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An anticipating stochastic integral with respect to mixed fractional Brownian motion

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Abstract. In this paper, we define a stochastic integral of an anticipating integrand based on Ayed and Kuo's approach [1]. This provides a new concept of stochastic integration of non-adapted processes. In addition, under some conditions, we prove that our anticipating integral is a near-martingale. Furthermore, we deal with some particular cases when the Hurst parameter $H > \frac{3}{4}$.

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1 Introduction

Let $B(t)$ be a Brownian motion and let $\{\mathcal{F}_t; 0 \le t \le T\}$ denote a filtration such that:

1. $f(t)$ is an $\mathcal{F}_t$-adapted stochastic process, i.e. $f(t)$ is $\mathcal{F}_t$-measurable for each $0 \le t \le T$.
2. $g(t)$ is instantly independent with respect to $\mathcal{F}_t$, i.e. $g(t)$ and $\{\mathcal{F}_t\}$ are independent for each $0 \le t \le T$.

Ayed and Kuo [1] defined the anticipating stochastic integral of the product $f(t)g(t)$ as:

$$\int_0^T f(t)g(t)dB(t) = \lim_{\|\Delta_n\| \to 0} \sum_{i=1}^{n} f(t_{i-1})g(t_i)(B(t_i) - B(t_{i-1})) \tag{1}$$

provided that the convergence in probability exists, where $\Delta_n = \{0 = t_0 < t_1 < .... < t_n = T\}$ is the partition of interval $[0, T]$.
Notice that the evaluation points are the left endpoints of subintervals for the first process and the right endpoints for the second one.

This new approach has attracted the attention of many researchers. The study of a class of stochastic differential equations with anticipating initial conditions was treated in Khalifa et al. [7]. After that, the concept of near-martingale property of anticipating stochastic integral was introduced in Kuo et al. [8]. It has been proved that both $\int_0^t f(B(s))g(B(T) - B(s))dB(t)$ and $\int_t^T f(B(s))g(B(T)-B(s))dB(t)$ are near-martingales with respect to the forward filtration $\mathcal{F}_t = \sigma\{B(s); 0 \le s \le t\}$ and the backward filtration $\mathcal{F}^{(t)} = \sigma\{B(T) - B(s); 0 \le s \le t\}$, respectively. Interesting literature on the near martingale property can be found in Hwang et al. [6] and Hibino et al. [5]. Recently, Belhadj et al. [2] introduced the anticipating stochastic integral with respect to sub-fractional Brownian motion and discussed the conditions under which this integral satisfies the near-martingale property.

Next, we consider the process

$$M^H(t) = M_t^H(a, b) = aB(t) + bB^H(t), t \in \mathbb{R}_+, \qquad a, b \in \mathbb{R}^*, \tag{2}$$

where B and B^H are independent standard and fractional Brownian motions, respectively. The latter is the centered Gaussian process with a Hurst parameter $H \in (0,1)$ and covariance function:

$$R^H(s,t) = \frac{1}{2}[t^{2H} + s^{2H} + |t-s|^{2H}], \qquad s,t \geq 0. \tag{3}$$

The linear combination M^H is the so-called mixed fractional Brownian motion (mfBm). This process has been firstly introduced by Chridito [3] to present an interesting stochastic model in financial markets (by taking $b = 1$). The stochastic properties of mfBm have been studied by Zili [13].

It is worth pointing out that, for $H > \frac{3}{4}$, the process M^H is a semimartingale which is equivalent (in distribution) to aB (Chredito [3]), and for $H < \frac{1}{4}$, M^H is equivalent (in distribution) to a bB^H (Van Zanten [10]). Furthermore, we mention that for $H < \frac{3}{4}$, the mixed fBm is not a semi-martingale. Therefore, the techniques of stochastic calculus with respect to fBm should be employed while dealing with a mixed fBm. In the case where $H > \frac{1}{2}$, we can use the pathwise approach that allows us to write the integral as a limit of Riemann sum (Young [11], Zähle[12], and Feyel and Pradelle [4] and the references therein). In our study, we use this approach in order to give a definition of the anticipating integral with respect to a mixed fractional Brownian motion M^H and study the near-martingale property.

1.1 Practical application of our research work

Our study has a notable application in finance and economy. For instance, we consider a financial stock market where the process $f(t)$ is a quantity of the stock at time t, adapted to $\mathcal{F}_t$, the σ-field represents information available by time t, and $B(t)$ (the standard Brownian motion) characterizes the stock price at time t. The integral $\int_0^T f(t)dB(t)$ describes the change of the stock market wealth over the trading period $[0,T]$. By dividing the time integral into the subintervals $[t_{i-1}, t_i]$, $\int_0^T f(t)dB(t)$ can be computed as a limit of Riemann-like sums of $f(t_{i-1})(B(t_i) - B(t_{i-1}))$. The use of the left endpoint of subintervals comes from the fact that $f(t)$ depends on the past and present but not the future. If one comes across the case where the quantity of stock $f(t)$ is independent of past and present, i.e for each $t \in [0,T])$, $f(t)$ is $\mathcal{F}_t$-independent then the future change in stocks can be known and one can use the right endpoint t_i as an evaluation point for the above stochastic integral. On the other hand, it has been interesting, in recent years, to divide the noise of stock price into two parts: the first describes the stochastic behavior of stock markets which is considered as a white noise, the other one represents the random state of the stock price which has a long memory, this motivates researchers to take such a situation into consideration and to provide a mixture of processes in accordance with the requirements of the phenomena.

Furthermore, over the past, there has been an extensive studies on option pricing. It has been shown that the distributions of logarithmic returns of financial assets generally exhibit properties of self-similarity and long-term dependence, and since the fractional Brownian motion has these two important properties, it has the ability to capture the behavior of the underlying asset price. The Black-Scholes model supposed that the volatility of the underlying security is constant, while stochastic volatility models classified the price of the underlying security as a random variable or, more generally, a stochastic process. In turn, the dynamics of this stochastic process can be driven by another process (usually by Brownian motion), see Thao et al. [9]. In a stochastic volatility model, the volatility randomly changes according to stochastic processes. In our paper, the process used is the mixture between fBm (fractional Brownian motion) and Bm(Brownian motion). The current study helps to solve the stochastic differential equations (SDEs) driven by a mixed fractional Brownian motion in the case of no adapted integrands which contributes to the resolution of the phenomena linked to volatility in the above situations.

This paper is arranged as follows. In Section 2, we present some preliminaries on mixed fractional integral as well as pathwise integral with respect to mfBm. In Section 3, we introduce a definition of stochastic integral of a product of instantly independent process and adapted process with respect to M^H, $H > \frac{1}{2}$ as a Riemann sum. Then, we discuss the near-martingale property of our anticipating integral. Section 4 is devoted to some particular cases when $H > \frac{3}{4}$. We conclude the paper in Section 5.

2 Preliminaries on mixed fractional Brownian motion

The fBm $(B^H(t); t \geq 0)$ with a Hurst parameter $H \in (0, 1)$ is a centered Gaussian process with covariance function given by Equation (3). The main properties of B^H are self-similarity and the stationary of its increments, it presents a long-range dependence when $H > \frac{1}{2}$. For $H = \frac{1}{2}$, B^H coincides with the standard Brownian motion.
Note that the mixture M^H reserves several properties of the fBm. We recall in

this section some basic facts on mixed fractional Brownian motion, the proofs are detailed in Zili [13].

Lemma 1 *(Zili [13]). The mfBm satisfies the following properties:*

- M^H *is a centered gaussian process;*
- *Second moment: for all* $t \in \mathbb{R}_+$*;* $\mathbb{E}((M_t^H(a,b))^2) = a^2 t + b^2 t^{2H}$.
- *Covariance function: for all* $t, s \geq 0$*;*

$$\mathrm{Cov}(M_t^H(a,b), M_s^H(a,b)) = a^2 \min(t,s) + \frac{b^2}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right).$$

- *The increments of the mfBm are stationary.*
- *Mixed self similarity:* $\left(M_{\alpha t}^H(a,b)\right)_{t\geq 0}$ *and* $\left(M_t^H(a\alpha^{\frac{1}{2}}, b\alpha)\right)_{t\geq 0}$ *have the same distribution.*
- *Hölder continuity: for all* $T > 0$ *and* $\beta < \frac{1}{2} \wedge H$*, the mfBm has a modification which sample paths having a Hölder continuity, with order* β *on the interval* $[0;T]$ *such that, for every* $\alpha > 0$ *:*

$$\mathbb{E}\left(\left|M^H(t) - M^H(s)\right|^{\alpha}\right) \leq C_\alpha |t-s|^{\alpha(\frac{1}{2}\wedge H)}, \qquad t, s \in [0;T],$$

where C_α *is a positive constant.*

Feyel and Pradelle [4] showed that if f is α-Hölder, g is β-Hölder with $\alpha + \beta > 1$, then the Riemann-Stieltjes integral $\int_0^T f(s)dg(s)$ exists and is β-Hölder. Moreover, for every $0 < \varepsilon < \alpha + \beta - 1$, we have

$$\left|\int_0^T f(s)dg(s)\right| \leq C(\alpha,\beta) \parallel f \parallel_{[0,T],\alpha} \parallel g \parallel_{[0,T],\beta} T^{1+\varepsilon}. \tag{4}$$

Since mfBm has Hölder paths, then it is possible to define the stochastic integral for processes with respect to it in pathwise sense. Particularly, if a process $(u_t)_{t\in[0,T]}$ has α-Hölder paths for some $\alpha > 1 - H$, then the Riemann-Stieltjes integral $\int_0^t u_r dM_r^H$ is well defined and has β-Hölder paths, for every $\beta < H$ (see Young [11] and Zähle [12]).

3 New anticipating integral

Based on the concept presented above, we give a definition of the stochastic integral of the product $f(t)g(t)$, following Definition 2.2 given in Kuo and Ayed [1], by taking the mfBm M^H as an integrator. Formally, we have

Definition 1 *Let* $M^H(t)$, $H > \frac{1}{2}$ *be a mixed fractional Brownian motion and let* $\mathcal{F}_t$ *denote the* σ*-field generated by* $\{M^H(t), t \geq 0\}$*. For an adapted stochastic process* $f(t)$ *with respect to the filtration* $\mathcal{F}_t$*, and an instantly independent stochastic process* $g(t)$ *with respect to the same filtration. We define the stochastic integral of* $f(t)g(t)$ *as:*

$$\int_0^T f(t)g(t)dM^H(t) = \lim_{\|\Delta_n\|\to 0} \sum_{i=1}^{n} f(t_{i-1})g(t_i)(M^H(t_i) - M^H(t_{i-1})) \qquad (5)$$

provided that the convergence in probability exists.

It is quite clear that the anticipating integral (5) is not a $\mathcal{F}_t$-martingale. Thus, we have to check if this latter satisfies the near-martingale property presented in Kuo et al. [8].

Definition 2 *(Kuo et al. [8]). Let* $E|X_t| < \infty$ *for all* t*. We will say that* X_t *is a near-martingale with respect to a forward filtration* $\{\mathcal{F}_t\}$ *if*

$$E[X_t - X_s/\mathcal{F}_s] = 0, \qquad \forall s < t. \qquad (6)$$

On the other hand, we say that X_t *is a near-martingale with respect to a backward filtration* $\{\mathcal{F}^{(t)}\}$ *if*

$$E[X_t - X_s/\mathcal{F}^{(t)}] = 0, \qquad \forall s < t. \qquad (7)$$

Next, we have to prove that the processes X_t and Y_t defined by (8) and (13) respectively, are near-martingales for an adapted process $f(t)$ and centered instantly independent process $g(t)$ with respect to the forward filtration

$$\mathcal{F}_t = \sigma\{B(s), M^H(s); 0 \leq s \leq t\},$$

Theorem 1 *Let* $\mathcal{F}_t$ *be a forward filtration and let* $f(x)$ *and* $g(x)$ *be continuous functions such that:*

1. $E\Big[\int_0^T f(B(t))g(B(T) - B(t))dM^H(t)\Big] < +\infty,$

2. $E\big[g(B(T)-B(t))\big]=0.$

Then,

$$X_t=\int_0^t f(B(s))g(B(T)-B(s))dM^H(s);\qquad 0\le t\le T \tag{8}$$

exists and is a near-martingale with respect to the forward filtration $\mathcal{F}_t$*.*

Proof. We need to verify that $E[X_t-X_s/\mathcal{F}_s]=0$, for $0\le s\le t$. Notice that

$$X_t-X_s=\int_s^t f(B(u))g(B(T)-B(u))dM_u^H.$$

Let $\Delta_n=\{s=t_0<t_1<...<t_{n-1}<t_n=t\}$ be a partition of the interval $[s,t]$ and let $\Delta M_i^H=M^H(t_i)-M^H(t_{i-1})$. Then, we have:

$$E[X_t-X_s/\mathcal{F}_s]=E\left[\int_s^t f(B(u))g(B(T)-B(u))dM^H(u)/\mathcal{F}_s\right]. \tag{9}$$

Making use of Definition 1, we get

$$\begin{aligned}E[X_t-X_s/\mathcal{F}_s]&=E\left[\lim_{\|\Delta_n\|\to 0}\sum_{i=1}^n f(B(t_{i-1}))g(B(T)-B(t_i))\Delta M_i^H/\mathcal{F}_s\right]\\&=\lim_{\|\Delta_n\|\to 0}\sum_{i=1}^n E\left[f(B(t_{i-1}))g(B(T)-B(t_i))\Delta M_i^H/\mathcal{F}_s\right].\end{aligned} \tag{10}$$

It is sufficient to show that every component of the last sum is zero. Recall that $f(B(t_{i-1}))$ is $\mathcal{F}_{t_{i-1}}$-measurable and $g(B(T)-B(t_i))$ is independent of $\mathcal{F}_{t_{i-1}}$. Using the properties of conditional expectation, we obtain

$$\begin{aligned}&E\left[f(B(t_{i-1}))g(B(T)-B(t_i))\Delta M_i^H/\mathcal{F}_s\right]\\&\qquad=E\left[E\big[f(B(t_{i-1}))g(B(T)-B(t_i))\Delta M_i^H/\mathcal{F}_{t_i}\big]/\mathcal{F}_s\right]\\&\qquad=E\left[f(B(t_{i-1}))\Delta M_i^H E\big[g(B(T)-B(t_i))/\mathcal{F}_{t_i}\big]/\mathcal{F}_s\right].\end{aligned} \tag{11}$$

Making use of the independence of Brownian increments and the zero expectation of $g(B(T) - B(t_i))$, we get

$$\begin{aligned} E\left[f(B(t_{i-1}))g(B(T)-B(t_i))\Delta M_i^H/\mathcal{F}_s\right] \\ &= E\left[f(B(t_{i-1}))\Delta M_i^H E\big[g(B(T)-B(t_i))\big]/\mathcal{F}_s\right] \\ &= E\big[g(B(T)-B(t_i))\big]E\left[f(B(t_{i-1}))\Delta M_i^H/\mathcal{F}_s\right] \\ &= 0. \end{aligned} \tag{12}$$

Thus, X_t is a near-martingale with respect to $\mathcal{F}_t$. □

Theorem 2 *Let $\mathcal{F}_t$ be a forward filtration and let $f(x)$ and $g(x)$ be continuous functions such that:*

1. $E\big[\int_0^T f(B(t))g(B(T)-B(t))dM^H(t)\big] < +\infty$
2. $E\big[g(B(T)-B(t))\big] = 0.$

Then,

$$Y_t = \int_t^T f(B(s))g(B(T)-B(s))dM^H(s), \quad 0 \le t \le T \tag{13}$$

exists and is a near-martingale with respect to the forward filtration $\mathcal{F}_t$.

Proof. For $0 \le s < t \le T$, we have

$$Y_t - Y_s = -\int_s^t f(B(u))g(B(T)-B(u))dM^H(u) = -(X_t - X_s),$$

where X_t is given in Equation (8). Thus, Y_t is a near-martingale with respect to $\mathcal{F}_t$. □

Next, we prove that X_t and Y_t given in Equations (14) and (17) respectively, are near-martingales for a centered adapted process $f(t)$, and instantly independent process $g(t)$ with respect to the backward filtration

$$\mathcal{F}^{(t)} = \sigma\{B(T)-B(s), M^H(T)-M^H(s), \ 0 \le s \le t\}.$$

Theorem 3 *Let $\mathcal{F}^{(t)}$ be a backward filtration and let $f(x)$ and $g(x)$ be continuous functions such that:*

1. $E\big[\int_0^T f(B(t))g(B(T)-B(t))dM^H(t)\big] < +\infty,$

2. $E\big[f(B(t))\big] = 0.$

Then,

$$X_t = \int_0^t f(B(s))g(B(T)-B(s))dM^H(s), \;\; 0 \le t \le T \tag{14}$$

exists and is a near-martingale with respect to the backward filtration $\mathcal{F}^{(t)}$.

Proof. According to the proof of Theorem 1, we have to show that

$$E\Big[f(B(t_{i-1}))g(B(T)-B(t_i))\Delta M_i^H/\mathcal{F}^{(t)}\Big] = 0,$$

where $0 \le s < t \le T$ and $s = t_0 < t_1 < .... < t_{n-1} < t_n = t$.

It is well known that the increments $M_T^H - M_{t_{i-1}}^H$ and $M_T^H - M_{t_i}^H$ are $\mathcal{F}^{(t_{i-1})}$-measurable, then we have

$$\Delta M_i^H = (M_T^H - M_{t_{i-1}}^H) - (M_T^H - M_{t_i}^H) \in \mathcal{F}^{(t_{i-1})}.$$

By the $\mathcal{F}^{(t_{i-1})}$- measurability of ΔM_i^H and the conditional expectation properties, we obtain

$$\begin{aligned} &E\Big[f(B(t_{i-1}))g(B(T)-B(t_i))\Delta M_i^H/\mathcal{F}^{(t)}\Big] \\ &\qquad = E\Big[E\big[f(B(t_{i-1}))g(B(T)-B(t_i))\Delta M_i^H/\mathcal{F}^{(t_{i-1})}\big]/\mathcal{F}^{(t)}\Big] \\ &\qquad = E\Big[g(B(T)-B(t_i))\Delta M_i^H E\big[f(B(t_{i-1}))/\mathcal{F}^{(t_{i-1})}\big]/\mathcal{F}^{(t)}\Big]. \end{aligned} \tag{15}$$

Furthermore, $B(T)-B(s)$ is independent of $\mathcal{F}_{t_{i-1}}$ and measurable with respect to $\mathcal{F}^{(t_{i-1})}$ for each $s > t_{i-1}$. This involves the independence of $\mathcal{F}^{(t_{i-1})}$ and $\mathcal{F}_{t_{i-1}}$. Consequently, $f(B(t_{i-1}))$ is independent of $\mathcal{F}^{(t_{i-1})}$ since it is $\mathcal{F}_{t_{i-1}}$ measurable. Hence,

$$\begin{aligned} &E\Big[f(B(t_{i-1}))g(B(T)-B(t_i))\Delta M_i^H/\mathcal{F}^{(t)}\Big] \\ &\qquad = E\Big[g(B(T)-B(t_i))\Delta M_i^H E\big[f(B(t_{i-1}))\big]/\mathcal{F}^{(t)}\Big] \\ &\qquad = E\big[f(B(t_{i-1}))\big]E\Big[g(B(T)-B(t_i))\Delta M_i^H/\mathcal{F}^{(t)}\Big] \\ &\qquad = 0. \end{aligned} \tag{16}$$

□

Theorem 4 *Let $\mathcal{F}^{(t)}$ be a backward filtration and let $f(x)$ and $g(x)$ be continuous functions such that:*

1. $E\Big[\int_0^T f(B(t))g(B(T)-B(t))dM^H(t)\Big] < +\infty$,
2. $E\big[f(B(t))\big] = 0$.

Then,

$$Y_t = \int_t^T f(B(s))g(B(T)-B(s))dM^H(s), \quad 0 \le t \le T \tag{17}$$

exists and is a near-martingale with respect to the backward filtration $\mathcal{F}^{(t)}$.

Proof. From Theorem 3, we have $Y_t - Y_s = -(X_t - X_s)$. This completes the proof of the Theorem. □

4 Some results in the case where $H \in \left(\frac{3}{4}, 1\right)$

This section presents some results establishing the relationship between standard Bm and mixed-fBm in the case where $H > \frac{3}{4}$. We show that our anticipating integral with respect to M^H can be written as a Riemann sum depending on standard Bm satisfying the near martingale property.

Proposition 1 *Let $M^H(t); H > \frac{3}{4}$ be a mixed fractional Brownian motion and $\mathcal{F}_t = \sigma\{M^H(t), t \ge 0\}$. For an $\mathcal{F}_t$-adapted stochastic process $f(t)$ and an $\mathcal{F}_t$-instantly independent stochastic process $g(t)$, we have*

$$\int_0^T f(t)g(t)dM^H(t) = a \lim_{\|\Delta_n\|\to 0} \sum_{i=1}^{n} f(t_{i-1})g(t_i)(B^H(t_i) - B^H(t_{i-1})) \tag{18}$$

provided that the convergence in probability exists.

Proof. The proof is a direct result of Theorem 1.7 of Cheridito [3]. □

Proposition 2 *Let $\mathcal{F}_t$ be a forward filtration, $\mathcal{F}^{(t)}$ denotes the backward filtration and let $f(x)$ and $g(x)$ be continuous functions such that:*

$$E\Big[\int_0^T f(B(t))g(B(T)-B(t))dM^H(t)\Big] < +\infty.$$

Then,

$$X_t = \int_0^t f(B(s))g(B(T) - B(s))dM^H(s); \qquad 0 \le t \le T, \tag{19}$$

and

$$Y_t = \int_t^T f(B(s))g(B(T) - B(s))dM^H(s); \qquad 0 \le t \le T \tag{20}$$

exist and are near-martingales with respect to $\mathcal{F}_t$ *and* $\mathcal{F}^{(t)}$ *respectively.*

Proof. The proof of this proposition is based on Theorem 1.7 in Chridito [3] and Theorems 3.5-3.8 given in Kuo et al. [8]. □

In what follows, we give some examples at which we evaluate some anticipating stochastic integrals with respect to mixed fractional Brownian motion when $H > \frac{3}{4}$, using the result obtained in the Proposition 1.

Example 1 *Consider the following integral*

$$\int_0^t B(T)^2 dM^H(s), \qquad 0 \le t \le T. \tag{21}$$

The integrand $B(T)^2$ *is decomposed as*

$$B(1)^2 = [(B(T) - B(s))]^2 + 2B(s)[B(T) - B(s)] + B(s)^2. \tag{22}$$

In addition, the integral converges in probability to

$$\sum_{i=1}^{n} \left([(B(T)-B(s_i))]^2 + 2B(s_{i-1})[B(T)-B(s_i)] + B(s_{i-1})^2\right)(M^H(s_i) - M^H(s_{i-1})).$$

As M^H *and* aB *are equivalent (in law), then the above sum can be expressed as*

$$a\sum_{i=1}^{n} \left([(B(T) - B(s_i))]^2 + 2B(s_{i-1})[B(T) - B(s_i)] + B(s_{i-1})^2\right)(B(s_i) - B(s_{i-1})).$$

Therefore, we have

$$\int_0^t B(T)^2 dM^H(s) = aB(T)^2 B(t) - 2aB(T)t, \qquad 0 \le t \le T.$$

In general, for any $n \in \mathbb{N}^*$, *it is easy to check that*

$$\int_0^t B(T)^n dM^H(s) = aB(T)^n B(t) - anB(T)^{n-1}t, \qquad 0 \le t \le T.$$

Example 2 *Consider the integrand* $B(s)B(T)$, *equivalently,*

$$B(s)(B(T) - B(s)) + B(s)^2.$$

Then,

$$\begin{aligned}
&\int_0^t B(s)B(T)dM^H(s) \\
&\qquad = a \lim_{\|\Delta_n\|\to 0} \sum_{i=1}^{n} \big(B(s_{i-1})(B(T) - B(s_i)) + B(s_{i-1})^2\big)(B(s_i) - B(s_{i-1})) \\
&\qquad = \frac{a}{2}B(T)(B(t)^2 - t) - a\int_0^t B(s)ds, \qquad 0 \le t \le T.
\end{aligned} \tag{23}$$

In the same manner, an integrand of the form $\phi(B(s))B(T)$ *can be decomposed as*

$$\phi(B(s))(B(T) - B(s)) + \phi(B(s))B(s),$$

for any continuous function $\phi(x)$. *Therefore, the integral*

$$\int_0^t \phi(B(s))B(T)dM^H(s), \qquad 0 \le t \le T$$

converges in probability to

$$aB(T)\sum_{i=1}^{n}(\phi(B(s_{i-1}))(B(s_i) - B(s_{i-1})) - a\sum_{i=1}^{n}\phi(B(s_{i-1}))(B(s_i) - B(s_{i-1}))^2,$$

which is equivalent to

$$aB(T)\int_0^t \phi(B(s))dB(s) - a\int_0^t \phi(B(s))ds.$$

Example 3 *The integral*

$$\int_0^t e^{B(T)}dM^H(s), \qquad 0 \le t \le T \tag{24}$$

is the limit of the sum

$$e^{B(T)}\sum_{i=1}^{n} e^{(B(s_i)-B(s_{i-1}))}(M(s_i) - M(s_{i-1})).$$

Using Taylor series expansions of exponential function, Equation (24) converges in probability to

$$ae^{B(T)}\sum_{i=1}^{n}\big(1-(B(s_i)-B(s_{i-1}))-\frac{1}{2}(B(s_i)-B(s_{i-1}))^2$$
$$+o((B(s_i)-B(s_{i-1}))^2)(B(s_i)-B(s_{i-1})).$$

Consequently,

$$\int_0^t e^{B(T)}dM^H(s)=ae^{B(T)}(B(t)-t),\qquad 0\le t\le T.$$

5 Conclusion

In this paper, we introduced an anticipating stochastic integral with respect to a mixed fractional Brownian motion (mfBm) in the case where $H>\frac{1}{2}$, based on Ayed and Kuo's approach [1]. This gives a new concept of stochastic integration of non-adapted processes. Under some conditions, we showed that our anticipating integral turns out to be a near-martingale. In addition, few specific cases when $H>\frac{3}{4}$ have been treated. The present study has a useful application in many areas including finance and economy. For further works, it will be interesting to deal with anticipating stochastic integrals with respect to a weighted fractional Brownian motion and Lévy fractional Brownian motion.

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On stammering p-adic Ruban continued fractions

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Abstract. We establish a new transcendence criterion of Ruban p-adic continued fractions and we prove that a p-adic number whose sequence of partial quotients is bounded in $\mathbb{Q}_p$ and has a stammering continued fraction expansion is either quadratic or transcendental where p is a prime number.

1 Introduction

Continued fractions have a crucial role in number theory. In fact, the continued fraction expansion of algebraic numbers is considered an open and difficult problem, as mentioned by Khintchine [5] in his conjecture, which is classified among the complicated questions in number theory. It remains difficult to give explicit and total answers, however, many authors have been able to establish several continued fraction transcendence criteria. As an example, the first author to give examples of transcendental continued fractions having bounded partial quotients was Maillet [8]. After that, Baker [2] improved Maillet's results using conditions that are simpler and more explicit.

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Throughout this paper, $\mathcal{A}$ denotes a countable set, called the *alphabet.* A sequence $\mathbf{a} = (a_n)_{n\geq 1}$ whose elements are from $\mathcal{A}$ is identified by the infinite word $a_1 \dots a_n \dots$. The number of letters composing a finite word W on the alphabet $\mathcal{A}$ is called the *length* of W which is denoted by $|W|$.

In 2013, Bugeaud [3] studied the case of stammering continued fractions given by the following theorem:

Theorem 1 *Let $\boldsymbol{a} = (a_n)_{n\geq 1}$ be a sequence of positive integers not ultimately periodic, $(U_n)_{n\geq 1}$, $(V_n)_{n\geq 1}$ and $(W_n)_{n\geq 1}$ three sequences of finite words such that:*

i) For every $n \geq 1$, the word $W_nU_nV_nU_n$ is a prefix of the word $\boldsymbol{a}$;

ii) The sequence $\left(\frac{|V_n|}{|U_n|}\right)_{n\geq 1}$ is bounded;

iii) The sequence $\left(\frac{|W_n|}{|U_n|}\right)_{n\geq 1}$ is bounded;

iv) The sequence $(|U_n|)_{n\geq 1}$ is increasing.

Let $\left(\dfrac{p_n}{q_n}\right)_{n\geq 1}$ denote the sequence of convergents to the real number $\alpha = [0, a_1, \dots, a_n, \dots]$. Assume that the sequence $(q_n^{\frac{1}{n}})_{n\geq 1}$ is bounded. Then, α is transcendental.

There exists a similar theory of continued fractions in the p-adic number field $\mathbb{Q}_p$. In 1968, Schneider [12] gave an algorithm of p-adic continued fraction expansion. After two years, Ruban [10] defined another definition which is more alike the real case. Ever since, a lot of authors studied properties of Ruban's continued fractions. For instance, Ubolsri, Laohakosol, Deze and Wang [7, 4, 13, 14] established multiple Ruban continued fractions transcendence criteria. Add to that, Ooto [9] showed that, for the Ruban continued fractions, the analogue of Lagrange's theorem is not true.

The aim of this paper is to study Bugeaud result's analogue, previously stated, for the p-adic continued fractions. The structure of this paper is as follows. In Section 2, we introduce the field of p-adic numbers $\mathbb{Q}_p$, the p-adic absolute value, the Ruban continued fractions and describe some of their fundamental properties. In Section 3, we give our transcendance criterion in $\mathbb{Q}_p$ as well as its proof. Finally, we close by giving an example to highlight the significance and influence of our finding.

2 Continued fractions of p-adic numbers

Let p be a prime number. We denote by $\mathbb{Q}_p$ the *field* of p-adic numbers with

$$\mathbb{Q}_p = \left\{ \sum_{i \geq j} b_i p^i / b_i \in \{0, \ldots, p-1\}, j \in \mathbb{Z} \right\}.$$

We also denote by $\mathbb{Z}_p$ the *ring* of the p-adic integers of $\mathbb{Q}_p$ where

$$\mathbb{Z}_p = \left\{ \sum_{i \geq 0} b_i p^i / b_i \in \{0, \ldots, p-1\} \right\}.$$

The p-*adic valution* ν_p is defined as follows

$$\begin{aligned} \nu_p : \mathbb{Q} &\longrightarrow \mathbb{Z} \cup \{+\infty\} \\ \alpha &\mapsto \begin{cases} +\infty & \text{if } \alpha = 0, \\ \inf\{i / b_i \neq 0\} & \text{otherwise.} \end{cases} \end{aligned}$$

The field of p-adic numbers $\mathbb{Q}_p$ is equipped with the p-adic absolute value, called *ultrametric absolute value*, normalized to satisfy $|p|_p = \frac{1}{p}$ and defined by $|\alpha|_p = \dfrac{1}{p^{\nu_p(\alpha)}}$ and $|0|_p = 0$.
Let $\alpha \in \mathbb{Q}_p$. Then, α can be written in the form

$$\alpha = b_{-m} \frac{1}{p^m} + b_{-m+1} \frac{1}{p^{m-1}} + \ldots + b_0 + b_1 p + \ldots$$

with $m \in \mathbb{Z}$, $b_{-m} \neq 0$ and $b_i \in \{0, \ldots, p-1\}$.
Define

$$[\alpha]_p = b_{-m} \frac{1}{p^m} + b_{-m+1} \frac{1}{p^{m-1}} + \ldots + b_0,$$

as the p-*adic floor part* of α.
Set $a_0 = [\alpha]_p$. If $\alpha \neq a_0$, then α becomes

$$\alpha = \alpha_0 = a_0 + \frac{1}{\alpha_1},$$

where $\alpha_1 \in \mathbb{Q}_p$, $|\alpha_1|_p \geq p$ and $[\alpha_1]_p \neq 0$. In the same way, if $\alpha_1 \neq a_1$, with $a_1 = [\alpha_1]_p$, then

$$\alpha_1 = a_1 + \frac{1}{\alpha_2},$$

where $\alpha_2 \in \mathbb{Q}_p$. We continue the above process provided $\alpha_n \neq a_n := [\alpha_n]_p$. Finally, it follows that α can be written as

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_{n-1} + \cfrac{1}{\alpha_n}}}},$$

where $a_k = [\alpha_k]_p$ is called a *partial quotient* of α and α_n is the n^{th} *complete quotient* of α.
We note $\alpha = [a_0, \ldots, a_n]_p$ which is defined as the *finite Ruban continued fraction.*
Otherwise, if we have $\alpha = [a_0, \ldots, a_n, \ldots]_p$ then it is called an *infinite Ruban continued fraction*, where $a_0 \in \mathbb{Z}\left[\frac{1}{p}\right] \cap [0, p)$ and $a_n \in \mathbb{Z}\left[\frac{1}{p}\right] \cap (0, p), \forall\ n \geq 1$.
For an infinite Ruban continued fraction $\alpha = [a_0, \ldots, a_n, \ldots]_p$, we define non-negative rational numbers p_n and q_n by using recurrence equations as follows

$$p_{-1} = 1, \quad q_{-1} = 0, \quad p_0 = a_0, \quad q_0 = 1$$

and for any $n \geq 1$, we have

$$\begin{cases} p_n = a_n p_{n-1} + p_{n-2}, \\ q_n = a_n q_{n-1} + q_{n-2}. \end{cases}$$

In $\mathbb{Q}_p$, p_n and q_n are not integers. Thus, we introduce the following notations:

Notation 1 *For any $n \geq 1$, we set*

$$\begin{cases} p'_n = |p_n|_p p_n, \\ q'_n = |q_n|_p q_n. \end{cases}$$

It is clear that p'_n and q'_n are both integers.

The Ruban continued fraction has the following properties, for all $n \geq 0$, we have

- $\dfrac{p_n}{q_n} = [a_0, \ldots, a_n]_p$,
- $\alpha = [a_0, \ldots, a_{n-1}, \alpha_n]_p = \dfrac{\alpha_n p_{n-1} + p_{n-2}}{\alpha_n q_{n-1} + q_{n-2}}$,
- $p_{n-1} q_n - p_n q_{n-1} = (-1)^n$.

$\frac{p_n}{q_n}$ is called the n^{th} *convergent* of α and we have in $\mathbb{Q}_p$, $\lim_{n\to+\infty} \frac{p_n}{q_n} = \alpha$.

Lemma 1 *[13] Let $\alpha = [a_0, \dots, a_n, \dots]_p$ be a p-adic number. Let $\left(\frac{p_n}{q_n}\right)_{n\geq 0}$ denote the sequence of convergents of α. Then, we have*

- $|q_n|_p = |a_1 \dots a_n|_p, \ \forall\ n \geq 1,$
- $\begin{cases} |p_n|_p = |a_0 \dots a_n|_p \ \forall\ n \geq 1, & \textit{if } a_0 \neq 0, \\ |p_1|_p = 1, |p_n|_p = |a_2 \dots a_n|_p \ \forall\ n \geq 2, & \textit{otherwise} \end{cases}$
- $|q_n|_p \geq p^n, \ \forall n \geq 1,$
- $\begin{cases} |p_n|_p \geq p^n, & \textit{if } a_0 \neq 0, \\ |p_n|_p \geq p^{n-1}, & \textit{otherwise} \end{cases} \quad \forall n \geq 1$
- $\begin{cases} |q_n|_p < |q_{n+1}|_p, \\ |p_n|_p < |p_{n+1}|_p, \end{cases} \quad \forall n \geq 0$
- $\left| \alpha - \frac{p_n}{q_n} \right|_p < |q_n|_p^{-2}, \ \forall\ n \geq 0.$

Lemma 2 *[9] If $\beta = [b_0, \dots, b_n, \dots]_p$ is a Ruban continued fraction having the same first $(n+1)$ partial quotients as $\alpha = [a_0, \dots, a_n, \dots]_p$, then*

$$|\alpha - \beta|_p \leq |q_n|_p^{-2}.$$

Wang [13] and Laohakosol [6] gave a characterization of rational numbers having Ruban continued fractions as follows.

Proposition 1 *Let α be a p-adic number. Then α is rational if and only if its Ruban continued fraction expansion is finite or ultimately periodic with a period equal to $p - p^{-1}$.*

3 Results

The purpose of our main result is to deal with stammering p-adic continued fractions with bounded partial quotients in $\mathbb{Q}_p$.

Let $\mathbf{a} = (a_i)_{i\geq 1}$ be a sequence of elements from an alphabet $\mathcal{A}$. We say that $\mathbf{a}$ satisfies Condition $(\star)$ if $\mathbf{a}$ is not ultimately periodic and if there exist two sequences of finite words $(U_n)_{n\geq 1}$ and $(V_n)_{n\geq 1}$ such that:

i) For every $n \geq 1$, the word $U_n V_n U_n$ is a prefix of the word $\mathbf{a}$;

ii) The sequence $\left(\frac{|V_n|}{|U_n|}\right)_{n\geq 1}$ is bounded;

iii) The sequence $(|U_n|)_{n\geq 1}$ is increasing.

We denote by $A = \max\{a_i / i \geq 1\}$ and $B = \frac{A+\sqrt{A^2+4}}{2}$.

We begin now by our main result given by the following theorem:

Theorem 2 *Let p be a prime number. Let $\boldsymbol{a} = (a_i)_{i\geq 1}$ be a sequence of rational numbers in $\mathbb{Z}\left[\frac{1}{p}\right] \cap (0, p)$ not ultimately periodic satisfying Condition $(\star)$ and $\alpha = [0, a_1, \ldots, a_i, \ldots]_p$ be a p-adic number. Assume that $-\nu_p(a_i)$ is bounded. If*

$$\frac{\log B}{\log p} < \frac{1}{4}$$

then α is either quadratic or transcendental.

We show an immediate consequence of Theorem 2.

Corollary 1 *Let $p \geq 7$ be a prime number. Let $\boldsymbol{a} = (a_i)_{i\geq 1}$ be a sequence of rational numbers in $\mathbb{Z}\left[\frac{1}{p}\right] \cap (0, p)$ not ultimately periodic such that $-\nu_p(a_i)$ is bounded. If $A \leq 1$ then the p-adic number $\alpha = [0, a_1, \ldots, a_i, \ldots]_p$ is either quadratic or transcendental.*

The primary tool used for the proof of Theorem 2 is the following version of the Schmidt Subspace Theorem, established by Schlickewei [11], which is recalled below:

Theorem 3 *[11] Let p be a prime number, $L_{1,\infty}, \ldots, L_{m,\infty}$ be m linearly independent forms in the variable $\boldsymbol{x} = (x_1, \ldots, x_m)$ with real algebraic coefficients. Let $L_{1,p}, \ldots, L_{m,p}$ be m linearly independent forms with algebraic p-adic coefficients and in the same variable $\boldsymbol{x} = (x_1, \ldots, x_m)$ and let $\varepsilon > 0$ be a real number. Then, the set of solutions $\boldsymbol{x} \in \mathbb{Z}^m$ of the inequality:*

$$\prod_{i=1}^{m} (|L_{i,\infty}(\boldsymbol{x})||L_{i,p}(\boldsymbol{x})|_p) \leq (\max\{|x_1|, \ldots, |x_m|\})^{-\varepsilon}$$

lies in finitely many proper subspaces of $\mathbb{Q}^m$.

The following lemma is also required for the proof of Theorem 2.

Lemma 3 *[1] Suppose that* $a_i \in \mathbb{Q}^*{}_+$ *and* $\{a_i/i \in \mathbb{N}\}$ *is bounded. Set* $A = \max\{a_i/i \in \mathbb{N}\}$ *and* $B = \frac{A+\sqrt{A^2+4}}{2}$*. Then, for all* $n \geq 0$*, we have*

$$p_n \leq B^{n+1} \text{ and } q_n \leq B^n.$$

Proof. By induction on n. □

Proof of Theorem 2. Assume that the p-adic number $\alpha = [0, a_1, \ldots, a_i, \ldots]_p$ is algebraic of degree at least three. Set $u_n = |U_n|$ and $v_n = |V_n|$, for $n \geq 1$. α admits infinitely many good quadratic approximants, set then the quadratic number $\alpha_n = [0, \overline{U_n, V_n}]_p = [0, U_n, V_n, U_n, V_n, \ldots]_p$.
Since α and α_n have the same first $(2u_n + v_n + 1)$ partial quotients, then we get from Lemma 2 that

$$|\alpha - \alpha_n|_p \leq |q_{2u_n+v_n}|_p^{-2}. \tag{1}$$

Furthermore, α_n is a root of the polynomial

$$P_n(X) = q_{u_n+v_n}X^2 - (p_{u_n+v_n} - q_{u_n+v_n-1})X - p_{u_n+v_n-1}.$$

Since $|\alpha|_p \leq 1$ and $|\alpha_n|_p \leq 1$ then $|p_i|_p \leq |q_i|_p$, for $i \geq 1$. We have

$$|q_{u_n+v_n}\alpha - p_{u_n+v_n}|_p < |q_{u_n+v_n}|_p^{-1} \tag{2}$$

likewise,

$$|q_{u_n+v_n-1}\alpha - p_{u_n+v_n-1}|_p < |q_{u_n+v_n-1}|_p^{-1}. \tag{3}$$

Because α_n is a root of the polynomial $P_n(X)$ then $P_n(\alpha_n) = 0$. Using (1), (2) and (3), we obtain

$$\begin{aligned}
|P_n(\alpha)|_p &= |P_n(\alpha) - P_n(\alpha_n)|_p \\
&= |(q_{u_n+v_n}(\alpha - \alpha_n)(\alpha + \alpha_n) - (p_{u_n+v_n} - q_{u_n+v_n-1})(\alpha - \alpha_n)|_p \\
&= |\alpha - \alpha_n|_p|(q_{u_n+v_n}(\alpha + \alpha_n) - (p_{u_n+v_n} - q_{u_n+v_n-1})|_p \\
&\leq |q_{u_n+v_n}|_p|q_{2u_n+v_n}|_p^{-2}.
\end{aligned} \tag{4}$$

We consider the four following independent linear forms with algebraic p-adic coefficients

$$\begin{cases}
L_{1,p}(X_1, X_2, X_3, X_4) = \alpha^2 X_1 - \alpha(X_2 - X_3) - X_4, \\
L_{2,p}(X_1, X_2, X_3, X_4) = \alpha X_1 - X_2, \\
L_{3,p}(X_1, X_2, X_3, X_4) = \alpha X_3 - X_4, \\
L_{4,p}(X_1, X_2, X_3, X_4) = X_3,
\end{cases}$$

and the following independent linear forms with algebraic real coefficients

$$L_{i,\infty}(X_1, X_2, X_3, X_4) = X_i, \text{ for } 1 \le i \le 4.$$

Keeping Notations 1, we evaluate the product of these linear forms on the quadruple $X_n = (X_1, X_2, X_3, X_4)$ with $X_1 = \mathfrak{q}'_{u_n+v_n}$, $X_2 = \mathfrak{p}'_{u_n+v_n}$, $X_3 = \mathfrak{q}'_{u_n+v_n-1}$ and $X_4 = \mathfrak{p}'_{u_n+v_n-1}$, we get from (2), (3) and (4)

$$\prod_{i=1}^{4} |L_{i,p}(X_n)|_p \le \frac{1}{|q_{u_n+v_n}|_p^4 |q_{2u_n+v_n}|_p^2}.$$

Hence, from Lemma 1 we get

$$\prod_{i=1}^{4} |L_{i,p}(X_n)|_p \le \frac{1}{|q_{u_n+v_n}|_p^4 p^{2(2u_n+v_n)}} \le \frac{1}{|q_{u_n+v_n}|_p^4 p^{u_n+v_n}}.$$

In addition, we have

$$\begin{aligned}\prod_{i=1}^{4} |L_{i,\infty}(X_n)|_\infty &= |\mathfrak{q}'_{u_n+v_n}|_\infty |\mathfrak{p}'_{u_n+v_n}|_\infty |\mathfrak{q}'_{u_n+v_n-1}|_\infty |\mathfrak{p}'_{u_n+v_n-1}|_\infty \\ &\le |q_{u_n+v_n}|_p^4 q_{u_n+v_n}^4.\end{aligned}$$

By Lemma 3, we obtain

$$\prod_{i=1}^{4} |L_{i,\infty}(X_n)|_\infty \le |q_{u_n+v_n}|_p^4 B^{4(u_n+v_n)}.$$

This easily implies that

$$\prod_{i=1}^{4} (|L_{i,\infty}(X_n)|_\infty \, |L_{i,p}(X_n)|_p) \le \frac{B^{4(u_n+v_n)}}{p^{u_n+v_n}}.$$

Since $-v_p(a_i) \le k, \forall\, i \ge 1$, then we have

$$|X_n|_\infty^\varepsilon = |q_{u_n+v_n}|_p^\varepsilon \; q_{u_n+v_n}^\varepsilon \le p^{k\varepsilon(u_n+v_n)} B^{\varepsilon(u_n+v_n)}.$$

It follows then that

$$|X_n|_\infty^\varepsilon \prod_{i=1}^{4} (|L_{i,\infty}(X_n)|_\infty \, |L_{i,p}(X_n)|_p) \le \left(\frac{B^{4+\varepsilon}}{p^{1-k\varepsilon}}\right)^{u_n+v_n} \le \left[\left(\frac{B^{4+\varepsilon}}{p^{1-k\varepsilon}}\right)^{1+\frac{v_n}{u_n}}\right]^{u_n}.$$

From the hypothesis of Theorem 2, by choosing $\varepsilon = \frac{1}{k^2}$ and the fact that $\frac{4k^2+1}{k(k-1)}$ decreases to 4 as k grows, we can choose k large enough in such a way that $-\nu_p(a_i) \leq k, \forall\, i \geq 1$ and $\frac{\log p}{\log B} > \frac{4k^2+1}{k(k-1)}$. Therefore, we obtain

$$\prod_{i=1}^{4} (|L_{i,\infty}(X_n)|_\infty \, |L_{i,p}(X_n)|_p) \leq |X_n|_\infty^{-\varepsilon}.$$

It follows then from Theorem 3 that the points $X_n = (X_1, X_2, X_3, X_4)$ lie in a finite number of proper subspaces of $\mathbb{Q}^4$. Hence, there exist a non-zero integer quadruple (x_1, x_2, x_3, x_4) and an infinite set of distinct positive integers $\mathcal{N}_1$ such that

$$x_1 X_1 + x_2 X_2 + x_3 X_3 + x_4 X_4 = 0.$$

By this equation, we get

$$x_1 q_{u_n+v_n} + x_2 p_{u_n+v_n} + x_3 q_{u_n+v_n-1} + x_4 p_{u_n+v_n-1} = 0. \tag{5}$$

It is clear that $(x_1, x_2) \neq (0, 0)$ since otherwise, by letting n tend to infinity along $\mathcal{N}_1$, we would get that α is rational.
Dividing (5) by $q_{u_n+v_n}$, we get

$$x_1 + x_2 \frac{p_{u_n+v_n}}{q_{u_n+v_n}} + x_3 \frac{q_{u_n+v_n-1}}{q_{u_n+v_n}} + x_4 \frac{p_{u_n+v_n-1}}{q_{u_n+v_n-1}} . \frac{q_{u_n+v_n-1}}{q_{u_n+v_n}} = 0. \tag{6}$$

By letting n tend to infinity along $\mathcal{N}_1$, we obtain from (6) that

$$x_1 + x_2 \alpha + x_3 \beta + x_4 \alpha \beta = 0,$$

with $\beta := \lim\limits_{n \to +\infty} \frac{q_{u_n+v_n-1}}{q_{u_n+v_n}}$. We can observe that β is irrational since otherwise, α would be rational.
For every sufficiently large integer n in $\mathcal{N}_1$, we obtain

$$|q_{u_n+v_n} \beta - q_{u_n+v_n-1}|_p \leq |q_{u_n+v_n-1}|_p^{-1}. \tag{7}$$

Let us consider now the six linearly independent forms with algebraic real and p-adic coefficients

$$\begin{cases} L'_{1,p}(Y_1, Y_2, Y_3) = \alpha Y_1 - Y_3, \\ L'_{2,p}(Y_1, Y_2, Y_3) = \beta Y_1 - Y_2, \\ L'_{3,p}(Y_1, Y_2, Y_3) = Y_2. \\ L'_{i,\infty}(Y_1, Y_2, Y_3) = Y_i, \ \text{for } 1 \leq i \leq 3, \end{cases}$$

Keeping Notations 1, we evaluate the product of these linear forms on the triple $Y_n = (Y_1, Y_2, Y_3)$ with $Y_1 = q'_{u_n+v_n}$, $Y_2 = q'_{u_n+v_n-1}$ and $Y_3 = p'_{u_n+v_n}$. We get from (7) and Lemma 3 that

$$|Y_n|_\infty^\varepsilon \prod_{i=1}^{3}(\left|L'_{i,\infty}(Y_n)\right|_\infty \left|L'_{i,p}(Y_n)\right|_p) \leq \left(\frac{B^{3+\varepsilon}}{p^{1-k\varepsilon}}\right)^{u_n+v_n} \leq \left[\left(\frac{B^{4+\varepsilon}}{p^{1-k\varepsilon}}\right)^{1+\frac{v_n}{u_n}}\right]^{u_n}.$$

From the hypothesis of Theorem 2, we can choose $\varepsilon = \frac{1}{k^2}$ such that for n large enough, we get

$$\prod_{i=1}^{3}(\left|L'_{i,\infty}(Y_n)\right|_\infty \left|L'_{i,p}(Y_n)\right|_p) \leq |Y_n|_\infty^{-\varepsilon}.$$

It follows then from Theorem 3 that the points $Y_n = (Y_1, Y_2, Y_3)$ lie in a finite number of proper subspaces of $\mathbb{Q}^3$. Hence, there exist a non-zero integer triple (y_1, y_2, y_3) and an infinite set of distinct positive integers $\mathcal{N}_2 \subset \mathcal{N}_1$ such that

$$y_1 Y_1 + y_2 Y_2 + y_3 Y_3 = 0.$$

From this equation, we obtain

$$y_1 q_{u_n+v_n} + y_2 q_{u_n+v_n-1} + y_3 p_{u_n+v_n} = 0. \tag{8}$$

Dividing (8) by $q_{u_n+v_n}$ and letting n tend to infinity along $\mathcal{N}_2$, we obtain

$$y_1 + y_2\beta + y_3\alpha = 0. \tag{9}$$

To get another equation connecting α and β, let us consider the six linearly independent forms with algebraic real and p-adic coefficients

$$\begin{cases} L''_{1,p}(Z_1, Z_2, Z_3) = \alpha Z_2 - Z_3, \\ L''_{2,p}(Z_1, Z_2, Z_3) = \beta Z_1 - Z_2, \\ L''_{3,p}(Z_1, Z_2, Z_3) = Z_2. \\ L''_{i,\infty}(Z_1, Z_2, Z_3) = Z_i, \ \text{for } 1 \leq i \leq 3, \end{cases}$$

Keeping Notations 1, we evaluate the product of these linear forms on the triple $Z_n = (Z_1, Z_2, Z_3)$ with $Z_1 = q'_{u_n+v_n}$, $Z_2 = q'_{u_n+v_n-1}$ and $Z_3 = p'_{u_n+v_n-1}$. We get from (7) and Lemma 3 that

$$|Z_n|_\infty^\varepsilon \prod_{i=1}^{3}(\left|L''_{i,\infty}(Z_n)\right|_\infty \left|L''_{i,p}(Z_n)\right|_p) \leq \left(\frac{B^{3+\varepsilon}}{p^{1-k\varepsilon}}\right)^{u_n+v_n} \leq \left[\left(\frac{B^{4+\varepsilon}}{p^{1-k\varepsilon}}\right)^{1+\frac{v_n}{u_n}}\right]^{u_n}.$$

From the hypothesis of Theorem 2, we can choose $\varepsilon = \frac{1}{k^2}$ such that for n large enough, we obtain

$$\prod_{i=1}^{3}(\left|L''_{i,\infty}(Z_n)\right|_\infty \left|L''_{i,p}(Z_n)\right|_p) \leq |Z_n|_\infty^{-\varepsilon}.$$

It follows then from Theorem 3 that the points $Z_n = (Z_1, Z_2, Z_3)$ lie in a finite number of proper subspaces of $\mathbb{Q}^3$. Hence, there exist a non-zero integer triple (z_1, z_2, z_3) and an infinite set of distinct positive integers $\mathcal{N}_3 \subset \mathcal{N}_2$ such that

$$z_1 Z_1 + z_2 Z_2 + z_3 Z_3 = 0.$$

By this equation, we get

$$z_1 q_{u_n+v_n} + z_2 q_{u_n+v_n-1} + z_3 p_{u_n+v_n-1} = 0. \tag{10}$$

Dividing (10) by $q_{u_n+v_n-1}$ and letting n tend to infinity along $\mathcal{N}_3$, we obtain

$$\frac{z_1}{\beta} + z_2 + z_3\alpha = 0. \tag{11}$$

We deduce from (9) and (11) that

$$(y_3\alpha + y_1)(z_3\alpha + z_2) = y_2 z_1.$$

As we have β is irrational, we obtain from (9) and (11) that $y_3 z_3 \neq 0$. Thus, α is an algebraic number of degree at most two, which is a contradiction with the assumption that α has a degree at least three. Consequently, α is transcendental and the proof of Theorem 2 is reached.

Proof of Corollary 1.

We have $p \geq 7$, therefore $p > \phi^4 \simeq 6,85$, with ϕ is the golden ratio.
Besides, we have $A \leq 1$, then $B \leq \phi$. Thus we obtain from Theorem 2 that $4 < \dfrac{\log p}{\log B}$. This brings us to the end of the proof.

Example 1 *Let $p = 7$. Let $(A_n)_{n\geq 0}$ be a sequence of blocks defined as follows:*

$$\begin{cases} A_0 = 1 \ \frac{1}{p}, \\ A_n = A_{n-1} \ A_{n-1} \ \underbrace{1 \ldots 1}_{n \ times} \ A_{n-1}. \end{cases}$$

A_{n-1} *is a prefix of* A_n*, then set* $\mathbf{A} = \lim_{n \to +\infty} A_n$*. As stated in Corollary 1,* $(A_n)_{n \geq 0}$ *satisfies Condition (*$\star$*). Therefore,* $\alpha = [0, \mathbf{A}]_7$ *is either quadratic or transcendental in* $\mathbb{Q}_7$*.*

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On auto-nilpotent polygroups

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Abstract. The purpose of this paper is to introduce the concept of autonilpotent polygroups and investigate their properties concerning the automorphism of polygroups. To realize the article's goals, we present the notation of $\mathfrak{m}$-very thin polygroups and construct the (non) commutative very thin polygroups on every (infinite) finite non–empty set, where $\mathfrak{m} \in \mathbb{N}$. As a result of the research, is to show that the set of automorphism of some very thin polygroups is equal to the set of automorphism of special groups. The paper includes implications for the development of automorphism of polygroups, and shows that under some conditions very thin polygroups are autonilpotent polygroups and investigates the connection between of autonilpotent polygroups and nilpotent polygroups. The new conception of autonilpotent polygroups was broached for the in this paper the first time.

1 Introduction

The hyper compositional structure theory as an extension of classic structures, was firstly introduced, by F. Marty in 1934 [16]. In algebraic hyper compositional system, output from the hyperoperation on elements is a set and so any

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algebraic system is an algebraic hypercompositional system. Marty extended the concept of groups to hypergroups and other researchers presented the algebra hypercompositional structures concepts such as hyperring, hypermodule, hyperfield, hypergraph, polygroup, multiring, etc in this similar way [18]. Algebraic hypercompositional structures are applied in several branches of sciences such as artificial intelligence and (hyper) complex networks [7]. Polygroup, as an important subclass of hypergroups is introduced by Bonansinga and Corsini citebc and is discussed by many scholars [1, 3, 20]. Comer used playgroups to study color algebra [3, 4] and considered some algebraic and combinatorial properties of playgroups [5, 6]. Further materials regarding polygroups and hypergroup such as permutation polygroups [9], isomorphism in polygroups [10], weak polygroups [11], rough subpolygroups in factor polygroup [12], automorphism group of very thin H_v–groups [13], divisible groups derived from divisible hypergroups [14] etc are investigated. An important class of groups as the concept of autonilpotent groups introduced by Moghaddam et. al [17, 19]. Recently Hamidi et al. introduced the concept of auto-Engel polygroups via the heart of hypergroups and investigated the relation between auto–Engel polygroups and auto-nilpotent polygroups. They showed that the concept of the heart of hypergroups plays an important role in the construction of auto-engel polygroups and proved the heart of hypergroups is a characteristic set in hypergroups [15].

This paper introduces the concept of $\mathfrak{m}$-very thin polygroups and constructs finite and infinite very thin polygroups, where $\mathfrak{m} \in \mathbb{N}$. We compute the number of very thin polygroups up to isomorphic. The motivation of our work is the generalization of nilpotent playgroups to autonilpotent polygroups. So we introduce the concept of autonilpotent polygroups and investigate the automorphism of $\mathfrak{m}$–very thin polygroups. It considered some properties of autonilpotent polygroups and connected the autonilpotent polygroups and the quotient of autonilpotent polygroups via the fundamental relations. This study considers the relation between autonilpotent polygroups and nilpotent polygroups and extends the autonilpotent polygroups by the quotient of autonilpotent ploygroup and the direct product of autonilpotent polygroups.

2 Preliminaries

In this section, we review some definitions and results from [8, 20], which we need in what follows. Assume that $H \neq \emptyset$ be an arbitrary set and $P^*(H) = \{G \mid \emptyset \neq G \subseteq H\}$. Each map $\rho : H^2 \longrightarrow P^*(H)$ is said to be a *hyperop-*

eration, hyperstructure (H, ρ) is called a *hypergroupoid* and for every $\emptyset \neq A, B \subseteq H, \rho(A, B) = \bigcup_{a \in A, b \in B} \rho(a, b)$. A *hypergroupoid* (H, ρ) together with an associative binary hyperoperation is said a *semihypergroup* and a semihypergroup (H, ρ) is called a *hypergroup* if for any $x \in H, \rho(x, H) = \rho(H, x) = H$(*reproduction axiom*).

Definition 1 [8] *A semihypergroup* (H, ρ) *is said to be a polygroup, if* (i) *there exists* $e \in H$ *such that for all* $x \in H$, $\rho(e, x) = \rho(x, e) = \{x\}$, (ii) $x \in \rho(y, z)$ *concludes that* $y \in \rho(x, \vartheta(z))$ *and* $z \in \rho(\vartheta(y), x)$, *where* ϑ *is an unitary operation on* H($it follows that for all $x \in H$ *there exists a uiniqe* $\vartheta(x) \in H$ *i.e* $e \in (\rho(x, \vartheta(x)) \cap (\rho(\vartheta(x), x)), \vartheta(e) = e, \vartheta(\vartheta(x)) = x$) *and is denoted by* (H, ρ, e, ϑ) *or* $(H, \cdot, e,^{-1})$, *for simplify. A set* $\emptyset \neq K \subseteq H$ *is said to be a subpolygroup of* H, *if for all* $x, y \in K, \rho(x, \vartheta(y)) \subseteq K$ *and it is denoted by* $K \leq H$.

Definition 2 [20] *Suppose that* (H, ρ) *is a hypergroup. For any given an equivalence relation* ω *on* H, *a hyperoperation* σ *on* $\frac{H}{\omega}$ *is defined by* $\sigma(\omega(a), \omega(b)) = \{\omega(c) \mid c \in \rho(\omega(a), \omega(b))\}$.

Theorem 1 [2] *Let* (H, ρ) *be a hypergroup. Then* $\left(\frac{H}{\omega}, \sigma\right)$ *is a hypergroup if and only if* ω *is a regular equivalence relation and* $\left(\frac{H}{\omega}, \sigma\right)$ *is a group if and if only* ω *is a strongly regular equivalence relation.*

One of famous algebraic relation on any given hypergroup is β which is defined by $a\beta b$ if and only if there exists $u \in \mathcal{U}(H)$ s.t $\{a, b\} \subseteq u$, where $\mathcal{U}(H)$ is denoted by the set of all finite product of elements of H. The smallest transitive relation in a way contains β is denoted by β^* and it means the *transitive closure* of β and $\left(\frac{H}{\beta^*}, \sigma\right)$ is said the *fundamental group* of (H, ρ) [20].

Definition 3 [20] *A map* $f : H_1 \to H_2$ *is called a homomorphism of hypergroups if* $\forall\, x, y \in H_1$, *we have* $f(\rho_1(x, y)) = \rho_2(f(x), f(y))$ *and it is said to be an isomorphism if it is a one to one and onto homomorphism. In similar to algebraic system,* $Aut(H) = \{f : H \to H \mid f$ *is an isomorphism on hypergroup* $H\}$ *is defined. Assume that* $\varphi : H \longrightarrow H/\beta^*$ *by* $\varphi(x) = \beta^*(x)$ *is the canonical homomorphism, then* $w_H = \{x \in H \mid \varphi(x) = 1\}$ *means heart of* H.

Definition 4 [8] *For each* $\emptyset \neq X \subseteq H$, *a subpolygroup generated by* X *is the intersection of all subpolygroups of* H *which contain* X *and is denoted by* $< X >$.

(i) *In every hypergroup* H*, a commutator of* $x, y \in H$ *is shown by* $[x, y] = \{h \in H \mid \rho(x, y) \cap \rho(h, y, x) \neq \emptyset\}$ *and* $H = L_0(H) \supseteq L_1(H) \supseteq \cdots$ *is called a lower series of* H*, where for any* $n \in \mathbb{N}^*$, $L_{n+1}(H) = \{h \in [x, y] \mid x \in L_n(H), y \in H\}$.

(ii) *In every hypergroup* H*,* $H = \Gamma_0(H) \supseteq \Gamma_1(H) \supseteq \cdots$ *is called a derived series of* H*, where for each* $n \in \mathbb{N}^*$, $\Gamma_{n+1}(H) = \{h \in [x, y] \mid x, y \in \Gamma_n(H)\}$. *A polygroup* (H, ρ, e, ϑ) *means a nilpotent polygroup, if for some given integer* $n \in \mathbb{N}, \rho(l_n(H), w_H) = w_H$*, where* $l_{n+1}(H) = \langle\{h \in [x, y] \mid x \in l_n(H), y \in H\}\rangle$ *and* $l_0(H) = H$ *(if there exists a smallest integer* c *in a way that* $\rho(l_c(H), w_H) = w_H$*, and* c *is called the nilpotency class for* H*).*

(iii) *In every hypergroup* H*, for each given* $n \in \mathbb{N}$*, define* $H' = H^{(1)} = \langle\Gamma_1(H)\rangle$ *and* $H^{(n+1)} = (H^{(n)})'$.

3 Automorphism of very thin polygroups

In this section, we introduce the concept of very thin polygroups and for given an arbitrary set constructed at least a very thin polygroup. Moreover, we obtain the number of automorphism group of some very thin polygroups.

Proposition 1 *Let* $(G, \cdot, e)$ *be a polygroup. If for all* $x \in G$ *we have* $x \cdot x^{-1} = \{e\}$*, then* G *is a group.*

Proof. Let $x, y \in G$ and $|x \cdot y| \geq 2$. Then there exists $z_1, z_2 \in x \cdot y$. It follows that $y \in x^{-1} \cdot z_2$ and so $z_1 \in x \cdot y \subseteq x \cdot (x^{-1} \cdot z_2) = (x \cdot (x^{-1}) \cdot z_2 = e \cdot z_2 = \{z_2\}$. Hence G is a group. □

Let $(G, \cdot, e, ^{-1})$ be a polygroup, where $n, r \in \mathbb{N}$ and $|G| = n$. Consider $\mathcal{A}^{(r)} = \{x \cdot y \mid r = |x \cdot y| \text{ and } x, y \in G\}$ and $\mathcal{A} = \bigcup_{1 \leq r \leq n} \mathcal{A}^{(r)}$.

Definition 5 *A polygroup* $(G, \cdot, e, ^{-1})$ *is said to be an* m*–very thin polygroup if* $|\mathcal{A}^{(m)}| = 1$*, where* $m \in \mathbb{N}$.

It is clear that every 1–very thin polygroup is isomorphic to a group and we consider any 2–very thin polygroup as a very thin polygroup.

Example 1 *Let* $(G, \cdot, e)$ *be a (non)commutative group and* $g \notin G$*. Define a hyperoperation "*$\cdot_g$*" on* G' *as follows:*

$$x \cdot_g y = \begin{cases} \{x \cdot y\} & x, y \in G, x \neq y^{-1} \\ \{e, g\} & x = y^{-1} \in G \setminus \{e\}, \\ \{e\} & x = y = g, \\ y & x = g, y \in G \setminus \{e\}, \\ x & y = g, x \in G \setminus \{e\} \end{cases}$$

and $e \cdot_g g = g \cdot_g e = g$*. Some modifications and computations show that* $(G', \cdot_g, e)$ *is a (non)commutative very thin polygroup.*

Definition 6 *Let* $(G, \cdot)$ *be a polygroup. Then* G *is called a cyclic polygoup, if there exists* $g \in G$ *in such a way that* $G = \bigcup_{n \in \mathbb{Z}} g^n$*, where* $g^0 = g \cdot g^{-1}$*,* $g^n = \underbrace{g \cdot g \cdot \ldots \cdot g}_{n-\text{times}}$ *and it is denoted by* $G = \langle g \rangle$*.*

Example 2 (i) *Let* $G = \{0, a\}, r \notin G$*, and* $G' = G \cup \{r\}$*. Then for* $1 \leq i \leq 3$*,* $(G', +_r^{(i)}, 0, -)$ *are cyclic polygroups as follows:*

$+_r^{(1)}$	0	a	r
0	$\{0\}$	$\{a\}$	$\{r\}$
a	$\{a\}$	$\{0, r\}$	$\{a\}$
r	$\{r\}$	$\{a\}$	$\{0\}$

,

$+_r^{(2)}$	0	a	r
0	$\{0\}$	$\{a\}$	$\{r\}$
a	$\{a\}$	$\{0\}$	$\{r\}$
r	$\{r\}$	$\{r\}$	$\{0, a\}$

and

$+_r^{(3)}$	0	a	r
0	$\{0\}$	$\{a\}$	$\{r\}$
a	$\{a\}$	$\{0, r\}$	$\{a\}$
r	$\{r\}$	$\{a\}$	$\{0, r\}$

.

(ii) *Let* $G = \{e, a, b, c\}$*. Then* $(G, \cdot, e,^{-1})$ *is a cyclic polygroup as follows:*

$\cdot$	e	a	b	c
e	$\{e\}$	$\{a\}$	$\{b\}$	$\{c\}$
a	$\{a\}$	$\{a\}$	G	c
b	$\{b\}$	$\{e, a, b\}$	$\{b\}$	$\{b, c\}$
c	$\{c\}$	$\{a, c\}$	$\{c\}$	G

,

where $G = \langle c \rangle$*.*

The above Example, shows that cyclic polygroups necessarily are not commutative polygroup.

Corollary 1 *Let* $n \in \mathbb{N}$*. Then there exists a cyclic polygroup* $(G, \cdot)$ *in such a way that* $|G| = n$*.*

Proof. Let $(H, \cdot)$ be a cyclic group and $g \notin H$ and $G = H \cup \{g\}$. Hence similar to Example 1, we can see that G is a cyclic very thin polygroup, where $G = \bigcup_{n\in\mathbb{Z}} a^n$ and $a^0 = a \cdot a^{-1}$. □

Theorem 2 *Let $G = \langle a \rangle$ be a cyclic polygroup, G' be a polygroup and $f : G \longrightarrow G'$ be an on to homomorphism, where $a \in G$. Then*

(i) G' *is a cyclic polygroup,*

(ii) G/β^* *is a cyclic group,*

(iii) *if $K \leq G$ and K be a complete part of G, then K is a cyclic subpolygroup of G.*

Proof. (i) *Consider $G' = \langle f(a) \rangle$ and so G' is a cyclic polygroup.*

(ii) *Consider $G/\beta^* = \langle \beta^*(a) \rangle$ and so G/β^* is a cyclic group.*

(iii) *Consider $K = \langle a^n \rangle$, where n is the smallest natural number such that $a^n \cap K \neq \emptyset$. Since $x \in K$ implies that there exists $m \in \mathbb{N}$ such that $x \in a^m$, then $a^m \cap K \neq \emptyset$ and so $S = \{l \in \mathbb{N} \mid a^l \cap K \neq \emptyset\} \neq \emptyset$ and so there exist the smallest natural number such that $a^n \cap K \neq \emptyset$. But K is a complete part of G, then $a^n \subseteq K$ and for $m \in \mathbb{N}$ we have $(a^n)^m \subseteq K$. In addition, if $x \in K$, then there exists $m \in \mathbb{N}$ such that $x \in a^m$. If there exists $0 \leq r < n \leq m$, where $m = nq + r$ and $r \neq 0$, then we have*

$$K = x \cdot K \subseteq a^m \cdot K \subseteq (a^{nq} \cdot a^r) \cdot K = a^r \cdot K.$$

It follows that $a^r \cap K \neq \emptyset$, which is a contradiction and so $K = \langle a^n \rangle$. □

The concept of strong homomorphism is defined in [8]. Let $(G_1, \cdot_1, e_1)$ and $(G_2, \cdot_2, e_2)$ be polygroups and $\alpha : G_1 \longrightarrow G_2$ be a map. We define α is a homomorphism, if for all $x, y \in G_1, \alpha(x \cdot_1 y) = \alpha(x) \cdot_2 \alpha(y)$.

Lemma 1 *Let G_1 and G_2 be polygroups and $\alpha : G_1 \longrightarrow G_2$ be a homomorphism. Then*

(i) $e_2 \in \operatorname{Im}(\alpha)$ *if and only if* $\alpha(e_1) = e_2$,

(ii) *for all $x \in G_1$, $\alpha(e_1) = e_2$ implies that $\alpha(x^{-1}) = (\alpha(x))^{-1}$.*

Proof. (i) Since $e_2 \in \operatorname{Im}(\alpha)$, there exists $x \in G_1$ in such a way that $e_2 = \alpha(x)$. So $e_2 = \alpha(x) = \alpha(e_1 \cdot_1 x) = \alpha(e_1) \cdot_2 \alpha(x) = \alpha(e_1) \cdot_2 e_2 = \alpha(e_1)$.

(ii) By definition and the item (i), is obtained. □

Table 1: polygroup G

$\cdot$	a_0	a_1	a_2	a_3	a_4	a_5	a_6
a_0	a_0	a_1	a_2	a_3	a_4	a_5	a_6
a_1	a_1	T	a_3	a_2	a_1	a_1	a_1
a_2	a_2	a_3	T	a_1	a_2	a_2	a_2
a_3	a_3	a_2	a_1	T	a_3	a_3	a_3
a_4	a_4	a_1	a_2	a_3	T	$T \setminus \{a_0\}$	$T \setminus \{a_0\}$
a_5	a_5	a_1	a_2	a_3	$T \setminus \{a_0\}$	T	$T \setminus \{a_0\}$
a_6	a_6	a_1	a_2	a_3	$T \setminus \{a_0\}$	$T \setminus \{a_0\}$	T

Theorem 3 *Let* $(G, \cdot)$ *be a polygroup,* $g \notin G$ *and* $G' = G \cup \{g\}$*. Then*

(i) *If* $\alpha \in Aut(G', ._g, ^{-1}, e)$*, then* $\alpha(e) = e$ *and* $\alpha(g) = g$,

(ii) $Aut(G', ._g, ^{-1}, e) \cong Aut(G, ., ^{-1}, e)$.

Proof. (i) By Lemma 1, $\alpha(e) = e$. Clearly, $e = \alpha(e) = \alpha(g \cdot g) = \alpha(g) \cdot \alpha(g)$, so $\alpha(g) = e$ or $\alpha(g) = g$. If $\alpha(g) = e$, it follows that $\alpha(g) = \alpha(e)$. Since α is an one to one map, we get $g = e$ that is a contradiction. Thus $\alpha(g) = g$.

(ii) Let $\alpha \in Aut(G', ._g, ^{-1}, e)$. Then $f : Aut(G', ._g, ^{-1}, e) \to Aut(G, ., ^{-1}, e)$ by $f(\alpha) = \alpha|_G$ is an isomorphism and so $Aut(G', ._g, ^{-1}, e) \cong Aut(G, ., ^{-1}, e)$. $\square$

Corollary 2 *Let* $n \in \mathbb{N}$*. Then*

(i) $|Aut(\mathbb{Z}_n, +_g, -, \bar{0})| = \varphi(n)$.

(ii) $|Aut(\mathbb{Z}, +_g, -, 0)| = 2$.

(iii) $|Aut(D_{2n}, ._g, -, \bar{0})| = |Aut(D_{2n})| = n\varphi(n)$.

(iv) $|Aut(S_n, ._g, -, \bar{0})| = |Aut(S_n)| = n!$.

Example 3 *Let* $G = \{a_0, a_1, a_2, a_3, a_4, a_5, a_6\}$*. Then* $(G, \cdot)$ *is a polygroup in Table 1, where* $T = \{a_0, a_4, a_5, a_6\}$*. Simple computations show that* $Aut(G) = \{xy \mid x \in S_3, y \in S_X,$ *where* $X = \{4, 5, 6\}\}, S_3 \trianglelefteq Aut(G), S_X \trianglelefteq Aut(G), S_3 \cap S_X = \{e\}$ *for all* $\alpha \in Aut(G)$ *we have ,* $\alpha(T) = T$ *and so* $Aut(G, \cdot) \cong S_3 \times S_3$.

Let G be a non–empty set. We denote the set of all very thin polygroups on G by $VP(G)$, the number of all very thin polygroups on G by $|VP(G)|$ and the number of all very thin polygroups up to isomorphic on G by $\|VP(G)\|$.

Theorem 4 *Let* $m \in \mathbb{N}$, $(G, \cdot, e)$ *be an* m*–very thin polygroup and* $x, y \in G$. *If* $|x \cdot y| \neq 1$, *then* $e \in x \cdot y$.

Proof. Let $e \notin x \cdot y$ and there exists $a \in G$ such that $|a \cdot a^{-1}| \neq 1$. Since $(G, \cdot, e)$ is an m–very thin polygroup $e \in a^{-1} \cdot a = x \cdot y$, which is a contradication. Thus for any $a \in G, |a \cdot a^{-1}| = 1$, so G must be a group, which is a contradiction. □

Corollary 3 *Let* $m \in \mathbb{N}$ *and* (G, e) *be an* m*–very thin polygroup. Then*

(i) *there exist* $a_1, a_2, \ldots a_{m-1}, x \in G$ *such that* $x \cdot x^{-1} = \{e, a_1, a_2, \ldots a_{m-1}\}$,

(ii) *if* $m = 2$ *and* $a_1 \in x \cdot x^{-1}$, *then for all* $\alpha \in Aut(G)$ *we have* $\alpha(a_1) = a_1$.

Proof. (i) By definition is clear. (ii) Let $\alpha \in Aut(G)$. If $x \cdot x^{-1} = \{e, a_1\}$, we have $\alpha(x) \cdot \alpha^{-1}(x) = \alpha(x) \cdot \alpha(x^{-1}) = \alpha(x \cdot x^{-1}) = \alpha(\{e, a_1\}) = \{e, \alpha(a_1)\}$. Hence $\alpha(a_1) \neq e$ and $\{e, a_1\} = \{e, \alpha(a_1)\}$, imply that $\alpha(a_1) = a_1$. □

Theorem 5 *Let* G *be a non–empty set and* $e \in G$.

(i) *If* $|G| = 2$, *then* $|VP(G)| = 2$ *and* $\|VP(G)\| = 1$.

(ii) *If* $|G| = 3$, *then* $|VP((G, e))| = 4$ *and* $\|VP((G, e))\| = 2$,

(iii) *If* $|G| = 3$, *then* $|VP(G)| = 12$ *and* $\|VP(G)\| = 2$.

Proof. (ii), (iii) Let $G = \{e, a, b\}$. For all $x, y \in G, |x \cdot y| \neq 1$ implies that $|x \cdot y| = 2$. If (G, e) is a very thin polygroup, then $a \cdot a = \{e\}, b \cdot b = \{e, a\}$ or $a \cdot a = b \cdot b = \{e, a\}$ or $a \cdot a = b \cdot b = \{e, b\}$, or $b \cdot b = \{e\}, a \cdot a = \{e, b\}$. So $|VP((G, e))| = 4$ and $\|VP((G, e))\| = 3$. □

Let G be a non–empty set. We can compute the set of all very thin polygroups on G and the number of all very thin polygroups up to isomorphic on G, whence $|G| \leq 3$. But can't compute for $|G| \geq 4$.

Open Problem 1 *Let* $m, n \in \mathbb{N}$, $|G| = n$ *and* G *be an* m*–very thin polygroup. Then* $|VP(G)| = ?$ *and* $\|VP(G)\| = ?$

Theorem 6 *Let* $|G| \in \{2, 3\}$. *If* G *is a very thin polygroup, then it is a commutative very thin polygroup and* $|Aut(G)| = 1$.

Proof. It is obtained by Corollary 3 and Theorem 5. □

Example 4 *Consider the very thin polygroup* $G = \mathbb{Z}_3 \cup \{g\}$, *where* $g \notin \mathbb{Z}_3$. *By Corollary 2, we have* $|Aut(G)| = 2$. *It shows that if* G *is a very thin polygroup and* $|G| \geq 4$, *then necessarily* $|Aut(G)| \neq 1$.

Proposition 2 *Let* G *be a hypergroup,* $x \in G$, $\overline{G} = G/\beta$ *and* $\alpha \in Aut(G)$. *Then*

(i) $\overline{\alpha} \in Aut(\overline{G})$, *where* $\overline{\alpha}(\overline{x}) = \overline{\alpha(x)}$ *and* $\overline{x} = \beta^*(x)$,

(ii) $\overline{Aut(G)} \subseteq Aut(\overline{G})$, *where* $\overline{Aut(G)} = \{\overline{\alpha} \mid \alpha \in Aut(G)\}$,

Proof. (i) Let $\overline{x} = \overline{y}$. Then there exists $u \in \mathcal{U}$ in such a way that $\{x, y\} \subseteq u$ and so $\{\alpha(x), \alpha(y)\} \subseteq \alpha(u) \in \mathcal{U}$. Hence, $\overline{\alpha}(\overline{x}) = \overline{\alpha}(\overline{y})$ and then $\overline{\alpha}$ is a well-defined map. In similar a way one can see that $\overline{\alpha}$ is an isomorphism. □

Corollary 4 *Let* G *be a hypergroup,* $\overline{G} = G/\beta^*$ *and* $\alpha, \theta \in Aut(G)$. *Then*

(i) $\overline{\alpha}^{-1} = \overline{\alpha^{-1}}$,

(ii) $\overline{\alpha \circ \theta} = \overline{\alpha} \circ \overline{\theta}$,

(iii) $\overline{Aut(G)} \leq Aut(\overline{G})$.

Proof. Let $x \in G$. Define $\overline{\alpha}(\beta^*(x)) = \beta^*(\alpha(x))$. So the proof is obtained. □

4 Autonilpotent polygroups

In this section, we introduce the concept of autonilpotent polygroup and via the fundamental relations and regular relations consider some conditions to construct autonilpotent groups and autonilpotent polygroups.

Let G be a hypergroup, $x \in G$ and $\alpha \in Aut(G)$. Define $[x, \alpha] = \{g \in G \mid x \in g \cdot \alpha(x)\}$ and will call an autocommutator of x and α. Inductively, for all $\alpha_1, \alpha_2, \ldots, \alpha_n \in Aut(G), [x, \alpha_1, \alpha_2, \ldots, \alpha_n] = \big[x, \alpha_1, \alpha_2, \ldots, \alpha_{n-1}], \alpha_n\big]$ is an autocommutator of $x, \alpha_1, \alpha_2, \ldots, \alpha_n$ of weight $n+1$, where for all $X \subseteq G$ we have $[X, \alpha] = \bigcup_{x \in X} [x, \alpha]$. Let $K_0(G) = G$ and for every $n \in \mathbb{N}^*$, consider $K_{n+1}(G) = \{g \in [x, \alpha] \mid x \in K_n(G), \alpha \in Aut(G)\}$.

Definition 7 *Let* $n \in \mathbb{N}$, G *be a polygroup. Then* G *is called an autonilpotent polygroup of class at most* n, *if* $K_n(G) \subseteq \omega_G$.

Proposition 3 *Let* (G, e) *be a polygroup,* $x \in G, n \in \mathbb{N}$ *and* $\alpha \in Aut(G)$.

(i) $[x, id] = [e, x] = x \cdot x^{-1}$ *and* $[e, \alpha] = e$,

(ii) $[x, \alpha] = x \cdot \alpha(x^{-1})$,

(iii) $\beta^*([x, \alpha]) = [\beta^*(x), \overline{\alpha}]$,

(iv) $[x, \alpha]^{-1} = [\alpha(x), \alpha^{-1}]$,

(v) $K_n(G) = \{h \in [g, \alpha_1, \alpha_2, \ldots, \alpha_n] \mid g \in G, \alpha_1, \alpha_2, \ldots, \alpha_n \in Aut(G)\}$,

(vi) $K_{n+1}(G) \subseteq K_n(G)$.

Proof. Since $[x, \alpha] = \{g \in G \mid x \in g \cdot \alpha(x)\} = \{g \in G \mid g \in x \cdot \alpha(x^{-1})\}$ and $\beta([x, \alpha]) = \beta(x\alpha(x^{-1})) = [\beta(x), \alpha^{-1}]$, we get the results. □

Example 5 (i) *Consider the very thin polygroup* $G = (\mathbb{Z}, +_g, -, 0)$. *Routine computations show that* $K_1(G) = 2\mathbb{Z} \cup \{g\}, K_2(G) = 2^2\mathbb{Z} \cup \{g\}$ *and for every* $n \in \mathbb{N}$ *we have* $K_n(G) = 2^n\mathbb{Z} \cup \{g\}$, *while* $w_G = \{0, g\}$. *Hence* G *is not an autonilpotent polygroup.*

(ii) *Consider the very thin polygroup* $G = (D_6, +_g, -, 0)$. *Routine computations show that for every* $n \in \mathbb{N}$ *we have* $K_n(G) = \{id, (1, 2, 3), (1, 3, 2), g\}$ *while* $w_G = \{0, g\}$. *Hence* G *is not an autonilpotent polygroup.*

Theorem 7 *Let* $(G, \cdot, e, ^{-1})$ *be a group and* $g \notin G$. *Then*

(i) $(G', \cdot_g, e, ^{-1})/\beta^* \cong (G, \cdot, e, ^{-1})$ *and* $\overline{Aut(G')} = Aut(\overline{G'})$,

(ii) *for all* $n \in \mathbb{N}^*$, *we have* $K_n(G) \cup \{g\} = K_n(G')$,

(iii) $(G', \cdot_g, e, ^{-1})$ *is an autonilpotent polygroup if and only if* $(G, \cdot, e, ^{-1})$ *is an autonilpotent group.*

Proof. (i) It is easy to see that $\overline{e} = \overline{g} = \{e, g\}$ and for all $x \notin \overline{e}$ we have $\overline{x} = x$.

(ii) Obviously we have $K_0(G) \cup \{g\} = K_0(G')$ and by induction one can see that $K_n(G) \cup \{g\} = K_n(G')$.

(iii) Since $w_{G'} = \{e, g\}$ and by the item (ii) the proof is obtained. □

Example 6 *Let* $G = \{e, a, b\}$. *Then* $(G, \cdot, e, ^{-1})$ *is a polygroup as follows:*

$\cdot$	e	a	b
e	$\{e\}$	$\{a\}$	$\{b\}$
a	$\{a\}$	$\{e, b\}$	$\{a, b\}$
b	$\{b\}$	$\{a, b\}$	$\{e, a\}$

.

It is easy to see that $Aut(G) = \{id, \alpha = (a\ b)\}$ *and* $\overline{Aut(G)} = Aut(\overline{G})$. *In addition for every* $n \in \mathbb{N}^*, K_n(G) = w_G = G$, *implies that* G *is an autonilpotent polygroup.*

Theorem 8 *Let* $n \in \mathbb{N}$, $k = 2^n$, $g \notin G'$ *be a cyclic group and* $|G'| = k$. *Then* $G = (G' \cup \{g\}, \cdot_g, 0)$ *is an autonilpotent polygroup of class at most* n.

Proof. Let $G' = \langle a \rangle$ and $a \in G$. By Corollary 2, $|Aut(G)| = 2^n - 2^{n-1}$. Let $\alpha \in Aut(G)$. Then $\alpha(a) = a^r$, where $r \in S = \{1, 3, 5, 2^n - 1\}$. So

$$K_1(G) = \{a^{r-1}, g \mid r \in S\}, K_2(G) = \{a^{2(r-1)}, g \mid r \in S\},$$
$$K_3(G) = \{a^{4(r-1)}, g \mid r \in S\}, \dots \quad \text{and} \quad K_n(G) = \{a^{2^{n-1}(r-1)}, g \mid r \in S\} = \{0, g\}.$$

Hence $w_G = \{0, g\} = K_n(G)$ and so G is an autonilpotent polygroup.

□

Corollary 5 *Let* $k \in \mathbb{N}$, $n = 2^k$. *Then* $G = (\mathbb{Z}_n \cup \{\sqrt{2}\}, \cdot_{\sqrt{2}}, \overline{0})$ *is an autonilpotent polygroup.*

Proof. Let $k \in \mathbb{N}$, $n = 2^k$. Since $\mathbb{Z}_{2^k}$ is an autonilpotent group, by definition of G, $G = (\mathbb{Z}_n \cup \{\sqrt{2}\}, \cdot_{\sqrt{2}}, \overline{0})$ is an autonilpotent polygroup. □

Definition 8 *Let* G *be a polygroup. Define* $Z_0(G) = w_G$, *for every* $n \in \mathbb{N}^*, Z_{n+1}(G) = \{x \mid [x, \alpha] \subseteq Z_n(G), \forall \alpha \in Aut(G)\}$ *and we called it by absolute center of* G.

Theorem 9 *Let* G *be a polygroup,* $x \in G$ *and* $n \in \mathbb{N}^*$. *Then*

(i) $Z_n \subseteq Z_{n+1}$ *and so* $w_G \subseteq Z_n$,

(ii) $Z_n(G)$ *is a complete part of* G,

(iii) $[x, id] \subseteq Z_n(G)$,

(iv) *if* $|Aut(G)| = 1$, *then* G *is autonilpotent.*

Proof. (i) Let $\alpha \in Aut(G)$. Since $\alpha(w_G) \subseteq w_G$, we get that $Z_0(G) \subseteq Z_1(G)$ and so by induction the proof is obtained.

(ii) By item (i), $w_G \subseteq Z_n$ implies that $C(Z_n(G)) = Z_n(G) \cdot w_G = Z_n(G)$. Thus $Z_n(G)$ is a complete part of G.

(iii) It is obtained by item (i).

(iv) Let $x \in K_n(G)$ and $h \in [x, id]$. Then by definition we have $h \in x \cdot x^{-1} \subseteq w_G$ that it follows $K_{n+1}(G) \subseteq w_G$. Thus G is autonilpotent. □

Corollary 6 *Let* G *be a very thin polygroup. If* $|G| \leq 3$*, then* G *is an autonilpotent polygroup.*

Proof. It is obtained from Theorems 6 and 9. □

Theorem 10 *Let* G *be a polygroup and* $n \in \mathbb{N}$*.* $K_n(G) \subseteq \omega_G$ *if and only if* $Z_n(G) = G$*.*

Proof. Let $Z_n(G) = G$. Then by induction on i, we have $K_i(G) \subseteq Z_{n-i}(G)$. Now for $i = n$ we obtain that $K_n(G) \subseteq Z_0(G) = \omega_G$.

Conversely, if $K_n(G) \subseteq \omega_G$, then by induction we conclude $K_{n-i}(G) \subseteq Z_i(G)$. Letting $i = n$ implies that $G = K_0(G) \subseteq Z_n(G) \subseteq G$. □

Example 7 (i) *Let* $G = \mathbb{Z}_4 \cup \{g\}$*. Then for all* $n \geq 2$*, we have* $Z_n(G) = Z_n(\mathbb{Z}_4) \cup \{g\} = G$ *and so it is an autonilpotent polygroup.*

(ii) *Let* $G = S_3 \cup \{g\}$*. Then* $Z_n(G) = Z_n(S_3) \cup \{g\} = \{e, g\}$ *and so it is not an autonilpotent polygroup.*

Corollary 7 *Let* G *be a polygroup.* G *is an autonilpotent polygroup if and only if there exists some* $n \in \mathbb{N}$ *in such a way that* $Z_n(G) = G$*.*

Theorem 11 *Let* $G \neq \{e\}$ *be an autonilpotent group. Then* $Z_1(G) \neq \{e\}$*.*

Proof. Since G is an autonilpotent group, by Corollary 7, there exists some $n \in \mathbb{N}$ in such a way that $Z_n(G) = G$. Let $Z_1(G) = \{e\}$. Then for all $n \geq 2$ we obtain that $Z_n(G) = \{e\}$, which is a contradiction. □

Theorem 12 *Let* G *be a polygroup,* $\mathrm{Aut}(G)$ *be a commutative group and* $\alpha \in \mathrm{Aut}(G)$*. Then*

(i) $\alpha(K_n(G)) \subseteq K_n(G)$,

(ii) *if* $x \in K_n(G)$*, then* $x^{-1} \in K_n(G)$,

(iii) *if for all* $x \in G, x \cdot x^{-1} = \{e\}$*, then* $L_n(G) \subseteq K_n(G)$,

(iv) *if* G *is an autonilpotent group, then it is an nilpotent group.*

Proof. (i) Let $h \in K_{n+1}(G)$ and $f \in \mathrm{Aut}(G)$. Then there exist $x \in K_n(G)$ and $\alpha \in \mathrm{Aut}(G)$ such that $h \in [x, \alpha]$ and so $f(h) \in f(x \cdot \alpha(x^{-1}))) = f(x) \cdot f(\alpha(x^{-1})) = f(x)\alpha(f(x^{-1})) = [f(x), \alpha]$. So by induction hypothesis, $x \in K_n(G)$ implies that $f(x) \in K_n(G)$ and so $f(h) \in K_{n+1}(G)$.

(ii) Let $h \in K_{n+1}(G)$. Then there exist $x \in K_n(G)$ and $\alpha \in Aut(G)$ in such a way that $h \in [x, \alpha]$. Using Proposition 3, we have $h^{-1} \in [\alpha(x), \alpha^{-1}]$ and by the item (i), we obtain that $h^{-1} \in K_{n+1}(G)$.

$(iii), (iv)$ It is obtained by induction. Let $h \in [x, y]$ and $x \in L_n(G)$. Then induction assumption, implies that $x \in K_n(G)$. Let $y \in G$ and $\varphi_y \in Inn(G)$, where for all $a \in G$, we have $\varphi_y(a) = y \cdot a \cdot y^{-1}$. Thus $h = x \cdot y \cdot x^{-1} \cdot y^{-1} = [x, \varphi_y]$. Hence by the item (ii), we conclude so $h \in K_{n+1}(G)$. □

Theorem 13 *Let* G *be a hypergroup,* $n \in \mathbb{N}, \overline{G} = G/\beta^*$ *and* $Aut(\overline{G}) \subseteq \overline{Aut(G)}$*. Then*

(i) $K_n(\overline{G}) = \{\overline{t} \mid t \in K_n(G)\}$,

(ii) G *is an autonilpotent polygroup if and only if* $\overline{G}$ *is an autonilpotent group.*

Proof.

(i) Let $\overline{a} \in K_{n+1}(\overline{G})$. Then there exist $\alpha \in Aut(\overline{G})$ and $\overline{x} \in K_n(\overline{G})$ in such a way that $\overline{a} = [\overline{x}, \alpha]$. Thus there exists $\alpha_0 \in Aut(G)$ such that $\overline{\alpha_0} = \alpha$. Using induction hypotheses implies that there exists $t \in K_n(G)$ such that $\overline{x} = \overline{t}$. If $b \in [t, \alpha_0]$, then $b \in K_{n+1}(G)$ and $\overline{b} = [\overline{x}, \overline{\alpha_0}] = [\overline{x}, \alpha] = \overline{a}$. The converse is clear.

(ii) Let G be an autonilpotent polygroup. Then there exists $n \in \mathbb{N}$ in such a way that $K_n(G) \subseteq w_G$. It follows $K_n(\overline{G}) = \{e\}$. The converse is similarly. □

In [8], it is shown that every polygroup G is a nilpotent polygroup if and only if G/β^* is a nilpotent group. So we have the following theorem.

Theorem 14 *Let* G *be an autonilpotent polygroup and* $Aut(G/\beta^*) \subseteq \overline{Aut(G)}$*. Then* G *is a nilpotent polygroup.*

Proof. Applying, Theorem 13, G/β^* is an autonilptent group, so there exists $n \in \mathbb{N}$ in such a way that $K_n(G/\beta^*) = \{\beta^*(e)\}$. By Theorem 12, $L_n(G/\beta^*) = \{\beta^*(e)\}$ and so it is a nilpotent group. Therefore, G is a nilpotent polygroup. □

For any given autonilpotent polygroup G, we can't prove that prove or disprove that it is a nilpotent polygroup, so we give up it as the following open problem.

Open Problem 2 *Let* G *be an autonilpotent polygroup. Then it is a nilpotent polygroup.*

Example 8 *Consider the polygroup* $G = \mathbb{Z}_6 \cup \{g\}$*, where* $g \notin G$*. Thus the converse of Theorem 14 it is not necessarily true.*

Theorem 15 *Let* G_1, G_2 *be hypergroups. Then*

(i) $K_n(G_1) \times K_n(G_2) \subseteq K_n(G_1 \times G_2)$,

(ii) $w_{G_1 \times G_2} = w_{G_1} \times w_{G_2}$,

(iii) *if* $K_n(G_1) \times K_n(G_2) \subseteq w_{G_1} \times w_{G_2}$, *then* $K_n(G_1) \subseteq w_{G_1}$ *and* $K_n(G_2) \subseteq w_{G_2}$.

Proof. (i), (ii) *We prove by induction. Let* $(h_1, h_2) \in K_{n+1}(G_1) \times K_{n+1}(G_2)$. *Then there exist* $x_1 \in K_(G_1), x_2 \in\in K_(G_2), \alpha_1 \in \mathrm{Aut}(G_1)$ *and* $\alpha_2 \in \mathrm{Aut}(G_2)$ *in such a way that* $h_1 \in [x_1, \alpha_1]$ *and* $h_2 \in [x_2, \alpha_2]$. *Define* $\alpha = (\alpha_1, \alpha_2)$ *by* $\alpha(x, y) = (\alpha_1(x), \alpha_2(y))$. *Clearly* $\alpha \in \mathrm{Aut}(G_1 \times G_2)$ *and so by induction assumption,* $(x_1, x_2) \in K_n(G_1) \times K_n(G_2) \subseteq K_n(G_1 \times G_2)$. *So* $(h_1, h_2) \in [(x_1, x_2), (\alpha_1, \alpha_2)] \subseteq K_{n+1}(G_1 \times G_2)$.

(ii) [8]. □

Example 9 *Consider the polygroup* $G_1 = G_2 = \mathbb{Z}_2$. *It is easy to see that* $\{(\bar{0}, \bar{0})\} = K_1(G_1) \times K_1(G_1) \subset K_1(G_1 \times G_2)$. *So necessarily for all* $n \in \mathbb{N}, K_n(G_1) \times K_n(G_2) \neq K_n(G_1 \times G_2)$.

Theorem 16 *Let* G_1 *and* G_2 *be polygroups. If* $G_1 \times G_2$ *is an autonilpotent polygroup, then* G_1 *and* G_2 *are autonilpotent polygroups.*

Proof. Since $G_1 \times G_2$ is an autonilpotent polygroup, then there exists $n \in \mathbb{N}$ such that $K_n(G_1 \times G_2) \subseteq w_{G_1 \times G_2} = w_{G_1} \times w_{G_2}$. Applying Theorem 15, we have

$$K_n(G_1) \times K_n(G_2) \subseteq K_n(G_1 \times G_2) \subseteq w_{G_1} \times w_{G_2}.$$

It follows that $K_n(G_1) \subseteq w_{G_1}$ and $K_n(G_2) \subseteq w_{G_2}$. Hence G_1 and G_2 are autonilpotent polygroups. □

Example 10 *Let* $G = \mathbb{Z}_2 \times \mathbb{Z}_2$. *Then* $\mathrm{Aut}(G) \cong S_3$ *and* $K_1(G) = G$. *So* G *is not an autonilpotent polygroup, whlie* $K_1(\mathbb{Z}_2) = \{\bar{0}\}$ *implies that* $\mathbb{Z}_2$ *is an autonilpotent polygroup.*

The Example 10, shows that the convese of Theorem 16, is not necessarily true.

5 Application

We refer to some applications of our work to sample as follows.

Economic Hypernetwork: Let $G = \{a_0, a_1, a_2, a_3, a_4, a_5, a_6\}$ be a set of some peoples that want to share in an economic benefit and follow the instructions of this partnership based on their abilities. We assume that the instructions are based on the axioms of polygroups and so construct a polygroup as Table 1. This polygroup shows that the people a_0, a_4, a_5, a_6 must be only help in the investment of each person so that the result of the work can be balanced as $T = \{a_0, a_4, a_5, a_6\}$ and sometimes the person a_0 must be removed in this regards.

Artificial Hypernetwork: Let $G = \{a_0, a_1, a_2, a_3, a_4, a_5, a_6\}$ be a set of 7 computers that are used in a intelligent hypernetwork. We want to input layers and output layers of our data satisfy in a certain law, so we put this law in the form of axioms of a polygroup as Table 1 and use their automorphisms as information transfer. Thus we have for all $\alpha \in Aut(G)$ we have, $\alpha(T) = T$ and $Aut(G, \cdot) \cong S_3 \times S_3$ and so we can do this work in 36 ways.

6 Conclusion and discussion

The current paper introduced the notion of $m-$ very thin polygroups, the concept of autonilpotent polygroups and investigated some properties of autonilpotent polygroups. Such as:

(i) For any non–empty set, (non)commutative very thin polygroups are constructed.

(ii) Using the concept of homomorphisms, we obtain the set of autonilpotent of very thin polygroups.

(iii) We show that the set of automorphism of very thin polygroups are equal to set of automorphism of some groups.

(iv) With respect to the concept of nilpotent polygroups, we investigated the relation between of autonilpotent polygroups and nilpotent polygroups.

(v) Through the concept of direct product of autonilpotent polygroups, we extend the autonilpotent polygroups.

We hope that these results are helpful for furthers studies in autonilpotent polygroups. In our future studies, we hope to obtain more results regarding autonilpotent polygroups, autosolvable polygroups, nilpotent polygroups, solvable polygroups and their applications.

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A unified study of the Fourier series involving the Aleph-function of two variables

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Abstract. In 2016, authors have studied Fourier series involving the Aleph-function. In this paper, we make an application of integrals involving sine function, exponential function, the product of Kampé de Fériet functions and the Aleph-function of two variables to evaluate Fourier series. We also develop a multiple integral involving the Aleph-function of two variables to make its application to derive a multiple exponential Fourier series. Some interesting particular cases and remarks are also given.

1 Introduction and Preliminaries

Recently, I-function of two variables, [18], has been studied as a generalization of the H-function of two variables developed by Gupta and Mittal [4] (see also [13]). Singh and Joshi [20] investigated certain double integrals involving the H-function of two variables. These integrals are of a highly general nature and can be specialized to derive numerous known and new integral formulas,

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which hold significant importance in mathematical analysis and are potentially useful in solving various boundary value problems. Srivastava and Singh [25] extended these results to the I-function of two variables as determined by Sharma and Mishra [18].

More recently, Kumar [7] has introduced the Aleph-function of two variables (see also [19]), which is an extension of the I-function of two variables by Sharma and Mishra [18]. The Aleph-function of two variables also generalizes the Aleph-function of one variable introduced by Südland et al. [26]. Systematic studies on the Aleph-function of one variable have been conducted by Ram and Kumar [14], Kumar et al. [8, 9, 10, 11], Saxena et al. [16, 17] and others.

The Aleph-function of two variables is defined using a double Mellin-Barnes type integral as follows:

$$\begin{aligned}\aleph(z_1,z_2) &= \aleph^{0,\mathbf{n}:U}_{P_i,Q_i,\tau_i;r:V}\left(\begin{array}{c|c} z_1 & \mathbb{A} \\ z_2 & \mathbb{B}\end{array}\right) \\ &= \frac{1}{(2\pi\omega)^2}\int_{L_1}\int_{L_2}\theta(s_1,s_2)\prod_{j=1}^{2}\phi_j(s_j)\, z_1^{s_1}\, z_2^{s_2}\, ds_1\, ds_2,\end{aligned} \tag{1}$$

where:

$$\begin{aligned}\mathbb{A} &= (a_j;\alpha_j,A_j)_{1,\mathbf{n}},[\tau_i(a_{ji};\alpha_{ji},A_{ji})]_{\mathbf{n}+1,P_i};(c_j,\gamma_j)_{1,n_1},\left[\tau_{i'}\left(c_{ji'},\gamma_{ji'}\right)\right]_{n_1+1,P_{i'}}; \\ &(e_j,E_j)_{1,n_2},\left[\tau_{i''}\left(e_{ji''},E_{ji''}\right)\right]_{n_2+1,P_{i''}},\end{aligned} \tag{2}$$

$$\begin{aligned}\mathbb{B} &= [\tau_i(b_{ji};\beta_{ji},B_{ji})]_{1,Q_i};(d_j,\delta_j)_{1,m_1},\left[\tau_{i'}\left(d_{ji'},\delta_{ji'}\right)\right]_{m_1+1,Q_{i'}}; \\ &(f_j,F_j)_{1,m_2},\left[\tau_{i''}\left(f_{ji''},F_{ji''}\right)\right]_{m_2+1,Q_{i''}},\end{aligned} \tag{3}$$

$$U = m_1, n_1 : m_2, n_2, \tag{4}$$

$$V = P_{i'}, Q_{i'}, \tau_{i'}; r' : P_{i''}, Q_{i''}, \tau_{i''}; r''. \tag{5}$$

$\theta(s_1,s_2)$ and $\phi_j(s_j)$ are defined by K. Sharma [19] (see also, [7]). The conditions for the existence of equation (1) are provided as follows:

$$\Omega = \tau_i\sum_{j=1}^{P_i}\alpha_{ji} - \tau_i\sum_{j=1}^{Q_i}\beta_{ji} + \tau_{i'}\sum_{j=1}^{P_{i'}}\gamma_{ji} - \tau_{i'}\sum_{j=1}^{Q_{i'}}\delta_{ji'} < 0, \tag{6}$$

$$\Delta = \tau_i\sum_{j=1}^{P_i}A_{ji} - \tau_i\sum_{j=1}^{Q_i}B_{ji} + \tau_{i''}\sum_{j=1}^{P_{i''}}E_{ji''} - \tau_{i''}\sum_{j=1}^{Q_{i''}}F_{ji''} < 0, \tag{7}$$

The conditions for absolute convergence of double Mellin-Barnes type contour integral (1) are as follows:

$$|\arg(z_1)| < \frac{\pi}{2}\Theta \quad \text{and} \quad |\arg(z_2)| < \frac{\pi}{2}\Lambda,$$

where

$$\Theta = \sum_{j=1}^{n} \alpha_j - \tau_i \sum_{j=n+1}^{P_i} \alpha_{ji} - \tau_i \sum_{j=1}^{Q_i} \beta_{ji} + \sum_{j=1}^{n_1} \gamma_j - \tau_{i'} \sum_{j=n_1+1}^{P_{i'}} \gamma_{ji'} + \sum_{j=1}^{n_2} E_j - \tau_{i''} \sum_{j=n_2+1}^{P_{i''}} \gamma_{ji''} > 0, \tag{8}$$

and

$$\Lambda = \sum_{j=1}^{n} A_j - \tau_i \sum_{j=n+1}^{P_i} A_{ji} - \tau_i \sum_{j=1}^{Q_i} B_{ji} + \sum_{j=1}^{m_1} \delta_j - \tau_{i'} \sum_{j=m_1+1}^{Q_{i'}} \delta_{ji'} + \sum_{j=1}^{m_2} F_j - \tau_{i''} \sum_{j=m_2+1}^{Q_{i''}} F_{ji''} > 0. \tag{9}$$

Remark 1 *If* $\tau_i, \tau_{i'}, \tau_{i''} \to 1$*, the Aleph-function of two variables reduces to the* I*-function of two variables due to Sharma and Mishra [18].*

Remark 2 *If* $\tau_i, \tau_{i'}, \tau_{i''} \to 1$ *and* $r = r' = r'' = 1$*, the Aleph-function reduces to the* H*-function of two variables introduced by Gupta and Mittal [4] (see also, [13]).*

The Kampé de Fériet hypergeometric function is represented as follows [1].

$$K^{E;F;F'}_{G;H;H'}\left(\begin{array}{c} x \\ y \end{array} \middle| \begin{array}{c} (e);(f);(f') \\ (g);(h);(h') \end{array}\right) = \sum_{r,t=0}^{\infty} \frac{\prod_{k=1}^{E}(e_k)_{r+t} \prod_{k=1}^{F}(f_k)_r \prod_{k=1}^{F'}(f'_k)_t}{\prod_{k=1}^{G}(g_k)_{r+t} \prod_{k=1}^{H}(h_k)_r \prod_{k=1}^{H'}(h'_k)_t} \frac{x^r y^t}{r!t!} \tag{10}$$

For further details, see Appell and Kampé de Fériet [1]. For brevity, we shall use the following notations.

$$\epsilon = \frac{\prod_{k=1}^{E}(e_k)_{r+t} \prod_{k=1}^{F}(f_k)_r \prod_{k=1}^{F'}(f'_k)_t}{\prod_{k=1}^{G}(g_k)_{r+t} \prod_{k=1}^{H}(h_k)_r \prod_{k=1}^{H'}(h'_k)_t}, \tag{11}$$

$$\epsilon_1 = \frac{\prod_{k_1=1}^{E_1} (e_{1k_1})_{r_1+t_1} \prod_{k_1=1}^{F_1} (f_{1k_1})_{r_1} \prod_{k_1=1}^{F_1'} \left(f'_{1k_1}\right)_{t_1}}{\prod_{k_1=1}^{G_1} (g_{1k_1})_{r_1+t_1} \prod_{k_1=1}^{H_1} (h_{1k_1})_{r_1} \prod_{k_1=1}^{H_1'} \left(h'_{1k_1}\right)_{t_1}}, \tag{12}$$

$$\epsilon_n = \frac{\prod_{k_n=1}^{E_n} (e_{nk_n})_{r_n+t_n} \prod_{k_n=1}^{F_n} (f_{nk_n})_{r_n} \prod_{k_n=1}^{F_n'} \left(f'_{nk_n}\right)_{t_n}}{\prod_{k_n=1}^{G_n} (g_{nk_n})_{r_n+t_n} \prod_{k_n=1}^{H_n} (h_{nk_n})_{r_n} \prod_{k_n=1}^{H_n'} \left(h'_{nk_n}\right)_{t_n}}. \tag{13}$$

Mishra [12] has evaluated the following integral:

Lemma 1

$$\int_0^\pi (\sin x)^{w-1} e^{imx} {}_pF_q\left(\begin{matrix} (\alpha_p); \\ (\beta_q); \end{matrix} \; C(\sin x)^{2h} \right) dx$$
$$= \frac{\pi e^{im\pi/2}}{2^{w-1}} \sum_{r=0}^{\infty} \frac{(\alpha_p)_r \; C^r \, \Gamma(w+2hr)}{(\beta_q)_r \; 4^{hr} \, \Gamma\left(\frac{w+2hr\pm m+1}{2}\right) \; r!} \tag{14}$$

where $(\alpha)_p$ *denotes* $\alpha_1, \cdots, \alpha_p$*;* $\Gamma(a \pm b)$ *represents* $\Gamma(a+b), \Gamma(a-b)$*;* h *is a positive integer:* $p < q$ *and* $\mathrm{Re}(w) > 0$*. We have the following results:*

$$\int_0^\pi e^{i(m-n)x} \, dx = \pi\delta_{m,n}; \quad \int_0^\pi e^{imx} \sin nx \, dx = i\frac{\pi}{2}\delta_{m,n} \tag{15}$$

where $\delta_{m,n} = 1$ *if* $m = n, 0$ *else.*

$$\int_0^\pi e^{imx} \cos nx \, dx = \pi \, \epsilon_{m,n}, \tag{16}$$

where $\epsilon_{m,n} = \frac{1}{2}$ *if* $m = n \neq 0, 1$ *if* $m = n = 0,\ 0$ *else.*

2 Main results

The integrals to be evaluate are:

Theorem 1

$$\int_0^\pi (\sin x)^{w-1} \; e^{imx} \, K_{G;H;H'}^{E;F;F'} \left(\begin{matrix} \alpha(\sin x)^{2\rho} \\ \beta(\sin x)^{2\gamma} \end{matrix} \; \middle| \; \begin{matrix} (e);(f);(f') \\ (g);(h);(h') \end{matrix} \right) \aleph \left(\begin{matrix} z_1 (\sin x)^{\sigma_1} \\ z_2 (\sin x)^{\sigma_2} \end{matrix} \right) dx$$
$$= \frac{\pi \, e^{im\pi/2}}{2^{w-1}} \sum_{r,t=0}^{\infty} E \, \frac{\alpha^r \, \beta^t}{4^{(\rho r+\gamma t)} \; r! \; t!}$$
$$\aleph_{p_i+1,q_i+2,\tau_i;r:V}^{0,\mathbf{n}+1:U} \left[\begin{matrix} 4^{-\sigma_1} z_1 \\ 4^{-\sigma_2} z_2 \end{matrix} \; \middle| \; \begin{matrix} (1-w-2\rho r-2\gamma t; 2\sigma_1, 2\sigma_2), \mathbb{A} \\ \left(\frac{1-w-2\rho r-2\gamma t\pm m}{2}; \sigma_1, \sigma_2\right), \mathbb{B} \end{matrix} \right], \tag{17}$$

provided that $\Re(w) > 0, \rho > 0, \gamma > 0, \sigma_1 > 0, \sigma_2 > 0$, $|\arg z_1| < \frac{\pi}{2}\Theta$ *and* $|\arg z_2| < \frac{\pi}{2}\Lambda$, *where* Θ *and* Λ *are defined respectively by* (8) *and* (9).

Theorem 2

$$\int_0^{\pi}\cdots\int_0^{\pi}\prod_{j=1}^{n}(\sin x)^{w_j-1}\, e^{im_jx_j}\, K^{E_j;F_j;F_j'}_{G_j;H_j;H_j'}\left(\begin{array}{c} \alpha_j(\sin x_j)^{2\rho_j} \\ \beta_j(\sin x_j)^{2\gamma_j}\end{array}\middle|\begin{array}{c}(e_j),(f_j),\left(f_j'\right)\\ (g_j),(h_j),\left(h_j'\right)\end{array}\right)$$
$$\times\ \aleph\left(\begin{array}{c} z_1\prod_{j=1}^{n}(\sin x_j)^{\sigma_j'} \\ z_2\prod_{j=1}^{n}(\sin x_j)^{\sigma_j''}\end{array}\right)dx_1\cdots dx_r$$
$$=\sum_{r_1,t_1,\cdots,r_n,t_n=0}^{\infty}\prod_{j=1}^{n}E_j\,\frac{\pi e^{im_j\pi/2}}{2^{w_j-1}}\,\frac{\alpha_j^{r_j}\,\beta_j^{t_j}}{4^{(\rho_jr_j+\gamma_jt_j)}\,r_j!\,t_j!}$$
$$\times\ \aleph^{0,\mathbf{n}+n:U}_{p_i+n,q_i+2n,\tau_i;r:V}\left[\begin{array}{c} z_14^{-\sum_{j=1}^{n}\sigma_j'} \\ z_24^{-\sum_{j=1}^{n}\sigma_j''}\end{array}\middle|\begin{array}{c}\left(1-w_j-2\rho_jr_j-2\gamma_jt_j;2\sigma_j',2\sigma_j''\right)_{1,n},\mathbb{A}\\ \left(\frac{1-w_j-2\rho r_j-2\gamma t_j\pm m_j}{2};\sigma_j',\sigma_j''\right)_{1,n},\mathbb{B}\end{array}\right],\tag{18}$$

provided that $\Re(w_j) > 0, \rho_j > 0, \gamma_j > 0, \sigma_j' > 0, \sigma_j'' > 0$ *for* $j = 1,\cdots,n$, $|\arg z_1| < \frac{\pi}{2}\Theta$ *and* $|\arg z_2| < \frac{\pi}{2}\Lambda$.

Proof. To prove (17), we express the Aleph-function of two variables into the Mellin-Barnes contour integral with the help of (1) and the Kampé de Fériet function in double series with the help of (10). Now, we change the order of integration and summation, which is permissible under the conditions stated with the integral and we evaluate the inner integral with the help of 1. Now Interpreting the Mellin-Barnes contour integral in Aleph-function of two variables, we obtain the desired result (17). The integral (18) is obtained by the similar procedure. □

3 Exponential Fourier series

In this section, we give the exponential Fourier series of the product of Kampé de Fériet hypergeometric function and the Aleph-function of two variables by using the orthogonality property of exponential function.
Let

$$f^{(1)}(x) = (\sin x)^{w-1}\,K^{E;F;F'}_{G;H;H'}\left(\begin{array}{c}\alpha(\sin x)^{2\rho}\\ \beta(\sin x)^{2\gamma}\end{array}\middle|\begin{array}{c}(e);(f);(f')\\ (g);(h);(h')\end{array}\right)\aleph\left(\begin{array}{c}z_1(\sin x)^{\sigma_1}\\ z_2(\sin x)^{\sigma_2}\end{array}\right)$$

$$= \sum_{p=-\infty}^{\infty} A_p \, e^{-ipx} \tag{19}$$

$f(x)$ is a continuous function and bounded variation with interval $(0, \pi)$. Now, multiplied by e^{imx} both sides in (19) and integrating it with respect x from 0 to π and then making an appeal to (15) and (17), we get

$$A_p = \frac{e^{ip\pi/2}}{2^{w-1}} \sum_{r,t=0}^{\infty} E \, \frac{\alpha^r \beta^t}{4^{(\rho r+\gamma t)} r!\, t!} \aleph^{0,\mathbf{n}+1:U}_{p_i+1,q_i+2,\tau_i;r:V} \left(\begin{array}{c} 4^{-\sigma_1} z_1 \\ 4^{-\sigma_2} z_2 \end{array} \middle| \begin{array}{c} (1-w-2\rho r-2\gamma t; 2\sigma_1, 2\sigma_2)\,,\, \mathbb{A} \\ \left(\frac{1-w-2\rho r-2\gamma t\pm m}{2}; \sigma_1, \sigma_2\right)\,,\, \mathbb{B} \end{array} \right). \tag{20}$$

Using (19) and (20), we obtain the following exponential Fourier series:

Theorem 3

$$(\sin x)^{w-1} \, K^{E;F;F'}_{G;H;H'} \left(\begin{array}{c} \alpha\,(\sin x)^{2\rho} \\ \beta\,(\sin x)^{2\gamma} \end{array} \middle| \begin{array}{c} (e)\,;(f)\,;(f') \\ (g)\,;(h)\,;(h') \end{array} \right) \aleph \left(\begin{array}{c} z_1\,(\sin x)^{\sigma_1} \\ z_2\,(\sin x)^{\sigma_2} \end{array} \right)$$

$$= \sum_{p=-\infty}^{\infty} \sum_{r,t=0}^{\infty} E e^{ip(\pi/2-x)} \tag{21}$$

$$\times \frac{\alpha^r \beta^t}{4^{(\rho r+\gamma t)}\, r!\, t!} \aleph^{0,\mathbf{n}+1:U}_{p_i+1,q_i+2,\tau_i;r:V} \left(\begin{array}{c} 4^{-\sigma_1} z_1 \\ 4^{-\sigma_2} z_2 \end{array} \middle| \begin{array}{c} (1-w-2\rho r-2\gamma t; 2\sigma_1, 2\sigma_2)\,,\, \mathbb{A} \\ \left(\frac{1-w-2\rho r-2\gamma t\pm m}{2}; \sigma_1, \sigma_2\right)\,,\, \mathbb{B} \end{array} \right),$$

under the same conditions as (17).

4 Cosine Fourier series

In this section, we obtain the cosine Fourier series of the product of Kampé de Fériet hypergeometric function and the Aleph-function of two variables by using the orthogonality property (15) and (16).

$$f^{(2)}(x) = (\sin x)^{w-1} \, K^{E;F;F'}_{G;H;H'} \left(\begin{array}{c} \alpha\,(\sin x)^{2\rho} \\ \beta\,(\sin x)^{2\gamma} \end{array} \middle| \begin{array}{c} (e)\,;(f)\,;(f') \\ (g)\,;(h)\,;(h') \end{array} \right) \aleph \left(\begin{array}{c} z_1\,(\sin x)^{\sigma_1} \\ z_2\,(\sin x)^{\sigma_2} \end{array} \right)$$

$$= \frac{B_0}{2} + \sum_{p=1}^{\infty} B_p \, \cos px. \tag{22}$$

Integrating it with respect x from 0 to π, we have

$$\frac{B_0}{2} = \frac{1}{\pi^{\frac{1}{2}}} \sum_{r,t=0}^{\infty} E \, \frac{\alpha^r \beta^t}{r!\, t!} \, \frac{B_0}{2}$$
$$\aleph^{0,\mathbf{n}+1:U}_{p_i+1,q_i+2,\tau_i;r:V} \left(\begin{array}{c} z_1 \\ z_2 \end{array} \,\middle|\, \begin{array}{c} \left(1 - \frac{w}{2} - 2\rho r - 2\gamma t; 2\sigma', 2\sigma''\right), \mathbb{A} \\ \left(\frac{1-w}{2} - \rho r - \gamma t; \sigma', \sigma''\right), \mathbb{B} \end{array} \right). \tag{23}$$

Multiplying both sides in (22) by e^{imx} and integrating it withrespect x from 0 to π and use the equations (15), (16) and (17), we obtain

$$B_p = \frac{e^{ip\pi/2}}{2^{w-1}} \sum_{r,t=0}^{\infty} E \, \frac{\alpha^r \beta^t}{4^{(\rho r+\gamma t)} r!\, t!}$$
$$\aleph^{0,\mathbf{n}+1:U}_{p_i+1,q_i+2,\tau_i;r:V} \left(\begin{array}{c} 4^{-\sigma_1} z_1 \\ 4^{-\sigma_2} z_2 \end{array} \,\middle|\, \begin{array}{c} (1 - w - 2\rho r - 2\gamma t; 2\sigma_1, 2\sigma_2), \mathbb{A} \\ \left(\frac{1-w-2\rho r-2\gamma t \pm m}{2}; \sigma_1, \sigma_2\right), \mathbb{B} \end{array} \right). \tag{24}$$

Using the equations (22), (23) and (24), we obtain the following cosine Fourier series:

Theorem 4

$$(\sin x)^{w-1} \, K^{E;F;F'}_{G;H;H'} \left(\begin{array}{c} \alpha \, (\sin x)^{2\rho} \\ \beta \, (\sin x)^{2\gamma} \end{array} \,\middle|\, \begin{array}{c} (e);(f);(f') \\ (g);(h);(h') \end{array} \right) \aleph \left(\begin{array}{c} z_1 \, (\sin x)^{\sigma_1} \\ z_2 \, (\sin x)^{\sigma_2} \end{array} \right)$$
$$= \frac{1}{\pi^{\frac{1}{2}}} \sum_{r,t=0}^{\infty} E \frac{\alpha^r \beta^t}{r!\, t!} \, \frac{B_0}{2} \, \aleph^{0,\mathbf{n}+1:U}_{p_i+1,q_i+1,\tau_i;r:V} \left(\begin{array}{c} z_1 \\ z_2 \end{array} \,\middle|\, \begin{array}{c} \left(1 - \frac{w}{2} - 2\rho r - 2\gamma t; 2\sigma', 2\sigma''\right), \mathbb{A} \\ \left(\frac{1-w}{2} - \rho r - \gamma t; \sigma', \sigma''\right), \mathbb{B} \end{array} \right)$$
$$+ \sum_{p=1}^{\infty} \sum_{r,t=0}^{\infty} E \, e^{ip\pi/2} \cos px \frac{\alpha^r \beta^t}{4^{(\rho r+\gamma t)} r!\, t!}$$
$$\times \; \aleph^{0,\mathbf{n}+1:U}_{p_i+1,q_i+2,\tau_i;r:V} \left(\begin{array}{c} 4^{-\sigma_1} z_1 \\ 4^{-\sigma_2} z_2 \end{array} \,\middle|\, \begin{array}{c} (1 - w - 2\rho r - 2\gamma t; 2\sigma_1, 2\sigma_2), \mathbb{A} \\ \left(\frac{1-w-2\rho r-2\gamma t \pm m}{2}; \sigma_1, \sigma_2\right), \mathbb{B} \end{array} \right) \tag{25}$$

under the same conditions as (17).

5 Sine Fourier series

In this section, we obtain the sine Fourier series of the product of Kampé de Fériet hypergeometric function and the Aleph-function of two variables by using the orthogonality property (15).

$$f^{(3)}(x) = (\sin x)^{w-1} \, K^{E;F;F'}_{G;H;H'} \left(\begin{array}{c} \alpha (\sin x)^{2\rho} \\ \beta \, (\sin x)^{2\gamma} \end{array} \,\middle|\, \begin{array}{c} (e);(f);(f') \\ (g);(h);(h') \end{array} \right) \aleph \left(\begin{array}{c} z_1 \, (\sin x)^{\sigma_1} \\ z_2 \, (\sin x)^{\sigma_2} \end{array} \right)$$

$$= \sum_{p=-\infty}^{\infty} C_p \sin px. \tag{26}$$

Multiplying both sides in (26) e^{imx} and integrating it with respect x from 0 π and use the equations (15), and (17), we obtain

$$C_p = \frac{e^{ip\pi/2}}{i2^{w-2}} \sum_{r,t=0}^{\infty} E \frac{\alpha^r \beta^t}{4^{(\rho r+\gamma t)} r!\, t!}$$
$$\aleph^{0,\mathbf{n}+1:U}_{p_i+1,q_i+2,\tau_i;r:V} \left(\begin{array}{c} 4^{-\sigma_1} z_1 \\ 4^{-\sigma_2} z_2 \end{array} \middle| \begin{array}{c} (1-w-2\rho r-2\gamma t; 2\sigma_1, 2\sigma_2), \mathbb{A} \\ \left(\frac{1-w-2\rho r-2\gamma t\pm m}{2}; \sigma_1, \sigma_2\right), \mathbb{B} \end{array} \right). \tag{27}$$

Using the equations (26) and (27), we get the following sine Fourier series:

Theorem 5

$$f^{(3)}(x) = (\sin x)^{w-1} K^{E;F;F'}_{G;H;H'} \left(\begin{array}{c} \alpha (\sin x)^{2\rho} \\ \beta (\sin x)^{2\gamma} \end{array} \middle| \begin{array}{c} (e);(f);(f') \\ (g);(h);(h') \end{array} \right) \aleph \left(\begin{array}{c} z_1 (\sin x)^{\sigma_1} \\ z_2 (\sin x)^{\sigma_2} \end{array} \right)$$
$$= -2i \sum_{p=-\infty}^{\infty} \sum_{r,t=0}^{\infty} E\, e^{ip\pi/2} \sin px \frac{\alpha^r \beta^t}{4^{(\rho r+\gamma t)} r!\, t!}$$
$$\times \aleph^{0,\mathbf{n}+1:U}_{p_i+1,q_i+2,\tau_i;r:V} \left(\begin{array}{c} 4^{-\sigma_1} z_1 \\ 4^{-\sigma_2} z_2 \end{array} \middle| \begin{array}{c} (1-w-2\rho r-2\gamma t; 2\sigma_1, 2\sigma_2), \mathbb{A} \\ \left(\frac{1-w-2\rho r-2\gamma t\pm m}{2}; \sigma_1, \sigma_2\right), \mathbb{B} \end{array} \right), \tag{28}$$

under the same conditions as (17).

6 Multiple exponential Fourier series

In this section, we obtain the multiple exponential Fourier series of the product of Kampé de Fériet hypergeometric function and the Aleph-function of two variables.
$f(x_1, \cdots, x_n)$ is a function that is continuous and of bounded variation in the domain $\underbrace{(0,\pi) \times \cdots \times (0,\pi)}_{n}$.

$$f(x_1, \cdots, x_n) = \prod_{j=1}^{n} (\sin x)^{w_j-1} K^{E_j;F_j;F'_j}_{G_j;H_j;H'_j} \left(\begin{array}{c} \alpha_j (\sin x_j)^{2\rho_j} \\ \beta_j (\sin x_j)^{2\gamma_j} \end{array} \middle| \begin{array}{c} (e_j), (f_j), \left(f'_j\right) \\ (g_j), (h_j), \left(h'_j\right) \end{array} \right)$$
$$\times \aleph \left(\begin{array}{c} z_1 \prod_{j=1}^{n} (\sin x_j)^{\sigma'_j} \\ z_2 \prod_{j=1}^{n} (\sin x_j)^{\sigma''_j} \end{array} \right) = \sum_{p_1,\cdots,p_n=-\infty}^{\infty} A_{p_1,\cdots,p_n} e^{-i(p_1x_1+\cdots p_nx_n)}. \tag{29}$$

We fix $x_1, \cdots, x_{n-1}$ and multiplying both sides in (29) by $e^{im_n x_n}$ and integrating with respect to x_n from 0 to π, we obtain

$$\prod_{j=1}^{n-1} (\sin x)^{w_j-1} \; K^{E_j;F_j;F'_j}_{G_j;H_j;H'_j}\left(\begin{array}{c} \alpha_j (\sin x_j)^{2\rho_j} \\ \beta_j (\sin x_j)^{2\gamma_j} \end{array} \middle| \begin{array}{c} (e_j), (f_j), \left(f'_j\right) \\ (g_j), (h_j), \left(h'_j\right) \end{array} \right)$$
$$\aleph \left(\begin{array}{c} z_1 \prod_{j=1}^{n} (\sin x_j)^{\sigma'_j} \\ z_2 \prod_{j=1}^{n} (\sin x_j)^{\sigma''_j} \end{array} \right) \tag{30}$$
$$= \sum_{p_1=-\infty}^{\infty} \cdots \sum_{p_{n-1}=-\infty}^{\infty} e^{-i(p_1x_1+\cdots+p_nx_n)} + \sum_{p_n=-\infty}^{\infty} \int_0^{\pi} e^{i(m_n-p_n)} \, dx_n,$$

using the first relation of (15) and (17), from (30), we get

$$A_{p_1,\cdots,p_n} = \sum_{r_1,t_1,\cdots,r_n,t_n=0}^{\infty} \prod_{j=1}^{n} E_j \frac{e^{ip_j\pi/2}}{2^{w_j-1}} \frac{\alpha_j^{r_j} \beta_j^{t_j}}{4^{(\rho_j r_j+\gamma_j t_j)} r_j!\, t_j!}$$
$$\aleph^{0,\mathbf{n}+n:U}_{p_i+n,q_i+2n,\tau_i;r:V} \left(\begin{array}{c} z_1 4^{-\sum_{j=1}^{n} \sigma'_j} \\ z_2 4^{-\sum_{j=1}^{n} \sigma''_j} \end{array} \middle| \begin{array}{c} \left(1-w_j-2\rho_j r_j-2\gamma_j t_j; 2\sigma'_j, 2\sigma''_j\right)_{1,n}, \mathbb{A} \\ \left(\frac{1-w_j-2\rho r_j-2\gamma t_j \pm m_j}{2}; \sigma'_j, \sigma''_j\right)_{1,n}, \mathbb{B} \end{array} \right). \tag{31}$$

Using (29) and (31), we obtain the multiple exponential Fourier series.

Theorem 6

$$\prod_{j=1}^{n} (\sin x)^{w_j-1} \; K^{E_j;F_j;F'_j}_{G_j;H_j;H'_j}\left(\begin{array}{c} \alpha_j (\sin x_j)^{2\rho_j} \\ \beta_j (\sin x_j)^{2\gamma_j} \end{array} \middle| \begin{array}{c} (e_j), (f_j), \left(f'_j\right) \\ (g_j), (h_j), \left(h'_j\right) \end{array} \right)$$
$$\aleph \left(\begin{array}{c} z_1 \prod_{j=1}^{n} (\sin x_j)^{\sigma'_j} \\ z_2 \prod_{j=1}^{n} (\sin x_j)^{\sigma''_j} \end{array} \right) \tag{32}$$
$$= \sum_{p_1,\cdots,p_n=-\infty}^{\infty} \sum_{r_1,t_1,\cdots,r_n,t_n=0}^{\infty} \prod_{j=1}^{n} E_j \frac{e^{ip_j(\pi/2-x)}}{2^{w_j-1}} \frac{\alpha_j^{r_j} \beta_j^{t_j}}{4^{(\rho_j r_j+\gamma_j t_j)} r_j!\, t_j!}$$
$$\times \; \aleph^{0,\mathbf{n}+n:U}_{p_i+n,q_i+2n,\tau_i;r:V} \left(\begin{array}{c} z_1 4^{-\sum_{j=1}^{n} \sigma'_j} \\ z_2 4^{-\sum_{j=1}^{n} \sigma''_j} \end{array} \middle| \begin{array}{c} \left(1-w_j-2\rho_j r_j-2\gamma_j t_j; 2\sigma'_j, 2\sigma''_j\right)_{1,n}, \mathbb{A} \\ \left(\frac{1-w_j-2\rho r_j-2\gamma t_j \pm m_j}{2}; \sigma'_j, \sigma''_j\right)_{1,n}, \mathbb{B} \end{array} \right),$$

under the same conditions that (18).

7 Particular cases

By setting $\beta_1, \cdots, \beta_n = 0$ in equations (18) and (32), we respectively obtain the following multiple integral and multiple exponential Fourier series.

Corollary 1

$$\int_0^\pi \cdots \int_0^\pi \prod_{j=1}^n (\sin x)^{w_j-1} e^{im_j x_j} {}_{E_j+F_j}K_{G_j+H_j}\left(\alpha_j (\sin x_j)^{2\rho_j} \left| \begin{matrix} (e_j),(f_j) \\ (g_j),(h_j) \end{matrix} \right.\right)$$
$$\times \aleph \begin{pmatrix} z_1 \prod_{j=1}^n (\sin x_j)^{\sigma_j'} \\ z_2 \prod_{j=1}^n (\sin x_j)^{\sigma_j''} \end{pmatrix} dx_1 \cdots dx_r = \sum_{r_1,\cdots,r_n=0}^\infty \prod_{j=1}^n \frac{e^{ip_j\pi/2}}{2^{w_j-1}} \mathfrak{E}_j \frac{\alpha_j^{r_j}}{4^{\rho_j r_j} r_j!}$$
$$\times \aleph^{0,\mathbf{n}+n:U}_{p_i+n,q_i+2n,\tau_i;r:V}\left(\begin{matrix} z_1 4^{-\sum_{j=1}^n \sigma_j'} \\ z_2 4^{-\sum_{j=1}^n \sigma_j''} \end{matrix} \left| \begin{matrix} \left(1-w_j-2\rho_j r_j; 2\sigma_j', 2\sigma_j''\right)_{1,n}, \mathbb{A} \\ \left(\frac{1-w_j-2\rho r_j \pm m_j}{2}; \sigma_j', \sigma_j''\right)_{1,n}, \mathbb{B} \end{matrix} \right.\right), \tag{33}$$

under the same conditions that (18) *with* $\beta_1, \cdots, \beta_n = 0$, *and*

$$\mathfrak{E}_j = \frac{\prod_{k_j=1}^{E_j} (e_{jk_j})_{r_j} \prod_{k_j=1}^{F_j} (f_{jk_j})_{r_j}}{\prod_{k_j=1}^{G_j} (g_{jk_j})_{r_j} \prod_{k_j=1}^{H_j} (h_{jk_j})_{r_j}}, \qquad j = 1, \cdots, n.$$

Corollary 2

$$\prod_{j=1}^n (\sin x)^{w_j-1} {}_{E_j+F_j}K_{G_j+H_j}\left(\alpha_j (\sin x_j)^{2\rho_j} \left| \begin{matrix} (e_j),(f_j) \\ (g_j),(h_j) \end{matrix} \right.\right) \aleph \begin{pmatrix} z_1 \prod_{j=1}^n (\sin x_j)^{\sigma_j'} \\ z_2 \prod_{j=1}^n (\sin x_j)^{\sigma_j''} \end{pmatrix}$$
$$= \sum_{p_1,\cdots,p_n=-\infty}^\infty \sum_{r_1,\cdots,r_n=0}^\infty \prod_{j=1}^n E_j \frac{e^{ip_j(\pi/2-x)}}{2^{w_j-1}} \frac{\alpha_j^{r_j} \beta_j^{t_j}}{4^{(\rho_j r_j+\gamma_j t_j)} r_j!\, t_j!}$$
$$\times \aleph^{0,\mathbf{n}+n:U}_{p_i+n,q_i+2n,\tau_i;r:V}\left(\begin{matrix} z_1 4^{-\sum_{j=1}^n \sigma_j'} \\ z_2 4^{-\sum_{j=1}^n \sigma_j''} \end{matrix} \left| \begin{matrix} \left(1-w_j-2\rho_j r_j; 2\sigma_j', 2\sigma_j''\right)_{1,n}, \mathbb{A} \\ \left(\frac{1-w_j-2\rho r_j \pm m_j}{2}; \sigma_j', \sigma_j''\right)_{1,n}, \mathbb{B} \end{matrix} \right.\right), \tag{34}$$

under the same conditions that (18) *with* $\beta_1, \cdots, \beta_n = 0$.

If $\alpha_1 = \cdots = \alpha_n = 0$ in equation (33), we obtain the following multiple integral, defined as Corollary 3:

Corollary 3

$$\int_0^{\pi}\cdots\int_0^{\pi}\prod_{j=1}^{n}(\sin x)^{w_j-1}\ e^{im_jx_j}\,\aleph\begin{pmatrix} z_1\prod_{j=1}^{n}(\sin x_j)^{\sigma_j'}\\ z_2\prod_{j=1}^{n}(\sin x_j)^{\sigma_j''}\end{pmatrix}dx_1\cdots dx_r=\prod_{j=1}^{n}\frac{\pi e^{im_j\pi/2}}{2^{w_j-1}}$$

$$\times\ \aleph^{0,\mathbf{n}+n:U}_{p_i+n,q_i+2n,\tau_i;r:V}\left(\begin{matrix} z_1 4^{-\sum_{j=1}^{n}\sigma_j'}\\ z_2 4^{-\sum_{j=1}^{n}\sigma_j''}\end{matrix}\ \middle|\ \begin{matrix}\left(1-w_1;2\sigma_j',2\sigma_j''\right)_{1,n},\mathbb{A}\\ \left(\frac{1-w_1\pm m_j}{2};\sigma_j',\sigma_j''\right)_{1,n},\mathbb{B}\end{matrix}\right), \tag{35}$$

under the same conditions that (18) with $\beta_1,\cdots,\beta_n=0$ and $\alpha_1=\cdots=\alpha_n=0$.

Remark 3 *We can also obtain the similar formulas*
(i) *with the multivariable* H*-function defined by Srivastava and Panda [23, 24], for more details see also [3].*
(ii) *with the Aleph-function of one variable defined by Südland et al. [26], see Ayant and Kumar [2].*
(iii) *with the* $\bar{\mathsf{H}}$*-function defined by Inayat-Hussain [5, 6], see R.C. Singh and Khan [21].*
(iv) *with the* I*-function defined by Saxena [15], see Singh and Khan [22].*

8 Concluding Remarks

The Aleph-function of two variables and the Kampe de Fériet function presented in this paper are fundamentally simple in nature. By specializing the parameters of these functions, we can derive various Fourier series expansions related to other special functions, such as the I-function of two variables, the H-function of two variables, the I-function, Fox's H-function, Meijer's G-function, Wright's generalized Bessel function, Wright's generalized hypergeometric function, MacRobert's E-function, generalized hypergeometric functions, the Bessel function of the first kind, the modified Bessel function, the Whittaker function, the exponential function, the binomial function, and more. Consequently, numerous unified integral representations can be obtained as special cases of our results.

Conflicts of Interest: The authors declare that they have no conflicts of interest.

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Some generalisations and minimax approximants of D'Aurizio trigonometric inequalities

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Abstract. In this paper, we generalise J. Sándor's results on D'Aurizio's trigonometric inequalities using stratified families of functions.

1 Introduction and preliminaries

In this paper, we give some generalisations of the following results of József Sándor [1] concerning D'Aurizio's trigonometric inequalities:

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Theorem 1 (*J. Sándor*) *For* $0 < |x| < \pi/2$

$$1 - \frac{4}{\pi^2}x^2 < \frac{\cos x}{\cos \frac{x}{2}} < 1 - \frac{3}{8}x^2 \tag{1}$$

holds.

Theorem 2 (*J. Sándor*) *For* $0 < |x| < \pi/2$

$$2 - \frac{1}{4}x^2 < \frac{\sin x}{\sin \frac{x}{2}} < 2 - \frac{4(2-\sqrt{2})}{\pi^2}x^2 \tag{2}$$

holds.

The improved results are obtained using concepts presented in [2]. In this section, the important theorems from [2], which are necessary for further proofs, are listed.

Let

$$\varphi_p(x) : (a, b) \longrightarrow \mathbb{R}$$

be a family of functions with a variable $x \in (a, b)$ and a parameter $p \in \mathbb{R}^+$. In this paper, we call $\sup\limits_{x \in (a,b)} |\varphi_p(x)|$ an error and denote it by:

$$d^{(p)} = \sup_{x \in (a,b)} |\varphi_p(x)|. \tag{3}$$

In [2], the conditions for the existence of the unique value p_0 of the parameter, for which an infimum of an error (as a positive real number) is attained, are explored. Such infimum is denoted by:

$$d_0 = \inf_{p \in \mathbb{R}^+} \sup_{x \in (a,b)} |\varphi_p(x)|. \tag{4}$$

For such a value p_0, the function $\varphi_{p_0}(x)$ is called *the minimax approximant* on (a, b).

A family of functions $\varphi_p(x)$ is increasingly stratified if $p' > p'' \iff \varphi_{p'}(x) > \varphi_{p''}(x)$ for any $x \in (a, b)$ and, conversely, it is decreasingly stratified if $p' > p'' \iff \varphi_{p'}(x) < \varphi_{p''}(x)$ for any $x \in (a, b)$ $(p', p'' \in \mathbb{R}^+)$.

Based on Theorem 1 and Theorem 1' from [2], we can conclude that for stratified families of functions, the following theorem is true:

Theorem 3 *Let* $\varphi_p(x)$ *be an increasingly (decreasingly) stratified family of functions (for* $p \in \mathbb{R}^+$*) that are continuous with respect to* $x \in (a, b)$ *for each* $p \in \mathbb{R}^+$*, and let* c, d *be in* $\mathbb{R}^+$ *such that* $c < d$*. If:*

(a) $\varphi_c(x) < 0$ $(\varphi_c(x) > 0)$ *and* $\varphi_d(x) > 0$ $(\varphi_d(x) < 0)$ *for all* $x \in (a, b)$*, and at the endpoints* $\varphi_c(a+) = \varphi_d(a+) = 0$*,* $\varphi_c(b-) = 0$ $(\varphi_d(b-) = 0)$ *and* $\varphi_d(b-) \in \mathbb{R}^+$ $(\varphi_c(b-) \in \mathbb{R}^+)$ *hold;*

(b) *the functions* $\varphi_p(x)$ *are continuous with respect to* $p \in (c, d)$ *for each* $x \in (a, b)$ *and* $\varphi_p(b-)$ *is continuous with respect to* $p \in (c, d)$ *too;*

(c) *for all* $p \in (c, d)$*, there exists a right neighbourhood of point* a *in which* $\varphi_p(x) < 0$ *holds and a left neighbourhood of point* b *in which* $\varphi_p(x) > 0$ *holds;*

(d) *for all* $p \in (c, d)$*, the function* $\varphi_p(x)$ *has exactly one extremum* $t^{(p)}$ *on* (a, b)*, which is minimum;*

then there exists exactly one solution p_0*, for* $p \in \mathbb{R}^+$*, of the following equation*

$$|\varphi_p(t^{(p)})| = \varphi_p(b-)$$

and for $d_0 = |\varphi_{p_0}(t^{(p_0)})| = \varphi_{p_0}(b-)$ *we have*

$$d_0 = \inf_{p \in \mathbb{R}^+} \sup_{x \in (a, b)} |\varphi_p(x)|.$$

Remark 1 *Theorem* 1 *in* [2] *considers the case of an increasingly stratified family of functions, while Theorem* $1'$ *is analogous and considers the case of a decreasingly stratified family of functions. In this paper, both Theorems are unified in Theorem* 3 *and improved. Specifically, in condition* (c)*, we have added that there exists a left neighbourhood of point* b *in which* $\varphi_p(x) > 0$ *holds. Although the theorems in* [2] *were correct, this addition uniquely defines the function* $\varphi_c(x)$ *in Theorem* 1 *and the function* $\varphi_d(x)$ *in Theorem* $1'$ *from the paper* [2].

Exploring the fulfillment of the conditions for Theorem 3 is often reduced to the following statement [2]:

Theorem 4 (Nike theorem) *Let* $f : (0, c) \longrightarrow \mathbb{R}$ *be* m *times differentiable function (for some* $m \geq 2$*,* $m \in \mathbb{N}$*) satisfying the following conditions*:

(a) $f^{(m)}(x) > 0$ *for* $x \in (0, c)$;

(b) *there is a right neighbourhood of zero in which the following inequalities are true*:

$$f < 0,\ f' < 0,\ \ldots,\ f^{(m-1)} < 0;$$

(c) *there is a left neighbourhood of* c *in which the following inequalities are true*:

$$f > 0,\ f' > 0,\ \ldots,\ f^{(m-1)} > 0.$$

Then the function f *has exactly one zero* $x_0 \in (0, c)$, *and* $f(x) < 0$ *for* $x \in (0, x_0)$ *and* $f(x) > 0$ *for* $x \in (x_0, c)$. *Also, the function* f *has exactly one local minimum* t *on the interval* $(0, c)$. *More precisely, there is exactly one point* $t \in (0, c)$ $\big($*in fact* $t \in (0, x_0)\big)$ *such that* $f(t) < 0$ *is the smallest value of the function* f *on the interval* $(0, c)$ *and particularly on* $(0, x_0)$.

2 Main results

In this section, some generalisations of Theorems 1 and 2 are given.

Generalisation of Theorem 1

First, we give some auxiliary results.

Lemma 1 *The family of functions*

$$\varphi_p(x) = 1 - \frac{\cos x}{\cos \frac{x}{2}} - p\,x^2 \qquad \Big(\textit{for } x \in (0, \pi/2)\Big)$$

is decreasingly stratified with respect to parameter $p \in \mathbb{R}^+$.

The family of functions $\varphi_p(x)$, introduced in the previous lemma, is formed based on the double inequality from Theorem 1 for parameter values $p = \frac{4}{\pi^2}$ and $p = \frac{3}{8}$, as will be discussed in the following analysis. With that aim, we introduce the function

$$g(x) = \frac{-2\cos^2 \frac{x}{2} + \cos \frac{x}{2} + 1}{x^2 \cos \frac{x}{2}} \qquad \Big(\text{for } x \in (0, \pi/2)\Big)$$

which is strictly increasing, while $g(0+) = \dfrac{3}{8}$ and $g(\pi/2) = \dfrac{4}{\pi^2}$ hold [1]. Obviously,

$$\varphi_p(x) = 0 \Leftrightarrow p = g(x)\,.$$

Now we give the main results for the first generalisation:

Statement 1 *Let*

$$A = \frac{3}{8} = 0.375 \quad \textit{and} \quad B = \frac{4}{\pi^2} = 0.40528\ldots\,.$$

$(\mathbf{i})$ *If* $p \in (0, A]$*, then*

$$x \in \left(0, \frac{\pi}{2}\right) \implies \frac{\cos x}{\cos \frac{x}{2}} < 1 - A\,x^2 < 1 - p\,x^2\,.$$

$(\mathbf{ii})$ *If* $p \in (A, B)$*, then* $\varphi_p(x)$ *has exactly one zero* $x_0^{(p)}$ *on* $(0, \pi/2)$*. Also,*

$$x \in \left(0, x_0^{(p)}\right) \implies \frac{\cos x}{\cos \frac{x}{2}} > 1 - p\,x^2$$

and

$$x \in \left(x_0^{(p)}, \frac{\pi}{2}\right) \implies \frac{\cos x}{\cos \frac{x}{2}} < 1 - p\,x^2$$

hold.

$(\mathbf{iii})$ *If* $p \in [B, \infty)$*, then*

$$x \in \left(0, \frac{\pi}{2}\right) \implies \frac{\cos x}{\cos \frac{x}{2}} > 1 - B\,x^2 > 1 - p\,x^2\,.$$

Proof. The function $g(x)$ is increasing, continuous and surjection on (A, B), see [1]. It is obvious that

$$g(x) - p = \frac{\varphi_p(x)}{x^2}$$

holds. Therefore, $g(x) \neq p$ (i.e. $\varphi_p(x) \neq 0$) holds on $(0, \frac{\pi}{2})$ if $p \in (0, A]$ or $p \in [B, +\infty)$. We can easily see that $\varphi_A(\pi/2) > 0$ and $\varphi_B(\pi/3) < 0$. Hence, $\varphi_A(x) > 0$ for $x \in (0, \frac{\pi}{2})$ and $\varphi_B(x) < 0$ for $x \in (0, \frac{\pi}{2})$. Then (i) and (iii) follow from the decreasing stratification of the family $\varphi_p(x)$. Furthermore, for $p \in (A, B)$, the equation $g(x) = p$ has exactly one solution, which we denote by $x_0^{(p)}$, while $g(x) < p$ for $x \in (0, x_0^{(p)})$ and $g(x) > p$ for $x \in (x_0^{(p)}, \frac{\pi}{2})$. Hence, (ii) is true. □

Corollary 1 *For any* $0 < x < \pi/2$

$$1 - \frac{4}{\pi^2}x^2 < \frac{\cos x}{\cos\dfrac{x}{2}} < 1 - \frac{3}{8}x^2$$

holds, with the best possible constants $A = \dfrac{3}{8} = 0.375$ *and* $B = \dfrac{4}{\pi^2} = 0.40528\ldots$.

Statement 2 *Let*

$$\varphi_p(x) = 1 - \frac{\cos x}{\cos\dfrac{x}{2}} - p\,x^2 \quad \textit{for } x \in (0, \tfrac{\pi}{2}) \textit{ and } p \in \mathbb{R}^+.$$

$(\mathbf{i})$ *For* $p \in (A, B)$, *there exists only one extremum of this function on* $(0, \frac{\pi}{2})$ *at* $t^{(p)}$ *and that extremum is minimum.*

$(\mathbf{ii})$ *There is exactly one solution to the equation*

$$\left|\varphi_p\left(t^{(p)}\right)\right| = \varphi_p\left(\frac{\pi}{2}-\right)$$

with the respect to parameter $p \in (A, B)$, *which can be determined numerically as*

$$p_0 = 0.39916\ldots$$

For the value

$$d_0 = \left|\varphi_{p_0}\left(t^{(p_0)}\right)\right| = \varphi_{p_0}\left(\frac{\pi}{2}-\right) = 0.015109\ldots,$$

the following result

$$d_0 = \inf_{p \in \mathbb{R}^+} \sup_{x \in (0,\pi/2)} |\varphi_p(x)|$$

holds.

$(\mathbf{iii})$ *The minimax approximant of the family* $\varphi_p(x)$ *is*

$$\varphi_{p_0}(x) = 1 - \frac{\cos x}{\cos\dfrac{x}{2}} - p_0\,x^2,$$

which determines the corresponding minimax approximation

$$\frac{\cos x}{\cos\dfrac{x}{2}} \approx 1 - 0.39916\,x^2.$$

Proof. For $p \in (A, B)$, functions $\varphi_p(x)$ fulfill the conditions of Theorem 4 (Nike theorem):

(a) For $m = 3$

$$\varphi_p'''(x) = \frac{d^3\varphi_p}{dx^3} = \frac{1}{8}\frac{\left(6 - 2\cos^4\frac{x}{2} - \cos^2\frac{x}{2}\right)\sin\frac{x}{2}}{\cos^4\frac{x}{2}} > 0 \qquad \big(x\in(0,\pi/2)\big).$$

(b) Based on the Taylor expansions of the functions $\varphi_p(x)$ around $x{=}0$:

$$\varphi_p(x) = \left(\frac{3}{8} - p\right)x^2 + \frac{1}{128}x^4 + o(x^4), \tag{5}$$

there exists a right neighbourhood $\mathcal{U}_0$ of the point 0 such that

$$\varphi_p(x), \varphi_p'(x) = \frac{d\varphi_p}{dx}, \varphi_p''(x) = \frac{d^2\varphi_p}{dx^2} < 0 \qquad \big(x\in\mathcal{U}_0\big).$$

(c) Based on the Taylor expansions of the functions $\varphi_p(x)$ around $x{=}\dfrac{\pi}{2}$:

$$\begin{aligned}\varphi_p(x) = &\left(1 - \frac{p\,\pi^2}{4}\right) + \left(-p\,\pi + \sqrt{2}\right)\left(x - \frac{\pi}{2}\right) + \\ &+ \left(\frac{\sqrt{2}}{2} - p\right)\left(x - \frac{\pi}{2}\right)^2 + \frac{5\sqrt{2}}{24}\left(x - \frac{\pi}{2}\right)^3 + o\left(\left(x - \frac{\pi}{2}\right)^3\right),\end{aligned} \tag{6}$$

there exists a left neighbourhood $\mathcal{U}_{\pi/2}$ of the point $\pi/2$ such that

$$\varphi_p(x), \varphi_p'(x) = \frac{d\varphi_p}{dx}, \varphi_p''(x) = \frac{d^2\varphi_p}{dx^2} > 0 \qquad \big(x\in\mathcal{U}_{\pi/2}\big).$$

Based on Theorem 3, for $p \in (A, B)$, we can conclude that each function $\varphi_p(x)$ has exactly one extremum $t^{(p)}$, which is minimum, on $(0, \frac{\pi}{2})$ $\big($and thus exactly one zero $x_0^{(p)}$ on $(0, \frac{\pi}{2})\big)$.

The family of functions $\varphi_p(x)$, for values $p \in (A, B)$, fulfills the conditions of Theorem 3, thereby there exists a minimax approximant. Numerical determination of the minimax approximant and the error can be calculated in Maple in the manner we present here. Let $f(x, p) := \varphi_p(x)$ and $F(x, p) := \varphi_p'(x)$. With Maple code

```
fsolve({F(x,p) = 0, abs(f(x,p)) = f(π/2,p)}, {x = 0..π/2, p = A..B});
```

we have numerical values

$$\{p = 0.399161163, x = 1.069252853\}.$$

For the value $p_0 = 0.39916\ldots$, we have the minimax approximant of the family

$$\varphi_{p_0}(x) = 1 - \frac{\cos x}{\cos \frac{x}{2}} - p_0\, x^2$$

and numerical value of minimax error

$$d_0 = f(\pi/2, p_0) = 0.015109\ldots. \qquad \square$$

Figure 1 illustrates the stratified family of functions from Lemma 1 for $p \in \mathbb{R}^+$.

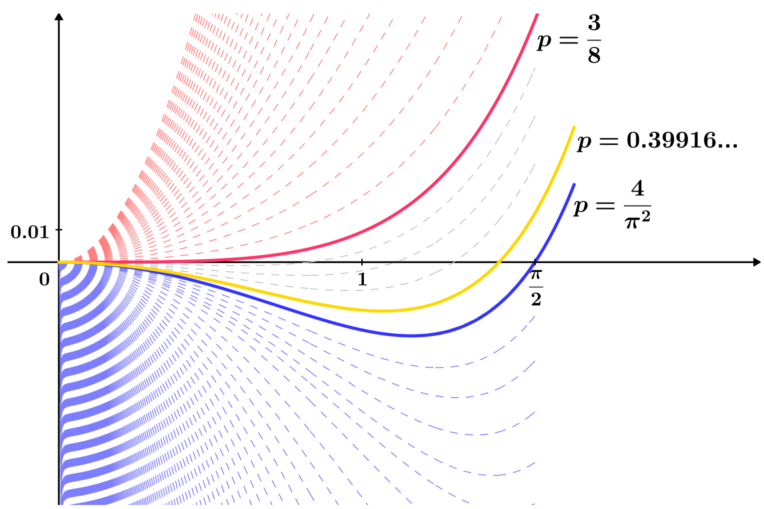

Figure 1: Stratified family of functions from Lemma 1

Generalisation of Theorem 2

First, we give some auxiliary results.

Lemma 2 *The family of functions*

$$\varphi_p(x) = -2 + \frac{\sin x}{\sin \frac{x}{2}} + p\, x^2 \qquad \Big(\textit{for } x \in (0, \pi/2)\Big)$$

is increasingly stratified with respect to parameter $p \in \mathbb{R}^+$.

The family of functions $\varphi_p(x)$, introduced in the previous lemma, is formed based on the double inequality from Theorem 2 for parameter values $p = \frac{1}{4}$ and $p = \frac{8-4\sqrt{2}}{\pi^2}$, as will be discussed in the following analysis. With that aim, we introduce the function

$$g(x) = \frac{2\left(1-\cos\frac{x}{2}\right)}{x^2} \qquad \Big(\text{for } x \in (0, \pi/2)\Big)$$

which is strictly decreasing, while $g(0+) = \dfrac{1}{4}$ and $g(\pi/2-) = \dfrac{8-4\sqrt{2}}{\pi^2}$ hold. Further,

$$\varphi_p(x) = 0 \Leftrightarrow p = g(x)\,.$$

holds, as in the previous case.

Now we give the main results for the second generalisation:

Statement 3 *Let*

$$A = \frac{8-4\sqrt{2}}{\pi^2} = 0.23741\ldots \quad \textit{and} \quad B = \frac{1}{4} = 0.25.$$

(i) *If* $p \in (0, A]$, *then*

$$x \in \left(0, \frac{\pi}{2}\right) \implies \frac{\sin x}{\sin\frac{x}{2}} < 2 - A\,x^2 < 2 - p\,x^2\,.$$

(ii) *If* $p \in (A, B)$, *then* $\varphi_p(x)$ *has exactly one zero* $x_0^{(p)}$ *on* $(0, \frac{\pi}{2})$. *Also,*

$$x \in \left(0, x_0^{(p)}\right) \implies \frac{\sin x}{\sin\frac{x}{2}} < 2 - p\,x^2$$

and

$$x \in \left(x_0^{(p)}, \frac{\pi}{2}\right) \implies \frac{\sin x}{\sin\frac{x}{2}} > 2 - p\,x^2$$

hold.

(iii) *If* $p \in [B, \infty)$, *then*

$$x \in \left(0, \frac{\pi}{2}\right) \implies \frac{\sin x}{\sin\frac{x}{2}} > 2 - B\,x^2 > 2 - p\,x^2\,.$$

Proof. The function $g(x)$ is increasing, continuous and surjection on (A, B), and

$$g(x) - p = -\frac{\varphi_p(x)}{x^2}$$

holds. Hence, $\varphi_p(x) \neq 0$ holds on $(0, \frac{\pi}{2})$ if $p \in (0, A]$ or $p \in [B, +\infty)$. It can be checked that $\varphi_A(\pi/3) < 0$ and $\varphi_B(\pi/2) > 0$, which means that $\varphi_A(x) < 0$ for $x \in (0, \frac{\pi}{2})$ and $\varphi_B(x) > 0$ for $x \in (0, \frac{\pi}{2})$. Then (i) and (iii) follow from the increasing stratification of the family $\varphi_p(x)$. For $p \in (A, B)$, the equation $g(x) = p$ has exactly one solution which we denote by $x_0^{(p)}$, while $g(x) > p$ for $x \in (0, x_0^{(p)})$ and $g(x) < p$ for $x \in (x_0^{(p)}, \frac{\pi}{2})$. Hence, (ii) holds. □

Corollary 2 *For any* $0 < x < \pi/2$

$$2 - \frac{1}{4}x^2 < \frac{\sin x}{\sin \frac{x}{2}} < 2 - \frac{4(2-\sqrt{2})}{\pi^2}x^2.$$

holds, with the best possible constants $A = \dfrac{8-4\sqrt{2}}{\pi^2} = 0.23741\ldots$ *and* $B = \dfrac{1}{4} = 0.25$.

Statement 4 *Let*

$$\varphi_p(x) = -2 + \frac{\sin x}{\sin \frac{x}{2}} + p\,x^2 \ \ \textit{for } x \in (0, \tfrac{\pi}{2}) \textit{ and } p \in \mathbb{R}^+.$$

$(\mathbf{i})$ *For* $p \in (A, B)$, *there exists only one extremum of this function on* $(0, \frac{\pi}{2})$ *at* $t^{(p)}$ *and that extremum is minimum.*
$(\mathbf{ii})$ *There is exactly one solution to the equation*

$$\left|\varphi_p\left(t^{(p)}\right)\right| = \varphi_p\left(\frac{\pi}{2}-\right),$$

where $t^{(p)}$ *is a unique local minimum of* $\varphi_p(x)$, *by parameter* $p \in (A, B)$, *which we determine numerically as*

$$p_0 = 0.23955\ldots$$

For the value

$$d_0 = \left|\varphi_{p_0}\left(t^{(p_0)}\right)\right| = \varphi_{p_0}\left(\frac{\pi}{2}-\right) = 0.0052842\ldots,$$

the following result

$$d_0 = \inf_{p\in\mathbb{R}^+} \sup_{x\in(0,\pi/2)} |\varphi_p(x)|$$

holds.

(**iii**) *The minimax approximant of the family* $\varphi_p(x)$ *is*

$$\varphi_{p_0}(x) = -2 + \frac{\sin x}{\sin\frac{x}{2}} + p_0\, x^2,$$

which determines the corresponding minimax approximation

$$\frac{\sin x}{\sin\frac{x}{2}} \approx 2 - 0.23955\, x^2\,.$$

Proof. For $p \in (A, B)$, functions $\varphi_p(x)$ fulfill the conditions of Theorem 4 (Nike theorem):

(a) For $m = 3$

$$\varphi_p'''(x) = \frac{d^3\varphi_p}{d\,x^3} = \frac{1}{4}\sin\frac{x}{2} > 0 \qquad \big(x\in(0,\pi/2)\big).$$

(b) Based on the Taylor expansion of the functions $\varphi_p(x)$ around $x=0$:

$$\varphi_p(x) = \left(-\frac{1}{4} + p\right)x^2 + \frac{1}{192}x^4 + o(x^4) \tag{7}$$

there exists a right neighbourhood $\mathcal{U}_0$ of the point 0 such that

$$\varphi_p(x), \varphi_p'(x) = \frac{d\varphi_p}{d\,x}, \varphi_p''(x) = \frac{d^2\varphi_p}{d\,x^2} < 0 \qquad \big(x\in\mathcal{U}_0\big).$$

(c) Based on the Taylor expansion of the functions $\varphi_p(x)$ around $x=\frac{\pi}{2}$:

$$\begin{aligned}\varphi_p(x) = \left(-2 + \sqrt{2} + \frac{p\,\pi^2}{4}\right) + \left(p\,\pi - \frac{\sqrt{2}}{2}\right)\left(x - \frac{\pi}{2}\right) +\\ + \left(p - \frac{\sqrt{2}}{8}\right)\left(x - \frac{\pi}{2}\right)^2 + \frac{\sqrt{2}}{48}\left(x - \frac{\pi}{2}\right)^3 + o\left(\left(x - \frac{\pi}{2}\right)^3\right)\end{aligned} \tag{8}$$

there exists a left neighbourhood $\mathcal{U}_{\pi/2}$ of the point $\pi/2$ such that it is

$$\varphi_p(x), \varphi'_p(x) = \frac{d\varphi_p}{d\,x}, \varphi''_p(x) = \frac{d^2\varphi_p}{d\,x^2} > 0 \qquad \big(x \in \mathcal{U}_{\pi/2}\big).$$

Based on Theorem 3, for $p \in (A, B)$, we can conclude that functions $\varphi_p(x)$ has exactly one extremum $t^{(p)}$, which is minimum, on $(0, \frac{\pi}{2})$ $\big($and thus exactly one zero $x_0^{(p)}$ on $(0, \frac{\pi}{2})\big)$.

The family of functions $\varphi_p(x)$, for values $p \in (A, B)$, fulfills the conditions of Theorem 3, thereby there exists a minimax approximant. Numerical determination of the minimax approximant and the error can be calculated in Maple in the manner we present here. Let $f(x, p) := \varphi_p(x)$ and $F(x, p) := \varphi'_p(x)$. With Maple code

$$\texttt{fsolve}\big(\{F(x,p) = 0, \texttt{abs}\big(f(x,p)\big) = f(\pi/2, p)\}, \{x = 0..\pi/2, p = A..B\}\big);$$

we have numerical values

$$\{p = 0.2395519170, x = 1.007887451\}.$$

For the value $p_0 = 0.23955\ldots$, we have the minimax approximant of the family

$$\varphi_{p_0}(x) = -2 + \frac{\sin x}{\sin \frac{x}{2}} + p_0\, x^2$$

and numerical value of minimax error

$$d_0 = f(\pi/2, p_0) = 0.0052842\ldots.$$

□

Figure 2 illustrates the stratified family of functions from Lemma 2 for $p \in \mathbb{R}^+$.

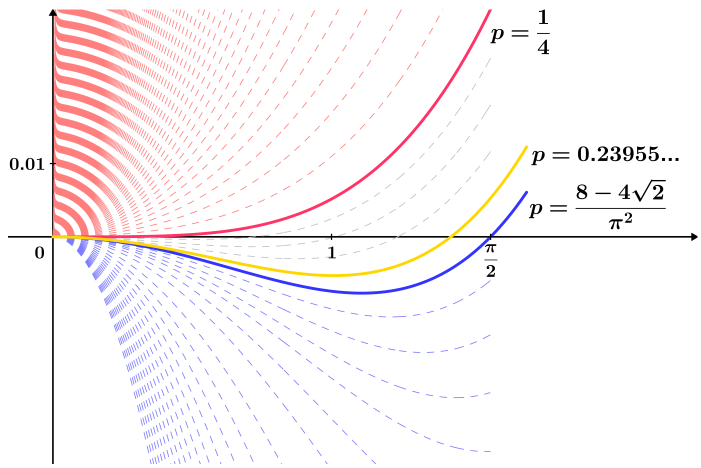

Figure 2: Stratified family of functions from Lemma 2

3 Conclusion

This paper specifies the results of J. Sándor [1] related to D'Aurizio's trigonometric inequality [8] using concepts from the paper [2]. Additionally, Theorems 1 and 1' from [2] were improved. Let us emphasize that the paper [2] presents one method for possible improvements of existing results in the Theory of analytic inequalities in terms of determining the corresponding minimax approximants for many inequalities from reviewed papers [6], [7], and books [3]-[5]. The concept of stratification is used in recent research to improve and generalise some inequalities, see [11]-[14], and can be used to improve many more from [3]-[5], [10], [15]-[21]. In further papers, the subject of our studies will be to determine the appropriate minimax approximants for papers [9] and [10] relating to the generalizations of D'Aurizio's trigonometric inequalities.

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A generalized (ψ, φ)- weak contraction in metric spaces

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Abstract. In this paper, we introduce weakly generalized (ψ, φ)-weak quasi contraction for four self-maps and establish a common fixed point theorem using weak compatible property.

1 Introduction

In 1997, Alber and Guerre-Delabrier [2] defined the concept of weak contraction as a generalization of contraction and established the existence of fixed points for a self-map in Hilbert space. In 2001, Rhoades [9] extended this concept to metric spaces. A mapping $T : X \to X$ is said to be a *weak contraction* if there exists a function $\varphi : [0, \infty) \to [0, \infty)$, $\varphi(t) > 0$ for all $t > 0$ and $\varphi(0) = 0$ such that

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y)) \; \forall \, x, y \in X. \tag{1}$$

As weak contractions are defined through φ, these are referred as φ-*weak contraction*.

Rhoades [9] established that every φ-weak contraction has a unique fixed point in complete metric space when φ is continuous.

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Key words and phrases: common fixed point, (ψ, φ)-weak contraction condition, weakly compatible maps, metric spaces

Theorem 1 *Let (X,d) be a nonempty complete metric space, and let $T : X \to X$ be a φ-weak contraction on X. If $\varphi(t) > 0$, for all $t > 0$ and $\varphi(0) = 0$, then T has a unique fixed point.*

Afterwards, Dutta and Choudhury [4] generalized the concept of weak contraction and proved the following theorem.

Theorem 2 *[4] Let (X,d) be a nonempty complete metric space, and let T be a self-map on X, satisfying*

$$\psi(d(Tx,Ty)) \leq \psi(d(x,y)) - \varphi(d(x,y)) \tag{2}$$

for each $x,y,\in X$, where, $\psi,\varphi : \mathfrak{R}^+ \to \mathfrak{R}^+$ are both continuous and non-decreasing function with $\psi(t) = \varphi(t) = 0$ iff $t = 0$. Then T has a unique fixed point in X.

Throughout this paper, we denote
$\Psi = \{\psi : \mathfrak{R}^+ \to \mathfrak{R}^+$(i) ψ is continuous (ii) ψ is non-decreasing (iii) $\psi(t) = 0 \Leftrightarrow t = 0\}$
$\Phi = \{\varphi : \mathfrak{R}^+ \to \mathfrak{R}^+$ (i) lower semi-continuous for all $t > 0$ and φ is discontinuous at $t = 0$ with $\varphi(0) = 0$ $\}$.
In fact, the function Ψ is called the altering distance function and it was introduced by Khan, Swaleh and Sessa [7].

In 2009, Doric [3] introduced generalized (ψ,φ)-weak contraction for a pair of self-maps as follows.

Definition 1 *[3] Let (X,d) be a metric space. Let S and T be self-maps in X. If there exist $\psi \in \Psi$ and $\varphi \in \Phi$ such that*

$$\psi(d(Sx,Ty)) \leq \psi(M(x,y)) - \varphi(M(x,y)) \tag{3}$$

for each $x,y,\in X$, where

$$M(x,y) = \max\left\{d(x,y), d(Tx,x), d(Sy,y), \frac{1}{2}[d(y,Tx) + d(x,Sy)]\right\}$$

then we say that S and T satisfy generalized (ψ,φ)-weak contraction condition.

Theorem 3 *[3] Let (X,d) be a nonempty metric space. Let S and T be self-maps of X, satisfying generalized (ψ,φ)-weak contraction condition. Then S and T have a unique common fixed point in X.*

In 2010, Abbas and Doric [1] extended the concept of generalized (ψ,φ)-weak contraction for a pair of self-maps to four self-maps in the following way.

Definition 2 *[1] Let* (X,d) *be a metric space. Let A, B, S and T be self-maps in X. If there exist* $\psi\in\Psi$ *and* $\varphi\in\Phi$ *such that*

$$\psi(d(Ax,By))\leq\psi(M(x,y))-\varphi(M(x,y)), \tag{4}$$

for each $x,y,\in X$*, where*

$$M(x,y)=\max\left\{d(Sx,Ty),d(Ax,Sx),d(By,Ty),\frac{1}{2}[d(Sx,By)+d(Ax,Ty)]\right\},$$

then we say that A, B, S and T satisfy generalized (ψ,φ)*-weak contraction condition.*

Theorem 4 *[1] Let* (X,d) *be a complete metric space and A, B, S and T be self-maps of X satisfying generalized* (ψ,φ)*-weak contraction condition. Suppose that* $A(X)\subseteq T(X)$*,* $B(X)\subseteq S(X)$ *and that the pairs* (A,S) *and* (B,T) *are weakly compatible. Then A, B, S and T have a unique common fixed point in X, provided one of the range spaces* $A(X),B(X),S(X)$ *and* $T(X)$ *are closed in X.*

In 2015, P.P. Murthy et al, [8] extended the concept of generalized (ψ,φ)-weak contraction condition in a complete metric space by using a weaker condition than the (1.2) in complete metric space.

Theorem 5 *[8] Let* (X,d) *be a complete metric space, and A, B, S and* $T:X\to X$*be a continuous mapping satisfying*

$$\psi(d(Ax,By))\leq\psi(M(x,y))-\varphi(N(x,y)), \tag{5}$$

for all $x,y,\in X$*, with* $x\neq y$*, for some* $\psi\in\Psi$ *and* $\varphi\in\Phi$

$$M(x,y)=\max\left\{d(Sx,Ty),\frac{1}{2}[d(Sx,Ax)+d(Ty,By)],\frac{1}{2}[d(Sx,By)+d(Ty,Ax)]\right\},$$

and

$$N(x,y)=\min\left\{d(Sx,Ty),\frac{1}{2}[d(Sx,Ax)+d(Ty,By)],\frac{1}{2}[d(Sx,By)+d(Ty,Ax)]\right\},$$

$$A(X)\subseteq T(X)\text{ and }B(X)\subseteq S(X) \tag{6}$$

$$(A,S)\text{ and }(B,T)\text{ are weak compatible pairs.} \tag{7}$$

Then A, B, S and T have a unique common fixed point in X.

Definition 3 *[5] (i) Let S and T be mappings of a metric space* (X, d) *into itself. The mappings S and T are said to be compatible*

$$\lim_{n\to\infty} d(STx_n, TSx_n) = 0,$$

whenever $\{x_n\}$ *is a sequence in X such that*

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z,$$

for some $z \in X$.

Definition 4 *[6] (i) A pair of self-mapping S and T of a metric space* (X, d) *is said to be weakly compatible if they commute at their coincidence points i.e if* $Ax = Bx$ *for some* $x \in X$*, then* $ABx = BAx$*,*
(ii) be occasionally weakly compatible (owc) [10] if $TSx = STx$ *for some* $x \in X$*.*

Remark. Every compatible map are weakly compatible but the converse is not true [6].

In this paper, we introduce weakly generalized (ψ,φ)-weak quasi- contraction condition and establish a common fixed point theorem by using weakly compatible pairs in metric space.

Definition 5 *Let* (X, d) *be a metric space, and A, B, S and* $T : X \to X$*be a mappings satisfying*

$$\psi(d(Ax, By)) \leq \psi(M(x, y)) - \varphi(N(x, y)), \tag{8}$$

for all $x, y, \in X$*, with* $x \neq y$*, for some* $\psi \in \Psi$ *and* $\varphi \in \Phi$

$$M(x, y) = max\left\{d(Sx, Ty), d(Sx, Ax), d(Ty, By), \frac{1}{2}[d(Sx, By) + d(Ty, Ax)]\right\},$$

and

$$N(x, y) = min\left\{d(Sx, Ty), d(Sx, Ax), d(Ty, By), \frac{1}{2}[d(Sx, By) + d(Ty, Ax)]\right\}.$$

Then we say that A, B, S and T satisfy weakly generalized (ψ,φ)*-weak quasi contraction condition.*

Remark. If ψ and φ in (5) satisfy '(ψ,φ) is non-decreasing' then the inequality (5) implies that inequality (8). But its converse need not be true. The following example shows that there exist maps A, B, S and T which are weakly generalized (ψ,φ)-weak quasi-contraction condition, but they do not satisfy the condition (5).

Example 1 *Let* $X = [0, 2)$ *be endowed with the Euclidean metric* $d(x, y) = |x - y|$, *and let A, B, S and* $T \to X$ *be defined by*

$$A(X) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x \neq 0 \end{cases} \quad B(X) = \begin{cases} 0 & \text{if } x = 0 \\ \frac{1}{4} & \text{if } x \neq 0 \end{cases}$$

$$S(X) = \begin{cases} 0 & \text{if } x = 0 \\ \frac{3}{2} & \text{if } x \neq 0 \end{cases} \quad T(X) = \begin{cases} 0 & \text{if } x = 0 \\ \frac{5}{4} & \text{if } x \neq 0 \end{cases}$$

where $x, y \in X$, *defined as* $\psi \in \Psi$ *and* $\varphi \in \Phi$, *by*

$$\psi(t) = \frac{t}{2} \text{ and } \varphi(t) = \begin{cases} 0 & \text{if } t = 0 \\ \frac{t}{16} & \text{if } t > 0 \end{cases}$$

In particular, $x \neq 0$ and $y \neq 0$, the inequality (5) does not hold

$$\psi(d(Ax, By)) = \psi(\frac{3}{4}) \leq \psi(\frac{3}{4}) - \varphi(\frac{1}{4})$$

$$\frac{3}{8} \leq \frac{3}{8} - \frac{1}{64}.$$

But, these mappings satisfy the condition (8) in all possible cases.

2 Fixed point theorems in metric space

Before stating the main result we prove the following lemma.

Lemma 1 *Let* (X, d) *be a metric space, and A, B, S and* $T : X \to X$*be a mapping satisfying the condition* (6) *and* (7), *weakly generalized* (ψ, φ)*- weak quasi contraction condition. Then the sequence* $\{y_n\}$ *is a Cauchy sequence.*

Proof. Let $x_0 \in X$, from (6), there exists a point $x_1 \in X$ such that $y_0 = Ax_0 = Tx_1$, for this x_1, there exists a point $x_2 \in X$ such that $y_0 = Bx_1 = Sx_2$. In general $\{y_n\}$ is defined by

$$y_{2n} = Ax_{2n} = Tx_{2n+1}, \tag{9}$$

$$y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}. \tag{10}$$

Now, we suppose that

$$y_{2n} \neq y_{2n+1} \ \forall n \tag{11}$$

For this suppose that $x = x_{2n}, y = x_{2n+1}$ in (8), we have

$$\begin{aligned}\psi(d(Ax_{2n}, Bx_{2n+1})) &\le \psi(M(x_{2n}, x_{2n+1})) - \phi(N(x_{2n}, x_{2n+1}))\\ &= \psi\big(max\{d(Sx_{2n}, Tx_{2n+1}), d(Sx_{2n}, Ax_{2n}), d(Tx_{2n+1}, Bx_{2n+1}),\\ &\frac{1}{2}[d(Sx_{2n}, Bx_{2n+1}) + d(Tx_{2n+1}, Ax_{2n})]\}\big) - \varphi(N(x_{2n}, x_{2n+1})).\end{aligned} \tag{12}$$

Using (9), (10) in (12), then we get

$$\begin{aligned}\psi(d(y_{2n}, y_{2n+1})) &\le \psi\big(max\{d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1}),\\ &\frac{1}{2}[d(y_{2n-1}, y_{2n+1}) + d(y_{2n}, y_{2n})]\}\big) - \varphi(N(x_{2n}, x_{2n+1})),\\ &\le \psi\big(max\{d(y_{2n-1}, y_{2n}), d(y_{2n+1}, y_{2n}),\\ &\frac{1}{2}[d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1})]\}\big) - \varphi(N(x_{2n}, x_{2n+1})).\end{aligned} \tag{13}$$

If $y_{2n+1} \neq y_{2n+2}$ $\forall n$ then taking $x = x_{2n+2}$ and $y = x_{2n+1}$ in (8), and applying the above process, then we get

$$\begin{aligned}\psi(d(y_{2n+2}, y_{2n+1})) &\le \psi\big(max\{d(y_{2n+1}, y_{2n+2}), d(y_{2n}, y_{2n+1}),\\ &\frac{1}{2}[d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n+2})]\}\big) - \varphi(N(x_{2n+1}, x_{2n+2})).\end{aligned} \tag{14}$$

From (13) and (14) for any n, then we have

$$\begin{aligned}\psi(d(y_n, y_{n+1})) \le \psi\Big(&max\{d(y_{n-1}, y_n), d(y_{n+1}, y_n),\\ &\frac{1}{2}[d(y_{n-1}, y_n) + d(y_n, y_{n+1})]\}\Big) - \varphi(N(x_n, x_{n+1})).\end{aligned} \tag{15}$$

If

$$d(y_{n-1}, y_n) < d(y_n, y_{n+1}). \tag{16}$$

Then inequality (15) reduces to

$$\psi(d(y_n, y_{n+1})) < \psi(d(y_n, y_{n+1})) - \varphi(N(x_n, x_{n+1})).$$

On taking liminf as $n \to \infty$ on both sides, then we have

$$\underline{\lim}_{n\to\infty}\psi(d(y_n, y_{n+1})) < \underline{\lim}_{n\to\infty}\psi(d(y_n, y_{n+1})) - \underline{\lim}_{n\to\infty}\varphi(N(x_n, x_{n+1})). \tag{17}$$

The right-hand side is positive due to the property of Φ, therefore inequality (17), change the form

$$\underline{\lim}_{n\to\infty}\psi(d(y_n, y_{n+1})) < \underline{\lim}_{n\to\infty}\psi(d(y_n, y_{n+1})),$$

a contradiction. From (15) we have

$$\begin{aligned}\psi(d(y_n, y_{n+1})) &\leq \psi(d(y_n, y_{n-1})) - \varphi(N(x_n, x_{n+1})) \\ &< \psi(d(y_n, y_{n-1})).\end{aligned} \tag{18}$$

Therefore by the property of ψ, we get

$$d(y_n, y_{n+1}) < d(y_n, y_{n-1}). \tag{19}$$

Hence, the sequence $\{d(y_n, y_{n+1})\}$ is a non increasing sequence of nonnegative real number and hence it converges to some real number r (say), $r \geq 0$. Suppose $r > 0$, on taking liminf as $n \to \infty$ on (18), we have

$$\underline{\lim}_{n\to\infty}\psi(d(y_n, y_{n+1})) < \underline{\lim}_{n\to\infty}\psi(d(y_n, y_{n-1}))$$

The right term $\underline{\lim}_{n\to\infty}\varphi(N(x_n, x_{n+1})) > 0$, due to the property of φ. Hence

$$\psi(r) < \psi(r),$$

a contradiction. Thus

$$\underline{\lim}_{n\to\infty}\psi d(y_n, y_{n+1}) = 0,$$

and then

$$\underline{\lim}_{n\to\infty}d(y_n, y_{n+1}) = 0. \tag{20}$$

Next, we prove that $\{y_n\}$ is a Cauchy sequence. It is enough to show that the sub-sequence $\{y_{2n}\}$ of $\{y_n\}$ is a Cauchy sequence. Suppose that $\{y_{2n}\}$ is not a Cauchy sequence, then there exist $\epsilon > 0$ and the sequence of natural numbers $\{2m(k)\}$ and $\{2n(k)\}$ such that $2m(k) > 2n(k) > 2k$ for $k \in N$ and

$$d(y_{2m(k)}, y_{2n(k)}) > \epsilon, \tag{21}$$

For each k, let $2m(k)$ be the least positive integer exceeding $2n(k)$ and satisfying (21). Then we have

$$d(y_{2m(k)}, y_{2n(k)}) > \epsilon \text{ and } d(y_{2n(k)}, y_{2m(k)-2}) \leq \epsilon. \tag{22}$$

We have

$$\begin{aligned}\epsilon < (d(y_{2m(k)}, y_{2n(k)})) &\leq (d(y_{2m(k)}, y_{2m(k)-1})) + d(y_{2m(k)-1}, y_{2m(k)-2}) \\ &\quad + d(y_{2m(k)-2}, y_{2n(k)}) \\ &\leq (d(y_{2m(k)}, y_{2m(k)-1})) + d(y_{2m(k)-1}, y_{2m(k)-2}) + \epsilon\end{aligned}$$

by taking the liminf as $k \to \infty$ and using (21), we get

$$\epsilon < \underline{\lim}_{k\to\infty}(d(y_{2m(k)}, y_{2n(k)})) \leq \epsilon.$$

Therefore

$$\underline{\lim}_{k\to\infty}(d(y_{2m(k)}, y_{2n(k)})) = \epsilon.$$

Using triangular inequality

$$|d(y_{2m(k)}, y_{2n(k)}) - d(y_{2m(k)-1}), y_{2n(k)+1})| \leq d(y_{2m(k)}, y_{2m(k)-1}) + d(y_{2n(k)}, y_{2n(k)+1}).$$

We take the limit $k \to \infty$, on both sides, we get

$$\lim_{k\to\infty} d(y_{2m(k)-1}, y_{2n(k)+1}) = \epsilon. \tag{23}$$

Again using triangular inequality

$$|d(y_{2m(k)}, y_{2n(k)}) - d(y_{2m(k)-1}, y_{2n(k)})| \leq d(y_{2m(k)}, y_{2m(k)-1}),$$

on taking $\lim_{k\to\infty}$, on both sides

$$\lim_{k\to\infty} d(y_{2m(k)-1}, y_{2n(k)}) = \epsilon. \tag{24}$$

Now consider

$$\begin{aligned}\psi(d(y_{2m(k)}, y_{2n(k)})) &\leq \psi d(y_{2n(k)}, y_{2n(k)+1}) + \psi(d(y_{2n(k)+1}, y_{2m(k)})) \\ &= \psi d(y_{2n(k)}, y_{2n(k)+1}) + \psi(d(Ax_{2m(k)}, Bx_{2n(k)+1}))\end{aligned} \tag{25}$$

Then

$$\begin{aligned}&\psi(d(Ax_{2m(k)}, Bx_{2n(k)+1}) \leq \psi(M(x_{2m(k)}, x_{2n(k)+1})) - \varphi(N(x_{2m(k)}, x_{2n(k)+1})) \\ &= \psi\big(max\{d(Sx_{2m(k)}, Tx_{2n(k)+1}), d(Sx_{2m(k)}, Ax_{2m(k)}), d(Tx_{2n(k)+1}), Bx_{2n(k)+1}), \\ &\frac{1}{2}[d(Sx_{2m(k)}, Bx_{2n(k)+1}) + d(Tx_{2n(k)+1}, Ax_{2m(k)})]]\big) - \varphi(N(x_{2m(k)}, x_{2n(k)+1}))\end{aligned}$$

$$\begin{aligned}&= \psi\big(max\{d(y_{2m(k)-1}, y_{2n(k)}), d(y_{2m(k)-1}, y_{2m(k)}), d(y_{2n(k)}, y_{2n(k)+1}), \\ &\frac{1}{2}[d(y_{2m(k)-1}, y_{2n(k)-1}) + d(y_{2n(k)}, y_{2m(k)})]]\}\big) - \varphi(N(x_{2m(k)}, x_{2n(k)+1}))\end{aligned} \tag{26}$$

Using (25) and (26) and taking $\underline{\lim}_{k\to\infty}$, on both side we get

$$\psi(\epsilon) \le \psi(\epsilon) - \underline{\lim}_{k\to\infty}\varphi(N(x_{2m(k)}, x_{2n(k)+1})).$$

We observe that the last term on the right-hand side of the above inequality is non-zero. Thus we arrive at a contradiction. Therefore $\{y_{2n}\}$ is a Cauchy so that $\{y_n\}$ is a Cauchy sequence □

Theorem 6 *Let* (X, d) *be a metric space, and A, B, S and* $T : X \to X$ *be a mapping satisfying the condition* (6)*,* (7) *and weakly generalized* (ψ, φ) *weak quasi contraction condition. Then A, B, S and T have a unique common fixed point in X, provided any one of the ranges* $A(X), B(X), S(X), T(X)$ *is a closed subspace of X.*

Proof. Since $\{y_n\}$ is a Cauchy sequence and assumes that $S(X)$ is a closed subspace of X, $\{y_{2n}\}$ is sub-sequence of $\{y_n\}$, we get

$$\lim_{n\to\infty} y_{2n} = \lim_{n\to\infty} Ax_{2n} = z, \tag{27}$$

where $z \in X$. Since $\{y_n\}$ is a Cauchy sequence it follows that $\lim_{n\to\infty} y_n = z$, therefore

$$\lim_{n\to\infty} y_{2n} = \lim_{n\to\infty} y_{2n+1} = z. \tag{28}$$

Consequently, the subsequence also converges to z in X.
Therefore

$$\lim_{n\to\infty} Ax_{2n} = \lim_{n\to\infty} Tx_{2n+1} = \lim_{n\to\infty} Bx_{2n+1} = \lim_{n\to\infty} Sx_{2n+2} = z \ \forall z \in X. \tag{29}$$

Since $S(X)$ is closed. Then, there exists a $u \in X$ such that

$$z = Su. \tag{30}$$

We claim that $Au = z$. Suppose not, putting $x = u$ and $y = x_{2n+1}$ then in inequality (8), we get

$$\begin{aligned}\psi(d(Au, Bx_{2n+1})) &\le \psi(M(u, x_{2n+1})) - \varphi(N(u, x_{2n+1}))\\ &= \psi\Big(\max\big\{d(Su, Tx_{2n+1}), d(Su, Au), d(Tx_{2n+1}, Bx_{2n+1}),\\ &\quad \frac{1}{2}[d(Su, Bx_{2n+1}) + d(Tx_{2n+1}, Au)]\big\}\Big) - \varphi(N(u, x_{2n+1}))\end{aligned}$$

on taking the $\underline{\lim}_{n\to\infty}$, we get

$$\psi(d(Au, z)) \leq \psi\left(\max\left\{d(z, z), d(z, Au), \frac{1}{2}[d(z, z) + d(z, Au)]\right\}\right) - \underline{\lim}_{n\to\infty}\varphi(N(u, x_{2n+1})) \tag{31}$$

we obtain that the last term on the right side of the inequality (31) is non-zero by the property of φ, then we get

$$\psi(d(Au, z)) < \psi(d(Au, z)) \tag{32}$$

a contradiction.

$$Au = z. \tag{33}$$

Therefore from (30) and (33), we get

$$Au = Su = z. \tag{34}$$

Since the pair (A, S) is weakly compatible, then we get

$$Au = Su \Rightarrow ASu = SAu \Rightarrow Az = Sz. \tag{35}$$

We shall show that z is a common fixed point of A and S.
If $Az \neq z$, then we take $x = z$ and $y = x_{2n+1}$ in (8), we have

$$\begin{aligned}\psi(d(Az, Bx_{2n+1})) &\leq \psi(M(z, x_{2n+1})) - \varphi(N(z, x_{2n+1}))\\ &= \psi\Big(\max\{d(Sz, Tx_{2n+1}), d(Sz, Az), d(Tx_{2n+1}, Bx_{2n+1}),\\ &\quad \frac{1}{2}[d(Sz, Bx_{2n+1}) + d(Tx_{2n+1}, Az)]\}\Big) - \varphi(N(z, x_{2n+1})),\end{aligned}$$

on taking $\underline{\lim}_{n\to\infty}$, we have

$$\underline{\lim}_{n\to\infty}\psi(d(Az, Bx_{2n+1})) \leq \underline{\lim}_{n\to\infty}\psi(M(z, x_{2n+1})) - \underline{\lim}_{n\to\infty}\varphi(N((z, x_{2n+1})). \tag{36}$$

It is clear that from the condition of φ right-hand side term

$$\underline{\lim}_{n\to\infty}\varphi(N((z, x_{2n+1}))$$

is non-zero, then we get

$$\psi(d(Az, z)) < \psi(d(Az, z))$$

a contradiction. Thus, we have

$$\psi(d(Az, z)) < \psi(d(Az, z)),$$

which implies that

$$Az = z. \tag{37}$$

From (35) and (37), we get

$$Az = Sz = z. \tag{38}$$

Since $A(X) \subset T(X)$, there is a point $v \in X$ such that $Az = Tv$.
Thus from (38), we have

$$Az = Sz = Tv = z. \tag{39}$$

Suppose that $Bv \neq z$. On taking $x = x_{2n}$ and $y = v$ in inequality (8), we have

$$\begin{aligned} \psi(d(Ax_{2n}, Bv)) &\leq \psi(M(x_{2n}, v)) - \varphi(N(x_{2n}, v)) \\ &= \psi\Big(max\big\{ d(Sx_{2n}, Tv), d(Sx_{2n}, Ax_{2n}), d(Tv, Bv), \\ &\frac{1}{2}[d(Sx_{2n}, Bv) + d(Tv, Ax_{2n})]\big\}\Big) - \varphi(N(x_{2n}, v)), \end{aligned} \tag{40}$$

on taking the liminf as $n \to \infty$ and using (39)

$$\begin{aligned} \underline{\lim}_{n\to\infty}\psi(d(Ax_{2n}, Bv)) \leq \underline{\lim}_{n\to\infty}\psi(M(z, Bv)) - \underline{\lim}_{n\to\infty}\varphi(N(x_{2n}, v)) \\ \underline{\lim}_{n\to\infty}\psi\Big(max\big\{ d(Sz, Tv), d(Sz, Av), d(Tv, Bv), \\ \frac{1}{2}[d(Sz, Bv) + d(Tv, Az)]\big\}\Big) - \underline{\lim}_{n\to\infty}\varphi(N(x_{2n}, v)), \end{aligned}$$

by the property of φ function, $\underline{\lim}_{n\to\infty}\varphi(N(x_{2n}, v))$ is positive, then we have

$$\psi(d(z, Bv)) < \psi(d(z, Bv)),$$

by monotone properties of ψ, we get

$$Bv = z. \tag{41}$$

From (39) and (41), we get

$$Az = Sz = Bv = Tv = z. \tag{42}$$

Since (B, T) is weakly compatible, then

$$\begin{aligned} z = Bv = Tv &\Rightarrow BTv = TBv \\ &\Rightarrow Bz = Tz. \end{aligned} \tag{43}$$

Finally, we have to show that z is a common fixed point of B and T.
Taking $x = x_{2n}$ and $y = z$ in inequality (8), then we have

$$\begin{aligned} \psi(d(Ax_{2n}, Bz)) &\leq \psi(M(x_{2n}, z)) - \varphi(N(x_{2n}, z)) \\ &= \psi\Big(max\big\{d(Sx_{2n}, Tz), d(Sx_{2n}, Ax_{2n}), d(Tz, Bz), \\ &\frac{1}{2}[d(Sx_{2n}, Bz) + d(Tz, Ax_{2n})]\big\}\Big) - \varphi(N(x_{2n}, z)), \end{aligned} \tag{44}$$

on taking the liminf as $n \to \infty$, using (42) and (43)

$$\begin{aligned} \underline{\lim}_{n\to\infty}\psi(d(Ax_{2n}, Bz)) &\leq \underline{\lim}_{n\to\infty}\psi(M(x_{2n}, z)) - \underline{\lim}_{n\to\infty}\varphi(N(x_{2n}, z)) \\ &\leq \underline{\lim}_{n\to\infty}\psi\Big(max\big\{d(Sx_{2n}, Tz), d(Sx_{2n}, Ax_{2n}), d(Tz, Bz), \\ &\frac{1}{2}[d(Sx_{2n}, Bz) + d(Tz, Ax_{2n})]\big\}\Big) - \underline{\lim}_{n\to\infty}\varphi(N(x_{2n}, z)), \end{aligned}$$

by the property of φ function, $\underline{\lim}_{n\to\infty}\varphi(N(x_{2n}, z))$ is positive, then we have

$$\psi(d(z, Bz)) < \psi(d(z, Bz)),$$

by monotone properties of ψ, we have

$$Bz = z. \tag{45}$$

By using (42), (43) and (45), we get

$$Az = Sz = Bz = Tz = z. \tag{46}$$

Hence A, B, S and T have a common fixed point in X.
Similarly, we can take $A(X), B(X), T(X)$ is a closed subspace of X.
Uniqueness follow easily from (5). □

Theorem 7 *Let* (X, d) *be a metric space, and A, B, S and* $T : X \to X$*be a mapping satisfying the condition weakly generalized* (ψ, φ)*- weak quasi contraction condition. And* (6)*, the pairs* (A, S) *and* (B, T) *satisfying occasionally weakly compatible. Then A, B, S and T have a unique common fixed point in X, provided any one of the ranges* $A(X), B(X), S(X), T(X)$ *is a closed subspace of X.*

We get the following corollaries.

Corollary 1 *Let* (X, d) *be a complete metric space, and A, B, S and* $T : X \to X$*be a continuous mapping satisfying* (6)

$$\psi(d(Ax, By)) \leq \psi(M(x, y)) - \varphi(M(x, y)) \tag{47}$$

for all $x, y, \in X$*, with* $x \neq y$ *and*

$$M(x, y) = \max\left\{d(Sx, Ty), d(Sx, Ax), d(Ty, By), \frac{1}{2}(d(Sx, By) + d(Ty, Ax))\right\},$$

where $\psi \in \Psi$ *and* $\varphi \in \Phi$*. Then A, B, S and T have a unique common fixed point in X.*

Now, the following example is support of our main result.

Example 2 *Let* $X = [0, 3)$ *be endowed with the Euclidean metric* $d(x, y) = |x - y|$*, and let A, B, S and* $T \to X$ *be defined by*

$$A(X) = \begin{cases} 0 & \text{if} \quad x = 0 \\ \frac{x}{5} + 1 & \text{if} \quad x \neq 0 \end{cases} \quad B(X) = \begin{cases} 0 & \text{if} \quad x = 0 \\ \frac{x}{4} + 1 & \text{if} \quad x \neq 0 \end{cases}$$

$$S(X) = \begin{cases} 0 & \text{if} \quad x = 0 \\ \frac{x}{2} + 1 & \text{if} \quad x \neq 0 \end{cases} \quad T(X) = \begin{cases} 0 & \text{if} \quad x = 0 \\ \frac{2x}{3} + 1 & \text{if} \quad x \neq 0 \end{cases}$$

where $x, y \in X$

$$A(X) = \{0\} \cup \left[1, \frac{8}{5}\right) \subset \{0\} \cup [1, 3) = T(X)$$

and

$$B(X) = \{0\} \cup \left[1, \frac{7}{4}\right) \subset \{0\} \cup \left[0, \frac{5}{2}\right) = S(X).$$

Define $\psi(t)$ *and* φ *as follows:*

$$\psi(t) = t^2 \;\forall\, t \in \mathfrak{R}^+,$$

and

$$\varphi(t) = \begin{cases} 0 & \text{if} \quad t = 0 \\ 1 + \frac{t}{2} & \text{if} \quad t > 0 \end{cases}$$

Case 1: *If* $x = 0$ *and* $y = 0$

$$\psi(d(Ax, By)) = 0, \psi(M(x, y)) = 0, \varphi(N(x, y)) = 0,$$

hence equation(8) satisfied.
Case 2: *If* $x = 0$ *and* $y \neq 0$

$$\psi(d(Ax, By)) = \left(\frac{y}{4} + 1\right)^2,$$

and

$$M(x,y) = \max\left\{\left|\frac{2y}{3} + 1\right|, 0, \left|\frac{2y}{3} - \frac{y}{4}\right|, \left|\frac{2y}{3} + 1\right|\right\}$$

$$M(x,y) = \left|\frac{2y}{3} + 1\right|,$$

and

$$N(x,y) = \left|\frac{2y}{3} - \frac{y}{4}\right|,$$

$$\psi(M(x,y)) - \varphi(N(x,y)) = \left(\frac{2y}{3} + 1\right)^2 - \left(1 + \frac{5y}{48}\right)$$

$$\psi(M(x,y)) - \varphi(N(x,y)) \geq \psi(d(Ax, By)).$$

Case 3: *If* $x \neq 0$ *and* $y = 0$

$$\psi(d(Ax, By)) = \left(\frac{x}{5} + 1\right)^2,$$

and

$$M(x,y) = \max\left\{\left|\frac{2x}{5} + 1\right|, \left|\frac{x}{2} - \frac{x}{5}\right|, 0, \frac{1}{2}\left|\frac{x}{2} + \frac{x}{5}\right|\right\}$$

$$M(x,y) = \left|\frac{2x}{5} + 1\right|,$$

and

$$N(x,y) = \left|\frac{x}{2} - \frac{x}{5}\right|,$$

$$\psi(M(x,y)) - \varphi(N(x,y)) = \left(\frac{2x}{5} + 1\right)^2 - \left(1 + \frac{3x}{20}\right)$$

$$\psi(M(x,y)) - \varphi(N(x,y)) \geq \psi(d(Ax, By)).$$

Case 4: *If* $x \neq 0$ *and* $y \neq 0$

$$\psi(d(Ax, By)) = \left(\frac{x}{5} - \frac{y}{4}\right)^2,$$

and

$$M(x, y) = \max\left\{\left|\frac{2x}{5} - \frac{5y}{3}\right|, \left|\frac{x}{5}\right|, \left|\frac{5y}{12}\right|, \frac{1}{2}\left[\left|\frac{x}{2} - \frac{y}{4}\right| + \left|\frac{2y}{3} - \frac{x}{5}\right|\right]\right\}$$

$$M(x, y) = \left|\frac{5y}{12}\right|,$$

and

$$N(x, y) = \frac{1}{2}\left[\left|\frac{x}{2} - \frac{y}{4}\right| + \left|\frac{2y}{3} - \frac{x}{5}\right|\right],$$

$$\psi(M(x, y)) - \varphi(N(x, y)) = \left(\frac{5y}{12}\right)^2 - \left(1 + \frac{18x + 25y}{240}\right)$$

$$\psi(M(x, y)) - \varphi(N(x, y)) \geq \psi(d(Ax, By)).$$

Hence the inequality holds in each of the cases.

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Inequalities for rational functions with poles in the Half plane

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Abstract. In this paper we prove certain Bernstein-type inequalities for rational functions with poles in the right half plane. We also deduce some estimates for the maximum modulus of polar derivative of a polynomial on the imaginary axis in terms of the modulus of the polynomial.

1 Introduction

Let $\mathcal{P}_n$ denote the class of all complex polynomials $p(z) := \sum_{j=0}^{n} c_j z^j$ of degree at most n. For every $p \in \mathcal{P}_n$, the following inequality is due to Bernstein [4]:

$$\max_{|z|=1} |p'(z)| \leq n \max_{|z|=1} |p(z)|.$$

It was conjectured by Erdös and proved by Lax [6] that if all the zeros of p lie outside the open unit disk, then

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{2} \max_{|z|=1} |p(z)|.$$

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Later Turán [11] proved that if all the zeros of p lie inside the closed unit disk, then

$$\max_{|z|=1}|p'(z)| \geq \frac{n}{2}\max_{|z|=1}|p(z)|.$$

There have been many refinements and generalisations of the results of Lax and Turan (see [9], [10]). Li, Mohapatra and Rodriguez [7] extended the above inequalities to rational functions r with poles outside the closed unit disk and proved the following results:

Theorem 1 *Suppose* $r(z) = \dfrac{p(z)}{\prod_{j=1}^{n}(z-a_j)}$, *where* $p \in \mathcal{P}_n$ *and* $|a_j| > 1$, *for all* $1 \leq j \leq n$. *Then for* $|z| = 1$

$$|r'(z)| \leq |B'(z)|\max_{|z|=1}|r(z)|. \tag{1}$$

where $B(z) = \prod_{j=1}^{n}\left(\dfrac{1-\overline{a_j}z}{z-a_j}\right)$ *is the Blashke Product for unit disk.*

They also proved:

Theorem 2 *Suppose* $r(z) = \dfrac{p(z)}{\prod_{j=1}^{n}(z-a_j)}$, *where* $p \in \mathcal{P}_n$ *and* $|a_j| > 1$, *for all* $1 \leq j \leq n$*and all the zeroes of* r *lie outside open unit disk. Then for* $|z| = 1$

$$|r'(z)| \leq \frac{1}{2}|B'(z)|\max_{|z|=1}|r(z)|. \tag{2}$$

Theorem 3 *Suppose* $r(z) = \dfrac{p(z)}{\prod_{j=1}^{n}(z-a_j)}$, *where* $p \in \mathcal{P}_n$ *and* $|a_j| > 1$, *for all* $1 \leq j \leq n$*and all the zeroes of* r *lie inside closed unit disk. Then for* $|z| = 1$

$$|r'(z)| \geq \frac{1}{2}(|B'(z)| - (n-m)\max_{|z|=1}|r(z)|). \tag{3}$$

where m *is the number of zeros of* r.

Following the paper by Li, Mohapatra and Rodriguez [7], there have been many generalizations of Theorems 1, 2 and 3 (For details see [2], [3], [5], [8]). In all the cases, it is assumed that the poles of the rational function r are either inside or outside of the unit circle in the complex plane. In this paper, instead of assuming that the poles of r are inside/outside unit circle we consider the case when the poles are in the left/right half of the complex plane and derive the corresponding inequalities on the imaginary axis. So we derive these estimates on a line which is an unbounded set unlike the boundary of a disk. Further, we obtain certain estimates of the maximum modulus of the polar derivative $D_\zeta p(z)$ of a polynomial $p(z)$ in terms of the maximum modulus of $p(z)$ on the imaginary axis. We start with the following notations and definitions:

Let $\mathbb{I} := \{z \in \mathbb{C} : \ \Re(z) = 0\}$, $\mathbb{I}^+ := \{z \in \mathbb{C} : \ \Re(z) > 0\}$ and $\mathbb{I}^- := \{z \in \mathbb{C} : \Re(z) < 0\}$. For $a_j \in \mathbb{I}^+, j = 1, 2, \ldots, n$, let

$$w(z) := \prod_{j=1}^{n} (z - a_j),$$

and
$$\mathcal{R}_n = R_n(a_1, a_2, \ldots, a_n) := \left\{ \frac{p(z)}{w(z)} : \quad p \in \mathcal{P}_n \right\}.$$

Thus $\mathcal{R}_n$ is the set of all rational functions with poles $a_1, a_2, \ldots, a_n$ in the open right half plane and with finite limit at ∞. We define the corresponding Blashke product $B(z)$ for the half plane

$$B(z) := \prod_{j=1}^{n} \left(\frac{z + \overline{a_j}}{z - a_j} \right).$$

Clearly $B(z) \in \mathcal{R}_n$.

We also define for $p(z) = \sum_{j=0}^{n} c_j z^j$, the *conjugate transpose*(reciprocal) p^* of p as

$$p^*(z) := (-1)^n \overline{p(-\overline{z})} = \overline{c_n} z^n - \overline{c_{n-1}} z^{n-1} + \cdots + (-1)^n \overline{c_0}.$$

For $r(z) = \dfrac{p(z)}{w(z)} \in \mathcal{R}_n$, we define $r^*(z) := B(z)\overline{r(-\overline{z})}$. Note that if $r = \dfrac{p}{w} \in \mathcal{R}_n$, then $r^* = \dfrac{p^*}{w}$ and hence $r^* \in \mathcal{R}_n$. Further, we define the polar derivative $D_\zeta p(z)$ of a polynomial $p(z)$ with respect to ζ as

$$D_\zeta p(z) := np(z) - (z - \zeta)p'(z).$$

It is clear that $D_\zeta p(z)$ is a polynomial of degree atmost $n - 1$ and

$$\lim_{\zeta \to \infty} \left(\frac{D_\zeta p(z)}{\zeta} \right) = p'(z).$$

For details regarding Bernstein-type inequalities for polar derivatives on unit circle (see [1], [12]).

2 Main results

In this paper we assume that all the poles a_j, $j = 1, 2, \ldots, n$ lie in open right half plane $\mathbb{I}^+$. For the case when all the poles are in open left half plane $\mathbb{I}^-$, we obtain analogous results with suitable modifications. We first prove:

Theorem 4 *Let* i *be the imaginary unit, then* $B(z) = i$ *has exactly n simple roots, say* $t_1, t_2, \ldots, t_n$ *and all lie on the imaginary axis* $\mathbb{I}$*. Further, if* $r \in \mathcal{R}_n$ *and* $z \in \mathbb{I}$*, then*

$$r'(z)(B(z) - i) - B'(z)r(z) = (B(z) - i)^2 \sum_{k=1}^{n} \frac{u_k r(t_k)}{|z - t_k|^2}, \tag{4}$$

where

$$\frac{1}{u_k} = B'(t_k) = i \sum_{j=1}^{n} \frac{2\Re(a_j)}{|t_k - a_j|^2}, 1 \leq k \leq n. \tag{5}$$

Moreover for $z \in \mathbb{I}$

$$\frac{B'(z)}{B(z)} = \sum_{j=1}^{n} \frac{2\Re(a_j)}{|z - a_j|^2}. \tag{6}$$

From Theorem 1 we can deduce the following:

Corollary 1 *Let* t_k, $k = 1, 2, \ldots, n$ *be as defined in Theorem 4 and* s_k, $k = 1, 2, \ldots, n$ *be the roots of* $B(z) = -i$*, then for* $z \in \mathbb{I}$

$$|r'(z)| \leq \frac{1}{2}|B'(z)|[\max_{1 \leq k \leq n} |r(t_k)| + \max_{1 \leq k \leq n} |r(s_k)|]. \tag{7}$$

Corollary 1 immediately gives us the following:

Corollary 2 *If* $z \in \mathbb{I}$*, then*

$$\max_{z \in \mathbb{I}} |r'(z)| \leq |B'(z)| \max_{z \in \mathbb{I}} |r(z)| \tag{8}$$

The inequality is sharp in the sense that the equality holds if we take $r(z) = \lambda B(z)$ *for some* $\lambda \in \mathbb{C}$*.*

This is the Bernstein-type inequality for $\mathcal{R}_n$, the rational functions with all the poles in open right half plane and is identical to Theorem 1.

Theorem 5 *If* $r \in \mathcal{R}_n$ *and* $z \in \mathbb{I}$, *then*

$$|(r^*(z))'| - |r'(z)| \leq |B'(z)||r(z)|. \tag{9}$$

Theorem 6 *Suppose* $r \in \mathcal{R}_n$
(i) If r *has all its zeros in the closed left half plane* $\mathbb{I} \cup \mathbb{I}^-$, *then for* $z \in \mathbb{I}$

$$\Re\left(\frac{r'(z)}{r(z)}\right) \geq \frac{1}{2}|B'(z)|. \tag{10}$$

(ii) If r *has all its zeros in the closed right half plane* $\mathbb{I} \cup \mathbb{I}^+$, *then for* $z \in \mathbb{I}$

$$\Re\left(\frac{r'(z)}{r(z)}\right) \leq \frac{1}{2}|B'(z)|. \tag{11}$$

The inequalities are sharp and the equality holds if all the zeros of r *lie on the imaginary axis* $\mathbb{I}$.

If we set $a_1 = a_2 = \cdots = a_n = \alpha$ in Theorem 4, then we get the following estimates for the polar derivative of a polynomial $p \in \mathcal{P}_n$:

Theorem 7 *If* $p \in \mathcal{P}_n$ *and* $\alpha \in \mathbb{I}^+$, *then there exists a* $z_0 \in \mathbb{I}$ *such that*

$$|D_\alpha p(z)| \leq 2n \left|\frac{z-\alpha}{z_0-\alpha}\right|^n |p(z_0)| \quad \text{for } z \in \mathbb{I}. \tag{12}$$

Theorem 8 *If* $p \in \mathcal{P}_n$, *then for* $\alpha \in \mathbb{I}^+$

$$|D_\alpha p^*(z)| - |D_\alpha p(z)| \leq 2n|P(z)| \quad \text{for } z \in \mathbb{I}. \tag{13}$$

Theorem 9 *Suppose* $p \in \mathcal{P}_n$
(i) If p *has all its zeros in the closed left half plane* $\mathbb{I} \cup \mathbb{I}^-$, *then for* $\alpha \in \mathbb{I}^+$

$$\Re\left(\frac{D_\alpha p(z)}{(\alpha - z)p(z)}\right) \geq \frac{n\Re(\alpha)}{|z-\alpha|^2} \quad \text{for } z \in \mathbb{I}. \tag{14}$$

(ii) If p *has all its zeros in the closed right half plane* $\mathbb{I} \cup \mathbb{I}^+$, *then for* $\alpha \in \mathbb{I}^+$

$$\Re\left(\frac{D_\alpha p(z)}{(\alpha - z)p(z)}\right) \leq \frac{n\Re(\alpha)}{|z-\alpha|^2} \quad \text{for } z \in \mathbb{I}. \tag{15}$$

The inequalities are sharp and equality holds for a polynomial p *having all the zeros on the imaginary axis* $\mathbb{I}$.

Proofs:

Proof of the Theorem 4. Suppose

$$B(z) - i = 0. \tag{16}$$

Then $w^*(z) - iw(z) = 0$, which is clearly a polynomial of degree n and therefore it has n zeros.
We claim that

$$z \in \mathbb{I} \text{ if and only if } |B(z)| = 1. \qquad (*)$$

Indeed, we have $\left|\dfrac{z+\overline{a_j}}{z-a_j}\right|^2 - 1 = \dfrac{4\Re(z)\Re(a_j)}{|z-a_j|^2}$. Therefore if $\Re(z) = 0$, then $\left|\dfrac{z+\overline{a_j}}{z-a_j}\right| = 1$ for all $j = 1, 2, \cdots, n$ and we get $|B(z)| = \prod_{j=1}^{n}\left|\dfrac{z+\overline{a_j}}{z-a_j}\right| = 1.$ Conversely if $|B(z)| = 1$, then $\Re(z) > 0$, gives us

$$\left|\frac{z+\overline{a_j}}{z-a_j}\right|^2 - 1 = \frac{4\Re(z)\Re(a_j)}{|z-\alpha|^2} > 0 \text{ for all } j = 1, 2, \cdots, n.$$

This in particular gives $|B(z)| > 1$, a contradiction. There will be a similar contradiction, if we assume that $\Re(z) < 0$. Hence $z \in \mathbb{I}$.

By $(*)$ all the roots of (16) lie on $\mathbb{I}$ and $w(z) \neq 0$ on $\mathbb{I}$. So the n zeros of $w^*(z) - iw(z)$ are the n roots (say) $t_1, t_2, \cdots, t_n$ of (16), which lie on the imaginary axis. We show that all t_k, $k = 1, 2, \cdots, n$ are distinct. We have

$$B(z) = \frac{\prod_{j=1}^{n}(z+\overline{a_j})}{\prod_{j=1}^{n}(z-a_j)}.$$

Therefore

$$\begin{aligned}\frac{B'(z)}{B(z)} &= \sum_{j=1}^{n}\left(\frac{1}{z+\overline{a_j}} - \frac{1}{z-a_j}\right) \\ &= \sum_{j=1}^{n}\frac{2\Re(a_j)}{|z-a_j|^2} \quad \text{for} \quad z \in \mathbb{I}.\end{aligned}$$

This proves (6) and hence for all $t_k, k = 1, 2, \cdots, n$, we get

$$B'(t_k) = i\sum_{j=1}^{n}\frac{2\Re(a_j)}{|t_k-a_j|^2}.$$

Since $\Re(a_j) > 0$, for all $j = 1, 2, \cdots, n$, $B'(t_k)$ is a non-zero (purely imaginary) number for all $k = 1, 2, \cdots, n$. Hence $t_k,\ k = 1, 2, \cdots, n$ are all distinct roots of (16). Now let

$$\begin{aligned} q(z) &= w^*(z) - iw(z) \\ &= w(z)\left(B(z) - i\right) \\ &= a\prod_{k=1}^{n}(z - t_k), \quad a \neq 0. \end{aligned}$$

Then $q \in \mathcal{P}_n$. Now for $r(z) = \dfrac{p(z)}{w(z)} \in \mathcal{R}_n$, let $p(z) = cz^n + \ldots$ Then $p(z) - \dfrac{c}{a}q(z)$ is a polynomial of degree atmost $n-1$. Since $t_k, k = 1, 2, \cdots, n$ are n distinct numbers, by Lagrange's interpolation formula

$$p(z) - \frac{c}{a}q(z) = \sum_{k=1}^{n}\frac{(p(t_k) - \frac{c}{a}q(t_k))q(z)}{(z - t_k)q'(t_k)}.$$

This implies

$$\frac{p(z)}{q(z)} - \frac{c}{a} = \sum_{k=1}^{n}\frac{p(t_k)}{(z - t_k)q'(t_k)},$$

which on differentiation gives

$$\left(\frac{p(z)}{q(z)}\right)' = -\sum_{k=1}^{n}\frac{p(t_k)}{(z - t_k)^2 q'(t_k)}. \tag{17}$$

Now $p(z) = w(z)r(z)$ and $q(z) = w(z)(B(z) - i)$ gives $\dfrac{p(z)}{q(z)} = \dfrac{r(z)}{B(z) - i}$ and hence

$$\left(\frac{p(z)}{q(z)}\right)' = \frac{(B(z) - i)r'(z) - r(z)B'(z)}{(B(z) - i)^2}.$$

Also $p(t_k) = w(t_k)r(t_k)$ and

$$\begin{aligned} q'(t_k) &= w'(t_k)(B(t_k) - i) + w(t_k)B'(t_k) \\ &= w(t_k)B'(t_k). \end{aligned}$$

Therefore from (17), we have

$$\frac{(B(z) - i)r'(z) - r(z)B'(z)}{(B(z) - i)^2} = -\sum_{k=1}^{n}\frac{r(t_k)}{(z - t_k)^2 B'(t_k)}$$

$$= \sum_{k=1}^{n} \frac{r(t_k)}{|z-t_k|^2 B'(t_k)}, \quad \text{for} \quad z \in \mathbb{I}$$

Hence

$$(B(z)-i)r'(z) - r(z)B'(z) = (B(z)-i)^2 \sum_{k=1}^{n} \frac{u_k r(t_k)}{|z-t_k|^2} \tag{18}$$

where

$$\frac{1}{u_k} = B'(t_k) = i \sum_{j=1}^{n} \frac{2\Re(a_j)}{|t_k - a_j|^2}.$$

This proves (4) and (5).

Remark 1 *Note that* u_k, $(k = 1, 2, \cdots, n)$ *are purely imaginary numbers with negative imaginary part under our assumption* $\Re(a_j) > 0$ *for all* $j = 1, 2, \cdots, n$.

Proof of Corollary 1. By the same argument as in Theorem 4 applied to $B(z) = -i$ instead of $B(z) = i$, we get

$$(B(z)+i)r'(z) - r(z)B'(z) = (B(z)+i)^2 \sum_{k=1}^{n} \frac{v_k r(s_k)}{|z-s_k|^2}, \tag{19}$$

where

$$\frac{1}{v_k} = B'(t_k) = -i \sum_{j=1}^{n} \frac{2\Re(a_j)}{|s_k - a_j|^2}.$$

Subtracting (18) from (19) we have

$$2ir'(z) = (B(z)+i)^2 \sum_{k=1}^{n} \frac{v_k r(s_k)}{|z-s_k|^2} - (B(z)-i)^2 \sum_{k=1}^{n} \frac{u_k r(t_k)}{|z-t_k|^2}. \tag{20}$$

Taking $r(z) \equiv 1$ in (18) and (19) we get

$$B'(z) = -(B(z)-i)^2 \sum_{k=1}^{n} \frac{u_k}{|z-t_k|^2}$$

$$B'(z) = -(B(z)+i)^2 \sum_{k=1}^{n} \frac{v_k}{|z-s_k|^2}$$

and hence

$$|B'(z)| = |B(z) - i|^2 \left| \sum_{k=1}^{n} \frac{u_k}{|z - t_k|^2} \right| \tag{21}$$

$$|B'(z)| = |B(z) + i|^2 \left| \sum_{k=1}^{n} \frac{v_k}{|z - s_k|^2} \right|. \tag{22}$$

Now from (20)

$$|2r'(z)| \le |(B(z)+i)|^2 \left| \sum_{k=1}^{n} \frac{v_k r(s_k)}{|z - s_k|^2} \right| + |(B(z)-i)|^2 \left| \sum_{k=1}^{n} \frac{u_k r(t_k)}{|z - t_k|^2} \right|.$$

Using (21) and (22), we get for $z \in \mathbb{I}$

$$|r'(z)| \le \frac{1}{2}|B'(z)|[\max_{1\le k\le n} |r(t_k)| + \max_{1\le k\le n} |r(s_k)|]$$

Proof of Theorem 5. We have

$$r^*(z) = B(z)\overline{r(\overline{-z})}.$$

Therefore

$$\begin{aligned} (r^*(z))' &= B'(z)\overline{r(\overline{-z})} - B(z)\overline{r'(\overline{-z})} \\ &= B'(z)\overline{r(z)} - B(z)\overline{r'(z)} \quad \text{for } z \in \mathbb{I} \end{aligned}$$

This implies that

$$\begin{aligned} |(r^*(z))'| &\le |B'(z)| \left|\overline{r(z)}\right| + |B(z)| \left|\overline{r'(z)}\right| \\ &= |B'(z)||r(z)| + |B(z)||r'(z)|. \end{aligned}$$

Since $|B(z)| = 1$ on imaginary axis, it follows that for $z \in \mathbb{I}$

$$|(r^*(z))'| - |r'(z)| \le |B'(z)||r(z)|.$$

Proof of Theorem 6. Let $b_1, b_2, \ldots, b_m, m \le n.$ be the zeros of r.
(i) Suppose $\Re(b_j) \le 0$ for all $j = 1, 2, \ldots, m$. Then $p(z) = c\prod_{j=1}^{m}(z - b_j)$ with $c \ne 0$ and we have

$$r(z) = \frac{p(z)}{w(z)} = \frac{c\prod_{j=1}^{m}(z - b_j)}{\prod_{j=1}^{n}(z - a_j)}.$$

Taking logarithms on both sides and differentiating we get

$$\frac{r'(z)}{r(z)} = \sum_{j=1}^{m} \frac{1}{z-b_j} - \sum_{j=1}^{n} \frac{1}{z-a_j}. \tag{23}$$

Now for $\Re(z) = 0$

$$\Re\left(\frac{1}{z-b_j}\right) = \frac{-\Re(b_j)}{|z-b_j|^2} \geq 0 \quad \text{for all} \quad j = 1, 2, \ldots, m$$

and therefore

$$\sum_{j=1}^{m} \Re\left(\frac{1}{z-b_j}\right) \geq 0.$$

Hence from (23) and by using (6) we have

$$\begin{aligned}
\Re\left(\frac{r'(z)}{r(z)}\right) &= \sum_{j=1}^{m} \Re\left(\frac{1}{z-b_j}\right) - \sum_{j=1}^{n} \Re\left(\frac{1}{z-a_j}\right) \\
&\geq -\sum_{j=1}^{n} \Re\left(\frac{1}{z-a_j}\right) \\
&= -\sum_{j=1}^{n} \frac{\Re(z-a_j)}{|z-a_j|^2} \\
&= \sum_{j=1}^{n} \frac{\Re(a_j)}{|z-a_j|^2} \quad \text{for } \Re(z) = 0 \\
&= \frac{1}{2}\left|\frac{B'(z)}{B(z)}\right|.
\end{aligned}$$

Since $|B(z)| = 1$ for $z \in \mathbb{I}$, we conclude

$$\Re\left(\frac{r'(z)}{r(z)}\right) \geq \frac{1}{2}|B'(z)|.$$

(ii) Suppose $\Re(b_j) \geq 0$ for all $j = 1, 2, \ldots, m$. Then for $\Re(z) = 0$

$$\Re\left(\frac{1}{z-b_j}\right) = \frac{-\Re(b_j)}{|z-b_j|^2} \leq 0 \quad \text{for all} \quad j = 1, 2, \cdots, m.$$

This in particular gives

$$\sum_{j=1}^{m} \Re\left(\frac{1}{z-b_j}\right) \le 0.$$

Thus as in part (i), we get for $\Re(z) = 0$

$$\begin{aligned}
\Re\left(\frac{r'(z)}{r(z)}\right) &= \sum_{j=1}^{m} \Re\left(\frac{1}{z-b_j}\right) - \sum_{j=1}^{n} \Re\left(\frac{1}{z-a_j}\right) \\
&\le -\sum_{j=1}^{n} \Re\left(\frac{1}{z-a_j}\right) \\
&= -\sum_{j=1}^{n} \frac{\Re(z-a_j)}{|z-a_j|^2} \\
&= \sum_{j=1}^{n} \frac{\Re(a_j)}{|z-a_j|^2} \\
&= \frac{1}{2}\left|\frac{B'(z)}{B(z)}\right|.
\end{aligned}$$

That is

$$\Re\left(\frac{r'(z)}{r(z)}\right) \le \frac{1}{2}|B'(z)|,$$

Proof of Theorem 7. Let s_k and t_k, $k = 1, 2, \ldots, n$ be as defined in Corollary 1 and Let
$z_0 \in \{t_1, t_2, \ldots, t_n, s_1, s_2, \ldots, s_n\}$, be such that
$|r(z_0)| = max\{|r(t_1)|, |r(t_2)|, \ldots, |r(t_n)|, |r(s_1)|, |r(s_2)|, \ldots, |r(s_n)|\}$. By Corollary 1

$$|r'(z)| \le |B'(z)||r(z_0)| \tag{24}$$

For $a_1 = a_2 = \ldots = a_n = \alpha$, $r(z) = \dfrac{p(z)}{(z-\alpha)^n}$ and $B(z) = \dfrac{(z+\overline{\alpha})^n}{(z-\alpha)^n}$

$$\begin{aligned}
\text{so that } r'(z) &= \left(\frac{p(z)}{(z-\alpha)^n}\right)' \\
&= \frac{(z-\alpha)^n p'(z) - p(z)n(z-\alpha)^{n-1}}{(z-\alpha)^{2n}} \\
&= \frac{1}{-(z-\alpha)^{n+1}} D_\alpha p(z).
\end{aligned}$$

Also from (6)

$$|B'(z)| = \frac{2n\Re(\alpha)}{|z-\alpha|^2} \quad \text{for } z \in \mathbb{I}.$$

Substituting in (24), we get for $z \in \mathbb{I}$

$$\begin{aligned}\left|\frac{1}{(z-\alpha)^{n+1}} D_\alpha p(z)\right| &\leq \frac{2n\Re(\alpha)}{|z-\alpha|^2}\left|\frac{P(z_0)}{(z_0-\alpha)^n}\right| \\ &= \frac{2n\Re(\alpha-z)}{|z-\alpha|^2}\left|\frac{P(z_0)}{(z_0-\alpha)^n}\right|\end{aligned}$$

and hence

$$\left|\frac{1}{(z-\alpha)^{n+1}} D_\alpha p(z)\right| \leq \frac{2n\Re(\alpha-z)}{|z-\alpha|^2}\left|\frac{P(z_0)}{(z_0-\alpha)^n}\right|.$$

Proof of Theorem 8. We have from Theorem 5 for every $z \in \mathbb{I}$

$$|(r^*(z))'| - |r'(z)| \leq |B'(z)||r(z)| \tag{25}$$

Taking $r(z) = \dfrac{p(z)}{(z-\alpha)^n}$, so that

$$r'(z) = \frac{1}{-(z-\alpha)^{n+1}} D_\alpha p(z).$$

Also

$$r^*(z) = \frac{p^*(z)}{(z-\alpha)^n}$$

gives

$$(r^*(z))' = \frac{1}{-(z-\alpha)^{n+1}} D_\alpha p^*(z).$$

Further from (6), we have for $z \in \mathbb{I}$

$$|B'(z)| = \frac{2n\Re(\alpha)}{|z-\alpha|^2}.$$

Therefore from (25), for $z \in \mathbb{I}$

$$\left|\frac{1}{(z-\alpha)^{n+1}} D_\alpha p^*(z)\right| - \left|\frac{1}{(z-\alpha)^{n+1}} D_\alpha p(z)\right| \leq \frac{2n\Re(\alpha)}{|z-\alpha|^2}\left|\frac{p(z)}{(z-\alpha)^n}\right|. \tag{26}$$

Also for $z \in \mathbb{I}$

$$\begin{aligned}\Re(\alpha) &= \Re(\alpha - z)\\ &\leq |\alpha - z|\\ &= |z - \alpha|.\end{aligned}$$

Thus from (26) we get

$$\left|\frac{1}{(z-\alpha)^{n+1}} D_\alpha p^*(z)\right| - \left|\frac{1}{(z-\alpha)^{n+1}} D_\alpha p(z)\right| \leq \frac{2n|z-\alpha|}{|z-\alpha|^2}\left|\frac{p(z)}{(z-\alpha)^n}\right| .$$

This gives

$$|D_\alpha p^*(z)| - |D_\alpha p(z)| \leq 2n|p(z)| \quad \text{for } z \in \mathbb{I}.$$

Proof of Theorem 9. Let $b_1, b_2, \ldots, b_m, m \leq n$, be the zeros of p.
(i) Suppose $\Re(b_j) \leq 0$ for all $j = 1, 2, \ldots, m$. Taking $r(z) = \frac{p(z)}{(z-\alpha)^n}$ in Theorem 6 (i), we get by using the fact that $r'(z) = \frac{-D_\alpha p(z)}{(z-\alpha)^{n+1}}$ and $|B'(z)| = \frac{2n\Re(\alpha)}{|z-\alpha|^2}$ for $z \in \mathbb{I}$,

$$\Re\left(\frac{D_\alpha p(z)}{(\alpha - z)p(z)}\right) \geq \frac{n\Re(\alpha)}{|z-\alpha|^2} \quad \text{for } z \in \mathbb{I}. \tag{27}$$

(ii) Suppose $\Re(b_j) \geq 0$ for all $j = 1, 2, \ldots, m$. Then taking $r(z) = \frac{p(z)}{w(z)}$ in Theorem 6 (ii).

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Affine factorable surfaces of finite type in the 3-dimensional Euclidean $\mathbb{E}^3$ and Lorentzian $\mathbb{L}^3$ spaces

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Abstract. In this paper we study affine factorable surfaces in the 3-dimensional Euclidean space $\mathbb{E}^3$ and Lorentzian $\mathbb{L}^3$ under the condition $\Delta r_i = \lambda_i r_i$, where $\lambda_i \in \mathbb{R}$ and Δ denotes the Laplace operator with respect to the first fundamental form.

1 Introduction

Let M^2 be a connected non-degenerate submanifold in the three-dimensional Lorentz– Minkowski space $\mathbb{E}^3_1$ and $r: M^2 \to \mathbb{E}^3_1$ be a parametric representation of a surface in the $\mathbb{E}^3_1$ equipped with the induced metric. Then the position vector of M^2 in $\mathbb{E}^3_1$ satisfies [6]

$$\Delta r = -2\mathbf{H}, \tag{1}$$

where $\mathbf{H}$ is the mean curvature vector of M^2 in $\mathbb{E}^3_1$. It follows from (1) that M^2 is minimal ($\mathbf{H} = 0$) in $\mathbb{E}^3_1$ if and only if the immersion r is harmonic $\Delta r = 0$. The notion of finite type immersion of submanifolds of a Euclidean space has been widely used in classifying and characterizing well known Riemannian submanifolds [5]. B.-Y. Chen posed the problem of classifying the finite type

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submanifolds in the 3-dimensional Euclidean space $\mathbb{E}^3$. These can be regarded as a generalization of minimal submanifolds.

A well known result due to Takahashi [14] states that minimal surfaces and spheres are the only surfaces in $\mathbb{E}^3$ satisfying the condition

$$\Delta r = \lambda r, \ \lambda \in \mathbb{R}.$$

For the 3-dimensional Lorentz-Minkowski space $\mathbb{L}^3$, Alias, Ferrandez and Lucas proved that the only such surfaces are minimal surfaces and open pieces of Lorentz circular cylinders, hyperbolic cylinders, Lorentz hyperbolic cylinders, hyperbolic spheres or pseudo-spheres [1].
The authors [3] classified factorable surfaces in the three-dimensional Euclidean and Lorentzian spaces, whose component functions are eigenfunctions of their Laplace operator. The authors [4] studied translation surfaces in the 3-dimensional Euclidean and Lorentz-Minkowski spaces under the condition

$$\Delta^{III} r_i = \mu_i r_i, \ \mu_i \in \mathbb{R}, \tag{2}$$

where Δ^{III} denotes the Laplacian of the surface with respect to the third fundamental form III. They showed that in both spaces a translation surface satisfying (2) is a surface of Scherk. For the third fundamental form surfaces of revolution were studied in [2]. H. Liu and S.D. Jung [8] obtained the affine translation surfaces in Euclidean 3-space of constant mean curvature. They classified these surfaces with null Gaussian and mean curvature.

Senoussi et al. [11, 12, 13] have derived a classification of THA- surfaces in the 3 dimensional Euclidean and Galilean spaces.

In [7, 9, 15], the authors study translation surfaces and affine translation surfaces in Euclidean and Minkowski 3-spaces.

In this paper we study affine factorable surfaces in the three-dimensional Euclidean and Lorentzian spaces satisfying the condition

$$\Delta r_i = \lambda_i r_i, \ \ \lambda_i \in \mathbb{R}. \tag{3}$$

2 Preliminaries

A submanifold M^2 of a 3-dimensional Euclidean space $\mathbb{E}^3$ is said to be of finite type if each component of its position vector field r can be written as a finite sum of eigenfunctions of the Laplacian Δ of M^2, that is, if

$$r = r_0 + \sum_{i=1}^{k} r_i,$$

where r_0 is a fixed vector and r_i are $\mathbb{E}^3$ – vector valued eigenfunctions of the Laplacian of (M^2, r) [5]:

$$\Delta r_i = \lambda_i r_i, \ \lambda_i \in \mathbb{R}, \ i = 1, 2, .., k.$$

If λ_i are different, then M^2 is said to be of k-type.

The coefficients of the first fundamental form and the second fundamental form are

$$E = < r_u, r_u >, \quad F = < r_u, r_v >, \qquad G = < r_v, r_v >;$$
$$L = < r_{uu}, \mathbf{N} >, \ M = < r_{uv}, \mathbf{N} >, \ N = < r_{vv}, \mathbf{N} >,$$

where $< ., . >$ is the Euclidean inner product, $r_u = \frac{\partial r}{\partial u}$, $r_v = \frac{\partial r}{\partial v}$ and $\mathbf{N}$ is the unit normal vector to M^2.

It is well known in terms of local coordinates $\{u, v\}$ of M^2 the Laplacian operator Δ of the first fundamental form on M^2 is defined by

$$\Delta\varphi = \frac{-1}{\sqrt{EG - F^2}} \left[\frac{\partial}{\partial u} \left(\frac{G\varphi_u - F\varphi_v}{\sqrt{EG - F^2}} \right) + \frac{\partial}{\partial v} \left(\frac{E\varphi_v - F\varphi_u}{\sqrt{EG - F^2}} \right) \right]. \tag{4}$$

Proposition 1 *[10] (Gauss Equations) Since the vectors r_u, r_v and $\boldsymbol{N}$ are linearly independent, we can write*

$$r_{uu} = \Gamma_{11}^1 r_u + \Gamma_{11}^2 r_v + L\boldsymbol{N}$$
$$r_{uv} = \Gamma_{12}^1 r_u + \Gamma_{12}^2 r_v + M\boldsymbol{N}$$
$$r_{vv} = \Gamma_{22}^1 r_u + \Gamma_{22}^2 r_v + N\boldsymbol{N},$$

where

$$\Gamma_{11}^1 = \frac{1}{W^2} \det \begin{pmatrix} \frac{E_u}{2} & F \\ F_u - \frac{E_v}{2} & G \end{pmatrix}, \qquad \Gamma_{11}^2 = \frac{1}{W^2} \det \begin{pmatrix} E & \frac{E_u}{2} \\ F & F_u - \frac{E_v}{2} \end{pmatrix},$$
$$\Gamma_{12}^1 = \frac{1}{W^2} \det \begin{pmatrix} \frac{E_v}{2} & F \\ \frac{G_u}{2} & G \end{pmatrix}, \qquad \Gamma_{12}^2 = \frac{1}{W^2} \det \begin{pmatrix} E & \frac{E_v}{2} \\ F & \frac{G_u}{2} \end{pmatrix},$$
$$\Gamma_{22}^1 = \frac{1}{W^2} \det \begin{pmatrix} F_v - \frac{G_u}{2} & F \\ \frac{G_v}{2} & G \end{pmatrix}, \qquad \Gamma_{22}^2 = \frac{1}{W^2} \det \begin{pmatrix} E & F_v - \frac{G_u}{2} \\ F & \frac{G_v}{2} \end{pmatrix}.$$

The six $(\Gamma_{ij}^k)_{1 \le i,j,k \le 2}$ coefficients in these formulas are called Christoffel symbols of the second kind.

Lemma 1 *Laplacian operator* Δ *of the first fundamental form on* M^2 *is defined by*

$$\Delta\varphi = -\frac{1}{W^2}[G\varphi_{uu} - 2F\varphi_{uv} + E\varphi_{vv} - P(u,v)\varphi_u - Q(u,v)\varphi_v], \tag{5}$$

where

$$P(u,v) = G\Gamma_{11}^1 - 2F\Gamma_{12}^1 + E\Gamma_{22}^1, \quad Q(u,v) = G\Gamma_{11}^2 - 2F\Gamma_{12}^2 + E\Gamma_{22}^2, \quad W^2 = EG - F^2.$$

Proof. From (4), we have

$$\begin{aligned}\Delta\varphi &= \frac{-1}{W}\left[\frac{\partial}{\partial u}\left(\frac{G\varphi_u - F\varphi_v}{W}\right) - \frac{\partial}{\partial v}\left(\frac{F\varphi_u - E\varphi_v}{W}\right)\right] \\ &= -\frac{1}{2W^4}[2W^2(G\varphi_{uu} - 2F\varphi_{uv} + E\varphi_{vv}) + \Lambda_1\varphi_u + \Lambda_2\varphi_v],\end{aligned} \tag{6}$$

where

$$\Lambda_1 = (G_u - F_v)2W^2 - G^2E_u - EGG_u + FG(2F_u + E_v) + FEG_v - 2F^2F_v,$$

$$\Lambda_2 = (E_v - F_u)2W^2 - E^2G_v - EGE_v + FE(2F_v + G_u) + FGE_u - 2F^2F_u.$$

After some manipulations, we get

$$\begin{aligned}\Lambda_1 &= -2W^2(G\Gamma_{11}^1 - 2F\Gamma_{12}^1 + E\Gamma_{22}^1) \\ &= -2W^2P(u,v),\end{aligned} \tag{7}$$

$$\begin{aligned}\Lambda_2 &= -2W^2(G\Gamma_{11}^2 - 2F\Gamma_{12}^2 + E\Gamma_{22}^2) \\ &= -2W^2Q(u,v).\end{aligned} \tag{8}$$

By using (7) and (8) in (6), we find (5). □

The mean curvature H and the Gauss curvature K_G are, respectively, defined by

$$H = \frac{1}{2}(k_1 + k_2) = \frac{EN + GL - 2FM}{2(EG - F^2)}$$

and

$$K_G = k_1k_2 = \frac{LN - M^2}{EG - F^2},$$

where k_1 and k_2 are called the principal curvatures.

A surface is said to be flat (resp. minimal) if its Gaussian (resp. mean) curvature vanishes.

3 Affine factorable surfaces in $\mathbb{E}^3$

P. Zong, L. Xiao, H. L. Liu [16] investigated the affine factorable surfaces with constant mean curvature or constant Gauss curvature.

Definition 1 *A affine factorable surface* M^2 *in Euclidean space* $\mathbb{E}^3$ *is a surface that is a graph of a function [16]*

$$z = f(x)g(ax + y), \tag{9}$$

where f *and* g *are two smooth functions.*

The coefficients of the first fundamental form of M^2 are:

$$E = 1 + (f'g + afg')^2,\ F = fg'(f'g + ag'f),\ G = 1 + f^2g'^2. \tag{10}$$

The unit normal vector of M^2 is given by

$$\mathbf{N} = \left(\frac{-(f'g + ag'f)}{W}, \frac{-fg'}{W}, \frac{1}{W}\right).$$

Then the coefficients of the second fundamental form of M^2 are:

$$L = \frac{f''g + a^2fg'' + 2af'g'}{W},\ M = \frac{f'g' + afg''}{W},\ N = \frac{fg''}{W}, \tag{11}$$

where $W = \sqrt{(f'g + afg')^2 + f^2g'^2 + 1}$.

From these we find that the mean curvature H and the curvature K_G of (9) are given by

$$H = \frac{H_1}{2W^3}, \tag{12}$$

where

$$H_1 = f''g + (1 + a^2)fg'' + 2af'g' + f^2gf''g'^2 + ff'^2g^2g'' - 2fgf'^2g'^2$$

and

$$K_G = \frac{fgf''g'' - f'^2g'^2}{W^4}. \tag{13}$$

By a transformation

$$\begin{cases} u = x \\ v = ax + y, \end{cases} \tag{14}$$

and $\frac{\partial(x,y)}{\partial(u,v)} \neq 0$, from (14) we have

$$E = 1 + a^2 + f'^2 g^2,\ F = -a + fgf'g',\ G = 1 + f^2 g'^2;$$

$$L = \frac{f''g}{W},\ M = \frac{f'g'}{W},\ N = \frac{fg''}{W}.$$

From (12) and (13) we get

$$H = \frac{H_2}{2W^3}, \tag{15}$$

where

$$H_2 = f''g(1 + f^2 g'^2) + (1 + a^2 + f'^2 g^2) fg'' + 2(a - ff'gg')f'g'$$

and

$$K_G = \frac{fgf''g'' - f'^2 g'^2}{W^4}, \tag{16}$$

where $W = \sqrt{(f'g + afg')^2 + f^2 g'^2 + 1}$.

4 Affine factorable surfaces in $\mathbb{L}^3$

Let $\mathbb{L}^3$ be a Lorentz-Minkowski 3-space with the scalar product of index 1 given by

$$g_L(X, Y) = -x_1 y_1 + x_2 y_2 + x_3 y_3,$$

where $X = (x_1, x_2, x_3), Y = (y_1, y_2, y_3) \in \mathbb{R}^3$.

For two vectors $V = (v_1, v_2, v_3)$ and $W = (w_1, w_2, w_3)$ in $\mathbb{L}^3$ the Lorentz cross product of V and W is defined by

$$V \wedge_L W = (v_3 w_2 - v_2 w_3, v_3 w_1 - v_1 w_3, v_1 w_2 - v_2 w_1).$$

Affine factorable surfaces in $\mathbb{L}^3$

$$r(x, y) = (x,\ y,\ z = f(x) g(ax + y)). \tag{17}$$

By a transformation

$$\begin{cases} u = x \\ v = ax + y, \end{cases} \tag{18}$$

from (18) we have

$$E = -1 + a^2 + f'^2 g^2,\ F = -a + fgf'g',\ G = 1 + f^2 g'^2;$$

$$\mathbf{N} = \left(\frac{f'g + ag'f}{W}, \frac{-fg'}{W}, \frac{1}{W}\right);$$
$$L = \frac{f''g}{W},\ M = \frac{f'g'}{W},\ N = \frac{fg''}{W},$$

where $W = \sqrt{\varepsilon((f'g + afg')^2 - f^2g'^2 - 1)}$ and $\varepsilon = g_L(r_x \wedge_L r_y, r_x \wedge_L r_y) = \pm 1$.

The mean curvature H and the curvature K_G of (17) are given by

$$H = \frac{H_3}{2W^3}, \tag{19}$$

where

$$H_3 = f''g(1 + f^2g'^2) + (-1 + a^2 + f'^2g^2)fg'' + 2(a - ff'gg')f'g'$$

and

$$K_G = g_L(\mathbf{N}, \mathbf{N})\frac{fgf''g'' - f'^2g'^2}{W^4}. \tag{20}$$

5 Affine factorable surfaces in $\mathbb{E}^3$ satisfying $\Delta r_i = \lambda_i r_i$

The factorable surfaces $(a = 0)$ in the Euclidean space and the pseudo Euclidean space satisfying the condition (3) have been studied in [3].

By a direct computation with the help of (4), the Laplacian Δ on M^2 is given by

$$\Delta\varphi = \frac{-(G\varphi_{uu} + E\varphi_{vv} - 2F\varphi_{uv})}{W^2} + \frac{Q_1(u,v)}{W}\varphi_u + \frac{P_1(u,v)}{W}\varphi_v, \tag{21}$$

where

$$Q_1(u,v) = 2H\left(afg' + f'g\right),\quad P_1(u,v) = 2H\left(agf' + (1 + a^2)fg'\right).$$

Applying (21) on the coordinate functions u, $v - au$ and $z(u,v) = f(u)g(v)$ of the position vector r we find

$$\begin{cases} \Delta(u) = \frac{2H(afg'+f'g)}{W} \\ \Delta(v - au) = \frac{2H(fg')}{W} \\ \Delta(fg) = -\frac{2H}{W}. \end{cases} \tag{22}$$

By using (3) and (22) we have the following equations

$$\frac{2H(afg' + f'g)}{W} = \lambda_1 u \tag{23}$$

$$\frac{2Hfg'}{W} = \lambda_2(v - au) \tag{24}$$

$$-\frac{2H}{W} = \lambda_3 fg. \tag{25}$$

Next we study it according to the constants λ_1, λ_2, λ_3.

Case 1. Let $\lambda_3 = 0$.

Then, the equation (25) gives rise to $H = 0$, which means that the surfaces are minimal. We get also, by the equations (23) and (24), $\lambda_1 = \lambda_2 = 0$.

Case 2. Let $\lambda_3 \neq 0$.

In this case we have four possibilities:

2.1) If $\lambda_1 = 0$ and $\lambda_2 \neq 0$ equations (23), (24) and (25) imply that

$$H(afg' + f'g) = 0 \tag{26}$$

$$\frac{2Hfg'}{W} = \lambda_2(v - au) \tag{27}$$

$$-\frac{2H}{W} = \lambda_3 fg. \tag{28}$$

2-1-1) If $H = 0$, then $\lambda_3 = 0$ and $\lambda_2 = 0$. So we get a contradiction.

2-1-2) If $afg' + f'g = 0$, then

$$ag' = \mu g, \quad f' = -\mu f, \quad \mu \in \mathbb{R}. \tag{29}$$

Substituting (28) into (27), we get

$$-\lambda_3 f^2 g g' = \lambda_2(v - au).$$

Using equation (29) we get

$$\omega f^2 g^2 = v - au, \quad \omega = \frac{-\lambda_3 \mu}{a\lambda_2}. \tag{30}$$

Differentiating (30) with respect to u we get $g' = 0$, then $\lambda_2 = 0$. So we get a contradiction.

2-2) If $\lambda_1 \neq 0$ and $\lambda_2 = 0$. In this case the system (23), (24) and (25) is reduced equivalently to

$$\begin{cases} \frac{2H(afg'+f'g)}{W} = \lambda_1 u \\ Hfg' = 0 \\ -\frac{2H}{W} = \lambda_3 fg. \end{cases} \tag{31}$$

2-2-1) If $f = 0$, then $\lambda_1 = 0$. So we get a contradiction.
2-2-2) If $g' = 0$. Then $g(v) = \alpha$, $\alpha \in \mathbb{R}$.
Now (31) reduces to

$$\frac{\alpha^2 f' f''}{(1+\alpha^2 f'^2)^2} = \lambda_1 u \tag{32}$$

$$\frac{-f''}{(1+\alpha^2 f'^2)^2} = \lambda_3 f. \tag{33}$$

By using equations (33) and (32), we get the relation between f and f' such as $\lambda_1 u + \lambda_3 \alpha^2 f' f = 0$, or equivalently,

$$f(u) = \varepsilon \frac{\sqrt{2}}{\alpha\sqrt{|\lambda_3|}} \sqrt{\left| c + \frac{\lambda_1}{2} u^2 \right|}, \quad \varepsilon = \pm 1, \quad c \in \mathbb{R}.$$

Hence the Gaussian and the mean curvature of M^2 are given by

$$K_G = 0, \quad H = -\frac{1}{2\alpha f'} \left(\frac{1}{\sqrt{1+\alpha^2 f'^2}} \right)'. \tag{34}$$

From (34) M^2 is flat surface.
2-3) If $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$. Substituting (25) into (24), we get

$$\lambda_3 f^2 g g' = -\lambda_2 (v - au). \tag{35}$$

Take successively derivatives of (35) with respect to u and v, obtaining

$$\lambda_3 f f' (g g')' = 0.$$

2-3-1) If $ff' = 0$. Then $f(u) = \gamma$, $\gamma \in \mathbb{R}$. In this case (35) is reduced equivalently to

$$\lambda_3 \gamma^2 g g' = -\lambda_2 (v - au). \tag{36}$$

Differentiating (36) with respect to u we get $\lambda_2 = 0$. So we get a contradiction.

2-3-2) If $(gg')' = 0$. Then $(gg')(v) = \delta$, $\delta \in \mathbb{R}$. In this case (35) is reduced equivalently to

$$\lambda_3 f^2 \delta = -\lambda_2 (v - au). \tag{37}$$

Differentiating (37) with respect to v we get $\lambda_2 = 0$. So we get a contradiction.

Theorem 1 *Let M^2 be a affine factorable surface given by (9) in $\mathbb{E}^3$. Then M^2 satisfies the equation $\Delta r_i = \lambda_i r_i$ $(i = 1, 2, 3)$ if and only if one of the following statement is true:*
1) M^2 has zero mean curvature everywhere.
2) M^2 is parametrized as

$$r(u, v) = \left(u, v,\ \varepsilon \frac{\sqrt{2}}{\sqrt{|\lambda_3|}} \sqrt{\left| c + \frac{\lambda_1}{2} u^2 \right|} \right), \quad \varepsilon = \pm 1.$$

6 Affine factorable surfaces in $\mathbb{L}^3$ satisfying $\Delta r_i = \lambda_i r_i$

We explore the classification of the affine factorable surfaces satisfying the relation (3).

We distinguish two cases according to whether $EG - F^2 > 0$ or $EG - F^2 < 0$.

6.1 Spacelike affine factorable surfaces in $\mathbb{L}^3$

Now we will investigate the spacelike affine factorable surfaces of $\mathbb{L}^3$.

If we use (4), the Laplacian Δ on M^2 is given by

$$\Delta \varphi = \frac{-(G\varphi_{uu} + E\varphi_{vv} - 2F\varphi_{uv})}{W^2} + \frac{Q_2(u, v)}{W}\varphi_u + \frac{P_2(u, v)}{W}\varphi_v, \tag{38}$$

where

$$Q_2(u, v) = 2H\left(afg' + f'g\right), \quad P_2(u, v) = 2H\left(agf' + (a^2 - 1)fg'\right)$$

and $W^2 = (f'g + afg')^2 - f^2 g'^2 - 1$.

Assume that $EG - F^2 = (f'g + afg')^2 - f^2 g'^2 - 1 > 0$, the metric of M^2 is spacelike.

Applying (38) on the coordinate functions u, $v - au$ and $z(u, v) = f(u)g(v)$ of the position vector r we find

$$\begin{cases} \Delta(u) = \frac{2H(afg'+f'g)}{W} \\ \Delta(v - au) = -\frac{2H(fg')}{W} \\ \Delta(fg) = \frac{2H}{W}. \end{cases} \tag{39}$$

By using (3) and (39) we have the following equations

$$\frac{2H(afg' + f'g)}{W} = \lambda_1 u \tag{40}$$

$$\frac{-2Hfg'}{W} = \lambda_2(v - au) \tag{41}$$

$$\frac{2H}{W} = \lambda_3 fg. \tag{42}$$

Case 1. Let $\lambda_3 = 0$.
Then, the equation (42) gives rise to $H = 0$, which means that the surfaces are minimal. We get also, by the equations (40) and (41), $\lambda_1 = \lambda_2 = 0$.
Case 2. Let $\lambda_3 \neq 0$.
In this case we have four possibilities:
2.1) If $\lambda_1 = 0$ and $\lambda_2 \neq 0$ equations (40), (41) and (42) imply that

$$2H(afg' + f'g) = 0 \tag{43}$$

$$-\frac{2Hfg'}{W} = \lambda_2(v - au) \tag{44}$$

$$\frac{2H}{W} = \lambda_3 fg. \tag{45}$$

2-1-1) If $H = 0$, then $\lambda_3 = 0$ and $\lambda_2 = 0$. So we get a contradiction.
2-1-2) If $afg' + f'g = 0$, then

$$ag' = kg, \quad f' = -kf, k \in \mathbb{R}. \tag{46}$$

Substituting (45) into (44), we get

$$-\lambda_3 f^2 gg' = \lambda_2(v - au).$$

Using equation (46) we get

$$Af^2g^2 = v - au, \quad A = \frac{-\lambda_3 k}{a\lambda_2}.$$

Differentiating now with respect to u, we get $g' = 0$, then $\lambda_2 = 0$. So we get a contradiction.

2-2) If $\lambda_1 \neq 0$ and $\lambda_2 = 0$. In this case the system (40), (41) and (42) is reduced equivalently to

$$\begin{cases} \frac{2H(afg'+f'g)}{W} = \lambda_1 u \\ 2Hfg' = 0 \\ \frac{2H}{W} = \lambda_3 fg. \end{cases} \tag{47}$$

2-2-1) If $f = 0$, then $\lambda_1 = 0$. So we get a contradiction.
2-2-2) If $g' = 0$. Then $g(v) = \alpha$, $\alpha \in \mathbb{R}$.
Therefore, (47) is equivalently reduced to

$$\frac{\alpha^2 f' f''}{(1+\alpha^2 f'^2)^2} = \lambda_1 u \tag{48}$$

$$\frac{f''}{(1+\alpha^2 f'^2)^2} = \lambda_3 f. \tag{49}$$

Multiply (49) with $\alpha^2 f'$ and then sum (48), obtaining $\lambda_1 u - \lambda_3 \alpha^2 f' f = 0$, or equivalently,

$$f(u) = \varepsilon \frac{\sqrt{2}}{\alpha\sqrt{|\lambda_3|}} \sqrt{\left| c + \frac{\lambda_1}{2} u^2 \right|}, \quad \varepsilon = \pm 1, \quad c \in \mathbb{R}.$$

2-3) If $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$. Substituting (42) into (41), we get

$$\lambda_3 f^2 g g' = -\lambda_2 (v - au). \tag{50}$$

Take successively derivatives of (50) with respect to u and v, obtaining

$$\lambda_3 f f' (g g')' = 0.$$

2-3-1) If $ff' = 0$. Then $f(u) = c$, $c \in \mathbb{R}$. In this case (50) is reduced equivalently to

$$\lambda_3 c^2 g g' = -\lambda_2 (v - au). \tag{51}$$

Differentiating (51) with respect to u we get $\lambda_2 = 0$. So we get a contradiction.

2-3-2) If $(gg')' = 0$. Then $(gg')(v) = c$, $c \in \mathbb{R}$. In this case (50) is reduced equivalently to

$$\lambda_3 f^2 c = -\lambda_2 (v - au). \tag{52}$$

Differentiating (52) with respect to v we get $\lambda_2 = 0$. So we get a contradiction.

Theorem 2 *Let M^2 be a spacelike affine factorable surface given by (9) in $\mathbb{L}^3$. Then M^2 satisfies the equation $\Delta r_i = \lambda_i r_i$, $(i = 1, 2, 3)$ if and only if one of the following statement is true:*
1) M^2 has zero mean curvature everywhere.
2) M^2 is parametrized as

$$r(u,v) = \left(u, v, \ \varepsilon \frac{\sqrt{2}}{\sqrt{|\lambda_3|}} \sqrt{\left| c + \frac{\lambda_1}{2} u^2 \right|} \right), \quad \varepsilon = \pm 1.$$

6.2 Timelike affine factorable surfaces in $\mathbb{L}^3$

Now we will investigate the timelike affine factorable surfaces of $\mathbb{L}^3$.

If we use (4), the Laplacian Δ on M^2 is given by

$$\Delta\varphi = \frac{-(G\varphi_{uu} + E\varphi_{vv} - 2F\varphi_{uv})}{W^2} + \frac{Q_2(u,v)}{W}\varphi_u + \frac{P_2(u,v)}{W}\varphi_v, \tag{53}$$

where

$$Q_2(u,v) = -2H\left(afg' + f'g\right), \quad P_2(u,v) = -2H\left(agf' + (a^2-1)fg'\right)$$

and $W^2 = F^2 - EG = 1 + f^2g'^2 - (f'g + afg')^2$.
Assume that $EG - F^2 = f'^2g^2 - f^2g'^2 - 1 < 0$, the metric of M^2 is timelike.

Applying (38) on the coordinate functions u, $v - au$ and $z(u,v) = f(u)g(v)$ of the position vector r we find

$$\begin{cases} \Delta(u) = -\frac{2H(afg'+f'g)}{W} \\ \Delta(v - au) = \frac{2H(fg')}{W} \\ \Delta(fg) = -\frac{2H}{W}. \end{cases} \tag{54}$$

By using (3) and (39) we have the following equations

$$\frac{2H(afg' + f'g)}{W} = -\lambda_1 u \tag{55}$$

$$\frac{2Hfg'}{W} = \lambda_2(v - au) \tag{56}$$

$$\frac{2H}{W} = -\lambda_3 fg. \tag{57}$$

Here the proofs are also similar.

Case 1. Let $\lambda_3 = 0$.

Then, the equation (57) gives rise to $H = 0$, which means that the surfaces are minimal. We get also, by the equations (55) and (56), $\lambda_1 = \lambda_2 = 0$.

Case 2. Let $\lambda_3 \neq 0$.

In this case we have four possibilities:

2.1) If $\lambda_1 = 0$ and $\lambda_2 \neq 0$ equations (55), (56) and (57) imply that

$$H(afg' + f'g) = 0 \tag{58}$$

$$\frac{2Hfg'}{W} = \lambda_2(v - au) \tag{59}$$

$$-\frac{2H}{W} = \lambda_3 fg. \tag{60}$$

2-1-1) If $H = 0$, then $\lambda_3 = 0$ and $\lambda_2 = 0$. So we get a contradiction.

2-1-2) If $afg' + f'g = 0$, then

$$ag' = kg, \quad f' = -kf, \;\; k \in \mathbb{R}. \tag{61}$$

Substituting (60) into (59), we get

$$-\lambda_3 f^2 gg' = \lambda_2(v - au).$$

Using equation (61) we get

$$Af^2g^2 = v - au, \;\; A = \frac{-\lambda_3 k}{a\lambda_2}. \tag{62}$$

Differentiating (62) with respect to u we get $g' = 0$, then $\lambda_2 = 0$. So we get a contradiction.

2-2) If $\lambda_1 \neq 0$ and $\lambda_2 = 0$. In this case the system (55), (56) and (57) is reduced equivalently to

$$\begin{cases} \frac{2H(afg'+f'g)}{W} = -\lambda_1 u \\ Hfg' = 0 \\ \frac{2H}{W} = -\lambda_3 fg. \end{cases} \tag{63}$$

2-2-1) If $f = 0$, then $\lambda_1 = 0$. So we get a contradiction.
2-2-2) If $g' = 0$. Then $g(v) = \alpha$, $\alpha \in \mathbb{R}$.
Therefore, (63) is equivalently reduced to

$$\frac{\alpha^2 f' f''}{(1+\alpha^2 f'^2)^2} = -\lambda_1 u \tag{64}$$

$$\frac{f''}{(1+\alpha^2 f'^2)^2} = -\lambda_3 f. \tag{65}$$

Substituting (65) into (64) gives

$$\lambda_1 u - \lambda_3 \alpha^2 f' f = 0. \tag{66}$$

By solving (66), we conclude the following

$$f(u) = \varepsilon \frac{\sqrt{2}}{\alpha\sqrt{|\lambda_3|}} \sqrt{\left| c + \frac{\lambda_1}{2} u^2 \right|}, \quad \varepsilon = \pm 1, \quad c \in \mathbb{R}$$

for nonzero constants λ_1, λ_3.
2-3) If $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$. Substituting (57) into (56), we get

$$\lambda_3 f^2 g g' = -\lambda_2 (v - au). \tag{67}$$

Take successively derivatives of (67) with respect to u and v, obtaining

$$\lambda_3 f f' (g g')' = 0.$$

2-3-1) If $ff' = 0$. Then $f(u) = c$, $c \in \mathbb{R}$. In this case (67) is reduced equivalently to

$$\lambda_3 c^2 g g' = -\lambda_2 (v - au). \tag{68}$$

Differentiating (68) with respect to u we get $\lambda_2 = 0$. So we get a contradiction.

2-3-2) If $(gg')' = 0$. Then $(gg')(v) = c$, $c \in \mathbb{R}$. In this case (67) is reduced equivalently to

$$\lambda_3 f^2 c = -\lambda_2 (v - au). \tag{69}$$

Differentiating (69) with respect to v we get $\lambda_2 = 0$. So we get a contradiction.

Theorem 3 *Let* M^2 *be a timelike affine factorable surface given by (9) in* $\mathbb{L}^3$*. Then* M^2 *satisfies the equation* $\Delta r_i = \lambda_i r_i$, $(i = 1, 2, 3)$ *if and only if one of the following statement is true:*

1) M^2 *has zero mean curvature everywhere.*

2) M^2 *is parametrized as*

$$r(u, v) = \left(u, v, \ \varepsilon \frac{\sqrt{2}}{\sqrt{|\lambda_3|}} \sqrt{\left| c + \frac{\lambda_1}{2} u^2 \right|} \right), \quad \varepsilon = \pm 1.$$

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