infty,-\sqrt{5}]|4,\dots,[\sqrt{5},\infty)|4,[\sqrt{5},\infty)|5,\dots,(-\infty,-\sqrt{3}]|3,\dots\right\}.$$

So, $I-\liminf H=\sqrt{5}|4$.

Theorem 5. *Let $H=\{x_n|c_n\}$ be a multiset sequence and I be a nontrivial proper admissible ideal on $\mathbb{N}$. Then, $I-\limsup H$ and $I-\liminf H$ are unique.*

Proof. Proof is straightforward, so omitted. □

Theorem 6. *Let $H=\{x_n|c_n\}$ be a multiset sequence. If $I-\limsup H=p|r$, then for every $\varepsilon>0$,*

(1) $\left\{n\in\mathbb{N}:\sqrt{x_n^2+(c_n-1)^2}>\sqrt{(p-\varepsilon)^2+(r-1)^2}\right\}\notin I$.

(2) $\left\{n\in\mathbb{N}:\sqrt{x_n^2+(c_n-1)^2}>\sqrt{(p+\varepsilon)^2+(r-1)^2}\right\}\in I$.

Proof. Since $I-\limsup H=p|r$. Then, r is the greatest multiplicities in B_H and p is the supremum of all real numbers whose multiplicity is r. Let $\varepsilon>0$. So, there exists a real number q with $(p-\varepsilon)<q$ and $q|r\in B_H$. Since $\sqrt{(p-\varepsilon)^2+(r-1)^2}<\sqrt{q^2+(r-1)^2}$, $\left\{n\in\mathbb{N}:\sqrt{x_n^2+(c_n-1)^2}>\sqrt{q^2+(r-1)^2}\right\}\subset\{n\in\mathbb{N}:\sqrt{x_n^2+(c_n-1)^2}>\sqrt{(p-\varepsilon)^2+(r-1)^2}\right\}$. This shows that

$$\left\{n\in\mathbb{N}:\sqrt{x_n^2+(c_n-1)^2}>\sqrt{(p-\varepsilon)^2+(r-1)^2}\right\}\notin I.$$

On the other hand for $\varepsilon>,0$ if

$$\left\{n\in\mathbb{N}:\sqrt{x_n^2+(c_n-1)^2}>\sqrt{(p+\varepsilon)^2+(r-1)^2}\right\}\notin I,$$

then $(p+\varepsilon)|r\in B_H$, and this will contradict the fact that p is the supremum of all real numbers whose multiplicity is r. So,

$$\left\{n\in\mathbb{N}:\sqrt{x_n^2+(c_n-1)^2}>\sqrt{(p+\varepsilon)^2+(r-1)^2}\right\}\in I.$$

□

Theorem 7. *Let $H = \{x_n|c_n\}$ be a multiset sequence. If $I - \liminf H = p|r$, then for every $\varepsilon > 0$,*

(1) $\left\{n \in \mathbb{N} : \sqrt{x_n^2 + (c_n - 1)^2} > \sqrt{(p+\varepsilon)^2 + (r-1)^2}\right\} \notin I$.

(2) $\left\{n \in \mathbb{N} : \sqrt{x_n^2 + (c_n - 1)^2} > \sqrt{(p-\varepsilon)^2 + (r-1)^2}\right\} \in I$.

Proof. Proof is similar to Theorem 6, so omitted. □

4. I-CAUCHY AND I^*-CAUCHY MULTISET SEQUENCES

In this section, we introduce and study the notions of I-Cauchy and I^*-Cauchy sequences. Furthermore, we establish relationships between the notions of I-convergence, I^*-convergence, I-Cauchy, and I^*-Cauchy multiset sequences.

Definition 17. Let $H = \{x_n|c_n\}$ be a multiset sequence. Then, $H = \{x_n|c_n\}$ is said to be I-Cauchy if for each $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\left\{n \in \mathbb{N} : \sqrt{(x_n - x_N)^2 + (c_n - c_N)^2} \geq \varepsilon\right\} \in I.$$

Example 2. Let I be an admissible ideal on $\mathbb{N}$ such that $A \in I$, where A is an infinite subset of $\mathbb{N}$. Let $\{x_n\}$ be any I-Cauchy sequence of real numbers. Define a sequence of positive integers $\{c_n\}$ as follows:

$$c_n = \begin{cases} n, & \text{if } n \in A \\ 10, & \text{otherwise.} \end{cases}$$

Clearly, the multiset sequence $H = \{x_n|c_n\}$ is I-Cauchy.

Remark 1. Since I is an admissible ideal, every multiset Cauchy sequence is I-Cauchy. In particular, if I_f is the ideal of all finite subsets of $\mathbb{N}$, then the notions of multiset Cauchy sequences and I_f-Cauchy sequences are equivalent. Let I_d be the collection of all density zero subsets of $\mathbb{N}$. Then, I_d is a non-trivial admissible ideal on $\mathbb{N}$. A multiset sequence is said to be statistically Cauchy if and only if it is I_d-Cauchy.

Theorem 8. *A multiset sequence $H = \{x_n|c_n\}$ is I-convergent if and only if it is I-Cauchy.*

Proof. Let $H = \{x_n|c_n\}$ is convergent to $l|c$. Then, for any $\varepsilon > 0$,

$$A\left(\frac{\varepsilon}{2}\right) = \left\{n \in \mathbb{N} : \sqrt{(x_n - l)^2 + (c_n - c)^2} \geq \frac{\varepsilon}{2}\right\} \in I.$$

Choose $N \notin A\left(\frac{\varepsilon}{2}\right)$ and fix it. Then, $\sqrt{(x_N - l)^2 + (c_N - c)^2} < \frac{\varepsilon}{2}$. Also, for all $n \notin A\left(\frac{\varepsilon}{2}\right)$, we have $\sqrt{(x_n - l)^2 + (c_n - c)^2} < \frac{\varepsilon}{2}$. Then, by the property (triangle inequality) of the usual norm in $\mathbb{R}^2$, we have

$$\sqrt{(x_n - x_N)^2 + (c_n - c_N)^2} \leq \sqrt{(x_n - l)^2 + (c_n - c)^2} + \sqrt{(x_N - l)^2 + (c_N - c)^2} < \varepsilon$$

for all $n \notin A\left(\frac{\varepsilon}{2}\right)$. Consequently, $\left\{n \in \mathbb{N} : \sqrt{(x_n - x_N)^2 + (c_n - c_N)^2} \geq \varepsilon\right\} \subset A\left(\frac{\varepsilon}{2}\right)$. Since $A\left(\frac{\varepsilon}{2}\right) \in I$, $\left\{n \in \mathbb{N} : \sqrt{(x_n - x_N)^2 + (c_n - c_N)^2} \geq \varepsilon\right\} \in I$. Hence, $H = \{x_n | c_n\}$ is I-Cauchy.

Conversely, let $H = \{x_n | c_n\}$ be I-Cauchy. Then, for any $\varepsilon > 0$, there exists a positive integer k_ε such that

$$A(\varepsilon) = \left\{n \in \mathbb{N} : \sqrt{(x_n - x_{k_\varepsilon})^2 + (c_n - c_{k_\varepsilon})^2} \geq \varepsilon\right\} \in I.$$

Set $\varepsilon_m = \frac{1}{2^m}$ for $m \in \mathbb{N}$. Then, for each $m \in \mathbb{N}$ there exists $k_m \in \mathbb{N}$ such that

$$A(m) = \left\{n \in \mathbb{N} : \sqrt{(x_n - x_{k_m})^2 + (c_n - c_{k_m})^2} \geq \frac{\varepsilon_m}{2}\right\} \in I.$$

Define recursively, $B_1 = \mathrm{cl}\, B(x_{k_1}, \frac{\varepsilon_1}{2})$, $E_1 = \mathrm{cl}\, B(c_{k_1}, \frac{\varepsilon_1}{2})$,

$$B_{m+1} = B_m \cap \mathrm{cl}\, B(x_{k_{m+1}}, \frac{\varepsilon_{m+1}}{2}),$$

and

$$E_{m+1} = E_m \cap \mathrm{cl}\, B(c_{k_{m+1}}, \frac{\varepsilon_{m+1}}{2})$$

for $m \in \mathbb{N}$. We claim that both B_m and E_m are nonempty for each $m \in \mathbb{N}$.

Indeed, since $A(1) \in I$, for each $n \notin A(1)$ we have $\sqrt{(x_n - x_{k_1})^2 + (c_n - c_{k_1})^2} < \frac{\varepsilon_1}{2}$. Since both $|x_n - x_{k_1}|$ and $|c_n - c_{k_1}|$ are less than or equal to $\sqrt{(x_n - x_{k_1})^2 + (c_n - c_{k_1})^2}$, $x_n \in B_1$ and $c_n \in E_1$. Similarly, $x_n \in \mathrm{cl}\, B(x_{k_{m+1}, \frac{\varepsilon_{m+1}}{2}})$ and $c_n \in \mathrm{cl}\, B(c_{k_{m+1}, \frac{\varepsilon_{m+1}}{2}})$ for all $n \notin A(m+1)$ and $m \in \mathbb{N}$. Let $m \in \mathbb{N}$ and $C \in I$ such that $x_n \in B_m$ and $c_n \in E_m$ for each $n \notin C$. Then, $x_n \in B_{m+1}$ and $c_n \in E_{m+1}$ for all $n \notin C \cup A(m+1)$. Furthermore, since $\{B_m\}$ and $\{E_m\}$ are nested sequences of closed sets of real numbers with diameters tend to zero, there exist $l, c \in \mathbb{R}$ such that $\bigcap_m B_m = \{l\}$ and $\bigcap_m E_m = \{c\}$. We show that $H = \{x_n | c_n\}$ is I-convergent to $l|c$. Let $\delta > 0$ be given. Then, there exists $m \in \mathbb{N}$ such that $\varepsilon_m < \frac{\delta}{4}$. Let

$$B(\delta) = \left\{n \in \mathbb{N} : \sqrt{(x_n - l)^2 + (c_n - c)^2} \geq \delta\right\}.$$

If $B(\delta)$ is empty, then $B(\delta) \in I$. Let $B(\delta) \neq \varnothing$. Then, for all $n \in B(\delta)$, we have $\sqrt{(x_n - l)^2 + (c_n - c)^2} \geq \delta$. Since

$$\sqrt{(x_n - l)^2 + (c_n - c)^2} \leq \sqrt{(x_n - x_{k_m})^2 + (c_n - c_{k_m})^2} + \sqrt{(l - x_{k_m})^2 + (c - c_{k_m})^2},$$

either $\sqrt{(x_n - x_{k_m})^2 + (c_n - c_{k_m})^2} \geq \frac{\delta}{2} > 2\varepsilon_m$ or $\sqrt{(l - x_{k_m})^2 + (c - c_{k_m})^2} \geq \frac{\delta}{2} > 2\varepsilon_m$. But $l \in \mathrm{cl}B(x_{k_m}, \frac{\varepsilon_m}{2})$ and $c \in \mathrm{cl}B(c_{k_m}, \frac{\varepsilon_m}{2})$, that is, $|x_{k_m} - l| < \varepsilon_m$ and $|c_{k_m} - c| < \varepsilon_m$. Therefore, $\sqrt{(l - x_{k_m})^2 + (c - c_{k_m})^2} \leq |x_{k_m} - l| + |c_{k_m} - c| < 2\varepsilon_m$. Thus,

$$B(\delta) \subset \left\{ n \in \mathbb{N} : \sqrt{(x_n - x_{k_m})^2 + (c_n - c_{k_m})^2} > 2\varepsilon_m \right\} \subset A_m \in I.$$

Hence, the multiset sequence $H = \{x_n|c_n\}$ is I-convergent to $l|c$. □

Definition 18. A multiset sequence $H = \{x_n|c_n\}$ is said to be I^*-Cauchy if there exists a set $M = \{m_1 < m_2 < \cdots < m_k < \cdots\} \in \mathcal{F}(I)$ such that the submultiset sequence $\{x_{m_k}|c_{m_k}\}$ is Cauchy.

Proposition 2. *A multiset sequence $H = \{x_n|c_n\}$ is I^*-convergent if and only if it is I^*-Cauchy.*

Theorem 9. *If a multiset sequence $H = \{x_n|c_n\}$ is I^*-Cauchy, then it is I-Cauchy.*

Proof. Since $H = \{x_n|c_n\}$ is I^*-Cauchy, there exists a set

$$M = \{m_1 < m_2 < \cdots < m_k < \cdots\} \in \mathcal{F}(I)$$

such that for any $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that for all $k > n_\varepsilon$, we have

$$\sqrt{(x_{m_k} - x_{m_{n_\varepsilon}})^2 + (c_{m_k} - c_{m_{n_\varepsilon}})^2} < \varepsilon.$$

Set $N = m_{n_\varepsilon} + 1$ and

$$A(\varepsilon) = \left\{ n \in \mathbb{N} : \sqrt{(x_n - x_N)^2 + (c_n - c_N)^2} \geq \varepsilon \right\}.$$

Clearly, $A(\varepsilon) \subset (\mathbb{N} \setminus M) \cup \{m_1, m_2, \cdots, m_{n_\varepsilon}\}$. Since the latter set is in I, so does $A(\varepsilon)$. Hence, $H = \{x_n|c_n\}$ is I-Cauchy. □

The following example shows that the notions of I-Cauchy and I^*-Cauchy of multiset sequences are not equivalent in general:

Example 3. Let $\mathbb{N} = \bigcup_j A_j$, be a decomposition of $\mathbb{N}$ such that each A_j is infinite and $A_i \cap A_j = \varnothing$ for $i \neq j$. Let I be the class of all subsets A of $\mathbb{N}$ which intersects at most finite number of A_j's. It is easy to verify that I is a non-trivial admissible ideal of $\mathbb{N}$. Construct a multiset sequence $H = \{x_n|c_n\}$ as follows: $x_n = \frac{1}{j}$ if $n \in A_j$ and $c_n = 10$ for all $n \in \mathbb{N}$. Let $\varepsilon > 0$ be given. Then, there exists $j \in \mathbb{N}$ such that $\frac{2}{j+1} < \varepsilon$. Let m_0 be the least positive integer such that $m_0 \in A_{j+1}$. Let $C(\varepsilon) = \left\{ n \in \mathbb{N} : \sqrt{(x_n - x_{m_0})^2 + (c_n - c_{m_0})^2} \geq \varepsilon \right\}$. Clearly, $C(\varepsilon) \subset A_1 \cup A_2 \cup \cdots \cup A_j$. Since the latter set belongs to I, so does $C(\varepsilon)$. Therefore, the multiset sequence $H = \{x_n|c_n\}$ is I-Cauchy.

Now we show that $H = \{x_n|c_n\}$ is not I^*-Cauchy. Suppose, on contrary $H = \{x_n|c_n\}$ is not I^*-Cauchy. Then, there exists a set $M = \{m_1 < m_2 < \cdots < m_k < \cdots\} \in$

$\mathcal{F}(I)$ such that for any $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that for all $k > n_\varepsilon$, we have $\sqrt{(x_{m_k} - x_{m_{n_\varepsilon}})^2 + (c_{m_k} - c_{m_{n_\varepsilon}})^2} < \varepsilon$. From the definition of I, we have $\mathbb{N} \setminus M \subset A_1 \cup A_2 \cup \cdots \cup A_l$ for some $l \in \mathbb{N}$. Thus, $A_i \subset M$ for all $i > l$. Therefore, there are infinitely many terms of the sequence $\{x_{m_k}\}$ equal to $\frac{1}{l+1}$ and $\frac{1}{l+2}$. Let $0 < \varepsilon_0 < \frac{1}{(l+1)(l+2)}$. Then, there does not exit any $n_{\varepsilon_0} \in \mathbb{N}$ such that $\sqrt{(x_{m_k} - x_{m_{n_{\varepsilon_0}}})^2 + (c_{m_k} - c_{m_{n_{\varepsilon_0}}})^2} < \varepsilon_0$ holds for all $k > n_{\varepsilon_0}$, which is a contradiction. Hence, $H = \{x_n|c_n\}$ is not I^*-Cauchy.

We now introduce the following definition to provide a sufficient condition for a multiset I-Cauchy sequence to be an I^*-Cauchy.

Definition 19. An admissible ideal I on $\mathbb{N}$ is said to have the weakly additive property (WAP) if for every sequence of mutually disjoint sets $\{P_i\}$ in I, there exists a set $P \in \mathcal{F}(I)$ such that $P \setminus P_i$ is finite for all $i \in \mathbb{N}$.

From [16, Lemma 4], we observe that if an admissible ideal has the property (AP), then it has the property (WAP).

Theorem 10. *Let I be an admissible ideal and it has the property (WAP). If a multiset sequence $H = \{x_n|c_n\}$ is I-Cauchy, then it is I^*-Cauchy.*

Proof. Since $H = \{x_n|c_n\}$ is I-Cauchy, for any $\varepsilon > 0$, there exists a positive integer m_ε such that

$$A(\varepsilon) = \left\{ n : \mathbb{N} : \sqrt{(x_n - x_{m_\varepsilon})^2 + (c_n - c_{m_\varepsilon})^2} \geq \varepsilon \right\} \in I.$$

Set $P_i = \left\{ n \in \mathbb{N} : \sqrt{(x_n - x_{m_i})^2 + (c_n - c_{m_i})^2} < \frac{1}{i} \right\}$ for all $i \in \mathbb{N}$. Clearly, $P_i \in \mathcal{F}(I)$ for all $i \in \mathbb{N}$. Since I has the property (WAP), we have a set $P \in \mathcal{F}(I)$ such that $P \setminus P_i$ is finite for all $i \in \mathbb{N}$. Let $\delta > 0$ be given. Then, there exists $j \in \mathbb{N}$ such that $\frac{1}{j} < \frac{\delta}{2}$. Since $P \setminus P_j$ is finite, there exists a positive integer k_j such that $m, n \in P_j$ for all $m, n \in P$ and $m, n > k_j$. Therefore,

$$\sqrt{(x_m - x_{m_j})^2 + (c_m - c_{m_j})^2} < \frac{1}{j}$$

and

$$\sqrt{(x_n - x_{m_j})^2 + (c_n - c_{m_j})^2} < \frac{1}{j}$$

for all $m, n \in P$ and $m, n > k_j$. Hence, by a property (triangle property) of the usual norm of $\mathbb{R}^2$, we have $\sqrt{(x_m - x_n)^2 + (c_m - c_n)^2} < \frac{2}{j} < \delta$ for all $m, n \in P$ and $m, n > k_j$. This proves that $H = \{x_n|c_n\}$ is I^*-Cauchy. □

Corollary 1. *Let I be an admissible ideal and it has the property (WAP). Then, the notions of I-Cauchy, I-convergent, I^*-Cauchy, and I^*-convergent of multiset sequences are equivalent.*

CONCLUSION

In this paper, we introduced and studied the notions of I-limit points and I-cluster points for multiset sequences. Also, we introduced and studied the notions of I-Cauchy and I^*-Cauchy multiset sequences. We proved some basic results. We observed that if I is an admissible ideal and has weak additive property (WAP), then the notions of I-Cauchy, I-convergent, I^*-Cauchy, and I^*-convergent of multiset sequences are equivalent. Let (Z,d) be a metric space. On a multiset M whose elements are from Z, in this paper, we considered the metric

$$d_M(x_n|c_n, y_n|d_n) = \sqrt{(x_n - y_n)^2 + (c_n - d_n)^2}$$

on M. However, one may start with a probabilistic metric space and consider a multiset whose elements are from the probabilistic metric space. Now the question is, can we define a probabilistic metric on the multiset? In addition, we leave the following question to the interested reader: If the notions of I-Cauchy, I-convergent, I^*-Cauchy, and I^*-convergent of multiset sequences are equivalent for an admissible ideal I, then can we say that I has the property (WAP)?

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THE (NON-)EXTENDABILITY OF EMBRY'S THEOREM

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Abstract. Let A, P and T be bounded linear operators on a Hilbert space such that P is an orthogonal projection commuting with A, and zero is not in the numerical range of T. We prove that if $PT = TA$, then $A = P$. As a consequence, Embry's Theorem on the similarity of normal operators follows easily from our result. Furthermore, we demonstrate that this theorem cannot be extended to quasinormal operators, thereby providing a negative answer to a conjecture posed by Mortad in [9].

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1. INTRODUCTION

Let $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be a complex Hilbert space, and let $\mathfrak{B}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. For an operator $T \in \mathfrak{B}(\mathcal{H})$, we denote its null space and range by $\mathcal{N}(T)$ and $\mathcal{R}(T)$, respectively. For the spectrum of T we use the notation $\sigma(T)$, while the numerical range will be denoted by $\mathcal{W}(T)$. Recall that

$$\mathcal{W}(T) = \{\langle Tx, x\rangle : \|x\| = 1\}.$$

The adjoint of T is denoted by T^*. An operator T is called positive, written $T \geq 0$, if $\langle Tx, x\rangle \geq 0$ for all $x \in \mathcal{H}$, and it is called self-adjoint (or Hermitian) if $T = T^*$. An operator $P \in \mathfrak{B}(\mathcal{H})$ is called an orthogonal projection if it satisfies $P = P^* = P^2$. It is clear that any orthogonal projection is a positive operator.

An operator T is said to be normal if $T^*T = TT^*$. The theory of normal operators has been studied in great depth, largely due to the powerful structure afforded by the Spectral Theorem, which serves as a foundational tool in their analysis. For a comprehensive introduction to Spectral Theorem and its applications, see [11].

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A fundamental result in the theory of normal operators is the Fuglede–Putnam Theorem [5, 13], which asserts that if $A,B \in \mathfrak{B}(\mathcal{H})$ are normal operators and $T \in \mathfrak{B}(\mathcal{H})$ satisfies $AT = TB$, then it necessarily follows that $A^*T = TB^*$. This theorem highlights the rigidity of normal operators under intertwining and has profound implications in spectral theory. For the different proofs of Fuglede-Putnam Theorem, and in different settings, see [5, 7, 13–15, 17].

Fuglede-Putnam Theorem has many fundamental consequences. For example, it is used to show that the product of two commuting normal operators must be normal (cf. [8]).

Another very interesting result relying on the Fuglede-Putnam Theorem is the following theorem by Embry [4].

Theorem 1 (Embry's Theorem)**.** *[4] Let $A,B \in \mathfrak{B}(\mathcal{H})$ be commuting normal operators. If there exists $T \in \mathfrak{B}(\mathcal{H})$ such that $0 \notin \mathcal{W}(T)$ and $AT = TB$, then $A = B$.*

The importance of this theorem lies in its elegant combination of spectral theory, the geometry of the numerical range, and operator similarity. In the same paper, Embry also demonstrated that neither the commutativity condition on A and B, nor the numerical range condition $0 \notin \mathcal{W}(T)$, can be omitted. Moreover, the same counterexample reveals that the condition $0 \notin \mathcal{W}(T)$ cannot be weakened to $0 \notin \sigma(T)$. For an interesting comparison between Embry's Theorem and Fuglede-Putnam Theorem, see [10].

This paper revisits Embry's Theorem and explores its potential generalizations beyond the class of normal operators.

2. Main results

Our approach begins with a seemingly elementary, yet powerful, characterization involving orthogonal projections and the numerical range.

Theorem 2. *Let $A,P,T \in \mathfrak{B}(\mathcal{H})$ be such that P is an orthogonal projection commuting with A and $0 \notin \mathcal{W}(T)$. If $PT = TA$, then $A = P$.*

Proof. Without loss of generality, we may assume that $P \notin \{0, I\}$. With respect to the decomposition $\mathcal{H} = \mathcal{R}(P) \oplus \mathcal{N}(P)$, we can write

$$P = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \quad \text{and} \quad T = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}.$$

Since P and A commute, we immediately have that $A_2 = 0$ and $A_3 = 0$, i.e.,

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_4 \end{bmatrix}.$$

Condition $PT = TA$ is now equivalent with the system of equations

$$T_1(I - A_1) = 0, \quad T_2(I - A_4) = 0, \quad T_3A_1 = 0, \quad T_4A_4 = 0. \tag{2.1}$$

Next, note that T_1 and T_4 are one-to-one. Indeed, let $x \in \mathcal{N}(T_1) \subseteq \mathcal{R}(P)$, and let $x' = \begin{bmatrix} x \\ 0 \end{bmatrix} \in \mathcal{H}$. Then,

$$\langle Tx', x' \rangle = \left\langle \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} \begin{bmatrix} x \\ 0 \end{bmatrix}, \begin{bmatrix} x \\ 0 \end{bmatrix} \right\rangle = \langle T_1 x, x \rangle = 0,$$

and so it must be $x = 0$. Similarly, we can show that $\mathcal{N}(T_4) = \{0\}$. The first and the last equation in (2.1) now imply that $A_1 = I$ and $A_4 = 0$, which clearly demonstrates that $A = P$. □

Remark 1. It is easy to see that the previous theorem also holds if P is any scalar multiple of an orthogonal projection.

As a consequence, we recover Embry's Theorem.

Proof of Theorem 1. Let E_A and E_B be the spectral measures corresponding to A and B, respectively, and let Δ be an arbitrary Borel set in the complex plane. Since $AT = TB$, the Fuglede-Putnam Theorem gives

$$E_A(\Delta)T = TE_B(\Delta).$$

Moreover, since A and B commute, the Fuglede Theorem also ensures that

$$E_A(\Delta)E_B(\Delta) = E_B(\Delta)E_A(\Delta).$$

Theorem 2 now yields that $E_A(\Delta) = E_B(\Delta)$. Since Δ was arbitrary, we conclude that $A = B$. □

A natural attempt to generalize Embry's Theorem and extend it to some super-classes of normal operators is to follow the path of generalizing Fuglede-Putnam Theorem. The crucial insight that allowed certain generalizations is to replace one of the normal operators with its adjoint. Recall that $T \in \mathfrak{B}(\mathcal{H})$ is called p-hyponormal for some $0 < p \leq 1$ if $(TT^*)^p \leq (T^*T)^p$. If $p = 1$, T is simply called a hyponormal operator. Clearly, any normal operator is p-hyponormal. For example, we have the following generalization of the Fuglede-Putnam Theorem:

Theorem 3. *[3, Theorem 7] Let $A, B \in \mathfrak{B}(\mathcal{H})$ be such that A and B^* are p-hyponormal operators. If*

$$AX = XB$$

for some $X \in \mathfrak{B}(\mathcal{H})$, then

$$A^*X = XB^*.$$

For many other generalizations of the Fuglede-Putnam Theorem, see [12] and the references therein.

We may try to extend Embry's Theorem in the same way. However, such a generalization would not fundamentally change the original theorem, as the following discussion shows. We say that $(A,B) \in \mathfrak{B}(\mathcal{H})^2$ has the FP-property if for any $X \in \mathfrak{B}(\mathcal{H})$,

$$AX = XB \quad \Longrightarrow \quad A^*X = XB^*.$$

Also, recall the following result.

Corollary 1. *[16, Corollary 1] Suppose that $(A,B) \in \mathfrak{B}(\mathcal{H})^2$ has the FP-property. If there exists a quasi-affinity $X \in \mathfrak{B}(\mathcal{H})$ (X is one-to-one and has dense range) such that $AX = XB$, then A and B are unitarily equivalent normal operators.*

Theorem 4. *Suppose that a commuting pair $(A,B) \in \mathfrak{B}(\mathcal{H})^2$ has the FP-property. If there exists $T \in \mathfrak{B}(\mathcal{H})$ such that $0 \notin \mathcal{W}(T)$ and $AT = TB$, then A and B are normal, and $A = B$.*

Proof. Since the condition $0 \notin \mathcal{W}(T)$ implies that T is quasi-affinity, the conclusion follows immediately from Corollary 1 and Theorem 1. □

In particular, Theorem 3 yields the following:

Theorem 5. *Let $A,B \in \mathfrak{B}(\mathcal{H})$ be such that A and B^* are p-hyponormal operators, and A and B commute. If there exists $T \in \mathfrak{B}(\mathcal{H})$ such that $0 \notin \mathcal{W}(T)$ and $AT = TB$, then A and B are normal, and $A = B$.*

Another attempt to generalize Embry's Theorem is the following conjecture which appeared in [9] (cf. [12]).

Conjecture 1. *Let $A,B \in \mathfrak{B}(\mathcal{H})$ be two commuting hyponormal (subnormal, quasinormal) operators. If there exists $T \in \mathfrak{B}(\mathcal{H})$ such that $0 \notin \mathcal{W}(T)$ and $AT = TB$, then $A = B$.*

Recall that $T \in \mathfrak{B}(\mathcal{H})$ is called subnormal if it has a normal extension, and quasinormal if it commutes with T^*T. It is also well-known that

$$\text{normal} \Longrightarrow \text{quasinormal} \Longrightarrow \text{subnormal} \Longrightarrow \text{hyponormal}.$$

The following example shows that the Conjecture 1 does not hold even for quasinormal operators.

Example 1. Let $P \in \mathfrak{B}(\mathcal{H})$ be a non-zero positive operator, and let $0 < q < 1$. Define operators $A,B,T \in \mathfrak{B}(\bigoplus_{i=1}^{\infty} \mathcal{H})$ as follows:

$$A = \begin{bmatrix} 0 & & & & \\ qP & 0 & & & \\ 0 & qP & 0 & & \\ & 0 & qP & 0 & \\ & & \ddots & \ddots & \ddots \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & & & & \\ P & 0 & & & \\ 0 & P & 0 & & \\ & 0 & P & 0 & \\ & & \ddots & \ddots & \ddots \end{bmatrix},$$

$$T = \begin{bmatrix} I & 0 & & & \\ 0 & qI & 0 & & \\ & 0 & q^2 I & 0 & \\ & & \ddots & \ddots & \ddots \end{bmatrix}.$$

By Brown's characterization of quasinormal operators (see [1] or [2, Chapter 2, Theorem 3.2]), or by direct verification, we have that A and B are quasinormal, and it is evident that $AB = BA$. Using [6, Chapter 1, Proposition 1.8] and the obvious fact that

$$\mathcal{W}(q^n I) = \{q^n\} \quad (n \geq 0),$$

we have that $\mathcal{W}(T)$ is the convex hull of the set $\{q^n : n \geq 0\}$, which is clearly $(0,1]$. Thus, $0 \notin \mathcal{W}(T)$. Finally,

$$AT = \begin{bmatrix} 0 & & & & \\ qP & 0 & & & \\ 0 & q^2 P & 0 & & \\ & 0 & q^3 P & 0 & \\ & & \ddots & \ddots & \ddots \end{bmatrix} = TB,$$

while $A \neq B$.

3. Conclusion

In conclusion, our analysis underscores the intrinsic connection between Embry's Theorem and the class of normal operators. Under the current restriction on the numerical range, any meaningful generalization of the theorem appears implausible. Nevertheless, as illustrated in Example 1, although $0 \notin \mathcal{W}(T)$, it is in fact the case that $0 \in \overline{\mathcal{W}(T)}$. This observation suggests that a variant of Embry's Theorem under the stronger condition $0 \notin \overline{\mathcal{W}(T)}$ might be worth investigating.

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INTEGRAL CHARACTERIZATIONS OF UNIFORM H-DICHOTOMY IN MEAN FOR DISCRETE-TIME STOCHASTIC SKEW-EVOLUTION SEMIFLOWS

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Abstract. The paper provides integral characterizations for the concept of uniform dichotomy in the mean with growth rates for discrete-time stochastic skew-evolution semiflows in Banach spaces. More precisely, necessary and sufficient conditions are given using both invariant projection families and strongly invariant projection families to the discrete-time stochastic skew-evolution semiflows. As a consequence, we obtain integral characterizations for uniform exponential dichotomy in mean.

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1. INTRODUCTION

Occurrences in the real world, within domains such as biology, economics, and environmental sciences, happen at specific points in time rather than continuously. Consequently, the discrete-time approach has become essential. By utilizing stochastic skew-evolution semiflows, we aim to develop a framework that enhances the analysis of discrete dynamical systems, providing a deeper understanding of their behavior and properties.

The concept of dichotomy is a key focus in the study of asymptotic behavior for evolution equations. O. Perron [21] introduced the concept of exponential dichotomy for linear differential equations. This concept was further explored in the monograph by J. L. Daleckii and M. G. Krein [10], as well as in a more recent work by Dragičević, Sasu, and Sasu [12]. In their study, the authors introduce new admissibility conditions for uniform exponential dichotomy and provide novel characterizations of polynomial dichotomy through double admissibilities.

An alternative perspective on studying dichotomic behavior focuses on cases where the asymptotic behaviours are of polynomial type. In this context, we address the concepts of nonuniform polynomial dichotomy, initially introduced by L. Barreira and C. Valls in [3] for the continuous case of evolution operators, and subsequently extended by A.J.G. Bento and C. Silva in [4] for discrete-time systems. Additional results related to polynomial behavior are discussed in [5],[14],[15].

Several significant papers have addressed the problem of the existence of stochastic semiflows for stochastic evolution equations. Notable examples of stochastic evolution semiflows arise from these equations, and readers can refer to the monographs by Arnold [1] and Prato and Zabczyk [9] for more information. The exponential dichotomy in a stochastic setting was discussed by many authors, such as A. M. Ateiwi in [2], T. Caraballo et al. in [8] or D. Stoica and M. Megan in [23].

The concept of skew-evolution semiflow was introduced by Megan and Stoica for the continuous case in [18] and for the discrete case in [17]. This research was further developed by M. Megan and C. Stoica in [19], as well as by C. Stoica in [22].

This investigation aims to outline several characterisations of uniform h-dichotomy in mean of discrete-time stochastic skew-evolution semiflows in Banach spaces, where $h : \mathbb{N} \to [1,\infty)$ acts as a growth rate function. Specifically, h is a non-decreasing and bijective function with the property that $\lim_{m\to\infty} h(m) = \infty$. For recent contributions, we refer to the works [6,7], [13], [16], [20], and [24].

This paper builds on the foundational work of Datko [11] who provided an integral characterization of uniform exponential stability for evolution operators. Expanding on Datko's results, and the aforementioned studies, we present integral characterizations for the concept of uniform dichotomy in mean with growth rates for discrete-time stochastic skew-evolution semiflows considering invariant projections families and, respectively, strongly invariant projection families.

2. Definitions and notations

Let $(\Omega,\mathcal{B},\mu)$ be a probability space. Let $\tilde{\Delta}$ be the set defined by $\tilde{\Delta} = \{(m,n) \in \mathbb{N}^2 : m \geq n \geq 0\}$ and let $\tilde{T}$ be the set defined by $\tilde{T} = \{(m,n,p) \in \mathbb{N}^3 : m \geq n \geq p\}$. For a real or complex Banach space X we denote by $\mathcal{B}(X)$ the Banach algebra of all bounded linear operators on X.

Definition 1. A measurable random field $\varphi : \tilde{\Delta} \times \Omega \to \Omega$ is said to be a *discrete-time stochastic evolution semiflow* on Ω if the following properties hold:

(es_1) $\varphi(m,m,\omega) = \omega$, for all $(m,\omega) \in \mathbb{N} \times \Omega$,

(es_2) $\varphi(m,n,\varphi(n,p,\omega)) = \varphi(m,p,\omega)$, for all $m \geq n \geq p \geq 0$ and all $\omega \in \Omega$.

Definition 2. Let $\Phi : \Delta \times \Omega \to \mathcal{B}(X)$ be a measurable map. We say that Φ is a *discrete-time stochastic evolution cocycle* associated to the stochastic evolution semiflow $\varphi : \tilde{\Delta} \times \Omega \to \Omega$ if the following conditions hold:

(ec_1) $\Phi(m,m,\omega) = I$ (the identity operator on X), for all $(m,\omega) \in \mathbb{N} \times \Omega$,

(ec_2) $\Phi(m,n,\varphi(n,p,\omega))\Phi(n,p,\omega) = \Phi(m,p,\omega)$, for all $m \geq n \geq p \geq 0$ and all $\omega \in \Omega$.

If Φ represents a discrete-time stochastic evolution cocycle over a discrete-time stochastic evolution semiflow φ, then the pair $\tilde{C} = (\Phi, \varphi)$ is referred to as a *discrete-time stochastic skew-evolution semiflow.*

Definition 3. A map $\tilde{P} : \mathbb{N} \times \Omega \to \mathcal{B}(X)$ with the property $\tilde{P}^2(n,\omega) = \tilde{P}(n,\omega)$ for all $(n,\omega) \in \mathbb{N} \times \Omega$ is called *projections family* on X.

Remark 1. If $\tilde{P} : \mathbb{N} \times \Omega \to \mathcal{B}(X)$ is a projections family, then the map $\tilde{Q} : \mathbb{N} \times \Omega \to \mathcal{B}(X)$ defined as $\tilde{Q}(n,\omega) = I - \tilde{P}(n,\omega)$ is called a projection family. This is referred to as the *complementary projections family of* $\tilde{P}$.

Definition 4. A projections family $\tilde{P} : \mathbb{N} \times \Omega \to \mathcal{B}(X)$ is said to be *invariant* to the discrete-time stochastic skew-evolution semiflow $\tilde{C} = (\Phi, \varphi)$ if

$$\Phi(m,n,\omega)\tilde{P}(n,\omega) = \tilde{P}(m,\varphi(m,n,\omega))\Phi(m,n,\omega),$$

for all $(m,n,\omega) \in \tilde{\Delta} \times \Omega$.

If $\tilde{P}$ remains invariant for $\tilde{C} = (\Phi, \varphi)$, we denote by $\Phi_{\tilde{P}} : \Delta \times \Omega \to \mathcal{B}(X)$ the map defined by $\Phi_{\tilde{P}}(m,n,\omega) = \Phi(m,n,\omega)\tilde{P}(n,\omega)$.

From Definitions 2 and 4, it immediately follows:

Proposition 1. *The properties of the map $\Phi_{\tilde{P}}$ are as follows:*

(i) $\Phi_{\tilde{P}}(m,n,\omega) = \tilde{P}(m,\varphi(m,n,\omega))\Phi_{\tilde{P}}(m,n,\omega)$, $\forall\ (m,n,\omega) \in \tilde{\Delta} \times \Omega$;

(ii) $\Phi_{\tilde{P}}(m,m,\omega) = \tilde{P}(m,\omega)$, $\forall\ (m,\omega) \in \mathbb{N} \times \Omega$;

(iii) $\Phi_{\tilde{P}}(m,p,\omega) = \Phi_{\tilde{P}}(m,n,\varphi(n,p,\omega))\Phi_{\tilde{P}}(n,p,\omega)$, $\forall (m,n,p,\omega) \in \tilde{T} \times \Omega$.

Proof. The properties (i) and (ii) are immediate from Definition 4 and Definition 2. For (iii) we observe that

$$\begin{aligned}\Phi_{\tilde{P}}(m,p,\omega) = \Phi(m,p,\omega)\tilde{P}(p,\omega) &= \Phi(m,n,\varphi(n,p,\omega))\tilde{P}(n,\varphi(n,p,\omega))\Phi_{\tilde{P}}(n,p,\omega) \\ &= \Phi_{\tilde{P}}(m,n,\varphi(n,p,\omega))\Phi_{\tilde{P}}(n,p,\omega),\end{aligned}$$

for all $(m,n,p,\omega) \in \tilde{T} \times \Omega$. □

Remark 2. If the projections family $\tilde{P}$ is invariant to $\tilde{C} = (\Phi, \varphi)$ then its complementary $\tilde{Q}(n,\omega) = I - \tilde{P}(n,\omega)$ is also invariant to $\tilde{C}$. Thus, for all $(m,n,\omega) \in \tilde{\Delta} \times \Omega$ we have that $\Phi(m,n,\omega)(Range\ \tilde{Q}(n,\omega)) \subset Range\ \tilde{Q}(m,\varphi(m,n,\omega))$.

Definition 5. The projections family $\tilde{P} : \mathbb{N} \times \Omega \to \mathcal{B}(X)$ is said to be *strongly invariant* to $\tilde{C} = (\Phi, \varphi)$ if it is invariant to C and for all $(m,n,\omega) \in \tilde{\Delta} \times \Omega$, the map $\Phi(m,n,\omega)$ is an isomorphism from *Range* $\tilde{Q}(n,\omega)$ to *Range* $\tilde{Q}(m,\varphi(m,n,\omega))$.

Remark 3. If the projections family $\tilde{P} : \mathbb{N} \times \Omega \to \mathcal{B}(X)$ is strongly invariant to the discrete-time stochastic skew-evolution semiflow $\tilde{C} = (\Phi, \varphi)$, then there exists $\Psi : \tilde{\Delta} \times \Omega \to \mathcal{B}(X)$ such that for all $(m,n,\omega) \in \tilde{\Delta} \times \Omega$, $\Psi(m,n,\omega)$ is an isomorphism from *Range* $\tilde{Q}(m,\varphi(m,n,\omega))$ to *Range* $\tilde{Q}(n,\omega)$.

We will use the following notation:

$$\Psi_{\tilde{Q}}(m,n,\omega) = \Psi(m,n,\omega)\tilde{Q}(m,\varphi(m,n,\omega)).$$

Proposition 2. *If the projections family $P : \mathbb{N}\times\Omega \to \mathcal{B}(X)$ is strongly invariant to the discrete-time stochastic skew-evolution semiflow $\tilde{C} = (\Phi,\varphi)$ then the map $\Psi_{\tilde{Q}}$ has the following properties:*

(*i*) $\Phi_{\tilde{Q}}(m,n,\omega)\Psi_{\tilde{Q}}(m,n,\omega) = \tilde{Q}(m,\varphi(m,n,\omega))$, *for all* $(m,n,\omega)\in\tilde{\Delta}\times\Omega$*;*
(*ii*) $\Psi_{\tilde{Q}}(m,n,\omega)\Phi_{\tilde{Q}}(m,n,\omega) = \tilde{Q}(n,\omega)$, *for all* $(m,n,\omega)\in\tilde{\Delta}\times\Omega$*;*
(*iii*) $\Psi_{\tilde{Q}}(m,n,\omega) = \tilde{Q}(n,\omega)\Psi_{\tilde{Q}}(m,n,\omega)$, *for all* $(m,n,\omega)\in\tilde{\Delta}\times\Omega$.
(*iv*) $\Psi_{\tilde{Q}}(m,m,\omega) = \tilde{Q}(m,\omega)$, *for all* $(m,\omega)\in\mathbb{N}\times\Omega$*;*
(*v*) $\Psi_{\tilde{Q}}(m,p,\omega) = \Psi_{\tilde{Q}}(n,p,\omega)\Psi_{\tilde{Q}}(m,n,\varphi(n,p,\omega))$, *for all* $(m,n,p,\omega)\in\tilde{T}\times\Omega$.

Proof. The properties (i) and (ii) are immediate from Definition 5 and Remark 3. To prove (iii), we will use the first two conditions and we have

$$\begin{aligned}\tilde{Q}(n,\omega)\Psi_{\tilde{Q}}(m,n,\omega) &= \Psi_{\tilde{Q}}(m,n,\omega)\Phi_{\tilde{Q}}(m,n,\omega)\Psi_{\tilde{Q}}(m,n,\omega)\\ &= \Psi_{\tilde{Q}}(m,n,\omega)\tilde{Q}(m,\varphi(m,n,\omega)) = \Psi_{\tilde{Q}}(m,n,\omega),\end{aligned}$$

for all $(m,n,\omega)\in\tilde{\Delta}\times\Omega$.

The condition (iv) it follows from (i) by taking $n = m$

$$\Psi_{\tilde{Q}}(m,m,\omega)\Psi_{\tilde{Q}}(m,m,\omega) = \tilde{Q}(m,\varphi(m,m,\omega))$$

which is equivalent with $\Psi_{\tilde{Q}}(m,m,\omega) = \tilde{Q}(m,\omega)$ and more $\tilde{Q}(m,\omega)\Psi_{\tilde{Q}}(m,m,\omega) = \tilde{Q}(m,\omega)$. Follows that $\Psi_{\tilde{Q}}(m,m,\omega) = \tilde{Q}(m,\omega)$.

To prove (v), using the properties (i)-(iv) we obtain

$$\begin{aligned}&\Psi_{\tilde{Q}}(m,p,\omega)\tilde{Q}(p,\omega)\Psi_{\tilde{Q}}(m,p,\omega)\\ &\quad= \Psi_{\tilde{Q}}(n,p,\omega)\Phi_{\tilde{Q}}(n,p,\omega)\Psi_{\tilde{Q}}(m,p,\omega)\\ &\quad= \Psi_{\tilde{Q}}(n,p,\omega)\tilde{Q}(n,\varphi(n,p,\omega))\Phi_{\tilde{Q}}(n,p,\omega)\Psi_{\tilde{Q}}(m,p,\omega)\\ &\quad= \Psi_{\tilde{Q}}(n,p,\omega)\Psi_{Q}(m,n,\varphi(n,p,\omega))\Phi_{\tilde{Q}}(m,n,\varphi(n,p,\omega))\Phi_{\tilde{Q}}(n,p,\omega)\Psi_{\tilde{Q}}(m,p,\omega)\\ &\quad= \Psi_{\tilde{Q}}(n,p,\omega)\Psi_{\tilde{Q}}(m,n,\varphi(n,p,\omega))\Phi_{\tilde{Q}}(m,p,\omega)\Psi_{\tilde{Q}}(m,p,\omega)\\ &\quad= \Psi_{\tilde{Q}}(n,p,\omega)\Psi_{\tilde{Q}}(m,n,\varphi(n,p,\omega))\tilde{Q}(m,\varphi(m,p,\omega))\\ &\quad= \Psi_{\tilde{Q}}(n,p,\omega)\Psi_{\tilde{Q}}(m,n,\varphi(n,p,\omega)),\end{aligned}$$

for all $(m,n,p,\omega)\in\tilde{T}\times\Omega$. □

Definition 6. A nondecreasing map $h : \mathbb{N}\to[1,\infty)$ with $\lim_{m\to\infty} h(m) = \infty$ is called a *growth rate.*

Let $\tilde{C} = (\Phi,\varphi)$ be a strongly measurable discrete-time stochastic skew-evolution semiflow, $\tilde{P}$ an invariant projections family for $\tilde{C}$ and $h : \mathbb{N}\to[1,\infty)$ a growth rate.

In the following, $L(\Omega, X, \mu)$ denotes the Banach space of all Bochner-measurable functions $f : \Omega \to X$ such that $\int_\Omega \|f(\omega)\| d\mu(\omega) < \infty$.

Definition 7. Let $\tilde{C} = (\Phi, \varphi)$ be a discrete-time stochastic skew-evolution semiflow. We say that $\tilde{C}$ is *strongly measurable* if, for all $(p, x) \in \mathbb{N} \times L(\Omega, X, \mu)$, the map $n \mapsto \int_\Omega \|\Phi(n, p, \omega)x(\omega)\| d\mu(\omega)$, is measurable on $\mathbb{N}$, for all $n \geq p$.

Definition 8. The pair $(\tilde{C}, \tilde{P})$ is said to be *uniformly h-dichotomic in mean (u.h. d.m.)* if there are some constants $N \geq 1$ and $\nu > 0$ such that

(uhd$_1$m)

$$h(m)^\nu \int_\Omega \|\Phi_{\tilde{P}}(m, p, \omega)x(\omega)\| d\mu(\omega) \leq N h(n)^\nu \int_\Omega \|\Phi_{\tilde{P}}(n, p, \omega)x(\omega)\| d\mu(\omega);$$

(uhd$_2$m)

$$h(m)^\nu \int_\Omega \|\Phi_{\tilde{Q}}(n, p, \omega)x(\omega)\| d\mu(\omega) \leq N h(n)^\nu \int_\Omega \|\Phi_{\tilde{Q}}(m, p, \omega)x(\omega)\| d\mu(\omega),$$

for all $(m, n, p, \omega) \in \tilde{T} \times \Omega$ and $x \in L(\Omega, X, \mu)$;

When we examine the specific cases where $h(m) = e^m$ and $h(m) = m + 1$, we infer the concepts of *uniform exponential dichotomy in mean* and *uniform polynomial dichotomy in mean* respectively.

Remark 4. The pair $(\tilde{C}, \tilde{P})$ is uniformly h-dichotomic in mean if and only if there exist $N \geq 1$ and $\nu > 0$ with

(uhd$_1'$m)

$$h(m)^\nu \int_\Omega \|\Phi_{\tilde{P}}(m, n, \omega)x(\omega)\| d\mu(\omega) \leq N h(n)^\nu \int_\Omega \|\tilde{P}(n, \omega)x(\omega)\| d\mu(\omega);$$

(uhd$_2'$m)

$$h(m)^\nu \int_\Omega \|\tilde{Q}(n, \omega)x(\omega)\| d\mu(\omega) \leq N h(n)^\nu \int_\Omega \|\Phi_{\tilde{Q}}(m, n, \omega)(n, \omega)x(\omega)\| d\mu(\omega),$$

for all $(m, n, \omega) \in \tilde{\Delta} \times \Omega$ and $x \in L(\Omega, X, \mu)$.

Proposition 3. *The pair $(\tilde{C}, \tilde{P})$ is uniformly h-dichotomic in mean if and only if there are $N \geq 1$ and $\nu > 0$ such that*

($\mathbf{uhd_1''m}$)

$$h(m)^{\nu}\int_{\Omega}\|\Phi_{\tilde{P}}(m,n,\omega)x(\omega)\|d\mu(\omega) \leq Nh(n)^{\nu}\int_{\Omega}\|\tilde{P}(n,\omega)x(\omega)\|d\mu(\omega);$$

($\mathbf{uhd_2''m}$)

$$h(m)^{\nu}\int_{\Omega}\|\Psi_{\tilde{Q}}(m,n,\omega)x(\omega)\|d\mu(\omega) \leq Nh(n)^{\nu}\int_{\Omega}\|\tilde{Q}(m,\varphi(m,n,\omega))x(\omega)\|d\mu(\omega),$$

for all $(m,n,\omega)\in\tilde{\Delta}\times\Omega$ *and* $x\in L(\Omega,X,\mu)$.

Proof. It follows immediately from Remark 4 and Proposition 2. □

Theorem 1. *The pair* $(\tilde{C},\tilde{P})$ *is uniformly h-dichotomic in mean if and only if there exist some constants* $N\geq 1$ *and* $\nu>0$ *with*

($\mathbf{uhd_1'''m}$)

$$h(m)^{\nu}\int_{\Omega}\|\Phi_{\tilde{P}}(m,p,\omega)x(\omega)\|d\mu(\omega) \leq Nh(n)^{\nu}\int_{\Omega}\|\Phi_{\tilde{P}}(n,p,\omega)x(\omega)\|d\mu(\omega);$$

($\mathbf{uhd_2'''m}$)

$$h(n)^{\nu}\int_{\Omega}\|\Psi_{\tilde{Q}}(m,p,\omega)x(\omega)\|d\mu(\omega) \leq Nh(p)^{\nu}\int_{\Omega}\|\Psi_{\tilde{Q}}(m,n,\varphi(n,p,\omega)) x(\omega)\|d\mu(\omega),$$

for all $(m,n,p,\omega)\in\tilde{T}\times\Omega$ *and* $x\in L(\Omega,X,\mu)$.

Proof. It arises from Definition 8, Proposition 2 and Proposition 3. □

Definition 9. The pair $(\tilde{C},\tilde{P})$ is said to be with *uniform h-growth in mean (u.h.g.m.)* if there exist constants $M\geq 1$ and $\alpha>0$ such that:

($\mathbf{uhg_1m}$)

$$h(n)^{\alpha}\int_{\Omega}\|\Phi_{\tilde{P}}(m,p,\omega)x(\omega)\|d\mu(\omega) \leq Mh(m)^{\alpha}\int_{\Omega}\|\Phi_{\tilde{P}}(n,p,\omega)x(\omega)\|d\mu(\omega);$$

($\mathbf{uhg_2m}$)

$$h(n)^{\alpha}\int_{\Omega}\|\Phi_{\tilde{Q}}(n,p,\omega)x(\omega)\|d\mu(\omega)$$

$$\leq Mh(m)^{\alpha} \int_{\Omega} \|\Phi_{\tilde{Q}}(m,p,\omega)x(\omega)\| d\mu(\omega),$$

for all $(m,n,p,\omega) \in \tilde{T} \times \Omega$ and $x \in L(\Omega,X,\mu)$.

As specific cases we note that when the growth rate is e^m, this establishes the concept of *uniform exponential growth in mean* and if the growth rate is $m+1$, then we arrive at the concept of *uniform polynomial growth in mean* respectively.

Remark 5. The pair $(\tilde{C},\tilde{P})$ has uniform h-growth in mean if and only if there exist $M \geq 1$ and $\alpha > 0$ with

($\mathbf{uhg_1'm}$)

$$h(n)^{\alpha} \int_{\Omega} \|\Phi_{\tilde{P}}(m,n,\omega)x(\omega)\| d\mu(\omega) \leq Mh(m)^{\alpha} \int_{\Omega} \|\tilde{P}(n,\omega)x(\omega)\| d\mu(\omega);$$

($\mathbf{uhg_2'm}$)

$$h(n)^{\alpha} \int_{\Omega} \|\tilde{Q}(n,\omega)x(\omega)\| d\mu(\omega) \leq Mh(m)^{\alpha} \int_{\Omega} \|\Phi_{\tilde{Q}}(m,n,\omega)x(\omega)\| d\mu(\omega),$$

for all $(m,n,\omega) \in \tilde{\Delta} \times \Omega$ and $x \in L(\Omega,X,\mu)$.

Proposition 4. *The pair $(\tilde{C},\tilde{P})$ is said to be with uniform h-growth in mean if and only if there exist two constants $M \geq 1$ and $\alpha > 0$ such that*

*(**$\mathbf{uhg_1''m}$**)*

$$h(n)^{\alpha} \int_{\Omega} \|\Phi_{\tilde{P}}(m,n,\omega)x(\omega)\| d\mu(\omega) \leq Mh(m)^{\alpha} \int_{\Omega} \|\tilde{P}(n,\omega)x(\omega)\| d\mu(\omega);$$

*(**$\mathbf{uhg_2''m}$**)*

$$h(n)^{\alpha} \int_{\Omega} \|\Psi_{\tilde{Q}}(m,n,\omega)x(\omega)\| d\mu(\omega) \leq Mh(m)^{\alpha} \int_{\Omega} \|\tilde{Q}(m,\varphi(m,n,\omega)) x(\omega)\| d\mu(\omega),$$

for all $(m,n,\omega) \in \tilde{\Delta} \times \Omega$ and $x \in L(\Omega,X,\mu)$.

Proof. It follows a similar approach as Proposition 3. □

Theorem 2. *The pair $(\tilde{C},\tilde{P})$ has uniform h-growth in mean if and only if there are $M \geq 1$ and $\alpha > 0$ with*

($\mathbf{uhg_1^{'''}m}$*)*

$$h(n)^{\alpha}\int_{\Omega}\|\Phi_{\tilde{P}}(m,p,\omega)x(\omega)\|d\mu(\omega)$$
$$\leq Mh(m)^{\alpha}\int_{\Omega}\|\Phi_{\tilde{P}}(n,p,\omega)x(\omega)\|d\mu(\omega);$$

($\mathbf{uhg_2^{'''}m}$*)*

$$h(p)^{\alpha}\int_{\Omega}\|\Psi_{\tilde{Q}}(m,p,\omega)x(\omega)\|d\mu(\omega)$$
$$\leq Mh(n)^{\alpha}\int_{\Omega}\|\Psi_{\tilde{Q}}(m,n,\varphi(n,p,\omega))$$
$$x(\omega)\|d\mu(\omega), \textit{ for all } (m,n,p,\omega)\in\tilde{T}\times\Omega \textit{ and } x\in L(\Omega,X,\mu).$$

Proof. The proof uses the same technique demonstrated in Theorem 1. □

3. Main results

We denote by $\mathcal{H}_1$ the set of all functions $h:\mathbb{N}\to[1,\infty)$ with the following properties:

•$\mathcal{H}_1$ the set of all functions $h:\mathbb{N}\to[1,\infty)$ with the property that for all $\beta<0$, exists $H_1>1$ such that $h(m+1)\leq H_1h(m)$, for all $m\geq 0$ and

$$\sum_{k=n}^{\infty}h(k)^{\beta}\leq H_1\,h(n)^{\beta},\quad\text{for all}\quad n\geq 0.$$

•$\mathcal{H}_2$ the set of functions $h:\mathbb{N}\to[1,\infty)$ with the property that for all $\beta>0$, exists $H_2>1$ such that $h(m+1)\leq H_2h(m)$ and

$$\sum_{j=n}^{m}h(j)^{\beta}\leq H_2\,h(m)^{\beta},\quad\text{for all}\quad m\geq 0.$$

Remark 6. If $h(m)=e^m$, then $h\in\mathcal{H}_\infty$.

Theorem 3. *We assume that $\tilde{C}=(\Phi,\varphi)$ is a strongly measurable discrete-time stochastic skew-evolution semiflow, $(\tilde{C},\tilde{P})$ with uniform h-growth in mean and $h\in\mathcal{H}_1$. The pair $(\tilde{C},\tilde{P})$ is uniformly h-dichotomic in mean if and only if there exist constants $D\geq 1$ and $d\in(0,1)$ such that*

$$(\mathbf{uhD_1^1m_d})\ \sum_{k=n}^{\infty}h(k)^d\left(\int_{\Omega}\|\Phi_{\tilde{P}}(k,p,\omega)x(\omega)\|d\mu(\omega)\right)$$
$$\leq D\,h(n)^d\left(\int_{\Omega}\|\Phi_{\tilde{P}}(n,p,\omega)x(\omega)\|d\mu(\omega)\right);$$

for all $(n,p,\omega)\in\tilde{\Delta}\times\Omega$ and $x\in L(\Omega,X,\mu)$.

$$(\mathbf{uhD_2^1m_d})\ \sum_{k=n}^{\infty}\frac{h(k)^d}{\int_\Omega\|\Phi_{\tilde{Q}}(k,p,\omega)x(\omega)\|d\mu(\omega)}\leq\frac{D\,h(n)^d}{\int_\Omega\|\Phi_{\tilde{Q}}(n,p,\omega)x(\omega)\|d\mu(\omega)},$$

for all $(n,p,\omega)\in\tilde{\Delta}\times\Omega$ *and* $x\in L(\Omega,X,\mu)$
with $\int_\Omega\|\Phi_Q(n,p,\omega)x(\omega)\|\neq 0.$

Proof. Necessity. Let $d\in(0,\nu)$. For $(uhd_1m)\implies(uhD_1^1m_d)$ we have

$$\begin{aligned}
&\sum_{k=n}^{\infty}h(k)^d\left(\int_\Omega\|\Phi_{\tilde{P}}(k,p,\omega)x(\omega)\|d\mu(\omega)\right)\\
&\leq N\sum_{k=n}^{\infty}\left(\frac{h(k)}{h(n)}\right)^{-\nu}h(k)^d\left(\int_\Omega\|\Phi_{\tilde{P}}(n,p,\omega)x(\omega)\|d\mu(\omega)\right)\\
&=Nh(n)^\nu\int_\Omega\|\Phi_{\tilde{P}}(n,p,\omega)x(\omega)\|d\mu(\omega)\sum_{k=n}^{\infty}h(k)^{d-\nu}\\
&\leq Nh(n)^\nu\int_\Omega\|\Phi_{\tilde{P}}(n,p,\omega)x(\omega)\|d\mu(\omega)H_1h(n)^{d-\nu}\\
&=NH_1h(n)^d\int_\Omega\|\Phi_{\tilde{P}}(n,p,\omega)x(\omega)\|d\mu(\omega)\leq Dh(n)^d\int_\Omega\|\Phi_{\tilde{P}}(n,p,\omega)x(\omega)\|d\mu(\omega),
\end{aligned}$$

where $D=1+NH_1$.

Analogously, we have to prove $(uhd_2m)\implies(uhD_2^1m_d)$

$$\begin{aligned}
&\sum_{k=n}^{\infty}\frac{h(k)^d}{\int_\Omega\|\Phi_{\tilde{Q}}(k,p,\omega)x(\omega)\|d\mu(\omega)}\\
&\leq N\sum_{k=n}^{\infty}\left(\frac{h(k)}{h(n)}\right)^{-\nu}\frac{h(k)^d}{\int_\Omega\|\Phi_{\tilde{Q}}(n,p,\omega)x(\omega)\|d\mu(\omega)}\\
&=\frac{Nh(n)^\nu}{\int_\Omega\|\Phi_{\tilde{Q}}(n,p,\omega)x(\omega)\|d\mu(\omega)}\sum_{k=n}^{\infty}h(k)^{d-\nu}\\
&\leq\frac{Nh(n)^\nu}{\int_\Omega\|\Phi_{\tilde{Q}}(n,p,\omega)x(\omega)\|d\mu(\omega)}H_1h(n)^{d-\nu}\\
&\leq\frac{Dh(n)^d}{\int_\Omega\|\Phi_{\tilde{Q}}(n,p,\omega)x(\omega)\|d\mu(\omega)},
\end{aligned}$$

where $D=1+NH_1$, for all $(n,p,\omega)\in\tilde{\Delta}\times\Omega$ and $x\in L(\Omega,X,\mu)$,
with $\int_\Omega\|\Phi_{\tilde{Q}}(n,p,\omega)x(\omega)\|d\mu(\omega)\neq 0.$

Sufficiency. To establish $(uhD_1^1m_d)\implies(uhd_1m)$, we need to examine the following cases:

Case $I.1$. If $m > p+1$ we have

$$\begin{aligned}
&h(m)^d\left(\int_\Omega \|\Phi_{\tilde{P}}(m,p,\omega)x(\omega)\|d\mu(\omega)\right)\\
&\quad = h(m)^d\frac{1}{2}\sum_{k=m-1}^{m}\left(\int_\Omega \|\Phi_{\tilde{P}}(m,p,\omega)x(\omega)\|d\mu(\omega)\right)\\
&\quad \le \frac{1}{2}\sum_{k=m-1}^{m} Mh(m)^d\left(\frac{h(m)}{h(k)}\right)^{\alpha}\left(\int_\Omega \|\Phi_{\tilde{P}}(k,p,\omega)x(\omega)\|d\mu(\omega)\right)\\
&\quad = \frac{M}{2}\sum_{k=m-1}^{m} h(k)^d\left(\frac{h(m)}{h(k)}\right)^{\alpha+d}\left(\int_\Omega \|\Phi_{\tilde{P}}(k,p,\omega)x(\omega)\|d\mu(\omega)\right)\\
&\quad \le \frac{M}{2}H_1^{\alpha+d}\sum_{k=n}^{\infty} h(k)^d\left(\int_\Omega \|\Phi_{\tilde{P}}(k,p,\omega)x(\omega)\|d\mu(\omega)\right)\\
&\quad \le \frac{M}{2}DH_1^{\alpha+d}h(n)^d\int_\Omega \|\Phi_{\tilde{P}}(n,p,\omega)x(\omega)\|d\mu(\omega)
\end{aligned}$$

Therefore,

$$\begin{aligned}
&h(m)^d\left(\int_\Omega \|\Phi_{\tilde{P}}(m,p,\omega)x(\omega)\|d\mu(\omega)\right)\\
&\qquad \le \frac{M}{2}DH_1^{\alpha+d}h(n)^d\int_\Omega \|\Phi_{\tilde{P}}(n,p,\omega)x(\omega)\|d\mu(\omega),
\end{aligned}$$

for all $(n,p,\omega)\in\tilde{\Delta}\times\Omega$ and $x\in L(\Omega,X,\mu)$.

Case $I.2$. If $n\in[p,p+1)$ we have

$$\begin{aligned}
&h(m)^d\left(\int_\Omega \|\Phi_{\tilde{P}}(m,p,\omega)x(\omega)\|d\mu(\omega)\right)\\
&\quad \le Mh(m)^d\left(\frac{h(m)}{h(n)}\right)^{\alpha}\int_\Omega \|\Phi_{\tilde{P}}(n,p,\omega)x(\omega)\|d\mu(\omega)\\
&\quad = M\left(\frac{h(m)}{h(n)}\right)^{\alpha+d}h(n)^d\int_\Omega \|\Phi_{\tilde{P}}(n,p,\omega)x(\omega)\|d\mu(\omega)\\
&\quad \le MH_1^{\alpha+d}h(n)^d\int_\Omega \|\Phi_{\tilde{P}}(n,p,\omega)x(\omega)\|d\mu(\omega)
\end{aligned}$$

Thus,

$$\begin{aligned}
&h(m)^d\left(\int_\Omega \|\Phi_{\tilde{P}}(m,p,\omega)x(\omega)\|d\mu(\omega)\right)\\
&\qquad \le MH_1^{\alpha+d}h(n)^d\int_\Omega \|\Phi_{\tilde{P}}(n,p,\omega)x(\omega)\|d\mu(\omega)
\end{aligned}$$

From Case $I.1$. and Case $I.2$. it results that there exists $N = MDH_1^{\alpha+d} + 1$ with

$$h(m)^d \left(\int_\Omega \|\Phi_{\tilde{P}}(m,p,\omega)x(\omega)\| d\mu(\omega) \right) \leq Nh(n)^d \int_\Omega \|\Phi_{\tilde{P}}(n,p,\omega)x(\omega)\| d\mu(\omega),$$

for all $(n,p,\omega) \in \tilde{\Delta} \times \Omega$ and $x \in L(\Omega, X, \mu)$.

For the second relation $(uhD_2^1 m_d) \implies (uhD_2 m)$, we initially consider $(m,n,p,\omega) \in \tilde{T} \times \Omega$ and $\int_\Omega \|\Phi_{\tilde{Q}}(m,p,\omega)x(\omega)\| d\mu(\omega) \neq 0$. Moreover, there are two cases to be considered:

Case $II.1$. When $n \geq p+1$ we obtain

$$\begin{aligned}
\frac{h(m)^d}{\int_\Omega \|\Phi_{\tilde{Q}}(m,p,\omega)x(\omega)\| d\mu(\omega)} &= \frac{1}{2} \sum_{k=m-1}^{m} \frac{h(m)^d}{\int_\Omega \|\Phi_{\tilde{Q}}(m,p,\omega)x(\omega)\| d\mu(\omega)} \\
&\leq \frac{M}{2} \sum_{k=m-1}^{m} \left(\frac{h(m)}{h(k)} \right)^\alpha \frac{h(m)^d}{\int_\Omega \|\Phi_{\tilde{Q}}(m,p,\omega)x(\omega)\| d\mu(\omega)} \\
&= \frac{M}{2} \sum_{k=m-1}^{m} \left(\frac{h(m)}{h(k)} \right)^{\alpha+d} \frac{h(k)^d}{\int_\Omega \|\Phi_{\tilde{Q}}(m,p,\omega)x(\omega)\| d\mu(\omega)} \\
&\leq \frac{M}{2} H_1^{\alpha+d} \sum_{k=n}^{\infty} \frac{h(k)^d}{\int_\Omega \|\Phi_{\tilde{Q}}(n,p,\omega)x(\omega)\| d\mu(\omega)} d \\
&\leq \frac{M}{2} MDH_1^{\alpha+d} \frac{h(n)^d}{\int_\Omega \|\Phi_{\tilde{Q}}(n,p,\omega)x(\omega)\| d\mu(\omega)},
\end{aligned}$$

So,

$$h(m)^d \int_\Omega \|\Phi_{\tilde{Q}}(n,p,\omega)x(\omega)\| d\mu(\omega) \leq \frac{M}{2} DH_1^{\alpha+d} h(n)^d \int_\Omega \|\Phi_{\tilde{Q}}(m,p,\omega)x(\omega)\| d\mu(\omega),$$

for all $(n,p,\omega) \in \tilde{\Delta} \times \Omega$ and $x \in L(\Omega, X, \mu)$.

Case $II.2$. If $n \in [p, p+1)$ with $\int_\Omega \|\Phi_{\tilde{Q}}(m,p,\omega)x(\omega)\| d\mu(\omega) \neq 0$

$$\begin{aligned}
h(m)^d \int_\Omega \|\Phi_{\tilde{Q}}(n,p,\omega)x(\omega)\| d\mu(\omega) & \\
&\leq Mh(m)^d \left(\frac{h(m)}{h(n)} \right)^\alpha \int_\Omega \|\Phi_{\tilde{Q}}(m,p,\omega)x(\omega)\| d\mu(\omega) \\
&= M \left(\frac{h(m)}{h(n)} \right)^{\alpha+d} h(n)^d \int_\Omega \|\Phi_{\tilde{Q}}(m,p,\omega)x(\omega)\| d\mu(\omega) \\
&\leq MH_1^{\alpha+d} h(n)^d \int_\Omega \|\Phi_{\tilde{Q}}(m,p,\omega)x(\omega)\| d\mu(\omega)
\end{aligned}$$

Thus,

$$h(m)^d\left(\int_\Omega \|\Phi_{\tilde{Q}}(n,p,\omega)x(\omega)\|d\mu(\omega)\right)\le MH_1^{\alpha+d}h(n)^d\int_\Omega \|\Phi_{\tilde{Q}}(m,p,\omega)x(\omega)\|d\mu(\omega)$$

From Case $II.1.$ and Case $II.2.$ it follows that there exists $N=MDH_1^{\alpha+d}+1$ with

$$h(m)^d\left(\int_\Omega \|\Phi_{\tilde{Q}}(n,p,\omega)x(\omega)\|d\mu(\omega)\right)\le N_1h(n)^d\int_\Omega \|\Phi_{\tilde{Q}}(m,p,\omega)x(\omega)\|d\mu(\omega),$$

for all $(m,p,\omega)\in\tilde{\Delta}\times\Omega$ and $x\in L(\Omega,X,\mu)$. □

Corollary 1. *We suppose that $\tilde{C}=(\Phi,\varphi)$ is a strongly measurable discrete-time stochastic skew-evolution semiflow, $(\tilde{C},\tilde{P})$ with uniform exponential growth in mean. The pair $(\tilde{C},\tilde{P})$ is uniformly exponentially dichotomic in mean if and only if there exist constants $D\ge 1$ and $d\in(0,1)$ with*

$(\mathbf{ueD_1^1m_d})$

$$\sum_{k=n}^{\infty}e^{dk}\left(\int_\Omega \|\Phi_{\tilde{P}}(k,p,\omega)x(\omega)\|d\mu(\omega)\right)\le D\,e^{dn}\left(\int_\Omega \|\Phi_{\tilde{P}}(n,p,\omega)x(\omega)\|d\mu(\omega)\right);$$

for all $(m,n,p,\omega)\in\tilde{T}\times\Omega$ and $x\in L(\Omega,X,\mu)$.

$(\mathbf{ueD_2^1m_d})$ $$\sum_{k=n}^{\infty}\frac{e^{dk}}{\int_\Omega \|\Phi_{\tilde{Q}}(k,p,\omega)x(\omega)\|d\mu(\omega)}\le\frac{D\,e^{dn}}{\int_\Omega \|\Phi_{\tilde{Q}}(n,p,\omega)x(\omega)\|d\mu(\omega)},$$

for all $(m,n,p,\omega)\in\tilde{T}\times\Omega$ and $x\in L(\Omega,X,\mu)$ with $\int_\Omega \|\Phi_{\tilde{Q}}(n,p,\omega)x(\omega)\|\neq 0$.

Proof. It follows from Theorem 3 for $h(m)=e^m$. □

Theorem 4. *Consider $\tilde{C}=(\Phi,\varphi)$ as a strongly measurable discrete-time stochastic skew-evolution semiflow, $(\tilde{C},\tilde{P})$ has uniform h-growth in mean and $h\in\mathcal{H}_2$. The pair $(\tilde{C},\tilde{P})$ is uniformly h-dichotomic in mean if and only if there exist constants $D\ge 1$ and $d\in(0,1)$ such that*

$(\mathbf{uhD_1^2m_d})$ $$\sum_{j=n}^{m}\frac{h(j)^{-d}}{\int_\Omega \|\Phi_{\tilde{P}}(j,p,\omega)x(\omega)\|d\mu(\omega)}\le\frac{D\,h(m)^{-d}}{\int_\Omega \|\Phi_{\tilde{P}}(m,p,\omega)x(\omega)\|d\mu(\omega)},$$

for all $(m,n,p,\omega)\in\tilde{T}\times\Omega$ and $x\in L(\Omega,X,\mu)$, with $\int_\Omega \|\Phi_{\tilde{P}}(m,p,\omega)x(\omega)\|d\mu(\omega)\neq 0$.

$(\mathbf{uhD_2^2m_d})$

$$\sum_{j=n}^{m}h(j)^{-d}\left(\int_\Omega \|\Phi_{\tilde{Q}}(j,p,\omega)x(\omega)\|d\mu(\omega)\right)$$

$$\leq D\, h(m)^{-d} \left(\int_\Omega \|\Phi_{\tilde{Q}}(m,p,\omega)x(\omega)\| d\mu(\omega) \right),$$

for all $(m,p,\omega) \in \tilde{\Delta} \times \Omega$ *and* $x \in L(\Omega, X, \mu)$.

Proof. Necessity. Let $d \in (0,\nu)$. For $(uhd_1m) \implies (uhD_1^2m_d)$ we have

$$\sum_{j=p}^{m} \frac{h(j)^{-d}}{\int_\Omega \|\Phi_{\tilde{P}}(j,p,\omega)x(\omega)\| d\mu(\omega)}$$
$$\leq \sum_{j=p}^{m} N \left(\frac{h(j)}{h(m)}\right)^{\nu} \frac{h(j)^{-d}}{\int_\Omega \|\Phi_{\tilde{P}}(m,p,\omega)x(\omega)\| d\mu(\omega)}$$
$$= \frac{Nh(m)^{-\nu}}{\int_\Omega \|\Phi_{\tilde{P}}(m,p,\omega)x(\omega)\| d\mu(\omega)} \sum_{j=p}^{m} h(j)^{\nu-d}$$
$$\leq \frac{NH_2h(m)^{-\nu}}{\int_\Omega \|\Phi_{\tilde{P}}(m,p,\omega)x(\omega)\| d\mu(\omega)} h(m)^{\nu-d}$$
$$= \frac{Dh(m)^{-d}}{\int_\Omega \|\Phi_{\tilde{P}}(m,p,\omega)x(\omega)\| d\mu(\omega)},$$

where $D = 1 + NH_2$, with $\int_\Omega \|\Phi_{\tilde{P}}(m,p,\omega)x(\omega)\| d\mu(\omega) \neq 0$

For the implication $(uhd_2m) \implies (uhD_2^2m_d)$ we have

$$\sum_{j=p}^{m} \frac{\int_\Omega \|\Phi_{\tilde{Q}}(j,p,\omega)x(\omega)\| d\mu(\omega)}{h(j)^d}$$
$$\leq \sum_{j=p}^{m} N \left(\frac{h(j)}{h(m)}\right)^{\nu} \frac{\int_\Omega \|\Phi_{\tilde{Q}}(m,p,\omega)x(\omega)\| d\mu(\omega)}{h(j)^d}$$
$$\leq \frac{N\int_\Omega \|\Phi_{\tilde{Q}}(m,p,\omega)x(\omega)\| d\mu(\omega)}{h(m)^{\nu}} \sum_{j=p}^{m} h(j)^{\nu-d}$$
$$\leq NH_2h(m)^{\nu-d} \cdot \frac{\int_\Omega \|\Phi_{\tilde{Q}}(m,p,\omega)x(\omega)\| d\mu(\omega)}{h(m)^{\nu}}$$
$$\leq \frac{D\, \int_\Omega \|\Phi_{\tilde{Q}}(m,p,\omega)x(\omega)\| d\mu(\omega)}{h(m)^d},$$

where $D = 1 + NH_2$.

Sufficiency. For $(uhD_2^1m_d') \implies (uhd_1m)$ we have two cases:
Case $I.1$. If $n \geq p+1$ we have

$$\frac{h(n)^{-d}}{\int_\Omega \|\Phi_{\tilde{P}}(j,n,\omega)x(\omega)\| d\mu(\omega)}$$

$$= \frac{1}{2}\sum_{j=p}^{p+1} \frac{h(n)^{-d}}{\int_\Omega \|\Phi_{\tilde{P}}(j,n,\omega)x(\omega)d\mu(\omega)}$$
$$\le \frac{1}{2}\sum_{j=p}^{p+1} h(n)^{-d} M \left(\frac{h(j)}{h(n)}\right)^{\alpha} \frac{h(j)^{-d}\, h(j)^{d}}{\int_\Omega \|\Phi_{\tilde{P}}(j,p,\omega)x(\omega)\|d\mu(\omega)}$$
$$= \frac{M}{2}\sum_{j=p}^{p+1} \left(\frac{h(j)}{h(n)}\right)^{\alpha+d} \frac{h(j)^{-d}}{\int_\Omega \|\Phi_{\tilde{P}}(j,p,\omega)x(\omega)\|d\mu(\omega)}$$
$$\le \frac{M}{2} H_2^{\alpha+d} \sum_{j=p}^{m} \frac{h(j)^{-d}}{\int_\Omega \|\Phi_{\tilde{P}}(j,p,\omega)x(\omega)\|d\mu(\omega)}$$
$$\le \frac{M}{2} D H_2^{\alpha+d} \frac{h(m)^{-d}}{\int_\Omega \|\Phi_{\tilde{P}}(m,p,\omega)x(\omega)\|d\mu(\omega)}$$

We obtain

$$h(m)^d \int_\Omega \|\Phi_{\tilde{P}}(m,p,\omega)x(\omega)\|d\mu(\omega) \le D\frac{M}{2} H_2^{\alpha+d} h(n)^d \int_\Omega \|\Phi_{\tilde{P}}(n,p,\omega)x(\omega)\|d\mu(\omega),$$

for all $(n,p,\omega) \in \tilde{\Delta}\times\Omega$ and $x \in L(\Omega,X,\mu)$.

Case $I.2$. If $m \in [p,p+1)$ and $\int_\Omega \|\Phi_{\tilde{P}}(m,p,\omega)x(\omega)\|d\mu(\omega) \neq 0$

$$\int_\Omega \|\Phi_{\tilde{P}}(m,p,\omega)x(\omega)\|d\mu(\omega)$$
$$= \int_\Omega \|\Phi(m,n,\varphi(n,p,\omega))\Phi_{\tilde{P}}(n,p,\omega)x(\omega)\|d\mu(\omega)$$
$$\le M\left(\frac{h(m)}{h(n)}\right)^{\alpha} \int_\Omega \|\tilde{P}(n,\varphi(n,p,\omega))\Phi_{\tilde{P}}(n,p,\omega)x(\omega)\|d\mu(\omega)$$
$$\le M\left(\frac{h(m)}{h(n)}\right)^{\alpha+d} \left(\frac{h(m)}{h(n)}\right)^{-d} \int_\Omega \|\Phi_{\tilde{P}}(n,p,\omega)x(\omega)\|d\mu(\omega)$$
$$\le M H_2^{\alpha+d} \left(\frac{h(m)}{h(n)}\right)^{-d} \int_\Omega \|\Phi_{\tilde{P}}(n,p,\omega)x(\omega)\|d\mu(\omega),$$

for all $(n,p,\omega) \in \tilde{\Delta}\times\Omega$ and $x \in L(\Omega,X,\mu)$.

From Case $I.1$. and Case $I.2$. it follows that there exists $N = MDH_2^{\alpha+d}+1$ with

$$h(m)^d \int_\Omega \|\Phi_{\tilde{P}}(m,p,\omega)x(\omega)\|d\mu(\omega) \le MDH_2^{\alpha+d} h(n)^d \int_\Omega \|\Phi_{\tilde{P}}(n,p,\omega)x(\omega)\|d\mu(\omega),$$

for all $(n,p,\omega) \in \tilde{\Delta}\times\Omega$ and $x \in L(\Omega,X,\mu)$.

For the relation $(uhD_2^2 m_d) \implies (uhd_2 m)$ we also consider two cases:

Case $II.1$. When $n \ge p+1$ we obtain

$$h(n)^{-d} \int_\Omega \|\Phi_{\tilde{Q}}(n,p,\omega)x(\omega)\|d\mu(\omega)$$

$$
\begin{aligned}
&= \frac{1}{2}\sum_{j=p}^{p+1} h(n)^{-d}\left(\int_\Omega \|\Phi_{\tilde{Q}}(n,p,\omega)x(\omega)\|d\mu(\omega)\right)\\
&\leq \frac{1}{2}\sum_{j=p}^{p+1} h(n)^{-d}M\left(\frac{h(j)}{h(n)}\right)^{\alpha}\int_\Omega \|\Phi_{\tilde{Q}}(j,p,\omega)x(\omega)\|d\mu(\omega)\\
&= \frac{M}{2}\sum_{j=p}^{p+1}\left(\frac{h(j)}{h(n)}\right)^{\alpha+d} h(j)^{-d}\int_\Omega \|\Phi_{\tilde{Q}}(j,p,\omega)x(\omega)\|d\mu(\omega)\\
&\leq \frac{M}{2}H_2^{\alpha+d}\sum_{j=p}^{m} h(j)^{-d}\int_\Omega \|\Phi_{\tilde{Q}}(j,p,\omega)x(\omega)\|d\mu(\omega)\\
&\leq \frac{M}{2}DH_2^{\alpha+d}h(m)^{-d}\int_\Omega \|\Phi_{\tilde{Q}}(m,p,\omega)x(\omega)\|d\mu(\omega),
\end{aligned}
$$

So, we obtain

$$
h(m)^d\int_\Omega \|\Phi_{\tilde{Q}}(n,p,\omega)x(\omega)\|d\mu(\omega) \leq \frac{M}{2}DH_2^{\alpha+d}h(n)^d\int_\Omega \|\Phi_{\tilde{Q}}(m,p,\omega)x(\omega)\|d\mu(\omega),
$$

for all $(m,p,\omega)\in\tilde{\Delta}\times\Omega$ and $x\in L(\Omega,X,\mu)$.

Case $II.2$. If $t\in[p,p+1)$ we have

$$
\begin{aligned}
&\frac{h(m)^d}{\int_\Omega \|\Phi_{\tilde{Q}}(m,p,\omega)x(\omega)\|d\mu(\omega)}\\
&\qquad\leq M\left(\frac{h(m)}{h(n)}\right)^{\alpha}\frac{h(m)^d}{\int_\Omega \|\Phi_{\tilde{Q}}(n,p,\omega)x(\omega)\|d\mu(\omega)}\\
&\qquad= M\left(\frac{h(m)}{h(n)}\right)^{\alpha+d}\frac{h(n)^d}{\int_\Omega \|\Phi_{\tilde{Q}}(n,p,\omega)x(\omega)\|d\mu(\omega)}\\
&\qquad\leq MH_2^{\alpha+d}\frac{h(n)^d}{\int_\Omega \|\Phi_{\tilde{Q}}(n,p,\omega)x(\omega)\|d\mu(\omega)}
\end{aligned}
$$

Therefore,

$$
h(m)^d\int_\Omega \|\Phi_{\tilde{Q}}(n,p,\omega)x(\omega)\|d\mu(\omega) \leq MH_2^{\alpha+d}h(n)^d\int_\Omega \|\Phi_{\tilde{Q}}(m,p,\omega)x(\omega)\|d\mu(\omega),
$$

for all $(m,p,\omega)\in\tilde{\Delta}\times\Omega$ and $x\in L(\Omega,X,\mu)$.
From Case $II.1$. and Case $II.2$. it follows that there exists $N=MDH_2^{\alpha+d}+1$ with

$$
h(m)^d\int_\Omega \|\Phi_{\tilde{Q}}(n,p,\omega)x(\omega)\|d\mu(\omega) \leq Nh(n)^d\int_\Omega \|\Phi_{\tilde{Q}}(m,p,\omega)x(\omega)\|d\mu(\omega),
$$

for all $(m,p,\omega)\in\tilde{\Delta}\times\Omega$ and $x\in L(\Omega,X,\mu)$. □

Corollary 2. *Let $\tilde{C}=(\Phi,\varphi)$ as a strongly measurable discrete-time stochastic skew-evolution semiflow, $(\tilde{C},\tilde{P})$ has uniform exponential growth in mean. The pair*

$(\tilde{C},\tilde{P})$ is uniformly exponentially dichotomic in mean if and only if there exist constants $D\geq 1$ and $d\in(0,1)$ with

(ueD$_1^2$m$_d$) $$\sum_{j=n}^{m}\frac{e^{-dj}}{\int_\Omega\|\Phi_{\tilde{P}}(j,p,\omega)x(\omega)\|d\mu(\omega)}\leq\frac{D\,e^{-dm}}{\int_\Omega\|\Phi_{\tilde{P}}(m,p,\omega)x(\omega)\|d\mu(\omega)},$$
for all $(m,n,p,\omega)\in\tilde{T}\times\Omega$ and $x\in L(\Omega,X,\mu)$, with
$$\int_\Omega\|\Phi_{\tilde{P}}(m,p,\omega)x(\omega)\|d\mu(\omega)\neq 0.$$

(ueD$_2^2$m$_d$)
$$\sum_{j=n}^{m}e^{-dj}\left(\int_\Omega\|\Phi_{\tilde{Q}}(j,p,\omega)x(\omega)\|d\mu(\omega)\right)\leq D\,e^{-dm}\left(\int_\Omega\|\Phi_{\tilde{Q}}(m,p,\omega)x(\omega)\|d\mu(\omega)\right),$$
for all $(m,p,\omega)\in\tilde{\Delta}\times\Omega$ and $x\in L(\Omega,X,\mu)$.

Proof. It follows from Theorem 4 for $h(m)=e^m$. □

Theorem 5. *Let $\tilde{C}=(\Phi,\varphi)$ be a strongly measurable discrete-time stochastic skew-evolution semiflow, $(\tilde{C},\tilde{P})$ with strong uniform h-growth in mean and $h\in\mathcal{H}_1$. The pair $(\tilde{C},\tilde{P})$ is uniformly h-dichotomic in mean if and only if there are two constants $D\geq 1$ and $d\in(0,1)$ such that*

(**uhD$_1^3$m$_d$**)
$$\sum_{k=p}^{\infty}h(k)^d\left(\int_\Omega\|\Phi_{\tilde{P}}(k,p,\omega)x(\omega)\|d\mu(\omega)\right)\leq Dh(n)^d\int_\Omega\|\Phi_{\tilde{P}}(n,p,\omega)x(\omega)\|d\mu(\omega);$$
for all $(m,n,p,\omega)\in\tilde{T}\times\Omega$ and $x\in L(\Omega,X,\mu)$.

(uhD$_2^3$m$_d$)
$$\sum_{k=p}^{\infty}\frac{h(k)^d}{\int_\Omega\|\Psi_{\tilde{Q}}(m,k,\varphi(k,p,\omega))x(\omega)\|d\mu(\omega)}\leq\frac{Dh(p)^d}{\int_\Omega\|\Psi_{\tilde{Q}}(m,p,\omega)x(\omega)\|d\mu(\omega)},$$
for all $(m,n,p,\omega)\in\tilde{T}\times\Omega$ and $x\in L(\Omega,X,\mu)$.

Proof. Necessity. The relation $(uhd_1m)\Longrightarrow(uhD_2^1m_d)$ is similar with the proof of Theorem 3. To prove the relation $(uhd_2m)\Longrightarrow(uhD_2^3m_d)$ we suppose that $(\tilde{C},\tilde{P})$ has u.h.g.m. and we obtain
$$\sum_{k=p}^{\infty}\frac{h(k)^d}{\int_\Omega\|\Psi_{\tilde{Q}}(m,k,\varphi(k,p,\omega))x(\omega)\|d\mu(\omega)}$$

$$\leq \sum_{k=p}^{\infty} N\left(\frac{h(k)}{h(p)}\right)^{-\nu} \frac{h(k)^{d}}{\int_{\Omega}\|\Psi_{\tilde{Q}}(m,p,\omega)x(\omega)\|d\mu(\omega)}$$

$$\leq \frac{Nh(p)^{\nu}}{\int_{\Omega}\|\Psi_{\tilde{Q}}(m,p,\omega)x(\omega)\|d\mu(\omega)} \sum_{k=p}^{\infty} h(k)^{d-\nu}$$

$$\leq NH_1 \frac{h(p)^{d}}{\int_{\Omega}\|\Psi_{\tilde{Q}}(m,p,\omega)x(\omega)\|d\mu(\omega)},$$

where $D = 1 + NH_1$.

Sufficiency. The implication $(uhD_1^3 m_d') \implies (uhd_1 m)$ is similar with the proof of Theorem 3. For the relation $(uhD_2^3 m_d) \implies (uhd_2 m)$ we have two cases:

Case $I.1.$ $n \geq p+1$ and $\int_{\Omega}\|\Psi_{\tilde{Q}}(m,p,\omega)x(\omega)\|d\mu(\omega) \neq 0.$

$$\frac{1}{\int_{\Omega}\|\tilde{Q}(m,\varphi(m,p,\omega))x(\omega)\|d\mu(\omega)} = \frac{1}{2}\sum_{m-1}^{m} \frac{1}{\int_{\Omega}\|\Phi_{\tilde{Q}}(m,p,\omega)\Psi_{\tilde{Q}}(m,p,\omega)x(\omega)\|d\mu(\omega)}$$

$$\leq \frac{1}{2}\sum_{m-1}^{m} M\left(\frac{h(n)}{h(k)}\right)^{\alpha} \frac{1}{\int_{\Omega}\left\|\Phi_{\tilde{Q}}(k,p,\omega)\Psi_{\tilde{Q}}(m,p,\omega)x(\omega)\right\|d\mu(\omega)}$$

$$\leq \frac{M}{2}\sum_{m-1}^{m} \left(\frac{h(n)}{h(k)}\right)^{\alpha+d}\left(\frac{h(n)}{h(p)}\right)^{-d}\left(\frac{h(k)}{h(p)}\right)^{d} \frac{1}{\int_{\Omega}\left\|\Psi_{\tilde{Q}}(m,k,\varphi(k,p,\omega))x(\omega)\right\|d\mu(\omega)}$$

$$\leq \frac{M}{2}H_1^{\alpha+d}\left(\frac{h(n)}{h(p)}\right)^{-d}\sum_{k=p}^{\infty}\left(\frac{h(k)}{h(p)}\right)^{d} \frac{1}{\int_{\Omega}\left\|\Psi_{\tilde{Q}}(m,k,\varphi(k,p,\omega))x(\omega)\right\|d\mu(\omega)}$$

$$\leq \frac{DMH_1^{\alpha+d}}{2}\left(\frac{h(n)}{h(p)}\right)^{-d} \frac{1}{\int_{\Omega}\left\|\Psi_{\tilde{Q}}(m,p,\omega)x(\omega)\right\|d\mu(\omega)};$$

Case $I.2.$ $n \in [p, p+1)$.

$$\int_{\Omega}\left\|\Psi_{\tilde{Q}}(m,p,\omega)x(\omega)\right\|d\mu(\omega)$$

$$\leq M\left(\frac{h(n)}{h(p)}\right)^{\alpha}\int_{\Omega}\left\|\tilde{Q}(m,\varphi(m,p,\omega))x(\omega)\right\|d\mu(\omega)$$

$$\leq M\left(\frac{h(n)}{h(p)}\right)^{\alpha+d}\left(\frac{h(n)}{h(p)}\right)^{-d}\int_{\Omega}\left\|\tilde{Q}(m,\varphi(m,p,\omega))x(\omega)\right\|d\mu(\omega)$$

$$\leq MH_1^{\alpha+d}\left(\frac{h(n)}{h(p)}\right)^{-d}\int_{\Omega}\left\|\tilde{Q}(m,\varphi(m,p,\omega))x(\omega)\right\|d\mu(\omega).$$

Combining Case $I.1.$ with Case $I.2.$, we can conclude that there exist $N = 1 + MH_1^{\alpha+d}D$ and $\nu = d$ such that $(uhd_2 m)$ holds for all $(m,p,\omega) \in \tilde{\Delta} \times \Omega$ and all $x \in L(\Omega, X, \mu)$. Hence, we have shown that $(\tilde{C}, \tilde{P})$ is u.h.d.m., completing the proof. □

Corollary 3. *We assume that $\tilde{C} = (\Phi, \varphi)$ be a strongly measurable discrete-time stochastic skew-evolution semiflow, $(\tilde{C}, \tilde{P})$ with strong uniform exponential growth in mean. The pair $(\tilde{C}, \tilde{P})$ is uniformly exponentially dichotomic in mean if and only if there are two constants $D \geq 1$ and $d \in (0,1)$ with*

$(\mathbf{ueD_1^3m_d})$

$$\sum_{k=p}^{\infty} e^{dk} \left(\int_{\Omega} \|\Phi_{\tilde{P}}(k,p,\omega)x(\omega)\| d\mu(\omega) \right) \leq De^{dn} \int_{\Omega} \|\Phi_{\tilde{P}}(n,p,\omega)x(\omega)\| d\mu(\omega);$$

for all $(m,n,p,\omega) \in \tilde{T} \times \Omega$ and $x \in L(\Omega, X, \mu)$.

$(\mathbf{ueD_2^3m_d})$

$$\sum_{k=p}^{\infty} \frac{e^{dk}}{\int_{\Omega} \|\Psi_{\tilde{Q}}(m,k,\varphi(k,p,\omega))x(\omega)\| d\mu(\omega)} \leq \frac{De^{dp}}{\int_{\Omega} \|\Psi_{\tilde{Q}}(m,p,\omega)x(\omega)\| d\mu(\omega)},$$

for all $(m,n,p,\omega) \in \tilde{T} \times \Omega$ and $x \in L(\Omega, X, \mu)$.

Proof. It follows from Theorem 5 for $h(m) = e^m$. □

Theorem 6. *We assume that $\tilde{C} = (\Phi, \varphi)$ be a strongly measurable stochastic skew-evolution semiflow, $(\tilde{C}, \tilde{P})$ with strong uniform h-growth in mean and $h \in \mathcal{H}_2$. The pair $(\tilde{C}, \tilde{P})$ is uniformly h-dichotomic in mean if and only if there exist $D \geq 1$ and $d \in (0,1)$ such that*

$(\mathbf{uhD_1^4m_d})$

$$\sum_{j=p}^{m} \frac{h(j)^{-d}}{\int_{\Omega} \|\Phi_{\tilde{P}}(j,p,\omega)x(\omega)\| d\mu(\omega)} \leq \frac{D\,h(m)^{-d}}{\int_{\Omega} \|\Phi_{\tilde{P}}(m,p,\omega)x(\omega)\| d\mu(\omega)},$$

for all $(m,n,p,\omega) \in \tilde{T} \times \Omega$ and $x \in L(\Omega, X, \mu)$, with $\int_{\Omega} \|\Phi_{\tilde{P}}(m,p,\omega)x(\omega)\| d\mu(\omega) \neq 0.$

$(\mathbf{uhD_2^4m_d})$

$$\sum_{j=p}^{m} \frac{h(j)^{d}}{\int_{\Omega} \|\Psi_{\tilde{Q}}(m,j,\varphi(j,p,\omega))x(\omega)\| d\mu(\omega)} \leq \frac{D\,h(m)^{d}}{\int_{\Omega} \|\Psi_{\tilde{Q}}(m,p,\omega)x(\omega)\| d\mu(\omega)},$$

for all $(m,n,p,\omega)\in\tilde{T}\times\Omega$ *and* $x\in L(\Omega,X,\mu)$, *with* $\int_\Omega\|\Psi_{\tilde{Q}}(m,p,\omega)x(\omega)\|d\mu(\omega)\neq 0$.

Proof. Necessity. The relation $(\mathbf{uhd_1m})\Longrightarrow(\mathbf{uhD_4^1m_d})$ is similar with the proof of Theorem 4. To prove the relation $(uhd_2m)\Longrightarrow(uhD_2^4m_d)$ we suppose that $(\tilde{C},\tilde{P})$ has u.h.g.m. and we obtain

$$\begin{aligned}
&\sum_{j=p}^{m}h(j)^{-d}\left(\int_\Omega\|\Psi_{\tilde{Q}}(m,j,\varphi(j,p,\omega))x(\omega)\|d\mu(\omega)\right)\\
&\quad\leq\sum_{j=p}^{m}h(j)^{-d}N\left(\frac{h(j)}{h(m)}\right)^{\nu}\left(\int_\Omega\|\tilde{Q}(m,\varphi(m,p,\omega))x(\omega)\|d\mu(\omega)\right)\\
&\quad=Nh(m)^{-\nu}\int_\Omega\|\tilde{Q}(m,\varphi(m,p,\omega)))x(\omega)\|d\mu(\omega)\sum_{j=p}^{m}h(j)^{\nu-d}\\
&\quad\leq Nh(m)^{-\nu}\int_\Omega\|\tilde{Q}(m,\varphi(m,p,\omega))x(\omega)\|d\mu(\omega)H_2h(m)^{\nu-d}\\
&\quad\leq NH_2h(m)^{-d}\int_\Omega\|\tilde{Q}(m,\varphi(m,p,\omega))x(\omega)\|d\mu(\omega),
\end{aligned}$$

where $D=1+NH_2$.

Sufficiency. The implication $(uhD_1^4m_d)\Longrightarrow(uhd_1m)$ is similar with the proof of Theorem 4. For the relation $(uhD_2^4m_d)\Longrightarrow(uhd_2m)$ we have two cases:

Case $I.1$. $n\geq p+1$ and

$$\begin{aligned}
&\int_\Omega\|\Psi_{\tilde{Q}}(m,p,\omega)x(\omega)\|d\mu(\omega)\\
&\quad=\frac{1}{2}\sum_{j=p}^{p+1}\left(\int_\Omega\|\Psi_{\tilde{Q}}(m,p,\omega)x(\omega)\|d\mu(\omega)\right)\\
&\quad\leq\frac{1}{2}\sum_{j=p}^{p+1}M\left(\frac{h(j)}{h(p)}\right)^{\alpha}\left(\int_\Omega\|\Psi_{\tilde{Q}}(m,j,\varphi(j,p,\omega))x(\omega)\|d\mu(\omega)\right)\\
&\quad\leq\frac{M}{2}\left(\frac{h(p)}{h(n)}\right)^{d}\sum_{j=p}^{p+1}\left(\frac{h(j)}{h(p)}\right)^{\alpha+d}\left(\frac{h(j)}{h(n)}\right)^{-d}\left(\int_\Omega\|\Psi_{\tilde{Q}}(m,j,\varphi(j,p,\omega))x(\omega)\|d\mu(\omega)\right)\\
&\quad\leq\frac{M}{2}H_2^{\alpha+d}\left(\frac{h(p)}{h(n)}\right)^{d}\sum_{j=p}^{p+1}\left(\frac{h(j)}{h(n)}\right)^{-d}\left(\int_\Omega\|\Psi_{\tilde{Q}}(m,j,\varphi(j,p,\omega))x(\omega)\|d\mu(\omega)\right)\\
&\quad\leq MDH_2^{\alpha+d}\left(\frac{h(p)}{h(n)}\right)^{d}\int_\Omega\|\tilde{Q}(m,\varphi(m,p,\omega))x(\omega)\|d\mu(\omega).
\end{aligned}$$

Case $I.2$. $n\in[p,p+1)$.

$$\int_\Omega\|\Psi_{\tilde{Q}}(m,p,\omega)x(\omega)\|d\mu(\omega)$$

$$\le M\left(\frac{h(m)}{h(p)}\right)^{\alpha}\int_{\Omega}\|\tilde{Q}(m,\varphi(m,p,\omega))x(\omega)\|d\mu(\omega)$$
$$\le \left(\frac{h(p)}{h(m)}\right)^{d}\left(\frac{h(n)}{h(p)}\right)^{\alpha+d}\int_{\Omega}\|\tilde{Q}(m,\varphi(m,p,\omega))x(\omega)\|d\mu(\omega)$$
$$\le MH_2^{\alpha+d}\left(\frac{h(p)}{h(m)}\right)^{d}\int_{\Omega}\|\tilde{Q}(m,\varphi(m,p,\omega))x(\omega)\|d\mu(\omega).$$

From Case $I.1.$ with Case $I.2.$, it results that there exist $N = 1+MH_2^{\alpha+d}D$ and $\nu = d$ such that (uhd_2m) holds for all $(m,p,\omega)\in\tilde{\Delta}\times\Omega$ and all $x\in L(\Omega,X,\mu)$. Hence, we have shown that $(\tilde{C},\tilde{P})$ is u.h.d.m., completing the proof. □

Corollary 4. *We suppose that $\tilde{C} = (\Phi,\varphi)$ be a strongly measurable stochastic skew-evolution semiflow, $(\tilde{C},\tilde{P})$ with strong uniform exponential growth in mean. The pair $(\tilde{C},\tilde{P})$ is uniformly exponentially dichotomic in mean if and only if there exist $D\ge 1$ and $d\in(0,1)$ with*

$(\mathbf{ueD_1^4m_d})$ $$\sum_{j=p}^{m}\frac{e^{-dj}}{\int_{\Omega}\|\Phi_{\tilde{P}}(j,p,\omega)x(\omega)\|d\mu(\omega)}\le\frac{D\,e^{-dm}}{\int_{\Omega}\|\Phi_{\tilde{P}}(m,p,\omega)x(\omega)\|d\mu(\omega)},$$
for all $(m,n,p,\omega)\in\tilde{T}\times\Omega$ and $x\in L(\Omega,X,\mu)$,
with $\int_{\Omega}\|\Phi_{\tilde{P}}(m,p,\omega)x(\omega)\|d\mu(\omega)\neq 0.$

$(\mathbf{ueD_2^4m_d})$
$$\sum_{j=p}^{m}\frac{e^{dj}}{\int_{\Omega}\|\Psi_{\tilde{Q}}(m,j,\varphi(j,p,\omega))x(\omega)\|d\mu(\omega)}\le\frac{D\,e^{dm}}{\int_{\Omega}\|\Psi_{\tilde{Q}}(m,p,\omega)x(\omega)\|d\mu(\omega)},$$
for all $(m,n,p,\omega)\in\tilde{T}\times\Omega$ and $x\in L(\Omega,X,\mu)$,
with $\int_{\Omega}\|\Psi_{\tilde{Q}}(m,p,\omega)x_0(\omega)\|d\mu(\omega)\neq 0.$

Proof. It follows from Theorem 6 for $h(m) = e^m$. □

CONCLUSIONS

This study broadens several well-known results on uniform exponential dichotomy in mean by applying them to the more general framework of uniform h-dichotomy in mean. Integral characterizations for this generalized concept are provided for discrete-time stochastic skew-evolution semiflows, considering both invariant projection families and strongly invariant projection families. From these characterizations, necessary and sufficient conditions are derived for the specific case of uniform exponential dichotomy in mean. Future research will focus on unifying the analysis of discrete and continuous asymptotic properties for stochastic skew-evolution semiflows. Additionally, efforts will be directed toward extending these findings to the nonuniform case.

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A REMARK ON DIMENSIONALITY REDUCTION IN DISCRETE SUBGROUPS OF $\mathbb{R}^d$

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Abstract. In this short note, we prove a version of the Johnson-Lindenstrauss flattening Lemma for point sets taking values in discrete subgroups. More precisely, given $d, \lambda_0, N_0 \in \mathbb{N}$ and $\varepsilon \in \left(0, \frac{1}{2}\right)$ suitably chosen, we show there exists a natural number $k = k(d, \varepsilon) = O\left(\frac{1}{\varepsilon^2}\log d\right)$, such that for every sufficiently large scaling factor $\lambda \in \mathbb{N}$ and any point set $\mathcal{D} \subset \frac{\lambda}{\lambda_0}\mathbb{Z}^d \cap B(0, \lambda N_0)$ with cardinality d, there exists an embedding $F : \mathcal{D} \to \frac{1}{\lambda_0}\mathbb{Z}^k$, with distortion at most $\left(1 + \varepsilon + \frac{\varepsilon}{\lambda\lambda_0}\right)$.

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1. INTRODUCTION

The renowned Johnson-Lindenstrauss Lemma [4,7,8] (JL-Lemma for short) establishes that, given a point set $\mathcal{D} = \{x_1, \ldots, x_d\} \subset \mathbb{R}^d$ and a positive number $\varepsilon \in (0,1)$, there exists a (linear) embedding $\Phi : \mathbb{R}^d \to \mathbb{R}^k$, where $k = k(d, \varepsilon) = O(\log d/\varepsilon^2)$, that maps $\mathcal{D}$ into $\mathbb{R}^k$ with distortion at most $(1+\varepsilon)$. Although Johnson and Lindenstrauss proved their lemma to tackle a problem concerning extensions of Lipschitz maps, the computer and data science communities realized the potential of this lemma in reducing the dimension of high-dimensional data while preserving its key features up to a constant multiplicative error ~ 1; we refer the reader to [1,3,5] and the references therein for a broader discussion.

Albeit JL-Lemma has become a powerful tool for "flattening" high-dimensional data, represented by vectors in $\mathbb{R}^d$ for some $d \gg 1$, without distorting the distances too much, in the author's opinion a more realistic scenario for a computer model-space of d-dimensional vectors is the set $\frac{1}{\lambda_0}\mathbb{Z}^d \cap B_{N_0}$, where λ_0, N_0 are fixed positive integers and B_{N_0} denotes the euclidean ball with radius N_0 centered at the origin; this is because we cannot consider vectors with arbitrarily large entries (in absolute value) or as many decimals as we want. Thus, an interesting question is, fixed some suitable error term $\varepsilon \in (0,1)$, whether we can reduce the number of variables of data in $\frac{1}{\lambda_0}\mathbb{Z}^d \cap B_{N_0}$ by embedding them into the grid $\frac{1}{\lambda_0}\mathbb{Z}^k \cap B_N$ in the same spirit as JL-Lemma, for some positive integer N, and $k \lesssim_\varepsilon \log d$.

A naive approach is to proceed as follows: after applying the JL-Lemma to a data point set $\mathcal{D} \subset \frac{1}{\lambda_0}\mathbb{Z}^d \cap B_{N_0}$ with $|\mathcal{D}| = d$, we obtain a point set $\mathcal{D}_{\mathsf{flat}} = \{y_1, \ldots, y_k\} \subset \mathbb{R}^k$, where k is given as in the conclusions of JL-Lemma; then define $\widetilde{\mathcal{D}}_{\mathsf{flat}} = \{\widetilde{z}_1, \ldots, \widetilde{z}_d\}$, where $\widetilde{z}_i$ is the closest point of $\frac{1}{\lambda_0}\mathbb{Z}^k$ from y_i. A priori, for any $\lambda > 0$ we can only ensure that

$$d\left(\lambda \mathcal{D}_{\mathsf{flat}}, \frac{1}{\lambda_0}\mathbb{Z}^k\right) \leq \frac{\sqrt{k}}{\lambda_0}.$$

Even in the best case, by uniform distribution modulo 1 (see the proof of Lemma 2 below), we could find a sequence of positive integers $(n_l)_{l\geq 1}$ such that

$$(\forall l \geq 1): \qquad d\left(n_l \mathcal{D}_{\mathsf{flat}}, \frac{1}{\lambda_0}\mathbb{Z}^k\right) < \varepsilon;$$

this means that (a priori) only a subsequence of dilations of $\mathcal{D}_{\mathsf{flat}}$ are close-enough to $\frac{1}{\lambda_0}\mathbb{Z}^k$. In view of the previous discussion, we aim to prove the following: given $d, \lambda_0 \in \mathbb{N}$ and $\varepsilon \in (0, 1/2)$ appropriately chosen, then *every* sufficiently separated point set in $\frac{1}{\lambda_0}\mathbb{Z}^d$ can be flattened into a k-dimensional sub-lattice $\frac{1}{\lambda_0}\mathbb{Z}^k$ with distortion $\sim_{\varepsilon,\lambda_0} 1$ (i.e., with distortion factor close to 1 and depending on ε and λ_0), for some $k = k(d,\varepsilon) \ll d$.

Proposition 1 (Main Proposition). *Let $d, \lambda_0, N_0 \in \mathbb{N}$ and $\varepsilon \in \left(0, \frac{1}{\lambda_0+1}\right)$ be given. There exists $c = c(\varepsilon) > 0$ such that the following holds for every positive integer $k \geq c \log d$: there is a scaling factor $\lambda_1 = \lambda_1(\lambda_0, \varepsilon, k, N_0) \in \mathbb{N}$ such that for every $\lambda \geq \lambda_1$ and for every point set $\mathcal{D} \subset \frac{\lambda}{\lambda_0}\mathbb{Z}^d \cap B_{\lambda N_0}$ with $|\mathcal{D}| = d$, there is a mapping $F : \mathcal{D} \to \frac{1}{\lambda_0}\mathbb{Z}^k$ such that*

$$(\forall x, y \in \mathcal{D}): \qquad \left(1 - \varepsilon - \frac{\varepsilon}{\lambda\lambda_0}\right)\|x-y\| \leq \|F(x) - F(y)\| \leq \left(1 + \varepsilon + \frac{\varepsilon}{\lambda\lambda_0}\right)\|x-y\|. \tag{1.1}$$

This version of JL-Lemma has the advantage that, after rescaling a data set by λ, we can reduce its dimension with small distortion while keeping the number of decimals and the magnitude of the flattened data bounded.

Besides the JL-Lemma itself, the following remarkable Theorem due to Tamar Ziegler [9, Theorem 1.3] (see also [2] for an elementary proof in the case of even dimensions) plays a crucial role in the proof of the Proposition 1.

Theorem 1 (Ziegler's Theorem). *Let $\mathcal{D} := \{x_1, \ldots, x_n\}$ be a set of n vectors in $\mathbb{R}^d$ and $\varepsilon > 0$ be given. Then there is $l_0 = l_0(\varepsilon, \mathcal{D}) > 0$ such that for any $l \geq l_0$, there* *exists a rotation* $\rho = \rho(l) \in SO(d)$ *satisfying that*

$$(\forall i = 1, \dots, n): \qquad d(\rho(l \cdot x_i), \mathbb{Z}^d) \leq \varepsilon.$$

2. Proof of Main Proposition

The following result is a key ingredient to determine the dependence of the parameter λ.

Lemma 1. *Let* $t \in (0,1) \setminus \mathbb{Q}$*,* $N \in \mathbb{N}$*, and* $\varepsilon > 0$ *be fixed. Then there exists* $\lambda_1 = \lambda_1(t, \varepsilon, N) \in \mathbb{N}$ *such that for every* $\lambda \geq \lambda_1$ *and* $\mathcal{D} \subset t\mathbb{Z}^k \cap B_N$*, there must exist a rotation* $\rho = \rho(\lambda) \in SO(k)$ *such that*

$$d(\rho(\lambda\mathcal{D}), \mathbb{Z}^k) < \varepsilon.$$

Proof. Since $t \in (0,1) \setminus \mathbb{Q}$, the sequence $(nt)_{n \geq 1}$ is uniformly distributed modulo 1 (see for instance [6]); in particular, given $\varepsilon > 0$, there exist $n_1 = n_1(\varepsilon, N) \in \mathbb{N}$ and $p \in \mathbb{Z}$ such that

$$|n_1 t - p| < \frac{\varepsilon}{N} \leq \varepsilon.$$

Thus, for every $l = 1, \dots, k$ and $q = (q_1, \dots, q_k) \in \mathbb{Z}^k \cap B_N$, we obtain:

$$|n_1 t q_l - p q_l| < \frac{\varepsilon |q_l|}{N} \leq \varepsilon. \tag{2.1}$$

Hence from (2.1), for every subset $\mathcal{D}$ of $t\mathbb{Z}^k \cap B_N$ there holds that $d(n_1\mathcal{D}, \mathbb{Z}^k) < \varepsilon$. The rest of the proof follows the very same lines as in [2]. ☐

Proof of the Proposition 1. Let $\mathcal{D}_{\mathsf{flat}} := \Phi(\mathcal{D})$, where $\Phi : \mathbb{R}^d \to \mathbb{R}^k$ is the linear embedding given by the Johnson-Lindenstrauss Lemma, and write $y_i := \Phi(x_i)$ where $\mathcal{D} = \{x_1, \dots, x_d\}$; the proof is quite direct if k is a perfect square, and so we assume that $\sqrt{k}$ is an irrational number. Since a translation by a vector is an isometry, we can assume that the origin of $\mathbb{R}^k$ is the circumcenter of $\mathcal{D}_{\mathsf{flat}}$. Moreover, by [7, Theorem 3.1] (see also [8, Theorem 1.35]), we can consider $\mathcal{D}_{\mathsf{flat}}$ as a subset of $\frac{1}{\lambda_0\sqrt{k}}\mathbb{Z}^k$, since Φ takes the form:

$$\Phi(x) = \frac{1}{\sqrt{k}} R x^{\mathsf{T}},$$

where R is a $d \times k$ (random) matrix with entries taking values in $\{0,1\}$. In particular, by linearity, for every $t > 0$ we have that

$$t\mathcal{D}_{\mathsf{flat}} = \Phi(t\mathcal{D}),$$

and in consequence, we get that $\Phi \circ \mathsf{dil}_t : \mathcal{D} \to t\mathcal{D}_{\mathsf{flat}} \subset \mathbb{R}^k$ is an $(1+\varepsilon)$-embedding, where dil_t stands for the dilation by t.

By Theorem 1 and Lemma 2, there exists a scaling factor $\lambda_1 = \lambda_1(\varepsilon, k, N_0) \in \mathbb{N}$ such that for every $\lambda \geq \lambda_1$ there is a rotation $\rho = \rho(\lambda) \in SO(k)$, such that

$$d\left(\rho(\lambda\mathcal{D}_{\mathsf{flat}}), \frac{1}{\lambda_0}\mathbb{Z}^k\right) \leq \frac{\varepsilon}{\lambda_0} \leq \frac{\varepsilon}{2},$$

and thus for each $i \in \{1,\dots,d\}$, there exists $z_i \in \mathbb{Z}^k$ such that

$$\left\| \rho(\lambda y_i) - \frac{1}{\lambda_0} z_i \right\| < \frac{\varepsilon}{2}. \tag{2.2}$$

Firstly we shall prove that for $i \neq j$, there holds that $z_i \neq z_j$; indeed, by (2.2), the linearity of Φ, the triangle inequality, the definition of $\mathcal{D}_{\mathtt{flat}}$, and since the x_i's belong to $\frac{1}{\lambda_0}\mathbb{Z}^d$, we have that

$$\frac{(1-\varepsilon)}{\lambda_0} \leq (1-\varepsilon)\|x_i - x_j\| \leq \|y_i - y_j\| \leq \frac{\varepsilon}{\lambda} + \frac{1}{\lambda\lambda_0}\|z_i - z_j\|,$$

and thus, by the choice of ε, we have that

$$\frac{1}{\lambda\lambda_0}\|z_i - z_j\| \geq \frac{1}{\lambda_0} - \varepsilon\left(1 + \frac{1}{\lambda_0}\right) > 0.$$

Now we claim that the mapping $F : \lambda x_i \mapsto \frac{1}{\lambda_0} z_i$ verifies (1.1). By triangle inequality, the fact the JL-embedding Φ has distortion at most $(1+\varepsilon)$, and that the x_i's belong to $\frac{1}{\lambda_0}\mathbb{Z}^d$, and since $\rho \in SO(k)$ is an isometry, we get

$$\begin{aligned} \frac{1}{\lambda_0}\|z_i - z_j\| &\leq \varepsilon + \|\rho(\lambda y_i) - \rho(\lambda y_j)\| \leq \varepsilon + \lambda(1+\varepsilon)\|x_i - x_j\| \\ &\leq \left(1 + \varepsilon + \frac{\varepsilon}{\lambda\lambda_0}\right)\|\lambda x_i - \lambda x_j\|. \end{aligned} \tag{2.3}$$

Analogously, we have that

$$\left(1 - \varepsilon - \frac{\varepsilon}{\lambda\lambda_0}\right)\|x_i - x_j\| \leq \frac{1}{\lambda_0}\|z_i - z_j\|. \tag{2.4}$$

Therefore, (1.1) follows from (2.3) and (2.4). This finishes the proof. □

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DYNAMICS OF TRAVELING WAVE SOLUTIONS AND CHAOS FOR GENERALIZED KDV-BURGERS EQUATION

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Abstract. This paper applies the qualitative theory of differential equations to explore the global structure of the generalized KdV-Burgers equation, including a complete description of the phase portrait at infinity and the amplitude estimation of the oscillating shock wave. And based on KCC theory, the deviation curvature tensor of the generalized KdV-Burgers equation is given. After determining the Jacobi stability at any point on the traveling wave solution, this paper applies the Melnikov method to further discuss the chaotic behavior when the dissipative term receiving periodic perturbations.

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Keywords: generalized KdV-Burgers equation, dynamics, chaos, Kosambi-Cartan-Chern (KCC) theory, Melnikov method

1. INTRODUCTION

For nonlinear partial differential equations, it is very difficult to find its analytical solutions. A common way to analysis the partial differential equation is to transform the equation into two-dimensional ordinary differential equations through traveling wave transformation, and then analyze the traveling wave solutions of the equivalent system. Traveling wave solutions play a crucial role in nonlinear science, as they can effectively describe various natural phenomena such as vibrations [4] and propagating waves [20].

In this paper, we investigate the dynamics of the generalized KdV-Burgers equation in the following form [24].

$$\gamma\frac{\partial^3 u}{\partial x^3}+\beta\frac{\partial^2 u}{\partial x^2}+\alpha u^n\frac{\partial u}{\partial x}+\frac{\partial u}{\partial t}=0, \tag{1.1}$$

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where $\gamma \neq 0$ is the dispersion coefficient, β is the dissipation coefficient and $\alpha \neq 0$ is the nonlinear term. This equation with $n \geq 1$ arises in modeling waves generated by a wavemaker in a channel and the waves incoming from deep water into nearshore zones [18]. In fact, if one takes different values for α, β, γ, n, the equation (1.1) includes many equations. For example, the KdV-Burgers equation with $n = 1$ [1], the Gardner equation with $n = 2$ [8] and the generalized KdV equation with a closed orbit when $\beta = 0$ [6].

The generalized KdV-Burgers equation also has two different traveling wave solutions. One of which is a monotonic shock wave [11]. The other one is an oscillating shock wave which is difficult to calculate the exact solution. Therefore, further research on the oscillation behavior of the oscillating shock wave is needed. This paper further explores the case of $n = 1$. The generalized KdV-Burgers equation is a more general case, leading to more general conclusions.

KCC theory (or Jacobi analysis) [2, 5, 13] is a geometric method for studying the stability of dynamic systems. As one of the KCC invariants, deviation curvature tensor is used to obtain Jacobi stability of the second-order ordinary differential equation. Nowadays, KCC theory has been applied to many three-dimensional systems, such as Rabinovich system [16], Yang-Chen system [17], Chen system [12, 19] and other unusual systems [3, 7, 14, 22, 23]. In these literatures, there is a close connection between the deviating curvature tensor and the chaotic behavior of trajectories. Besides, KCC theory has also been applied in traveling wave solutions [15]. This provides a new point for studying the dynamic behavior of traveling wave solutions. It helps to analyze the stability of the traveling wave solution at any point.

The rest of this paper is organized as follows. In Sec. 2, dynamics of the equivalent plane system near equilibria and at infinity is discussed. A more accurate amplitude estimation of the oscillating shock wave solution is given for (1.1) and the global structure diagram is presented. In Sec. 3, Jacobi stability and the deviation vector of the generalized KdV-Burgers equation is discussed. In Sec. 4, the chaotic behavior of traveling wave solutions when the dissipative term receiving periodic perturbations is analytically confirmed by Melnikov method. Conclusions are given in the last section.

2. Global analysis

This section will analyse the global dynamic behavior of the traveling wave solutions of the generalized KdV-Burgers equation.

2.1. *equilibrium points*

By traveling wave transformation $\zeta = x - ct$, $c \neq 0$, integrating both sides and making the constant of integration zero, the generalized KdV-Burgers equation becomes

$$\gamma u_{2\zeta} + \beta u_{\zeta} + \frac{\alpha}{n+1} u^{n+1} - cu = 0. \tag{2.1}$$

One can formalize (2.1) into the equivalent planar system.

$$\frac{du}{d\zeta} = v,$$
$$\frac{dv}{d\zeta} = -\frac{\beta}{\gamma}v - \frac{\alpha}{(n+1)\gamma}u^{n+1} + \frac{c}{\gamma}u. \tag{2.2}$$

When n is an odd number, the plane system (2.2) has two equilibrium points, $E_0 = (0,0)$ and $E_1 = \left(\frac{2c}{\alpha}, 0\right)$. If n is an even number, there has one equilibrium point $E_0 = (0,0)$ for $\frac{c}{\alpha} > 0$. For $\frac{c}{\alpha} < 0$, there has three equilibrium points $E_0 = (0,0)$, $E_1\left(\left(\frac{(n+1)c}{\alpha}\right)^{\frac{1}{n}}, 0\right)$ and $E_2\left(-\left(\frac{(n+1)c}{\alpha}\right)^{\frac{1}{n}}, 0\right)$.

The Jacobi matrix corresponding to E_0 is

$$J(E_0) = \begin{pmatrix} 0 & 1 \\ \frac{c}{\gamma} & -\frac{\beta}{\gamma} \end{pmatrix},$$

and the eigenvalues are $\lambda_{1,2} = \frac{\beta}{2\gamma} \pm \frac{\sqrt{\beta^2+4c\gamma}}{2|\gamma|}$.

Similarly, the system (2.2) has the same Jacobi matrixes at E_1 and E_2.

$$J(E_1) = J(E_2) = \begin{pmatrix} 0 & 1 \\ -\frac{nc}{\gamma} & -\frac{\beta}{\gamma} \end{pmatrix},$$

and the eigenvalues are $\lambda_{3,4} = \frac{\beta}{2\gamma} \pm \frac{\sqrt{\beta^2-4c\gamma}}{2|\gamma|}$.

Local phase diagrams of the equilibrium points of (2.2) are depicted in Fig. 1.

If n is an even number, and $\frac{c}{\alpha} > 0$, system (2.2) has three equilibrium points E_0 and $E_{1,2}$, their stability is shown as the table 1.

If n is an even number, and $\frac{c}{\alpha} < 0$, system (2.2) has only one equilibrium point E_0, the stability is shown as the table 2.

If n is an odd number, system (2.2) has two equilibrium points E_0 and E_1, their stability conclusions are consistent with Table 1. But there is no equilibrium point E_2.

TABLE 1. Parameter Condition and Equilibrium Point Type for $\frac{c}{\alpha} > 0$

Condition			E_0	$E_{1,2}$
$\gamma > 0$	$c > 0$	$\beta \geq 2\sqrt{nc\gamma}$	saddle point	stable node
		$0 < \beta < 2\sqrt{nc\gamma}$		stable focus
		$\beta = 0$		center
		$-2\sqrt{nc\gamma} < \beta < 0$		unstable focus
		$\beta \leq -2\sqrt{nc\gamma}$		unstable node
	$c < 0$	$\beta \geq 2\sqrt{-c\gamma}$	stable focus	saddle point
		$0 < \beta < 2\sqrt{-c\gamma}$	stable node	
		$\beta = 0$	center	
		$-2\sqrt{-c\gamma} < \beta < 0$	unstable node	
		$\beta \leq -2\sqrt{-c\gamma}$	unstable focus	
$\gamma < 0$	$c > 0$	$\beta \geq 2\sqrt{-c\gamma}$	unstable node	saddle point
		$0 < \beta < 2\sqrt{-c\gamma}$	unstable focus	
		$\beta = 0$	center	
		$-2\sqrt{-c\gamma} < \beta < 0$	stable focus	
		$\beta \leq -2\sqrt{c\gamma}$	stable node	
	$c < 0$	$\beta \geq 2\sqrt{nc\gamma}$	saddle point	unstable focus
		$0 < \beta < 2\sqrt{nc\gamma}$		unstable node
		$\beta = 0$		center
		$-2\sqrt{nc\gamma} < \beta < 0$		stable node
		$\beta \leq -2\sqrt{nc\gamma}$		stable focus

2.2. *Dynamics at infinity*

In this subsection, one will discuss dynamics at infinity and give the global structure diagram of the generalized KdV-Burgers equation.

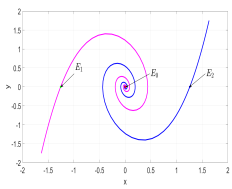
For the limit cycle, it can be proved that there is no limit cycle in the system (2.2). Due to the equation system (2.1), one has

$$v_u + \left(-\frac{\beta}{\gamma}v + \frac{c}{\gamma}u - \frac{\alpha}{(n+1)\gamma}u^{n+1}\right)_v = -\frac{\beta}{\gamma}.$$

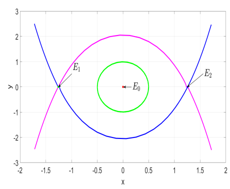
According to the Bendixson criterion, when $\beta \neq 0$, the system does not have any closed orbits. Thus, there is no limit cycle.

TABLE 2. Equilibrium Point Type for $\frac{c}{\alpha} < 0$

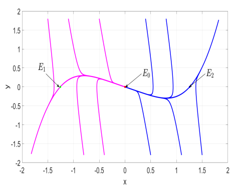
Condition			E_0
$\gamma > 0$	$c > 0$	For any β	saddle point
	$c < 0$	$\beta \geq 2\sqrt{-c\gamma}$	stable focus
		$0 < \beta < 2\sqrt{-c\gamma}$	stable node
		$\beta = 0$	center
		$-2\sqrt{-c\gamma} < \beta < 0$	unstable node
		$\beta \leq -2\sqrt{-c\gamma}$	unstable focus
$\gamma < 0$	$c > 0$	$\beta \geq 2\sqrt{-c\gamma}$	unstable node
		$0 < \beta < 2\sqrt{-c\gamma}$	unstable focus
		$\beta = 0$	center
		$-2\sqrt{-c\gamma} < \beta < 0$	stable focus
		$\beta \leq -2\sqrt{c\gamma}$	stable node
	$c < 0$	For any β	saddle point



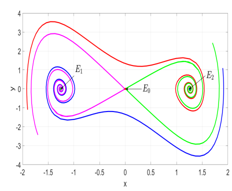
(a) $\alpha = 8, \beta = -3.26, \gamma = 1, c = 4$



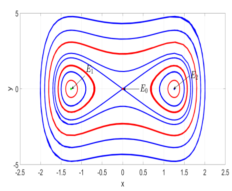
(b) $\alpha = 8, \beta = 0, \gamma = 1, c = 4$



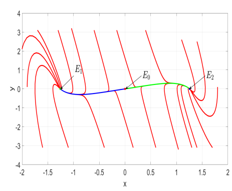
(c) $\alpha = 8, \beta = -4.62, \gamma = 1, c = 4$



(d) $\alpha = 8, \beta = -3.26, \gamma = -1, c = 4$



(e) $\alpha = 8, \beta = 0, \gamma = -1, c = 4$



(f) $\alpha = 8, \beta = -4.62, \gamma = -1, c = 4$

FIGURE 1. (a) Saddle point - focus - Saddle point (b) Saddle point - center - Saddle point (c) Saddle point - node - Saddle point (d) focus - Saddle point - focus (e) center - Saddle point - center (f) node - Saddle point - node

For the infinite singularities, Poincaré Compaction Technology is a common tool to analyze it.

Making $u = \frac{1}{z}$, $v = \frac{w}{z}$, $dT = \frac{d\zeta}{z^n}$, system (2.2) becomes

$$\begin{aligned} \frac{dz}{dT} &= -z^{n+1}w, \\ \frac{dw}{dT} &= -w^2 z^n - \frac{\beta}{\gamma} w z^n - \frac{\alpha}{(n+1)\gamma} + \frac{c z^n}{\gamma}. \end{aligned} \tag{2.3}$$

When $z = 0$, the system (2.3) has no equilibrium point.

Let $u = \frac{w}{z}$, $v = \frac{1}{z}$, $dT = \frac{d\zeta}{z}$, and $z' = z^n$, it is obtained that

$$\begin{aligned} \frac{dz'}{dT} &= \frac{z'^2 n\beta}{\gamma} + \frac{w^{n+1} z' n\alpha}{(n+1)\gamma} - \frac{cnwz'^2}{\gamma}, \\ \frac{dw}{dT} &= \frac{wz'\beta}{\gamma} + \frac{w^{n+2}\alpha}{(n+1)\gamma} - \frac{cw^2 z'}{\gamma} + z'. \end{aligned} \tag{2.4}$$

The origin is an equilibrium point of the system (2.4). Suppose

$$p_2 = \frac{wz'\beta}{\gamma} + \frac{w^{n+2}\alpha}{(n+1)\gamma} - \frac{cw^2 z'}{\gamma}, q_2 = \frac{z'^2 n\beta}{\gamma} + \frac{w^{n+1} z' n\alpha}{(n+1)\gamma} - \frac{cnwz'^2}{\gamma}.$$

Assuming that the equation

$$z + p_2 = 0 \text{ has the solution } z = \varphi(w) = c_1 w + c_2 w^2 + \cdots,$$

one has

$$c_1 w + \left(c_2 + \frac{\beta}{\gamma} c_1\right) w^2 + \left(c_3 + \frac{\beta}{\gamma} c_2 - \frac{c}{\gamma} c_1\right) w^3 + \cdots = 0.$$

One can get

$$c_1 = c_2 = c_3 = \cdots = c_{n+1} = 0, c_{n+2} = -\frac{\alpha}{(n+1)\gamma}.$$

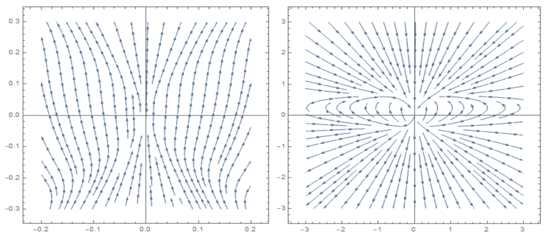
Thus, there is $z' = -\frac{\alpha}{(n+1)\gamma} w^{n+2} + \cdots$.

The equation (2.4) becomes

$$\frac{dz'}{dT} = \frac{z'^2 n\beta}{\gamma} + \frac{w^{n+1} z' n\alpha}{(n+1)\gamma} - \frac{cnwz'^2}{\gamma} = -\frac{n\alpha^2}{(n+1)^2\gamma^2} w^{2n+3} + \cdots.$$

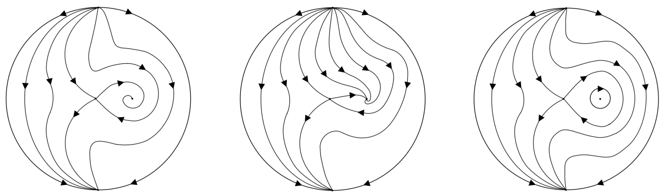
One obtains that

$$\begin{aligned} [(p_2)_w + (q_2)_{z'}] &= \frac{\beta}{\gamma} z' + \frac{(n+2)\alpha}{(n+1)\gamma} w^{n+1} - \frac{2c}{\gamma} wz' + \frac{2n\beta}{\gamma} z' + \frac{n\alpha}{(n+1)\gamma} w^{n+1} - \frac{2nc}{\gamma} wz' \\ &= \left(\frac{2\alpha}{\gamma}\right) w^{n+1} + \cdots, \end{aligned}$$



(a) (b)

FIGURE 2. For $\beta = 1, \gamma = 1, n = 4$, the phase diagrams near the singularity at infinity: (a)$c = 4$, $\alpha = 8$; (b)$c = -4$, $\alpha = -8$.



(a) $c > \frac{\beta^2}{4\gamma}$ (b) $c \leq \frac{\beta^2}{4\gamma}$ (c) $\beta = 0$

FIGURE 3. The global structure diagram of system (2.2) when n is an odd number.

and

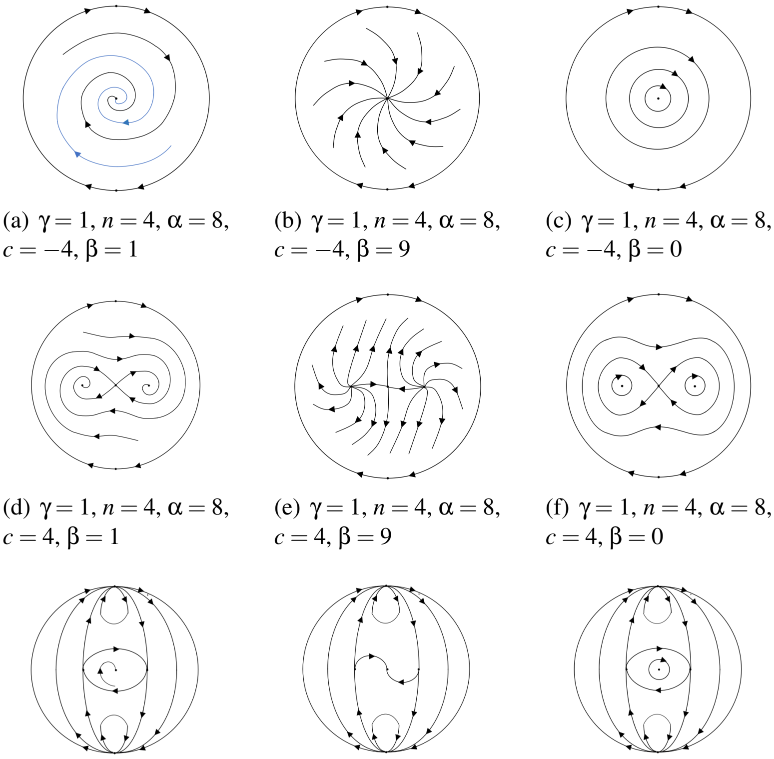
$$\lambda = b_{n+1}^2 + 4(m+1)a_{2n+3} = \frac{4\alpha^2}{\gamma^2}\left[1 - \frac{n(n+2)}{(n+1)^2}\right] > 0.$$

Therefore, the singularity is an unstable node when n is an odd number, as shown in Fig. 2. For n is an even number, the sufficient small domain of the singularity consists of a hyperbolic sector and an elliptical sector. If α and γ have the same sign, the hyperbolic sector is above the elliptical sector. Otherwise, the elliptical sector is above the hyperbolic sector. Due to n being an even number, the image below the V-axis is symmetrical from the image above the V-axis.

Combining the conclusions of Section 2.2 and Theorem 1, the global phase diagram corresponding to each situation is shown in Fig. 3 and Fig. 4.

2.3. *Amplitude estimation*

In order to understand the dynamic behavior of the generalized KdV-Burgers equation, one further analyzes the amplitude of the oscillating shock wave.



(a) $\gamma = 1, n = 4, \alpha = 8, c = -4, \beta = 1$

(b) $\gamma = 1, n = 4, \alpha = 8, c = -4, \beta = 9$

(c) $\gamma = 1, n = 4, \alpha = 8, c = -4, \beta = 0$

(d) $\gamma = 1, n = 4, \alpha = 8, c = 4, \beta = 1$

(e) $\gamma = 1, n = 4, \alpha = 8, c = 4, \beta = 9$

(f) $\gamma = 1, n = 4, \alpha = 8, c = 4, \beta = 0$

(g) $\gamma = 1, n = 4, \alpha = -8, c = -4, \beta = 1$

(h) $\gamma = 1, n = 4, \alpha = -8, c = -4, \beta = 9$

(i) $\gamma = 1, n = 4, \alpha = -8, c = -4, \beta = 0$

FIGURE 4. The global structure diagram of system (2.2) when n is an even number.

For convenience, one fixes $\gamma > 0, \beta > 0,\ c > \frac{\beta^2}{4n\gamma}$ and transfers the focus to the origin. Let $v(s) = u(\zeta) - \left(\frac{(n+1)c}{\alpha}\right)^{\frac{1}{n}}$, we have

$$\gamma v_{2s} + \beta v_s + \frac{\alpha}{n+1}\left(v + \left(\frac{(n+1)c}{\alpha}\right)^{\frac{1}{n}}\right)^{n+1} - c\left(v + \left(\frac{(n+1)c}{\alpha}\right)^{\frac{1}{n}}\right) = 0. \tag{2.5}$$

Let expand $\left(v + \left(\frac{(n+1)c}{\alpha}\right)^{\frac{1}{n}}\right)^{n+1}$,

$$\left(v + \left(\frac{(n+1)c}{\alpha}\right)^{\frac{1}{n}}\right)^{n+1} = v^{n+1} + \cdots + C_{n+1}^{n} v \frac{(n+1)c}{\alpha} + \left(\frac{(n+1)c}{\alpha}\right)^{\frac{n+1}{n}}. \tag{2.6}$$

Substitute equation (2.6) into equation (2.5),

$$\gamma v_{2s}+\beta v_s+\frac{\alpha}{n+1}f(v)+ncv=0, \tag{2.7}$$

where

$$f(v)=v^{n+1}+\cdots+C_{n+1}^{n-1}v^2\frac{(n+1)c}{\alpha}^{\frac{n+1}{n-1}}+\left(\frac{(n+1)c}{\alpha}\right)^{\frac{n+1}{n}}<\left(v+\left(\frac{(n+1)c}{\alpha}\right)^{\frac{1}{n}}\right)^{n+1}.$$

Then $v(s)$ has a series of peaks and valleys. So one can set the first and largest wave peak u_0 and obtain the initial value condition

$$\begin{cases} v(0) & =u_0-\left(\frac{(n+1)c}{\alpha}\right)^{\frac{1}{n}}, \\ v_s(0) & =0. \end{cases} \tag{2.8}$$

Therefore, equation (1.1) becomes an initial value problem

$$\begin{cases} \gamma v_{2s}+\beta v_s+ncv & =-\frac{\alpha}{n+1}f(v), \\ v(0) & =u_0-\left(\frac{(n+1)c}{\alpha}\right)^{\frac{1}{n}}, v_s(0)=0. \end{cases} \tag{2.9}$$

On the one hand, the problem

$$\begin{cases} \gamma w^{oo}(s)_{2s}+\beta w^{oo}(s)_s+ncw & =0, \\ w^{oo}(s)(0) & =u_0-\left(\frac{(n+1)c}{\alpha}\right)^{\frac{1}{n}}, w_s^{oo}(0)=0, \end{cases} \tag{2.10}$$

has a solution $w^o(s)$. Because of the parameter condition $c>\frac{\beta^2}{4n\gamma}$, the solution is

$$w^{oo}(s)=e^{-\frac{\beta}{2\gamma}s}\left[(u_0-(\frac{(n+1)c}{\alpha})^{\frac{1}{n}})\cos\frac{\sqrt{4nc\gamma-\beta^2}}{2\gamma}s\right.$$
$$\left.+\left(\frac{\beta(u_0-\left(\frac{(n+1)c}{\alpha}\right)^{\frac{1}{n}})}{\sqrt{4nc\gamma-\beta^2}}\right)\sin\frac{\sqrt{4nc\gamma-\beta^2}}{2\gamma}s\right].$$

On the other hand, the nonlinear initial value problem

$$\begin{cases} \gamma w_{2s}^{**}+\beta w_s^{**}+cw^{**} & =0, \\ w^{**}(0)=0, w_s^{**}(0) & =-\frac{\alpha}{n+1}f(v), \end{cases} \tag{2.11}$$

has the solution

$$w^{**}(s,\iota)=-e^{-\frac{\beta}{2\gamma}(s-\iota)}\frac{2\alpha\gamma}{(n+1)\sqrt{4nc\gamma-\beta^2}}f(v(\iota))\sin\frac{\sqrt{4nc\gamma-\beta^2}}{2\gamma}(s-\iota).$$

The solution of the initial value problem (2.9) is

$$w(s) = w^{oo}(s) + \int_0^s w^{**}(s,\iota)d\iota,$$

To obtain the upper bound of $w(s)$, one discusses the upper bound of $w^{oo}(s)$ and $\int_0^s w^{**}(s,\iota)d\iota$.

For $w^{oo}(s)$, one takes the derivative of the part within the parentheses,

$$-\left(u_0 - \left(\frac{(n+1)c}{\alpha}\right)^{\frac{1}{n}}\right)\frac{\sqrt{4nc\gamma-\beta^2}}{2\gamma}\sin\frac{\sqrt{4nc\gamma-\beta^2}}{2\gamma}s$$
$$+\beta\frac{u_0 - \left(\frac{(n+1)c}{\alpha}\right)^{\frac{1}{n}}}{2\gamma}\cos\frac{\sqrt{4nc\gamma-\beta^2}}{2\gamma}s = 0.$$

After simplification, it can be obtained that $\tan\frac{\sqrt{4nc\gamma-\beta^2}}{2\gamma}s = \frac{\beta}{\sqrt{4nc\gamma-\beta^2}}$.

From this, it can be obtained that

$$\sin\frac{\sqrt{4c\gamma-\beta^2}}{2\gamma}s = \frac{\beta}{\sqrt{4nc\gamma}} \text{ and } \cos\frac{\sqrt{4nc\gamma-\beta^2}}{2\gamma}s = \frac{\sqrt{4nc\gamma-\beta^2}}{\sqrt{4nc\gamma}}.$$

So the maximum value of $w^{oo}(s)$ is

$$|w^{oo}(s)| \le e^{-\frac{\beta}{2\gamma}s}\left[\left(u_0 - \left(\frac{(n+1)c}{\alpha}\right)^{\frac{1}{n}}\right)\frac{\sqrt{4nc\gamma-\beta^2}}{\sqrt{4nc\gamma}} + \frac{u_0 - \left(\frac{(n+1)c}{\alpha}\right)^{\frac{1}{n}}}{\sqrt{4nc\gamma-\beta^2}}\frac{\beta^2}{\sqrt{4nc\gamma}}\right]. \tag{2.12}$$

Let $f(v) < (v + (\frac{(n+1)c}{\alpha})^{\frac{1}{n}})^{n+1} < u_0^{n+1}$, the upper bound of $\int_0^s w^{**}(s,\iota)d\iota$ is

$$\int_0^s w^{**}(s,\iota)d\iota \le u_0^n\frac{2\alpha\gamma}{(n+1)\sqrt{4nc\gamma-\beta^2}}e^{-\frac{\beta}{2\gamma}s}\int_0^s |v(\iota)|e^{\frac{\beta}{2\gamma}\iota}d\iota. \tag{2.13}$$

According to the existence and uniqueness of the solution, one has

$$|w(s)| \equiv |v(s)| \le |w^{oo}(s)| + \int_0^s w^{**}(s,\iota)d\iota. \tag{2.14}$$

According to Gronwall inequality and returning to the original situation, one can get

$$|u(s)| \le \exp\left\{u_0^n\frac{4\alpha\gamma^2}{(n+1)\beta\sqrt{4nc\gamma-\beta^2}}\left[1-\exp\left(-\frac{\beta}{2\gamma}s\right)\right] - \frac{\beta}{2\gamma}s\right\}$$
$$\times\left[\frac{\left(u_0 - (\frac{(n+1)c}{\alpha})^{\frac{1}{n}}\right)\sqrt{4nc\gamma}}{\sqrt{4nc\gamma-\beta^2}}\right] + \left(\frac{(n+1)c}{\alpha}\right)^{\frac{1}{n}}.$$

The amplitude of the generalized KdV-Burgers equation is

$$|f(s)| \leq \exp\left\{u_0^n \frac{4\alpha\gamma^2}{(n+1)\beta\sqrt{4nc\gamma-\beta^2}}\left[1-\exp\left(-\frac{\beta}{2\gamma}s\right)\right]-\frac{\beta}{2\gamma}s\right\} \times\left[\frac{(u_0-\left(\frac{(n+1)c}{\alpha}\right)^{\frac{1}{n}})\sqrt{4nc\gamma}}{\sqrt{4nc\gamma-\beta^2}}\right].$$

It can be seen that the oscillating shock wave is exponentially stable, and the spiral will converge to the equilibrium point at this speed.

Remark 1. When n becomes a variable, the calculation process is different from the case when $n=1$. This paper makes changes in the selection of the maximum wave amplitude and calculates the amplitude estimation of the KdV-Burgers equation with n as a variable.

3. Jacobi analysis

In this section, Jacobi stability of the traveling wave solutions at any point of the trajectory is analysed on the basis of KCC theory. In addition, the deviation vector is calculated.

3.1. *KCC theory and Jacobi stability*

In the subsection, one introduces the basic concepts of the KCC theory, including nonlinear connection, Berwald connection, deviation curvature tensor and the definition of Jacobi stability, et.al. [9][21]. The Einstein summation convention is used throughout.

Let $(x^i)=(x^1,x^2,\ldots,x^n)\in\mathbf{R}^n$ and $(y^i)=(y^1,y^2,\ldots,y^n)\in\mathbf{R}^n$. The coordinates y^i are defined by $y^i=dx^i/d\zeta$. One considers second-order differential equations of the form

$$\frac{d^2x^i}{d\zeta^2}+2G^i(x,y,\zeta)=0, i=1,2,\cdots,n, \tag{3.1}$$

where $(x^i,y^i,\zeta)\in\Omega\subset\mathbf{R}^n\times\mathbf{R}^n\times\mathbf{R}$, Ω is an open connected set, and each function $G^i(x,y,\zeta)$ is C^∞ in a neighborhood at any point.

The coefficient of the nonlinear connection N_j^i is defined by $N_j^i=\partial G^i/\partial y^j$. The value of N are closely related to the linear stability of the equilibrium point [10].

One considers the disturbed trajectories $x^i(\zeta)$ of the system (3.1) as

$$\tilde{x}^i(\zeta)=x^i(\zeta)+\eta\xi^i(\zeta), \tag{3.2}$$

where $|\eta|$ is a small parameter, and $\xi^i(\zeta)$ are the components of a contravariant vector field defined along the path $x^i(\zeta)$. Substituting Eq. (3.2) into Eq. (3.1) and taking the limit $\eta \to 0$, one obtains the equation in the form

$$\frac{d^2\xi^i}{d\zeta^2} + 2N^i_j \frac{d\xi^j}{d\zeta} + 2\frac{\partial G^i}{\partial x^j}\xi^j = 0. \tag{3.3}$$

Converting to the KCC covariant differential, Eq. (3.3) becomes

$$\frac{D^2\xi^i}{d\zeta^2} = P^i_j \xi^j, \tag{3.4}$$

where

$$P^i_j = -2\frac{\partial G^i}{\partial x^j} - 2G^l G^i_{jl} + y^l \frac{\partial N^i_j}{\partial x^l} + N^i_l N^l_j + \frac{\partial N^i_j}{\partial \zeta}, \tag{3.5}$$

and $G^i_{jl} = \partial N^i_j / y^l$ is Berwald connection coefficients. The tensor P^i_j is called the second KCC invariant or the deviation curvature tensor. The trajectories of (3.1) are Jacobi stable if and only if the real parts of the eigenvalues of the deviation curvature tensor P^i_j are strictly negative everywhere. Otherwise, the trajectories are Jacobi unstable.

According to the formula (3.5), one can get KCC differential invariants:

$$N = \frac{\beta}{2\gamma},\ G = 0,\ P = -\frac{\alpha}{\gamma}u^n + \frac{\beta^2 + 4c\gamma}{4\gamma^2}.$$

Therefore, the following propositions can be proved.

Proposition 1.

(1) *If β and γ have the same sign, the equilibrium points are linear stable.*
(2) *If β and γ have different signs, the equilibrium points are linear unstable.*

From equation (3.5), the Jacobi stability of any point on the system (2.2) orbit can be obtained. When n is an odd, we can obtain Proposition 2.

Proposition 2.

(1) *If $\frac{\alpha}{\gamma} > 0$ and $u < \left(\frac{\beta^2+4c\gamma}{2\alpha\gamma}\right)^{\frac{1}{n}}$, the orbit of the system (2.2) is* Jacobi *unstable. Otherwise, the orbit is* Jacobi *stable.*
(2) *If $\frac{\alpha}{\gamma} < 0$ and $u < \left(\frac{\beta^2+4c\gamma}{2\alpha\gamma}\right)^{\frac{1}{n}}$, the orbit of the system (2.2) is* Jacobi *unstable. Otherwise, the orbit is* Jacobi *stable.*

When n is an even, we can obtain Proposition 3.

Proposition 3.

(1) *If $\frac{\alpha}{\gamma} > 0$, $\beta^2 + 4c\gamma > 0$ and $-\left(\frac{\beta^2+4c\gamma}{2\alpha\gamma}\right)^{\frac{1}{n}} < u < \left(\frac{\beta^2+4c\gamma}{2\alpha\gamma}\right)^{\frac{1}{n}}$, the orbits of the system (2.2) are* Jacobi *unstable. Otherwise, the orbits are* Jacobi *stable.*
(2) *If $\frac{\alpha}{\gamma} > 0$, $\beta^2 + 4c\gamma < 0$, the orbits of the system (2.2) are* Jacobi *stable.*
(3) *If $\frac{\alpha}{\gamma} < 0$, $\beta^2 + 4c\gamma > 0$, the orbits of the system (2.2) are* Jacobi *unstable.*

(4) *If* $\frac{\alpha}{\gamma} < 0$, $\beta^2 + 4c\gamma < 0$, *and* $-\left(\frac{\beta^2+4c\gamma}{2\alpha\gamma}\right)^{\frac{1}{n}} < u < \left(\frac{\beta^2+4c\gamma}{2\alpha\gamma}\right)^{\frac{1}{n}}$, *the orbits of the system* (2.2) *are* Jacobi *stable. Otherwise, the orbits are* Jacobi *unstable.*

Remark 2. When α and c have different signs, system (2.2) does not have bounded traveling wave solutions.

One discusses the deviation curvature tensor at the equilibrium points.

Proposition 4. *At the point* E_0*, it is obtained that* $P = \frac{\beta^2+4c\gamma}{4\gamma^2}$.

(1) *If* $\gamma > 0$ *and* $c \geq -\frac{\beta^2}{4\gamma}$*, the equilibrium point* E_0 *is Jacobi unstable.*
(2) *If* $\gamma > 0$ *and* $c < -\frac{\beta^2}{4\gamma}$*, the equilibrium point* E_0 *is Jacobi stable.*
(3) *If* $\gamma < 0$ *and* $c \geq -\frac{\beta^2}{4\gamma}$*, the equilibrium point* E_0 *is Jacobi stable.*
(4) *If* $\gamma < 0$ *and* $c < -\frac{\beta^2}{4\gamma}$*, the equilibrium point* E_0 *is Jacobi unstable.*

Similarly, one can analyze the Jacobi stability of equilibrium point $E_{1,2}$.

Proposition 5. *At the point* $E_{1,2}$*, it is obtained that* $P = \frac{\beta^2-4nc\gamma}{4\gamma^2}$.

(1) *If* $\gamma > 0$ *and* $c > \frac{\beta^2}{4n\gamma}$*, the equilibrium point* $E_{1,2}$ *is Jacobi stable.*
(2) *If* $\gamma > 0$ *and* $c \leq \frac{\beta^2}{4n\gamma}$*, the equilibrium point* $E_{1,2}$ *is Jacobi unstable.*
(3) *If* $\gamma < 0$ *and* $c > \frac{\beta^2}{4n\gamma}$*, the equilibrium point* $E_{1,2}$ *is Jacobi unstable.*
(4) *If* $\gamma < 0$ *and* $c > \frac{\beta^2}{4n\gamma}$*, the equilibrium point* $E_{1,2}$ *is Jacobi stable.*

When n is an odd, the boundary line $P = 0$ will go through the following stages. In the first stage $\beta \leq -2\sqrt{cn\gamma}$, the boundary line appears to the right side of the two equilibrium points. As β gradually increases to $2\sqrt{cn\gamma} > \beta > -2\sqrt{cn\gamma}$, the boundary moves between two equilibrium points. Specially, when β increases to 0, the linear stability of equilibrium points changes. As β continues to increase to $\beta \geq 2\sqrt{cn\gamma}$, the boundary line will move back to the right of the equilibrium points.

When n is even, in the first stage $\beta \leq -2\sqrt{cn\gamma}$, the boundary line appears outside the two equilibrium points. As β gradually increases to $2\sqrt{cn\gamma} > \beta > -2\sqrt{cn\gamma}$, the boundary moves between the equilibrium points $E_{1,2}$ and E_0. Specially, when β increases to 0, the linear stability of equilibrium points changes. As β continues to increase to $\beta \geq 2\sqrt{cn\gamma}$, the boundary line will move back to the outside of the equilibrium points. This conclusion can correspond the type of equilibrium points mentioned earlier.

3.2. *Dynamic behavior of deviation vectors*

In this subsection, the dynamic behavior of deviation vectors is analyzed. According to the equation (3.3), the dynamical equation of the deviation vector is

$$\frac{d^2\xi}{d\zeta^2}+\frac{\beta}{\gamma}\frac{d\xi}{d\zeta}+\left(\frac{\alpha}{\gamma}u^n-\frac{c}{\gamma}\right)\xi=0. \tag{3.6}$$

For convenience, one makes $\gamma>0,\beta>0$. About other situations, conclusions can be obtained similarly.

3.2.1. *Dynamic behavior of deviation vectors at E_0*

For equilibrium point $E_0(0,0)$, according to expression (3.6), it can be obtained that

$$\frac{d^2\xi_0}{d\zeta^2}+\frac{\beta}{\gamma}\frac{d\xi_0}{d\zeta}-\frac{c}{\gamma}\xi_0=0, \tag{3.7}$$

According to the initial conditions $\xi_0(0)=0,\dot{\xi}_0(0)=\xi_{01}$, The solution of the equation (3.2) is

$$\xi_0=\begin{cases}\exp\left(\frac{-\beta+\sqrt{\beta^2+4c\gamma}}{2\gamma}\zeta\right)-\exp\left(\frac{-\beta-\sqrt{\beta^2+4c\gamma}}{2\gamma}\zeta\right), & c>-\frac{\beta^2}{4\gamma},\\ \zeta\exp\left(-\frac{\beta}{2\gamma}\zeta\right), & c=-\frac{\beta^2}{4\gamma},\\ \exp\left(-\frac{\beta}{2\gamma}\zeta\right)\sin(-\beta^2-4c\gamma)\zeta, & c<-\frac{\beta^2}{4\gamma}.\end{cases} \tag{3.8}$$

3.2.2. *Dynamic behavior of deviation vectors at $E_{1,2}$*

For equilibrium point $E_{1,2}(\frac{2c}{\alpha},0)$, it can be obtained that

$$\frac{d^2\xi_1}{d\zeta^2}+\frac{\beta}{\gamma}\frac{d\xi_1}{d\zeta}+\frac{nc}{\gamma}\xi_1=0. \tag{3.9}$$

Similarly, one utilizes initial conditions $\xi_1(0)=0,\dot{\xi}_1(0)=\xi_{11}$. The solution of the equation (3.4) is

$$\xi_1=\begin{cases}\exp\left(\frac{-\beta+\sqrt{\beta^2-4nc\gamma}}{2\gamma}\zeta\right)-\exp\left(\frac{-\beta-\sqrt{\beta^2-4nc\gamma}}{2\gamma}\zeta\right), & c>\frac{\beta^2}{4\gamma},\\ \zeta\exp\left(-\frac{\beta}{2\gamma}\zeta\right), & c=\frac{\beta^2}{4\gamma},\\ \exp\left(-\frac{\beta}{2\gamma}\zeta\right)\sin(-\beta^2+4nc\gamma)\zeta & c<\frac{\beta^2}{4\gamma}.\end{cases} \tag{3.10}$$

4. Melnikov Analysis of Chaotic Behaviors

In this section, one begins by the equation (2.1) when the dissipation term is periodically disturbed.

$$\gamma u_{2\zeta}+\beta(1+\delta\cos\omega\zeta)u_\zeta+\frac{\alpha}{n+1}u^{n+1}-cu=0. \tag{4.1}$$

one transforms the equation (4.1) into an equivalent planar system.

$$\begin{aligned} \frac{du}{d\zeta} &= v \\ \frac{dv}{d\zeta} &= -\frac{\beta}{\gamma}(1+\delta\cos\omega\zeta)v - \frac{\alpha}{(n+1)\gamma}u^{n+1} + \frac{c}{\gamma}u \end{aligned} \tag{4.2}$$

When β is small enough, one can define $\varepsilon = -\frac{\beta}{\gamma}$ as disturbance amplitude. So the system (3.6) can be transformed into the following form.

$$\begin{aligned} \frac{du}{d\zeta} &= v, \\ \frac{dv}{d\zeta} &= -\frac{\alpha}{(n+1)\gamma}u^{n+1} + \frac{c}{\gamma}u + \varepsilon(\delta\cos\omega\zeta v - v). \end{aligned} \tag{4.3}$$

For equation (4.3) without disturbance, one obtains Hamiltonian function

$$H = \frac{v^2}{2} - \frac{c}{2\gamma}u^2 + \frac{\alpha}{(n+1)(n+2)\gamma}u^{n+2}.$$

Let $H = 0$, the explicit expression for the homoclinic orbit can be obtained as

$$q_0(\zeta) = \left(\left(\frac{(n+1)(n+2)c}{2\alpha}\right)^{\frac{1}{n}} \operatorname{sech}^{\frac{2}{n}}\zeta, -\frac{2}{n}\left(\frac{(n+1)(n+2)c}{2\alpha}\right)^{\frac{1}{n}} \operatorname{sech}^{\frac{2}{n}}\zeta \tanh\zeta \right).$$

Therefore, Melnikov function for the homoclinic orbit $q_0(\zeta)$ becomes

$$M = \int_{-\infty}^{\infty} v(q_0)[\delta\cos\omega\zeta v(q_0) - v(q_0)]d\zeta$$

where

$$\begin{aligned} v(q_0) &= -\frac{2}{n}\left(\frac{(n+1)(n+2)c}{2\alpha}\right)^{\frac{1}{n}}\operatorname{sech}^{\frac{2}{n}}\zeta\tanh\zeta. \\ M &= \frac{4}{n^2}\left(\frac{(n+1)(n+2)c}{2\alpha}\right)^{\frac{2}{n}}\left[\int_{-\infty}^{\infty}\delta\operatorname{sech}^{\frac{4}{n}}\zeta\tanh^2\zeta\cos\omega(\zeta+\zeta_0)d\zeta \right. \\ &\quad \left. - \int_{-\infty}^{\infty}\operatorname{sech}^{\frac{4}{n}}\zeta\tanh^2\zeta d\zeta\right]. \end{aligned}$$

It is clear that the homoclinic Melnikov function M has simple zeros for $\cos\omega(\zeta+\zeta_0) = \frac{1}{\delta}$. This means the equation (4.1) has chaos in the sense of Smale horseshoe.

5. Conclusion

This paper analyzes the cases where n is an odd and an even respectively and devotes to the dynamic behavior of traveling wave solutions of the generalized KdV-Burgers equation. The global structure diagrams of the system (2.2) under different parameters are sketched. An amplitude estimation of the generalized KdV-Burgers equation (1.1) is obtained. The KdV-Burgers equation is a special case when $n = 1$. Therefore, the obtained results are more accurate and universal. The conditions of Jacobi stability of the system (2.1) at any point on the trajectory are obtained. When $\gamma > 0$ and $c < -\frac{\beta^2}{4\gamma}$, or $\gamma < 0$ and $c \geq -\frac{\beta^2}{4\gamma}$, the equilibrium point E_0 is Jacobi stable, and for $\gamma > 0$ and $c > \frac{\beta^2}{4n\gamma}$, or $\gamma < 0$ and $c \leq \frac{\beta^2}{4n\gamma}$, the equilibrium points $E_{1,2}$ are Jacobi stable. For the equation (2.1) with the dissipative term receiving periodic perturbations, it is discovered that the system is chaotic in the sense of Smale horseshoe when $|\delta| > 1$.

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IMPULSIVE BASSET FRACTIONAL DIFFERENTIAL EQUATION WITH NONLINEAR BOUNDARY CONDITION

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Abstract. In this paper, we consider the existence of solutions for a class of impulsive Basset fractional differential equation with nonlinear boundary condition. Using the comparison principle established and Schauder's fixed point theorem, we show that the problem has at least a solution between the upper and lower solution under appropriate conditions. Meanwhile, the existence of extreme solutions is obtained by means of quasilinearization technique. Finally, two examples are presented to illustrate the applicability of our main results.

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Keywords: Basset fractional differential equation, impulse, upper and lower solutions

1. INTRODUCTION

In the paper, we consider the following nonlinear Basset fractional differential equation with impulses

$$\begin{cases} x'(t)+MD^{\alpha}x(t)=f(t,x(t)), & t\in(0,1],\ t\neq t_k,\\ \Delta x(t_k)=I_k(x(t_k)), & k=1,2,\ldots,p,\\ g(x(0), \quad x(1))=0, \end{cases} \tag{1.1}$$

where $0<\alpha<1, M\in\mathbb{R}$, $0=t_0<t_1<t_2<\cdots<t_p<t_{p+1}=1$, $\Delta x(t_k)=x(t_k^+)-x(t_k^-)$ denotes the jump of $x(t)$ at $t=t_k$, $x(t_k^+)$ and $x(t_k^-)$ represent the right and left limits of $x(t)$ at $t=t_k$ respectively, and $D^{\alpha}x={}_0D_t^{\alpha}x$ is the Caputo fractional derivative.

Fractional integrals and derivatives are vital for modeling phenomena across engineering, physics, and biology [8, 11, 18, 20]. As such, the widespread application of fractional differential equations has led to significant and growing research interest, as seen in [5, 9, 13, 15, 25, 26] and related works.

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Significant attention has been directed towards fractional differential equations that blend classical and fractional derivatives for modeling specialized physical phenomena. A foundational example is the Basset fractional differential equation, which originated from Basset's study of a sphere under gravity [4]. He introduced a special hydraulic force, now known as the "Basset force," which Mainardi [16] later interpreted as being proportional to the fractional derivative of order $1/2$ of the particle's relative velocity. Consequently, this model incorporates both a first-order derivative and a fractional derivative. Staněk [22] considered the general Basset fractional equation

$$\begin{cases} u'(t) = AD^{\alpha}u(t) + f(t,u(t)), \\ u(0) = u(T), \end{cases} \tag{1.2}$$

where $0 < \alpha < 1$. Under appropriate conditions, the author showed the existence of solution for (1.2) by using the Leray-Schauder degree method.

In modelling the motion of a rigid body immersed in Newtonian fluid, Torvik and Bagley [23] introduced the following fractional differential equation

$$Au''(t) + BD^{\frac{3}{2}}u(t) + Cu(t) = f(t),$$

where A, B, C are real numbers, f is the known function, which is referred to as Bagley-Torvik equation by later literature.

Fazli, Sun, Aghchi and Nieto [6] studied the following fractional differential equation with the nonlinear conditions

$$\begin{cases} u^{(m)}(t) + MD^{\delta}u(t) = f(t,u(t)), & 0 < t \le T, \\ g_k(u^{(k)}(t_0), u^{(k)}(t_1), \ldots, u^{(k)}(t_r)) = 0, \end{cases}$$

where $m-1 < \delta < m$, $0 = t_0 < t_1 < \cdots < t_r = T$, $k = 0, 1, \ldots, m-1$. The authors obtained the existence of extremal solutions by establishing one comparison theorem and applying the monotone iterative method. The other results about those equations, we refer the reader to [1–3, 14, 17, 19] and the references therein.

Impulsive perturbation originates from external disturbances in the process of time evolution and is commonly present in practical problems in modern technology. Impulsive fractional differential equation has also attracted the interest of many researchers, see [7, 12, 21]. In [10], Guo and Jiang studied the fractional impulsive problem

$$\begin{cases} D^q u(t) = f(t,u(t)), & t \in [0,T]/\{t_1, t_2, \ldots, t_m\},\ 0 < q < 1, \\ \Delta u(t_k) = I_k(u(t_k)), & k = 1, 2, \ldots, m, \\ au(0) + bu(T) = c, \end{cases} \tag{1.3}$$

where $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, and $f\colon J \times \mathbb{R} \to \mathbb{R}$ is continuous. They obtained some existence results by using fixed point method and generalized singular Gronwall's inequality. In [24], Yang and Chen studied the following impulsive fractional differential equation

$$\begin{cases} D^{\alpha}u(t) = f(t,u(t)), & t \in [0,1]/\{t_1,t_2,\dots,t_m\}, \\ \Delta u(t_k) = I_k(u(t_k)), \quad \Delta D^{\beta}u(t_k) = J_k(u(t_k)), & k = 1,2,\dots,m, \\ u(0) + K_1 D^{\beta}u(1) = \theta_1, \\ D^{\beta}u(0) + K_2 u(1) = \theta_2, \end{cases} \tag{1.4}$$

where $f\colon [0,1]\times\mathbb{R}\to\mathbb{R}, \alpha\in(1,2], \beta\in(0,1]$, $K_1,K_2,\theta_1,\theta_2$ are constants. By applying Krasnoselskii's fixed point theorem and contraction mapping principle, the authors showed the existence of solution of (1.4) under appropriate conditions.

The model (1.1) investigated in this work possesses several distinctive features when compared to existing formulations. Unlike (1.2), it incorporates impulsive perturbations and a nonlinear boundary condition. Furthermore, it differs from (1.3) by including a first-order derivative term and a more complex boundary structure. To the best of our knowledge, the solvability of impulsive Basset equations remains an unexplored area. Therefore, this paper aims to establish the existence of solutions to (1.1) employing the method of lower and upper solutions.

The paper is organized as follows. In section 2, we establish a comparison principle related to the problem (1.1). In section 3, the concept of lower and upper solution of (1.1) is introduced. By using fixed point theorem and the approach of quasilinearization, we obtain the existence results of (extreme) solution for (1.1).

2. Preliminaries

Let $J_0 = [0,t_1), J_1 = (t_1,t_2),\dots,J_{p-1} = (t_{p-1},t_p), J_p = (t_p,1], J = [0.1]$,

$$PC(J) = \big\{x\colon J\to\mathbb{R} \mid x\in C(J_k),\ k=0,1,\dots,p,\ x(t_i^+), x(t_i^-)\text{exist,}, \\ x(t_i^-) = x(t_i),\ i=1,2,\dots,p\big\},$$

then $PC(J)$ is Banach spaces with the norm $\|x\| = \sup_{t\in J}|x(t)|$. A function $x\in\Lambda :=$ $\{u\in PC(J)\cap C^1(J^*)\colon u'\in L^1(0,1)\}$ is called a solution of (1.1) if it satisfies (1.1), where $J^* = J/\{t_1,t_2,\dots,t_p\}$.

Definition 1 ([11, (3.1) of Chapter 3])**.** Two-parameter Mittag-Leffler function

$$E_{a,b}(\xi) = \sum_{i=0}^{\infty}\frac{\xi^i}{\Gamma(ia+b)}, \qquad \xi\in\mathbb{R},\ a>0,\ b\in\mathbb{R}.$$

Definition 2 ([11, Definition 2.1])**.** Let $x\in L^1(a,b)$, its Riemann-Liouville fractional integral ${}_aI_t^{\gamma}x$ of order $\gamma>0$ is defined as

$${}_aI_t^{\gamma}x(t) = \frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}x(s)\mathrm{d}s.$$

Definition 3 ([11, Definition 2.3])**.** Let $x \in L^1(a,b)$, its Caputo fractional derivative ${}_aD_t^\gamma x$ of order $n-1<\gamma\le n$ is defined by

$$ {}_aD_t^\gamma x(t) = ({}_aI_t^{n-\gamma}x^{(n)})(t) = \frac{1}{\Gamma(n-\gamma)}\int_a^t (t-s)^{n-1-\gamma}x^{(n)}(s)\mathrm{d}s, $$

provided that the right-hand side integral exists and is finite.

The Caputo fractional derivative ${}_aD_t^\gamma x$ of x can also be defined by

$$ {}_aD_t^\gamma x(t) = D^n({}_aI_t^{n-\gamma})\left(x(t)-\sum_{k=0}^{n-1}\frac{x^{(k)}(0)}{k!}t^k\right), \qquad n-1<\gamma\le n, $$

provided that the right-hand side integral exists and is finite, see [6].

Lemma 1. *Assume that $x \in \Lambda$, then its Caputo fractional derivative of order α exists and*

$$ D^\alpha x(t) \in C(J^*)\cap L^1(0,1). $$

Proof. Without losing generality, we assume that $p=1$. Since ${}_0I_t^{1-\alpha}$ is bounded on $L^p(0,1)$ for any $1\le p\le\infty$ (see Theorem 2.2(i) of [11]), $x'\in L^1(0,1)$ and

$$ D^\alpha x(t) = ({}_0I_t^{1-\alpha}x')(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}x'(s)\mathrm{d}s, $$

$D^\alpha x$ exists almost everywhere and $D^\alpha x \in L^1(0,1)$.Using Theorem 2.2(iii) of [11], from that fact that $x'\in C[0,t_1)$, we have

$$ D^\alpha x(t) \in C[0,\theta] $$

for any $[0,\theta]\subset[0,t_1)$. For $t_1<t\le 1$,

$$ \begin{aligned} D^\alpha x(t) &= \frac{1}{\Gamma(1-\alpha)}\int_0^{t_1}(t-s)^{-\alpha}x'(s)\mathrm{d}s + \frac{1}{\Gamma(1-\alpha)}\int_{t_1}^{t}(t-s)^{-\alpha}x'(s)\mathrm{d}s \\ &= \frac{1}{\Gamma(1-\alpha)}\int_0^{t_1}(t-s)^{-\alpha}x'(s)\mathrm{d}s + ({}_{t_1}I_t^{1-\alpha}x')(t) := h_1(t)+h_2(t), \end{aligned} $$

where

$$ h_1(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^{t_1}(t-s)^{-\alpha}x'(s)\mathrm{d}s,\ \ h_2(t) = ({}_{t_1}I_t^{1-\alpha}x')(t) = {}_{t_1}D_t^\alpha x(t). $$

Since $(t-s)^{-\alpha}x'(s)$ is continuous in $t\in(t_1,1]$ and integrable in $s\in(0,t_1)$, $h_1\in C(t_1,1]$. Similar to the case $x\in C[0,t_1)$, $h_2\in C(t_1,1]$. Hence, $D^\alpha x(t)\in C(J^*)$. □

Consider the linear equation

$$ \begin{cases} x'(t)+MD^\alpha x(t) = h(t), & t\neq t_k, \\ \Delta x(t_k) = d_k, & k=1,2,\ldots,p, \\ x(0)=x_0, \end{cases} \tag{2.1} $$

where $h\in C(J^*)\cap L^1(0,1)$ and $M, d_k(1\le k\le p), x_0\in\mathbb{R}$.

Lemma 2. *The function $\tilde{x} \in \Lambda$ is the solution of if and only if*

$$\tilde{x}(t) = \int_0^t E_{1-\alpha,1}\left(-M(t-s)^{1-\alpha}\right) h(s)\mathrm{d}s + x_0 + \sum_{0<t_k<t} d_k.$$

Proof. Assume that x_1, x_2 are two solutions of (2.1) and $y = x_1 - x_2$, then

$$\begin{cases} y'(t) + MD^{\alpha}y(t) = 0, & t \neq t_k, \\ \Delta y(t_k) = 0, & k = 1,2,\ldots,p, \\ y(0) = 0. \end{cases}$$

Since $y \in C^1[0,t_1)$, by employing Laplace transform, one can obtain that $y \equiv 0$ in $[0,t_1)$ and thus $y(t_1) = 0$ since y is left continuous. For $t \in (t_1,t_2)$, we have

$$y'(t) + MD^{\alpha}y(t) = y'(t) + M_0 D_t^{\alpha} y(t) = y'(t) + M_{t_1} D_t^{\alpha} y(t) = 0.$$

Therefore $y \equiv 0$ in $(t_1,t_2]$. Similarly, $y \equiv 0$ in $[t_i,t_{i+1}], i = 2,\ldots,p$. Hence, the solution of (2.1) is unique.

Let $g(t) = \int_0^t E_{1-\alpha,1}(-M(t-s)^{1-\alpha})h(s)\mathrm{d}s$ and

$$\psi(t) = h(t) - M(1-\alpha)\int_0^t E^{(1)}_{1-\alpha,1}(-M(t-s)^{1-\alpha})(t-s)^{-\alpha}h(s)\mathrm{d}s,$$

where $E^{(1)}_{1-\alpha,1}(t) = (E_{1-\alpha,1}(t))'$, which is continuous in $\mathbb{R}$. It follows from $h \in C(J^*) \cap L^1(0,1)$ that $g \in C[0,1]$. Hence,

$$\tilde{x}(0) = x_0, \quad \tilde{x}(t_k^-) = \tilde{x}(t_k), \quad \Delta\tilde{x}(t_k) = d_k, \quad k = 1,2,\ldots,p.$$

For $0 < y < x < 1$, we have

$$\begin{aligned}
&\int_y^x \psi(t)\mathrm{d}t = \int_y^x h(t)\mathrm{d}t - M(1-\alpha)\int_y^x \mathrm{d}r \int_0^r E^{(1)}_{1-\alpha,1}(-M(r-s)^{1-\alpha})(t-s)^{-\alpha}h(s)\mathrm{d}s \\
&= \int_y^x h(t)\mathrm{d}t - M(1-\alpha)[\int_0^y h(s)\mathrm{d}s \int_y^x E^{(1)}_{1-\alpha,1}(-M(r-s)^{1-\alpha})(r-s)^{-\alpha}\mathrm{d}r \\
&\quad + \int_y^x h(s)\mathrm{d}s \int_s^x E^{(1)}_{1-\alpha,1}(-M(r-s)^{1-\alpha})(r-s)^{-\alpha}\mathrm{d}r] \\
&= \int_y^x h(t)\mathrm{d}t + \int_0^y [E_{1-\alpha,1}(-M(x-s)^{1-\alpha}) - E_{1-\alpha,1}(-M(y-s)^{1-\alpha})]h(s)\mathrm{d}s \\
&\quad + \int_y^x [E_{1-\alpha,1}(-M(x-s)^{1-\alpha}) - E_{1-\alpha,1}(0)]h(s)\mathrm{d}s \\
&= g(x) - g(y),
\end{aligned}$$

which implies that $g'(t) = \psi(t)$.

We show that $\psi \in C(J^*)$. For simplicity, we assume that $p = 1$. Let $H(t,s) = E^{(1)}_{1-\alpha,1}(-M(t-s)^{1-\alpha})(t-s)^{-\alpha}h(s)$. If $0 < t < t_1$,

$$\int_0^t H(t,s)\mathrm{d}s = \int_0^t E^{(1)}_{1-\alpha,1}(-Ms^{1-\alpha})s^{-\alpha}h(t-s)\mathrm{d}s.$$

Since $E^{(1)}_{1-\alpha,1}(-Ms^{1-\alpha})s^{-\alpha}h(t-s)$ is continuous in t and integrable in $s\in[0,t]$, $\int_0^t H(t,s)\mathrm{d}s\in C[0,t_1)$. If $t_1<t<1$, taking $t-t_1<\tau<t$, we have

$$\int_0^t H(t,s)\mathrm{d}s=\int_0^\tau H(t,s)\mathrm{d}s+\int_0^{t-\tau}E^{(1)}_{1-\alpha,1}(-Ms^{1-\alpha})s^{-\alpha}h(t-s)\mathrm{d}s:=g_1(t)+g_2(t).$$

Noting that $E^{(1)}_{1-\alpha,1}(-M(t-s)^{1-\alpha})(t-s)^{-\alpha}h(s)$ is continuous in t and integrable in $s\in[0,\tau]$, $E^{(1)}_{1-\alpha,1}(-Ms^{1-\alpha})s^{-\alpha}h(t-s)$ is continuous in t and integrable in $s\in[0,t-\tau]$, we get that $g_1,g_2\in C(t_1,1]$. Hence, $\psi\in C(J^*)$ and $g\in C^1(J^*)$.

It follows from the fact $|E^{(1)}_{1-\alpha,1}(-M(t-s)^{1-\alpha})|\le E^{(1)}_{1-\alpha,1}(M)$ in region $\{(t,s)|0\le s\le t\le 1\}$ that there exists $C>0$ such that

$$\begin{aligned}\left|\int_0^t E^{(1)}_{1-\alpha,1}(-M(t-s)^{1-\alpha})(t-s)^{-\alpha}h(s)\mathrm{d}s\right| &\le C\int_0^t (t-s)^{-\alpha}|h(s)|\mathrm{d}s\\ &= C\Gamma(1-\alpha)_0I_t^{1-\alpha}|h(t)|.\end{aligned}$$

Since ${}_0I_t^{1-\alpha}$ is bounded on $L^1(0,1)$, we have

$$\int_0^t E^{(1)}_{1-\alpha,1}(-M(t-s)^{1-\alpha})(t-s)^{-\alpha}h(s)\mathrm{d}s\in L^1(0,1),$$

which implies that $\psi\in L^1(0,1)$ and thus $g'\in L^1(0,1)$. It follows from Lemma 1 that $D^\alpha g(t)$ exists. In addition, for $0<t,l<1$,

$$\begin{aligned}
&M[{}_0I_t^{1-\alpha}g(t)-{}_0I_l^{1-\alpha}g(l)]\\
&=\frac{M}{\Gamma(1-\alpha)}\int_0^t (t-r)^{-\alpha}\int_0^r E_{1-\alpha,1}(-M(r-s)^{1-\alpha})h(s)\mathrm{d}s\mathrm{d}r\\
&\quad-\frac{M}{\Gamma(1-\alpha)}\int_0^l (l-r)^{-\alpha}\int_0^r E_{1-\alpha,1}(-M(r-s)^{1-\alpha})h(s)\mathrm{d}s\mathrm{d}r\\
&=\frac{M}{\Gamma(1-\alpha)}\int_0^t h(s)\mathrm{d}s\int_s^t (t-r)^{-\alpha}E_{1-\alpha,1}(-M(r-s)^{1-\alpha})\mathrm{d}r\\
&\quad-\frac{M}{\Gamma(1-\alpha)}\int_0^l h(s)\mathrm{d}s\int_s^l (l-r)^{-\alpha}E_{1-\alpha,1}(-M(r-s)^{1-\alpha})\mathrm{d}r\\
&=\frac{M}{\Gamma(1-\alpha)}\int_0^t h(s)\mathrm{d}s\int_s^t (t-r)^{-\alpha}\sum_{i=0}^{\infty}\frac{(-M)^i(r-s)^{(1-\alpha)i}}{\Gamma((1-\alpha)i+1)}\mathrm{d}r\\
&\quad-\frac{M}{\Gamma(1-\alpha)}\int_0^l h(s)\mathrm{d}s\int_s^l (l-r)^{-\alpha}E_{1-\alpha,1}(-M(r-s)^{1-\alpha})\mathrm{d}r\\
&=\frac{M}{\Gamma(1-\alpha)}\int_0^t h(s)\sum_{i=0}^{\infty}\frac{(-M)^i\int_s^t(t-r)^{-\alpha}(r-s)^{(1-\alpha)i}dr}{\Gamma((1-\alpha)i+1)}\mathrm{d}s\\
&\quad-\frac{M}{\Gamma(1-\alpha)}\int_0^l h(s)\mathrm{d}s\int_s^l (l-r)^{-\alpha}E_{1-\alpha,1}(-M(r-s)^{1-\alpha})\mathrm{d}r
\end{aligned}$$

$$
\begin{aligned}
&= M\int_0^t h(s)\sum_{i=0}^{\infty}\frac{(-M)^i(t-s)^{(1-\alpha)(i+1)}}{\Gamma((1-\alpha)(i+1)+1)}\mathrm{d}s \\
&\quad - M\int_0^l h(s)\mathrm{d}s\int_s^l (l-r)^{-\alpha}E_{1-\alpha,1}(-M(r-s)^{1-\alpha})\mathrm{d}r \\
&= -\int_0^t [E_{1-\alpha,1}(-M(t-s)^{1-\alpha})-1]h(s)\mathrm{d}s \\
&\quad + \int_0^l [E_{1-\alpha,1}(-M(l-s)^{1-\alpha})-1]h(s)\mathrm{d}s \\
&= g(l)-g(t)+\int_l^t h(s)\mathrm{d}s,
\end{aligned}
$$

where we use the formula

$$\int_s^t (t-r)^{-\alpha}(r-s)^{(1-\alpha)i}\mathrm{d}r = \frac{(t-s)^{(1-\alpha)(i+1)}\Gamma(1-\alpha)\Gamma((1-\alpha)i+1)}{\Gamma((1-\alpha)(i+1)+1)}.$$

Hence,

$$\int_l^t g'(s)\mathrm{d}s + M\int_l^t ({}_0I_s^{1-\alpha}g(s))'\mathrm{d}s = \int_l^t h(s)\mathrm{d}s,$$

$$\begin{cases} g'(t)+MD^{\alpha}g(t) = h(t), & a.e, \\ g(0)=0. \end{cases} \tag{2.2}$$

It follows from (2.2) and the fact $\tilde{x}' = g' \in C(J^*)$ for $t \neq t_k$ and $D^{\alpha}\tilde{x}(t) = {}_0I_t^{1-\alpha}\tilde{x}'(t)$ that $\tilde{x}$ is the solution of (2.1). The proof is completed. □

Remark 1. If $h \in C(J^*)\cap L^1(0,1)$ and $a_k, c_k, x_0 \in \mathbb{R}$, then the problem

$$\begin{cases} x'(t)+MD^{\alpha}x(t) = h(t), & t\neq t_k, \\ \Delta x(t_k) = a_k x(t_k)+c_k, & k=1,2,\dots,p, \\ x(0)=x_0 \end{cases} \tag{2.3}$$

has a unique solution

$$
\begin{aligned}
x(t) &= \int_0^t E_{1-\alpha,1}\left(-M(t-s)^{1-\alpha}\right)h(s)\mathrm{d}s \\
&\quad + x_0\prod_{0<t_k<t}(1+a_k) + \sum_{0<t_k<t} c_k \prod_{t_k<t_j<t}(1+a_j) \\
&\quad + \sum_{0<t_k<t} a_k \prod_{t_k<t_j<t}(1+a_j)\int_0^{t_k} E_{1-\alpha,1}\left(-M(t_k-s)^{1-\alpha}\right)h(s)\mathrm{d}s.
\end{aligned} \tag{2.4}
$$

Lemma 3. *If $a\in[0,1]$, $b\geq a$, then*

$$E_{a,b}(x)\geq 0, \quad \forall x\in\mathbb{R}.$$

Proof. By Corollary 3.2 of [11], $E_{a,b}(-x)$ is completely monotone on $\mathbb{R}_+$, which implies that $E_{a,b}(-x) \geq 0$ for $x > 0$. In addition, it follows from the definition of $E_{a,b}$ that $E_{a,b}(x) \geq 0$ for $x \geq 0$. □

Using (2.4) and Lemma 3, we have

Lemma 4. *If* $h \geq 0$, $a_k \geq -1, c_k \geq 0 (1 \leq k \leq p)$ *and* $x_0 \geq 0$, *then the solution* x *of* (2.3) *satisfies*

$$x \geq 0, \quad t \in J.$$

3. Main results

Definition 4. The function $u \in \Lambda$ is said to be the lower solution of (1.1) if

$$\begin{cases} u'(t) + MD^{\alpha}u(t) \leq f(t,u(t)), & t \neq t_k, \\ \Delta u(t_k) \leq I_k(u(t_k)), & k = 1,2,\ldots,p, \\ g(u(0),u(1)) \leq 0 \end{cases}$$

and it is an upper solution of (1.1) if the above inequalities are reverted.

We list the following assumptions.

(H_1) (1.1) has the lower solution u_0, the upper solution v_0 and $u_0 \leq v_0$ for $t \in J$.

(H_2) $f\colon J \times [\gamma_1,\gamma_2] \to \mathbb{R}$ is continuous, here $\gamma_1 = \min\{\min_{t\in J} u_0, \min_{t\in J} v_0\}$ and $\gamma_2 = \max\{\max_{t\in J} u_0, \max_{t\in J} v_0\}$. $I_k(u)$ is continuous in $u \in [u_0(t_k), v_0(t_k)]$ for $k = 1,2,\ldots,p$.

(H_3) $g\colon \mathbb{R}^2 \to \mathbb{R}$ is continuous and $g(\cdot,v)$ is nonincreasing in $v \in [\gamma_1,\gamma_2]$.

(H_4) For $u_0 \leq v \leq u \leq v_0$ and $t \in J$, $f(t,u) \geq f(t,v)$. There exist $b_k \leq 1 (k = 1,2,\ldots,p)$ such that $I_k(\xi) + b_k\xi \geq I_k(\zeta) + b_k\zeta$ for $u_0(t_k) \leq \zeta \leq \xi \leq v_0(t_k)$ and $k = 1,2,\ldots,p$.

(H_5) $g\colon \mathbb{R}^2 \to \mathbb{R}$ is continuous and there exist constants $\lambda > 0, \mu \geq 0$ such that

$$g(\bar{y}_1,\bar{y}_2) - g(y_1,y_2) \leq \lambda(\bar{y}_1 - y_1) - \mu(\bar{y}_2 - y_2)$$

for $\gamma_1 \leq y_i \leq \bar{y}_i \leq \gamma_2, i = 1,2$.

Theorem 1. *Assume that* $(H_1) - (H_4)$ *are satisfied, then* (1.1) *has one solution* $x \in [u_0, v_0] = \{u \in PC(J) : u_0 \leq u \leq v_0, t \in J\}$.

Proof. Let $n(t,x) = \max\{u_0(t), \min\{x, v_0(t)\}\}, F(t,x) = f(t,n(t,x))$ and $I_k^*(x) = b_k n(t_k,x) + I_k(n(t_k,x))$. Consider the equation

$$\begin{cases} x'(t) + MD^{\alpha}x(t) = F(t,x(t)), & t \neq t_k, \\ \Delta x(t_k) = -b_k x(t_k) + I_k^*(x(t_k)), & k = 1,2,\ldots,p, \\ x(0) = n(0,x(0) - g(x(0),x(1))). \end{cases} \tag{3.1}$$

According to Remark 1, the solution of (3.1) satisfies

$$\begin{aligned} x(t) = & \int_0^t E_{1-\alpha,1}(-M(t-s)^{1-\alpha})F(s,x(s))\mathrm{d}s + \sum_{0<t_k<t} I_k^*(x(t_k)) \prod_{t_k<t_j<t}(1-b_j) \\ & - \sum_{0<t_k<t} b_k \prod_{t_k<t_j<t}(1-b_j)\int_0^{t_k} E_{1-\alpha,1}\left(-M(t_k-s)^{1-\alpha}\right)F(s,x(s))\mathrm{d}s \\ & + n(0,x(0)-g(x(0),x(1)))\prod_{0<t_k<t}(1-b_k) =: (Ax)(t). \end{aligned}$$

Let the right-hand part of the above equality be Ax so that we define the operator A in $PC(J)$. Clearly, the fixed point of A in $PC(J)$ is the solution of (3.1).

By the continuity of f, g, I_k and the definition of n, there is $L > 0$ such that for any $x \in PC(J)$,

$$|F(t,x)| < L, \quad |n(0,x(0)-g(x(0),x(1)))| < L, \quad |I_k^*(x(t_k))| < L,$$

which imply that there exists $D > 0$ such that $||Ax|| \leq D$ for any $x \in PC(J)$.

Let $\Omega = \{u \in PC(J) \colon ||u|| \leq D\}$, then $A\colon \Omega \to \Omega$. It is obvious that $A\colon \Omega \to \Omega$ is continuous. Let $t_*, t^* \in (t_k, t_{k+1}]$ and $t_* < t^*$, then for $x \in \Omega$,

$$\begin{aligned} |(Ax)(t^*) - (Ax)(t_*)| \leq & L\int_0^{t_*} |E_{1-\alpha,1}(-M(t^*-s)^{1-\alpha}) - E_{1-\alpha,1}(-M(t_*-s)^{1-\alpha})|\mathrm{d}s \\ & + L\int_{t_*}^{t^*} |E_{1-\alpha,1}(-M(t^*-s)^{1-\alpha})|\mathrm{d}s, \end{aligned}$$

which implies that $|(Ax)(t^*) - (Ax)(t_*)| \to 0$ if $|t^* - t_*| \to 0$ and thus $A\colon \Omega \to \Omega$ is completely continuous. It follows from Schauder's fixed point theorem that there exists $x \in \Omega$ such that $Ax = x$. Moreover, $x \in \Lambda$ since x is the solution of (3.1).

Next, we show that $u_0 \leq x \leq v_0$. Let $y = x - u_0$, from the definition of lower solution and (H_4), we have

$$\begin{aligned} y' + MD^\alpha y &= f(t,n(t,x(t))) - u_0'(t) - MD^\alpha u_0(t) \geq f(t,n(t,x(t))) - f(t,u_0(t)) \geq 0, \\ \Delta y(t_k) &= -b_k x(t_k) + I_k^*(x(t_k)) - \Delta u_0(t_k) \\ &\geq -b_k x(t_k) + I_k(u_0(t_k)) + b_k u_0(t_k) - \Delta u_0(t_k) \geq -b_k y(t_k), \\ y(0) &= x(0) - u_0(0) \geq 0. \end{aligned}$$

Clearly, $f(t,n(t,x(t))), u_0', D^\alpha u_0 \in C(J^*) \cap L^1(0,1)$. Using Lemma 4, we obtain that $x \geq u_0$ for all $t \in J$. Similarly, $x \leq v_0$ for all $t \in J$. Hence,

$$\begin{cases} x'(t) + MD^\alpha x(t) = f(t,x(t)), & t \neq t_k, \\ \Delta x(t_k) = I_k(x(t_k)), & k = 1,2,\ldots,p. \end{cases}$$

Finally, we show that $g(x(0),x(1)) = 0$. We only need to show that $u_0(0) \leq x(0) - g(x(0),x(1)) \leq v_0(0)$. If $x(0) - g(x(0),x(1)) < u_0(0)$, then $x(0) = n(0,x(0) -$

$g(x(0),x(1))) = u_0(0)$ and thus $g(x(0),x(1)) > 0$. From the definition of lower solution and (H_3), we have

$$g(u_0(0),u_0(1)) \le 0 < g(x(0),x(1)) = g(u_0(0),x(1)) \le g(u_0(0),u_0(1)) \le 0,$$

which is a contradiction. Hence $x(0) - g(x(0),x(1)) \ge u_0(0)$. Similarly, $x(0) - g(x(0),x(1)) \le v_0(0)$. Hence, $x(0) = n(0,x(0) - g(x(0),x(1))) = x(0) - g(x(0),x(1))$ and thus $g(x(0),x(1)) = 0$. x is a solution of (1.1) and $x \in [u_0, v_0]$. □

Theorem 2. *Assume that $(H_1) - (H_2)$ and $(H_4) - (H_5)$ are satisfied, then there exist sequences $\{u_i\},\{v_i\} \subseteq \Lambda$ such that $\lim_{i\to\infty} u_i = u^*, \lim_{i\to\infty} v_i = v^*$ and $u^*, v^* \in [u_0, v_0]$ are minimal and maximal solutions of (1.1), respectively.*

Proof. The proof is divided into four steps.

Step 1: Constructing sequences $\{u_i\},\{v_i\}$. Consider the following linear equation

$$\begin{cases} W'_{i+1}(t) + MD^{\alpha}W_{i+1}(t) = f(t,W_i(t)), & t \in J,\ t \neq t_k, \\ \Delta W_{i+1}(t_k) = -b_k W_{i+1}(t_k) + I_k^*(W_i(t_k)), & k = 1,2,\dots,p, \\ W_{i+1}(0) = W_i(0) - \frac{1}{\lambda} g(W_i(0),W_i(1)), \end{cases} \tag{3.2}$$

where $W_0 = u_0$ or $W_0 = v_0$. From Remark 1, (3.2) has a unique solution

$$\begin{aligned} W_{i+1}(t) = &\int_0^t E_{1-\alpha,1}\left(-M(t-s)^{1-\alpha}\right) f(t,W_i(t))\mathrm{d}s \\ &+ (W_i(0) - \lambda^{-1} g(W_i(0),W_i(1))) \prod_{0<t_k<t} (1-b_k) \\ &- \sum_{0<t_k<t} b_k \prod_{t_k<t_j<t} (1-b_j) \int_0^{t_k} E_{1-\alpha,1}\left(-M(t_k-s)^{1-\alpha}\right) f(t,W_i(t))\mathrm{d}s \\ &+ \sum_{0<t_k<t} I_k^*(W_i(t_k)) \prod_{t_k<t_j<t} (1-b_j). \end{aligned} \tag{3.3}$$

Setting $W_i = u_i$ if $W_0 = u_0$, $W_i = v_i$ if $W_0 = v_0$, we obtain two sequences $\{u_i\}$ and $\{v_i\}$ and $u_i, v_i \in \Lambda$ for $i = 1,2,\dots$.

Step 2: Monotone property of sequences $\{u_i\},\{v_i\}$:

$$u_0 \le u_1 \le u_2 \le \dots \le u_i \le u_{i+1} \le v_{i+1} \le v_i \le \dots \le v_1 \le v_0.$$

Let $z = u_1 - u_0$, we obtain that

$$\begin{aligned} z'(t) + MD^{\alpha}z(t) &\ge f(t,u_0(t)) - f(t,u_0(t)) = 0, \\ \Delta z(t_k) &= -b_k u_1(t_k) + I_k^*(u_0(t_k)) - \Delta u_0(t_k) \\ &\ge -b_k(u_1(t_k) - u_0(t_k)) + I_k(u_0(t_k)) - \Delta u_0(t_k) \\ &\ge -b_k(u_1(t_k) - u_0(t_k)), \\ z(0) &= -\frac{1}{\lambda} g(u_0(0),u_0(1)) \ge 0. \end{aligned}$$

It follows from Lemma 4 that $z \geq 0$, so $u_0 \leq u_1$ for all $t \in J$. Similarly, one can prove that $v_1 \leq v_0$ for all $t \in J$. Now, let $\omega = v_1 - u_1$, using (H_4) and (H_5), we obtained

$$\begin{aligned}
&\omega'(t) + MD^{\alpha}\omega(t) \geq 0,\\
&\Delta\omega(t_k) = -b_k v_1(t_k) + I_k^*(v_0(t_k)) + b_k u_1(t_k) - I_k^*(u_0(t_k))\\
&\qquad \geq -b_k(v_1(t_k) - u_1(t_k)) + b_k(v_0(t_k) - u_0(t_k)) + I_k(v_0(t_k)) - I_k(u_0(t_k))\\
&\qquad \geq -b_k(v_1(t_k) - u_1(t_k)) = -b_k w(t_k),\\
&\omega(0) \geq \frac{\mu}{\lambda}(v_0(1) - u_0(1)) \geq 0.
\end{aligned}$$

Hence, $v_1 \geq u_1$. From (H_4) and (H_5), we obtain that

$$\begin{aligned}
u_1'(t) + MD^{\alpha}u(t) &\leq f(t, u_1(t)),\\
\Delta u_1(t_k) &= -b_k u_1(t_k) + I_k^*(u_0(t_k)) = -b_k(u_1(t_k) - u_0(t_k)) + I_k(u_0(t_k))\\
&\leq I_k(u_1(t_k)),\\
g(u_1(0), u_1(1)) &\leq g(u_0(0), u_0(1)) + \lambda(u_1(0) - u_0(0)) - \mu(u_1(1) - u_0(1))\\
&= -\mu(u_1(1) - u_0(1)) \leq 0.
\end{aligned}$$

Therefore, u_1 is the lower solution of (1.1). Similarly, v_1 is the upper solution of (1.1). Using the similar argument, we can show that $u_i \leq u_{i+1} \leq v_{i+1} \leq v_i$ for $i \geq 1$.

Step 3: According to Step 2, the sequences $\{u_i\}, \{v_i\}$ are monotonic and bounded. Therefore, the pointwise limits exist and we assume that

$$\lim_{i\to\infty} u_i = u^*, \quad \lim_{i\to\infty} v_i = v^*,$$

where $u^*, v^* \in [u_0, v_0]$.

From (H_2), (H_5) and $\lim_{i\to\infty} W_i = W \in [u_0, v_0]$, where $W = u^*$ or v^*, let $i \to \infty$ in (3.3) and applying the dominated convergence theorem, we obtain that

$$\begin{aligned}
W(t) = &\int_0^t E_{1-\alpha,1}\left(-M(t-s)^{1-\alpha}\right) f(t, W(t))\mathrm{d}s + \sum_{0<t_k<t} I_k^*(W(t_k)) \prod_{t_k<t_j<t}(1-b_j)\\
&- \sum_{0<t_k<t} b_k \prod_{t_k<t_j<t}(1-b_j) \int_0^{t_k} E_{1-\alpha,1}\left(-M(t_k-s)^{1-\alpha}\right) f(t, W(t))\mathrm{d}s\\
&+ (W(0) - \lambda^{-1}g(W(0), W(1))) \prod_{0<t_k<t}(1-b_k).
\end{aligned}$$

Through simple calculation, we have

$$\begin{cases}
W'(t) + MD^{\alpha}W(t) = f(t, W(t)), & t \neq t_k,\\
\Delta W(t_k) = -b_k W(t_k) + I_k^*(W(t_k)) = I_k(W(t_k)), & k = 1, 2, \ldots, p,\\
W(0) = W(0) - \lambda^{-1}g(W(0), W(1)).
\end{cases}$$

Therefore, u^*, v^* are solutions of (1.1).

Step 4: u^*, v^* are the extremal solutions of (1.1) in $[u_0, v_0]$. Assume that $u \in [u_0, v_0]$ is a solution of (1.1), we suppose that $u_i \le u \le v_i$ for some $i \in \mathbb{N}$. By (H_4), we have

$$f(t, u_i(t) \le f(t, u(t)) \le f(t, v_i(t)),$$

$$\begin{aligned}\Delta(u(t_k) - u_{i+1}(t_k)) &= I_k(u(t_k)) + b_k u_{i+1} - I_k^*(u_i(t_k)) \\ &\le b_k(u_{i+1}(t_k) - u_i(t_k)) + I_k(u(t_k)) - I_k u_i(t_k) \\ &\le -b_k(u(t_k) - u_{i+1}(t_k)).\end{aligned}$$

Similarly, $\Delta(v_{i+1}(t_k) - u(t_k)) \ge -b_k(v_{i+1}(t_k) - u(t_k))$.

Using (H_5), we have

$$\begin{aligned}u_{i+1}(0) &= u_i(0) - \frac{1}{\lambda} g(u_i(0), u_i(1)) \\ &= u_i(0) + \frac{1}{\lambda} g(u(0), u(1)) - \frac{1}{\lambda} g(u_i(0), u_i(1)) \\ &\le u(0) - \frac{\mu}{\lambda}(u(1) - u_i(1)) \le u(0).\end{aligned}$$

Similarly, $u(0) \le v_{i+1}(0)$. It follows from Lemma 4 that $u_{i+1} \le u \le v_{i+1}$. Therefore,

$$u_j \le u \le v_j, \quad j = i, i+1, i+2, \ldots. \tag{3.4}$$

Taking limit in (3.4) as $j \to \infty$, we get that $u^* \le u \le v^*$. Therefore, u^*, v^* are the extremal solutions of (1.1) in $[u_0, v_0]$.

□

Example 1. Consider the equation

$$\begin{cases} x'(t) + \kappa D^{\frac{1}{2}} x(t) = t(1 + x^{\beta}(t)), & t \ne \frac{1}{2}, \\ \Delta x(0.5) = 0.1 - \sin x(0.5), \\ x^4(0) \sin x(0) - x^2(1) - x(1) = 0, \end{cases} \tag{3.5}$$

where $\beta > 0$ and κ is a positive parameter.

We claim that for any $l \in \mathbb{N}$, (3.5) has at least l solutions for $\kappa \ge 2(2\pi l + 2)^{\beta}$. In fact,

$$f(t, s) = t(1 + s^{\beta}), \quad I_1(s) = 0.1 - \sin s, \quad g(u, v) = u^4 \sin u - v^2 - v.$$

Let $U_j = 2j\pi$, $V_j = 2j\pi + 1 + t$, $j = 1, \ldots, l$, then

$$\begin{cases} U_j'(t) + \kappa D^{\frac{1}{2}} U_j(t) = 0 \le t(1 + U_j^{\beta}(t)), & t \ne 0.5, \\ \Delta U_j(0.5) = 0 < 0.1 - \sin U_j(0.5) = 0.1, \\ U_j^4(0) \sin U_j(0) - U_j^2(1) - U_j(1) = -(2j\pi)^2 - 2j\pi < 0, \end{cases}$$

$$\begin{cases} V_j'(t)+\kappa D^{\frac{1}{2}}V_j(t)=1+\frac{2\kappa\sqrt{t}}{\sqrt{\pi}}\geq t(1+V_j^{\beta}(t)), & t\neq 0.5,\\ \Delta V_j(0.5)=0>0.1-\sin V_j(0.5)=0.1-\sin(1.5), \\ V_j^4(0)\sin V_j(0)-V_j^2(1)-V_j(1) \\ =(2j\pi+1)^4\sin(1.5)-(2j\pi+2)^2-2j\pi-2>0, \end{cases}$$

which imply that U_j and V_j are the lower and upper solutions of (3.5), respectively. Hence, (H_1) holds. Obviously, $f\colon J\times\mathbb{R}\to\mathbb{R}$, $I_1\colon\mathbb{R}\to\mathbb{R}$ and $g\colon\mathbb{R}^2\to\mathbb{R}$ are continuous. In addition, $f(\cdot,s)$ is nondecreasing in $(0,+\infty)$ and $g(\cdot,v)$ is nonincreasing in $(0,+\infty)$. Moreover, there exists $b_1=1$ such that

$$I_1(\xi)+b_1\xi\geq I_1(\zeta)+b_1\zeta$$

for $\xi\geq\zeta$. Therefore, $(H_2)-(H_4)$ holds. It follows from Theorem 1 that (3.5) has solutions $x_j\in[U_j,V_j](j=1,\ldots,l)$.

Example 2. Consider the equation

$$\begin{cases} x'(t)-\frac{1}{2}D^{\frac{1}{2}}x(t)=\frac{t}{4}\left[\frac{2+2x(t)}{2+x(t)}-\frac{1}{40}x^2(t)\right], & t\neq t_1,t_2,\\ \Delta x(t_k)=\frac{1}{4+k}\ln(1+x^2(t_k)), & k=1,2,\\ 100x(0)+x^3(1)-15x(1)=0, \end{cases}\tag{3.6}$$

where $0<t_1<t_2<1$.

In fact,

$$f(t,s)=\frac{t}{4}\left[\frac{2+2s}{2+s}-\frac{s^2}{40}\right],\quad I_k(s)=\frac{1}{4+k}\ln(1+s^2),\quad g(u,v)=100u+v^3-15v.$$

Let

$$U(t)=0,\quad V(t)=\begin{cases}\frac{1}{4}+t, & 0\leq t\leq t_1,\\ \frac{1}{2}+t, & t_1<t\leq t_2,\\ 1+t, & t_2<t\leq 1.\end{cases}$$

Clearly, U is a lower solution of (3.6). In addition,

$$V'(t)-\frac{1}{2}D^{\frac{1}{2}}V(t)=1-\frac{\sqrt{t}}{\sqrt{\pi}}>\frac{t}{3}\geq f(t,V(t)),$$

$$\Delta V(t_1)=\frac{1}{4}>\frac{1}{5}\ln(1+(0.25+t_1)^2),\quad \Delta V(t_2)=\frac{1}{2}>\frac{1}{6}\ln(1+(0.5+t_2)^2),$$

$$g(V(0),V(1))=3.$$

So, V is a upper solution of (3.6) and (H_1) holds. Moreover, $f\colon J\times\mathbb{R}\to\mathbb{R}$ and $I_1,I_2\colon\mathbb{R}\to\mathbb{R}$ are continuous. For f, we have

$$f_s(t,s)=\frac{t}{4}\left[\frac{40-s(2+s)^2}{20(2+s)^2}\right]\geq 0,\qquad \forall s\in[0,2],\ t\in[0,1].$$

Hence, $f(t,s)$ is nondecreasing in $s \in [0,2]$. There exist $b_1 = b_2 = 0$ such that for $0 \le \zeta \le \xi \le 2$, $k = 1,2$,

$$I_k(\xi) + b_k\xi = \frac{1}{4+k}\ln(1+\xi^2) \ge I_k(\zeta) + b_k\zeta = \frac{1}{4+k}\ln(1+\zeta^2).$$

Hence, (H_2) and (H_4) are satisfied.

In addition, $g\colon \mathbb{R}^2 \to \mathbb{R}$ are continuous and

$$\begin{aligned} g(\overline{y_1},\overline{y_2}) - g(y_1,y_2) &= 100(\overline{y_1} - y_1) + (\overline{y_2}^2 + y_2\overline{y_2} + y_2^2 - 15)(\overline{y_2} - y_2) \\ &\le 100(\overline{y_1} - y_1) - 3(\overline{y_2} - y_2) \end{aligned}$$

for $0 \le y_i \le \overline{y_i} \le 2, i = 1,2$. Therefore, (H_5) holds. It follows from Theorem 2 that there exist monotone iterative sequences $\{u_j\},\{v_j\}$ which converge to the extremal solutions u^*,v^* of (3.6),respectively.

Remark 2. Even if $I_k \equiv 0$ and $g(u,v) = u - v$, our results are also new because one of the prerequisites of [22] is that $A > 0$ in (1.2), which is equivalent to the condition $M < 0$ in (1.1). Our conditions are different from those in [22].

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CHARACTERIZATION OF LIPSCHITZ FUNCTIONS VIA COMMUTATORS OF MAXIMAL OPERATORS ON SLICE SPACES

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Abstract. Let $0 \leq \alpha < n$, M_α be the fractional maximal operator, $M^\sharp$ be the sharp maximal operator and b be the locally integrable function. Denote by $[b, M_\alpha]$ and $[b, M^\sharp]$ be the commutators of the fractional maximal operator M_α and the sharp maximal operator $M^\sharp$. In this paper, we show some necessary and sufficient conditions for the boundedness of the commutators $[b, M_\alpha]$ and $[b, M^\sharp]$ on slice spaces when the function b is the Lipschitz function, by which some new characterizations of the non-negative Lipschitz function are obtained.

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Keywords: fractional maximal operator, sharp maximal operator, commutator, Lipschitz function, slice space

1. INTRODUCTION AND MAIN RESULTS

Let T be the classical singular integral operator and b be the locally integrable function, the commutator $[b, T]$ is defined by

$$[b, T]f(x) = bTf(x) - T(bf)(x).$$

The well-known result of Coifman, Rochberg and Weiss[6] showed that the commutator $[b, T]$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$ if and only if $b \in BMO(\mathbb{R}^n)$. The bounded mean oscillation space $BMO(\mathbb{R}^n)$ was introduced by John and Nirenberg [14], which is defined as the set of all locally integrable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{BMO(\mathbb{R}^n)} := \sup_Q \frac{1}{|Q|} \int_Q |f(x) - f_Q| \, dx < \infty,$$

where the supremum is taken over all cubes in $\mathbb{R}^n$ and $f_Q := \frac{1}{|Q|} \int_Q f(x)dx$. In 1978, Janson[12] obtained some characterizations of the Lipschitz space $\dot{\Lambda}_\beta(\mathbb{R}^n)$ via the commutator $[b, T]$ and proved that $[b, T]$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ if and

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only if $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)\,(0 < \beta < 1)$, where $1 < p < n/\beta$ and $1/p - 1/q = \beta/n$ (see also Paluszyński[17]). Recently, the commutators have been studied intensively by many authors, which plays an important role in harmonic analysis and partial differential equations (see, for example, [1,5,10,11,19,21]).

As usual, a cube $Q \subset \mathbb{R}^n$ always means its sides parallel to the coordinate axes. Denote by $|Q|$ the Lebesgue measure of Q and χ_Q the characteristic function of Q. For $1 \le p \le \infty$, we denote by p' the conjugate index of p, namely, $p' = p/(p-1)$. We always denote by C a positive constant which is independent of the main parameters, but it may vary from line to line. The symbol $f \lesssim g$ means that $f \le Cg$. If $f \lesssim g$ and $g \lesssim f$, we then write $f \sim g$.

Let $0 \le \alpha < n$, for a locally integrable function f, the maximal operator M_α is given by

$$M_\alpha(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|^{1-\alpha/n}} \int_Q |f(y)|dy,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ containing x.

When $\alpha = 0$, M_0 is the classical Hardy-Littlewood maximal operator M, and M_α is the classical fractional maximal operator when $0 < \alpha < n$.

The sharp maximal operator $M^\sharp$ was introduced by Fefferman and Stein [9], which is defined as

$$M^\sharp f(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q|\,dy,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ containing x.

The maximal commutator of the fractional maximal operator M_α with the locally integrable function b is given by

$$M_{\alpha,b}(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|^{1-\alpha/n}} \int_Q |b(x) - b(y)||f(y)|dy,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ containing x.

The nonlinear commutators of the fractional maximal operator M_α and sharp maximal operator $M^\sharp$ with the locally integrable function b are defined as

$$[b, M_\alpha]\,(f)(x) = b(x)M_\alpha(f)(x) - M_\alpha(bf)(x)$$

and

$$[b, M^\sharp](f)(x) = b(x)M^\sharp(f)(x) - M^\sharp(bf)(x).$$

When $\alpha = 0$, we simply write by $[b,M] = [b,M_0]$ and $M_b = M_{0,b}$. We also remark that the commutators $M_{\alpha,b}$ and $[b,M_\alpha]$ essentially differ from each other. For example, maximal commutator $M_{\alpha,b}$ is positive and sublinear, but nonlinear commutators $[b,M_\alpha]$ and $\left[b,M^\sharp\right]$ are neither positive nor sublinear. The study of the mapping properties of commutators of maximal operators has been widely explored, we refer the readers to see [8,18,20,22,23,25] and therein references.

To state our results, we first present some definitions and notations.

Definition 1. Let $0 < \beta < 1$, we say a function b belongs to the Lipschitz space $\dot{\Lambda}_\beta(\mathbb{R}^n)$ if there exists a constant C such that for all $x, y \in \mathbb{R}^n$,

$$|b(x) - b(y)| \leq C|x-y|^\beta.$$

The smallest such constant C is called the $\dot{\Lambda}_\beta$ norm of the function b and is denoted by $\|b\|_{\dot{\Lambda}_\beta}$.

In 2019, Auscher and Mourgoglou [2] introduced the slice space $(E_2^p)_t(\mathbb{R}^n)$ with $0 < t < \infty$ and $1 < p < \infty$, they studied the weak solutions of boundary value problems with a t-independent elliptic systems in the upper half plane. Recently, Auscher and Prisuelos-Arribas[3] obtained the boundedness of some classical operators on the slice space $(E_r^p)_t(\mathbb{R}^n)$ with $0 < t < \infty$ and $1 < p, r < \infty$.

For $0 < p < \infty$, the Lebesgue space $L^p(\mathbb{R}^n)$ is defined as the set of all measurable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{L^p(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} |f(x)|^p \, \mathrm{d}x\right)^{\frac{1}{p}} < \infty.$$

Definition 2. Let $0 < t < \infty$ and $1 < r, p < \infty$. The slice space $(E_r^p)_t(\mathbb{R}^n)$ is defined as the set of all locally r-integrable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{(E_r^p)_t(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} \left(\frac{1}{|Q(x,t)|}\int_{Q(x,t)} |f(y)|^r dy\right)^{\frac{p}{r}} dx\right)^{\frac{1}{p}} < \infty.$$

If we take $r = p$, then the slice space $(E_r^p)_t(\mathbb{R}^n)$ is the Lebesgue space $L^p(\mathbb{R}^n)$. For a cube Q, we denote by $\|f\|_{(E_r^p)_t(Q)} = \|f\chi_Q\|_{(E_r^p)_t(\mathbb{R}^n)}$.

For a fixed cube Q and $0 \leq \alpha < n$, the maximal operator with respect to Q of a function f is given by

$$M_{\alpha,Q}(f)(x) = \sup_{Q \supseteq Q_0 \ni x} \frac{1}{|Q_0|^{1-\alpha/n}} \int_{Q_0} |f(y)| dy,$$

where the supremum is taken over all the cubes Q_0 with $Q_0 \subseteq Q$ and $Q_0 \ni x$. Moreover, we denote by $M_Q = M_{0,Q}$ when $\alpha = 0$.

In 2017, Zhang [24] showed some characterizations via the boundedness of the commutator $[b, M]$ on Lebesgue spaces, where the function b belongs to the Lipschitz space $\dot{\Lambda}_\beta(\mathbb{R}^n)$ and $b \geq 0$.

Theorem 1. *[24] Let $0 < \beta < 1$ and b be a locally integrable function. If $1 < p < n/\beta$ and $1/q = 1/p - \beta/n$, then the following statements are equivalent:*

(1) $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ *and* $b \geq 0$.
(2) $[b, M]$ *is bounded from* $L^p(\mathbb{R}^n)$ *to* $L^q(\mathbb{R}^n)$.

(3) *there exists a constant* $C > 0$ *such that*

$$\sup_Q \frac{1}{|Q|^{\beta/n}} \left(\frac{1}{|Q|} \int_Q |b(x) - M_Q(b)(x)|^q \, \mathrm{d}x \right)^{1/q} \leq C.$$

Next, we recall the result of [22], which showed some characterizations via the boundedness of the commutator $[b, M]$ on slice spaces, where the function b belongs to the Lipschitz space $\dot{\Lambda}_\beta(\mathbb{R}^n)$ and $b \geq 0$.

Theorem 2. *[22] Let* $0 < \beta < 1$, $0 < t < \infty$ *and* b *be a locally integrable function. If* $1 < p < r < \infty$, $1 < q < s < \infty$ *and* $\beta/n = 1/p - 1/r = 1/q - 1/s$, *then the following statements are equivalent:*

(1) $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ *and* $b \geq 0$.
(2) $[b, M]$ *is bounded from* $(E_p^q)_t(\mathbb{R}^n)$ *to* $(E_r^s)_t(\mathbb{R}^n)$.
(3) *There exists a constant* $C > 0$ *such that*

$$\sup_Q \frac{1}{|Q|^{\beta/n+1/s}} \|b(\cdot) - M_Q(b)(\cdot)\|_{(E_r^s)_t(Q)} \leq C.$$

Our first result can be stated as follows.

Theorem 3. *Let* $0 < \beta < 1$, $0 \leq \alpha < n$, $0 < \alpha + \beta < n$, $0 < t < \infty$ *and* b *be a locally integrable function. If* $1 < p < r < \infty$, $1 < q < s < \infty$ *and* $(\alpha + \beta)/n = 1/p - 1/r = 1/q - 1/s$, *then the following statements are equivalent:*

(1) $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ *and* $b \geq 0$.
(2) $[b, M_\alpha]$ *is bounded from* $(E_p^q)_t(\mathbb{R}^n)$ *to* $(E_r^s)_t(\mathbb{R}^n)$.
(3) *There exists a constant* $C > 0$ *such that*

$$\sup_Q \frac{1}{|Q|^{\beta/n+1/s}} \|b(\cdot) - M_Q(b)(\cdot)\|_{(E_r^s)_t(Q)} \leq C. \tag{1.1}$$

(4) *There exists a constant* $C > 0$ *such that*

$$\sup_Q \frac{1}{|Q|^{1+\beta/n}} \int_Q |b(x) - M_Q(b)(x)| \, dx \leq C. \tag{1.2}$$

Here is the second result.

Theorem 4. *Let* $0 < \beta < 1$, $0 \leq \alpha < n$, $0 < \alpha + \beta < n$, $0 < t < \infty$ *and* b *be a locally integrable function. If* $1 < p < r < \infty$, $1 < q < s < \infty$ *and* $(\alpha + \beta)/n = 1/p - 1/r = 1/q - 1/s$, *then the following statements are equivalent:*

(1) $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$.
(2) $M_{\alpha,b}$ *is bounded from* $(E_p^q)_t(\mathbb{R}^n)$ *to* $(E_r^s)_t(\mathbb{R}^n)$.
(3) *There exists a constant* $C > 0$ *such that*

$$\sup_Q \frac{1}{|Q|^{\beta/n+1/s}} \|b(\cdot) - b_Q\|_{(E_r^s)_t(Q)} \leq C. \tag{1.3}$$

(4) *There exists a constant $C > 0$ such that*

$$\sup_Q \frac{1}{|Q|^{1+\beta/n}} \int_Q |b(x) - b_Q|\, dx \le C. \tag{1.4}$$

Finally, we obtain the following result.

Theorem 5. *Let $0 < \beta < 1$, $0 < t < \infty$ and b be a locally integrable function. If $1 < p < r < \infty$, $1 < q < s < \infty$ and $\beta/n = 1/p - 1/r = 1/q - 1/s$, then the following statements are equivalent:*

(1) $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ *and* $b \ge 0$.

(2) $\left[b, M^\sharp\right]$ *is bounded from* $(E_p^q)_t(\mathbb{R}^n)$ *to* $(E_r^s)_t(\mathbb{R}^n)$.

(3) *There exists a constant $C > 0$ such that*

$$\sup_Q \frac{1}{|Q|^{\beta/n+1/s}} \left\| b(\cdot) - 2M^\sharp(b\chi_Q)(\cdot) \right\|_{(E_r^s)_t(Q)} \le C. \tag{1.5}$$

(4) *There exists a constant $C > 0$ such that*

$$\sup_Q \frac{1}{|Q|^{1+\beta/n}} \int_Q \left| b(x) - 2M^\sharp(b\chi_Q)(x) \right| dx \le C. \tag{1.6}$$

2. Preliminaries

To prove our results, we give some necessary lemmas in this section. It is well-known that the Lipschitz space $\dot{\Lambda}_\beta(\mathbb{R}^n)$ coincides with some Morrey-Companato spaces (see [13] for example) and can be characterized by mean oscillation as the following lemma, which is due to DeVore and Sharpley [7] and Paluszyński [17].

Lemma 1. *Let $0 < \beta < 1$ and $1 \le q < \infty$. The space $\dot{\Lambda}_{\beta,q}(\mathbb{R}^n)$ is defined as the set of all locally integrable functions f such that*

$$\|f\|_{\dot{\Lambda}_{\beta,q}} = \sup_Q \frac{1}{|Q|^{\beta/n}} \left(\frac{1}{|Q|} \int_Q |f(x) - f_Q|^q\, dx \right)^{1/q} < \infty.$$

Then, for all $0 < \beta < 1$ and $1 \le q < \infty$, $\dot{\Lambda}_\beta(\mathbb{R}^n) = \dot{\Lambda}_{\beta,q}(\mathbb{R}^n)$ with equivalent norms.

Lemma 2. *[26] Let $0 \le \alpha < n$, Q be a cube in $\mathbb{R}^n$ and f be locally integrable. Then*

$$M_\alpha(f\chi_Q)(x) = M_{\alpha,Q}(f)(x), \text{ for all } x \in Q.$$

The following lemma is given by Lu, Wang and Zhou[15], they obtained that the boundedness of the fractional maximal operator M_α on slice spaces.

Lemma 3. *Let $0 < t < \infty$, $1 < p < r < \infty$ and $1 < q < s < \infty$ with $\alpha/n = 1/p - 1/r = 1/q - 1/s$ for $0 \le \alpha < n$. If $f \in (E_p^q)_t(\mathbb{R}^n)$, then*

$$\|M_\alpha f\|_{(E_r^s)_t(\mathbb{R}^n)} \le C\|f\|_{(E_p^q)_t(\mathbb{R}^n)},$$

where the positive constant C is independent of f and t.

Lemma 4. *[16] Let $0 < t < \infty$, $1 < p, r < \infty$ and Q be a cube in $\mathbb{R}^n$. Then*

$$\|\chi_Q\|_{(E_r^p)_t(\mathbb{R}^n)} \sim |Q|^{1/p}.$$

Lemma 5. *[4] For any fixed cube Q, let $E = \{x \in Q : b(x) \le b_Q\}$ and $F = \{x \in Q : b(x) > b_Q\}$. Then the following equality is true:*

$$\int_E |b(x) - b_Q|\,dx = \int_F |b(x) - b_Q|\,dx.$$

3. PROOFS OF THEOREMS 3-5

Proof of Theorem 3. (1) $\Rightarrow$ (2): Assume $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ and $b \ge 0$. For any locally integral function f, we have

$$\begin{aligned} |[b, M_\alpha](f)(x)| &= |b(x)M_\alpha(f)(x) - M_\alpha(bf)(x)| \\ &\le \sup_{Q \ni x} \frac{1}{|Q|^{1-\alpha/n}} \int_Q |b(x) - b(y)||f(y)|dy \\ &\le C\|b\|_{\dot{\Lambda}_\beta} \sup_{Q \ni x} \frac{1}{|Q|^{1-(\alpha+\beta)/n}} \int_Q |f(y)|dy \\ &\le C\|b\|_{\dot{\Lambda}_\beta} M_{\alpha+\beta}(f)(x). \end{aligned}$$

By Lemma 3, we obtain that $[b, M_\alpha]$ is bounded from $(E_p^q)_t(\mathbb{R}^n)$ to $(E_r^s)_t(\mathbb{R}^n)$.

(2) $\Rightarrow$ (3): We divide the proof into two cases based on the scope of α.

Case 1. Assume $0 < \alpha < n$. For any fixed cube Q,

$$\begin{aligned} &\frac{1}{|Q|^{\beta/n+1/s}} \|b(\cdot) - M_Q(b)(\cdot)\|_{(E_r^s)_t(Q)} \\ &\le \frac{1}{|Q|^{\beta/n+1/s}} \left\| b(\cdot) - |Q|^{-\alpha/n} M_{\alpha,Q}(b)(\cdot) \right\|_{(E_r^s)_t(Q)} \\ &\quad + \frac{1}{|Q|^{\beta/n+1/s}} \left\| |Q|^{-\alpha/n} M_{\alpha,Q}(b)(\cdot) - M_Q(b)(\cdot) \right\|_{(E_r^s)_t(Q)} \\ &:= I + II. \end{aligned}$$

For I. By the definition of $M_{\alpha,Q}$, we can see

$$M_{\alpha,Q}(\chi_Q)(x) = |Q|^{\alpha/n}, \text{ for all } x \in Q. \tag{3.1}$$

Using Lemma 2, for any $x \in Q$, we have

$$M_\alpha(\chi_Q)(x) = M_{\alpha,Q}(\chi_Q)(x) = |Q|^{\alpha/n}, M_\alpha(b\chi_Q)(x) = M_{\alpha,Q}(b)(x).$$

Thus, for any $x \in Q$,

$$\begin{aligned} b(x) - |Q|^{-\alpha/n} M_{\alpha,Q}(b)(x) &= |Q|^{-\alpha/n} \left(b(x)|Q|^{\alpha/n} - M_{\alpha,Q}(b)(x) \right) \\ &= |Q|^{-\alpha/n} \left(b(x) M_\alpha(\chi_Q)(x) - M_\alpha(b\chi_Q)(x) \right) \end{aligned}$$

$$= |Q|^{-\alpha/n}\,[b,M_\alpha]\,(\chi_Q)\,(x).$$

Since $[b,M_\alpha]$ is bounded from $(E_p^q)_t(\mathbb{R}^n)$ to $(E_r^s)_t(\mathbb{R}^n)$, then by Lemma 4 and noting that $(\alpha+\beta)/n = 1/q-1/s$, we have

$$\begin{aligned} I &= |Q|^{-\beta/n-1/s}\left\|b(\cdot)-|Q|^{-\alpha/n}M_{\alpha,Q}(b)(\cdot)\right\|_{(E_r^s)_t(Q)} \\ &= |Q|^{-(\alpha+\beta)/n-1/s}\,\|[b,M_\alpha]\,(\chi_Q)\,(\cdot)\|_{(E_r^s)_t(Q)} \\ &\le C|Q|^{-(\alpha+\beta)/n-1/s}\,\|\chi_Q\|_{(E_p^q)_t(\mathbb{R}^n)} \\ &\le C. \end{aligned}$$

Next, we estimate II. Similar to (3.1), by Lemma 2.3 and noting that

$$M_Q\,(\chi_Q)\,(x) = \chi_Q(x),\ \text{for all } x\in Q,$$

it is easy to see

$$M\,(\chi_Q)\,(x) = \chi_Q(x) \text{ and } M\,(b\chi_Q)\,(x) = M_Q(b)(x), \text{for any } x\in Q. \tag{3.2}$$

Then, by (3.1) and (3.2), for any $x\in Q$, we obtain

$$\begin{aligned} &\left||Q|^{-\alpha/n}M_{\alpha,Q}(b)(x)-M_Q(b)(x)\right| \\ &\quad\le |Q|^{-\alpha/n}\,|M_\alpha\,(b\chi_Q)\,(x)-|b(x)|M_\alpha\,(\chi_Q)\,(x)| \\ &\qquad + |Q|^{-\alpha/n}||b(x)\,|M_\alpha\,(\chi_Q)\,(x)-M_\alpha\,(\chi_Q)\,(x)M\,(b\chi_Q)\,(x)| \\ &\quad= |Q|^{-\alpha/n}\,|M_\alpha\,(|b|\chi_Q)\,(x)-|\,b(x)\,|M_\alpha\,(\chi_Q)\,(x)| \\ &\qquad + |Q|^{-\alpha/n}M_\alpha\,(\chi_Q)\,(x)||b(x)\,|M\,(\chi_Q)\,(x)-M\,(b\chi_Q)\,(x)| \\ &\quad= |Q|^{-\alpha/n}\,|[|b|,M_\alpha](\chi_Q)(x)| + |[|b|,M]\,(\chi_Q)\,(x)|\,. \end{aligned}$$

Since $[b,M_\alpha]$ is bounded from $(E_p^q)_t(\mathbb{R}^n)$ to $(E_r^s)_t(\mathbb{R}^n)$ and we can see that $b\in\dot{\Lambda}_\beta\,(\mathbb{R}^n)$ implies $|b|\in\dot{\Lambda}_\beta\,(\mathbb{R}^n)$. By the definitions of $[b,M_\alpha]$ and M_α, we have, for any $x\in Q$,

$$\begin{aligned} |[|b|,M_\alpha]\,(\chi_Q)\,(x)| &\le \sup_{Q'\ni x}\frac{1}{|Q'|^{1-\alpha/n}}\int_{Q'}|b(x)-b(y)||\chi_Q(y)|\,dy \\ &\le \|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)}\sup_{Q'\ni x}\frac{1}{|Q'|^{1-(\alpha+\beta)/n}}\int_{Q'}|\chi_Q(y)|\,dy \\ &\le \|b\|_{\dot{\Lambda}_\beta}M_{\alpha+\beta}\,(\chi_Q)\,(x) \\ &= \|b\|_{\dot{\Lambda}_\beta}|Q|^{(\alpha+\beta)/n}\chi_Q(x). \end{aligned}$$

Similarly, we can see

$$|[|b|,M](\chi_Q)(x)| \le \|b\|_{\dot{\Lambda}_\beta}|Q|^{\beta/n}\chi_Q(x),\ \text{for any } x\in Q.$$

Thus, for any $x \in Q$,

$$\left| |Q|^{-\alpha/n} M_{\alpha,Q}(b)(x) - M_Q(b)(x) \right| \le C\|b\|_{\dot{\Lambda}_\beta} |Q|^{\beta/n} \chi_Q(x).$$

Then, by Lemma 4, we have

$$\begin{aligned} II &= |Q|^{-\beta/n-1/s} \left\| |Q|^{-\alpha/n} M_{\alpha,Q}(b)(\cdot) - M_Q(b)(\cdot) \right\|_{(E_r^s)_t(Q)} \\ &\le C|Q|^{-1/s} \|\chi_Q\|_{(E_r^s)_t(Q)} \le C. \end{aligned}$$

This gives the desired estimate

$$|Q|^{-\beta/n-1/s} \|b(\cdot) - M_Q(b)(\cdot)\|_{(E_r^s)_t(Q)} \le C,$$

which leads us to (1.1) since Q is arbitrary and the constant C is dependent of Q.

Case 2. Assume $\alpha = 0$. For any fixed cube Q and any $x \in Q$, by (3.2), we can see

$$b(x) - M_Q(b)(x) = b(x)M(\chi_Q)(x) - M(b\chi_Q)(x) = [b,M](\chi_Q)(x).$$

Assume that $[b,M]$ is bounded from $(E_p^q)_t(\mathbb{R}^n)$ to $(E_r^s)_t(\mathbb{R}^n)$ and $\beta/n = 1/q - 1/s$, then by Lemma 4, we have

$$\begin{aligned} |Q|^{-\beta/n-1/s} \|b(\cdot) - M_Q(b)(\cdot)\|_{(E_r^s)_t(Q)} &= |Q|^{-\beta/n-1/s} \|[b,M](\chi_Q)(\cdot)\|_{(E_r^s)_t(Q)} \\ &\le C|Q|^{-\beta/n-1/s} \|\chi_Q\|_{(E_r^s)_t(\mathbb{R}^n)} \le C, \end{aligned}$$

which implies (1.1).

(3) $\Rightarrow$ (4): Assume (1.1) holds, then for any fixed cube Q, by Hölder's inequality and (1.1), we can see

$$\begin{aligned} &\frac{1}{|Q|^{1+\beta/n}} \int_Q |b(x) - M_Q(b)(x)|\,dx \\ &\quad \le \frac{C}{|Q|^{1+\beta/n}} \|b(\cdot) - M_Q(b)(\cdot)\|_{(E_r^s)_t(Q)} \|\chi_Q\|_{(E_{r'}^{s'})_t(\mathbb{R}^n)} \\ &\quad \le \frac{C}{|Q|^{\beta/n+1/s}} \|b(\cdot) - M_Q(b)(\cdot)\|_{(E_r^s)_t(Q)} \le C, \end{aligned}$$

where the constant C is independent of Q. Thus we have (1.2).

(4) $\Rightarrow$ (1): To prove $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$, by Lemma 1, it suffices to show that there is a constant $C > 0$ such that for any fixed cube Q,

$$\frac{1}{|Q|^{1+\beta/n}} \int_Q |b(x) - b_Q|\,dx \le C.$$

For any fixed cube Q, let $E = \{x \in Q : b(x) \le b_Q\}$ and $F = \{x \in Q : b(x) > b_Q\}$. Since for any $x \in E$, we have $b(x) \le b_Q \le M_Q(b)(x)$, then

$$|b(x) - b_Q| \le |b(x) - M_Q(b)(x)|. \tag{3.3}$$

By Lemma 5 and (3.3), we obtain

$$\begin{aligned}\frac{1}{|Q|^{1+\beta/n}}\int_Q |b(x)-b_Q|\,\mathrm{d}x &= \frac{2}{|Q|^{1+\beta/n}}\int_E |b(x)-b_Q|\,\mathrm{d}x\\ &\le \frac{2}{|Q|^{1+\beta/n}}\int_E |b(x)-M_Q(b)(x)|\,\mathrm{d}x\\ &\le \frac{2}{|Q|^{1+\beta/n}}\int_Q |b(x)-M_Q(b)(x)|\,\mathrm{d}x \le C.\end{aligned}$$

Thus we obtain $b\in\dot{\Lambda}_\beta(\mathbb{R}^n)$. Next, we will prove $b\ge 0$, it suffices to show $b^-=0$, where $b^-=-\min\{b,0\}$. Let $b^+=|b|-b^-$, then $b=b^+-b^-$. For any fixed cube Q and $x\in Q$, we observe that

$$0\le b^+(x)\le |b(x)|\le M_Q(b)(x),$$

then it is easy to see

$$0\le b^-(x)\le M_Q(b)(x)-b^+(x)+b^-(x)=M_Q(b)(x)-b(x).$$

Combining with the above estimates and (1.2), we obtain

$$\begin{aligned}\frac{1}{|Q|}\int_Q b^-(x)\mathrm{d}x &\le \frac{1}{|Q|}\int_Q |M_Q(b)(x)-b(x)|\\ &\le |Q|^{\beta/n}\left(\frac{1}{|Q|^{1+\beta/n}}\int_Q |b(x)-M_Q(b)(x)|\,dx\right)\le C|Q|^{\beta/n}.\end{aligned}$$

Thus, $b^-=0$ follows from Lebesgue's differentiation theorem.

This completes the proof of Theorem 3. □

Proof of Theorem 4. (1) ⇒ (2): Assume $b\in\dot{\Lambda}_\beta(\mathbb{R}^n)$. For any fixed cube $Q\subset\mathbb{R}^n$, we have

$$\begin{aligned}M_{\alpha,b}(f)(x) &= \sup_{Q\ni x}\frac{1}{|Q|^{1-\alpha/n}}\int_Q |b(x)-b(y)||f(y)|dy\\ &\le C\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)}M_{\alpha+\beta}f(x).\end{aligned}$$

By Lemma 3, we obtain that $M_{\alpha,b}$ is bounded from $(E_p^q)_t(\mathbb{R}^n)$ to $(E_r^s)_t(\mathbb{R}^n)$.

(2) ⇒ (3): For any fixed cube $Q\subset\mathbb{R}^n$ and all $x\in Q$, we have

$$\begin{aligned}|b(x)-b_Q| &\le \frac{1}{|Q|}\int_Q |b(x)-b(y)|dy\\ &= \frac{1}{|Q|^{\alpha/n}}\frac{1}{|Q|^{1-\alpha/n}}\int_Q |b(x)-b(y)|\chi_Q(y)dy\\ &\le |Q|^{-\alpha/n}M_{\alpha,b}(\chi_Q)(x).\end{aligned}$$

Since $M_{\alpha,b}$ is bounded from $(E_p^q)_t(\mathbb{R}^n)$ to $(E_r^s)_t(\mathbb{R}^n)$, then by Lemma 4 and noting that $(\alpha+\beta)/n = 1/q - 1/s$, we obtain

$$\begin{aligned}\frac{1}{|Q|^{\beta/n+1/s}}\|b(\cdot)-b_Q\|_{E_r^s)_t(Q)} &\le |Q|^{-(\alpha+\beta)/n-1/s}\|M_{\alpha,b}(\chi_Q)(\cdot)\|_{(E_r^s)_t(Q)}\\ &\le C|Q|^{-(\alpha+\beta)/n-1/s}\|\chi_Q\|_{(E_p^q)_t(\mathbb{R}^n)} \le C,\end{aligned}$$

which implies (1.3) since the cube $Q \subset \mathbb{R}^n$ is arbitrary.

(3) $\Rightarrow$ (4): Assume (1.3) holds, we will prove (1.4). For any fixed cube Q, by Hölder's inequality and Lemma 4, it is easy to see

$$\begin{aligned}\frac{1}{|Q|^{1+\beta/n}}\int_Q |b(x)-b_Q|\,dx &\le \frac{C}{|Q|^{1+\beta/n}}\|b(\cdot)-b_Q\|_{(E_r^s)_t(Q)}\|\chi_Q\|_{(E_{r'}^{s'})_t(\mathbb{R}^n)}\\ &\le \frac{C}{|Q|^{\beta/n+1/s}}\|b(\cdot)-b_Q\|_{(E_r^s)_t(Q)} \le C.\end{aligned}$$

(4) $\Rightarrow$ (1): It follows from Lemma 1 directly, thus we omit the details.

The proof of Theorem 4 is finished. □

Proof of Theorem 5. (1) $\Rightarrow$ (2): Assume $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ and $b \ge 0$. For any locally integral function f, the following estimate was given in [25]:

$$\left|[b,M^\sharp]f(x)\right| \le C\|b\|_{\dot{\Lambda}_\beta} M_\beta(f)(x).$$

Then, by Lemma 3, we obtain that $\left[b,M^\sharp\right]$ is bounded from $(E_p^q)_t(\mathbb{R}^n)$ to $(E_r^s)_t(\mathbb{R}^n)$.

(2) $\Rightarrow$ (3): Assume $[b,M^\sharp]$ is bounded from $(E_p^q)_t(\mathbb{R}^n)$ to $(E_r^s)_t(\mathbb{R}^n)$, we will prove (1.5). For any fixed cube Q, we have (see [4] for details)

$$M^\sharp(\chi_Q)(x) = 1/2, \text{ for all } x \in Q.$$

Then, for all $x \in Q$,

$$\begin{aligned}b(x) - 2M^\sharp(b\chi_Q)(x) &= 2\left(b(x)M^\sharp(\chi_Q)(x) - M^\sharp(b\chi_Q)(x)\right)\\ &= 2[b,M^\sharp](\chi_Q)(x).\end{aligned}$$

Since $[b,M^\sharp]$ is bounded from $(E_p^q)_t(\mathbb{R}^n)$ to $(E_r^s)_t(\mathbb{R}^n)$, then applying Lemma 4 and noting that $\beta/n = 1/q - 1/s$, we obtain

$$\begin{aligned}|Q|^{-\beta/n-1/s}\left\|b(\cdot)-2M^\sharp(b\chi_Q)(\cdot)\right\|_{(E_r^s)_t(Q)} &= 2|Q|^{-\beta/n-1/s}\left\|[b,M^\sharp](\chi_Q)\right\|_{(E_r^s)_t(Q)}\\ &\le C|Q|^{-\beta/n-1/s}\|\chi_Q\|_{(E_p^q)_t(\mathbb{R}^n)} \le C,\end{aligned}$$

where the constant C is independent of Q. Then we achieve (1.5).

(3) $\Rightarrow$ (4): Assume (1.5) holds, we will prove (1.6). For any fixed cube Q, it follows from Hölder's inequality and (1.5) that

$$\frac{1}{|Q|^{1+\beta/n}}\int_Q \left|b(x)-2M^\sharp(b\chi_Q)(x)\right|dx$$

$$\leq C|Q|^{-\beta/n-1/s}\left\|b(\cdot)-2M^{\sharp}(b\chi_Q)(\cdot)\right\|_{(E^q_p)_t(Q)}\leq C,$$

which implies (1.6) since the constant C is independent of Q.

(4) $\Rightarrow$ (1): We first prove $b\in\dot{\Lambda}_\beta(\mathbb{R}^n)$. For any fixed cube Q, the following estimate was given in [4]:

$$\frac{1}{|Q|}\int_Q|b(x)-b_Q|\,dx\leq\frac{2}{|Q|}\int_Q\left|b(x)-2M^{\sharp}(b\chi_Q)(x)\right|dx.$$

Then by (1.6), we have

$$\frac{1}{|Q|^{1+\beta/n}}\int_Q|b(x)-b_Q|\,dx\leq\frac{2}{|Q|^{1+\beta/n}}\int_Q\left|b(x)-2M^{\sharp}(b\chi_Q)(x)\right|dx\leq C,$$

which leads to $b\in\dot{\Lambda}_\beta(\mathbb{R}^n)$ by Lemma 1.

Now, let us prove $b\geq 0$. It suffices to show $b^-=0$, where $b^-=-\min\{b,0\}$ and let $b^+=|b|-b^-$. For any fixed cube Q, we have (see [4] for details)

$$|b_Q|\leq 2M^{\sharp}(b\chi_Q)(x),\text{ for any }x\in Q.$$

Then, for all $x\in Q$,

$$2M^{\sharp}(b\chi_Q)(x)-b(x)\geq|b_Q|-b(x)=|b_Q|-b^+(x)+b^-(x).$$

By (1.6), we obtain

$$|b_Q|-\frac{1}{|Q|}\int_Q b^+(x)dx+\frac{1}{|Q|}\int_Q b^-(x)dx\leq C|Q|^{\beta/n},\tag{3.4}$$

where the constant C is independent of Q.

Let the side length of Q tends to 0 (then $|Q|\to 0$) with $x\in Q$. By Lebesgue's differentiation theorem, we obtain that the limit of the left-hand side of (3.4) equals to

$$|b(x)|-b^+(x)+b^-(x)=2b^-(x)=2\left|b^-(x)\right|.$$

Moreover, the right-hand side of (3.4) tends to 0. Thus, we have $b^-=0$.

The proof of Theorem 5 is completed. □

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HOMOLOGICAL CHARACTERIZATIONS OF G-KRULL DOMAINS AND G-DEDEKIND DOMAINS

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Abstract. Gorenstein Krull domains (G-Krull domains) are defined as domains R that satisfy the following three conditions: (1) For each prime ideal $\mathfrak{p}$ of R of height one, $R_{\mathfrak{p}}$ is a Gorenstein ring. (2) $R = \bigcap R_{\mathfrak{p}}$, where $\mathfrak{p}$ ranges over all prime ideals of R of height one. (3) Any nonzero element of R lies in only a finite number of prime ideals of height one. In this paper, we aim to characterize G-Krull domains from the perspective of Gorenstein homological algebra, similar to Gorenstein Dedekind domains (G-Dedekind domains). To achieve this objective, we introduce the notion of w-locally Gorenstein projective modules (G-projective modules). An R-module M is called w-locally Gorenstein projective if $M_{\mathfrak{m}}$ is G-projective for any maximal w-ideal $\mathfrak{m}$ of R. We show that a domain R is G-Krull if and only if R is a strong Mori domain and every w-ideal of R is w-locally G-projective. Additionally, we establish that a domain R is G-Dedekind if and only if R is a Noetherian domain and every maximal ideal of R is G-projective.

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1. INTRODUCTION

All rings considered in this paper are commutative with identity. In [7], Qiao and Wang introduced Gorenstein Krull domains (for short, G-Krull domains). A domain R is called a *G-Krull domain* if R satisfies the following three conditions: (1) For each prime ideal $\mathfrak{p}$ of R of height one, $R_{\mathfrak{p}}$ is a Gorenstein ring. (2) $R = \bigcap R_{\mathfrak{p}}$, where $\mathfrak{p}$ ranges over all prime ideals of R of height one. (3) Any nonzero element of R lies in only a finite number of prime ideals of height one ([7, Page 48, Definition]). R is said to be *Gorenstein* if $\mathrm{id}_{R_{\mathfrak{m}}} R_{\mathfrak{m}} < \infty$ (the injective dimension of $R_{\mathfrak{m}}$ as an $R_{\mathfrak{m}}$-module)

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for any maximal ideal $\mathfrak{m}$ of R (refer to [9, Definition 4.6.12]). They characterized G-Krull domains in the context of various conditions, including the following: a domain R is G-Krull if and only if R is an SM domain, and any nonzero w-ideal of R is a v-ideal. Recently, Xing ([12]) characterized G-Krull domains in terms of w-factor rings: an integral domain is G-Krull if and only if, for any nonzero nonunit, the Gorenstein global dimension of its w-factor ring is zero. SM domains are the analog of Noetherian domains. A domain R is called *strong Mori* (for short, SM) if R satisfies the ascending chain condition on w-ideals of R ([10, Definition 4]). Note that a domain R is a G-Dedekind domain if and only if R is a Noetherian domain and any nonzero ideal of R is a v-ideal ([1, Proposition 1.5]). Thus, G-Krull domains can be viewed as the w-version of G-Dedekind domains. G-Dedekind domains were introduced by Mahdou and Tamekkante in [6]. Recall that a domain R is called *G-Dedekind* if every submodule of a projective R-module is G-projective ([6, Definition 2.1(1)]). An R-module M is said to be *Gorenstein projective* (for short, G-projective) if there exists an exact sequence $\cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \dots$ of projective R-modules with $M = \mathrm{Ker}(P^0 \to P^0)$ such that $\mathrm{Hom}_R(-,Q)$ leaves the sequence exact whenever Q is a projective R-module.

In this paper, we aim to characterize G-Krull domains from the Gorenstein homological algebra point of view, similar to G-Dedekind domains. To attain this objective, for convenience, we introduce the notion of w-locally G-projective modules. An R-module M is called *w-locally G-projective* if $M_\mathfrak{m}$ is G-projective for any maximal w-ideal $\mathfrak{m}$ of R. Then we obtain that a domain R is G-Krull if and only if R is an SM domain and every w-ideal of R is w-locally G-projective; if and only if R is an SM domain and every prime (or maximal) w-ideal of R is w-locally G-projective.

Now recall some definitions and notations from [3] and [9]. Let R be an integral domain with quotient field K and let $F(R)$ denote the set of all fractional ideals of R. A *star operation* on R is a mapping $*\colon F(R) \to F(R)$ satisfying: for any $A,B \in F(R)$ and $0 \neq a \in K$, we have

(1) $(aR)_* = aR$ and $(aA)_* = aA_*$;
(2) If $A \subseteq B$, then $A_* \subseteq B_*$;
(3) $A \subseteq A_*$ and $(A_*)_* = A_*$.

For any fractional ideal A of R, A is called a *fractional $*$-ideal* if $A_* = A$, and A is called a *$*$-ideal* if A is an ideal of R and $A_* = A$. For the introduction of star operations, one may consult [3, Section 32 and 34] or [9, Section 7.2]. Examples of classical star operations include the v-operation, where $A_v = (A^{-1})^{-1}$ and $A^{-1} = \{x \in K \mid xA \subseteq R\}$, the t-operation, where

$$A_t = \bigcup\{B_v \mid B \text{ is taken over all finitely generated fractional subideals of } A\},$$

and the w-operation, where

$$A_w = \{x \in A \otimes K \mid xJ \subseteq A \text{ for some finitely generated ideal } J \text{ of } R \text{ with } J^{-1} = R\}.$$

A nonzero ideal $\mathfrak{p}$ of R is said to be *a prime w-ideal* if $\mathfrak{p}$ is both a prime ideal and a w-ideal, denoted by $\mathfrak{p} \in w\text{-Spec}(R)$; and *a maximal w-ideal* if $\mathfrak{p}$ is maximal in the set of all proper w-ideals of R, denoted by $\mathfrak{p} \in w\text{-Max}(R)$. Each maximal w-ideal is prime. The *w-Krull dimension or w-dimension* of R, denoted by $w\text{-dim}(R)$, is the supremum of the heights of all maximal w-ideals of R.

Any unexplained terminology is standard as in [9].

2. Main results

To characterize G-Krull domains in terms of w-locally G-projective modules, we begin with the following lemmas.

Lemma 1 ([8], Proposition 1.8)**.** *Let S be a multiplicative subset of R, M be an R-module, and N be an R_S-module. Then the natural R_S-homomorphism*

$$\theta\colon \mathrm{Hom}_R(M,N) \to \mathrm{Hom}_{R_S}(M_S,N)$$

is an isomorphism.

Lemma 2. *Let S be a multiplicative subset of R, M be an R-module, and N be an R_S-module. Then the natural R_S-homomorphism*

$$\theta\colon \mathrm{Ext}^1_R(M,N) \to \mathrm{Ext}_{R_S}(M_S,N)$$

is an isomorphism.

Proof. This is followed by setting $T := R_S$, $A := M$, $M := T$, and $X := N$ in [9, Theorem 3.3.11]. □

Lemma 3 ([9], Theorem 7.4.13)**.** *A domain R is an SM domain if and only if $R_\mathfrak{m}$ is a Noetherian domain for any maximal w-ideal $\mathfrak{m}$ of R, and each nonzero element of R lies in only finitely many maximal w-ideals of R.*

Theorem 1. *The following statements are equivalent for a domain R.*

(1) *R is an SM domain and* $\mathrm{id}_R R_\mathfrak{m} \le 1$ *for any maximal w-ideal* $\mathfrak{m}$ *of R.*
(2) *R is an SM domain and* $\mathrm{id}_{R_\mathfrak{m}} R_\mathfrak{m} \le 1$ *for any maximal w-ideal* $\mathfrak{m}$ *of R.*
(3) *R is an SM domain and every w-ideal of R is w-locally G-projective.*
(4) *R is an SM domain and every prime w-ideal of R is w-locally G-projective.*
(5) *$R_\mathfrak{m}$ is a G-Dedekind domain for any maximal w-ideal $\mathfrak{m}$ of R, and each nonzero element of R lies in only finitely many maximal w-ideals of R.*
(6) *R is a G-Krull domain.*

Proof. (1) ⇒ (2) Let $\mathfrak{m}$ be a maximal w-ideal of R. Then $0 \to R_\mathfrak{m} \to K \to K/R_\mathfrak{m} \to 0$ is an exact sequence of both $R_\mathfrak{m}$-modules and R-modules. Since $\mathrm{id}_R R_\mathfrak{m} \le 1$, we can get that $K/R_\mathfrak{m}$ is an injective R-module. Let $I_\mathfrak{m}$ be an ideal of $R_\mathfrak{m}$, where I is an ideal of R. Then $\mathrm{Ext}^1_{R_\mathfrak{m}}(R_\mathfrak{m}/I_\mathfrak{m}, K/R_\mathfrak{m}) \cong \mathrm{Ext}^1_R(R/I, K/R_\mathfrak{m}) = 0$ by Lemma 2, which implies that $K/R_\mathfrak{m}$ is an injective $R_\mathfrak{m}$-module. Thus $\mathrm{id}_{R_\mathfrak{m}} R_\mathfrak{m} \le 1$.

(2) $\Rightarrow$ (1) Let $\mathfrak{m}$ be a maximal w-ideal of R. If $\mathrm{id}_{R_\mathfrak{m}} R_\mathfrak{m} \leq 1$, then $K/R_\mathfrak{m}$ is an injective $R_\mathfrak{m}$-module. Thus, $K/R_\mathfrak{m}$ is an injective R-module by [9, Exercise 3.16]. So, $\mathrm{id}_R R_\mathfrak{m} \leq 1$.

(2) $\Leftrightarrow$ (5) It follows by [9, Theorem 11.7.7] and Lemma 3.

(3) $\Leftrightarrow$ (5) Note that $I_\mathfrak{m} = (I_w)_\mathfrak{m}$ for any maximal w-ideal $\mathfrak{m}$ of R. Then it follows by [9, Theorem 11.7.7] and Lemma 3.

(3) $\Rightarrow$ (4) It is trivial.

(4) $\Rightarrow$ (3) Let

$$\Gamma = \{I \mid I \text{ is a } w\text{-ideal of } R \text{ and } I \text{ is not a } w\text{-locally G-projective } R\text{-module}\}.$$

If Γ is not empty, then Γ has a maximal element by [9, Theorem 6.8.5]. Let $\mathfrak{p}$ be a maximal element of Γ. If $\mathfrak{p}$ is a prime ideal of R, then $\mathfrak{p}$ is a w-locally G-projective R-module by (4), a contradiction with $\mathfrak{p} \in \Gamma$. Therefore, Γ is empty, i.e., every w-ideal of R is w-locally G-projective.

Next, we show that $\mathfrak{p}$ is a prime ideal of R. If not, then there exist some $a, b \in R \setminus \mathfrak{p}$ such that $ab \in \mathfrak{p}$. Then $\mathfrak{p} \subsetneq (\mathfrak{p} :_R Ra)$ and $\mathfrak{p} \subsetneq (\mathfrak{p} + Ra)_w$. Note that $(\mathfrak{p} :_R Ra)$ is a w-ideal of R. Then both $(\mathfrak{p} :_R Ra)$ and $(\mathfrak{p} + Ra)_w$ are w-locally G-projective R-modules. Consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & (\mathfrak{p} :_R Ra) & = & (\mathfrak{p} :_R Ra) & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathfrak{p} & \longrightarrow & L & \longrightarrow & R & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow \varphi & & \\
0 & \longrightarrow & \mathfrak{p} & \longrightarrow & (\mathfrak{p} + Ra)_w & \longrightarrow & (\mathfrak{p} + Ra)_w/\mathfrak{p} & \longrightarrow & 0,
\end{array}$$

where $\varphi(r) = ra + \mathfrak{p}$ for any $r \in R$. Let $\mathfrak{m}$ be a maximal w-ideal. Then $(\mathfrak{p} :_R Ra)_\mathfrak{m} = (\mathfrak{p}_\mathfrak{m} :_{R_\mathfrak{m}} aR_\mathfrak{m})$ and $((\mathfrak{p} + Ra)_w/\mathfrak{p})_\mathfrak{m} = (\mathfrak{p}_\mathfrak{m} + aR_\mathfrak{m})/\mathfrak{p}_\mathfrak{m}$. Note that

$$0 \longrightarrow (\mathfrak{p}_\mathfrak{m} :_{R_\mathfrak{m}} aR_\mathfrak{m}) \longrightarrow R_\mathfrak{m} \xrightarrow{\varphi_\mathfrak{m}} \frac{\mathfrak{p}_\mathfrak{m} + aR_\mathfrak{m}}{\mathfrak{p}_\mathfrak{m}} \longrightarrow 0$$

is an exact sequence. Then $0 \to (\mathfrak{p} :_R Ra)_\mathfrak{m} \to L_\mathfrak{m} \to (\mathfrak{p} + Ra)_\mathfrak{m} \to 0$ is an exact sequence. Since $(\mathfrak{p} :_R Ra)_\mathfrak{m}$ and $(\mathfrak{p} + Ra)_\mathfrak{m}$ are G-projective $R_\mathfrak{m}$-modules, we can conclude that $L_\mathfrak{m}$ is a G-projective $R_\mathfrak{m}$-module. Note that $0 \to \mathfrak{p}_\mathfrak{m} \to L_\mathfrak{m} \to R_\mathfrak{m} \to 0$ is a split exact sequence of $R_\mathfrak{m}$-modules. Therefore, $\mathfrak{p}_\mathfrak{m}$ is a G-projective $R_\mathfrak{m}$-module. Thus, $\mathfrak{p}$ is a w-locally G-projective R-module, a contradiction with $\mathfrak{p} \in \Gamma$. Hence, $\mathfrak{p}$ is a prime ideal of R.

(5) ⇒ (6) Since $R_{\mathfrak{m}}$ is a G-Dedekind domain for any maximal w-ideal $\mathfrak{m}$ of R, we can conclude that $\dim(R_{\mathfrak{m}}) \leq 1$ by [9, Corollary 11.7.8]. Thus, w-$\dim(R) \leq 1$. Therefore, w-Max(R) is precisely the set of all prime ideals of R of height one. Hence, $R = \bigcap_{\mathfrak{m}\in w\text{-Max}(R)} R_{\mathfrak{m}}$. Therefore, R is a G-Krull domain by (5).

(6) ⇒ (5) If R is a G-Krull domain, then w-$\dim(R) = 1$ by [7, Corollary 4.5]. Thus, $X^1(R) = w$-Max(R), where $X^1(R)$ denotes the set of height-one prime ideals of R. Then (5) holds by the definition of G-Krull domains. □

It is well known that a G-Dedekind domain is of finite character, i.e., each nonzero element of a G-Dedekind domain R lies in only finitely many maximal ideals of R. Next, we show that a domain R satisfying that $R_{\mathfrak{m}}$ is a G-Dedekind domain for any maximal w-ideal $\mathfrak{m}$ of R is not necessarily of w-finite character, i.e., some nonzero element of such a domain R may be contained in infinitely many maximal w-ideals of R.

In [4], Kang called a domain R *t-almost Dedekind* if $R_{\mathfrak{m}}$ is a discrete valuation ring for each maximal t-ideal $\mathfrak{m}$ of R [4, Page 166]. Note that maximal t-ideals and maximal w-ideals coincide [9, Theorem 7.3.4]. Then a domain R is a t-almost Dedekind domain if and only if $R_{\mathfrak{m}}$ is a discrete valuation ring for each maximal w-ideal $\mathfrak{m}$. Note that a domain R is a Krull domain if and only if R is a t-almost Dedekind domain and R is an SM domain by [9, Theorem 7.9.3].

Example 1. Let R be a non-Krull t-almost Dedekind domain. Such a detailed example can be referenced in [4, Page 167, Remark]. Then, for any maximal w-ideal $\mathfrak{m}$ of R, $R_{\mathfrak{m}}$ is a discrete valuation ring, thus a G-Dedekind domain. However, R does not satisfy that each nonzero element of R lies in only finitely many maximal w-ideals of R. If not, we would get that R is an SM domain by Lemma 3. Thus, R would be a Krull domain, a contradiction.

For many cases, if every prime ideal of R satisfies the property, then every ideal of R satisfies such a property. For example, if every prime ideal of R is finitely generated, then every ideal of R is finitely generated, i.e., R is Noetherian. This is the well-known Cohen theorem. However, if every maximal ideal of R satisfies some property, it is not necessary that every ideal of R does. For example, let $R = \mathbb{Z}+\mathbb{Q}[X]$, where $\mathbb{Z}$ denotes the ring of integers and $\mathbb{Q}$ denotes the rational number field. Then the maximal ideals of R are those of the form pR, where p is a prime element of $\mathbb{Z}$, and the principal ideals $f(X)R$, where $f(X)$ is irreducible in $\mathbb{Q}[X]$ and $f(0) = 1$ ([2, Theorem 4.21]). In this case, every maximal ideal of R is finitely generated, but R is not Noetherian.

It is noteworthy that if a domain R is Noetherian and every maximal ideal of R is G-projective, then every ideal of R is G-projective, i.e., R is G-Dedekind (Corollary 1). Based on the above result, we also can conclude that a domain R is G-Krull if and only if R is an SM domain and every maximal w-ideal of R is w-locally G-projective (Theorem 3).

Lemma 4 ([5], Theorem 211)**.** *Let $(R,\mathfrak{m})$ be a local Noetherian domain, $\mathfrak{p}$ a prime ideal of R with $\mathfrak{p} \subsetneq \mathfrak{m}$, and M a finitely generated R-module. If $\operatorname{Ext}_R^{i+1}(R/Q,M)=0$ for any prime ideal Q properly containing $\mathfrak{p}$, then $\operatorname{Ext}_R^{i}(R/\mathfrak{p},M)=0$.*

Theorem 2. *Let $(R,\mathfrak{m})$ be a local Noetherian domain. If $\mathfrak{m}$ is a G-projective R-module, then R is a G-Dedekind domain.*

Proof. Since $\mathfrak{m}$ is a G-projective R-module, we have $\operatorname{Ext}_R^{i\geq 1}(\mathfrak{m},R)=0$ by [9, Corollary 11.1.3]. Then $\operatorname{Ext}_R^{i\geq 2}(R/\mathfrak{m},R)=0$. By the Generalized Principal Ideal Theorem of Noetherian rings ([9, Theorem 4.3.12]), the height of $\mathfrak{m}$ is finite. Assume that $\operatorname{ht}(\mathfrak{m})=t$. For any prime ideal $\mathfrak{p}$ of R with $\operatorname{ht}(\mathfrak{p})=t-1$, $\operatorname{Ext}_R^{i\geq 2}(R/\mathfrak{p},R)=0$ by Lemma 4. Then for any prime ideal $\mathfrak{p}$ of R with $\operatorname{ht}(\mathfrak{p})=t-2$, $\operatorname{Ext}_R^{i\geq 2}(R/\mathfrak{p},R)=0$, again by Lemma 4. Continuing this process, we get that $\operatorname{Ext}_R^{i\geq 2}(R/\mathfrak{p},R)=0$ for any prime ideal $\mathfrak{p}$ of R. Let M be a finitely generated R-module. Then there exists an ascending chain of submodules of M

$$0=M_0\subseteq M_1\subseteq M_2\subseteq\cdots\subseteq M_{n-1}\subseteq M_n=M$$

such that $M_{i+1}/M_i\cong R/\mathfrak{p}_{i+1}$ for some prime ideal $\mathfrak{p}_{i+1}$ of R, $i=1,2,\ldots,n-1$. For the exact sequence $0\to R/\mathfrak{p}_1\to M_2\to R/\mathfrak{p}_2\to 0$, we get the exact sequence

$$\operatorname{Ext}_R^2(R/\mathfrak{p}_2,R)\to\operatorname{Ext}_R^2(M_2,R)\to\operatorname{Ext}_R^2(R/\mathfrak{p}_1,R).$$

Then $\operatorname{Ext}_R^2(M_2,R)=0$. By the same method, we get that $\operatorname{Ext}_R^2(M,R)=0$. Thus, $\operatorname{id}_R(R)\leq 1$. Therefore, $\operatorname{Ext}_R^1(I,R)=0$. Thus, R is a G-Dedekind domain by [9, Theorem 11.7.7]. □

Corollary 1. *If R is a Noetherian domain and every maximal ideal of R is G-projective, then R is G-Dedekind.*

Proof. Let $\mathfrak{m}$ be a maximal ideal of R. Then $(R_\mathfrak{m},\mathfrak{m}R_\mathfrak{m})$ is a local Noetherian domain. Since $\mathfrak{m}$ is G-projective over R and $\mathfrak{m}$ is super finitely presented (i.e., there exists an exact sequence $\cdots\to P_n\to\cdots\to P_1\to P_0\to\mathfrak{m}\to 0$, where each P_i is a finitely generated projective R-module), we get that $\mathfrak{m}R_\mathfrak{m}$ is a G-projective $R_\mathfrak{m}$-module by [9, Theorem 11.6.17]. Then $R_\mathfrak{m}$ is G-Dedekind by Theorem 2. Hence, the Krull dimension of $R_\mathfrak{m}$ is less than or equal to 1. The same holds for the Krull dimension of R, which implies that every nonzero prime ideal of R is maximal. Thus, R is G-Dedekind by [9, Theorem 11.7.7]. □

Theorem 3. *The following statements are equivalent for a domain R.*

(1) *R is a G-Krull domain.*
(2) *R is an SM domain and every maximal w-ideal of R is w-locally G-projective.*

Proof. (1) $\Rightarrow$ (2) It follows by Theorem 1.

(2) $\Rightarrow$ (1) Let $\mathfrak{m}$ be a maximal w-ideal of R. If R is an SM domain, then $R_\mathfrak{m}$ is a Noetherian domain and each nonzero element of R lies in only finitely many maximal

w-ideals of R by Lemma 3. By (2), $\mathfrak{m}R_{\mathfrak{m}}$ is a G-projective $R_{\mathfrak{m}}$-module. Then $R_{\mathfrak{m}}$ is a G-Dedekind domain by Theorem 2. Thus, R is a G-Krull domain by Theorem 1. □

In Theorem 1, it is shown that if R is an SM domain and $\mathrm{id}_R R_{\mathfrak{m}} \leq 1$ for any maximal w-ideal $\mathfrak{m}$ of R, then R is a G-Krull domain. Next, we show that if R is an SM domain and $\mathrm{id}_R R \leq 1$, then R is just G-Dedekind (Theorem 4).

First, we recall the notion of w-modules. The w-operation on domains was introduced by Wang and McCasland in [10], and then considered for commutative rings by Yin et al. in [13]. Let J be a finitely generated ideal of R. If the natural homomorphism $\varphi : R \to J^* = \mathrm{Hom}_R(J,R)$ is an isomorphism, then J is called a GV-*ideal*, denoted by $J \in \mathrm{GV}(R)$. Let M be an R-module. Define

$$\mathrm{tor}_{\mathrm{GV}}(M) = \{x \in M \mid Jx = 0 \text{ for some } J \in \mathrm{GV}(R)\}.$$

Thus, $\mathrm{tor}_{\mathrm{GV}}(M)$ is a submodule of M. And M is said to be GV-*torsion* (resp., GV-*torsion-free*) if $\mathrm{tor}_{\mathrm{GV}}(M) = M$ (resp., $\mathrm{tor}_{\mathrm{GV}}(M) = 0$). Clearly, R is a GV-torsion-free R-module ([13, Corollary 1.5]). A GV-torsion-free module M is called a *w-module* if $\mathrm{Ext}^1_R(R/J,M) = 0$ for any $J \in \mathrm{GV}(R)$ ([13, Definition 2.2]). The w-envelope of a GV-torsion-free module M is the set given by

$$M_w = \{x \in E(M) \mid Jx \subseteq M \text{ for some } J \in \mathrm{GV}(R)\},$$

where $E(M)$ is the injective hull of M. And M is a w-module if and only if $M_w = M$. A w-module M is *of finite type* if $M = N_w$ for some finitely generated submodule N of M ([13, p. 216]). Recall that a GV-torsion-free R-module M is said to be a *strong w-module* if $\mathrm{Ext}^i_R(N,M) = 0$ for each integer $i \geq 1$ and for any GV-torsion R-module N ([11, p. 1918, Definition]).

Proposition 1. *If $\mathrm{id}_R R \leq 1$, then R is a strong w-module.*

Proof. Since R is a w-module, we have that $\mathrm{Ext}^1_R(N,R) = 0$ for any GV-torsion module N by [9, Theorem 6.2.7]. Given that $\mathrm{id}_R R \leq 1$, we also have $\mathrm{Ext}^i_R(N,R) = 0$ for any GV-torsion module N and for any integer $i \geq 2$. Thus, R is a strong w-module. □

Proposition 2. *Let R be a domain. If $\mathrm{id}_R R \leq 1$, then R is a DW domain; that is, every ideal of R is a w-ideal.*

Proof. Let I be a nonzero finitely generated ideal of R. Then there exists an exact sequence

$$0 \to K \to F \to I \to 0,$$

where F is a finitely generated free R-module. Since $\mathrm{id}_R R \leq 1$, we have

$$\mathrm{Ext}^1_R(I,R) \cong \mathrm{Ext}^2_R(R/I,R) = 0.$$

Thus, we obtain the exact sequence

$$0 \to I^* \to F^* \to K^* \to 0,$$

where $M^* = \mathrm{Hom}_R(M,R)$ for any R-module M. Hence, K^* is a finitely generated torsion-free R-module. Therefore, there exists an exact sequence

$$0 \to K^* \to F_1 \to C \to 0,$$

with F_1 a free R-module. Consequently,

$$\mathrm{Ext}_R^1(K^*,R) \cong \mathrm{Ext}_R^2(C,R) = 0,$$

which implies that the sequence

$$0 \to K^{**} \to F^{**} \to I^{**} \to 0$$

is exact. Now consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & F & \longrightarrow & I & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \rho_1} & & \downarrow{\scriptstyle \rho_2} & & \downarrow{\scriptstyle \rho_3} & & \\ 0 & \longrightarrow & K^{**} & \longrightarrow & F^{**} & \longrightarrow & I^{**} & \longrightarrow & 0 \end{array}$$

By [9, Proposition 2.1.29(1)], both ρ_1 and ρ_3 are monomorphisms, and ρ_2 is an isomorphism by [9, Theorem 2.3.7(3)]. Then, by the Five Lemma ([9, Theorem 1.9.9]), ρ_1 is also an epimorphism. Hence, ρ_1 is an isomorphism, and it follows that I is a w-ideal of R.

Now let A be a nonzero ideal of R. Then

$$A_w = \bigcup_i B_w,$$

where B ranges over the set of finitely generated subideals of A. Since each B is a w-ideal, we have $B_w = B$, and thus $A_w = A$. Therefore, every ideal of R is a w-ideal, and R is a DW domain. □

Theorem 4. *The following statements are equivalent for a domain R with quotient field K.*

(1) *R is an SM domain and* $\mathrm{id}_R R \le 1$.
(2) *R is a Noetherian domain and* $\mathrm{id}_R R \le 1$.
(3) *R is an SM domain and* $\mathrm{Ext}_R^1(M,R) = 0$ *for every submodule M of a free module.*
(4) *R is an SM domain, K/R is a w-module, and* $\mathrm{Ext}_R^1(M,R) = 0$ *for every finite type submodule M of a free module.*
(5) *R is an SM domain, K/R is a w-module, and* $\mathrm{Ext}_R^1(I,R) = 0$ *for every w-ideal I of R.*
(6) *R is an SM domain and* $\mathrm{Ext}_R^1(I,R) = 0$ *for every ideal I of R.*
(7) *R is an SM domain and every nonzero ideal I of R is a v-ideal.*
(8) *R is a G-Dedekind domain.*

Proof. (1) $\Leftrightarrow$ (2) follows from Proposition 2.

(2) $\Leftrightarrow$ (8) $\Rightarrow$ (3), and (2) $\Rightarrow$ (7), follow from [9, Theorem 11.7.7].

(3) $\Rightarrow$ (4) It suffices to show that K/R is a w-module. Let J be a GV-ideal of R. Then the exact sequence

$$0 \to J \to R \to R/J \to 0$$

yields $\mathrm{Ext}_R^1(J,R) \cong \mathrm{Ext}_R^2(R/J,R)$. By (3), we obtain $\mathrm{Ext}_R^1(J,R) = 0$, so $\mathrm{Ext}_R^2(R/J,R) = 0$. Now, from the exact sequence

$$0 \to R \to K \to K/R \to 0,$$

we have the long exact sequence

$$\mathrm{Ext}_R^1(R/J,K) \to \mathrm{Ext}_R^1(R/J,K/R) \to \mathrm{Ext}_R^2(R/J,R).$$

Since the last term is zero, it follows that $\mathrm{Ext}_R^1(R/J,K/R) = 0$. Hence, K/R is a w-module.

(4) $\Rightarrow$ (5) This is clear.

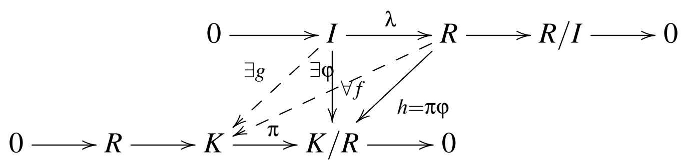
(5) $\Rightarrow$ (1) To show $\mathrm{id}_R R \leq 1$, it suffices to prove that K/R is injective over R, since we have the exact sequence

$$0 \to R \to K \to K/R \to 0$$

and K is injective over R. By (5), for any w-ideal I of R, the sequence

$$0 \to \mathrm{Hom}_R(I,R) \to \mathrm{Hom}_R(I,K) \to \mathrm{Hom}_R(I,K/R) \to 0$$

is exact. Consider the following commutative diagram:



For any $f \in \mathrm{Hom}_R(I,K/R)$, there exists $g \in \mathrm{Hom}_R(I,K)$ such that $f = \pi g$. Since K is injective, there exists $\varphi \in \mathrm{Hom}_R(R,K)$ with $g = \varphi\lambda$. Then $h = \pi\varphi \in \mathrm{Hom}_R(R,K/R)$ satisfies $h\lambda = f$. Hence, $\mathrm{Ext}_R^1(R/I,K/R) = 0$. Since K/R is a w-module, it is injective over R by [9, Theorem 6.8.26], so $\mathrm{id}_R R \leq 1$.

(1) $\Leftrightarrow$ (6) follows from the isomorphism $\mathrm{Ext}_R^1(I,R) \cong \mathrm{Ext}_R^2(R/I,R)$ for every ideal I of R.

(7) $\Rightarrow$ (8) If R is an SM domain and every nonzero ideal I of R is a v-ideal, then each such ideal is also a w-ideal. Thus, R is a DW domain, and hence Noetherian. Therefore, R is a G-Dedekind domain by [9, Theorem 11.7.7]. $\square$

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