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Talenti's comparison theorem on Finsler manifolds with nonnegative Ricci curvature

Ágnes Mester
Babeş-Bolyai University,
Faculty of Mathematics and Computer Science,
Cluj-Napoca, Romania
email: agnes.mester@ubbcluj.ro

Abstract. We establish a Talenti-type comparison theorem for the Dirichlet problem associated with Poisson's equation on complete non-compact Finsler manifolds having nonnegative Ricci curvature and Euclidean volume growth. The proof relies on anisotropic symmetrization arguments and leverages the sharp isoperimetric inequality recently established by Manini [*Preprint, arXiv:2212.05130, 2022*]. In addition, we characterize the rigidity of the comparison principle under the additional assumption that the reversibility constant of the Finsler manifold is finite. As application, we prove a Faber-Krahn inequality for the first Dirichlet eigenvalue of the Finsler-Laplacian.

1 Introduction

Talenti's comparison theorem [23] is a fundamental result that establishes a relationship between the solutions of two elliptic boundary value problems: the Poisson equation with Dirichlet boundary condition and a so-called 'symmetrized' problem of similar kind. More precisely, given a bounded domain

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$\Omega \subset \mathbb{R}^n$ and a nonnegative function $f \in L^2(\Omega)$, one might consider the Dirichlet problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

and its 'symmetrized' counterpart

$$\begin{cases} -\Delta v = f^\star & \text{in } \Omega^\star, \\ v = 0 & \text{on } \partial\Omega^\star, \end{cases} \tag{2}$$

where $\Omega^\star$ denotes the Euclidean open ball centered at the origin and having the same Lebesgue measure as Ω, while $f^\star : \Omega^\star \to \mathbb{R}$ is the Schwarz rearrangement of f, see Kesavan [13, Chapter 1].

According to Talenti [23], if u and v are the weak solutions of the problems (1) and (2), respectively, then one has that

$$u^\star(x) \leq v(x), \quad \text{a.e. } x \in \Omega^\star,$$

where $u^\star : \Omega^\star \to \mathbb{R}$ is the Schwarz rearrangement of u.

The key ingredient of Talenti's proof is the classical technique known as Schwarz symmetrization, which turns out to be an invaluable method in addressing numerous isoperimetric and variational problems in the Euclidean space. For example, with the help of this symmetrization procedure, Talenti's comparison principle has been extended to several boundary value problems, see e.g., Alvino, Ferone and Trombetti [3], Alvino, Lions and Trombetti [2], Alvino, Nitsch and Trombetti [3], and Talenti [24]. For a comprehensive introduction to Talenti's technique and its countless applications, we refer to Kesavan [13] and references therein.

Recently, there has been an increasing endeavor to study similar comparison results on complete Riemannian manifolds having Ricci curvature bounded from below, see Chen and Li [8], Chen, Li and Wei [9], Colladay, Langford and McDonald [11], and Mondino and Vedovato [16].

In particular, in 2023 Chen and Li [8] extended Talenti's original comparison result to complete noncompact Riemannian manifolds having nonnegative Ricci curvature and Euclidean volume growth. In their proof, they applied a Schwarz-type symmetrization method 'from the manifold (M, g) to the Euclidean space $(\mathbb{R}^n, |\cdot|)$', obtaining a comparison result between the solution of the Dirichlet problem

$$\begin{cases} -\Delta_g u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3}$$

defined in (M, g), where $\Omega \subset M$ is a bounded domain, $f \in L^2(\Omega)$ is a nonnegative function and Δ_g denotes the Laplace-Beltrami operator induced by the Riemannian metric g, and the solution of the 'symmetrized' problem

$$\begin{cases} -\Delta v = f^\star & \text{in } \Omega^\star, \\ v = 0 & \text{on } \partial\Omega^\star, \end{cases} \tag{4}$$

which is defined on the Euclidean symmetric rearrangement of Ω, namely $\Omega^\star \subset \mathbb{R}^n$.

More precisely, for the given bounded domain $\Omega \subset M$ from (3), one can consider the Euclidean open ball $\Omega^\star \subset \mathbb{R}^n$, which is centered at the origin and $\mathrm{Vol}_g(\Omega) = \mathsf{AVR}_g \mathrm{Vol}_e(\Omega^\star)$. Here, Vol_g and Vol_e stand for the Riemannian volume induced by the metric g and the canonical Euclidean volume, while AVR_g denotes the positive asymptotic volume ratio of (M, g). Furthermore, $f^\star : \Omega^\star \mapsto [0, \infty)$ stands for the Euclidean rearrangement function of f.

Then, due to Chen and Li [8, Theorem 1.1], if u and v are the weak solutions of problems (3) and (4), respectively, then $u^\star(x) \leq v(x)$, a.e. $x \in \Omega^\star$, where $u^\star : \Omega^\star \mapsto \mathbb{R}$ is the Euclidean rearrangement of u. Moreover, equality holds for a.e. $x \in \Omega^\star$ if and only if (M, g) is isometric to the Euclidean space $\mathbb{R}^n$ endowed with its canonical metric and Ω is isoperimetric to the Euclidean ball $\Omega^\star$.

In these types of symmetrization results, a crucial element lies in the application of the sharp isoperimetric inequality. This inequality was recently established for Riemannian manifolds having nonnegative Ricci curvature and Euclidean volume growth by Brendle [6] and, alternatively, by Balogh and Kristály [4], facilitating the application of symmetrization arguments on these curved spaces.

The sharp isoperimetric inequality was newly extended to potentially nonreversible Finsler manifolds with nonnegative n-Ricci curvature and Euclidean volume growth by Manini [15], who also characterized the inequality's rigidity property. This breakthrough enables the utilization of rearrangement arguments within the broader framework of Finsler geometry. However, in order to fully integrate the general Finslerian context, the rearrangement needs to be performed 'from the Finsler manifold to a Minkowski normed space', laying the foundation for a so-called anisotropic (or convex) symmetrization.

In light of these results, the main objective of the present paper is to extend the Talenti-type comparison result of Chen and Li [8] to complete Finsler manifolds having nonnegative n-Ricci curvature and Euclidean volume growth. This is achieved by the adaptation of the usual Euclidean rearrangement technique to the Finslerian context and the application of the sharp isoperimetric inequality available on Finsler manifolds. Moreover, we also prove a rigidity property of the comparison principle in the spirit of Manini [15].

2 Main results

To present our findings, let (M, F) be a noncompact, complete $n(\geq 2)$-dimensional Finsler manifold with $\mathsf{Ric}_n \geq 0$, equipped with the induced Finsler metric $d_F : M \times M \to \mathbb{R}$ and the Busemann-Hausdorff volume form $d\nu_F$. Let $r_F \in [1, \infty]$ denote the reversibility constant of (M, F), see Section 3.

The asymptotic volume ratio of (M, F) is expressed as

$$\mathsf{AVR}_F = \lim_{r \to \infty} \frac{\mathrm{Vol}_F(B_F(x, r))}{\omega_n r^n},$$

where $B_F(x, r) = \{z \in M : d_F(x, z) < r\}$ denotes the forward geodesic ball centered at a fixed $x \in M$ with radius $r > 0$, $\omega_n = \pi^{\frac{n}{2}}/\Gamma(1 + \frac{n}{2})$ denotes the volume of the n-dimensional Euclidean open unit ball, and $\mathrm{Vol}_F(S) = \int_S d\nu_F$, for any measurable set $S \subset M$. Since $\mathsf{Ric}_n \geq 0$, due to the Bishop-Gromov volume comparison theorem (see Shen [22, Theorem 1.1]), we have that $\mathsf{AVR}_F \in [0, 1]$.

We suppose that (M, F) exhibits Euclidean volume growth, i.e., $\mathsf{AVR}_F > 0$.

On a bounded domain $\Omega \subset M$, we consider the Dirichlet problem

$$\begin{cases} -\Delta_F u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{$\mathcal{P}$}$$

where Δ_F is the Finsler-Laplace operator defined on (M, F) and $f \in L^2(\Omega)$ is a nonnegative function.

Now let $(\mathbb{R}^n, H)$ be a reversible Finsler manifold equipped with the canonical volume form $d\nu_H$, such that H is a normalized Minkowski norm, i.e., the set

$$W_H(1) := \{x \in \mathbb{R}^n : H(x) < 1\}$$

has measure $\mathrm{Vol}_H(W_H(1)) = \mathrm{Vol}_e(W_H(1)) = \omega_n$.

We consider the anisotropic rearrangement of $\Omega \subset M$ with respect to (w.r.t.) the norm H, which is defined as a Wulff-shape

$$\Omega_H^\star = \{x \in \mathbb{R}^n : H(x) < R\}$$

for some $R > 0$ such that $\mathrm{Vol}_F(\Omega) = \mathsf{AVR}_F \mathrm{Vol}_H(\Omega_H^\star)$.

Our main result is a Talenti-type comparison principle concerning the solution of problem ($\mathcal{P}$) and the Dirichlet problem

$$\begin{cases} -\Delta_{H^*} v = f_H^\star & \text{in } \Omega_H^\star, \\ v = 0 & \text{on } \partial\Omega_H^\star, \end{cases} \tag{$\mathcal{P}^\star$}$$

where $H^* : \mathbb{R}^n \to \mathbb{R}$,

$$H^*(x) = \sup_{\xi \in \mathbb{R}^n \setminus \{0\}} \frac{\langle \xi, x \rangle}{H(\xi)}$$

is the dual norm (i.e., the polar transform) of H, Δ_{H^*} is the Finsler-Laplace operator associated to H^*, and $f_H^\star : \Omega_H^\star \to [0, \infty)$ is the anisotropic rearrangement of f w.r.t. the norm H. Namely,

$$f_H^\star(x) = w\left(\mathsf{AVR}_F \omega_n H(x)^n\right)$$

for some nonincreasing function $w : [0, \mathrm{Vol}_F(\Omega)] \to [0, \infty)$, such that for every $t \geq 0$,

$$\mathrm{Vol}_F\big(\{x \in \Omega : f(x) > t\}\big) = \mathsf{AVR}_F \cdot \mathrm{Vol}_H\big(\{x \in \Omega_H^\star : f_H^\star(x) > t\}\big)$$

holds true, see Section 4.

Specifically, we have the following theorem.

Theorem 1 *Let* (M, F) *be a noncompact, complete* n*-dimensional Finsler manifold with* $\mathsf{Ric}_n \geq 0$, $n \geq 2$ *and* $\mathsf{AVR}_F > 0$. *Assume that* $\Omega \subset M$ *is a bounded domain and* $f \in L^2(\Omega)$ *is a nonnegative function. Let* $H : \mathbb{R}^n \to [0, \infty)$ *be an absolutely homogeneous, normalized Minkowski norm, and* $H^* : \mathbb{R}^n \to [0, \infty)$ *be its dual norm. Finally, let* $\Omega_H^\star$ *and* $f_H^\star$ *be the anisotropic rearrangement w.r.t.* H *of the set* Ω *and the function* f. *If* $u : \Omega \to \mathbb{R}$ *and* $v : \Omega_H^\star \to \mathbb{R}$ *are the weak solutions to problems* ($\mathcal{P}$) *and* ($\mathcal{P}^\star$), *respectively, then we have that*

$$u_H^\star(x) \leq v(x), \qquad \text{a.e. } x \in \Omega_H^\star, \tag{5}$$

where $u_H^\star : \Omega_H^\star \to \mathbb{R}$ *is the anisotropic rearrangement of* u *w.r.t. the norm* H.

If, in addition, we suppose that $r_F < \infty$ *and for all* $x_1, x_2 \notin \partial M$ *and for all geodesics* $\gamma : [0, 1] \to M$ *with* $\gamma(0) = x_1$ *and* $\gamma(1) = x_2$, *it holds that* $\gamma(t) \notin \partial M$, *for all* $t \in [0, 1]$, *then we have the following rigidity property.*

If equality holds in (5), then there exists (a unique) $x_0 \in M$ *such that, up to a negligible set,* $\Omega = B_F(x_0, r)$ *with* $r = \left(\frac{\mathrm{Vol}_F(\Omega)}{\mathsf{AVR}_F \omega_n}\right)^{\frac{1}{n}}$.

Moreover, the Busemann-Hausdorff measure $\mathrm{d}\nu_F$ *has the following representation:*

$$\mathrm{d}\nu_F = \int_{\partial B_F(x_0,r)} \mathfrak{m}_\alpha \mathfrak{q}(\mathrm{d}\alpha), \quad \mathfrak{q} \in \mathcal{P}\left(\partial B_F(x_0, r)\right), \quad \mathfrak{m}_\alpha \in \mathcal{M}_+(M),$$

where $\mathfrak{m}_\alpha$ *is concentrated on the geodesic ray from* x_0 *through* α*, and* $\mathfrak{m}_\alpha$ *can be identified with* $n\omega_n \mathsf{AVR}_F t^{n-1}\mathcal{L}^1 \llcorner_{[0,\infty)}$.

The proof relies on an anisotropic rearrangement argument similar to the one outlined in Kristály, Mester and Mezei [14], along with the sharp and rigid isoperimetric inequality due to Manini [15]. Although the general strategy of the Schwarz-type symmetrization is well-established (see Talenti [23] or Chen and Li [8]), a meticulous adaptation is necessary in order to fully characterize the Finslerian setting.

As a consequence of Theorem 1, we derive the following Faber-Krahn-type inequality concerning the first Dirichlet eigenvalue of the Finsler-Laplacian Δ_F. For similar eigenvalue comparison results, refer to the works of Ge and Shen [12] and Yin and He [25]. Here, we introduce an alternative approach by using anisotropic rearrangement and the Talenti comparison result. In addition, we also provide a characterization of the equality case, which follows directly from Theorem 1.

Theorem 2 *Let* (M, F) *be a noncompact, complete* n*-dimensional Finsler manifold with* $\mathsf{Ric}_n \geq 0$, $n \geq 2$ *and* $\mathsf{AVR}_F > 0$*. Assume that* $\Omega \subset M$ *is a bounded domain. Let* $H : \mathbb{R}^n \to [0, \infty)$ *be an absolutely homogeneous, normalized Minkowski norm,* $H^* : \mathbb{R}^n \to [0, \infty)$ *its dual norm, and* $\Omega_H^\star \subset \mathbb{R}^n$ *the anisotropic rearrangement of* Ω *w.r.t. the norm* H.

Let us consider the eigenvalue problem

$$\begin{cases} -\Delta_F u = \lambda_1(\Omega) u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{$\mathcal{EP}$}$$

where $\lambda_1(\Omega)$ *denotes the first Dirichlet eigenvalue of the Finsler-Laplacian* Δ_F.

Then, we have that

$$\lambda_1\left(\Omega_H^\star\right) \leq \lambda_1(\Omega), \tag{6}$$

where $\lambda_1\left(\Omega_H^\star\right)$ *is the first eigenvalue associated with the eigenvalue problem*

$$\begin{cases} -\Delta_{H^*} v = \lambda_1(\Omega_H^\star) v, & \text{in } \Omega_H^\star, \\ v = 0, & \text{on } \partial\Omega_H^\star. \end{cases} \tag{$\mathcal{EP}^\star$}$$

If, in addition, we suppose that $r_F < \infty$ *and for all* $x_1, x_2 \notin \partial M$ *and for all geodesics* $\gamma : [0,1] \to M$ *with* $\gamma(0) = x_1$ *and* $\gamma(1) = x_2$*, it holds that* $\gamma(t) \notin \partial M$*, for all* $t \in [0,1]$*, then we have the following rigidity property.*

If equality holds in (6), then there exists (a unique) $x_0 \in M$ *such that, up to a negligible set,* $\Omega = B_F(x_0, r)$ *with* $r = \left(\frac{\mathrm{Vol}_F(\Omega)}{\mathsf{AVR}_F \omega_n}\right)^{\frac{1}{n}}$. *Moreover, the Busemann-Hausdorff measure* $d\nu_F$ *has the following representation:*

$$d\nu_F = \int_{\partial B_F(x_0,r)} \mathfrak{m}_\alpha \mathfrak{q}(d\alpha), \quad \mathfrak{q} \in \mathcal{P}\left(\partial B_F(x_0, r)\right), \quad \mathfrak{m}_\alpha \in \mathcal{M}_+(M),$$

where $\mathfrak{m}_\alpha$ *is concentrated on the geodesic ray from* x_0 *through* α*, and* $\mathfrak{m}_\alpha$ *can be identified with* $n\omega_n \mathsf{AVR}_F t^{n-1} \mathcal{L}^1 \llcorner_{[0,\infty)}$.

The paper is organized as follows. In Section 3 we briefly present the fundamental notions of Finsler geometry that are used throughout the paper. Section 4 recalls the sharp isoperimetric inequality due to Manini [15], then presents the anisotropic rearrangement method applied in our arguments. Finally, in Section 5 we present the proof of Theorem 1 and 2.

3 Preliminaries on Finsler geometry

This section summarizes the fundamental notions of Finsler geometry necessary for our further developments. For a comprehensive presentation of the subject, see Bao, Chern and Shen [5], Ohta and Sturm [18] and Shen [21].

Let M be a connected n-dimensional differentiable manifold and $TM = \cup_{x \in M}\left\{(x, y) : y \in T_x M\right\}$ the tangent bundle of M.

The pair (M, F) is called a Finsler manifold if $F : TM \to [0, \infty)$ is a continuous function such that

(i) F is C^∞ on $TM \setminus \{0\}$;

(ii) $F(x, \lambda y) = \lambda F(x, y)$ for all $\lambda \geq 0$ and all $(x, y) \in TM$;

(iii) the $n \times n$ Hessian matrix $\left(g_{ij}(x, y)\right) := \left(\frac{1}{2} \frac{\partial^2}{\partial y^i \partial y^j} F^2(x, y)\right)$ is positive definite for all $(x, y) \in TM \setminus \{0\}$.

Note that in general, $F(x, y) \neq F(x, -y)$. If (M, F) is a Finsler manifold such that $F(x, \lambda y) = |\lambda| F(x, y)$, for every $\lambda \in \mathbb{R}$ and $(x, y) \in TM$, we say that the Finsler manifold is reversible. Otherwise, (M, F) is called nonreversible.

The reversibility constant of (M, F) is defined by the number

$$r_F = \sup_{x\in M} \sup_{y\in T_xM\setminus\{0\}} \frac{F(x,y)}{F(x,-y)} \in [1,\infty],$$

measuring how much the manifold deviates from being reversible, see Rademacher [19]. Specifically, $r_F = 1$ if and only if (M, F) is a reversible Finsler manifold.

A smooth curve $\gamma : [a, b] \to M$ is called a geodesic if its velocity field $\dot\gamma$ is parallel along the curve, i.e., $D_{\dot\gamma}\dot\gamma = 0$, where D denotes the covariant derivative induced by the Chern connection, see Bao, Chern and Shen [5, Chapter 2]. (M, F) is said to be complete if every geodesic $\gamma : [a, b] \to M$ can be extended to a geodesic defined on $\mathbb{R}$.

The Finslerian distance function $d_F : M \times M \to [0,\infty)$ is defined by

$$d_F(x_1, x_2) = \inf_{\gamma\in\Gamma(x_1,x_2)} \int_a^b F\big(\gamma(t), \dot\gamma(t)\big)dt,$$

where $\Gamma(x_1, x_2)$ denotes the set of all piecewise differentiable curves $\gamma : [a, b] \to M$ such that $\gamma(a) = x_1$ and $\gamma(b) = x_2$. Clearly, $d_F(x_1, x_2) = 0$ if and only if $x_1 = x_2$, and d_F verifies the triangle inequality. However, in general, d_F is not symmetric. In fact, we have that d_F is symmetric if and only if (M, F) is a reversible Finsler manifold.

For a point $x \in M$ and a number $r > 0$, the forward open geodesic ball with center x and radius r is defined as

$$B_F(x, r) = \{z \in M : d_F(x, z) < r\}.$$

For a fixed point $x \in M$ let $y, v \in T_xM$ be two linearly independent tangent vectors. The flag curvature is defined as

$$K^y(y, v) = \frac{g_y(R(y, v)v, y)}{g_y(y, y)g_y(v, v) - g_y(y, v)^2},$$

where g is the fundamental tensor induced by the Hessian matrices (g_{ij}) and R is the Chern curvature tensor, see Bao, Chern and Shen [5, Chapter 3].

The Ricci curvature at the point $x \in M$ and in the direction $y \in T_xM$ is defined by

$$\mathrm{Ric}_x(y) = F^2(x, y) \sum_{i=1}^{n-1} K^y(y, e_i),$$

where $\{e_1, \cdots, e_{n-1}, \frac{1}{F(x,y)} y\}$ is an orthonormal basis of $T_x M$ with respect to g.

The density function $\sigma_F : M \to [0, \infty)$ is defined by

$$\sigma_F(x) = \frac{\omega_n}{\mathrm{Vol}_e(B(x,1))},$$

where $\omega_n = \pi^{\frac{n}{2}} / \Gamma(1 + \frac{n}{2})$ is the volume of the n-dimensional Euclidean open unit ball, Vol_e denotes in the sequel the canonical Euclidean volume, and

$$B(x,1) = \left\{ (y^i) \in \mathbb{R}^n : \ F\left(x, \sum_{i=1}^{n} y^i \frac{\partial}{\partial x^i} \right) < 1 \right\} \subset \mathbb{R}^n.$$

The canonical Busemann-Hausdorff volume form on (M, F) is defined as

$$dv_F(x) = \sigma_F(x) dx^1 \wedge \cdots \wedge dx^n,$$

see Shen [21, Section 2.2]. Note that in the following we may omit the parameter x for the sake of brevity. The Finslerian volume of a measurable set $\Omega \subset M$ is given by $\mathrm{Vol}_F(\Omega) = \int_\Omega dv_F$.

The mean distortion of (M, F) is defined by $\mu : TM \setminus \{0\} \to (0, \infty)$,

$$\mu(x,y) = \frac{\sqrt{\det\left[g_{ij}(x,y) \right]}}{\sigma_F(x)},$$

while the mean covariation is given by $\mathbf{S} : TM \setminus \{0\} \to \mathbb{R}$,

$$\mathbf{S}(x,y) = \frac{d}{dt} \Big(\ln\, \mu\big(\gamma(t), \dot{\gamma}(t)\big) \Big) \Big|_{t=0},$$

where γ is the geodesic with $\gamma(0) = x$ and $\dot{\gamma}(0) = y$.

We say that (M, F) has nonnegative n-Ricci curvature, denoted by $\mathsf{Ric}_n \geq 0$, if $\mathrm{Ric}_x(y) \geq 0$ for all $(x, y) \in TM$ and the mean covariation $\mathbf{S}$ is identically zero. Examples of Finsler manifolds with vanishing mean covariation include the so-called Berwald spaces, which, in particular, contain both Riemannian manifolds and Minkowski spaces, see Shen [22].

As previously introduced, the asymptotic volume ratio of (M, F) is defined by

$$\mathsf{AVR}_F = \lim_{r \to \infty} \frac{\mathrm{Vol}_F(B_F(x,r))}{\omega_n r^n},$$

where $x \in M$ is arbitrarily fixed. Note that AVR_F is well-defined, being independent of the choice of the point $x \in M$.

On the one hand, an n-dimensional Finsler manifold (M, F) equipped with the canonical volume form dv_F satisfies the condition that for every $x \in M$,

$$\lim_{r \searrow 0} \frac{\mathrm{Vol}_F(B_F(x,r))}{\omega_n r^n} = 1.$$

On the other hand, if (M, F) is a complete Finsler manifold having $\mathsf{Ric}_n \geq 0$, the Bishop-Gromov volume comparison principle asserts that the function $r \mapsto \frac{\mathrm{Vol}_F(B_F(x,r))}{r^n}$ is nonincreasing on $(0, \infty)$, see Shen [22, Theorem 1.1]. Consequently, it follows that if $\mathsf{Ric}_n \geq 0$, then $\mathsf{AVR}_F \in [0, 1]$.

The polar transform $F^* : T^*M \to [0, \infty)$ is defined as the dual metric of F, namely

$$F^*(x, \alpha) = \sup_{y \in T_xM \setminus \{0\}} \frac{\alpha(y)}{F(x, y)},$$

where $T^*M = \cup_{x \in M} \{(x, \alpha) : \alpha \in T_x^*M\}$ is the cotangent bundle of M and T_x^*M is the dual space of T_xM.

The Legendre transform is defined by $J^* : T^*M \to TM$,

$$J^*(x, \alpha) = \sum_{i=1}^{n} \frac{\partial}{\partial \alpha^i} \left(\frac{1}{2} F^{*2}(x, \alpha) \right) \frac{\partial}{\partial x^i},$$

for every $\alpha = \sum_{i=1}^{n} \alpha^i dx^i \in T_x^*M$. Note that if $J^*(x, \alpha) = (x, y)$, then

$$F(x, y) = F^*(x, \alpha) \quad \text{and} \quad \alpha(y) = F^*(x, \alpha) F(x, y).$$

Let $u : M \to \mathbb{R}$ be a differentiable function in the distributional sense. The gradient of u is defined as $\nabla_F u(x) = J^*(x, Du(x))$, where $Du(x) \in T_x^*M$ denotes the (distributional) derivative of u at the point $x \in M$.

Using the properties of the Legendre transform, it follows that

$$F(x, \nabla_F u(x)) = F^*(x, Du(x)) \quad \text{and} \quad Du(x)\big(\nabla_F u(x)\big) = F^*(x, Du(x))^2.$$

In local coordinates, one has that

$$Du(x) = \sum_{i=1}^{n} \frac{\partial u}{\partial x^i}(x) dx^i \quad \text{and} \quad \nabla_F u(x) = \sum_{i,j=1}^{n} g_{ij}^*(x, Du(x)) \frac{\partial u}{\partial x^i}(x) \frac{\partial}{\partial x^j},$$

where (g_{ij}^*) is the Hessian matrix $\left(g_{ij}^*(x, \alpha)\right) = \left(\frac{1}{2} \frac{\partial^2}{\partial \alpha^i \partial \alpha^j} F^{*2}(x, \alpha)\right)$, see Ohta and Sturm [18, Lemma 1.1]. In general, the gradient operator ∇_F is not linear.

Given a vector field V on M, the divergence of V is defined in local coordinates as $\mathrm{div}V(x) = \frac{1}{\sigma_F(x)} \sum_{i=1}^{n} \frac{\partial}{\partial x^i}(\sigma_F(x)V^i(x))$.

The Finsler-Laplace operator is defined by

$$\Delta_F u(x) = \mathrm{div}\left(\nabla_F u(x)\right).$$

Note that Δ_F is usually not linear. However, in the particular case when $(M, F) = (M, g)$ is a Riemannian manifold, Δ_F coincides with the usual Laplace-Beltrami operator Δ_g.

The divergence theorem implies that

$$\int_M \varphi(x)\Delta_F u(x)\, dv_F = -\int_M D\varphi(x)\big(\nabla_F u(x)\big)\, dv_F, \tag{7}$$

for all $\varphi \in C_0^\infty(M)$, see Ohta and Sturm [18].

In the specific case when $(\mathbb{R}^n, H)$ is a reversible Finsler manifold, then H is actually a smooth, absolutely homogeneous norm on $\mathbb{R}^n$. Consequently, the polar transform of H is in fact its dual norm $H^* : \mathbb{R}^n \to [0, \infty)$,

$$H^*(x) = \sup_{\xi \in \mathbb{R}^n \setminus \{0\}} \frac{\langle \xi, x\rangle}{H(\xi)}.$$

In this case, the Finsler-Laplace operator Δ_{H^*} associated with the norm H^* is given by

$$\Delta_{H^*} u = \mathrm{div}\left(H^*(\nabla u)\nabla_\xi H^*(\nabla u)\right),$$

where ∇_ξ stands for the gradient operator with respect to the variables $\xi \in \mathbb{R}^n$.

Due to Cianchi and Salani [10, Lemma 3.1], we have the following relation between the norms H and H^*:

$$H^*\left(\nabla_\xi H(\xi)\right) = 1, \quad \text{for all } \xi \in \mathbb{R}^n \setminus \{0\}. \tag{8}$$

Finally, let $\Omega \subset M$ be an open subset. The Sobolev space on Ω associated with the Finsler structure F and the Busemann-Hausdorff measure dv_F is defined by

$$W_F^{1,2}(\Omega) = \left\{ u \in W_{\mathrm{loc}}^{1,2}(\Omega) : \int_\Omega F^*(x, Du(x))^2\, dv_F < +\infty \right\},$$

while $W_{0,F}^{1,2}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{W_F^{1,2}(\Omega)} = \left(\int_\Omega |u(x)|^2\, dv_F + \int_\Omega F^*(x, Du(x))^2\, dv_F \right)^{\frac{1}{2}}.$$

4 Anisotropic symmetrization

In the following, let (M, F) be a noncompact, complete $n(\geq 2)$-dimensional Finsler manifold having $\mathsf{Ric}_n \geq 0$, equipped with the induced Finsler metric d_F and the Busemann-Hausdorff volume form dv_F. In this case, (M, d_F, dv_F) is a metric measure space which satisfies the $\mathsf{CD}(0, n)$ condition, see Ohta [17].

We further suppose that (M, F) has Euclidean volume growth, i.e., $\mathsf{AVR}_F > 0$. For this geometric setting, the sharp isoperimetric inequality has been recently established by Manini [15, Theorem 1.3]. In particular, for every bounded open set $\Omega \subset M$, one has the following isoperimetric inequality:

$$\mathcal{P}_F(\partial\Omega) \geq n\omega_n^{\frac{1}{n}} \mathsf{AVR}_F^{\frac{1}{n}} \, \mathrm{Vol}_F(\Omega)^{\frac{n-1}{n}}. \tag{9}$$

Here $\mathcal{P}_F(\partial\Omega)$ denotes the anisotropic perimeter of Ω, defined as $\mathcal{P}_F(\partial\Omega) = \int_{\partial\Omega} d\sigma_F$, where $d\sigma_F$ is the $(n-1)$-dimensional Lebesgue measure induced by dv_F. It is noteworthy that inequality (9) holds true in the general Finslerian setting, unrestricted by any reversibility assumption regarding the Finsler structure F.

Moreover, due to Manini's rigidity result [15, Theorem 1.4], the equality in (9) can be characterized by introducing the additional assumption that the reversibility constant r_F of (M, F) is finite. More precisely, one has the following theorem.

Theorem 3 ([15, Theorem 1.4]) *Let $(M, F, \mathfrak{m})$ be a Finsler manifold (possibly with boundary) satisfying the $\mathsf{CD}(0, n)$ condition for some $n > 1$, such that $\mathsf{AVR}_F > 0$, $r_F < \infty$ and all closed forward geodesic balls are compact. Assume that for all $x_1, x_2 \notin \partial M$ and for all geodesics $\gamma : [0, 1] \to M$ such that $\gamma(0) = x_1$ and $\gamma(1) = x_2$, it holds that $\gamma(t) \notin \partial M$, for every $t \in [0, 1]$. Let $\Omega \subset X$ be a bounded Borel set that saturates the isoperimetric inequality*

$$\mathcal{P}(\partial\Omega) \geq n\omega_n^{\frac{1}{n}} \mathsf{AVR}_F^{\frac{1}{n}} \, \mathfrak{m}(\Omega)^{\frac{n-1}{n}}.$$

Then there exists (a unique) $x \in M$ such that, up to a negligible set,

$$\Omega = B_F(x, r) \quad \text{with} \quad r = \left(\frac{\mathfrak{m}(\Omega)}{\mathsf{AVR}_F \omega_n} \right)^{\frac{1}{n}}.$$

Moreover, the measure $\mathfrak{m}$ has the following representation:

$$\mathfrak{m} = \int_{\partial B_F(x,r)} \mathfrak{m}_\alpha \mathfrak{q}(d\alpha), \quad \mathfrak{q} \in \mathcal{P}\left(\partial B_F(x, r)\right), \quad \mathfrak{m}_\alpha \in \mathcal{M}_+(M),$$

with $\mathfrak{m}_\alpha$ *concentrated on the geodesic ray from* x *through* α*, and* $\mathfrak{m}_\alpha$ *can be identified with* $n\omega_n \mathsf{AVR}_F t^{n-1}\mathcal{L}^1 \llcorner_{[0,\infty)}$.

In particular, if the measure chosen on (M,F) is the Busemann-Hausdorff measure $d\nu_F$, then, by Theorem 3, an extremizer set of the isoperimetric inequality (9) satisfies that

$$\Omega = B_F(x,r), \quad \text{where} \quad r = \left(\frac{\mathrm{Vol}_F(\Omega)}{\mathsf{AVR}_F\omega_n}\right)^{\frac{1}{n}}. \tag{10}$$

In order to leverage Manini's results, we adapt the classical Schwarz symmetrization technique (see e.g., Kesavan [13]) to accommodate the Finslerian context. For similar concepts of anisotropic (or convex) rearrangements, we refer to Alvino, Ferone, Trombetti and Lions [1], Kristály, Mester and Mezei [14] and Schaftingen [20].

Let us consider a reversible Finsler manifold $(\mathbb{R}^n, H)$ endowed with the canonical volume form $d\nu_H$. In addition, we assume, without loss of generality, that the set

$$\mathcal{W}_H(1) \coloneqq \{x \in \mathbb{R}^n : H(x) < 1\}$$

has measure $\mathrm{Vol}_e(\mathcal{W}_H(1)) = \omega_n$, i.e., H is a normalized Minkowski norm. Consequently, we have that the density function of $(\mathbb{R}^n, H)$, $\sigma_H = 1$ is constant, which yields $d\nu_H(x) = dx$. Accordingly, it turns out that $\mathrm{Vol}_H(\mathcal{W}_H(1)) = \mathrm{Vol}_e(\mathcal{W}_H(1)) = \omega_n$ and $\mathsf{AVR}_H = 1$.

Our goal is to apply a so-called anisotropic (or convex) rearrangement technique 'from the Finsler manifold (M,F) to the Minkowski normed space $(\mathbb{R}^n, H)$'.

In the following, let $\Omega \subset M$ be a bounded domain.

The anisotropic rearrangement of Ω w.r.t. the normalized Minkowski norm H is a Wulff-shape

$$\Omega_H^\star = \{z \in \mathbb{R}^n : H(z) < R\} \eqqcolon \mathcal{W}_H(R),$$

where $R > 0$ is determined such that

$$\mathrm{Vol}_F(\Omega) = \mathsf{AVR}_F \mathrm{Vol}_H(\Omega_H^\star) = \mathsf{AVR}_F \mathrm{Vol}_e(\Omega_H^\star).$$

Remark 1 *Clearly, in the particular case when* H *is the standard Euclidean norm* $|\cdot|$*, the anisotropic rearrangement of* Ω *w.r.t.* $|\cdot|$ *is precisely the usual Euclidean symmetric rearrangement, which is an* n*-dimensional open Euclidean* *ball centered at the origin and having radius*

$$\mathsf{R} = \left(\frac{\mathrm{Vol}_F(\Omega)}{\mathsf{AVR}_F\omega_n}\right)^{\frac{1}{n}}.$$

In this case, if $\Omega \subset M$ *is an extremizer of the isoperimetric inequality* (9)*, it turns out that the metric balls* Ω *and* $\Omega^\star_{|\cdot|}$ *have equal radii, see* (10)*.*

Now let $u : \Omega \to \mathbb{R}$ be a nonnegative, measurable function.

The distribution function of $u : \Omega \to [0,\infty)$ is defined by the function $\mu_u : [0,\infty) \to [0, \mathrm{Vol}_F(\Omega)]$,

$$\mu_u(t) = \mathrm{Vol}_F\big(\{x \in \Omega : u(x) > t\}\big).$$

It can be seen that μ_u is decreasing and $\mu_u(t) = 0$, for all $t \geq \operatorname{ess\,sup}_\Omega u$.

The decreasing rearrangement of u is defined by $u^\sharp : [0, \mathrm{Vol}_F(\Omega)] \to [0,\infty)$,

$$u^\sharp(s) = \begin{cases} \operatorname{ess\,sup}_\Omega u, & \text{if } s = 0, \\ \inf\{t : \mu_u(t) \leq s\}, & \text{if } 0 < s \leq \mathrm{Vol}_F(\Omega). \end{cases}$$

Finally, the anisotropic rearrangement of u w.r.t. the normalized Minkowski norm H is given by $u^\star_H : \Omega^\star_H \to [0,\infty)$,

$$u^\star_H(x) = u^\sharp(\mathsf{AVR}_F\omega_n H(x)^n). \tag{11}$$

By the previous definition, it follows that for every $t \geq 0$, one has that

$$\mathrm{Vol}_F\big(\{x \in \Omega : u(x) > t\}\big) = \mathsf{AVR}_F \cdot \mathrm{Vol}_H\big(\{x \in \Omega^\star_H : u^\star_H(x) > t\}\big),$$

which implies that $\mu_u(t) = \mathsf{AVR}_F \cdot \mu_{u^\star_H}(t)$, for all $t \geq 0$.

By the layer cake representation, it follows that

$$\|u\|_{L^p(\Omega)} = \|u^\sharp\|_{L^p(0,\mathrm{Vol}_F(\Omega))} = \mathsf{AVR}_F^{\frac{1}{p}}\|u^\star_H\|_{L^p(\Omega^\star_H)},$$

for every $p \in [1,\infty]$.

Similarly to Kesavan [13, Theorem 1.2.2], one can prove the following Hardy-Littlewood-Pólya-type inequality: if $u, g \in L^2(\Omega)$ are nonnegative functions, then

$$\int_\Omega u(x)g(x)dv_F \leq \int_0^{\mathrm{Vol}_F(\Omega)} u^\sharp(s)g^\sharp(s)ds = \int_{\Omega^\star_H} u^\star_H(x)g^\star_H(x)dv_H.$$

In particular, one has that

$$\int_{\{x\in\Omega:u(x)>t\}} g(x)dv_F \leq \int_0^{\mu_u(t)} g^\sharp(s)ds, \tag{12}$$

for any $t \in [0,\infty)$ fixed.

Remark 2 *For a fixed function $u : \Omega \to [0,\infty)$, one can define multiple anisotropic rearrangements of u w.r.t. various Minkowski norms. However, by definition, these all will be equimeasurable in the following sense: if $u^\star_{H_1}$ and $u^\star_{H_2}$ are the anisotropic rearrangements of u w.r.t. two different absolutely homogeneous, normalized Minkowski norms H_1 and H_2, then for any $p \in [1,\infty]$,*

$$\|u^\star_{H_1}\|_{L^p(\Omega^\star_{H_1})} = \|u^\star_{H_2}\|_{L^p(\Omega^\star_{H_2})}.$$

In the particular case when $(M,F) = (M,g)$ is a Riemannian manifold and $H(x) = |x|$ is the standard Euclidean norm, $u^\star_{|\cdot|}$ turns out to be the classical radially symmetric rearrangement of u. This type of rearrangement is employed by Chen and Li [8] in their comparison result on Riemannian manifolds. Therefore, our findings effectively extend the results of Chen and Li [8] to the broader, Finslerian framework.

In the Finslerian case, however, it is indicated to substitute the classical Euclidean symmetrization with the anisotropic rearrangement (11). This choice is motivated by the fact that the minimizers of the isoperimetric inequality (9), when analyzed within a Minkowski space $(\mathbb{R}^n, H)$, correspond to Wulff-shapes associated with H (up to translations), see Cabré, Ros-Oton, and Serra [7, Theorem 1.2] *or Manini [15,* Theorem 1.5]*.*

5 Proof of Theorems 1&2

This section contains the proof of the Talenti comparison principle from Theorem 1 and the Faber-Krahn inequality from Theorem 2. The key ingredients are the anisotropic rearrangement technique presented in Section 4 and the sharp isoperimetric inequality (9). For the characterization of the equality case, we use the rigidity result from Theorem 3.

Proof of Theorem 1.

Step 1. We start by studying the solution of the Dirichlet problem defined on (M, F), namely,

$$\begin{cases} -\Delta_F u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{$\mathcal{P}$}$$

Let $u : \Omega \subset M \to \mathbb{R}$ be the weak solution of $(\mathcal{P})$. Since $f \in L^2(\Omega)$ is a nonnegative function, by the maximum principle, it follows that u is nonnegative on Ω.

We consider the distribution function $\mu_u : [0, \infty) \to [0, \mathrm{Vol}_F(\Omega)]$ of u, and for any $0 \leq t \leq \operatorname{ess\,sup}_\Omega u$ fixed, we define the level sets

$$\Omega_t = \{x \in \Omega : u(x) > t\} \quad \text{and} \quad \Gamma_t = \{x \in \Omega : u(x) = t\}.$$

Note that by Sard's theorem, we have that $\Gamma_t = \partial\Omega_t$.

Then, we have that $\mu_u(t) = \mathrm{Vol}_F(\Omega_t)$ and $\mathcal{P}_F(\Gamma_t) = -\mu_u'(t)$, see Section 4.

Applying the co-area formula given by Shen [21, Theorem 3.3.1], we can prove that

$$\int_{\Gamma_t} F^*(x, Du(x)) d\sigma_F = -\frac{d}{dt} \int_{\Omega_t} F^*(x, Du(x))^2 dv_F \tag{13}$$

and

$$\int_{\Gamma_t} \frac{1}{F^*(x, Du(x))} d\sigma_F = -\frac{d}{dt} \int_{\Omega_t} dv_F = -\mu_u'(t) \ . \tag{14}$$

Since u is the weak solution of $(\mathcal{P})$, by (7) we have that

$$\int_\Omega D\varphi(x)\big(\nabla_F u(x)\big)\, dv_F = \int_\Omega f(x)\varphi(x)\, dv_F, \tag{15}$$

for every test function $\varphi \in W^{1,2}_{0,F}(\Omega)$.

For a fixed $t > 0$ and $h > 0$, we define the function

$$\varphi_h(x) = \begin{cases} 0, & \text{if} \quad 0 \leq u(x) \leq t, \\ \frac{u(x)-t}{h}, & \text{if} \quad t < u(x) \leq t + h, \\ 1, & \text{if} \quad u(x) > t + h. \end{cases}$$

By choosing φ_h as test funcion in (15) and taking the limit $h \to 0$, we obtain

$$-\frac{d}{dt} \int_{\Omega_t} F^*(x, Du(x))^2 dv_F = \int_{\Omega_t} f(x) dv_F. \tag{16}$$

Let $f^\sharp : [0, \mathrm{Vol}_F(\Omega)] \to [0,\infty)$ be the decreasing rearrangement of f. Combining (13), (16) and the Hardy-Littlewood-Pólya-type inequality (12), it follows that

$$\int_{\Gamma_t} F^*(x, Du(x)) d\sigma_F = \int_{\Omega_t} f(x) d\nu_F \le \int_0^{\mu_u(t)} f^\sharp(s) ds. \tag{17}$$

By applying the isoperimetric inequality (9) to the set Ω_t, then using the Cauchy-Schwarz inequality and relations (14) and (17), we obtain that

$$\begin{aligned} n^2(\omega_n \mathsf{AVR}_F)^{\frac{2}{n}} \mathrm{Vol}_F(\Omega_t)^{2-\frac{2}{n}} &\le \mathcal{P}_F(\Gamma_t)^2 = \left(\int_{\Gamma_t} d\sigma_F\right)^2 \\ &\le \int_{\Gamma_t} \frac{1}{F^*(x, Du(x))} d\sigma_F \cdot \int_{\Gamma_t} F^*(x, Du(x)) d\sigma_F \\ &\le -\mu_u'(t) \int_0^{\mu_u(t)} f^\sharp(s) ds. \end{aligned} \tag{18}$$

Hence, since $\mu_u(t) = \mathrm{Vol}_F(\Omega_t)$, we have that

$$1 \le n^{-2}(\omega_n \mathsf{AVR}_F)^{-\frac{2}{n}} \mu_u(t)^{\frac{2}{n}-2} \left(-\mu_u'(t)\right) \int_0^{\mu_u(t)} f^\sharp(s) ds.$$

Integrating from 0 to t and applying a change of variables yields

$$\begin{aligned} t &\le n^{-2}(\omega_n \mathsf{AVR}_F)^{-\frac{2}{n}} \int_0^t \mu_u(\tau)^{\frac{2}{n}-2} \left(-\mu_u'(\tau)\right) \int_0^{\mu_u(\tau)} f^\sharp(s) ds d\tau \\ &= n^{-2}(\omega_n \mathsf{AVR}_F)^{-\frac{2}{n}} \int_{\mu_u(t)}^{\mathrm{Vol}_F(\Omega)} \eta^{\frac{2}{n}-2} \int_0^\eta f^\sharp(s) ds d\eta. \end{aligned}$$

Using the definition of the decreasing rearrangement $u^\sharp : [0, \mathrm{Vol}_F(\Omega)] \to [0,\infty)$ of u, we obtain that

$$u^\sharp(\xi) \le n^{-2}(\omega_n \mathsf{AVR}_F)^{-\frac{2}{n}} \int_\xi^{\mathrm{Vol}_F(\Omega)} \eta^{\frac{2}{n}-2} \int_0^\eta f^\sharp(s) ds d\eta. \tag{19}$$

Step 2. Now we turn to the Dirichlet problem defined on $(\mathbb{R}^n, H)$, namely,

$$\begin{cases} -\Delta_{H^*} v = f_H^\star & \text{in } \Omega_H^\star \\ v = 0 & \text{on } \partial\Omega_H^\star, \end{cases} \tag{$\mathcal{P}^\star$}$$

where $\Omega_H^\star \subset \mathbb{R}^n$ is a Wulff-shape such that

$$\mathrm{Vol}_F(\Omega) = \mathsf{AVR}_F \mathrm{Vol}_H(\Omega_H^\star), \tag{20}$$

while $f_H^\star : \Omega_H^\star \to [0,\infty)$ is the anisotropic rearrangement of f w.r.t. the norm H, i.e.,

$$f_H^\star(x) = f^\sharp(\mathsf{AVR}_F \omega_n H(x)^n),$$

where $f^\sharp$ is the decreasing rearrangement of f.

We can associate to ($\mathcal{P}^\star$) the energy functional $\mathcal{E} : W_{0,H}^{1,2}(\Omega_H^\star) \to \mathbb{R}$, defined as

$$\mathcal{E}(v) = \frac{1}{2}\int_{\Omega_H^\star} H^*(\nabla v(x))^2 dv_H - \int_{\Omega_H^\star} f_H^\star(x) v(x)\, dv_H.$$

Let $v : \Omega_H^\star \to \mathbb{R}$ be the weak solution of problem ($\mathcal{P}^\star$). By the maximum principle, we have that v is nonnegative on $\Omega_H^\star$.

Since the anisotropic rearrangements $\Omega_H^\star$ and $f_H^\star$ are constructed w.r.t. the Minkowski norm H, we can suppose (by abuse of notation) that the solution of ($\mathcal{P}^\star$) satisfies

$$v(x) = v(H(x)) \text{ on } \Omega_H^\star.$$

Then, by relation (8), we have that

$$H^*(\nabla v(x)) = H^*\big(v'(H(x))\nabla H(x)\big) = -v'(H(x))H^*(\nabla H(x)) = -v'(H(x)).$$

Consequently, we obtain that

$$\begin{aligned}\mathcal{E}(v) &= \frac{1}{2}\int_{\Omega_H^\star} v'(H(x))^2 dv_H - \int_{\Omega_H^\star} f^\sharp(\mathsf{AVR}_F\omega_n H(x)^n) v(H(x))\, dv_H \\ &= n\omega_n\left\{\frac{1}{2}\int_0^R v'(\rho)^2\rho^{n-1}d\rho - \int_0^R f^\sharp(\mathsf{AVR}_F\omega_n\rho^n)v(\rho)\rho^{n-1}\, d\rho\right\},\end{aligned}$$

where $R > 0$ is determined such that the Wulff-shape $\Omega_H^\star = W_H(R)$ satisfies (20).

Since v is the critical point of $\mathcal{E}$, it follows that v satisfies the ordinary differential equation

$$(v'(\rho)\rho^{n-1})' + f^\sharp(\mathsf{AVR}_F\omega_n\rho^n)\rho^{n-1} = 0, \tag{21}$$

together with the boundary conditions

$$v(R) = v'(0) = 0.$$

Integrating (21) from 0 to r and applying a change of variables yields

$$-r^{n-1}v'(r) = \int_0^r f^\sharp(\mathsf{AVR}_F\omega_n\rho^n)\rho^{n-1}d\rho$$

$$= (\mathsf{AVR}_F n\omega_n)^{-1} \int_0^{\mathsf{AVR}_F \omega_n r^n} f^\sharp(s) ds.$$

Then, integrating from r to R and using a change of variable again yields that

$$\begin{aligned} v(r) &= (\mathsf{AVR}_F n\omega_n)^{-1} \int_r^R \rho^{1-n} \int_0^{\mathsf{AVR}_F \omega_n \rho^n} f^\sharp(s) ds d\rho \\ &= n^{-2}(\omega_n \mathsf{AVR}_F)^{-\frac{2}{n}} \int_{\mathsf{AVR}_F \omega_n r^n}^{\mathsf{AVR}_F \mathrm{Vol}_H(\Omega_H^\star)} \eta^{\frac{2}{n}-2} \int_0^\eta f^\sharp(s) ds d\eta. \end{aligned}$$

Hence, we obtain that $v = v_H^\star$ and

$$v^\sharp(\xi) = n^{-2}(\omega_n \mathsf{AVR}_F)^{-\frac{2}{n}} \int_{\mathsf{AVR}_F \xi}^{\mathsf{AVR}_F \mathrm{Vol}_H(\Omega_H^\star)} \eta^{\frac{2}{n}-2} \int_0^\eta f^\sharp(s) ds d\eta.$$

where $v^\sharp : [0, \mathrm{Vol}_H(\Omega_H^\star)] \to [0, \infty)$ is the decreasing rearrangement of v.

Step 3. Using relations (19) and (20), we obtain that

$$u^\sharp(\mathsf{AVR}_F \xi) \leq v^\sharp(\xi), \text{ a.e. } \xi \in [0, \mathrm{Vol}_H(\Omega_H^\star)]. \tag{22}$$

Keeping in mind the definitions of the anisotropic rearrangements $u_H^\star$ and $v_H^\star = v$ (see (11)), it follows that

$$u_H^\star(x) = u^\sharp(\mathsf{AVR}_F \omega_n H(x)^n) \leq v^\sharp(\omega_n H(x)^n) = v_H^\star(x),$$

a.e. $x \in \Omega_H^\star$, which concludes the proof of inequality (5).

Step 4. If $u_H^\star(x) = v(x)$, for a.e. $x \in \Omega_H^\star$, it follows that we have equality in (22), which in turn implies that equality holds in (19), as well. Consequently, we obtain that equality is achieved in the isoperimetric inequality (18) for every level set Ω_t. In particular, for $t = 0$ we have that

$$\mathcal{P}_F(\partial\Omega) = n\omega_n^{\frac{1}{n}} \mathsf{AVR}_F^{\frac{1}{n}} \mathrm{Vol}_F(\Omega)^{\frac{n-1}{n}}.$$

Therefore, we can apply Theorem 3, which completes the proof.

Proof of Theorem 2.

Let $u : \Omega \subset M \to \mathbb{R}$ be the eigenfunction associated with the first eigenvalue $\lambda_1(\Omega)$ of $(\mathcal{EP})$, and consider the anisotropic rearrangement function of u w.r.t. the norm H, i.e., $u_H^\star : \Omega_H^\star \subset \mathbb{R}^n \to \mathbb{R}$.

Given $\lambda_1(\Omega)$ and $u_H^\star$, we can define the Dirichlet problem

$$\begin{cases} -\Delta_{H^*} v = \lambda_1(\Omega) u_H^\star, & \text{in } \Omega_H^\star, \\ v = 0, & \text{on } \partial\Omega_H^\star. \end{cases} \tag{23}$$

If $v : \Omega_H^\star \to \mathbb{R}$ is a solution of problem (23), then, by Theorem 1, it follows that

$$u_H^\star(x) \le v(x), \quad a.e.\ x \in \Omega_H^\star. \tag{24}$$

Consequently, we have that

$$\int_{\Omega_H^\star} u_H^\star(x) v(x) dv_H \le \int_{\Omega_H^\star} v(x)^2 dv_H. \tag{25}$$

Multiplying by v the equation from (23), then integrating on $\Omega_H^\star$ and using relation (7), we obtain that

$$\int_{\Omega_H^\star} H^*(\nabla v(x))^2 dv_H = \lambda_1(\Omega) \int_{\Omega_H^\star} u_H^\star(x) v(x) dv_H.$$

Therefore, by applying (25) and the variational characterization of the first eigenvalue of problem ($\mathcal{EP}^\star$) (see Shen [21, page 176]), it follows that

$$\lambda_1(\Omega) = \frac{\displaystyle\int_{\Omega_H^\star} H^*(\nabla v(x))^2 dv_H}{\displaystyle\int_{\Omega_H^\star} u_H^\star(x) v(x) dv_H} \ge \frac{\displaystyle\int_{\Omega_H^\star} H^*(\nabla v(x))^2 dv_H}{\displaystyle\int_{\Omega_H^\star} v(x)^2 dv_H} \ge \lambda_1\left(\Omega_H^\star\right),$$

which completes the proof of (6).

If equality holds in (6), then we have equalities in all of the above inequalities. In particular, we have equality in (24). Thus we can apply the rigidity result of Theorem 1, which concludes the proof. □

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On the spectral radius of D_α-matrix of a connected graph

Abdollah Alhevaz
Faculty of Mathematical Sciences,
Shahrood University of Technology,
Shahrood, Iran
email: a.alhevaz@shahroodut.ac.ir

Maryam Baghipur
Department of Mathematics,
University of Hormozgan,
Bandar Abbas, Iran
email: maryamb8989@gmail.com

Hilal A. Ganie
Department of School Education,
JK Govt. Kashmir, India
email: hilahmad1119kt@gmail.com

Gui-Xian Tian
Department of Mathematics,
Zhejiang Normal University,
Jinhua, Zhejiang, P.R. China
email: guixiantian@163.com

Abstract. In this paper, we further study the convex combinations $D_\alpha(G)$ of $Tr(G)$ and $D(G)$, defined as $D_\alpha(G) = \alpha Tr(G)+(1-\alpha)D(G)$, $0 \leq \alpha \leq 1$, where $D(G)$ and $Tr(G)$ denote the distance matrix and diagonal matrix of the vertex transmissions of a simple connected graph G, respectively. We obtain some upper and lower bounds for the spectral radius of the generalized distance matrix, in terms of various graph parameters and characterize the extremal graphs. We also obtain a lower bound for the generalized distance spectral radius of a graph with given edge connectivity, in terms of the order n, the edge connectivity r and the parameter α. Further, we obtain a lower bound for the generalized distance spectral radius of a tree, in terms of the order n, the diameter d and the parameter α. We characterize the extremal graphs for some values of diameter d.

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1 Introduction

In this paper, we consider only connected, undirected, simple and finite graphs. A graph is denoted by $G = (V(G), E(G))$, where $V(G) = \{v_1, v_2, \ldots, v_n\}$ is its vertex set and $E(G)$ is its edge set. The *order* of G is the number $n = |V(G)|$ and its *size* is the number $m = |E(G)|$. The set of vertices adjacent to $v \in V(G)$, denoted by $N(v)$, refers to the *neighborhood* of v. The *degree* of v, denoted by $d_G(v)$ (we simply write d_v if it is clear from the context) means the cardinality of $N(v)$. A graph is called *regular* if each of its vertex has the same degree. The *distance* between two vertices $u, v \in V(G)$, denoted by d_{uv}, is defined as the length of a shortest path between u and v in G. The *diameter* of G is the maximum distance between any two vertices of G. The *distance matrix* of G, denoted by $D(G)$ is defined as $D(G) = (d_{uv})_{u,v \in V(G)}$. We direct the interested reader to consult the survey [6] for some spectral properties of the distance matrix of graphs. The *transmission* $Tr_G(v)$ of a vertex v is defined as the sum of the distances from v to all other vertices in G, that is, $Tr_G(v) = \sum\limits_{u \in V(G)} d_{uv}$. A graph G is said to be k-*transmission regular* if $Tr_G(v) = k$, for each $v \in V(G)$. The *transmission* of a graph G, denoted by $W(G)$, is the sum of distances between all unordered pairs of vertices in G. Clearly, $W(G) = \frac{1}{2} \sum\limits_{v \in V(G)} Tr_G(v)$. For any vertex $v_i \in V(G)$, the transmission $Tr_G(v_i)$ is called the *transmission degree*, shortly denoted by Tr_i and the sequence $\{Tr_1, Tr_2, \ldots, Tr_n\}$ is called the *transmission degree sequence* of the graph G. The *second transmission degree* of v_i, denoted by T_i is given by $T_i = \sum\limits_{j=1}^{n} d_{ij} Tr_j$.

Let $Tr(G) = diag(Tr_1, Tr_2, \ldots, Tr_n)$ be the diagonal matrix of vertex transmissions of G. M. Aouchiche and P. Hansen [7, 8, 9] introduced the Laplacian and the signless Laplacian for the distance matrix of a connected graph. The matrix $D^L(G) = Tr(G) - D(G)$ is called the *distance Laplacian matrix* of G, while the matrix $D^Q(G) = Tr(G) + D(G)$ is called the *distance signless Laplacian matrix of* G. The spectral properties of $D(G), D^L(G)$ and $D^Q(G)$ have attracted much more attention of the researchers and a large number of papers have been published regarding their spectral properties, like spectral radius, second largest eigenvalue, smallest eigenvalue, etc. For some recent works we refer to [1, 6, 7, 8, 9, 15, 16, 18] and the references therein.

In [11], Cui et al. introduced the *generalized distance matrix* $D_\alpha(G)$ defined as $D_\alpha(G) = \alpha Tr(G) + (1-\alpha)D(G)$, for $0 \leq \alpha \leq 1$. Since $D_0(G) = D(G)$, $2D_{\frac{1}{2}}(G) = D^Q(G)$, $D_1(G) = Tr(G)$ and $D_\alpha(G) - D_\beta(G) = (\alpha -$

$\beta)D^L(G)$, any result regarding the spectral properties of generalized distance matrix, has its counterpart for each of these particular graph matrices, and these counterparts follow immediately from a single proof. In fact, this matrix reduces to merging the distance spectral and distance signless Laplacian spectral theories. Since the matrix $D_\alpha(G)$ is real symmetric, all its eigenvalues are real. Therefore, we can arrange them as $\partial_1 \geq \partial_2 \geq \cdots \geq \partial_n$. The largest eigenvalue ∂_1 of the matrix $D_\alpha(G)$ is called the *generalized distance spectral radius* of G (From now onwards, we will denote $\partial_1(G)$ by $\partial(G)$). As $D_\alpha(G)$ is non-negative and irreducible, by the Perron-Frobenius theorem, $\partial(G)$ is a simple (with multiplicity one) eigenvalue and there is a unique positive unit eigenvector X corresponding to $\partial(G)$, which is called the *generalized distance Perron vector* of G.

A column vector $X = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$ can be considered as a function defined on $V(G)$ which maps vertex v_i to x_i, i.e., $X(v_i) = x_i$ for $i = 1, 2, \ldots, n$. Then,

$$X^T D_\alpha(G) X = \alpha \sum_{i=1}^{n} Tr(v_i) x_i^2 + 2(1-\alpha) \sum_{1 \leq i < j \leq n} d(v_i, v_j) x_i x_j,$$

and λ is an eigenvalue of $D_\alpha(G)$ corresponding to the eigenvector X if and only if $X \neq \mathbf{0}$ and,

$$\lambda x_{v_i} = \alpha Tr(v_i) x_i + (1-\alpha) \sum_{j=1}^{n} d(v_i, v_j) x_j.$$

These equations are called the (λ, x)-*eigenequations* of G. For some spectral properties of the generalized distance matrix of graphs, we direct the interested reader to consult the papers [2, 3, 11, 12, 19, 20, 14].

The remainder of the paper is organized as follows. In Section 2, we obtain some upper and lower bounds for the spectral radius of the matrix $D_\alpha(G)$, involving different graph parameters, and characterize the extremal graphs. In Section 3, we obtain a lower bound for the generalized distance spectral radius of a tree, in terms of the order n, the diameter d and the parameter α. We also characterize the extremal graphs for some values of diameter d. Finally, in Section 4, we obtain a lower bound for the generalized distance spectral radius of a graph with given edge connectivity, in terms of the order n, the edge connectivity r and the parameter α.

2 Bounds on the generalized distance spectral radius of graphs

In this section, we obtain upper and lower bounds for the generalized distance spectral radius of a connected graph G, in terms of various graph parameters associated with the structure of the graph. We characterize the extremal graphs attaining these bounds.

We start by mentioning two previously known results that will be needed in the sequel. The following lemmas can be found in [11].

Lemma 1 (See [11]) *Let G be a connected graph of order n. Then,*

$$\partial(G) \geq \frac{2W(G)}{n},$$

with equality if and only if G is a transmission regular graph.

Lemma 2 (See [11]) *Let G be a connected graph of order n and let $\frac{1}{2} \leq \alpha \leq 1$. If G' is a connected graph obtained from G by deleting an edge, then for any $1 \leq i \leq n$,*

$$\partial_i(G') \geq \partial_i(G).$$

The following gives a lower bound for the generalized distance spectral radius $\partial(G)$, in terms of the order n and the size m of the graph G.

Theorem 1 *Let G be a connected graph of order $n \geq 2$ and size m. Then*

$$\partial(G) \geq 2(n-1) - \frac{2m}{n}, \tag{1}$$

with equality if and only if $G \cong K_n$ or G is a transmission regular graph with diameter two.

Proof. We know that the transmission of each vertex $v \in V(G)$ is

$$Tr(v) \geq d(v) + 2(n-1-d(v)) = 2n - d(v) - 2,$$

where $d(v)$ denotes the degree of v in G, with equality if and only if the maximal distance from v to other vertices in G is at most two. With this we have

$$W(G) = \frac{1}{2}\sum_{v\in V(G)} Tr_G(v) \geq \frac{1}{2}\sum_{v\in V(G)} (2n - d(v) - 2) = n(n-1) - m,$$

with equality if and only if G is of diameter at most two. Using Lemma 1, and the above observation, the result follows. Suppose equality holds in (1), then equality holds in Lemma 1 and $W(G) = m + 2\left(\frac{n(n-1)}{2} - m\right) = n(n-1) - m$. Which is possible, if G is transmission regular and the diameter of G is at most two, that is, $G \cong K_n$ or G is a transmission regular graph of diameter two.

Conversely, if $G \cong K_n$ or G is a transmission regular graph of diameter two, then it is easy to see that (1) is an equality. □

The following gives a lower bound for the generalized distance spectral radius $\partial(G)$ of triangle-free and quadrangle-free graphs.

Corollary 1 *Let G be a triangle-free and quadrangle-free connected graph of order $n \geq 2$ and size m. Then*

$$\partial(G) \geq 3(n-1) - \frac{1}{n}\sum_{i=1}^{n} d^2(v_i) - \frac{2m}{n}, \tag{2}$$

with equality if and only if G is a transmission regular graph and the diameter of G is at most three.

Proof. For a connected graph G, which is triangle-free and quadrangle-free it is shown in [21] that

$$W(G) \geq \frac{3n(n-1)}{2} - \frac{1}{2}\sum_{i=1}^{n} d^2(v_i) - m,$$

with equality holding if and only if the diameter is at most three. Now, the result follows from Lemma 1. □

The following gives an upper bound for the generalized distance spectral radius $\partial(G)$, in terms of the transmission degrees Tr_i, the second transmission degrees T_i and the parameter α.

Theorem 2 *Let G be a connected graph of order $n \geq 2$ and let $\alpha \in [0,1)$. Let $\{Tr_1, Tr_2, \ldots, Tr_n\}$ be the transmission degree sequence and $\{T_1, T_2, \ldots, T_n\}$ be the second transmission degree sequence of the graph G. Then*

$$\partial(G) \leq \frac{1}{2}\max_{1\leq i\neq j\leq n}\left\{\alpha(Tr_i + Tr_j) + \sqrt{\alpha^2(Tr_i - Tr_j)^2 + 4(1-\alpha)^2\left(\frac{T_i}{Tr_i}\right)\left(\frac{T_j}{Tr_j}\right)}\right\}. \tag{3}$$

Moreover, if $\frac{1}{2} \leq \alpha < 1$, the equality holds if and only if G is a transmission regular graph.

Proof. Let $\mathsf{Tr} = \mathsf{Tr}(G)$ be the diagonal matrix of vertex transmissions of the connected graph G, then the matrix Tr^{-1} exists. Since the matrices $D_\alpha(G)$ and $\mathsf{Tr}^{-1}D_\alpha(G)\mathsf{Tr}$ are similar and similar matrices have same spectrum, it follows that $\partial(G)$ is the spectral radius of the matrix $\mathsf{Tr}^{-1}D_\alpha(G)\mathsf{Tr}$. Let $X = (x_1, x_2, \ldots, x_n)^T$ be an eigenvector of $\mathsf{Tr}^{-1}D_\alpha(G)\mathsf{Tr}$ corresponding to $\partial(G)$. Suppose $x_s = \max\{x_i | \ i = 1, 2, \ldots, n\}$ and $x_t = \max\{x_i | \ x_i \neq x_s, \ i = 1, 2, \ldots, n\}$. Now, the (i, j)-th entry of $\mathsf{Tr}^{-1}D_\alpha(G)\mathsf{Tr}$ is $\alpha\mathsf{Tr}_i$ if $i = j$ and $\frac{\mathsf{Tr}_j}{\mathsf{Tr}_i}(1-\alpha)d_{ij}$ if $i \neq j$. We have

$$\mathsf{Tr}^{-1}D_\alpha(G)\mathsf{Tr}X = \partial(G)X. \tag{4}$$

From the s-th equation of (4), we have

$$\begin{aligned}(\partial(G) - \alpha\mathsf{Tr}_s)x_s &= \sum_{i=1}^{n} \frac{\mathsf{Tr}_i}{\mathsf{Tr}_s}(1-\alpha)d_{si}x_i \\ &\leq \frac{(1-\alpha)x_t}{\mathsf{Tr}_s}\sum_{i=1}^{n} d_{si}\mathsf{Tr}_i = \frac{(1-\alpha)T_s}{\mathsf{Tr}_s}x_t.\end{aligned} \tag{5}$$

Similarly, from the t-th equation of (4), we have

$$\begin{aligned}(\partial(G) - \alpha\mathsf{Tr}_t)x_t &= \sum_{i=1}^{n} \frac{\mathsf{Tr}_i}{\mathsf{Tr}_t}(1-\alpha)d_{ti}x_i \\ &\leq \frac{(1-\alpha)x_s}{\mathsf{Tr}_t}\sum_{i=1}^{n} d_{ti}\mathsf{Tr}_i = \frac{(1-\alpha)T_t}{\mathsf{Tr}_t}x_s.\end{aligned} \tag{6}$$

Combining (5) and (6) we get,

$$(\partial(G) - \alpha\mathsf{Tr}_s)(\partial(G) - \alpha\mathsf{Tr}_t)x_s x_t \leq \frac{(1-\alpha)T_s}{\mathsf{Tr}_s}\frac{(1-\alpha)T_t}{\mathsf{Tr}_t}x_t x_s,$$

which implies that

$$\partial^2(G) - \alpha(\mathsf{Tr}_s + \mathsf{Tr}_t)\partial(G) + \alpha^2\mathsf{Tr}_s\mathsf{Tr}_t - \left(\frac{(1-\alpha)T_s}{\mathsf{Tr}_s}\right)\left(\frac{(1-\alpha)T_t}{\mathsf{Tr}_t}\right) \leq 0.$$

Thus, we have

$$\partial(G) \leq \frac{1}{2}\Bigg(\alpha(\mathsf{Tr}_s + \mathsf{Tr}_t) + \sqrt{\alpha^2(\mathsf{Tr}_s - \mathsf{Tr}_t)^2 + 4(1-\alpha)^2\left(\frac{T_s}{\mathsf{Tr}_s}\right)\left(\frac{T_t}{\mathsf{Tr}_t}\right)}\Bigg).$$

From this the result follows. Assume that G is a k-transmission regular graph. Then $Tr_i = k, T_i = k^2$ for all $i = 1, 2, \dots, n$, and $\partial(G) = k$. It is now easy to see that equality in (3) holds.

Conversely, suppose that equality holds in (3), then all the inequalities in the above argument must hold as equalities. In particular, from (5) and (6), we have $x_1 = x_2 = \cdots = x_n$. Hence, $\partial(G) = \alpha Tr_1 + \frac{(1-\alpha)T_1}{Tr_1} = \alpha Tr_2 + \frac{(1-\alpha)T_2}{Tr_2} = \cdots = \alpha Tr_n + \frac{(1-\alpha)T_n}{Tr_n}$. Let $Tr_{\max}$ and $Tr_{\min}$ denote the maximum and minimum vertex transmission, respectively. Without loss of generality, assume that $Tr_i = Tr_{\max}$ and $Tr_j = Tr_{\min}$. Therefore, $\alpha Tr_{\max} + \frac{(1-\alpha)T_i}{Tr_{\max}} = \alpha Tr_{\min} + \frac{(1-\alpha)T_j}{Tr_{\min}}$. Since $T_i \geq Tr_{\max}Tr_{\min}$ and $T_j \leq Tr_{\max}Tr_{\min}$, we have

$$\begin{aligned}\alpha Tr_{\max} + (1-\alpha)Tr_{\min} &\leq \alpha Tr_{\max} + \frac{(1-\alpha)T_i}{Tr_{\max}} = \alpha Tr_{\min} + \frac{(1-\alpha)T_j}{Tr_{\min}} \\ &\leq (1-\alpha)Tr_{\max} + \alpha Tr_{\min},\end{aligned}$$

which implies that $Tr_{\max} = Tr_{\min}$ for $\frac{1}{2} \leq \alpha < 1$. Hence, G is a transmission regular graph. This completes the proof. $\square$

The following gives a lower bound for the generalized distance spectral radius $\partial(G)$, in terms of the transmission degrees Tr_i, the second transmission degrees T_i and the parameter α.

Theorem 3 *Let G be a connected graph of order $n \geq 2$ and let $\alpha \in [0, 1)$. Let $\{Tr_1, Tr_2, \dots, Tr_n\}$ be the transmission degree sequence and $\{T_1, T_2, \dots, T_n\}$ be the second transmission degree sequence of the graph G. Then*

$$\partial(G) \geq \frac{1}{2} \min_{1 \leq i \neq j \leq n} \left\{ \alpha(Tr_i + Tr_j) + \sqrt{\alpha^2(Tr_i - Tr_j)^2 + 4(1-\alpha)^2 \left(\frac{T_i}{Tr_i}\right)\left(\frac{T_j}{Tr_j}\right)} \right\}.$$

Moreover, if $\frac{1}{2} \leq \alpha < 1$, the equality holds if and only if G is a transmission regular graph.

Proof. Let $X = (x_1, x_2, \dots, x_n)^T$ be an eigenvector of $Tr^{-1}D_\alpha(G)Tr$ corresponding to $\partial(G)$. Suppose $x_s = \min\{x_i |\ i = 1, 2, \dots, n\}$ and $x_t = \min\{x_i |\ x_i \neq x_s,\ i = 1, 2, \dots, n\}$. The rest of the proof is similar to that of Theorem 2 and is therefore omitted. $\square$

The following lemma can be found in [17].

Lemma 3 *If A is an $n \times n$ non-negative matrix with the spectral radius $\lambda(A)$ and row sums $r_1, r_2, \dots, r_n$, then $\min_{1\leq i\leq n} r_i \leq \lambda(A) \leq \max_{1\leq i\leq n} r_i$. Moreover, if A is irreducible, then one of the equalities holds if and only if the row sums of A are all equal.*

The following gives an upper bound for the generalized distance spectral radius $\partial(G)$, in terms of the maximum transmission degree $Tr_{\max}$, the second maximum transmission degree $T_{\max}$ and the parameter α.

Theorem 4 *Let G be a connected graph of order $n \geq 2$ and let $\alpha \in [0, 1)$. Let $Tr_{\max}$ and $T_{\max}$ be respectively the maximum transmission degree and the second maximum transmission degree of the graph G. Then*

$$\partial(G) \leq \frac{\alpha Tr_{\max} + \sqrt{\alpha^2 Tr_{\max}^2 + 4(1-\alpha)T_{\max}}}{2},$$

Moreover, the equality holds if and only if G is a transmission regular graph.

Proof. For a graph matrix M, let $r_{v_i}(M)$ be the sum of the entries in the row corresponding to the vertex v_i, for $1 \leq i \leq n$. We have $D_\alpha(G) = \alpha Tr(G) + (1-\alpha)D(G)$, by a simple calculation, it can be seen that $r_{v_i}(D_\alpha(G)) = Tr_i$ and $r_{v_i}(D(G)Tr) = r_{v_i}(D^2(G)) = \sum_{j=1}^{n} d_{ij}Tr_j = T_i$. Then

$$\begin{aligned}
r_{v_i}(D_\alpha^2(G)) &= r_{v_i}\Big(\alpha Tr(G) + (1-\alpha)D(G)\Big)^2 \\
&= r_{v_i}\Big(\alpha^2 Tr^2 + \alpha(1-\alpha)TrD(G) + \alpha(1-\alpha)D(G)Tr + (1-\alpha)^2 D^2(G)\Big) \\
&= r_{v_i}\Big(\alpha Tr(\alpha Tr + (1-\alpha)D(G))\Big) + r_{v_i}\Big(\alpha(1-\alpha)D(G)Tr\Big) \\
&\quad + r_{v_i}\Big((1-\alpha)^2 D^2(G)\Big) \\
&= \alpha Tr_i r_{v_i}(D_\alpha(G)) + (1-\alpha)T_i \\
&\leq \alpha Tr_{\max} r_{v_i}(D_\alpha(G)) + (1-\alpha)T_{\max}.
\end{aligned}$$

So, we have

$$r_{v_i}\left(D_\alpha^2(G) - \alpha Tr_{\max} D_\alpha(G)\right) \leq (1-\alpha)T_{\max}.$$

Using Lemma 3, we get

$$\partial^2(G) - \alpha Tr_{\max}\partial(G) - (1-\alpha)T_{\max} \leq 0,$$

from this the result now follows. In order to get the equality, all inequalities in the above argument should be equalities. That is, $Tr_i = Tr_{\max}$ and $T_i = T_{\max}$ holds for any vertex v_i. So, by Lemma 3, it follows that G is a transmission regular graph.

Conversely, if G is transmission regular, then it is easy to check that the equality holds. □

The following gives a lower bound for the generalized distance spectral radius $\partial(G)$, in terms of the minimum transmission degree Tr_{min}, the second minimum transmission degree T_{min} and the parameter α.

Theorem 5 *Let G be a connected graph of order $n \geq 2$ and let $\alpha \in [0, 1)$. Let Tr_{min} and T_{min} be respectively the minimum transmission degree and the second minimum transmission degree of the graph G. Then*

$$\partial(G) \geq \frac{\alpha Tr_{min} + \sqrt{\alpha^2 Tr_{min}^2 + 4(1-\alpha)T_{min}}}{2}.$$

Equality holds if and only if G is transmission regular.

Proof. Proceeding similar to Theorem 4, we arrive at

$$\begin{aligned} r_{v_i}(D_\alpha^2(G)) &= \alpha Tr_i r_{v_i}(D_\alpha(G)) + (1-\alpha)T_i \\ &\geq \alpha Tr_{min} r_{v_i}(D_\alpha(G)) + (1-\alpha)T_{min}. \end{aligned} \tag{7}$$

Since (7) is true for all v_i, in particular it is true for v_{min}, where v_{min} is the vertex corresponding the row with minimum row sum. Therefore, from (7), we get

$$r_{v_{min}}\Big(D_\alpha^2(G) - \alpha Tr_{min} D_\alpha(G)\Big) - (1-\alpha)T_{min} \geq 0.$$

Now, using Lemma 3, we get

$$\partial^2(G) - \alpha Tr_{min}\partial(G) - (1-\alpha)T_{min} \geq 0,$$

from this the result now follows. The equality case be discussed similarly as in Theorem 4. □

The following gives a lower bound for the generalized distance spectral radius $\partial(G)$, in terms of the order n, the maximum degree Δ and the parameter α.

Theorem 6 *Let G be a connected graph of order $n \geq 2$ and let $\alpha \in [0, 1)$. If $\Delta = \Delta(G)$ is the maximum degree of the graph G, then*

$$\partial(G) \geq \frac{\alpha(2n-\Delta-2) + \sqrt{\alpha^2(2n-\Delta-2)^2 + 4(1-\alpha)(2n-2-\Delta)^2}}{2}, \tag{8}$$

with equality if and only if G is a regular graph with diameter less than or equal to 2.

Proof. Let G be a connected graph of order n and let $d_i = d(v_i)$ be the degree of the vertex v_i, for $1 \leq i \leq n$. It is well known that $Tr_i = Tr(v_i) \geq d_i + 2(n-1-d_i) = 2n-2-d_i$, for all $1 \leq i \leq n$, with equality if and only if G is a degree regular graph of diameter less than or equal to 2. Similar to the Theorem 4, we have

$$\begin{aligned} r_{v_i}(D_\alpha^2(G)) &= \alpha Tr_i r_{v_i}(D_\alpha(G)) + (1-\alpha)T_i \\ &\geq \alpha Tr_i r_{v_i}(D_\alpha(G)) + (1-\alpha)(2n-d_j-2)\sum_{j=1}^{n} d_{ij} \\ &\geq \alpha(2n-d_i-2)r_{v_i}(D_\alpha(G)) + (1-\alpha)(2n-2-\Delta)^2 \\ &\geq \alpha(2n-\Delta-2)r_{v_i}(D_\alpha(G)) + (1-\alpha)(2n-2-\Delta)^2, \end{aligned}$$

where we have used the fact that $Tr_i \geq 2n-2-d_i \geq 2n-2-\Delta$. Thus it follows that for each $v_i \in V(G)$, we have

$$r_{v_i}((D_\alpha)^2) \geq r_{v_i}(\alpha(2n-\Delta-2)D_\alpha) + (1-\alpha)(2n-2-\Delta)^2. \tag{9}$$

Since (9) is true for all v_i, in particular it is true for $v_{\min}$, where $v_{\min}$ is the vertex corresponding the row with minimum row sum. So, from (9), we get

$$r_{v_{\min}}\Big(D_\alpha^2(G) - \alpha(2n-2-\Delta)D_\alpha(G)\Big) - (1-\alpha)(2n-2-\Delta)^2 \geq 0.$$

Now, using Lemma 3, it follows that

$$\partial^2(G) - \alpha(2n-\Delta-2)\partial(G) - (1-\alpha)(2n-2-\Delta)^2 \geq 0,$$

which gives that

$$\partial(G) \geq \frac{\alpha(2n-\Delta-2) + \sqrt{\alpha^2(2n-\Delta-2)^2 + 4(1-\alpha)(2n-\Delta-2)^2}}{2}.$$

This proves the first part of the proof.

Suppose that equality holds in inequality (8), then all the inequalities hold as equalities in the above argument. Since the equality holds in $Tr_i \geq 2n-2-d_i \geq 2n-2-\Delta$ if G is Δ-regular graph of diameter less than or equal to 2 and equality holds in Lemma 3 if G is a transmission regular graph. It follows that equality holds in (8) if G is Δ-regular graph of diameter less than or equal to 2. Conversely, it is easily seen that $\partial(G) = \frac{\alpha(2n-\Delta-2)+\sqrt{\alpha^2(2n-\Delta-2)^2+4(1-\alpha)(2n-2-\Delta)^2}}{2}$ if G is a regular graph with diameter less than or equal to 2. $\square$

We conclude this section with the following remark.

Remark 1 *As mentioned in the introduction that* $D_0(G) = D(G)$ *and* $2D_{\frac{1}{2}}(G) = D^Q(G)$*, it follows that from the bounds obtained in this section for* $\partial(G)$*, we can obtain the corresponding bounds for the distance spectral radius* $\rho_1^D(G)$ *and the distance signless Laplacian spectral radius* $\rho_1^Q(G)$ *by taking* $\alpha = 0$ *and* $\alpha = \frac{1}{2}$*, respectively.*

3 Lower bounds for the generalized distance spectral radius of a tree

In this section, we obtain a lower bound for the generalized distance spectral radius $\partial(G)$ of a tree, in terms of the order n, diameter d and the parameter α.

The following gives the generalized distance spectrum of the complete bipartite graph $K_{r,s}$, where $r + s = n$, and can be found in [20].

Lemma 4 *The generalized distance spectrum of complete bipartite graph* $K_{r,s}$ *consists of eigenvalue* $\alpha(2r + s) - 2$ *with multiplicity* $r - 1$*, the eigenvalue* $\alpha(2s + r) - 2$ *with multiplicity* $s - 1$ *and the remaining two eigenvalues as* x_1, x_2*, where* $x_1, x_2 = \frac{\alpha(s+r)+2(s+r)-4\pm\sqrt{(r^2+s^2)(\alpha-2)^2+2rs(\alpha^2-2)}}{2}$.

Suppose a graph G has a special kind of symmetry so that its associated matrix is written in the form

$$M = \begin{pmatrix} X & \beta & \cdots & \beta & \beta \\ \beta^t & B & \cdots & C & C \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ \beta^t & C & \cdots & B & C \\ \beta^t & C & \cdots & C & B \end{pmatrix}, \tag{10}$$

where $X \in R^{t\times t}$, $\beta \in R^{t\times s}$ and $B, C \in R^{s\times s}$, such that $n = t + cs$, where c is the number of copies of B. Then the spectrum of this matrix can be obtained as the union of the spectrum of smaller matrices using the following technique given in [13]. In the statement of the following lemma, $\sigma^{[k]}(Y)$ indicates the multi-set formed by k copies of the spectrum of Y, denoted by $\sigma(Y)$.

Lemma 5 *Let* M *be a matrix of the form given in* (10)*, with* $c \geq 1$ *copies of the block* B*. Then*

(i) $\sigma^{[c-1]}(B - C) \subseteq \sigma(M)$*;*

(ii) $\sigma(M)\backslash\sigma^{[c-1]}(B-C) = \sigma(M')$ *is the set of the remaining* $t+s$ *eigenvalues of* M, *where* $M' = \begin{pmatrix} X & \sqrt{c}.\beta \\ \sqrt{c}.\beta^t & B+(c-1)C \end{pmatrix}$.

Let $T_{a,b}$, with $a+b=n-2$ and $a \geq b \geq 1$ be the tree obtained by joining an edge between the root vertices of stars $K_{1,a}$ and $K_{1,b}$(the vertex of degree greater than one in a star is called root vertex). It is clear that a tree with diameter $d=3$ is always of the form $T_{a,b}$. The following gives the generalized distance spectrum of $T_{a,b}$.

Lemma 6 *The generalized distance spectrum of* $T_{a,b}$ *is*

$$\{\alpha(h_1+2)-2^{[b-1]}, \alpha(h_2+2)-2^{[a-1]}, x_1, x_2, x_3, x_4\},$$
$$h_1 = 2a+3b+1, h_2 = 2b+3a+1,$$

where $x_1 \geq x_2 \geq x_3 \geq x_4$ *are the eigenvalues of the matrix*

$$M_2 = \begin{pmatrix} \alpha(2a+b+1) & 1-\alpha & 2(1-\alpha)\sqrt{a} & (1-\alpha)\sqrt{b} \\ 1-\alpha & \alpha(2b+a+1) & (1-\alpha)\sqrt{a} & 2(1-\alpha)\sqrt{b} \\ 2(1-\alpha)\sqrt{a} & (1-\alpha)\sqrt{a} & \alpha h_1 + 2(1-\alpha)(a-1) & 3(1-\alpha)\sqrt{ab} \\ (1-\alpha)\sqrt{b} & 2(1-\alpha)\sqrt{b} & 3(1-\alpha)\sqrt{ab} & \alpha h_2 + 2(1-\alpha)(b-1) \end{pmatrix}.$$

Proof. Let $V(K_{1,b}) = \{v_1, u_1, \ldots, u_b\}$ and $V(K_{1,a}) = \{v_2, w_1, \ldots, w_a\}$. Then, the vertex set of $T_{a,b}$ is $V(T_{a,b}) = \{v_1, v_2, u_1, \ldots, u_b, w_1, \ldots, w_a\}$. It is easy to see that $Tr(v_1) = 2a+b+1$, $Tr(v_2) = 2b+a+1$, $Tr(u_i) = 2b+3a+1 = h_2$ and $Tr(w_j) = 2a+3b+1 = h_1$, for $i = 1, 2, \ldots, b$ and $j = 1, 2, \ldots, a$. With this labeling, the generalized distance matrix of $T_{a,b}$ takes the form

$$D_\alpha(T_{a,b}) = \begin{pmatrix} X & \beta & \beta & \cdots & \beta \\ \beta^t & \alpha h_1 & 2(1-\alpha) & \cdots & 2(1-\alpha) \\ \beta^t & 2(1-\alpha) & \alpha h_1 & \cdots & 2(1-\alpha) \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \beta^t & 2(1-\alpha) & 2(1-\alpha) & \cdots & \alpha h_1 \end{pmatrix}, \text{ where } \beta = \begin{pmatrix} 2 \\ 1 \\ 3 \\ \vdots \\ 3 \end{pmatrix}$$

and

$$X = \begin{pmatrix} \alpha(2a+b+1) & 1-\alpha & 1-\alpha & \cdots & 1-\alpha \\ 1-\alpha & \alpha(2b+a+1) & 2(1-\alpha) & \cdots & 2(1-\alpha) \\ 1-\alpha & 2(1-\alpha) & \alpha h_2 & \cdots & 2(1-\alpha) \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1-\alpha & 2(1-\alpha) & 2(1-\alpha) & \cdots & \alpha h_2 \end{pmatrix}.$$

Using Lemma 5 with $B = [\alpha h_1]$, $C = [2(1-\alpha)]$ and $c = a$, it follows that $\sigma(D_\alpha(T_{a,b})) = \sigma^{[a-1]}(B-C) \cup \sigma(M_1) = \sigma^{[a-1]}([\alpha(h_1+2)-2]) \cup \sigma(M_1)$, where

$M_1 = \begin{pmatrix} X & \sqrt{a}\beta \\ \sqrt{a}\beta & \alpha h_1 + 2(1-\alpha)(a-1) \end{pmatrix}$. Interchanging the third and last column of M_1 and then third and last row of the resulting matrix, we obtain a matrix similar to M_1. In the resulting matrix taking

$$X = \begin{pmatrix} \alpha(2a+b+1) & 1-\alpha & 2(1-\alpha)\sqrt{a} \\ 1-\alpha & \alpha(2b+a+1) & (1-\alpha)\sqrt{a} \\ 2(1-\alpha)\sqrt{a} & (1-\alpha)\sqrt{a} & \alpha h_1 + 2(1-\alpha)(a-1) \end{pmatrix},$$

$$\beta = \begin{pmatrix} 1-\alpha \\ 2(1-\alpha) \\ 3(1-\alpha)\sqrt{a} \end{pmatrix},$$

$B = [\alpha h_2]$, $C = [2(1-\alpha)]$ and $c = b$ in Lemma 5. It follows that $\sigma(M_1) = \sigma^{[b-1]}(B-C) \cup \sigma(M_2) = \sigma^{[b-1]}([\alpha(h_2+2)-2]) \cup \sigma(M_2)$, where M_2 is the matrix given in the statement. That completes the proof. □

The following gives a lower bound for the generalized distance spectral radius of a tree, in terms of the order n, the diameter d and the parameter α.

Theorem 7 *Let T be a tree of order $n \geq 2$ having diameter d. If $d = 1$, then $\partial(T) = 1$; if $d = 2$, then $\partial(T) = \frac{(\alpha+2)n-4+\sqrt{\phi}}{2}$, $\phi = n^2\alpha^2 - (n^2+2-2n)4\alpha + 4(n^2-3n+3)$; if $d = 3$, then $\partial(T) = x_1$, where x_1 is the largest eigenvalue of the matrix M_2 defined in Lemma 6. For $d \geq 4$, let $P = v_1v_2\dots v_dv_{d+1}$ be a diametral path of G, such that there are a_1, a_2 pendent vertices at v_2, v_d, respectively. Then*

$$\partial(T) \geq \max_{a_1,a_2}\left\{\frac{6n + d(d-7) + (a_1+a_2)(d-4) + 2 + \sqrt{\theta}}{2}\right\},$$

where $\theta = \alpha^2(a_2-a_1)^2(d-2)^2 + 4(1-\alpha)^2d^2$.

Proof. If T is a tree of diameter $d = 1$, then $T \cong K_2$ and so $\partial(T) = 1$. If T is a tree of diameter $d = 2$, then $T \cong K_{1,n-1}$ and so using Lemma 4, it follows that $\partial(T) = \frac{(\alpha+2)n-4+\sqrt{\phi}}{2}$, where $\phi = n^2\alpha^2 - (n^2+2-2n)4\alpha + 4(n^2-3n+3)$. If T is a tree of diameter $d = 3$, then $T \cong T_{a,b}$ and so using Lemma 6, it follows that $\partial(T) = x_1$, where x_1 is the largest eigenvalue of the matrix M_2 defined in Lemma 6. So, suppose that diameter of tree T is at least 4, then $n \geq 5$. Let $v_1v_2\dots v_{d+1}$ be a diametral path of T, and let a_1 and a_2 be the number of pendent neighbors of v_2 and v_d, respectively. We have

$$Tr(v_1) \geq 2(a_1-1) + 1 + 2 + \dots + (d-1) + da_2 + 3(n - a_1 - a_2 - d + 1)$$

$$= \quad 3n - a_1 + a_2(d-3) - 3d + 1 + \frac{d(d-1)}{2}.$$

Similarly,

$$Tr(v_{d+1}) \geq 3n - a_2 + a_1(d-3) - 3d + 1 + \frac{d(d-1)}{2}.$$

Let M be the principal submatrix of $D_\alpha(T)$ indexed by the vertices v_1 and v_{d+1}. Then

$$M = \begin{pmatrix} \alpha Tr(v_1) & (1-\alpha)d \\ (1-\alpha)d & \alpha Tr(v_{d+1}) \end{pmatrix},$$

thus

$$\begin{aligned} \partial(M) &= \frac{\alpha(Tr(v_1) + Tr(v_{d+1})) + \sqrt{\alpha^2(Tr(v_1) - Tr(v_{d+1}))^2 + 4(1-\alpha)^2 d^2}}{2} \\ &\geq \frac{\alpha(6n + d(d-7) + (a_1 + a_2)(d-4) + 2)) + \sqrt{\alpha^2(a_2 - a_1)^2(d-2)^2 + 4(1-\alpha)^2 d^2}}{2}. \end{aligned}$$

Now, by Interlacing Theorem [10], we have $\partial(T) \geq \partial(M)$. From this the result follows. That completes the proof. □

The following observation follows from Theorem 7.

Corollary 2 *Let T be a tree of order n having diameter $d \geq 4$. Then*

$$\partial(T) \geq \frac{1}{2}\Big(\alpha(6n + d^2 - 9d + 2) + 2d\Big).$$

Proof. Using $a_1, a_2 \geq 0$ in Theorem 7, the result follows. □

Taking $\alpha = 0$ in Theorem 7, we have the following observation, which gives a lower bound for the distance spectral radius $\rho^D(T)$ of a tree T.

Corollary 3 *Let T be a tree of order $n \geq 2$ having diameter d. If $d = 1$, then $\rho^D(T) = 1$; if $d = 2$, then $\rho^D(T) = n - 2 + \sqrt{n^2 - 3n + 3}$; if $d = 3$, then $\rho^D(T) = x_1$, where x_1 is the largest eigenvalue of the matrix M_2 (with $\alpha = 0$) defined in Lemma 6. For $d \geq 4$, let $P = v_1 v_2 \dots v_d v_{d+1}$ be a diametral path of G, such that there are a_1, a_2 pendent vertices at v_2, v_d, respectively. Then*

$$\rho^D(T) \geq \max_{a_1, a_2} \left\{ \frac{6n + d(d-5) + (a_1 + a_2)(d-4) + 2}{2} \right\}.$$

Taking $\alpha = \frac{1}{2}$ in Theorem 7 and using the fact $2\partial(T) = \rho_1^Q(T)$, we have the following observation, which gives a lower bound for the distance signless Laplacian spectral radius $\rho^Q(T)$ of a tree T.

Corollary 4 *Let T be a tree of order $n \geq 2$ having diameter d. If $d = 1$, then $\rho^Q(T) = 1$; if $d = 2$, then $\rho^Q(T) = \frac{5n-8+\sqrt{9n^2-32n+32}}{2}$; if $d = 3$, then $\rho^Q(T) = 2x_1$, where x_1 is the largest eigenvalue of the matrix M_2 (with $\alpha = \frac{1}{2}$) defined in Lemma 6. For $d \geq 4$, let $P = v_1v_2 \ldots v_dv_{d+1}$ be a diametral path of G, such that there are a_1, a_2 pendent vertices at v_2, v_d, respectively. Then*

$$\rho^Q(T) \geq \max_{a_1,a_2}\left\{6n + d(d-7) + (a_1+a_2)(d-4) + 2 + \sqrt{t}\right\},$$

where $t = \frac{(a_2-a_1)^2}{4} + 2d^2 - 4d + 4$.

4 Lower bounds for the generalized distance spectral radius for a graph with given edge connectivity

In this section, we obtain a lower bound for the generalized distance spectral radius $\partial(G)$ for the family of graphs with fixed edge connectivity, in terms of the order n and the parameter α.

The *edge connectivity* of a connected graph is the minimum number of edges whose removal disconnects the graph. Let $G(n,r)$ be the set of all connected graphs of order n and edge connectivity r. It is clear that, $G(n, n-1) = K_n$. It is well known that $\partial(K_n) = n-1$, therefore we will consider $r \leq n-2$.

The following gives a lower bound for the generalized distance spectral radius of a graph belonging to the family $G(n,r)$, in terms of the order n, the edge connectivity r and the parameter α.

Theorem 8 *Let $G \in G(n,r)$ with $1 \leq r \leq n-2$ and $\frac{1}{2} \leq \alpha \leq 1$. If the degree of every vertex of G is greater than r, then*

$$\partial(G) \geq \frac{\alpha(4n-2r-2) + \sqrt{4\alpha^2(n_2-n_1)^2 + 36(1-\alpha)^2}}{2},$$

where n_1 and n_2 are the cardinalities of the components of graph obtained from G by deleting r edges.

Proof. Let $G \in G(n,r)$, then every vertex of G is of degree greater or equal to r. Let us suppose that every vertex of G has degree at least $r+1$. Let E_c be an edge cut of G with r edges. Let G_1 and G_2 be the two components of $G - E_c$ (the graph obtained from G by deleting the edges from E_c). Let $n_i = |V(G_i)|$ for $i = 1, 2$. We claim that $\min\{n_1, n_2\} \geq r+2$. Suppose that

$\min\{n_1, n_2\} \le r+1$. Without loss of generality, we assume that $n_2 \ge n_1$. Then we have $n_1 \le r+1$. If $n_1 = r+1$, then there exists a vertex of G_1 which is not incident with any edge in E_c, and thus its degree in G is at most $n_1 - 1 = r$, which is a contradiction. On the other hand, if $n_1 \le r$, then there exists a vertex of G_1 whose degree in G is at most $n_1 - 1 + \frac{r}{n_1} \le (n_1 - 1)\frac{r}{n_1} + \frac{r}{n_1} = r$, again a contradiction. Therefore, we must have $\min\{n_1, n_2\} \ge r+2$. Thus, there exists a vertex u of G_1 (v of G_2, respectively) which is not adjacent to any vertex of G_2 (G_1, respectively).

Let G' be the graph obtained from G by adding all possible edges in G_1 and G_2. Then $G' - E_c = K_{n_1} \cup K_{n_2}$. Obviously, $G' \in G(n, r)$. Let t be the number of vertices of G' which are at a distance of 2 from u. Note that $t \le r$. Since the diameter of G' is 3, we have

$$\begin{aligned} Tr_{G'}(u) &= n_1 - 1 + 2t + 3(n_2 - t) = n_1 + 3n_2 - 1 - t \\ &\ge n_1 + 3n_2 - 1 - r. \end{aligned}$$

Similarly,

$$Tr_{G'}(v) \ge n_2 + 3n_1 - 1 - r.$$

Let M be the principal submatrix of $D_\alpha(G')$ indexed by u and v. Then

$$M = \begin{pmatrix} \alpha Tr_{G'}(u) & 3(1-\alpha) \\ 3(1-\alpha) & \alpha Tr_{G'}(v) \end{pmatrix},$$

thus

$$\begin{aligned} \partial(M) &= \frac{\alpha(Tr(u) + Tr(v)) + \sqrt{\alpha^2(Tr(u) - Tr(v))^2 + 36(1-\alpha)^2}}{2} \\ &\ge \frac{\alpha(4n - 2r - 2) + \sqrt{4\alpha^2(n_2 - n_1)^2 + 36(1-\alpha)^2}}{2}. \end{aligned}$$

Now, using Lemma 2 and Interlacing Theorem [10], we have $\partial(G) \ge \partial(G') \ge \partial(M)$. From this the result follows. □

The following observation follows from Theorem 8.

Corollary 5 *Let* $G \in G(n, r)$ *with* $1 \le r \le n-2$ *and* $\frac{1}{2} \le \alpha \le 1$. *If the degree of every vertex of* G *is greater than* r, *then*

$$\partial(G) \ge \alpha(2n - r - 1) + 3(1-\alpha).$$

Proof. Using $(n_2 - n_1)^2 \geq 0$ in Theorem 8, the result follows. □

Taking $\alpha = \frac{1}{2}$ in Theorem 8 and using the fact $2\partial(G) = \rho_1^Q(G)$, we have the following observation, which gives a lower bound for the distance signless Laplacian spectral radius $\rho^Q(G)$ of a graph $G \in G(n, r)$.

Corollary 6 *Let $G \in G(n, r)$ with $1 \leq r \leq n-2$. If the degree of every vertex of G is greater than r, then*

$$\rho_1^Q(G) \geq 2n - r - 1 + \sqrt{(n_2 - n_1)^2 + 9},$$

where n_1 and n_2 are the cardinalities of the components of graph obtained from G by deleting r edges.

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Uniform approximation by smooth Picard multivariate singular integral operators revisited

George A. Anastassiou
Department of Mathematical Sciences,
University of Memphis,
Memphis, TN 38152, U.S.A.
email: ganastss@memphis.edu

Abstract. In this article we reexamine the uniform approximation properties of smooth Picard multivariate singular integral operators over $\mathbb{R}^N$, $N \geq 1$. We establish their convergence to the unit operator with rates. The estimates are pointwise and uniform. The established inequalities involve the multivariate first modulus of continuity. Our approach is based on a new multivariate trigonometric Taylor formula. At first we present in detail the general theory of uniform approximation by general smooth multivariate singular integral operators, which then is applied to the Picard operators case.

1 Introduction

The rate of convergence of univariate and multivariate singular integral operators has been studied extensively in [1]-[3] and [5], [6] and [8]. All these motivate our current work. In particular we studied the smooth singular integral operators in [1]-[3] and [6], which are not in general positive ones.

Here we continue the study of the last ones at the multivariate level, at first in general, and then apply our theory to the smooth Picard ones. The

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main tool here, we are based on, is a new trigonometric multivariate Taylor formula from [4]. Our quantitative estimates are pointwise and uniform, using the multivariate first modulus of continuity.

2 Results

Here $r \in \mathbb{N}$, $m \in \mathbb{Z}_+$, and define

$$\alpha_j := \alpha_{j,r}^{[m]} := \begin{cases} (-1)^{r-j} \binom{r}{j} j^{-m}, & \text{if } j = 1, 2, ..., r, \\ 1 - \sum\limits_{j=1}^{r} (-1)^{r-j} \binom{r}{j} j^{-m}, & \text{if } j = 0. \end{cases} \tag{1}$$

See that

$$\sum_{j=0}^{r} \alpha_{j,r}^{[m]} = 1, \tag{2}$$

and

$$-\sum_{j=1}^{r} (-1)^{r-j} \binom{r}{j} = (-1)^r \binom{r}{0}. \tag{3}$$

Let μ_{ξ_n} be a probability Borel measure on $\mathbb{R}^N$, $N \geq 1$, $\xi_n > 0$, $n \in \mathbb{N}$.
We now define the multiple smooth singular integral operators

$$\theta_n(f; x_1, ..., x_N) := \theta_{r,n}^{[m]}(f; x_1, ..., x_N) :=$$

$$\sum_{j=0}^{r} \alpha_{j,r}^{[m]} \int_{\mathbb{R}^N} f(x_1 + s_1 j, x_2 + s_2 j, ..., x_N + s_N j)\, d\mu_{\xi_n}(s), \tag{4}$$

where $s := (s_1, ..., s_N)$, $x := (x_1, ..., x_N) \in \mathbb{R}^N$; $n, r \in \mathbb{Z}$, $m \in \mathbb{Z}_+$, $f : \mathbb{R}^N \to \mathbb{R}$ is a Borel measurable function, and also $(\xi_n)_{n \in \mathbb{N}}$ is a bounded sequence of positive real numbers, we take $0 < \xi_n \leq 1$.

Remark 1 *The operators $\theta_{r,n}^{[m]}$ are not in general positive, see [2], p. 2.*

We observe that

Lemma 1 *It holds*

$$\theta_{r,n}^{[m]}(c; x_1, ..., x_n) = c,$$

where c is a constant.

We need

Definition 1 *Let* $f \in C\left(\mathbb{R}^N\right)$, $N \geq 1$. *We define the first modulus of continuity of* f *as*

$$\omega_1(f, \delta) := \sup_{\substack{x, y \in \mathbb{R}^N: \\ \|x-y\|_\infty \leq \delta}} |f(x) - f(y)|, \quad \delta > 0, \tag{5}$$

where $\|\cdot\|_\infty$ *is the max norm in* $\mathbb{R}^N$. *The functional* $\omega_1(f, \delta)$ *is bounded for* f *being bounded or uniformly continuous, and* $\omega_1(f, \delta) \to 0$ *as* $\delta \to 0$, *in the case of* f *being uniformly continuous.*

We present the main general approximation result regarding the operator θ_n.

Theorem 1 *Here* $f \in C^2\left(\mathbb{R}^N\right)$ *and let all* $\alpha_i \in \mathbb{Z}^+$, $i = 1, ..., N$, $N \geq 1$, $|\alpha| := \sum_{i=1}^{N} \alpha_i = 2$; $x \in \mathbb{R}^N$, *and all the partials* f_α *of order* 2, *along with* $f \in C_B\left(\mathbb{R}^N\right)$ *(continuous and bounded functions); or all* f_α *of order* 2, $f \in C_U\left(\mathbb{R}^N\right)$ *(uniformly continuous functions). Let* μ_{ξ_n} *be a Borel probability measure on* $\mathbb{R}^N$, *for* $0 < \xi_n \leq 1$, $n \in \mathbb{N}$.

Suppose that for all $\alpha := (\alpha_1, ..., \alpha_N)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, ..., N$, $|\alpha| = \sum_{i=1}^{N} \alpha_i = 2$, $j = 0, 1, ..., r$, *we have that both*

$$I_{1j}(\alpha) := \int_{\mathbb{R}^N} \left(1 + \frac{j\|s\|_1}{3\xi_n}\right) \left(\prod_{i=1}^{N} |s_i|^{\alpha_i}\right) d\mu_{\xi_n}(s), \tag{6}$$

$$I_{2j}(\alpha) := \int_{\mathbb{R}^N} \left(1 + \frac{j\|s\|_1}{3\xi_n}\right) d\mu_{\xi_n}(s), \tag{7}$$

are uniformly bounded in $\xi_n \in (0, 1]$.

Denote $(n \in \mathbb{N})$

$$\Delta_n(x) := \theta_n(f, x) - f(x) - \left(\sum_{j=0}^{r} \alpha_j j\right) \sin(1) \left[\sum_{i=1}^{N} \frac{\partial f(x)}{\partial x_i} \left(\int_{\mathbb{R}^N} s_i d\mu_{\xi_n}(s)\right)\right]$$

$$-2\left(\sum_{j=0}^{r} \alpha_j j^2\right) \sin^2\left(\frac{1}{2}\right) \left\{\sum_{i=1}^{N} \left(\int_{\mathbb{R}^N} s_i^2 d\mu_{\xi_n}(s)\right) \frac{\partial^2 f(x)}{\partial x_i^2} + \right. \tag{8}$$

$$\left.\sum_{\substack{i\neq j^*,\\ i,j^*\in\{1,\dots,N\}}}\left(\int_{\mathbb{R}^N} s_i s_{j^*}\,d\mu_{\xi_n}(s)\right)\frac{\partial^2 f(x)}{\partial x_i\partial x_{j^*}}\right\}.$$

Then

(i)

$$|\Delta_n(x)|\leq\|\Delta_n(x)\|_\infty\leq\sum_{j=0}^{r}|\alpha_j|$$

$$\left[\left[j^2\sum_{\substack{\alpha_i\in\mathbb{Z}^+,\\ \alpha:|\alpha|=2}}\left(\frac{1}{\prod_{i=1}^{N}\alpha_i!}\right)\omega_1(f_\alpha,\xi_n)\int_{\mathbb{R}^N}\left(1+\frac{j\|s\|_\infty}{3\xi_n}\right)\left(\prod_{i=1}^{N}|s_i|^{\alpha_i}\right)d\mu_{\xi_n}(s)\right]+\right.$$

$$\left.\frac{1}{2}\omega_1(f,\delta)\int_{\mathbb{R}^N}\left(1+\frac{j\|s\|_\infty}{3\xi_n}\right)d\mu_{\xi_n}(s)\right]=:\varphi_{\xi_n}. \tag{9}$$

In case of all f_α *of order* 2 *and* $f\in C_U(\mathbb{R}^N)$ *and* $\xi_n\to 0$*, as* $n\to\infty$*, then* $\Delta_n(x)$*,* $\|\Delta_n(x)\|_\infty\to 0$ *with rates.*

(ii) *If* $\frac{\partial f(x)}{\partial x_i}=0$*,* $i=1,\dots,N$*, and* $f_\alpha(x)=0$*,* $\alpha_i\in\mathbb{Z}^+$*,* $i=1,\dots,N$*, with* $|\alpha|=2$*, then*

$$|\theta_n(f,x)-f(x)|\leq\varphi_{\xi_n}. \tag{10}$$

And $\theta_n(f,x)\to f(x)$ *in the uniformly continuous case.*

(iii) *Additionally assume all partials of order* ≤ 2 *are bounded. Hence*

$$\|\theta_n(f)-f\|_\infty\leq\left(\sum_{j=0}^{r}|\alpha_j|\,j\right)(0.8414)\left[\sum_{i=1}^{N}\left\|\frac{\partial f}{\partial x_i}\right\|_\infty\left(\int_{\mathbb{R}^N}s_i\,d\mu_{\xi_n}(s)\right)\right]+$$

$$\left(\sum_{j=0}^{r}|\alpha_j|\,j^2\right)(0.4596)\left\{\sum_{i=1}^{N}\left(\int_{\mathbb{R}^N}s_i^2\,d\mu_{\xi_n}(s)\right)\left\|\frac{\partial^2 f}{\partial x_i^2}\right\|_\infty+\right.$$

$$\left.\sum_{\substack{i\neq j^*,\\ i,j^*\in\{1,\dots,N\}}}\left(\int_{\mathbb{R}^N}|s_i|\,|s_{j^*}|\,d\mu_{\xi_n}(s)\right)\left\|\frac{\partial^2 f}{\partial x_i\partial x_{j^*}}\right\|_\infty\right\}+\varphi_{\xi_n}. \tag{11}$$

If all $\int_{\mathbb{R}^N}s_i^2\,d\mu_{\xi_n}(s)$ *and* $\int_{\mathbb{R}^N}|s_i|\,|s_{j^*}|\,d\mu_{\xi_n}(s)$ *converge to zero, as* $n\to\infty$*, with* $\xi_n\to 0$*, and all* f_α *of order* 2*,* $f\in C_U(\mathbb{R}^N)$*, then*

$$\|\theta_n(f)-f\|_\infty\to 0 \text{ with rates, as } \xi_n\to 0,\ n\to+\infty.$$

Proof. Let $s := (s_1, ..., s_N)$, $x := (x_1, ..., x_N)$, $z := (z_1, ..., z_N) := (x_1 + s_1 j, x_2 + s_2 j, ... x_N + s_N j) = x + sj$; $j = 0, 1, ..., r$, and $x := x_0 = (x_{01}, ..., x_{0N}) = (x_1, ..., x_N)$, all in $\mathbb{R}^N$.

Here $f \in C^2(\mathbb{R}^N)$, $N \in \mathbb{N}$, and clearly all the mixed partials commnute.

Consider

$$g_{x+sj}(t) := f(x + t(sj)), \quad 0 \le t \le 1. \tag{12}$$

Notice that $g_{x+sj}(0) = f(x)$, $g_{x+sj}(1) = f(x + sj)$.

We have (by [4])

$$f(x+sj) - f(x) = g_{x+sj}(1) - g_{x+sj}(0) =$$

$$g'_{x+sj}(0)\sin(1) + 2g''_{x+sj}(0)\sin^2\left(\frac{1}{2}\right) +$$

$$\int_0^1 \left[\left(g''_{x+sj}(t) + g_{x+sj}(t)\right) - \left(g''_{x+sj}(0) + g_{x+sj}(0)\right)\right]\sin(1-t)\,dt =$$

$$\left(\sum_{i=1}^N (s_i j)\frac{\partial f}{\partial x_i}(x)\right)\sin(1) +$$

$$2\left\{\left[\left(\sum_{i=1}^N (s_i j)\frac{\partial}{\partial x_i}\right)^2 f\right](x)\right\}\sin^2\left(\frac{1}{2}\right) +$$

$$\int_0^1 \left\{\left\{\left[\left(\sum_{i=1}^N (s_i j)\frac{\partial}{\partial x_i}\right)^2 f\right](x + t(sj)) + f(x + t(sj))\right\} - \right. \tag{13}$$

$$\left.\left\{\left[\left(\sum_{i=1}^N (s_i j)\frac{\partial}{\partial x_i}\right)^2 f\right](x) + f(x)\right\}\right\}\sin(1-t)\,dt.$$

Denote the remainder $(j = 0, 1, ..., r)$

$$R_j := \int_0^1 \left\{\left\{\left[\left(\sum_{i=1}^N (s_i j)\frac{\partial}{\partial x_i}\right)^2 f\right](x + tsj) + f(x + tsj)\right\}\right.$$

$$\left. - \left\{\left[\left(\sum_{i=1}^N (s_i j)\frac{\partial}{\partial x_i}\right)^2 f\right](x) + f(x)\right\}\right\}\sin(1-t)\,dt = \tag{14}$$

$$\int_0^1 \left\{ \sum_{\substack{\alpha:=(\alpha_1,\ldots,\alpha_N),\alpha_i\in\mathbb{Z}_+ \\ i=1,\ldots,N,|\alpha|:=\sum_{i=1}^N \alpha_i=2}} \left(\frac{2}{\prod_{i=1}^N \alpha_i!} \right) \left(\prod_{i=1}^N (js_i)^{\alpha_i} \right) [f_\alpha (x + t(js)) - f_\alpha (x)] \right.$$

$$\left. + (f(x + t(js)) - f(x)) \right\} \sin(1-t)\, dt.$$

Therefore it holds

$$|R_j| \leq \int_0^1 \left\{ \sum_{\substack{\alpha:=(\alpha_1,\ldots,\alpha_N),\alpha_i\in\mathbb{Z}_+ \\ i=1,\ldots,N,|\alpha|:=\sum_{i=1}^N \alpha_i=2}} \left(\frac{2}{\prod_{i=1}^N \alpha_i!} \right) \right.$$

$$\left. \left(\prod_{i=1}^N (j|s_i|)^{\alpha_i} \right) |f_\alpha(x + tsj) - f_\alpha(x)| + |f(x + tsj) - f(x)| \right\} |\sin(1-t)|\, dt \leq \tag{15}$$

$$\int_0^1 \left\{ \sum_{\substack{\alpha:=(\alpha_1,\ldots,\alpha_N),\alpha_i\in\mathbb{Z}_+ \\ i=1,\ldots,N,|\alpha|:=\sum_{i=1}^N \alpha_i=2}} \left(\frac{2}{\prod_{i=1}^N \alpha_i!} \right) \left(\prod_{i=1}^N (j|s_i|)^{\alpha_i} \right) \omega_1 (f_\alpha, tj\|s\|_\infty) \right. \tag{16}$$

$$\left. + \omega_1 (f, tj\|s\|_\infty) \right\} |\sin(1-t)|\, dt \leq$$

$(0 < \xi_n \leq 1)$

$$\int_0^1 \left\{ \sum_{\substack{\alpha:=(\alpha_1,\ldots,\alpha_N),\alpha_i\in\mathbb{Z}_+ \\ i=1,\ldots,N,|\alpha|:=\sum_{i=1}^N \alpha_i=2}} \left(\frac{2}{\prod_{i=1}^N \alpha_i!} \right) \left(\prod_{i=1}^N (j|s_i|)^{\alpha_i} \right) \omega_1 (f_\alpha, \xi_n) \left(1 + \frac{tj\|s\|_\infty}{\xi_n} \right) \right.$$

$$\left. + \omega_1 (f, \xi_n) \left(1 + \frac{tj\|s\|_\infty}{\xi_n} \right) \right\} |\sin(1-t)|\, dt =$$

$$\left\{\left[2\sum_{\substack{\alpha:=(\alpha_1,\ldots,\alpha_N),\alpha_i\in\mathbb{Z}_+\\ i=1,\ldots,N,|\alpha|:=\sum_{i=1}^{N}\alpha_i=2}}\left(\frac{1}{\prod_{i=1}^{N}\alpha_i!}\right)\left(\prod_{i=1}^{N}(j\,|s_i|)^{\alpha_i}\right)\omega_1(f_\alpha,\xi_n)\right]+\omega_1(f,\xi_n)\right\}$$

$$\left[\int_0^1\left(1+\frac{tj\,\|s\|_\infty}{\xi_n}\right)|\sin(1-t)|\,dt\right]\leq \tag{17}$$

$$\left\{\left[2\sum_{\substack{\alpha:=(\alpha_1,\ldots,\alpha_N),\alpha_i\in\mathbb{Z}_+\\ i=1,\ldots,N,|\alpha|:=\sum_{i=1}^{N}\alpha_i=2}}\left(\frac{1}{\prod_{i=1}^{N}\alpha_i!}\right)\left(\prod_{i=1}^{N}(j\,|s_i|)^{\alpha_i}\right)\omega_1(f_\alpha,\xi_n)\right]+\omega_1(f,\xi_n)\right\}$$

$$\left[\int_0^1\left(1+\frac{tj\,\|s\|_\infty}{\xi_n}\right)(1-t)\,dt\right].$$

So far we have proved that

$$|R_j|\leq\left\{\left[2\sum_{\substack{\alpha_i\in\mathbb{Z}_+,\\ |\alpha|:=\sum_{i=1}^{N}\alpha_i=2}}\left(\frac{1}{\prod_{i=1}^{N}\alpha_i!}\right)\left(\prod_{i=1}^{N}(j\,|s_i|)^{\alpha_i}\right)\omega_1(f_\alpha,\xi_n)\right]+\omega_1(f,\xi_n)\right\}$$

$$\left[\int_0^1\left(1+\frac{tj\,\|s\|_\infty}{\xi_n}\right)(1-t)\,dt\right], \tag{18}$$

$j=0,1,\ldots,r;\ 0<\xi_n\leq 1.$

So, we have found that

$$|R_j|\leq\left[\left[j^2\sum_{\substack{\alpha_i\in\mathbb{Z}_+,\\ |\alpha|=2}}\left(\frac{1}{\prod_{i=1}^{N}\alpha_i!}\right)\left(\prod_{i=1}^{N}|s_i|^{\alpha_i}\right)\omega_1(f_\alpha,\xi_n)\right]+\frac{1}{2}\omega_1(f,\xi_n)\right] \tag{19}$$

$$\left[1+\frac{j\,\|s\|_\infty}{3\xi_n}\right],\quad j=0,1,\ldots,r;\ 0<\xi_n\leq 1.$$

Next we can write

$$\sum_{j=0}^{r}\alpha_j\left[f\left(x+sj\right)-f\left(x\right)\right]-\left(\sum_{j=0}^{r}\alpha_j j\right)\left(\sum_{i=1}^{N}s_i\frac{\partial f}{\partial x_i}\left(x\right)\right)\sin\left(1\right)- \tag{20}$$

$$2\left(\sum_{j=0}^{r}\alpha_j j^2\right)\left\{\left[\left(\sum_{i=1}^{N}s_i\frac{\partial}{\partial x_i}\right)^2 f\right]\left(x\right)\right\}\sin^2\left(\frac{1}{2}\right)=\sum_{j=0}^{r}\alpha_j R_j.$$

Call

$$R:=\sum_{j=0}^{r}\alpha_j R_j. \tag{21}$$

Hence it holds

$$|R|\leq\sum_{j=0}^{r}|\alpha_j|\,|R_j|\leq\sum_{j=0}^{r}|\alpha_j|$$

$$\left[\left[j^2\sum_{\left\{\substack{\alpha:=(\alpha_1,\dots,\alpha_N),\alpha_i\in\mathbb{Z}_+\\ i=1,\dots,N,|\alpha|:=\sum_{i=1}^{N}\alpha_i=2}\right\}}\left(\frac{1}{\prod_{i=1}^{N}\alpha_i!}\right)\left(\prod_{i=1}^{N}|s_i|^{\alpha_i}\right)\omega_1\left(f_\alpha,\xi_n\right)\right]+\frac{1}{2}\omega_1\left(f,\xi_n\right)\right] \tag{22}$$

$$\left[1+\frac{j\left\|s\right\|_\infty}{3\xi_n}\right]=$$

$$\sum_{j=0}^{r}|\alpha_j|\left[\left[j^2\sum_{\alpha:|\alpha|=2}\left(\frac{1}{\prod_{i=1}^{N}\alpha_i!}\right)\omega_1\left(f_\alpha,\xi_n\right)\left[1+\frac{j\left\|s\right\|_\infty}{3\xi_n}\right]\left(\prod_{i=1}^{N}|s_i|^{\alpha_i}\right)\right]\right.$$

$$\left.+\frac{1}{2}\omega_1\left(f,\xi_n\right)\left[1+\frac{j\left\|s\right\|_\infty}{3\xi_n}\right]\right], \tag{23}$$

$0<\xi_n\leq 1.$

See that

$$\begin{aligned}\sin 1 &\cong 0.8414\\(\sin 0.5)^2 &\cong (0.4794)^2 \cong 0.2298.\end{aligned}$$

Clearly, it holds

$$\theta_n(f,x) - f(x) = \sum_{j=0}^{r} \alpha_j \int_{\mathbb{R}^N} (f(x+sj) - f(x))\, d\mu_{\xi_n}(s). \tag{24}$$

We observe that

$$\Delta_n(x) = \sum_{j=0}^{r} \alpha_j \int_{\mathbb{R}^N} R_j\, d\mu_{\xi_n}(s). \tag{25}$$

We have that

$$\sum_{j=0}^{r} |\alpha_j| \int_{\mathbb{R}^N} |R_j|\, d\mu_{\xi_n}(s) \le$$

$$\sum_{j=0}^{r} |\alpha_j| \left[\left[j^2 \sum_{\alpha:|\alpha|=2} \left(\frac{1}{\prod_{i=1}^{N} \alpha_i!} \right) \omega_1(f_\alpha, \xi_n) \int_{\mathbb{R}^N} \left(1 + \frac{j\|s\|_\infty}{3\xi_n}\right) \left(\prod_{i=1}^{N} |s_i|^{\alpha_i} \right) d\mu_{\xi_n}(s) \right]\right.$$

$$\left. + \frac{1}{2}\omega_1(f,\xi_n) \int_{\mathbb{R}^N} \left(1 + \frac{j\|s\|_\infty}{3\xi_n}\right) d\mu_{\xi_n}(s) \right] = \varphi_{\xi_n}. \tag{26}$$

To remind (see also (6), (7))

$$\|s\|_\infty \le \sum_{i=1}^{N} |s_i| =: \|s\|_1, \tag{27}$$

hence the integrals in (9) ans (26) are uniformly bounded in $\xi_n \in (0,1]$.

Notice also that $(j = 0, 1, ..., r)$

$$\begin{gathered}\int_{\mathbb{R}^N} |s_i|\,|s_{j^*}|\, d\mu_{\xi_n}(s) \le I_{1j}(\alpha) < \infty,\\ \int_{\mathbb{R}^N} s_i^2 d\mu_{\xi_n}(s) \le I_{1j}(\alpha) < \infty,\end{gathered} \tag{28}$$

and

$$\int_{\mathbb{R}^N} |s_i|\, d\mu_{\xi_n}(s) \le \left(\int_{\mathbb{R}^N} s_i^2 d\mu_{\xi_n}(s) \right)^{\frac{1}{2}} < \infty, \tag{29}$$

by Hölder's inequality, and all of them are uniformly bounded in $\xi_n \in (0,1]$.

Thus, in the uniformly continuous case of f_α, $|\alpha| = 2$, and f we get $\varphi_{\xi_n} \to 0$, as $\xi_n \to 0$.

That is $\|\Delta_n(x)\|_\infty \to 0$, as $\xi_n \to 0$.

The proof of the theorem is now completed. □

We make

Remark 2 *Next we will apply Theorem 1 to specific multivariate smooth Picard singular integral operators*

$$P_n(f; x_1, ..., x_N) := P_{r,n}^{[m]}(f; x_1, ..., x_N) :=$$

$$\frac{1}{(2\xi_n)^N} \sum_{j=0}^{r} \alpha_{j,r}^{[m]} \int_{\mathbb{R}^N} f(x_1 + s_1 j, x_2 + s_2 j, ..., x_N + s_N j)\, e^{-\frac{\sum_{i=1}^{N} |s_i|}{\xi_n}} ds_1 ... ds_N, \quad (30)$$

$r, n \in \mathbb{N}$, $m \in \mathbb{Z}_+$, $0 < \xi_n \leq 1$.

Clearly here it is

$$d\mu_{\xi_n}(s) = \frac{1}{(2\xi_n)^N} e^{-\frac{\sum_{i=1}^{N} |s_i|}{\xi_n}} ds_1 ... ds_N, \quad s \in \mathbb{R}^N. \quad (31)$$

We observe that

$$\frac{1}{(2\xi_n)^N} \int_{\mathbb{R}^N} e^{-\frac{\sum_{i=1}^{N} |s_i|}{\xi_n}} ds_1 ... ds_N = 1, \quad (32)$$

see [3], Chap. 22, p. 356.

Here $\alpha_i \in \mathbb{Z}^+$, $i = 1, ..., N : |\alpha| = \sum_{i=1}^{N} \alpha_i = 2$. *We notice that*

$$\int_{\mathbb{R}^N} \left(\prod_{i=1}^{N} |s_i|^{\alpha_i} \right) e^{-\frac{\sum_{i=1}^{N} |s_i|}{\xi_n}} ds_1 ... ds_N \leq 4^N \xi_n^{N+2} \leq 4^N, \quad (33)$$

by [3], p. 364.

So (6), (7) are confirmed for $j = 0$ *when* $d\mu_{\xi_n}$ *is as in (31).*

We need

Theorem 2 *Let* $N \in \mathbb{N}$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, ..., N : |\alpha| = \sum_{i=1}^{N} \alpha_i = 2$, $\xi_n \in (0,1]$, $n \in \mathbb{N}$*;* $j = 1, 2, ..., r$. *Then*

$$I_{1j}^{*}(\alpha) := \frac{1}{(2\xi_n)^N} \int_{\mathbb{R}^N} \left(1 + \frac{j\,\|s\|_1}{3\xi_n}\right) \left(\prod_{i=1}^{N} |s_i|^{\alpha_i}\right) e^{-\frac{\sum_{i=1}^{N}|s_i|}{\xi_n}} ds_1 ... ds_N \leq \tag{34}$$

$$\xi_n^2 \left[\left(1 + \frac{j}{3}N\right) + \left(1 + \frac{j}{3}\right) \prod_{i=1}^{N} \left(\frac{\lfloor e(\alpha_i + 1)!\rfloor}{e}\right)\right] \leq$$

$$\left[\left(1 + \frac{j}{3}N\right) + \left(1 + \frac{j}{3}\right)\left(\frac{16}{e}\right)^N\right] < +\infty,$$

are uniformly bunded in $\xi_n \in (0,1]$*, fulfilling* (6)*. Above* $\lfloor\cdot\rfloor$ *is the integral part of the number symbol.*

Proof. Let $j = 1, ..., r$, then

$$I_{1j}^{*}(\alpha) = \frac{1}{(2\xi_n)^N} \int_{\mathbb{R}^N} \left(1 + \frac{j\,\|s\|_1}{3\xi_n}\right) \left(\prod_{i=1}^{N} |s_i|^{\alpha_i}\right) e^{-\frac{\sum_{i=1}^{N}|s_i|}{\xi_n}} ds_1 ... ds_N =$$

$$\frac{1}{\xi_n^N} \int_{\mathbb{R}_+^N} \left(1 + \frac{j\left(\sum_{i=1}^{N} s_i\right)}{3\xi_n}\right) \left(\prod_{i=1}^{N} s_i^{\alpha_i}\right) e^{-\frac{\sum_{i=1}^{N} s_i}{\xi_n}} ds_1 ... ds_N =$$

$$\xi_n^2 \int_{\mathbb{R}_+^N} \left(1 + \frac{j}{3}\left(\sum_{i=1}^{N} \frac{s_i}{\xi_n}\right)\right) \left(\prod_{i=1}^{N} \left(\frac{s_i}{\xi_n}\right)^{\alpha_i}\right) e^{-\sum_{i=1}^{N} \frac{s_i}{\xi_n}} d\left(\frac{s_1}{\xi_n}\right) ... d\left(\frac{s_N}{\xi_n}\right) = \tag{35}$$

$$\xi_n^2 \int_{\mathbb{R}_+^N} \left(1 + \frac{j}{3}\left(\sum_{i=1}^{N} z_i\right)\right) \left(\prod_{i=1}^{N} z_i^{\alpha_i}\right) e^{-\sum_{i=1}^{N} z_i} dz_1 ... dz_N =$$

$$\xi_n^2 \left[\int_{[0,1]^N} \left(1 + \frac{j}{3}\left(\sum_{i=1}^{N} z_i\right)\right) \left(\prod_{i=1}^{N} z_i^{\alpha_i}\right) e^{-\sum_{i=1}^{N} z_i} dz_1 ... dz_N + \right.$$

$$\left. \int_{(\mathbb{R}_+ - [0,1])^N} \left(1 + \frac{j}{3}\left(\sum_{i=1}^{N} z_i\right)\right) \left(\prod_{i=1}^{N} z_i^{\alpha_i}\right) e^{-\sum_{i=1}^{N} z_i} dz_1 ... dz_N\right] \leq$$

$$\xi_n^2 \left[\left(1 + \frac{j}{3} N \right) + \int_{(\mathbb{R}_+ - [0,1])^N} \left(1 + \frac{j}{3} \left(\sum_{i=1}^N z_i \right) \right) \left(\prod_{i=1}^N z_i^{\alpha_i} \right) e^{-\sum\limits_{i=1}^N z_i} dz_1 ... dz_N \right] \leq \tag{36}$$

$$\xi_n^2 \left[\left(1 + \frac{j}{3} N \right) + \left(1 + \frac{j}{3} \right) \int_{(\mathbb{R}_+ - [0,1])^N} \left(\sum_{i=1}^N z_i \right) \left(\prod_{i=1}^N z_i^{\alpha_i} \right) e^{-\sum\limits_{i=1}^N z_i} dz_1 ... dz_N \right] \leq$$

$$\xi_n^2 \left[\left(1 + \frac{j}{3} N \right) + \left(1 + \frac{j}{3} \right) \int_{(\mathbb{R}_+ - [0,1])^N} \left(\prod_{i=1}^N z_i \right) \left(\prod_{i=1}^N z_i^{\alpha_i} \right) \left(\prod_{i=1}^N e^{-z_i} \right) \left(\prod_{i=1}^N dz_i \right) \right] =$$

$$\xi_n^2 \left[\left(1 + \frac{j}{3} N \right) + \left(1 + \frac{j}{3} \right) \prod_{i=1}^N \int_1^\infty z_i^{\alpha_i + 1} e^{-z_i} dz_i \right] =$$

$$\xi_n^2 \left[\left(1 + \frac{j}{3} N \right) + \left(1 + \frac{j}{3} \right) \prod_{i=1}^N \int_1^\infty z_i^{(\alpha_i + 2) - 1} e^{-z_i} dz_i \right]$$

(by [7], p. 348)

$$\xi_n^2 \left[\left(1 + \frac{j}{3} N \right) + \left(1 + \frac{j}{3} \right) \prod_{i=1}^N \Gamma\left((\alpha_i + 2), 1 \right) \right], \tag{37}$$

where $\Gamma(\cdot, \cdot)$ is the upper incomplete gamma function.

We have proved that, $j = 1, ..., r$, that

$$I_{1j}^* (\alpha) \leq \xi_n^2 \left[\left(1 + \frac{j}{3} N \right) + \left(1 + \frac{j}{3} \right) \prod_{i=1}^N \Gamma\left((\alpha_i + 2), 1 \right) \right] \leq \tag{38}$$

$$\xi_n^2 \left[\left(1 + \frac{j}{3} N \right) + \left(1 + \frac{j}{3} \right) \prod_{i=1}^N \left(\frac{\lfloor e (\alpha_i + 1)! \rfloor}{e} \right) \right] \leq$$

$$\left[\left(1 + \frac{j}{3} N \right) + \left(1 + \frac{j}{3} \right) \left(\frac{16}{e} \right)^N \right] < +\infty,$$

therefore $I_{1j}^* (\alpha)$ are uniformly bounded.

Above we used the formula

$$\Gamma(s + 1, 1) = \frac{\lfloor es! \rfloor}{e}, \quad s \in \mathbb{N}. \tag{39}$$

Here $\alpha_i + 2 \in \mathbb{N}$, hence

$$\Gamma\left(\left(\alpha_i + 2\right), 1\right) = \frac{\lfloor e\left(\alpha_i + 1\right)!\rfloor}{e} \leq \frac{\lfloor e3!\rfloor}{e} = \frac{\lfloor 6e\rfloor}{e} = \frac{\lfloor 16.30968\rfloor}{e} = \frac{16}{e}. \tag{40}$$

The claim is proved. ☐

It follows

Theorem 3 *Let* $N \in \mathbb{N}$, $\xi_n \in (0,1]$, $n \in \mathbb{N}$; $j = 1, 2, ..., r$. *Then*

$$I_{2j}^{*}\left(\alpha\right) := \frac{1}{\left(2\xi_n\right)^N} \int_{\mathbb{R}^N} \left(1 + \frac{j\left(\sum_{i=1}^{N} |s_i|\right)}{3\xi_n}\right) e^{-\frac{\sum_{i=1}^{N}|s_i|}{\xi_n}} ds_1 ... ds_N \leq \tag{41}$$

$$\left[\left(1 + \frac{j}{3}N\right) + \left(1 + \frac{j}{3}\right)\left(\frac{2}{e}\right)^N\right] < +\infty,$$

are uniformly bounded in $\xi_n \in (0,1]$, *fulfilling* (7).

Proof. We have

$$I_{2j}^{*}\left(\alpha\right) = \frac{1}{\left(2\xi_n\right)^N} \int_{\mathbb{R}^N} \left(1 + \frac{j\left(\sum_{i=1}^{N} |s_i|\right)}{3\xi_n}\right) e^{-\frac{\sum_{i=1}^{N}|s_i|}{\xi_n}} ds_1 ... ds_N =$$

$$\frac{1}{\xi_n^N} \int_{\mathbb{R}_+^N} \left(1 + \frac{j\left(\sum_{i=1}^{N} s_i\right)}{3\xi_n}\right) e^{-\frac{\sum_{i=1}^{N} s_i}{\xi_n}} ds_1 ... ds_N = \tag{42}$$

$$\int_{\mathbb{R}_+^N} \left(1 + \frac{j}{3}\left(\sum_{i=1}^{N} z_i\right)\right) e^{-\sum_{i=1}^{N} z_i} dz_1 ... dz_N =$$

$$\int_{[0,1]^N} \left(1 + \frac{j}{3}\left(\sum_{i=1}^{N} z_i\right)\right) e^{-\sum_{i=1}^{N} z_i} dz_1 ... dz_N +$$

$$\int_{\left(\mathbb{R}_+ - [0,1]\right)^N} \left(1 + \frac{j}{3}\left(\sum_{i=1}^{N} z_i\right)\right) e^{-\sum_{i=1}^{N} z_i} dz_1 ... dz_N \leq$$

$$\left(1+\frac{j}{3}N\right)+\int_{(\mathbb{R}_{+}-[0,1])^{N}}\left(1+\frac{j}{3}\left(\sum_{i=1}^{N}z_{i}\right)\right)e^{-\sum_{i=1}^{N}z_{i}}dz_{1}...dz_{N}\leq$$

$$\left(1+\frac{j}{3}N\right)+\left(1+\frac{j}{3}\right)\int_{(\mathbb{R}_{+}-[0,1])^{N}}\left(\sum_{i=1}^{N}z_{i}\right)e^{-\sum_{i=1}^{N}z_{i}}dz_{1}...dz_{N}\leq$$

$$\left(1+\frac{j}{3}N\right)+\left(1+\frac{j}{3}\right)\int_{(\mathbb{R}_{+}-[0,1])^{N}}\left(\prod_{i=1}^{N}z_{i}\right)\left(\prod_{i=1}^{N}e^{-z_{i}}\right)\left(\prod_{i=1}^{N}dz_{i}\right)=$$

$$\left(1+\frac{j}{3}N\right)+\left(1+\frac{j}{3}\right)\prod_{i=1}^{N}\int_{1}^{\infty}z_{i}e^{-z_{i}}dz_{i}= \tag{43}$$

$$\left(1+\frac{j}{3}N\right)+\left(1+\frac{j}{3}\right)\prod_{i=1}^{N}\int_{1}^{\infty}z_{i}^{2-1}e^{-z_{i}}dz_{i}=$$

$$\left(1+\frac{j}{3}N\right)+\left(1+\frac{j}{3}\right)\prod_{i=1}^{N}\Gamma(2,1)=$$

$$\left(1+\frac{j}{3}N\right)+\left(1+\frac{j}{3}\right)\left(\frac{\lfloor e\rfloor}{e}\right)^{N}=$$

$$\left(1+\frac{j}{3}N\right)+\left(1+\frac{j}{3}\right)\left(\frac{2}{e}\right)^{N}<+\infty.$$

□

We make

Remark 3 *By* (28), (29), *Remark* (2), *and Theorem 2, we observe that* ($j=0,1,...,r$)

$$\frac{1}{(2\xi_{n})^{N}}\int_{\mathbb{R}^{N}}|s_{i}|\,|s_{j^{*}}|\,e^{-\frac{\sum_{i=1}^{N}|s_{i}|}{\xi_{n}}}ds_{1}...ds_{N},$$

$$\frac{1}{(2\xi_{n})^{N}}\int_{\mathbb{R}^{N}}s_{i}^{2}e^{-\frac{\sum_{i=1}^{N}|s_{i}|}{\xi_{n}}}ds_{1}...ds_{N}, \tag{44}$$

$$\frac{1}{(2\xi_{n})^{N}}\int_{\mathbb{R}^{N}}|s_{i}|\,e^{-\frac{\sum_{i=1}^{N}|s_{i}|}{\xi_{n}}}ds_{1}...ds_{N},$$

are uniformly bounded in $\xi_n\in(0,1]$ *and they converge to zero as* $\xi_n\to 0$.

Based on Theorem 1, Remark 2, Theorem 2, Theorem 3 and Remark 3, we present our major result about approximation properties of smooth Picard singular integral operators P_n, see (30).

Theorem 4 *Here* $f \in C^2\left(\mathbb{R}^N\right)$ *and let* $\alpha_i \in \mathbb{Z}^+$, $i = 1, ..., N$, $N \geq 1$, $|\alpha| := \sum_{i=1}^{N} \alpha_i = 2$; $x \in \mathbb{R}^N$, *and* f_α *of order* 2, $f \in C_B\left(\mathbb{R}^N\right) \cup C_U\left(\mathbb{R}^N\right)$, *and* $0 < \xi_n \leq 1$, $n \in \mathbb{N}$.

Denote ($n \in \mathbb{N}$)

$$\overline{\Delta}_n(x) := P_n(f,x) - f(x) -$$

$$\left(\sum_{j=0}^{r} \alpha_j j\right) \sin(1) \left[\sum_{i=1}^{N} \frac{\partial f(x)}{\partial x_i} \left(\frac{1}{(2\xi_n)^N} \int_{\mathbb{R}^N} s_i e^{-\frac{\sum_{i=1}^{N}|s_i|}{\xi_n}} ds_1 ... ds_N\right)\right] \tag{45}$$

$$-2\left(\sum_{j=0}^{r} \alpha_j j^2\right) \sin^2\left(\frac{1}{2}\right) \left\{\sum_{i=1}^{N} \left(\frac{1}{(2\xi_n)^N} \int_{\mathbb{R}^N} s_i^2 e^{-\frac{\sum_{i=1}^{N}|s_i|}{\xi_n}} ds_1 ... ds_N\right) \frac{\partial^2 f(x)}{\partial x_i^2}\right.$$

$$\left. + \sum_{\substack{i \neq j^*, \\ i, j^* \in \{1,...,N\}}} \left(\frac{1}{(2\xi_n)^N} \int_{\mathbb{R}^N} s_i s_{j^*} e^{-\frac{\sum_{i=1}^{N}|s_i|}{\xi_n}} ds_1 ... ds_N\right) \frac{\partial^2 f(x)}{\partial x_i \partial x_{j^*}}\right\}.$$

Then

(i)

$$\left|\overline{\Delta}_n(x)\right| \leq \left\|\overline{\Delta}_n(x)\right\|_\infty \leq$$

$$\sum_{j=0}^{r} |\alpha_j| \left[\left[\left[j^2 \sum_{\substack{\alpha_i \in \mathbb{Z}^+, \\ \alpha : |\alpha| = 2}} \left(\frac{1}{\prod_{i=1}^{N} \alpha_i!}\right) \omega_1(f_\alpha, \xi_n) \frac{1}{(2\xi_n)^N}\right.\right.\right.$$

$$\left.\int_{\mathbb{R}^N} \left(1 + \frac{j\|s\|_\infty}{3\xi_n}\right) \left(\prod_{i=1}^{N} |s_i|^{\alpha_i}\right) e^{-\frac{\sum_{i=1}^{N}|s_i|}{\xi_n}} ds_1 ... ds_N\right] + \tag{46}$$

$$\left.\frac{1}{2}\omega_1(f,\delta) \frac{1}{(2\xi_n)^N} \int_{\mathbb{R}^N} \left(1 + \frac{j\|s\|_\infty}{3\xi_n}\right) e^{-\frac{\sum_{i=1}^{N}|s_i|}{\xi_n}} ds_1 ... ds_N\right] =: \overline{\varphi}_{\xi_n}.$$

In case of all f_α *of order* 2 *and* $f \in C_U\left(\mathbb{R}^N\right)$ *and* $\xi_n \to 0$*, as* $n \to \infty$*, then* $\overline{\Delta}_n(x)$*,* $\left\|\overline{\Delta}_n(x)\right\|_\infty \to 0$ *with rates.*

(ii) *If* $\frac{\partial f(x)}{\partial x_i} = 0$*,* $i = 1, ..., N$*, and* $f_\alpha(x) = 0$*,* $\alpha_i \in \mathbb{Z}^+$*,* $i = 1, ..., N$*, with* $|\alpha| = 2$*, then*

$$|P_n(f, x) - f(x)| \le \overline{\varphi}_{\xi_n}. \tag{47}$$

And $P_n(f, x) \to f(x)$ *in the uniformly continuous case.*

(iii) *Additionally assume that all partials of order* ≤ 2 *are bounded. Hence*

$$\|P_n(f) - f\|_\infty \le \left(\sum_{j=0}^{r} |\alpha_j|\, j\right)(0.8414)$$

$$\left[\sum_{i=1}^{N} \left\|\frac{\partial f}{\partial x_i}\right\|_\infty \frac{1}{(2\xi_n)^N}\left(\int_{\mathbb{R}^N} |s_i|\, e^{-\frac{\sum_{i=1}^{N}|s_i|}{\xi_n}} ds_1 ... ds_N\right)\right]$$

$$+ \left(\sum_{j=0}^{r} |\alpha_j|\, j^2\right)(0.4596)$$

$$\left\{\sum_{i=1}^{N} \frac{1}{(2\xi_n)^N}\left(\int_{\mathbb{R}^N} s_i^2 e^{-\frac{\sum_{i=1}^{N}|s_i|}{\xi_n}} ds_1 ... ds_N\right)\left\|\frac{\partial^2 f}{\partial x_i^2}\right\|_\infty + \right.$$

$$\left.\sum_{\substack{i \ne j^*, \\ i, j^* \in \{1, ..., N\}}} \frac{1}{(2\xi_n)^N}\left(\int_{\mathbb{R}^N} |s_i|\, |s_{j^*}|\, e^{-\frac{\sum_{i=1}^{N}|s_i|}{\xi_n}} ds_1 ... ds_N\right)\left\|\frac{\partial^2 f}{\partial x_i \partial x_{j^*}}\right\|_\infty\right\} + \overline{\varphi}_{\xi_n}. \tag{48}$$

If all f_α *of order* 2*,* $f \in C_U\left(\mathbb{R}^N\right)$*, then*

$$\|P_n(f) - f\|_\infty \to 0 \text{ with rates, as } \xi_n \to 0,\ n \to +\infty.$$

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On the spectra of quasi join of graphs and families of integral graphs

Manash Protim Borah
Department of Mathematics,
LTK College, Lakhimpur, Assam, India
email: mpborah36@gmail.com

Karam Ratan Singh
Department of BAS, National Institute of Technology, Arunachal Pradesh, India
email: karamratan7@gmail.com

S. Pirzada
Department of Mathematics,
University of Kashmir, Srinagar, India
email: pirzadasd@kashmiruniversity.ac.in

Abstract. The subdivision graph $S(G)$ of a graph G is formed by adding a new vertex into every edge of G. The *quasi-corona subdivision-vertex join* of two graphs G and G' is a graph derived from $S(G)$ and G' by choosing a copy of $S(G)$ and n_1 copies of G' and then connecting those vertices of $S(G)$ which were in G to all the vertices of G'. The *quasi-corona subdivision-edge join* of two graphs G and G' is a graph derived from $S(G)$ and G' by choosing a copy of $S(G)$ and n_1 copies of G' and then connecting those vertices of $S(G)$ which were not in G to all the vertices of G'. The adjacency, Laplacian and signless Laplacian spectrum of these graphs are determined. As a consequence, we obtain some families of integral graphs, infinite families of cospectral graphs, the number of spanning trees and the Kirchhoff index.

1 Introduction

Let G and G' be two simple and finite graphs. Let n_1 and m_1, respectively, be the order and the size of G, and n_2 and m_2, respectively, be the order and

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the size of G'. Let $A(G) = [a_{ij}]$ be the *adjacency matrix* of G, whose vertex set is $V = \{v_1, v_2, \ldots, v_n\}$, with $a_{ij} = 1$ if the vertices v_i and v_j are adjacent ($v_i \sim v_j$) and 0 otherwise. Let $D(G)$ be the diagonal matrix. Let $Q(G) = D(G) - A(G)$ be the *Laplacian matrix* and $L(G) = D(G) + A(G)$ be the *signless Laplacian matrix* of G. Let $P_G(A : x) = \det(xI - A(G))$, $P_G(L : x) = \det(xI - L(G))$ and $P_G(Q : x) = \det(xI - Q(G))$, respectively, be the characteristic polynomials of $A(G)$, $L(G)$ and $Q(G)$, where I is the identity matrix. The roots of these characteristic equations are called the eigenvalues of $A(G)$, $L(G)$ and $Q(G)$. Let $\lambda_j(G)$, $\mu_j(G)$ and $\nu_j(G)$ denote the eigenvalues of $A(G)$, $L(G)$ and $Q(G)$, respectively, where $j = 1, 2, \ldots, n$. The set of eigenvalues with their multiplicities of $A(G)$, $L(G)$ or $Q(G)$ are respectively, called the A-spectrum, the L-spectrum and the Q-spectrum. If two graphs have the same A, L or Q spectrum, they are called A, L or Q-cospectral, respectively. The number of the *spanning trees* of G with n_1 vertices is given by $t(G) = \frac{\mu_2(G)\mu_3(G),\ldots,\mu_{n_1}(G)}{n_1}$. The *Kirchhoff index* of G is given by $Kf(G) = n_1 \sum_{i=2}^{n_1} \frac{1}{\mu_i(G)}$. A graph G is said to be an *integral graph* [4] if all of its eigenvalues are integers. More on definitions and notations from graph theory, we refer to [14].

Formulating the characteristic equations and obtaining the spectra of graphs are fundamental works in spectral graph theory. The spectra of graph operations including *complement, union, joins, corona operations* and *graph product* have been explored and obtained in [2, 3, 7, 9, 15] . The *subdivision graph* $S(G)$ [3] is formed by adding a new vertex to every edge of G.

Indulal [8] determined A-spectra of the subdivision-vertex and edge join of two regular graphs and obtained numerous infinite families of integral graphs. In [10], Liu and Zhang extended their findings by determining the A-spectra, L- spectra, and signless L-spectra of these joins of two graphs. Furthermore, they obtained the *spanning trees* and *Kirchhoff's index*. Subdivision graph-based corona operation have been discussed and established. The A-spectra, L-spectra and signless L-spectra have been investigate in [7, 11, 12]. Morover, the generalized distance spectrum of the join of graphs can be seen in [1]. Hou et al.[6] defined the *quasi-corona SG -vertex join* and the *multiple SG-vertex join* of the graph and obtained their adjacency spectra for two regular graphs. However, Hou et al. [6] considered R-*graph* [3] instead of *subdivision graph* $S(G)$ in the discussion.

We define two graphs corresponding to the above work: *quasi-corona subdivision-vertex join* and *quasi-corona subdivision-edge join.*

Definition 1 *The quasi-corona subdivision-vertex join of two graphs* G *and* G'*, represented by* $G\dot{\sqcup}G'$*, is the graph derived from* $S(G)$ *and* G' *by choosing* *a copy of* $S(G)$ *and* n_1 *copies of* G' *and then connecting those vertices of* $S(G)$ *which were in* G *to all the vertices of* G'. *This graph has* $n_1 + n_1m_1 + n_1n_2$ *vertices and* $n_1 + n_1m_2 + n_1^2n_2$ *edges.*

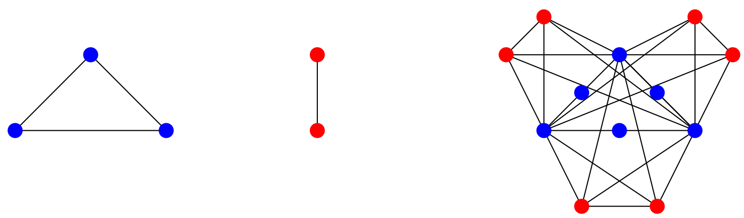

Figure 1: $K_3\dot{\sqcup}K_2$

Definition 2 *The quasi-corona subdivision-edge join of two graphs* G *and* G', *represented by* $G\underline{\sqcup}G'$, *is the graph derived from* $S(G)$ *and* G' *by choosing a copy of* $S(G)$ *and* n_1 *copies of* G' *and then connecting those vertices of* $S(G)$ *which were not in* G *to all the vertices of* G'. *This graph has* $n_1 + n_1m_1 + n_1n_2$ *vertices and* $n_1 + n_1m_2 + m_1^2n_2$ *edges.*

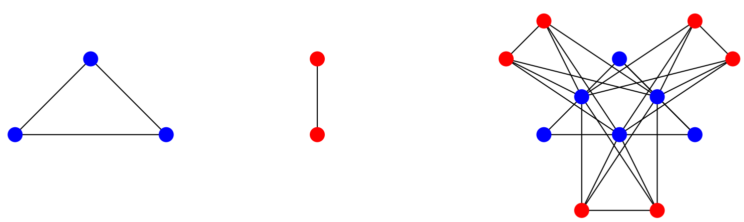

Figure 2: $K_3\underline{\sqcup}K_2$

The *Kronecker product* of two matrices $P = (p_{ij})$ of order $p_1 \times p_2$ and Q of order $q_1 \times q_2$ denoted by $P \otimes Q$, is defined to be the $p_1q_1 \times p_2q_2$ matrix $(p_{ij}Q)$[5]. For any four matrices R, S, T and U, $RS \otimes TU = (R \otimes T)(S \otimes U)$. Also, $(R \otimes S)^{-1} = R^{-1} \otimes S^{-1}$ if R and S are non-singular matrices. Moreover, if P and Q are $p \times p$ and $q \times q$ square matrices, then $\det(P \otimes Q) = (\det P)^p(\det Q)^q$. Let $B(G) = [b_{ij}]$ be the incident matrix of order $n_1 \times m_1$ with $b_{ij} = 1$ if v_i is incident with e_j, where $i, j = 1, 2, \ldots, n$, and 0 otherwise. Choosing $B(G) = B$, then $BB^T = A(Line(G)) + 2I_{m_1}$ and $BB^T = A(G) + 2I_{n_1}$, where $Line(G)$ is the line graph.

The M-Coronal $\Gamma_M(x)$ [13] is defined on the $n \times n$ matrix of M such that $\Gamma_M(x) = J_n^T(xI_n - M)^{-1}J_n$, where J_n is the $n \times 1$ matrix with all 1 entries.

If t is the constant of each row sum of matrix M, then from [13], we have $\Gamma_M(x) = \frac{n}{x-t}$.

If $L(G)$ is the Laplacian matrix, then $\Gamma_L(x) = \frac{n}{x}$, from [13].

Also, from [10], $\det(M + \gamma J_{n\times n}) = \det(M) + \gamma J_{n\times 1}^T \mathrm{adj}(M) J_{n\times 1}$, where $\mathrm{adj}(M)$ is the adjoint of M and γ is a real number. The following lemmas will be used in the sequel.

Lemma 1 [10] If M is an real matrix of $n \times n$, then

$$\det(xI_n - M - \gamma J_n) = (1 - \gamma\Gamma_M(x))\det(xI_n - M)$$

Lemma 2 [Schur complement] [3] Let N_1, N_2, N_3 and N_4 be four matrices, where N_1 and N_4 are non-singular square matrices, then

$$\det\begin{pmatrix} N_1 & N_2 \\ N_3 & N_4 \end{pmatrix} = \det(N_1).\det(N_4 - N_3 {N_1}^{-1} N_2) = \det(N_4).\det(N_1 - N_2 {N_4}^{-1} N_3)$$

The rest of the paper is organized as follows. In Section 2, we determine the A, L and Q-spectra of quasi-corona subdivision-vertex join of two graphs. In Section 3, we obtain the A, L and Q-spectra of quasi-corona subdivision-edge join of two graphs. In Section 4, we obtain infinite number of cospectral graphs, integral graphs, the number of the spanning trees and the Kirchhoff index.

2 Adjacency, Laplacian and signless Lapacian spectra of quasi-corona subdivision-vertex join

We begin this section with the following result on adjacency spectra.

Theorem 1 *If G is an r_1-regular graph and G' is any graph, then*

$$P_{G\dot{\sqcup}G'}(A : x) = x^{m_1 - n_1}\{x^2 - 2r_1 - \Gamma_{A(G')\otimes I_{n_1}}(x) n_1 x\} \prod_{j=2}^{n_2}\{x - \lambda_j(G')\}^{n_1}$$
$$\prod_{j=2}^{n_1}\left(x^2 - \lambda_j(G) - r_1\right).$$

Proof. After proper labelling, A-matrix of $G\dot{\sqcup}G'$ is

$$A(G\dot{\sqcup}G') = \begin{pmatrix} 0_{n_1\times n_1} & B & J_{n_1\times n_2}\otimes J_{n_1}^T \\ B^T & 0_{m_1\times m_1} & 0_{m_1\times n_2}\otimes J_{n_1}^T \\ J_{n_2\times n_1}\otimes J_{n_1} & 0_{n_2\times m_1}\otimes J_{n_1} & A(G')\otimes I_{n_1}. \end{pmatrix}$$

The characteristic polynomial is

$$\begin{aligned} P_{G\dot{\sqcup}G'}(A:x) &= \det(xI_{n_1 n_2+n_1+m_1} - A(G \rhd G')) \\ &= \det\begin{pmatrix} xI_{n_1} & -B & -J_{n_1\times n_2}\otimes J_{n_1}^T \\ -B^T & xI_{m_1} & 0_{m_1\times n_2}\otimes J_{n_1}^T \\ -J_{n_2\times n_1}\otimes J_{n_1} & 0_{n_2\times m_1}\otimes J_{n_1} & \{xI_{n_2}-A(G')\}\otimes I_{n_1} \end{pmatrix} \\ &= \det\{(xI_{n_2}-A(G'))\otimes I_{n_1}\}\det S \\ &= \det\{(xI_{n_2}-A(G')\}^{n_1}\det S, \end{aligned}$$

where

$$\begin{aligned} S &= \begin{pmatrix} xI_{n_1} & -B \\ -B^T & xI_{m_1} \end{pmatrix} - \begin{pmatrix} -J_{n_1\times n_2}\otimes J_{n_1}^T \\ 0\otimes J_{n_1}^T \end{pmatrix} \\ &\quad .\left(\{xI_{n_2}-A(G')\}^{-1}\otimes I_{n_1}\right)\begin{pmatrix} -J_{n_2\times n_1}\otimes J_{n_1} & 0_{n_2\times m_1}\otimes J_{n_1} \end{pmatrix} \\ &= \begin{pmatrix} xI_{n_1} & -B \\ -B^T & xI_{m_1} \end{pmatrix} - \begin{pmatrix} \Gamma_{A(G')\otimes I_{n_1}}(x)J_{n_1\times n_1} & 0_{n_1\times m_1} \\ 0_{m_1\times n_1} & 0_{m_1\times m_1} \end{pmatrix} \\ &= \begin{pmatrix} xI_{n_1} - \Gamma_{A(G')\otimes I_{n_1}}(x)J_{n_1\times n_1} & -B \\ -B^T & xI_{m_1} \end{pmatrix}. \end{aligned}$$

Thus,

$$\begin{aligned} \det S &= \det(xI_{m_1})\det\left\{xI_{n_1} - \Gamma_{A(G')\otimes I_{n_1}}(x)J_{n_1\times n_1} - B(xI_{m_1})^{-1}B^T\right\} \\ &= x^{m_1}\det\left\{xI_{n_1} - \Gamma_{A(G')\otimes I_{n_1}}(x)J_{n_1\times n_1} - \frac{BB^T}{x}\right\}. \end{aligned}$$

Therefore,

$$P_{G\dot{\sqcup}G'}(A:x) = x^{m_1}\det\{xI_{n_2}-A(G')\}^{n_1}\det\left\{xI_{n_1} - \Gamma_{A(G')\otimes I_{n_1}}(x)J_{n_1\times n_1} - \frac{BB^T}{x}\right\}$$

$$\begin{aligned} P_{G\dot{\sqcup}G'}(A:x) = x^{m_1}\det\{xI_{n_2}-A(G')\}^{n_1}&\left\{1 - \Gamma_{A(G')\otimes I_{n_1}}(x)\Gamma_{\frac{BB^T}{x}}(x)\right\} \\ &\det\left(xI_{n_1} - \frac{BB^T}{x}\right). \end{aligned}$$

Putting $\Gamma_{\frac{BB^T}{x}}(x) = \frac{n_1 x}{x^2 - 2r_1}$ and $BB^T = A(G) + rIn_1$, we get

$$P_{G\dot{\sqcup}G'}(A : x) = x^{m_1 - n_1}\left\{x^2 - 2r_1 - \Gamma_{A(G')\otimes I_{n_1}}(x) n_1 x\right\}\prod_{j=1}^{n_2}\{x - \lambda_j(G')\}^{n_1}$$
$$\prod_{j=1}^{n_1}\left(x^2 - \lambda_j(G) - r_1\right). \tag{1}$$

Using $\lambda_1(G) = r_1$ and $\lambda_1(G') = r_2$ in equation (1), the result follows. □

Remark. We have $\Gamma_{A(G')\otimes I_{n_1}}(x) = \frac{n_1 n_2}{x - r_2}$ if G' is a r_2 regular graph. Therefore, from equation (1), by using $\lambda_1(G') = r_2$, we get

$$P_{G\dot{\sqcup}G'}(A : x) = x^{m_1 - n_1}\{x^3 - r_2 x^2 - (2r_1 + n_1^2 n_2)x + 2r_1 r_2\}\prod_{j=2}^{n_2}\{x - \lambda_j(G')\}^{n_1}$$
$$\prod_{j=2}^{n_1}\left(x^2 - \lambda_j(G) - r_1\right). \tag{2}$$

Observations. From equation (2), we have the following. When G is an r_1 regular graph and G' is an r_2 regular graph, the A-spectrum of $G\dot{\sqcup}G'$ contains 0 with multiplicity $m_1 - n_1$, $\pm\sqrt{\lambda_j(G) + r_1}$, for each $j = 2, 3, \ldots, n_1$, $\lambda_j(G')$, for $j = 2, 3, \ldots, n_2$ with multiplicity n_1 and the roots of $x^3 - r_2 x^2 - (2r_1 + n_1^2 n_2)x + 2r_1 r_2 = 0$.

When G is an r_1-regular graph and $G' = K_{a,b}$, the A-spectrum of $G\dot{\sqcup}K_{a,b}$ contains 0 with multiplicity $m_1 + n_1(n_2 - 3)$, $\pm\sqrt{ab}$ with multiplicity $n_1 - 1$, $\pm\sqrt{r_1 + \lambda_j(G)}$, for $j = 2, 3, \ldots, n_1$ and the roots of $x^2 - 2r_1 - \Gamma_{A(K_{a,b})\otimes I_{n_1}}(x) n_1 x = 0$.

If F_1 and F_2 are both regular A-cospectral and F is any graph, then $F_1\dot{\sqcup}F$ and $F_2\dot{\sqcup}F$ are also A-cospectral. If F is a regular graph, and F_1 and F_2 are any two graphs that are both A-cospectral, then $F\dot{\sqcup}F_1$ and $F\dot{\sqcup}F_2$ are also cospectral. Now, we have the following result for the characteristic polynomial of the Laplacian matrix.

Theorem 2 *If G is an r_1-regular graph and G' is any graph, then*

$$P_{G\dot{\sqcup}G'}(L : x) = \prod_{j=2}^{n_2}\{x - n_1 - \mu_j(G')\}^{n_1}\prod_{j=2}^{n_1}\{x^2 - (r_1 + n_1 n_2 + 2)x + 2n_1 n_2 + \mu_j(G)\}$$
$$(x - 2)^{m_1 - n_1} x\{x^2 - (2 + r_1 + n_1 + n_1 n_2)x + 2n_1 + n_1 r_1 + 2n_1 n_2\}.$$

Proof. L-matrix of $G\dot{\sqcup}G'$ is

$$L(G\dot{\sqcup}G') = \begin{pmatrix} (r_1+n_1n_2)I_{n_1} & -B & -J_{n_1\times n_2}\otimes J_{n_1}^T \\ -B^T & 2I_{m_1} & 0_{m_1\times n_2}\otimes J_{n_1}^T \\ -J_{n_2\times n_1}\otimes J_{n_1} & 0_{n_2\times m_1}\otimes J_{n_1} & n_1I_{n_2}+L(G')\otimes I_{n_1} \end{pmatrix}.$$

The characteristic polynomial is $P_{G\dot{\sqcup}G'}(L:x)$

$$\begin{aligned} &= \det(xI_{n_1n_2+n_1+m_1} - L(G \rhd G')) \\ &= \det\begin{pmatrix} (x-r_1-n_1n_2)I_{n_1} & B & J_{n_1\times n_2}\otimes J_{n_1}^T \\ B^T & (x-2)I_{m_1} & 0_{m_1\times n_2}\otimes J_{n_1}^T \\ J_{n_2\times n_1}\otimes J_{n_1} & 0_{n_2\times m_1}\otimes J_{n_1} & ((x-n_1)I_{n_2}-L(G'))\otimes I_{n_1} \end{pmatrix} \\ &= \det\{((x-n_1)I_{n_2}-L(G'))\otimes I_{n_1}\}\det S, \end{aligned} \tag{3}$$

where

$$\begin{aligned} S &= \begin{pmatrix} (x-r_1-n_1n_2)I_{n_1} & B \\ B^T & (x-2)I_{m_1} \end{pmatrix} - \begin{pmatrix} J_{n_1\times n_2}\otimes J_{n_1}^T \\ 0_{m_1\times n_2}\otimes J_{n_1}^T \end{pmatrix} \\ &\quad .\begin{pmatrix} ((x-n_1)I_{n_2}-L(G'))^{-1}\otimes I_{n_1} \end{pmatrix}\begin{pmatrix} -J_{n_2}\otimes J_{n_1}^T & 0_{n_2\times n_1}\otimes J_{n_1}^T \end{pmatrix} \\ &= \begin{pmatrix} (x-r_1-n_1n_2)I_{n_1} & B \\ B^T & (x-2)I_{m_1} \end{pmatrix} - \begin{pmatrix} \Gamma_{L(G')\otimes I_{n_1}}(x-n_1)J_{n_1\times n_1} & 0_{n_1\times m_1} \\ 0_{m_1\times n_1} & 0_{m_1\times m_1} \end{pmatrix} \\ &= \begin{pmatrix} (x-r_1-n_1n_2)I_{n_1} - \Gamma_{L(G')\otimes I_{n_1}}(x-n_1)J_{n_1\times n_1} & B \\ B^T & (x-2)I_{m_1} \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} \det S &= \det\{(x-2)I_{m_1}\}\det\{(x-r_1-n_1n_2)I_{n_1} - \Gamma_{L(G')\otimes I_{n_1}}(x-n_1)J_{n_1\times n_1} - \\ &\qquad \frac{1}{x-2}BB^T\}. \\ &= \det\{(x-2)I_{m_1}\}\{1-\Gamma_{L(G')\otimes I_{n_1}}(x-n_1)\Gamma_{\frac{BB^T}{x-2}}(x-r_1-n_1n_2)\} \\ &\qquad \det\{(x-r_1-n_1n_2)I_{n_1} - \frac{1}{x-2}BB^T\}. \end{aligned} \tag{4}$$

Since

$$\Gamma_{\frac{BB^T}{x-2}}(x-r_1-n_1n_2) = \frac{n_1(x-2)}{x^2-(2+r_1+n_1n_2)x+2n_1n_2}$$

and

$$\Gamma_{L(G')\otimes I_{n_1}}(x-n_1) = \frac{n_1n_2}{x-n_1},$$

therefore, from equation (4), we get

$$\begin{aligned}
\det S &= \det\{(x-2)I_{m_1}\}\{1-\frac{n_1n_2}{x-n_1}.\frac{n_1(x-2)}{x^2-(2+r_1+n_1n_2)x+2n_2}\}\\
&\quad \det\left\{(x-r_1-n_1n_2)I_{n_1}-\frac{1}{x-2}(A(G)+r_1I_{n_1})\right\}\\
&= (x-2)^{m_1-n_1}\{x^3-(2+r_1+n_1+n_1n_2)x^2+(2n_1+n_1r_1+2n_1n_2)x\}\\
&\quad \prod_{j=1}^{n_1}\{x^2-(r_1+n_2+2)+2r_1+2n_2-A(G)+r_1\}.
\end{aligned}$$

In the above equation, applying the fact that $\lambda_i(G)=r_1-\mu_i(G)$, $\mu_1(G)=0$ and $\mu_1(G')=0$, then putting the value of $\det S$ in equation (3), we get the result. □

Observations. We have the following observations from Theorem (2). When G is an r_1- regular graph and G' is any graph, the L-spectrum of $G\dot{\sqcup}G'$ contains 0, 2 with multiplicity m_1-n_1, $n_1+\mu_j(G')$, for each $j=2,3,\ldots,n_2$ with multiplicity n_1, roots of $x^2-(r_1+n_1n_2+2)x+2n_1n_2+\mu_j(G)=0$, $j=2,3,\ldots,n_1$ and roots of $x^2-(2+r_1+n_1+n_1n_2)x+(2n_1+n_1r_1+2n_1n_2)=0$. The L-spectrum of $G\dot{\sqcup}K_{n_2}$, when G is an r_1- regular graph and $G'=K_{n_2}$, contains 0, 2 with multiplicity m_1-n_1, n_1 with multiplicities n_1, n_1+1 with multiplicity n_1n_2, roots of $x^2-(r_1+n_1n_2+2)x+2n_1n_2+\mu_j(G)=0$, $j=2,3,4,\ldots,n_1$ and roots of $x^2-(2+r_1+n_1+n_1n_2)x+(2n_1+n_1r_1+2n_1n_2)=0$.

If F_1 and F_2 are two regular L-cospectral, and F is any graph, then $F_1\dot{\sqcup}F$ and $F_2\dot{\sqcup}F$ are also L-cospectral. If F is a regular graph, and F_1 and F_2 are two any graphs that are L-cospectral, then $F\dot{\sqcup}F_1$ and $F\dot{\sqcup}F_2$ are also L- cospectral. If F_1 and F_2 are two regular L-cospectral, and W_1 and W_2 are two any L-cospectral, then $F_1\dot{\sqcup}W_1$ and $F_2\dot{\sqcup}W_2$ are also L-cospectral graphs.

Theorem 3 *If G is an r_1-regular graph and G' is a r_2-regular graph, then*

$$\begin{aligned}
P_{G\dot{\sqcup}G'}(Q:x) &= (x-2)^{m_1-n_1}(x^3-ax^2+bx-4n_1n_2r_2)\prod_{j=1}^{n_2-1}\{x-r_1-n_1n_2-\\
&\quad \nu_j(G')\}^{n_1}\prod_{j=1}^{n_1-1}\{x^2-(r_1+n_1n_2+2)x+2n_1n_2+2r_1-\nu_j(G)\}
\end{aligned}$$

where $a=(2+r_1+n_1+2r_2+n_1n_2)$, *and* $b=2n_1+n_1r_1+2n_1n_2+n_1^2n_2+4r_2+2r_1r_2+2n_1n_2r_2-n_1^2n_2$

Proof. Q-matrix of $G\dot{\sqcup}G'$ is

$$Q(G\dot{\sqcup}G') = \begin{pmatrix} (r_1+n_1n_2)I_{n_1} & B & J_{n_1\times n_2}\otimes J_{n_1}^T \\ B^T & 2I_{m_1} & 0_{m_1\times n_2}\otimes J_{n_1}^T \\ J_{n_2\times n_1}\otimes J_{n_1} & 0_{n_2\times m_1}\otimes J_{n_1} & n_1I_{n_2}+Q(G')\otimes I_{n_1} \end{pmatrix}.$$

Remaining part of the proof is as same as above. □

Observations. If F_1 and F_2 are two regular Q-cospectral, and F is any regular graph, then $F_1\dot{\sqcup}F$ and $F_2\dot{\sqcup}F$ are also Q-cospectral. If F is a regular graph, and F_1 and F_2 are two any graphs that are Q-cospectral, then $F\dot{\sqcup}F_1$ and $F\dot{\sqcup}F_2$ are also Q- cospectral.

3 Adjacency, Laplacian and signless Laplacian spectra of quasi-corona subdivision-edge join

We start with the computation of the characteristic polynomial of adjacency matrix of quasi-corona subdivision-edge join.

Theorem 4 *If G is an r_1- regular graph and G' is any graph, then*

$$P_{G\underline{\sqcup}G'}(A:x) = x^{m_1-n_1}\{x^2-2r_1-\Gamma_{A(G')\otimes I_{n_1}}(x)m_1x\}\prod_{j=2}^{n_1}\left(x^2-\lambda_j(G)-r_1\right)$$
$$\prod_{j=2}^{n_2}\{x-\lambda_j(G')\}^{n_1}.$$

Proof. $A(G\underline{\sqcup}G')$ can be written as

$$\begin{pmatrix} 0_{n_1\times n_1} & B & 0_{n_1\times n_2}\otimes J_{n_1}^T \\ B^T & 0_{m_1\times m_1} & J_{m_1\times n_2}\otimes J_{n_1}^T \\ 0_{n_2\times n_1}\otimes J_{n_1} & J_{n_2\times m_1}\otimes J_{n_1} & A(G')\otimes I_{n_1} \end{pmatrix}.$$

The characteristic polynomial is

$$\begin{aligned} P_{G\underline{\sqcup}G'}(A:x) &= \det\begin{pmatrix} xI_{n_1} & -B & 0_{n_1\times n_2}\otimes J_{n_1}^T \\ -B^T & xI_{m_1} & -J_{m_1\times n_2}\otimes J_{n_1}^T \\ 0_{n_2\times n_1}\otimes J_{n_1} & -J_{n_2\times m_1}\otimes J_{n_1} & (xI_{n_2}-A(G'))\otimes I_{n_1} \end{pmatrix} \\ &= \det\{(xI_{n_2}-A(G'))\otimes I_{n_1}\}\det S \\ &= \det\{xI_{n_2}-A(G')\}^{n_1}\det S, \qquad (5) \end{aligned}$$

where

$$S = \begin{pmatrix} xI_{n_1} & -B \\ -B^T & xI_{m_1} \end{pmatrix} - \begin{pmatrix} 0_{n_1\times n_2} \otimes J_{n_1}^T \\ -J_{m_1\times n_2} \otimes J_{n_1}^T \end{pmatrix}$$
$$\left((xI_{n_2} - A(G'))^{-1} \otimes I_{n_1}\right)\left(0_{n_2\times n_1} \otimes J_{n_1} \quad - J_{n_2\times m_1} \otimes J_{n_1}\right)$$
$$= \begin{pmatrix} xI_{n_1} & -B \\ -B^T & xI_{m_1} - \Gamma_{A(G')\otimes I_{n_1}}(x) J_{m_1\times m_1} \end{pmatrix}$$

and

$$\det S = \det(xI_{n_1})\left\{1 - \Gamma_{A(G')\otimes I_{n_1}}(x)\Gamma_{\frac{BB^T}{x}}(x)\right\}\det\left(xI_{m_1} - \frac{BB^T}{x}\right).$$

Since

$$\Gamma_{\frac{BB^T}{x}}(x) = \frac{m_1 x}{x^2 - 2r_1}$$

and A-spectrum of the line graph $Line(G)$ is $\lambda_j(G) + r_1 - 2$, $j = 1, 2, \ldots, n_1$, therefore

$$\det S = x^{m_1 - n_1}\{x^2 - 2r_1 - \Gamma_{A(G')\otimes I_{n_1}}(x) m_1 x\}\prod_{i=2}^{n_1}\{x^2 - r_1 - \lambda_j(G)\}.$$

From equation (5), we get

$$P_{G\sqcup G'}(A : x) = x^{m_1 - n_1}\left\{x^2 - 2r_1 - \Gamma_{A(G')\otimes I_{n_1}}(x) m_1 x\right\}\prod_{j=1}^{n_2}\{x - \lambda_j(G')\}^{n_1}$$
$$\prod_{j=2}^{n_1}\left(x^2 - \lambda_j(G) - r_1\right). \tag{6}$$

□

Remark. If G' is an r_2-regular graph, then $\Gamma_{A(G')\otimes I_{n_1}}(x) = \frac{n_1 n_2}{x - r_1}$. Putting the above in equation (6), we get

$$P_{G\sqcup G'}(A : x) = x^{m_1 - n_1}\prod_{j=2}^{n_2}\{x - \lambda_j(G')\}^{n_1}\left(x^3 - r_2 x^2 - (2r_1 + n_1 n_2 m_1)x + 2r_1 r_2\right)$$
$$\prod_{j=2}^{n_1}\left(x^2 - \lambda_j(G) - r_1\right). \tag{7}$$

From equation (6), we have the following observations. The A-spectrum of $G\sqcup K_{a,b}$ contains the following eigenvalues when G is an r_1-regular graph and $G' = K_{a,b}$. 0 with multiplicity $m_1+n_1(a+b-3)$, $\pm\sqrt{ab}$ with multiplicity n_1-1, $\pm\sqrt{r_1+\lambda_j(G)}$, for each $j = 2, 3, \ldots, n_1$, roots of $x^2-2r_1-\Gamma_{A(K_{a,b})\otimes I_{n_1}}(x)m_1x = 0$. Similarly, the A-spectrum of $G\sqcup G'$ contains the following eigenvalues when G is an r_1 regular graph and G' is an r_2- regular graph. 0 with multiplicities $m_1 - n_1$, $\lambda_j(G')$, for each $j = 2, 3, \ldots, n_2$ with multiplicity n_1, roots of $x^2 - \lambda_j(G) - r_1 = 0$, where $j = 2, 3, \ldots, n_1$, roots of $x^3 - r_2x^2 - (2r_1 + n_1n_2m_1)x + 2r_1r_2 = 0$.

If F_1 and F_2 are both regular A-cospectral, and F is any graph, then $F_1\sqcup F$ and $F_2\sqcup F$ are also A-cospectral. If F is a regular graph, and F_1 and F_2 are two any graphs that are both A-cospectral, then $F\sqcup F_1$ and $F\sqcup F_2$ are also cospectral.

Now, we have the following theorem.

Theorem 5 *If G is an r_1- regular graph and G' be any graph, then*

$$P_{G\sqcup G'}(L:x) = x.(x-r_1)^{m_1-n_1}\{x^2-(2+n_1n_2+r_1+m_1)x+2m_1+r_1m_1+r_1n_1n_2\}$$
$$\prod_{j=2}^{n_1}\{x^2-(2+r_1+n_1n_2)x+r_1n_1n_2+\mu_j(G)\}\prod_{j=2}^{n_2}\{x-m_1-\mu_j(G')\}^{n_1}.$$

Proof. $L(G\sqcup G')$ can be expressed as

$$L(G\sqcup G') = \begin{pmatrix} r_1I_{n_1} & -B & 0_{n_1\times n_2}\otimes J_{n_1}^T \\ -B^T & (2+n_1n_2)I_{m_1} & -J_{m_1\times n_2}\otimes J_{n_1}^T \\ 0_{n_2\times n_1}\otimes J_{n_1} & -J_{n_2\times m_1}\otimes J_{n_1} & m_1I_{n_2}+L(G')\otimes I_{n_1}. \end{pmatrix}$$

The characteristic polynomial is $P_{G\sqcup G'}(L:x)$

$$\begin{aligned} &= \det\begin{pmatrix} (x-r_1)I_{n_1} & B & J_{n_1\times n_2}\otimes 0_{n_1}^T \\ B^T & (x-n_1n_2-2)I_{m_1} & J_{m_1\times n_2}\otimes J_{n_1}^T \\ 0_{n_2\times n_1}\otimes J_{n_1} & J_{n_2\times m_1}\otimes J_{n_1} & ((x-m_1)I_{n_2}-L(G'))\otimes I_{n_1} \end{pmatrix} \\ &= \det\{((x-m_1)I_{n_2}-L(G'))\otimes I_{n_1}\}\det S, \end{aligned} \tag{8}$$

where

$$S = \begin{pmatrix} (x-r_1)I_{n_1} & B \\ B^T & (x-n_1n_2-2)I_{m_1} \end{pmatrix} - \begin{pmatrix} 0_{n_1\times n_2}\otimes J_{n_1}^T \\ J_{m_1\times n_2}\otimes J_{n_1}^T \end{pmatrix}$$
$$.\Big(((x-m_1)I_{n_2}-L(G'))^{-1}\otimes I_{n_1}\Big)\begin{pmatrix} 0_{n_2\times n_1}\otimes J_{n_1}^T & J_{n_2\times m_1}\otimes J_{n_1}^T \end{pmatrix}$$

$$= \begin{pmatrix} (x-r_1)I_{n_1} & B \\ B^T & (x-n_1n_2-2)I_{m_1} - \Gamma_{L(G')\otimes I_{n_1}}(x-m_1)J_{m_1\times m_1} \end{pmatrix}.$$

Thus,

$$\begin{aligned} \det S &= \det\{(x-r_1)I_{n_1}\}\det\{(x-2-n_1n_2)I_{m_1} - \Gamma_{L(G')\otimes I_{n_1}}(x-m_1)J_{m_1\times m_1} - \\ &\qquad \frac{1}{x-r_1}BB^T\} \\ &= \det\{(x-r_1)I_{n_1}\}\{1-\Gamma_{L(G')\otimes I_{n_1}}(x-m_1)\Gamma_{\frac{BB^T}{x-r_1}}(x-2-n_1n_2)\} \\ &\qquad \det\{(x-2-n_1n_2)I_{m_1} - \frac{1}{x-r_1}BB^T\}. \end{aligned}$$

Since

$$\Gamma_{\frac{BB^T}{x-r_1}}(x-2-n_1n_2) = \frac{m_1(x-r_1)}{x^2-(2+r_1+n_1n_2)x+r_1n_1n_2}$$

and

$$\Gamma_{L(G')\otimes I_{n_1}}(x-m_1) = \frac{n_1n_2}{x-m_1},$$

therefore

$$\begin{aligned} \det S &= \det\{(x-r_1)I_{n_1}\}\{1-\frac{n_1n_2}{x-m_1}.\frac{m_1(x-r_1)}{x^2-(2+r_1+n_1n_2)x+r_1n_1n_2}\} \\ &\qquad \det\{(x-2-n_1n_2)I_{m_1} - \frac{1}{x-r_1}(A(L(G)+2I_{m_1})\}. \end{aligned}$$

Simplifying the above equation, we get

$$\begin{aligned} \det S &= x.(x-r_1)^{m_1-n_1}\{x^2-(2+n_1n_2+r_1+m_1)x+2m_1+r_1m_1+r_1n_1n_2\} \\ &\qquad \prod_{j=1}^{n_1}\{x^2-(2+r_1+n_1n_2)x+r_1n_1n_2+\mu_j(G)\}. \end{aligned} \tag{9}$$

Putting (9) in (8) and using the facts $\mu_1(G)=0$, $\mu_1(G')=0$ gives the result.□

From equation (9), we have the following observations. The L-spectrum of $G\sqcup G'$ contains the following eigenvalues when G is an r_1-regular graph and G' is any graph. 0, r_1 with multiplicity m_1-n_1, $m_1+\mu_j(G')$, where $j=2,3,\dots,n_2$ with multiplicity n_1, roots of $x^2-(2+r_1+n_1n_2)x+r_1n_1n_2+\mu_j(G))=0$, where $j=2,3,\dots,n_1$ and roots of $x^2-(2+n_1n_2+r_1+m_1)x+2m_1+r_1m_1+r_1n_1n_2)=0$.

The L-spectrum of $G \sqcup K_{n_2}$ contains the following eigenvalues if G is an r_1 regular graph and $G' = K_{n_2}$. 0, 2 with multiplicity $m_1 - n_1$, n_1 with multiplicity n_1, n_1+1 with multiplicity $n_1 n_2$, roots of $x^2-(r_1+n_1n_2+2)x+2n_1n_2+\mu_j(G) = 0$, where $j = 2, 3, \ldots, n_1$ and roots of $x^2-(2+n_1n_2+r_1+m_1)x+2m_1+r_1m_1+r_1n_1n_2 = 0$.

If F_1 and F_2 are two regular L-cospectral , and F is any graph, then $F_1 \sqcup F$ and $F_2 \sqcup F$ are also L-cospectral. If F is a regular graph, and F_1 and F_2 are two any graphs that are L-cospectral, then $F \sqcup F_1$ and $F \sqcup F_2$ are also L- cospectral. If F_1 and F_2 are two regular L-cospectral, and W_1 and W_2 are two any L-cospectral, then $F_1 \sqcup W_1$ and $F_2 \sqcup W_2$ are also L-cospectral.

Theorem 6 *If G is an r_1-regular graph and G' is a r_2-regular graph, then*

$$P_{G \sqcup G'}(Q:x) = (x-2-n_1n_2)^{m_1-n_1}(x^3-ax^2+bx+c)\prod_{j=1}^{n_2-1}\{x-m_1-\nu_j(G')\}^{n_1}\}$$
$$\prod_{j=1}^{n_1-1}\{x^2-(r_1+n_1n_2+2)x+r_1n_1n_2+2r_1-\nu_j(G)$$

where $a = 2+r_1+m_1+2r_2+n_1n_2$, $b = 2m_1+m_1n_1n_2+m_1r_1+4r_2+2n_1n_2r_2+2r_1r_2-n_1^2n_2$
and $c = r_1n_1n_2-2r_1r_2n_1n_2+n_1^2n_2r_1-m_1r_1n_1n_2$

Proof. Q-matrix of $G \sqcup G'$ is

$$Q(G \sqcup G') = \begin{pmatrix} r_1 I_{n_1} & B & 0_{n_1\times n_2}\otimes J_{n_1}^T \\ B^T & (2+n_1n_2)I_{m_1} & J_{m_1\times n_2}\otimes J_{n_1}^T \\ 0_{n_2\times n_1}\otimes J_{n_1} & J_{n_2\times m_1}\otimes J_{n_1} & m_1 I_{n_2}+Q(G')\otimes I_{n_1} \end{pmatrix}.$$

Remaining part of the proof is as same as above. □

Observations. If F_1 and F_2 are two regular Q-cospectral, and F is any regular graph, then $F_1 \dot{\sqcup} F$ and $F_2 \dot{\sqcup} F$ are also Q-cospectral. If F is a regular graph, and F_1 and F_2 are two any graphs that are Q-cospectral, then $F \dot{\sqcup} F_1$ and $F \dot{\sqcup} F_2$ are also Q- cospectral.

From the above observations, we show the existence of families of integral graphs.

Let G be an r_1-regular graph and G' be an r_2-regular graph. Then adjacency spectrum of $G \dot{\sqcup} G'$ is integral if and only if G' is integral. The eigenvalues are $\pm\sqrt{r_1+\lambda_i(G)}, i = 2, 3, 4, \ldots n$ and the three roots of $x^3-r_2x^2-(2r_1+n_1^2n_2)x+2r_1r_2 = 0$.

Let G be a r_1-regular graph and $K_{a,b}$ be complete bipartite graph. Then adjacency spectrum of $G\dot{\sqcup}K_{a,b}$ is integral with eigenvalues as $\pm\sqrt{r_1+\lambda_i(G)}$, $i = 2,3,4\dots n$ and the four solutions of the equation $x^4-\{n^2(a+b)+ab+2r\}x^2-2n^2xab+2rab=0$.

Let G be an r_1-regular graph and G' be an r_2-regular graph. Then adjacency spectrum of $G\underline{\sqcup}G'$ is integral and eigenvalues are $\pm\sqrt{r_1+\lambda_i(G)}$, $i = 2,3,4,\dots n$ and the three roots of $x^3-r_2x^2-(2r_1+n_1n_2m_1)x+2r_1r_2=0$.

Let G be an r-regular graph and $K_{a,b}$ be the complete bipartite graph. Then the adjacency spectrum of $G\underline{\sqcup}K_{a,b}$ is integral with eigenvalues $\pm\sqrt{ab}$, $\pm\sqrt{r+\lambda_i(G)}$,$i = 2,3,4\dots n$ and the four solutions of the equation $x^4-\{nm(a+b)+ab+2r\}x^2-2nmxab+2rab=0$.

Let G' be any totally disconnected graph, then $r_2=0$ and A-spectrum of $G\dot{\sqcup}G'$ has eigenvalues as $\pm\sqrt{r_1+\lambda_i(G)}$, $i=2,3,4,\dots n_1$, 0 with multiplicities m_1-n_1+1 and $\pm\sqrt{2r_1+n_1^2n_2}$.

Let G' be a totally disconnected graph with n_2 vertices, then $r_2=0$ and A-spectrum of $G\underline{\sqcup}G'$ has eigenvalues as $\pm\sqrt{r_1+\lambda_i(G)}, i=2,3,\dots n_1$, 0 with multiplicities m_1-n_1+1 and $\pm\sqrt{2r_1+n_1n_2m_1}$.

In particular if $G=K_{n_1}$ and $G'=\overline{K}_{n_2}$ then we have the following.

1. $K_{n_1}\rhd\overline{K}_{n_2}$ is integral if and only if $\pm\sqrt{r_1+\lambda_i(K_{n_1})}$, $i=2,3,\dots n_1$ and $\pm\sqrt{2r_1+n_1^2n_2}$ are integers.

2. $K_{n_1}\unrhd\overline{K}_{n_2}$ is integral if and only if $\pm\sqrt{r_1+\lambda_i(K_{n_1})}$, $i=2,3,\dots n_1$ and $\pm\sqrt{2r_1+n_1n_2m_1}$ are integers.

Also, if $G=K_{n_1,n_1}$ and $G'=\overline{K}_{n_2}$ then we have the following.

1. $K_{n_1n_1}\rhd\overline{K}_{n_2}$ is integral if and only if $\pm\sqrt{r_1+\lambda_i(K_{n_1n_1})}$, $i=2,3,\dots n_1$ and $\pm\sqrt{2r_1+n_1^2n_2}$ are integers.

2. $K_{n_1n_1}\unrhd\overline{K}_{n_2}$ is integral if and only if $\pm\sqrt{r_1+\lambda_i(K_{n_1n_1})}$, $i=2,3,\dots n_1$ and $\pm\sqrt{2r_1+n_1n_2m_1}$ are integers.

Some integral graphs are $K_3\rhd\overline{K}_5$, $K_3\rhd\overline{K}_{13}$, $K_3\unrhd\overline{K}_5$, $K_3\unrhd\overline{K}_{13}$ and $K_{8,8}\rhd\overline{K}_2$.

Kirchhoff index and spanning trees are also obtained from the earlier observations.

If G is an r_1- regular and G' is any graph. Then

1. $t(G\dot{\sqcup}G')=\frac{2^{m_1-n_1}(2n_1+n_1r_1+2n_1n_2)\prod_{j=2}^{n_1}\left(2n_1n_2+\mu_j(G)\right)\prod_{j=2}^{n_2}(n_1+\mu_j(G'))^{n_1}}{n_1+m_1+n_1n_2}$

2. $Kf(G\dot{\sqcup}G') = (m_1+n_1+m_1n_1)\left(\frac{m_1-n_1}{2}+\frac{2+r_1+n_1+n_1n_2}{2n_1+2n_1n_2+n_1r_1}+\sum_{i=2}^{n_2}\frac{n_1}{n_1+\mu_j(G')}+\right.$
$\left.\sum_{i=2}^{n_1}\frac{r_1+2+n_1n_2}{2n_1n_2+\mu_j(G)}\right)$.

If G is an r_1 regular graph and G' is any graph, then

1. $t(G\underline{\sqcup}G') = \frac{r_1^{m_1-n_1}(2m_1+m_1r_1+r_1n_1n_2)\prod_{j=2}^{n_2}(m_1+\mu_j(G'))^{n_1}\prod_{j=2}^{n_1}\left(r_1n_1n_2+\mu_j(G)\right)}{n_1+m_1+n_1n_2}$

2. $Kf(G\underline{\sqcup}G') = (m_1+n_1+m_1n_1)\left(\frac{m_1-n_1}{r_1}+\frac{2+r_1+m_1+n_1n_2}{2m_1+r_1n_1n_2+m_1r_1}+\sum_{i=2}^{n_2}\frac{n_1}{m_1+\mu_j(G')}\right.$
$\left.+\sum_{i=2}^{n_1}\frac{r_1+2+n_1n_2}{r_1n_1n_2+\mu_j(G)}\right)$.

4 Conclusion

The main findings of the paper is based on certain graph operations of two graphs so that the adjacency, Laplacian and signless Laplacian spectra are obtained. As an applications some families of integral graphs, co spectral graphs, spanning trees and Kirchhoff index are determined by using the results. Thus one may search for some other graph operations.

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On Hankel determinant problems of functions associated to the lemniscate of Bernoulli and involving conjugate points

Syed Zakar Hussain Bukhari
Department of Mathematics,
Mirpur University of Science and Technology (MUST),
Mirpur-10250(AJK), Pakistan
email: fatmi@must.edu.pk

Imran Khan
Department of Mathematics,
Mirpur University of Science and Technology (MUST),
Mirpur-10250(AJK), Pakistan
email: imranmaths24@gmail.com

Asma Munir
Department of Mathematics,
Mirpur University of Science and Technology
(MUST), Mirpur-10250(AJK), Pakistan
email: asmamunir.maths@must.edu.pk

Abstract. The notion of Hankel determinant H_q in univalent functions theory is initiated by Noonan and Thomas while studying it for areally mean multivalent mappings. This determinant has significant role while dealing with singularities and particularly it's important for analyzing integral coefficient. Fekete-Szegö functional used in the study of the area theorem is a particular case of this determinant. We explore a known class of holomorphic mappings which is related with the various classes of functions with conjugate symmetric points We also study upper bounds in different settings of the coefficients of these mappings. We also relate our exploration with the existing literature of the subject.

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1 Introduction

Suppose that an analytic function f is expressed in the following series form:

$$\mathsf{f}(z) = z + \sum_{j=2}^{\infty} \eta_j z^j, z \in \mathbb{E}_1^0 \tag{1}$$

where $\mathbb{E}_1^0 \subset \mathbb{E}_r^{z_0} = \{|z \in \mathbb{C} : z - z_0| < r\}$. We use $\mathcal{A}$ to represent the family of these functions. Also $\mathcal{S} \subset \mathcal{A}$ deputize for the family of one-to-one or univalent functions defined in $\mathbb{E}_1^0$. Let $\mathcal{Q}$ stand for the collection of functions $\hbar$ such that

$$\hbar(z) = 1 + \sum_{j=1}^{\infty} \vartheta_j z^j \ : \mathrm{Re}\,\hbar(z) > 0, z \in \mathbb{E}_1^0. \tag{2}$$

If for a Schwarz mapping w, we write $\mathsf{f}(z) = \mathsf{g}(w(z))$, where f and g are analytic in $\mathbb{E}_1^0$, then it is said that f is subordinate g, and mathematically, we write $\mathsf{f} \prec \mathsf{g}$.

A large number of subfamilies are related with the class $\mathcal{P}$ and some of its generalizations. These may include the family $\mathcal{S}^*$ of starlike and a related family $\mathcal{C}$ of convex mappings. These families are further studied with the order and arguments or in such a way that the function f maps on to the right half plane as well as some specific plane region. Ma and Minda as seen in [8] introduced two classes of analytic functions namely;

$$\mathcal{S}^*(\psi) = \left\{ \mathsf{g} \in \mathcal{A} : \frac{z\mathsf{g}'(z)}{\mathsf{g}(z)} \prec \psi(z) \quad (z \in \mathbb{E}_1^0) \right\}$$

and

$$\mathcal{C}(\psi) = \left\{ \mathsf{g} \in \mathcal{A} : \phi\,(z) = \frac{z\mathsf{g}'(z)}{\mathsf{g}(z)} \prec \psi(z) \quad (z \in \mathbb{E}_1^0 \right\},$$

where the function ψ is an analytic univalent function such that $\Re(\psi) > 0$ in $\mathbb{U}$ with $\psi(0) = 1$, $\psi'(0) > 0$ and g maps $z \in \mathbb{E}_1^0$ onto a region starlike with respect to 1 and the symbol $\prec$ denotes the subordination between two analytic functions ϕ and ψ. By varying the function ψ, several familiar families will be deduced as seen below:

(i) For $\psi = \frac{1+Az}{1+Bz}$ $(-1 \leq B \leq A \leq 1)$, we get the family $\mathcal{S}^*(A, B)$, see [5].

(ii) For $A = 1 - 2\alpha$ and $B = -1$, the family $\mathcal{S}^*(\alpha)$ is studied at large as seen in [11].

(iii) In case $\psi = 1 + \frac{2}{\pi^2}\left(\log \frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2$, the desired family is studied in [12].

Recently as seen in [7] and by choosing a particular function for ψ as above, inequalities related with coefficient bounds of some subfamiles of univalent functions have been discussed extensively.

A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{S}^{\ell B}$, if and only if

$$\left|\left(\frac{zf'(z)}{f(z)}\right)^2 - 1\right| < 1, z \in \mathbb{E}_1^0. \tag{3}$$

For $f \in \mathcal{S}^{\ell B}$, $\frac{zf'(z)}{f(z)}$ is bounded by the lemniscate of Bernoulli

$$\left\{\psi \in \mathbb{C} \text{ with } \operatorname{Re}(\psi) > 0 : \left|\psi^2 - 1\right| < 1\right\} \tag{4}$$

in the right half of the $w-$plane. In term of subordination, we say that $f \in \mathcal{S}^{\ell B}$, if and only if

$$\frac{zf'(z)}{f(z)} \prec \sqrt{1+z}, z \in \mathbb{E}_1^0. \tag{5}$$

The known family of functions starlike with respect to symmetric points were introduced by Sakaguchi. Subsequently, we make use the same idea along with (6) or (7) and define the family $\mathcal{S}_{SP}^{\ell B}$ of Sakaguchi functions associated with the lemniscate of the Bernoulli as:

$$\left|\left(\frac{zf'(z)}{f(z) - f(-z)}\right)^2 - 1\right| < 1, z \in \mathbb{E}_1^0. \tag{6}$$

Thus w maps $\mathbb{E}_1^0$ onto the the right half of the lemniscate of Bernoulli defined by the inequality $\operatorname{Re}(\psi) > 0 : |\psi^2 - 1| < 1$. It is obvious that $f \in \mathcal{S}_{SP}^{\ell B}$, iff

$$\frac{2zf'(z)}{f(z) - f(-z)} \prec \sqrt{1+z},\ z \in \mathbb{E}_1^0. \tag{7}$$

Let $f \in \mathcal{A}$. Then the family $\mathcal{S}_{SP}^{\ell B}$ is defined by

$$\left|\left(\frac{2zf'(z)}{f(z) + \overline{f(\overline{z})}}\right)^2 - 1\right| < 1,\ z \in \mathbb{E}_1^0, \tag{8}$$

where $\operatorname{Re}\frac{2zf'(z)}{f(z)+\overline{f(\overline{z})}} > 0$. Thus a mapping $f \in \mathcal{S}_{SCP}^{\ell B}$, if $\frac{2zf'(z)}{f(z)+\overline{f(\overline{z})}}$ lies to the right half of the lemniscate of Bernoulli as defined by (4). It is evident that $f \in \mathcal{S}_{SCP}^{\ell B}$, if it satisfies

$$\frac{2zf'(z)}{f(z)+\overline{f(\overline{z})}} \prec \sqrt{1+z},\ z \in \mathbb{E}_1^0, \tag{9}$$

where Re $\frac{2zf'(z)}{f(z)+\overline{f(\overline{z})}} > 0$. Sokol and Stankiewicz [15] and other [1, 15] introduced the same structure of other related families of these functions.

The coefficient bounds problem plays a significant role in dealing with the geometrical aspects of complex mappings. Hankel matrices or catalecticant matrices are square matrices, where ascending skew-diagonals from left to right are constants. These matrices are obtained for a sequence of outputs, when a realization of a hidden Markov model or a state-space model is required. Some decomposition of such matrices provide a mean of computing those matrices which define these realizations. This matrix is also obtained when signals are assumed useful for separation of non-stationary signals along with time-frequency representation. Certain techniques used in polynomial distributions are leading to the Hankel matrix and it results in obtaining weight parameters of their approximations.

The qth Hankel determinant $H_d(q,j)$ is studied in [9] and it can be defined as:

$$H_d(q,j) = \begin{vmatrix} \eta_j & \eta_{j+1} & \cdot & \cdot & \eta_{j+q-1} \\ \eta_{j+1} & \eta_{j+2} & \cdot & \cdot & \eta_{j+q-2} \\ \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \eta_{j+q-1} & \eta_{j+q-2} & \cdot & \cdot & \eta_{j+2q-2} \end{vmatrix},$$

where $q \geq 1$, $\eta_j : j = 2, 3, ...$ are the complex coefficients of an analytic function $f \in \mathcal{A}$. This determinant is also significant in the study of singularities, see [3]. This is particularly significant when analyzing integral coefficient in a power series, for detail, again we refer [3]. We also find its applications in the study of meromorphic functions. Fekete-Szegö problem $H_d(2,1) = \eta_3 - \eta_2^2$ is a particular form of the generalized functional $\eta_3 - \tau\eta_2^2$ for some τ real or complex. For τ real and $f \in \mathcal{S}$, the family of injective, one-to-one or univalent functions, Fekete and Szegö provided sharp estimates for $|\eta_3 - \tau\eta_2^2|$. As seen, it is just a combination of the first two coefficients that describe the known Gronwall's area problems. In addition, we know that the functional $\eta_2\eta_4 - \eta_3^2$ is equivalent to $H_d(2,2)$. For a few subclasses of holomorphic functions, this determinant $H_d(2,2)$ has been lately investigated and many authors have looked into the bounds of the functional $\eta_2\eta_4 - \eta_3^2$, see [3, 6]. Babalola [2] also investigated $H_d(3,1)$ for few other classes involving analytic mappings. Using a well-known

Toeplitz determinants, we find the upper bounds of $H_d(3,1)$ for functions connected to the lemniscate of Bernoulli $\Gamma_{\ell\beta}$.

2 Preliminaries

The following lemmas are used in our major results. In the subsequent lemma as seen in [8] on page 162, Section 4, we find bounds on $\vartheta_2 - \tau\vartheta_1^2$.

Lemma 1 *Let $\hbar(z) = 1 + \vartheta_1 z + \vartheta_2 z^2 + \ldots \in \mathcal{Q}$ be represented by (2). Then we have*

$$|\vartheta_2 - \tau\vartheta_1^2| \leq \begin{cases} -4\tau + 2, & \tau < 0, \\ 2, & 0 \leq \tau \leq 1, \\ 4\tau - 2, & \tau > 1. \end{cases}$$

For $\tau < 0$ or $\tau > 1$, we have the equality iff $\hbar(z) = \frac{1+z}{1-z}$ and $0 < \tau < 1$, we have the equality iff $\hbar(z) = \frac{1+z^2}{1-z^2}$ or its any rotation. If $\tau = 0$, the equality holds iff

$$\hbar(z) = (\frac{1}{2} + \frac{\eta}{2})\frac{1+z}{1-z} + (\frac{1}{2} - \frac{\eta}{2})\frac{1-z}{1+z}, \; 0 \leq \eta \leq 1$$

or its any rotation. However, the previous upper bound can be improved for

$$\left|\vartheta_2 - \tau\vartheta_1^2\right| + \tau|\vartheta_1|^2 \leq 2, \; 0 < \tau \leq \frac{1}{2}$$

and

$$\left|\vartheta_2 - \tau\vartheta_1^2\right| + (1+\tau)|\vartheta_1|^2 \leq 2, \frac{1}{2} < \tau \leq 1.$$

The following lemma also deals with the coefficients bounds for the functions in class $\mathcal{Q}$, when $\tau \in \mathbb{C}$.

Lemma 2 *If $\hbar(z) = 1 + \vartheta_1 z + \vartheta_2 z^2 + \ldots \in \mathcal{Q}$, then for $\tau \in \mathbb{C}$, we have*

$$|\vartheta_2 - \tau\vartheta_1^2| \leq 2\max\{1, |2\tau - 1|\}$$

This inequality is sharp. The equality is concerned with the function

$$\hbar_1(z) = \frac{1+z}{1-z} \quad or \quad \hbar_2(z) = \frac{1+z^2}{1-z^2}.$$

For the reference of aforementioned lemma, see [4]. The subsequently given lemma also addresses the estimation of the coefficients under specific constraints.

Lemma 3 *If $\hbar \in \mathcal{Q}$, then for some $x : |x| \le 1$, we have*

$$2\vartheta_2 = x(4-\vartheta_1^2)+\vartheta_1^2$$

and also for some $z : |z| \le 1$, we obtain

$$4\vartheta_3 = (\vartheta_1^2-4)\vartheta_1 x^2+\vartheta_1^3-2(\vartheta_1^2-4)\vartheta_1 x-2(\vartheta_1^2-4)(1-|x|^2)z.$$

For reference, see [13].

3 Discussions

In this section, we study some Hankel determinant related problems. The theorem below describes bounds estimates of the Fekete-Szegö functional $\eta_3 - \tau\eta_2^2$.

Theorem 1 *Let $f \in \mathcal{S}^{\ell B}_{SCP}$ be represented by (8) or equivalently, we have (9). Then the bounds on the Fekete-Szegö functional $\eta_3 - \tau\eta_2^2$ can be written as:*

$$|\eta_3-\tau\eta_2^2| \le \begin{cases} -\frac{1}{8}(2\tau+1), & \tau < -\frac{5}{2} \\ \frac{1}{2}, & -\frac{5}{2} \le \tau \le \frac{3}{2} \\ \frac{1}{8}(2\tau+1), & \tau > \frac{3}{2} \end{cases}.$$

Moreover, we can see that

$$\left|\eta_3-\tau\eta_2^2\right| + (2\tau+5)\,|\eta_2|^2 \le \frac{1}{2},\ -\frac{5}{2} < \tau \le -\frac{1}{2}$$

and

$$\left|\eta_3-\tau\eta_2^2\right| + (3-\tau)\,|\eta_2|^2 \le \frac{1}{2},\ \ -\frac{1}{2} < \tau \le \frac{3}{2}$$

These above results are sharp.

Proof. For the mapping $f \in \mathcal{S}^{\ell B}_{SCP}$, from the definition which is equivalent to (8), we see that $\frac{zf'(z)}{f(z)+\overline{f(\bar{z})}} \prec \frac{1}{2}\phi(z)$, when $\phi(z)=\sqrt{1+z}$. Assuming a functional $\hbar$ such that

$$\hbar(z) = \frac{1+\vartheta(z)}{1-\vartheta(z)} = 1+\vartheta_1 z+\vartheta_2 z^2+\ldots$$

Obviously $\vartheta(z) = \frac{\hbar(z)-1}{\hbar(z)+1}$. Thus, $\frac{2zf'(z)}{f(z)+\overline{f(\bar{z})}} = \phi\left(\vartheta(z)\right)$ or $\phi\left(\vartheta(z)\right) = \left(\frac{2\hbar(z)}{\hbar(z)+1}\right)^{\frac{1}{2}}$. Now we see that

$$\left(\frac{2\hbar(z)}{\hbar(z)+1}\right)^{\frac{1}{2}} = 1+\frac{1}{4}\vartheta_1 z+\left(\frac{1}{4}\vartheta_2-\frac{5}{32}\vartheta_1^2\right)z^2+\left(\frac{1}{4}\vartheta_3-\frac{5}{16}\vartheta_1\vartheta_2+\frac{13}{128}\vartheta_1^3\right)z^3+\ldots.$$

Similarly, we can write

$$\frac{2zf'(z)}{f(z)+\overline{f(\overline{z})}} = 1 + \eta_2 z + \eta_3 z^2 + \eta_4 z^3 + \ldots$$

Therefore, we conclude that

$$\eta_2 = \frac{1}{4}\vartheta_1, \tag{10}$$

$$\eta_3 = \frac{1}{4}\vartheta_1 - \frac{5}{32}\vartheta_1^2 \tag{11}$$

and also we see that

$$\eta_4 = \frac{1}{4}\vartheta_3 - \frac{5}{16}\vartheta_1\vartheta_2 + \frac{13}{128}\vartheta_1^3. \tag{12}$$

This implies that

$$\left|\eta_3 - \tau\eta_2^2\right| = \frac{1}{4}\left|\vartheta_2 - \frac{1}{8}(2\tau+5)\vartheta_1^2\right|.$$

Applying Lemma 1, we obtain the required result. The equality follows from the functions $\hbar_j(z)$, $j = 1, 2, 3, 4$, such that

$$\frac{z\hbar'(z)}{\hbar(z)} = \begin{cases} \sqrt{1+z} \text{ if } \tau < \frac{-5}{2} \text{ or } \tau > \frac{3}{2}, \\ \sqrt{1+z^2} \text{ if } \frac{-5}{2} < \tau < \frac{3}{2} \\ \sqrt{1+\phi(z)} \text{ if } \tau = \frac{-5}{2}, \\ \sqrt{1-\phi(z)}, \text{ if } \tau = \frac{3}{2}. \end{cases}$$

where $\phi(z) = \frac{z(z+\eta)}{1+\eta}$ with $0 \le \eta \le 1$. □

The subsequent theorem describes $|\eta_3 - \tau\eta_2^2|$, when τ is a complex number.

Theorem 2 *Let* $f \in \mathcal{S}_{\mathrm{SCP}}^{\ell\mathrm{B}}$ *and* τ *be a complex number. Then*

$$|\eta_3 - \tau\eta_2^2| \le \frac{1}{2}\max\left\{1; \frac{1}{4}\,|2\tau+1|\right\}.$$

Proof. From (10) and (12), we observe that

$$|\eta_3 - \tau\eta_2^2| \le \frac{1}{4}\left|\vartheta_2 - \frac{1}{8}(\tau+5)\vartheta_1^2\right|.$$

Thus application of Lemma 2 leads to the desired result. This result is sharp and equality holds for the functions

$$\frac{2zf'(z)}{f(z)+\overline{f(\overline{z})}} = \sqrt{1+z}$$

or

$$\frac{2zf'(z)}{f(z)+\overline{f(\overline{z})}} = \sqrt{1+z^2}.$$

□

Remark 1 *For* $\tau = 1$*,* $\mathrm{Hd}_2(1) = \alpha_3 - \alpha_2^2$ *and for* $f \in \mathcal{S}^*_{\ell\beta}$*,* $|\alpha_3 - \alpha_2^2| \leq \frac{1}{2}$.

In context of the lemniscate of Bernoulli and in the view of above Lemma 3, we state that:

Theorem 3 *Let* $f \in \mathcal{S}^{\ell B}_{SCP}$*. Then* $|\eta_2\eta_4 - \eta_3^2| \leq \frac{1}{4}$.

Proof. Keeping in view the values for η_2, η_3 and η_4 as given in (10), (11) and (12) respectively, we calculate $\eta_2\eta_4 - \eta_3^2$ as:

$$\begin{aligned}
\eta_2\eta_4 - \eta_3^2 &= \frac{1}{16}\left(\vartheta_1\vartheta_3 - \frac{5}{4}\vartheta_1^2\vartheta_2 + \frac{13}{32}\vartheta_1^4\right) - \left(\frac{1}{4}\vartheta_2 - \frac{5}{32}\vartheta_1^2\right)^2 \\
&= \frac{1}{16}\vartheta_1\vartheta_3 + \frac{1}{1024}\vartheta_1^4 - \frac{1}{16}\vartheta_2^2.
\end{aligned}$$

By taking $C = \left|\eta_2\eta_4 - \eta_3^2\right|, (4-\vartheta_1^2) = c$ and then assuming that $t = \vartheta_1 \in (0,2]$ and using the value of ϑ_2 and ϑ_3 in term of t, from Lemma 3, we write

$$C = \frac{1}{1024}\left|16t\left\{t^3 + 2ctx - ctx^2 + 2c(1-|x|^2)z - 16\{t^2 + xc\}^2 + t_1^4\right\}\right|.$$

After some simplification, we apply triangular inequality and replace $|x|$ by ρ to obtain

$$C = \frac{1}{1024}\left[t^4 + \{16t^2 + 16(4-t^2)\}(4-t^2)\rho^2 + 32t(4-t^2)(1-\rho^2)\right] = F(t,\rho).$$

On differentiating partially with ρ, we see that $\frac{\partial F(t,\rho)}{\partial \rho}$ is positive which means that the multivariable function $F(t,\rho)$ is increasing on the compact set $[0,1]$. Thus the greatest value occurs at $\rho = 1$. Therefore, we take $\max F(t,\rho) = G(t)$.

Considering G, we calculate G' and G'' and find that $G' > 0$ along with $G''(z) < 0$ for $t = 0$. Thus the max $G(t)$ occurs at $t = 0$. Therefore, we can write

$$|\eta_2\eta_4 - \eta_3^2| \leq \frac{1}{4}.$$

This is a sharp result and equality holds for the functions $\frac{zf'(z)}{f(z)+\overline{f(\overline{z})}} = \frac{1}{2}\sqrt{1+z^2}$ or $\frac{1}{2}\sqrt{1+z}$. □

In context of the lemniscate of Bernoulli, we determine the value of the modulus of $\eta_2\eta_3 - \eta_4$:

Theorem 4 *For* $f \in \mathcal{S}_{\mathrm{SCP}}^{\ell\mathrm{B}}$, *we have* $|\eta_2\eta_3 - \eta_4| \leq \frac{1}{2}$.

Proof. For $f \in \mathcal{S}_{\mathrm{SCP}}^{\ell\mathrm{B}}$, we can write □

$$\eta_2 = \frac{1}{4}\vartheta_1,\ \eta_3 = \frac{1}{4}\vartheta_2 - \frac{5}{32}\vartheta_1^2 \text{ and } \eta_4 = \frac{1}{4}\vartheta_3 - \frac{5}{16}\vartheta_1\vartheta_2 + \frac{1}{4}\vartheta_1^3,$$

which leads to

$$\begin{aligned}\eta_2\eta_4 - \eta_3^2 &= \frac{1}{16}\left(\vartheta_1\vartheta_2 - \frac{5}{8}\vartheta_1^3\right) - \left(\frac{1}{4}\vartheta_2 - \frac{5}{16}\vartheta_1\vartheta_2 + \frac{13}{128}\vartheta_1^3\right)\\ &= \frac{3}{8}\vartheta_1\vartheta_2 + \frac{1}{4}\vartheta_3 - \frac{9}{64}\vartheta_1^3\\ &= \frac{1}{64}(24\vartheta_1\vartheta_2 + 16\vartheta_3 - 9\vartheta_1^3).\end{aligned}$$

Therefore, in view of Lemma 3, we note that

$$\left|\eta_2\eta_4 - \eta_3^2\right| = \frac{1}{64}\left|\vartheta_1^3 + 2c\vartheta_1 x - c\vartheta_1 x^2 + 2c(1-|x|^2)z - 12\vartheta_1\{\vartheta_1^2 + xc\} + 9\vartheta_1^3\right|$$

where $4 - \vartheta_1^2 = c$. Applying triangle inequality, replacing $|x|$ with ρ, $|z|$ by 1 and assuming that $t > 0$, such that $\vartheta_1 = t \in [0, 2]$, we can write

$$\left|\eta_2\eta_4 - \eta_3^2\right| \leq \frac{1}{64}\left\{t^3 + 4(4-t^2)t\rho + 4(4-t^2)t\rho^2 + 8(4-t^2)(1-\rho^2)\right\}.$$

Let us consider that

$$F(t,\rho) = \frac{1}{64}\left\{t^3 + 4(4-t^2)t\rho + 4(4-t^2)t\rho^2 + 8(4-t^2)(1-\rho^2)\right\}. \tag{13}$$

We further suppose the upper bounds exist in the interior of $[0,2]\times[0,1]$. Differentiating 13 partially with ρ, we see that

$$\frac{\partial}{\partial\rho}\left(F(t,\rho)\right)=\frac{1}{64}\left\{4t(4-t^2)+8\rho(t-2)(4-t^2)\right\}.$$

For $0<\rho<1$ and fixed $t\in[0,2]$, we see that $\frac{\partial F(\omega,\rho)}{\partial\rho}<0$.This shows that $F(t,\rho)$ is decreasing which contradicts to our supposition. Hence, $\max F(t,\rho)=F(t,0)=G(t)$ and

$$G(t)=\frac{1}{64}[t^3-8t^2+32], G'(t)=\frac{1}{64}[3t^2-16t],$$

which shows that $G''(t)=\frac{1}{64}[6t-16]<0$ for $t=0$. Therefore, at $t=0$ a maximum is achieved. Hence, we obtain the required proof.

4 Conclusion

The Fekete-Szegö inequality denoted as F-S inequality is one of the inequalities involving certain coefficients related to the Bieberbach conjecture and associated with this inequality is the Hankel determinant, which is used in the investigations of the singularities and determination of integral coefficients. In this investigation, we studied F-S inequalities for certain mappings f as defined by (8) for which the image domain is related with the lemniscate of Bernoulli.

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Unipotent similarity for matrices over commutative domains

Grigore Călugăreanu
Department of Mathematics,
Babeş-Bolyai University
email: calu@math.ubbcluj.ro

Abstract. A unit u of a ring is called unipotent if $u-1$ is nilpotent. We characterize the similarity of 2×2 matrices over commutative domains, realized by unipotent matrices, i.e., $B = U^{-1}AU$ with unipotent matrix U.

1 Introduction

In this note R denotes an associative ring with identity (for short, *unital* ring), $U(R)$ the group of units, $N(R)$ the set of nilpotent elements and $\mathbb{M}_n(R)$ the corresponding matrix ring (i.e., the set of all $n \times n$ matrices with entries in R). An element u of a ring is called *unipotent* if $u-1$ is nilpotent. That is, $u = 1+t$ for some nilpotent t. Over any (unital) ring it is easy to check that unipotents are units.

Two elements $a, b \in R$ are called *conjugate* if there is a unit $u \in U(R)$ such that $b = u^{-1}au$. Two square matrices $A, B \in \mathbb{M}_n(R)$ are called *similar* if these are conjugate in the matrix ring $\mathbb{M}_n(R)$. In the sequel we consider the following

Definition. Two elements a, b of a ring R are *unipotent conjugate* if there is a unipotent u such that $b = u^{-1}au$, that is, if there is a nilpotent $t \in N(R)$ such that $b = (1+t)^{-1}a(1+t)$.

Clearly, unipotent conjugate elements are conjugate. Examples will show that the converse fails, even for special classes of elements (i.e., idempotents,

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nilpotents or units). Two square matrices $A, B \in \mathbb{M}_n(R)$ will be called *unipotent similar* if these are unipotent conjugate in the matrix ring $\mathbb{M}_n(R)$.

Our goal in this note is to find a *criterion* (Theorem 1) for two matrices in order to be unipotent similar. According to the above definition, two (square) matrices A, B are unipotent similar if we can find a unipotent matrix U such that $AU = UB$, or equivalently, a nilpotent matrix T, such that $A(I_n + T) = (I_n + T)B$. If we denote the entries of T by t_{ij}, $1 \le i, j \le n$, it is readily seen that this equality amounts to a linear system in the unknown entries of T. Hence, if the base ring R is a field, our problem is easy to solve using basic linear algebra. The divisibility relations in the characterization provided by Proposition 1, are no longer an issue.

However, we want to find a more general environment (that is, a larger class of rings) in which we still can use some basic linear algebra methods.

The first necessary restriction, in order to be able to use *determinants*, is that we suppose the ring R is commutative. The reader can use the excellent book of William K. Brown, "Matrices over commutative rings" in order to have a complete look of what remains true when passing from fields to arbitrary commutative rings (including a suitable notion of rank, solving linear systems of equations etc).

The second necessary restriction, in order to have a *known form of the nilpotent matrices*, is that we suppose the commutative ring R to be a domain (i.e., an integral domain). For $n = 2$, a matrix is nilpotent if and only if it has zero determinant and zero trace. For $n \ge 3$ there are conditions which characterize the nilpotent matrices, but more complicated (e.g., see [2]).

As mentioned in [1] (**4.13**), if R is a commutative domain with quotient field F, the rank of a matrix A over R (see first paragraph of Section 2) is just the classical rank of A when A is viewed as a matrix over F. Thus, when solving a linear system of equations over R, we can solve this over F and then find the conditions which assure the solution belongs to R. Of course, over F, we can use the Kronecker (Rouché) - Capelli theorem and Cramer's rule too.

In this note we describe the unipotent similarity for 2×2 matrices over (commutative) domains. As customarily, $[A, \mathbf{b}]$ denotes the augmented matrix.

2 The 2×2 matrix unipotent similarity

For a commutative ring R and any positive integer m, the ideal $D_m(A)$ of R generated by the $m \times m$ minors of a matrix A was called the m-*th determinantal ideal* of A and we put $D_0(A) = R$. These are used in order to define a rank notion for matrices over any commutative ring (the analogue of the maximum order of nonzero minors). Namely (see [1], chapter 4), these ideals form an ascending sequence of ideals

$$(0) = D_{n+1}(A) \subseteq D_n(A) \subseteq ... \subseteq D_1(A) \subseteq D_0(A) = R$$

and the *rank of* A is $\mathrm{rk}(A) := \max\{m : \mathrm{ann}_R(D_m(A)) = (0)\}$. Here, for an ideal I of R, $\mathrm{ann}_R(I)$ is the *annihilator* of I, that is, $\{a \in R : aI = (0)\}$.

As already mentioned in the introduction, if F is the quotient field of R and $A \in \mathbb{M}_n(R)$ then $\mathrm{rk}(A) = \mathrm{rank}_F(A)$.

Let R be a commutative domain, $A, B \in \mathbb{M}_2(R)$ and let T be a nilpotent 2×2 matrix. Then $T = \begin{bmatrix} x & y \\ z & -x \end{bmatrix}$ with $x^2 + yz = 0$ (that is, has zero trace and zero determinant) and A, B are unipotent similar if and only if there is a matrix T of the previous form such that $A(I_2 + T) = (I_2 + T)B$. We denote $A = [a_{ij}]$, $B = [b_{ij}]$. Since unipotent similar matrices are similar and so have the same determinant and the same trace, we first prove the following

Proposition 1 *Let* R *be a commutative domain and let* $A, B \in \mathbb{M}_2(R)$ *be such that* $\det(A) = \det(B)$ *and* $\mathrm{Tr}(A) = \mathrm{Tr}(B)$. *There exists a zero trace matrix* T *such that* $A(I_2 + T) = (I_2 + T)B$ *if and only if any of the following three conditions is fulfilled*

(i) *there exists* z *such that* $a_{21} + b_{21}$ *divides* $b_{21} - a_{21} - z(b_{22} - a_{11})$ *and* $2(a_{11} - b_{11}) + z(a_{12} + b_{12})$*;*

(ii) *there exists* y *such that* $a_{12} + b_{12}$ *divides* $a_{12} - b_{12} - y(b_{22} - a_{11})$ *and* $2(b_{11} - a_{11}) + y(a_{21} + b_{21})$*;*

(iii) *there exists* x *such that* $b_{22} - a_{11}$ *divides* $a_{12} - b_{12} - x(a_{12} + b_{12})$ *and* $b_{21} - a_{21} - x(a_{21} + b_{21})$.

Proof. We start with an unknown zero trace matrix $T = \begin{bmatrix} x & y \\ z & -x \end{bmatrix}$ and write $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 1+x & y \\ z & 1-x \end{bmatrix} = \begin{bmatrix} 1+x & y \\ z & 1-x \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$. This equality is equivalent to a linear system of 4 equations and 3 unknowns which we write

$$MX = \begin{bmatrix} a_{11} - b_{11} & -b_{21} & a_{12} \\ a_{12} + b_{12} & b_{22} - a_{11} & 0 \\ a_{21} + b_{21} & 0 & a_{22} - b_{11} \\ a_{22} - b_{22} & -a_{21} & b_{12} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_{11} - a_{11} \\ a_{12} - b_{12} \\ b_{21} - a_{21} \\ a_{22} - b_{22} \end{bmatrix} = N.$$

We consider this system over the quotient field F. Using $\det(A) = a_{11}a_{22} - a_{12}a_{21} = b_{11}b_{22} - b_{12}b_{21} = \det(B)$ and $\mathrm{Tr}(A) = a_{11} + a_{22} = b_{11} + b_{22} = \mathrm{Tr}(B)$, it can be shown that the system matrix M and the augmented matrix

$$[M|N] = \left[\begin{array}{ccc|c} a_{11}-b_{11} & -b_{21} & a_{12} & b_{11}-a_{11} \\ a_{12}+b_{12} & b_{22}-a_{11} & 0 & a_{12}-b_{12} \\ a_{21}+b_{21} & 0 & a_{22}-b_{11} & b_{21}-a_{21} \\ a_{22}-b_{22} & -a_{21} & b_{12} & a_{22}-b_{22} \end{array}\right],$$

both have rank 2.

For M we have just four 3×3 minors to check and for $[M|N]$ we have another twelve 3×3 minors to check. We skip the easy calculations.

According to Kronecker (Rouché) - Capelli theorem, the system is solvable and using Cramer's rule we choose an independent unknown and solve the system for the other two dependent unknowns. The initial linear system is equivalent to any two independent equations.

For instance, by Cramer's rule, for (i) we choose the first two equations

$$\begin{array}{rcl} (a_{11}-b_{11})x - b_{21}y & = & b_{11}-a_{11}-a_{12}z \\ (a_{12}+b_{12})x + (b_{22}-a_{11})y & = & a_{12}-b_{12} \end{array}.$$

By elimination we get $x\Delta = \Delta_x$, $y\Delta = \Delta_y$, with the determinant

$\Delta = \det\begin{bmatrix} a_{11}-b_{11} & -b_{21} \\ a_{12}+b_{12} & b_{22}-a_{11} \end{bmatrix} = a_{12}(a_{21}+b_{21})$,

$\Delta_x = \det\begin{bmatrix} b_{11}-a_{11}-a_{12}z & -b_{21} \\ a_{12}-b_{12} & b_{22}-a_{11} \end{bmatrix} = a_{12}[b_{21}-a_{21}-z(b_{22}-a_{11})]$ and

$\Delta_y = \det\begin{bmatrix} a_{11}-b_{11} & b_{11}-a_{11}-a_{12}z \\ a_{12}+b_{12} & a_{12}-b_{12} \end{bmatrix} = a_{12}[2(a_{11}-b_{11})+z(a_{12}+b_{12})]$.

That is, if $\Delta \neq 0$, the system is equivalent to

$$\begin{array}{rcl} a_{12}(a_{21}+b_{21})x & = & a_{12}[b_{21}-a_{21}-z(b_{22}-a_{11})] \\ a_{12}(a_{21}+b_{21})y & = & a_{12}[2(a_{11}-b_{11})+z(a_{12}+b_{12})] \end{array}.$$

If $a_{12} \neq 0$, by cancellation a solution (x, y, z) exists iff the condition (i) holds.

If $a_{12} = 0$ the (initial) system has the solution $x = -1$, $y = 0$ and $z = \dfrac{2(b_{11}-a_{11})}{b_{12}} = \dfrac{2b_{21}}{b_{22}-a_{11}}$ iff b_{12} divides $2(b_{11}-a_{11})$ or equivalently, $b_{22}-a_{11}$ divides $2b_{21}$.

Choosing other pairs of independent equations from the (initial) system we obtain the conditions (ii) and (iii), respectively. $\square$

Remarks. 1) If all $a_{21}+b_{21}=a_{12}+b_{12}=b_{22}-a_{11}=0$ then $B=adj(A)$, the adjugate. The conditions show that $A=B$ are diagonal with equal entries on the diagonal, i.e., $A=B=a_{11}I_2$, obviously unipotent similar.

2) In particular, if any of the divisibilities below hold, we can choose $z=0$ (resp. $y=0$ resp. $x=0$) for a solution (x,y,z).

(i) $a_{21}+b_{21}$ divides both $b_{21}-a_{21}$ and $2(a_{11}-b_{11})$;

(ii) $a_{12}+b_{12}$ divides both $b_{12}-a_{12}$ and $2(a_{11}-b_{11})$;

(iii) $b_{22}-a_{11}$ divides both $a_{12}-b_{12}$ and $b_{21}-a_{21}$.

Formally using fractions, accordingly, we have

(i) $x=\dfrac{b_{21}-a_{21}}{a_{21}+b_{21}}, y=\dfrac{2(a_{11}-b_{11})}{a_{21}+b_{21}}$, or

(ii) $x=\dfrac{a_{12}-b_{12}}{a_{12}+b_{12}}, z=\dfrac{2(b_{11}-a_{11})}{a_{12}+b_{12}}$, or

(iii) $y=\dfrac{a_{12}-b_{12}}{b_{22}-a_{11}}, z=\dfrac{b_{21}-a_{21}}{b_{22}-a_{11}}$.

Only one more condition is necessary in order to describe the unipotent similarity for 2×2 matrices over commutative domains.

Theorem 1 *Let R be a commutative domain and let $A, B \in \mathbb{M}_2(R)$ be such that $\det(A)=\det(B)$ and $\mathrm{Tr}(A)=\mathrm{Tr}(B)$. The matrices A, B are unipotent similar if and only if any of the conditions (i), (ii) or (iii) in Proposition 1 holds and for the corresponding solution, the quadratic equation in z (resp. y or x) $x^2+yz=0$ is solvable. Accordingly, any of the corresponding quadratic equations should be solvable*

(i) $[b_{21}-a_{21}-z(b_{22}-a_{11})]^2+[2(a_{11}-b_{11})+z(a_{12}+b_{12})](a_{21}+b_{21})z=0$, *or*

(ii) $[a_{12}-b_{12}-y(b_{22}-a_{11})]^2+[2(b_{11}-a_{11})+y(a_{21}+b_{21})](a_{21}+b_{21})y=0$, *or*

(iii) $(b_{22}-a_{11})^2x^2-[a_{12}-b_{12}-x(a_{12}+b_{12})][a_{21}-b_{21}+x(a_{21}+b_{21})]=0$.

3 Examples

In this section, using the characterization proved in the previous section, we mainly give examples of *similar matrices which are not unipotent similar*. The examples are over the integers. Among these examples we choose idempotents, nilpotents and units. In the next five examples, to simplify the exposition, the *similarity* of the pair of 2×2 matrices is given by $U = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$, with $U^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. Actually, we check the solvability of the quadratic equations (Theorem 1) on the examples below.

(1) $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = U^{-1}AU = \begin{bmatrix} 6 & -2 \\ 2 & -1 \end{bmatrix}$.

The linear system reduces to $5x = 2z - 1$, $y = -2$ and so (for example (i); equivalently, (ii) or (iii)) $25(x^2 + yz) = (2z - 1)^2 - 50z = 0$ has *no* integer solutions. According to Theorem 1, *these similar matrices are not unipotent similar over* $\mathbb{Z}$. Here A, B have no special property: $\det(A) = \det(B) = -2$, $\mathrm{Tr}(A) = \mathrm{Tr}(B) = 5$.

(2) $E = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$, $F = U^{-1}EU = \begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix}$.

The linear system (for example (ii)) reduces to $x = 3 + 2y$, $z = 2 + 2y$ and $x^2 + yz = (3y+2)^2 + 2y(y+1) = 11y^2 + 14y + 4 = 0$ with *no* rational solutions. According to Theorem 1, *these similar idempotents (indeed, zero determinants and traces* $= 1$) *are not unipotent similar over* $\mathbb{Z}$.

(3) $N = E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $N_1 = U^{-1}NU = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$.

The linear system (for example (iii)) reduces to $y = -2$, $z = -1 + x$ and $x^2 + yz = x^2 - 2x + 2 = (x-1)^2 + 1 = 0$ with *no* real solutions. According to Theorem 1, *these similar nilpotents (indeed, zero determinants and zero traces) are not unipotent similar over* $\mathbb{Z}$.

(4) $V = E_{12} + E_{21} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $V_1 = U^{-1}VU = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$.

The linear system (for example (i)) reduces to $2x = z$, $2y = -2 + z$ and $4(x^2 + yz) = z(3z - 4) = 0$, which, over any integral domain where 3 is not a unit (e.g., over $\mathbb{Z}$), has only the solution $z = 0$. Accordingly $x = 0$ and $y = -1$ and indeed, for $T = -E_{12}$, that is $I_2 + T = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$, $V(I_2 + T) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} = (I_2 + T)V_1$.

Therefore (as the matrix equality $V(I_2+T) = (I_2+T)V_1$, recorded in (i) holds over any unital ring) *these two similar units (indeed, determinants* $= -1$) *are also unipotent similar over any (unital) ring.*

(5) $W = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$, $W_1 = U^{-1}WU = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$.

The linear system (for example (ii)) reduces to $3x = 2y + 1$, $z = y$ and $9(x^2 + yz) = (2y + 1)^2 + 9y^2 = 0$ has *no* real solutions. According to Theorem 1, *these similar units* ($\textit{indeed, determinants} = 1$) *are not unipotent similar over* $\mathbb{Z}$.

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On Fefferman's inequality. A simple proof

René Erlin Castillo
Departamento de Matemáticas,
Universidad Nacional de Colombia,
Bogotá, Colombia
email: recastillo@unal.edu.co

Abstract. In this short note we shall give a simple proof of the so called Fefferman's inequality allowing the potential V belong to L_p with $1 < p < \infty$.

1 Introduction

In his celebrated paper Charles Fefferman [6] prove the inequality

$$\int_B |u(x)|^p |V(x)|\,dx \leq C \int_B |\nabla u(x)|^p\,dx \tag{1}$$

for all $u \in C_c^\infty$, in case $p = 2$, assuming the potential V belong to the class $L^{r,n-2r}$, with $1 < r \leq \frac{n}{2}$.

In latter work, Chiarenza and Frasca [3] extended Fefferman's result with a different proof, assuming the potential V in $L^{r,n-pr}$ with $1 < r \leq \frac{n}{p}$ and $1 < p < n$.

In [4] Danielli, Garofallo and Nhice introduced a suitable version of Morrey Spaces adapted to the Carnot-Carahéodory (C-C) metric and proved the same inequality with V in the Morrey Space $L^{1,\lambda}$ for $\lambda > 0$.

A different approach to inequality (1) was started by Schecter in [7] where he proved the inequality with $p = 2$ and V in the Stummed-Kato Class.

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At the beginning of the 21st century in [8] inequality (1) was proved with $1 < p < n$ and V in a more general class of potentials, namely non-linear Kato class for details in this class see [2]. In [5] inequality (1) was proved by replacing the gradient in the right hand side of (1) by energy associated to an arbitrary system of vector fields, and the function V was take in an appropriate Stummed-Kato class, defined via the Carnot-Carathéodory metric associated to the vector fields in a metric space.

In [1] inequality (1) was proved allowing $V \in A_1 \cap L_{\frac{n}{p}} \cap C_c^2$ with $1 < p < \frac{n}{p}$.

In section 2 of this note we shall prove (1) allowing $V \in L_p$ with $1 < p < \infty$.

2 Main result

After Fefferman gave the proof of (1) for $p = 2$, all subsequent authors who have proved (1) have used the following Lemma, which is the cornerstone in the proof of the aforementioned inequality (1) in that sense (1) deserve to have a name and so we will call it the workhorse Lemma. In order to make this note self contained we will give its proof as well.

Lemma 1 (The workhorse Lemma) *Let $u \in C^1(\mathbb{R}^n)$ suppose that u and its partial derivatives of first order are integrable on $\mathbb{R}^n$. Then*

$$|u(x)| \leq \frac{1}{n\omega_n} \int_{\mathbb{R}^n} \frac{|\nabla u(y)|}{|x-y|^{n-1}}\, dy$$

for $x \in \mathbb{R}^n$ where ω_n is the Lebesgue measure of the unit ball in $\mathbb{R}^n$.

Proof. Observe first that

$$\frac{(x-y) \cdot \nabla u(y)}{|x-y|^n}$$

is integrable on $\mathbb{R}^n$ as function of y; actually for $r > 0$, we have

$$\begin{aligned}
\int_{\mathbb{R}^n} \frac{|(x-y)\cdot \nabla u(y)|}{|x-y|^n}\, dy &\leq \int_{B_r(x)} \frac{|\nabla u(y)|}{|x-y|^{n-1}}\, dy + \int_{\mathbb{R}^n \setminus B_r(x)} \frac{|\nabla u(y)|}{|x-y|^{n-1}}\, dy \\
&\leq \sup_{y \in B_r(x)} |\nabla u(y)| \int_{B_r(x)} \frac{dy}{|x-y|^{n-1}} \\
&\quad + \frac{1}{r^{n-1}} \int_{\mathbb{R}^n} |\nabla u(y)|\, dy < \infty.
\end{aligned}$$

Next, since $u \in C_c^1(\mathbb{R}^n)$ we also have

$$u(x) = -\int_0^\infty \frac{\partial}{\partial r} u(x + rz)\, dr \tag{2}$$

where $z \in S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$. Integrating (2) over the whole unit sphere surface S^{n-1} yields

$$\begin{aligned}
\omega_{n-1} u(x) &= \int\limits_{S^{n-1}} u(x)\, d\sigma(z) \\
&= -\int\limits_{S^{n-1}} \int_0^\infty \frac{\partial}{\partial r} u(x + rz)\, dr d\sigma(z) \\
&= -\int\limits_{S^{n-1}} \int_0^\infty \nabla u(x + rz) \cdot z\, dr d\sigma(z) \\
&= -\int_0^\infty \int\limits_{S^{n-1}} \nabla u(x + rz) \cdot z\, dr d\sigma(z).
\end{aligned}$$

Changing variables $y = x + rz$, $d\sigma(z) = r^{n-1}\, d\sigma(y)$ and

$$z = \frac{y - x}{|x - y|} \quad \text{and} \quad r = |x - y|,$$

hence we get

$$\begin{aligned}
\omega_{n-1} u(x) &= -\int_0^\infty \int\limits_{\partial B(x,r)} \nabla u(y) \cdot \frac{y - x}{|x - y|^n}\, d\sigma(y) dr \\
&= \int\limits_{\mathbb{R}^n} \nabla u(y) \cdot \frac{x - y}{|x - y|^n}\, dy,
\end{aligned}$$

which implies that

$$|u(x)| \leq \frac{1}{n\omega_n} \int\limits_{\mathbb{R}^n} \frac{|\nabla u(y)|}{|x - y|^{n-1}}\, dy,$$

as we wish to prove. □

Theorem 1 (Fefferman's inequality) *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded set and* $V \in L_p(\Omega)$ *for* $1 \leq p < \infty$. *Then*

$$\int\limits_\Omega |u(x)|^p |V(x)|\, dx \leq C(n, p, q) \|V\|_{L_p(\Omega)} \int\limits_\Omega |\nabla u(x)|^p\, dx.$$

Proof. For any $u \in C_c^\infty(\mathbb{R}^n)$, let us consider a ball B such that $u \in C_c^\infty(B)$. By Lemma (2) and Hölder's inequality we have

$$\begin{aligned} |u(x)| \leq & C_n \left(\int_B |\nabla u(y)|^p \, dy \right)^{\frac{1}{p}} \left(\int_B \frac{dy}{|x-y|^{q(n-1)}} \right)^{\frac{1}{q}} \\ = & C_n C_q \left(\int_B |\nabla(y)|^p \, dy \right)^{\frac{1}{p}} . \end{aligned}$$

Thus

$$|u(x)|^p \leq (C_n C_q)^p \int_B |\nabla u(y)|^p \, dy. \tag{3}$$

Next, multiplying by $|V(x)|$ at both side of (3) and integrating with respect to x and invoking one more time the Hölder inequality we obtain

$$\begin{aligned} \int_B |u(x)|^p |V(x)| \, dx \leq & (C_n C_q)^p \int_B |V(x)| \left(\int_\Omega |\nabla u(y)|^p \, dy \right) \, dx \\ \leq & (C_n C_q)^p [m(B)]^{\frac{1}{q}} \left(\int_\Omega |V(x)|^p \, dx \right)^{\frac{1}{p}} \left(\int_\Omega |\nabla u(y)|^p \, dy \right) \\ = & C(n,p,q) \|V\|_{L_p(\Omega)} \int_\Omega |\nabla u(y)|^p \, dy. \end{aligned}$$

Finally

$$\begin{aligned} \int_\Omega |u(x)|^p |V(x)| \, dx = & \int_B |u(x)|^p |V(x)| \, dx \\ \leq & C(n,p,q) \|V\|_{L_p(\Omega)} \int_\Omega |\nabla u(x)|^p \, dx, \end{aligned}$$

as we announced. □

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Embedding topological manifolds into L^p spaces

Yu-Lin Chou
freelance author, Hsinchu County, Taiwan (R.O.C.)
email: chou.y.l.edu@gmail.com

Abstract. With a simple argument, we show as a main note that, for every given $1 \leq p \leq +\infty$, every locally compact second-countable Hausdorff space is topologically embeddable into some L^p space with respect to some finite nonzero Borel measure, where the embedding may be chosen so that its range is included in some open proper subset of the L^p space.

Throughout, a manifold is always assumed to be a topological manifold, i.e. a second-countable Hausdorff space where every point has some neighborhood homeomorphic to some (fixed) Euclidean space. And an embedding is always assumed to be a topological embedding, i.e. a homeomorphism acting between a topological space and a subspace of a topological space.

In addition to the existing embedding results for various types of manifolds, we wish to show with a simple elementary proof that, given any $1 \leq p \leq +\infty$, every manifold is embeddable into some L^p space with some additional properties.

Our main result is more general:

Theorem 1 *If $1 \leq p \leq +\infty$, then every locally compact second-countable Hausdorff space is embeddable into some L^p space such that i) the underlying measure may be chosen to be a finite nonzero Borel one, and ii) the embedding may be chosen so that its range is included in some open proper subset of the L^p space chosen in i).*

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Proof. Let M be a locally compact second-countable Hausdorff space. If M_∞ denotes the Alexandroff (one-point) compactification of M, possibly without denseness of M, then, since M is sigma-compact, the space M_∞ is in addition second-countable and hence metrizable by the usual Urysohn construction.

Upon choosing a metric d for M_∞, define for every $x \in M_\infty$ the continuous function $f_x : M_\infty \to \mathbb{R}, y \mapsto d(x, y)$; the functions f_x are evidently a version of the Kuratowski construction. Since M_∞ is compact, it suffices to work with f_x in the simplified form.

On the other hand, let μ be a weighted sum of Dirac measures (restricted to the Borel sigma-algebra of M_∞) over M_∞ concentrated respectively at the points of a chosen countable dense subset of M_∞ with the property that $\mu(M_\infty) = 1$; such a choice of μ is always possible by considering for example the coefficients $2^{-1}, 2^{-2}, \ldots$. Then μ is a finite nonzero Borel probability measure over M_∞.

Identify two functions in the real Banach space $L^p(\mu)$ that are μ-almost everywhere equal with each other. Then, as every f_x is bounded and hence lies in $L^p(\mu)$, the map $F : x \mapsto f_x$ is continuous with respect to the L^p-norm; indeed, if $1 \leq p < +\infty$ then

$$\left(\int_{M_\infty} |f_x - f_z|^p \, d\mu \right)^{1/p} \leq d(x, z)$$

for all $x, z \in M_\infty$, and

$$|f_x - f_z|_{L^\infty} \leq |d(x, z)|_{L^\infty} \leq d(x, z)$$

for all $x, z \in M_\infty$. Moreover, since $d(x, \cdot) = d(z, \cdot)$ implies $0 = d(z, x)$, and since the equivalence class of f_x is $\{f_x\}$ by the construction of μ for every $x \in M_\infty$, the map F is an injection; the compactness of M_∞ and the continuity of F then jointly imply that F is a closed map and hence embeds M_∞ into $L^p(\mu)$.

Since M is by construction a subspace of M_∞, the composition $\Phi : M \to L^p(\mu)$ of $F|_M$ circ the inclusion map $\mathrm{id}_{M_\infty}|_M$ serves as an embedding.

As M is by construction open in M_∞, with ∞ denoting the additional element of M_∞ there is some open $V \subset L^p(\mu)$ such that

$$\Phi^{1)}(M) = V \cap F^{1)}(M_\infty) = (V \cap \Phi^{1)}(M)) \cup (V \cap \{F(\infty)\}).$$

Since $V \cap \{F(\infty)\}$ is then empty, it follows that V is a proper subset of $L^p(\mu)$ and

$$\Phi^{1)}(M) \subset V.$$

The above argument proves for the noncompact case; by a manifest slight modification it also works for M compact (e.g., adjoining a single point to M as an isolated point). This completes the proof. □

Given the importance of L^2 spaces as Hilbert spaces, the case where $p = 2$ in Theorem 1 would be of particular interest.

We will use the phrase "locally Euclidean" in the following sense: A second-countable Hausdorff space is called locally Euclidean if and only if for every point of it there are some neighborhood of the point and some $n \in \mathbb{N}$ such that the neighborhood is homeomorphic to the Euclidean space $\mathbb{R}^n$.

Since every locally Euclidean space is evidently also locally compact, we summarize for ease of reference the intended corollaries in the following

Corollary 1 *Let $1 \leq p \leq +\infty$. Then every locally Euclidean space is embeddable into some L^p space in the way described in Theorem 1.*

In particular, every manifold, and hence every Euclidean space $\mathbb{R}^n$, is embeddable into some L^p space in the same way.

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Characterization of spectral elements in non-archimedean Banach algebras

Jawad Ettayb
Regional Academy of Education and Training of Casablanca-Settat,
Hamman Al-Fatawaki collegiate High School, Had Soualem, Berrechid Province, Morocco
email: jawad.ettayb@gmail.com

Abstract. Let $\mathcal{A}$ be a non-archimedean Banach algebra with unit e over an algebraically closed field. In this paper, we give a generalization of results of the paper [2] and we establish a new necessary and sufficient condition on the resolvent of an element $a \in \mathcal{A}$ such that for all $n \in \mathbb{N}$, $\|a^n\| \leq 1$.

1 Introduction and preliminaries

Throughout this paper, $\mathcal{A}$ is a non-archimedean Banach algebra with unit e $\Big(\|e\| = 1\Big)$ over a non trivially complete non-archimedean valued field $\mathbb{K}$ which is also algebraically closed with valuation $|\cdot|$, $\mathbb{Q}_p$ is the field of p-adic numbers equipped with p-adic valuation $|.|_p$ and $\mathbb{Z}_p$ denotes the ring of p-adic integers of $\mathbb{Q}_p$. For more details, we refer to [6] and [8]. We denote the completion of algebraic closure of $\mathbb{Q}_p$ under the p-adic valuation $|\cdot|_p$ by $\mathbb{C}_p$ ([6]). Let $r > 0$ and let Ω_r be the clopen ball of $\mathbb{K}$ centred at 0 with radius $r > 0$, that is $\Omega_r = \{t \in \mathbb{K} : |t| < r\}$. A non-archimedean normed algebra is a non-archimedean normed space with linear associative multiplication satisfying for all $a, b \in \mathcal{A}$, $\|ab\| \leq \|a\|\|b\|$. A non-archimedean complete normed algebra is

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called a non-archimedean Banach algebra, moreover, if there is $e \in \mathcal{A}$ such that for all $a \in \mathcal{A}$, $ae = ea = a$ and $\|e\| = 1$, $\mathcal{A}$ is said to be a non-archimedean Banach algebra with unit e. For more details, we refer to [1], [3], [8] and [10]. We have the following lemma.

Lemma 1 **([8])** *Let $\mathcal{A}$ be a non-archimedean Banach algebra with unit e, let $a \in \mathcal{A}$ such that $\|a\| < 1$, then $e-a$ is invertible in $\mathcal{A}$ and $(e-a)^{-1} = \sum_{k=0}^{\infty} a^k$.*

let $a \in \mathcal{A}$, we set $\sigma(a) = \{\lambda \in \mathbb{K} : a - \lambda e \text{ is not invertible}\}$.

Definition 1 **([9])** *Let $\mathcal{A}$ be a non-archimedean Banach algebra with unit e. Set $r(a) = \inf_n \|a^n\|^{\frac{1}{n}} = \lim_n \|a^n\|^{\frac{1}{n}}$, a is said to be a spectral element if $\sup\{|\lambda| : \lambda \in \sigma(a)\} = r(a)$. For $a \in \mathcal{A}$, set*

$$U_a = \{\lambda \in \mathbb{K} : (e - \lambda a)^{-1} \textit{ exists in } \mathcal{A}\}.$$

$\left(U_a \textit{ is open and } 0 \in U_a\right)$ *and*

$$C_a = \{\alpha \in \mathbb{K} : B(0, |\beta|) \subset U_a \textit{ for some } \beta \in \mathbb{K},\ |\beta| > |\alpha|\}.$$

We generalize the Proposition 6.6 of [9] as follows.

Proposition 1 *[9] Let $\mathcal{A}$ be a non-archimedean Banach algebra with unit e, then the following are equivalent.*

(i) a is a spectral element.

(ii) For all $\lambda \in C_a$, $(e - \lambda a)^{-1} = \sum_{n=0}^{\infty} \lambda^n a^n$.

(iii) For each $\alpha \in C_a^*$, the function $\lambda \mapsto (e - \lambda a)^{-1}$ is analytic on $B(0, |\alpha|)$.

2 Main results

In the rest of this paper, for an element $a \in \mathcal{A}$ such that for all $n \in \mathbb{N}$, $\|a^n\| \leq 1$, we assume that $U_a = \Omega_1$ where for all $\lambda \in U_a$, $R(\lambda, a) = (e - \lambda a)^{-1}$.

Proposition 2 *Let $\mathcal{A}$ be a non-archimedean Banach algebra over $\mathbb{K}$ with unit e, let a be a spectral element such that $\sup_{n \in \mathbb{N}} \|a^n\| \leq 1$. Then,*

$$\textit{for all } \lambda \in C_a,\ \|R(\lambda, a)\| \leq 1.$$

Proof. From Proposition 1, for each $\lambda \in C_a$, $\lim_{n\to\infty} |\lambda|^n \|a^n\| = 0$, then

$$\begin{aligned} \|R(\lambda, a)\| &= \left\| \sum_{n=0}^{\infty} \lambda^n a^n \right\| \\ &\leq \max_{n\in\mathbb{N}} |\lambda^n| \\ &= 1. \end{aligned}$$

□

Proposition 3 *Let $\mathcal{A}$ be a non-archimedean Banach algebra over $\mathbb{K}$ with unit e, let a be a spectral element such that $\sup\limits_{n\in\mathbb{N}} \|a^n\| \leq 1$. Then, for all $\lambda, \mu \in C_a$,*

$$\lambda R(\lambda, a) - \mu R(\mu, a) = (\lambda - \mu) R(\lambda, a) R(\mu, a).$$

Proof. If $\lambda, \mu \in C_a$, then

$$\lambda R(\lambda, a)(e - \mu a) R(\mu, a) - \mu R(\lambda, a)(e - \lambda a) R(\mu, a) \tag{1}$$

and

$$\begin{aligned} (1) &= \lambda R(\lambda, a) R(\mu, a) - \lambda\mu R(\lambda, a) a R(\mu, a) - \mu R(\lambda, a) R(\mu, a) \\ &\quad + \lambda\mu R(\lambda, a) a R(\mu, a) \\ &= \lambda R(\lambda, a) R(\mu, a) - \mu R(\lambda, a) R(\mu, a) \\ &= (\lambda - \mu) R(\lambda, a) R(\mu, a). \end{aligned}$$

□

Proposition 4 *Let $\mathcal{A}$ be a non-archimedean Banach algebra over $\mathbb{K}$ with unit e, let a be a spectral element such that $\sup\limits_{n\in\mathbb{N}} \|a^n\| \leq 1$. Then for all $\lambda \in C_a$, $\|R(\lambda, a) - e\| \leq |\lambda|$.*

Proof. Since $a \in \mathcal{A}$ is a spectral element, we get for each $\lambda \in C_a$, $R(\lambda, a) = \sum_{n=0}^{\infty} \lambda^n a^n$. Then, for any $\lambda \in C_a$,

$$\begin{aligned} \|R(\lambda, a) - e\| &= \|\sum_{n=1}^{\infty} \lambda^n a^n\| \qquad (2) \\ &\leq \sup_{n\geq 1} \|\lambda^n a^n\| \qquad (3) \\ &\leq |\lambda|. \qquad (4) \end{aligned}$$

□

Proposition 5 *Let $\mathcal{A}$ be a non-archimedean Banach algebra over $\mathbb{K}$ with unit* e, *let* a *be a spectral element such that* $\sup_{n\in\mathbb{N}} \|a^n\| \le 1$. *Then for any* $n \in \mathbb{N}, \alpha \in C_a^*$, $\lambda \in \Omega_{|\alpha|}$,

$$R^{(n)}(\lambda, a) = \frac{n!(R(\lambda, a) - e)^n R(\lambda, a)}{\lambda^n}.$$

Proof. From Proposition 3, for each $\lambda, \mu \in \Omega_{|\alpha|}$ with $\alpha \in C_a^*$,

$$\Big(\lambda e + (\mu - \lambda)e + (\lambda - \mu)R(\lambda, a)\Big)R(\mu, a) = \lambda R(\lambda, a). \tag{5}$$

Thus

$$\Big(e - \frac{1}{\lambda}(\mu - \lambda)(R(\lambda, a) - e)\Big)R(\mu, a) = R(\lambda, a). \tag{6}$$

The quantity in square brackets on the left of this equation is invertible for $|\lambda|^{-1}|\mu - \lambda|\|R(\lambda, a) - e\| < 1$. Then

$$R(\mu, a) = \sum_{n=0}^{\infty} \frac{(R(\lambda, a) - e)^n R(\lambda, a)}{\lambda^n}(\mu - \lambda)^n. \tag{7}$$

But it follows by Proposition 1 that $R(\mu, a)$ is analytic on $B(\lambda, |\alpha|)$. Since $a \in \mathcal{A}$ is a spectral element, we get for all $\lambda, \mu \in \Omega_{|\alpha|}$, $R(\mu, a)$ can be written as follows:

$$R(\mu, a) = \sum_{n=0}^{\infty} \frac{R^{(n)}(\lambda, a)}{n!}(\mu - \lambda)^n.$$

Then, for any $n \in \mathbb{N}$, $\lambda \in \Omega_{|\alpha|}$,

$$R^{(n)}(\lambda, a) = \frac{n!(R(\lambda, a) - e)^n R(\lambda, a)}{\lambda^n}.$$

□

We have the following theorem.

Theorem 1 *Let $\mathcal{A}$ be a non-archimedean Banach algebra over $\mathbb{K}$ with unit* e, *let* a *be a spectral element. Then for all* $n \in \mathbb{N}$, $\|a^n\| \le 1$ *if and only if*

$$\Big\|\Big(R(\lambda, a) - e\Big)^n R(\lambda, a)\Big\| \le |\lambda|^n, \tag{8}$$

for all $\lambda \in \Omega_{|\alpha|}$ *where* $\alpha \in C_a^*$ *and* $R(\lambda, a) = (e - \lambda a)^{-1}$.

Proof. Suppose that for each $n \in \mathbb{N}, \|a^n\| \leq 1$, let $\alpha \in C_a^*$, from Proposition 1, $R(\lambda, a) = (e - \lambda a)^{-1} = \sum_{k=0}^{\infty} \lambda^k a^k$ is analytic on $\Omega_{|\alpha|}$. By Proposition 5, for any $n \in \mathbb{N}, \lambda \in \Omega_{|\alpha|}$,

$$R^{(n)}(\lambda, a) = \frac{n!(R(\lambda, a) - e)^n R(\lambda, a)}{\lambda^n} \tag{9}$$

and

$$R^{(n)}(\lambda, a) = \sum_{k=n}^{\infty} k(k-1)\cdots(k-n+1)\lambda^{k-n}a^k = \sum_{k=n}^{\infty} n!\binom{k}{n}\lambda^{k-n}a^k.$$

Hence for each $n \in \mathbb{N}$ and for all $\lambda \in \Omega_{|\alpha|}$,

$$\begin{aligned}
\left\|\frac{R^{(n)}(\lambda, a)}{n!}\right\| &= \left\|\sum_{k=n}^{\infty}\binom{k}{n}\lambda^{k-n}a^k\right\| && (10)\\
&\leq \sup_{k \geq n}\left|\binom{k}{n}\right||\lambda|^{k-n}\|a^k\| && (11)\\
&\leq \sup_{k \geq n}|\lambda|^{k-n}\|a^k\| && (12)\\
&\leq 1. && (13)
\end{aligned}$$

Then, for any $n \in \mathbb{N}$ and $\lambda \in \Omega_{|\alpha|}$,

$$\left\|\frac{R^{(n)}(\lambda, a)}{n!}\right\| \leq 1. \tag{14}$$

From (9) and (14), we have for any $n \in \mathbb{N}$, $\lambda \in \Omega_{|\alpha|}$,

$$\|(R(\lambda, a) - e)^n R(\lambda, a)\| \leq |\lambda|^n. \tag{15}$$

Conversely, we assume that (8) holds. From a is spectral, we have for any $\lambda \in \Omega_{|\alpha|}, R(\lambda, a) = \sum_{n=0}^{\infty} \lambda^n a^n$. Put for any $\lambda \in \Omega_{|\alpha|}$, $k \in \mathbb{N}$, $S_k(\lambda) = \lambda^{-k}(R(\lambda, a) - e)^k R(\lambda, a)$, then for any $\lambda \in \Omega_{|\alpha|}, k \in \mathbb{N}$, $\|S_k(\lambda)\| \leq 1$. Since a and $R(\lambda, a)$ commute, we have

$$\begin{aligned}
S_k(\lambda) &= \lambda^{-k}\Big(\big(e - (e - \lambda a)\big)R(\lambda, a)\Big)^k R(\lambda, a), && (16)\\
&= \lambda^{-k}(\lambda a R(\lambda, a))^k R(\lambda, a), && (17)\\
&= a^k R(\lambda, a)^{k+1}. && (18)
\end{aligned}$$

Then for each $\lambda \in \Omega_{|\alpha|}$ and for all $k \in \mathbb{N}$,

$$\begin{aligned}
\|a^k\| &= \|(e-\lambda a)^{k+1} S_k(\lambda)\|, && (19)\\
&\leq \|(e-\lambda a)^{k+1}\| \|S_k(\lambda)\|, && (20)\\
&\leq \left\| \sum_{j=0}^{k+1} \binom{k+1}{j} (-\lambda a)^j \right\|, && (21)\\
&\leq \max\{1, \|\lambda a\|, \|\lambda^2 a^2\|, \cdots, \|\lambda^{k+1} a^{k+1}\|\}, && (22)
\end{aligned}$$

for $\lambda \to 0$, we get for all $k \in \mathbb{N}, \|a^k\| \leq 1$. □

We generalize the result of [4] in non-archimedean Banach algebra as follows.

Theorem 2 *Let $\mathcal{A}$ be a non-archimedean Banach algebra over $\mathbb{K}$ with unit e, let $a \in \mathcal{A}$ be a spectral element with $U_a = \Omega_1$, then for all $n \geq 1$, $\|a^n\| \leq 1$ if and only if*

$$\left\| \Big(R(\lambda, a) - e \Big)^k \right\| \leq |\lambda|^k, \tag{23}$$

for all $\lambda \in \Omega_{|\alpha|}$, $k \geq 1$ where $\alpha \in C_a^*$ and $R(\lambda, a) = (e-\lambda a)^{-1}$.

Proof. Assume that for any $n \in \mathbb{N}, \|a^n\| \leq 1$, let $\alpha \in C_a^*$, then $R(\lambda, a) = (e-\lambda a)^{-1} = \sum_{k=0}^{\infty} \lambda^k a^k$ is analytic on $\Omega_{|\alpha|}$. Using $R(\lambda, a) - e = \lambda a R(\lambda, a)$ and Proposition 5, we have

$$(R(\lambda, a) - e)^{n+1} = \lambda a (R(\lambda, a) - e)^n R(\lambda, a) = \frac{\lambda^{n+1}}{n!} a R^{(n)}(\lambda, a)$$

and

$$R^{(n)}(\lambda, a) = \sum_{k=n}^{\infty} k(k-1)\cdots(k-n+1)\lambda^{k-n} a^k = \sum_{k=n}^{\infty} n! \binom{k}{n} \lambda^{k-n} a^k.$$

Thus

$$(R(\lambda, a) - e)^{n+1} = \sum_{k=n}^{\infty} \binom{k}{n} (\lambda a)^{k+1}.$$

Then for all $n \in \mathbb{N}$ and for any $\lambda \in \Omega_{|\alpha|}$,

$$\begin{aligned}
\left\| (R(\lambda, a) - e)^{n+1} \right\| &= \left\| \sum_{k=n}^{\infty} \binom{k}{n} (\lambda a)^{k+1} \right\| \\
&\leq \sup_{k \geq n} \left| \binom{k}{n} \right| |\lambda|^{k+1} \|a^{k+1}\|
\end{aligned}$$

$$
\begin{aligned}
&\leq \sup_{k\geq n}|\lambda|^{k+1}\|a^{k+1}\|\\
&\leq |\lambda|^{n+1}.
\end{aligned}
$$

Conversely, we assume that (23) holds. Since a is a spectral element, then for all $\lambda \in \Omega_{|\alpha|}, R(\lambda, a) = \sum_{n=0}^{\infty}\lambda^n a^n$. Put for any $\lambda \in \Omega_{|\alpha|}$, $k \in \mathbb{N}$, $S_k(\lambda) = \lambda^{-k-1}(R(\lambda, a) - e)^{k+1}$, then for all $\lambda \in \Omega_{|\alpha|}$, $k \in \mathbb{N}$, $\|S_k(\lambda)\| \leq 1$. Since a and $R(\lambda, a)$ commute. From $R(\lambda, a) - e = \lambda a R(\lambda, a)$, we get $S_k(\lambda) = (aR(\lambda, a))^{k+1}$, hence:

$$a^{k+1} = (e - \lambda a)^{k+1} S_k(\lambda).$$

Then for all $\lambda \in \Omega_{|\alpha|}$ and for each $k \in \mathbb{N}$,

$$
\begin{aligned}
\|a^{k+1}\| &= \|(e-\lambda a)^{k+1} S_k(\lambda)\|\\
&\leq \|(e-\lambda a)^{k+1}\|\|S_k(\lambda)\|\\
&\leq \left\|\sum_{j=0}^{k+1}\binom{k+1}{j}(-\lambda a)^j\right\|\\
&\leq \max\{1, \|\lambda a\|, \|\lambda^2 a^2\|, \cdots, \|\lambda^{k+1}a^{k+1}\|\},
\end{aligned}
$$

for $\lambda \to 0$, we get for any $k \in \mathbb{N}$, $\|a^{k+1}\| \leq 1$. □

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Asymptotic properties of a nonparametric conditional distribution function estimator in the local linear estimation for functional data via a functional single-index model

Fadila Benaissa
University of science and technology
Mohammed Boudiaf.
Oran 31000, Algeria
email: fadila.benaissa24@gmail.com

Abdelhak Chouaf
Laboratory of Statistics and Stochastic Processes
Djillali Liabes University
Sidi Bel Abbes 22000, Algeria
email: abdo_stat@yahoo.fr

Abstract. This paper deals with the conditional distribution function estimator of a real response variable given a functional random variable (i.e takes values in an infinite dimensional space). Specifically, we focus on the functional index model, this approach represents a good compromise between nonparametric and parametric models. Then we give under general conditions and when the variables are independent, the quadratic error and asymptotic normality of estimator by local linear method, based on the single-index structure. Moreover, as an application, the asymptotic $(1-\gamma)$ confidence interval of the conditional distribution function is given for $0 < \gamma < 1$.

Keywords: Mean squared error, single functional index , conditional distribution function, nonparametric estimation, local linear estimation, Asymptotic normality, functional data.

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1 Introduction

The estimation of the conditional cumulative distribution function has great importance. In fact, it is involved in many applications, such as reliability, survival analysis (see Zamanzade and all. [31], Tabti and Ait Saadi [28]), ... Moreover, there are several prediction tools in the nonparametric statistics branch, for instance the conditional mode, the conditional median or the conditional quantiles, which are based on the preliminary estimation of this nonparametric model. In the nonfunctional case, the local polynomial fitting has been the subject of considerable studies and key references on this topic are Fan and Yao [14], Fan [16], Fan and Gijbels [15] and references therein. However, only few results are available for the local linear modeling in the functional statistics setup. Indeed, the first results, in this direction, were established in Baillo and Grané [6]. These papers focus on the local linear estimation of the regression operator when the explanatory variable takes values in a Hilbert space. The general case, where the regressors do not belong to a Hilbert space but just to a semi-metric space, has been considered in Barrientos-Marin et al. [7] and El Methni and Rachdi [13]. In these works, authors obtained the almost-complete convergence (a.co.), with rates, of the proposed estimator. Other alternative versions of the local linear modeling for functional data were investigated (see Boj et al.[8]; Baillo and Grané [6]; El Methni and Rachdi [13]), for the regression operator and Demongeot et al.[10]; Demongeot et al.[12], Xiong and al. [30], for the conditional density function, Demongeot et al.[11] for the conditional distribution function). in the case of spatial data Laksaci et al.[23] they established pointwise almost complete convergence with rate.
Furthermore, the functional index model plays a major role in statistics. The interest of this approach comes from its use to reduce the dimension of the data by projection in fractal space. The literature on this topic is closely limited, the first work which was interested in the single-index model on the nonparametric estimation is Ferraty et al.[17] they stated for i.i.d. variables and obtained the almost complete convergence under some conditions. Based on the cross-validation procedure, Ait Saidi et al. [2] proposed an estimator of this parameter, where the functional single-index is unknown. See Ait Saidi et al. [1] for the dependant case. Attaoui et al.[4] considered the nonparametric estimation of the conditional density in the single functional model. They established its pointwise and uniform almost complete convergence (a.co.) rates. In the same topic, Attaoui et al.[5] proved the asymptotic results of a nonparametric conditional cumulative distribution estimator for time series data. Ait Saadi and Mecheri [3], established the pointwise and the uniform almost

complete convergence (with the rate) of the kernel estimate of of the conditional cumulative distribution function of a scalar response variable Y given a Hilbertian random variable X when the observations are linked via a single-index structure. Ferraty and al. [21] proposed an estimator based on the idea of functional derivative estimation of a single index parameter. Hamdaoui and al. [22] established The asymptotic normality of the conditional distribution kernel estimator.

Tabti and al. [28] obtained the almost complete convergence and the uniform almost complete convergence of a kernel estimator of the hazard function with quasi-association condition when the observations are linked with functional single-index structure. In this paper, we focus on the local linear estimation with the single-index structure to compute under some conditions, the quadratic error of the conditional distribution function estimator. In practice, this study has great importance, because, it permits to construct a prediction method based on the maximum risk estimation with a single functional index.

In Section 2, We introduce the estimator of our model in the single-functional index. In Section 3 we introduce assumptions and asymptotic properties are given.

Finally, Section 5 is devoted to the proofs of the results.

2 The model

Let $\{(X_i, Y_i),\ 1 \le i \le n\}$ be n random variables, independent and identically distributed as the random pair (X, Y) with values in $\mathcal{H} \times \mathbb{R}$, where $\mathcal{H}$ is a separable real Hilbert space with the norm $\| \,.\, \|$ generated by an inner product $< .,. >$. We consider the semi-metric d_θ associated to the single index $\theta \in \mathcal{H}$ defined by $\forall x_1, x_2 \in \mathcal{H} : d_\theta(x_1, x_2) := |< x_1 - x_2, \theta >|$. Assume that the explanation of Y given X is done through a fixed functional index θ in $\mathcal{H}$. In the sense that, there exists a θ in $\mathcal{H}$ (unique up to a scale normalization factor) such that: $\mathbb{E}[Y|X] = \mathbb{E}[Y| < \theta, X >]$. The conditional probability distribution of Y given $X = x$ denoted by $F_\theta(.|x)$ exists and is given by $\forall y \in \mathbb{R},\ F_\theta(y|x) := F(y| < x, \theta >)$. In the following, we denote by $F(\theta, ., x)$, the conditional distribution function of Y given $< x, \theta >$ and we define the local linear estimator for single-index structure $\widehat{F}(\theta, ., x)$ of $F(\theta, ., x)$ by:

$$\widehat{F}(\theta, y, x) = \frac{\sum\limits_{1 \le i,j \le n} W_{ij}(\theta, x) H({h_H}^{-1}(y - Y_j))}{\sum\limits_{1 \le i,j \le n} W_{ij}(\theta, x)} = \frac{\sum\limits_{1 \le j \le n} \Omega_j K_j H_j}{\sum\limits_{1 \le j \le n} \Omega_j K_j},$$

with

$$W_{ij}(\theta,x)=\beta_\theta(X_i,x)\Big(\beta_\theta(X_i,x)-\beta_\theta(X_j,x)\Big)K(h_K^{-1}d_\theta(x,X_i))K(h_K^{-1}d_\theta(x,X_j)),$$

and $\Omega_j K_j = \sum_{i=1}^{n} W_{ij}$ with $\beta_\theta(X_i,x)$ is a known bi-functional operator from $\mathcal{H}^2$ into $\mathbb{R}$ where K is a kernel, H is a cumulative distribution function and $h_K := h_{n,K}$ (resp $h_H := h_{n,H}$) is a sequence that decrease to zero as n goes to infinity.

3 Assumptions and Mains results

All along the paper, we will denote by C, C' and $C_{\theta,x}$ some strictly positive generic constants and by $K_i(\theta,x) := K(h_K^{-1}d_\theta(x,X_i))$, $\forall x\in\mathcal{H}, i=1,...,n$, $H_j := H({h_H}^{-1}(y-Y_j)), \forall y\in\mathbb{R}, j=1,...,n.$, $\beta_{\theta,i} := \beta_\theta(X_i,x)$, $W_{ij}(\theta,x) := W_{\theta,ij}$ and we will use the notation $B_\theta(x,h_K) := \{x_1\in\mathcal{H}:\ 0<|<x-x_1,\theta>|<h_K\}$, the ball centered at x with radius h_K. Moreover, for find the results in our paper we denote: for any $l\in\{0,2\}$ $\psi_l(.,y) := \dfrac{\partial^l F(.,y,.)}{\partial y^l}$,
$\Phi_l(s)=\mathbb{E}[\psi_l(X,y)-\psi_l(x,y)|\beta_\theta(x,X)=s]$,
and $\phi_{\theta,x}(r_1,r_2)=\mathbb{P}(r_1\le d_\theta(x,X)\le r_2)$.
In order to study our asymptotic results we need the following assumptions:

(H1) **(i)** $\mathbb{P}(X\in B_\theta(x,h_K)) =: \phi_{\theta,x}(h_K)>0$,

(ii) assume that there exists a function $\chi_{\theta,x}(\cdot)$ such that

$$\forall s\in[-1,1]\ \lim_{h_K\to 0}\frac{\phi_{\theta,x}(sh_K,h_K)}{\phi_{\theta,x}(h_K)}=\chi_{\theta,x}(s).$$

(iii) For any $l\in\{0,2\}$, the quantities $\Phi_l'(0)$ and $\Phi_l^{(2)}(0)$ exist, where Φ_l' (resp.$\Phi_l^{(2)}$) denotes the first (resp. the second) derivative of Φ_l

(H2) The conditional distribution function $F(\theta,y,x)$ satisfies that there exist some positive constants b_1 and b_2, such that for all (x_1,x_2,y_1,y_2)

$$|F(\theta,y_1,x)-F(\theta,y_2,x)|\le C(|d_\theta(x_1,x_2)|^{b_1}+|y_1-y_2|^{b_2})$$

(H3) The bi-functional $\beta_\theta(.,.)$ satisfies:

(i) $\forall x'\in\ \mathcal{F},\ C_1\,d_\theta(x,x')\le|\beta_\theta(x,x')|\le C_2\,d_\theta(x,x'),$ where $C_1,C_2>0$,

(ii) $\sup_{u\in B(x,r)} |\beta_\theta(u,x) - d_\theta(x,u)| = o(r)$,

(iii) $h_K \int_{B(x,h_K)} \beta_\theta(u,x) dP(u) = o\left(\int_{B(x,h_K)} \beta_\theta^2(u,x)\, dP(u)\right)$

Where $B_\theta(x,r) = \{x' \in \mathcal{H} / |d_\theta(x,x') \le r\}$ and $dP(x)$ is the cumulative distribution of X.

(H4) **(i)** The kernel K is a positive function, which is supported within $[-1,1]$, and $K(1) > 0$.

(ii) The kernel K is a differentiable function and its derivative K' satisfies

$$K^2(1) - \int_{-1}^{1} (K^2(u))' \chi_{\theta,x}(u) du > 0$$

(H5) The kernel H is a differentiable function and bounded, such that:

$$\int H^{(1)}(t)dt = 1 \ , \ \int |t|^{b_2} H^{(1)}(t)dt < \infty \text{ and } \int H^2(t)dt < \infty.$$

(H6) The bandwidths h_K,h_H satisfies:

(i) $\lim_{n\to\infty} h_K = 0, \ \lim_{n\to\infty} h_H = 0 \ \text{ and } \lim_{n\to\infty} \dfrac{\log\log n}{n\phi_{\theta,x}(h_K)}$,

(ii) $\exists \eta_0 \in \mathbb{N}, \forall \eta > \eta_0, \ \frac{1}{\phi_{\theta,x}(h_K)} \int_{-1}^{1} \phi_{\theta,x}(th_K, h_K) \frac{d}{dt}(t^2K(t))dt > C_3 > 0$

Comments on assumptions: The first part of assumption (H1) characterizes the concentration property of the probability measure of the functional variable X, which permits to control the effect of the topological structure in the asymptotic results (see Ferraty et al. [19]), the second part of assumption is known as (for small h) the concentration assumption acting on the distribution of X in infinite dimensional spaces.The function χ_x plays a determinant role. It is possible to specify this function in the above examples by
1. $\chi_0(u) = \delta_1(u)$; where $\delta_1(.)$ is Dirac function,
2. $\chi_0(u) = 1_{]0;1]}(u)$.
The third part of (H1) characterizes the functional space of our model, it is obvious that this condition is closely related to the existence of the functions, ψ_l and Φ_l, (see Ferraty et al.[20], for more discussions on the link between their derivatives). Moreover, this condition is used in order to keep the usual form of the quadratic error (see Vieu, 1991 [29]). However, if we replace the

third part of assumption (H1), by the following Lipschitz condition (where $\mathcal{N}_z$ denotes a neighborhood of z):
$\forall(y_1, y_2) \in \mathcal{N}_y \times \mathcal{N}_y$ and $\forall(x_1, x_2) \in \mathcal{N}_x \times \mathcal{N}_x$

$$|F(\theta, y_1, x) - F(\theta, y_2, x)| \leq C(|d_\theta(x_1, x_2)|^{b_1} + |y_1 - y_2|^{b_2})$$

which is less restrictive than assumption (H2), then Theorem 3.1's final result becomes as follows:

$$\mathbb{E}\left[\widehat{F}(\theta, y, x) - F(\theta, y, x)\right]^2 = o(h_H^4 + h_K^2) + o\left(\frac{1}{n\phi_{\theta,x}(h_K)}\right).$$

Such expression of the rate of convergence of our estimator is inaccurate and cannot be useful to determine the smoothing parameters. In other words, the third part of assumption (H1) on the differentiability of the conditional density permits to determine the unknown constants in the mean squared error (MSE). Thus, the third part of assumption (H1) may be considered as a good compromise permitting to obtain an asymptotically exact expression of the convergence rate of $\hat{F}(\theta, x, y)$. while the assumption (H2) is a regularity condition which characterizes the functional space, of our model, and is needed to evaluate the bias term in the asymptotic results. Then, assumption (H3) has been introduced and commented, first, in Barrientos et al. [7] and it plays an important role in our methodology, particularly when we will compute exact constant terms involved in the asymptotic result. The second part of the condition (H3) is verified, for instance, if $d_\theta(\cdot,\cdot) = \beta_\theta(\cdot,\cdot)$, moreover if

$$\lim_{d_\theta(x,u)\to 0}\left|\frac{\beta_\theta(u, x)}{d_\theta(x, u)} - 1\right| = 0.$$

Moreover, assumption (H6) is classicaly used and is standard in the context of the quadratic error determination in functional statistics and is common in the setting of functional local linear fitting (see for instance Laksaci et al. [23] and Rachdi et al. [26]). The rest of the hypotheses are imposed for a sake of brevity of our results's proofs. Moreover, one could find in Ferraty and Vieu [18] some examples of kernels K and H satisfying assumptions (H4) and (H5). The small ball probability effects are really inherent to our infinite dimensional context, as exemples, we can cite diffusion processes and Gaussian processes (see F.Ferraty, A.Laksaci and P. Vieu [19]).

3.1 Mean square convergence

In this part, we are going to show the asymptotic results of quadratic-mean convergence

Theorem 1 *Under assumptions (H1)-(H6), we obtain:*

$$\mathbb{E}\left[\widehat{F}(\theta,y,x)-F(\theta,y,x)\right]^2 = B_H^2(\theta,x,y)h_H^4 + B_K^2(\theta,x,y)h_K^4 + \frac{V_{HK}(\theta,x,y)}{n\phi_{\theta,x}(h_K)} + o(h_H^4) + o(h_K^4) + o\left(\frac{1}{n\phi_{\theta,x}(h_K)}\right),$$

where

$$B_H(\theta,x,y) = \frac{1}{2}\frac{\partial^2 F(\theta,y,x)}{\partial y^2}\int t^2 H^{(1)}(t)dt, \quad B_K(\theta,x,y) = \frac{1}{2}\Phi_0^{(2)}(0)\frac{M_0}{M_1} + o(h_K^2),$$

and

$$V_{HK}(\theta,x,y) = \frac{M_2}{M_1^2}F(\theta,y,x)(1-F(\theta,y,x)),$$

with

$$M_0 = K(1) - \int_{-1}^{1} s^2 K'(s)\chi_{\theta,x}(s)ds \quad \textit{and}$$

$$M_j = K^j(1) - \int_{-1}^{1}(K^j)'(s)\chi_{\theta,x}(s)ds \ \textit{for} \ \ j=1,2.$$

we set

$$\widehat{F}(\theta,y,x) = \frac{\widehat{F}_N(\theta,y,x)}{\widehat{F}_D(\theta,x)}.$$

where

$$\widehat{F}_N(\theta,y,x) = \frac{1}{n(n-1)\mathbb{E}[W_{12}(\theta,x)]}\sum_{1\le i\ne j\le n} W_{ij}(\theta,x)H(h_H^{-1}(y-Y_j)),$$

and

$$\widehat{F}_D(\theta,x) = \frac{1}{n(n-1)\mathbb{E}[W_{12}(\theta,x)]}\sum_{1\le i\ne j\le n} W_{ij}(\theta,x),$$

The following lemmas will be useful for proof of Theorem 1.

Lemma 1 *Under the assumptions of Theorem 1, we obtain:*

$$\mathbb{E}\left[\widehat{F}_N(\theta,y,x)\right] - F(\theta,y,x) = B_H(\theta,x,y)h_H^2 + B_K(\theta,x,y)h_K^2 + o(h_H^2) + o(h_K^2).$$

Lemma 2 *Under the assumptions of Theorem 1, we obtain:*

$$\mathrm{Var}\left[\widehat{F}_N(\theta, y, x)\right] = \frac{V_{HK}(\theta, x, y)}{n\phi_{\theta,x}(h_K)} + o\left(\frac{1}{n\phi_{\theta,x}(h_k)}\right).$$

Lemma 3 *Under the assumptions of Theorem 1, we get:*

$$\mathrm{Cov}(\widehat{F}_N(\theta, y, x), \widehat{F}_D(\theta, x)) = O\left(\frac{1}{n\phi_{\theta,x}(h_K)}\right).$$

Lemma 4 *Under the assumptions of Theorem 1, we get:*

$$\mathrm{Var}\left[\widehat{F}_D(\theta, x)\right] = O\left(\frac{1}{n\phi_{\theta,x}(h_K)}\right).$$

Comments

Since the mean squart error depend on the bias and variance, The idea of the proof of both variance term, and bias term is to treat separately the numerator and the denominator of the estimator. Lemma 3.2 is auxiliary result wish allow us to determine the bias of the estimator, wile Lemma 3.3-3.5, allow us to determine the variance of our estimator, by means the variance decomposition of Sarda and Vieu [27] and Lecoutre [24], see also Ferraty and al. [20]. As all asymptotic result in functional statistic, the dispersion term is related to the "dimensionality" of the functional variable in sense that the variance term depends on the function $\phi_x(h_K)$ which is closely linked on bi-functional operator δ and the latter can be related to the topological structure on the functional space $\mathcal{H}$.
Another way to highlight the interest of our asymptotic result is to show how the exact calculation of the leading terms in the quadratic error leads to the build of confidence intervals. Indeed, it is well known that the computation of the bias and the variance terms is commonly a preliminary result permitting to obtain the asymptotic normality result of the estimator.

3.2 Asymptotic normality

This section contains results on the asymptotic normality of $\widehat{F}(\theta, y, x)$. Before announcing our main results, we introduce the quantity $N(a, b)$, which will appear in the bias and variance dominant terms:

$$N(a, b) = K^a(1) - \int_{-1}^{1} (u^b K^a(u))' \chi_x(u) du \text{ for all } a > 0 \text{ and } b = 2, 4$$

Then, we have the following theorem:

Theorem 2 *Under assumptions (H1)-(H6), we obtain:*

$$\sqrt{n\phi_{\theta,x}(h_K)}(\widehat{F}(\theta,y,x) - F(\theta,y,x) - \mathbb{B}_n(\theta,x,y)) \xrightarrow{D} \mathcal{N}(0, V_{HK}(\theta,x,y)) \quad (1)$$

where,

$$V_{HK}(\theta,x,y) = \frac{M_2}{M_1^2} F(\theta,y,x)(1 - F(\theta,y,x)) \quad (2)$$

and

$$\mathbb{B}_n(\theta,x,y) = \frac{\mathbb{E}(\widehat{F}_N(\theta,y,x)(y))}{\mathbb{E}(\widehat{F}_D(\theta,x))} - F(\theta,y,x) \quad (3)$$

with $\xrightarrow{D}$ *denoting the convergence in distribution.*

Proof of Theorem 2.
Inspired by the decomposition given in Masry [25], we set.

$$\begin{aligned}\widehat{F}(\theta,y,x) &- F(\theta,y,x) - \mathbb{B}_n(\theta,x,y)\\ &= \frac{\widehat{F}_N(\theta,y,x) - F(\theta,y,x)\widehat{F}_D(\theta,x) - \widehat{F}_D(\theta,x)\mathbb{B}_n(\theta,x,y)}{\widehat{F}_D(\theta,x)}\end{aligned}$$

If we denote by

$$\begin{aligned}Q_n(\theta,x,y) = \widehat{F}_N(\theta,y,x) &- F(\theta,y,x)\widehat{F}_D(\theta,x) - \mathbb{E}(\widehat{F}_N(\theta,y,x)\\ &- F(\theta,y,x)\widehat{F}_D(\theta,x) = \widehat{F}_N(\theta,y,x)\\ &- F(\theta,y,x)\widehat{F}_D(\theta,x) - \mathbb{B}_n(\theta,x,y)\end{aligned} \quad (4)$$

since

$$\widehat{F}_N(\theta,y,x) - F(\theta,y,x)\widehat{F}_D(\theta,x) = Q_n(\theta,x,y) + \mathbb{B}_n(\theta,x,y)$$

then the proof of this theorem will be completed from the following expression

$$\begin{aligned}\widehat{F}(\theta,y,x) &- F(\theta,y,x) - \mathbb{B}_n(\theta,x,y)\\ &= \frac{Q_n(\theta,x,y) - \mathbb{B}_n(\theta,x,y)(\widehat{F}_D(\theta,x) - \mathbb{E}(\widehat{F}_D(\theta,x)))}{\widehat{F}_D(\theta,x)}\end{aligned} \quad (5)$$

and the following auxiliary results which play a main role and for which proofs are given in the appendix.

Lemma 5 *Under assumptions (H1)-(H5), we have*

$$\widehat{F}_D(\theta, x) \xrightarrow{P} \mathbb{E}(\widehat{F}_D(\theta, x)) = 1$$

where $\xrightarrow{P}$ *denotes the convergence in probability.*

Lemma 6 *Under assumptions (H2), (H4) and (H5), as* $n \to \infty$*, we have*

$$\mathbb{E}\left(K_1^2 var\left(H\left(\frac{y - Y_1}{h}\right)|X_1\right)\right) \to \mathbb{E}(K_1^2)F(\theta, y, x)(1 - F(\theta, y, x))$$

So, Lemma 5, implies that $\widehat{F}_D(\theta, x) \to 1$. Moreover, $\mathbb{B}_n(\theta, x, y) = o(1)$ as $n \to \infty$ because of the continuity of $F(\theta, ., x)$. Then, we obtain that

$$\widehat{F}(\theta, y, x) - F(\theta, y, x) - \mathbb{B}_n(\theta, x, y) = \frac{Q_n(\theta, x, y)}{\widehat{F}_D(\theta, x)}(1 + o_p(1))$$

Lemma 7 *Under assumptions (H1)-(H5), we have*

$$\sqrt{n\phi_{\theta,x}(h_K)}Q_n(\theta, x, y) \xrightarrow{D} \mathcal{N}(0, V_{HK}(\theta, x, y)), \tag{6}$$

where $V_{HK}(\theta, x, y)$ is defined by (2).

If we take advantage of the following assumptions,
(H7) $\lim_{n\to+\infty} \sqrt{nh_H\phi_{\theta,x}(h_K)}\mathbb{B}_n(\theta, x, y) = 0$, we can cancel the bias term and obtain the following corollary.

Corollary 1 *Under the assumptions of Theorem 2, we get*

$$\sqrt{\frac{nh_H\hat{\phi}_{\theta,x}(h_K)}{V_{HK}(\theta, x, y)}}(\widehat{F}(\theta, y, x) - F(\theta, y, x)) \to \mathcal{N}(0, 1)$$

Indeed: by the additional assumption (H7), we firstly obtain,

$$\sqrt{n\phi_{\theta,x}(h_K)}(\widehat{F}(\theta, y, x) - F(\theta, y, x)) \xrightarrow{D} \mathcal{N}(0, V_{HK}(\theta, x, y)),$$

to avoid estimating the constants in this last expression, one may consider the simple uniform kernel ($M_1 = M_2 = 1$) and get the above corollary (Corollary 3.11). So the practical utilization of our result in confidence intervals construction requires only the estimation of the function $\phi_{\theta,x}(t)$. This last can be empirically estimated by:

$$\hat{\phi}_{\theta,x}(t) = \frac{\sharp\{i : |d(X_i, x)| \leq t\}}{n},$$

where $\sharp(A)$ denote the cardinality of the set A.
Finally, for $\gamma \in (0,1)$, we obtain the following $(1-\gamma)$ confidence interval for $F(\theta,x,y)$:

$$\widehat{F}(\theta,y,x) \underset{-}{+} t_{1-\frac{\gamma}{2}} \times \frac{\widehat{\sigma}(\theta,x)}{\sqrt{n\widehat{\varphi}_{\theta,x}(h_K)}},$$

where $t_{1-\frac{\gamma}{2}}$ is the quantile of standard normal distribution, and $\widehat{\sigma}^2(x,y)$ denote the estimators of $V_{HK}(\theta,x,y)$.

Discussion on the importance of our model and on impacts of our results

It is well known that, the conditional distribution function (cdf) has the advantage of completely characterizing the conditional law of the considered random variables. In fact, the determination of the cdf allows to obtain the conditional density, the conditional hazard and the conditional quantile functions. Thus, even if the estimation of the conditional distribution has an interest in its own right, it is moreover of great aid in estimating various conditional models. On the other hand, the asymptotic results, obtained here, would have a great impact on the theoretical as well as on the practical aspects. The determination of the bias and of the variance terms of the estimator is a basic ingredient to obtain its asymptotic normality. This question is a natural way to extend results of this work. Notice also that this asymptotic property is very interesting to make statistical tests. The convergence in mean square wich study the L^2-consistency of $\widehat{F}(\theta,x,y)$ is one of the most useful/practical accuracy measures in the nonparametric smoothing estimation.

Remark 1 *The generalisation to multi-index model as mentioned by the reviewer, is an interesting subject, and a good prospect, to do that we consider* θ_D *as a matrix* $D \times D$ *of vectors* $(\theta_{jD})_{j=1,D}$ *of* $\mathcal{H}$*, where the direction* D *can be chosen by cross validation, and the inner product can be defined by* $|<\theta_D, x>| = \sum_j \theta_{jD} x_j$.

Remark 2 *Being independent refers to how the process of collectiong the sample was performed and it assures the representation fairness of the sampling. Dependent samples introduce bias into the results. From computational point of view independency significantly simplifies operation. If the random variables are not independents, the complexity of the problem explodes and we can not be able to use several results that need the random variables to be independent, in* *this case we can use mixing coeifficient to measure the dependency, this work is one of our goals to prepare another paper for submission.*

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4 Appendix

Proof of Theorem 1. We know the theorem is a consequence of a separate computes two quantities (bias and variance) of $\widehat{F}(\theta, y, x)$, we have

$$\mathbb{E}\left[\widehat{F}(\theta, y, x) - F(\theta, y, x)\right]^2 = \left[\mathbb{E}\left(\widehat{F}(\theta, y, x)\right) - F(\theta, y, x)\right]^2 + Var\left[\widehat{F}(\theta, y, x)\right]$$

By classical calculations, we obtain

$$\begin{aligned}\widehat{F}(\theta, y, x) - F(\theta, y, x) &= \left(\widehat{F}_N(\theta, y, x) - f(\theta, y, x)\right) - \widehat{F}_N(\theta, y, x)\left(\widehat{F}_D(\theta, x) - 1\right)\\ &- \mathbb{E}[\widehat{F}_N(\theta, y, x)]\left(\widehat{F}_D(\theta, x) - 1\right)\\ &- \mathbb{E}[\widehat{F}_N(\theta, y, x)]\left(\widehat{F}_D(\theta, x) - 1\right) + \left(\widehat{F}_D(\theta, x) - 1\right)^2\widehat{F}(\theta, y, x).\end{aligned}$$

which implies that:

$$\begin{aligned}\mathbb{E}\left[\widehat{F}(\theta, y, x)\right] - F(\theta, y, x) &= \left(\mathbb{E}[\widehat{F}_N(\theta, y, x)] - F(\theta, y, x)\right)\\ &- Cov\left(\widehat{F}_N(\theta, y, x), \widehat{F}_D(\theta, x)\right)\\ &+ \mathbb{E}\left[\left(\widehat{F}_D(\theta, x) - \mathbb{E}[\widehat{F}_D(\theta, x)]\right)^2\widehat{F}(\theta, y, x)\right].\end{aligned}$$

Hence:

$$\begin{aligned}\mathbb{E}\left[\widehat{F}(\theta, y, x)\right] - F(\theta, y, x) &= \left(\mathbb{E}[\widehat{F}_N(\theta, y, x)] - F(\theta, y, x)\right)\\ &- Cov\left(\widehat{F}_N(\theta, y, x), \widehat{F}_D(\theta, x)\right)\\ &+ Var\left[\widehat{F}_D(\theta, x)\right] O(h_H^{-1}).\end{aligned}$$

Now, by similar technics as those Sarda and Vieu [27] and by Bosq and Lecoutre [9], the variance term is

$$\begin{aligned} Var\left[\widehat{F}(\theta,y,x)\right] &= Var\left[\widehat{F}_N(\theta,y,x)\right] - 2\mathbb{E}[\widehat{F}_N(\theta,y,x)]Cov\left(\widehat{F}_N(\theta,y,x),\widehat{F}_D(\theta,x)\right) \\ &\quad + \left(\mathbb{E}[\widehat{F}_N(\theta,y,x)]\right)^2 Var\left(\widehat{F}_D(\theta,x)\right) + o\left(\frac{1}{n\phi_{\theta,x}(h_K)}\right). \end{aligned}$$

Proof of Lemma 1. We have:

$$\begin{aligned} \mathbb{E}[\widehat{F}_N(\theta,y,x)] &= \mathbb{E}\left[\frac{1}{n(n-1)\mathbb{E}[W_{12}(\theta,x)]}\sum_{1\le i\ne j\le n} W_{ij}(\theta,x)H(h_H^{-1}(y-Y_j))\right] \\ &= \frac{1}{\mathbb{E}[W_{\theta,12}]}\mathbb{E}\left[W_{\theta,12}\mathbb{E}[H_2|X_2]\right]. \end{aligned}$$

We use an integration by part to show that:

$$\mathbb{E}[H_2|X_2] = h_H^{-1}\int_{\mathbb{R}} H^{(1)}(h_H^{-1}(y-z))F(\theta,z,x)dz$$

Now the change of variable $t = \frac{y-z}{h_H}$ allows to write:

$$|\mathbb{E}[H_2|X_2] + \int_{\mathbb{R}} H^{(1)}(t)|F(\theta,y-th_H,x)$$

By using a Taylor's expansion and under assumption (H5), we have

$$\mathbb{E}[H_2|X_2] = F(\theta,y,X_2) + \frac{h_H^2}{2}\left(\int t^2H^{(1)}(t)dt\right)\frac{\partial^2F(\theta,y,X_2)}{\partial y^2} + o(h_H^2).$$

Now, we can re-written as:

$$\mathbb{E}[H_2|X_2] = \psi_0(X_2,y) + \frac{h_H^2}{2}\left(\int t^2H^{(1)}(t)dt\right)\psi_2(X_2,y) + o(h_H^2).$$

Thus, we obtain

$$\begin{aligned} \mathbb{E}\left[\widehat{F}_N(\theta,y,x)\right] &= \frac{1}{E[W_{\theta,12}]}\mathbb{E}\left[W_{\theta,12}\psi_0(X_2,y)\right] \\ &\quad + \frac{1}{E[W_{\theta,12}]}\left(\int t^2H^{(1)}(t)dt\right)\mathbb{E}\left[W_{\theta,12}\psi_2(X_2,y)\right] + o(h_H^2). \end{aligned}$$

Accordingly to Ferraty et al. [20], for $l \in \{0, 2\}$, we show that

$$\begin{aligned}\mathbb{E}[W_{\theta,12}\psi_l(X_2,y)] &= \psi_l(x,y)\mathbb{E}[W_{\theta,12}] + \mathbb{E}[W_{\theta,12}(\psi_l(X_2,y)-\psi_l(x,y))]\\ &= \psi_l(x,y)\mathbb{E}[W_{\theta,12}] + \mathbb{E}[W_{\theta,12}\mathbb{E}[\psi_l(X_2,y)-\psi_l(x,y)|\beta_\theta(X_2,x)]]\\ &= \psi_l(x,y)\mathbb{E}[W_{\theta,12}] + \mathbb{E}[W_{\theta,12}\Phi_l(\beta_\theta(X_2,x))].\end{aligned}$$

Since $\mathbb{E}[\beta_{\theta,2}W_{\theta,12}] = 0$ and $\Phi_l(0) = 0$, for $l \in 0, 2$, we obtain

$$\mathbb{E}[W_{\theta,12}\Phi_l(\beta_\theta(X_2,x))] = \frac{1}{2}\Phi_l^{(2)}(0)\mathbb{E}[\beta_\theta^2(X_2,x)W_{\theta,12}] + o(\mathbb{E}\,[\beta_\theta(X_2,x)W_{\theta,12}]).$$

Then,

$$\begin{aligned}\mathbb{E}\left[\widehat{F}_N(\theta,y,x)\right] &= F(\theta,y,x) + \frac{h_H^2}{2}\frac{\partial^2 F(\theta,y,x)}{\partial y^2}\int t^2 H^{(1)}(t)dt\\ &+ o\left(h_H^2\frac{\mathbb{E}\left[\beta_\theta^2(X_2,x)W_{\theta,12}\right]}{\mathbb{E}[W_{\theta,12}]}\right) + \frac{1}{2}\Phi_l^{(2)}(0)\frac{\mathbb{E}\left[\beta_\theta^2(X_2,x)W_{\theta,12}\right]}{\mathbb{E}[W_{\theta,12}]}\\ &+ o\left(\frac{\mathbb{E}\left[\beta_\theta^2(X_2,x)W_{\theta,12}\right]}{\mathbb{E}[W_{\theta,12}]}\right).\end{aligned}$$

Therefore, it remains to determine the quantities $E\left[\beta_\theta^2(X_2,x)W_{\theta,12}\right]$ and $E[W_{\theta,12}]$. According to the definition of $W_{\theta,12}$, the behaviours of the two quantities $E\left[\beta_\theta^2(X_2,x)W_{\theta,12}\right]$ and $E[W_{\theta,12}]$ are based on the asymptotic evaluation of $E[K_1^a\beta_1^b]$. To do that, we treat firstly, the case $b = 1$. For this case,we use the assumptions (H3) and (H4) to get

$$h_K\mathbb{E}[K_1^a\beta_{\theta,1}] = o\left(\int_{B(x,h_K)}\beta_\theta^2(u,x)dP(u)\right) = o(h_K^2\phi_{\theta,x}(h_K)).$$

So, we obtain that,

$$\mathbb{E}[K_1^a\beta_{\theta,1}] = o(h_K\phi_{\theta,x}(h_K)). \tag{7}$$

Morever, for all $b > 1$, and after simplifications of the expressions, permits to write that

$$\mathbb{E}[K_1^a\beta_{\theta,1}^b] = \mathbb{E}[K_1^a d_\theta^b(x,X)] + o(h_K^b\phi_{\theta,x}(h_K)).$$

Concerning the first term, we write

$$\begin{aligned}
h_K^{-b}\mathbb{E}[K_1^a d_\theta^b] &= \int v^b K^a(v) dP^{h_K^{-1} d_\theta(x,X)}(v) \\
&= \int_{-1}^{1} \left[K^a(1) - \int_v^1 \left((s^b K^a(s))'\right) du\right] dP^{h_K^{-1} d_\theta(x,X)}(v) \\
&= \left(K(1)\phi_{\theta,x}(h_K) - \int_{-1}^{1} (s^b K^a(s))^{(1)} \phi_{\theta,x}(sh_K, h_K) ds\right) \\
&= \phi_{\theta,x}(h_K)\left(K(1) - \int_{-1}^{1} (s^b K^a(s))' \frac{\phi_{\theta,x}(sh_K, h_K)}{\phi_{\theta,x}(h_K)} ds\right).
\end{aligned}$$

Finally, under assumptions (H1), we get

$$\mathbb{E}[K_1^a \beta_{\theta,1}^b] = h_K^b \phi_{\theta,x}(h_K)\left(K(1) - \int_{-1}^{1} (s^b K^a(u))' \chi_{\theta,x}(s) ds\right) + o(h_K^b \phi_{\theta,x}(h_K)). \tag{8}$$

On other hand, by following the same steps in Ferraty and al. [20], we have

$$\mathbb{E}[W_{\theta,12}] = O(h_K^2 \phi_{\theta,x}^2(h_K)), \tag{9}$$

and

$$\mathbb{E}(K_{\theta,1}^j) = M_j \phi_{\theta,x}(h_K) \text{ for } j = 1, 2 \tag{10}$$

So,

$$\frac{\mathbb{E}[\beta_\theta^2(X_2,x) W_{\theta,12}]}{\mathbb{E}[W_{\theta,12}]} = h_K^2\left(\frac{K(1) - \int_{-1}^{1}(s^2 K(s))' \chi_{\theta,x}(s) ds}{K(1) - \int_{-1}^{1}(K'(u)\chi_{\theta,x}(s) ds}\right) + o(h_K^2).$$

Hence,

$$\begin{aligned}
\mathbb{E}\left[\widehat{F}_N(\theta,y,x)\right] &= F(\theta,y,x) + \frac{h_H^2}{2}\frac{\partial^2 F(\theta,y,x)}{\partial y^2}\int t^2 H^{(1)}(t) dt + o(h_H^2) \\
&+ h_K^2 \Phi_0^{(2)}(0) \frac{\left(K(1) - \int_{-1}^{1}(s^2 K(s))' \chi_{\theta,x}(s) ds\right)}{2\left(K(1) - \int_{-1}^{1} K'(s)\chi_{\theta,x}(s) ds\right)} + o(h_K^2).
\end{aligned}$$

Proof of Lemma 2. We know

$$\mathrm{Var}\left(\widehat{F}_N(\theta,y,x)\right) = \frac{1}{(n(n-1)(\mathbb{E}[W_{\theta,12}]))^2} \mathrm{Var}\left(\sum_{1\le i\ne j\le n} W_{\theta,ij} H_j\right)$$

$$= \frac{1}{(n(n-1)(\mathbb{E}[W_{\theta,12}]))^2}\Big[n(n-1)\mathbb{E}[W^2_{\theta,12}H^2_2] + n(n-1)\mathbb{E}[W_{\theta,12}W_{\theta,21}H_2H_1]$$
$$+n(n-1)(n-2)\mathbb{E}[W_{\theta,12}W_{\theta,13}H_2H_3] + n(n-1)(n-2)\mathbb{E}[W_{\theta,12}W_{\theta,23}H_2H_3]$$
$$+n(n-1)(n-2)\mathbb{E}[W_{\theta,12}W_{\theta,31}H_2H_1] + n(n-1)(n-2)\mathbb{E}[W_{\theta,12}W_{\theta,32}H^2_2]$$
$$-n(n-1)(4n-6)(\mathbb{E}[W_{\theta,12}H_2])^2\Big]. \tag{11}$$

By direct calculations, we get

$$\begin{cases} \mathbb{E}[W^2_{\theta,12}H^2_2] = O(h^4_K\phi^2_{\theta,x}(h_K)), \quad \mathbb{E}[W_{\theta,12}W_{\theta,21}H_2H_1] = O(h^4_K\phi^2_{\theta,x}(h_K)), \\ \mathbb{E}[W_{\theta,12}W_{\theta,13}H_2H_3] = (F(\theta,y,x))^2\mathbb{E}[\beta^4_{1,\theta}K^2_{\theta,1}](\mathbb{E}[K_{\theta,1}])^2 + o(h^4_K\phi^3_{\theta,x}(h_K)) \\ \mathbb{E}[W_{\theta,12}W_{\theta,23}H_2H_3] = (F(\theta,y,x))^2\mathbb{E}[\beta^2_{1,\theta}K_{\theta,1}]\mathbb{E}[\beta^2_{1,\theta}K^2_{\theta,1}]\mathbb{E}[K_{\theta,1}] + o(h^4_K\phi^3_{\theta,x}(h_K)) \\ \mathbb{E}[W_{\theta,12}W_{\theta,31}H_2H_3] = (F(\theta,y,x))^2\mathbb{E}[\beta^2_{1,\theta}K_{\theta,1}]\mathbb{E}[\beta^2_{1,\theta}K^2_{\theta,1}]\mathbb{E}[K_{\theta,1}] + o(h^4_K\phi^3_{\theta,x}(h_K)) \\ \mathbb{E}[W_{\theta,12}W_{\theta,32}H^2_2] = F(\theta,y,x)\mathbb{E}^2[\beta^2_1K_1]\mathbb{E}[K^2_1] + o(h^4_K\phi^3_{\theta,x}(h_K)). \\ \mathbb{E}[W_{\theta,12}H_1] = O(h^2_K\phi^2_{\theta,x}(h_K)) \end{cases}$$

By equation (7), equation (8), (9) and (10)

$$Var\left(\widehat{F}_N(\theta,y,x)\right) = \frac{F(\theta,y,x)(1-F(\theta,y,x))}{n\phi_{\theta,x}(h_K)}\left[\frac{\left(K^2(1)-\int_{-1}^1(K^2(s))'\chi_{\theta,x}(s)ds\right)}{\left(K(1)-\int_{-1}^1(K(s))'\chi_{\theta,x}(s)ds\right)^2}\right]$$
$$+o\left(\frac{1}{n\phi_{\theta,x}(h_K)}\right) = \frac{M_2}{M_1^2 n\phi_{\theta,x}(h_K)}F(\theta,y,x)(1-F(\theta,y,x)) + o\left(\frac{1}{n\phi_{\theta,x}(h_K)}\right).$$

Proof of Lemma 3. The proof of this Lemma it's similar to Lemma 2 proof, it permits to write (with $I = \{(i,j) : 1 \le i \ne j \le n\}$)

$$Cov\left(\widehat{F}_N(\theta,y,x),\widehat{F}_D(\theta,x)\right) = \frac{1}{(n(n-1)\mathbb{E}[W_{\theta,12}])^2}Cov\Big(\sum_{i,j\in I}W_{\theta,ij}H_j, \sum_{k,l\in I}W_{\theta,kl}\Big)$$
$$= \frac{1}{(n(n-1)\mathbb{E}[W_{\theta,12}])^2}\Big[n(n-1)\mathbb{E}[W^2_{\theta 12}H_2] + n(n-1)\mathbb{E}[W_{\theta,12}W_{\theta,21}H_2]$$
$$+n(n-1)(n-2)\mathbb{E}[W_{\theta,12}W_{\theta,13}H_2] + n(n-1)(n-2)\mathbb{E}[W_{\theta,12}W_{\theta,23}H_2]$$
$$+n(n-1)(n-2)\mathbb{E}[W_{\theta,12}W_{\theta,31}H_2] + n(n-1)(n-2)\mathbb{E}[W_{\theta,12}W_{\theta,32}H_2]$$
$$-n(n-1)(4n-6)(\mathbb{E}[W_{\theta,12}H_2]\mathbb{E}[W_{\theta,12}]\Big].$$

By direct calculations, we get

$$\begin{cases} \mathbb{E}[W_{\theta,12}^2 H_2] = O(h_K^4 \phi_{\theta,x}^2(h_K)), \quad \mathbb{E}[W_{\theta,12}W_{\theta,21}H_2] = O(h_K^4 \phi_{\theta,x}^2(h_K)), \\ \mathbb{E}[W_{\theta,12}W_{\theta,13}H_2] = (F(\theta,y,x))\mathbb{E}[\beta_{1,\theta}^4 K_{\theta,1}^2](\mathbb{E}[K_{\theta,1}])^2 + o(h_K^4 \phi_{\theta,x}^3(h_K)) \\ \mathbb{E}[W_{\theta,12}W_{\theta,23}H_2] = (F(\theta,y,x))\mathbb{E}[\beta_{1,\theta}^2 K_{\theta,1}]\mathbb{E}[\beta_{1,\theta}^2 K_{\theta,1}^2]\mathbb{E}[K_{\theta,1}] + o(h_K^4 \phi_{\theta,x}^3(h_K)) \\ \mathbb{E}[W_{\theta,12}W_{\theta,31}H_2] = (F(\theta,y,x))\mathbb{E}[\beta_{1,\theta}^2 K_{\theta,1}]\mathbb{E}[\beta_{1,\theta}^2 K_{\theta,1}^2]\mathbb{E}[K_{\theta,1}] + o(h_K^4 \phi_{\theta,x}^3(h_K)) \\ \mathbb{E}[W_{\theta,12}W_{\theta,32}H_2] = F(\theta,y,x)\mathbb{E}^2[\beta_{\theta,1}^2 K_1]\mathbb{E}[K_{\theta,1}^2] + o(h_K^4 \phi_{\theta,x}^3(h_K)). \\ \mathbb{E}[W_{\theta,12}H_1] = O(h_K^2 \phi_{\theta,x}^2(h_K)) \end{cases}$$

By equation (7), equation (8), (9) and (10), we obtain

$$\mathrm{Cov}\left(\widehat{F}_N(\theta,y,x), \widehat{F}_D(\theta,x)\right) = O\left(\frac{1}{n\phi_{\theta,x}(h_K)}\right).$$

Proof of Lemma 4. We have that

$$Var(\widehat{f}_D(\theta,x)) = \frac{1}{(n(n-1)\mathbb{E}[W_{\theta,12}])^2} Var\left(\sum_{1\le i\ne j\le n} W_{\theta,ij}\right).$$

That is

$$\begin{aligned} Var\left(\widehat{F}_D(\theta,x)\right) &= \frac{1}{(n(n-1)(\mathbb{E}[W_{\theta,12}]))^2}\Big[n(n-1)\mathbb{E}[W_{\theta,12}^2] + n(n-1)\mathbb{E}[W_{\theta,12}W_{\theta,21}] \\ &\quad + n(n-1)(n-2)\mathbb{E}[W_{\theta,12}W_{\theta,13}] + n(n-1)(n-2)\mathbb{E}[W_{\theta,12}W_{\theta,23}] \\ &\quad + n(n-1)(n-2)\mathbb{E}[W_{\theta,12}W_{\theta,31}] + n(n-1)(n-2)\mathbb{E}[W_{\theta,12}W_{\theta,32}] \\ &\quad - n(n-1)(4n-6)(\mathbb{E}[W_{\theta,12}])^2\Big]. \end{aligned} \tag{12}$$

and similarly to the previous cases

$$\begin{cases} \mathbb{E}[W_{\theta,12}^2] = O(h_K^4 \phi_{\theta,x}^2(h_K)), \quad \mathbb{E}[W_{\theta,12}W_{\theta,21}] = O(h_K^4 \phi_{\theta,x}^2(h_K)), \\ \mathbb{E}[W_{\theta,12}W_{\theta,13}] = \mathbb{E}[\beta_{1,\theta}^4 K_{\theta,1}^2](\mathbb{E}[K_{\theta,1}])^2 + o(h_K^4 \phi_{\theta,x}^3(h_K)) \\ \mathbb{E}[W_{\theta,12}W_{\theta,23}] = \mathbb{E}[\beta_{1,\theta}^2 K_{\theta,1}]\mathbb{E}[\beta_{1,\theta}^2 K_{\theta,1}^2]\mathbb{E}[K_{\theta,1}] + o(h_K^4 \phi_{\theta,x}^3(h_K)) \\ \mathbb{E}[W_{\theta,12}W_{\theta,31}] = \mathbb{E}[\beta_{1,\theta}^2 K_{\theta,1}]\mathbb{E}[\beta_{1,\theta}^2 K_{\theta,1}^2]\mathbb{E}[K_{\theta,1}] + o(h_K^4 \phi_{\theta,x}^3(h_K)) \\ \mathbb{E}[W_{\theta,12}W_{\theta,32}] = \mathbb{E}^2[\beta_{\theta,1}^2 K_1]\mathbb{E}[K_{\theta,1}^2] + o(h_K^4 \phi_{\theta,x}^3(h_K)). \\ \mathbb{E}[W_{\theta,12}] = O(h_K^2 \phi_{\theta,x}^2(h_K)) \end{cases}$$

By the same arguments used in the previous lemmas, we can write:

$$\begin{aligned} Var\left(\widehat{F}_D(\theta,x)\right) &= \frac{M_2\phi_{\theta,x}(h_K)}{n(M_1\phi_{\theta,x}(h_K))^2} + o\left(\frac{1}{n\phi_{\theta,x}(h_K)}\right) \\ &= O\left(\frac{1}{n\phi_{\theta,x}(h_K)}\right). \end{aligned}$$

Proof of Lemma 5. By applying the Bienaym'e-Tchebychev's inequality, as $n \to +\infty$,, we obtain, for all $\varepsilon > 0$,

$$\begin{aligned}
\mathbb{P}(|\widehat{F}_D(\theta,x)) - \mathbb{E}(\widehat{F}_D(\theta,x)|) \geq \varepsilon) &< \frac{var(\widehat{F}_D(\theta,x))}{\varepsilon^2} \\
&< \frac{1}{\varepsilon^2} O\Big(\frac{1}{n\phi_{\theta,x}(h_K)}\Big) \\
&= \frac{o(1)}{\varepsilon^2} \\
&\to 0
\end{aligned}$$

Proof of Lemma 6. We have,

$$\begin{aligned}
\mathbb{E}\left(K_1^2 var\left(H\left(\frac{y-Y_1}{h}\right)|X_1\right)\right) &= \mathbb{E}\left(K_1^2 \mathbb{E}\left(\left(H\left(\frac{y-Y_1}{h}\right)\right)^2 |X_1\right)\right) \\
&\quad - \mathbb{E}\left(K_1^2 \mathbb{E}^2\left(H\left(\frac{y-Y_1}{h}\right)|X_1\right)\right)
\end{aligned}$$

By an integration par parts, followed by a change of variable, we get

$$\begin{aligned}
\mathbb{E}(H^2\left(\frac{y-Y_1}{h_H}\right)|X_1) &= \frac{1}{h_H}\int H^2(t) dF(\theta, y - th_H, X_1) \\
&= 2\int H^{(1)} H(t)(F(\theta, y - th_H, X_1) - F(\theta, y, x)) dt \\
&+ 2\int H^{(1)} H(t) F(\theta, y, x) dt
\end{aligned}$$

Since

$$2\int H^{(1)} H(t) F(\theta, y, x) dt = F(\theta, y, x) \text{ as } n \to +\infty,$$

we deduce that, as $n \to +\infty$, we have

$$\mathbb{E}(K_{\theta,1}^2 H^2\left(\frac{y-Y_1}{h_H}\right)|X_1) \to \mathbb{E}(K_{\theta,1}^2] F(\theta, y, x)$$

and

$$\mathbb{E}(H\left(\frac{y-Y_1}{h_H}\right)|X_1) - F(\theta, y, x) \to 0,$$

so

$$\mathbb{E}(K_{\theta,1}^2 \mathbb{E}^2(H^2\left(\frac{y-Y_1}{h_H}\right)|X_1)) \to \mathbb{E}(K_{\theta,1}^2] F^2(\theta, y, x)$$

finally, we obtain:

$$\mathbb{E}\left(K_1^2 var\left(H\left(\frac{y-Y_1}{h}\right)|X_1\right)\right) \to \mathbb{E}(K_1^2)F(\theta,y,x)(1-F(\theta,y,x))$$

Proof of Lemma 7.
We have

$$\begin{aligned}\sqrt{n\phi_{\theta,x}(h_K)}Q_n(\theta,x,y) &= \frac{\sqrt{n\phi_{\theta,x}(h_K)}}{n\mathbb{E}(\Omega_1 K_1)}\sum_{j=1}^{n}\Omega_j K_j(H_j - F(\theta,y,x)) \\ &- \frac{\sqrt{n\phi_{\theta,x}(h_K)}}{n\mathbb{E}(\Omega_1 K_1)}\mathbb{E}(\sum_{j=1}^{n}\Omega_j K_j(H_j - F(\theta,y,x))\end{aligned}$$

then, combined with (4) implies that

$$\begin{aligned}&\sqrt{n\phi_{\theta,x}(h_K)}\ Q_n(\theta,x,y) = \frac{1}{n\mathbb{E}(\beta_1^2 K_1)}\sum_{i=1}^{n}\beta_i^2 K_i\frac{\sqrt{n\phi_{\theta,x}(h_K)}\mathbb{E}(\beta_1^2 K_1)}{\mathbb{E}(\Omega_1 K_1)} \\ &\times\sum_{j=1}^{n}K_j(H_j - F(\theta,y,x)) \\ &- \frac{1}{n\mathbb{E}(\beta_1 K_1)}\sum_{i=1}^{n}\beta_i K_i\frac{\sqrt{n\phi_{\theta,x}(h_K)}\mathbb{E}(\beta_1 K_1)}{\mathbb{E}(\Omega_1 K_1)}\sum_{j=1}^{n}\beta_j K_j(H_j - F(\theta,y,x)) \\ &- \mathbb{E}\left(\frac{1}{n\mathbb{E}(\beta_1^2 K_1)}\sum_{i=1}^{n}\beta_i^2 K_i\frac{\sqrt{n\phi_{\theta,x}(h_K)}\mathbb{E}(\beta_1^2 K_1)}{\mathbb{E}(\Omega_1 K_1)}\sum_{j=1}^{n}K_j(H_j - F(\theta,y,x))\right) \\ &+ \mathbb{E}\left(\frac{1}{n\mathbb{E}(\beta_1 K_1)}\sum_{i=1}^{n}\beta_i K_i\frac{\sqrt{n\phi_{\theta,x}(h_K)}\mathbb{E}(\beta_1 K_1)}{\mathbb{E}(\Omega_1 K_1)}\sum_{j=1}^{n}\beta_j K_j(H_j - F(\theta,y,x))\right)\end{aligned}$$

Denote by

$$S_1 = \frac{1}{n\mathbb{E}(\beta_1^2 K_1)}\sum_{i=1}^{n}\beta_i^2 K_i \quad , \quad S_2 = \frac{\sqrt{n\phi_{\theta,x}(h_K)}\mathbb{E}(\beta_1^2 K_1)}{\mathbb{E}(\Omega_1 K_1)}\sum_{j=1}^{n}K_j(H_j - F(\theta,y,x))$$

$$S_3 = \frac{1}{n\mathbb{E}(\beta_1 K_1)}\sum_{i=1}^{n}\beta_i K_i \quad \text{and} \quad S_4 = \frac{\sqrt{n\phi_{\theta,x}(h_K)}\mathbb{E}(\beta_1 K_1)}{\mathbb{E}(\Omega_1 K_1)}\sum_{j=1}^{n}\beta_j K_j(H_j - F(\theta,y,x))$$

It remains to show that,

$$\begin{aligned}\sqrt{n\phi_{\theta,x}(h_K)}Q_n(\theta,x,y) &= S_1S_2 - S_3S_4 - \mathbb{E}(S_1S_2 - S_3S_4) \\ &= (S_1S_2 - \mathbb{E}(S_1S_2)) - (S_3S_4 - \mathbb{E}(S_3S_4))\end{aligned} \tag{13}$$

Hence by the Slutsky's theorem, to show (13), it suffices to prove the following two claims:

$$S_1S_2 - \mathbb{E}(S_1S_2) \xrightarrow{D} \mathcal{N}(0, V_{HK}(\theta,x,y)) \tag{14}$$

$$S_3S_4 - \mathbb{E}(S_3S_4) \xrightarrow{P} 0, \tag{15}$$

Proof of (14) We can write that

$$S_1S_2 - \mathbb{E}(S_1S_2) = S_2 - \mathbb{E}(S_2) + (S_1-1)S_2 - \mathbb{E}((S_1-1)S_2).$$

by the Slutsky's theorem, we get the following intermediate results,

$$(S_1-1)S_2 - \mathbb{E}((S_1-1)S_2) \xrightarrow{P} 0 \tag{16}$$

and

$$S_2 - \mathbb{E}(S_2) \xrightarrow{D} \mathcal{N}(0, V_{HK}(\theta,x,y)) \tag{17}$$

Concerning the proof of (16), by applying the Bienaymé-Tchebychv's inequality, we obtain for all $\epsilon > 0$

$$\mathbb{P}(|(S_1-1)S_2 - \mathbb{E}((S_1-1)S_2)| > \epsilon) \le \frac{\mathbb{E}(|(S_1-1)S_2 - \mathbb{E}(S_1-1)S_2)|)}{\epsilon}.$$

Then, the Cauchy-Schwarz inequality implies that

$$\mathbb{E}(|(S_1-1)S_2 - \mathbb{E}((S_1-1)S_2)|) \le 2\mathbb{E}(|(S_1-1)S_2)|) \le 2\sqrt{\mathbb{E}((S_1-1)^2)}\sqrt{\mathbb{E}((S_2)^2)}$$

On one side, by using equations (7) and (8), we obtain

$$\begin{aligned}\mathbb{E}((S_1-1)^2) &= var(S_1) = \frac{1}{n^2\mathbb{E}^2(\beta_1^2K_1)} n\,var(\beta_1^2K_1) \\ &\le \frac{1}{nO(h_K^4\phi_{\theta,x}^2(h_K))}\mathbb{E}(\beta_1^4K_1^2) = O\left(\frac{1}{n\phi_{\theta,x}(h_K)}\right).\end{aligned}$$

and on the other side, we obtain

$$\begin{aligned}\mathbb{E}((S_2)^2) &= \frac{n\phi_{\theta,x}(h_K)\mathbb{E}^2(\beta_1^2K_1)}{\mathbb{E}^2(\Omega_1K_1)}\mathbb{E}\left(\sum_{j=1}^{n} K_j(H_j - F(\theta,y,x))\right)^2 \\ &= \frac{n}{(n-1)^2O(\phi_{\theta,x}(h_K))}(nO(\phi_{\theta,x}(h_K)) + n(n-1)o(\phi_{\theta,x}^2(h_K))) \\ &= O(1) + o(n\phi_{\theta,x}(h_K)).\end{aligned}$$

Thus

$$\begin{aligned}\mathbb{E}(|(S_1-1)S_2-\mathbb{E}((S_1-1)S_2)|) &\leq 2\sqrt{\mathbb{E}((S_1-1)^2)}\sqrt{\mathbb{E}((S_2)^2)}\\ &2\sqrt{O\left(\frac{1}{n\phi_{\theta,x}(h_K)}\right)(O(1)+o(n\phi_{\theta,x}(h_K)))}\\ &=o(1),\end{aligned}$$

which implies that $(S_1-1)S_2-\mathbb{E}(S_1-1)S_2)=o_p(1)$. Then, as $n\to\infty$, we get

$$\mathbb{P}(|(S_1-1)S_2-\mathbb{E}(S_1-1)S_2)|)>\epsilon)\leq\frac{\mathbb{E}(|(S_1-1)S_2-\mathbb{E}(S_1-1)S_2)|)}{\epsilon}\to 0.$$

Concerning the proof of (17), we denote

$$\begin{aligned}P_n &= S_2-\mathbb{E}(S_2)\\ &= \frac{\sqrt{n\phi_{\theta,x}(h_K)}\mathbb{E}(\beta_1^2K_1)}{\mathbb{E}(\Omega_1K_1)}\sum_{j=1}^{n}K_j(H_j-F(\theta,y,x))-\mathbb{E}(K_j(H_j-F(\theta,y,x)))\\ &= \frac{\sqrt{n\phi_{\theta,x}(h_K)}\mathbb{E}(\beta_1^2K_1)}{\mathbb{E}(\Omega_1K_1)}\sum_{j=1}^{n}\mu_{nj}(x,y),\end{aligned}$$

where

$$\mu_{nj}(x,y)=K_j(H_j-F(\theta,y,x))-\mathbb{E}(K_j(H_j-F(\theta,y,x)))$$

By the fact that $\mu_{nj}(x,y)$ are i.i.d., it follows that

$$\begin{aligned}var(P_n(x,y)) &= \frac{n^2\phi_{\theta,x}(h_K)\mathbb{E}^2(\beta_1^2K_1)}{\mathbb{E}^2(\Omega_1K_1)}var(\mu_{n1}(x,y))\\ &= \frac{n^2\phi_{\theta,x}(h_K)\mathbb{E}^2(\beta_1^2K_1)}{\mathbb{E}^2(\Omega_1K_1)}\mathbb{E}(\mu_{n1}^2(x,y))\end{aligned}$$

Thus

$$\begin{aligned}var(P_n(x,y)) = \frac{n^2\phi_{\theta,x}(h_K)\mathbb{E}^2(\beta_1^2K_1)}{\mathbb{E}^2(\Omega_1K_1)}&(\mathbb{E}(K_1^2(H_1\\ &-F(\theta,y,x))^2)-(\mathbb{E}(K_1(H_1-F(\theta,y,x)))^2).\end{aligned} \tag{18}$$

Concerning the second term on the right hand side of (18), we have

$$\begin{aligned}(\mathbb{E}(K_1(H_1-F(\theta,y,x)))^2 &= (\mathbb{E}(\mathbb{E}(K_1(H_1-F(\theta,y,x))|X_1))^2\\ &= (\mathbb{E}(K_1\mathbb{E}((H_1|X_1)-F(\theta,y,x))))^2,\end{aligned}$$

where

$$\frac{1}{h_H}\mathbb{E}((H_1|X_1) - F(\theta, y, x)) \to 0 \quad as \quad n \to \infty \tag{19}$$

Now let us return to the first term of the right hand of (18). We have

$$\begin{aligned}
&\frac{n^2\phi_{\theta,x}(h_K)\mathbb{E}^2(\beta_1^2K_1)}{\mathbb{E}^2(\Omega_1K_1)}(\mathbb{E}(K_1^2(H_1 - F(\theta, y, x))^2)\\
&= \frac{n^2\phi_{\theta,x}(h_K)\mathbb{E}^2(\beta_1^2K_1)}{\mathbb{E}^2(\Omega_1K_1)}(\mathbb{E}(\mathbb{E}((H_1 - F(\theta, y, x))^2|X_1)K_1^2)\\
&= \frac{n^2\phi_{\theta,x}(h_K)\mathbb{E}^2(\beta_1^2K_1)}{\mathbb{E}^2(\Omega_1K_1)}\mathbb{E}(var(H_1|X_1)K_1^2)\\
&+ \frac{n^2\phi_{\theta,x}(h_K)\mathbb{E}^2(\beta_1^2K_1)}{\mathbb{E}^2(\Omega_1K_1)}(\mathbb{E}(\mathbb{E}((H_1|X_1) - F(\theta, y, x))^2)K_1^2)
\end{aligned}$$

By using (19), that allows to have, as $n \to \infty$

$$\frac{n^2\phi_{\theta,x}(h_K)\mathbb{E}^2(\beta_1^2K_1)}{\mathbb{E}^2(\Omega_1K_1)}(\mathbb{E}(\mathbb{E}((H_1|X_1) - F(\theta, y, x))^2)K_1^2) \to 0$$

Combining equations (7), (8) and (10), with lemma 6, we obtain as $n \to \infty$

$$\begin{aligned}
&\mathbb{E}(var(H_1|X_1)K_1^2) \to \mathbb{E}(K_1^2)F(\theta, y, x)(1 - F(\theta, y, x))\\
&= M_2F(\theta, y, x)(1 - F(\theta, y, x))\phi_{\theta,x}(h_K).
\end{aligned}$$

Therefore, by using equations (7), (8) and (10) , equation (18) becomes

$$\begin{aligned}
var(P_n(x, y)) &= \frac{n^2\phi_{\theta,x}(h_K)(N(1, 2)h_K^2\phi_{\theta,x}(h_K))^2}{((n - 1)N(1, 2)M_1h_K^2\phi_{\theta,x}(h_K))^2}\\
&\quad M_2F(\theta, y, x)(1 - F(\theta, y, x))\phi_{\theta,x}(h_K)\\
&= \frac{n^2M_2}{(n - 1)^2M_1^2}F(\theta, y, x)(1 - F(\theta, y, x))\\
&\to \frac{M_2}{M_1^2}F(\theta, y, x)(1 - F(\theta, y, x)) = V_{HK}(\theta, x, y) \quad as \quad n \to \infty
\end{aligned}$$

Now, in order to end the proof of (17), we focus on the central limit theorem. So, the proof of (14) is completed if the Lindberg's condition is verified. In fact, the Lindberg's condition holds since, for any $\eta > 0$

$$\sum_{j=1}^{n}\mathbb{E}(\mu_{nj}^2\mathbb{1}_{(|\mu_{nj}|>\eta)}) = n\mathbb{E}(\mu_{n1}^2\mathbb{1}_{(|\mu_{n1}|>\eta)}) = \mathbb{E}((\sqrt{n}\mu_{n1})^2\mathbb{1}_{(|\sqrt{n}\mu_{n1}|>\sqrt{n}\eta)})$$

as

$$\mathbb{E}((\sqrt{n}\mu_{n1})^2) = n\mathbb{E}(\mu_{n1}^2) \to \frac{M_2}{M_1^2}F(\theta,y,x)(1-F(\theta,y,x)).$$

Proofs of (15). To use the same arguments as those invoked to prove (14), let us write

$$S_3S_4 - \mathbb{E}(S_3S_4) = S_4 - \mathbb{E}(S_4) + (S_3-1)S_4 - \mathbb{E}(S_3-1)S_4)).$$

By applying the Bienaymé-Tchebychv's inequality, we obtain for all $\epsilon > 0$

$$\mathbb{P}(|S_3S_4 - \mathbb{E}(S_3S_4)|) > \epsilon) \le \frac{\mathbb{E}(|S_3S_4 - \mathbb{E}(S_3S_4)|)}{\epsilon}.$$

and the Cauchy-Schwarz inequality implies that

$$\mathbb{E}(|(S_3-1)S_4 - \mathbb{E}((S_3-1)S_4)|) \le 2\mathbb{E}(|(S_3-1)S_4)|) \le 2\sqrt{\mathbb{E}((S_3-1)^2)}\sqrt{\mathbb{E}((S_4)^2)}$$

Taking into account the equations (9) and (10), we get

$$\begin{aligned}
\mathbb{E}((S_3-1)^2 &= var(S_3) = \frac{n}{n^2\mathbb{E}^2(\beta_1K_1)}var(\beta_1K_1)\\
&\le \frac{1}{nO(h_K^4\phi_{\theta,x}^2(h_K))}\mathbb{E}(\beta_1^4K_1^2) = O\left(\frac{1}{n\phi_{\theta,x}(h_K)}\right).
\end{aligned}$$

On the other hand

$$\begin{aligned}
\mathbb{E}((S_4)^2) &= \frac{n\phi_{\theta,x}(h_K)\mathbb{E}^2(\beta_1K_1)}{\mathbb{E}^2(\Omega_1K_1)}\mathbb{E}\left(\sum_{j=1}^{n}\beta_jK_j(H_j - F(\theta,y,x))\right)^2\\
&= \frac{n\phi_{\theta,x}(h_K)O(h_K^2\phi_{\theta,x}^2(h_K))}{(n-1)^2O(h_K^4\phi_{\theta,x}^4(h_K))}(n\mathbb{E}(\beta_1K_1(H_1 - F(\theta,y,x))))^2\\
&\qquad +n(n-1)\mathbb{E}^2(\beta_1K_1(H_1 - F(\theta,y,x)))\\
&= o(1) + o(n\phi_{\theta,x}(h_K))
\end{aligned}$$

It remains to show

$$\mathbb{E}(|(S_3-1)S_4 - \mathbb{E}((S_3-1)S_4)|) \le 2\sqrt{\mathbb{E}((S_3-1)^2)}\sqrt{\mathbb{E}((S_4)^2)} = o(1)$$

which implies that

$$|(S_3-1)S_4 - \mathbb{E}((S_3-1)S_4)| = o_p(1)$$

Therefore,

$$\mathbb{P}(|S_3S_4 - \mathbb{E}(S_3S_4)|) > \epsilon) \leq \frac{\mathbb{E}(|S_3S_4 - \mathbb{E}(S_3S_4)|)}{\epsilon} \to 0 \quad as \quad n \to \infty.$$

So, to prove (15), it suffices to show $S_4 - \mathbb{E}(S_4) = o(1)$, while

$$\mathbb{E}(S_4 - \mathbb{E}(S_4))^2 = var(S_4) = \frac{n^2\phi_{\theta,x}(h_K)\mathbb{E}^2(\beta_1 K_1)}{\mathbb{E}^2(\Omega_1 K_1)} var(\beta_1 K_1(H_1 - F(\theta, y, x)))$$

We arrive finally at

$$var(\beta_1 K_1(H_1 - F(\theta, y, x))) = F(\theta, y, x)(1 - F(\theta, y, x))\mathbb{E}(\beta_1^2 K_1^2)$$

This last result together equation (7), (8) and (10), lead directly to

$$\begin{aligned}
\mathbb{E}(S_4 - \mathbb{E}(S_4))^2 &= \frac{n^2\phi_{\theta,x}(h_K)\mathbb{E}^2(\beta_1 K_1)}{\mathbb{E}^2(\Omega_1 K_1)} F(\theta, y, x)(1 - F(\theta, y, x))\mathbb{E}(\beta_1^2 K_1^2) \\
&= (F(\theta, y, x)(1 - F(\theta, y, x)))o(1),
\end{aligned}$$

which allows to finish the proof of Theorem.

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Applications of equi-statistical convergence and Korovkin-type theorem

Swati Jasrotia
School of Mathematics Shri Mata Vaishno Devi University, Katra-182320, J & K (India)
email: swatijasrotia12@gmail.com

Uday Pratap Singh
Department of Mathematics Central University of Jammu, Samba J & K (India)
email: usinghiitg@gmail.com

Kuldip Raj
School of Mathematics Shri Mata Vaishno Devi University, Katra-182320, J & K (India)
email: kuldipraj68@gmail.com

Abstract. The paper aims to introduce and investigate equi-statistical convergence, pointwise statistical convergence and uniform statistical convergence for a sequence of real-valued functions via deferred Nörlund and deferred Euler statistical convergence. Based on the above defined method, we study different results with compelling instances to illustrate the findings. Moreover, we give an illustrative example that proves that our Korovkin-type theorem is stronger version of the classical theorem. Finally, we study rates of deferred Nörlund and deferred Euler equi-statistical convergence through modulus of continuity.

1 Introduction and Preliminaries

Classical convergence has got numerous applications in the field of science and engineering where the convergence of a sequence requires almost all elements to satisfy the convergence condition. This means that all the elements of the

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sequence need to be in an arbitrarily small neighborhood of the limit. However, such type of limitations is relaxed in statistical convergence, where the validity of the convergence condition is achieved only for a majority of elements. Recently, statistical convergence has been a dynamic research area since it is more general than classical convergence. Moreover, such theory is discussed in the study of Fourier analysis, Number theory, Approximation theory, etc. The credit of statistical convergence goes to Fast [6] and Schoenberg [25] for detecting this idea independently. The concept of statistical convergence was further investigated by Connor [4], Fridy [7], Miller and Orhan [16], Jena et al. [9], İnce and Karaçal [8], Kadak and Mohiuddine [11], Mursaleen et al. [14], Raj and Choudhary [18], Srivastava et al. [21] and many more. For recent studies involving the notion of statistical convergence one may refer ([10], [19], [22]).

Suppose that $T \subset \mathbb{N}$ such that $T_m = \{n : n \leq m \text{ and } n \in T\}$. A sequence $z = (z_m)$ is called statistically convergent to z_0 if the set T_m has zero natural density [7] i.e., for each $\varepsilon > 0$,

$$\lim_{m\to\infty} \frac{1}{m}|\{n : n \leq m \text{ and } |z_n - z_0| \geq \varepsilon\}| = 0.$$

Here, vertical bar implies the cardinality of the enclosed set. In such case, we write

$$\mathrm{St}\lim_{m\to\infty} z_m = z_0.$$

Further, Balcerzak et al. [2] define and introduced the concept of equi-statistical convergence. After that, Edely et al. [5] studied the Korovkin-type approximation theorem by virtue of $\lambda-$statistical convergence. Srivastava et al. [24] define another variant of equi-statistical convergence with the help of non-decreasing sequence (λ_n) and called it $\lambda-$equi-statistical convergence wherein they have proved Korovkin and Voronovskaya-type approximation theorems. Despite the above work, Karakųs et al. [12] also obtained several interesting results involving the idea of equi-statistical convergence, including Korovkin and Voronovskaya-type theorems. For detailed study, one may refer ([3], [15], [20], [23], [26]). Suppose that (x_m) and (y_m) are the sequences of non-negative integers fulfilling

$$x_m < y_m, \ \forall \ m \ \in \mathbb{N} \qquad \text{and} \qquad \lim_{x\to\infty} y_m = \infty. \tag{1}$$

In 1932, R. P. Agnew [1] defined deferred Cesaro mean of sequences of real numbers as

$$(D_{x,y}z)_m = \frac{1}{y_m - x_m}\sum_{n=x_m+1}^{y_m} z_n.$$

In 2016, Küçükaslan and Yilmazturk [13] introduce the idea of deferred statistically convergence. A sequence (z_n) is called as deferred statistically convergent to z, if for every $\varepsilon > 0$,

$$\lim_{m\to\infty} \frac{1}{y_m - x_m} |\{x_m < n \le y_m; |z_n - z| \ge \varepsilon\}|.$$

Let (e_m) and (f_m) be two sequences of non-negative real numbers such that

$$\mathcal{E}_m = \sum_{n=x_m+1}^{y_m} e_n \quad \text{and} \quad \mathcal{F}_m = \sum_{n=x_m+1}^{y_m} f_n. \tag{2}$$

The convolution of (2) is given as

$$\mathcal{R}_m = \sum_{\nu=x_m+1}^{y_m} e_\nu f_{y_m-\nu}.$$

As introduced by Srivastava et al. in [22], the deferred Nörlund (DN) mean is defined as

$$t_m = \frac{1}{\mathcal{R}_m} \sum_{n=x_m+1}^{y_m} e_{y_m-n} f_n z_n.$$

Next, the deferred Euler (DE) mean [17] of r^{th} order is given as

$$s_m = \frac{1}{(1+r)^{y_m}} \sum_{n=x_m+1}^{y_m} \binom{y_m}{n} r^{y_m-n} z_n,$$

$\forall\ m \in \mathbb{N}$ and $r > 0$.

Definition 1 *A sequence* $z = (z_m)$ *is known to be deferred Euler statistically convergent to* z, *if for each* $\varepsilon > 0$

$$E_m = \{n : n \le (1+r)^{y_m} \textit{ and } r^{y_m-n}|z_m - z| \ge \varepsilon\}$$

has zero natural density, i.e.,

$$\lim_{m\to\infty} \frac{|E_m|}{(1+r)^{y_m}} = 0.$$

Inspired by the above mentioned investigations, we investigate and study the concept of statistical product convergence via deferred Nörlund and deferred Euler product means. We explore the concept of equi-statistical, pointwise statistical and uniform statistical convergence via deferred Nörlund and deferred Euler statistical (D(NE)S) convergence. As an application to our newly formed sequence space, we introduce Korovkin-type approximation theorem using deferred Nörlund and deferred Euler equi-statistical ($e\mathsf{D}(\mathsf{NE})\mathsf{S}$) convergence.

Now, we define the product of means obtained by deferred Nörlund (DN) and deferred Euler (DE) as follows

$$\omega_m = (t_m s_m) = (ts)_m = \frac{1}{\mathcal{R}_m(1+r)^{y_m}} \sum_{n=x_m+1}^{y_m} \binom{y_m}{n} e_{y_m-n} f_n r^{y_m-n} z_n.$$

Further, the sequence (ω_m) is said to be summable to z by the product $\mathsf{D}(\mathsf{NE})$ summability mean if

$$\lim_{m\to\infty} \omega_m = z.$$

2 Deferred Nörlund and deferred Euler equistatistical convergence

Suppose that $C(\mathcal{T})$ be the space of all continuous real valued functions defined on a compact subset $\mathcal{T}$ of real numbers. The space $C(\mathcal{T})$ is a Banach space with the norm

$$||z||_\infty = \sup_{t\in\mathcal{T}}\{|z(t)|\}, \quad z \in C(\mathcal{T}).$$

Throughout, $(z_m) \in C(\mathcal{T})$ is a sequence of continuous functions.

Definition 2 *Suppose that* (x_m) *and* (y_m) *are the sequences fulfilling conditions* (1) *and* $(e_m), (f_m)$ *are sequences satisfying* (2). *A sequence* $(z_m) \in C(\mathcal{T})$ *is said to be deferred Nörlund and deferred Euler pointwise statistically* $(p\mathsf{D}(\mathsf{NE})\mathsf{S})$ *convergent to* z, *if* $\forall\, t \in \mathcal{T}$ *and* $\varepsilon > 0$,

$$\lim_{m\to\infty} \frac{\Theta(t,\varepsilon)}{\mathcal{R}_m(1+r)^{y_m}} = 0$$

where

$$\Theta(t,\varepsilon) = |\{n : n \le (1+r)^{y_m}\mathcal{R}_m \text{ and } e_{y_m-n} f_n r^{y_m-n} |z_n(t) - z(t)| \ge \varepsilon\}|.$$

We write

$$z_m \to z \ \ (pD(NE)S).$$

Definition 3 *Suppose that* (x_m) *and* (y_m) *are the sequences fulfilling conditions* (1) *and* $(e_m), (f_m)$ *are sequences satisfying* (2)*. A sequence* (z_m) *is said to be deferred Nörlund and deferred Euler equi-statistical* $(eD(NE)S)$ *convergent to* z *if* $\forall\ \varepsilon > 0$,

$$\lim_{m\to\infty} \frac{\Theta(t,\varepsilon)}{\mathcal{R}_m(1+r)^{y_m}} = 0,$$

uniformly with respect to $t \in C(\mathcal{T})$*, i.e.,*

$$\lim_{m\to\infty} \frac{\|\Theta_m(t,\varepsilon)\|_{C(\mathcal{T})}}{\mathcal{R}_m(1+r)^{y_m}} = 0,$$

where

$$\Theta_m(t,\varepsilon) \ = \ |\{n : n \leq (1+r)^{y_m}\mathcal{R}_m \text{ and } e_{y_m-n} f_n r^{y_m-n}|z_n(t) - z(t)| \geq \varepsilon\}|.$$

We write

$$z_m \twoheadrightarrow z \ \ (eD(NE)S).$$

Definition 4 *Suppose that* (x_m) *and* (y_m) *are the sequences fulfilling conditions* (1) *and* $(e_m), (f_m)$ *are sequences satisfying* (2)*. A sequence* (z_m) *is said to be deferred Nörlund and deferred Euler uniform statistically* $(uD(NE)S)$ *convergent to* z *if* $\forall\ \varepsilon > 0$,

$$\lim_{m\to\infty} \frac{\Phi_m(t,\varepsilon)}{\mathcal{R}_m(1+r)^{y_m}} = 0,$$

where

$$\Phi_m(t,\varepsilon) \ = \ |\{n : n \leq (1+r)^{y_m}\mathcal{R}_m \text{ and } e_{y_m-n} f_n r^{y_m-n}\|z_n(t) - z(t)\|_{C(\mathcal{T})} \geq \varepsilon\}|.$$

We write

$$z_m \rightrightarrows z \ \ (uD(NE)S).$$

Lemma 1 *The following implications are true*

$$z_m \rightrightarrows z(uD(NE)S) \Rightarrow z_m \twoheadrightarrow z(eD(NE)S) \Rightarrow z_m \to z(pD(NE)S). \quad (3)$$

Proof. (1) Since $z_m \rightrightarrows z(uD(NE)S)$, From Definition 4

$$\lim_{m\to\infty} \frac{1}{\mathcal{R}_m(1+r)^{y_m}} |\{n : n \leq (1+r)^{y_m}\mathcal{R}_m \text{ and } e_{y_m-n}f_n r^{y_m-n}\|z_n(t) - z(t)\|_{C(\mathcal{T})} \geq \varepsilon\}| = 0,$$

for $\varepsilon > 0$ be arbitrary small positive real number. Now,

$$|\{n : n \leq (1+r)^{y_m}\mathcal{R}_m \text{ and } e_{y_m-n}f_n r^{y_m-n}|z_n(t) - z(t)| \geq \varepsilon\}| \subseteq |\{n : n \leq (1+r)^{y_m}\mathcal{R}_m \text{ and } e_{y_m-n}f_n r^{y_m-n}\|z_n(t) - z(t)\|_{C(\mathcal{T})} \geq \varepsilon\}|.$$

This implies that

$$\lim_{m\to\infty} \frac{1}{\mathcal{R}_m(1+r)^{y_m}} |\{n : n \leq (1+r)^{y_m}\mathcal{R}_m \text{ and } e_{y_m-n}f_n r^{y_m-n}|z_n(t) - z(t)|_{C(\mathcal{T})} \geq \varepsilon\}| = 0.$$

Thus,

$$z_m \rightrightarrows z(uD(NE)S) \Rightarrow z_m \twoheadrightarrow z(eD(NE)S).$$

(2) Since $z_m \twoheadrightarrow z(eD(NE)S)$, from Definition 3

$$\lim_{m\to\infty} \frac{1}{\mathcal{R}_m(1+r)^{y_m}} |\{n : n \leq (1+r)^{y_m}\mathcal{R}_m \text{ and } e_{y_m-n}f_n r^{y_m-n}|z_n(t) - z(t)| \geq \varepsilon\}| = 0,$$

for $\varepsilon > 0$ be arbitrary small positive real number. Now,

$$\lim_{m\to\infty} \frac{\Theta_m(t,\varepsilon)}{\mathcal{R}_m(1+r)^{y_m}} \subseteq \frac{\|\Theta_m(t,\varepsilon)\|_{C(\mathcal{T})}}{\mathcal{R}_m(1+r)^{y_m}},$$

for

$$\Theta_m(t,\varepsilon) = |\{n : n \leq (1+r)^{y_m}\mathcal{R}_m \text{ and } e_{y_m-n}f_n r^{y_m-n}|z_n(t) - z(t)| \geq \varepsilon\}|.$$

This implies that

$$\lim_{m\to\infty} \frac{1}{\mathcal{R}_m(1+r)^{y_m}} |\{n : n \leq (1+r)^{y_m}\mathcal{R}_m \text{ and } e_{y_m-n}f_n r^{y_m-n}|z_n(t) - z(t)|_{C(\mathcal{T})} \geq \varepsilon\}| = 0.$$

Thus,

$$z_m \twoheadrightarrow z(eD(NE)S) \Rightarrow z_m \to z(pD(NE)S).$$

□

Furthermore, in general the reverse implications do not hold true. Such an example is given as follows:

Example 1 *Suppose that* $x_m = 2m-1, y_m = 4m-1$*. Also, consider* $e_{y_m - m} = 2m, f_m = 1$ *and* $z_m : \mathcal{T} = [0,1] \to \mathbb{R}, (m \in \mathbb{N})$ *is given as*

$$z_m(t) = \begin{cases} \frac{\left[(\frac{1}{m+1})^2 - t^2\right](m+1)^2}{1+t^2}, \ x \in \left[0, \frac{1}{m+1}\right] \\ \\ 0, \ \textit{otherwise}. \end{cases}$$

Thus, $\forall\ \varepsilon > 0$ *we get*

$$\frac{1}{(1+r)^{y_m}\mathcal{R}_m}\left|\left\{n : n \le (1+r)^{y_m}\mathcal{R}_m \textit{ and } e_{y_m - n} f_n r^{y_m - n}|z_n(t) - z(t)| \ge \varepsilon\right\}\right|$$

$$\le \frac{1}{(1+r)^{y_m}}$$
$$\to 0 \textit{ as } m \to \infty,$$

uniformly on $\mathcal{T}$*. Thus, we get* $z_m \twoheadrightarrow z$*. But*

$$\sup_{t\in[0,1]} |z_m(t)| = 1.$$

Hence, $z_m \rightrightarrows z$ *does not holds.*

Example 2 *To prove second inclusion in* (3)*, we let* $x_m = 2m - 1$ *and* $y_m = 4m-1,\ e_{y_m - n} = 2m, f_m = 1$ *and* $z_m : \mathcal{T} = [0,1] \to \mathbb{R}, (m \in \mathbb{N})$ *given as*

$$z_m(t) = t^m.$$

Suppose that

$$\lim_{m\to\infty} z_m(t) = z(t), \quad t \in [0,1].$$

Therefore,

$$z_m(t) = z(t), \ (pD(NE)S).$$

Let $\varepsilon = \frac{1}{3}$. *Then,* $\forall\ m \in \mathbb{N}, \exists\ s > m$ *such that* $s \in [2m+1, 4m]$ *and for each* $t \in [\sqrt[m]{\frac{1}{3}}, 1]$, *we get,*

$$|z_s(t)| = |t^s| > \left|\left(\sqrt[m]{\frac{1}{3}}\right)^s\right| > \left|\left(\sqrt[s]{\frac{1}{3}}\right)^s\right| = \frac{1}{3}.$$

Thus, $z_m = 0\ \ (eD(NE)S)$ *does not holds.*

3 Korovkin-type approximation theorem

This particular section consists of the concept of $(eD(NE)S)$ to demonstrate Korovkin-type approximation theorem.

Consider $C[c,d]$ be the space of all real valued continuous functions on $[c,d]$. The space $C[c,d]$ is a Banach space with norm

$$||z||_\infty = \sup_{t\in[c,d]} |z(t)|,\ \forall z \in C[c,d].$$

Suppose $\rho : C[c,d] \to C[c,d]$ is a positive linear operator (PLO), i.e. $\rho(z) \geq 0\ \forall\ z \geq 0$. By $\rho(z,t)$, we mean the value of $\rho(z)$ at a point t.

Theorem 1 *Let* ρ_n, $(n \in \mathbb{N})$ *be the sequence of PLO from* $C[c,d]$ *onto itself. Then* $\forall\ z \in C[c,d]$,

$$\rho_n(z,t) \twoheadrightarrow z(t),\ \ (eD(NE)S) \tag{4}$$

iff

$$\rho_n(z_k,t) \twoheadrightarrow z_k(t),\ \ (eD(NE)S), \tag{5}$$

where

$$z_0(t) = 1, z_1(t) = t, z_2(t) = t^2.$$

Proof. Since each of the function $z_k(t) = t^k \in C[c,d],\ (k = 0,1,2)$ is continuous. This means that condition (4) implies condition (5). Let us consider that (5) holds and also $z \in C[c,d]$, then $\exists$ a constant C such that

$$|z(t)| \leq C,\ \forall(t \in [c,d]),$$

which means that

$$|z(m) - z(t)| \leq 2C,\ \forall m, t \in [c,d].$$

For given $\varepsilon > 0,\ \exists \delta > 0$ s.t.

$$|z(m) - z(t)| \leq \varepsilon \tag{6}$$

whenever $|m-t|<\delta$. Set $\phi=\phi(m,t)=(m-t)^2$. If $m-t\geq\delta$, then

$$|z(m)-z(t)|<\frac{2C}{\delta^2}\phi(m,t). \tag{7}$$

From (6) and (7), we have

$$|z(m)-z(t)|\leq\varepsilon+\frac{2C}{\delta^2}\phi(m,t).$$

Which implies

$$-\varepsilon-\frac{2C}{\delta^2}\phi(m,t)\leq z(m)-z(t)\leq\varepsilon+\frac{2C}{\delta^2}\phi(m,t).$$

By monotonicity and linearity of the operators $\rho_n(1,t)$, we get

$$\begin{aligned}\rho_n(1,t)\Big(-\varepsilon-\frac{2C}{\delta^2}\phi(m,t)\Big)&\leq\rho_n(1,t)[z(m)-z(t)]\\&\leq\rho_n(1,t)\Big(\varepsilon+\frac{2C}{\delta^2}\phi(m,t)\Big).\end{aligned}$$

Consider that t is fixed, so $z(t)$ is constant. Thus,

$$-\varepsilon\rho_n(1,t)-\frac{2C}{\delta^2}\rho_n(\phi,t)\leq\rho_n(z,t)-z(t)\rho_n(1,t)\leq\varepsilon\rho_n(1,t)+\frac{2C}{\delta^2}\rho_n(\phi,t).$$

But

$$\rho_n(z,t)-z(t)=[\rho_n(z,t)-z(t)\rho_n(1,t)]+z(t)[\rho_n(1,t)-1],$$

which gives

$$\rho_n(z,t)-z(t)<\varepsilon\rho_n(1,t)+\frac{2C}{\delta^2}\rho_n(\phi,t)+z(t)[\rho_n(1,t)-1]. \tag{8}$$

Next, we suppose that $\rho_n(\phi,t)$ as

$$\rho_n(\phi,t)=\rho_n((m-t)^2,t)=\rho_n(m^2-2mt+t^2,t)$$

$$\begin{aligned}&=\rho_n(m^2,t)-2t\rho_n(m,t)+t^2\rho_n(1,t)\\&=[\rho_n(m^2,t)-t^2]-2t[\rho_n(m,t)-t]+t^2[\rho_n(1,t)-1].\end{aligned}$$

From (8) we get

$$\rho_n(z,t)-z(t)<\varepsilon\rho_n(1,t)+\frac{2C}{\delta^2}\Big\{[\rho_n(m^2,t)-t^2]-2t[\rho_n(m,t)-t]+$$

$$t^2[\rho_n(1,t)-1]\Big\} + z(t)[\rho_n(1,t)-1]$$
$$= \varepsilon[\rho_n(1,t)-1] + \varepsilon + \frac{2C}{\delta^2}\Big\{[\rho_n(m^2,t)-t^2] - 2t[\rho_n(m,t)-t] +$$
$$t^2[\rho_n(1,t)-1]\Big\} + z(t)[\rho_n(1,t)-1].$$

Since $\varepsilon > 0$, we get

$$\begin{aligned}|\rho_n(z,t)-z(t)| &\leq \Big(\varepsilon + \frac{2C}{\delta^2} + C\Big)|\rho_n(1,t)-1| + \frac{4C}{\delta^2}|\rho_n(m,t)-t| + \\ &\quad \frac{2C}{\delta^2}|\rho_n(m^2,t)-t^2| \\ &\leq \mathcal{S}(|\rho_n(1,t)-1| + |\rho_n(m,t)-t| + |\rho_n(m^2,t)-t^2|),\end{aligned}$$

where

$$\mathcal{S} = \max\Big\{\varepsilon + \frac{2C}{\delta^2} + C, \frac{2C}{\delta^2}, \frac{4C}{\delta^2}\Big\}.$$

Now, for given $s > 0, \exists \varepsilon > 0$ such that $0 < \varepsilon < s$. Now, we define the following sets

$$\Psi_n(t,s) = \{n : n \leq (1+r)^{y_m}\mathcal{R}_m \text{ and } e_{y_m - n} f_n r^{y_m - n}|\rho_n(z,t) - z(t)| \geq s\}$$

and

$$\begin{aligned}\Psi_{k,n}(t,s) &= \{n : n \leq (1+r)^{y_m}\mathcal{R}_m \text{ and } e_{y_m - n} f_n r^{y_m - n}|\rho_n(z_k,t) - z_k(t)| \\ &\geq \frac{s-\varepsilon}{3C}\}.\end{aligned}$$

For $k = 0, 1, 2$, we get

$$\Psi_n(t,s) \leq \sum_{k=0}^{2} \Psi_{k,n}(t,s).$$

Thus,

$$\frac{||\Psi_n(t,s)||_{C[c,d]}}{(1+r)^{y_m}\mathcal{R}_m} \leq \sum_{k=0}^{2} \frac{||\Psi_{k,n}(t,s)||_{C[c,d]}}{(1+r)^{y_m}\mathcal{R}_m}. \tag{9}$$

By using Definition 2.2 and the supposition about the implication in (5), the R.H.S of (9) tends to zero as $m \to \infty$. Hence,

$$\lim_{m\to\infty} \frac{||\Psi_n(t,s)||_{C[c,d]}}{(1+r)^{y_m}\mathcal{R}_m} = 0, s > 0$$

Therefore, (4) holds true. $\square$

Example 3 *Suppose that $\mathcal{T} = [0,1]$, the Mayer-König and Zeller operators (Altin, Doğru and Taşdelen 2005) $\mathcal{M}_m(z,t)$ on $C[0,1]$ is given as*

$$\mathcal{M}_m(z,t) = \sum_{l=0}^{\infty} z\Big(\frac{l}{l+m+1}\Big)\binom{m+l}{l} t^l (1-t)^{m-l},\ (t \in [0,1]).$$

Now, we define a operator $\rho_m : C[0,1] \to C[0,1]$ by

$$\rho_m(z,t) = [1+z_m(t)]t(1+tD)\mathcal{M}_n(z,t),\ \ z \in C[0,1], \tag{10}$$

where the sequence $(Z_m(t))$ of functions as described in Example 2.1. Now,

$$\rho_m(z_0,t) \ = \ [1+z_m(t)]t(1+tS)z_0(t) = [1+z_m(t)]t,$$

$$\rho_m(z_1,t) \ = \ [1+z_m(t)]t(1+tS)z_1(t) = [1+z_m(t)]t[1+t],$$

and

$$\begin{aligned}\rho_m(z_2,t) &= [1+z_m(t)]t(1+tD)\Big\{z_2(t)\Big(\frac{m+2}{m+1}\Big) + \Big(\frac{t}{m+1}\Big)\Big\} \\ &= [1+z_m(t)]\Big\{t^2\Big[\Big(\frac{m+2}{m+1}\Big)t + 2\Big(\frac{1}{m+1}\Big) + 2t\Big(\frac{m+2}{m+1}\Big)\Big]\Big\}.\end{aligned}$$

Since $z_m \twoheadrightarrow z = 0$, on $[0,1]$. Thus, we get

$$\rho_m(z_k,t) \twoheadrightarrow z_k,\ \text{on } [0,1]\ \ \forall\, k = 0,1,2.$$

Hence, from Theorem 3.1, we get

$$\rho_m(z,x) \twoheadrightarrow z$$

$\forall\, z \in C[0,1]$. *Since (z_m) is not $uD(NE)S$ convergent to z on $[0,1]$. Thus, we can say that work in Karakus et al. [12] are not true for our operator described in (10) whereas our Theorem 3.1 still works for (10).*

Definition 5 *Let (s_m) be a positive non-increasing sequence of real numbers. The sequence (z_m) of functions is known to be deferred Nörlund and deferred Euler equi-statistically $(eD(NE)S)$ convergent to z with the convergence rate $o(s_m)$, if $\forall \varepsilon > 0$,*

$$\lim_{m\to\infty} \frac{\Theta_m(t,\varepsilon)}{s_m \mathcal{R}_m (1+r)^{y_m}} = 0$$

uniformly with respect to $t \in \mathcal{T}$, *i.e.,*

$$\lim_{m\to\infty} \frac{\|\Theta_m(t,\varepsilon)\|_{C(\mathcal{T})}}{s_m \mathcal{R}_m (1+r)^{y_m}} = 0$$

where

$$\Theta_m(t,\varepsilon) = \{n : n \leq (1+r)^{y_m}\mathcal{R}_m \text{ and } e_{y_m-n} f_n r^{y_m-n}|z_n(t) - z(t)| \geq \varepsilon\}.$$

We write

$$z_m - z = o(s_m).$$

Lemma 2 *Suppose that* (s_m) *and* (w_m) *are two positive decreasing sequences of real numbers. Also* (z_m) *and* (x_m) *be two sequences of function in* $C[\mathcal{T}]$ *such that*

$$z_m(t) - z(t) = o(s_m), \quad (eD(NE)S)$$

and

$$x_m(t) - x(t) = o(w_m), \quad (eD(NE)S).$$

Then, we get

1. $[z_m(t) + x_m(t)] - [z(t) + x(t)] = o(c_m), \ (eD(NE)S)$

2. $[z_m(t) - z(t)][x_m(t) - x(t)] = o(s_m w_m), \ (eD(NE)S)$

3. $\eta[z_m(t) - z(t)] = o(s_m)(eD(NE)S)$ *for any scalar* η,

4. $\sqrt{|z_m(t) - z(t)|} = o(s_m)(eD(NE)S)$,

where $c_m = \max\{s_m, w_m\}$.

Proof. Let $z_m(t) - z(t) = o(s_m)$, $(eD(NE)S)$ and $x_m(t) - x(t) = o(w_m)$, $(eD(NE)S)$. Now we construct the following sets for given $\varepsilon > 0$ and $t \in \mathcal{T}$ as

$$\begin{aligned}\mathcal{N}_m(t,\varepsilon) &= \{n : n \leq (1+r)^{y_m}\mathcal{R}_m \text{ and } e_{y_m-n} f_n r^{y_m-n} \\ &\qquad |(z_n + x_n)(t) - (z+x)(t)| \geq \varepsilon\}, \\ \mathcal{N}_{1,m}(t,\varepsilon) &= \left\{n : n \leq (1+r)^{y_m}\mathcal{R}_m \text{ and } e_{y_m-n} f_n r^{y_m-n}|z_n(t) - z(t)| \geq \frac{\varepsilon}{2}\right\},\end{aligned}$$

and

$$\mathcal{N}_{2,m}(t,\varepsilon) = \Big\{n : n \le (1+r)^{y_m}\mathcal{R}_m \text{ and } e_{y_m-n} f_n r^{y_m-n}|x_n(t)-x(t)| \ge \frac{\varepsilon}{2}\Big\}.$$

Thus, we get

$$\mathcal{N}_m(t,\varepsilon) \subseteq \mathcal{N}_{1,m}(t,\varepsilon) \subseteq \mathcal{N}_{2,m}(t,\varepsilon).$$

Therefore, we get

$$\frac{||\mathcal{N}_m(t,\varepsilon)||_{C(\mathcal{T})}}{c_m(1+r)^{y_m}\mathcal{R}_m} \le \frac{||\mathcal{N}_{1,m}(t,\varepsilon)||_{C(\mathcal{T})}}{s_m(1+r)^{y_m}\mathcal{R}_m} + \frac{||\mathcal{N}_{2,m}(t,\varepsilon)||_{C(\mathcal{T})}}{w_m(1+r)^{y_m}\mathcal{R}_m}$$

where $c_m = \max\{s_m, w_m\}$. Taking $m \to \infty$ in above inequality, we get

$$\frac{||\mathcal{N}_m(t,\varepsilon)||_{C(\mathcal{T})}}{c_m(1+r)^{y_m}\mathcal{R}_m} = 0$$

which satisfies (1). The other part can be proved through similar method.

□

We define modulus of continuity of $z \in C(\mathcal{T})$ as

$$g(z;\psi) = \sup_{t,m\in\mathcal{T},|t-m|\le\psi} |z(m)-z(t)|,$$

for any $\psi > 0$ which fulfills

$$|z(m)-z(t)| \le \Big(1+\frac{|t-m|}{\psi}\Big)g(z,\psi).$$

Theorem 2 *Consider* (ρ_n) *be the sequence of PLO from* $C(\mathcal{T})$ *onto itself. Suppose that*

1. $\rho_n(1,t)-1 = o(s_m), \quad (eD(NE)S)$

2. $g(z,\psi) = o(w_m), \quad (eD(NE)S)$

where

$$\psi_m(t) = \sqrt{\rho_n(\phi^2,t)} \text{ and } \phi(m) = m-t.$$

Then,

$$\rho_n(z,t) - z = o(c_m), \quad (eD(NE)S),\ \forall\, z \in C(\mathcal{T}) \tag{11}$$

where $c_m = \max\{s_m, w_m\}$.

Proof. Consider $z \in C(\mathcal{T})$ and $t \in \mathcal{T}$. Then, we get

$$|\rho_n(z,t) - z(t)| \leq \mathcal{M}|\rho_n(1,t) - 1| + \Big(\rho_n(1,t) + \sqrt{\rho_n(1,t)}g(z,\psi_m)\Big),$$

where $\mathcal{M} = \|z\|_{C(\mathcal{T})}$. This means

$$\begin{aligned} |\rho_n(z,t) - z(t)| &\leq \mathcal{M}|\rho_n(1,t) - 1| \\ +2g(z,\psi_m) + g(z,\psi_m)|\rho_n(1,t) - 1| &+ g(z,\psi_m)\sqrt{\rho_n(1,t) - 1}. \end{aligned} \tag{12}$$

By using condition (1), (2), (12) and Lemma 2, we get (11). □

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Uniqueness of an entire function sharing a small function with its linear differential polynomial with non-constant coefficients

Imrul Kaish
Department of Mathematics and Statistics, Aliah University, Kolkata, West Bengal 700160, India
email: imrulksh3@gmail.com

Md Majibur Rahaman
Department of Mathematics and Statistics, Aliah University, Kolkata, West Bengal 700160, India
email: majiburjrf107@gmail.com

Abstract. The uniqueness problems of entire functions sharing at least two values with their derivatives or linear differential polynomials have been studied and many results on this topic have been obtained. In our paper, we study the uniqueness of an entire function when it shares a small function with its first derivative and two linear differential polynomials of different orders. Here we consider the differential polynomial with non-constant coefficients. In particular, the result of the paper improves the results due to P. Li [7], I. Kaish and Md. M. Rahaman [4].

1 Introduction, definitions and results

Let us consider a non-constant meromorphic function f in the open complex plane $\mathbb{C}$. For a meromorphic function $a = a(z)$ defined in $\mathbb{C}$, we denote by $E(a; f)$ the set of zeros of $f - a$, counted with multiplicities and by $\overline{E}(a; f)$, the set of distinct zeros of $f - a$.

The investigation of uniqueness of an entire function sharing two values has been introduced by L. A. Rubel and C. C. Yang [9] in 1977 by the following result.

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Theorem A [9] Let f be a non-constant entire function satisfying $E(a;f) = E(a;f^{(1)})$ and $E(b;f) = E(b;f^{(1)})$, for distinct finite complex numbers a and b. Then $f \equiv f^{(1)}$.

If for two meromorphic functions f and g, $E(a;f) = E(a;g)$ then we say that f and g share a CM and if $\overline{E}(a;f) = \overline{E}(a;g)$ then we say that f and g share a IM. In Theorem A, f and $f^{(1)}$ share a and b CM.

In 1979 considering IM sharing, E. Mues and N. Steinmetz [8] proved the following result.

Theorem B [8] Let f be a non-constant entire function satisfying $\overline{E}(a;f) = \overline{E}(a;f^{(1)})$ and $\overline{E}(b;f) = \overline{E}(b;f^{(1)})$. Then $f = f^{(1)}$.

From the following example we see that the two values cannot be replaced by a single value.

Example 1 *Let* $f(z) = \exp(e^z)\int_0^z \exp(-e^t)(1-e^t)dt$. *Then* $f^{(1)}-1 = e^z(f-1)$ *and so* $E(1;f) = E(1;f^{(1)})$ *but* $f \neq f^{(1)}$.

Considering a single shared value G. Jank, E. Mues and L. Volkmann [3] established the following result.

Theorem C [3] Let f be a non-constant entire function satisfying $\overline{E}(a;f) = \overline{E}(a;f^{(1)}) \subset \overline{E}(a;f^{(2)})$, for a non-zero constant a. Then $f = f^{(1)}$.

J. Chang and F. Fang [1] extended Theorem C by considering shared fixed points. Their result may be stated as follows.

Theorem D [1] Let f be a non-constant entire function satisfying $\overline{E}(z;f) = \overline{E}(z;f^{(1)}) \subset \overline{E}(z;f^{(2)})$, then $f = f^{(1)}$.

In 2009, I. Lahiri and G.K. Ghosh [5] extended Theorem D and proved the following theorem.

Theorem E [5] Let f be a non-constant entire function and $a(z) = \alpha z + \beta$, where $\alpha(\neq 0)$, β are constants. If $E(a;f) \subset E(a;f^{(1)})$ and $E(a;f^{(1)}) \subset E(a;f^{(2)})$, then either $f = A\exp(z)$ or $f = \alpha z + \beta + (\alpha z + \beta - 2\alpha)\exp\{\frac{\alpha z+\beta-2\alpha}{\alpha}\}$, where A is a non-zero constant.

For further discussion we need the following notation.

Let f be a non-constant meromorphic function and A be a set of complex numbers. For any meromorphic function $a = a(z)$, the integrated counting function $N_A(r, a; f)$ of zeros of $f - a$ which lie in $A \cap \{z : |z| \leq r\}$ is defined as

$$N_A(r, a; f) = \int_0^r \frac{n_A(t, a; f) - n_A(0, a; f)}{t} dt + n_A(0, a; f) \log r,$$

where $n_A(t, a; f)$ is the number of zeros of $f - a$, counted according to their multiplicities in $A \cap \{z : |z| \leq r\}$ and $n_A(0, a; f)$ be the multiplicity of the zeros of $f - a$ at origin. $T(r, f)$ be the characteristic function of f and $S(r, f)$ is any quantity satisfying $S(r, f) = o\{T(r, f)\}$ as $r \to \infty$ possibly outside a set of finite linear measure. A meromorphic function $a = a(z)$ defined in $\mathbb{C}$ is called a small function of f if $T(r, a) = S(r, f)$. For standard definitions and notations we refer the reader to [2] and [10].

For two subsets A and B of $\mathbb{C}$, we denote by $A \triangle B$ the symmetric difference of A and B i.e. $A \triangle B = (A - B) \cup (B - A)$.

I. Lahiri and I. Kaish [6] extended Theorem E in the following way.

Theorem F [6] Let f be a non-constant entire function and $a = a(z)$ be a polynomial. Suppose that $A = \overline{E}(a; f) \triangle \overline{E}(a; f^{(1)})$ and $B = \overline{E}(a; f^{(1)}) \setminus \{\overline{E}(a; f^{(n)}) \cap \overline{E}(a; f^{(n+1)})\}$. If

(i) $deg(a) \neq deg(f)$,

(ii) $N_A(r, a; f) + N_A(r, a; f^{(1)}) = O\{logT(r, f)\}$ and $N_B(r, a; f^{(1)}) = S(r, f)$,

(iii) each common zero of $f - a$ and $f^{(1)} - a$ has the same multiplicity, then $f = \lambda e^z$, where $\lambda(\neq 0)$ is a constant.

Suppose that f be a non-constant entire function and $a_1, a_2, ..., a_n(\neq 0)$ are complex numbers. Then

$$L = L(f) = a_1 f^{(1)} + a_2 f^{(2)} + ... + a_n f^{(n)}, \tag{1}$$

is called a linear differential polynomial generated by f.

In 1999 P. Li [7] extended Theorem C by considering a linear differential polynomial and they prove the following theorem.

Theorem G [7] Let f be a non-constant entire function and L be defined by (1). Suppose that a be a non-zero finite value. If $\overline{E}(a; f) = \overline{E}(a; f^{(1)})$ and $\overline{E}(a; f) \subset \overline{E}(a; L) \cap \overline{E}(a; L^{(1)})$, then $f = f^{(1)} = L$.

In 2018 I. Kaish and Md.M. Rahaman [4] improved Theorem F and Theorem G in the following way.

Theorem H [4] Let f be a non-constant entire function and $L = a_2 f^{(2)} + a_3 f^{(3)} + ... + a_n f^{(n)}$, where $a_2, a_3, ..., a_n (\neq 0)$ are constants , and $n(\geq 2)$ be an integer. Also let $a(z) \not\equiv 0$ be a polynomial with $deg(a) \neq deg(f)$. Suppose that $A = \overline{E}(a; f) \triangle \overline{E}(a; f^{(1)})$ and $B = \overline{E}(a; f^{(1)}) \setminus \{\overline{E}(a; L) \cap \overline{E}(a; L^{(1)})\}$. If

(1) $N_A(r, a; f) + N_A(r, a; f^{(1)}) = O\{\log T(r, f)\}$,

(2) $N_B(r, a; f^{(1)}) = S(r, f)$, and

(3) each common zero of $f - a$ and $f^{(1)} - a$ has the same multiplicity,

then $f = L = \lambda e^z$, where $\lambda(\neq 0)$ is a constant .

In this paper we consider a linear differential polynomial of an entire function f whose coefficients are small functions of f and we improve Theorem G and Theorem H by considering small function sharing by an entire function and its differential polynomials of various orders. The following theorem is our main result in the paper.

Theorem 1 *Let f be a non-constant entire function and $a = a(z)(\neq 0, \infty)$ be a small function of f with $a \not\equiv a^{(1)}$. Suppose that $A = \overline{E}(a; f) \triangle \overline{E}(a; f^{(1)})$, $B = \overline{E}(a; f^{(1)}) \setminus \{\overline{E}(a; L^{(p)}) \cap \overline{E}(a; L^{(q)})\}$, and $L = a_1(z) f^{(1)}(z) + a_2(z) f^{(2)}(z) + ... + a_n(z) f^{(n)}(z)$, where $a_1(z), a_2(z), ..., a_n(z)(\neq 0)$ are small functions of f and n, p, q are positive integers, $q > p \geq 0$. If*

(i) $E_{1)}(a; f) \subset \overline{E}(a; f^{(1)})$,

(ii) $N_A(r, a; f) + N_{A\cup B}(r, a; f^{(1)}) = S(r, f)$, *and*

(iii) *each common zero of $f - a$ and $f^{(1)} - a$ has the same multiplicity,*

then $f = L = \delta e^z$, where $\delta(\neq 0)$ is a constant.

Putting $A = B = \emptyset$, we get the following corollary.

Corollary 1 *Let f be a non-constant entire function and $a = a(z)(\neq 0, \infty)$ be a small function of f with $a \not\equiv a^{(1)}$. If $E(a; f) = E(a; f^{(1)})$ and $\overline{E}(a; f^{(1)}) \subset \{\overline{E}(a; L^{(p)}) \cap \overline{E}(a; L^{(q)})\}$, where L is defined as in Theorem 1, then $f = L = \delta e^z$, where $\delta(\neq 0)$ is a constant.*

In Corollary 1, if we consider that a is a constant and L be a linear differential polynomial with constant coefficient then it is a particular case of Theorem G. Also in corollary if we consider that a is a polynomial with $\deg(a) \neq \deg(f)$, L is a linear differential polynomial with constant coefficient and $a_1 = 0$, $p = 0$, $q = 1$ then it is Theorem H.

We assume the following:

1. The degree of a transcendental entire function is infinity.
2. The order of a differential polynomial of f is the order of the highest ordered derivative of f presented in the polynomial.

2 Lemmas

In this section we give some necessary lemmas.

Lemma 1 [1] *Let f be a meromorphic function and k be a positive integer. Suppose that f is a solution of the following differential equation : $a_0 w^{(k)} + a_1 w^{(k-1)} + \ldots + a_k w = 0$, where $a_0 (\neq 0), a_1, a_2, \ldots, a_k$ are constants. Then $T(r, f) = O(r)$. Furthermore, if f is transcendental, then $r = O(T(r, f))$.*

Lemma 2 [1] *Let f be a meromorphic function and n be a positive integer. If there exists meromorphic functions $a_0 (\not\equiv 0), a_1, a_2, \ldots, a_n$ such that*

$$a_0 f^n + a_1 f^{n-1} + \ldots + a_{n-1} f + a_n \equiv 0,$$

then

$$m(r, f) \quad \leq \quad nT(r, a_0) + \sum_{j=1}^{n} m(r, a_j) + (n-1)\log 2.$$

Lemma 3 ([2], p. 68). *Let f be a transcendental meromorphic function and $f^n P(z) = Q(z)$, where $P(z), Q(z)$ are differential polynomials generated by f and the degree of Q is at most n. Then $m(r, P) = S(r, f)$.*

Lemma 4 ([2], p. 69). *Let f be a non-constant meromorphic function and*

$$g(z) = f^n(z) + P_{n-1}(f),$$

where $P_{n-1}(f)$ is a differential polynomial generated by f and of degree at most $n-1$.

If $N(r, \infty; f) + N(r, 0; g) = S(r, f)$, then $g(z) = h^n(z)$, where $h(z) = f(z) + \frac{a(z)}{n}$ and $h^{n-1}(z)a(z)$ is obtained by substituting $h(z)$ for $f(z)$, $h^{(1)}(z)$ for $f^{(1)}(z)$ etc. in the terms of degree $n-1$ in $P_{n-1}(f)$.

Lemma 5 ([2], p. 57). *Suppose that* g *be a non-constant meromorphic function and* $\psi = \sum_{\nu=0}^{l} a_\nu g^{(\nu)}$, *where* $a_\nu' s$ *are meromorphic functions satisfying* $T(r, a_\nu) = S(r, g)$ *for* $\nu = 1, 2, \ldots, l$. *If* ψ *is non-constant, then*

$$T(r, g) \leq \overline{N}(r, \infty; g) + N(r, 0; g) + \overline{N}(r, 1; \psi) + S(r, g).$$

Lemma 6 *Let* f *be a non-constant meromorphic function and* $a = a(z)$ *be a small function of* f *with* $a \not\equiv a^{(1)}$. *Then*

$$T(r, f) \leq \overline{N}(r, \infty; f) + N(r, a; f) + \overline{N}(r, a; f^{(1)}) + S(r, f).$$

Proof. Lemma follows from Lemma 5 for $g = f - a$ and $\psi = \frac{g^{(1)}}{a - a^{(1)}}$. □

Lemma 7 *Let* f *be a non-constant entire function and* $a = a(z)(\neq 0, \infty)$ *be a small function of* f *with* $a \not\equiv a^{(1)}$. *If*

(i) $N_A(r, a; f) + N_A(r, a; f^{(1)}) = S(r, f)$, *where* $A = \overline{E}(a; f) \triangle \overline{E}(a; f^{(1)})$,

(ii) *each common zero of* $f - a$ *and* $f^{(1)} - a$ *has the same multiplicity,*

then $T(r, f) \leq 2\overline{N}(r, a; f) + S(r, f)$.

Proof. Let z_0 be a zero of $f - a$ and $f^{(1)} - a$ with multiplicity $q(\geq 2)$. Then z_0 is a zero of $f^{(1)} - a^{(1)}$ with multiplicity $q - 1$. Hence z_0 is a zero of $a - a^{(1)} = (f^{(1)} - a^{(1)}) - (f^{(1)} - a)$ with multiplicity $q - 1$.

Then we have

$$\begin{aligned} N_{(2}(r, a; f) &\leq 2N(r, 0; a - a^{(1)}) + N_A(r, a; f) \\ &= S(r, f). \end{aligned} \tag{2}$$

Again

$$\begin{aligned} \overline{N}(r, a; f^{(1)}) &\leq \overline{N}(r, a; f) + N_A(r, a; f^{(1)}) + S(r, f) \\ &= \overline{N}(r, a; f) + S(r, f). \end{aligned} \tag{3}$$

Now using (2) and (3) and from Lemma 6, we get

$$T(r, f) \leq 2\overline{N}(r, a; f) + S(r, f).$$

□

Lemma 8 ([2], p.47). *Let* f *be a non-constant meromorphic function and* a_1, a_2, a_3 *be three distinct meromorphic functions satisfying* $T(r, a_\nu) = S(r, f)$ *for* $\nu = 1, 2, 3$. *Then*

$$T(r, f) \leq \sum_{\nu=1}^{3} \overline{N}(r, a_\nu; f) + S(r, f).$$

Lemma 9 ([10], p.92). *Suppose that* $f_1, f_2, \ldots, f_n (n \geq 3)$ *are meromorphic functions which are not constants except for* f_n. *Furthermore, let* $\sum_{j=1}^{n} f_j \equiv 1$.

If $f_n \not\equiv 0$ *and* $\sum_{j=1}^{n} N(r, 0; f_j) + (n-1) \sum_{j=1}^{n} \overline{N}(r, \infty; f_j) < \{\lambda + o(1)\} T(r, f_k)$, *where* $r \in I$, $k = 1, 2, \ldots, n-1$ *and* $\lambda < 1$, *then* $f_n \equiv 1$.

Lemma 10 *Let* f *be a non-constant entire function and* $a = a(z) (\neq 0, \infty)$ *be a small function of* f *with* $a \not\equiv a^{(1)}$. *Suppose that* $A = \overline{E}(a; f) \triangle \overline{E}(a; f^{(1)})$, $B = \overline{E}(a; f^{(1)}) \setminus \{\overline{E}(a; L^{(p)}) \cap \overline{E}(a; L^{(q)})\}$, *where* L *is defined in Theorem 1 and* $q > p \geq 0$. *If*

(i) $E_{1)}(a; f) \subset \overline{E}(a; f^{(1)})$,

(ii) $N_A(r, a; f) + N_{A \cup B}(r, a; f^{(1)}) = S(r, f)$, *and*

(iii) *each common zero of* $f - a$ *and* $f^{(1)} - a$ *has the same multiplicity,*

then the function $h = \frac{f^{(1)} - a}{f - a}$ *is a small function of* f.

Proof. Let $F = f - a$. Then from

$$h = \frac{f^{(1)} - a}{f - a}, \tag{4}$$

we get

$$\begin{aligned} F^{(1)} = f^{(1)} - a^{(1)} &= f^{(1)} - a + (a - a^{(1)}) \\ &= hF + (a - a^{(1)}) \\ &= b_1 F + c_1, \end{aligned} \tag{5}$$

where $b_1 = h$, $c_1 = a - a^{(1)} = b$ (say).

Differentiating (5) and then using (5), we get

$$\begin{aligned} F^{(2)} &= b_1 F^{(1)} + b_1^{(1)} F + c_1^{(1)} \\ &= b_1 (b_1 F + c_1) + b_1^{(1)} F + c_1^{(1)} \end{aligned}$$

$$\begin{aligned} &= (b_1 b_1 + b_1^{(1)})F + b_1 c_1 + c_1^{(1)} \\ &= b_2 F + c_2, \end{aligned}$$

where $b_2 = b_1 b_1 + b_1^{(1)}$ and $c_2 = b_1 c_1 + c_1^{(1)}$.

Similarly,

$$F^{(k)} = b_k F + c_k, \tag{6}$$

where $b_{k+1} = b_1 b_k + b_k^{(1)}$ and $c_{k+1} = c_1 b_k + c_k^{(1)}$.

If h is a constant then $T(r,h) = S(r,f)$ i.e. h is a small function of f. So we suppose h is non-constant.

Clearly from the hypothesis, we can obtain

$$\begin{aligned} N(r,0;h) + N(r,\infty;h) &\leq N_A(r,a;f) + N_A(r,a;f^{(1)}) \\ &= S(r,f). \end{aligned} \tag{7}$$

Now putting $k = 1$ in $b_{k+1} = b_1 b_k + b_k^{(1)}$, we get

$$b_2 = b_1 b_1 + b_1^{(1)} = h^2 + h^{(1)} = h^2 + h d_1,$$

where $d_1 = \frac{h^{(1)}}{h}$.

Again putting $k = 2$ in $b_{k+1} = b_1 b_k + b_k^{(1)}$, we have

$$\begin{aligned} b_3 &= b_1 b_2 + b_2^{(1)} \\ &= h^3 + 3 d_1 h^2 + d_2 h, \end{aligned}$$

where $d_2 = d_1^{(1)} + d_1^2$.

Similarly,

$$b_4 = h^4 + 6 d_1 h^3 + (d_2 + 6 d_1^2 + 3 d_1^{(1)}) h^2 + (d_2^{(1)} + d_1 d_2) h.$$

Therefore in general, we get for $k \geq 2$

$$b_k = h^k + \sum_{j=1}^{k-1} \alpha_j h^j, \tag{8}$$

where $T(r,\alpha_j) = O(\overline{N}(r,0;h) + \overline{N}(r,\infty;h)) + S(r,h) = S(r,f)$, for $j = 1, 2, \ldots, k-1$.

Again putting $k = 1$ in $c_{k+1} = c_1 b_k + c_k^{(1)}$, we have

$$c_2 = c_1 b_1 + c_1^{(1)} = hb + b^{(1)}.$$

Also putting $k = 2$ in $c_{k+1} = c_1 b_k + c_k^{(1)}$, we can obtain

$$c_3 = bh^2 + (b^{(1)} + 2bd_1)h + b^{(2)}.$$

Similarly,

$$c_4 = bh^3 + (5hd_1 + b^{(1)})h^2 + (3b^{(1)}d_1 + 4bd_1^{(1)} + b^2 + d_2 b)h + b^{(3)}.$$

Therefore in general, we get for $k \geq 2$

$$c_k = \sum_{j=1}^{k-1} \beta_j h^j + b^{(k-1)}, \tag{9}$$

where $T(r, \beta_j) = O(\overline{N}(r, 0; h) + \overline{N}(r, \infty; h)) + S(r, h) = S(r, f)$, for $j = 1, 2, \ldots, k-1$.

Case 1. In this case we suppose that either $n \geq 2$ or $n = 1$, $a_1 \neq 1$ and $p \geq 0$ or $n = 1$, $a_1 = 1$ and $p > 0$.

We put

$$\Psi = \frac{(a - L^{(p)}(a))(f^{(1)} - a^{(1)}) - (a - a^{(1)})(L^{(p)}(f) - L^{(p)}(a))}{f - a} \tag{10}$$

Then by lemma of the logarithmic derivative, we get $m(r, \Psi) = S(r, f)$.

Also from the hypothesis

$$\begin{aligned} N(r, \Psi) &\leq N_{(2}(r, a; f) + N_A(r, a; f) + N_B(r, a; f^{(1)}) + N(r, \infty; a_k) \\ &= S(r, f). \end{aligned}$$

Therefore $T(r, \Psi) = S(r, f)$.

Now from (11), we have

$$\begin{aligned} \Psi F &= (a - L^{(p)}(a))F^{(1)} - bL^{(p)}(F) \\ &= (a - L^{(p)}(a))(hF + b) - b\sum_{k=1}^{n} a_k F^{(k+p)}, \quad \text{using (5)} \\ &= (a - L^{(p)}(a))(hF + b) - b\sum_{k=1}^{n} a_k [b_{k+p}F + c_{k+p}], \quad \text{using (6)} \\ &= (a - L^{(p)}(a))(hF + b) - b\sum_{k=1}^{n} a_k \left\{ h^{k+p} + \sum_{j=1}^{k+p-1} \alpha_j h^j \right\} F \end{aligned}$$

$$-b\sum_{k=1}^{n}a_k\left\{\sum_{j=1}^{k+p-1}\beta_j h^j + b^{(k+p-1)}\right\}, \quad \text{using (8), using (9)}$$

Or,

$$\begin{aligned}&\left[\Psi - h(a-L^{(p)}(a)) + b\sum_{k=1}^{n}a_k\left\{h^{k+p}+\sum_{j=1}^{k+p-1}\alpha_j h^j\right\}\right]F\\ &+\left[b\sum_{k=1}^{n}a_k\left\{\sum_{j=1}^{k+p-1}\beta_j h^j + b^{(k+p-1)}\right\} - b(a-L^{(p)}(a))\right]=0.\end{aligned} \tag{11}$$

Or,

$$\Delta_1 F + \Delta_2 = 0, \tag{12}$$

where

$$\Delta_1 = \Psi - h(a-L^{(p)}(a)) + b\sum_{k=1}^{n}a_k\left\{h^{k+p}+\sum_{j=1}^{k+p-1}\alpha_j h^j\right\}$$

and

$$\Delta_2 = b\sum_{k=1}^{n}a_k\left\{\sum_{j=1}^{k+p-1}\beta_j h^j + b^{(k+p-1)}\right\} - b(a-L^{(p)}(a)).$$

If $\Delta_1 \equiv 0$, then by Lemma 2 we get $m(r,h) = S(r,f)$ and from (7), $T(r,h) = S(r,f)$.

Therefore we suppose $\Delta_1 \not\equiv 0$.

From (12) we get

$$F = -\frac{\Delta_2}{\Delta_1}. \tag{13}$$

From First Fundamental theorem and the properties of characteristic function, we can obtain

$$T(r,F) = O(T(r,h)) + S(r,f)$$

i.e.

$$\begin{aligned}T(r,f) &= T(r,F) + S(r,f)\\ &= O(T(r,h)) + S(r,f).\end{aligned} \tag{14}$$

Here Δ_1 is a polynomial of h of degree $n+p$ and Δ_2 is a polynomial of h of degree $n+p-1$. Also the coefficients of both the polynomials are small functions of h .

Without loss of generality we may suppose F is irreducible if not cancelling the common factor it can be made irreducible.

Since $N(r,F) = S(r,f)$, from (13) and (14), we get

$$N(r,0;\Delta_1) = S(r,h).$$

Also from (7) and (14), we have

$$N(r,\infty;h) = S(r,f) = S(r,h).$$

Then by Lemma 4, we get

$$\Delta_1 = \left(h + \frac{c}{n+p}\right)^{n+p}, \tag{15}$$

where c is the coefficient of h^{n+p-1} in Δ_1.

If $c \neq 0$ then from Lemma 8, we can obtain

$$\begin{aligned} T(r,h) &\leq \overline{N}(r,0;h) + \overline{N}(r,\infty;h) + \overline{N}\left(r, -\frac{c}{n+p}; h\right) + S(r,h) \\ &= \overline{N}(r,0;\Delta_1) + S(r,h) \\ &= S(r,h), \end{aligned}$$

a contradiction.

Therefore $c = 0$ and from (15), $\Delta_1 = h^{n+p}$ and from (13), $F = -\frac{\Delta_2}{h^{n+p}}$.

Differentiating, we have

$$\begin{aligned} F^{(1)} &= -\frac{h^{n+p}\Delta_2^{(1)} - (n+p)h^{n+p-1}\Delta_2}{(h^{n+p})^2} h^{(1)} \\ &= d_1 \frac{(n+p)\Delta_2 - h\Delta_2^{(1)}}{h^{n+p}}, \end{aligned}$$

where $d_1 = \frac{h^{(1)}}{h}$.

From the properties of characteristic function, we get

$$T(r,F^{(1)}) = (n+p)T(r,h) + S(r,h). \tag{16}$$

Again

$$F^{(1)} = hF + b = -\frac{\Delta_2}{h^{n+p-1}} + b,$$

Therefore

$$T(r, F^{(1)}) = (n + p - 1)T(r, h) + S(r, h). \tag{17}$$

From (16) and (17) we get $T(r, h) = S(r, h)$, which is a contradiction. Therefore

$$T(r, h) = S(r, f).$$

Case 2. In this case we suppose $n = 1$, $a_1 = 1$ and $p = 0$. Then $L^{(p)} = L = f^{(1)}$.

We put

$$\Psi_1 = \frac{(a - L^{(q)}(a))(f^{(1)} - a^{(1)}) - (a - a^{(1)})(L^{(q)}(f) - L^{(q)}(a))}{f - a}. \tag{18}$$

From the hypothesis

$$\begin{aligned} N(r, \Psi_1) &\leq N_A(r, a; f) + N_B(r, a; f^{(1)}) + N_{(2}(r, a; f) + N(r, \infty; a_k) \\ &= S(r, f). \end{aligned}$$

Also $m(r, \Psi_1) = S(r, f)$.
Therefore $T(r, \Psi_1) = S(r, f)$.
Now following the similar arguments of case-1 and using (18), we can prove

$$T(r, h) = S(r, f).$$

This proves the lemma. □

3 Proof of the main theorem

Proof.

To prove the theorem let us consider h as defined in Lemma 10.
That is,

$$h = \frac{f^{(1)} - a}{f - a}. \tag{19}$$

By Lemma 10, $T(r, h) = S(r, f)$.

Now from (19), we have

$$\begin{aligned} f^{(1)} &= hf + a(1 - h) \\ &= \xi_1 f + \eta_1, \end{aligned} \tag{20}$$

where $\xi_1 = h$ and $\eta_1 = a(1 - h)$.

Differentiating (20) and then using it, we get

$$\begin{aligned} f^{(2)} &= \xi_1^{(1)} f + \xi_1 f^{(1)} + \eta_1^{(1)} \\ &= \xi_1^{(1)} f + \xi_1(\xi_1 f + \eta_1) + \eta_1^{(1)} \\ &= (\xi_1^{(1)} + \xi_1 \xi_1) f + \xi_1 \eta_1 + \eta_1^{(1)} \\ &= \xi_2 f + \eta_2, \end{aligned} \tag{21}$$

where $\xi_2 = \xi_1^{(1)} + \xi_1 \xi_1$ and $\eta_2 = \eta_1^{(1)} + \xi_1 \eta_1$.

Similarly

$$f^{(k)} = \xi_k f + \eta_k, \tag{22}$$

where $\xi_{k+1} = \xi_k^{(1)} + \xi_1 \xi_k$ and $\eta_k = \eta_k^{(1)} + \eta_1 \xi_k$.

Since $T(r, h) = S(r, f)$, we see that

$$T(r, \xi_k) + T(r, \eta_k) = S(r, f), \tag{23}$$

for $k = 1, 2, \dots$.

Now

$$\begin{aligned} L^{(p)} &= \sum_{k=1}^{n} a_k f^{(k+p)} \\ &= \sum_{k=1}^{n} a_k (\xi_{k+p} f + \eta_{k+p}) \\ &= \left(\sum_{k=1}^{n} a_k \xi_{k+p} \right) f + \left(\sum_{k=1}^{n} a_k \eta_{k+p} \right) \\ &= \mu_1 f + \nu_1, \end{aligned} \tag{24}$$

where

$$\mu_1 = \sum_{k=1}^{n} a_k \xi_{k+p} \quad \text{and} \quad \nu_1 = \sum_{k=1}^{n} a_k \eta_{k+p}.$$

Since each a_k is a small function of f and from (23), $T(r, \mu_1) + T(r, \nu_1) = S(r, f)$.

Similarly

$$L^{(q)} = \mu_2 f + \nu_2, \tag{25}$$

where

$$\mu_2 = \sum_{k=1}^{n} a_k \xi_{k+q} \quad \text{and} \quad \nu_2 = \sum_{k=1}^{n} a_k \eta_{k+q}.$$

Also $T(r,\mu_2)+T(r,\nu_2)=S(r,f)$.

Let z_1 be a zero of $f-a$ such that $z_1\notin A\cup B$. Then $f(z_1)=f^{(1)}(z_1)=L^{(p)}(z_1)=L^{(q)}(z_1)=a(z_1)$.

From (24) and (25), we get

$$\mu_1(z_1)a(z_1)+\nu_1(z_1)-a(z_1)=0$$

and

$$\mu_2(z_1)a(z_1)+\nu_2(z_1)-a(z_1)=0.$$

If $\mu_1(z)a(z)+\nu_1(z)-a(z)\not\equiv 0$, then

$$\begin{aligned}\overline{N}(r,a;f) &\leq N_A(r,a;f)+N(r,0;\mu_1a+\nu_1-a)+S(r,f)\\ &= S(r,f).\end{aligned}$$

From Lemma 7, $T(r,f)=S(r,f)$, a contradiction. Therefore

$$\mu_1(z)a(z)+\nu_1(z)\equiv a(z). \tag{26}$$

Again if $\mu_2(z)a(z)+\nu_2(z)-a(z)\not\equiv 0$, then

$$\begin{aligned}\overline{N}(r,a;f) &\leq N_A(r,a;f)+N(r,0;\mu_2a+\nu_2-a)+S(r,f)\\ &= S(r,f).\end{aligned}$$

From Lemma 7, $T(r,f)=S(r,f)$, a contradiction. Therefore

$$\mu_2(z)a(z)+\nu_2(z)\equiv a(z). \tag{27}$$

From (26) and (27), we see that $\mu_1(z)\equiv\mu_2(z)\equiv 1$ and $\nu_1(z)\equiv\nu_2(z)\equiv 0$.
Therefore from (24) and (25), we have $L^{(p)}\equiv L^{(q)}\equiv f$.

Let $q-p=r$. Then

$$L^{(p+r)}\equiv L^{(q)}\quad\text{or}\quad f^{(r)}\equiv f. \tag{28}$$

Solving (28), we get

$$f=p_1e^{\alpha_1 z}+p_2e^{\alpha_2 z}+...+p_te^{\alpha_t z}, \tag{29}$$

where α_1, α_2,...,α_t are distinct roots of $z^r-1=0$ and $p_1,p_2,...,p_t$ are constants or polynomials.

Differentiating (29), we have

$$f^{(1)} = (p_1\alpha_1 + p_1^{(1)})e^{\alpha_1 z} + (p_2\alpha_2 + p_2^{(1)})e^{\alpha_2 z} + \dots + (p_t\alpha_t + p_t^{(1)})e^{\alpha_t z}. \tag{30}$$

Now from (19), (29) and (30), we can obtain

$$hf - f^{(1)} = a(h-1).$$

Or,

$$\sum_{j=1}^{t}(hp_j - p_j\alpha_j - p_j^{(1)})e^{\alpha_j z} = a(h-1). \tag{31}$$

If $h \not\equiv 1$, then from (31) we get

$$\sum_{j=1}^{t}\frac{(hp_j - p_j\alpha_j - p_j^{(1)})}{a(h-1)}e^{\alpha_j z} \equiv 1. \tag{32}$$

Also we note that $T(r,f) = O(T(r,e^{\alpha_j z}))$ for $j = 1, 2, \dots, t$.

If the left hand side of (32) contains more than two terms, then from Lemma 9 we get

$$\frac{(hp_j - p_j\alpha_j - p_j^{(1)})}{a(h-1)}e^{\alpha_j z} \equiv 1, \tag{33}$$

for one value of $j \in \{1, 2, \dots, t\}$.

From (33), we see that

$$T(r, e^{\alpha_j z}) = S(r,f) = S(r, e^{\alpha_j z}),$$

a contradiction.

Now we suppose that the left hand side of (32) contains exactly two terms.

Suppose (32) is of the form

$$\frac{(hp_l - p_l\alpha_l - p_l^{(1)})}{a(h-1)}e^{\alpha_l z} + \frac{(hp_m - p_m\alpha_m - p_m^{(1)})}{a(h-1)})e^{\alpha_m z} \equiv 1, \tag{34}$$

where $1 \le l, m \le t$.

From Lemma 8, we have

$$T(r, e^{\alpha_l z}) \quad \le \quad \overline{N}(r, 0; e^{\alpha_l z}) + \overline{N}(r, \infty; e^{\alpha_l z})$$

$$
\begin{aligned}
&+\overline{N}\left(r, \frac{a(h-1)}{(hp_l - p_l\alpha_l - p_l^{(1)})}; e^{\alpha_l z}\right) + S(r, e^{\alpha_l z}\Big) \\
= \quad & \overline{N}(r, 0; e^{\alpha_m z}) + S(r, e^{\alpha_l z}) \\
= \quad & S(r, e^{\alpha_l z}),
\end{aligned}
$$

which is a contradiction.

Finally we suppose that the left hand side of (32) contains exactly one term, say, of the form

$$\frac{(hp_l - p_l\alpha_l - p_l^{(1)})}{a(h-1)} e^{\alpha_l z} \equiv 1.$$

This implies $T(r, e^{\alpha_l z}) = S(r, e^{\alpha_l z})$, a contradiction.

Therefore $h \equiv 1$. i.e., $f^{(1)} \equiv f$.
This implies $f = \delta e^z$, where $\delta(\neq 0)$ is a constant. Now

$$
\begin{aligned}
L^{(p)} &= \sum_{k=1}^{n} a_k f^{(p+k)} \\
&= \left(\sum_{k=1}^{n} a_k\right) \delta e^z.
\end{aligned}
$$

Since $L^{(p)} \equiv f$ i.e.,

$$\left(\sum_{k=1}^{n} a_k\right) \delta e^z \equiv \delta e^z, \quad \sum_{k=1}^{n} a_k \equiv 1.$$

Therefore

$$
\begin{aligned}
L &= \sum_{k=1}^{n} a_k f^{(k)} \\
&= \left(\sum_{k=1}^{n} a_k\right) \delta e^z = \delta e^z.
\end{aligned}
$$

Hence $f = L = \delta e^z$, where $\delta(\neq 0)$ is a constant. This completes the proof. □

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Composition of continued fractions convergents to $\sqrt[3]{2}$

Mitja Lakner
Žaucerjeva ulica 17, 1000 Ljubljana, Slovenia
email: mlakner@gmail.com

Peter Petek
University of Ljubljana,
Faculty of Education,
Kardeljeva ploščad 16,
1000 Ljubljana, Slovenia
email: peter.petek@guest.arnes.si

Marjeta Škapin Rugelj
University of Ljubljana,
Faculty of Civil and Geodetic Engineering,
Jamova 2, 1000 Ljubljana, Slovenia
email:
marjeta.skapin-rugelj@fgg.uni-lj.si

Abstract. Applying geometrical construction in the 3-dim space, we compose all good convergents of $\sqrt[3]{2}$. The problem tackled in this paper is the nature of the continued fraction expansion of $\sqrt[3]{2}$: are the partial quotients bounded or not.

1 Introduction

The present paper uses some notations and results of [5] and [3].

We investigate $\sqrt[3]{2}$ and its adjunction ring. It is a common belief that the partial quotients in C.F.E. of $\sqrt[3]{2}$ that begins with

[1,3,1,5,1,1,4,1,1,8,1,14,1,10,2,1,4,12,2,3,2,1,3,4,1,1,2,14,3,12,1,15,3,1,4,534,1,...]

are not bounded, as supported by extensive computations, but there is no proof [4].

In the adjunction ring, we have the unit $\rho = 1 + \sqrt[3]{2} + \sqrt[3]{4}$ and its inverse $\sigma = -1 + \sqrt[3]{2}$. Multiplicative norm is defined in $\mathbb{Z}[\sqrt[3]{2}]$. Let $a = x + y\sqrt[3]{2} + z\sqrt[3]{4}$, its norm is $N(a) = x^3 + 2y^3 + 4z^3 - 6xyz$.

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2 Ambient vector space $\mathcal{V}$ and its geometry

Now, let $\mathcal{V} = \mathbb{R}^3$ be the 3-dimensional space endowed with the usual scalar product $\langle \mathbf{a}, \mathbf{b} \rangle$ and cross product $\mathbf{a} \times \mathbf{b}$. We define a linear mapping

$$\eta \colon \mathbb{Z}[\sqrt[3]{2}] \to \mathcal{V}$$

by $\eta(x + y\sqrt[3]{2} + z\sqrt[3]{4}) = (x, y, z)$, the resulting image consisting of all vectors with integer entries, multiplication inherited from $\mathbb{Z}[\sqrt[3]{2}]$.

Multiplication with σ will prove very important and we observe

$$\eta(\sigma \cdot a) = S\eta(a)$$

where S is the matrix

$$\begin{bmatrix} -1 & 0 & 2 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}.$$

If $\mathbf{s}_j = \eta(\sigma^j)$, then we have $\mathbf{s}_{j+1} = S\mathbf{s}_j$,

$$\mathbf{s}_0 = (1, 0, 0), \quad \mathbf{s}_1 = (-1, 1, 0), \quad \mathbf{s}_2 = (1, -2, 1), \quad \mathbf{s}_3 = (1, 3, -3).$$

With the aid of diagonalization we can write

$$\mathbf{s}_j = \sigma^j \mathbf{h} + \rho^{\frac{j}{2}} (\mathbf{g} \cos(j\theta) + \mathbf{k} \sin(j\theta)) \tag{1}$$

where $\mathbf{h}$ and $\mathbf{g} \pm i\mathbf{k}$ are eigenvectors of matrix S

$$\begin{aligned} \mathbf{g} &= \frac{1}{6}(4, -\sqrt[3]{4}, -\sqrt[3]{2}), \\ \mathbf{k} &= \frac{\sqrt{3}}{6}(0, \sqrt[3]{4}, -\sqrt[3]{2}), \\ \mathbf{h} &= \frac{1}{6}(2, \sqrt[3]{4}, \sqrt[3]{2}) \end{aligned}$$

and the rotation angle is

$$\theta = \pi - \arctan \frac{\sqrt{3}\sqrt[3]{2}}{2 + \sqrt[3]{2}} \doteq 146.2^0.$$

Remark: Formula (1) can be extended for noninteger $t \in \mathbb{R}$

$$\mathbf{s}_t = \sigma^t \mathbf{h} + \rho^{\frac{t}{2}} (\mathbf{g} \cos(t\theta) + \mathbf{k} \sin(t\theta)) \tag{2}$$

Plane P, spanned by $\mathbf{g}, \mathbf{k}$, is the eigenplane, invariant for S, and together with the line of $\mathbf{h}$ forms the locus of zero norm.

The basic vectors $\mathbf{s}_j$ with increasing positive j are approaching the invariant plane and are for negative j almost collinear to eigenvector $\mathbf{h}$.

For each real N we consider the funnel

$$F_N = \{(x, y, z) \in \mathcal{V}; x^3 + 2y^3 + 4z^3 - 6xyz = N\},$$

i.e. points of norm $= N$. The positive funnels lie "above" the invariant plane $P : x + y\sqrt[3]{2} + z\sqrt[3]{4} = 0$, the negative ones "below". Figure 1 shows the funnel F_1 containing all the above units $\mathbf{s_j}$. The funnel flattens towards the invariant plane P spanned by vectors $\mathbf{g}$, $\mathbf{k}$, and embraces the line of $\mathbf{h}$.

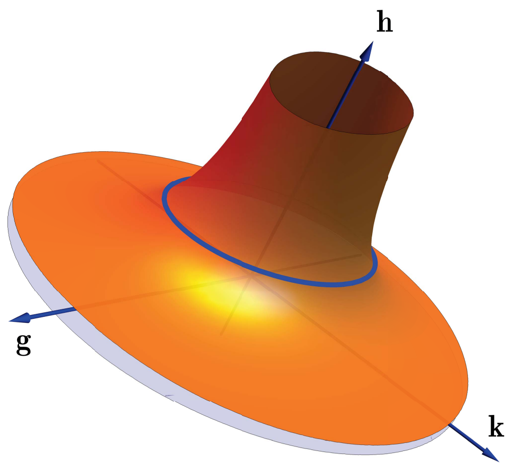

Figure 1: Funel F_1 with collar $\mathbf{c}_\phi$ and vectors $\mathbf{g}$, $\mathbf{k}$, $\mathbf{h}$.

3 Shortest vector algorithm

Definition 1 *We denote by* M_j *the lattice of integral vectors, orthogonal to* $\mathbf{s}_j$*.*

$$M_j = \{(x, y, z) \in \mathbb{Z}^3; \langle (x, y, z), \mathbf{s}_j \rangle = 0\}.$$

Using (1) we get a result on orthogonality

Lemma 1

$$T\mathbf{s}_{-j+1} \times T\mathbf{s}_{-j} = \mathbf{s}_j$$

where transposition T *is the linear transformation*

$$\mathsf{T}(x, y, z) = (z, y, x).$$

Thus, vectors orthogonal to $\mathbf{s}_j$ are $\mathsf{T}\mathbf{s}_{-j}$ and $\mathsf{T}\mathbf{s}_{-j+1}$ and they form a basis for lattice M_j.

Lemma 2

$$M_j = \{m\mathsf{T}\mathbf{s}_{-j+1} + n\mathsf{T}\mathbf{s}_{-j}; m, n \in \mathbb{Z}\}.$$

Proof. Let (x, y, z) be point from M_j. Then $(x, y, z) = \alpha\mathsf{T}\mathbf{s}_{-j+1} + \beta\mathsf{T}\mathbf{s}_{-j}$ for some real α and β. Applying transformations T and S^j to this equation, we get

$$S^j(z, y, x) = \alpha\mathbf{s}_1 + \beta\mathbf{s}_0 = \alpha(-1, 1, 0) + \beta(1, 0, 0)$$

□

To prove our theorem, the length of vectors that form a basis of the lattice M_j is crucial to get good estimates. Therefore, we need the shortest basis vectors $\mathbf{u}_j$, $\mathbf{v}_j$ of lattice M_j. In [1] we find the construction called the *shortest vector algorithm* SVA, which gives the shortest lattice vectors $\mathbf{u}_j$, $\mathbf{v}_j$ and cross product preserved by construction

$$\mathbf{u}_j \times \mathbf{v}_j = \mathbf{s}_j. \tag{3}$$

Computations of the shortest vectors can be done inductively, because vectors $\left(S^{\mathsf{T}}\right)^{-1}\mathbf{u}_j$, $\left(S^{\mathsf{T}}\right)^{-1}\mathbf{v}_j$ form the basis of lattice M_{j+1}. This essentially reduces the SVA algorithm.

4 Multiplications in $\mathcal{V}$

We shall endow the 3-dim vector space $\mathcal{V}$ with some additional structures. We already know the usual scalar and vector products. The multiplication can also be inherited from the immersion of $\mathbb{Z}[\sqrt[3]{2}]$.

Definition 2 $(x, y, z) \otimes (a, b, c) = (ax + 2cy + 2bz, bx + ay + 2cz, cx + by + az)$.

If we allow for any real entries, the multiplication retains its favorable properties of commutativity, associativity and distibutivity.

Function $\gamma \colon \mathcal{V} \to \mathbb{R}$, $\gamma(x, y, z) = x + \sqrt[3]{2}y + \sqrt[3]{4}z$ is multiplicative with respect to the $\otimes$ product.

5 Collar and collar coordinates

First we shall define the collar in F_1, which is a topological circle of points $\mathbf{c}_\phi$ near the origin

$$\mathbf{c}_\phi = \mathbf{h} + \mathbf{g}\cos\phi + \mathbf{k}\sin\phi.$$

We shall prove a uniqueness theorem.

Theorem 1 *For every point* $(x, y, z) \in \mathcal{V}$*, which does not lie on the invariant plane or the invariant line, we have a unique representation*

$$(x, y, z) = \sqrt[3]{N}\mathbf{c}_\phi \otimes \mathbf{s}_t$$

for some $\phi \in [0, 2\pi)$*,* $t \in \mathbb{R}$ *and* N *is the norm of the given point.*

Proof. Since multiplication with $\sqrt[3]{N}$ moves points from F_1 to F_N, we can suppose $(x, y, z) \in F_1$ and try to solve the equation

$$(x, y, z) = \mathbf{c}_\phi \otimes \mathbf{s}_t \tag{4}$$

uniquely for $\phi \in [0, 2\pi)$, $t \in \mathbb{R}$.

Function γ is positive on F_1 and $\gamma(\mathbf{c}_\phi \otimes \mathbf{s}_t) = \sigma^t$, so $t = \log_\sigma \gamma(x, y, z)$ is defined. Point $T_0 = (x, y, z) \otimes \mathbf{s}_{-t}$ lies on F_1 and has development

$$T_0 = \mathbf{h} + \alpha\mathbf{g} + \beta\mathbf{k},$$

with $\alpha^2 + \beta^2 = 1$, and (4) holds for some $\phi \in [0, 2\pi)$.

Uniqueness is the consequence of identity

$$\mathbf{c}_\phi \otimes \mathbf{s}_t = \sigma^t\mathbf{h} + \rho^{t/2}(\mathbf{g}\cos(\phi + t\theta) + \mathbf{k}\sin(\phi + t\theta)).$$

□

Corollary 1 *Every point* $(x, y, z) \in \mathcal{V}$ *has a unique representation*

$$(x, y, z) = \sqrt[3]{N}\mathbf{c}_\phi \otimes \mathbf{s}_j \otimes \mathbf{s}_\kappa$$

where j *is integer,* $\kappa \in [-0.723, 0.277)$*,* $\phi \in [0, 2\pi)$ *and* N *is the norm of the point.*

In continuation of the article, *Mathematica* [6] is used to get some crucial numerical not sharp estimates of smooth elementary functions on compact interval or rectangle.

6 Some technical lemmas

Lemma 3

$$\rho^{\frac{\kappa}{4}} |\mathbf{c}_\phi \otimes \mathbf{s}_\kappa| \leq 1.152$$

for all $\phi \in [0, 2\pi]$ *and* $\kappa \in [-0.723, 0.277]$.

The chosen interval of unit length gives optimal inequality.

Lemma 4

$$0.5773 < |\mathbf{g} \cos \phi + \mathbf{k} \sin \phi| < 0.7534$$

for all $\phi \in [0, 2\pi]$.

Lemma 5

$$|\mathbf{s}_j| \geq \rho^{\frac{j}{2}} 0.576$$

for $j \geq 4$.

Proof. Estimate is the consequence of (1), Lemma 4 and

$$\begin{aligned} |\mathbf{s}_j| &= \rho^{\frac{j}{2}} \left| \sigma^{\frac{3j}{2}} \mathbf{h} + \mathbf{g} \cos(j\theta) + \mathbf{k} \sin(j\theta) \right| \\ &\geq \rho^{\frac{j}{2}} \left(|\mathbf{g} \cos(j\theta) + \mathbf{k} \sin(j\theta)| - \sigma^{\frac{3j}{2}} |\mathbf{h}| \right). \end{aligned}$$

□

Lemma 6

$$K = 1 + \frac{\delta}{\sqrt[3]{2} q_n^2} + \frac{\delta^2}{3 \sqrt[3]{4} q_n^4} < 1.0032$$

for $|\delta| < 0.196$ *and* $q_n \geq 7$.

Lemma 7

$$|\mathbf{u}_j| < 0.9328 \rho^{\frac{j}{4}}$$

for $j \geq 5$.

Proof. Because the angle $\varphi = \sphericalangle(\mathbf{u}_j, \mathbf{v}_j) \in [\pi/3, \pi/2]$, [1], we use (1), (3) and Lemma 4

$$\begin{aligned} |\mathbf{u}_j|^2 \frac{\sqrt{3}}{2} &\leq |\mathbf{u}_j| \, |\mathbf{v}_j| \sin \varphi = \left| \sigma^j \mathbf{h} + \rho^{\frac{j}{2}} (\mathbf{g} \cos(j\theta) + \mathbf{k} \sin(j\theta)) \right| \\ &\leq \sigma^j |\mathbf{h}| + \rho^{\frac{j}{2}} 0.7534 \leq \rho^{\frac{j}{2}} 0.7535 \end{aligned}$$

and Lemma folows. □

From this lemma and (1) we see that the length of vector $\mathbf{u}_j$ is of the order of the fourth root of the length of basis vectors $T\mathbf{s}_{-j+1}$, $T\mathbf{s}_{-j}$ of M_j.

Lemma 8 *On the unit sphere* $|\gamma(x, y, z)| \leq 1 + \sqrt[3]{2}$.

Lemma 9 *On the unit sphere* $\sqrt{N\gamma}(x, y, z) < 2.627$.

7 Representation with the shortest vector

Take now a n-th C.F. convergent $\frac{p_n}{q_n}$. As usual, we say

$$\frac{p_n}{q_n} - \sqrt[3]{2} = \frac{\delta}{q_n^2}.$$

We express the norm of the vector $(p_n, -q_n, 0)$ as

$$N = N(p_n, -q_n, 0) = p_n^3 - 2q_n^3 = \left(q_n \sqrt[3]{2} + \frac{\delta}{q_n}\right)^3 - 2q_n^3 = 3\sqrt[3]{4}q_n\delta K \quad (5)$$

where K is from Lemma 6.

Apply the collar representation

$$(p_n, -q_n, 0) = \sqrt[3]{N}\mathbf{c}_\phi \otimes \mathbf{s}_j \otimes \mathbf{s}_\kappa \quad (6)$$

and we shall first express q_n computing γ of the above equation:

$$\begin{aligned} \gamma(p_n, -q_n, 0) = p_n - \sqrt[3]{2}q_n &= \sqrt[3]{N}\gamma(\mathbf{c}_\phi)\gamma(\mathbf{s}_j)\gamma(\mathbf{s}_\kappa), \\ \frac{\delta}{q_n} &= \left(3\sqrt[3]{4}q_n\delta K\right)^{\frac{1}{3}} 1\sigma^j\sigma^\kappa. \end{aligned}$$

We get

$$q_n = \sqrt{|\delta|}\left(3\sqrt[3]{4}\right)^{-\frac{1}{4}} K^{-\frac{1}{4}}\rho^{\frac{3j}{4}}\rho^{\frac{3\kappa}{4}}. \quad (7)$$

Now, let in the representation (6) be $\mathbf{a} = \sqrt[3]{N}\mathbf{c}_\phi \otimes \mathbf{s}_\kappa$ and calculate its length

$$|\mathbf{a}| = \left(\sqrt[3]{|N|}\rho^{-\frac{\kappa}{4}}\right)\left(\rho^{\frac{\kappa}{4}}|\mathbf{c}_\phi \otimes \mathbf{s}_\kappa|\right). \quad (8)$$

We transform the first factor in (8) using (5) and (7)

$$\sqrt[3]{|N|}\rho^{-\frac{\kappa}{4}} = \left(3\sqrt[3]{4}\right)^{\frac{1}{4}}\sqrt{|\delta|}K^{\frac{1}{4}}\rho^{\frac{j}{4}}. \quad (9)$$

On the other hand, since $\mathbf{u}_j \times \mathbf{v}_j = \mathbf{s}_j$ by (3) and using Lemma 5 we get

$$|\mathbf{v}_j||\mathbf{v}_j| \geq |\mathbf{u}_j||\mathbf{v}_j| > 0.576\rho^{\frac{j}{2}},$$

$$|\mathbf{v}_j| > 0.758\rho^{\frac{j}{4}}.$$

In the representation $(p_n, -q_n, 0) = \mathbf{a} \otimes \mathbf{s}_j$, the last component of $\otimes$ product is scalar product $\langle T\mathbf{a}, \mathbf{s}_j \rangle$, thus $T\mathbf{a} \perp \mathbf{s}_j$, i.e. $T\mathbf{a} \in M_j$. As $\mathbf{v}_j$ is the second shortest basis vector, we have

Lemma 10 *From* $|\mathbf{a}| < 0.758\rho^{\frac{j}{4}}$, *follows* $T\mathbf{a} = \pm\mathbf{u}_j$.

We are now prepared to formulate and prove

Theorem 2 *Let a C.F. convergent* $\frac{p_n}{q_n}$ *have* $|\delta| < 0.196$*. Then, for some* $j \in \mathbb{N}$*, there exists a representation*

$$(p_n, -q_n, 0) = T\mathbf{u}_j \otimes \mathbf{s}_j.$$

Proof. In equation (8) we estimate factors one by one using (9), Lemma 3 and Lemma 6

$$\begin{aligned}
|\mathbf{a}| &< \left(3\sqrt[3]{4}\right)^{\frac{1}{4}} \sqrt{|\delta|}K^{\frac{1}{4}}\rho^{\frac{j}{4}} \left(\rho^{\frac{\kappa}{4}}|\mathbf{c}_\phi \otimes \mathbf{s}_\kappa|\right) \\
&< 1.478 \cdot 0.443 \cdot 1.0008 \cdot \rho^{\frac{j}{4}} \cdot 1.152 \\
&< 0.755\rho^{\frac{j}{4}}.
\end{aligned}$$

This yields the condition of Lemma 10 and thus proves the theorem.

Conditions used in lemmas are satisfied for all convergents, which have $|\delta| < 0.196$, except convergent $\frac{5}{4}$, which has $j = 3$ and the theorem is true by inspection. □

From the first line of the proof we get an estimate of $\mathbf{u}_j$ in terms oh δ.

Corollary 2

$$|\mathbf{u}_j|^2 < 2.91|\delta|\rho^{\frac{j}{2}}.$$

If $\frac{p}{q}$ is convergent and B next partial quotient, then we have [2]

$$\frac{1}{q_n(B+2)} < \left|p_n - q_n\sqrt[3]{2}\right| < \frac{1}{qB}.$$

From this it follows that integer part of $\frac{1}{|\delta|}$ is B or $B+1$. Our Theorem covers all partial quotiens with B greater than 5. This may prove useful in search of big partial quotients.

Let as before, u_j be the shortes lattice vector of M_j. Then we have $T\mathbf{u}_j \otimes \mathbf{s}_j$ of the form $(p, -q, 0)$, $\frac{p}{q}$ not necessarily a C.F. convergent. Still it is a good approximation as the Theorem 3, some sort of converse of the Theorem 2 shows.

Theorem 3 *Let* j *be at least 5 and* $(p,-q,0) = T\mathbf{u}_j \otimes \mathbf{s}_j$. *Then it holds*

$$|p - q\sqrt[3]{2}| < 2.11\sigma^{\frac{3j}{4}} \tag{10}$$

and for $\delta = q(p - q\sqrt[3]{2})$

$$|\delta| < 1.054. \tag{11}$$

Proof. We use Lemmas 7 and 8

$$|p - q\sqrt[3]{2}| = |\gamma(p,-q,0)| = |\gamma(T\mathbf{u}_j)\gamma(\mathbf{s}_j)| = |T\mathbf{u}_j| \left|\gamma\left(\frac{T\mathbf{u}_j}{|T\mathbf{u}_j|}\right)\right| \sigma^j$$
$$< 0.9328\rho^{\frac{j}{4}}(1+\sqrt[3]{2})\sigma^j < 2.11\sigma^{\frac{3j}{4}}$$

and (10) is proved.

From inequality (10) we have for some constant $|c| < 2.11$

$$p = q\sqrt[3]{2} + c\sigma^{\frac{3j}{4}}.$$

Function $R = \sqrt{N/\gamma}$ is defined outside the invariant plane $\gamma = 0$, where it is $\otimes$ multiplicative.

$$\begin{aligned} R^2(p,-q,0) &= \frac{p^3 - 2q^3}{p - q\sqrt[3]{2}} = p^2 + pq\sqrt[3]{2} + q^2\sqrt[3]{4} \\ &= 3\sqrt[3]{4}q^2\left(1 + \frac{c}{q\sqrt[3]{2}}\sigma^{\frac{3j}{4}} + \frac{c^2}{3q^2\sqrt[3]{4}}\sigma^{\frac{3j}{2}}\right) \\ &= 3\sqrt[3]{4}q^2\hat{K}^2 \end{aligned}$$

and $|q| = \frac{|R|}{\sqrt{3}\sqrt[3]{2}\hat{K}}$.

We have

$$R(p,-q,0) = R(T\mathbf{u}_j)R(\mathbf{s}_j) = |T\mathbf{u}_j|R(\mathbf{b})\rho^{\frac{j}{2}},$$
$$\gamma(p,-q,0) = \gamma(T\mathbf{u}_j)\gamma(\mathbf{s}_j) = |T\mathbf{u}_j|\gamma(\mathbf{b})\sigma^j,$$

where vector $\mathbf{b}$ is from the unit sphere. Using these equalities, Lemmas 7 and 9, we estimate

$$|\delta| = |q||p - q\sqrt[3]{2}| = \frac{|R(p,-q,0)|}{\sqrt{3}\sqrt[3]{2}\hat{K}}|\gamma(p,-q,0)| = \frac{|T\mathbf{u}_j|^2}{\sqrt{3}\sqrt[3]{2}\hat{K}}\sqrt{N(\mathbf{b})\gamma(\mathbf{b})}\sigma^{\frac{j}{2}}$$
$$< \frac{0.9328^2 2.627}{\sqrt{3}\sqrt[3]{2}\hat{K}} < \frac{1.048}{\hat{K}} < \frac{1.048}{\sqrt{0.9892}} < 1.054.$$

We have used inequality

$$\hat{K}^2 > 1 - \frac{2.11}{\sqrt[3]{2}\cdot 1}\sigma^{\frac{3\cdot 5}{4}} - \frac{2.11^2}{3\cdot 1^2\sqrt[3]{4}}\sigma^{\frac{3\cdot 5}{2}} > 0.9892.$$

□

Thus $\frac{p}{q}$ is a good rational approximation to $\sqrt[3]{2}$. If $|\delta| < 0.5$, then $\frac{p}{q}$ is C.F. convergent.
From the proof we get an estimate of δ in terms of $\mathbf{u}_j$.

Corollary 3

$$|\delta| < 1.22|\mathbf{u}_j|^2\sigma^{\frac{j}{2}}.$$

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The conditional quantile function in the single-index

Kheira Mecheri
LMA Applied Mathematics Laboratory,
University of Bejaia, Algeria.
email: kheira.mecheri@univ-bejaia.dz.

Abstract. The main contribution of the present paper is to give the conditional quantile estimator and we establish the pointwise and the almost complete convergence of the kernel estimate of this model in the functional single-index model.

1 Introduction

In recent years, nonparametric statistics have undergone a very important development. As well as the single fictional index models which are used in different fields, namely, medical, economic, epidemiology, and others. In the literature, the prediction problem has been widely studied when the two variables are of finite dimensions and in the case of functional variables. When the explanatory variable is functional and the response is still real. Note that the modeling of functional data is becoming more and more popular. In 1985, Härdle et al. The first who are interested in the nonparametric estimations of the regression functions [21] and in 2005, Ferraty and vieu gave a good synthesis on the conditional models using the nucleus method [15]. The approach of single-index is to widely applied in econometrics as a reasonable compromise between nonparametric and parametric models. A number of works dealing

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with index model can be found in the literature when the explanatory variable is multivariate. Without claiming to be exhaustive, we quote for example Härdle et al. [20], Hristache et al. [23]. A first work linking the single-index model and the nonparametric regression models for functional random variables for independent observations can be found in [13]. Their results were extended to dependent case by Ait saidi et al. [2, 3]. Concerning the conditional density estimate, Attaoui et al. [4] studied the estimation of the single functional index and established some asymptotic results. Their work extend, in different way, the works of Delecroix et al.[10]. In 2017, Hamdaoui and al. have studied the asymptotic normality of the conditional distribution function in the single index model [19].

Much has been done on conditional quantile estimates. For example, Berlitet et al. [6, 5], have studied proprits and normality asymptottic od the conditional quantile and Dabo-Niang et al. have also studied the estimation of the quantile regression [8, 9]. We can also cite the work of Ezzahrioui and Ould Said (see [11]) and Honda (see [22]) carried out the study on estimator in the α mixing case. We also have Ferraty and al. in the case of dependent data [14]. we can also cite other works, such as those of Gannoun and al. on the median and the quantile [16], Koenker on the quantile linked to the regression [17, 18], Laksaci and Maref [24] and Wang and Zhao [26].

In this work, we consider the problem of estimating the conditional quantile function of a scalar response variable Y given a Hilbertian random variable X when the explanation of Y given X is done through its projection on one functional direction. Following this study we can build a prediction method based on the conditional quantile estimation with simple functional index. This alternative method is more robust than the conditional method. This result allows us to calculate the prediction expectation which is very sensitive to the errors of the observations when the data of heteroscedasty, or asymmetry and in the case where the distribution is bimodal. Ait saidi et al. ([3]) studied the expectation when we regress a real random variable on a functional random variable (in the case of infinite dimension).

In this article, we are first interested in the estimation of the conditional quantile by the kernel method for the functional single index model. Subsequently, we study the pointwise convergence and almost complete convergence of the estimate of the kernel of this model in the functional single index model.

2 Model and estimator

Let (X, Y) be a couple of random variables taking its values in $\mathcal{H} \times \mathbf{R}$, where $\mathcal{H}$ is a separable real Hilbert space with the inner product $\langle ., . \rangle$. Consider now the sample $(X_i, Y_i)_{i=1,\dots,n}$ of n independent pairs identically distributed as the pair (X, Y). Assume that the conditional cumulative distribution function (c.d.f.) of Y given X has a single-index structure. Such structure supposes that the explanation of Y from X is done through a fixed functional index θ in $\mathcal{H}$. More precisely, we suppose that the conditional c.d.f. of Y given $X = x$, denoted by $F(. \mid x)$, is given by

$$\forall y \in \mathbf{R} \quad F(y \mid x) = F(y \mid \langle \theta,\ x \rangle).$$

The functional index θ appears as a filter allowing the extraction of the part of X explaining the response Y and represents a functional direction which reveals pertinent explanation of the response variable. Concerning the identifiability of this model, we consider the same conditions as those in Ferraty et al. [13] on the regression operator. In other words, we assume that the F is differentiable with respect to x and θ such that $\langle \theta, e_1 \rangle = 1$, where e_1 is the first vector of an orthonormal basis of $\mathcal{H}$. Clearly, under this condition, we have, for all $x \in \mathcal{H}$,

$$F_1(. \mid \langle \theta_1,\ x \rangle) = F_2(. \mid \langle \theta_2,\ x \rangle) \Longrightarrow \theta_1 = \theta_2 \ \text{ and } \ F_1 \equiv F_2.$$

We consider the semi-metric d_θ, associated to the single-index $\theta \in \Theta_{\mathcal{H}} \subset \mathcal{H}$ defined by $\forall x_1, x_2 \in \mathcal{H} : d_\theta\,(x_1, x_2) = |\langle x_1 - x_2,\ \theta \rangle|$. In what follows we denote by $F_\theta(., x)$ the conditional c.d.f. of Y given $\langle \theta, x \rangle$ and we we define the Kernel estimator $\widehat{F}_\theta(y, x)$ of $F_\theta(y, x)$ by

$$\widehat{F}_\theta(y, x) = \frac{\sum_{i=1}^{n} K\left(\frac{d_\theta(X_i, x)}{h}\right) H\left(\frac{y - Y_i}{g}\right)}{\sum_{i=1}^{n} K\left(\frac{d_\theta(X_i, x)}{h}\right)}, \quad \forall y \in \mathbb{R} \tag{1}$$

With the convention $0/0 = 0$, where H is defined by :

$$\forall u \in \mathbb{R} \quad H(u) = \int_{-\infty}^{u} K_0(v) dv.$$

The function K is a kernel of type I or of type II and the function K_0 is a kernel of type 0 and $h = h(n)$ (resp. $g = g(n)$) is a sequence of positive real

numbers which goes to zero as n tends to infinity. This estimate extend, in different way, the works of Samanta [25]) in the real case and Ferraty et al. [12] in the functional case.
Recall that a function K from $\mathbb{R}$ into $\mathbb{R}^+$ such that $\int K = 1$ is called kernel of type I if there exist two real constants $0 < C_1 < C_2 < \infty$ such that

$$C_1 1_{[0,1]} \leq K \leq C_2 1_{[0,1]}.$$

It is called kernel of type II if its support is $[0, 1]$ and if its derivative $K^{'}$ exists on $[0, 1]$ and satisfies for two real constants $-\infty < C_4 < C_3 < 0$:

$$C_4 \leq K^{'} \leq C_3.$$

A function K_0 from $\mathbb{R}$ into $\mathbb{R}^+$ such that $\int K_0 = 1$ is called kernel of type 0 if its compact support is $[-1, 1]$ and such that $\forall u \in (0, 1)\,,\ K(u) > 0$.

Let $\alpha \in]0, 1[$, the conditional quantile function of Y given $X = x$, denoted by $Q_{\theta,\ \alpha}(x)$, is given by

$$Q_{\theta,\ \alpha}(x) = \inf\{y \in \mathbb{R}, F_\theta(y, x) \geq \alpha\} \tag{2}$$

and we can write That:

$$F_\theta(Q_{\theta,\ \alpha}(x), x) = \alpha$$

The fact that the conditional c.d.f $F_\theta\,(y, x)$ is strictly increasing, insures the existence and unicity of the conditional quantile c.q.f.
The kernel estimate $\widehat{Q}_{\theta,\ \alpha}(x)$, of the conditional quantile $Q_{\theta,\ \alpha}(x)$ is defined by

$$\widehat{F}_\theta(\widehat{Q}_{\theta,\ \alpha}(x), x) = \alpha$$

3 Main results

All along the paper, we will denote by C and $C^{'}$ some strictly positive generic constants.

3.1 Pointwise almost complete convergence

Let x (resp. y) be a fixed element of $\mathcal{H}$ (resp. $\mathbb{R}$), let $\mathcal{N}_x \subset \mathcal{H}$ be a neighborhood of x and $\mathbb{S}$ be a fixed compact subset of $\mathbb{R}$. In order to establish the almost complete (a.co.) convergence of our estimate we will introduce some hypotheses.

(H_1) The probability of the functional variable on a small ball is non null:

$$P\left(d_\theta\left(X,x\right) < h\right) = \varphi_{\theta,x}(h) > 0, \tag{3}$$

(H_2) h is a sequence of positive numbers satisfying

$$\lim_{n\to\infty} \frac{\log n}{n\varphi_{\theta,x}(h)} = 0, \tag{4}$$

(H_3) About the small ball conditional probability $\varphi_{\theta,x}(.)$, we assume that :

$$\exists C > 0,\ \exists \varepsilon_0,\ \forall \varepsilon < \varepsilon_0,\ \int_0^\varepsilon \varphi_{\theta,x}(u)du > C\varepsilon\varphi_{\theta,x}(\varepsilon), \tag{5}$$

(H_4) Now, we suppose that the operator F_θ satisfy the following Hölder-type condition :

$$\begin{cases} \exists C_{\theta,x} > 0 \text{ such that } \forall\left(y_1,y_2\right) \in \mathbb{S}^2,\ \forall\left(x_1,x_2\right) \in \mathcal{N}_x \times \mathcal{N}_x, \\ |F_\theta(y_1,x_1) - F_\theta(y_2,x_2)| \le C_{\theta,x}\left(d_\theta^{\beta_1}\left(x_1,x_2\right) + |y_1-y_2|^{\beta_2}\right),\ \beta_1 > 0,\ \beta_2 > 0. \end{cases} \tag{6}$$

Theorem 1 *Under the hypotheses (H_1)-(H_4), as n goes to infinity, we have*

$$\left|\widehat{Q}_{\theta,\ \alpha}(y,x) - Q_{\theta,\ \alpha}(y,x)\right| = \bigcirc(h^{\beta_1}) + \bigcirc(g^{\beta_2}) + \bigcirc_{a.co.}\left(\sqrt{\frac{\log n}{n\varphi_{\theta,x}(h)}}\right). \tag{7}$$

Recall that a sequence $\left(X_n\right)_{n\in\mathbb{N}^*}$ of a real-valued random variables is said to converge almost completely (a.co.) to a real-valued varaible X if and only if

$$\forall\epsilon > 0,\ \sum_{n\in\mathbb{N}^*} P\left(|X_n - X| > \epsilon\right) < \infty).$$

This mode of convergence implies both almost sure and in probability convergence (see for instance Bosq and Lecoutre, [7])

Proof. Since, we have $\lim\limits_{n\to\infty} g = 0$ and K_0 is a kernel of type 0 the estimated conditional c.d.f. $\widehat{F}_\theta(.,x)$ is continuous and strictly increasing. So, the function $\widehat{F}_\theta^{-1}(.,x)$ exists and is continuous. The continuity property of $\widehat{F}_\theta(.,x)$ at point $\widehat{F}_\theta(Q_{\theta,\ \alpha}(x),x)$ can be written as: $\forall\epsilon > 0,\ \exists\delta\left(\epsilon\right) > 0, \forall y,$

$$\left|\widehat{F}_\theta(y,x) - \widehat{F}_\theta(Q_{\theta,\ \alpha}(x),x)\right| \le \delta\left(\epsilon\right) \Longrightarrow |y - Q_{\theta,\ \alpha}(x)| \le \epsilon.$$

In the special case when $y = \widehat{Q}_\theta(x)$, we have: $\forall\epsilon > 0,\ \exists\delta(\epsilon) > 0$,

$$\left|\widehat{F}_\theta(\widehat{Q}_{\theta,\ \alpha}(x), x) - \widehat{F}_\theta(Q_{\theta,\ \alpha}(x), x)\right| \leq \delta(\epsilon) \Longrightarrow \left|\widehat{Q}_{\theta,\ \alpha}(x) - Q_{\theta,\ \alpha}(x)\right| \leq \epsilon,$$

in such a way that we arrive at: $\forall\epsilon > 0,\ \exists\delta(\epsilon) > 0$,

$$\begin{aligned} P\left(\left|\widehat{Q}_{\theta,\ \alpha}(x) - Q_{\theta,\ \alpha}(x)\right| > \epsilon\right) &\leq P\left(\left|\widehat{F}_\theta(\widehat{Q}_{\theta,\ \alpha}(x), x) - \widehat{F}_\theta(Q_{\theta,\ \alpha}(x), x)\right| > \delta(\epsilon)\right) \\ &\leq P\left(\left|F_\theta(Q_{\theta,\ \alpha}(x), x) - \widehat{F}_\theta(Q_{\theta,\ \alpha}(x), x)\right| > \delta(\epsilon)\right), \end{aligned}$$

the last inequality following from the simple observation that

$$F_\theta(Q_{\theta,\ \alpha}(x), x) = \widehat{F}_\theta(\widehat{Q}_{\theta,\ \alpha}(x), x) = \alpha. \tag{8}$$

The pointwise almost complete convergence of the kernel conditional c.d.f. estimate $\widehat{F}_\theta(., x)$ given by Ait Aidi and Mecheri (2016) (see Theorem 1 [1]),we get the result directly:

$$\forall\epsilon > 0,\ \sum_{n=1}^{\infty} P\left(\left|\widehat{Q}_{\theta,\ \alpha}(x) - Q_{\theta,\ \alpha}(x)\right| > \epsilon\right) < \infty, \tag{9}$$

Finally, we get the result. □

3.2 Pointwise almost complete rate of convergence

The aim of this section we study the rate of convergence of our conditional quantile estimator $Q_{\theta,\ \alpha}(x)$. As it is usual in conditional quantiles estimation, the rate of convergence can be linked with the flatness of the cond-cdf $F(.|x)$ around the conditional quantile $Q_{\theta,\ \alpha}(x)$.However, the behavior of the conditional quantiles estimation depends on the flatness of F_θ around the point $Q_{\theta,\ \alpha}(x)$.
In order to study the rate of convergence of this conditional estimator, we must introduce other hypotheses.

(H_5) $F(.|x)$ is j-times continuously differentiable in some neighbourhood of $Q_{\theta,\ \alpha}(x)$,

(H_6) $F_\theta(y, x)$ is strictly increasing and if we suppose that exists $l \in \{1, ..., j\}$ such that $F_\theta^{(l)}(., x)$ is Lipschitz continuous of order β_0 :

$$\exists C \in (0, \infty),\ \forall\left(y, y'\right) \in \mathbb{R}^2,\ \left|F_\theta^{(l)}(y, x) - F_\theta^{(l)}\left(y', x\right)\right| \leq C\left|y - y'\right|^{\beta_0}, \tag{10}$$

(H_7) $\exists j > 0, \forall l = 1, ..., j-1,$

$$\begin{cases} F_\theta^{(l)}(Q_{\theta,\ \alpha}(x), x) = 0, \\ F_\theta^{(j)}(Q_{\theta,\ \alpha}(x), x) > 0. \end{cases} \tag{11}$$

(H_8) The coumulative kernel H is j-times continuously differentiable.

$$\lim_{n \longrightarrow \infty} \frac{\log n}{n g^{2j-1} \varphi_{\theta,x}(h)} = 0, \tag{12}$$

Theorem 2 *Under the conditions (H_1)-(H_8), we have:*

$$\widehat{Q}_{\theta,\ \alpha}(x) - Q_{\theta,\ \alpha}(x) = \bigcirc\left((h^{\beta_1} + g^{\beta_2})^{\frac{1}{j}}\right) + \bigcirc_{a.co.}\left(\left(\frac{\log n}{\varphi_{\theta,x}(h)}\right)^{\frac{1}{2j}}\right). \tag{13}$$

Proof. Taylor expansion of the function $\widehat{F}_\theta$ leads the existence of some $Q^*_{\theta,\ \alpha}$ between $\widehat{Q}_{\theta,\ \alpha}(x)$ and $Q_{\theta,\ \alpha}(x)$ such that :

$$\widehat{F}_\theta(Q_{\theta,\ \alpha}(x), x) - \widehat{F}_\theta(\widehat{Q}_{\theta,\ \alpha}(x), x) = \sum_{l=1}^{j-1} \frac{\left(Q_{\theta,\ \alpha}(x) - \widehat{Q}_{\theta,\ \alpha}(x)\right)^l}{l!} \widehat{F}_\theta^{(l)}(Q_{\theta,\ \alpha}(x), x)$$

$$+ \frac{\left(Q_{\theta,\ \alpha}(x) - \widehat{Q}_{\theta,\ \alpha}(x)\right)^j}{j!} \widehat{F}_\theta^{(j)}(Q^*_{\theta,\ \alpha}, x).$$

Because of (9), this can be rewritten as :

$$\widehat{F}_\theta(Q_{\theta,\ \alpha}(x), x) - \widehat{F}_\theta(\widehat{Q}_{\theta,\ \alpha}(x), x) = \sum_{l=1}^{j-1} \frac{(Q_{\theta,\ \alpha}(x) - \widehat{Q}_{\theta,\ \alpha}(x))^l}{l!} \times$$

$$\left(\widehat{F}_\theta^{(l)}(Q_\nu(x), x) - F_\theta^{(l)}(Q_{\theta,\ \alpha}(x), x)\right) + \frac{(Q_{\theta,\ \alpha}(x) - \widehat{Q}_{\theta,\ \alpha}(x))^j}{j!} \widehat{F}_\theta^{(j)}(Q^*_{\theta,\ \alpha}, x).$$

Because $\widehat{F}_\theta(\widehat{Q}_{\theta,\ \alpha}(x), x) = F_\theta(Q_{\theta,\ \alpha}(x), x) = \alpha$, we have

$$(Q_{\theta,\ \alpha}(x) - \widehat{Q}_{\theta,\ \alpha}(x))^j \widehat{F}_\theta^{(j)}(Q^*_{\theta,\ \alpha}, x) = \bigcirc\left(\widehat{F}_\theta(Q_{\theta,\ \alpha}(x), x) - F_\theta(Q_{\theta,\ \alpha}(x), x)\right)$$

$$+ \bigcirc\left(\sum_{l=1}^{j-1} (Q_\theta(x) - \widehat{Q}_{\theta,\ \alpha}(x))^l \left(\widehat{F}_\theta^{(l)}(Q_{\theta,\ \alpha}(x), x) - F_\theta^{(l)}(Q_{\theta,\ \alpha}(x), x)\right)\right)$$

By combining the results of following Lemma 1 and Theorem 2, together with the fact that $Q^*_{\theta,\ \alpha}$ is lying between $\widehat{Q}_{\theta,\ \alpha}(x)$ and $Q_{\theta,\ \alpha}(x)$, it follows that

$$\lim_{n\longrightarrow\infty}\widehat{F}^{(j)}_{\theta}(Q^*_{\theta,\ \alpha},x)=F^{(j)}_{\theta}(Q_{\theta,\ \alpha}(x),x),\ \text{a.co.}$$

Lemma 1 (*See Ait Saidi and Mecheri (2016)* [1] *and Ferraty et al. (2005)* [14])

Let be an integer $l\in\{1,...,j\}$. *Under the conditions of theorem 2, we have*

$$\lim_{n\longrightarrow\infty}\widehat{F}^{(l)}_{\theta}(y,x)=F^{(l)}_{\theta}(y,x),\ \text{a.co.} \tag{14}$$

In addition the function $F^{(l)}_{\theta}(.,x)$ *is Lipschitz continuous of order* β_0, *that is if*

$$\exists C\in(0,+\infty),\ \forall\left(y,y'\right)\in\mathbb{R}^2,\ \left|F^{(l)}_{\theta}(y,x)-F^{(l)}_{\theta}(y',x)\right|\leq C\left|y-y'\right|^{\beta_0}, \tag{15}$$

then we have

$$F^{(l)}_{\theta}(y,x)-\widehat{F}^{(l)}_{\theta}(y,x)=\bigcirc(h^{\beta_1})+\bigcirc(g^{\beta_0})+\bigcirc_{a.co.}\left(\sqrt{\frac{\log n}{ng^{2l-1}\varphi_{\theta,x}(h)}}\right). \tag{16}$$

Proof of Lemma 1

The proof is given by Ait Saidi and Mecheri (2016) (See [1]).

Because the second part of assumption (12) is insuring that this limit is not 0, it follows by using proposition A.6-ii Ferraty and Vieu (2006) [15] that :

$$\begin{aligned}(Q_{\theta,\ \alpha}(x)-\widehat{Q}_{\theta,\ \alpha}(x))^j&=\bigcirc_{a.co.}\left(\widehat{F}_{\theta}(Q_{\theta,\ \alpha}(x),x)-F_{\theta}(Q_{\theta,\ \alpha}(x),x)\right)\\&+\bigcirc_{a.co.}\left(\sum_{l=1}^{j-1}(Q_{\theta,\ \alpha}(x)-\widehat{Q}_{\theta,\ \alpha}(x))\right)^{l}\left(\widehat{F}^{(l)}_{\theta}(Q_{\theta,\ \alpha}(x),x)-F^{(l)}_{\theta}(Q_{\theta,\ \alpha}(x),x)\right).\end{aligned}$$

Because (9), for all $l\in\{0,1,...,j\}$ and for all y in a neighborhood of $Q_{\theta,\ \alpha}(x)$, it exists $Q^*_{\theta,\ \alpha}$ between y and $Q_{\theta,\ \alpha}$ such that :

$$F^{(l)}_{\theta}(y,x)-F^{(l)}_{\theta}(Q_{\theta,\ \alpha}(x),x)=\frac{(y-Q_{\theta,\ \alpha}(x))^{j-l}}{(j-l)\,!}F^{(j)}_{\theta}(Q^*_{\theta,\ \alpha},x)$$

wich implies that $F^{(l)}_{\theta}$ is Lipschitz continuous around $Q_{\theta,\ \alpha}(x)$ with order $j-l$. So, by using now Theorem 1 of Ait Saidi and Mecheri (2016) (See [1]) together with the following Lemma with the suitable Lipschitz orders, one get:

$$(Q_{\theta,\ \alpha}(x)-\widehat{Q}_{\theta,\ \alpha}(x))^{j}=\bigcirc(h^{\beta_1})+\bigcirc(g^{\beta_2})+\bigcirc_{a.co.}\left(\sqrt{\frac{\log n}{n\varphi_{\theta,x}(h)}}\right)$$
$$+\bigcirc_{a.co.}\left(\sum_{l=1}^{j-1}A_{n,l}\right)+\bigcirc_{a.co.}\left(\sum_{l=1}^{j-1}B_{n,l}\right), \tag{17}$$

where

$$A_{n,l}=(Q_{\theta,\ \alpha}(x)-\widehat{Q}_{\theta,\ \alpha}(x))^{l}\left(\sqrt{\frac{\log n}{ng^{2l-1}\varphi_{\theta,x}(h)}}\right)$$

and

$$B_{n,l}=(Q_{\theta,\ \alpha}(x)-\widehat{Q}_{\theta,\ \alpha}(x))^{l}g^{j-l}.$$

- Now we suppose that it exists $l\in\{1,...,j-1\}$ such that $(Q_{\theta,\ \alpha}(x)-\widehat{Q}_{\theta,\ \alpha}(x))^{j}=\bigcirc\ (A_{n,l})$, we can write that:

$$\left|Q_{\theta,\ \alpha}(x)-\widehat{Q}_{\theta,\ \alpha}(x)\right|^{j}\leq C\left|Q_{\theta,\ \alpha}(x)-\widehat{Q}_{\theta,\ \alpha}(x)\right|^{l}\left(\sqrt{\frac{\log n}{ng^{2l-1}\varphi_{\theta,x}(h)}}\right),$$

which implies that

$$\left|Q_{\theta,\ \alpha}(x)-\widehat{Q}_{\theta,\ \alpha}(x)\right|^{j-l}\leq C\left(\sqrt{\frac{\log n}{ng^{2l-1}\varphi_{\theta,x}(h)}}\right)$$

and

$$\left|Q_{\theta,\ \alpha}(x)-\widehat{Q}_{\theta,\ \alpha}(x)\right|^{j}\leq C\left(\frac{\log n}{ng^{2l-1}\varphi_{\theta,x}(h)}\right)^{\frac{j}{2(j-l)}}.$$

So, because (10), as soon as it exists l such that $\left(Q_{\theta,\ \alpha}(x)-\widehat{Q}_{\theta,\ \alpha}(x)\right)^{j}=\bigcirc\ (A_{n,l})$, then we have

$$\left(Q_{\theta,\ \alpha}(x)-\widehat{Q}_{\theta,\ \alpha}(x)\right)^{j}=\bigcirc\left(\sqrt{\frac{\log n}{n\varphi_{\theta,x}(h)}}\right). \tag{18}$$

- In the same way, we will deny that if $l\in\{1,...,j-1\}$ such that $(Q_{\theta,\ \alpha}(x)-\widehat{Q}_{\theta,\ \alpha}(x))^{j}=\bigcirc\ (B_{n,l})$, we have :

$$\left|Q_{\theta,\ \alpha}(x)-\widehat{Q}_{\theta,\ \alpha}(x)\right|^{j}\leq C\left|Q_{\theta,\ \alpha}(x)-\widehat{Q}_{\theta,\ \alpha}(x)\right|^{l}g^{j-l},$$

which implies that
$$\left|Q_{\theta,\ \alpha}(x)-\widehat{Q}_{\theta,\ \alpha}(x)\right|^{j}\leq Cg^{j}.$$

So, as soon as it exists l such that $\left(Q_{\theta,\ \alpha}(x)-\widehat{Q}_{\theta,\ \alpha}(x)\right)^{j}=\bigcirc\ (B_{n,l})$, then we have
$$\left(Q_{\theta,\ \alpha}(x)-\widehat{Q}_{\theta,\ \alpha}(x)\right)^{j}=\bigcirc\left(g^{\beta_1}\right). \tag{19}$$

Finally, we get the result
$$\left(Q_{\theta,\ \alpha}-\widehat{Q}_{\theta,\ \alpha}(x)\right)^{j}=\bigcirc(h^{\beta_1})+\bigcirc(g^{\beta_2})+\bigcirc_{a.co.}\left(\sqrt{\frac{\log n}{n\varphi_{\theta,x}(h)}}\right).$$

The proof is finished. □

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Some equalities and inequalities in 2-inner product spaces

Fahimeh Sultanzadeh
Department of Mathematics, Mashhad Branch, Islamic Azad University, Iran
email: fsultanzadeh@gmail.com

Mahmoud Hassani
Department of Mathematics, Mashhad Branch, Islamic Azad University, Iran
email: mhassanimath@gmail.com

Mohsen Erfanian Omidvar
Department of Mathematics, Mashhad Branch, Islamic Azad University, Iran
email: math.erfanian@gmail.com

Rajab Ali kamyabi Gol
Department of Pure Mathematics, Ferdowsi University of Mashhad, Mashhad, Iran
email: kamyabi@um.ac.ir

Abstract. We obtain some new results concerning some equalities and inequalities in a 2-inner product space. These inequalities are a generalization of the Cauchy–Schwarz inequality. Also a reverse of Cauchy–Schwarz's inequality in this space is given.

1 Introduction

In 1964, Gähler [10] introduced the concepts of 2-norm and 2-inner product spaces as a generalization of norm and inner product spaces, respectively, which have been intensively studied by many authors in the last four decades. A presentation of the results related to the theory of 2-inner product spaces can be found in [2]. The Cauchy–Schwarz inequality is one of the many inequalities related to inner product spaces. The theory of such inequalities plays an important role in modern mathematics together with numerous applications

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for the nonlinear analysis, approximation and optimization theory, numerical analysis, probability theory, statistics, and other fields.
The Cauchy–Schwarz inequality has been frequently used for obtaining bounds or estimating the errors in various approximation formulas occurring in the above domains. Thus, any new advantages will have a number of important consequences in the mathematical fields, where inequalities are basic tools. Cho, Matic, and Pecaric [3] proved the Cauchy–Schwarz inequality in 2-inner product spaces. In this paper, we obtain some equalities and inequalities in a 2-inner product space, and then we get the Cauchy–Schwarz inequality. Finally, we state a reverse Cauchy–Schwarz inequality in a 2-inner product space.

2 Notation and preliminary results

In this section, we recall some basic notations, definitions, and some important properties, which will be used. For more detailed information, one can see [2, 3].

2.1 2-inner Product space

Let $\mathcal{X}$ be a linear space of dimension greater than 1 over the field $\mathcal{K}$, where $\mathcal{K}$ is the real or complex numbers field. Suppose that $(\cdot,\cdot|\cdot)$ is a $\mathcal{K}$-valued function defined on $\mathcal{X}\times\mathcal{X}\times\mathcal{X}$ satisfying the following conditions:

(I1) $(x,x|z)\geq 0$ and $(x,x|z)=0$ if and only if x and z are linearly dependent,

(I2) $(x,x|z)=(z,z|x)$,

(I3) $(y,x|z)=\overline{(x,y|z)}$,

(I4) $(\alpha x,y|z)=\alpha(x,y|z)$ for any scalar $\alpha\in\mathcal{K}$,

(I5) $(x+x',y|z)=(x,y|z)+(x',y|z)$,

where $x,x',y,z\in\mathcal{X}$.
Indeed, $(\cdot,\cdot|\cdot)$ is called a 2-inner product on $\mathcal{X}$ and $(\mathcal{X},(\cdot,\cdot|\cdot))$ is called a 2-inner product space (or 2-pre-Hilbert space). Some properties of 2-inner product $(\cdot,\cdot|\cdot)$ can be obtained as follows:

(1) If $\mathcal{K}=\mathcal{R}$, then (I3) reduces to

$$(y,x|z)=(x,y|z).$$

(2) From (I3) and (I4), we have

$$(0, y|z) = 0, \quad (x, 0|z) = 0$$

and also

$$(x, \alpha y|z) = \overline{\alpha}(x, y|z). \tag{1}$$

(3) Using (I3)–(I5), we have

$$(z, z|x \pm y) = (x \pm y, x \pm y|z) = (x, x|z) + (y, y|z) \pm 2\mathrm{Re}(x, y|z)$$

and

$$\mathrm{Re}(x, y|z) = \frac{1}{4}\left[(z, z|x + y) - (z, z|x - y)\right]. \tag{2}$$

In the real case $\mathcal{K} = \mathcal{R}$, equation (2) reduces to

$$(x, y|z) = \frac{1}{4}\left[(z, z|x + y) - (z, z|x - y)\right], \tag{3}$$

and using this formula, it is easy to see that, for any $\alpha \in \mathcal{R}$,

$$(x, y|\alpha z) = \alpha^2(x, y|z). \tag{4}$$

In the complex case, using (1) and (2), we have

$$\mathrm{Im}(x, y|z) = \mathrm{Re}[-i(x, y|z)] = \frac{1}{4}\left[(z, z|x + iy) - (z, z|x - iy)\right],$$

which, in combination with (2), yields

$$(x, y|z) = \frac{1}{4}\left[(z, z|x + y) - (z, z|x - y)\right] + \frac{i}{4}\left[(z, z|x + iy) - (z, z|x - iy)\right]. \tag{5}$$

Using the above formula and (1), for any $\alpha \in \mathcal{C}$, we have

$$(x, y|\alpha z) = |\alpha|^2(x, y|z). \tag{6}$$

Moreover, for $\alpha \in \mathcal{R}$, equation (6) reduces to (4). Also, it follows from (6) that

$$(x, y|0) = 0.$$

(4) For any three given vectors $x, y, z \in \mathcal{X}$, consider the vector $u = (y, y \mid z)x - (x, y \mid z)y$. We know that $(u, u \mid z) \geq 0$ with the equality if and only if u and z are linearly dependent. The inequality $(u, u \mid z) \geq 0$ can be rewritten as follows:

$$(y, y \mid z)\left[(x, x \mid z)(y, y \mid z) - \mid (x, y \mid z) \mid^2\right] \geq 0. \tag{7}$$

If $x = z$, then (7) becomes

$$-(y, y \mid z) \mid (z, y \mid z) \mid^2 \geq 0,$$

which implies that

$$(z, y \mid z) = (y, z \mid z) = 0, \tag{8}$$

provided y and z are linearly independent. Obviously, when y and z are linearly dependent, (8) holds too.

Thus (8) is true for any two vectors $y, z \in \mathcal{X}$. Now, if y and z are linearly independent, then $(y, y \mid z) > 0$, and from (7), it follows the Cauchy–Bunyakovsky–Schwarz inequality (CBS-inequality, for short) for 2-inner products:

$$\mid (x, y \mid z) \mid^2 \leq (x, x \mid z)(y, y \mid z). \tag{9}$$

Using (8), it is easy to check that (9) is trivially fulfilled when y and z are linearly dependent. Therefore, inequality (9) holds for any three vectors $x, y, z \in \mathcal{X}$ and is strict unless the vectors $u = (y, y \mid z)x - (x, y \mid z)y$ and z are linearly dependent. In fact, we have the equality in (9) if and only if the three vectors x, y, and z are linearly dependent.
In any given 2-inner product space $(\mathcal{X}, (\cdot, \cdot|\cdot))$, we can define a function $\|\cdot, \cdot\|$ on $\mathcal{X} \times \mathcal{X}$ by

$$\|x, z\| = (x, x|z)^{\frac{1}{2}} \tag{10}$$

for all $x, z \in \mathcal{X}$.

It is easy to see that, this function satisfies the following conditions:

(N1) $\|x, z\| \geq 0$ and $\|x, z\| = 0$ if and only if x and z are linearly dependent,

(N2) $\|x, z\| = \|z, x\|$,

(N3) $\|\alpha x, z\| = |\alpha|\|x, z\|$ for any scalar $\alpha \in \mathcal{K}$,

(N4) $\|x + x', z\| \leq \|x, z\| + \|x', z\|$.

The function $\|\cdot,\cdot\|$ defined on $\mathcal{X}\times\mathcal{X}$ and satisfying the above conditions is called a 2-norm on $\mathcal{X}$, and $(\mathcal{X},\|\cdot,\cdot\|)$ is called a linear 2-normed space; see [8]. Whenever a 2-inner product space $(\mathcal{X},(\cdot,\cdot|\cdot))$ is given, we consider it as a linear 2-normed space $(\mathcal{X},\|\cdot,\cdot\|)$ with 2-norm defined by (11).
In terms of 2-norms, the (CBS)-inequality (9) can be written as

$$|\,(x,y\mid z)\,|^2\leq \|x,z\|^2\|y,z\|^2. \tag{11}$$

The equality in (11) holds if and only if x,y, and z are linearly dependent.
For recent inequalities, see [1, 7, 9, 11].

3 Main results

We present some equalities and inequalities in a 2-inner product space. The first part is devoted to illustrate these equalities and inequalities that the Cauchy–Schwarz inequality is one of them. Thereafter in the second part, we show a reverse of Cauchy–Schwarz inequality in 2-inner product spaces.

3.1 Some equalities and inequalities in 2-inner product space

Let $\mathcal{X}$ be a 2-inner product space over the field $\mathcal{K}$, where $\mathcal{K}$ is the field of real numbers $\mathcal{R}$ or complex numbers $\mathcal{C}$. The 2-inner product space $(\cdot,\cdot|\cdot)$ induces an associated norm, given by $\|x,z\|=(x,x|z)^{\frac{1}{2}}$, for all $x,z\in\mathcal{X}$, thus $\mathcal{X}$ is a linear 2-normed space. In this section, we establish several new results related to the equalities and inequalities in a 2-inner product space that lead to the Cauchy–Schwarz inequality. Then we deduce some relations.

Theorem 1 *In a 2-Inner product space $\mathcal{X}$ over the field of complex numbers $\mathcal{C}$, we have*

$$\begin{aligned}&\frac{1}{\|y,z\|^2}\left(\alpha y-x,x-\beta y\mid z\right)\\ &\quad=\left[\alpha-\frac{(x,y\mid z)}{\|y,z\|^2}\right]\left[\frac{\overline{(x,y\mid z)}}{\|y,z\|^2}-\overline{\beta}\right]-\frac{1}{\|y,z\|^2}\left\|x-\frac{(x,y\mid z)}{\|y,z\|^2}y,z\right\|^2\end{aligned}$$

for every $\alpha,\beta\in\mathcal{C}$ and for all $x,y,z\in\mathcal{X}$, where y and z are linearly independent.

Proof. The proof is obtained from the following:

$$\frac{1}{\|y,z\|^2}\left(\alpha y-x,x-\beta y\mid z\right)$$

$$
\begin{aligned}
&= \frac{1}{\|y,z\|^2}\left[(\alpha y, x \mid z) - (\alpha y, \beta y \mid z) - (x, x \mid z) + \overline{\beta}\,(x, y \mid z)\right] \\
&= \alpha \frac{\overline{(x,y \mid z)}}{\|y,z\|^2} - \alpha\overline{\beta} - \frac{\|x,z\|^2}{\|y,z\|^2} + \overline{\beta}\frac{(x,y \mid z)}{\|y,z\|^2} \\
&= \left[\alpha - \frac{(x,y \mid z)}{\|y,z\|^2}\right]\left[\frac{\overline{(x,y \mid z)}}{\|y,z\|^2} - \overline{\beta}\right] - \frac{1}{\|y,z\|^2}\left[\|x,z\|^2 - \frac{|(x,y \mid z)|^2}{\|y,z\|^2}\right] \\
&= \left[\alpha - \frac{(x,y \mid z)}{\|y,z\|^2}\right]\left[\frac{\overline{(x,y \mid z)}}{\|y,z\|^2} - \overline{\beta}\right] - \frac{1}{\|y,z\|^2}\left\|x - \frac{(x,y \mid z)}{\|y,z\|^2}y, z\right\|^2.
\end{aligned}
$$

□

Corollary 1 *In a 2-Inner product space $\mathcal{X}$ over the field of real numbers $\mathcal{R}$, we have*

$$
\begin{aligned}
\frac{1}{\|y,z\|^2}(\alpha y - x, x - \beta y \mid z) = &\left[\alpha - \frac{(x,y \mid z)}{\|y,z\|^2}\right]\left[\frac{(x,y \mid z)}{\|y,z\|^2} - \beta\right] \\
&- \frac{1}{\|y,z\|^2}\left\|x - \frac{(x,y \mid z)}{\|y,z\|^2}y, z\right\|^2
\end{aligned}
$$

for every $\alpha, \beta \in \mathcal{R}$ and for all $x, y, z \in \mathcal{X}$, where y and z are linearly independent.

Corollary 2 *In a 2-Inner product space X over the field of complex numbers $\mathcal{C}$, we have*

$$
\|x - \alpha y, z\|^2 = \left|\alpha\|y,z\| - \frac{(x,y \mid z)}{\|y,z\|}\right|^2 + \left\|x - \frac{(x,y \mid z)}{\|y,z\|^2}y, z\right\|^2 \tag{12}
$$

for every $\alpha \in \mathcal{C}$ and for all $x, y, z \in \mathcal{X}$, where y and z are linearly independent.

Proof. Putting $\alpha = \beta$ in Theorem 1, we get

$$
\begin{aligned}
\frac{1}{\|y,z\|^2}(\alpha y - x, x - \alpha y \mid z) = &-\frac{1}{\|y,z\|^2}\|x - \alpha y, z\|^2 \\
= &\left[\alpha - \frac{(x,y \mid z)}{\|y,z\|^2}\right]\left[\frac{\overline{(x,y \mid z)}}{\|y,z\|^2} - \overline{\alpha}\right] \\
&- \frac{1}{\|y,z\|^2}\left\|x - \frac{(x,y \mid z)}{\|y,z\|^2}y, z\right\|^2.
\end{aligned}
$$

Now multiplying this equality by $-\|y,z\|^2$ we get the desired result. □

Corollary 3 *In a 2-Inner product space $\mathcal{X}$ over the field of complex numbers $\mathcal{C}$, we have*

$$\|x-\alpha y, z\| \geq \left\| x - \frac{(x, y \mid z)}{\|y, z\|^2} y, z \right\| \tag{13}$$

and

$$\|x-\alpha y, z\| \geq \left| \alpha \|y, z\| - \frac{(x, y \mid z)}{\|y, z\|} \right|, \tag{14}$$

for every $\alpha \in \mathcal{C}$ and for all $x, y, z \in \mathcal{X}$, where y and z are linearly independent.

Proof. In the proof of (12), we see that

$$\frac{1}{\|y, z\|^2} \left\| x - \frac{(x, y \mid z)}{\|y, z\|^2} y, z \right\|^2 \geq 0.$$

Hence, (13) and (14) are obtained. □

Remark 1 *If y and z are linearly independent, then from relation (14), for $\alpha = 0$, we obtain the following inequality of Cauchy–Schwartz:*

$$|(x, y \mid z)| \leq \|x, z\| \ \|y, z\|.$$

Now we are ready to state the following result.

Corollary 4 *In a 2-inner product space $\mathcal{X}$ over the field of complex numbers $\mathcal{C}$, we have*

$$\|x, z\|^2 \|y, z\|^2 = |(x, y \mid z)|^2 + \left\| \ \|y, z\| x - \frac{(x, y \mid z)}{\|y, z\|} y, z \right\|^2, \tag{15}$$

$$\begin{aligned} &\|x, z\|^2 \|y, z\|^2 \left\| \frac{x}{\|x, z\|} - \frac{y}{\|y, z\|}, z \right\|^2 \\ &\quad = |\|x, z\| \ \|y, z\| - (x, y \mid z)|^2 + \|x, z\|^2 \|y, z\|^2 - |(x, y \mid z)|^2 \end{aligned} \tag{16}$$

for all $x, y, z \in \mathcal{X}$, where y and z are linearly independent and x and z are linearly independent, too.

$$\operatorname{Re}(x, y \mid z) = \frac{1}{2} \left(\|x, z\|^2 + \|y, z\|^2 - \|x - y, z\|^2 \right), \tag{17}$$

$$\operatorname{Im}(x, y \mid z) = \frac{1}{2} \left(\|x, z\|^2 + \|y, z\|^2 - \|x - iy, z\|^2 \right), \tag{18}$$

for all $x, y, z \in \mathcal{X}$.

Proof. From (12), for $\alpha = 0$, we have

$$\|x,z\|^2 = \frac{|(x,y \mid z)|^2}{\|y,z\|^2} + \left\| x - \frac{(x,y \mid z)}{\|y,z\|^2} y, z \right\|^2 .$$

Therefore

$$\|x,z\|^2 \|y,z\|^2 = |(x,y \mid z)|^2 + \left\| \ \|y,z\| x - \frac{(x,y \mid z)}{\|y,z\|} y, z \right\|^2 ,$$

and (15) is obtained.

Also, for $\alpha = \dfrac{\|x,z\|}{\|y,z\|}$ in (12), it follows that

$$\begin{aligned} &\|x\|y,z\| - \|x,z\|y,z\|^2 \\ &= | \ \|x,z\| \ \|y,z\| - (x,y \mid z)|^2 + \left\| \ \|y,z\| x - \frac{(x,y \mid z)}{\|y,z\|} y, z \right\|^2 , \end{aligned}$$

which is equivalent to

$$\begin{aligned} &\|x,z\|^2 \|y,z\|^2 \| \frac{x}{\|x,z\|} - \frac{y}{\|y,z\|}, z\|^2 \\ &= | \ \|x,z\| \ \|y,z\| - (x,y \mid z)|^2 + \left\| \ \|y,z\| x - \frac{(x,y \mid z)}{\|y,z\|} y, z \right\|^2 . \end{aligned}$$

Indeed, by (15), we have $\|x,z\|^2 \|y,z\|^2 - |(x,y \mid z)|^2 = \left\| \ \|y,z\| x - \dfrac{(x,y \mid z)}{\|y,z\|} y, z \right\|^2$, therefore (16) holds.

For $\alpha = 1$, (12) implies that

$$\|x - y, z\|^2 = \left| \ \|y,z\| - \frac{(x,y \mid z)}{\|y,z\|} \right|^2 + \left\| x - \frac{(x,y \mid z)}{\|y,z\|^2} y, z \right\|^2 . \tag{19}$$

On the other hand, we have

$$\left| \ \|y,z\| - \frac{(x,y \mid z)}{\|y,z\|} \right|^2 = \left(\|y,z\| - \frac{\mathrm{Re}(x,y \mid z)}{\|y,z\|} \right)^2 + \frac{\mathrm{Im}^2(x,y \mid z)}{\|y,z\|^2} . \tag{20}$$

Indeed

$$\left| \ \|y,z\| - \frac{(x,y \mid z)}{\|y,z\|} \right|^2 = \left(\|y,z\| - \frac{(x,y \mid z)}{\|y,z\|} \right) \left(\|y,z\| - \frac{\overline{(x,y \mid z)}}{\|y,z\|} \right)$$

$$
\begin{aligned}
&= \|y,z\|^2 - (y,x \mid z) - (x,y \mid z) + \frac{|(x,y \mid z)|^2}{\|y,z\|^2} \\
&= \|y,z\|^2 - 2\mathrm{Re}(x,y \mid z) + \frac{|(x,y \mid z)|^2}{\|y,z\|^2} \\
&= \|y,z\|^2 - 2\mathrm{Re}(x,y \mid z) + \frac{\mathrm{Re}^2(x,y \mid z)}{\|y,z\|^2} + \frac{\mathrm{Im}^2(x,y \mid z)}{\|y,z\|^2} \\
&= \left(\|y,z\| - \frac{\mathrm{Re}(x,y \mid z)}{\|y,z\|}\right)^2 + \frac{\mathrm{Im}^2(x,y \mid z)}{\|y,z\|^2}.
\end{aligned}
$$

Also,

$$
\left\|x - \frac{(x,y \mid z)}{\|y,z\|^2}y, z\right\|^2 = \|x,z\|^2 - \frac{|(x,y \mid z)|^2}{\|y,z\|^2}, \tag{21}
$$

since

$$
\begin{aligned}
\left\|x - \frac{(x,y \mid z)}{\|y,z\|^2}y, z\right\|^2 &= \left(x - \frac{(x,y \mid z)}{\|y,z\|^2}y, x - \frac{(x,y \mid z)}{\|y,z\|^2}y \mid z\right) \\
&= (x,x \mid z) - \left(x, \frac{(x,y \mid z)}{\|y,z\|^2}y \mid z\right) - \left(\frac{(x,y \mid z)}{\|y,z\|^2}y, x \mid z\right) \\
&\quad + \left(\frac{(x,y \mid z)}{\|y,z\|^2}y, \frac{(x,y \mid z)}{\|y,z\|^2}y \mid z\right) \\
&= \|x,z\|^2 - \frac{\overline{(x,y \mid z)}}{\|y,z\|^2}(x,y \mid z) - \frac{(x,y \mid z)}{\|y,z\|^2}\overline{(x,y \mid z)} \\
&\quad + \frac{(x,y \mid z)}{\|y,z\|^2}\frac{\overline{(x,y \mid z)}}{\|y,z\|^2}(y,y) \\
&= \|x,z\|^2 - \frac{|(x,y \mid z)|^2}{\|y,z\|^2}.
\end{aligned}
$$

Now by substitutions (20) and (21) in (19), it follows that

$$
\begin{aligned}
\|x-y,z\|^2 &= \left|\ \|y,z\| - \frac{(x,y \mid z)}{\|y,z\|}\right|^2 + \left\|x - \frac{(x,y \mid z)}{\|y,z\|^2}y, z\right\|^2 \\
&= \left(\|y,z\| - \frac{\mathrm{Re}(x,y \mid z)}{\|y,z\|}\right)^2 + \frac{\mathrm{Im}^2(x,y \mid z)}{\|y,z\|^2} + \|x,z\|^2 - \frac{|(x,y \mid z)|^2}{\|y,z\|^2} \\
&= \|y,z\|^2 + \frac{\mathrm{Re}^2(x,y \mid z)}{\|y,z\|^2} - 2\mathrm{Re}(x,y \mid z) \\
&\quad + \frac{\mathrm{Im}^2(x,y \mid z)}{\|y,z\|^2} + \|x,z\|^2 - \frac{|(x,y \mid z)|^2}{\|y,z\|^2}
\end{aligned}
$$

$$= \|x, z\|^2 + \|y, z\|^2 - 2\mathrm{Re}(x, y \mid z).$$

Hence, $\mathrm{Re}(x, y \mid z) = \frac{1}{2}\left(\|x, z\|^2 + \|y, z\|^2 - \|x - y, z\|^2\right)$ and (17) is proved. Using (12) for $\alpha = i$, we obtain

$$\|x - iy, z\|^2 = \left| i\|y, z\| - \frac{(x, y \mid z)}{\|y, z\|} \right|^2 + \left\| x - \frac{(x, y \mid z)}{\|y, z\|^2} y, z \right\|^2. \tag{22}$$

On the other hand, we have

$$\left| i\|y, z\| - \frac{(x, y \mid z)}{\|y, z\|} \right|^2 = \frac{\mathrm{Re}^2(x, y \mid z)}{\|y, z\|^2} + \left(\|y, z\| - \frac{\mathrm{Im}(x, y \mid z)}{\|y, z\|} \right)^2, \tag{23}$$

since

$$\begin{aligned}
\left| i\|y, z\| - \frac{(x, y \mid z)}{\|y, z\|} \right|^2 &= \left(i\|y, z\| - \frac{(x, y \mid z)}{\|y, z\|} \right) \left(i\|y, z\| - \frac{\overline{(x, y \mid z)}}{\|y, z\|} \right) \\
&= \|y, z\|^2 - i(y, x \mid z) - i(x, y \mid z) + \frac{|(x, y \mid z)|^2}{\|y, z\|^2} \\
&= \|y, z\|^2 - 2\mathrm{Im}(x, y \mid z) + \frac{|(x, y \mid z)|^2}{\|y, z\|^2} \\
&= \|y, z\|^2 - 2\mathrm{Im}(x, y \mid z) + \frac{\mathrm{Re}^2(x, y \mid z)}{\|y, z\|^2} + \frac{\mathrm{Im}^2(x, y \mid z)}{\|y, z\|^2} \\
&= \frac{\mathrm{Re}^2(x, y \mid z)}{\|y, z\|^2} + \left(\|y, z\| - \frac{\mathrm{Im}(x, y \mid z)}{\|y, z\|} \right)^2.
\end{aligned}$$

From (21), we have $\|x - \frac{(x, y \mid z)}{\|y, z\|^2} y, z\|^2 = \|x, z\|^2 - \frac{|(x, y \mid z)|^2}{\|y, z\|^2}$, so we put (23) and (21) in (22) and it follows that

$$\begin{aligned}
\|x - iy, z\|^2 &= \left| i\|y, z\| - \frac{(x, y \mid z)}{\|y, z\|} \right|^2 + \left\| x - \frac{(x, y \mid z)}{\|y, z\|^2} y, z \right\|^2 \\
&= \frac{\mathrm{Re}^2(x, y \mid z)}{\|y, z\|^2} + \left(\|y, z\| - \frac{\mathrm{Im}(x, y \mid z)}{\|y, z\|} \right)^2 + \|x, z\|^2 - \frac{|(x, y \mid z)|^2}{\|y, z\|^2} \\
&= \frac{\mathrm{Re}^2(x, y \mid z)}{\|y, z\|^2} + \|y, z\|^2 + \frac{\mathrm{Im}^2(x, y \mid z)}{\|y, z\|^2} \\
&\quad - 2\mathrm{Im}(x, y \mid z) + \|x, z\|^2 - \frac{|(x, y \mid z)|^2}{\|y, z\|^2}
\end{aligned}$$

$$= \|x,z\|^2 + \|y,z\|^2 - 2\mathrm{Im}(x,y \mid z).$$

Therefore (18) is obtained. $\square$

3.2 A new reverse of the Cauchy–Schwarz inequality in 2-inner product spaces

Reverses of Cauchy–Schwarz inequality in 2-inner product spaces, usually establish upper bounds for one of the following nonnegative quantities:

$$\|x,z\|\|y,z\| - \mid (x,y \mid z) \mid, \quad \|x,z\|^2\|y,z\|^2 - \mid (x,y \mid z) \mid^2,$$
$$\frac{\|x,z\|\|y,z\|}{\mid (x,y \mid z) \mid}, \quad \frac{\|x,z\|^2\|y,z\|^2}{\mid (x,y \mid z) \mid^2}.$$

Classical examples of such inequalities can be found in [4, 5, 6, 12]. Otachel [13] obtained several reverses of the Cauchy–Schwarz inequality in inner product space. In this section, we present a reverse of the Cauchy–Schwarz inequality in 2-inner product space.
First we fix some notations. Suppose that $(\mathcal{X}, (\cdot,\cdot \mid \cdot))$ is a 2-inner product space over the field $\mathcal{K}$, where $\mathcal{K}$ is the real or complex numbers field, and that $z \in \mathcal{X}$. For given $A, a, B, b \in \mathcal{X}$, we define $M = \|A-a,z\| + \dfrac{\|A+a,z\|}{\|B+b,z\|}\min\{\|B+b,z\|, \|B-b,z\|\}$ if B+b and z are linearly independent.
Now we prove the following Lemma.

Lemma 1 *Let* $A, a \in \mathcal{X}$*. Then*

$$\mathrm{Re}(A-x, x-a \mid z) \geq 0 \quad \textit{if and only if} \quad \|x - \frac{A+a}{2}, z\| \leq \frac{1}{2}\|A-a,z\| \tag{24}$$

for $x \in \mathcal{X}$*.*

Proof. Suppose that $\mathrm{Re}(A-x, x-a \mid z) \geq 0$. Using (2), we have $\frac{1}{4}[(z,z \mid A-a) - (z,z \mid A-2x+a)] \geq 0$, that is, $(z,z \mid A-2x+a) \leq (z,z \mid A-a)$. Therefore, $\|A-2x+a,z\| \leq \|A-a,z\|$ or $\left\|-2(x-\frac{A+a}{2}), z\right\| \leq \|A-a,z\|$. It follows that $\left\|x - \frac{A+a}{2}, z\right\| \leq \frac{1}{2}\|A-a,z\|$. $\square$

Theorem 2 *Let* $A, a, B, b \in \mathcal{X}$ *with* $A+a, B+b \in \mathrm{span}\{v\}$ *for a certain* $0 \neq v \in \mathcal{X}$*.*
If

$$\mathrm{Re}(A-x, x-a \mid z) \geq 0 \quad \text{and} \quad \mathrm{Re}(B-y, y-b \mid z) \geq 0 \tag{25}$$

for x,y,$z \in \mathcal{X}$, *then the following inequalities hold:*

$$0 \leq \|x,z\|^2\|y,z\|^2 - \mid (x,y \mid z) \mid^2$$
$$\leq \begin{cases} \frac{1}{4}M^2 \min\{\|x,z\|^2, \|y,z\|^2\}, & B+b \textit{ and } z \textit{ are linearly independent,} \\ \frac{1}{4}\min\{\|A-a,z\|^2\|y,z\|^2, \|B-b,z\|^2\|x,z\|^2\}, & A+a \textit{ and } z \textit{ also } B+b \textit{ and } z \textit{ are linearly dependent.} \end{cases}$$

where M *is a real number, which does not depend from* x *and* y.

Proof. If x and z, or y and z are linearly dependent, the inequalities hold. Let y and z be linearly independent. For any $c \in \mathcal{K}$, we have

$$\left\|x - \frac{(x,y \mid z)}{\|y,z\|^2}y, z\right\| \leq \|x - cy, z\| = \left\|x - \frac{1}{2}(A+a) + \frac{1}{2}(A+a) - cy, z\right\|$$
$$\leq \left\|x - \frac{1}{2}(A+a), z\right\| + \left\|\frac{1}{2}(A+a) - cy, z\right\|.$$

Since $A+a, B+b \in \operatorname{span}\{v\}$ with $v \neq 0$, $B+b$ and z are linearly independent. Put
$A+a = \varepsilon\left(\frac{\|A+a,z\|}{\|B+b,z\|}\right)(B+b)$, for a certain $\varepsilon \in \mathcal{K}$ with $\mid \varepsilon \mid = 1$.
Hence, letting $c = \frac{(A+a,y \mid z)}{2\|y,z\|^2}$, we obtain

$$\left\|x - \frac{(x,y \mid z)}{\|y,z\|^2}y, z\right\| \leq \left\|x - \frac{1}{2}(A+a), z\right\| + \left\|\frac{1}{2}(A+a) - \frac{(A+a,y \mid z)}{2\|y,z\|^2}y, z\right\|$$
$$= \left\|x - \frac{1}{2}(A+a), z\right\| + \frac{\|A+a,z\|}{\|B+b,z\|}\left\|\frac{1}{2}(B+b) - \frac{(B+b,y \mid z)}{2\|y,z\|^2}y, z\right\|.$$

Obviously, $\left\|\frac{1}{2}(B+b) - \frac{(B+b,y \mid z)}{2\|y,z\|^2}y, z\right\| \leq \left\|\frac{1}{2}(B+b) - \tilde{c}y, z\right\|$ for any $\tilde{c} \in \mathcal{K}$.

Next, substituting consecutively $\tilde{c} = 0$ and $\tilde{c} = 1$, we obtain

$$\left\|\frac{1}{2}(B+b) - \frac{(B+b,y \mid z)}{2\|y,z\|^2}y, z\right\| \leq \min\left\{\frac{1}{2}\|B+b,z\|, \left\|\frac{1}{2}(B+b) - y, z\right\|\right\}.$$

Consequently, by using (24) and (25), we have

$$\left\|x - \frac{(x,y \mid z)}{\|y,z\|^2}y, z\right\| \leq \left\|x - \frac{1}{2}(A+a), z\right\|$$

$$+\frac{\|A+a,z\|}{\|B+b,z\|}\min\left\{\frac{1}{2}\|B+b,z\|,\left\|\frac{1}{2}(B+b)-y,z\right\|\right\}$$
$$\leq\frac{1}{2}\left(\|A-a,z\|+\frac{\|A+a,z\|}{\|B+b,z\|}\min\{\|B+b,z\|,\|B-b,z\|\}\right).$$

Finally,

$$\|x,z\|^2-\frac{|\,(x,y\mid z)\,|^2}{\|y,z\|^2}=\left\|x-\frac{(x,y\mid z)}{\|y,z\|^2}y,z\right\|^2$$
$$\leq\frac{1}{4}\left(\|A-a,z\|+\frac{\|A+a,z\|}{\|B+b,z\|}\min\{\|B+b,z\|,\|B-b,z\|\}\right)^2.$$

Multiplying the both of sides by $\|y,z\|^2>0$, we have

$$\|x,z\|^2\|y,z\|^2-|\,(x,y\mid z)\,|^2\leq\frac{1}{4}M^2\|y,z\|^2.$$

Similarly, we obtain

$$\|x,z\|^2\|y,z\|^2-|\,(x,y\mid z)\,|^2\leq\frac{1}{4}M^2\|x,z\|^2.$$

Hence

$$\|x,z\|^2\|y,z\|^2-|\,(x,y\mid z)\,|^2\leq\frac{1}{4}M^2\min\{\|x,z\|^2,\|y,z\|^2\}.$$

If $A+a$ and z and also $B+b$ and z are linearly dependent, then the right side inequality (24) takes the form $\|x,z\|\leq\frac{1}{2}\|A-a,z\|$ and $\|y,z\|\leq\frac{1}{2}\|B-b,z\|$. Hence, we have

$$\|x,z\|^2\|y,z\|^2-|\,(x,y\mid z)\,|^2\leq\|x,z\|^2\|y,z\|^2\leq\begin{cases}\frac{1}{4}\|A-a,z\|^2\|y,z\|^2,\\ \frac{1}{4}\|B-b,z\|^2\|x,z\|^2.\end{cases}$$

Therefore,

$$\|x,z\|^2\|y,z\|^2-|\,(x,y\mid z)\,|^2\leq\frac{1}{4}\min\{\|A-a,z\|^2\|y,z\|^2,\|B-b,z\|^2\|x,z\|^2\}.$$

□

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