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NEW GENERALIZED FOURIER TRANSFORMS AND THEIR APPLICATIONS TO ORDINARY, PARTIAL AND FRACTIONAL DIFFERENTIAL EQUATIONS

ENES ATA AND İ. ONUR KIYMAZ

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Abstract. This article presents new generalized definitions of Fourier, Fourier sine, Fourier cosine, inverse Fourier, inverse Fourier sine and inverse Fourier cosine transforms, which encompass various studies on the generalized Fourier transforms in the existing literature. We also give some fundamental properties such that linearity, shifting, differentiability and convolution. Moreover, the solutions to the ordinary electric current differential equation and the fractional motion differential equation are obtained through the use of the generalized Fourier and inverse Fourier transforms. Subsequently, the solutions to the partial diffusion differential equation are obtained through the use of the generalized Fourier sine, inverse Fourier sine, Fourier cosine, and inverse Fourier cosine transforms. Furthermore, we illustrate the relations of the new generalized Fourier transforms with other the generalized Fourier transforms available in the literature. Finally, we provide tables of the new generalized Fourier transforms, and then graphs of the approximate behaviours of the solution of the ordinary electric current differential equation.

2010 *Mathematics Subject Classification:* 42A38; 26A33; 34A12; 35E05; 34A08

Keywords: Fourier transform, fractional derivatives and integrals, ordinary differential equations, partial differential equations, fractional differential equations

1. INTRODUCTION

The Fourier transform is an integral transformation that is popular in many fields such as medicine, astronomy, physics, chemistry, biology and engineering sciences. In the most general sense, the Fourier and inverse Fourier transforms in [12] for the function u belonging to the Schwartz space, respectively, are defined by

$$\mathfrak{F}[u(t)](w) = \hat{u}(w) = \int_{\mathbb{R}} \exp(iwt)u(t)\mathrm{d}t, \quad (w \in \mathbb{R})$$

and

$$\mathfrak{F}^{-1}[\hat{u}(w)](t) = \frac{1}{2\pi}\int_{\mathbb{R}} \exp(-iwt)\hat{u}(w)\mathrm{d}w, \quad (t \in \mathbb{R}).$$

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The Fourier transform has become a powerful technique due to its multi-purpose use in various branches of science. For example, each person's voice can be expressed as a sum of sine and cosine. Since the frequency spectrum of each sound is different, the sum of the sines and cosines will have a different frequency. Thus, with the Fourier transform, we can find out who the sound belongs to. Similarly, the Fourier transform can be used in phenomena such as light waves, tides in the ocean, solar motion, electric current and thermal conductivity. Again, the Fourier transform can also be used for solutions of ordinary, partial and fractional differential equations.

For studies on the generalization of the Fourier transform in the literature (see for example [4, 5, 8–12, 14] and reference therein).

The objective and importance of this article is to introduce new generalized Fourier, inverse Fourier, Fourier sine, inverse Fourier sine, Fourier cosine and inverse Fourier cosine transforms, covering the existing literature on various generalized Fourier transform studies. Subsequently, solutions of miscellaneous ordinary, partial and fractional differential equations are obtained with the help of the newly defined generalized Fourier transforms, thereby extending the application areas of integral transforms.

We set the rest of this article as follows: Section 2 presents the fundamental concepts that are essential to the article. In Section 3, the new generalized Fourier, Fourier sine, and Fourier cosine transforms with and their inverse transforms are defined, and subsequently some of their properties are presented. In Section 4, solutions to the ordinary electric current differential equation, fractional motion differential equation, and partial diffusion differential equation are obtained through the use of the new generalized Fourier transforms. Section 5 elucidates the relationships between the generalized Fourier transforms that have been previously identified in the literature and the new generalized Fourier transforms that are introduced in this article. Additionally, at the end of the article, tables of the new generalized Fourier transforms and approximate graphs of the solutions to the ordinary electric current differential equation are presented.

2. PRELIMINARIES

This section presents the fundamental concepts that is required throughout the article. This includes the following: the Schwartz space, Lizorkin space, convolution, Dirac delta function and fractional integral and derivative operators.

The Schwartz space [15] is defined by

$$S(\mathbb{R}) = \left\{ u \in C^{\infty} : \ \sup \left| t^m \frac{d^n}{dt^n} u(t) \right| < \infty, \quad \forall m,n \in \mathbb{N}_0 \right\}.$$

The $V(\mathbb{R})$ set be as follows:

$$V(\mathbb{R}) = \left\{ v \in S(\mathbb{R}) : \ v^{(n)}(0) = 0, \quad n \in \mathbb{N}_0 \right\}.$$

Then the Lizorkin space [13] is given by

$$\phi(\mathbb{R}) = \{\varphi \in S(\mathbb{R}) : \mathfrak{F}[\varphi] \in V(\mathbb{R})\}.$$

The convolution [2] of functions u and v for $t \in \mathbb{R}$ is defined by

$$u(t) * v(t) = \int_{\mathbb{R}} u(t-\tau)v(\tau)\mathrm{d}\tau$$

or

$$v(t) * u(t) = \int_{\mathbb{R}} v(t-\tau)u(\tau)\mathrm{d}\tau.$$

The Dirac delta function [3] is given by

$$\delta(t) = \begin{cases} 0, & \text{for } t \neq 0, \\ \infty, & \text{for } t = 0. \end{cases}$$

Also the following equation [3] holds true:

$$\int_{\mathbb{R}} u(t)\delta(t)\mathrm{d}t = u(0). \tag{2.1}$$

The Riemann-Liouville fractional integral (RLFI) [7] of order β is defined by

$${}_{-\infty}I_t^{\beta}u(t) = \frac{1}{\Gamma(\beta)}\int_{-\infty}^{t}(t-\tau)^{\beta-1}u(\tau)\mathrm{d}\tau, \quad (t \in \mathbb{R};\ \Re(\beta) > 0).$$

Let $m-1 < \Re(\beta) < m$ for $m \in \mathbb{N}$. The Riemann-Liouville fractional derivative (RLFD) [7] of order β is given by

$${}_{-\infty}D_t^{\beta}u(t) = \frac{1}{\Gamma(m-\beta)}\frac{d^m}{dt^m}\int_{-\infty}^{t}(t-\tau)^{m-\beta-1}u(\tau)\mathrm{d}\tau, \quad (t \in \mathbb{R};\ \Re(\beta) > 0).$$

Let $m-1 < \Re(\beta) < m$ for $m \in \mathbb{N}$. The Caputo fractional derivative (CFD) [7] of order β is defined by

$${}_{-\infty}^{c}D_t^{\beta}u(t) = \frac{1}{\Gamma(m-\beta)}\int_{-\infty}^{t}(t-\tau)^{m-\beta-1}u^{(m)}(\tau)\mathrm{d}\tau, \quad (t \in \mathbb{R};\ \Re(\beta) > 0).$$

3. New generalized Fourier transforms and their properties

This section introduces new generalisations of the Fourier, inverse Fourier, Fourier sine, inverse Fourier sine, Fourier cosine and inverse Fourier cosine transforms, and presents some of their fundamental properties.

Definition 1. Let $u \in \phi(\mathbb{R})$, $p : \mathbb{R} \to \mathbb{R} - \{0\}$, $q : \mathbb{R} \to \mathbb{R}^+$ and $w, t \in \mathbb{R}$. Then, the generalized Fourier and inverse Fourier transforms, respectively, are defined as:

$${}_p\mathfrak{F}_q[u(t)](w) = {}_p\hat{u}_q(w) = p(\alpha)\int_{\mathbb{R}} \exp\left(iw^{q(\alpha)}t\right)u(t)\mathrm{d}t \tag{3.1}$$

and

$$ {}_p\mathfrak{F}_q^{-1}\left[{}_p\hat{u}_q(w)\right](t) = \frac{q(\alpha)}{p(\alpha)}\frac{1}{2\pi}\int_{\mathbb{R}} \exp\left(-iw^{q(\alpha)}t\right) {}_p\hat{u}_q(w)w^{q(\alpha)-1}\mathrm{d}w, \tag{3.2} $$

where

$$ \exp\left(iw^{q(\alpha)}t\right) := \begin{cases} \exp\left(-i|w|^{q(\alpha)}t\right), & \text{for } w \le 0, \\ \exp\left(i|w|^{q(\alpha)}t\right), & \text{for } w \ge 0. \end{cases} $$

Definition 2. Let $u \in \phi(\mathbb{R})$, $p\colon \mathbb{R} \to \mathbb{R}-\{0\}$, $q\colon \mathbb{R} \to \mathbb{R}^+$ and $w,t \in \mathbb{R}_0^+$. Then, the generalized Fourier sine and inverse Fourier sine transforms, respectively, are defined as:

$$ {}_p^s\mathfrak{F}_q[u(t)](w) = {}_p^s\hat{u}_q(w) = p(\alpha)\int_{\mathbb{R}_0^+} \sin\left(w^{q(\alpha)}t\right) u(t)\mathrm{d}t \tag{3.3} $$

and

$$ {}_p^s\mathfrak{F}_q^{-1}\left[{}_p^s\hat{u}_q(w)\right](t) = \frac{q(\alpha)}{p(\alpha)}\frac{2}{\pi}\int_{\mathbb{R}_0^+} \sin\left(w^{q(\alpha)}t\right) {}_p^s\hat{u}_q(w)w^{q(\alpha)-1}\mathrm{d}w. \tag{3.4} $$

Definition 3. Let $u \in \phi(\mathbb{R})$, $p\colon \mathbb{R} \to \mathbb{R}-\{0\}$, $q\colon \mathbb{R} \to \mathbb{R}^+$ and $w,t \in \mathbb{R}_0^+$. Then, the generalized Fourier cosine and inverse Fourier cosine transforms, respectively, are defined as:

$$ {}_p^c\mathfrak{F}_q[u(t)](w) = {}_p^c\hat{u}_q(w) = p(\alpha)\int_{\mathbb{R}_0^+} \cos\left(w^{q(\alpha)}t\right) u(t)\mathrm{d}t \tag{3.5} $$

and

$$ {}_p^c\mathfrak{F}_q^{-1}\left[{}_p^c\hat{u}_q(w)\right](t) = \frac{q(\alpha)}{p(\alpha)}\frac{2}{\pi}\int_{\mathbb{R}_0^+} \cos\left(w^{q(\alpha)}t\right) {}_p^c\hat{u}_q(w)w^{q(\alpha)-1}\mathrm{d}w. \tag{3.6} $$

Remark 1. If we take $p(\alpha) = q(\alpha) = 1$ in the equations (3.1), (3.2), (3.3), (3.4), (3.5) and (3.6), we obtain classic Fourier, inverse Fourier, Fourier sine, inverse Fourier sine, Fourier cosine and inverse Fourier cosine transforms in [3], respectively.

It should also be noted that throughout this article the generalized Fourier, inverse Fourier, Fourier sine, inverse Fourier sine, Fourier cosine and inverse Fourier cosine transforms will be denoted by the symbols ${}_p\mathfrak{F}_q$, ${}_p\mathfrak{F}_q^{-1}$, ${}_p^s\mathfrak{F}_q$, ${}_p^s\mathfrak{F}_q^{-1}$, ${}_p^c\mathfrak{F}_q$ and ${}_p^c\mathfrak{F}_q^{-1}$ respectively.

Theorem 1. *Let* $u,v \in \phi(\mathbb{R})$, $p\colon \mathbb{R} \to \mathbb{R}-\{0\}$, $q\colon \mathbb{R} \to \mathbb{R}^+$ *and* λ_1,λ_2, $w,t \in \mathbb{R}$. *Then,*

$$ {}_p\mathfrak{F}_q\left[\lambda_1 u(t) + \lambda_2 v(t)\right](w) = \lambda_1\ {}_p\mathfrak{F}_q[u(t)](w) + \lambda_2\ {}_p\mathfrak{F}_q[v(t)](w). $$

Proof. Using the ${}_p\mathfrak{F}_q$ transform, we have

$$ {}_p\mathfrak{F}_q\left[\lambda_1 u(t) + \lambda_2 v(t)\right](w) = \lambda_1\left(p(\alpha)\int_{\mathbb{R}} \exp\left(iw^{q(\alpha)}t\right) u(t)\mathrm{d}t\right) $$

$$+\lambda_2\left(p(\alpha)\int_{\mathbb{R}}\exp\left(iw^{q(\alpha)}t\right)v(t)\mathrm{d}t\right)$$
$$=\lambda_1\ {}_p\mathfrak{F}_q[u(t)](w)+\lambda_2\ {}_p\mathfrak{F}_q[v(t)](w). \qquad \square$$

Theorem 2. *Let $u\in\phi(\mathbb{R})$, $p\colon\mathbb{R}\to\mathbb{R}-\{0\}$, $q\colon\mathbb{R}\to\mathbb{R}^+$ and $w,t\in\mathbb{R}$. Then,*

$${}_p\mathfrak{F}_q[u(t-x)](w)=\exp\left(iw^{q(\alpha)}x\right){}_p\mathfrak{F}_q[u(t)](w).$$

Proof. Using the ${}_p\mathfrak{F}_q$ transform, we have

$$\begin{aligned}{}_p\mathfrak{F}_q[u(t-x)](w)&=p(\alpha)\int_{\mathbb{R}}\exp\left(iw^{q(\alpha)}t\right)u(t-x)\mathrm{d}t,\quad (t-x=\tau)\\&=p(\alpha)\int_{\mathbb{R}}\exp\left(iw^{q(\alpha)}(x+\tau)\right)u(\tau)\mathrm{d}\tau\\&=\exp\left(iw^{q(\alpha)}x\right)p(\alpha)\int_{\mathbb{R}}\exp\left(iw^{q(\alpha)}\tau\right)u(\tau)\mathrm{d}\tau\\&=\exp\left(iw^{q(\alpha)}x\right){}_p\mathfrak{F}_q[u(t)](w). \qquad \square\end{aligned}$$

Theorem 3. *Let $u\in\phi(\mathbb{R})$, $p\colon\mathbb{R}\to\mathbb{R}-\{0\}$, $q\colon\mathbb{R}\to\mathbb{R}^+$ and $w,t\in\mathbb{R}$. Then,*

$${}_p\mathfrak{F}_q\left[u^{(n)}(t)\right](w)=\left(-iw^{q(\alpha)}\right)^n{}_p\mathfrak{F}_q[u(t)](w). \tag{3.7}$$

Proof. Let us prove the result by contradiction. For $n=1$, we have

$$\begin{aligned}{}_p\mathfrak{F}_q\left[u'(t)\right](w)&=p(\alpha)\int_{\mathbb{R}}\exp\left(iw^{q(\alpha)}t\right)u'(t)\mathrm{d}t\\&=\left(-iw^{q(\alpha)}\right)p(\alpha)\int_{\mathbb{R}}\exp\left(iw^{q(\alpha)}t\right)u(t)\mathrm{d}t\\&=\left(-iw^{q(\alpha)}\right){}_p\mathfrak{F}_q[u(t)](w).\end{aligned}$$

For $n=k$, let the equation be true:

$${}_p\mathfrak{F}_q\left[u^{(k)}(t)\right](w)=\left(-iw^{q(\alpha)}\right)^k{}_p\mathfrak{F}_q[u(t)](w). \tag{3.8}$$

For $n=k+1$, considering the equation (3.8), we obtain

$$\begin{aligned}{}_p\mathfrak{F}_q\left[u^{(k+1)}(t)\right](w)&=p(\alpha)\int_{\mathbb{R}}\exp\left(iw^{q(\alpha)}t\right)u^{(k+1)}(t)\mathrm{d}t\\&=\left(-iw^{q(\alpha)}\right)p(\alpha)\int_{\mathbb{R}}\exp\left(iw^{q(\alpha)}t\right)u^{(k)}(t)\mathrm{d}t\\&=\left(-iw^{q(\alpha)}\right)\left(-iw^{q(\alpha)}\right)^k{}_p\mathfrak{F}_q[u(t)](w)\\&=\left(-iw^{q(\alpha)}\right)^{k+1}{}_p\mathfrak{F}_q[u(t)](w). \qquad \square\end{aligned}$$

Theorem 4. *Let $u,v \in \phi(\mathbb{R})$, $p\colon \mathbb{R} \to \mathbb{R}-\{0\}$, $q\colon \mathbb{R} \to \mathbb{R}^+$ and $w,t \in \mathbb{R}$. Then,*

$$ {}_p\mathfrak{F}_q[u(t)*v(t)](w) = \frac{{}_p\mathfrak{F}_q[u(t)](w)\ {}_p\mathfrak{F}_q[v(t)](w)}{p(\alpha)}. \tag{3.9}$$

Proof. Using the convolution and ${}_p\mathfrak{F}_q$ transform, we have

$$\begin{aligned} {}_p\mathfrak{F}_q[u(t)*v(t)](w) &= p(\alpha)\int_{\mathbb{R}} \exp\left(iw^{q(\alpha)}t\right)\left(u(t)*v(t)\right)\mathrm{d}t \\ &= p(\alpha)\int_{\mathbb{R}} \exp\left(iw^{q(\alpha)}t\right)\int_{\mathbb{R}} u(t-x)v(x)\mathrm{d}x\mathrm{d}t. \end{aligned}$$

Considering Fubini's theorem and taking $\tau = t-x$, we get

$$ {}_p\mathfrak{F}_q[u(t)*v(t)](w) = p(\alpha)\int_{\mathbb{R}} v(x)\int_{\mathbb{R}} \exp\left(iw^{q(\alpha)}(x+\tau)\right)u(\tau)\mathrm{d}\tau\mathrm{d}x. \tag{3.10}$$

Multiplying and dividing the right side of (3.10) by function $p(\alpha)$, we obtain

$$\begin{aligned} {}_p\mathfrak{F}_q[u(t)*v(t)](w) &= p(\alpha)\int_{\mathbb{R}} \exp\left(iw^{q(\alpha)}x\right)v(x)\mathrm{d}x\ \frac{p(\alpha)}{p(\alpha)}\int_{\mathbb{R}} \exp\left(iw^{q(\alpha)}\tau\right)u(\tau)\mathrm{d}\tau \\ &= \frac{{}_p\mathfrak{F}_q[u(t)](w)\ {}_p\mathfrak{F}_q[v(t)](w)}{p(\alpha)}. \end{aligned}$$

□

Theorem 5. *Let $p\colon \mathbb{R} \to \mathbb{R}-\{0\}$, $q\colon \mathbb{R} \to \mathbb{R}^+$ and $w,t \in \mathbb{R}$. Then,*

$$ {}_p\mathfrak{F}_q\left[\delta(t)\right](w) = p(\alpha). \tag{3.11}$$

Proof. Considering equation (2.1) in the ${}_p\mathfrak{F}_q$ transform, we have

$$\begin{aligned} {}_p\mathfrak{F}_q[\delta(t)](w) &= p(\alpha)\int_{\mathbb{R}} \exp\left(iw^{q(\alpha)}t\right)\delta(t)\mathrm{d}t \\ &= p(\alpha). \end{aligned}$$

□

Theorem 6. *Let $u \in \phi(\mathbb{R})$, $p\colon \mathbb{R} \to \mathbb{R}-\{0\}$, $q\colon \mathbb{R} \to \mathbb{R}^+$ and $w,t \in \mathbb{R}_0^+$. Then,*

$$\begin{aligned} {}^s_p\mathfrak{F}_q[u'(t)](w) &= -w^{q(\alpha)}\ {}^c_p\mathfrak{F}_q[u(t)](w), \\ {}^s_p\mathfrak{F}_q[u''(t)](w) &= -w^{2q(\alpha)}\ {}^s_p\mathfrak{F}_q[u(t)](w) + w^{q(\alpha)}p(\alpha)u(0). \end{aligned} \tag{3.12}$$

Proof. Using the function $u'(t)$ in the ${}^s_p\mathfrak{F}_q$ transform, we have

$$\begin{aligned} {}^s_p\mathfrak{F}_q[u'(t)](w) &= p(\alpha)\int_{\mathbb{R}_0^+} \sin\left(w^{q(\alpha)}t\right)u'(t)\mathrm{d}t \\ &= -w^{q(\alpha)}p(\alpha)\int_{\mathbb{R}_0^+} \cos\left(w^{q(\alpha)}t\right)u(t)\mathrm{d}t \\ &= -w^{q(\alpha)}\ {}^c_p\mathfrak{F}_q[u(t)](w). \end{aligned}$$

Using the function $u''(t)$ in the ${}^s_p\mathfrak{F}_q$ transform, we get

$$ {}^s_p\mathfrak{F}_q[u''(t)](w) = p(\alpha)\int_{\mathbb{R}_0^+} \sin\left(w^{q(\alpha)}t\right)u''(t)\mathrm{d}t$$

$$= -w^{q(\alpha)} p(\alpha) \int_{\mathbb{R}_0^+} \cos\left(w^{q(\alpha)} t\right) u'(t) \mathrm{d}t$$
$$= -w^{2q(\alpha)} p(\alpha) \int_{\mathbb{R}_0^+} \sin\left(w^{q(\alpha)} t\right) u(t) \mathrm{d}t + w^{q(\alpha)} p(\alpha) u(0)$$
$$= -w^{2q(\alpha)} \, {}_p^s\mathfrak{F}_q[u(t)](w) + w^{q(\alpha)} p(\alpha) u(0). \qquad \square$$

Theorem 7. *Let $u \in \phi(\mathbb{R})$, $p\colon \mathbb{R} \to \mathbb{R} - \{0\}$, $q\colon \mathbb{R} \to \mathbb{R}^+$ and $w,t \in \mathbb{R}_0^+$. Then,*

$$ {}_p^c\mathfrak{F}_q[u'(t)](w) = w^{q(\alpha)} \, {}_p^s\mathfrak{F}_q[u(t)](w) - p(\alpha) u(0),$$
$$ {}_p^c\mathfrak{F}_q[u''(t)](w) = -w^{2q(\alpha)} \, {}_p^c\mathfrak{F}_q[u(t)](w) - p(\alpha) u'(0). \tag{3.13}$$

Proof. Using the function $u'(t)$ in the ${}_p^c\mathfrak{F}_q$ transform, we have

$$ {}_p^c\mathfrak{F}_q[u'(t)](w) = p(\alpha) \int_{\mathbb{R}_0^+} \cos\left(w^{q(\alpha)} t\right) u'(t) \mathrm{d}t$$
$$= w^{q(\alpha)} p(\alpha) \int_{\mathbb{R}_0^+} \sin\left(w^{q(\alpha)} t\right) u(t) \mathrm{d}t - p(\alpha) u(0)$$
$$= w^{q(\alpha)} \, {}_p^s\mathfrak{F}_q[u(t)](w) - p(\alpha) u(0).$$

Using the function $u''(t)$ in the ${}_p^c\mathfrak{F}_q$ transform, we get

$$ {}_p^c\mathfrak{F}_q[u''(t)](w) = p(\alpha) \int_{\mathbb{R}_0^+} \cos\left(w^{q(\alpha)} t\right) u''(t) \mathrm{d}t$$
$$= w^{q(\alpha)} p(\alpha) \int_{\mathbb{R}_0^+} \sin\left(w^{q(\alpha)} t\right) u'(t) \mathrm{d}t - p(\alpha) u'(0)$$
$$= -w^{2q(\alpha)} p(\alpha) \int_{\mathbb{R}_0^+} \cos\left(w^{q(\alpha)} t\right) u(t) \mathrm{d}t - p(\alpha) u'(0)$$
$$= -w^{2q(\alpha)} \, {}_p^c\mathfrak{F}_q[u(t)](w) - p(\alpha) u'(0). \qquad \square$$

Theorem 8. *Let $p\colon \mathbb{R} \to \mathbb{R} - \{0\}$, $q\colon \mathbb{R} \to \mathbb{R}^+$ and $w,t \in \mathbb{R}_0^+$. Then,*

$$ {}_p^c\mathfrak{F}_q^{-1}\left[\exp\left(-aw^{2q(\alpha)}\right)\right](t) = \frac{1}{p(\alpha)} \frac{1}{\sqrt{a\pi}} \exp\left(\frac{-t^2}{4a}\right). \tag{3.14}$$

Proof. Using the function $\exp\left(-aw^{2q(\alpha)}\right)$ in the ${}_p^c\mathfrak{F}_q^{-1}$ transform, we have

$$ {}_p^c\mathfrak{F}_q^{-1}\left[\exp\left(-aw^{2q(\alpha)}\right)\right](t) = \frac{q(\alpha)}{p(\alpha)} \frac{2}{\pi} \int_{\mathbb{R}_0^+} \cos\left(w^{q(\alpha)} t\right) w^{q(\alpha)-1} \exp\left(-aw^{2q(\alpha)}\right) \mathrm{d}w.$$

Using the formula $\cos\left(w^{q(\alpha)} t\right) = \frac{1}{2}\left[\exp\left(iw^{q(\alpha)} t\right) + \exp\left(-iw^{q(\alpha)} t\right)\right]$, we get

$$ {}_p^c\mathfrak{F}_q^{-1}\left[\exp\left(-aw^{2q(\alpha)}\right)\right](t)$$
$$= \frac{q(\alpha)}{p(\alpha)} \frac{1}{\pi} \int_{\mathbb{R}_0^+} \exp\left(-aw^{2q(\alpha)}\right) \left(\exp\left(iw^{q(\alpha)} t\right) + \exp\left(-iw^{q(\alpha)} t\right)\right) w^{q(\alpha)-1} \mathrm{d}w.$$

By making the necessary arrangements, we obtain

$$
\begin{aligned}
{}_p^c\mathfrak{F}_q^{-1}\left[\exp\left(-aw^{2q(\alpha)}\right)\right](t) &= \frac{q(\alpha)}{p(\alpha)}\frac{1}{\pi}\exp\left(-\frac{t^2}{4a}\right)\\
&\quad\times\left[\int_{\mathbb{R}_0^+}\exp\left(-a\left(w^{q(\alpha)}-\frac{it}{2a}\right)^2\right)w^{q(\alpha)-1}\mathrm{d}w\right.\\
&\quad\left.+\int_{\mathbb{R}_0^+}\exp\left(-a\left(w^{q(\alpha)}+\frac{it}{2a}\right)^2\right)w^{q(\alpha)-1}\mathrm{d}w\right].
\end{aligned}
$$

Taking $y=\left(w^{q(\alpha)}-\frac{it}{2a}\right)$ and $z=\left(w^{q(\alpha)}+\frac{it}{2a}\right)$, we have

$$
\begin{aligned}
&{}_p^c\mathfrak{F}_q^{-1}\left[\exp\left(-aw^{2q(\alpha)}\right)\right](t)\\
&\qquad=\frac{1}{p(\alpha)}\frac{1}{\pi}\exp\left(-\frac{t^2}{4a}\right)\left[\int_{\mathbb{R}_0^+}\exp\left(-ay^2\right)dy+\int_{\mathbb{R}_0^+}\exp\left(-az^2\right)\mathrm{d}z\right].
\end{aligned}
$$

Considering the formula $\int_{\mathbb{R}}\exp\left(-ax^2\right)dx=\sqrt{\frac{\pi}{a}}$, we get

$$
{}_p^c\mathfrak{F}_q^{-1}\left[\exp\left(-aw^{2q(\alpha)}\right)\right](t)=\frac{1}{p(\alpha)}\frac{1}{\sqrt{a\pi}}\exp\left(\frac{-t^2}{4a}\right). \qquad \square
$$

Theorem 9. *Let $p\colon\mathbb{R}\to\mathbb{R}-\{0\}$, $q\colon\mathbb{R}\to\mathbb{R}^+$ and $w,t\in\mathbb{R}_0^+$. Then,*

$$
{}_p^s\mathfrak{F}_q^{-1}\left[w^{q(\alpha)}\exp\left(-aw^{2q(\alpha)}\right)\right](t)=\frac{1}{p(\alpha)}\frac{t}{2\sqrt{\pi}}\frac{1}{a^{3/2}}\exp\left(\frac{-t^2}{4a}\right). \tag{3.15}
$$

Proof. Using the function $\exp\left(-aw^{2q(\alpha)}\right)$ in the ${}_p^c\mathfrak{F}_q^{-1}$ transform, we have

$$
{}_p^c\mathfrak{F}_q^{-1}\left[\exp\left(-aw^{2q(\alpha)}\right)\right](t)=\frac{q(\alpha)}{p(\alpha)}\frac{2}{\pi}\int_{\mathbb{R}_0^+}\cos\left(w^{q(\alpha)}t\right)w^{q(\alpha)-1}\exp\left(-aw^{2q(\alpha)}\right)\mathrm{d}w.
$$

Considering equation (3.14), we get

$$
\frac{1}{p(\alpha)}\frac{1}{\sqrt{a\pi}}\exp\left(\frac{-t^2}{4a}\right)=\frac{q(\alpha)}{p(\alpha)}\frac{2}{\pi}\int_{\mathbb{R}_0^+}\cos\left(w^{q(\alpha)}t\right)w^{q(\alpha)-1}\exp\left(-aw^{2q(\alpha)}\right)\mathrm{d}w.
$$

Differentiating both sides of the last equation according to the parameter t, we obtain

$$
{}_p^s\mathfrak{F}_q^{-1}\left[w^{q(\alpha)}\exp\left(-aw^{2q(\alpha)}\right)\right](t)=\frac{1}{p(\alpha)}\frac{t}{2\sqrt{\pi}}\frac{1}{a^{3/2}}\exp\left(\frac{-t^2}{4a}\right). \qquad \square
$$

Theorem 10. *Let $v\in\phi(\mathbb{R})$, $p\colon\mathbb{R}\to\mathbb{R}-\{0\}$, $q\colon\mathbb{R}\to\mathbb{R}^+$ and $w,t\in\mathbb{R}$. Then,*

$$
{}_p\mathfrak{F}_q\left[{}_{-\infty}I_t^{\beta}v(t)\right](w)=\left(-iw^{q(\alpha)}\right)^{-\beta}{}_p\hat{v}_q(w),\quad(0<\Re(\beta)<1). \tag{3.16}
$$

Proof. There is a relationship between the convolution of the functions $u_+(t)$ and $v(t)$ and the RLFI as follows:

$$ {}_{-\infty}I_t^{\beta} v(t) = u_+(t) * v(t). \tag{3.17}$$

Here the function $u_+(t)$ is as follows:

$$u_+(t) = \begin{cases} \frac{t^{\beta-1}}{\Gamma(\beta)}, & \text{for } t > 0, \\ 0, & \text{for } t \leq 0. \end{cases}$$

Applying the ${}_p\mathfrak{F}_q$ transform to the equation (3.17) and considering the equation (3.9), we have

$$\begin{aligned} {}_p\mathfrak{F}_q \left[{}_{-\infty}I_t^{\beta} v(t) \right](w) &= {}_p\mathfrak{F}_q \left[u_+(t) * v(t) \right](w) \\ &= \frac{{}_p\mathfrak{F}_q \left[u_+(t) \right](w) \; {}_p\mathfrak{F}_q [v(t)](w)}{p(\alpha)} \\ &= \left(-iw^{q(\alpha)} \right)^{-\beta} {}_p\hat{v}_q(w). \end{aligned}$$

□

Theorem 11. *Let $v \in \phi(\mathbb{R})$, $p\colon \mathbb{R} \to \mathbb{R} - \{0\}$, $q\colon \mathbb{R} \to \mathbb{R}^+$ and $w,t \in \mathbb{R}$, $m \in \mathbb{N}$, $m-1 < \Re(\beta) < m$. Then,*

$$ {}_p\mathfrak{F}_q \left[{}_{-\infty}D_t^{\beta} v(t) \right](w) = \left(-iw^{q(\alpha)} \right)^{\beta} {}_p\hat{v}_q(w), \quad (\Re(\beta) > 0). $$

Proof. The relationship between the RLFD and the RLFI is as follows:

$$ {}_{-\infty}D_t^{\beta} v(t) = \frac{d^n}{dt^n} \left({}_{-\infty}I_t^{(n-\beta)} v(t) \right). $$

By choosing ${}_{-\infty}I_t^{(n-\beta)} v(t) = g(t)$, we have

$$ {}_{-\infty}D_t^{\beta} v(t) = g^{(n)}(t), $$

and then applying the ${}_p\mathfrak{F}_q$ transform and considering equation (3.7), we get

$$\begin{aligned} {}_p\mathfrak{F}_q \left[{}_{-\infty}D_t^{\beta} v(t) \right](w) &= {}_p\mathfrak{F}_q \left[g^{(n)}(t) \right](w) \\ &= \left(-iw^{q(\alpha)} \right)^{n} {}_p\hat{g}_q(w). \end{aligned} \tag{3.18}$$

Applying the ${}_p\mathfrak{F}_q$ transform to the equation $g(t) = {}_{-\infty}I_t^{(n-\beta)} v(t)$, we obtain

$$ {}_p\mathfrak{F}_q \left[g(t) \right](w) = {}_p\mathfrak{F}_q \left[{}_{-\infty}I_t^{(n-\beta)} v(t) \right](w), $$

and then considering equation (3.16), we have

$$ {}_p\hat{g}_q(w) = \left(-iw^{q(\alpha)} \right)^{-(n-\beta)} {}_p\hat{v}_q(w). \tag{3.19}$$

Using equation (3.19) in equation (3.18), we get

$$ {}_p\mathfrak{F}_q\left[{}_{-\infty}D_t^{\beta}v(t)\right](w)=\left(-iw^{q(\alpha)}\right)^{\beta}{}_p\hat{v}_q(w). \qquad \square $$

Theorem 12. *Let $v\in\phi(\mathbb{R})$, $p\colon\mathbb{R}\to\mathbb{R}-\{0\}$, $q\colon\mathbb{R}\to\mathbb{R}^{+}$ and $w,t\in\mathbb{R}$, $m\in\mathbb{N}$, $m-1<\Re(\beta)<m$. Then,*

$$ {}_p\mathfrak{F}_q\left[{}_{-\infty}^{c}D_t^{\beta}v(t)\right](w)=\left(-iw^{q(\alpha)}\right)^{\beta}{}_p\hat{v}_q(w),\quad(\Re(\beta)>0). $$

Proof. The relationship between the CFD and the RLFI is as follows:

$$ {}_{-\infty}^{c}D_t^{\beta}v(t)={}_{-\infty}I_t^{(n-\beta)}v^{(n)}(t). $$

By choosing $v^{(n)}(t)=g(t)$, we have

$$ {}_{-\infty}^{c}D_t^{\beta}v(t)={}_{-\infty}I_t^{(n-\beta)}g(t). $$

Applying the ${}_p\mathfrak{F}_q$ transform and considering equations (3.7) and (3.16), we get

$$ \begin{aligned} {}_p\mathfrak{F}_q\left[{}_{-\infty}^{c}D_t^{\beta}v(t)\right](w)&=\left(-iw^{q(\alpha)}\right)^{-(n-\beta)}{}_p\hat{g}_q(w)\\ &=\left(-iw^{q(\alpha)}\right)^{-(n-\beta)}\left(-iw^{q(\alpha)}\right)^{n}{}_p\hat{v}_q(w)\\ &=\left(-iw^{q(\alpha)}\right)^{\beta}{}_p\hat{v}_q(w). \end{aligned} \qquad \square $$

Corollary 1. *The ${}_p\mathfrak{F}_q$ transform of the RLFD and CFD operators overlap.*

4. Applications

In this section, solutions to ordinary electric current differential equation, partial diffusion differential equation and fractional motion differential equation are obtained by means of the ${}_p\mathfrak{F}_q$, ${}_p\mathfrak{F}_q^{-1}$, ${}_p^{s}\mathfrak{F}_q$, ${}_p^{s}\mathfrak{F}_q^{-1}$, ${}_p^{c}\mathfrak{F}_q$ and ${}_p^{c}\mathfrak{F}_q^{-1}$ transforms.

Problem 1. *Let the ordinary electric current differential equation be given as:*

$$ L\frac{\mathrm{d}I(t)}{\mathrm{d}t}+RI(t)=E(t), $$

where L is inductance, I is current, R is resistance and E is applied electromagnetic force.

Taking $E(t)=\delta(t)$ and applying the ${}_p\mathfrak{F}_q$ transform to the ordinary electric current differential equation and considering the equation (3.11), we have

$$ \begin{aligned} L\,{}_p\mathfrak{F}_q\left[\frac{\mathrm{d}I(t)}{\mathrm{d}t}\right](w)+R\,{}_p\mathfrak{F}_q\left[I(t)\right](w)&={}_p\mathfrak{F}_q\left[\delta(t)\right](w)\\ -iLw^{q(\alpha)}{}_p\mathfrak{F}_q\left[I(t)\right](w)+R\,{}_p\mathfrak{F}_q\left[I(t)\right](w)&=p(\alpha). \end{aligned} $$

That is,

$$ {}_p\mathfrak{F}_q\left[I(t)\right](w) = \frac{p(\alpha)}{R - iLw^{q(\alpha)}}. \tag{4.1} $$

Applying the ${}_p\mathfrak{F}_q^{-1}$ transform to equation (4.1), we get

$$ I(t) = \frac{q(\alpha)}{L}\frac{1}{2\pi i}\int_{\mathbb{R}} \frac{\exp\left(-iw^{q(\alpha)}t\right)w^{q(\alpha)-1}}{\left(\frac{R}{iL} - w^{q(\alpha)}\right)}\mathrm{d}w. \tag{4.2} $$

Since $w = \left(\frac{R}{iL}\right)^{\frac{1}{q(\alpha)}}$ pole point in equation (4.2), using the Cauchy's residue theorem [16], we obtain

$$ \begin{aligned} I(t) &= \frac{q(\alpha)}{L}\frac{2\pi i}{2\pi i}\mathrm{Res}\left(w = \left(\frac{R}{iL}\right)^{\frac{1}{q(\alpha)}}\right) \\ &= \frac{q(\alpha)}{L}\lim_{w\to\left(\frac{R}{iL}\right)^{\frac{1}{q(\alpha)}}} \frac{\left(\frac{R}{iL} - w^{q(\alpha)}\right)\exp\left(-iw^{q(\alpha)}t\right)w^{q(\alpha)-1}}{\left(\frac{R}{iL} - w^{q(\alpha)}\right)} \\ &= \frac{q(\alpha)}{L}\exp\left(\frac{-Rt}{L}\right)\left(\frac{R}{iL}\right)^{\frac{q(\alpha)-1}{q(\alpha)}}. \end{aligned} $$

Problem 2. *Let the fractional motion differential equation be given as:*

$$ y''(t) + \lambda_1\left({}_{-\infty}D_t^{\beta}y(t)\right) + \lambda_2 y(t) = f(t), $$

where $\lambda_1, \lambda_2 \in \mathbb{R}$ *and* $0 < \Re(\beta) < 1$.

Applying the ${}_p\mathfrak{F}_q$ transform to the fractional motion differential equation, we have

$$ \begin{aligned} {}_p\mathfrak{F}_q\left[f(t)\right](w) &= {}_p\mathfrak{F}_q\left[y''(t)\right](w) + \lambda_1\ {}_p\mathfrak{F}_q\left[{}_{-\infty}D_t^{\beta}y(t)\right](w) + \lambda_2\ {}_p\mathfrak{F}_q\left[y(t)\right](w) \\ &= \left(-iw^{q(\alpha)}\right)^2\ {}_p\mathfrak{F}_q\left[y(t)\right](w) + \lambda_1\left(-iw^{q(\alpha)}\right)^{\beta}\ {}_p\mathfrak{F}_q\left[y(t)\right](w) \\ &\quad + \lambda_2\ {}_p\mathfrak{F}_q\left[y(t)\right](w). \end{aligned} $$

That is,

$$ {}_p\mathfrak{F}_q\left[y(t)\right](w) = \frac{{}_p\mathfrak{F}_q\left[f(t)\right](w)}{\left(-iw^{q(\alpha)}\right)^2 + \lambda_1\left(-iw^{q(\alpha)}\right)^{\beta} + \lambda_2}. $$

By choosing $A(w) = \left(-iw^{q(\alpha)}\right)^2 + \lambda_1\left(-iw^{q(\alpha)}\right)^{\beta} + \lambda_2$, we get

$$ {}_p\mathfrak{F}_q\left[y(t)\right](w) = \frac{{}_p\mathfrak{F}_q\left[f(t)\right](w)}{A(w)}. $$

Taking ${}_p\mathfrak{F}_q\left[g(t)\right] = \frac{1}{A(w)}$, we obtain

$$ {}_p\mathfrak{F}_q\left[y(t)\right](w) = {}_p\mathfrak{F}_q\left[f(t)\right](w)\,{}_p\mathfrak{F}_q\left[g(t)\right](w), $$

and then applying the ${}_p\mathfrak{F}_q^{-1}$ transform, we have

$$y(t) = {}_p\mathfrak{F}_q^{-1}\left[{}_p\mathfrak{F}_q\left[f(t)\right](w)\,{}_p\mathfrak{F}_q\left[g(t)\right](w)\right](t). \tag{4.3}$$

Applying the ${}_p\mathfrak{F}_q^{-1}$ transform to equation (3.9), we get

$$f(t)*g(t) = \frac{{}_p\mathfrak{F}_q^{-1}\left[{}_p\mathfrak{F}_q[f(t)](w)\;{}_p\mathfrak{F}_q[g(t)](w)\right](t)}{p(\alpha)}. \tag{4.4}$$

Multiplying and dividing the right side of (4.3) by function $p(\alpha)$, we obtain

$$y(t) = \frac{p(\alpha)}{p(\alpha)}\,{}_p\mathfrak{F}_q^{-1}\left[{}_p\mathfrak{F}_q\left[f(t)\right](w)\;{}_p\mathfrak{F}_q\left[g(t)\right](w)\right](t). \tag{4.5}$$

Using equation (4.4) in equation (4.5), we have

$$y(t) = p(\alpha)\big(f(t)*g(t)\big),$$

and then using the convolution, we get

$$y(t) = p(\alpha)\int_{\mathbb{R}} f(t-\tau)g(\tau)\mathrm{d}\tau. \tag{4.6}$$

Applying the ${}_p\mathfrak{F}_q^{-1}$ transform to the equation ${}_p\mathfrak{F}_q\left[g(t)\right](w) = \frac{1}{A(w)}$, we obtain

$$g(t) = \frac{q(\alpha)}{p(\alpha)}\frac{1}{2\pi}\int_{\mathbb{R}} \exp\left(-iw^{q(\alpha)}t\right)\frac{w^{q(\alpha)-1}}{A(w)}\mathrm{d}w. \tag{4.7}$$

Using equation (4.7) in equation (4.6), we have

$$y(t) = \frac{q(\alpha)}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} f(t-\tau)\exp\left(-iw^{q(\alpha)}\tau\right)\frac{w^{q(\alpha)-1}}{\left(-iw^{q(\alpha)}\right)^2+\lambda_1\left(-iw^{q(\alpha)}\right)^{\beta}+\lambda_2}\mathrm{d}w\mathrm{d}\tau.$$

Problem 3. *The initial and boundary conditions*

(a) $u(t,0) = 0, \quad 0 < t < \infty,$
(b) $u(0,\xi) = f(\xi), \quad \xi \geq 0, \quad u(t,\xi) \to 0 \quad \text{as} \quad t \to \infty,$
(c) $u_t(0,\xi) = f(\xi), \quad \xi \geq 0, \quad u(t,\xi) \to 0 \quad \text{as} \quad t \to \infty,$

and partial diffusion differential equation

$$\frac{\partial u}{\partial \xi} = \kappa\frac{\partial^2 u}{\partial t^2}, \quad (0 < t < \infty,\ \xi > 0)$$

are given. Here κ *is constant.*

The ${}_p^s\mathfrak{F}_q$ transform of a function $u(t,\xi)$ is given by

$${}_p^s\mathfrak{F}_q[u(t,\xi)](w) = {}_p^s\hat{u}_q(w,\xi) = p(\alpha)\int_{\mathbb{R}_0^+}\sin\left(w^{q(\alpha)}t\right)u(t,\xi)\mathrm{d}t.$$

Considering equation (3.12) and applying the ${}_p^s\mathfrak{F}_q$ transform to the partial diffusion differential equation and the initial condition (a), we have

$$\frac{d}{d\xi}\left\{{}_p^s\hat{u}_q(w,\xi)\right\} = \kappa\left(w^{q(\alpha)}p(\alpha)u(0,\xi) - w^{2q(\alpha)}\ {}_p^s\hat{u}_q(w,\xi)\right)$$
$${}_p^s\hat{u}_q(w,0) = 0.$$

Using the boundary condition (b), we get

$$\frac{d}{d\xi}\left\{{}_p^s\hat{u}_q(w,\xi)\right\} = \kappa w^{q(\alpha)}p(\alpha)f(\xi) - \kappa w^{2q(\alpha)}\ {}_p^s\hat{u}_q(w,\xi).$$

The solution of the last equation is obtained by

$${}_p^s\hat{u}_q(w,\xi) = \kappa w^{q(\alpha)}p(\alpha)\int_0^\xi f(\tau)\exp\left(-\kappa w^{2q(\alpha)}(\xi-\tau)\right)d\tau.$$

Applying the ${}_p^s\mathfrak{F}_q^{-1}$ transform, we have

$$u(t,\xi) = \kappa p(\alpha)\int_0^\xi f(\tau)\ {}_p^s\mathfrak{F}_q^{-1}\left[w^{q(\alpha)}\exp\left(-\kappa w^{2q(\alpha)}(\xi-\tau)\right)\right](t)d\tau.$$

By choosing $a = \kappa(\xi-\tau)$ in equation (3.15) and using in the last equation above, the solution of the given partial diffusion differential equation with the help of the ${}_p^s\mathfrak{F}_q$ transform is obtained by

$$u(t,\xi) = \frac{t}{2\sqrt{\kappa\pi}}\int_0^\xi \frac{f(\tau)}{(\xi-\tau)^{3/2}}\exp\left(\frac{-t^2}{4\kappa(\xi-\tau)}\right)d\tau.$$

The ${}_p^c\mathfrak{F}_q$ transform of a function $u(t,\xi)$ is given by

$${}_p^c\mathfrak{F}_q[u(t,\xi)](w) = {}_p^c\hat{u}_q(w,\xi) = p(\alpha)\int_{\mathbb{R}_0^+}\cos\left(w^{q(\alpha)}t\right)u(t,\xi)dt.$$

Considering equation (3.13) and applying the ${}_p^c\mathfrak{F}_q$ transform to the partial diffusion differential equation and the initial condition (a), we have

$$\frac{d}{d\xi}\left\{{}_p^c\hat{u}_q(w,\xi)\right\} = \kappa\left(-p(\alpha)u'(0,\xi) - w^{2q(\alpha)}\ {}_p^c\hat{u}_q(w,\xi)\right)$$
$${}_p^c\hat{u}_q(w,0) = 0.$$

Using the boundary condition (c), we get

$$\frac{d}{d\xi}\left\{{}_p^c\hat{u}_q(w,\xi)\right\} = -\kappa p(\alpha)f(\xi) - \kappa w^{2q(\alpha)}\ {}_p^c\hat{u}_q(w,\xi).$$

The solution of the last equation is obtained by

$${}_p^c\hat{u}_q(w,\xi) = -\kappa p(\alpha)\int_0^\xi f(\tau)\exp\left(-\kappa w^{2q(\alpha)}(\xi-\tau)\right)d\tau.$$

Applying the ${}_p^c\mathfrak{F}_q^{-1}$ transform, we have

$$u(t,\xi) = -\kappa p(\alpha) \int_0^{\xi} f(\tau)\ {}_p^c\mathfrak{F}_q^{-1}\left[\exp\left(-\kappa w^{2q(\alpha)}(\xi-\tau)\right)\right](t)\mathrm{d}\tau.$$

By choosing $a = \kappa(\xi-\tau)$ in equation (3.14) and using in the last equation above, the solution of the given partial diffusion differential equation with the help of the ${}_p^c\mathfrak{F}_q$ transform is obtained by

$$u(t,\xi) = -\sqrt{\frac{\kappa}{\pi}} \int_0^{\xi} \frac{f(\tau)}{\sqrt{(\xi-\tau)}} \exp\left(\frac{-t^2}{4\kappa(\xi-\tau)}\right) \mathrm{d}\tau.$$

5. Conclusion

In this article, we introduced the generalized Fourier (${}_p\mathfrak{F}_q$, ${}_p^s\mathfrak{F}_q$ and ${}_p^c\mathfrak{F}_q$) and inverse Fourier (${}_p\mathfrak{F}_q^{-1}$, ${}_p^s\mathfrak{F}_q^{-1}$ and ${}_p^c\mathfrak{F}_q^{-1}$) transforms and examine their certain properties. Our motivation in doing this was to define integral transforms that has a more general form than many Fourier-like integral transforms found in the literature. We also presented tables of ${}_p\mathfrak{F}_q$, ${}_p^s\mathfrak{F}_q$ and ${}_p^c\mathfrak{F}_q$ transformations in Table 1, Table 2 and Table 3, respectively. Then, we use generalized Fourier, Fourier sine, Fourier cosine and inverse Fourier, Fourier sine, Fourier cosine transforms to arrive the analytical solutions of ordinary electric current, partial diffusion and fractional motion problems. Finally, we examined the behavior for different $q(\alpha)$ values of the approximate solutions of the ordinary electric current problem in Figure 1.

We should say that the solutions of the application problems we obtained with the generalized Fourier and inverse Fourier transforms are in full agreement with the results obtained in the literature for $q(\alpha) = 1$ (see for example [1, 3, 6]). As a result of these applications, we found that the newly defined generalized Fourier transforms are much more general and are quite compatible with both ordinary, partial and fractional problems. We would like to note that in this research generalized Fourier transforms play a very important role in finding analytical solutions for both ordinary, partial and fractional problems, and therefore the results presented in this article are very important for their implementation.

We conclude this research by presenting some generalized Fourier transforms that have been recently presented to the literature and their relations with new generalized Fourier transforms (${}_p\mathfrak{F}_q$, ${}_p^s\mathfrak{F}_q$ and ${}_p^c\mathfrak{F}_q$) below.

- The generalized Fourier transform defined by Luchko et al. [9] is as follows:

$$(\mathcal{F}_\alpha u)(w) = \int_{\mathbb{R}} \exp_\alpha(w,t)\, u(t)\mathrm{d}t, \quad (u \in \phi(\mathbb{R}),\ w \in \mathbb{R},\ 0 < \alpha \leq 1)$$

and its relationship with the ${}_p\mathfrak{F}_q$ transform is as follows:

$${}_1\mathfrak{F}_{\frac{1}{\alpha}}[u](w) = (\mathcal{F}_\alpha u)(w).$$

- The generalized Fourier transform defined by Romero and Luque [12] is as follows:

$$\mathfrak{F}_\alpha[u](w) = \int_{\mathbb{R}} \exp\left(iw^{\frac{1}{\alpha}}t\right) u(t)\mathrm{d}t, \quad (u \in \phi(\mathbb{R}),\ w \in \mathbb{R},\ 0 < \alpha \leq 1)$$

and its relationship with the ${}_p\mathfrak{F}_q$ transform is as follows:

$${}_1\mathfrak{F}_{\frac{1}{\alpha}}[u](w) = \mathfrak{F}_\alpha[u](w).$$

- The generalized Fourier transform defined by Kumar [8] is as follows:

$$\Omega_\mu[u](w) = \int_{\mathbb{R}} \exp\left(iw^{\frac{\rho}{\mu}}t\right) u(t)\mathrm{d}t, \quad \left(u \in \phi(\mathbb{R}),\ w > 0,\ \mu \leq \rho;\ \mu,\rho \in \mathbb{R}^+\right)$$

and its relationship with the ${}_p\mathfrak{F}_q$ transform is as follows:

$${}_1\mathfrak{F}_{\frac{\rho}{\mu}}[u](w) = \Omega_\mu[u](w).$$

- The generalized Fourier sine and Fourier cosine transforms defined by Mahor et al. [10] are as follows:

$$\hat{f}_s^\alpha(w) = \int_{\mathbb{R}_0^+} \sin_\alpha(wt)^\alpha u(t)(\mathrm{d}t)^\alpha, \quad (w > 0,\ 0 < \alpha \leq 1),$$
$$\hat{f}_c^\alpha(w) = \int_{\mathbb{R}_0^+} \cos_\alpha(wt)^\alpha u(t)(\mathrm{d}t)^\alpha, \quad (w > 0,\ 0 < \alpha \leq 1)$$

and their relationship with the ${}_p^s\mathfrak{F}_q$ and ${}_p^c\mathfrak{F}_q$ transforms are as follows:

$${}_1^s\mathfrak{F}_1[u(t)](w) = \hat{f}_s^1(w),$$
$${}_1^c\mathfrak{F}_1[u(t)](w) = \hat{f}_c^1(w).$$

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TABLE 1. ${}_p\mathfrak{F}_q$ transform table

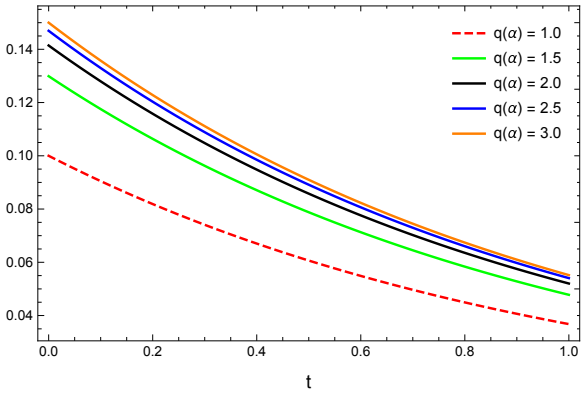
$u(t)$	${}_p\mathfrak{F}_q[u(t)](w) = p(\alpha)\int_{\mathbb{R}} \exp\left(iw^{q(\alpha)}t\right) u(t)\mathrm{d}t$	Conditions
$\exp\left(-at^2\right)$	$p(\alpha)\sqrt{\frac{\pi}{a}}\exp\left(\frac{-w^{2q(\alpha)}}{4a}\right)$	$a>0$
$\exp\left(-a\lvert t\rvert\right)$	$p(\alpha)\left(\frac{2a}{a^2+w^{2q(\alpha)}}\right)$	$a>0$
$H\left(a-\lvert t\rvert\right)$	$2p(\alpha)\left(\frac{\sin\left(aw^{q(\alpha)}\right)}{w^{q(\alpha)}}\right)$	
$\exp\left(-at\right)H(t)$	$\frac{p(\alpha)}{a-iw^{q(\alpha)}}$	$a>0$
$\delta(t)$	$p(\alpha)$	
$\sin(at)$	$-p(\alpha)i\pi\left[\delta\left(-w^{q(\alpha)}-a\right)-\delta\left(-w^{q(\alpha)}+a\right)\right]$	
$\cos(at)$	$p(\alpha)\pi\left[\delta\left(-w^{q(\alpha)}-a\right)+\delta\left(-w^{q(\alpha)}+a\right)\right]$	
c : constant	$2p(\alpha)c\pi\delta\left(w^{q(\alpha)}\right)$	
$u(t)*v(t)$	$\frac{{}_p\mathfrak{F}_q[u(t)](w)\ {}_p\mathfrak{F}_q[v(t)](w)}{p(\alpha)}$	
$u'(t)$	$\left(-iw^{q(\alpha)}\right){}_p\mathfrak{F}_q[u(t)](w)$	
$u''(t)$	$\left(-iw^{q(\alpha)}\right)^2{}_p\mathfrak{F}_q[u(t)](w)$	
$u^{(n)}(t)$	$\left(-iw^{q(\alpha)}\right)^n{}_p\mathfrak{F}_q[u(t)](w)$	
${}_{-\infty}I_t^{\beta}u(t)$	$\left(-iw^{q(\alpha)}\right)^{-\beta}{}_p\mathfrak{F}_q[u(t)](w)$	$0<\Re(\beta)<1$
${}_{-\infty}D_t^{\beta}u(t)$	$\left(-iw^{q(\alpha)}\right)^{\beta}{}_p\mathfrak{F}_q[u(t)](w)$	$\Re(\beta)>0$
${}_{-\infty}^{c}D_t^{\beta}u(t)$	$\left(-iw^{q(\alpha)}\right)^{\beta}{}_p\mathfrak{F}_q[u(t)](w)$	$\Re(\beta)>0$

TABLE 2. ${}^{s}_{p}\mathfrak{F}_{q}$ transform table

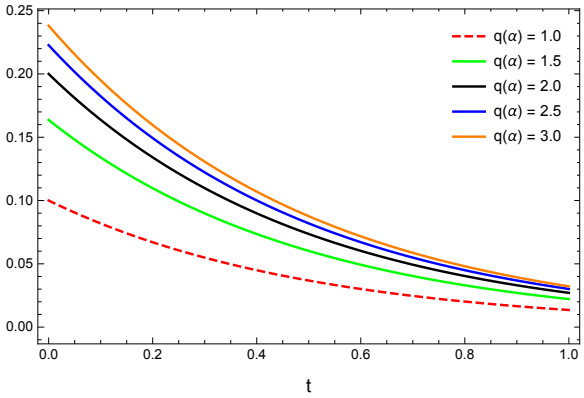
$u(t)$	${}^{s}_{p}\mathfrak{F}_{q}[u(t)](w) = p(\alpha)\int_{\mathbb{R}_0^+}\sin\left(w^{q(\alpha)}t\right)u(t)\mathrm{d}t$	Conditions
$\exp(-at)$	$p(\alpha)\left(\frac{w^{q(\alpha)}}{a^2+w^{2q(\alpha)}}\right)$	$a>0$
$t\exp(-at)$	$p(\alpha)\left(\frac{2aw^{q(\alpha)}}{\left(a^2+w^{2q(\alpha)}\right)^2}\right)$	$a>0$
$\frac{t}{a^2+t^2}$	$\frac{p(\alpha)\pi\exp\left(-aw^{q(\alpha)}\right)}{2}$	$a>0$
$H(a-t)$	$p(\alpha)\left(\frac{1-\cos\left(aw^{q(\alpha)}\right)}{w^{q(\alpha)}}\right)$	$a>0$
$u'(t)$	$-w^{q(\alpha)}\ {}^{c}_{p}\mathfrak{F}_{q}[u(t)](w)$	
$u''(t)$	$-w^{2q(\alpha)}\ {}^{s}_{p}\mathfrak{F}_{q}[u(t)](w)+w^{q(\alpha)}p(\alpha)u(0)$	

TABLE 3. ${}^{c}_{p}\mathfrak{F}_{q}$ transform table

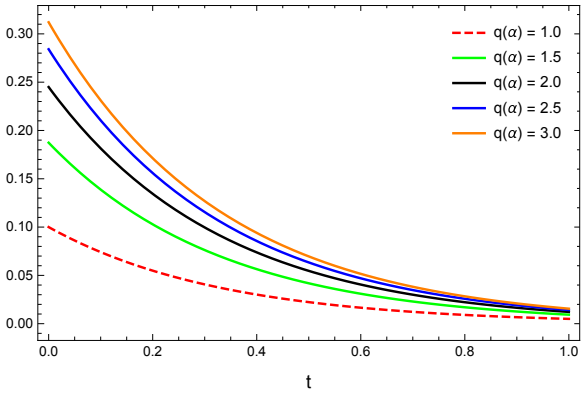
$u(t)$	${}^{c}_{p}\mathfrak{F}_{q}[u(t)](w) = p(\alpha)\int_{\mathbb{R}_0^+}\cos\left(w^{q(\alpha)}t\right)u(t)\mathrm{d}t$	Conditions
$\exp(-at)$	$p(\alpha)\left(\frac{a}{a^2+w^{2q(\alpha)}}\right)$	$a>0$
$t\exp(-at)$	$p(\alpha)\left(\frac{a^2-w^{2q(\alpha)}}{\left(a^2+w^{2q(\alpha)}\right)^2}\right)$	$a>0$
$\frac{1}{a^2+t^2}$	$\frac{p(\alpha)\pi\exp\left(-aw^{q(\alpha)}\right)}{2a}$	$a>0$
$H(a-t)$	$p(\alpha)\left(\frac{\sin\left(aw^{q(\alpha)}\right)}{w^{q(\alpha)}}\right)$	$a>0$
$u'(t)$	$w^{q(\alpha)}\ {}^{s}_{p}\mathfrak{F}_{q}[u(t)](w)-p(\alpha)u(0)$	
$u''(t)$	$-w^{2q(\alpha)}\ {}^{c}_{p}\mathfrak{F}_{q}[u(t)](w)-p(\alpha)u'(0)$	



(A) Approximate behaviours for $L = 10$ and $R = 10$.



(B) Approximate behaviours for $L = 10$ and $R = 20$.



(C) Approximate behaviours for $L = 10$ and $R = 30$.

FIGURE 1. Approximate behaviours corresponding to the different values of the function $q(\alpha)$ for solving the ordinary electric current differential equation.

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EXISTENCE OF SOLUTIONS FOR FRACTIONAL BOUNDARY VALUE PROBLEM VIA GLOBAL MINIMIZATION THEOREM

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Abstract. The goal of this paper is to investigate the existence of solutions for the following boundary value problem involving the Riemann-Liouville fractional derivative:

$$\begin{cases} D_{T^-}^{\alpha}\left(D_{0^+}^{\alpha}\left(D_{T^-}^{\alpha}\left(D_{0^+}^{\alpha}u(x)\right)\right)\right) = f\left(x,u(x)\right), & x \in [0,T], \\ u(0) = u(T) = 0, \\ D_{T^-}^{\alpha}\left(D_{0^+}^{\alpha}u(0)\right) = D_{T^-}^{\alpha}\left(D_{0^+}^{\alpha}u(T)\right) = 0, \end{cases}$$

where $0 < \alpha \leq 1$, $f\colon [0,T] \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and $D_{0^+}^{\alpha}, D_{T^-}^{\alpha}$ are the left and right fractional Riemann-Liouville derivatives of order α.

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Keywords: minimization theorem, Hilbert space, nonexpansive potential operator, BVP

1. INTRODUCTION

Many phenomena in engineering and technological discipline, that include various field of sciences, can be described by fractional differential equation; see for example [1, 2, 7, 8, 12, 13, 17, 18].

Numerous researchers have explored the intricate connections between fractional calculus and various scientific disciplines. For instance, the study conducted in [14] examines the relationship between fractional calculus, zeta functions, and Shannon entropy, offering valuable insights into information theory and thermodynamics. In the realm of mathematics, the work presented in [15] provides a deeper understanding of special functions and their analytical properties. Likewise, the research in [19] extends traditional complex differentiation to fractional orders, unveiling new structural properties and potential applications.

From a functional analysis standpoint, the study of commutators of fractional integral operators on null Morrey spaces [23] sheds light on significant operator-theoretic properties with implications for harmonic analysis and partial differential equations. Additionally, investigations into elliptic boundary value problems under

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zero mean oscillation assumptions establish connections between fractional differential operators and modern function space theory, leading to refined estimates for PDE solutions [22].

The existence of solutions of boundary value problems associated to a nonlinear fractional differential equation have been studied widely (see [3, 20, 24, 25]). The selection of the appropriate fractional derivative for a boundary value problem depends on various factors, including analytical properties, boundary conditions, and the specific requirements of the model under consideration. Several studies have investigated fractional-order differential equations using fixed point theory, yet only a limited number have explored them through variational methods. The Riemann-Liouville fractional derivative is particularly well-suited for such problems, as it aligns with fractional Sobolev spaces, making it a valuable tool in variational analysis and boundary value problems. It is worth noting that the study of equations involving both left and right fractional differential operators has attracted significant interest, leading to the development of various theorems and techniques. Follow the previous contributions in this direction and inspired by the theorem proposed in [11], Boucenna and Moussaoui proved [9] the existence of positive solutions for a following boundary value problem via a topological-variational theorem

$$\begin{cases} D_{T^-}^{\alpha}\left(D_{0^+}^{\alpha}u(x)\right) = f\left(x,u(x)\right), & x \in [0,T], \\ u(0) = u(T) = 0, \end{cases}$$

where $\frac{1}{2} < \alpha \leq 1$, $f\colon [0,T]\times\mathbb{R}$ a continuous function and $D_{0^+}^{\alpha}, D_{T^-}^{\alpha}$ are respectively the left and right fractional Riemann-Liouville derivatives of order α. A similar problem was studied in [16] by using Minimization Principle and Mountain Pass Theorem. Ricceri's critical point theorem was used in [6] to prove existence of infinitely many solutions for a perturbed nonlinear fractional boundary value problem depending on two parameters and involving both left and right Riemann-Liouville derivatives. Later on, in [10], the authors considered an appropriate Banach space where they have used Minimization Principle and Mountain Pass Theorem to prove the existence of at least one solution for the problem

$$\begin{cases} D_{T^-}^{\alpha}\left(D_{0^+}^{\alpha}\left(D_{T^-}^{\alpha}\left(D_{0^+}^{\alpha}u(x)\right)\right)\right) = f\left(x,u(x)\right), & x \in [0,T], \\ u(0) = u(T) = 0, \\ D_{T^-}^{\alpha}\left(D_{0^+}^{\alpha}u(0)\right) = D_{T^-}^{\alpha}\left(D_{0^+}^{\alpha}u(T)\right) = 0, \end{cases}$$

where $0 < \alpha \leq 1$ and $f\colon [0,T]\times\mathbb{R}\to\mathbb{R}$ is a continuous function.

Ayadi et al. [5] proved a new global minimization theorem using the notion of a nonexpansive potential operator. Our contribution consists in working in the Banach space introduced in [10] and applying the global minimization theorem proved in [5] to find the global minimum for a suitable functional. To this end, we will firstly give the needed tools for our proof and in the second section, we present the proof of our main result.

2. PRELIMINARIES

The definitions and properties of the fractional calculus which are used in the sequel are taken from [17, 21].

Definition 1 ([17, Section 2.1],[21, Section 2.3.2]). Let u be a function defined on $[a,b]$. The left and right Riemann-Liouville fractional integrals of order $\alpha > 0$ for a function u denoted by $I^{\alpha}_{a^+}u$ and $I^{\alpha}_{b^-}u$, respectively, are defined by

$$I^{\alpha}_{a^+}u(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}u(s)\mathrm{d}s, \qquad t > a$$

and

$$I^{\alpha}_{b^-}u(t) = \frac{1}{\Gamma(\alpha)}\int_t^b (s-t)^{\alpha-1}u(s)\mathrm{d}s, \quad t < b$$

provided that the right-hand side is pointwise defined on $[a,b]$, where $\Gamma(\alpha)$ is the gamma function.

Definition 2 ([17, Section 2.1], [21, Section 2.3.3]). Let u be a function defined on $[a,b]$. For $n-1 \le \alpha < n\,(n \in \mathbb{N}^*)$, the left and right Riemann-Liouville fractional derivatives of order $\alpha > 0$ for a function u denoted by $D^{\alpha}_{a^+}u$ and $D^{\alpha}_{b^-}u$, respectively, are defined by

$$\begin{aligned} D^{\alpha}_{a^+}u(t) &= \frac{\mathrm{d}^n}{\mathrm{d}t^n} I^{n-\alpha}_{a^+}u(t) \\ &= \frac{1}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int_a^t (t-s)^{n-\alpha-1}u(s)\mathrm{d}s, \qquad t > a \end{aligned}$$

and

$$\begin{aligned} D^{\alpha}_{b^-}u(t) &= (-1)^n\frac{\mathrm{d}^n}{\mathrm{d}t^n} I^{n-\alpha}_{b^-}u(t) \\ &= \frac{(-1)^n}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int_t^b (s-t)^{n-\alpha-1}u(s)\mathrm{d}s, \qquad t < b \end{aligned}$$

provided that the right-hand side is pointwise defined. In particular, for $\alpha = n$, $D^{\alpha}_{a^+}u(t) = D^n u(t)$ and $D^{\alpha}_{b^-}u(t) = (-1)^n D^n u(t)$, $t \in [a,b]$.

Property 1 ([17, property 2.1], [21, Section 2.3.3, Section 2.3.4]). *For* $\alpha > 0$ *and for some* $\beta > -1, \quad \alpha - 1 > 0$, *we have*

(1) $D^{\alpha}_{a^+}(t-a)^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(t-a)^{\beta-\alpha}$, $t > a$ *and*

$$D^{\alpha}_{b^-}(b-t)^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(b-t)^{\beta-\alpha}, \qquad t < b.$$

(2) $D^{\alpha-1}_{a^+}(t-a)^{\alpha-1} = \Gamma(\alpha)$, $t > a$ *and* $D^{\alpha-1}_{b^-}(b-t)^{\alpha-1} = \Gamma(\alpha)$, $t < b$.
(3) $D^{\alpha}_{a^+}(t-a)^{\alpha-j} = 0$, $t > a$ *and* $D^{\alpha}_{b^-}(b-t)^{\alpha-j} = 0$, $t < b$ *for* $j = 1,2,\ldots,[\alpha]+1$.

(4) *If C is a constant and $\beta = 0$, then*

$$D^{\alpha}_{a^+}C = \frac{C}{\Gamma(1-\alpha)}(t-a)^{-\alpha},\ t > a \ \text{ and } \ D^{\alpha}_{b^-}C = \frac{C}{\Gamma(1-\alpha)}(b-t)^{-\alpha}.$$

The following assertion shows that the fractional differentiation is an operation inverse to the fractional integration from the left.

Lemma 1 ([17, Lemma 2.4],[21, Section 2.3.3])**.** *If $\alpha > 0$ and $u \in L^p(a,b)$ $(1 \leq p \leq +\infty)$, then the following equalities*

$$D^{\alpha}_{a^+} I^{\alpha}_{a^+} u = u \quad \text{and} \quad D^{\alpha}_{b^-} I^{\alpha}_{b^-} u = u$$

hold almost everywhere on $[a,b]$.

Lemma 2 ([17, Corollary 2.1])**.** *Let $\alpha > 0$ and $n = [\alpha] + 1$.*

(1) *The equality $D^{\alpha}_{a^+} u(t) = 0$ is valid if and only if*

$$u(t) = \sum_{j=1}^{n} c_j (t-a)^{\alpha-j},$$

where $c_j \in \mathbb{R}$ $(j = 1,2,\ldots,n)$ are arbitrary constants.

(2) *The equality $D^{\alpha}_{b^-} u(t) = 0$ is valid if and only if*

$$u(t) = \sum_{j=1}^{n} d_j (b-t)^{\alpha-j},$$

where $d_j \in \mathbb{R}$ $(j = 1,2,\ldots,n)$ are arbitrary constants.

In this paper, we will use the Banach spaces defined in [10]. We point out that the reader can find more informations about this subject in [10] and references therein.

Definition 3 ([10, Definition 2])**.** Let $0 < \alpha \leq 1$ and $1 < p < \infty$. The fractional derivative space $E^{\alpha,p}_0$ is defined by the closure of $C^{\infty}_0([0,T],\mathbb{R})$ with respect to the norm

$$\|u\|_{\alpha,p} = \left(\|u\|^p_{L^p} + \|D^{\alpha}_{0^+} u\|^p_{L^p} \right)^{\frac{1}{p}}$$

for all $u \in E^{\alpha,p}_0$ with $\|u\|_{L^p} = (\int_0^T |u(t)|^p \mathrm{d}t)^{\frac{1}{p}}$. We put

$$E^{\alpha,p}_0 = \left\{ u \in L^p([0,T]);\ D^{\alpha}_{0^+} u \in L^p([0,T]);\ u(0) = u(T) = 0 \right\}.$$

Definition 4 ([10, Definition 3.2, Remark 3.3])**.** The fractional derivative space $E^{2\alpha,p}_0$ is defined by the closure of $C^{\infty}_0([0,T],\mathbb{R})$ with respect to the norm

$$\|u\|_{2\alpha,p} = \left(\|u\|^p_{L^p} + \|D^{\alpha}_{0^+} u\|^p_{L^p} + \|D^{\alpha}_{T^-}(D^{\alpha}_{0^+} u)\|^p_{L^p} \right)^{\frac{1}{p}}$$

for all $u \in E^{2\alpha,p}_0$. We put

$$E^{2\alpha,p}_0 = \{ u \in L^p([0,T]) \ :\ D^{\alpha}_{0^+} u \in L^p([0,T]),\ D^{\alpha}_{T^-}(D^{\alpha}_{0^+} u) \in L^p([0,T],\mathbb{R}),$$

$$u(0)=u(T)=0,\ D^{\alpha}_{T^-}(D^{\alpha}_{0^+}u(0))=D^{\alpha}_{T^-}(D^{\alpha}_{0^+}u(T))=0\}.$$

The space $E_0^{2\alpha,p}$ can be endowed with respect to the equivalent norm

$$\|u\|_{2\alpha,p}=\|D^{\alpha}_{T^-}(D^{\alpha}_{0^+}u)\|_{L^p}=\Big(\int_0^T|D^{\alpha}_{T^-}(D^{\alpha}_{0^+}u(t))|^p\mathrm{d}t\Big)^{\frac{1}{p}}.$$

The following properties which are taken from [10], are necessary for all proofs in this paper.

Lemma 3 ([10, Lemma 3.4]). *For any $u\in L^p([0,T],\mathbb{R})$, we have*

$$\|I^{\alpha}_{0^+}u\|_{L^p}\le\frac{T^{\alpha}}{\Gamma(\alpha+1)}\|u\|_{L^p}$$

and

$$\|I^{\alpha}_{T^-}u\|_{L^p}\le\frac{T^{\alpha}}{\Gamma(\alpha+1)}\|u\|_{L^p}.$$

Proposition 1 ([10, Proposition 3.5]). *For all $u\in E_0^{2\alpha,p}$, we have*

$$I^{\alpha}_{0^+}D^{\alpha}_{0^+}u=u\ \text{ and }\ I^{\alpha}_{T^-}D^{\alpha}_{T^-}u=u.$$

Proposition 2 ([10, Proposition 3.6]). *For any $u\in E_0^{\alpha,p}$, we have*

$$\|u\|_{L^p}\le\frac{T^{\alpha}}{\Gamma(\alpha+1)}\|D^{\alpha}_{0^+}u\|_{L^p}.$$

Moreover, if $\alpha>\frac{1}{p}$ and $\frac{1}{p}+\frac{1}{q}=1$, then

$$\|u\|_{\infty}\le\frac{T^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)((\alpha-1)q+1)^{\frac{1}{q}}}\|D^{\alpha}_{0^+}u\|_{L^p},$$

where $\|u\|_{\infty}=\sup\limits_{t\in[0,T]}|u(t)|$.

Proposition 3 ([10, Proposition 3.7]). *For any $u\in E_0^{2\alpha,p}$, we have*

$$\|D^{\alpha}_{0^+}u\|_{L^p}\le\frac{T^{\alpha}}{\Gamma(\alpha+1)}\|D^{\alpha}_{T^-}(D^{\alpha}_{0^+}u)\|_{L^p}.$$

Moreover, if $\alpha>\frac{1}{p}$ and $\frac{1}{p}+\frac{1}{q}=1$, then

$$\|u\|_{\infty}\le\frac{T^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)((\alpha-1)q+1)^{\frac{1}{q}}}\cdot\frac{T^{\alpha}}{\Gamma(\alpha+1)}\|D^{\alpha}_{T^-}(D^{\alpha}_{0^+}u)\|_{L^p}.$$

Proposition 4 ([10, Proposition 3.10]). *If $u\in E_0^{\alpha,p}$ and $v\in C_0^{\infty}([0,T])$, then*

$$\int_0^T D^{\alpha}_{0^+}u(t)v(t)\mathrm{d}t=\int_0^T u(t)D^{\alpha}_{T^-}v(t)\mathrm{d}t.$$

Proposition 5 ([10, Proposition 3.11]). *If $u \in E^{2\alpha,p}$ and $v \in C_0^\infty([0,T])$, then*

$$\int_0^T D_{T-}^{\alpha}(D_{0+}^{\alpha}u(t))v(t)\mathrm{d}t = \int_0^T u(t)D_{T-}^{\alpha}(D_{0+}^{\alpha}v(t))\mathrm{d}t.$$

We mention that the proof of our main result is based on the following global minimization theorem, see [5].

Let H a real Hilbert space and $(.,.)$ the scalar product.

Theorem 1 ([5, Theorem 4]). *Let $\varphi\colon H \longrightarrow \mathbb{R}$ be a functional such that:*

(1) *φ is twice Gâteaux differentiable on H.*
(2) *For all $u \in H$, $\|(I'-\varphi'')(u)\| \leq 1$.*
(3) *$(I-\varphi')(C) \subset C$ for some convex nonempty closed and bounded subset C of H.*

Then, φ has a global minimum on H. Indeed, there exists a $u_0 \in C$ such that

$$\varphi(u_0) = \inf_H \varphi.$$

In particular, $\varphi'(u_0) = 0$.

3. Main Results

We aim to use Theorem 1 to prove existence of at least a solution of the following problem:

$$\begin{cases} D_{T-}^{\alpha}\left(D_{0+}^{\alpha}\left(D_{T-}^{\alpha}\left(D_{0+}^{\alpha}u(x)\right)\right)\right) = f(x,u(x)), & x \in [0,T], \\ u(0) = u(T) = 0, \\ D_{T-}^{\alpha}\left(D_{0+}^{\alpha}u(0)\right) = D_{T-}^{\alpha}\left(D_{0+}^{\alpha}u(T)\right) = 0, \end{cases} \tag{3.1}$$

where $0 < \alpha \leq 1$ and $f\colon [0,T]\times\mathbb{R} \to \mathbb{R}$ is a Carathéodory function, $D_{0+}^{\alpha}, D_{T-}^{\alpha}$ are the left and right fractional Riemann-Liouville derivatives of order α, respectively. The idea is to reformulate the Problem (3.1) as a variational problem where we look for the critical point for some functional which will be defined in the sequel. Existence of such critical point is ensured by Theorem 1 when the nonlinear term f satisfies certain conditions. One can see that if u is a solution of the Problem ((3.1)), then u is a fixed point of the operator A defined by

$$\mathrm{A}u(x) = I_{0+}^{\alpha}\left(I_{T-}^{\alpha}\left(I_{0+}^{\alpha}\left(I_{T-}^{\alpha}f(x,u(x))\right)\right)\right) \tag{3.2}$$

and satisfies the problem

$$\begin{cases} D_{T-}^{\alpha}\left(D_{0+}^{\alpha}\left(D_{T-}^{\alpha}\left(D_{0+}^{\alpha}\mathrm{A}u(x)\right)\right)\right) = f(x,u(x)), & x \in [0,T], \\ \mathrm{A}u(0) = \mathrm{A}u(T) = 0, \\ D_{T-}^{\alpha}\left(D_{0+}^{\alpha}\mathrm{A}u(0)\right) = D_{T-}^{\alpha}\left(D_{0+}^{\alpha}\mathrm{A}u(T)\right) = 0. \end{cases} \tag{3.3}$$

On the other hand, a weak solution of the Problem (3.1) in the space $E_0^{2\alpha}$ satisfies

$$\int_0^T \left(D_{T^-}^{\alpha}\left(D_{0^+}^{\alpha}u(x)\right)\right)\left(D_{T^-}^{\alpha}\left(D_{0^+}^{\alpha}v(x)\right)\right)\mathrm{d}x - \int_0^T f\left(x,u(x)\right)v(x)\,\mathrm{d}x = 0 \tag{3.4}$$

for all $v \in E_0^{2\alpha}$.

Proposition 6 ([10, Proposition 4.2]). *All critical point of the functional* φ *defined on* $E_0^{2\alpha}$ *by*

$$\varphi(u) = \frac{1}{2}\int_0^T |D_{T^-}^{\alpha}(D_{0^+}^{\alpha}u(t))|^2\mathrm{d}t - \int_0^T F(t,u(t))\mathrm{d}t,$$

where $F(t,u) = \int_0^u f(t,s)\mathrm{d}s$ *is a solution of the boundary value problem* (3.1).

Let mention that by (3.3) (3.4), we have

$$\begin{aligned}
&\int_0^T \left(D_{T^-}^{\alpha}\left(D_{0^+}^{\alpha}u(x)\right)\right)\left(D_{T^-}^{\alpha}\left(D_{0^+}^{\alpha}v(x)\right)\right)\mathrm{d}x = \int_0^T f\left(x,u(x)\right)v(x)\,\mathrm{d}x\\
&= \int_0^T D_{T^-}^{\alpha}\left(D_{0^+}^{\alpha}\left(D_{T^-}^{\alpha}\left(D_{0^+}^{\alpha}\mathrm{A}u(x)\right)\right)\right)v(x)\,\mathrm{d}x,
\end{aligned}$$

then

$$\begin{aligned}
&\int_0^T \left(D_{T^-}^{\alpha}\left(D_{0^+}^{\alpha}u(x)\right)\right)\left(D_{T^-}^{\alpha}\left(D_{0^+}^{\alpha}v(x)\right)\right)\mathrm{d}x\\
&- \int_0^T D_{T^-}^{\alpha}\left(D_{0^+}^{\alpha}\left(D_{T^-}^{\alpha}\left(D_{0^+}^{\alpha}\mathrm{A}u(x)\right)\right)\right)v(x)\,\mathrm{d}x = 0.
\end{aligned}$$

We deduce that

$$\begin{aligned}
&\int_0^T \left(D_{T^-}^{\alpha}\left(D_{0^+}^{\alpha}u(x)\right)\right)\left(D_{T^-}^{\alpha}\left(D_{0^+}^{\alpha}v(x)\right)\right)\mathrm{d}x - \int_0^T f\left(x,u(x)\right)v(x)\,\mathrm{d}x = 0\\
&\Leftrightarrow \quad \langle u,v\rangle_{2\alpha} - \langle \mathrm{A}u,v\rangle_{2\alpha} = 0 \quad \Leftrightarrow \quad \langle u-\mathrm{A}u,v\rangle_{2\alpha} = 0.
\end{aligned}$$

Beside, it was proven in [10] that if $f\colon [0,T]\times\mathbb{R}\to\mathbb{R}$ a Carathéodory function such that

(C_1) $\left|f(x,y)\right| \le a(t)|y|^{\theta_1} + b(t)$, with $a(t),b(t),\in L^1\left([0,T]\right)$, $\theta_1 > 0$ hold,

then the functional

$$\varphi(u) = \frac{1}{2}\|u\|_{2\alpha} - \int_0^T F\left(x,u(x)\right)\mathrm{d}x$$

is well defined and Gâteaux differentiable on the Hilbert space $E_0^{2\alpha}$ with

$$\begin{aligned}
\varphi'(u).v &= \langle u,v\rangle_{2\alpha} - \int_0^T f\left(x,u(x)\right)v(x)\,\mathrm{d}x, \qquad \forall v \in E_0^{2\alpha},\\
&= \langle u-\mathrm{A}u,v\rangle_{2\alpha}.
\end{aligned}$$

That is

$$\varphi'(u) = (I-\mathrm{A})(u),$$

which means that a fixed point of A is a critical point of φ. In this sense, we will apply Theorem 1 to prove existence of critical point in $E_0^{2\alpha}$ for the functional φ. Let f' be a Carathéodory function, where f' is the derivative of f with respect to its second variable and satisfying:

(C_2) $\left|f'(x,y)\right| \leq p(t)|y|^{\theta_2} + q(t)$, with $p(t), q(t) \in L^1([0,T])$, $\theta_2 > 0$.

The following lemma is an important tool for our proof.

Lemma 4. *If $(C_1), (C_2)$ are satisfied, then φ is twice Gâteau differentiable and*

$$\left(\left(\varphi''(u)\right)v\right)z = \int_0^T \left(D_{T^-}^{\alpha}\left(D_{0^+}^{\alpha}v(x)\right)\right)\left(D_{T^-}^{\alpha}\left(D_{0^+}^{\alpha}z(x)\right)\right)\mathrm{d}x - \int_0^T f'\left(x,u(x)\right)v(x)z(x)\,\mathrm{d}x$$

for all $v, z \in E_0^{2\alpha}$.

Proof. Since (C_1) hold, then:

$$\varphi'(u).v = \langle u, v\rangle_{2\alpha} - \int_0^T f\left(x,u(x)\right)v(x)\,\mathrm{d}x,$$

for all $v \in E_0^{2\alpha}$. In a same way, if (C_2) hold, then φ′ is Gâteaux differentiable. Setting

$$\mathrm{K}(u) = \frac{1}{2}\|u\|_{2\alpha}, \quad \mathrm{J}(u) = \int_0^T F\left(x,u(x)\right)\mathrm{d}x$$

and by using the calculation rules in [4], we have

$$\begin{aligned}\left(\left(\varphi''(u)\right)v\right)z &= \frac{d}{\mathrm{d}t}\left(\frac{d}{\mathrm{d}s}\mathrm{K}(u+tv+sz)\right)\Big|_{t=s=0} - \frac{d}{\mathrm{d}t}\left(\frac{d}{\mathrm{d}s}\mathrm{J}(u+tv+sz)\right)\Big|_{t=s=0}\\ &= \langle z, v\rangle_{2\alpha} - \int_0^T f'(x,u(x))z(x)v(x)\,\mathrm{d}x, \quad \forall v, z \in E_0^{2\alpha}.\end{aligned}$$

□

By a subsolution and a supersolution of (3.4), we mean respectively $\mu, \zeta \in E_0^{2\alpha}$ satisfying respectively

$$\int_0^T \left(D_{T^-}^{\alpha}\left(D_{0^+}^{\alpha}\mu(x)\right)\right)\left(D_{T^-}^{\alpha}\left(D_{0^+}^{\alpha}v(x)\right)\right)\mathrm{d}x - \int_0^T f\left(x,\mu(x)\right)v(x)\,\mathrm{d}x \leq 0$$

and

$$\int_0^T \left(D_{T^-}^{\alpha}\left(D_{0^+}^{\alpha}\zeta(x)\right)\right)\left(D_{T^-}^{\alpha}\left(D_{0^+}^{\alpha}v(x)\right)\right)\mathrm{d}x - \int_0^T f\left(x,\zeta(x)\right)v(x)\,\mathrm{d}x \geq 0$$

for all $v \in E_0^{2\alpha}$, $v \geq 0$. Now, we assume that

(C_3) For each fixed $x \in [0,T]$, $f(x,y)$ is a nondecreasing function of y for

$$\mu(x) \leq y \leq \zeta(x).$$

(C_4) $|f'(x,y)| \leq \dfrac{(\Gamma(\alpha+1))^4}{T^{4\alpha}}, \quad \forall x \in [0,T], \forall y \in \mathbb{R}, \ 0 \leq \alpha < 1.$

The following theorem is our main result.

Theorem 2. *If $(C_1)-(C_4)$ hold, then Problem (3.1) has at least a weak solution in $E_0^{2\alpha}$.*

Proof. Let us consider the functional φ defined on $E_0^{2\alpha}$ by

$$\varphi(u) = \frac{1}{2}\|u\|_{2\alpha} - \int_0^T F(x,u(x))\,dx$$

with $\varphi' = I - A$, and A the operator defined by (3.2). We show that all conditions of Theorem 1 are satisfied.

(1) By Lemma 4, φ is twice Gâteaux differentiable.

(2) $\|(I'-\varphi'')u\| \leq 1$. Indeed, from Propositions 2, and 3 and Holder's inequality, we have

$$\begin{aligned}
\left\|(I'-\varphi'')(u)\right\| &= \sup_{\|v\|_{2\alpha}\leq 1, \|z\|_{2\alpha}\leq 1} \left|(I'-\varphi'')(u).v.z\right|, \\
&\leq \frac{(\Gamma(\alpha+1))^4}{T^{4\alpha}} \sup_{\|v\|_{2\alpha}\leq 1, \|z\|_{2\alpha}\leq 1} \left(\int_0^T |v(x)|^2 dx\right)^{\frac{1}{2}} \left(\int_0^T |z(x)|^2 dx\right)^{\frac{1}{2}} \\
&\leq \frac{(\Gamma(\alpha+1))^4}{T^{4\alpha}} \sup_{\|v\|_{2\alpha}\leq 1, \|z\|_{2\alpha}\leq 1} \|v\|_{L^2}.\|z\|_{L^2} \\
&\leq \frac{(\Gamma(\alpha+1))^4}{T^{4\alpha}} \left(\frac{T^\alpha}{\Gamma(\alpha+1)}\right)^2 \left(\frac{T^\alpha}{\Gamma(\alpha+1)}\right)^2 \\
&\quad \cdot \sup_{\|v\|_{2\alpha}\leq 1, \|z\|_{2\alpha}\leq 1} \|v\|_{2\alpha}.\|z\|_{2\alpha} \\
&\leq 1.
\end{aligned}$$

(3) Let

$$D = [\mu,\ \zeta] = \left\{u \in E_0^{2\alpha} : \mu(x) \leq u(x) \leq \zeta(x),\ \forall x \in [0,T]\right\}.$$

We can easily check that D is a bounded, closed and convex subset of the space $E_0^{2\alpha}$. Moreover,

$$\left(I - \varphi'(D)\right) \subset D.$$

Indeed, let $u \in D$, then from the assumption (C_3) and the definition of A, μ, ζ, we have

$$\langle \mu, v\rangle_{2\alpha} \leq \left((I-\varphi')(u), v\right) \leq \langle \zeta, v\rangle_{2\alpha}, \qquad \forall v \in E_0^{2\alpha}, \text{ and } v \geq 0.$$

Finally, by the maximum principle, we get

$$\mu \leq (I-\varphi')(u) \leq \zeta, \forall u \in D,$$

that is

$$(I-\varphi')(D)\subset D.$$

All conditions of Theorem 1 hold, then the functional φ has a critical point which is the weak solution of the Problem (3.1). □

Example 1. Let us consider the following boundary value problem:

$$\begin{cases} D^{\frac{1}{5}}_{\frac{\pi}{12}-}\left(D^{\frac{1}{5}}_{0+}\left(D^{\frac{1}{5}}_{\frac{\pi}{12}-}\left(D^{\frac{1}{5}}_{0+}u(x)\right)\right)\right)=f\left(x,u(x)\right), \qquad x\in\left[0,\dfrac{\pi}{12}\right], \\ u(0)=u\left(\dfrac{\pi}{12}\right)=0, \\ D^{\frac{1}{5}}_{\frac{\pi}{12}-}\left(D^{\frac{1}{5}}_{0+}u(0)\right)=D^{\frac{1}{5}}_{\frac{\pi}{12}-}\left(D^{\frac{1}{5}}_{0+}u\left(\dfrac{\pi}{12}\right)\right)=0, \end{cases} \tag{3.5}$$

where

$$f(x,y)=\begin{cases} \dfrac{1}{2}\dfrac{\left(\Gamma\left(\frac{6}{5}\right)\right)^4}{\left(\frac{\pi}{12}\right)^{\frac{4}{5}}}(y+1)\sin(x), & \text{if } x\in\left[0,\dfrac{\pi}{12}\right] \text{ and } y\le 0, \\ \dfrac{1}{2}\dfrac{\left(\Gamma\left(\frac{6}{5}\right)\right)^4}{\left(\frac{\pi}{12}\right)^{\frac{4}{5}}}(y-1)\sin(x), & \text{if } x\in\left[0,\dfrac{\pi}{12}\right] \text{ and } y>0. \end{cases}$$

For each fixed $x\in\left[0,\dfrac{\pi}{12}\right]$, $f(x,y)$ is a nondecreasing function of all $y\in\mathbb{R}$. Moreover, we can easily check that $\mu(x)=0$ and $\zeta(x)=\dfrac{\pi}{15}$ are subsolution and supersolution of (3.5), respectively. Indeed, we have

$$\begin{aligned} f(x,0)&=\frac{1}{2}\frac{\left(\Gamma\left(\frac{6}{5}\right)\right)^4}{\left(\frac{\pi}{12}\right)^{\frac{4}{5}}}\sin(x)\ge 0=D^{\frac{1}{5}}_{\frac{\pi}{12}-}\left(D^{\frac{1}{5}}_{0+}\left(D^{\frac{1}{5}}_{\frac{\pi}{12}-}\left(D^{\frac{1}{5}}_{0+}u(0)\right)\right)\right) \\ &=D^{\frac{1}{5}}_{\frac{\pi}{12}-}\left(D^{\frac{1}{5}}_{0+}\left(D^{\frac{1}{5}}_{\frac{\pi}{12}-}\left(D^{\frac{1}{5}}_{0+}\mu(x)\right)\right)\right). \end{aligned}$$

On the other hand,

$$f\left(x,\zeta(x)\right)=f\left(x,\frac{\pi}{15}\right)=\frac{1}{2}\frac{\left(\Gamma\left(\frac{6}{5}\right)\right)^4}{\left(\frac{\pi}{12}\right)^{\frac{4}{5}}}\left(\frac{\pi}{15}-1\right)\sin(x)<0.$$

By Property 1 and since $x\in\left[0,\dfrac{\pi}{12}\right]$, we have

$$D^{\frac{1}{5}}_{\frac{\pi}{12}-}\left(D^{\frac{1}{5}}_{0+}\left(D^{\frac{1}{5}}_{\frac{\pi}{12}-}\left(D^{\frac{1}{5}}_{0+}\zeta(x)\right)\right)\right)=\frac{\frac{\pi}{15}}{\Gamma\left(\frac{1}{5}\right)}(x)^{\frac{-4}{5}}>0,$$

then

$$D^{\frac{1}{5}}_{\frac{\pi}{12}-}\left(D^{\frac{1}{5}}_{0+}\left(D^{\frac{1}{5}}_{\frac{\pi}{12}-}\left(D^{\frac{1}{5}}_{0+}\zeta(x)\right)\right)\right)\ge f\left(x,\zeta(x)\right).$$

All the assumptions in the statement of Theorem 2 are satisfied, so Problem (3.5) has at least a nontrivial solution.

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UNIVERSAL DISTANCE DOMINATION IN RANDOM GRAPHS

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Abstract. Given a graph $G = (V,E)$, a *dominating set* is a subset $D \subseteq V$ such that every vertex in $V \setminus D$ is adjacent with at least one vertex in D. The *domination number* of G, traditionally denoted by $\gamma(G)$, is the minimum cardinality of a dominating set in G.

For any natural number k, a set D is *k-distance dominating*—called kdd set for short—if every vertex not in D is at distance at most d from some vertex in D. We define the *universal k-distance domination number* as

$$u_k(G) := \min\{d : \forall D \subseteq V \text{ with } |D| \geq d, D \text{ is a kdd set in } G\}.$$

In sharp contrast to most of the standard domination parameters, determining the universal k-distance domination number $u_k(G)$ turns out to be computable in polynomial time. We also characterize the graphs satisfying $\gamma(G) = u_1(G)$, and investigate the behavior of the function u_k on random graphs.

It remains a challenging open problem to characterize the equality of k-distance domination and universal k-distance domination numbers for a general k. A further problem is to sharpen the estimates on the edge probability in the auxiliary graph $H = G^k$.

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1. INTRODUCTION

The key notion studied here is k-distance domination, that involves vertex subsets in graphs, from which every vertex is at distance at most k apart. The history of distance domination began half a century ago, as introduced by Meir and Moon [13] (using the term "k-covering" at that time). Since then, more than 100 papers appeared on the subject. Beside theoretical interest, applications concerning various kinds of communication networks are also discussed in the literature; see, e.g., [8] and [9]. The interested reader can obtain a very detailed picture of the area by putting together the survey [15] and the (non-numbered) chapter [11] of a recent monograph.

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Those two works are complementary; the authors of the survey intentionally concentrated on the part of the literature not discussed in the book chapter. Beyond them, distance domination has been studied in vertex-partitioned graphs, as well; for results of that kind please see [10] and the references therein. Further issues concerning the relations between domination and distances are discussed in the papers [4–6], where the role of induced subgraphs is analyzed. [2] and [3] are also related to this paper.

A closer analysis of a proof in a paper by Wieland and Godbole [16] reveals that under some circumstances not only a k-distance dominating set of a certain size can be found, but in fact any vertex set of that size is k-distance dominating. This observation leads us to the new concept of universal distance domination, that we introduce in the present study. In this work both a deterministic and a random model of this concept will be discussed.

The decision version of the Set Cover problem is included in Karp's 21 NP-complete problems [12], and there is an easy direct reduction from it to (distance) domination. Hence, intractability of the latter is an immediate consequence from the early literature on the subject.

In sharp contrast to this, here we prove that determining the universal k-distance domination number $u_k(G)$, that we define in the next section, can be done in polynomial time, very efficiently. This follows from a result in Section 2 where we prove a minimax formula involving $u_k(G)$.

It is also a very interesting problem to characterize the graphs whose universal k-distance domination number is equal to their k-distance domination number. In Section 3 we solve the case $k = 1$, leaving the general question open. Let us note that a problem of similar flavor was recently studied by Boruzanlı Ekinci and Bujtás [14] who investigated the equality of domination and 2-domination numbers (the latter being the smallest size of a vertex set S such that every vertex not in S has at least two neighbors in S).

Concerning random graphs, in Section 4 we introduce a function defined recursively, and prove that it is an upper bound on $u_k(G)$ almost surely. Then the distribution of $u_k(G)$ for any given constant k is analyzed in Section 5.

2. Definitions and a minimax formula

Let us begin with quoting the definition of the fundamental notion of graph domination theory.

Definition 1. A vertex set D is *dominating* in the graph $G = (V,E)$ if for any $v \in V \setminus D$ there exists some $w \in D$ adjacent to v. The minimum cardinality of a dominating set is termed the *domination number*, and is denoted by $\gamma(G)$.

A natural generalization of this concept is the following.

Definition 2. A vertex set D is *k-distance dominating* in the graph $G = (V,E)$ if for any $v \in V \setminus D$ there exists some path of length at most k between some $w \in D$

and v. We use the shorthand kdd set to abbreviate "k-distance dominating set". We denote

$$\gamma_k(G) := \min\{|D| : D \text{ is a kdd set in } G\}.$$

In particular, we have

$$\gamma(G) := \gamma_1(G).$$

Motivated by Wieland and Godbole's work [16], here we introduce the following variant.

Definition 3. The *universal k-distance domination number* is

$$u_k(G) := \min\{d : \forall D \subseteq V \text{ with } |D| \geq d, D \text{ is a kdd set in } G\}$$

i.e. the minimum of d such that any vertex set of cardinality at least d vertices k-distance-dominates G.

Claim 1. Clearly, for any integer $k \geq 1$, every graph G satisfies the inequality

$$u_k(G) \geq \gamma_k(G).$$

As an illustration for $k = 1$, we mention the following fact.

Observation 1. *A graph G on n vertices has $u_1(G) \leq n-1$ if and only if there are no isolated vertices in G.*

Notation 1. For any two vertices u and v of a graph G, the distance between them will be denoted by $d_G(u,v)$, or by $d(u,v)$ for short when G is understood.

Let x be an arbitrary vertex of a graph G. The set $N_k[x]$ means $\{y \in V(G) : d(x,y) \leq k\}$, the *closed k-neighborhood* of x. It contains x, by definition. Let $C_k(x)$ be the vertex set $V(G) \setminus N_k[x]$ and $c_k(x) := 1 + |C_k(x)|$.

We state here a simple but useful minimax relation.

Theorem 1. *For the universal k-distance domination number $u_k(G)$ we have*

$$\begin{aligned} u_k(G) &= \min\{d : \forall D \subseteq V \text{ with } |D| \geq d, D \text{ is a kdd set in } G\} \\ &= \max\{c_k(x) : x \in V(G)\} \end{aligned} \tag{2.1}$$

for every graph G.

Proof. First, we prove that in (2.1), min $\geq$ max is valid.

Let us take a vertex x_0, yielding the maximum at the right-hand side. The set $C_k(x_0)$ does not k-distance dominate x_0. So it is not a kdd set, and the minimum $u_k(G)$ is strictly larger than $|C_k(x_0)|$. This means min $\geq$ max.

Second, we prove that the converse inequality min $\leq$ max also holds in (2.1).

By the definition of $u_k(G)$, there exists some vertex set W not k-distance dominating with $|W| = u_k(G) - 1$. This implies the existence of some vertex x_0 such that $W \cap N_k[x_0] = \varnothing$. Consequently, $|N_k[x_0]| \leq n - |W| \leq n - u_k(G) + 1$ and therefore

$$u_k(G) \leq 1 + n - |N_k[x_0]| = 1 + |C_k(x_0)| = c_k(x_0) = \max\{c_k(x) : x \in V(G)\}.$$

This completes the proof of the theorem. □

Theorem 1 makes it possible to design an efficient algorithm for the determination of *all* universal distance domination numbers of a graph.

Theorem 2. *Given a generic input graph $G = (V,E)$, the universal k-distance domination numbers $u_k(G)$ can be determined for all $k \geq 1$ in polynomial time (in at most $C * |V| * |E|$ steps, with an absolute constant C).*

Proof. First run any linear-time graph traversal algorithm that can identify the connected components of G. If G is disconnected, say it has components $G_1, \ldots, G_c$, then $u_k(G) = u_k(G_1) + \ldots + u_k(G_c)$, so the computation can be done separately on the components of G.

Assume now that the input graph $G = (V,E)$ is connected, and has n vertices. Then we run the standard Breadth-First Search algorithm n times, starting with each vertex $x \in V$ as root. During BFS we register the sizes $n_i(x) := |N_i[x]|$ of closed i-neighborhoods of x for $i = 1,2,\ldots$. Then, for any k, we have

$$u_k(G) = \max_{x \in V} c_k(x) = 1 + \left(n - \min_{x \in V} n_k(x) \right).$$

Since each run of BFS takes $O(|V| + |E|)$ time, and we need to make at most $O(n^2)$ comparisons at the end, the claimed total running time follows. □

3. On the equation $u_k(G) = \gamma_k(G)$

As noted in Claim 1, $u_k(G) \geq \gamma_k(G)$ holds for every graph G and every positive integer k. In this section we make the first step in characterizing the cases of equality.

Theorem 3. *A graph G satisfies the equality $u_1(G) = \gamma(G)$ if and only if G is complete or empty or it is the complement of a matching.*

Proof. We shall refer to the three types of graphs listed in the assertion as "sample graphs". It can easily be seen that every sample graph satisfies the eqality condition $u_1 = \gamma$.

To prove the "only if" part, let $G = (V,E)$ be any graph with $u_1(G) = \gamma(G)$, say on n vertices. As a useful notation, let us write

$$X(x) := \{y \in V : y \notin N_1[x]\},$$

where x is any vertex of G.

We first prove that G is regular. On the one hand, the vertex set $X(x) \cup \{x\}$ is dominating in G, which implies

$$\gamma(G) \leq n - d_G(x)$$

for every vertex x. On the other hand, the vertex set $X(x)$ is not dominating since it does not dominate x. Consequently, $u_1(G) > |X(x)| = n - d_G(x) - 1$. This fact and

the equality condition in the theorem imply

$$\gamma(G) \geq n - d_G(x)$$

for every x. Summarizing, the degree of each vertex is equal to $n-\gamma(G)$, thus G is regular.

Let us denote $w := u_1(G) - 1$. Note that the degree of regularity in G is equal to $n-1-w$, and w is in the range $0 \leq w \leq n-1$. Moreover, each set W of w vertices is non-dominating since $\gamma(G)$ is the minimum size of dominating vertex sets in the whole graph and, by assumption, we have $w < u_1(G) = \gamma(G)$.

Definition 4. For an arbitrary non-dominating vertex set Y, and for any vertex η not dominated by Y, we call η the *witness* of Y. (Certainly, Y may have several witnesses.)

Let us consider now a set W that has cardinality exactly w. This W is non-dominating. For any witness η of W, we have $W \subseteq X(\eta)$.

Suppose now that W is a *proper* subset of $X(\eta)$. In this case, there exists some $\xi \in X(\eta) \setminus W$ and adding it to W, we obtain a set Z of cardinality $u_1(G)$. By the definition of $u_1(G)$, Z is dominating. On the other hand, W does not dominate η, and ξ is also not adjacent to it. Thus, Z is not dominating, and we got a contradiction with the assumption "proper subset".

We have shown that, taking any vertex set W of cardinality w in the graph, it coincides with $X(\eta)$ for some vertex η. Consequently,

$$\binom{n}{w} \leq n\,.$$

This inequality is valid for $w = 0, 1, n-1$ only, that yields a clique, a co-matching and an empty graph, respectively. These are exactly the sample graphs. □

Here we prove an extension of Theorem 3 for general k.

We emphasize that Theorem 4 below is not an explicit structural characterization of those graphs G which satisfy $u_k(G) = \gamma_k(G)$.

Theorem 4. *A graph $G = (V,E)$ satisfies the equality $u_k(G) = \gamma_k(G)$ if and only if one of the following cases holds:*

- (*i*) *G has no edges, hence $u_k(G) = \gamma_k(G) = |V|$.*
- (*ii*) *G has diameter at most k, hence $u_k(G) = \gamma_k(G) = 1$.*
- (*iii*) *G has diameter $k+1$, and for every $x \in V$ there exists precisely one $\bar{x} \in V$ at distance $k+1$ apart from x. In this case, $u_k(G) = \gamma_k(G) = 2$.*

Proof. Here we introduce an auxiliary graph H.

Definition 5. In $H := G^k$, let $V(H) = V(G)$ and let $xy \in E(H)$ if and only if $d_G(x,y) \leq k$.

The definitions imply that the closed neighborhood of any vertex x in H coincides with $N_k[x]$ in G. Consequently, we have $u_1(H) = u_k(G)$ and $\gamma(H) = \gamma_k(G)$. Thus, $u_k(G) = \gamma_k(G)$ holds if and only if $u_1(H) = \gamma(H)$, and we may apply Theorem 3 for H.

For the cases when H is an empty graph or complete, respectively, we obtain the cases (i) and (ii).

For the third possibility, let (a_i, b_i), $i = 1, 2, \ldots, n/2$, be the edges of the perfect matching whose complementary graph is H. Then, in G, b_i is the unique vertex farther than k from a_i, and vice versa. Picking any neighbor u of a_i in G, we also see that a_i and b_i have distance

$$d_G(a_i, b_i) \le d_G(a_i, u) + d_G(u, b_i) \le 1 + k,$$

hence exactly $k+1$. This yields (iii). □

4. A PROBABILISTIC UPPER BOUND ON u_k

In this section we begin the study of the universal k-distance domination number u_k of the random graph $\mathbf{G}_{n,p}$ with n vertices and edge-probability p. We write $q := 1/(1-p)$. Let us consider now the function $f(n) := \lceil \log_q n + a_n \rceil$ where a_n is an arbitrary positive sequence tending to infinity, such that $f(n) < n$.

At first glance, the property that every vertex set of size at least d is k-dominating in the whole graph, seems to be too strong. However, Proposition 1 shows that its probability of having this property is relatively large.

Let us define the integer functions ϕ_k, $k \ge 1$, recursively: $\phi_1(n) := f(n)$, and $\phi_k(n) := f[\phi_{k-1}(n)]$ for all $k > 1$.

Proposition 1. *Let $k \ge 1$ be a fixed integer, and let $0 < p < 1$ be a fixed real number. Then*

$$\mathbb{P}(u_k(\mathbf{G}_{n,p}) \le \phi_k(n)) \to 1$$

as $n \to \infty$.

Proof. The argument works by induction on k. Essentially, we apply the proof of [16], page 2, where the assertion was proved for $k = 1$. This case serves as anchor for the induction.

Suppose that the proposition has been verified for $k-1$. Pick any set of vertices D with size $\phi_k(n)$ in $G := \mathbf{G}_{n,p}$. We will show that D is a kdd set in G, with high probability.

Let us extend D to a vertex set D' with size $\phi_{k-1}(n)$ arbitrarily. Consider

$$\mathbb{P}(D \text{ is a kdd set in } G) \ge \mathbb{P}(A_n B_n)$$

for the events

$$A_n := D' \text{ is } (k-1)\text{-distance dominating in } G$$

and

$$B_n := D \text{ is dominating in } D'.$$

We state the following two facts.
(i)

$$\mathbb{P}(A_n) \to 1,$$

it follows from the induction hypothesis.
(ii)

$$\mathbb{P}(B_n) \to 1.$$

Instead of n, in the proof of (ii) we use the variable ν, meaning the order of an arbitrary graph (possibly different from the original one), tending to ∞. Applying the abbreviations $M := \phi_{k-1}(\nu)$, $r := \phi_k(\nu)$, and recalling $q = 1/(1-p)$, from the Binomial Theorem we obtain

$$\rho := \mathbb{P}(D \text{ is dominating in } D') = [1-(1-p)^r]^{M-r}$$

$$\geq [1-(1-p)^r]^{\nu-r} \geq 1-(\nu-r)(1-p)^r,$$

$$\rho \geq 1-\nu(1-p)^r \geq 1-\nu(1-p)^{\log_q \nu + a_\nu},$$

because $(1-p)^{\log_q \nu} = 1/\nu$ and $\rho \geq 1-(1-p)^{a_\nu} \to 1$ as ν tends to ∞.

Clearly, (i) and(ii) together imply that $P(A_nB_n) \to 1$ holds, and from this

$$\mathbb{P}(D \text{ is } k\text{-distance dominating in } G) \to 1.$$

□

Since $u_k(G)$ is an upper bound on $\gamma_k(G)$ for every graph G and every positive integer k, Proposition 1 has the following immediate consequence.

Corollary 1. *For every integer $k \geq 1$ and every fixed edge probability p,*

$$\mathbb{P}(\gamma_k(\mathbf{G}_{n,p}) \leq \phi_k(n)) \to 1,$$

as $n \to \infty$.

5. On the distribution of u_k

In this section we analyze the asymptotic behavior of u_k in the random graph $\mathbf{G}_{n,p}$.

5.1. *The closed k-neighborhood*

For arbitrary fixed $k \geq 0$, let us consider the size $|N_k[x_0]|$ of the closed k-neighborhood for a vertex x_0 as a random variable. We compute here its distribution.

Recall that $d_G(u,v)$ or simply $d(u,v)$ denotes the distance between two vertices u, v of G.

We define now the *levels* with respect to x_0.

$$L_\mu(x_0) := \{y \in V(G) \mid d(y,x_0) = \mu\}$$

for $\mu = 0, 1, \ldots, k$.

The following is clearly true.

Observation 2. *For $z \in V(G)$ and $m \geq 1$, we have $d(x_0,z) = m+1$ (i.e., $z \in L_{m+1}$) if and only if*

(1) *z has at least one neighbor in L_m*
and
(2) *z has no neighbors in $N_{m-1}[x_0]$.*

For any sequence $j_1, \ldots, j_k$ of k nonnegative integers and any $1 \leq \mu \leq k$, let $\prod_\mu$ be the product of the following events: $|L_1(x_0)| = j_1, \ldots, |L_\mu(x_0)| = j_\mu$. For convenience we also put $j_0 := |L_0(x_0)| = 1$.

Let us consider the following events: $|L_1(x_0)| = j_1, \ldots, |L_\mu(x_0)| = j_\mu$. Let be Π_μ the product of these events.

We introduce a sequence $f_1, f_2, \ldots, f_k$ of auxuliary functions, that help in calculating the distribution of u_k. For $\mu = 1, \ldots, k$ the function f_μ has μ positive integer variables, and is defined as the probability $\mathbb{P}(\prod_\mu)$.

$$J_\mu := 1 + \sum_{\iota=1}^{\mu} j_\iota = \sum_{\iota=0}^{\mu} j_\iota \qquad (\mu = 1,2,\ldots k).$$

$$J_\mu := 1 + \sum_{\iota=1}^{\mu} j_\iota = \sum_{\iota=0}^{\mu} j_\iota \qquad (\mu = 1,2,\ldots k).$$

The meaning of J_μ is the number of vertices having distance at most μ from x_0.

We calculate these functions recursively.

Proposition 2. *For every sequence $j_1, \ldots, j_k$ the functions $f_1, f_2, \ldots, f_k$ can be computed as*

$$f_1(j_1) = \binom{n-1}{j_1} p^{j_1} (1-p)^{n-J_1},$$

$$f_2(j_1,j_2) = f_1(j_1) \binom{n-J_1}{j_2} (1-(1-p)^{j_1})^{j_2} (1-p)^{(n-J_2)j_1},$$

$$\vdots$$

$$f_\mu(j_1,\dots,j_\mu) = f_{\mu-1}(j_1,\dots,j_{\mu-1})\binom{n-J_{\mu-1}}{j_\mu}(1-(1-p)^{j_{\mu-1}})^{j_\mu}(1-p)^{(n-J_\mu)j_{\mu-1}},$$

$$\vdots$$

$$f_k(j_1,\dots,j_k) = f_{k-1}(j_1,\dots,j_{k-1})\binom{n-J_{k-1}}{j_k}(1-(1-p)^{j_{k-1}})^{j_k}(1-p)^{(n-J_k)j_{k-1}}.$$

Proof. We argue concerning the general step, applying Observation 2 with $\mu = m+1$. Having computed the probabilities and the positions of the levels up to $L_{\mu-1}$, there are $\binom{n-J_{\mu-1}}{j_\mu}$ ways to select the j_μ vertices of L_μ. Then each selected $x \in L_\mu$ has to be adjacent to at least one vertex in $L_{\mu-1}$; the probability that x satisfies this requirement is $1-(1-p)^{j_{\mu-1}}$. This is not only necessary but also sufficient for $x \in L_\mu$ because in the preceding steps we ensured that x does not belong to levels of smaller indices. The exponent j_μ of this expression in the recursive formula is due to the fact that for distinct vertices of L_μ these events are totally independent.

In addition, it has to be ensured that the unselected $n-J_\mu$ vertices are nonadjacent to $L_{\mu-1}$. This means that all edges of a complete bipartite graph with $n-J_\mu$ and $j_{\mu-1}$ vertices in its partite sets are missing from G, hence yielding the last factor in the formula, namely the power of $(1-p)$ with the claimed exponent. (Non-adjacency to levels of smaller indices were decided in earlier steps.) □

Now we are in the position to calculate the distibution we wanted.

Let us introduce the notation

$$W_{k,i} := \mathbb{P}(|N_k[x_0]| = i).$$

Theorem 5.

$$W_{k,i} = \sum_{\substack{j_1,j_2,\dots,j_k \geq 1 \\ j_1+j_2+\dots+j_k = i-1}} f_k(j_1,j_2,\dots,j_k).$$

Clearly, the following statement is valid.

Observation 3.

$$\max\{c_k(x) : x \in V(G)\} = n+1-\min\{|N_k[x]| : x \in V(G)\}.$$

For the distribution of the minimum in Observation 3, first it seems that the method in [1] can be applied. Unfortunately, the complete independence is supposed there. Here we show a simple case where even pairwise independence is not valid.

Example 1. Let $1 \leq k \leq n-1$ and $\xi_i := |N_k[x_i]|$. Then $P(\xi_0 = n) > 0$, $P(\xi_1 = 1) > 0$, but $P(\xi_0 = n \text{ and } \xi_1 = 1) = 0$.

Due to the lack of independence among the random variables in question, an alternative method is developed in the sequel.

5.2. *The new edge probability*

In Theorem 1 we reduced the problem concerning u_k to the minimum of $|N_k(x)|$. Now we go further, to the distribution of minimum degree in an auxiliary graph and finally reducing it to a significant result of Bollobás.

Here we present the key notion of the process, the auxiliary graph $H := G^k$. The aim of introducing it, is to take a quantity much simpler than $|N_k(x)|$.

By definition, let $xy \in E(H)$ if and only if $d_G(x,y) \le k$. We obtained H from a random graph with edge probability p and thus H is a random graph again, with some edge probability, say, $s = s(n,p,k)$.

We give recursive estimates on s. Having n and p fixed, let us introduce the notation $\pi_\kappa(u,v) := P(d_G(u,v) \le \kappa)$.

Proposition 3. *For $k \ge 2$, the following recursive lower and upper bounds are valid for $s(n,p,k) = \pi_k(x,y)$.*

(i) $\pi_k(x,y) \ge p * \max\{\pi_{k-1}(z,y) \mid z \in V(H) \setminus \{x,y\}\}$,

(ii) $\pi_k(x,y) \le p * \sum_{z \in V(H)\setminus\{x,y\}} \pi_{k-1}(z,y)$.

Proof. In order to verify the lower bound we observe that $\pi_k(x,y) = s(n,p,k) \ge \max\{P(xz \in E(G)|) * \pi_{k-1}(z,y) \mid z \in V(H) \setminus \{x,y\}\}$. Moreover, we have here disjoint edge sets and thus, independent events. The reason for the validity of the upper bound is similar. □

Remark 1. In fact, the proof of Proposition 3 is based on the following principle.

Observation 4. *The event $xy \in E(H)$ is the sum of the events $d_G(y,z) \le k-1$ for $z \in V(H) \setminus \{x,y\}$ and the event $xy \in E(G)$.*

5.3. *The minimum degree*

Supposing that the edge probability s in the auxuiliary random graph H is in our hand, it is enough to determine the distribution of minimum degree $\delta(H)$ in $H = G^k$.

Below, we abbreviate to G_s the notation for the random graph of order n and edge probability s, where $s = s(n,p,k)$ as defined above. From the theorem on page 65 of [7], the following formulas can be obtained.

Theorem 6 ([7], Theorem 3.5.)**.** *Let $0 < s < 1$ be given. Let us express it in the form*

$$s = (\log n + h \log\log n + y)/n,$$

where n is large enough, $h \ge 2$ is an integer and $y \in \mathbb{R}$ is a real number.

Then

$$P(\delta(G_s) = h) \to 1 - e^{-e^{-y/h!}}$$

and

$$P(\delta(G_s) = h+1) \to e^{-e^{-y/h!}},$$

if $n \to \infty$.

Remark 2. We can express y in terms of n, h and s:

$$y = ns - \log n - h \log\log n$$

5.4. *Summarizing*

Our original target was to determine the distribution of $u_k(G)$. First we calculated its exact value by a recursive formula in Subsection 5.1. Then we reduced distance domination to an auxiliary degree condition in Subsection 5.2. Finally in Subsection 5.3, we gave its limit distribution, using a classical result.

6. Concluding remarks

In this paper we introduced and studied universal k-distance dominating sets of graphs. Many new problems arise, here we mention those most related to the present study.

Problem 1. *Give a structural characterization of those graphs G for which $u_k(G) = \gamma_k(G)$ holds.*

Remark 3. Theorem 3 yields the answer for $k = 1$, and necessary and sufficient conditions are given in Theorem 4. In fact the latter is closely related to the long-standing open problem to characterize the so-called antipodal graphs. There are several variations of those graphs in the literature, and to the best of our knowledge, none of them has been characterized so far.

Problem 2. *Sharpen the estimates given in Proposition 3 concerning the asymptotics of the edge probability of the graph $H = (\mathbf{G}_{n,p})^k$.*

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SOME HERMITE–HADAMARD AND OSTROWSKI TYPE FRACTIONAL INTEGRAL INEQUALITIES VIA NEW GENERALIZED m-TYPE CONVEXITY

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Abstract. A new generalized class of m-type convex functions is presented. Some Hermite–Hadamard and Ostrowski type inequalities via Riemann–Liouville fractional integrals are obtained for these functions. Some particular cases are discussed as applications of the findings acquired in this study.

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1. INTRODUCTION

Let Δ be an interval of real numbers. If the following inequality

$$\varphi\left(\frac{\mu+\nu}{2}\right) \leq \frac{1}{\nu-\mu}\int_{\mu}^{\nu}\varphi(\tau)\,d\tau \leq \frac{\varphi(\nu)+\varphi(\mu)}{2} \tag{1.1}$$

holds for all $\mu,\nu\in\Delta$ with $\mu<\nu$, then φ is said to be a convex function on Δ. In the literature, inequality (1.1) is famed as the Hermite–Hadamard inequality for convex functions (see [13]).

A number of researchers have used fractional integrals, fractional quantum integrals, and other forms of integrals to establish the above inequality for various classes of convex functions. Sarikaya et al. [16], proposed the fractional integral version of (1.1), which is as follows

$$\varphi\left(\frac{\mu+\nu}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(\nu-\mu)^{\alpha}}\left(J^{\alpha}_{\mu^{+}}\varphi(\nu)+J^{\alpha}_{\nu^{-}}\varphi(\mu)\right) \leq \frac{\varphi(\nu)+\varphi(\mu)}{2}. \tag{1.2}$$

In addition, depending on (1.2), the authors have presented a trapezoid type inequality for convex functions in [16]. In the previous decade, many authors generalized

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inequality (1.2) and used it to derive a number of well-known fractional integral inequalities for different types of convex functions. For further detail on this, we refer the reader to [1,3,4,8,12,14,17,19–21,24,26–28].

G. Toader [22], defined a new class of m-convex functions, which gave rise to the definition of other classes of convex functions such as (α,m)-convex functions, (s,m)-convex functions and etc. [2,5,6,9,15,18] contains more information on extended classes of m-convex functions as well as certain well-known fractional integral inequalities for these functions, such as Hermite–Hadamard, Ostrowski type, and so forth.

Kashuri et al. [7], recently introduced a new generalized class of convex functions as well as several integral inequalities for these functions, leading to a number of known and unknown findings in the area of the modern theory of inequalities.

In this study, we define a new generalized class of m-type convex functions and derive some Hermite–Hadamard and Ostrowski type inequalities for these functions, based on the results and approaches in [7,16,18,22]. The previously known findings for numerous classes of convex functions as a particular case are included in our results.

2. Preliminaries

In this part, we revisit several previous definitions and lemmas which help us in proving our results.

Definition 1 ([13])**.** Let Δ be an interval of real numbers. A function $\varphi\colon \Delta \to \mathbb{R}$ is said to be convex on Δ if

$$\varphi(\mu\tau+(1-\tau)\nu) \le \mu\varphi(\tau)+(1-\tau)\varphi(\nu) \tag{2.1}$$

holds for all $\mu,\nu \in \Delta$, and $\tau \in [0,1]$.

Ω represents the convex set $\{\psi_1,\psi_2,\dots,\psi_k,\phi_1,\phi_2,\dots,\phi_l\}$ in the following definition, (see [7]).

Definition 2 ([7])**.** Let $\psi_{r_1},\phi_{r_2}\colon [0,1] \to [0,\infty)$ are continuous functions for all $r_1 = 1,2,\dots,k$ and $r_2 = 1,2,\dots,l$ where $k,l \in \mathbb{N}$ and $1 \le k \le l$. If a function $\varphi\colon \Delta \subseteq \mathbb{R} \to [0,\infty)$ satisfies the following inequality

$$\varphi(\mu\tau+(1-\tau)\nu) \le \left(\frac{1}{k}\sum_{r_1=1}^{k}\psi_{r_1}(\tau)\right)\varphi(\mu)+\left(\frac{1}{l}\sum_{r_2=1}^{l}\phi_{r_2}(\tau)\right)\varphi(\nu) \tag{2.2}$$

for all $\mu,\nu \in \Delta$ and $\tau \in [0,1]$, then it is said to be (k,l)-generalized convex with respect to Ω.

Definition 3 ([22])**.** Let $b > 0$. A function $\varphi\colon [0,b] \to \mathbb{R}$ is called m-convex on $[0,b]$ if

$$\varphi(\tau\mu+m(1-\tau)\nu) \le \tau\varphi(\mu)+m(1-\tau)\varphi(\nu) \tag{2.3}$$

holds for all $\mu,\nu \in [0,b]$, and $\tau \in [0,1]$.

Definition 4 ([5]). Let $(s,m) \in (0,1]^2$ and $b > 0$. A function $\varphi\colon [0,b] \to \mathbb{R}$ is said to be (s,m)-convex in the second sense on $[0,b]$ if

$$\varphi(\mu\tau + m(1-\tau)\nu) \leq \tau^s \varphi(\mu) + m(1-\tau)^s \varphi(\nu) \tag{2.4}$$

holds for all $\mu,\nu \in [0,b]$ and $\tau \in [0,1]$.

Definition 5 ([9]). Let $(\alpha,m) \in (0,1]^2$ and $b > 0$. A function $\varphi\colon [0,b] \to \mathbb{R}$ is said to be (α,m)-convex on $[0,b]$ if

$$\varphi(\mu\tau + m(1-\tau)\nu) \leq \tau^\alpha \varphi(\mu) + m(1-\tau^\alpha)\varphi(\nu) \tag{2.5}$$

is true for all $\mu,\nu \in \Delta$ and $\tau \in [0,1]$.

Definition 6 ([14]). Let $s \in [0,1]$, $n \in \mathbb{N}$ and Δ is an interval of real numbers. A function $\varphi\colon \Delta \to \mathbb{R}$ is called n-polynomial s-type convex if

$$\varphi(\mu\tau + (1-\tau)\nu) \leq \frac{1}{k}\sum_{r=1}^{k}\left[1-(s(1-\tau))^r\right]\varphi(\mu) + \frac{1}{k}\sum_{r=1}^{k}\left[1-(s\tau)^r\right]\varphi(\nu) \tag{2.6}$$

holds for all $\mu,\nu \in \Delta$ and $\tau \in [0,1]$.

Definition 7 ([25]). Let $h\colon [0,1] \to \mathbb{R}$ be a positive function and Δ is an interval of real numbers. A function $\varphi\colon \Delta \subseteq [0,\infty) \to \mathbb{R}$ is said to be h-convex, if

$$\varphi(\mu\tau + (1-\tau)\nu) \leq h(\tau)\varphi(\mu) + h(1-\tau)\varphi(\nu) \tag{2.7}$$

holds for all $\mu,\nu \in \Delta$ and $\tau \in [0,1]$.

In the following, we state the definition of Riemann–Liouville integrals, which are used in this study. For further information, one can see [10].

Definition 8. Assume that $\mu \geq 0$, $\alpha > 0$ and $\varphi \in L[\mu,\nu]$. The Riemann–Liouville integrals $J^\alpha_{\mu^+}\varphi$ and $J^\alpha_{\nu^-}\varphi$ of order α are defined by

$$J^\alpha_{\mu^+}\varphi(\xi) = \frac{1}{\Gamma(\alpha)}\int_\mu^\xi (\xi-\tau)^{\alpha-1}\varphi(\tau)\,d\tau\,,\ \xi > \mu$$

and

$$J^\alpha_{\nu^-}\varphi(\xi) = \frac{1}{\Gamma(\alpha)}\int_\xi^\nu (\tau-\xi)^{\alpha-1}\varphi(\tau)\,d\tau\,,\ \xi < \nu$$

respectively, where $\Gamma(\alpha) = \int_0^1 e^\tau \tau^{\alpha-1}\ d\tau$ and $J^0_{\mu^+}\varphi(\xi) = J^0_{\nu^-}\varphi(\xi) = \varphi(\xi)$.

The fractional integral is reduced to the classical integral when $\alpha = 1$.

Lemma 1 ([16]). *Assume that $\varphi\colon \Delta \to \mathbb{R}$ is a differentiable function on Δ°. If $\varphi' \in L[\mu,\nu]$, where $\mu,\nu \in \Delta$ with $\mu < \nu$, then the following equality*

$$\frac{\varphi(\mu)+\varphi(\nu)}{2} - \frac{\Gamma(1+\alpha)}{2(\nu-\mu)^\alpha}\left(J^\alpha_{\mu^+}\varphi(\nu) + J^\alpha_{\nu^-}\varphi(\mu)\right)$$

$$= \frac{(\nu-\mu)}{2}\int_0^1 \left[(1-\tau)^{\alpha}-\tau^{\alpha}\right]\varphi'\left(\mu\tau+(1-\tau)\nu\right)d\tau \tag{2.8}$$

is true for all $\alpha > 0$.

Lemma 2 ([18]). *Assume that* $\varphi\colon \Delta \to \mathbb{R}$ *is a differentiable function on* Δ°. *If* $\varphi' \in L[\mu,\nu]$, *where* $\mu,\nu \in \Delta$ *with* $\mu < \nu$, *then the following equality*

$$\frac{(\xi-\mu)^{\alpha}+(\nu-\xi)^{a}}{\nu-\mu}\varphi(\xi) - \frac{\Gamma(1+\alpha)}{\nu-\mu}\left(J^{\alpha}_{\mu^{+}}\varphi(\nu)+J^{\alpha}_{\nu^{-}}\varphi(\mu)\right) \tag{2.9}$$
$$= \frac{1}{\nu-\mu}\left((\xi-\mu)^{\alpha+1}\int_0^1 \tau^{a}\varphi'\left(\tau\xi+(1-\tau)\mu\right)d\tau - (\nu-\xi)^{\alpha+1}\int_0^1 \tau^{a}\varphi'\left(\tau\xi+(1-\tau)\nu\right)d\tau\right)$$

is valid for all $\xi \in [\mu,\nu]$ *and* $\alpha > 0$.

The Euler beta and incomplete beta functions are defined by

$$\beta(\omega,\rho) = \int_0^1 \tau^{\omega-1}(1-\tau)^{\rho-1}d\tau,$$

$$\beta(\xi;\omega,\rho) = \int_0^{\xi} \tau^{\omega-1}(1-\tau)^{\rho-1}d\tau,$$

respectively, where $\omega,\rho > 0$ and $0 < \xi < 1$, (see [11]).

Now, we merge Definition 3 and 2 to obtain the class of generalized m-type convex functions as follows.

Definition 9. Let $\psi_{r_1},\phi_{r_2}\colon [0,1] \to [0,\infty)$ are continuous functions for all $r_1 = 1,2,\ldots,k$ and $r_2 = 1,2,\ldots,l$ where $k,l \in \mathbb{N}$ and $1 \le k \le l$. If a function $\varphi\colon [0,b] \to [0,\infty)$, $b > 0$, satisfies the following inequality

$$\varphi(\mu\tau+m(1-\tau)\nu) \le \left(\frac{1}{k}\sum_{r_1=1}^{k}\psi_{r_1}(\tau)\right)\varphi(\mu)+m\left(\frac{1}{l}\sum_{r_2=1}^{l}\phi_{r_2}(\tau)\right)\varphi(\nu) \tag{2.10}$$

for all $\mu,\nu \in [0,b]$; $\tau \in [0,1]$, and $m \in (0,1]$, then it is said to be (k,l)-generalized m-type convex with respect to Ω.

Remark 1.

(1) Definition 9 is reduced to Definition 3 when $\psi_{r_1}(\tau) = \tau$, $\phi_{r_2} = 1-\tau$ and $k = l = 1$ in (2.10), and $\varphi\colon [0,b] \to [0,\infty)$, $b > 0$.
(2) Definition 9 is reduced to Definition 4 when $\psi_{r_1}(\tau) = \tau^{s}$, $\phi_{r_2} = (1-\tau)^{s}$, and $k = l = 1$ in (2.10), and $\varphi\colon [0,b] \to [0,\infty)$, $b > 0$.
(3) Definition 9 is reduced to Definition 5 when $\psi_{r_1}(\tau) = \tau^{\alpha}, \phi_{r_2} = 1-\tau^{\alpha}$ and $k = l = 1$ in (2.10), and $\varphi\colon [0,b] \to [0,\infty)$, $b > 0$.
(4) Definition 9 is reduced to Definition 6 when $\psi_{r_1}(\tau) = 1-(s(1-\tau))^{r_1}$, $\phi_{r_2} = 1-(s\tau)^{r_2}$, $m = 1$ and $r_1 = r_2 = 1,2,\ldots,k$ in (2.10), $\varphi\colon [0,b] \to [0,\infty)$, $b > 0$.
(5) Definition 9 is reduced to Definition 7 when $\psi_{r_1}(\tau) = h(\tau)$, $\phi_{r_2} = h(1-\tau)$ and $m = k = l = 1$ in (2.10).

By appropriately selecting the functions ψ_{r_1} and ϕ_{r_2}, the reader can obtain a number of known and unknown classes of m-convex functions.

3. MAIN RESULTS

In this section, we describe some algebraic properties of generalized m-type convex functions and investigate fractional integral inequalities of the Hermite–Hadamard and Ostrowski types for these functions.

Theorem 1. *Let $\varphi, \varphi_1, \varphi_2 \colon [0,b] \to [0,\infty)$, $b > 0$, are (k,l)-generalized m-type convex functions with respect to Ω. Then*

(1) *$\varphi_1 + \varphi_2$ is a (k,l)-generalized m-type convex with respect to Ω.*
(2) *$\gamma\varphi$ is a (k,l))-generalized m-type convex with respect to Ω, where γ is a positive real number.*

Proof. We ignore the proof of the theorem since it is straightforward. □

Theorem 2. *Let $\psi_{r_1}, \phi_{r_2} \colon [0,1] \to [0,\infty)$ are continuous functions for all $r_1 = 1,2,\ldots,k$ and $r_2 = 1,2,\ldots,l$ where $k,l \in \mathbb{N}$ and $1 \le k \le l$ and assume that $\varphi_2 \colon [0,b] \to [0,\infty)$, $b > 0$, is an increasing and (k,l)-generalized m-type convex function with respect to Ω and $\varphi_1 : \varphi_2([0,b]) \to [0,\infty)$ is an m-convex function, then $\varphi_1 \circ \varphi_2$ is a (k,l)-generalized m-type convex function with respect to Ω.*

Proof. The m-convexity of φ_2 and (k,l)-generalized m-type convexity of φ_1 give that

$$\begin{aligned}
\varphi_1 \circ \varphi_2(\mu\tau + m(1-\tau)\nu) &= \varphi_1(\varphi_2(\mu\tau + m(1-\tau)\nu)) \\
&\le \varphi_1(\tau\varphi_2(\mu) + m(1-\tau)\varphi_2(\nu)) \\
&\le \left(\frac{1}{k}\sum_{r_1=1}^{k}\psi_{r_1}(\tau)\right)\varphi_1(\varphi_2(\mu)) + m\left(\frac{1}{l}\sum_{r_2=1}^{l}\phi_{r_2}(\tau)\right)\varphi_1(\varphi_2(\nu)) \\
&= \left(\frac{1}{k}\sum_{r_1=1}^{k}\psi_{r_1}(\tau)\right)\varphi_1 \circ \varphi_2(\mu) + m\left(\frac{1}{l}\sum_{r_2=1}^{l}\phi_{r_2}(\tau)\right)\varphi_1 \circ \varphi_2(\nu)
\end{aligned}$$

is true for all $\mu, \nu \in [0,b]$ and $\tau \in [0,1]$. This ends the proof. □

Theorem 3. *Assume that $\psi_{r_1}, \phi_{r_2} \colon [0,1] \to [0,\infty)$ are continuous functions for all $r_1 = 1,2,\ldots,k$ and $r_2 = 1,2,\ldots,l$ where $k,l \in \mathbb{N}$ and $1 \le k \le l$ and let $(\varphi_i)_{i\in I}$ be a family, where $\varphi_i \colon [0,b] \to [0,\infty)$, $b > 0$, is a (k,l)-generalized m-type convex functions with respect to Ω for any $i \in I$.Then $\varphi(\omega) := \sup_i \varphi_i(\omega)$ is a (k,l)-generalized m-type convex function with respect to Ω and $\Upsilon = \{\omega \in [0,b] \colon \varphi(\omega) < +\infty\}$ is an interval.*

Proof. The (k,l)-generalized m-type convexity of φ_i with respect to Ω for any $i \in I$, give that

$$\varphi(\mu\tau + m(1-\tau)\nu) = \sup_i \varphi_i(\mu\tau + m(1-\tau)\nu)$$

$$\leq \sup_i \left(\left(\frac{1}{k} \sum_{r_1=1}^{k} \psi_{r_1}(\tau) \right) \varphi_i(\mu) + m \left(\frac{1}{l} \sum_{r_2=1}^{l} \phi_{r_2}(\tau) \right) \varphi_i(\nu) \right)$$
$$\leq \left(\frac{1}{k} \sum_{r_1=1}^{k} \psi_{r_1}(\tau) \right) \sup_i \varphi_i(\mu) + m \left(\frac{1}{l} \sum_{r_2=1}^{l} \phi_{r_2}(\tau) \right) \sup_i \varphi_i(\nu)$$
$$= \left(\frac{1}{k} \sum_{r_1=1}^{k} \psi_{r_1}(\tau) \right) \varphi(\mu) + m \left(\frac{1}{l} \sum_{r_2=1}^{l} \phi_{r_2}(\tau) \right) \varphi(\nu)$$

is valid for all $\mu, \nu \in \Upsilon$ and $\tau \in [0,1]$.

This ends the proof. □

Theorem 4. *Suppose that $\psi_{r_1}, \phi_{r_2} : [0,1] \to [0,\infty)$ are continuous functions for all $r_1 = 1,2,\ldots,k$ and $r_2 = 1,2,\ldots,l$ where $k,l \in \mathbb{N}$ and $1 \leq k \leq l$ and let $\varphi : [0,b] \to [0,\infty)$, $b > 0$, be a (k,l)-generalized convex function with respect to Ω. If $\mu, \nu \in [0,b]$ with $\mu < \nu$ and $\varphi \in L[\mu,\nu]$, then for any $\alpha > 0$ the following inequality holds*

$$\varphi\left(\frac{\mu+\nu}{2}\right) \leq \frac{\Gamma(1+\alpha)}{(\nu-\mu)} \left(\Psi\left(\frac{1}{2}\right) J^{\alpha}_{\mu^+} \varphi(\nu) + \Phi\left(\frac{1}{2}\right) J^{\alpha}_{\nu^-} \varphi(\mu) \right)$$
$$\leq \left[\left(\frac{\alpha\Psi\left(\frac{1}{2}\right)}{k} \sum_{r_1=1}^{k} \Psi_{r_1} + \frac{\alpha\Phi\left(\frac{1}{2}\right)}{l} \sum_{r_2=1}^{l} \Phi_{r_2} \right) \varphi(\mu) \right.$$
$$\left. + \left(\frac{\alpha\Phi\left(\frac{1}{2}\right)}{k} \sum_{r_1=1}^{k} \Psi_{r_1} + \frac{\alpha\Psi\left(\frac{1}{2}\right)}{l} \sum_{r_2=1}^{l} \Phi_{r_2} \right) \varphi(\nu) \right], \tag{3.1}$$

where

$$\Psi_{r_1} = \int_0^1 \tau^{\alpha-1} \psi_{r_1}(\tau) d\tau, \quad r_1 = 1,2,\ldots,k,$$

$$\Phi_{r_2} = \int_0^1 \tau^{\alpha-1} \phi_{r_2}(\tau) d\tau, \quad r_2 = 1,2,\ldots,l,$$

$$\Psi\left(\frac{1}{2}\right) = \frac{1}{k} \sum_{r_1=1}^{k} \psi_{r_1}\left(\frac{1}{2}\right),$$

$$\Phi\left(\frac{1}{2}\right) = \frac{1}{l} \sum_{r_2=1}^{l} \phi_{r_2}\left(\frac{1}{2}\right).$$

Proof. Since φ is a (k,l)-generalized m-type convex function with respect to Ω, then for all $\mu, \nu \in [0,b]$ with $\mu < \nu$, we get

$$\varphi(\mu\tau + (1-\tau)\nu) \leq \left(\frac{1}{k} \sum_{r_1=1}^{k} \psi_{r_1}(\tau) \right) \varphi(\mu) + \left(\frac{1}{l} \sum_{r_2=1}^{l} \phi_{r_2}(\tau) \right) \varphi(\nu). \tag{3.2}$$

Hence, we have that

$$\varphi\left(\frac{\zeta_1+\zeta_2}{2}\right) \leq \left(\frac{1}{k}\sum_{r_1=1}^{k}\psi_{r_1}\left(\frac{1}{2}\right)\right)\varphi(\zeta_1)+\left(\frac{1}{l}\sum_{r_2=1}^{l}\phi_{r_2}\left(\frac{1}{2}\right)\right)\varphi(\zeta_2)$$

is valid for any $\zeta_1,\zeta_2\in\Delta$ with $\zeta_1<\zeta_2$. If we set $\zeta_1=\tau\mu+(1-\tau)\nu$ and $\zeta_2=\nu\tau+(1-\tau)\mu$, where $\tau\in[0,1]$, then we obtain

$$\begin{aligned}\varphi\left(\frac{\mu+\nu}{2}\right) \leq{}& \left(\frac{1}{k}\sum_{r_1=1}^{k}\psi_{r_1}\left(\frac{1}{2}\right)\right)\varphi(\tau\mu+(1-\tau)\nu)\\ &+\left(\frac{1}{l}\sum_{r_2=1}^{l}\phi_{r_2}\left(\frac{1}{2}\right)\right)\varphi(\nu\tau+(1-\tau)\mu).\end{aligned} \tag{3.3}$$

Multiplying both sides of (3.3) by $\tau^{\alpha-1}$, then integrating the resulting inequality in terms of τ on $[0,1]$ yields

$$\begin{aligned}\frac{1}{\alpha}\varphi\left(\frac{\mu+\nu}{2}\right) &\leq \left(\frac{1}{k}\sum_{r_1=1}^{k}\psi_{r_1}\left(\frac{1}{2}\right)\right)\int_0^1\tau^{\alpha-1}\varphi(\tau\mu+(1-\tau)\nu)\,d\tau\\ &\quad+\left(\frac{1}{l}\sum_{r_2=1}^{l}\phi_{r_2}\left(\frac{1}{2}\right)\right)\int_0^1\tau^{\alpha-1}\varphi(\nu\tau+(1-\tau)\mu)\,d\tau\\ &=\left(\frac{1}{k}\sum_{r_1=1}^{k}\psi_{r_1}\left(\frac{1}{2}\right)\right)\frac{1}{(\mu-\nu)}\int_\nu^\mu\left(\frac{\nu-\omega}{\nu-\mu}\right)^{\alpha-1}\varphi(\omega)\,d\omega\\ &\quad+\left(\frac{1}{l}\sum_{r_2=1}^{l}\phi_{r_2}\left(\frac{1}{2}\right)\right)\frac{1}{(\nu-\mu)}\int_\mu^\nu\left(\frac{\varpi-\mu}{\nu-\mu}\right)^{\alpha-1}\varphi(\varpi)\,d\varpi\\ &=\frac{\Gamma(\alpha)}{(\nu-\mu)}\left(\left(\frac{1}{k}\sum_{r_1=1}^{k}\psi_{r_1}\left(\frac{1}{2}\right)\right)\frac{1}{\Gamma(\alpha)}\int_\nu^\mu\left(\frac{\nu-\omega}{\nu-\mu}\right)^{\alpha-1}\varphi(\omega)\,d\omega\right.\\ &\quad\left.+\left(\frac{1}{l}\sum_{r_2=1}^{l}\phi_{r_2}\left(\frac{1}{2}\right)\right)\frac{1}{\Gamma(\alpha)}\int_\mu^\nu\left(\frac{\varpi-\mu}{\nu-\mu}\right)^{\alpha-1}\varphi(\varpi)\,d\varpi\right)\\ &=\frac{\Gamma(\alpha)}{(\nu-\mu)}\left(\left(\frac{1}{k}\sum_{r_1=1}^{k}\psi_{r_1}\left(\frac{1}{2}\right)\right)J^{\alpha}_{\mu^+}\varphi(\nu)+\left(\frac{1}{l}\sum_{r_2=1}^{l}\phi_{r_2}\left(\frac{1}{2}\right)\right)J^{\alpha}_{\nu^-}\varphi(\mu)\right)\\ &=\frac{\Gamma(\alpha)}{(\nu-\mu)}\left(\Psi\left(\frac{1}{2}\right)J^{\alpha}_{\mu^+}\varphi(\nu)+\Phi\left(\frac{1}{2}\right)J^{\alpha}_{\nu^-}\varphi(\mu)\right).\end{aligned}$$

Thus we have

$$\varphi\left(\frac{\mu+\nu}{2}\right) \leq \frac{\Gamma(1+\alpha)}{(\nu-\mu)}\left(\Psi\left(\frac{1}{2}\right)J^{\alpha}_{\mu^+}\varphi(\nu)+\Phi\left(\frac{1}{2}\right)J^{\alpha}_{\nu^-}\varphi(\mu)\right). \tag{3.4}$$

Also, from the generalized convexity of φ on $[\mu,\nu]$, we have

$$\varphi(\mu\tau+(1-\tau)\nu)\leq\left(\frac{1}{k}\sum_{r_1=1}^{k}\psi_{r_1}(\tau)\right)\varphi(\mu)+\left(\frac{1}{l}\sum_{r_2=1}^{l}\phi_{r_2}(\tau)\right)\varphi(\nu) \tag{3.5}$$

and

$$\varphi(\nu\tau+(1-\tau)\mu)\leq\left(\frac{1}{k}\sum_{r_1=1}^{k}\psi_{r_1}(\tau)\right)\varphi(\nu)+\left(\frac{1}{l}\sum_{r_2=1}^{l}\phi_{r_2}(\tau)\right)\varphi(\mu). \tag{3.6}$$

Multiplying both sides of the inequalities (3.5) and (3.6) respectively by $\tau^{\alpha-1}\Psi\left(\frac{1}{2}\right)$ and $\tau^{\alpha-1}\Phi\left(\frac{1}{2}\right)$, then integrating the resulting inequalities in terms of τ on $[0,1]$, gives

$$\begin{aligned}\Psi\left(\frac{1}{2}\right)\int_0^1\tau^{\alpha-1}\varphi(\mu\tau+(1-\tau)\nu)\,d\tau\leq&\left(\frac{\Psi\left(\frac{1}{2}\right)}{k}\sum_{r_1=1}^{k}\int_0^1\tau^{\alpha-1}\psi_{r_1}(\tau)d\tau\right)\varphi(\mu)\\&+\left(\frac{\Psi\left(\frac{1}{2}\right)}{l}\sum_{r_2=1}^{l}\int_0^1\tau^{\alpha-1}\phi_{r_2}(\tau)d\tau\right)\varphi(\nu)\end{aligned}$$

and

$$\begin{aligned}\Phi\left(\frac{1}{2}\right)\int_0^1\tau^{\alpha-1}\varphi(\nu\tau+(1-\tau)\mu)\,d\tau\leq&\left(\frac{\Phi\left(\frac{1}{2}\right)}{k}\sum_{r_1=1}^{k}\int_0^1\tau^{\alpha-1}\psi_{r_1}(\tau)d\tau\right)\varphi(\nu)\\&+\left(\frac{\Phi\left(\frac{1}{2}\right)}{l}\sum_{r_2=1}^{l}\int_0^1\tau^{\alpha-1}\phi_{r_2}(\tau)d\tau\right)\varphi(\mu).\end{aligned}$$

Hence we find

$$\frac{\Psi\left(\frac{1}{2}\right)\Gamma(1+\alpha)}{(\nu-\mu)}J^{\alpha}_{\mu^+}\varphi(\nu)\leq\left(\frac{\alpha\Psi\left(\frac{1}{2}\right)}{k}\sum_{r_1=1}^{k}\Psi_{r_1}\right)\varphi(\mu)+\left(\frac{\alpha\Psi\left(\frac{1}{2}\right)}{l}\sum_{r_2=1}^{l}\Phi_{r_2}\right)\varphi(\nu) \tag{3.7}$$

and

$$\frac{\Phi\left(\frac{1}{2}\right)\Gamma(1+\alpha)}{(\nu-\mu)}J^{\alpha}_{\nu^-}\varphi(\mu)\leq\left(\frac{\alpha\Phi\left(\frac{1}{2}\right)}{k}\sum_{r_1=1}^{k}\Psi_{r_1}\right)\varphi(\nu)+\left(\frac{\alpha\Phi\left(\frac{1}{2}\right)}{l}\sum_{r_2=1}^{l}\Phi_{r_2}\right)\varphi(\mu). \tag{3.8}$$

Adding the inequalities (3.7) and (3.8) gives

$$\begin{aligned}&\frac{\Gamma(1+\alpha)}{(\nu-\mu)}\left[\Psi\left(\frac{1}{2}\right)J^{\alpha}_{\mu^+}\varphi(\nu)+\Phi\left(\frac{1}{2}\right)J^{\alpha}_{\nu^-}\varphi(\mu)\right]\\&\leq\left[\left(\frac{\alpha\Psi\left(\frac{1}{2}\right)}{k}\sum_{r_1=1}^{k}\Psi_{r_1}+\frac{\alpha\Phi\left(\frac{1}{2}\right)}{l}\sum_{r_2=1}^{l}\Phi_{r_2}\right)\varphi(\mu)\right.\end{aligned}$$

$$+\left(\frac{\alpha\Phi\left(\frac{1}{2}\right)}{k}\sum_{r_1=1}^{k}\Psi_{r_1}+\frac{\alpha\Psi\left(\frac{1}{2}\right)}{l}\sum_{r_2=1}^{l}\Phi_{r_2}\right)\varphi(\nu)\Bigg]. \tag{3.9}$$

Combining the inequalities (3.4) and (3.9) gives the required result.

This completes the proof. □

Remark 2. Theorem 4 leads to the following specific cases:

(1) If $\psi_{r_1}(\tau)=\tau$, $\phi_{r_2}(\tau)=1-\tau$ for each r_1 and r_2, and $k=l=1$, then (3.1) reduces to (1.2).
(2) If $\psi_{r_1}(\tau)=\tau^s$, $\phi_{r_2}(\tau)=(1-\tau)^s$ for each r_1 and r_2, and $k=l=1$, then we get [20, Theorem 3].
(3) If $\psi_{r_1}(\tau)=h(\tau)$, $\phi_{r_2}(\tau)=h(1-\tau)$ for each r_1 and r_2, and $k=l=1$, then we have [24, Theorem 2.1].
(4) If $\psi_{r_1}(\tau)=1-(s(1-\tau))^{r_1}$, $\phi_{r_2}(\tau)=1-(s\tau)^{r_2}$, where $r_1=r_2=1,2,\ldots,k$, $k\in\mathbb{N}$, and $s\in[0,1]$, then (3.1) reduces to [14, Theorem 2.1], in case $\kappa=1$.

Theorem 5. *Suppose that $\psi_{r_1},\phi_{r_2}:[0,1]\to[0,\infty)$ are continuous functions for all $r_1=1,2,\ldots,k$ and $r_2=1,2,\ldots,l$ where $k,l\in\mathbb{N}$ and $1\le k\le l$, and let for $0\le\mu<\nu$ and $m\in(0,1]$, $\varphi:[\mu,\frac{\nu}{m}]\to[0,\infty)$ be a differentiable function on $(\mu,\frac{\nu}{m})$ such that $\varphi'\in L[\mu,\nu]$. If $\left|\varphi'\right|$ is (k,l)-generalized m-type convex with respect to Ω, on $[\mu,\frac{\nu}{m}]$, then for any $\alpha>0$, we have*

$$\begin{aligned}&\left|\frac{\varphi(\mu)+\varphi(\nu)}{2}-\frac{\Gamma(1+\alpha)}{2(\nu-\mu)^{\alpha}}\left(J^{\alpha}_{\mu^+}\varphi(\nu)+J^{\alpha}_{\nu^-}\varphi(\mu)\right)\right|\\&\le\frac{(\nu-\mu)}{2}\left(\frac{\left|\varphi'(\mu)\right|}{k}\sum_{r_1=1}^{k}\left(\Psi^1_{r_1}+\Psi^2_{r_1}\right)+\frac{m\left|\varphi'\left(\frac{\nu}{m}\right)\right|}{l}\sum_{r_2=1}^{l}\left(\Phi^1_{r_1}+\Phi^2_{r_1}\right)\right)\end{aligned} \tag{3.10}$$

where

$$\begin{aligned}\Psi^1_{r_1}&=\int_0^{\frac{1}{2}}[(1-\tau)^{\alpha}-\tau^a]\psi_{r_1}(\tau)\mathrm{d}\tau,\\\Psi^2_{r_1}&=\int_{\frac{1}{2}}^{1}[\tau^a-(1-\tau)^{\alpha}]\psi_{r_1}(\tau)\mathrm{d}\tau,\\\Phi^1_{r_1}&=\int_0^{\frac{1}{2}}[(1-\tau)^{\alpha}-\tau^a]\phi_{r_2}(\tau)\mathrm{d}\tau,\\\Phi^2_{r_1}&=\int_{\frac{1}{2}}^{1}[\tau^a-(1-\tau)^{\alpha}]\phi_{r_2}(\tau)\mathrm{d}\tau.\end{aligned}$$

Proof. From Lemma 1 and the fact that the $\left|\varphi'\right|$ is (k,l)-generalized m-type convex with respect to Ω on $[\mu,\frac{\nu}{m}]$, we have

$$\left|\frac{\varphi(\mu)+\varphi(\nu)}{2}-\frac{\Gamma(1+\alpha)}{2(\nu-\mu)^{\alpha}}\left(J^{\alpha}_{\mu^+}\varphi(\nu)+J^{\alpha}_{\nu^-}\varphi(\mu)\right)\right|$$

$$
\begin{aligned}
&\leq \frac{(\nu-\mu)}{2}\int_0^1 \left|(1-\tau)^\alpha-\tau^a\right|\left|\varphi'\left(\mu\tau+(1-\tau)\nu\right)\right|d\tau \\
&= \frac{(\nu-\mu)}{2}\int_0^1 \left|(1-\tau)^\alpha-\tau^a\right|\left|\varphi'\left(\mu\tau+m(1-\tau)\frac{\nu}{m}\right)\right|d\tau \\
&\leq \frac{(\nu-\mu)}{2}\int_0^1 \left|(1-\tau)^\alpha-\tau^a\right|\left(\frac{1}{k}\sum_{r_1=1}^{k}\psi_{r_1}(\tau)\left|\varphi'(\mu)\right|+m\left(\frac{1}{l}\sum_{r_2=1}^{l}\phi_{r_2}(\tau)\right)\left|\varphi'\left(\frac{\nu}{m}\right)\right|\right)d\tau \\
&= \frac{(\nu-\mu)}{2}\left[\int_0^{\frac{1}{2}}[(1-\tau)^\alpha-\tau^a]\left(\frac{1}{k}\sum_{r_1=1}^{k}\psi_{r_1}(\tau)\left|\varphi'(\mu)\right|+m\left(\frac{1}{l}\sum_{r_2=1}^{l}\phi_{r_2}(\tau)\right)\left|\varphi'\left(\frac{\nu}{m}\right)\right|\right)d\tau\right. \\
&\quad \left.+\int_{\frac{1}{2}}^1[\tau^a-(1-\tau)^\alpha]\left(\frac{1}{k}\sum_{r_1=1}^{k}\psi_{r_1}(\tau)\left|\varphi'(\mu)\right|+m\left(\frac{1}{l}\sum_{r_2=1}^{l}\phi_{r_2}(\tau)\right)\left|\varphi'\left(\frac{\nu}{m}\right)\right|\right)d\tau\right] \\
&= \frac{(\nu-\mu)}{2}\left[\frac{\left|\varphi'(\mu)\right|}{k}\sum_{r_1=1}^{k}\int_0^{\frac{1}{2}}[(1-\tau)^\alpha-\tau^a]\psi_{r_1}(\tau)d\tau\right. \\
&\quad +\frac{m\left|\varphi'\left(\frac{\nu}{m}\right)\right|}{l}\sum_{r_2=1}^{l}\int_0^{\frac{1}{2}}[(1-\tau)^\alpha-\tau^a]\phi_{r_2}(\tau)d\tau \\
&\quad \left.+\frac{\left|\varphi'(\mu)\right|}{k}\sum_{r_1=1}^{k}\int_{\frac{1}{2}}^1[\tau^a-(1-\tau)^\alpha]\psi_{r_1}(\tau)d\tau+\frac{m\left|\varphi'\left(\frac{\nu}{m}\right)\right|}{l}\sum_{r_2=1}^{l}\int_{\frac{1}{2}}^1[\tau^a-(1-\tau)^\alpha]\phi_{r_2}(\tau)d\tau\right] \\
&= \frac{(\nu-\mu)}{2}\left(\frac{\left|\varphi'(\mu)\right|}{k}\sum_{r_1=1}^{k}\left(\Psi_{r_1}^1+\Psi_{r_1}^2\right)+\frac{m\left|\varphi'\left(\frac{\nu}{m}\right)\right|}{l}\sum_{r_2=1}^{l}\left(\Phi_{r_1}^1+\Phi_{r_1}^2\right)\right).
\end{aligned}
$$

This ends the proof. □

Remark 3.

(1) If $\psi_{r_1}(\tau)=\tau$, $\phi_{r_2}(\tau)=1-\tau$ for each r_1 and r_2,and $k=l=m=1$, then inequality (3.10) reduces to the trapezoid type fractional integral inequality for convex functions as

$$
\begin{aligned}
&\left|\frac{\varphi(\mu)+\varphi(\nu)}{2}-\frac{\Gamma(1+\alpha)}{2(\nu-\mu)^\alpha}\left(J_{\mu^+}^\alpha\varphi(\nu)+J_{\nu^-}^\alpha\varphi(\mu)\right)\right| \\
&\leq \frac{(\nu-\mu)(1-2^{-\alpha})}{2(\alpha+1)}\left\{\left|\varphi'(\mu)\right|+\left|\varphi'(\nu)\right|\right\}.
\end{aligned}
$$

(2) If $\psi_{r_1}(\tau)=\tau^s$, $\phi_{r_2}(\tau)=(1-\tau)^s$ for each r_1 and r_2,and $k=l=m=1$ where $s\in(0,1]$, then inequality (3.10) reduces to the trapezoid type fractional integral inequality for s-convex functions as

$$
\left|\frac{\varphi(\mu)+\varphi(\nu)}{2}-\frac{\Gamma(1+\alpha)}{2(\nu-\mu)^\alpha}\left(J_{\mu^+}^\alpha\varphi(\nu)+J_{\nu^-}^\alpha\varphi(\mu)\right)\right|
$$

$$\leq \frac{(\nu-\mu)\left(\left|\varphi'(\mu)\right|+\left|\varphi'(\nu)\right|\right)}{2} \times\left\{\left(\beta\left(\frac{1}{2};s+1,\alpha+1\right)-\beta\left(\frac{1}{2};\alpha+1,s+1\right)+\frac{1-2^{-\alpha-s}}{\alpha+s+1}\right)\right\}.$$

(3) If $\psi_{r_1}(\tau)=1-(1-\tau)^{r_1}$, $\phi_{r_2}(\tau)=1-(\tau)^{r_2}$, where $r_1=r_2=1,2,\ldots,k$, $k,\in \mathbb{N}$, and $\alpha=m=1$, then inequality (3.10) reduces to the trapezoid type integral inequality for n-polynomial convex functions as

$$\left|\frac{\varphi(\mu)+\varphi(\nu)}{2}-\frac{1}{(\nu-\mu)^{\alpha}}\int_{\mu}^{\nu}\varphi(\tau)\,d\tau\right| \leq \frac{(\nu-\mu)\left(\left|\varphi'(\mu)\right|+\left|\varphi'(\nu)\right|\right)}{2k}\sum_{r_1=1}^{k}\left[\frac{\left(r_1^2+r_1+2\right)2^{r_1}-2}{(r_1+1)(r_1+2)2^{r_1+1}}\right].$$

See also [16, 20, 23].

Theorem 6. *Assume that $\psi_{r_1},\phi_{r_2}\colon[0,1]\to[0,\infty)$ are continuous functions for all $r_1=1,2,\ldots,k$ and $r_2=1,2,\cdots,l$ where $k,l\in\mathbb{N}$ and $1\leq k\leq l$, and let for $0\leq\mu<\nu$ and $m\in(0,1]$, $\varphi\colon[\mu,\frac{\nu}{m}]\to[0,\infty)$ be a differentiable function on $(\mu,\frac{\nu}{m})$ such that $\varphi'\in L[\mu,\nu]$. If $\left|\varphi'\right|$ is (k,l)-generalized m-type convex with respect to Ω, on $[\mu,\frac{\nu}{m}]$, then for any $\xi\in[\mu,\nu]$ and $\alpha>0$, we have*

$$\left|\frac{(\xi-\mu)^{\alpha}+(\nu-\xi)^{a}}{\nu-\mu}\varphi(\xi)-\frac{\Gamma(1+\alpha)}{\nu-\mu}\left(J^{\alpha}_{\mu^{+}}\varphi(\nu)+J^{\alpha}_{\nu^{-}}\varphi(\mu)\right)\right| \leq \frac{1}{\nu-\mu}\left[(\xi-\mu)^{\alpha+1}\left(\frac{\left|\varphi'(\mu)\right|}{k}\sum_{r_1=1}^{k}\Psi^{3}_{r_1}+\frac{m\left|\varphi'\left(\frac{\xi}{m}\right)\right|}{l}\sum_{r_2=1}^{l}\Phi^{3}_{r_2}\right) + (\nu-\xi)^{\alpha+1}\left(\frac{\left|\varphi'(\xi)\right|}{k}\sum_{r_1=1}^{k}\Psi^{4}_{r_1}+\frac{m\left|\varphi'\left(\frac{\nu}{m}\right)\right|}{l}\sum_{r_2=1}^{l}\Phi^{4}_{r_2}\right)\right], \tag{3.11}$$

where

$$\Psi^{3}_{r_1}=\int_0^1(1-\tau)^{\alpha}\psi_{r_1}(\tau)d\tau,$$

$$\Phi^{3}_{r_2}=\int_0^1(1-\tau)^{\alpha}\phi_{r_2}(\tau)d\tau,$$

$$\Psi^{4}_{r_1}=\int_0^1\tau^{\alpha}\psi_{r_1}(\tau)d\tau,$$

$$\Phi^{4}_{r_2}=\int_0^1\tau^{\alpha}\phi_{r_2}(\tau)d\tau.$$

Proof. From Lemma 2 and using the (k,l)-generalized m-type convexity of $\left|\varphi'\right|$, we get

$$\begin{aligned}
&\left|\frac{(\xi-\mu)^{\alpha}+(\nu-\xi)^{a}}{\nu-\mu}\varphi(\xi)-\frac{\Gamma(1+\alpha)}{\nu-\mu}\left(J^{\alpha}_{\mu^{+}}\varphi(\nu)+J^{\alpha}_{\nu^{-}}\varphi(\mu)\right)\right| \\
&\leq \frac{(\xi-\mu)^{\alpha+1}}{\nu-\mu}\int_0^1 \tau^{a}\left|\varphi'(\tau\xi+(1-\tau)\mu)\right|d\tau+\frac{(\nu-\xi)^{\alpha+1}}{\nu-\mu}\int_0^1 \tau^{a}\left|\varphi'(\tau\xi+(1-\tau)\nu)\right|d\tau \\
&= \frac{(\xi-\mu)^{\alpha+1}}{\nu-\mu}\int_0^1 (1-\tau)^{a}\left|\varphi'(\mu\tau+(1-\tau)\xi)\right|d\tau+\frac{(\nu-\xi)^{\alpha+1}}{\nu-\mu}\int_0^1 \tau^{a}\left|\varphi'(\tau\xi+(1-\tau)\nu)\right|d\tau \\
&\leq \frac{(\xi-\mu)^{\alpha+1}}{\nu-\mu}\left(\frac{\left|\varphi'(\mu)\right|}{k}\sum_{r_1=1}^{k}\int_0^1 (1-\tau)^{\alpha}\psi_{r_1}(\tau)d\tau+\frac{m\left|\varphi'\left(\frac{\xi}{m}\right)\right|}{l}\sum_{r_2=1}^{l}\int_0^1 (1-\tau)^{\alpha}\phi_{r_2}(\tau)d\tau\right) \\
&\quad+\frac{(\nu-\xi)^{\alpha+1}}{\nu-\mu}\left(\frac{\left|\varphi'(\xi)\right|}{k}\sum_{r_1=1}^{k}\int_0^1 \tau^{\alpha}\psi_{r_1}(\tau)d\tau+\frac{m\left|\varphi'\left(\frac{\nu}{m}\right)\right|}{l}\sum_{r_2=1}^{l}\int_0^1 \tau^{\alpha}\phi_{r_2}(\tau)d\tau\right) \\
&= \frac{1}{\nu-\mu}\left[(\xi-\mu)^{\alpha+1}\left(\frac{\left|\varphi'(\mu)\right|}{k}\sum_{r_1=1}^{k}\Psi^{3}_{r_1}+\frac{m\left|\varphi'\left(\frac{\xi}{m}\right)\right|}{l}\sum_{r_2=1}^{l}\Phi^{3}_{r_2}\right)\right. \\
&\quad\left.+(\nu-\xi)^{\alpha+1}\left(\frac{\left|\varphi'(\xi)\right|}{k}\sum_{r_1=1}^{k}\Psi^{4}_{r_1}+\frac{m\left|\varphi'\left(\frac{\nu}{m}\right)\right|}{l}\sum_{r_2=1}^{l}\Phi^{4}_{r_2}\right)\right].
\end{aligned}$$

□

Corollary 1. *If we choose $\xi=\frac{\mu+\nu}{2}$ in Theorem 6, then we have*

$$\begin{aligned}
&\left|\varphi\left(\frac{\mu+\nu}{2}\right)-\frac{2^{\alpha-1}\Gamma(1+\alpha)}{(\nu-\mu)^{\alpha}}\left(J^{\alpha}_{\mu^{+}}\varphi(\nu)+J^{\alpha}_{\nu^{-}}\varphi(\mu)\right)\right| \\
&\leq \frac{\nu-\mu}{4}\left[\left(\frac{\left|\varphi'(\mu)\right|}{k}\sum_{r_1=1}^{k}\Psi^{3}_{r_1}+\frac{m\left|\varphi'\left(\frac{\mu+\nu}{2m}\right)\right|}{l}\sum_{r_2=1}^{l}\Phi^{3}_{r_2}\right)\right. \\
&\quad\left.+\left(\frac{\left|\varphi'\left(\frac{\mu+\nu}{2}\right)\right|}{k}\sum_{r_1=1}^{k}\Psi^{4}_{r_1}+\frac{m\left|\varphi'\left(\frac{\nu}{m}\right)\right|}{l}\sum_{r_2=1}^{l}\Phi^{4}_{r_2}\right)\right],
\end{aligned}$$

where

$$\Psi^{3}_{r_1}=\int_0^1 (1-\tau)^{\alpha}\psi_{r_1}(\tau)d\tau,$$

$$\Phi^{3}_{r_2}=\int_0^1 (1-\tau)^{\alpha}\phi_{r_2}(\tau)d\tau,$$

$$\Psi^{4}_{r_1}=\int_0^1 \tau^{\alpha}\psi_{r_1}(\tau)d\tau,$$

$$\Phi_{r_2}^4 = \int_0^1 \tau^\alpha \phi_{r_2}(\tau) d\tau.$$

Remark 4.

(1) If $\psi_{r_1}(\tau) = \tau^s$, $\phi_{r_2}(\tau) = (1-\tau)^s$ for each r_1 and r_2, and $k = l = m = 1$ where $s \in (0,1]$, then inequality (3.11) reduces to the Ostrowski fractional integral inequality for s-convex functions as

$$\left| \frac{(\xi-\mu)^\alpha + (\nu-\xi)^a}{\nu-\mu} \varphi(\xi) - \frac{\Gamma(1+\alpha)}{\nu-\mu} \left(J_{\mu^+}^\alpha \varphi(\nu) + J_{\nu^-}^\alpha \varphi(\mu) \right) \right|$$
$$\leq \frac{1}{\nu-\mu} \left\{ \begin{array}{c} (\xi-\mu)^{\alpha+1} \left(\left|\varphi'(\mu)\right| \beta(s+1,\alpha+1) + \left|\varphi'(\xi)\right| \beta(1,\alpha+s+1) \right) \\ +(\nu-\xi)^{\alpha+1} \left(\left|\varphi'(\xi)\right| \beta(\alpha+s+1,1) + \left|\varphi'(\nu)\right| \beta(\alpha+1,s+1) \right) \end{array} \right\}.$$

(2) If $\psi_{r_1}(\tau) = 1-(s(1-\tau))^{r_1}$, $\phi_{r_2}(\tau) = 1-(s\tau)^{r_2}$, where $r_1 = r_2 = 1,2,\ldots,k$, $k, \in \mathbb{N}$, $m = 1$ and $s \in [0,1]$, then inequality (3.11) reduces to the Ostrowski fractional integral inequality for n-polynomial s-type convex functions as

$$\left| \frac{(\xi-\mu)^\alpha + (\nu-\xi)^a}{\nu-\mu} \varphi(\xi) - \frac{\Gamma(1+\alpha)}{\nu-\mu} \left(J_{\mu^+}^\alpha \varphi(\nu) + J_{\nu^-}^\alpha \varphi(\mu) \right) \right|$$
$$\leq \frac{1}{\nu-\mu} \left[(\xi-\mu)^{\alpha+1} \left(\begin{array}{c} \frac{\left|\varphi'(\mu)\right|}{k} \sum_{r_1=1}^k \left(\beta(1,\alpha+1) - s^{r_1} \beta(1,\alpha+r_1+1) \right) \\ +\frac{\left|\varphi'(\xi)\right|}{k} \sum_{r_1=1}^k \left(\beta(1,\alpha+1) - s^{r_1} \beta(r_2+1,\alpha+1) \right) \end{array} \right) \right.$$
$$\left. +(\nu-\xi)^{\alpha+1} \left(\begin{array}{c} \frac{\left|\varphi'(\xi)\right|}{k} \sum_{r_1=1}^k \left(\beta(\alpha+1,1) - s^{r_1} \beta(\alpha+1,r_1+1) \right) \\ +\frac{\left|\varphi'(\nu)\right|}{k} \sum_{r_1=1}^k \left(\beta(\alpha+1,1) - s^{r_1} \beta(\alpha+r_2+1,1) \right) \end{array} \right) \right].$$

See also [14, 18].

Theorem 7. *Suppose that $\psi_{r_1}, \phi_{r_2} : [0,1] \to [0,\infty)$ are continuous functions for all $r_1 = 1,2,\ldots,k$ and $r_2 = 1,2,\ldots,l$ where $k,l \in \mathbb{N}$ and $1 \leq k \leq l$, and let for $0 \leq \mu < \nu$ and $m \in (0,1]$, $\varphi : [\mu, \frac{\nu}{m}] \to [0,\infty)$ be a differentiable function on $(\mu, \frac{\nu}{m})$ such that $\varphi' \in L[\mu,\nu]$. If $\left|\varphi'\right|^q$ is (k,l)-generalized m-type convex with respect to Ω, on $[\mu, \frac{\nu}{m}]$ for $q > 1$; $\frac{1}{p} + \frac{1}{q} = 1$, then for any $\xi \in [\mu,\nu]$ and $\alpha > 0$, the following inequality holds*

$$\left| \frac{(\xi-\mu)^\alpha + (\nu-\xi)^a}{\nu-\mu} \varphi(\xi) - \frac{\Gamma(1+\alpha)}{\nu-\mu} \left(J_{\mu^+}^\alpha \varphi(\nu) + J_{\nu^-}^\alpha \varphi(\mu) \right) \right|$$
$$\leq \frac{\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}}{\nu-\mu} \left[(\xi-\mu)^{\alpha+1} \left(\frac{\left|\varphi'(\mu)\right|}{k} \sum_{r_1=1}^k \Psi_{r_1}^5 + \frac{m\left|\varphi'\left(\frac{\xi}{m}\right)\right|}{l} \sum_{r_2=1}^l \Phi_{r_2}^5 \right)^{\frac{1}{q}} \right.$$

$$+(\nu-\xi)^{\alpha+1}\left(\frac{\left|\varphi'(\xi)\right|}{k}\sum_{r_1=1}^{k}\Psi_{r_1}^{5}+\frac{m\left|\varphi'\left(\frac{\nu}{m}\right)\right|}{l}\sum_{r_2=1}^{l}\Phi_{r_2}^{5}\right)^{\frac{1}{q}}\Bigg], \tag{3.12}$$

where

$$\Psi_{r_1}^{5}=\int_0^1\psi_{r_1}(\tau)\mathrm{d}\tau,$$

$$\Phi_{r_2}^{5}=\int_0^1\phi_{r_2}(\tau)\mathrm{d}\tau.$$

Proof. From Lemma 2, Hölder's inequality and the fact that $|\varphi'|^q$ is (k,l)-generalized m-type convex with respect to Ω on $[\mu,\frac{\nu}{m}]$, we have

$$\begin{aligned}
&\left|\frac{(\xi-\mu)^{\alpha}+(\nu-\xi)^{a}}{\nu-\mu}\varphi(\xi)-\frac{\Gamma(1+\alpha)}{\nu-\mu}\left(J_{\mu^+}^{\alpha}\varphi(\nu)+J_{\nu^-}^{\alpha}\varphi(\mu)\right)\right|\\
&\leq\frac{(\xi-\mu)^{\alpha+1}}{\nu-\mu}\int_0^1\tau^{a}\left|\varphi'(\tau\xi+(1-\tau)\mu)\right|\mathrm{d}\tau+\frac{(\nu-\xi)^{\alpha+1}}{\nu-\mu}\int_0^1\tau^{a}\left|\varphi'(\tau\xi+(1-\tau)\nu)\right|\mathrm{d}\tau\\
&\leq\left[\frac{(\xi-\mu)^{\alpha+1}}{\nu-\mu}\left(\int_0^1\tau^{p\alpha}\mathrm{d}\tau\right)^{\frac{1}{p}}\left(\int_0^1\left|\varphi'(\tau\xi+(1-\tau)\mu)\right|^{q}\mathrm{d}\tau\right)^{\frac{1}{q}}\right.\\
&\quad\left.+\frac{(\nu-\xi)^{\alpha+1}}{\nu-\mu}\left(\int_0^1\tau^{p\alpha}\mathrm{d}\tau\right)^{\frac{1}{p}}\left(\int_0^1\left|\varphi'(\tau\xi+(1-\tau)\nu)\right|^{q}\mathrm{d}\tau\right)^{\frac{1}{q}}\right]\\
&=\frac{\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}}{\nu-\mu}\left[(\xi-\mu)^{\alpha+1}\left(\int_0^1\left|\varphi'(\tau\mu+(1-\tau)\xi)\right|^{q}\mathrm{d}\tau\right)^{\frac{1}{q}}\right.\\
&\quad\left.+(\nu-\xi)^{\alpha+1}\left(\int_0^1\left|\varphi'(\tau\xi+(1-\tau)\nu)\right|^{q}\mathrm{d}\tau\right)^{\frac{1}{q}}\right]\\
&\leq\frac{\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}}{\nu-\mu}\left[(\xi-\mu)^{\alpha+1}\left(\frac{\left|\varphi'(\mu)\right|^{q}}{k}\sum_{r_1=1}^{k}\int_0^1\psi_{r_1}(\tau)\mathrm{d}\tau+\frac{m\left|\varphi'\left(\frac{\xi}{m}\right)\right|^{q}}{l}\sum_{r_2=1}^{l}\int_0^1\phi_{r_2}(\tau)\mathrm{d}\tau\right)^{\frac{1}{q}}\right.\\
&\quad\left.+(\nu-\xi)^{\alpha+1}\left(\frac{\left|\varphi'(\xi)\right|^{q}}{k}\sum_{r_1=1}^{k}\int_0^1\psi_{r_1}(\tau)\mathrm{d}\tau+\frac{m\left|\varphi'\left(\frac{\nu}{m}\right)\right|^{q}}{l}\sum_{r_2=1}^{l}\int_0^1\phi_{r_2}(\tau)\mathrm{d}\tau\right)^{\frac{1}{q}}\right]\\
&=\frac{\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}}{\nu-\mu}\left[(\xi-\mu)^{\alpha+1}\left(\frac{\left|\varphi'(\mu)\right|^{q}}{k}\sum_{r_1=1}^{k}\Psi_{r_1}^{5}+\frac{m\left|\varphi'\left(\frac{\xi}{m}\right)\right|^{q}}{l}\sum_{r_2=1}^{l}\Phi_{r_2}^{5}\right)^{\frac{1}{q}}\right.
\end{aligned}$$

$$+(\nu-\xi)^{\alpha+1}\left(\frac{\left|\varphi'(\xi)\right|^q}{k}\sum_{r_1=1}^{k}\Psi_{r_1}^5+\frac{m\left|\varphi'\left(\frac{\nu}{m}\right)\right|^q}{l}\sum_{r_2=1}^{l}\Phi_{r_2}^5\right)^{\frac{1}{q}}\Bigg].$$

This completes the proof. □

Corollary 2. *If we take* $\xi=\frac{\mu+\nu}{2}$ *in Theorem 7, then we get*

$$\begin{aligned}&\left|\varphi\left(\frac{\mu+\nu}{2}\right)-\frac{2^{\alpha-1}\Gamma(1+\alpha)}{(\nu-\mu)^{\alpha}}\left(J_{\mu^+}^{\alpha}\varphi(\nu)+J_{\nu^-}^{\alpha}\varphi(\mu)\right)\right|\\&\leq\frac{(\nu-\mu)\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}}{4}\Bigg[\left(\frac{\left|\varphi'(\mu)\right|^q}{k}\sum_{r_1=1}^{k}\Psi_{r_1}^5+\frac{m\left|\varphi'\left(\frac{\mu+\nu}{2m}\right)\right|^q}{l}\sum_{r_2=1}^{l}\Phi_{r_2}^5\right)^{\frac{1}{q}}\\&+\left(\frac{\left|\varphi'\left(\frac{\mu+\nu}{2}\right)\right|^q}{k}\sum_{r_1=1}^{k}\Psi_{r_1}^5+\frac{m\left|\varphi'\left(\frac{\nu}{m}\right)\right|^q}{l}\sum_{r_2=1}^{l}\Phi_{r_2}^5\right)^{\frac{1}{q}}\Bigg],\end{aligned}$$

where

$$\Psi_{r_1}^5=\int_0^1\psi_{r_1}(\tau)\mathrm{d}\tau,$$

$$\Phi_{r_2}^5=\int_0^1\phi_{r_2}(\tau)\mathrm{d}\tau.$$

Remark 5.

(1) If $\psi_{r_1}(\tau)=\tau^s$, $\phi_{r_2}(\tau)=(1-\tau)^s$ for each r_1 and r_2, and $k=l=m=1$, where $s\in(0,1]$, then inequality (3.12) reduces to the Ostrowski fractional integral inequality for s-convex functions as

$$\begin{aligned}&\left|\frac{(\xi-\mu)^{\alpha}+(\nu-\xi)^{a}}{\nu-\mu}\varphi(\xi)-\frac{\Gamma(1+\alpha)}{\nu-\mu}\left(J_{\mu^+}^{\alpha}\varphi(\nu)+J_{\nu^-}^{\alpha}\varphi(\mu)\right)\right|\\&\leq\frac{\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}}{(\nu-\mu)}\left\{\begin{array}{c}(\xi-\mu)^{\alpha+1}\left(\left|\varphi'(\mu)\right|^q+\left|\varphi'(\xi)\right|^q\right)^{\frac{1}{q}}\\+(\nu-\xi)^{\alpha+1}\left(\left|\varphi'(\xi)\right|^q+\left|\varphi'(\nu)\right|^q\right)^{\frac{1}{q}}\end{array}\right\}\end{aligned}$$

(2) If $\psi_{r_1}(\tau)=1-(s(1-\tau))^{r_1}$, $\phi_{r_2}(\tau)=1-(s\tau)^{r_2}$ for each r_1 and r_2, where $r_1=r_2=1,2,\ldots,k$, $k\in\mathbb{N}$, $m=1$, and, $s\in[0,1]$, then inequality (3.12) reduces to the Ostrowski fractional integral inequality for n-polynomial s-type convex functions as

$$\left|\frac{(\xi-\mu)^{\alpha}+(\nu-\xi)^{a}}{\nu-\mu}\varphi(\xi)-\frac{\Gamma(1+\alpha)}{\nu-\mu}\left(J_{\mu^+}^{\alpha}\varphi(\nu)+J_{\nu^-}^{\alpha}\varphi(\mu)\right)\right|$$

$$\leq \frac{\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left(\sum_{r_1=1}^{k}\frac{r_1+1-s^{r_1}}{r_1+1}\right)^{\frac{1}{q}}}{(\nu-\mu)}\left\{\begin{array}{l}(\xi-\mu)^{\alpha+1}\left(\left|\varphi'(\mu)\right|+\left|\varphi'(\xi)\right|\right)^{\frac{1}{q}}\\ +(\nu-\xi)^{\alpha+1}\left(\left|\varphi'(\xi)\right|+\left|\varphi'(\nu)\right|\right)^{\frac{1}{q}}\end{array}\right\}.$$

See also [14, 18].

Theorem 8. *Assume that $\psi_{r_1},\phi_{r_2}\colon[0,1]\to[0,\infty)$ are continuous functions for all $r_1=1,2,\ldots,k$ and $r_2=1,2,\ldots,l$ where $k,l\in\mathbb{N}$ and $1\leq k\leq l$, and let for $0<\mu<\nu$ and $m\in(0,1]$, $\varphi\colon[\mu,\frac{\nu}{m}]\to[0,\infty)$ be a differentiable function on $(\mu,\frac{\nu}{m})$ such that $\varphi'\in L[\mu,\nu]$. If $\left|\varphi'\right|^q$ is (k,l)-generalized m-type convex with respect to Ω, on $[\mu,\frac{\nu}{m}]$ for $q>1$,then for any $\xi\in[\mu,\nu]$ and $\alpha>0$, we have*

$$\begin{aligned}&\left|\frac{(\xi-\mu)^{\alpha}+(\nu-\xi)^{a}}{\nu-\mu}\varphi(\xi)-\frac{\Gamma(1+\alpha)}{\nu-\mu}\left(J^{\alpha}_{\mu^+}\varphi(\nu)+J^{\alpha}_{\nu^-}\varphi(\mu)\right)\right|\\ &\leq\frac{\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}}{\nu-\mu}\left[(\xi-\mu)^{\alpha+1}\left(\frac{\left|\varphi'(\mu)\right|^q}{k}\sum_{r_1=1}^{k}\Psi^3_{r_1}+\frac{m\left|\varphi'\left(\frac{\xi}{m}\right)\right|^q}{l}\sum_{r_2=1}^{l}\Phi^3_{r_2}\right)^{\frac{1}{q}}\right.\\ &\left.+(\nu-\xi)^{\alpha+1}\left(\frac{\left|\varphi'(\xi)\right|^q}{k}\sum_{r_1=1}^{k}\Psi^4_{r_1}+\frac{m\left|\varphi'\left(\frac{\nu}{m}\right)\right|^q}{l}\sum_{r_2=1}^{l}\Phi^4_{r_2}\right)^{\frac{1}{q}}\right]\end{aligned}\tag{3.13}$$

where $\Psi^3_{r_1},\Phi^3_{r_2},\Psi^4_{r_1}$ and $\Phi^4_{r_2}$ are defined in Theorem 6.

Proof. Using Lemma 2 and the power-mean inequality, we obtain

$$\begin{aligned}&\left|\frac{(\xi-\mu)^{\alpha}+(\nu-\xi)^{a}}{\nu-\mu}\varphi(\xi)-\frac{\Gamma(1+\alpha)}{\nu-\mu}\left(J^{\alpha}_{\mu^+}\varphi(\nu)+J^{\alpha}_{\nu^-}\varphi(\mu)\right)\right|\\ &\leq\frac{(\xi-\mu)^{\alpha+1}}{\nu-\mu}\int_0^1\tau^a\left|\varphi'(\tau\xi+(1-\tau)\mu)\right|d\tau\\ &\quad+\frac{(\nu-\xi)^{\alpha+1}}{\nu-\mu}\int_0^1\tau^a\left|\varphi'(\tau\xi+(1-\tau)\nu)\right|d\tau\\ &\leq\frac{(\xi-\mu)^{\alpha+1}}{\nu-\mu}\left(\int_0^1\tau^a d\tau\right)^{1-\frac{1}{q}}\left(\int_0^1\tau^a\left|\varphi'(\tau\xi+(1-\tau)\mu)\right|^q d\tau\right)^{\frac{1}{q}}\\ &\quad+\frac{(\nu-\xi)^{\alpha+1}}{\nu-\mu}\left(\int_0^1\tau^a d\tau\right)^{1-\frac{1}{q}}\left(\int_0^1\tau^a\left|\varphi'(\tau\xi+(1-\tau)\nu)\right|^q d\tau\right)^{\frac{1}{q}}\\ &=\frac{\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}}{\nu-\mu}\left[(\xi-\mu)^{\alpha+1}\left(\int_0^1(1-\tau)^a\left|\varphi'(\mu\tau+(1-\tau)\xi)\right|^q d\tau\right)^{\frac{1}{q}}\right.\end{aligned}$$

$$+(\nu-\xi)^{\alpha+1}\left(\int_0^1 \tau^a \left|\varphi'(\tau\xi+(1-\tau)\nu)\right|^q d\tau\right)^{\frac{1}{q}}\Bigg].$$

Since $\left|\varphi'\right|^q$ is a (k,l)-generalized m-type convex function with respect to Ω, we gain

$$\begin{aligned}
&\left|\frac{(\xi-\mu)^\alpha+(\nu-\xi)^a}{\nu-\mu}\varphi(\xi)-\frac{\Gamma(1+\alpha)}{\nu-\mu}\left(J^\alpha_{\mu^+}\varphi(\nu)+J^\alpha_{\nu^-}\varphi(\mu)\right)\right| \\
&\leq \frac{\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}}{\nu-\mu}\Bigg[(\xi-\mu)^{\alpha+1}\left(\begin{array}{c}\frac{\left|\varphi'(\mu)\right|^q}{k}\sum_{r_1=1}^k\int_0^1(1-\tau)^a\psi_{r_1}(\tau)d\tau \\ +\frac{m\left|\varphi'\left(\frac{\xi}{m}\right)\right|^q}{l}\sum_{r_2=1}^l\int_0^1(1-\tau)^a\phi_{r_2}(\tau)d\tau\end{array}\right)^{\frac{1}{q}} \\
&\quad+(\nu-\xi)^{\alpha+1}\left(\begin{array}{c}\frac{\left|\varphi'(\xi)\right|^q}{k}\sum_{r_1=1}^k\int_0^1\tau^a\psi_{r_1}(\tau)d\tau \\ +\frac{m\left|\varphi'\left(\frac{\nu}{m}\right)\right|^q}{l}\sum_{r_2=1}^l\int_0^1(\tau)^a\phi_{r_2}(\tau)d\tau\end{array}\right)^{\frac{1}{q}}\Bigg] \\
&=\frac{\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}}{\nu-\mu}\Bigg[(\xi-\mu)^{\alpha+1}\left(\frac{\left|\varphi'(\mu)\right|^q}{k}\sum_{r_1=1}^k\Psi^3_{r_1}+\frac{m\left|\varphi'\left(\frac{\xi}{m}\right)\right|^q}{l}\sum_{r_2=1}^l\Phi^3_{r_2}\right)^{\frac{1}{q}} \\
&\quad+(\nu-\xi)^{\alpha+1}\left(\frac{\left|\varphi'(\xi)\right|^q}{k}\sum_{r_1=1}^k\Psi^4_{r_1}+\frac{m\left|\varphi'\left(\frac{\nu}{m}\right)\right|^q}{l}\sum_{r_2=1}^l\Phi^4_{r_2}\right)^{\frac{1}{q}}\Bigg].
\end{aligned}$$

This ends the proof. □

Corollary 3. *If we select* $\xi=\frac{\mu+\nu}{2}$ *in Theorem 8, then we derive*

$$\begin{aligned}
&\left|\varphi\left(\frac{\mu+\nu}{2}\right)-\frac{2^{\alpha-1}\Gamma(1+\alpha)}{(\nu-\mu)^\alpha}\left(J^\alpha_{\mu^+}\varphi(\nu)+J^\alpha_{\nu^-}\varphi(\mu)\right)\right| \\
&\leq\frac{(\nu-\mu)\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}}{4}\Bigg[\left(\frac{\left|\varphi'(\mu)\right|^q}{k}\sum_{r_1=1}^k\Psi^3_{r_1}+\frac{m\left|\varphi'\left(\frac{\mu+\nu}{2m}\right)\right|^q}{l}\sum_{r_2=1}^l\Phi^3_{r_2}\right)^{\frac{1}{q}} \\
&\quad+\left(\frac{\left|\varphi'\left(\frac{\mu+\nu}{2}\right)\right|^q}{k}\sum_{r_1=1}^k\Psi^4_{r_1}+\frac{m\left|\varphi'\left(\frac{\nu}{m}\right)\right|^q}{l}\sum_{r_2=1}^l\Phi^4_{r_2}\right)^{\frac{1}{q}}\Bigg]
\end{aligned}$$

where $\Psi^3_{r_1},\Phi^3_{r_2},\Psi^4_{r_1}$ *and* $\Phi^4_{r_2}$ *are defined in Theorem 6.*

Remark 6.

(1) If $\psi_{r_1}(\tau)=\tau^s$, $\phi_{r_2}(\tau)=(1-\tau)^s$ for each r_1 and r_2, and $k=l=m=1$, where $s\in(0,1]$, then inequality (3.13) reduces to the Ostrowski fractional integral inequality for s-convex functions as

$$\left|\frac{(\xi-\mu)^{\alpha}+(\nu-\xi)^{a}}{\nu-\mu}\varphi(\xi)-\frac{\Gamma(1+\alpha)}{\nu-\mu}\left(J^{\alpha}_{\mu^{+}}\varphi(\nu)+J^{\alpha}_{\nu^{-}}\varphi(\mu)\right)\right|$$
$$\leq\frac{\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}}{\nu-\mu}\left[(\xi-\mu)^{\alpha+1}\left(\left|\varphi'(\mu)\right|^{q}\beta(s+1,\alpha+1)+\left|\varphi'(\xi)\right|^{q}\beta(1,\alpha+s+1)\right)^{\frac{1}{q}}\right.$$
$$\left.+(\nu-\xi)^{\alpha+1}\left(\left|\varphi'(\xi)\right|^{q}\beta(\alpha+s+1,1)+\left|\varphi'(\nu)\right|^{q}\beta(s+1,\alpha+1)\right)^{\frac{1}{q}}\right].$$

(2) If $\psi_{r_1}(\tau)=1-(s(1-\tau))^{r_1}$, $\phi_{r_2}(\tau)=1-(s\tau)^{r_2}$ for each r_1 and r_2, where $r_1=r_2=1,2,\cdots,k$, $k\in\mathbb{N}$, $m=1$, and, $s\in[0,1]$, then inequality (3.13) reduces to the Ostrowski fractional integral inequality for n-polynomial s-type convex functions as

$$\left|\frac{(\xi-\mu)^{\alpha}+(\nu-\xi)^{a}}{\nu-\mu}\varphi(\xi)-\frac{\Gamma(1+\alpha)}{\nu-\mu}\left(J^{\alpha}_{\mu^{+}}\varphi(\nu)+J^{\alpha}_{\nu^{-}}\varphi(\mu)\right)\right|$$
$$\leq\frac{\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}}{\nu-\mu}$$
$$\times\left[(\xi-\mu)^{\alpha+1}\left(\begin{array}{c}\frac{\left|\varphi'(\mu)\right|^{q}}{k}\sum_{r_1=1}^{k}\left(\beta(1,\alpha+1)-s^{r_1}\beta(1,\alpha+r_1+1)\right)\\+\frac{\left|\varphi'(\xi)\right|^{q}}{k}\sum_{r_1=1}^{k}\left(\beta(1,\alpha+1)-s^{r_1}\beta(\alpha+1,r_1+1)\right)\end{array}\right)^{\frac{1}{q}}\right.$$
$$\left.+(\nu-\xi)^{\alpha+1}\left(\begin{array}{c}\frac{\left|\varphi'(\xi)\right|^{q}}{k}\sum_{r_1=1}^{k}\left(\beta(1,\alpha+1)-s^{r_1}\beta(\alpha+1,r_1+1)\right)\\+\frac{\left|\varphi'(\nu)\right|^{q}}{k}\sum_{r_1=1}^{k}\left(\beta(1,\alpha+1)-s^{r_1}\beta(1,\alpha+r_1+1)\right)\end{array}\right)^{\frac{1}{q}}\right].$$

See also [14, 18].

4. Conclusion

The aims of the present study were to define a new class of generalized m-type convex functions and derive some Hermite–Hadamard and Ostrowski type fractional integral inequalities for them. Using the findings of this work and the right selection of functions ψ_{r_1}, and ϕ_{r_2}, where $r_1=1,2,\ldots,k$, $r_2=1,2,\ldots,l$ and $k,l\in\mathbb{N}$ where $1\leq k\leq l$, as special cases, several new and existing classes of m-convex functions, as well as some of the stated fractional integral inequalities for them, can be obtained. This illustrates that the notion of new generalized m-type convex functions, as well as the findings presented in this paper, are broad and enhance the number of results from the literature of modern inequalities theory. Furthermore, some particular cases

have been investigated. It is hoped that the ideas and approaches described in this paper will spur more study into the issue.

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CONVERGENCE OF MATRIX TRANSFORM MEANS WITH RESPECT TO VILENKIN SYSTEMS

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Abstract. In this paper we discuss the convergence of matrix transform means of Vilenkin-Fourier series of integrable functions in Vilenkin-Lebesgue points, where the elements of the matrix rows are monotonic. We also prove convergence in $L_1(G_m)$ norm concerning these means.

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1. NOTATIONS AND DEFINITIONS

We give a brief introduction to the theory of Vilenkin-Fourier analysis (for more details see [1, 24]). Denote by $\mathbb{P}$ the set of the positive integers, $\mathbb{N} := \mathbb{P} \cup \{0\}$. Let $m := (m_0, m_1, \dots)$ be a sequence of positive integers not less than 2. Denote by $\mathbb{Z}_{m_n} := \{0, 1, \dots, m_n - 1\}$ the additive group of integers modulo m_n. Define the group G_m as the complete direct product of the groups $\mathbb{Z}_{m_n}$ with the product of the discrete topologies of $\mathbb{Z}_{m_n}$'s.

The direct product μ of the measures

$$\mu_n(\{j\}) := 1/m_n \ (j \in \mathbb{Z}_{m_n})$$

is the Haar measure on G_m with $\mu(G_m) = 1$.

If the sequence m is bounded, then G_m is called a bounded Vilenkin group, otherwise, it is called an unbounded one. In this paper, we only deal with the bounded situation. For results on unbounded Vilenkin groups see e.g. [5, 15–17].

We mention, that in case of $m = (2, 2, \dots)$ we get G, the so-called Walsh group. The elements of G_m are represented by sequences

$$x := (x_0, x_1, \dots, x_n, \dots) \ (x_n \in \mathbb{Z}_{m_n}).$$

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It is easy to give a base for the neighbourhoods of G_m :

$$I_0(x) := G_m,$$

$$I_n(x) := \{y \in G_m \mid y_0 = x_0, \dots, y_{n-1} = x_{n-1}\} \ (x \in G_m, n \in \mathbb{P}).$$

We define the so-called generalized number system based on sequence m in the following way:

$$M_0 := 1, \ M_{n+1} := m_n M_n \ (n \in \mathbb{N}).$$

Then every $n \in \mathbb{N}$ can be uniquely expressed as $n = \sum_{k=0}^{\infty} n_k M_k$, where $n_k \in \mathbb{Z}_{m_k}$ $(k \in \mathbb{N})$ and only a finite number of n_k's differ from zero. Let us denote by $|n|$ a natural number such that $M_{|n|} \leq n < M_{|n|+1}$.

Let $L_p(G_m)$ denote the usual Lebesgue spaces on G_m with corresponding norms $\|.\|_p$ and $C(G_m)$ denote the space of continuous functions on G_m with the norm $\|f\|_\infty := \sup\{|f(x)| : x \in G_m\}$.

Next, we introduce on G_m an orthonormal system which is called Vilenkin system. At first, we define the complex-valued functions $r_k(x) : G_m \to \mathbb{C}$, the generalized Rademacher functions, by

$$r_k(x) := \exp(2\pi \imath x_k / m_k) \ \left(\imath^2 = -1, \ x \in G_m, \ k \in \mathbb{N}\right).$$

Let us define the Vilenkin system $\psi := \{\psi_n : n \in \mathbb{N}\}$ on G_m as:

$$\psi_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x) \ (n \in \mathbb{N}, \ x \in G_m).$$

Specifically, we call this system the Walsh-Paley system, when $m = (2, 2, \dots)$. The Vilenkin system is orthonormal and complete in $L_2(G_m)$ (see [25]). Exactly, the elements of the Vilenkin system are the characters of G_m. Namely, $f : G_m \to \mathbb{C}$ continuous,

$$f(x+y) = f(x)f(y)$$

and $|f(x)| = 1$ for all $x, y \in G_m$. Moreover, it holds if and only if $f(x) = \psi_n(x)$ for some $n \in \mathbb{N}$ (see [24]).

Let $\mathcal{P}_n$ be the collection of Vilenkin polynomials of order less than n, that is, functions of the form

$$P(x) = \sum_{k=0}^{n-1} a_k \psi_k(x),$$

where $n \in \mathbb{P}$ and $\{a_k\}$ is a sequence of complex numbers. Let $\mathcal{P}$ denote the set of Vilenkin polynomials.

The nth Vilenkin-Fourier-coefficient, the nth partial sum of the Vilenkin-Fourier series, the nth Vilenkin-Fejér mean and the nth Vilenkin-Dirichlet kernel is defined by

$$\hat{f}(n) := \int_{G_m} f \bar{\psi}_n \mathrm{d}\mu, \ \ S_n(f) := \sum_{k=0}^{n-1} \hat{f}(k) \psi_k, \ \ \sigma_n(f) := \frac{1}{n} \sum_{k=1}^{n} S_k(f), \ \ D_n := \sum_{k=0}^{n-1} \psi_k,$$

where $n \in \mathbb{P}$, and $D_0 := 0$. It is well-known, that

$$S_n(f;x) = \int_{G_m} f(u)D_n(x-u)\mathrm{d}\mu(u). \tag{1.1}$$

Fejér kernels are defined as the arithmetical means of Dirichlet kernels, that is,

$$K_n := \frac{1}{n}\sum_{k=1}^{n} D_k.$$

Let $\{q_k : k \in \mathbb{N}\}$ be a sequence of non-negative numbers. The nth Nörlund mean of the Vilenkin-Fourier series is defined by

$$\frac{1}{Q_n}\sum_{k=1}^{n} q_{n-k}S_k(f;x),$$

where $Q_n := \sum_{k=0}^{n-1} q_k$ $(n \in \mathbb{P})$. It is always assumed that $q_0 > 0$ and

$$\lim_{n\to\infty} Q_n = \infty.$$

In this case, the summability method generated by $\{q_k\}$ is regular (see [28]) if and only if

$$\lim_{n\to\infty} \frac{q_{n-1}}{Q_n} = 0.$$

Let $\{p_k : k \in \mathbb{P}\}$ be a sequence of nonnegative numbers. The nth weighted mean T_n of Vilenkin-Fourier series is defined by

$$\frac{1}{P_n}\sum_{k=1}^{n} p_k S_k(f;x),$$

where $P_n := \sum_{k=1}^{n} p_k$ $(n \in \mathbb{P})$. It is always assumed that $p_1 > 0$ and

$$\lim_{n\to\infty} P_n = \infty,$$

which is the condition for regularity.

In particular case Nörlund and weighted means are Fejér ones (for all k set $p_k := 1$ and $q_k := 1$).

Let $T := (t_{i,j})_{i,j=1}^{\infty}$ be a doubly infinite matrix of numbers. It is always supposed that matrix T is triangular. Let us define the nth linear mean (or matrix transform mean) determined by the matrix T

$$\sigma_n^T(f;x) := \sum_{k=1}^{n} t_{k,n}S_k(f;x).$$

Since, the nth column of the matrix T determines the linear mean σ_n^T and its definition contains only finite number of entries, for the simplicity we say $\{t_{k,n} : 1 \le k \le n,\ k \in \mathbb{P}\}$ is a finite sequence of numbers for each $n \in \mathbb{P}$. We also use the notation

$$\Delta t_{k,n} := t_{k,n} - t_{k+1,n}.$$

In the further part of this paper, let $\{t_{k,n} : 1 \leq k \leq n,\ k \in \mathbb{P}\}$ be a finite sequence of non-negative numbers for each $n \in \mathbb{P}$. The nth matrix transform kernel is defined by

$$K_n^T(x) := \sum_{k=1}^{n} t_{k,n} D_k(x).$$

Based on (1.1) it is easily seen that

$$\sigma_n^T(f;x) = \int_{G_m} f(u) K_n^T(x-u) \mathrm{d}\mu(u). \tag{1.2}$$

2. Historical overview

Matrix transform means are common generalizations of several well-known summation methods. It follows by simple consideration that the Nörlund and weighted means, the Fejér (or the $(C,1)$) and the (C,α) means are special cases of the matrix transform summation method introduced above. For matrix transform means with respect to trigonometric system see e.g. results of Chandra [13] and Leindler [20], to Vilenkin system see paper of Blyumin [12]. This paper [12] of Blyumin gives an estimate of the approximation by general linear means of Vilenkin-Fourier series in terms of best approximations and some integrals similar to Lebesgue constants.

In recent years several papers were published with respect to the matrix transform means based on work of Móricz and Siddiqi [22] on the Walsh-Nörlund summation method and the result of Móricz and Rhoades [21] on the weighted mean method. A common generalization of these two results of Móricz and Siddiqi [22] and Móricz and Rhoades [21] was given by K. Nagy and the first author [7]. For other results on Walsh-Paley system see [10, 11], on bounded Vilenkin systems [8, 9]. In their paper [19], Goginava and K. Nagy improved these results (for the Walsh-system), introducing the definition of the so-called matrix transform variation.

Other aspects of these methods with respect to Walsh-Fourier series are treated in the papers [14, 27]. Avdispahić and Pepić proved some results also for Vilenkin system in their paper [2].

In their paper [18] Goginava and Gogoladze introduced the concept of Vilenkin-Lebesgue point generalized the definition of Walsh-Lebesgue point which was introduced by Weisz [26]. A point $x \in G_m$ is a Vilenkin-Lebesgue point of $f \in L_1(G_m)$, if

$$\lim_{A\to\infty} \sum_{s=0}^{A-1} M_s \sum_{r_s=1}^{m_s-1} \int_{I_A(x-r_s e_s)} |f(t)-f(x)| \mathrm{d}\mu(t) = 0,$$

where $e_s := (0,\dots,0,1,0,\dots) \in G_m$ (only the sth coordinate is 1, the others are 0).

An important result of the Vilenkin-Fourier analysis, that if $f \in L_1(G_m)$, then $S_{M_n}(f) \to f$ and $\sigma_n(f) \to f$, as $n \to \infty$ almost everywhere on G_m. Moreover, it

is known for any integrable function that almost every point of G_m is a Vilenkin-Lebesgue point and for any such point x

$$S_{M_n}(f;x) \to f(x) \tag{2.1}$$

as $n \to \infty$ for any $f \in L_1(G_m)$.

Also with respect to Vilenkin systems D. Baramidze, Dvalashvili and Tutberidze [3] for Nörlund means, D. Baramidze, Gogolashvili and Nadirashvili [4] for weighted means proved theorems connected to Lebesgue points. Our main aim is to generalize statements of papers of D. Baramidze, Dvalashvili, Tutberidze [3] and of D. Baramidze, Gogolashvili, Nadirashvili [4], to matrix transform means.

3. Auxiliary results

It is well-known that if G_m is a bounded Vilenkin group, the $L_1(G_m)$ norm of the Fejér kernels are uniformly bounded. Namely, there exists a positive constant c that depends only on $\sup_{n\in\mathbb{N}} m_n$ such that

$$\|K_n\|_1 \leq c \tag{3.1}$$

holds for every $n \in \mathbb{P}$.

It is also a famous result, that the M_nth Dirichlet kernel has the following closed form

$$D_{M_n}(x) = \begin{cases} 0, & \text{if } x \notin I_n(0), \\ M_n, & \text{if } x \in I_n(0). \end{cases} \tag{3.2}$$

To prove our corollaries we also need the following result.

Theorem 1 (Goginava and Gogoladze [18], see also Nadirashvili, Persson, Tephnadze and Weisz [23])**.** *If $f \in L_1(G_m)$, then*

$$\lim_{n\to\infty} \sigma_n(f;x) = f(x)$$

holds for all Vilenkin-Lebesgue points of f.

For the Walsh version of the next Lemma (more general, with $\sum_{k=1}^{n} t_{k,n} = O(1)$ assumption instead of (3.3)) see [6]. See also [19].

Lemma 1. *For every $n \in \mathbb{P}$, $\{t_{k,n} : 1 \leq k \leq n\}$ be a finite sequence of non-negative numbers such that*

$$\sum_{k=1}^{n} t_{k,n} = 1 \tag{3.3}$$

is satisfied.

a) Let the finite sequences $\{t_{k,n} : 1 \leq k \leq n\}$ of nonnegative numbers be non-increasing for all n, or

b) let the finite sequences $\{t_{k,n}: 1 \le k \le n\}$ *of nonnegative numbers be non-decreasing for all n and we suppose that*

$$t_{n,n} = O\left(\frac{1}{n}\right) \tag{3.4}$$

holds.

Then in both cases there exists absolute constant c, for which

$$\left\|K_n^T\right\|_1 < c$$

holds for every $n \in \mathbb{P}$.

Proof. From Abel transformation we get

$$K_n^T = \sum_{k=1}^{n} t_{k,n} D_k = \sum_{k=1}^{n-1} (t_{k,n} - t_{k+1,n}) k K_k + t_{n,n} n K_n,$$

so using (3.1)

$$\left\|K_n^T\right\|_1 \le \sum_{k=1}^{n-1} |\Delta t_{k,n}| k \|K_k\|_1 + t_{n,n} n \|K_n\|_1 \le c \left(\sum_{k=1}^{n-1} |\Delta t_{k,n}| k + t_{n,n} n \right).$$

It is easy to see that

$$\sum_{k=1}^{n-1} \Delta t_{k,n} k = \sum_{k=1}^{n} t_{k,n} - n t_{n,n}. \tag{3.5}$$

In case a) from (3.3) and (3.5)

$$\sum_{k=1}^{n-1} |\Delta t_{k,n}| k + t_{n,n} n = \sum_{k=1}^{n} t_{k,n} = 1.$$

In case b), using conditions (3.3), (3.4) and (3.5) we have

$$\sum_{k=1}^{n-1} |\Delta t_{k,n}| k + t_{n,n} n = 2 n t_{n,n} - \sum_{k=1}^{n} t_{k,n} = 2 n t_{n,n} - 1 \le 2c - 1 \le c.$$

□

4. Convergence in continuity or Vilenkin-Lebesgue points, almost everywhere convergence

Theorem 2. *For every* $n \in \mathbb{P}$, $\{t_{k,n}: 1 \le k \le n\}$ *be a finite sequence of non-negative numbers such that*

$$\sum_{k=1}^{n} t_{k,n} = 1 \tag{4.1}$$

is satisfied. Suppose that $f \in L_1(G_m)$ *and for some* $x \in G_m$ $\lim_{n\to\infty} \sigma_n(f;x) = f(x)$.

a) Let the finite sequences $\{t_{k,n}: 1 \le k \le n\}$ *of nonnegative numbers be non-increasing for all n and condition*

$$\lim_{n\to\infty} t_{1,n} = 0 \tag{4.2}$$

is satisfied, or

b) let the finite sequences $\{t_{k,n}: 1 \le k \le n\}$ *of nonnegative numbers be non-decreasing for all n. We also suppose that*

$$t_{n,n} = O\left(\frac{1}{n}\right) \tag{4.3}$$

holds.

Then in both cases

$$\lim_{n\to\infty} \sigma_n^T(f;x) = f(x).$$

Proof. From Abel transformation it is easy to get equality

$$\sum_{k=1}^{l} t_{k,n} = \sum_{k=1}^{l-1} \Delta t_{k,n} k + t_{l,n} l, \tag{4.4}$$

where $1 \le l \le n$ and a connection between T and Fejér means

$$\sigma_n^T(f;x) = \sum_{k=1}^{n-1} \Delta t_{k,n} k \sigma_k(f;x) + t_{n,n} n \sigma_n(f;x). \tag{4.5}$$

Using (4.1), (4.4) and (4.5) we can conclude

$$\sigma_n^T(f;x) - f(x) = \sum_{k=1}^{n-1} \Delta t_{k,n} k (\sigma_k(f;x) - f(x)) + t_{n,n} n (\sigma_n(f;x) - f(x)).$$

Fix $f \in L_1(G_m)$ and $x \in G_m$, for which $\lim_{n\to\infty} \sigma_n(f;x) = f(x)$.

Let us see case a). Then $|\Delta t_{k,n}| = \Delta t_{k,n}$.

$$\left|\sigma_n^T(f;x) - f(x)\right| \le \sum_{k=1}^{n-1} \Delta t_{k,n} k |\sigma_k(f;x) - f(x)| + t_{n,n} n |\sigma_n(f;x) - f(x)| =: I_n + II_n.$$

Since $\lim_{n\to\infty} |\sigma_n(f;x) - f(x)| = 0$, for any $\varepsilon > 0$ there exits $n_{a,1} \in \mathbb{P}$ such that

$$|\sigma_n(f;x) - f(x)| < \frac{\varepsilon}{2}, \tag{4.6}$$

when $n > n_{a,1}$, and there exists $A > 0$ constant, that

$$|\sigma_n(f;x) - f(x)| \le A \tag{4.7}$$

for every n. Then

$$I_n = \sum_{k=1}^{n-1} \Delta t_{k,n} k |\sigma_k(f;x) - f(x)|$$

$$= \sum_{k=1}^{n_{a,1}-1} \Delta t_{k,n} k |\sigma_k(f;x) - f(x)| + \sum_{k=n_{a,1}}^{n-1} \Delta t_{k,n} k |\sigma_k(f;x) - f(x)|$$
$$=: I_{n,1} + I_{n,2}.$$

For the first part using (4.4), (4.7) and the non-increasing monotony of sequence

$$I_{n,1} \le A \left(\sum_{k=1}^{n_{a,1}} t_{k,n} - t_{n_{a,1},n} n_{a,1} \right) \le A n_{a,1} t_{1,n}.$$

From condition (4.2) it follows that for any $\varepsilon > 0$ there exits $n_{a,2} \in \mathbb{P}$ such that $t_{1,n} < \varepsilon/(2An_{a,1})$ as $n_{a,2} < n$, so if $n_a := \max(n_{a,1}, n_{a,2}) < n$, then $I_{n,1} \le \varepsilon/2$.

Using (4.4), (4.6) and condition (4.1)

$$\begin{aligned} I_{n,2} &= \sum_{k=n_{a,1}}^{n-1} \Delta t_{k,n} k |\sigma_k(f;x) - f(x)| \\ &\le \frac{\varepsilon}{2} \sum_{k=n_{a,1}}^{n-1} \Delta t_{k,n} k \\ &\le \frac{\varepsilon}{2} \sum_{k=1}^{n-1} \Delta t_{k,n} k \\ &= \frac{\varepsilon}{2} (1 - t_{n,n} n) \le \frac{\varepsilon}{2}. \end{aligned}$$

Summarising our result we get $0 \le I_n \le \varepsilon$, if $n_a < n$, so $\lim_{n\to\infty} I_n = 0$.

On the other side, since the sequence is non-increasing, we obtain from condition (4.1), that

$$0 \le II_n \le \sum_{k=1}^{n} t_{k,n} |\sigma_n(f;x) - f(x)| = |\sigma_n(f;x) - f(x)| \to 0$$

as $n \to \infty$, so $\lim_{n\to\infty} II_n = 0$.

It yields that $I_n + II_n = \left|\sigma_n^T(f;x) - f(x)\right|$ also tends to zero in every point $x \in G_m$, for which $\lim_{n\to\infty} \sigma_n(f;x) = f(x)$.

Let us see case b). Then $|\Delta t_{k,n}| = -\Delta t_{k,n}$.

Assumption (4.3) means, that there exists absolute constant c, for which $n t_{n,n} \le c$.

Now since $\lim_{n\to\infty} |\sigma_n(f;x) - f(x)| = 0$, for any $\varepsilon > 0$ there exits $n_{b,1} \in \mathbb{P}$ such that

$$|\sigma_n(f;x) - f(x)| < \frac{\varepsilon}{2c}, \tag{4.8}$$

when $n > n_{b,1}$.

$$\left|\sigma_n^T(f;x) - f(x)\right| \le -\sum_{k=1}^{n-1} \Delta t_{k,n} k |\sigma_k(f;x) - f(x)| + t_{n,n} n |\sigma_n(f;x) - f(x)| =: III_n + IV_n.$$

Then

$$\begin{aligned} III_n &= -\sum_{k=1}^{n-1} \Delta t_{k,n} k |\sigma_k(f;x) - f(x)| \\ &= -\sum_{k=1}^{n_{b,1}-1} \Delta t_{k,n} k |\sigma_k(f;x) - f(x)| + \left(-\sum_{k=n_{b,1}}^{n-1} \Delta t_{k,n} k |\sigma_k(f;x) - f(x)| \right) \\ &=: III_{n,1} + III_{n,2}. \end{aligned}$$

In this case b) $\Delta t_{k,n} \leq 0$ is satisfied for all $k \in \{1,\dots,n\}$, therefore $0 \leq III_{n,1}$ and $0 \leq III_{n,2}$.

For the first part using (4.3), (4.4), (4.7) and the non-decreasing monotony of sequence $t_{k,n}$

$$III_{n,1} \leq A \left(t_{n_{b,1},n} n_{b,1} - \sum_{k=1}^{n_{b,1}} t_{k,n} \right) \leq A n_{b,1} t_{n_{b,1},n} \leq A n_{b,1} t_{n,n} \leq \frac{c}{n} A n_{b,1}.$$

It means, that for any $\varepsilon > 0$ let $n_b := \max(n_{b,1}, n_{b,2})$, where $n_{b,2} := [2cAn_{b,1}/\varepsilon] + 1$. (Here $[y]$ denotes the integer part of the real number y.) Then $III_{n,1} \leq \varepsilon/2$ holds for every $n_b < n$.

Using (4.3), (4.4) and (4.8) we get

$$\begin{aligned} III_{n,2} &= -\sum_{k=n_{b,1}}^{n-1} \Delta t_{k,n} k |\sigma_k(f;x) - f(x)| \\ &\leq -\frac{\varepsilon}{2c} \sum_{k=n_{b,1}}^{n-1} \Delta t_{k,n} k \\ &\leq -\frac{\varepsilon}{2c} \sum_{k=1}^{n-1} \Delta t_{k,n} k \\ &= \frac{\varepsilon}{2c} (t_{n,n} n - 1) \leq \frac{\varepsilon}{2c} \frac{c}{n} n = \frac{\varepsilon}{2}. \end{aligned}$$

Summarising our result we get $0 \leq III_n \leq \varepsilon$, if $n_b < n$, so $\lim_{n\to\infty} III_n = 0$.

On the other side, since the sequence is non-decreasing satisfying (4.3), it yields

$$0 \leq IV_n = t_{n,n} n |\sigma_n(f;x) - f(x)| \leq c |\sigma_n(f;x) - f(x)| \to 0$$

as $n \to \infty$, so $\lim_{n\to\infty} IV_n = 0$.

Consequently $III_n + IV_n = \left|\sigma_n^T(f;x) - f(x)\right|$ also tends to zero in every appropriate points x. □

Corollary 1. *For every $n \in \mathbb{P}$, $\{t_{k,n} : 1 \leq k \leq n\}$ be a finite sequence of non-negative numbers such that*

$$\sum_{k=1}^{n} t_{k,n} = 1$$

is satisfied.

a) Let the finite sequences $\{t_{k,n}\colon 1 \le k \le n\}$ *of nonnegative numbers be non-increasing for all n and condition*

$$\lim_{n\to\infty} t_{1,n} = 0$$

is satisfied, or

b) let the finite sequences $\{t_{k,n}\colon 1 \le k \le n\}$ *of nonnegative numbers be non-decreasing for all n. We also suppose that*

$$t_{n,n} = O\left(\frac{1}{n}\right)$$

holds.

Then in both cases

$$\lim_{n\to\infty} \sigma_n^T(f;x) = f(x)$$

for all continuity or Vilenkin-Lebesgue points x of $f \in L_1(G_m)$.

Proof. This statement is a simple consequence of Theorem 1 and Theorem 2. □

Remark 1. We get immediately, that with the conditions of Corollary 1

$$\sigma_n^T(f) \to f$$

almost everywhere.

5. Convergence in norm

Theorem 3. *Suppose that* $f \in L_1(G_m)$ *and for every* $n \in \mathbb{N}$, $\{t_{k,n}\colon 1 \le k \le n\}$ *be a finite sequence of non-negative numbers such that*

$$\sum_{k=1}^{n} t_{k,n} = 1$$

is satisfied.

a) If the finite sequence $\{t_{k,n}\colon 1 \le k \le n\}$ *is non-increasing as a function of k for all fixed n and we also have*

$$\lim_{n\to\infty} t_{1,n} = 0$$

is satisfied, or

b) if the finite sequence $\{t_{k,n}\colon 1 \le k \le n\}$ *is non-decreasing as a function of k for all fixed n and the condition*

$$t_{n,n} = O\left(\frac{1}{n}\right),$$

then we have the L_1*-norm convergence* $\sigma_n^T(f) \to f$.

Proof. The proof is a simple consequence of Lemma 1 and the usual density argument.

The $\mathcal{P}$ set of Vilenkin polynomials is dense in $L_p(G_m)$ for each $1 \leq p < \infty$. Besides, for any Vilenkin polynomial $P \in \mathcal{P}$ we have $S_n(P) = P$ for sufficiently large n, say for $n \geq k_0$.

In both cases we have

$$\lim_{n\to\infty} t_{k,n} = 0$$

for any fixed k, so $t_{k,n} \to 0$ if $k < k_0$ and

$$\lim_{n\to\infty} \sum_{k=k_0}^{n} t_{k,n} = 1.$$

It follows

$$\sigma_n^T(P) = \sum_{k=1}^{k_0-1} t_{k,n} S_k(P) + \sum_{k=k_0}^{n} t_{k,n} P \to 0 + P = P$$

as $n \to \infty$, in norm and also everywhere. □

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ON RESULTS OF PARAMETERIZED INEQUALITIES ASSOCIATED WITH CONFORMABLE FRACTIONAL INTEGRALS

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Abstract. In the present article, an equality is established for the case of differentiable convex functions associated with the conformable fractional integrals. With the help of this equality, several parameterized inequalities are proved by using the conformable fractional integrals. More precisely, some inequalities are obtained by taking advantage of the convexity, the Hölder inequality, and the power mean inequality. Moreover, previously achieved results and new results are given by using special cases of the obtained theorems.

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1. INTRODUCTION & PRELIMINARIES

Convex theory is an effective methodology in the case of considering the great issues that origin in several areas of the pure and applied sciences. This theory has a rich history and also has been the motivation of special mathematical research. Moreover, convexity theory has a critical place in the advancement of the idea of inequality. There are several types of convexity in the literature.

Definition 1 ([17])**.** Let I be an interval of real numbers. Then, a function $\mathcal{F}: I \to \mathbb{R}$ is said to be convex, if

$$\mathcal{F}\left(\mu x+(1-\mu)y\right) \leq \mu f\left(x\right)+(1-\mu)\mathcal{F}\left(y\right)$$

is valid $\forall x, y \in I$ and $\forall \mu \in [0,1]$.

The inequalities, introduced by C. Hermite and J. Hadamard for the case of convex functions, are significant subject in the literature. These inequalities state that if $\mathcal{F}: I \to \mathbb{R}$ is a convex function on the interval I of real numbers and $\sigma, \delta \in I$ with

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$\sigma < \delta$, then the following double inequality holds:

$$\mathcal{F}\left(\frac{\sigma+\delta}{2}\right) \leq \frac{1}{\delta-\sigma}\int_{\sigma}^{\delta} \mathcal{F}(x)\mathrm{d}x \leq \frac{\mathcal{F}(\sigma)+\mathcal{F}(\delta)}{2}. \tag{1.1}$$

Trapezoid-type and midpoint-type inequalities for various types of convex functions have been studied extensively by many mathematicians. To be more precise, an upper bound for which is the right-hand side of inequality (1.1) first was established by Dragomir and Agarwal in paper [6]. An upper bound for which is the left-hand side of inequality (1.1) first was proved by Kırmacı in paper [16]. Qaisar and Hussain [18] investigated several generalized midpoint-type inequalities. Moreover, Sarikaya et al. and Iqbal et al. proved some fractional trapezoid-type and midpoint-type inequalities to the case of convex functions in papers [20] and [12], respectively.

Large numbers of papers are focused on the results of Simpson-type inequalities for the case of convex functions. For example, several Simpson-type inequalities proved by the case of s-convex differentiable functions [2]. Moreover, Sarikaya et al. [19] presented the new variants of Simpson-type inequalities based on differentiable convex function. Some recent results connected with these type of inequalities, see [3] and the references cited therein.

Bullen [4] proved the inequality which is known as Bullen-type inequalities for the case of convex function. Erden and Sarikaya presented the generalized Bullen-type inequalities for the case of local fractional integrals on fractal sets in paper [8]. Moreover, Du et al. [7] used the generalized fractional integrals to get Bullen-type inequalities. In paper [5], several generalizations of integral inequalities of Bullen-type inequalities were obtained for the case of twice differentiable functions including Riemann-Liouville fractional integrals.

Kilbas et al. [15] presented fractional integrals, also called Riemann-Liouville integrals as follows:

Definition 2 ([15])**.** Let $\mathcal{F} \in L_1[\sigma,\delta]$, $\sigma,\delta \in \mathbb{R}$ with $\sigma < \delta$. The Riemann-Liouville integrals $J^{\beta}_{\sigma^+}\mathcal{F}$ and $J^{\beta}_{\delta^-}\mathcal{F}$ of order $\beta > 0$ are defined by

$$J^{\beta}_{\sigma^+}\mathcal{F}(x) = \frac{1}{\Gamma(\beta)}\int_{\sigma}^{x}(x-\mu)^{\beta-1}\mathcal{F}(\mu)\mathrm{d}\mu, \quad x > \sigma \tag{1.2}$$

and

$$J^{\beta}_{\delta^-}\mathcal{F}(x) = \frac{1}{\Gamma(\beta)}\int_{x}^{\delta}(\mu-x)^{\beta-1}\mathcal{F}(\mu)\mathrm{d}\mu, \quad x < \delta, \tag{1.3}$$

respectively. Here, Γ denotes the Gamma function defined by $\Gamma(\beta) = \int_0^{\infty} e^{-u}u^{\beta-1}\mathrm{d}u$, where $\beta > 0$.

Jarad et al. [13] established the fractional conformable integral operators and they also provided some characteristics and relationships between these operators and several other fractional operators in the literature. The fractional conformable integral operators are defined as follows:

Definition 3 ([13])**.** Let $\beta > 0$ and $\alpha \in (0,1]$. For $\mathcal{F} \in L_1[\sigma,\delta]$, the generalized fractional Riemann-Liouville integrals ${}^{\beta}_{+}\mathcal{J}^{\alpha}_{\sigma}\mathcal{F}$ and ${}^{\beta}_{-}\mathcal{J}^{\alpha}_{\delta}\mathcal{F}$ are defined by

$$ {}^{\beta}_{+}\mathcal{J}^{\alpha}_{\sigma}\mathcal{F}(x) = \frac{1}{\Gamma(\beta)}\int_{\sigma}^{x}\left(\frac{(x-\sigma)^{\alpha}-(\mu-\sigma)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{\mathcal{F}(\mu)}{(\mu-\sigma)^{1-\alpha}}d\mu, \quad x>\sigma, \tag{1.4} $$

and

$$ {}^{\beta}_{-}\mathcal{J}^{\alpha}_{\delta}\mathcal{F}(x) = \frac{1}{\Gamma(\beta)}\int_{x}^{\delta}\left(\frac{(\delta-x)^{\alpha}-(\delta-\mu)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{\mathcal{F}(\mu)}{(\delta-\mu)^{1-\alpha}}d\mu, \quad x<\delta, \tag{1.5} $$

respectively.

If we assign $\alpha = 1$, then the fractional integral in (1.4) and (1.5) equals to the Riemann-Liouville fractional integral in (1.2) and (1.3), respectively.

Several midpoint-type inequalities are proved by using the conformable fractional integrals in paper [10]. Set et al. [21] establish a new generalization of Hermite-Hadamard inequality for fractional conformable integrals. In addition to these, conformable derivative is improved in paper [14] by using the derivative's fundamental limit formulation. Some important requirements that cannot be fulfilled by the Riemann-Liouville and Caputo definitions are fulfilled by the conformable derivative. In paper [1], Abdelhakim indicates that the conformable approach in [14] cannot yield good results when compared to the Caputo definition for the case of specific functions.

2. Main results

Throughout the paper, we assume that $\alpha \in (0,1]$, $\beta \in \mathbb{R}^{+}$, and $\lambda \in [0,1]$.

Lemma 1. *Let $\mathcal{F}:[\sigma,\delta]\to\mathbb{R}$ be a differentiable mapping on (σ,δ) with $\beta>0$ and $\alpha\in(0,1]$. If $\mathcal{F}'\in L[\sigma,\delta]$, then the following equality for fractional integrals hold:*

$$ \begin{aligned} &(1-\lambda)\,\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+\lambda\left(\frac{\mathcal{F}(\sigma)+\mathcal{F}(\delta)}{2}\right) \\ &\quad-\frac{\alpha^{\beta}2^{\alpha\beta-1}}{(\delta-\sigma)^{\alpha\beta}}\Gamma(\beta+1)\left[{}^{\beta}_{+}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}}\mathcal{F}(\delta)+{}^{\beta}_{-}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}}\mathcal{F}(\sigma)\right] \\ &=\frac{\alpha^{\beta}(\delta-\sigma)}{4}\left[\int_{0}^{1}\left(\frac{\lambda}{\alpha^{\beta}}-\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right)\mathcal{F}'\left(\frac{\mu}{2}\sigma+\frac{2-\mu}{2}\delta\right)d\mu\right. \end{aligned} $$

$$-\int_0^1 \left(\frac{\lambda}{\alpha^\beta}-\left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^\beta\right)\mathcal{F}'\left(\frac{2-\mu}{2}\sigma+\frac{\mu}{2}\delta\right)d\mu\Bigg], \tag{2.1}$$

where $\Gamma(\beta)$ *is Euler Gamma function.*

Proof. Assume that I is the integral on the right side of (2.1) and consider

$$I := I_1 - I_2.$$

Using integrating by parts and changing variables with $x = \frac{\mu}{2}\sigma + \frac{2-\mu}{2}\delta$, we get

$$\begin{aligned}
I_1 &= \int_0^1 \left(\frac{\lambda}{\alpha^\beta}-\left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^\beta\right)\mathcal{F}'\left(\frac{\mu}{2}\sigma+\frac{2-\mu}{2}\delta\right)d\mu \\
&= -\frac{2}{\delta-\sigma}\left(\frac{\lambda}{\alpha^\beta}-\left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^\beta\right)\mathcal{F}\left(\frac{\mu}{2}\sigma+\frac{2-\mu}{2}\delta\right)\Bigg|_0^1 \\
&\quad -\frac{2\beta}{\delta-\sigma}\int_0^1 \left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^{\beta-1}(1-\mu)^{\alpha-1}\mathcal{F}\left(\frac{\mu}{2}\sigma+\frac{2-\mu}{2}\delta\right)d\mu \\
&= \frac{2(1-\lambda)}{(\delta-\sigma)\alpha^\beta}\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+\frac{2\lambda}{(\delta-\sigma)\alpha^\beta}\mathcal{F}(\delta) \\
&\quad -\left(\frac{2}{\delta-\sigma}\right)^{1+\alpha\beta}\beta\int_{\frac{\sigma+\delta}{2}}^{\delta}\left(\frac{\left(\frac{\delta-\sigma}{2}\right)^\alpha-\left(x-\frac{\sigma+\delta}{2}\right)^\alpha}{\alpha}\right)^{\beta-1}\frac{\mathcal{F}(x)}{\left(x-\frac{\sigma+\delta}{2}\right)^{1-\alpha}}dx \\
&= \frac{2(1-\lambda)}{(\delta-\sigma)\alpha^\beta}\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+\frac{2\lambda}{(\delta-\sigma)\alpha^\beta}\mathcal{F}(\delta) \\
&\quad -\left(\frac{2}{\delta-\sigma}\right)^{1+\alpha\beta}\Gamma(\beta+1)\,{}^{\beta}_{+}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}}\mathcal{F}(\delta).
\end{aligned}$$

Similar to foregoing process, changing variables with $x = \frac{2-\mu}{2}\sigma + \frac{\mu}{2}\delta$ and integrating on $[0,1]$, we have

$$\begin{aligned}
I_2 &= \int_0^1 \left(\frac{\lambda}{\alpha^\beta}-\left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^\beta\right)\mathcal{F}'\left(\frac{2-\mu}{2}\sigma+\frac{\mu}{2}\delta\right)d\mu \\
&= \frac{2}{\delta-\sigma}\left(\frac{\lambda}{\alpha^\beta}-\left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^\beta\right)\mathcal{F}\left(\frac{2-\mu}{2}\sigma+\frac{\mu}{2}\delta\right)\Bigg|_0^1
\end{aligned}$$

$$
\begin{aligned}
&+\frac{2\beta}{\delta-\sigma}\int_0^1\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta-1}(1-\mu)^{\alpha-1}\,\mathcal{F}\left(\frac{2-\mu}{2}\sigma+\frac{\mu}{2}\delta\right)\mathrm{d}\mu\\
&=\frac{2(\lambda-1)}{(\delta-\sigma)\alpha^{\beta}}\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)-\frac{2\lambda}{(\delta-\sigma)\alpha^{\beta}}\mathcal{F}(\sigma)\\
&+\left(\frac{2}{\delta-\sigma}\right)^{1+\alpha\beta}\beta\int_{\sigma}^{\frac{\sigma+\delta}{2}}\left(\frac{\left(\frac{\delta-\sigma}{2}\right)^{\alpha}-\left(\frac{\sigma+\delta}{2}-x\right)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{\mathcal{F}(x)}{\left(\frac{\sigma+\delta}{2}-x\right)^{1-\alpha}}\mathrm{d}x\\
&=-\frac{2(1-\lambda)}{(\delta-\sigma)\alpha^{\beta}}\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)-\frac{2\lambda}{(\delta-\sigma)\alpha^{\beta}}\mathcal{F}(\sigma)\\
&+\left(\frac{2}{\delta-\sigma}\right)^{1+\alpha\beta}\Gamma(\beta+1)\,{}_{-}^{\beta}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}}\mathcal{F}(\sigma).
\end{aligned}
$$

Finally, if we multiply I_1-I_2 by $\frac{\alpha^{\beta}(\delta-\sigma)}{4}$, then we have (2.1). This completes the proof of Lemma 1. □

Remark 1. If we choose $\lambda=0$ in (2.1), then the equality reduces to

$$
\begin{aligned}
&\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)-\frac{\alpha^{\beta}2^{\alpha\beta-1}}{(\delta-\sigma)^{\alpha\beta}}\Gamma(\beta+1)\left[{}_{+}^{\beta}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}}\mathcal{F}(\delta)+{}_{-}^{\beta}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}}\mathcal{F}(\sigma)\right]\\
&=\frac{\alpha^{\beta}(\delta-\sigma)}{4}\left[\int_0^1\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\mathcal{F}'\left(\frac{2-\mu}{2}\sigma+\frac{\mu}{2}\delta\right)\mathrm{d}\mu\right.\\
&\qquad\left.-\int_0^1\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\mathcal{F}'\left(\frac{\mu}{2}\sigma+\frac{2-\mu}{2}\delta\right)\mathrm{d}\mu\right].
\end{aligned}
$$

This coincides with [9, Lemma 1].

Remark 2. Let us consider $\lambda=1$ in (2.1). Then, we have

$$
\begin{aligned}
&\frac{\mathcal{F}(\sigma)+\mathcal{F}(\delta)}{2}-\frac{\alpha^{\beta}2^{\alpha\beta-1}}{(\delta-\sigma)^{\alpha\beta}}\Gamma(\beta+1)\left[{}_{+}^{\beta}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}}\mathcal{F}(\delta)+{}_{-}^{\beta}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}}\mathcal{F}(\sigma)\right]\\
&=\frac{\alpha^{\beta}(\delta-\sigma)}{4}\left[\int_0^1\left(\frac{1}{\alpha^{\beta}}-\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right)\mathcal{F}'\left(\frac{\mu}{2}\sigma+\frac{2-\mu}{2}\delta\right)\mathrm{d}\mu\right.\\
&\qquad\left.-\int_0^1\left(\frac{1}{\alpha^{\beta}}-\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right)\mathcal{F}'\left(\frac{2-\mu}{2}\sigma+\frac{\mu}{2}\delta\right)\mathrm{d}\mu\right],
\end{aligned}
$$

which is investigated by E. Set et al. [22, Lemma 2.1 $\left(\text{for } x=\frac{a+b}{2}\right)$].

Corollary 1. *Consider* $\lambda = \frac{1}{3}$ *in* (2.1). *Then,* (2.1) *becomes to*

$$\frac{1}{6}\left[\mathcal{F}(\sigma)+\mathcal{F}(\delta)+4f\left(\frac{\sigma+\delta}{2}\right)\right]-\frac{\alpha^{\beta}2^{\alpha\beta-1}}{(\delta-\sigma)^{\alpha\beta}}\Gamma(\beta+1)\left[{}^{\beta}_{+}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}}\mathcal{F}(\delta)+{}^{\beta}_{-}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}}\mathcal{F}(\sigma)\right]$$
$$=\frac{\alpha^{\beta}(\delta-\sigma)}{4}\left[\int_0^1\left(\frac{1}{3\alpha^{\beta}}-\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right)\mathcal{F}'\left(\frac{\mu}{2}\sigma+\frac{2-\mu}{2}\delta\right)d\mu\right.$$
$$\left.+\int_0^1\left(\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{3\alpha^{\beta}}\right)\mathcal{F}'\left(\frac{2-\mu}{2}\sigma+\frac{\mu}{2}\delta\right)d\mu\right].$$

Remark 3. For $\lambda = \frac{1}{2}$, equality (2.1) reduces to

$$\frac{1}{2}\left[\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+\frac{\mathcal{F}(\sigma)+\mathcal{F}(\delta)}{2}\right]-\frac{\alpha^{\beta}2^{\alpha\beta-1}}{(\delta-\sigma)^{\alpha\beta}}\Gamma(\beta+1)\left[{}^{\beta}_{+}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}}\mathcal{F}(\delta)+{}^{\beta}_{-}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}}\mathcal{F}(\sigma)\right]$$
$$=\frac{\alpha^{\beta}(\delta-\sigma)}{4}$$
$$\cdot\int_0^1\left(\frac{1}{2\alpha^{\beta}}-\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right)\left[\mathcal{F}'\left(\frac{\mu}{2}\sigma+\frac{2-\mu}{2}\delta\right)-\mathcal{F}'\left(\frac{2-\mu}{2}\sigma+\frac{\mu}{2}\delta\right)\right]d\mu.$$

This coincides with [11, Lemma 1].

Theorem 1. *Assume that all assumptions of Lemma 1 hold. Moreover, let* $|\mathcal{F}'|$ *be a convex function on* $[\sigma,\delta]$. *Then, we have*

$$\left|(1-\lambda)\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+\lambda\left(\frac{\mathcal{F}(\sigma)+\mathcal{F}(\delta)}{2}\right)\right.$$
$$\left.-\frac{\alpha^{\beta}2^{\alpha\beta-1}}{(\delta-\sigma)^{\alpha\beta}}\Gamma(\beta+1)\left[{}^{\beta}_{+}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}}\mathcal{F}(\delta)+{}^{\beta}_{-}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}}\mathcal{F}(\sigma)\right]\right|$$
$$\leq\frac{\alpha^{\beta}(\delta-\sigma)}{4}A_1(\alpha,\beta,\lambda)\left(\left|\mathcal{F}'(\sigma)\right|+\left|\mathcal{F}'(\delta)\right|\right), \tag{2.2}$$

where

$$A_1(\alpha,\beta,\lambda)=\int_0^1\left|\frac{\lambda}{\alpha^{\beta}}-\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right|d\mu$$
$$=\frac{1}{\alpha^{\beta}}\left[\lambda(2C-1)+\frac{1}{\alpha}\left(\mathfrak{B}\left(\beta+1,\frac{1}{\alpha}\right)-2\mathscr{B}\left(\beta+1,\frac{1}{\alpha},\lambda^{\frac{1}{\beta}}\right)\right)\right]$$

with $C = 1 - \left(1 - \lambda^{\frac{1}{\beta}}\right)^{\frac{1}{\alpha}}$, *the functions* $\mathfrak{B}(\cdot,\cdot)$ *and* $\mathscr{B}(\cdot,\cdot,\cdot)$ *are the Beta function and the incomplete Beta function defined as*

$$\begin{cases} \mathfrak{B}(x,y) = \int_0^1 u^{x-1}(1-u)^{y-1}\mathrm{d}u, \\ \mathscr{B}(x,y,r) = \int_0^r u^{x-1}(1-u)^{y-1}\mathrm{d}u \end{cases}$$

for $x,y>0$ *and* $r \in [0,1]$.

Proof. By Lemma 1, integrating by parts and convexity of $|\mathcal{F}'|$, we get

$$\begin{aligned}
&\left|(1-\lambda)\,\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+\lambda\left(\frac{\mathcal{F}(\sigma)+\mathcal{F}(\delta)}{2}\right)\right. \\
&\left.-\frac{\alpha^{\beta}2^{\alpha\beta-1}}{(\delta-\sigma)^{\alpha\beta}}\Gamma(\beta+1)\left[{}^{\beta}_{+}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}}\mathcal{F}(\delta)+{}^{\beta}_{-}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}}\mathcal{F}(\sigma)\right]\right| \\
&\leq \frac{\alpha^{\beta}(\delta-\sigma)}{4}\left[\left|\int_0^1\left(\frac{\lambda}{\alpha^{\beta}}-\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right)\mathcal{F}'\left(\frac{\mu}{2}\sigma+\frac{2-\mu}{2}\delta\right)\mathrm{d}\mu\right|\right. \\
&\qquad\left.+\left|\int_0^1\left(\frac{\lambda}{\alpha^{\beta}}-\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right)\mathcal{F}'\left(\frac{2-\mu}{2}\sigma+\frac{\mu}{2}\delta\right)\mathrm{d}\mu\right|\right] \\
&\leq \frac{\alpha^{\beta}(\delta-\sigma)}{4}\left[\int_0^1\left|\frac{\lambda}{\alpha^{\beta}}-\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right|\left|\mathcal{F}'\left(\frac{\mu}{2}\sigma+\frac{2-\mu}{2}\delta\right)\right|\mathrm{d}\mu\right. \\
&\qquad\left.+\int_0^1\left|\frac{\lambda}{\alpha^{\beta}}-\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right|\left|\mathcal{F}'\left(\frac{2-\mu}{2}\sigma+\frac{\mu}{2}\delta\right)\right|\mathrm{d}\mu\right] \\
&\leq \frac{\alpha^{\beta}(\delta-\sigma)}{4}\left[\int_0^1\left|\frac{\lambda}{\alpha^{\beta}}-\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right|\left[\frac{\mu}{2}\left|\mathcal{F}'(\sigma)\right|+\frac{2-\mu}{2}\left|\mathcal{F}'(\delta)\right|\right]\mathrm{d}\mu\right. \\
&\qquad\left.+\int_0^1\left|\frac{\lambda}{\alpha^{\beta}}-\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right|\left[\frac{2-\mu}{2}\left|\mathcal{F}'(\sigma)\right|+\frac{\mu}{2}\left|\mathcal{F}'(\delta)\right|\right]\mathrm{d}\mu\right] \\
&= \frac{\alpha^{\beta}(\delta-\sigma)}{4}A_1(\alpha,\beta,\lambda)\left(\left|\mathcal{F}'(\sigma)\right|+\left|\mathcal{F}'(\delta)\right|\right).
\end{aligned}$$

That is the desired result. □

Remark 4. If we consider (2.2) as $\lambda = 0$, then the inequality reduces to

$$\left| \mathcal{F}\left(\frac{\sigma+\delta}{2}\right) - \frac{\alpha^{\beta} 2^{\alpha\beta-1}}{(\delta-\sigma)^{\alpha\beta}} \Gamma(\beta+1) \left[{}^{\beta}_{+}J^{\alpha}_{\frac{\sigma+\delta}{2}} \mathcal{F}(\delta) + {}^{\beta}_{-}J^{\alpha}_{\frac{\sigma+\delta}{2}} \mathcal{F}(\sigma) \right] \right|$$
$$\leq \frac{\delta-\sigma}{4\alpha} \mathfrak{B}\left(\beta+1, \frac{1}{\alpha}\right) \left(\left|\mathcal{F}'(\sigma)\right| + \left|\mathcal{F}'(\delta)\right| \right).$$

This coincides with [9, Theorem 2].

Remark 5. If we select $\lambda = 1$ in (2.2), then we obtain

$$\left| \frac{\mathcal{F}(\sigma)+\mathcal{F}(\delta)}{2} - \frac{\alpha^{\beta} 2^{\alpha\beta-1}}{(\delta-\sigma)^{\alpha\beta}} \Gamma(\beta+1) \left[{}^{\beta}_{+}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}} \mathcal{F}(\delta) + {}^{\beta}_{-}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}} \mathcal{F}(\sigma) \right] \right|$$
$$\leq \frac{\delta-\sigma}{4} \left[1 - \frac{1}{\alpha} \mathfrak{B}\left(\beta+1, \frac{1}{\alpha}\right) \right] \left(\left|\mathcal{F}'(\sigma)\right| + \left|\mathcal{F}'(\delta)\right| \right),$$

which is established by E. Set et al. [22, Theorem 2.1 $\left(\text{for } x = \frac{a+b}{2}\right)$].

Corollary 2. *Let us consider* $\lambda = \frac{1}{3}$ *in inequality* (2.2). *Then, it follows*

$$\left| \frac{1}{6} \left[\mathcal{F}(\sigma) + \mathcal{F}(\delta) + 4f\left(\frac{\sigma+\delta}{2}\right) \right] - \frac{\alpha^{\beta}\Gamma(\beta+1)}{2} \left(\frac{2}{\delta-\sigma}\right)^{\alpha\beta} \left[{}^{\beta}_{+}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}} \mathcal{F}(\delta) + {}^{\beta}_{-}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}} \mathcal{F}(\sigma) \right] \right|$$
$$\leq \frac{(\delta-\sigma)}{4} \left[\frac{2D-1}{3} + \frac{1}{\alpha} \left(\mathfrak{B}\left(\beta+1, \frac{1}{\alpha}\right) - 2\mathscr{B}\left(\beta+1, \frac{1}{\alpha}, \left(\frac{1}{3}\right)^{\frac{1}{\beta}}\right) \right) \right] \left(\left|\mathcal{F}'(\sigma)\right| + \left|\mathcal{F}'(\delta)\right| \right).$$

Here, $D = 1 - \left(1 - \left(\frac{1}{3}\right)^{\frac{1}{\beta}}\right)^{\frac{1}{\alpha}}$.

Remark 6. For $\lambda = \frac{1}{2}$, inequality (2.2) reduces to

$$\left| \frac{1}{2} \left[\mathcal{F}\left(\frac{\sigma+\delta}{2}\right) + \frac{\mathcal{F}(\sigma)+\mathcal{F}(\delta)}{2} \right] - \frac{\alpha^{\beta} 2^{\alpha\beta-1}}{(\delta-\sigma)^{\alpha\beta}} \Gamma(\beta+1) \left[{}^{\beta}_{+}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}} \mathcal{F}(\delta) + {}^{\beta}_{-}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}} \mathcal{F}(\sigma) \right] \right|$$
$$\leq \frac{\alpha^{\beta}(\delta-\sigma)}{4} \left\{ \frac{1}{\alpha^{\beta}} \left[\frac{1}{2} - \left(1 - \left(\frac{1}{2}\right)^{\frac{1}{\beta}}\right)^{\frac{1}{\alpha}} \right] \right.$$
$$\left. + \frac{1}{\alpha^{\beta+1}} \left[\mathfrak{B}\left(\beta+1, \frac{1}{\alpha}\right) - 2\mathscr{B}\left(\beta+1, \frac{1}{\alpha}, \left(\frac{1}{2}\right)^{\frac{1}{\beta}}\right) \right] \right\} \left(\left|\mathcal{F}'(\sigma)\right| + \left|\mathcal{F}'(\delta)\right| \right).$$

This coincides with [11, Theorem 1].

Theorem 2. *Suppose that all assumptions of Lemma 1 hold. Moreover, let* $|\mathcal{F}'|^q$ *be a convex function on* $[\sigma,\delta]$ *where* $\frac{1}{p} + \frac{1}{q} = 1$ *with* $p, q > 1$. *Then, we have*

$$\left| (1-\lambda)\mathcal{F}\left(\frac{\sigma+\delta}{2}\right) + \lambda\left(\frac{\mathcal{F}(\sigma)+\mathcal{F}(\delta)}{2}\right) \right.$$

$$
\begin{aligned}
&-\frac{\alpha^{\beta}2^{\alpha\beta-1}}{(\delta-\sigma)^{\alpha\beta}}\Gamma(\beta+1)\left[{}_{+}^{\beta}\mathcal{J}_{\frac{\sigma+\delta}{2}}^{\alpha}\mathcal{F}(\delta)+{}_{-}^{\beta}\mathcal{J}_{\frac{\sigma+\delta}{2}}^{\alpha}\mathcal{F}(\sigma)\right]\Bigg| \\
&\leq \frac{\alpha^{\beta}(\delta-\sigma)}{4}A_2^{\frac{1}{p}}(\alpha,\beta,p,\lambda) \\
&\quad\cdot\left\{\left[\frac{|\mathcal{F}'(\sigma)|^q+3|\mathcal{F}'(\delta)|^q}{4}\right]^{\frac{1}{q}}+\left[\frac{3|\mathcal{F}'(\sigma)|^q+|\mathcal{F}'(\delta)|^q}{4}\right]^{\frac{1}{q}}\right\} \\
&\leq \frac{\alpha^{\beta}(\delta-\sigma)}{4}\left[4A_2(\alpha,\beta,p,\lambda)\right]^{\frac{1}{p}}\left(\left|\mathcal{F}'(\sigma)\right|+\left|\mathcal{F}'(\delta)\right|\right), \qquad (2.3)
\end{aligned}
$$

where

$$
A_2(\alpha,\beta,p,\lambda)=\int_0^1\left|\frac{\lambda}{\alpha^{\beta}}-\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right|^p d\mu.
$$

Proof. If we consider Lemma 1, then we get

$$
\begin{aligned}
&\Bigg|(1-\lambda)\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+\lambda\left(\frac{\mathcal{F}(\sigma)+\mathcal{F}(\delta)}{2}\right) \\
&-\frac{\alpha^{\beta}2^{\alpha\beta-1}}{(\delta-\sigma)^{\alpha\beta}}\Gamma(\beta+1)\left[{}_{+}^{\beta}\mathcal{J}_{\frac{\sigma+\delta}{2}}^{\alpha}\mathcal{F}(\delta)+{}_{-}^{\beta}\mathcal{J}_{\frac{\sigma+\delta}{2}}^{\alpha}\mathcal{F}(\sigma)\right]\Bigg| \\
&\leq \frac{\alpha^{\beta}(\delta-\sigma)}{4}\left[\int_0^1\left|\frac{\lambda}{\alpha^{\beta}}-\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right|\left|\mathcal{F}'\left(\frac{\mu}{2}\sigma+\frac{2-\mu}{2}\delta\right)\right|d\mu\right. \\
&\qquad\left.+\int_0^1\left|\frac{\lambda}{\alpha^{\beta}}-\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right|\left|\mathcal{F}'\left(\frac{2-\mu}{2}\sigma+\frac{\mu}{2}\delta\right)\right|d\mu\right]. \qquad (2.4)
\end{aligned}
$$

Now, we consider the integrals on the right side of (2.4). Using the convexity of $|\mathcal{F}'|^q$ and the well-known Hölder inequality, we have

$$
\begin{aligned}
&\int_0^1\left|\frac{\lambda}{\alpha^{\beta}}-\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right|\left|\mathcal{F}'\left(\frac{\mu}{2}\sigma+\frac{2-\mu}{2}\delta\right)\right|d\mu \\
&\leq\left(\int_0^1\left|\frac{\lambda}{\alpha^{\beta}}-\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right|^p d\mu\right)^{\frac{1}{p}}\left(\int_0^1\left|\mathcal{F}'\left(\frac{\mu}{2}\sigma+\frac{2-\mu}{2}\delta\right)\right|^q d\mu\right)^{\frac{1}{q}} \\
&\leq A_2^{\frac{1}{p}}(\alpha,\beta,p,\lambda)\left(\int_0^1\left(\frac{\mu}{2}\left|\mathcal{F}'(\sigma)\right|^q+\frac{2-\mu}{2}\left|\mathcal{F}'(\delta)\right|^q\right)d\mu\right)^{\frac{1}{q}}
\end{aligned}
$$

$$= A_2^{\frac{1}{p}}(\alpha,\beta,p,\lambda)\left[\frac{|\mathcal{F}'(\sigma)|^q + 3|\mathcal{F}'(\delta)|^q}{4}\right]^{\frac{1}{q}} \tag{2.5}$$

and similarly

$$\begin{aligned}
&\int_0^1 \left|\frac{\lambda}{\alpha^\beta} - \left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^\beta\right| \left|\mathcal{F}'\left(\frac{2-\mu}{2}\sigma + \frac{\mu}{2}\delta\right)\right| d\mu \\
&\leq \left(\int_0^1 \left|\frac{\lambda}{\alpha^\beta} - \left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^\beta\right|^p d\mu\right)^{\frac{1}{p}} \left(\int_0^1 \left|\mathcal{F}'\left(\frac{2-\mu}{2}\sigma + \frac{\mu}{2}\delta\right)\right|^q d\mu\right)^{\frac{1}{q}} \\
&\leq A_2^{\frac{1}{p}}(\alpha,\beta,p,\lambda)\left(\int_0^1 \left(\frac{2-\mu}{2}|\mathcal{F}'(\sigma)|^q + \frac{\mu}{2}|\mathcal{F}'(\delta)|^q\right) d\mu\right)^{\frac{1}{q}} \\
&= A_2^{\frac{1}{p}}(\alpha,\beta,p,\lambda)\left[\frac{3|\mathcal{F}'(\sigma)|^q + |\mathcal{F}'(\delta)|^q}{4}\right]^{\frac{1}{q}}.
\end{aligned} \tag{2.6}$$

If we consider (2.5) and (2.6) in (2.4), then we get

$$\begin{aligned}
&\left|(1-\lambda)\mathcal{F}\left(\frac{\sigma+\delta}{2}\right) + \lambda\left(\frac{\mathcal{F}(\sigma)+\mathcal{F}(\delta)}{2}\right)\right. \\
&\left. - \frac{\alpha^\beta 2^{\alpha\beta-1}}{(\delta-\sigma)^{\alpha\beta}}\Gamma(\beta+1)\left[{}_+^\beta \mathcal{J}^\alpha_{\frac{\sigma+\delta}{2}}\mathcal{F}(\delta) + {}_-^\beta \mathcal{J}^\alpha_{\frac{\sigma+\delta}{2}}\mathcal{F}(\sigma)\right]\right| \\
&\leq \frac{\alpha^\beta(\delta-\sigma)}{4} A_2^{\frac{1}{p}}(\alpha,\beta,p,\lambda) \\
&\quad \cdot \left\{\left[\frac{|\mathcal{F}'(\sigma)|^q + 3|\mathcal{F}'(\delta)|^q}{4}\right]^{\frac{1}{q}} + \left[\frac{3|\mathcal{F}'(\sigma)|^q + |\mathcal{F}'(\delta)|^q}{4}\right]^{\frac{1}{q}}\right\}.
\end{aligned}$$

The remaining part of Theorem 2 can be easily seen by setting $a_1 = 3|\mathcal{F}'(\sigma)|^q$, $a_2 = |\mathcal{F}'(\sigma)|^q$, $b_1 = 3|\mathcal{F}'(\delta)|^q$, $b_2 = |\mathcal{F}'(\delta)|^q$ and applying the well-known inequality

$$\sum_{k=1}^n (a_k + b_k)^s \leq \sum_{k=1}^n a_k^s + \sum_{k=1}^n b_k^s$$

with $0 \leq s < 1$. This finishes the proof of Theorem 2. □

Corollary 3. *If we take* $\lambda = 1$ *in* (2.3)*, then we obtain*

$$\left|\frac{\mathcal{F}(\sigma)+\mathcal{F}(\delta)}{2} - \frac{\alpha^\beta 2^{\alpha\beta-1}}{(\delta-\sigma)^{\alpha\beta}}\Gamma(\beta+1)\left[{}_+^\beta \mathcal{J}^\alpha_{\frac{\sigma+\delta}{2}}\mathcal{F}(\delta) + {}_-^\beta \mathcal{J}^\alpha_{\frac{\sigma+\delta}{2}}\mathcal{F}(\sigma)\right]\right|$$

$$\leq \frac{\alpha^{\beta}(\delta-\sigma)}{4} A_2^{\frac{1}{p}}(\alpha,\beta,p,1)\left\{\left[\frac{|\mathcal{F}'(\sigma)|^q+3|\mathcal{F}'(\delta)|^q}{4}\right]^{\frac{1}{q}}+\left[\frac{3|\mathcal{F}'(\sigma)|^q+|\mathcal{F}'(\delta)|^q}{4}\right]^{\frac{1}{q}}\right\}$$

$$\leq \frac{\alpha^{\beta}(\delta-\sigma)}{4}\left[4A_2(\alpha,\beta,p,1)\right]^{\frac{1}{p}}\left[\left|\mathcal{F}'(\sigma)\right|+\left|\mathcal{F}'(\delta)\right|\right],$$

where

$$A_2(\alpha,\beta,p,1)=\int_0^1\left|\frac{1}{\alpha^{\beta}}-\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right|^p d\mu.$$

Remark 7. If we consider (2.3) as $\lambda=0$, then (2.3) reduces to

$$\left|\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)-\frac{\alpha^{\beta}2^{\alpha\beta-1}}{(\delta-\sigma)^{\alpha\beta}}\Gamma(\beta+1)\left[{}_{+}^{\beta}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}}\mathcal{F}(\delta)+{}_{-}^{\beta}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}}\mathcal{F}(\sigma)\right]\right|$$

$$\leq \frac{\alpha^{\beta}(\delta-\sigma)}{4} A_2^{\frac{1}{p}}(\alpha,\beta,p,0)\left\{\left[\frac{|\mathcal{F}'(\sigma)|^q+3|\mathcal{F}'(\delta)|^q}{4}\right]^{\frac{1}{q}}+\left[\frac{3|\mathcal{F}'(\sigma)|^q+|\mathcal{F}'(\delta)|^q}{4}\right]^{\frac{1}{q}}\right\}$$

$$\leq \frac{\alpha^{\beta}(\delta-\sigma)}{4}\left(4\frac{1}{\alpha^{\beta p+1}}\mathfrak{B}\left(\beta p+1,\frac{1}{\alpha}\right)\right)^{\frac{1}{p}}\left(\left|\mathcal{F}'(\sigma)\right|+\left|\mathcal{F}'(\delta)\right|\right).$$

This coincides with [9, Theorem 4].

Corollary 4. *Note that the inequalities* (2.3) *become to*

$$\left|\frac{1}{6}\left[\mathcal{F}(\sigma)+\mathcal{F}(\delta)+4f\left(\frac{\sigma+\delta}{2}\right)\right]-\frac{\alpha^{\beta}\Gamma(\beta+1)}{2}\left(\frac{2}{\delta-\sigma}\right)^{\alpha\beta}\left[{}_{+}^{\beta}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}}\mathcal{F}(\delta)+{}_{-}^{\beta}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}}\mathcal{F}(\sigma)\right]\right|$$

$$\leq \frac{\alpha^{\beta}(\delta-\sigma)}{4}\left(\int_0^1\left|\frac{1}{3\alpha^{\beta}}-\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right|^p d\mu\right)^{\frac{1}{p}}\cdot\left\{\left[\frac{|\mathcal{F}'(\sigma)|^q+3|\mathcal{F}'(\delta)|^q}{4}\right]^{\frac{1}{q}}+\left[\frac{3|\mathcal{F}'(\sigma)|^q+|\mathcal{F}'(\delta)|^q}{4}\right]^{\frac{1}{q}}\right\}$$

$$\leq \frac{\alpha^{\beta}(\delta-\sigma)}{4}\left[4A_2\left(\alpha,\beta,p,\frac{1}{3}\right)\right]^{\frac{1}{p}}\left[\left|\mathcal{F}'(\sigma)\right|+\left|\mathcal{F}'(\delta)\right|\right]$$

for $\lambda=\frac{1}{3}$.

Remark 8. For $\lambda=\frac{1}{2}$, the inequalities (2.3) equal to

$$\left|\frac{1}{2}\left[\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+\frac{\mathcal{F}(\sigma)+\mathcal{F}(\delta)}{2}\right]-\frac{\alpha^{\beta}2^{\alpha\beta-1}}{(\delta-\sigma)^{\alpha\beta}}\Gamma(\beta+1)\left[{}_{+}^{\beta}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}}\mathcal{F}(\delta)+{}_{-}^{\beta}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}}\mathcal{F}(\sigma)\right]\right|$$

$$\leq \frac{\alpha^{\beta}(\delta-\sigma)}{4} A_2^{\frac{1}{p}}\left(\alpha,\beta,p,\frac{1}{2}\right)\left\{\left[\frac{|\mathcal{F}'(\sigma)|^q+3|\mathcal{F}'(\delta)|^q}{4}\right]^{\frac{1}{q}}+\left[\frac{3|\mathcal{F}'(\sigma)|^q+|\mathcal{F}'(\delta)|^q}{4}\right]^{\frac{1}{q}}\right\}$$

$$\leq \frac{\alpha^{\beta}(\delta-\sigma)}{4}\left(4\int_0^1\left|\frac{1}{2\alpha^{\beta}}-\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right|^p d\mu\right)^{\frac{1}{p}}\left[\left|\mathcal{F}'(\sigma)\right|+\left|\mathcal{F}'(\delta)\right|\right],$$

which is presented in [11, Theorem 2].

Theorem 3. *Let us consider that all assumptions of the Lemma 1 hold. If $|\mathcal{F}'|^q$ is convex on $[\sigma,\delta]$ where $q\geq 1$, then we have the following inequality*

$$\begin{aligned}
&\left|(1-\lambda)\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+\lambda\left(\frac{\mathcal{F}(\sigma)+\mathcal{F}(\delta)}{2}\right)\right.\\
&\left.-\frac{\alpha^{\beta}2^{\alpha\beta-1}}{(\delta-\sigma)^{\alpha\beta}}\Gamma(\beta+1)\left[{}_{+}^{\beta}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}}\mathcal{F}(\delta)+{}_{-}^{\beta}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}}\mathcal{F}(\sigma)\right]\right|\\
&\leq \frac{\alpha^{\beta}(\delta-\sigma)}{4}A_1^{1-\frac{1}{q}}(\alpha,\beta,\lambda)\left\{\left(A_3(\alpha,\beta,\lambda)\left|\mathcal{F}'(\sigma)\right|^q+A_4(\alpha,\beta,\lambda)\left|\mathcal{F}'(\delta)\right|^q\right)^{\frac{1}{q}}\right.\\
&\left.+\left(A_4(\alpha,\beta,\lambda)\left|\mathcal{F}'(\sigma)\right|^q+A_3(\alpha,\beta,\lambda)\left|\mathcal{F}'(\delta)\right|^q\right)^{\frac{1}{q}}\right\},
\end{aligned}\tag{2.7}$$

where A_1 is defined in Theorem 1 and

$$\begin{aligned}
A_3(\alpha,\beta,\lambda)&=\int_0^1\frac{\mu}{2}\left|\frac{\lambda}{\alpha^{\beta}}-\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right|d\mu\\
&=\frac{1}{2\alpha^{\beta}}\left\{\lambda\left(C^2-\frac{1}{2}\right)-\frac{1}{\alpha}\left[2\mathscr{B}\left(\beta+1,\frac{1}{\alpha},\lambda^{\frac{1}{\beta}}\right)-2\mathscr{B}\left(\beta+1,\frac{2}{\alpha},\lambda^{\frac{1}{\beta}}\right)\right.\right.\\
&\quad\left.\left.-\mathfrak{B}\left(\beta+1,\frac{1}{\alpha}\right)+\mathfrak{B}\left(\beta+1,\frac{2}{\alpha}\right)\right]\right\},\\
A_4(\alpha,\beta,\lambda)&=\int_0^1\frac{2-\mu}{2}\left|\frac{\lambda}{\alpha^{\beta}}-\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right|d\mu=A_1(\alpha,\beta,\lambda)-A_3(\alpha,\beta,\lambda)\\
&=\frac{1}{2\alpha^{\beta}}\left\{\lambda\left(4C-C^2-\frac{3}{2}\right)-\frac{1}{\alpha}\left[2\mathscr{B}\left(\beta+1,\frac{1}{\alpha},\lambda^{\frac{1}{\beta}}\right)+2\mathscr{B}\left(\beta+1,\frac{2}{\alpha},\lambda^{\frac{1}{\beta}}\right)\right.\right.\\
&\quad\left.\left.-\mathfrak{B}\left(\beta+1,\frac{1}{\alpha}\right)-\mathfrak{B}\left(\beta+1,\frac{2}{\alpha}\right)\right]\right\}.
\end{aligned}$$

Proof. If we consider the convexity of $|\mathcal{F}'|^q$ and power mean inequality, then we obtain

$$\begin{aligned}
&\int_0^1 \left|\frac{\lambda}{\alpha^\beta}-\left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^\beta\right|\left|\mathcal{F}'\left(\frac{\mu}{2}\sigma+\frac{2-\mu}{2}\delta\right)\right|d\mu \\
&\leq \left(\int_0^1 \left|\frac{\lambda}{\alpha^\beta}-\left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^\beta\right|d\mu\right)^{1-\frac{1}{q}} \\
&\quad\cdot\left(\int_0^1 \left|\frac{\lambda}{\alpha^\beta}-\left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^\beta\right|\left|\mathcal{F}'\left(\frac{\mu}{2}\sigma+\frac{2-\mu}{2}\delta\right)\right|^q d\mu\right)^{\frac{1}{q}} \\
&= A_1^{1-\frac{1}{q}}(\alpha,\beta,\lambda)\left(\int_0^1 \left|\frac{\lambda}{\alpha^\beta}-\left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^\beta\right|\left|\mathcal{F}'\left(\frac{\mu}{2}\sigma+\frac{2-\mu}{2}\delta\right)\right|^q d\mu\right)^{\frac{1}{q}} \\
&\leq A_1^{1-\frac{1}{q}}(\alpha,\beta,\lambda)\left(\int_0^1 \left|\frac{\lambda}{\alpha^\beta}-\left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^\beta\right|\left[\frac{\mu}{2}\left|\mathcal{F}'(\sigma)\right|^q+\frac{2-\mu}{2}\left|\mathcal{F}'(\sigma)\right|^q\right]d\mu\right)^{\frac{1}{q}} \\
&= A_1^{1-\frac{1}{q}}(\alpha,\beta,\lambda)\left(A_3(\alpha,\beta,\lambda)\left|\mathcal{F}'(\sigma)\right|^q+A_4(\alpha,\beta,\lambda)\left|\mathcal{F}'(\delta)\right|^q\right)^{\frac{1}{q}} \qquad (2.8)
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1 \left|\frac{\lambda}{\alpha^\beta}-\left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^\beta\right|\left|\mathcal{F}'\left(\frac{2-\mu}{2}\sigma+\frac{\mu}{2}\delta\right)\right|d\mu \\
&\leq \left(\int_0^1 \left|\frac{\lambda}{\alpha^\beta}-\left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^\beta\right|d\mu\right)^{1-\frac{1}{q}} \\
&\quad\cdot\left(\int_0^1 \left|\frac{\lambda}{\alpha^\beta}-\left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^\beta\right|\left|\mathcal{F}'\left(\frac{2-\mu}{2}\sigma+\frac{\mu}{2}\delta\right)\right|^q d\mu\right)^{\frac{1}{q}} \\
&= A_1^{1-\frac{1}{q}}(\alpha,\beta,\lambda)\left(\int_0^1 \left|\frac{\lambda}{\alpha^\beta}-\left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^\beta\right|\left|\mathcal{F}'\left(\frac{\mu}{2}\sigma+\frac{2-\mu}{2}\delta\right)\right|^q d\mu\right)^{\frac{1}{q}} \\
&\leq A_1^{1-\frac{1}{q}}(\alpha,\beta,\lambda)\left(\int_0^1 \left|\frac{\lambda}{\alpha^\beta}-\left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^\beta\right|\left[\frac{\mu}{2}\left|\mathcal{F}'(\sigma)\right|^q+\frac{2-\mu}{2}\left|\mathcal{F}'(\sigma)\right|^q\right]d\mu\right)^{\frac{1}{q}}
\end{aligned}$$

$$= A_1^{1-\frac{1}{q}}(\alpha,\beta,\lambda)\left(A_4(\alpha,\beta,\lambda)\left|\mathcal{F}'(\sigma)\right|^q + A_3(\alpha,\beta,\lambda)\left|\mathcal{F}'(\delta)\right|^q\right)^{\frac{1}{q}}. \tag{2.9}$$

If we consider (2.8) and (2.9) in (2.4), then the proof is completed. □

Corollary 5. *If we take* $\lambda = 1$ *in* (2.7)*, then we obtain*

$$\begin{aligned}
&\left|\frac{\mathcal{F}(\sigma)+\mathcal{F}(\delta)}{2} - \frac{\alpha^{\beta}2^{\alpha\beta-1}}{(\delta-\sigma)^{\alpha\beta}}\Gamma(\beta+1)\left[{}_{+}^{\beta}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}}\mathcal{F}(\delta) + {}_{-}^{\beta}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}}\mathcal{F}(\sigma)\right]\right| \\
&\le \frac{\alpha^{\beta}(\delta-\sigma)}{4}A_1^{1-\frac{1}{q}}(\alpha,\beta,1)\left\{\left(A_3(\alpha,\beta,1)\left|\mathcal{F}'(\sigma)\right|^q + A_4(\alpha,\beta,1)\left|\mathcal{F}'(\delta)\right|^q\right)^{\frac{1}{q}}\right. \\
&\qquad\left. + \left(A_4(\alpha,\beta,1)\left|\mathcal{F}'(\sigma)\right|^q + A_3(\alpha,\beta,1)\left|\mathcal{F}'(\delta)\right|^q\right)^{\frac{1}{q}}\right\},
\end{aligned}$$

where

$$\begin{aligned}
A_1(\alpha,\beta,1) &= \frac{1}{\alpha^{\beta}}\left\{1 - \frac{1}{\alpha}\mathfrak{B}\left(\beta+1,\frac{1}{\alpha}\right)\right\}, \\
A_3(\alpha,\beta,1) &= \frac{1}{2\alpha^{\beta}}\left\{\frac{1}{2} - \frac{1}{\alpha}\left[\mathfrak{B}\left(\beta+1,\frac{1}{\alpha}\right) - \mathfrak{B}\left(\beta+1,\frac{2}{\alpha}\right)\right]\right\}, \\
A_4(\alpha,\beta,1) &= \frac{1}{2\alpha^{\beta}}\left\{\frac{3}{2} - \frac{1}{\alpha}\left[\mathfrak{B}\left(\beta+1,\frac{1}{\alpha}\right) + \mathfrak{B}\left(\beta+1,\frac{2}{\alpha}\right)\right]\right\}.
\end{aligned}$$

Remark 9. If we consider (2.7) as $\lambda = 0$, then inequality (2.7) reduces to

$$\begin{aligned}
&\left|\mathcal{F}\left(\frac{\sigma+\delta}{2}\right) - \frac{\alpha^{\beta}2^{\alpha\beta-1}}{(\delta-\sigma)^{\alpha\beta}}\Gamma(\beta+1)\left[{}_{+}^{\beta}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}}\mathcal{F}(\delta) + {}_{-}^{\beta}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}}\mathcal{F}(\sigma)\right]\right| \\
&\le \frac{\delta-\sigma}{4\alpha}\left[\mathfrak{B}\left(\beta+1,\frac{1}{\alpha}\right)\right]^{1-\frac{1}{q}} \\
&\cdot\left\{\left[\left(\frac{\mathfrak{B}\left(\beta+1,\frac{1}{\alpha}\right) - \mathfrak{B}\left(\beta+1,\frac{2}{\alpha}\right)}{2}\right)\left|\mathcal{F}'(\sigma)\right|^q + \left(\frac{\mathfrak{B}\left(\beta+1,\frac{1}{\alpha}\right) + \mathfrak{B}\left(\beta+1,\frac{2}{\alpha}\right)}{2}\right)\left|\mathcal{F}'(\delta)\right|^q\right]^{\frac{1}{q}}\right. \\
&\left. + \left[\left(\frac{\mathfrak{B}\left(\beta+1,\frac{1}{\alpha}\right) + \mathfrak{B}\left(\beta+1,\frac{2}{\alpha}\right)}{2}\right)\left|\mathcal{F}'(\sigma)\right|^q + \left(\frac{\mathfrak{B}\left(\beta+1,\frac{1}{\alpha}\right) - \mathfrak{B}\left(\beta+1,\frac{2}{\alpha}\right)}{2}\right)\left|\mathcal{F}'(\delta)\right|^q\right]^{\frac{1}{q}}\right\}.
\end{aligned}$$

This coincides with [9, Theorem 3].

Corollary 6. *For* $\lambda = \frac{1}{3}$*, inequality* (2.7) *becomes to*

$$\begin{aligned}
&\left|\frac{1}{6}\left[\mathcal{F}(\sigma)+\mathcal{F}(\delta)+4f\left(\frac{\sigma+\delta}{2}\right)\right] - \frac{\alpha^{\beta}\Gamma(\beta+1)}{2}\left(\frac{2}{\delta-\sigma}\right)^{\alpha\beta}\left[{}_{+}^{\beta}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}}\mathcal{F}(\delta) + {}_{-}^{\beta}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}}\mathcal{F}(\sigma)\right]\right| \\
&\le \frac{\alpha^{\beta}(\delta-\sigma)}{4}A_1^{1-\frac{1}{q}}\left(\alpha,\beta,\frac{1}{3}\right)\left\{\left(A_3\left(\alpha,\beta,\frac{1}{3}\right)\left|\mathcal{F}'(\sigma)\right|^q + A_4\left(\alpha,\beta,\frac{1}{3}\right)\left|\mathcal{F}'(\delta)\right|^q\right)^{\frac{1}{q}}\right. \\
&\qquad\left. + \left(A_4\left(\alpha,\beta,\frac{1}{3}\right)\left|\mathcal{F}'(\sigma)\right|^q + A_3\left(\alpha,\beta,\frac{1}{3}\right)\left|\mathcal{F}'(\delta)\right|^q\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

Here, $D = 1-\left(1-\left(\frac{1}{3}\right)^{\frac{1}{\beta}}\right)^{\frac{1}{\alpha}}$,

$$A_1\left(\alpha,\beta,\frac{1}{3}\right)=\frac{1}{\alpha^{\beta}}\left[\frac{2D-1}{3}+\frac{1}{\alpha}\left(\mathfrak{B}\left(\beta+1,\frac{1}{\alpha}\right)-2\mathscr{B}\left(\beta+1,\frac{1}{\alpha},\left(\frac{1}{3}\right)^{\frac{1}{\beta}}\right)\right)\right],$$

$$A_3\left(\alpha,\beta,\frac{1}{3}\right)=\frac{1}{2\alpha^{\beta}}\left\{\frac{2D^2-1}{6}-\frac{2}{\alpha}\mathscr{B}\left(\beta+1,\frac{1}{\alpha},\left(\frac{1}{3}\right)^{\frac{1}{\beta}}\right)+\frac{2}{\alpha}\mathscr{B}\left(\beta+1,\frac{2}{\alpha},\left(\frac{1}{3}\right)^{\frac{1}{\beta}}\right)+\frac{1}{\alpha}\mathfrak{B}\left(\beta+1,\frac{1}{\alpha}\right)-\frac{1}{\alpha}\mathfrak{B}\left(\beta+1,\frac{2}{\alpha}\right)\right\},$$

$$A_4\left(\alpha,\beta,\frac{1}{3}\right)=\frac{1}{2\alpha^{\beta}}\left\{\frac{-2D^2+8D-3}{6}-\frac{2}{\alpha}\mathscr{B}\left(\beta+1,\frac{1}{\alpha},\left(\frac{1}{3}\right)^{\frac{1}{\beta}}\right)-\frac{2}{\alpha}\mathscr{B}\left(\beta+1,\frac{2}{\alpha},\left(\frac{1}{3}\right)^{\frac{1}{\beta}}\right)+\frac{1}{\alpha}\mathfrak{B}\left(\beta+1,\frac{1}{\alpha}\right)+\frac{1}{\alpha}\mathfrak{B}\left(\beta+1,\frac{2}{\alpha}\right)\right\}.$$

Remark 10. For $\lambda=\frac{1}{2}$, inequality (2.7) reduces to

$$\left|\frac{1}{2}\left[\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+\frac{\mathcal{F}(\sigma)+\mathcal{F}(\delta)}{2}\right]-\frac{\alpha^{\beta}2^{\alpha\beta-1}}{(\delta-\sigma)^{\alpha\beta}}\Gamma(\beta+1)\left[{}_{+}^{\beta}\mathcal{J}_{\frac{\sigma+\delta}{2}}^{\alpha}\mathcal{F}(\delta)+{}_{-}^{\beta}\mathcal{J}_{\frac{\sigma+\delta}{2}}^{\alpha}\mathcal{F}(\sigma)\right]\right|$$
$$\leq\frac{\alpha^{\beta}(\delta-\sigma)}{4}A_1^{1-\frac{1}{q}}\left(\alpha,\beta,\frac{1}{2}\right)\left\{\left(A_3\left(\alpha,\beta,\frac{1}{2}\right)\left|\mathcal{F}'(\sigma)\right|^{q}+A_4\left(\alpha,\beta,\frac{1}{2}\right)\left|\mathcal{F}'(\delta)\right|^{q}\right)^{\frac{1}{q}}+\left(A_4\left(\alpha,\beta,\frac{1}{2}\right)\left|\mathcal{F}'(\sigma)\right|^{q}+A_3\left(\alpha,\beta,\frac{1}{2}\right)\left|\mathcal{F}'(\delta)\right|^{q}\right)^{\frac{1}{q}}\right\},$$

where

$$A_1\left(\alpha,\beta,\frac{1}{2}\right)=\frac{1}{\alpha^{\beta}}\left[\frac{1}{2}-\left(1-\left(\frac{1}{2}\right)^{\frac{1}{\beta}}\right)^{\frac{1}{\alpha}}\right]+\frac{1}{\alpha^{\beta+1}}\left[\mathfrak{B}\left(\beta+1,\frac{1}{\alpha}\right)-2\mathscr{B}\left(\beta+1,\frac{1}{\alpha},\left(\frac{1}{2}\right)^{\frac{1}{\beta}}\right)\right],$$

$$A_3\left(\alpha,\beta,\frac{1}{2}\right)=\frac{1}{2\alpha^{\beta}}\left[\left(1-\left(1-\left(\frac{1}{2}\right)^{\beta}\right)^{\frac{1}{\alpha}}\right)^{2}-\frac{1}{2}\right]+\frac{1}{\alpha^{\beta+1}}\left[\mathfrak{B}\left(\beta+1,\frac{1}{\alpha}\right)-\mathfrak{B}\left(\beta+1,\frac{2}{\alpha}\right)+2\mathscr{B}\left(\beta+1,\frac{2}{\alpha},\left(\frac{1}{2}\right)^{\beta}\right)-2\mathscr{B}\left(\beta+1,\frac{1}{\alpha},\left(\frac{1}{2}\right)^{\beta}\right)\right],$$

$$A_4\left(\alpha,\beta,\frac{1}{2}\right)=\frac{1}{2\alpha^{\beta}}\left[2\left(1-\left(1-\left(\frac{1}{2}\right)^{\beta}\right)^{\frac{1}{\alpha}}\right)-\frac{1}{2}\left(1-\left(1-\left(\frac{1}{2}\right)^{\beta}\right)^{\frac{1}{\alpha}}\right)^{2}-\frac{3}{4}\right]$$
$$+\frac{1}{\alpha^{\beta+1}}\left[-\mathfrak{B}\left(\beta+1,\frac{1}{\alpha}\right)-\mathfrak{B}\left(\beta+1,\frac{2}{\alpha}\right)\right.$$
$$\left.+2\mathscr{B}\left(\beta+1,\frac{2}{\alpha},\left(\frac{1}{2}\right)^{\beta}\right)+2\mathscr{B}\left(\beta+1,\frac{1}{\alpha},\left(\frac{1}{2}\right)^{\beta}\right)\right].$$

This coincides with [11, Theorem 3].

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A CHARACTERIZATION OF PERMUTABILITY OF 2-UNIFORM TOLERANCES ON POSETS

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Abstract. Tolerance relations were investigated by several authors in various algebraic structures, see e.g. the monograph [1]. Recently G. Czédli [5] proved that two 2-uniform tolerances on a lattice containing no infinite chain permute if and only if they are amicable. We extend this result to tolerances on posets. Since in posets we have no lattice operations, we modify the notion of amicability in such a way that in case of lattices it coincides with the original definition. With this new definition we can prove that two 2-uniform tolerances on a poset containing no infinite chain permute if and only if they are amicable.

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1. Introduction

By a *tolerance* on an algebra $\mathbf{A} = (A, F)$ is meant a reflexive and symmetric binary relation on A having the Substitution Property with respect to all operations of F. For the theory of tolerances see e.g. the monograph [1]. Important are in particular tolerances on lattices since G. Czédli showed in [4] that every lattice $\mathbf{L} = (L, \vee, \wedge)$ can be factorized in a natural way by any tolerance T on $\mathbf{L}$, i.e. the set L/T of all blocks of T forms a lattice again. Such a situation is rather exceptional and does not hold for other types of algebras in general, but many varieties whose members have this property were described in [2].

Recall that a *block* of a tolerance T on an algebra (A, F) is a maximal subset B of A satisfying $B^2 \subseteq T$.

Recently G. Czédli proved in [5] that so-called 2-uniform tolerances on a lattice containing no infinite chain permute if and only if they are amicable.

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The concept of a tolerance on a lattice was generalized to posets by the present authors [3]. Despite the fact that we do not have explicit operations on a poset, the concept was defined in a way that the blocks are convex again and that posets can be factorized by these tolerances. For other interesting properties of tolerances on posets the reader is referred to [3]. Hence the natural question arises if also the result of G. Czédli concerning the permutability of 2-uniform tolerances can be extended to posets. This is the topic of the present paper.

2. Basic concepts

At first we recall several concepts introduced in [3] and [5].

Let $\mathbf{P} = (P, \leq)$ be a poset. A *tolerance* on $\mathbf{P}$ is a reflexive and symmetric binary relation T on P satisfying the following conditions:

(1) If $(x,y),(z,u) \in T$ and $x \vee z$ and $y \vee u$ exist then $(x \vee z, y \vee u) \in T$.
(2) If $(x,y),(z,u) \in T$ and $x \wedge z$ and $y \wedge u$ exist then $(x \wedge z, y \wedge u) \in T$.
(3) If $x,y,z \in P$ and $(x,y),(y,z) \in T \neq P^2$ then there exist $u,v \in P$ with $u \leq x,y,z \leq v$ and $(u,y),(y,v) \in T$.
(4) If $(x,y) \in T \neq P^2$ then there exists some $(z,u) \in T$ with both $z \leq x, y \leq u$ and $(v,z),(v,u) \in T$ for all $v \in P$ with $(v,x),(v,y) \in T$.

Conditions (3) and (4) are quite natural since they are satisfied by every tolerance on a lattice. In condition (3) one can take $u := x \wedge y \wedge z$ and $v := x \vee y \vee z$, and in condition (4) one can take $z := x \wedge y$ and $u := x \vee y$.

A *block* of a tolerance T on $\mathbf{P}$ is a maximal subset B of P satisfying $B^2 \subseteq T$. Let P/T denote the set of all blocks of T. Clearly, $T = \bigcup_{B \in P/T} B^2$. In [3] we proved that every block of T is convex. According to Zorn's Lemma every subset A of P satisfying $A^2 \subseteq T$ is contained in a block of T. Following [5] a tolerance is called 2-*uniform* if every of its blocks consists of exactly two elements. Let $a,b \in P$. We call a a *lower T-neighbor of b* and b an *upper T-neighbor of a* if $a \prec b$ and $(a,b) \in T$.

3. A characterization of permutability of 2-uniform tolerances on posets

It is evident that 2-uniform tolerances on a poset are very specific. Their basic properties are as follows.

Lemma 1. *Let $\mathbf{P} = (P, \leq)$ be a poset, $a,b \in P$ and T be a 2-uniform tolerance on $\mathbf{P}$. Then the following holds:*

(i) The element a has at most one lower T-neighbor and at most one upper T-neighbor.
(ii) If $(a,b) \in T$ then $a = b$ or $a \prec b$ or $b \prec a$.

Proof. Without loss of generality, assume $|P| > 2$. Then $T \neq P^2$.

(i) Assume a to have two distinct lower T-neighbors c and d. Then $(c,a),(a,d) \in T$. According to (3) there exists some $e \in P$ with $e \leq c,a,d$ and $(e,a) \in T$. Let $B \in P/T$ with $e,a \in B$. Since $e \leq c,d \leq a$ and B is convex we conclude $c,d,a \in B$ contradicting $|B| = 2$. This shows that a can have at most one lower T-neighbor. The second assertion follows by duality.

(ii) Assume $(a,b) \in T$. First suppose $a \parallel b$. According to (4) there exist $c,d \in P$ with $(c,d) \in T$ and $c \leq a,b \leq d$. Let $B \in P/T$ with $c,d \in B$. Since B is convex we conclude $c,a,b \in B$. Now we have $a \parallel b$ and $c \leq a,b$ and hence $c \neq a,b$ contradicting $|B| = 2$. This shows $a \leq b$ or $b \leq a$. Let $C \in P/T$ with $a,b \in C$. Since C is convex we obtain $[a,b] \subseteq C$ if $a \leq b$ and $[b,a] \subseteq C$ if $b \leq a$. Now $|C| = 2$ implies $a = b$ or $a \prec b$ or $b \prec a$.

□

Let $\mathbf{P}$ be a poset, $a,b \in P$ and T,S be 2-uniform tolerances on $\mathbf{P}$.

In what follows we adopt some key concepts from [5] for 2-uniform tolerances on posets.

The element a is called a *split* (T,S)*-bottom* if there exists some upper T-neighbor b of a and some upper S-neighbor of a being different from b. Further, a is called an *adherent* (T,S)*-bottom* if there exists some common upper T-neighbor and S-neighbor of a. Finally, a is called a (T,S)*-bottom* if it is either a split (T,S)-bottom or an adherent (T,S)-bottom. The notions *split* (T,S)*-top*, *adherent* (T,S)*-top* and (T,S)*-top* are defined dually. A T*-top* (T*-bottom*) is an upper (a lower) T-neighbor of some element of P.

The tolerances T and S are called *amicable* if the following four conditions hold:

(5) If $a \neq b$ and there exists some lower T-neighbor of a being a lower S-neighbor of b then there exists some upper S-neighbor of a being an upper T-neighbor of b.

(6) If $a \neq b$ and there exists some upper T-neighbor of a being an upper S-neighbor of b then there exists some lower S-neighbor of a being a lower T-neighbor of b.

(7) If a is a (T,S)-top and b is either an upper T-neighbor or an upper S-neighbor of a then b is a (T,S)-top, too.

(8) If a is a (T,S)-bottom and b is either a lower T-neighbor or a lower S-neighbor of a then b is a (T,S)-bottom, too.

Let us note that any two 2-uniform tolerances on a lattice $(L,\vee,\wedge)$ satisfy (5) and (6). Namely, G. Czédli [5] showed that if $a \neq b$ and there exists some lower T-neighbor of a being a lower S-neighbor of b then $a \vee b$ is an upper S-neighbor of a being an upper T-neighbor of b, and if $a \neq b$ and there exists some upper T-neighbor of a being an upper S-neighbor of b then $a \wedge b$ is a lower S-neighbor of a being a lower T-neighbor of b.

Remark 1. Note that (7) and (8) coincide with conditions (A1) and (A2) from [5], respectively. Hence, for lattices the concept of amicable 2-uniform tolerances as defined in [5] coincides with the one defined above.

The proof of our main theorem will broadly use the ideas from [5].

Theorem 1. *Let T and S be 2-uniform tolerances on a poset $(P,\leq)$ containing no infinite chain. Then T and S permute if and only if they are amicable.*

Proof. Let $a,b\in P$. First assume T and S to permute.

(5) Suppose $a\neq b$ and suppose c to be some lower T-neighbor of a being a lower S-neighbor of b. Because of (ii) of Lemma 1 we have $a\parallel b$ and hence $(a,b)\notin S\cup T$ again according to (ii) of Lemma 1. Now $(a,b)\in T\circ S=S\circ T$. Hence there exists some S-neighbor d of a being a T-neighbor of b. Because of $a\parallel b$ we have $d<a,b$ or $d>a,b$. Now $d<a,b$ would imply $(a,b)=(a\vee d,c\vee b)\in T$ according to (1) since $(a,c),(d,b)\in T$. But this contradicts (ii) of Lemma 1. Therefore d is an upper S-neighbor of a being an upper T-neighbor of b proving (5).

(6) follows from (5) by duality.

(7) Suppose b to be either an upper T-neighbor or S-neighbor of a. Without loss of generality assume $(a,b)\in T$. First suppose a to be a split (T,S)-top. Let c and d denote the lower T-neighbor and S-neighbor of a, respectively. Then $c\parallel d$. Since $(d,b)\in S\circ T=T\circ S$ there exists some $e\in P$ with $(d,e)\in T$ and $(e,b)\in S$. Now $b\leq e$ would imply $d<a<b\leq e$ which together with $(d,e)\in T$ would contradict (ii) of Lemma 1. Again according to Lemma 1 we have $e\prec b$. Now a and e are a lower T-neighbor and S-neighbor of b, respectively, showing b to be a (T,S)-top. Next assume a to be an adherent (T,S)-top. Let c denote the common lower T-neighbor and S-neighbor of a. Since $(c,b)\in S\circ T=T\circ S$ there exists some $d\in P$ with $(c,d)\in T$ and $(d,b)\in S$. Now $d\leq c$ would imply $d\leq c<a<b$ which together with $(d,b)\in S$ would contradict (ii) of Lemma 1, and $b\leq d$ would imply $c<a<b\leq d$ which together with $(c,d)\in T$ would again contradict (ii) of Lemma 1. Thus a further application of (ii) of Lemma 1 yields $c\prec d\prec b$. Now a and d are a lower T-neighbor and S-neighbor of b, respectively, showing b to be a (T,S)-top.

(8) follows from (7) by duality.

Hence T and S are amicable.

Conversely, assume T and S to be amicable. Suppose $(a,b)\in T\circ S$. Then there exists some $c\in P$ with $(a,c)\in T$ and $(c,b)\in S$. We want to show $(a,b)\in S\circ T$. If $a=b$ then $(a,b)=(a,a)\in S\circ T$. If $c=a$ then $(a,b)=(c,b)\in S$ and $(b,b)\in T$ and hence $(a,b)\in S\circ T$. If $c=b$ then $(a,a)\in S$ and $(a,b)=(a,c)\in T$ and hence $(a,b)\in S\circ T$. So we can assume that a,b,c are mutually distinct. According to (ii) of Lemma 1 we consider the following four cases.

$\underline{c \prec a}$ **and** $\underline{c \prec b}$**:** Because of (5) there exists a common upper S-neighbor d of a and T-neighbor of b and hence $(a,d) \in S$ and $(d,b) \in T$ showing $(a,b) \in S \circ T$.

$\underline{a \prec c}$ **and** $\underline{b \prec c}$ **:** This case is dual to the previous one.

$\underline{a \prec c \prec b}$ **:** Put $a_0 := a$, $a_1 := c$ and $a_2 := b$. For $i \geq 3$ define a_i as follows: If i is odd and a_{i-1} is a T-bottom then define a_i to be the unique upper T-neighbor of a_{i-1}. If i is even and a_{i-1} is an S-bottom then define a_i to be the unique upper S-neighbor of a_{i-1}. Note that a_0 is a T-bottom, a_1 is the unique upper T-neighbor of a_0, a_1 is an S-bottom and a_2 is the unique upper S-neighbor of of a_1. Since $a_2 \prec a_3 \prec a_4 \prec \cdots$, but **P** has no infinite chain, there exists some $n \geq 2$ such that $a_2, \ldots, a_n$ are defined, but a_{n+1} is not. First assume n to be even. Then a_n is not a T-bottom. Since every element of P belongs to at least one 2-element block of T, a_n is a T-top. But it is also an S-top and therefore a (T,S)-top. If n is odd then a similar reasoning shows that a_n is a (T,S)-top. So a_n is a (T,S)-top in any case. Now assume a_n to be an adherent (T,S)-top. Then, according to (i) of Lemma 1, a_{n-1} is an adherent (T,S)-bottom. Now (8) shows that a_{n-2} is a (T,S)-bottom, too. Next assume a_n to be a split (T,S)-top. Suppose n is even. Then a_{n-1} is the unique lower S-neighbor of a_n, a_n has a unique lower T-neighbor d and $a_{n-1} \neq d$. According to (6) there exists some lower S-neighbor e of d being a lower T-neighbor of a_{n-1}. Since a_{n-2} is a lower T-neighbor of a_{n-1}, too, (i) of Lemma 1 yields $e = a_{n-2}$. Therefore a_{n-2} is a (T,S)-bottom. With roles of T and S interchanged, one can prove that a_{n-2} is a (T,S)-bottom also in the case when n is odd. Hence in any case a_{n-2} is a (T,S)-bottom. Applying (8) finitely many times yields that $a_0 = a$ is a (T,S)-bottom, too. First assume a to be a split (T,S)-bottom. Then there exists some upper S-neighbor f of a being distinct from c. According to (5) there exists some upper S-neighbor g of c being an upper T-neighbor of f. Since b is an upper S-neighbor of c, too, (i) of Lemma 1 yields $g = b$. Hence $(a,f) \in S$ and $(f,b) = (f,g) \in T$ showing $(a,b) \in S \circ T$. Now suppose a to be an adherent (T,S)-bottom. Then c is a (T,S)-top. According to (7), b is a (T,S)-top, too. Hence b has a unique lower T-neighbor h. Assume $c \neq h$. According to (6) there exists some lower S-neighbor i of h being a lower T-neighbor of c. Since a is a lower T-neighbor of c, too, (i) of Lemma 1 yields $i = a$. On the other hand, $i = a$ is a lower S-neighbor of h. Hence c and h are two distinct upper S-neighbors of a contradicting (i) of Lemma 1. This shows $c = h$. Now $(a,c) \in S$ and $(c,b) = (h,b) \in T$ proving $(a,b) \in S \circ T$.

$\underline{b \prec c \prec a}$ **:** This case is dual to the previous one.

Hence in any case we have $(a,b) \in S \circ T$. This shows $T \circ S \subseteq S \circ T$. By duality, we obtain $S \circ T \subseteq T \circ S$ and hence $T \circ S = S \circ T$, i.e. T and S permute. □

4. EXAMPLES

Example 1. Consider the poset **P** depicted in Figure 1

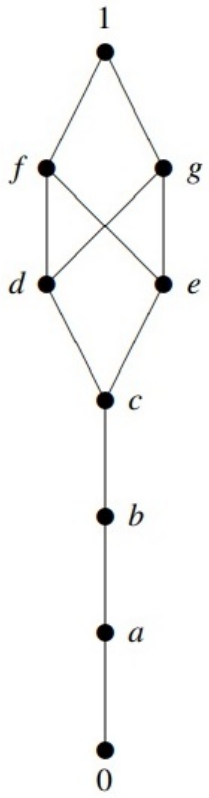

FIGURE 1. Poset

and put

$$T := \{0,a\}^2 \cup \{a,b\}^2 \cup \{c,e\}^2 \cup \{d,g\}^2 \cup \{f,1\}^2,$$
$$S := \{0,a\}^2 \cup \{a,b\}^2 \cup \{c,e\}^2 \cup \{d,f\}^2 \cup \{g,1\}^2.$$

Then T and S are permuting 2-uniform tolerances on **P** since

$$T \circ S = T \cup S \cup \{(0,b),(b,0),(d,1),(f,g),(g,f),(1,d)\} = S \circ T$$

and hence they are amicable according to Theorem 1.

Example 2. Consider the poset **P** visualized in Figure 2

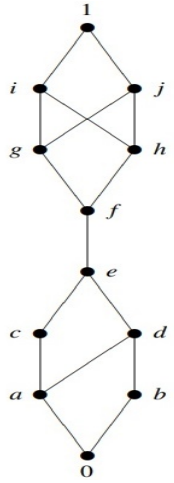

FIGURE 2. Poset

and put

$$T := \{0,a\}^2 \cup \{a,c\}^2 \cup \{b,d\}^2 \cup \{d,e\}^2 \cup \{f,h\}^2 \cup \{g,j\}^2 \cup \{i,1\}^2,$$
$$S := \{0,b\}^2 \cup \{a,d\}^2 \cup \{c,e\}^2 \cup \{f,g\}^2 \cup \{h,j\}^2 \cup \{i,1\}^2.$$

Then T and S are permuting 2-uniform tolerances on **P** since

$$T \circ S = T \cup S \cup \{(0,d),(a,b),(a,e),(b,a),(c,d),(d,0),$$
$$(d,c),(e,a),(f,j),(g,h),(h,g),(j,f)\} = S \circ T$$

and hence they are amicable according to Theorem 1.

Example 3. Consider the poset **P** depicted in Figure 3

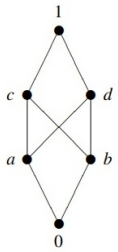

FIGURE 3. Poset

and put

$$T := \{0,a\}^2 \cup \{b,d\}^2 \cup \{c,1\}^2,$$
$$S := \{0,b\}^2 \cup \{a,c\}^2 \cup \{d,1\}^2.$$

Then T and S do not permute since $(a,b) \in (T \circ S) \setminus (S \circ T)$, and T and S are not amicable since 0 is a lower T-neighbor of a being a lower S-neighbor of b, but there exists no upper S-neighbor of a being an upper T-neighbor of b.

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INVERSE SPECTRAL PROBLEM FOR STURM-LIOUVILLE OPERATOR WITH HIGHER ORDER POLYNOMIALS OF SPECTRAL PARAMETER IN THE BOUNDARY CONDITION

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Abstract. This paper is concerned with an inverse scattering problem for Sturm-Liouville operator with polynomials of spectral parameter in boundary condition. We provide scattering data and present some spectral properties. In order to study inverse scattering problem, the main equation is derived. As a consequence, we show that the potential can be recovered uniquely from scattering data.

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Keywords: inverse problem, scattering data, Sturm-Liouville operator, nonlinear spectral parameter, main equation

1. INTRODUCTION

Inverse problems in spectral theory often appear in mathematics, mechanics, physics and other branches of natural sciences. The inverse scattering problem deals with the recovery of potential from their spectral characteristics, and besides, problems involving spectral parameter in equation and boundary conditions are an important part of spectral theory. Sturm-Liouville problems arise in studies of heat conduction problems and vibrating string problems when boundary conditions depend on spectral parameter. Cohen [2] presented a method to solve an initial boundary value problem arising in the theory of diffusion and heat flow. Many examples of spectral problems which appear in mechanical engineering and contain eigen parameter in the boundary conditions were presented in the book [5].

In this paper we consider the boundary value problem generated by Sturm-Liouville operator and spectral parameter dependent boundary condition:

$$\ell u(x) := -u''(x) + \left\{q(x) - \lambda^2\right\} u(x) = 0, \qquad (0 \le x < \infty), \tag{1.1}$$

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$$V(u) := \sum_{k=0}^{n} \left(\alpha_k u'(0) - \beta_k u(0)\right) (i\lambda)^k = 0, \tag{1.2}$$

where λ is a spectral parameter, $q(x)$ is a real valued function with the condition

$$\int_0^\infty (1+x)\,|q(x)|\,dx < \infty \tag{1.3}$$

and for $\alpha_i, \beta_j \in \mathbb{R}$ $(i, j = 1, \ldots, n)$

$$\begin{vmatrix} \alpha_i & \alpha_{i+2} \\ \beta_i & \beta_{i+2} \end{vmatrix} \geq 0, \quad \begin{vmatrix} \alpha_i & \alpha_{i+k} \\ \beta_i & \beta_{i+k} \end{vmatrix} = 0, \quad k \neq 2. \tag{1.4}$$

Marchenko [18] and Levitan [11] studied inverse scattering problem of the Sturm-Liouville operator on the half line with a boundary condition at the origin when there is no spectral parameter. Inverse problem for second order differential operator pencil on the axis was studied in [6–8, 20]. Problems with spectral parameter dependent boundary condition were examined in finite interval by several authors in [1,9,10,16, 17,21,22] and on the half line in [3,4,12–15,19].

The aim of this paper is to present inverse scattering problem for Sturm-Liouville operator with polynomials of spectral parameter in the boundary condition. In this view, we will extend the Marchenko method to the boundary value problem (1.1)-(1.3), which also introduces a process that can be applied to various other inverse problems where boundary condition contains polynomials of spectral parameter.

The remaining paper is organized as follows: In section (2), the scattering data corresponding to the boundary value problem (1.1)-(1.3) is provided and its properties are investigated. In Section (3), the main equation required for recovering potential uniquely is derived. Finally, we give the uniqueness of the solution of the main equation and the reconstruction of the potential of equation (1.1) in Section (4).

2. Scattering Data

The direct scattering problem concerns the determination of spectral characteristics for the boundary value problem when $q(x)$ is known. Therefore, we shall provide scattering data for (1.1)-(1.3) and analyze its properties.

It is well known in [18] that the equation (1.1) has a unique solution $e(\lambda, x)$ which satisfies the asymptotic behavior, for $\operatorname{Im} \lambda \geq 0$,

$$\lim_{x \to +\infty} e^{-i\lambda x} e(\lambda, x) = 1,$$

when the condition (1.3) is satisfied. It is called Jost solution and can be expressed by

$$e(\lambda, x) = e^{i\lambda x} + \int_x^\infty K(x,t)\, e^{i\lambda t}\, dt, \tag{2.1}$$

where the kernel function $K(x,t)$ satisfies the estimate

$$|K(x,t)| \le \frac{1}{2}\Omega\left(\frac{x+t}{2}\right)\exp\left\{\Omega_1(x)-\Omega_1\left(\frac{x+t}{2}\right)\right\}$$

and the functions $\Omega(x)$ and $\Omega_1(x)$ have the following notations:

$$\Omega(x) \equiv \int_x^\infty |q(t)|\,dt, \qquad \Omega_1(x) \equiv \int_x^\infty \Omega(x)\,dt.$$

In addition,

$$K(x,x) = \frac{1}{2}\int_x^\infty q(t)\,dt. \tag{2.2}$$

Moreover, $e(\lambda,x)$ is an analytic function of λ in the upper half plane ($\operatorname{Im}\lambda > 0$) and is continuous on the real line. The following estimates hold on the half plane $\operatorname{Im}\lambda \ge 0$:

$$|e(\lambda,x)| \le \exp\{-\operatorname{Im}\lambda x + \Omega_1(x)\},$$

$$\left|e(\lambda,x) - e^{i\lambda x}\right| \le \left\{\Omega_1(x) - \Omega_1\left(x+\frac{1}{|\lambda|}\right)\right\}\exp\{-\operatorname{Im}\lambda x + \Omega_1(x)\}, \tag{2.3}$$

$$\left|e'(\lambda,x) - i\lambda e^{i\lambda x}\right| \le \Omega(x)\exp\{-\operatorname{Im}\lambda x + \Omega_1(x)\}. \tag{2.4}$$

Let $W[f,g] = f'g - g'f$ denote the Wronskian. For real $\lambda \ne 0$, the functions $e(\lambda,x)$ and $\overline{e(\lambda,x)}$ form a fundamental system of solutions of equation (1.1) and their Wronskian is independent of x and equal to $2i\lambda$.

Let

$$p_1(\lambda) := \sum_{k=0}^{n} \alpha_k (i\lambda)^k, \qquad p_2(\lambda) := \sum_{k=0}^{n} \beta_k (i\lambda)^k$$

and denote by $\phi(\lambda,x)$ the solution of (1.1) satisfying the conditions

$$\phi(\lambda,0) = p_1(\lambda), \qquad \phi'(\lambda,0) = p_2(\lambda).$$

It is evident that the solution $\phi(\lambda,x)$ holds the boundary condition (1.2).

After stating the above preliminaries, we can now present the following lemmas.

Lemma 1. *The following identity is valid:*

$$\frac{2i\lambda\phi(\lambda,x)}{\Theta(\lambda)} = \overline{e(\lambda,x)} - S(\lambda)e(\lambda,x) \tag{2.5}$$

for all real $\lambda \ne 0$ *and*

$$S(\lambda) = \frac{p_1(\lambda)\overline{e'(\lambda,0)} - p_2(\lambda)\overline{e(\lambda,0)}}{p_1(\lambda)e'(\lambda,0) - p_2(\lambda)e(\lambda,0)}, \tag{2.6}$$

$$S(\lambda) = \overline{S(-\lambda)}, \qquad |S(\lambda)| = 1, \tag{2.7}$$

where

$$\Theta(\lambda) = p_1(\lambda) e'(\lambda,0) - p_2(\lambda) e(\lambda,0). \tag{2.8}$$

Proof. Because of that $e(\lambda,x)$ and $\overline{e(\lambda,x)}$ constitute the fundamental set of solutions of (1.1) for the real $\lambda \neq 0$, the function $\phi(\lambda,x)$ can be represented as a linear combination of these functions:

$$\phi(\lambda,x) = c_1(\lambda) e(\lambda,x) + c_2(\lambda) \overline{e(\lambda,x)},$$

where $c_1(\lambda)$ and $c_2(\lambda)$ are provided by evaluating the values of the Wronskians:

$$W[e(\lambda,x), \phi(\lambda,x)] = c_2(\lambda) 2i\lambda = p_1(\lambda) e'(\lambda,0) - p_2(\lambda) e(\lambda,0),$$
$$W\left[\overline{e(\lambda,x)}, \phi(\lambda,x)\right] = -c_1(\lambda) 2i\lambda = p_1(\lambda) \overline{e'(\lambda,0)} - p_2(\lambda) \overline{e(\lambda,0)}.$$

Thus, we have

$$\phi(\lambda,x) = -\frac{p_1(\lambda)\overline{e'(\lambda,0)} - p_2(\lambda)\overline{e(\lambda,0)}}{2i\lambda} e(\lambda,x) + \frac{p_1(\lambda) e'(\lambda,0) - p_2(\lambda) e(\lambda,0)}{2i\lambda} \overline{e(\lambda,x)}. \tag{2.9}$$

Let us define the function $\Theta(\lambda)$ in the form (2.8). Now, it is necessary to show $\Theta(\lambda) \neq 0$ for all real $\lambda \neq 0$. Assume the contrary, then there exists $\lambda_0 \in \mathbb{R}$, $\lambda_0 \neq 0$, such that

$$p_1(\lambda_0) e'(\lambda_0,0) = p_2(\lambda_0) e(\lambda_0,0).$$

Also,

$$W\left[e(\lambda_0,0), \overline{e(\lambda_0,0)}\right] = 2i\lambda_0$$

is satisfied. By using these relations, we obtain

$$\frac{|e(\lambda_0,0)|^2}{|p_1(\lambda_0)|^2} \sum_{j=0}^{n-2} (\alpha_j \beta_{j+2} - \alpha_{j+2}\beta_j) |\lambda_0|^{2j} \left(\overline{\lambda_0}^2 - \lambda_0^2\right) = 2i\lambda_0$$

and so $0 = 2i\lambda_0$, yielding contradiction since $\lambda_0 \neq 0$. Thus, by dividing equality (2.9) by $\frac{1}{2i\lambda}\Theta(\lambda)$, we obtain (2.5) where $S(\lambda)$ is defined with (2.6).

To complete proof, we need to show that $S(\lambda)$ satisfies the conditions in (2.7). Let us use the notation $\Theta_1(\lambda)$ for the numerator of $S(\lambda)$. Since $q(x)$ is real, it follows that $\overline{e(\lambda,0)} = e(-\lambda,0)$ for real $\lambda \neq 0$. Also, $\overline{p_1(\lambda)} = p_1(-\lambda)$ and $\overline{p_2(\lambda)} = p_2(-\lambda)$ hold for real $\lambda \neq 0$. Hence $\overline{\Theta(\lambda)} = \Theta(-\lambda)$ and $\overline{\Theta_1(\lambda)} = \Theta_1(-\lambda)$, which yields $S(\lambda) = \overline{S(-\lambda)}$.

By (2.6), we get for all real $\lambda \neq 0$

$$|S(\lambda)|^2 = S(\lambda) \cdot \overline{S(\lambda)}$$

$$= \frac{|p_1(\lambda)|^2 \cdot |e'(\lambda,0)|^2 + |p_2(\lambda)|^2 \cdot |e(\lambda,0)|^2 - 2\mathrm{Re}\left(p_1(\lambda) \cdot \overline{p_2(\lambda)} \cdot e(\lambda,0) \cdot \overline{e'(\lambda,0)}\right)}{|p_1(\lambda)|^2 \cdot |e'(\lambda,0)|^2 + |p_2(\lambda)|^2 \cdot |e(\lambda,0)|^2 - 2\mathrm{Re}\left(p_1(\lambda) \cdot \overline{p_2(\lambda)} \cdot e'(\lambda,0) \cdot \overline{e(\lambda,0)}\right)}.$$

Since the condition (1.4) holds and λ is real, we obtain

$$\begin{aligned}
&2\mathrm{Re}\left(p_1(\lambda) \cdot \overline{p_2(\lambda)} \cdot e'(\lambda,0) \cdot \overline{e(\lambda,0)}\right) - 2\mathrm{Re}\left(p_1(\lambda) \cdot \overline{p_2(\lambda)} \cdot e(\lambda,0) \cdot \overline{e'(\lambda,0)}\right) \\
&\quad = p_1(\lambda) \cdot \overline{p_2(\lambda)} \cdot e'(\lambda,0) \cdot \overline{e(\lambda,0)} - p_1(\lambda) \cdot \overline{p_2(\lambda)} \cdot e(\lambda,0) \cdot \overline{e'(\lambda,0)} \\
&\quad\quad - \overline{p_1(\lambda)} \cdot p_2(\lambda) \cdot e'(\lambda,0) \cdot \overline{e(\lambda,0)} + \overline{p_1(\lambda)} \cdot p_2(\lambda) \cdot e(\lambda,0) \cdot \overline{e'(\lambda,0)} \\
&\quad = \left[p_1(\lambda) \cdot \overline{p_2(\lambda)} - p_2(\lambda) \cdot \overline{p_1(\lambda)}\right] \cdot \left[e'(\lambda,0) \cdot \overline{e(\lambda,0)} - e(\lambda,0) \cdot \overline{e'(\lambda,0)}\right] \\
&\quad = \sum_{j=0}^{n-2} (\alpha_j \beta_{j+2} - \alpha_{j+2}\beta_j) |\lambda|^{2j} \left(\lambda^2 - \overline{\lambda}^2\right) \cdot W\left[e(\lambda,0), \overline{e(\lambda,0)}\right] \\
&\quad = 0 \cdot 2i\lambda = 0.
\end{aligned}$$

By combining the results, it follows that $|S(\lambda)|^2 = 1$, i.e., $|S(\lambda)| = 1$ for all real $\lambda \neq 0$.
The proof is completed. □

The function $S(\lambda)$ defined by (2.6) is called the *scattering function* of the boundary value problem (1.1)-(1.3).

Lemma 2. *The function $\Theta(\lambda)$ may have only a finite number of zeros on the half plane* ($\mathrm{Im}\,\lambda > 0$). *All the zeros are simple and lie on the imaginary axis.*

Proof. By the proof of Lemma 1, we have $\Theta(\lambda) \neq 0$ for all real $\lambda \neq 0$ and the point $\lambda = 0$ is the possible zero of $\Theta(\lambda)$. Since $e(\lambda,0)$ and $e'(\lambda,0)$ are analytic in the upper plane $\mathrm{Im}\,\lambda > 0$, it follows from the expression $\Theta(\lambda)$ that it has the same property. We now show that the set of zeros is bounded. Assume the contrary, i.e. let the set of zeros be unbounded. Then, there exists the numbers λ_k such that $\Theta(\lambda_k) = 0$ for $\mathrm{Im}\,\lambda_k > 0$ and $|\lambda_k| \to \infty$, and hence we have

$$e'(\lambda_k,0) = \frac{p_2(\lambda_k)}{p_1(\lambda_k)} e(\lambda_k,0).$$

With the help of (2.4)

$$\left|\frac{p_2(\lambda_k)}{p_1(\lambda_k)} e(\lambda_k,0) - i\lambda_k\right| \leq \Omega(0) \exp\{\Omega_1(0)\},$$

we have

$$|\lambda_k| \leq \left|\frac{p_2(\lambda_k)}{p_1(\lambda_k)} e(\lambda_k,0)\right| + \Omega(0) \exp\{\Omega_1(0)\}.$$

Since the equality (2.3) and $\lim_{k\to+\infty} e(\lambda_k,0) = 1$, the right side of the equality has a finite limit. We arrived at the contradiction, which yields that the set $\{\lambda_k\}$ is bounded.

Thus, the set of zeros of $\Theta(\lambda)$ is bounded and form at most countable set having zero the only possible limit point.

To show that zeros of $\Theta(\lambda)$ lie on the imaginary axis, we assume that λ_1 and λ_2 are zeros of $\Theta(\lambda)$, which follows that they satisfy the equation (1.1):

$$-e''(\lambda_1,x)+q(x)e(\lambda_1,x)=\lambda_1{}^2e(\lambda_1,x), \tag{2.10}$$

$$-\overline{e''(\lambda_2,x)}+q(x)\overline{e(\lambda_2,x)}=\overline{\lambda_2{}^2e(\lambda_2,x)}. \tag{2.11}$$

Let us multiply (2.10) by $\overline{e(\lambda_2,x)}$ and (2.11) by $e(\lambda_1,x)$, subtract the second from the first and finally integrate this result according to x from 0 to infinity. As a result, we obtain

$$\left(\lambda_1{}^2-\overline{\lambda_2}^2\right)\int_0^\infty e(\lambda_1,x)\overline{e(\lambda_2,x)}\mathrm{d}x-W\left[e(\lambda_1,x),\overline{e(\lambda_2,x)}\right]\Big|_{x=0}=0. \tag{2.12}$$

On the other hand, since λ_j's $(j=1,2)$ are zeros of $\Theta(\lambda)$, we have

$$W\left[e(\lambda_1,x),\overline{e(\lambda_2,x)}\right]\Big|_{x=0}=\left(\frac{p_2(\lambda_1)}{p_1(\lambda_1)}-\frac{\overline{p_2(\lambda_2)}}{\overline{p_1(\lambda_2)}}\right)e(\lambda_1,0)\overline{e(\lambda_2,0)}.$$

When we take $\lambda_1=\lambda_2=\mu$ and substitute in (2.12), the following result is obtained

$$\left(\mu^2-\overline{\mu}^2\right)\left[\frac{|e(\mu,0)|^2}{|p_1(\mu)|^2}\sum_{m=0}^{n-2}\left(\alpha_m\beta_{m+2}-\alpha_{m+2}\beta_m\right)|\mu|^{2m}+\int_0^\infty|e(\mu,x)|^2\,\mathrm{d}x\right]=0.$$

Thanks to the condition (1.4), the expression in the parentheses is positive, which implies $\mu^2=\overline{\mu}^2$. Since μ isn't real, we conclude that μ is pure imaginary, i.e. $\mu=i\lambda$, where $\lambda\geq 0$.

Next, let us prove that there are only finitely many zeros. Let δ denote the infimum of the distances between two neighboring zeros of $\Theta(\lambda)$ and show $\delta>0$. Assume the opposite and let $\{i\lambda_k\}$ and $\left\{i\widehat{\lambda}_k\right\}$ be two sequences of zeros of the function $\Theta(\lambda)$ such that

$$\lim_{k\to\infty}\left(\widehat{\lambda}_k-\lambda_k\right)=0,\quad 0<\lambda_k\leq\widehat{\lambda}_k,\quad \max_k\widehat{\lambda}_k<M.$$

It follows from the estimate (2.3) that, for A large enough, the inequality

$$e(i\lambda,x)>\frac{1}{2}e^{-\lambda x}$$

is satisfied uniformly with respect to $x\in[A,\infty)$ and $\lambda\in[0,\infty)$. Thus, we obtain

$$\int_A^\infty e\left(i\widehat{\lambda}_k,x\right)\overline{e(i\lambda_k,x)}\mathrm{d}x>\frac{1}{4}\frac{e^{-A\left(\widehat{\lambda}_k+\lambda_k\right)}}{\left(\widehat{\lambda}_k+\lambda_k\right)}>\frac{e^{-2AM}}{8M}. \tag{2.13}$$

On the other hand, the equality (2.12) yields

$$0=\int_0^{\infty} e\left(i\widehat{\lambda}_k,x\right)\overline{e\left(i\lambda_k,x\right)}\mathrm{d}x+\frac{e\left(i\widehat{\lambda}_k,0\right)\overline{e\left(i\lambda_k,0\right)}}{\overline{p_1\left(i\lambda_k\right)}p_1\left(i\widehat{\lambda}_k\right)}\sum_{m=0}^{n-2}\left(\alpha_m\beta_{m+2}-\alpha_{m+2}\beta_m\right)\left(\widehat{\lambda}_k\lambda_k\right)^m$$

$$=\int_0^{A} e\left(i\widehat{\lambda}_k,x\right)\left[\overline{e\left(i\lambda_k,x\right)}-\overline{e\left(i\widehat{\lambda}_k,x\right)}\right]\mathrm{d}x+\int_0^{A} e\left(i\widehat{\lambda}_k,x\right)\overline{e\left(i\widehat{\lambda}_k,x\right)}\mathrm{d}x$$

$$+\int_A^{\infty} e\left(i\widehat{\lambda}_k,x\right)\overline{e\left(i\lambda_k,x\right)}\mathrm{d}x+\frac{e\left(i\widehat{\lambda}_k,0\right)\overline{e\left(i\lambda_k,0\right)}}{\overline{p_1\left(i\lambda_k\right)}p_1\left(i\widehat{\lambda}_k\right)}\sum_{m=0}^{n-2}\left(\alpha_m\beta_{m+2}-\alpha_{m+2}\beta_m\right)\left(\widehat{\lambda}_k\lambda_k\right)^m$$

and letting $k\to\infty$, we get

$$\lim_{k\to\infty}\int_A^{\infty} e\left(i\widehat{\lambda}_k,x\right)\overline{e\left(i\lambda_k,x\right)}\mathrm{d}x\le 0 \tag{2.14}$$

since

$$\lim_{k\to\infty}\left[\overline{e\left(i\lambda_k,x\right)}-\overline{e\left(i\widehat{\lambda}_k,x\right)}\right]=0$$

holds uniformly with respect to $x\in[0,A]$. Comparing (2.13) and (2.14), we obtain a contradiction, yielding $\delta>0$ and hence we conclude that the function $\Theta(\lambda)$ has only a finite number of zeros: $i\lambda_k,\ k=1,\dots,N$.

Finally, let us show that all zeros of the function $\Theta(\lambda)$ are simple. By the derivation of the equation

$$-e''(\lambda,x)+q(x)e(\lambda,x)=\lambda^2 e(\lambda,x) \tag{2.15}$$

with respect to λ, we get

$$-\dot{e}''(\lambda,x)+q(x)\dot{e}(\lambda,x)=\lambda^2\dot{e}(\lambda,x)+2\lambda e(\lambda,x), \tag{2.16}$$

where the dot shows differentiation with respect to λ. Multiplying (2.15) by $\dot{e}(\lambda,x)$ and (2.16) by $e(\lambda,x)$ and subtracting the second from the first and integrating this relation with respect to x over $(0,\infty)$, we obtain

$$2\lambda\int_0^{\infty}\left[e(\lambda,x)\right]^2\mathrm{d}x+W\left[e(\lambda,x),\dot{e}(\lambda,x)\right]\Big|_{x=0}=0.$$

Let λ be a zero of the function $\Theta(\lambda)$. With the help of the expression of the function $\Theta(\lambda)$, the result is found that

$$W\left[e(\lambda,x),\dot{e}(\lambda,x)\right]\Big|_{x=0}$$

$$= -\frac{\dot{\Theta}(\lambda)\, e(\lambda,0)}{p_1(\lambda)} - \left(\frac{e(\lambda,0)}{p_1(\lambda)}\right)^2 i \sum_{k=1}^{n} k \sum_{m=0}^{n-k} (\alpha_m \beta_{m+k} - \alpha_{m+k}\beta_m)\,(i\lambda)^{2m+k-1}$$

and hence we conclude that

$$2\lambda \int_0^\infty [e(\lambda,x)]^2\, dx + \left(\frac{e(\lambda,0)}{p_1(\lambda)}\right)^2 2\lambda \sum_{m=0}^{n-2} (\alpha_m \beta_{m+2} - \alpha_{m+2}\beta_m)\,(i\lambda)^{2m}$$
$$= \frac{\dot{\Theta}(\lambda)\, e(\lambda,0)}{p_1(\lambda)}. \tag{2.17}$$

Substituting $\lambda = i\mu_k$, $\mu_k > 0$, in (2.17) and multiplying $-i$, we find that the left side of the equality is positive. Thus $\dot{\Theta}(i\mu_k) \neq 0$, i.e. the zeros of $\Theta(\lambda)$ are simple, which completes the proof. □

The numbers m_k, $k = 1,\ldots,N$ are defined by

$$m_k^{-2} \equiv \int_0^\infty |e(i\lambda_k,x)|^2\, dx + \frac{|e(i\lambda_k,0)|^2}{|p_1(i\lambda_k)|^2} \sum_{j=0}^{n-2} (\alpha_j \beta_{j+2} - \alpha_{j+2}\beta_j)\,\lambda_k^{2j}$$

and called *norming numbers* for the boundary value problem (1.1)-(1.3).

Definition 1. The collection $\left\{S(\lambda), \{i\lambda_k\}_{k=1}^N, \{m_k\}_{k=1}^N\right\}$ is called the scattering data of the boundary value problem (1.1)-(1.3).

Using (2.1) and substituting related expressions into the function $\Theta(\lambda)$, we get

$$\Theta(\lambda) = p_1(\lambda)\left\{i\lambda - K(0,0) + \int_0^\infty K_x(0,t)\, e^{i\lambda t}\, dt\right\} - p_2(\lambda)\left\{1 + \int_0^\infty K(0,t)\, e^{i\lambda t}\, dt\right\}$$
$$= (i\lambda)^{n+1}\left[\alpha_n + O\left(\frac{1}{\lambda}\right)\right]$$

if $\alpha_n \neq 0$ as $|\lambda| \to \infty$. In a similar way, we obtain

$$\Theta_1(\lambda) = p_1(\lambda)\left\{-i\lambda - K(0,0) + \int_0^\infty K_x(0,t)\, e^{-i\lambda t}\, dt\right\}$$
$$- p_2(\lambda)\left\{1 + \int_0^\infty K(0,t)\, e^{-i\lambda t}\, dt\right\} = (i\lambda)^{n+1}\left[-\alpha_n + O\left(\frac{1}{\lambda}\right)\right].$$

Taking these into account, we deduce that

$$-1 - S(\lambda) = O\left(\frac{1}{\lambda}\right), \qquad |\lambda| \to \infty.$$

In the case that $\alpha_n = 0$, we find that

$$\Theta(\lambda) = (i\lambda)^n \left[\alpha_{n-1} - \beta_n + O\left(\frac{1}{\lambda}\right)\right]$$

and

$$\Theta_1(\lambda) = (i\lambda)^n \left[-\alpha_{n-1} - \beta_n + O\left(\frac{1}{\lambda}\right)\right],$$

as $|\lambda| \to \infty$. Therefore, the following result is obtained

$$\frac{\beta_n + \alpha_{n-1}}{\beta_n - \alpha_{n-1}} - S(\lambda) = O\left(\frac{1}{\lambda}\right), \qquad |\lambda| \to \infty.$$

Let us define

$$S(\infty) = \begin{cases} \frac{\beta_n + \alpha_{n-1}}{\beta_n - \alpha_{n-1}}, & \alpha_n = 0, \\ -1, & \alpha_n \neq 0, \end{cases}$$

$S(\infty) - S(\lambda) \in L_2(-\infty, \infty)$ and hence the function

$$F_S(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (S(\infty) - S(\lambda)) e^{i\lambda x} d\lambda,$$

also belongs to the space $L_2(-\infty, \infty)$.

3. The main equation

In this section, we shall derive the main equation which is extremely important for discussions of inverse scattering problem.

Theorem 1. *For every fixed $x \geq 0$, the kernel $K(x,t)$ to the solution (2.1) satisfies the integral equation*

$$F(x+y) + K(x,y) + \int_x^{\infty} K(x,t) F(t+y)\, dt = 0, \qquad y > x, \tag{3.1}$$

where

$$F(x) = \sum_{k=1}^{N} m_k^2 e^{-\lambda_k x} + \frac{1}{2\pi} \int_{-\infty}^{\infty} (S(\infty) - S(\lambda)) e^{i\lambda x} d\lambda. \tag{3.2}$$

The integral equation (3.1) is called the *main equation* for the boundary value problem (1.1)-(1.3).

Proof. In order to obtain the main equation for the kernel $K(x,t)$ of the solution (2.1), we use the equality (2.5) provided in Lemma 1. Rewriting the identity (2.5), we obtain the following result

$$\frac{2i\lambda\phi(\lambda,x)}{\Theta(\lambda)} - e^{-i\lambda x} + S(\infty) e^{i\lambda x}$$

$$= [S(\infty) - S(\lambda)] e^{i\lambda x} + \int_x^\infty K(x,t) e^{-i\lambda t} dt$$
$$- \int_x^\infty S(\infty) K(x,t) e^{i\lambda t} dt + \int_x^\infty [S(\infty) - S(\lambda)] K(x,t) e^{i\lambda t} dt.$$

Multiplying both sides of this result by $\frac{1}{2\pi} e^{i\lambda y}$ and integrating it according to λ over $(-\infty, \infty)$, we find

$$\frac{1}{2\pi} \int_{-\infty}^\infty \left[\frac{2i\lambda \phi(\lambda, x)}{\Theta(\lambda)} - e^{-i\lambda x} + S(\infty) e^{i\lambda x} \right] e^{i\lambda y} d\lambda$$
$$= \frac{1}{2\pi} \int_{-\infty}^\infty [S(\infty) - S(\lambda)] e^{i\lambda(x+y)} d\lambda$$
$$+ \frac{1}{2\pi} \int_{-\infty}^\infty \int_x^\infty K(x,t) e^{-i\lambda(t-y)} dt d\lambda - \frac{1}{2\pi} \int_{-\infty}^\infty \int_x^\infty S(\infty) K(x,t) e^{i\lambda(t+y)} dt d\lambda$$
$$+ \frac{1}{2\pi} \int_{-\infty}^\infty \int_x^\infty [S(\infty) - S(\lambda)] K(x,t) e^{i\lambda(t+y)} dt d\lambda. \tag{3.3}$$

Here

$$\frac{1}{2\pi} \int_{-\infty}^\infty \int_x^\infty K(x,t) e^{-i\lambda(t-y)} dt d\lambda = \int_x^\infty K(x,t) \frac{1}{2\pi} \int_{-\infty}^\infty e^{-i\lambda(t-y)} d\lambda dt$$
$$= \int_x^\infty K(x,t) \delta(t-y) dt = K(x,y)$$

and since $K(x,-y) = 0$ for $y > x$, it follows that

$$\frac{1}{2\pi} \int_{-\infty}^\infty \int_x^\infty S(\infty) K(x,t) e^{i\lambda(t+y)} dt d\lambda = \int_x^\infty S(\infty) K(x,t) \frac{1}{2\pi} \int_{-\infty}^\infty e^{i\lambda(t+y)} d\lambda dt$$
$$= S(\infty) \int_x^\infty K(x,t) \delta(-t-y) dt = S(\infty) K(x,-y) = 0.$$

Therefore, on the right of (3.3), we obtain

$$K(x,y) + F_S(x+y) + \int_x^\infty K(x,t) F_S(t+y) dt, \qquad y > x,$$

where

$$F_S(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}[S(\infty)-S(\lambda)]e^{i\lambda x}d\lambda.$$

On the other side, using the residue theorem and Jordan's lemma, we get

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{2i\lambda\phi(\lambda,x)}{\Theta(\lambda)}-e^{-i\lambda x}+S(\infty)e^{i\lambda x}\right]e^{i\lambda y}d\lambda=-\sum_{k=1}^{N}\frac{2i\lambda_k\phi(i\lambda_k,x)}{\dot{\Theta}(i\lambda_k)}e^{-\lambda_k y}.$$

Taking formula (2.17) into account, we can transform this expression to the form

$$\begin{aligned}&-\sum_{k=1}^{N}\frac{2i\lambda_k\phi(i\lambda_k,x)}{\dot{\Theta}(i\lambda_k)}e^{-\lambda_k y}\\&=-\sum_{k=1}^{N}\frac{2i\lambda_k p_1(i\lambda_k)}{e(i\lambda_k,0)\dot{\Theta}(i\lambda_k)}e(i\lambda_k,x)e^{-\lambda_k y}=-\sum_{k=1}^{N}m_k^2e(i\lambda_k,x)e^{-\lambda_k y}\\&=-\sum_{k=1}^{N}m_k^2\left[e^{-\lambda_k(x+y)}+\int_x^{\infty}K(x,t)e^{-\lambda_k(t+y)}dt\right].\end{aligned}$$

If we substitute this value into the left side of (3.3), we provide integral equation (3.1) and $F(x)$ is defined by the formula (3.2). The theorem is proved. □

4. Solvability of the main equation

The inverse scattering problem deals with the recovery of the coefficient $q(x)$ from the scattering data of the boundary value problem (1.1)-(1.3). From the identity (2.2), it is obvious to determine $q(x)$ that it is sufficient to know $K(x,t)$. Therefore, we suppose that the collection $\left\{S(\lambda),\{i\lambda_k\}_{k=1}^{N},\{m_k\}_{k=1}^{N}\right\}$ is given and construct the equation (3.1) on the basis of this collection. In this equation, we can take kernel $K(x,t)$ as unknown and regard it as a Fredholm-type equation for every fixed x.

Theorem 2. *For every fixed $x\geq 0$, the main equation (3.1) has a unique solution $K(x,.)\in L_1[x,\infty)$.*

Proof. Let us take $g(y)=K(x,y)$ in the equation (3.1). In order to prove the unique solvability of the equation, it suffices to verify that corresponding homogeneous equation has only null solution:

$$g(y)+\int_x^{\infty}g(t)F(t+y)dt=0,\qquad g(y)\in L_1[x,\infty).\tag{4.1}$$

The auxiliary function $F(y)$ in the equation (4.1) has the same properties as the auxiliary function of the boundary value problem without spectral parameter in the

boundary conditions. Therefore, the operators which appear in this equation are compact (the proof of this fact is similar to that of Lemma 3.3.1 in [18]) and hence we obtain the proof of the theorem by applying Theorem 3.3.1 in [18] to equation (4.1). □

Theorem 3. *The scattering data of the boundary value problem* (1.1)-(1.3) *determines potential* $q(x)$ *in equation* (1.1) *uniquely.*

Proof. From given scattering data $\left\{S(\lambda), \{i\lambda_k\}_{k=1}^{N}, \{m_k\}_{k=1}^{N}\right\}$, the main equation (3.1) is established and we find a unique solution $K(x,y)$ for every $x \geq 0$ by solving the main equation with the help of Theorem 2. Thus, we have the kernel $K(x,y)$ of the Jost solution (2.1) and obtain the potential with

$$q(x) = -2\frac{d}{dx}K(x,x).$$

□

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A NOTE ON PROINOV CONTRACTION

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Abstract. Proinov contraction has appeared a several years ago as a new approach in the study of nonlinear contractions that unifies and extends several well-known classes of nonlinear contractions. What was not discussed is the relation of Proinov contraction and nonlinear contractions, especially Meir-Keeler contraction which also presents an important concept in Fixed point theory. The aim of this paper is to show that Proinov contraction is a Meir-Keeler contraction under some restrictions and that the obtained results under the scope of this contraction and its generalizations in this case are the direct consequence of the analogous results concerning Meir-Keeler contraction mappings. We will also present an example of a Meir-Keeler contraction that is not Proinov contraction and, additionally an example of Proinov contraction which is not a Boyd-Wong contraction, but is a Meir-Keeler contraction.

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1. Introduction

The beginnings of the research concerning nonlinear contractions in the Metric Fixed Point Theory date to 1960s and the first attempts of E. Rakotch and, later, F. E. Browder to replace an upper bound in a usual Banach's contractive condition including linear function $qd(x,y)$ with a nonlinear function acting on the set of all distances. (see [3, 14]) Only a year after Browder's result, two American mathematicians, D. W. Boyd and J. S. W. Wong [2] relaxed some assumptions concerning nonlinear function φ and retained the same conclusion, that a nonlinear contraction has a unique fixed point on a complete metric space. The main issue in both of those two concepts was a request of upper semicontinuity (continuity) on the right that should be fulfilled by the function φ. A decade later, Polish mathematician J. Matkowski in [11] used a different approach-he succeeded to omit any continuity assumption replacing it by the request that $(\varphi^n(t))$ is a zero sequence for any $t > 0$ (or t in a closure of a set

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of distances). Newly obtained nonlinear contraction presents an extension of Boyd-Wong contraction, and therefore of Browder contraction, as mentioned in [11], and, in more details, in [8]. Both Boyd-Wong and Matkowski contractions have been studied, generalized and transformed in the several differen settings like b-metric space, cone metric space, partial metric space, fuzzy metric space, etc.

Different approach to the study of nonlinear contractions was introduced in 1969. by A. Meir and E. Keeler [12] where we have only estimations regarding $d(x,y)$ and $d(Tx,Ty)$ based on which we conclude the existence and uniqueness of a fixed point of a mapping T on a complete metric space. This idea was also broadly researched, with several generalization in different settings. (see [5–7, 9, 10, 15])

In 2020. P. D. Proinov gave an extension of the concept of nonlinear contraction in [13] that includes, as a consequence, a Boyd-Wong result along with the results regarding F-contraction and θ-contraction on a complete metric space.

Due to the expansion of the papers concerning nonlinear contractions and their high number of citations, we are obliged to discuss on the relation among them. Do the newly constructed nonlinear contractions represent a scientific novelty or are they already included as one of the previously known types of nonlinear contractions? Are we inventing or transforming?

2. Preliminaries

For basic definitions and preliminaries in Fixed Point Theory we refer the reader to [1].

First extension of Banach fixed point theorem in the field of nonlinear contractions was made by E. Rakotch in [14].

Theorem 1 ([14, Corollary])**.** *Let (X,d) be a complete metric space and $T\colon X\mapsto X$ a mapping. If there exists a decreasing function $\lambda\colon (0,\infty)\mapsto[0,1)$ such that*

$$Tx\neq Ty\Longrightarrow d(Tx,Ty)\leq\lambda(d(x,y))\,d(x,y), \tag{2.1}$$

for any $x,y\in X$, then the mapping T has a unique fixed point and the iterative sequence (T^nx) converges to the fixed point of the mapping T for any $x\in X$.

Further, the function φ will be called increasing (decreasing) if $x\leq y\Longrightarrow\varphi(x)\leq\varphi(y)$ ($\varphi(x)\geq\varphi(y)$), for any $x,y\in X$, and strictly increasing (strictly decreasing) if $x<y\Longrightarrow\varphi(x)<\varphi(y)$ ($\varphi(x)>\varphi(y)$), for any $x,y\in X$.

The generalization of Banach fixed point theorem by Browder included the generalization of a contractive condition in a form:

$$d(Tx,Ty)\leq\varphi(d(x,y)),$$

for any $x,y\in X$ where (X,d) is a metric space and $\varphi\colon[0,\infty)\mapsto[0,\infty)$ is a function. What are the additional requests imposed on φ in order to obtain a unique fixed point for a mapping T on a complete metric space is partially answered in [3] and improved in [2].

Theorem 2 ([3, Theorem 1]). *Let (X,d) be a complete metric space and $T\colon X \mapsto X$ a mapping. If there exists an increasing, continuous on the right function $\varphi\colon [0,\infty) \mapsto [0,\infty)$ such that $\varphi(t) < t$, $t \in (0,\infty)$, and the inequality*

$$d(Tx,Ty) \leq \varphi\left(d(x,y)\right), \tag{2.2}$$

holds for any $x,y \in X$,then the mapping T has a unique fixed point and the iterative sequence $(T^n x)$ converges to the fixed point of the mapping T for any $x \in X$.

Boyd and Wong replaced the continuity on the right with the assumption of an upper semicontinuity from the right.

Theorem 3 ([2, Theorem 1]). *Let (X,d) be a complete metric space and $T\colon X \mapsto X$ a mapping. If there exists an upper semicontinuous from the right function $\varphi\colon [0,\infty) \mapsto [0,\infty)$ such that $\varphi(t) < t$, $t \in (0,\infty)$, and inequality*

$$d(Tx,Ty) \leq \varphi\left(d(x,y)\right), \tag{2.3}$$

holds for any $x,y \in X$, then the mapping T has a unique fixed point and the iterative sequence $(T^n x)$ convergesto the fixed point of the mapping T for any $x \in X$.

Boyd and Wong suggested that, instead of upper-semicontinuity from the right, we will obtain the existence and uniqueness result by assuming

$$\limsup_{s \to t+} \varphi(t) < t \qquad \text{for all } t > 0.$$

The another approach on omitting continuity assumption was given by Matkowski in [11].

Theorem 4 ([11, Theorem 1.2]). *Let (X,d) be a complete metric space and $T\colon X \mapsto X$ a mapping. If there exists an increasing function $\varphi\colon [0,\infty) \mapsto [0,\infty)$ such that $\lim_{n\to\infty} \varphi^n(t) = 0$, $t \geq 0$, and*

$$d(Tx,Ty) \leq \varphi\left(d(x,y)\right), \tag{2.4}$$

holds for all $x,y \in X$, then the mapping T has a unique fixed point and the iterative sequence $(T^n x)$ converges to the fixed point for any $x \in X$.

Evidently, φ is continuous at 0 and $\varphi(t) < t$ for any $t \in (0,\infty)$ in Theorems 2, 3 and 4. Observe that the conclusions of Theorem 3 and Theorem 4 hold if the domain of the function φ is restricted to the closure of the set of distances $d(x,y)$ for any $x,y \in X$.

Proinov's results [13] present a generalization of Boyd-Wong contraction without any continuity request but imposing monotonicity.

Theorem 5 ([13, Theorem 3.6]). *Let (X,d) be a complete metric space and $T\colon X \mapsto X$ be a mapping satisfying condition*

$$d(Tx,Ty) \neq 0 \Longrightarrow \psi\left(d(Tx,Ty)\right) \leq \varphi\left(d(x,y)\right), \tag{2.5}$$

for any $x,y \in X$, where the functions $\varphi,\psi\colon (0,\infty) \mapsto \mathbb{R}$ fulfill the following conditions:

(P_1) ψ *is increasing;*
(P_2) $\varphi(t) < \psi(t)$ *for any* $t > 0$*;*
(P_3) $\limsup_{t\to\varepsilon+} \varphi(t) < \liminf_{t\to\varepsilon+} \psi(t)$*, for any* $\varepsilon > 0$.

The mapping T has a unique fixed point and the iterative sequence $(T^n x)$ converges to the fixed point for every $x \in X$.

Another result of [13] is the following modification of Theorem 5.

Theorem 6 ([13, Theorem 3.7])**.** *Let (X,d) be a complete metric space and $T : X \mapsto X$ be a mapping satisfying the condition (2.5) for any $x,y \in X$, where the functions $\varphi,\psi\colon (0,\infty) \mapsto \mathbb{R}$ fulfill (P_2) and the following assumptions:*

(P_4) $\inf_{t>\varepsilon} \psi(t) > -\infty$ *for any* $\varepsilon > 0$*;*
(P_5) *if $(\psi(t_n))$ and $(\varphi(t_n))$ are convergent sequences with the same limit and $(\psi(t_n))$ is strictly decreasing, then* $\lim_{n\to\infty} t_n = 0$*;*
(P_6) $\limsup_{t\to\varepsilon+} \varphi(t) < \liminf_{t\to\varepsilon} \psi(t)$ *or* $\limsup_{t\to\varepsilon} \varphi(t) < \liminf_{t\to\varepsilon+} \psi(t)$ *for any* $\varepsilon > 0$*;*
(P_7) T *has a closed graph or* $\limsup_{t\to 0+} \varphi(t) < \liminf_{t\to\varepsilon} \psi(t)$ *for any* $\varepsilon > 0$.

The mapping T has a unique fixed point and the iterative sequence $(T_n x)$ converges to the fixed point for every $x \in X$.

In the following, whenever we talk about Proinov contraction we will address to the assumptions of Theorem 5 if it is not specified otherwise.

We recall basic definitions and theorems regarding the Meir-Keeler contraction introduced in [12].

Definition 1. Let (X,d) be a metric space and $T : X \mapsto X$ a mapping. If for every $\varepsilon > 0$, there exists some $\delta > 0$ such that

$$\varepsilon \le d(x,y) < \varepsilon + \delta \implies d(Tx,Ty) < \varepsilon, \tag{2.6}$$

then a mapping T is Meir-Keeler contraction.

Theorem 7 ([12, Theorem 1])**.** *If (X,d) is a complete metric space and $T : X \mapsto X$ a Meir-Keeler contraction, then it possesses a unique fixed point $x^* \in X$. Moreover, for any $x \in X$*

$$\lim_{n\to\infty} T^n x = x^*.$$

Meir-Keeler contraction is a contractive mapping since $d(Tx,Ty) < d(x,y)$ for $x \neq y$. Banach contraction is a Meir-Keeler contraction if for any $\varepsilon > 0$, we choose $\delta = \frac{1-q}{q}\varepsilon$ where q is a contractive constant of a Banach contraction. Many classes of nonlinear contractions are included in the class of Meir-Keeler contractions like F-contraction (see [4]).

3. Main results

Theorem 8. *If (X,d) is a metric space and $T\colon X \mapsto X$ is a Proinov contraction, i.e., (2.5) holds for any $x,y \in X$ such that $Tx \neq Ty$ where the functions $\varphi,\psi\colon (0,\infty) \mapsto \mathbb{R}$ fulfill $(P_1)-(P_3)$ and ψ is continuous from the right, then it is a Meir-Keeler mapping.*

Proof. Assume that (X,d) is a complete metric space and $T\colon X \mapsto X$ a Proinov contraction for some $\varphi,\psi\colon (0,\infty) \mapsto [0,\infty)$ satisfying

$$\psi(d(Tx,Ty)) \leq \varphi(d(x,y)),$$

for any $x,y \in X$ such that $Tx \neq Ty$.

Let $\varepsilon > 0$ be arbitrary. Assume contrary of what we intend to prove, that for each $\delta = \frac{1}{n}$, $n \in \mathbb{N}$, there exist some $x_n, y_n \in X$ such that

$$\varepsilon \leq d(x_n,y_n) < \varepsilon + \frac{1}{n} \text{ and } \varepsilon \leq d(Tx_n,Ty_n)$$

for any $n \in \mathbb{N}$. Firstly, we observe that $\lim_{n\to\infty} d(x_n,y_n) = \varepsilon$ and

$$\psi(d(Tx,Ty)) \leq \varphi(d(x,y)) < \psi(d(x,y)),$$

implies that $d(Tx_n,Ty_n) < d(x_n,y_n)$ for any $n \in \mathbb{N}$.

Consequently, we get $\lim_{n\to\infty} d(Tx_n,Ty_n) = \varepsilon$. Observe that both $(d(x_n,y_n))$ and $(d(Tx_n,Ty_n))$ converge to ε from above as $n \to \infty$.

Furthermore,

$$\psi(\varepsilon) \leq \psi(d(Tx_n,Ty_n)) \leq \varphi(d(x_n,y_n))$$

for any $n \in \mathbb{N}$, so by noticing that $(\varphi(d(x_n,y_n)))$ is bounded above by $\psi(\varepsilon+1)$, as $n \to \infty$ we get

$$\psi(\varepsilon) \leq \lim_{n\to\infty} \psi(d(Tx_n,Ty_n)) \leq \limsup_{n\to\infty} \varphi(d(x_n,y_n)). \tag{3.1}$$

Observe that the condition $\limsup_{t_n\to\varepsilon+} \varphi(t_n) < \liminf_{t\to\varepsilon+} \psi(t)$ contradicts the inequality (3.1) for $t_n = d(x_n,y_n)$, $n \in \mathbb{N}$. Therefore, for the arbitrary chosen $\varepsilon > 0$, there must exist some $\delta > 0$ such that (2.6) holds for any $x,y \in X$. Accordingly, Proinov contraction on a metric space is a Meir-Keeler contraction. □

Remark 1. Eventhough the functions φ and ψ are assumed to have as a codomain an arbitrary subset of the real line, that can be omitted with a slight improvement which will place the codomain among positive real numbers. Define $\Phi,\Psi\colon (0,\infty) \mapsto (0,\infty)$ by

$$\Phi(x) = e^{\varphi(x)} \text{ and } \Psi(x) = e^{\psi(x)} \quad \text{for any } x \in (0,\infty).$$

Due to the monotonicity of the exponential function, the condition (2.5) holds for Φ,Ψ and they are fulfilling assumptions $(P_1)-(P_3)$ if that is the case for φ and ψ.

The following example will illustrate that converse of Theorem 8 do not hold, i.e., there exist Meir-Keeler contraction mappings which are not in the class of Proinov contractions.

Example 1. Observe the complete metric space (X,d) where $X=\left[\frac{1}{4},1\right]$ and a metric d is defined by $d(x,y)=|x-y|$, for any $x,y\in X$ and a mapping $T:X\mapsto X$ defined by

$$Tx=\sqrt{x},\qquad x\in\left[\frac{1}{4},1\right].$$

The mapping T has a unique fixed point $x^*=1$ in X, but note that it is not a Banach contraction on X. Our intention is to prove that it is a Meir-Keeler contraction and not a Proinov contraction.

(1) Since the condition

$$\varepsilon\le d(x,y)<\varepsilon+\delta\Longrightarrow d(Tx,Ty)<\varepsilon.$$

should be satisfied on X and $d(Tx,Ty)\le\frac{1}{2}$ for any $x,y\in X$, it is sufficient to discuss on the case $\varepsilon\le\frac{1}{2}$.

Choose an arbitrary $\varepsilon\in\left(0,\frac{1}{2}\right]$ and, without a loss of generality, $x>y$. If $\delta=\varepsilon^2$, then assuming $\varepsilon\le x-y<\varepsilon+\delta$ will lead to

$$\begin{aligned}d(Tx,Ty)&=\sqrt{x}-\sqrt{y}<\sqrt{y+\varepsilon+\varepsilon^2}-\sqrt{y}\\&\le\sqrt{\frac{1}{4}+\varepsilon+\varepsilon^2}-\sqrt{\frac{1}{4}}=\left(\frac{1}{2}+\varepsilon\right)-\frac{1}{2}=\varepsilon.\end{aligned}$$

because of the fact that the maximum of the function $\sqrt{t+\varepsilon+\varepsilon^2}-\sqrt{t}$ on $\left[\frac{1}{4},1\right]$ is achieved in $t=\frac{1}{4}$. Thus, T is a Meir-Keeler contraction.

(2) Suppose that T is indeed a Proinov contraction for some adequately constructed functions φ and ψ. Let $x_n=\left(\frac{1}{2}+\frac{1}{n}\right)^2$ for any $n\ge2$, and $y=\frac{1}{16}$, then

$$d(Tx_n,Ty)=|\sqrt{x_n}-\sqrt{y}|=\frac{1}{4}+\frac{1}{n},$$

for any $n\in\mathbb{N}$. Apart from that, $d(x_n,y_n)=\frac{3}{16}+\frac{n+1}{n^2}$ for any $n\in\mathbb{N}$. Hence, for any natural number $n>1$,

$$\psi\left(\frac{1}{4}\right)\le\psi\left(d(Tx_n,Ty)\right)\le\varphi\left(d(x_n,y)\right),$$

further implying

$$\begin{aligned}\psi\left(\frac{1}{4}\right)&\le\liminf_{n\to\infty}\psi\left(\frac{1}{4}+\frac{1}{n}\right)\le\limsup_{n\to\infty}\varphi\left(\frac{3}{16}+\frac{n+1}{n^2}\right)\\&<\liminf_{n\to\infty}\psi\left(\frac{3}{16}+\frac{n+1}{n^2}\right)\le\psi\left(\frac{1}{4}\right),\end{aligned}$$

which is impossible. As the desired functions do not exist, we must conclude that T is not a Proinov contraction on a complete metric space (X,d).

In the original paper of Meir and Keeler [12] is given an example of a mapping which is a Meir-Keleer contraction, but not Boyd-Wong contraction. That example is inadequate in the case of Proinov contraction.

Example 2. Let $X=[0,1]\cup\{3n,3n+1 \mid n\in\mathbb{N}\}$ with the Euclidian distance d and $T: X\mapsto X$ defined by

$$Tx=\begin{cases}\frac{x}{2}, & \text{if } x\in[0,1];\\ 0, & \text{if } x=3n \text{ for some } n\in\mathbb{N};\\ 1-\frac{1}{n+2}, & \text{if } x=3n+1 \text{ for some } n\in\mathbb{N}.\end{cases}.$$

As observed in [12], the mapping T is a Meir-Keeler contraction and not a Boyd-Wong contraction. On the other hand, define $\varphi,\psi\colon (0,\infty)\mapsto\mathbb{R}$ such that

$$\varphi(x)=\begin{cases}\frac{x}{2}, & \text{if } x\in(0,1);\\ 1, & \text{otherwise.}\end{cases}$$

$$\psi(x)=\begin{cases}x, & \text{if } x\in(0,1);\\ 2, & \text{otherwise.}\end{cases}$$

In this way, T is a Proinov contraction. Indeed, $d(Tx,Ty)<1$ for any $x,y\in X$, but $d(x,y)\geq 1$ if $x\notin[0,1]$ and $y\notin[0,1]$, If $x,y\in[0,1]$, then $\psi(d(Tx,Ty))=|\frac{x-y}{2}|=\varphi(d(x,y))$ whenever $Tx\neq Ty$.

Remark 2. This example also testifies on the independence of Proinov contraction from Boyd-Wong contraction, since the defined mapping is a Proinov contraction, but not Boyd-Wong contraction.

Remark 3. The proof of Theorem 8 is relying on the properties of the functions φ and ψ, not related to the change of the type of generalized metric as long as its domain is in the real line and the definition of Meir-Keeler contraction is consistent to the one on a matric space. Therefore, we may state the analogous result for the several different classes of a generalized metric space.

Theorem 9. *Assume that $X\neq\varnothing$, $d\colon X\times X\mapsto[0,\infty)$ and $T\colon X\mapsto X$ is a Proinov contraction, i.e. for any $x,y\in X$ the condition (3.1) holds for some $\varphi,\psi\colon[0,\infty)\mapsto\mathbb{R}$ fulfilling $(P_1)-(P_3)$ with ψ being continuous from the right, then it is a Meir-Keeler mapping.*

Proof. The proof of the Theorem 8 is applicable in this case. □

In this way, by Theorem 9, we have covered a wide class of Proinov contractions in different types of generalized metric spaces-semimetric, partial metric, b-metric, fuzzy metric, etc.

The similar conclusion can be deduced for the contraction mappings fulfilling the assumptions of Theorem 6.

Theorem 10. *If (X,d) is a metric space and $T\colon X\mapsto X$ is such that (2.5) holds for any $x,y\in X$ satisfying $Tx\neq Ty$ where the functions $\varphi,\psi\colon (0,\infty)\mapsto\mathbb{R}$ fulfill (P_2) and $(P_4)-(P_7)$ and ψ being continuous from the right, then it is a Meir-Keeler mapping.*

Proof. Assume that (X,d) is a complete metric space and $T\colon X\mapsto X$ a Proinov contraction satisfying (2.5) for any $x,y\in X$ where $\varphi,\psi\colon [0,\infty)\mapsto\mathbb{R}$ fulfill (P_2) and $(P_4)-(P_7)$.

Let $\varepsilon>0$ be arbitrary. Assume contrary, for each $\delta=\frac{1}{n}$, $n\in\mathbb{N}$, there exist some $x_n,y_n\in X$ such that

$$\varepsilon\leq d(x_n,y_n)<\varepsilon+\frac{1}{n}$$

and $\varepsilon\leq d(Tx_n,Ty_n)$ for any $n\in\mathbb{N}$. Same as in the proof of Theorem 8, we get

$$\lim_{n\to\infty}d(x_n,y_n)=\lim_{n\to\infty}d(Tx_n,Ty_n)=\varepsilon.$$

Moreover,

$$\psi(\varepsilon)\leq\psi(d(Tx_n,Ty_n))\leq\varphi(d(x_n,y_n)),$$

for any $n\in\mathbb{N}$, so as $n\to\infty$, we get

$$\liminf_{n\to\infty}\psi(d(Tx_n,Ty_n))\leq\liminf_{n\to\infty}\varphi(d(x_n,y_n))\leq\limsup_{n\to\infty}\varphi(d(x_n,y_n)). \tag{3.2}$$

The condition (P_6) contradicts the inequality (3.2). Hence, the assumption is false, and the mapping T is a Meir-Keeler contraction on a metric space (X,d). □

Analogous extension can be derived as for Theorem 9.

Theorem 11. *Assume that $X\neq\varnothing$, $d\colon X\times X\mapsto[0,\infty)$ and $T\colon X\mapsto X$ is such that (2.5) holds for any $x,y\in X$ satisfying $Tx\neq Ty$ where the functions $\varphi,\psi\colon (0,\infty)\mapsto\mathbb{R}$ fulfill (P_2) and $(P_4)-(P_7)$ with ψ being continuous from the right, then it is a Meir-Keeler mapping.*

Proof. Analogous to the proof of Theorem 10. □

4. Conclusion

In summary, the main result shows that any Proinov contraction under the constraint that ψ is continuous from the right, is a Meir-Keeler contraction and not only in the setting of an arbitrary metric space, but also in various generalizations of a metric space. Therefore the existence and uniqueness of the fixed point of a this class of Proinov contraction on a complete metric space (analogously on any space (X,d) where $d\colon X\times X\mapsto[0,\infty)$) is a direct corollary of the theorem witnessing the existence and uniqueness of the fixed point of a Meir-Keeler contraction in the same setting. This could be transferred also for the Proinov and Meir-Keeler contractions in the setting of generalized metric space or cone metric space where the domain of

a generalized metric is not $\mathbb{R}$. Also, we believe that the same parallel could be drawn between different extensions of those two types of contractions. One more important question remains open and that is the relation of Proinov and Matkowski contraction.

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A NONLINEAR ULTRA-PARABOLIC DIFFERENTIAL INEQUALITY WITH BESSEL OPERATOR

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Abstract. In this paper, we investigate the existence and nonexistence of weak solutions to an ultra-parabolic differential inequality posed on the interval $(0,1)$, under an inhomogeneous Dirichlet boundary condition. The considered problem involves the singular Bessel operator

$$\frac{d^2}{dx^2}+\frac{\alpha}{x}\frac{d}{dx}$$

and a nonlinearity of the form $x^{-\sigma}|u|^p$. Our approach is based on nonlinear capacity estimates and an appropriate choice of test functions.

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1. Introduction

In this work, we are concerned with the study of existence and nonexistence of weak solutions to the two-time nonlinear differential inequality governed by the singular Bessel operator

$$u_t+(|u|^{m-1}u)_s-\left(u_{xx}+\frac{\alpha}{x}u_x\right)\geq x^{-\sigma}|u|^p,\quad t,s>0,\ 0<x<1, \tag{1.1}$$

where $u=u(t,s,x)$, $\alpha\in\mathbb{R}$, $p>m\geq 1$ and $\sigma\geq 0$. Problem (1.1) is studied under the inhomogeneous Dirichlet boundary condition

$$u(t,s,1)=\delta,\quad t,s>0, \tag{1.2}$$

where $\delta>0$ is a constant.

Problem (1.1), involving two-time variables, is a differential inequality of ultra-parabolic-type. Ultra-parabolic problems arise in many applications such as mathematical finance (see, e.g., [3]), the theory of the Brownian motion (see, e.g., [15]) and the kinetic theory of gases (see, e.g., [2, 10]). In the linear case, ultra-parabolic

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equations were first introduced by Kolmogoroff [7] in order to model the probability density of a system with 2D degrees of freedom. Since then, a lot of generalizations have been made by several authors. Nonlinear ultra-parabolic equations were first studied by Ugowski in [14], where differential inequalities of parabolic-type involving multidimensional time were investigated. For instance, a maximum principle which has several applications was established. A class of inverse problems for higher order ultraparabolic equations with unknown minor coefficients and a right-hand side function have been studied under specific constraints [12]. Recently, a class of ultra-parabolic Kolmogorov-type operators satisfying the Hörmander condition for the approximation and interpolation techniques was introduced by Pesce [11]. On the other hand, the Bessel operator arises as a result of solving the Helmholtz equation in spherical and cylindrical coordinates by means of separable variables. Therefore, Bessel operators are of great importance for many problems related to wave propagation and static potentials, see, eg., [8, 9].

The study of nonexistence of solutions to evolution problems involving multidimensional time was initiated by Kirane and his collaborators (see, e.g., [1, 5, 6]). Other contributions in this topic can be found in [4, 13]. In particular, Kerbal and Kirane [5] considered the two-time nonlinear equation

$$u_t + u_s - \Delta(|u|^m) = |u|^p, \quad t,s > 0, x \in \mathbb{R}^N, \tag{1.3}$$

where $u = u(t,s,x)$ and $p > m \geq 1$, subject to the initial conditions

$$u(t,0,x) = \varphi_1(t,x),\ u(0,s,x) = \varphi_2(s,x), \quad t,s > 0, x \in \mathbb{R}^N. \tag{1.4}$$

Namely, it was proved that, if

$$\int_0^\infty \int_{\mathbb{R}^N} \varphi_1(t,x)\,dx\,dt + \int_0^\infty \int_{\mathbb{R}^N} \varphi_2(s,x)\,dx\,ds > 0,$$

then (1.3)-(1.4) admits no weak solution, provided

$$1 \leq m < p \leq m + \frac{2m}{N+2}.$$

In the same reference [5], the authors extended the above result to systems of the form

$$\begin{cases} u_t + u_s - \Delta(|u|^m) &= |v|^p, \quad t,s > 0, x \in \mathbb{R}^N, \\ v_t + v_s - \Delta(|v|^n) &= |u|^q, \quad t,s > 0, x \in \mathbb{R}^N, \end{cases}$$

where $p > \max\{n,1\}$ and $q > \max\{m,1\}$.

To the best of our knowledge, the study of nonexistence of solutions to nonlinear ultra-parabolic problems in a bounded domain was not previously considered in the literature. Motivated by this fact, and the contribution [5], problem (1.1) under the Dirichlet boundary condition (1.2) is here investigated. Namely, using nonlinear capacity estimates, specifically adapted to the multidimensional time, the boundedness of the domain, the properties of the Bessel operator $\frac{\partial^2}{\partial x^2} + \frac{\alpha}{x}\frac{\partial}{\partial x}$ and the boundary

condition (1.2), we obtain optimal criteria for the nonexistence of solutions. The cases $\alpha < 1$ and $\alpha \geq 1$ are studied separately. In the case $\alpha < 1$, we show that, if $\sigma \leq 2 + (1-\alpha)(m-1)$, then for all $p > m$, (1.1)-(1.2) admits solutions for some $\delta > 0$. However, if $\sigma > 2+(1-\alpha)(m-1)$, then the dividing line with respect to existence and nonexistence is given by a critical exponent $p = p^*(\sigma, \alpha)$. In the case $\alpha \geq 1$, we show that the considered problem admits a different critical behavior. Namely, we show that $\sigma^* = 2$ is critical in the following sense: if $\sigma > \sigma^*$, then (1.1)-(1.2) admits no weak solution for all $p > m$; if $\sigma < \sigma^*$, then, for all $p > m$, (1.1)-(1.2) admits solutions for some $\delta > 0$.

The rest of the paper is organized as follows. In Section 2, we define weak solutions to the considered problem and state our main results. Some useful lemmas are established in Section 3. Finally, the proofs of our main results are given in Section 4.

Throughout the text, we shall use the following notations. For a positive real number ℓ, the notation $\ell \gg 1$ means that ℓ is sufficiently large. By C (or C_i), we mean a positive constant that is independent of the scaling parameters T, S, R and the solution u. The value of this constant is not important and is not necessarily the same from one line to another.

2. Main results

Before stating our main results, we need to define the notion of weak solution to (1.1)-(1.2). Let $Q = (0,\infty) \times (0,\infty) \times (0,1]$. We introduce the set of test functions

$$\Phi = \left\{ \varphi = \varphi(t,s,x) \in C^2(Q) : \sup(\varphi) \subset\subset Q, \varphi \geq 0, \varphi(\cdot,\cdot,1) \equiv 0 \right\}.$$

Here, by $\sup(\varphi) \subset\subset Q$, we mean that φ is compactly supported on Q.

Definition 1. We say that $u \in L^p_{loc}(Q)$ is a weak solution to (1.1)-(1.2) if

$$\begin{aligned} &\int_Q x^{-\sigma} |u|^p \varphi \, dx\, ds\, dt - \delta \int_{(0,\infty)\times(0,\infty)} \varphi_x(t,s,1)\, ds\, dt \\ &\qquad \leq -\int_Q u\varphi_t \, dx\, ds\, dt - \int_Q |u|^{m-1} u \varphi_s \, dx\, ds\, dt - \int_Q u \left(\varphi_{xx} - \alpha \left(\frac{\varphi}{x} \right)_x \right) dx\, ds\, dt \end{aligned} \tag{2.1}$$

for all $\varphi \in \Phi$.

Remark 1. Note that, if $u \in C^2((0,\infty)\times(0,\infty)\times(0,1])$ is a classical solution to (1.1)-(1.2) (i.e., u satisfies (1.1) for every $t, s > 0$, $0 < x < 1$, and u satisfies (1.2) for every $t, s > 0$), then, multiplying (1.1) by $\varphi \in \Phi$, integrating by parts over Q, and using (1.2), we obtain (2.1).

We first consider the case $\alpha < 1$.

Theorem 1. *Let* $\alpha < 1$ *and* $m \geq 1$.

(i) *If* $\delta > 0$ *and*

$$\sigma > 2 + (1-\alpha)(m-1), \ m < p < m + \frac{(\sigma - 2) - (1-\alpha)(m-1)}{1-\alpha}, \tag{2.2}$$

then (1.1)-(1.2) *admits no weak solution.*

(ii) *If*

$$p > \max\left\{m, m + \frac{(\sigma-2)-(1-\alpha)(m-1)}{1-\alpha}\right\},$$

then (1.1)-(1.2) *admits (stationary) solutions for some* $\delta > 0$.

Remark 2. Let $\alpha < 1$ and $m \geq 1$. From Theorem 1, we deduce the following facts:

(i) If $\sigma \leq 2 + (1-\alpha)(m-1)$, then for all $p > m$, (1.1)-(1.2) admits (stationary) solutions for some $\delta > 0$.

(ii) If $\sigma > 2 + (1-\alpha)(m-1)$, then

$$p^* := m + \frac{(\sigma-2)-(1-\alpha)(m-1)}{1-\alpha}$$

is a critical exponent for (1.1)-(1.2) in the following sense:

- If $\delta > 0$ and $m < p < p^*$, then (1.1)-(1.2) admits no weak solution.
- If $p > p^*$, then (1.1)-(1.2) admits (stationary) solutions for some $\delta > 0$.

Remark 3. When $\alpha < 1$ and $\sigma > 2 + (1-\alpha)(m-1)$, the critical case $p = p^*$ is left as an open question.

We next consider the case $\alpha \geq 1$.

Theorem 2. *Let* $\alpha \geq 1$ *and* $m \geq 1$.

(i) *If* $\delta > 0$ *and* $\sigma > 2$, *then, for all* $p > m$, (1.1)-(1.2) *admits no weak solution.*

(ii) *If* $\sigma < 2$, *then, for all* $p > m$, (1.1)-(1.2) *admits (stationary) solutions for some* $\delta > 0$.

Remark 4. If $\alpha \geq 1$ and $m \geq 1$, then it follows from Theorem 2 that (1.1)-(1.2) admits a new critical value $\sigma^* = 2$ in the following sense:

(i) If $\delta > 0$ and $\sigma > \sigma^*$, then, for all $p > m$, (1.1)-(1.2) admits no weak solution.

(ii) If $\sigma < \sigma^*$, then, for all $p > m$, (1.1)-(1.2) admits (stationary) solutions for some $\delta > 0$.

Remark 5. When $\alpha \geq 1$, the critical case $\sigma = \sigma^*$ is left as an open question.

3. Auxiliary results

Some useful lemmas for the proof of our main results are established in this section.

Let us consider a cut-off function $\zeta \in C^\infty([0,\infty))$ satisfying

$$0 \leq \zeta \leq 1, \ \sup(\zeta) \subset (0,1).$$

For $\ell \gg 1$ and $T, S > 0$, let

$$a_T(t) = \zeta^\ell\left(\frac{t}{T}\right), \quad t > 0, \tag{3.1}$$

and

$$a_S(s) = \zeta^{\ell}\left(\frac{s}{S}\right), \quad s > 0. \tag{3.2}$$

Lemma 1. *The following estimates hold:*

$$\int_0^{\infty} a_T(t)\,dt \le T, \tag{3.3}$$

and

$$\int_0^{\infty} a_S(s)\,ds \le S. \tag{3.4}$$

Proof. The inequalities (3.3) and (3.4) follow immediately from the properties of ζ, (3.1), and (3.2). □

Lemma 2. *The following inequality holds:*

$$\int_0^{\infty} a_T^{\frac{-1}{p-1}} \left|\frac{da_T}{dt}\right|^{\frac{p}{p-1}} dt \le CT^{1-\frac{p}{p-1}}. \tag{3.5}$$

Proof. For all $t > 0$, by (3.1) we have

$$\frac{da_T}{dt}(t) = \ell T^{-1}\zeta^{\ell-1}\left(\frac{t}{T}\right)\zeta'\left(\frac{t}{T}\right),$$

which implies that

$$a_T^{\frac{-1}{p-1}}(t)\left|\frac{da_T}{dt}(t)\right|^{\frac{p}{p-1}} = CT^{\frac{-p}{p-1}}\zeta^{\ell-\frac{p}{p-1}}\left(\frac{t}{T}\right)\left|\zeta'\left(\frac{t}{T}\right)\right|^{\frac{p}{p-1}}.$$

Then, by the properties of ζ, we get

$$\begin{aligned}\int_0^{\infty} a_T^{\frac{-1}{p-1}} \left|\frac{da_T}{dt}\right|^{\frac{p}{p-1}} dt &= CT^{\frac{-p}{p-1}} \int_0^{T} \zeta^{\ell-\frac{p}{p-1}}\left(\frac{t}{T}\right)\left|\zeta'\left(\frac{t}{T}\right)\right|^{\frac{p}{p-1}} dt \\ &\le CT^{\frac{-p}{p-1}} \int_0^{T} dt = CT^{1-\frac{p}{p-1}},\end{aligned}$$

which proves (3.5). □

Lemma 3. *The following inequality holds:*

$$\int_0^{\infty} a_S^{\frac{-m}{p-m}} \left|\frac{da_S}{ds}\right|^{\frac{p}{p-m}} ds \le CS^{1-\frac{p}{p-m}}. \tag{3.6}$$

Proof. For all $s > 0$, by (3.2) we have

$$\frac{da_S}{ds}(s) = \ell S^{-1}\zeta^{\ell-1}\left(\frac{s}{S}\right)\zeta'\left(\frac{s}{S}\right),$$

which implies that

$$a_S^{\frac{-m}{p-m}}(s)\left|\frac{da_S}{ds}(s)\right|^{\frac{p}{p-m}} = CS^{\frac{-p}{p-m}}\zeta^{\ell-\frac{p}{p-m}}\left(\frac{s}{S}\right)\left|\zeta'\left(\frac{s}{S}\right)\right|^{\frac{p}{p-m}}.$$

Then, by the properties of ζ, we get

$$\int_0^\infty a_S^{\frac{-m}{p-m}} \left|\frac{da_S}{ds}\right|^{\frac{p}{p-m}} ds = CS^{\frac{-p}{p-m}} \int_0^S \zeta^{\ell-\frac{p}{p-m}}\left(\frac{s}{S}\right)\left|\zeta'\left(\frac{s}{S}\right)\right|^{\frac{p}{p-m}} ds$$
$$\leq CS^{\frac{-p}{p-m}} \int_0^S ds = CS^{1-\frac{p}{p-m}},$$

which proves (3.6). □

For all $\alpha \in \mathbb{R}$, let us now introduce the function H_α defined in $(0,1]$ by

$$H_\alpha(x) = \begin{cases} x\left(1-x^{\alpha-1}\right) & \text{if} \quad \alpha > 1, \\ x\left(x^{\alpha-1}-1\right) & \text{if} \quad \alpha < 1, \\ -x\ln x & \text{if} \quad \alpha = 1. \end{cases} \tag{3.7}$$

It can be easily seen that

$$H_\alpha \geq 0, \ \frac{d^2H_\alpha}{dx^2} - \alpha\frac{d}{dx}\left(\frac{H_\alpha}{x}\right) = 0 \ \text{ in } (0,1) \tag{3.8}$$

and

$$\frac{dH_\alpha}{dx}(1) = \begin{cases} -|1-\alpha| & \text{if} \quad \alpha \neq 1, \\ -1 & \text{if} \quad \alpha = 1. \end{cases} \tag{3.9}$$

We also need a cut-off function $\xi \in C^\infty([0,\infty))$ satisfying

$$0 \leq \xi \leq 1, \ \xi \equiv 0 \text{ in } \left[0,\frac{1}{2}\right], \ \xi \equiv 1 \text{ in } [1,\infty).$$

For $\ell, R \gg 1$, let

$$\xi_R(x) = H_\alpha(x)\xi^\ell(Rx), \quad x \in (0,1],$$

that is,

$$\xi_R(x) = \begin{cases} 0 & \text{if} \quad 0 < x \leq \frac{1}{2R}, \\ H_\alpha(x)\xi^\ell(Rx) & \text{if} \quad \frac{1}{2R} \leq x \leq \frac{1}{R}, \\ H_\alpha(x) & \text{if} \quad \frac{1}{R} \leq x \leq 1. \end{cases} \tag{3.10}$$

Lemma 4. *The following inequality holds:*

$$\int_0^1 x^{\frac{\sigma}{p-1}} \xi_R(x)\,dx \leq C\left(\ln R + R^{-\left(\frac{\sigma}{p-1}+\alpha+1\right)}\right). \tag{3.11}$$

Proof. By the properties of ξ and (3.10), we have

$$\int_0^1 x^{\frac{\sigma}{p-1}} \xi_R(x)\,dx = \int_{\frac{1}{2R}}^1 x^{\frac{\sigma}{p-1}} H_\alpha(x)\xi^\ell(Rx)\,dx$$
$$\leq \int_{\frac{1}{2R}}^1 x^{\frac{\sigma}{p-1}} H_\alpha(x)\,dx. \tag{3.12}$$

We discuss three possible cases.

Case 1: $\alpha > 1$. In this case, by (3.7), we have

$$x^{\frac{\sigma}{p-1}} H_\alpha(x) \le x^{\frac{\sigma}{p-1}+1},$$

which implies by (3.12) that

$$\int_0^1 x^{\frac{\sigma}{p-1}} \xi_R(x)\,dx \le C. \tag{3.13}$$

Case 2: $\alpha < 1$. In this case, by (3.7), we have

$$x^{\frac{\sigma}{p-1}} H_\alpha(x) \le x^{\frac{\sigma}{p-1}+\alpha},$$

which implies by (3.12) that

$$\int_0^1 x^{\frac{\sigma}{p-1}} \xi_R(x)\,dx \le C\left(\ln R + R^{-\left(\frac{\sigma}{p-1}+\alpha+1\right)}\right). \tag{3.14}$$

Case 3: $\alpha = 1$. In this case, by (3.7), we have

$$x^{\frac{\sigma}{p-1}} H_\alpha(x) = x^{\frac{\sigma}{p-1}+1} \ln\left(\frac{1}{x}\right),$$

which implies by (3.12) that

$$\int_0^1 x^{\frac{\sigma}{p-1}} \xi_R(x)\,dx \le C \ln R. \tag{3.15}$$

Finally, (3.11) follows from (3.13), (3.14) and (3.15). □

Two more estimates will be needed.

Lemma 5. *The following inequality holds:*

$$\int_0^1 x^{\frac{\sigma m}{p-m}} \xi_R(x)\,dx \le C\left(\ln R + R^{-\left(\frac{\sigma m}{p-m}+\alpha+1\right)}\right).$$

Proof. Similar to that of Lemma 4. □

Lemma 6. *The following inequality holds:*

$$\int_0^1 x^{\frac{\sigma}{p-1}} \xi_R^{\frac{-1}{p-1}} \left|\frac{d^2\xi_R}{dx^2} - \alpha\frac{d}{dx}\left(\frac{\xi_R}{x}\right)\right|^{\frac{p}{p-1}} dx \le C R^{\frac{2p-\sigma}{p-1}-\min\{\alpha,1\}-1} \ln R. \tag{3.16}$$

Proof. By the definition of ξ_R, for all $x \in (0,1)$ we have

$$\begin{aligned}
\frac{d^2\xi_R}{dx^2} - \alpha\frac{d}{dx}\left(\frac{\xi_R}{x}\right) &= \frac{d^2}{dx^2}\left[H_\alpha(x)\xi^\ell(Rx)\right] - \alpha\frac{d}{dx}\left(\frac{H_\alpha}{x}\xi^\ell(Rx)\right) \\
&= \frac{d^2 H_\alpha}{dx^2}\xi^\ell(Rx) + H_\alpha(x)\frac{d^2}{dx^2}\left[\xi^\ell(Rx)\right] + 2\frac{dH_\alpha}{dx}\frac{d}{dx}\left[\xi^\ell(Rx)\right] \\
&\quad - \alpha\frac{d}{dx}\left(\frac{H_\alpha}{x}\right)\xi^\ell(Rx) - \alpha\frac{H_\alpha}{x}\frac{d}{dx}\left[\xi^\ell(Rx)\right]
\end{aligned}$$

$$= \xi^{\ell}(Rx)\left(\frac{d^2H_\alpha}{dx^2} - \alpha\frac{d}{dx}\left(\frac{H_\alpha}{x}\right)\right) + H_\alpha(x)\frac{d^2}{dx^2}\left[\xi^{\ell}(Rx)\right] + \frac{d}{dx}\left[\xi^{\ell}(Rx)\right]\left(2\frac{dH_\alpha}{dx} - \alpha\frac{H_\alpha}{x}\right),$$

which implies, by (3.8) and the properties of ξ, that

$$\frac{d^2\xi_R}{dx^2} - \alpha\frac{d}{dx}\left(\frac{\xi_R}{x}\right) = H_\alpha(x)\frac{d^2}{dx^2}\left[\xi^{\ell}(Rx)\right] + \frac{d}{dx}\left[\xi^{\ell}(Rx)\right]\left(2\frac{dH_\alpha}{dx} - \alpha\frac{H_\alpha}{x}\right) \tag{3.17}$$

and

$$\sup\left(\frac{d^2\xi_R}{dx^2} - \alpha\frac{d}{dx}\left(\frac{\xi_R}{x}\right)\right) \subset \left[\frac{1}{2R}, \frac{1}{R}\right].$$

Consequently, we have

$$\begin{aligned}&\int_0^1 x^{\frac{\sigma}{p-1}}\xi_R^{\frac{-1}{p-1}}\left|\frac{d^2\xi_R}{dx^2} - \alpha\frac{d}{dx}\left(\frac{\xi_R}{x}\right)\right|^{\frac{p}{p-1}}dx\\ &= \int_{\frac{1}{2R}}^{\frac{1}{R}} x^{\frac{\sigma}{p-1}}\xi_R^{\frac{-1}{p-1}}\left|\frac{d^2\xi_R}{dx^2} - \alpha\frac{d}{dx}\left(\frac{\xi_R}{x}\right)\right|^{\frac{p}{p-1}}dx.\end{aligned} \tag{3.18}$$

On the other hand, from the properties of ξ, we have

$$\left|\frac{d^2}{dx^2}\left[\xi^{\ell}(Rx)\right]\right| \le CR^2\xi^{\ell-2}(Rx), \ \left|\frac{d}{dx}\left[\xi^{\ell}(Rx)\right]\right| \le CR\xi^{\ell-2}(Rx), \tag{3.19}$$

$x \in \left(\frac{1}{2R}, \frac{1}{R}\right)$. Furthermore, by (3.7), for all $x \in \left(\frac{1}{2R}, \frac{1}{R}\right)$, we have

$$\frac{2\frac{dH_\alpha}{dx} - \alpha\frac{H_\alpha}{x}}{H_\alpha(x)} = \begin{cases} \dfrac{2-\alpha-\alpha x^{\alpha-1}}{x-x^\alpha} & \text{if} \quad \alpha > 1,\\ \dfrac{\alpha x^{\alpha-1}+\alpha-2}{x^\alpha - x} & \text{if} \quad \alpha < 1,\\ \dfrac{-\ln x - 2}{-x\ln x} & \text{if} \quad \alpha = 1,\end{cases}$$

which implies that

$$\frac{\left|2\frac{dH_\alpha}{dx} - \alpha\frac{H_\alpha}{x}\right|}{H_\alpha(x)} \le CR, \quad x \in \left(\frac{1}{2R}, \frac{1}{R}\right). \tag{3.20}$$

We also have by (3.7) that

$$H_\alpha(x) \le CR^{-\min\{\alpha,1\}}\ln R, x \in \left(\frac{1}{2R}, \frac{1}{R}\right). \tag{3.21}$$

Hence, it follows from (3.17), (3.19) and (3.20) that

$$\frac{\left|\frac{d^2\xi_R}{dx^2} - \alpha\frac{d}{dx}\left(\frac{\xi_R}{x}\right)\right|}{H_\alpha(x)} \le \left|\frac{d^2}{dx^2}\left[\xi^{\ell}(Rx)\right]\right| + \left|\frac{d}{dx}\left[\xi^{\ell}(Rx)\right]\right|\frac{\left|2\frac{dH_\alpha}{dx} - \alpha\frac{H_\alpha}{x}\right|}{H_\alpha(x)}$$

$$\leq CR^2\xi^{\ell-2}(Rx), \quad x\in\left(\frac{1}{2R},\frac{1}{R}\right),$$

which implies by the definition of ξ_R and (3.21) that

$$\begin{aligned} x^{\frac{\sigma}{p-1}}\xi_R^{\frac{-1}{p-1}}\left|\frac{d^2\xi_R}{dx^2}-\alpha\frac{d}{dx}\left(\frac{\xi_R}{x}\right)\right|^{\frac{p}{p-1}} &= x^{\frac{\sigma}{p-1}}\xi^{\frac{-\ell}{p-1}}(Rx)H_\alpha(x)\left(\frac{\left|\frac{d^2\xi_R}{dx^2}-\alpha\frac{d}{dx}\left(\frac{\xi_R}{x}\right)\right|}{H_\alpha(x)}\right)^{\frac{p}{p-1}} \\ &\leq Cx^{\frac{\sigma}{p-1}}\xi^{\frac{-\ell}{p-1}}(Rx)\xi^{\frac{(\ell-2)p}{p-1}}(Rx)R^{\frac{2p}{p-1}}R^{-\min\{\alpha,1\}}\ln R \\ &= Cx^{\frac{\sigma}{p-1}}R^{\frac{2p}{p-1}-\min\{\alpha,1\}}\ln R\,\xi^{\ell-\frac{2p}{p-1}}(Rx), \quad x\in\left(\frac{1}{2R},\frac{1}{R}\right). \end{aligned}$$

Integrating now over $\left(\frac{1}{2R},\frac{1}{R}\right)$ and using (3.18), we get

$$\begin{aligned} &\int_0^1 x^{\frac{\sigma}{p-1}}\xi_R^{\frac{-1}{p-1}}\left|\frac{d^2\xi_R}{dx^2}-\alpha\frac{d}{dx}\left(\frac{\xi_R}{x}\right)\right|^{\frac{p}{p-1}}dx \\ &\qquad\leq CR^{\frac{2p}{p-1}-\min\{\alpha,1\}}\ln R\int_{\frac{1}{2r}}^{\frac{1}{R}} x^{\frac{\sigma}{p-1}}\xi^{\ell-\frac{2p}{p-1}}(Rx)\,dx \\ &\qquad\leq CR^{\frac{2p}{p-1}-\min\{\alpha,1\}}\ln R\int_{\frac{1}{2r}}^{\frac{1}{R}} x^{\frac{\sigma}{p-1}}\,dx \leq CR^{\frac{2p}{p-1}-\min\{\alpha,1\}-\frac{\sigma}{p-1}-1}\ln R, \end{aligned}$$

which proves (3.16). □

For all $\ell,T,S,R\gg 1$, let

$$\varphi(t,s,x)=a_T(t)a_S(s)\xi_R(x), \quad (t,s,x)\in Q. \tag{3.22}$$

Lemma 7. *The function* φ *belongs to* Φ.

Proof. Follows immediately from the definitions of a_T, a_S, and ξ_R. □

We now introduce the integral terms

$$\omega_1(\varphi)=\int_Q x^{\frac{\sigma}{p-1}}\varphi^{\frac{-1}{p-1}}|\varphi_t|^{\frac{p}{p-1}}\,dx\,ds\,dt, \tag{3.23}$$

$$\omega_2(\varphi)=\int_Q x^{\frac{\sigma m}{p-m}}\varphi^{\frac{-m}{p-m}}|\varphi_s|^{\frac{p}{p-m}}\,dx\,ds\,dt, \tag{3.24}$$

$$\omega_3(\varphi)=\int_Q x^{\frac{\sigma}{p-1}}\varphi^{\frac{-1}{p-1}}\left|\varphi_{xx}-\alpha\left(\frac{\varphi}{x}\right)_x\right|^{\frac{p}{p-1}}dx\,ds\,dt. \tag{3.25}$$

We obtain the following estimates for $\omega_i(\varphi)$, $i=1,2,3$.

Lemma 8. *The following estimate of* $\omega_1(\varphi)$ *holds:*

$$\omega_1(\varphi)\leq CT^{1-\frac{p}{p-1}}S\left(\ln R+R^{-\left(\frac{\sigma}{p-1}+\alpha+1\right)}\right). \tag{3.26}$$

Proof. By (3.22) and (3.23), we have

$$\omega_1(\varphi) = \left(\int_0^\infty a_T^{\frac{-1}{p-1}} \left| \frac{da_T}{dt} \right|^{\frac{p}{p-1}} dt \right) \left(\int_0^\infty a_S(s)\, ds \right) \left(\int_0^1 x^{\frac{\sigma}{p-1}} \xi_R(x)\, dx \right).$$

We obtain (3.26) by using (3.4), and Lemma 2 and 4. □

Lemma 9. *The following estimate of* $\omega_2(\varphi)$ *holds:*

$$\omega_2(\varphi) \leq CTS^{1-\frac{p}{p-m}} \left(\ln R + R^{-\left(\frac{\sigma m}{p-m}+\alpha+1\right)} \right).$$

Proof. Similar to the proof of Lemma 8, using (3.3) and Lemma 3 and 5. □

Lemma 10. *The following estimate of* $\omega_3(\varphi)$ *holds:*

$$\omega_3(\varphi) \leq CTSR^{\frac{2p-\sigma}{p-1}-\min\{\alpha,1\}-1} \ln R.$$

Proof. Similar to the proof of Lemma 8, using (3.3), (3.4), and Lemma 6. □

4. Proofs of the main results

This section is devoted to the proofs of Theorems 1 and 2.

Proof of Theorem 1. (i) We argue by contradiction, supposing that $u \in L^p_{loc}(Q)$ is a weak solution to (1.1)-(1.2). For $\ell, T, S, R \gg 1$, let φ be the function defined by (3.22). By Lemma 7, we know that $\varphi \in \Phi$. Then, by (2.1), it follows that

$$\begin{aligned} &\int_Q x^{-\sigma} |u|^p \varphi\, dx\, ds\, dt - \delta \int_{(0,\infty)\times(0,\infty)} \varphi_x(t,s,1)\, ds\, dt \qquad (4.1)\\ &\quad \leq \int_Q |u||\varphi_t|\, dx\, ds\, dt + \int_Q |u|^m |\varphi_s|\, dx\, ds\, dt + \int_Q |u| \left| \varphi_{xx} - \alpha \left(\frac{\varphi}{x} \right)_x \right| dx\, ds\, dt. \end{aligned}$$

Using Young's inequality, we get

$$\begin{aligned} \int_Q |u||\varphi_t|\, dx\, ds\, dt &= \int_Q \left(x^{\frac{-\sigma}{p}} |u| \varphi^{\frac{1}{p}} \right) \left(x^{\frac{\sigma}{p}} \varphi^{\frac{-1}{p}} |\varphi_t| \right) dx\, ds\, dt \qquad (4.2)\\ &\leq \frac{1}{3} \int_Q x^{-\sigma} |u|^p \varphi\, dx\, ds\, dt + C\omega_1(\varphi), \end{aligned}$$

where $\omega_1(\varphi)$ is the integral term given by (3.23). Similarly, we obtain

$$\int_Q |u|^m |\varphi_s|\, dx\, ds\, dt \leq \frac{1}{3} \int_Q x^{-\sigma} |u|^p \varphi\, dx\, ds\, dt + C\omega_2(\varphi) \qquad (4.3)$$

and

$$\int_Q |u| \left| \varphi_{xx} - \alpha \left(\frac{\varphi}{x} \right)_x \right| dx\, ds\, dt \leq \frac{1}{3} \int_Q x^{-\sigma} |u|^p \varphi\, dx\, ds\, dt + C\omega_3(\varphi), \qquad (4.4)$$

where $\omega_2(\varphi)$ and $\omega_3(\varphi)$ are given respectively by (3.24) and (3.25). Then, it follows from (4.1), (4.2), (4.3), and (4.4), that

$$-\delta \int_{(0,\infty)\times(0,\infty)} \varphi_x(t,s,1)\,ds\,dt \le C\sum_{j=1}^{3}\omega_j(\varphi). \tag{4.5}$$

On the other hand, by (3.1), (3.2), (3.10), and (3.22), we have for all $t,s>0$ that

$$\varphi_x(t,s,1) = a_T(t)a_S(s)\frac{d\xi_R}{dx}(1) = \zeta^\ell\left(\frac{t}{T}\right)\zeta^\ell\left(\frac{s}{S}\right)\frac{dH_\alpha}{dx}(1).$$

Furthermore, by (3.7), we have

$$\frac{dH_\alpha}{dx}(1) = -C_\alpha,$$

where $C_\alpha>0$. Consequently, we get

$$\varphi_x(t,s,1) = -C_\alpha\zeta^\ell\left(\frac{t}{T}\right)\zeta^\ell\left(\frac{s}{S}\right),$$

which implies, by the properties of ζ, that

$$-\delta\int_{(0,\infty)\times(0,\infty)}\varphi_x(t,s,1)\,ds\,dt = C\delta TS. \tag{4.6}$$

Now, we use (4.5), (4.6), and Lemmas 8, 9, and 10, to obtain

$$\begin{aligned} C\delta \le{}& T^{\frac{-p}{p-1}}\ln R + T^{\frac{-p}{p-1}}R^{-\left(\frac{\sigma}{p-1}+\alpha+1\right)} + S^{\frac{-p}{p-m}}\ln R + S^{\frac{-p}{p-m}}R^{-\left(\frac{\sigma m}{p-m}+\alpha+1\right)} \\ &+ R^{\frac{2p-\sigma}{p-1}-\min\{\alpha,1\}-1}\ln R. \end{aligned} \tag{4.7}$$

Taking $S = T^{\frac{p-m}{p-1}}$, (4.7) reduces to

$$\begin{aligned} C\delta \le{}& T^{\frac{-p}{p-1}}\ln R + T^{\frac{-p}{p-1}}R^{-\left(\frac{\sigma}{p-1}+\alpha+1\right)} + T^{\frac{-p}{p-1}}\ln R + T^{\frac{-p}{p-1}}R^{-\left(\frac{\sigma m}{p-m}+\alpha+1\right)} \\ &+ R^{\frac{2p-\sigma}{p-1}-\min\{\alpha,1\}-1}\ln R. \end{aligned} \tag{4.8}$$

Taking $T = R^\theta$ and $\theta>0$, (4.8) reduces to

$$\delta \le R^{\mu_1(\theta)}\ln R + R^{\mu_2(\theta)} + R^{\mu_3(\theta)} + R^{\mu}\ln R, \tag{4.9}$$

where

$$\begin{aligned} \mu_1(\theta) &= -\frac{\theta p}{p-1}, \\ \mu_2(\theta) &= -\frac{\theta p}{p-1} - \left(\frac{\sigma}{p-1}+\alpha+1\right), \\ \mu_3(\theta) &= -\frac{\theta p}{p-1} - \left(\frac{\sigma m}{p-m}+\alpha+1\right), \\ \mu &= \frac{2p-\sigma}{p-1} - \min\{\alpha,1\} - 1. \end{aligned}$$

Since $\alpha < 1$, then

$$\mu = \frac{2p-\sigma}{p-1} - \alpha - 1.$$

Remark that $\mu_1(\theta) < 0$ for all $\theta > 0$. Furthermore, taking

$$\theta > \frac{p-1}{p} \max\left\{ -\left(\frac{\sigma}{p-1} + \alpha + 1\right), -\left(\frac{\sigma m}{p-m} + \alpha + 1\right), 0 \right\}, \tag{4.10}$$

we get $\mu_2(\theta) < 0$ and $\mu_3(\theta) < 0$. On the other hand, from (2.2), we have

$$p < m + \frac{(\sigma-2) - (1-\alpha)(m-1)}{1-\alpha} = \frac{\sigma - 1 - \alpha}{1-\alpha},$$

which is equivalent to $\mu < 0$. Finally, since $\mu_i(\theta) < 0$ for all $i = 1,2,3$ and $\mu < 0$, passing to the limit as $R \to \infty$ in (4.9), we obtain $\delta \leq 0$, which contradicts the positivity of δ. This proves part (i) of Theorem 1.

(ii) Let

$$p > \max\left\{ m, m + \frac{(\sigma-2) - (1-\alpha)(m-1)}{1-\alpha} \right\} = \max\left\{ m, \frac{\sigma - 1 - \alpha}{1-\alpha} \right\}. \tag{4.11}$$

For

$$\max\left\{ \frac{\sigma-2}{p-1}, 0 \right\} < \gamma < 1 - \alpha \tag{4.12}$$

and

$$0 < \varepsilon \leq \left[\gamma(1-\alpha-\gamma)\right]^{\frac{1}{p-1}}, \tag{4.13}$$

let

$$u(x) = \varepsilon x^{\gamma}, \quad 0 < x < 1. \tag{4.14}$$

Note that, due to (4.11), one has

$$\frac{\sigma-2}{p-1} < 1 - \alpha,$$

which implies, together with $\alpha < 1$, that the set of γ satisfying (4.12) is nonempty. Now, using (4.12), (4.13), and (4.14), we obtain

$$-u_{xx} - \frac{\alpha}{x} u_x = x^{-\sigma} |u(x)|^p \left[\varepsilon^{1-p} \gamma (1-\gamma-\alpha) x^{\sigma - 2 - \gamma(p-1)} \right] \geq x^{-\sigma} |u(x)|^p,$$

for all $x \in (0,1)$, which shows that u is a stationary solution to (1.1). Furthermore, u satisfies (1.2) with $\delta = \varepsilon > 0$. This completes the proof of part (ii) of Theorem 1. □

Proof of Theorem 2. (i) We also argue here by contradiction, supposing that $u \in L^p_{loc}(Q)$ is a weak solution to (1.1)-(1.2). Following exactly the same steps of the proof of part (i) of Theorem 1, for $R \gg 1$, we get (4.9). In this case, since $\alpha \geq 1$ and $\sigma > 2$, we have

$$\mu = \frac{2-\sigma}{p-1} < 0.$$

Then, for θ satisfying (4.10), we have $\mu_i(\theta) < 0$ for all $i = 1,2,3$ and $\mu < 0$. Hence, passing to the limit as $R \to \infty$ in (4.9), we reach a contradiction with $\delta > 0$. This proves part (i) of Theorem 2.

(ii) We consider separately the cases $\alpha > 1$ and $\alpha = 1$.

• Case 1: $\alpha > 1$. For

$$\max\left\{1-\alpha, \frac{\sigma-2}{p-1}\right\} < \beta < 0 \tag{4.15}$$

and

$$0 < \tau \le [\beta(1-\beta-\alpha)]^{\frac{1}{p-1}}, \tag{4.16}$$

let

$$u(x) = \tau x^{\beta}, \quad 0 < x < 1. \tag{4.17}$$

Note that since $\alpha > 1$ and $\sigma < 2$, the set of β satisfying (4.15) is nonempty. Using (4.15), (4.16), and (4.17), we obtain

$$-u_{xx} - \frac{\alpha}{x}u_x = x^{-\sigma}|u(x)|^p \left[\tau^{1-p}\beta(1-\beta-\alpha)x^{\sigma-2-\beta(p-1)}\right] \ge x^{-\sigma}|u(x)|^p,$$

for all $x \in (0,1)$, which shows that u is a stationary solution to (1.1). Furthermore, u satisfies (1.2) with $\delta = \tau > 0$.

• Case 2: $\alpha = 1$. For $0 < \eta < 1$, let us consider the function

$$f_\eta(x) = x^{\sigma-2}(1-\ln x)^{\eta(1-p)-2}, \quad 0 < x \le 1.$$

Observe that $f_\eta(x) > 0$ for all $x \in (0,1]$ and, since $\sigma < 2$,

$$\lim_{x\to 0^+} f_\eta(x) = +\infty,$$

which implies that

$$0 < m_\eta := \inf_{0<x\le 1} f_\eta(x) < +\infty. \tag{4.18}$$

For

$$0 < \eta < 1,\ 0 < \varepsilon \le [\eta(1-\eta)m_\eta]^{\frac{1}{p-1}}, \tag{4.19}$$

let

$$u(x) = \varepsilon(1-\ln x)^{\eta}, \quad 0 < x < 1. \tag{4.20}$$

By (4.18), (4.19), and (4.20), we obtain

$$\begin{aligned} -u_{xx} - \frac{\alpha}{x}u_x &= x^{-\sigma}|u(x)|^p \left[\varepsilon^{1-p}\eta(1-\eta)f_\eta(x)\right] \\ &\ge x^{-\sigma}|u(x)|^p \left[\varepsilon^{1-p}\eta(1-\eta)m_\eta\right] \ge x^{-\sigma}|u(x)|^p, \end{aligned}$$

for all $x \in (0,1)$, which shows that u is a stationary solution to (1.1). Furthermore, u satisfies (1.2) with $\delta = \varepsilon > 0$. The proof of part (ii) of Theorem 2 is complete. □

5. Conclusion

We investigated the existence and nonexistence of weak solutions to the two-time nonlinear differential inequality (1.1) under the inhomogeneous Dirichlet boundary condition (1.2). Using nonlinear capacity estimates, we obtained sharp criteria for the nonexistence of weak solutions. Namely, if $\alpha < 1$ and $\sigma > 2 + (1-\alpha)(m-1)$, we established that the dividing line with respect to existence and nonexistence is given by a critical exponent $p = p*$ that depends on σ and α. However, if $\alpha \geq 1$, then the problem admits a new critical value $\sigma^* = 2$ in the following sense: if $\sigma > \sigma^*$, then for all $p > m$, there is no weak solution; if $\sigma < \sigma^*$, then for all $p > m$, solutions exist for some $\delta > 0$.

In this work, when $\alpha < 1$ and $\sigma > 2 + (1-\alpha)(m-1)$, the critical case $p = p^*$ was not investigated. It would be interesting to study this case in order to decide whether a weak solution exists or not.

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A NOVEL METHOD FOR MILNE-TYPE INEQUALITIES USING PROPORTIONAL CAPUTO-HYBRID OPERATOR

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Abstract. In mathematics and the applied sciences, fractional calculus is a significant generalization and a very useful tool as it overcomes many limitations of classical analysis. More importantly, it is better to use the new hybrid fractional operator, which merges the proportional and Caputo operator. In this study, because of its numerous applications, we concentrate on the proportional Caputo-hybrid operator. First, we propose a new integral identity with the help of twice-differentiable mappings for the proportional Caputo-hybrid operator. Then, with the help of this newly derived identity, we establish several integral inequalities related to the Milne-type integral inequalities for proportional Caputo-hybrid operator. Also, we obtain various Milne-type inequalities for bounded mappings and mappings of bounded variation. Finally, we point out that the obtained results enhance and generalize some of the previous findings in the field of integral inequalities. These results are the first kind of such results in this direction.

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Keywords: Milne-type inequalities, convex mappings, bounded mappings, mappings of bounded variation, proportional Caputo-hybrid operator

1. INTRODUCTION

In advanced analysis, the theory of convexity holds an important and alluring position because of its many applications. With the help of this theory, we can effectively tackle a wide range of issues that come up in both pure and applied mathematics. Additionally, the use of integral inequalities and its applications has grown quickly, and it has influenced the various modern mathematical fields, including measure theory, approximation theory, and information theory, along with a number of fields within science and engineering. Also, the integral inequalities are an extremely useful tool for figuring out the error bounds of numerical integration formulas for a variety of mappings such as mappings with bounded variation, Lipschitz mappings, n-times differentiable mappings and bounded mappings. Some well-known inequalities like Simpson-type, Ostrowski-type, Hermite–Hadamard-type and Grüss-type analyze the

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remainder term and establish error bounds for quadrature rules. Due to the fact that the concept of convexity is a crucial property for improving these integral inequalities, the development of the theory of inequalities shares a strong connection with the theory of convexity. Because of the fundamental properties of these theories, many researchers have been quite interested in combining them, which helps to improve and generalize the integral inequalities.

One of the most significant and most commonly needed inequalities is Simpson's inequality, which is as follows:

$$\left|\frac{1}{3}\left[\frac{\psi(\varsigma)+\psi(\vartheta)}{2}+2\psi\left(\frac{\varsigma+\vartheta}{2}\right)\right]-\frac{1}{\vartheta-\varsigma}\int_{\varsigma}^{\vartheta}\psi(\varkappa)\mathrm{d}\varkappa\right|\leq\frac{(\vartheta-\varsigma)^4}{2880}\left\|\psi^{(4)}\right\|_{\infty},$$

where $\psi\colon[\varsigma,\vartheta]\to\mathbb{R}$ is four times continuously differantiable mapping on (ς,ϑ) and $\left\|\psi^{(4)}\right\|_{\infty}=\sup_{\varkappa\in(\varsigma,\vartheta)}\left|\psi^{(4)}(\varkappa)\right|<\infty$. This inequality establishes an upper limit on the error that occurs when estimating a definite integral using Simpson's rule. It is widely recognized that when the mapping ψ is not four times differentiable or when its fourth derivative $\psi^{(4)}$ is unbounded on the interval (ς,ϑ), the classical Simpson quadrature formula cannot be employed. Simpson-type inequalities provide boundaries and approximations, which aid in the solution of many geometric problems. Moreover, numerical integration, which is crucial in applied mathematics, engineering, and physics, can be estimated using Simpson-type inequalities. Therefore, in recent years, several authors have highlighted Simpson-type inequalities for various classes of mappings due to its abundant geometric importance and applications. For instance, in [14], Dragomir et al. demonstrated some recent advancements in Simpson's inequality, where the remaining part is expressed in terms of derivatives lower than the fourth order. In [2], Alomari introduced Simpson's type inequalities for s-convex functions. Sarikaya et al. gave some Simpson's type inequalities via twice differentiable functions in [27]. For the other results, one can refer to [8, 10, 17, 18, 20, 28].

According to the similar conditions as the Simpson inequality, the Milne inequality is the one that gives estimates of the error boundaries for the Milne formula:

$$\left|\frac{1}{3}\left[2\psi(\varsigma)-\psi\left(\frac{\varsigma+\vartheta}{2}\right)+2\psi(\vartheta)\right]-\frac{1}{\vartheta-\varsigma}\int_{\varsigma}^{\vartheta}\psi(\varkappa)\mathrm{d}\varkappa\right|\leq\frac{7(\vartheta-\varsigma)^4}{23040}\left\|\psi^{(4)}\right\|_{\infty},$$

where $\psi\colon[\varsigma,\vartheta]\to\mathbb{R}$ is a four times differentiable mapping on (ς,ϑ) and $\left\|\psi^{(4)}\right\|_{\infty}=\sup_{\varkappa\in(\varsigma,\vartheta)}\left|\psi^{(4)}(\varkappa)\right|<\infty$. In mathematical analysis, Milne inequality is essential, particularly when studying differential equations and dynamical systems. With the help of this inequality, researchers can derive bounds on solutions and establish significant results regarding the stability, convergence, and system control. Besides, the use of

the inequality can be seen in its applications to the proof of existence and uniqueness theorems for differential and integral equation solutions, which form the basis of numerous theoretical and applied fields. In the recent times, researchers' attention to the Milne inequality has been considerable. Alomari and Liu [3] established error estimations for the Milne's rule for mappings of bounded variation and for absolutely continuous mappings. Román-Flores et al. [22] proved some Milne type inequalities for interval-valued functions. Budak et al. [11] investigated Milne-type inequalities for bounded functions, Lipschitz functions and functions of bounded variation. Ali et al. [1] gave the fractional version of Milne's formula-type inequalities for differentiable convex functions and Riemann–Liouville fractional integrals. Many recent articles have been published on this subject, as in [7, 9, 19].

On the other hand, fractional calculus is a branch of mathematics that deals with derivatives and integrals of non-integer order. So, it plays a vital role in the generalization of classical calculus, modeling complex systems, solving fractional differential equations, analyzing fractal geometry and various scientific and engineering applications. Furthermore, it provides a framework for analyzing and understanding systems with fractional dynamics, allowing for a more comprehensive mathematical description of complex phenomena. Therefore, due to the new fractional integral and derivative such as Caputo-Fabrizio [12], Atangana-Baleanu [5] and tempered [23], this calculus has gained more importance and has found applications in various fields of science and engineering.

One of the significant definitions in fractional analysis is the following:

Definition 1 ([24])**.** Let $\rho > 0$ and $\rho \notin \{1,2,\dots\}$, $n = [\rho]+1$, $\psi \in AC^n[\varsigma,\vartheta]$, the space of functions having n-th derivatives absolutely continuous. The left-sided and right-sided Caputo fractional derivatives of order ρ are defined as follows:

$$ {}^{C}D^{\rho}_{\varsigma^+}\,\psi(\varkappa) = \frac{1}{\Gamma(n-\rho)}\int_{\varsigma}^{\varkappa}(\varkappa-\mathfrak{s})^{n-\rho-1}\psi^{(n)}(\mathfrak{s})\mathrm{d}\mathfrak{s}, \quad \varkappa > \varsigma, $$

and

$$ {}^{C}D^{\rho}_{\vartheta^-}\psi(\varkappa) = \frac{1}{\Gamma(n-\rho)}\int_{\varkappa}^{\vartheta}(\mathfrak{s}-\varkappa)^{n-\rho-1}\psi^{(n)}(\mathfrak{s})\mathrm{d}\mathfrak{s}, \quad \varkappa < \vartheta. $$

If $\rho = n \in \{1,2,3,\dots\}$ and usual derivative $\psi^{(n)}(\varkappa)$ of order n exists, then Caputo fractional derivative ${}^{C}D^{\rho}_{\varsigma^+}\,\psi(\varkappa)$ coincides with $\psi^{(n)}(\varkappa)$ whereas ${}^{C}D^{\rho}_{\vartheta^-}\,\psi(\varkappa)$ with exactness to a constant multiplier $(-1)^n$. For $n=1$ and $\rho=0$, we have ${}^{C}D^{\rho}_{\varsigma^+}\,\psi(\varkappa)$ $= {}^{C}D^{\rho}_{\vartheta^-}\,\psi(\varkappa) = \psi(\varkappa)$.

The Caputo derivative is defined as the application of a fractional integral to a standard derivative of the function whereas the Riemann–Liouville fractional derivative is obtained by differentiating the fractional integral of a function with respect to its independent variable of order n. The Caputo fractional derivative necessitates

more suitable initial conditions in contrast to the conventional Riemann–Liouville fractional derivative considering fractional differential equations [13]. Accordingly, when evaluating other fractional derivatives, the Caputo derivative is advantageous since it yields solutions that are more meaningful in a physical sense for the specific problems. Besides, the operator of proportional derivative denoted as ${}^{P}D_{\rho}\psi(\varkappa)$ is given by the equation [4] :

$$ {}^{P}D_{\rho}\psi(\varkappa) = K_1(\rho,\mathfrak{s})\psi(\mathfrak{s}) + K_0(\rho,\mathfrak{s})\psi'(\mathfrak{s}), $$

where K_1 and K_0 are the functions with respect to $\rho \in [0,1]$ and $\mathfrak{s} \in \mathbb{R}$ subject to certain conditions and also, the function ψ is differentiable with respect to $\mathfrak{s} \in \mathbb{R}$. This mathematical operator is commonly used in control systems and robotics. In recent years, there has been a notable increase in the importance of research conducted on both the Caputo derivative and the proportional derivative [15, 16, 21].

In [6], Baleanu et al. gave the following definition which they merge the concepts of Caputo derivative and proportional derivative in a novel manner, resulting in a hybrid fractional operator that can be represented as a linear combination of Caputo fractional derivative and Riemann–Liouville fractional integral.

Definition 2. Let $\psi\colon I \subset \mathbb{R}^{+} \to \mathbb{R}$ be a differentiable function on I° and ψ, ψ' are locally $L(I)$. Then, the proportional Caputo-hybrid operator may be defined as follows:

$$ {}^{C}D^{\rho}_{\varsigma^{+}}\,\psi(\mathfrak{s}) = \frac{1}{\Gamma(1-\rho)} \int_{0}^{\mathfrak{s}} \left[K_1(\rho,\tau)\psi(\tau) + K_0(\rho,\tau)\psi'(\tau)\right](\mathfrak{s}-\tau)^{-\rho}\mathrm{d}\tau, $$

where $\rho \in [0,1]$ and K_1 and K_0 are functions which satisfy the following conditions:

$$ \lim_{\rho\to 0^{+}} K_0(\rho,\tau) = 0; \quad \lim_{\rho\to 1} K_0(\rho,\tau) = 1; \;\; K_0(\rho,\tau) \neq 0, \;\; \rho \in (0,1]; $$
$$ \lim_{\rho\to 0} K_1(\rho,\tau) = 0; \quad \lim_{\rho\to 1^{-}} K_1(\rho,\tau) = 1; \;\; K_1(\rho,\tau) \neq 0, \;\; \rho \in [0,1). $$

Afterwards, Sarıkaya [25] presented a novel definition by employing distinct K_1 and K_0 functions based on Definition 2. Furthermore, Sarıkaya [25] derived the Hermite–Hadamard inequality utilizing his own new definition as presented below:

Definition 3. Let $\psi\colon I \subset \mathbb{R}^{+} \to \mathbb{R}$ be a differentiable function on I° and ψ, $\psi' \in L(I)$, where $L(I)$ denote the collection of all Riemann integrable functions on I. The left-sided and right-sided proportional Caputo-hybrid operator of order ρ are defined respectively as follows:

$$ {}^{PC}_{\varsigma^{+}}D^{\rho}_{\vartheta}\,\psi(\vartheta) = \frac{1}{\Gamma(1-\rho)} \int_{\varsigma}^{\vartheta} \left[K_1(\rho,\vartheta-\tau)\psi(\tau) + K_0(\rho,\vartheta-\tau)\psi'(\tau)\right](\vartheta-\tau)^{-\rho}\mathrm{d}\tau, $$

and

$$ {}^{PC}_{\vartheta^-}D^{\rho}_{\varsigma}\,\psi(\varsigma)=\frac{1}{\Gamma(1-\rho)}\int_{\varsigma}^{\vartheta}\left[K_1(\rho,\tau-\varsigma)\psi(\tau)+K_0(\rho,\tau-\varsigma)\psi'(\tau)\right](\tau-\varsigma)^{-\rho}\mathrm{d}\tau, $$

where $\rho\in[0,1]$ and $K_0(\rho,\tau)=(1-\rho)^2\tau^{1-\rho}$ and $K_1(\rho,\tau)=\rho^2\tau^{\rho}$.

Theorem 1. *Let $\psi\colon I\subset\mathbb{R}^+\to\mathbb{R}$ be a differentiable function on I°, the interior of the interval I, where $\varsigma,\vartheta\in I^\circ$ with $\varsigma<\vartheta$ and let ψ,ψ' be convex functions on I. Then, the following inequalities hold:*

$$ \begin{aligned} &\rho^2(\vartheta-\varsigma)^{\rho}\psi\left(\frac{\varsigma+\vartheta}{2}\right)+\frac{1}{2}(1-\rho)(\vartheta-\varsigma)^{1-\rho}\psi'\left(\frac{\varsigma+\vartheta}{2}\right)\\ &\le\ \frac{\Gamma(1-\rho)}{2(\vartheta-\varsigma)^{1-\rho}}\left[{}^{PC}_{\varsigma^+}D^{\rho}_{\vartheta}\psi(\vartheta)+{}^{PC}_{\vartheta^-}D^{\rho}_{\varsigma}\psi(\varsigma)\right]\\ &\le\ \rho^2(\vartheta-\varsigma)^{\rho}\left[\frac{\psi(\varsigma)+\psi(\vartheta)}{2}\right]+(1-\rho))(\vartheta-\varsigma)^{1-\rho}\left[\frac{\psi'(\varsigma)+\psi'(\vartheta)}{4}\right]. \end{aligned} $$

In [26], Sarıkaya also gave the following Simpson's type inequality using his own definition of the proportional Caputo operator:

Lemma 1. *Let $\psi\colon I\subset\mathbb{R}^+\to\mathbb{R}$ be differantiable function on I°, the interior of the interval I, where $\varsigma,\vartheta\in I^\circ$ with $\varsigma<\vartheta$, and $\psi',\psi''\in L[\varsigma,\vartheta]$. Then, the following identity holds:*

$$ \begin{aligned} S(\varsigma,\vartheta;\rho)=&\frac{\rho^2(\vartheta-\varsigma)^{\rho}}{6}\left[\psi(\varsigma)+4\psi\left(\frac{\varsigma+\vartheta}{2}\right)+\psi(\vartheta)\right]\\ &+\frac{(1-\rho)(\vartheta-\varsigma)^{2-\rho}}{12}\left[\psi'(\varsigma)+4\psi'\left(\frac{\varsigma+\vartheta}{2}\right)+\psi'(\vartheta)\right]\\ &-\frac{\Gamma(1-\rho)}{2(\vartheta-\varsigma)^{1-\rho}}\left[{}^{PC}_{\varsigma^+}D^{\rho}_{\vartheta}\psi(\vartheta)+{}^{PC}_{\vartheta^-}D^{\rho}_{\varsigma}\psi(\varsigma)\right], \end{aligned} $$

where

$$ \begin{aligned} S(\varsigma,\vartheta;\rho)=&\frac{\rho^2(\vartheta-\varsigma)^{1+\rho}}{2}\int_0^1 P(\mathfrak{s})[\psi'(\mathfrak{s}\varsigma+(1-\mathfrak{s})\vartheta)+\psi'(\mathfrak{s}\vartheta+(1-\mathfrak{s})\varsigma)]\,\mathrm{d}\mathfrak{s}\\ &+\frac{(1-\rho)(\vartheta-\varsigma)^{2-\rho}}{4}\int_0^1 Q(\mathfrak{s})[\psi''(\mathfrak{s}\varsigma+(1-\mathfrak{s})\vartheta)+\psi''(\mathfrak{s}\vartheta+(1-\mathfrak{s})\varsigma)]\,\mathrm{d}\mathfrak{s}, \end{aligned} $$

$$ P(\mathfrak{s})=\begin{cases}\frac{1}{6}-\mathfrak{s}, & 0\le\mathfrak{s}<\frac{1}{2},\\ \frac{5}{6}-\mathfrak{s}, & \frac{1}{2}\le\mathfrak{s}\le 1,\end{cases} $$

and

$$Q(\mathfrak{s})=\begin{cases}\frac{1}{6}-\mathfrak{s}^{2-2\rho}, & 0\leq\mathfrak{s}<\frac{1}{2},\\ \\ \frac{5}{6}-\mathfrak{s}^{2-2\rho}, & \frac{1}{2}\leq\mathfrak{s}\leq 1.\end{cases}$$

The motivation of this paper is to investigate the analogous forms of the Milne-type inequalities with regard to Riemann integrals by using the proportional Caputo-hybrid operator. In line with this purpose, firstly, we present an identity with the help of the newly defined proportional Caputo-hybrid operator. Then, we establish many important Milne-type inequalities by utilizing convexity, the Hölder inequality, and the power mean inequality. Moreover, we investigate Milne-type inequalities for bounded mappings and mappings of bounded variation. Through the consideration of appropriate assumptions of ρ, these results improve and generalize the inequalities derived in earlier works.

2. Main results

Our theorems rely primarily on the following lemma. So, we will first provide the proof of this lemma.

Lemma 2. *Let $\psi\colon I\subset\mathbb{R}^{+}\to\mathbb{R}$ be a twice differentiable function on I^{o}, the interior of the interval I, where $\varsigma,\vartheta\in I^{o}$ satisfying $\varsigma<\vartheta$ and let $\psi,\psi',\psi''\in L[\varsigma,\vartheta]$. Then, the following identity is satisfied:*

$$\begin{aligned}&\rho^{2}(\vartheta-\varsigma)^{\rho+1}2^{1-2\rho}\int_{0}^{1}\left(\frac{\mathfrak{s}}{2}+\frac{1}{6}\right)\left[\psi'\left(\frac{1-\mathfrak{s}}{2}\varsigma+\frac{1+\mathfrak{s}}{2}\vartheta\right)-\psi'\left(\frac{1+\mathfrak{s}}{2}\varsigma+\frac{1-\mathfrak{s}}{2}\vartheta\right)\right]d\mathfrak{s}\\ &\quad+\frac{(1-\rho)(\vartheta-\varsigma)^{2-\rho}}{2}\\ &\quad\times\int_{0}^{1}\left(\frac{\mathfrak{s}^{2-2\rho}}{2}+\frac{1}{6}\right)\left[\psi''\left(\frac{1-\mathfrak{s}}{2}\varsigma+\frac{1+\mathfrak{s}}{2}\vartheta\right)-\psi''\left(\frac{1+\mathfrak{s}}{2}\varsigma+\frac{1-\mathfrak{s}}{2}\vartheta\right)\right]d\mathfrak{s}\\ &=\frac{\rho^{2}(\vartheta-\varsigma)^{\rho}2^{2-2\rho}}{3}\left(2\psi(\varsigma)-\psi\left(\frac{\varsigma+\vartheta}{2}\right)+2\psi(\vartheta)\right)\\ &\quad+\frac{(1-\rho)(\vartheta-\varsigma)^{1-\rho}}{3}\left(2\psi'(\varsigma)-\psi'\left(\frac{\varsigma+\vartheta}{2}\right)+2\psi'(\vartheta)\right)\\ &\quad-\frac{2^{2-2\rho}\Gamma(1-\rho)}{(\vartheta-\varsigma)^{-\rho+1}}\left[{}^{PC}_{\varsigma^{+}}D^{\rho}_{\left(\frac{\varsigma+\vartheta}{2}\right)}\psi\left(\frac{\varsigma+\vartheta}{2}\right)+{}^{PC}_{\vartheta^{-}}D^{\rho}_{\left(\frac{\varsigma+\vartheta}{2}\right)}\psi\left(\frac{\varsigma+\vartheta}{2}\right)\right].\end{aligned}\tag{2.1}$$

Proof. Through the process of integration by parts, and using the change of the variable we get

$$\int_{0}^{1}\left(\frac{\mathfrak{s}}{2}+\frac{1}{6}\right)\psi'\left(\frac{1-\mathfrak{s}}{2}\varsigma+\frac{1+\mathfrak{s}}{2}\vartheta\right)d\mathfrak{s}$$

$$
\begin{aligned}
&= \frac{4}{3(\vartheta-\varsigma)}\psi(\vartheta) - \frac{2}{6(\vartheta-\varsigma)}\psi\left(\frac{\varsigma+\vartheta}{2}\right) - \frac{1}{\vartheta-\varsigma}\int_0^1 \psi\left(\frac{1-\mathfrak{s}}{2}\varsigma + \frac{1+\mathfrak{s}}{2}\vartheta\right)\mathrm{d}\mathfrak{s} \\
&= \frac{4}{3(\vartheta-\varsigma)}\psi(\vartheta) - \frac{2}{6(\vartheta-\varsigma)}\psi\left(\frac{\varsigma+\vartheta}{2}\right) - \frac{2}{(\vartheta-\varsigma)^2}\int_{\frac{\varsigma+\vartheta}{2}}^{\vartheta} \psi(\tau)\,\mathrm{d}\tau, \qquad (2.2)
\end{aligned}
$$

$$
\begin{aligned}
&\int_0^1 \left(\frac{\mathfrak{s}}{2}+\frac{1}{6}\right)\psi'\left(\frac{1+\mathfrak{s}}{2}\varsigma + \frac{1-\mathfrak{s}}{2}\vartheta\right)\mathrm{d}\mathfrak{s} \\
&= -\frac{4}{3(\vartheta-\varsigma)}\psi(\varsigma) + \frac{2}{6(\vartheta-\varsigma)}\psi\left(\frac{\varsigma+\vartheta}{2}\right) + \frac{1}{\vartheta-\varsigma}\int_0^1 \psi\left(\frac{1+\mathfrak{s}}{2}\varsigma + \frac{1-\mathfrak{s}}{2}\vartheta\right)\mathrm{d}\mathfrak{s} \\
&= -\frac{4}{3(\vartheta-\varsigma)}\psi(\varsigma) + \frac{2}{6(\vartheta-\varsigma)}\psi\left(\frac{\varsigma+\vartheta}{2}\right) + \frac{2}{(\vartheta-\varsigma)^2}\int_{\varsigma}^{\frac{\varsigma+\vartheta}{2}} \psi(\tau)\,\mathrm{d}\tau, \\
&\int_0^1 \left(\frac{\mathfrak{s}^{2-2\rho}}{2}+\frac{1}{6}\right)\psi''\left(\frac{1-\mathfrak{s}}{2}\varsigma + \frac{1+\mathfrak{s}}{2}\vartheta\right)\mathrm{d}\mathfrak{s} \\
&= \frac{4}{3(\vartheta-\varsigma)}\psi'(\vartheta) - \frac{2}{6(\vartheta-\varsigma)}\psi'\left(\frac{\varsigma+\vartheta}{2}\right) - \frac{2-2\rho}{\vartheta-\varsigma}\int_0^1 \mathfrak{s}^{1-2\rho}\psi'\left(\frac{1-\mathfrak{s}}{2}\varsigma + \frac{1+\mathfrak{s}}{2}\vartheta\right)\mathrm{d}\mathfrak{s} \\
&= \frac{4}{3(\vartheta-\varsigma)}\psi'(\vartheta) - \frac{2}{6(\vartheta-\varsigma)}\psi'\left(\frac{\varsigma+\vartheta}{2}\right) - \frac{2^{3-2\rho}(1-\rho)}{(\vartheta-\varsigma)^{3-2\rho}}\int_{\frac{\varsigma+\vartheta}{2}}^{\vartheta}\left(\tau - \frac{\varsigma+\vartheta}{2}\right)^{1-2\rho}\psi'(\tau)\,\mathrm{d}\tau
\end{aligned} \qquad (2.3)
$$

and

$$
\begin{aligned}
&\int_0^1 \left(\frac{\mathfrak{s}^{2-2\rho}}{2}+\frac{1}{6}\right)\psi''\left(\frac{1+\mathfrak{s}}{2}\varsigma + \frac{1-\mathfrak{s}}{2}\vartheta\right)\mathrm{d}\mathfrak{s} \\
&= -\frac{4}{3(\vartheta-\varsigma)}\psi'(\varsigma) + \frac{2}{6(\vartheta-\varsigma)}\psi'\left(\frac{\varsigma+\vartheta}{2}\right) + \frac{2-2\rho}{\vartheta-\varsigma}\int_0^1 \mathfrak{s}^{1-2\rho}\psi'\left(\frac{1+\mathfrak{s}}{2}\varsigma + \frac{1-\mathfrak{s}}{2}\vartheta\right)\mathrm{d}\mathfrak{s} \\
&= -\frac{4}{3(\vartheta-\varsigma)}\psi'(\varsigma) + \frac{2}{6(\vartheta-\varsigma)}\psi'\left(\frac{\varsigma+\vartheta}{2}\right) + \frac{2^{3-2\rho}(1-\rho)}{(\vartheta-\varsigma)^{3-2\rho}}\int_{\varsigma}^{\frac{\varsigma+\vartheta}{2}}\left(\frac{\varsigma+\vartheta}{2} - \tau\right)^{1-2\rho}\psi'(\tau)\,\mathrm{d}\tau.
\end{aligned}
$$

Multiplying (2.2) by $2^{1-2\rho}\rho^2(\vartheta-\varsigma)^{1+\rho}$ and (2.3) by $\frac{(1-\rho)(\vartheta-\varsigma)^{2-\rho}}{2}$, and combining them side by side, we obtain the following result:

$$\begin{aligned}
&2^{1-2\rho}\rho^2(\vartheta-\varsigma)^{1+\rho}\int_0^1\left(\frac{\mathfrak{s}}{2}+\frac{1}{6}\right)\psi'\left(\frac{1-\mathfrak{s}}{2}\varsigma+\frac{1+\mathfrak{s}}{2}\vartheta\right)d\mathfrak{s}\\
&\quad+\frac{(1-\rho)(\vartheta-\varsigma)^{2-\rho}}{2}\int_0^1\left(\frac{\mathfrak{s}^{2-2\rho}}{2}+\frac{1}{6}\right)\psi''\left(\frac{1-\mathfrak{s}}{2}\varsigma+\frac{1+\mathfrak{s}}{2}\vartheta\right)d\mathfrak{s}\\
&=\frac{2^{3-2\rho}\rho^2(\vartheta-\varsigma)^{\rho}}{3}\psi(\vartheta)-\frac{2^{2-2\rho}\rho^2(\vartheta-\varsigma)^{\rho}}{6}\psi\left(\frac{\varsigma+\vartheta}{2}\right)\\
&\quad+\frac{2(1-\rho)(\vartheta-\varsigma)^{1-\rho}}{3}\psi'(\vartheta)-\frac{(1-\rho)(\vartheta-\varsigma)^{1-\rho}}{6}\psi'\left(\frac{\varsigma+\vartheta}{2}\right)\\
&\quad-\frac{2^{2-2\rho}}{(\vartheta-\varsigma)^{1-\rho}}\int_{\frac{\varsigma+\vartheta}{2}}^{\vartheta}\left[\rho^2\left(\tau-\frac{\varsigma+\vartheta}{2}\right)^{\rho}\psi(\tau)\right.\\
&\quad\left.+(1-\rho)^2\left(\tau-\frac{\varsigma+\vartheta}{2}\right)^{1-\rho}\psi'(\tau)\right]\left(\tau-\frac{\varsigma+\vartheta}{2}\right)^{-\rho}d\tau. \qquad (2.4)
\end{aligned}$$

By employing similar steps, the following result emerges:

$$\begin{aligned}
&2^{1-2\rho}\rho^2(\vartheta-\varsigma)^{1+\rho}\int_0^1\left(\frac{\mathfrak{s}}{2}+\frac{1}{6}\right)\psi'\left(\frac{1+\mathfrak{s}}{2}\varsigma+\frac{1-\mathfrak{s}}{2}\vartheta\right)d\mathfrak{s}\\
&\quad+\frac{(1-\rho)(\vartheta-\varsigma)^{2-\rho}}{2}\int_0^1\left(\frac{\mathfrak{s}^{2-2\rho}}{2}+\frac{1}{6}\right)\psi''\left(\frac{1+\mathfrak{s}}{2}\varsigma+\frac{1-\mathfrak{s}}{2}\vartheta\right)d\mathfrak{s}\\
&=-\frac{2^{3-2\rho}\rho^2(\vartheta-\varsigma)^{\rho}}{3}\psi(\varsigma)+\frac{2^{2-2\rho}\rho^2(\vartheta-\varsigma)^{\rho}}{6}\psi\left(\frac{\varsigma+\vartheta}{2}\right)\\
&\quad-\frac{2(1-\rho)(\vartheta-\varsigma)^{1-\rho}}{3}\psi'(\varsigma)+\frac{(1-\rho)(\vartheta-\varsigma)^{1-\rho}}{6}\psi'\left(\frac{\varsigma+\vartheta}{2}\right)\\
&\quad+\frac{2^{2-2\rho}}{(\vartheta-\varsigma)^{1-\rho}}\int_{\varsigma}^{\frac{\varsigma+\vartheta}{2}}\left[\rho^2\left(\frac{\varsigma+\vartheta}{2}-\tau\right)^{\rho}\psi(\tau)\right.\\
&\quad\left.+(1-\rho)^2\left(\frac{\varsigma+\vartheta}{2}-\tau\right)^{1-\rho}\psi'(\tau)\right]\left(\frac{\varsigma+\vartheta}{2}-\tau\right)^{-\rho}d\tau. \qquad (2.5)
\end{aligned}$$

The required equality (2.1) is obtained by applying equations (2.4) and (2.5). □

Remark 1. Letting the limit as $\rho \to 1$ in Lemma 2, it follows that

$$\frac{(\vartheta-\varsigma)}{2}\int_0^1\left(\frac{\mathfrak{s}}{2}+\frac{1}{6}\right)\left[\psi'\left(\frac{1-\mathfrak{s}}{2}\varsigma+\frac{1+\mathfrak{s}}{2}\vartheta\right)-\psi'\left(\frac{1+\mathfrak{s}}{2}\varsigma+\frac{1-\mathfrak{s}}{2}\vartheta\right)\right]d\mathfrak{s}$$
$$=\frac{1}{3}\left(2\psi(\varsigma)-\psi\left(\frac{\varsigma+\vartheta}{2}\right)+2\psi(\vartheta)\right)-\frac{1}{\vartheta-\varsigma}\int_\varsigma^\vartheta\psi(\varkappa)d\varkappa,$$

which is the same with Lemma 1 provided by Budak et al. [11] under the condition $\rho=1$.

Corollary 1. *In the limiting case* $\rho=0$ *in Lemma 2, we obtain*

$$\frac{(\vartheta-\varsigma)}{2}\int_0^1\left(\frac{\mathfrak{s}^2}{2}+\frac{1}{6}\right)\left[\psi''\left(\frac{1-\mathfrak{s}}{2}\varsigma+\frac{1+\mathfrak{s}}{2}\vartheta\right)-\psi''\left(\frac{1+\mathfrak{s}}{2}\varsigma+\frac{1-\mathfrak{s}}{2}\vartheta\right)\right]d\mathfrak{s}$$
$$=\frac{1}{3}\left(2\psi'(\varsigma)-\psi'\left(\frac{\varsigma+\vartheta}{2}\right)+2\psi'(\vartheta)\right)-\frac{2\left(\psi(\vartheta)-\psi(\varsigma)\right)}{\vartheta-\varsigma}$$
$$-\frac{4}{(\vartheta-\varsigma)^2}\left(\int_\varsigma^{\frac{\varsigma+\vartheta}{2}}\psi(\varkappa)d\varkappa-\int_{\frac{\varsigma+\vartheta}{2}}^\vartheta\psi(\varkappa)d\varkappa\right).$$

Now, we present some important Milne-type inequalities by taking into account the convexity, power mean inequality and the Hölder inequality.

Theorem 2. *Let* $\psi: I\subset\mathbb{R}^+\to\mathbb{R}$ *be a twice differentiable function on* I^o*, the interior of the interval I, where* $\varsigma,\vartheta\in I^o$ *satisfying* $\varsigma<\vartheta$ *and let* $\psi,\psi',\psi''\in L[\varsigma,\vartheta]$*. If* $|\psi'|^q$ *and* $|\psi''|^q$ *are convex on* $[\varsigma,\vartheta]$ *for* $q\geq 1$*, then the following inequality holds:*

$$\left|\frac{\rho^2(\vartheta-\varsigma)^\rho 2^{2-2\rho}}{3}\left(2\psi(\varsigma)-\psi\left(\frac{\varsigma+\vartheta}{2}\right)+2\psi(\vartheta)\right)\right.$$
$$+\frac{(1-\rho)(\vartheta-\varsigma)^{1-\rho}}{3}\left(2\psi'(\varsigma)-\psi'\left(\frac{\varsigma+\vartheta}{2}\right)+2\psi'(\vartheta)\right)$$
$$\left.-\frac{2^{2-2\rho}\Gamma(1-\rho)}{(\vartheta-\varsigma)^{-\rho+1}}\left[{}^{PC}_{\varsigma^+}D^\rho_{\left(\frac{\varsigma+\vartheta}{2}\right)}\psi\left(\frac{\varsigma+\vartheta}{2}\right)+{}^{PC}_{\vartheta^-}D^\rho_{\left(\frac{\varsigma+\vartheta}{2}\right)}\psi\left(\frac{\varsigma+\vartheta}{2}\right)\right]\right|$$
$$\leq\ \rho^2(\vartheta-\varsigma)^{\rho+1}2^{1-2\rho}\left(\frac{5}{12}\right)^{1-\frac{1}{q}}\left[\left(\frac{1}{12}\left|\psi'(\varsigma)\right|^q+\frac{1}{3}\left|\psi'(\vartheta)\right|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{1}{3}\left|\psi'(\varsigma)\right|^q+\frac{1}{12}\left|\psi'(\vartheta)\right|^q\right)^{\frac{1}{q}}\right]+\frac{(1-\rho)(\vartheta-\varsigma)^{2-\rho}}{2}\left(\frac{1}{6}+\frac{1}{2(3-2\rho)}\right)^{1-\frac{1}{q}}$$
$$\times\left[\left(A(\rho)|\psi''(\varsigma)|^q+B(\rho)|\psi''(\vartheta)|^q\right)^{\frac{1}{q}}+\left(B(\rho)|\psi''(\varsigma)|^q+A(\rho)|\psi''(\vartheta)|^q\right)^{\frac{1}{q}}\right],\quad(2.6)$$

where

$$A(\rho)=\frac{1}{24}+\frac{1}{4(3-2\rho)}-\frac{1}{4(4-2\rho)},$$

and

$$B(\rho)=\frac{1}{8}+\frac{1}{4(3-2\rho)}+\frac{1}{4(4-2\rho)}.$$

Proof. First, let us examine the case in which $q=1$. Using the convexity of $|\psi'|$ and $|\psi''|$, we deduce from Lemma 2 that

$$\begin{aligned}
&\left|\frac{\rho^2(\vartheta-\varsigma)^{\rho}2^{2-2\rho}}{3}\left(2\psi(\varsigma)-\psi\left(\frac{\varsigma+\vartheta}{2}\right)+2\psi(\vartheta)\right)\right.\\
&\quad+\frac{(1-\rho)(\vartheta-\varsigma)^{1-\rho}}{3}\left(2\psi'(\varsigma)-\psi'\left(\frac{\varsigma+\vartheta}{2}\right)+2\psi'(\vartheta)\right)\\
&\quad\left.-\frac{2^{2-2\rho}\Gamma(1-\rho)}{(\vartheta-\varsigma)^{-\rho+1}}\left[{}^{PC}_{\varsigma^+}D^{\rho}_{\left(\frac{\varsigma+\vartheta}{2}\right)}\psi\left(\frac{\varsigma+\vartheta}{2}\right)+{}^{PC}_{\vartheta^-}D^{\rho}_{\left(\frac{\varsigma+\vartheta}{2}\right)}\psi\left(\frac{\varsigma+\vartheta}{2}\right)\right]\right|\\
&\leq\ \rho^2(\vartheta-\varsigma)^{\rho+1}2^{1-2\rho}\int_0^1\left(\frac{\mathfrak{s}}{2}+\frac{1}{6}\right)\left[\left(\frac{1-\mathfrak{s}}{2}\left|\psi'(\varsigma)\right|+\frac{1+\mathfrak{s}}{2}\left|\psi'(\vartheta)\right|\right)\right.\\
&\quad\left.+\left(\frac{1+\mathfrak{s}}{2}\left|\psi'(\varsigma)\right|+\frac{1-\mathfrak{s}}{2}\left|\psi'(\vartheta)\right|\right)\right]d\mathfrak{s}\\
&\quad+\frac{(1-\rho)(\vartheta-\varsigma)^{2-\rho}}{2}\int_0^1\left(\frac{\mathfrak{s}^{2-2\rho}}{2}+\frac{1}{6}\right)\left[\left(\frac{1-\mathfrak{s}}{2}\left|\psi''(\varsigma)\right|+\frac{1+\mathfrak{s}}{2}\left|\psi''(\vartheta)\right|\right)\right.\\
&\quad\left.+\left(\frac{1+\mathfrak{s}}{2}\left|\psi''(\varsigma)\right|+\frac{1-\mathfrak{s}}{2}\left|\psi''(\vartheta)\right|\right)\right]d\mathfrak{s}. \qquad (2.7)
\end{aligned}$$

Hence, from the fact that

$$\int_0^1\left(\frac{\mathfrak{s}}{2}+\frac{1}{6}\right)\frac{1-\mathfrak{s}}{2}d\mathfrak{s}=\frac{1}{12},\qquad\int_0^1\left(\frac{\mathfrak{s}}{2}+\frac{1}{6}\right)\frac{1+\mathfrak{s}}{2}d\mathfrak{s}=\frac{1}{3},$$

$$\int_0^1\left(\frac{\mathfrak{s}^{2-2\rho}}{2}+\frac{1}{6}\right)\frac{1-\mathfrak{s}}{2}d\mathfrak{s}=\frac{1}{24}+\frac{1}{4(3-2\rho)}-\frac{1}{4(4-2\rho)},$$

$$\int_0^1\left(\frac{\mathfrak{s}^{2-2\rho}}{2}+\frac{1}{6}\right)\frac{1+\mathfrak{s}}{2}d\mathfrak{s}=\frac{1}{8}+\frac{1}{4(3-2\rho)}+\frac{1}{4(4-2\rho)},$$

it follows that the expression on the right-hand side of inequality (2.7) is

$$\frac{5\rho^2(\vartheta-\varsigma)^{\rho+1}2^{1-2\rho}}{12}\left(\left|\psi'(\varsigma)\right|+\left|\psi'(\vartheta)\right|\right)$$

$$+\frac{(1-\rho)(\vartheta-\varsigma)^{2-\rho}}{2}\left(\frac{1}{6}+\frac{1}{2(3-2\rho)}\right)\left(\left|\psi''(\varsigma)\right|+\left|\psi''(\vartheta)\right|\right).$$

For $q\geq 1$, we can now achieve the following by using Lemma 2, the power mean inequality, and accounting for the convexity of $|\psi'|^q$ and $|\psi''|^q$:

$$\begin{aligned}
&\left|\frac{\rho^2(\vartheta-\varsigma)^{\rho}2^{2-2\rho}}{3}\left(2\psi(\varsigma)-\psi\left(\frac{\varsigma+\vartheta}{2}\right)+2\psi(\vartheta)\right)\right.\\
&\quad+\frac{(1-\rho)(\vartheta-\varsigma)^{1-\rho}}{3}\left(2\psi'(\varsigma)-\psi'\left(\frac{\varsigma+\vartheta}{2}\right)+2\psi'(\vartheta)\right)\\
&\quad\left.-\frac{2^{2-2\rho}\Gamma(1-\rho)}{(\vartheta-\varsigma)^{-\rho+1}}\left[{}^{PC}_{\varsigma^+}D^{\rho}_{\left(\frac{\varsigma+\vartheta}{2}\right)}\psi\left(\frac{\varsigma+\vartheta}{2}\right)+{}^{PC}_{\vartheta^-}D^{\rho}_{\left(\frac{\varsigma+\vartheta}{2}\right)}\psi\left(\frac{\varsigma+\vartheta}{2}\right)\right]\right|\\
&\leq \rho^2(\vartheta-\varsigma)^{\rho+1}2^{1-2\rho}\\
&\quad\times\left\{\left(\int_0^1\left(\frac{\mathfrak{s}}{2}+\frac{1}{6}\right)d\mathfrak{s}\right)^{\frac{1}{p}}\left(\int_0^1\left(\frac{\mathfrak{s}}{2}+\frac{1}{6}\right)\left[\frac{1-\mathfrak{s}}{2}|\psi'(\varsigma)|^q+\frac{1+\mathfrak{s}}{2}|\psi'(\vartheta)|^q\right]d\mathfrak{s}\right)^{\frac{1}{q}}\right.\\
&\quad\left.+\left(\int_0^1\left(\frac{\mathfrak{s}}{2}+\frac{1}{6}\right)d\mathfrak{s}\right)^{\frac{1}{p}}\left(\int_0^1\left(\frac{\mathfrak{s}}{2}+\frac{1}{6}\right)\left[\frac{1+\mathfrak{s}}{2}|\psi'(\varsigma)|^q+\frac{1-\mathfrak{s}}{2}|\psi'(\vartheta)|^q\right]d\mathfrak{s}\right)^{\frac{1}{q}}\right\}\\
&\quad+\frac{(1-\rho)(\vartheta-\varsigma)^{2-\rho}}{2}\left\{\left(\int_0^1\left(\frac{\mathfrak{s}^{2-2\rho}}{2}+\frac{1}{6}\right)d\mathfrak{s}\right)^{\frac{1}{p}}\right.\\
&\quad\times\left(\int_0^1\left(\frac{\mathfrak{s}^{2-2\rho}}{2}+\frac{1}{6}\right)\left[\frac{1-\mathfrak{s}}{2}|\psi''(\varsigma)|^q+\frac{1+\mathfrak{s}}{2}|\psi''(\vartheta)|^q\right]d\mathfrak{s}\right)^{\frac{1}{q}}\\
&\quad\left.+\left(\int_0^1\left(\frac{\mathfrak{s}^{2-2\rho}}{2}+\frac{1}{6}\right)d\mathfrak{s}\right)^{\frac{1}{p}}\left(\int_0^1\left(\frac{\mathfrak{s}^{2-2\rho}}{2}+\frac{1}{6}\right)\left[\frac{1+\mathfrak{s}}{2}|\psi''(\varsigma)|^q+\frac{1-\mathfrak{s}}{2}|\psi''(\vartheta)|^q\right]d\mathfrak{s}\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

Consequently, it follows that inequality (2.6) is true since

$$\int_0^1\left(\frac{\mathfrak{s}}{2}+\frac{1}{6}\right)d\mathfrak{s}=\frac{5}{12},\quad\int_0^1\left(\frac{\mathfrak{s}^{2-2\rho}}{2}+\frac{1}{6}\right)d\mathfrak{s}=\frac{1}{6}+\frac{1}{2(3-2\rho)}.$$

□

Remark 2. Letting the limit as $\rho\to 1$ and putting $q=1$ in Theorem 2, it follows that

$$\left|\frac{1}{3}\left(2\psi(\varsigma)-\psi\left(\frac{\varsigma+\vartheta}{2}\right)+2\psi(\vartheta)\right)-\frac{1}{\vartheta-\varsigma}\int_{\varsigma}^{\vartheta}\psi(\varkappa)d\varkappa\right|\leq\frac{5(\vartheta-\varsigma)}{24}\left(|\psi'(\varsigma)|+|\psi'(\vartheta)|\right),$$

which was proved by Budak et al. in [11]. Also, when ρ converges to 1 and with $q \geq 1$, the inequality described in Theorem 2 is given by

$$\left|\frac{1}{3}\left(2\psi(\varsigma)-\psi\left(\frac{\varsigma+\vartheta}{2}\right)+2\psi(\vartheta)\right)-\frac{1}{\vartheta-\varsigma}\int_{\varsigma}^{\vartheta}\psi(\varkappa)\mathrm{d}\varkappa\right|$$
$$\leq \frac{5(\vartheta-\varsigma)}{24}\left[\left(\frac{|\psi'(\varsigma)|^q+4|\psi'(\vartheta)|^q}{5}\right)^{1/q}+\left(\frac{4|\psi'(\varsigma)|^q+|\psi'(\vartheta)|^q}{5}\right)^{1/q}\right],$$

which was proved by Budak et al. in [11].

Corollary 2. *As* ρ *approaches* 0 *and for* $q \geq 1$ *in Theorem 2, we obtain*

$$\left|\frac{1}{3}\left(2\psi'(\varsigma)-\psi'\left(\frac{\varsigma+\vartheta}{2}\right)+2\psi'(\vartheta)\right)-\frac{2(\psi(\vartheta)-\psi(\varsigma))}{\vartheta-\varsigma}\right.$$
$$\left.-\frac{4}{(\vartheta-\varsigma)^2}\left(\int_{\varsigma}^{\frac{\varsigma+\vartheta}{2}}\psi(\varkappa)\mathrm{d}\varkappa-\int_{\frac{\varsigma+\vartheta}{2}}^{\vartheta}\psi(\varkappa)\mathrm{d}\varkappa\right)\right|$$
$$\leq \frac{(\vartheta-\varsigma)}{6}\left[\left(\frac{9}{48}|\psi''(\varsigma)|^q+\frac{39}{48}|\psi''(\vartheta)|^q\right)^{1/q}+\left(\frac{39}{48}|\psi''(\varsigma)|^q+\frac{9}{48}|\psi''(\vartheta)|^q\right)^{1/q}\right].$$

We present a concrete example to verify the validity of our theorem.

Example 1. Consider the function $\psi(\varkappa)=\varkappa^3$ defined on the interval $[0,2]$. Then, we can determine the right-hand side of inequality (2.6) for $q=1$ as follows:

$$5\rho^2 2^{2-\rho}+6(1-\rho)2^{2-\rho}\left(\frac{1}{6}+\frac{1}{2(3-2\rho)}\right):=\Psi_1.$$

Furthermore, it can be observed

$$\left|\frac{\rho^2(\vartheta-\varsigma)^\rho 2^{2-2\rho}}{3}\left(2\psi(\varsigma)-\psi\left(\frac{\varsigma+\vartheta}{2}\right)+2\psi(\vartheta)\right)\right.$$
$$+\frac{(1-\rho)(\vartheta-\varsigma)^{1-\rho}}{3}\left(2\psi'(\varsigma)-\psi'\left(\frac{\varsigma+\vartheta}{2}\right)+2\psi'(\vartheta)\right)$$
$$\left.-\frac{2^{2-2\rho}\Gamma(1-\rho)}{(\vartheta-\varsigma)^{-\rho+1}}\left[{}^{PC}_{\varsigma^+}D^{\rho}_{\left(\frac{\varsigma+\vartheta}{2}\right)}\psi\left(\frac{\varsigma+\vartheta}{2}\right)+{}^{PC}_{\vartheta^-}D^{\rho}_{\left(\frac{\varsigma+\vartheta}{2}\right)}\psi\left(\frac{\varsigma+\vartheta}{2}\right)\right]\right|$$
$$=2^{1-\rho}\left(6\rho^2+7-7\rho-\frac{3(1-\rho)(3-2\rho)}{\rho-2}\right):=\Psi_2.$$

Therefore, for all $0<\rho<1$ and $q=1$, the left side of the inequality (2.6) is constantly located below the right side of this inequality, as shown in Figure 1.

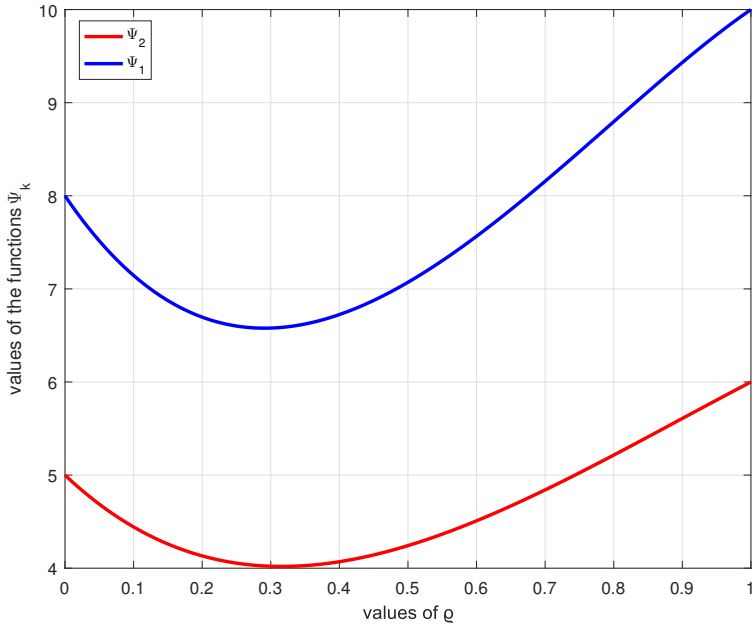

FIGURE 1. The graph of both sides of the inequality (2.6) depending on $0<\varrho<1$ and $q=1$, which is computed and drawn in MATLAB program.

Theorem 3. *Let $\psi: I\subset\mathbb{R}^+\to\mathbb{R}$ be a twice differentiable function on I^o, the interior of the interval I, where $\varsigma,\vartheta\in I^o$ satisfying $\varsigma<\vartheta$ and let $\psi,\psi',\psi''\in L[\varsigma,\vartheta]$. If $|\psi'|^q$ and $|\psi''|^q$ are convex on $[\varsigma,\vartheta]$ for $q>1$, then the following inequalities hold:*

$$\begin{aligned}
&\left|\frac{\varrho^2(\vartheta-\varsigma)^{\varrho}2^{2-2\varrho}}{3}\left(2\psi(\varsigma)-\psi\left(\frac{\varsigma+\vartheta}{2}\right)+2\psi(\vartheta)\right)\right.\\
&\quad+\frac{(1-\varrho)(\vartheta-\varsigma)^{1-\varrho}}{3}\left(2\psi'(\varsigma)-\psi'\left(\frac{\varsigma+\vartheta}{2}\right)+2\psi'(\vartheta)\right)\\
&\quad\left.-\frac{2^{2-2\varrho}\Gamma(1-\varrho)}{(\vartheta-\varsigma)^{-\varrho+1}}\left[{}^{PC}_{\varsigma^+}D^{\varrho}_{\left(\frac{\varsigma+\vartheta}{2}\right)}\psi\left(\frac{\varsigma+\vartheta}{2}\right)+{}^{PC}_{\vartheta^-}D^{\varrho}_{\left(\frac{\varsigma+\vartheta}{2}\right)}\psi\left(\frac{\varsigma+\vartheta}{2}\right)\right]\right|\\
&\leq\frac{\varrho^2(\vartheta-\varsigma)^{\varrho+1}2^{1-2\varrho}}{6}\left(\frac{2^{2p+3}-2}{6p+6}\right)^{\frac{1}{p}}\left[\left(\frac{3\left|\psi'(\varsigma)\right|^q+\left|\psi'(\vartheta)\right|^q}{4}\right)^{\frac{1}{q}}\right.\\
&\quad\left.+\left(\frac{\left|\psi'(\varsigma)\right|^q+3\left|\psi'(\vartheta)\right|^q}{4}\right)^{\frac{1}{q}}\right]+\frac{(1-\varrho)(\vartheta-\varsigma)^{2-\varrho}}{2}\left(\int_0^1\left(\frac{\mathfrak{s}^{2-2\varrho}}{2}+\frac{1}{6}\right)^p d\mathfrak{s}\right)^{\frac{1}{p}}
\end{aligned}$$

$$\times\left[\left(\frac{3\left|\psi''(\varsigma)\right|^q+\left|\psi''(\vartheta)\right|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{\left|\psi''(\varsigma)\right|^q+3\left|\psi''(\vartheta)\right|^q}{4}\right)^{\frac{1}{q}}\right]$$

$$\leq\frac{\rho^2(\vartheta-\varsigma)^{\rho+1}2^{1-2\rho}}{6}\left(\frac{2^{2p+5}-2^3}{6p+6}\right)^{\frac{1}{p}}\left(\left|\psi'(\varsigma)\right|+\left|\psi'(\vartheta)\right|\right)$$

$$+\frac{(1-\rho)(\vartheta-\varsigma)^{2-\rho}}{2}\left(4\int_0^1\left(\frac{\mathfrak{s}^{2-2\rho}}{2}+\frac{1}{6}\right)^p d\mathfrak{s}\right)^{\frac{1}{p}}\left(\left|\psi''(\varsigma)\right|+\left|\psi''(\vartheta)\right|\right),$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. By applying the well-known Hölder's inequality and taking into account the convexity of $|\psi'|^q$ and $|\psi''|^q$, based on Lemma 2, we obtain

$$\begin{aligned}
&\left|\frac{\rho^2(\vartheta-\varsigma)^{\rho}2^{2-2\rho}}{3}\left(2\psi(\varsigma)-\psi\left(\frac{\varsigma+\vartheta}{2}\right)+2\psi(\vartheta)\right)\right.\\
&\quad+\frac{(1-\rho)(\vartheta-\varsigma)^{1-\rho}}{3}\left(2\psi'(\varsigma)-\psi'\left(\frac{\varsigma+\vartheta}{2}\right)+2\psi'(\vartheta)\right)\\
&\quad\left.-\frac{2^{2-2\rho}\Gamma(1-\rho)}{(\vartheta-\varsigma)^{-\rho+1}}\left[{}^{PC}_{\varsigma^+}D^{\rho}_{\left(\frac{\varsigma+\vartheta}{2}\right)}\psi\left(\frac{\varsigma+\vartheta}{2}\right)+{}^{PC}_{\vartheta^-}D^{\rho}_{\left(\frac{\varsigma+\vartheta}{2}\right)}\psi\left(\frac{\varsigma+\vartheta}{2}\right)\right]\right|\\
&\leq\rho^2(\vartheta-\varsigma)^{1+\rho}2^{1-2\rho}\left(\int_0^1\left(\frac{\mathfrak{s}}{2}+\frac{1}{6}\right)^p d\mathfrak{s}\right)^{\frac{1}{p}}\\
&\quad\times\left[\left(\int_0^1\left|\psi'\left(\frac{1-\mathfrak{s}}{2}\varsigma+\frac{1+\mathfrak{s}}{2}\vartheta\right)\right|^q d\mathfrak{s}\right)^{\frac{1}{q}}+\left(\int_0^1\left|\psi'\left(\frac{1+\mathfrak{s}}{2}\varsigma+\frac{1-\mathfrak{s}}{2}\vartheta\right)\right|^q d\mathfrak{s}\right)^{\frac{1}{q}}\right]\\
&\quad+\frac{(1-\rho)(\vartheta-\varsigma)^{2-\rho}}{2}\left(\int_0^1\left(\frac{\mathfrak{s}^{2-2\rho}}{2}+\frac{1}{6}\right)^p d\mathfrak{s}\right)^{\frac{1}{p}}\\
&\quad\times\left[\left(\int_0^1\left|\psi''\left(\frac{1-\mathfrak{s}}{2}\varsigma+\frac{1+\mathfrak{s}}{2}\vartheta\right)\right|^q d\mathfrak{s}\right)^{\frac{1}{q}}+\left(\int_0^1\left|\psi''\left(\frac{1+\mathfrak{s}}{2}\varsigma+\frac{1-\mathfrak{s}}{2}\vartheta\right)\right|^q d\mathfrak{s}\right)^{\frac{1}{q}}\right].
\end{aligned}\tag{2.8}$$

When we assess the integrals in the aforementioned inequality, we arrive at

$$\int_0^1\left(\frac{\mathfrak{s}}{2}+\frac{1}{6}\right)^p d\mathfrak{s}=\frac{2^{2p+3}-2}{6^{p+1}(p+1)},$$

$$\int_0^1 \left[\frac{1-\mathfrak{s}}{2}\left|\psi'(\varsigma)\right|^q+\frac{1+\mathfrak{s}}{2}\left|\psi'(\vartheta)\right|^q\right]\mathrm{d}\mathfrak{s}=\frac{|\psi'(\varsigma)|^q+3|\psi'(\vartheta)|^q}{4},$$

$$\int_0^1 \left[\frac{1+\mathfrak{s}}{2}\left|\psi'(\varsigma)\right|^q+\frac{1-\mathfrak{s}}{2}\left|\psi'(\vartheta)\right|^q\right]\mathrm{d}\mathfrak{s}=\frac{3|\psi'(\varsigma)|^q+|\psi'(\vartheta)|^q}{4}.$$

$$\int_0^1 \left[\frac{1-\mathfrak{s}}{2}\left|\psi''(\varsigma)\right|^q+\frac{1+\mathfrak{s}}{2}\left|\psi''(\vartheta)\right|^q\right]\mathrm{d}\mathfrak{s}=\frac{|\psi''(\varsigma)|^q+3|\psi''(\vartheta)|^q}{4},$$

$$\int_0^1 \left[\frac{1+\mathfrak{s}}{2}\left|\psi''(\varsigma)\right|^q+\frac{1-\mathfrak{s}}{2}\left|\psi''(\vartheta)\right|^q\right]\mathrm{d}\mathfrak{s}=\frac{3|\psi''(\varsigma)|^q+|\psi''(\vartheta)|^q}{4}.$$

Thus, the first inequality of (2.8) can be obtained by replacing the calculated integral results. Also, it is known that we get the property

$$\sum_{k=1}^{n}(\varsigma_k+\vartheta_k)^s\leq\sum_{k=1}^{n}\varsigma_k^s+\sum_{k=1}^{n}\vartheta_k^s,$$

for $0\leq s<1$ and $\varsigma_k,\ \vartheta_k\geq 0$ with $k\in\{1,2,\ldots,n\}$. Therefore, for the proof of the second inequality, if we take $\varsigma_1=|\psi'(\varsigma)|^q$, $\vartheta_1=3\,|\psi'(\vartheta)|^q$, $\varsigma_2=3\,|\psi'(\varsigma)|^q$, $\vartheta_2=|\psi'(\vartheta)|^q$, $\varsigma_1'=|\psi''(\varsigma)|^q$, $\vartheta_1'=3\,|\psi''(\vartheta)|^q$, $\varsigma_2'=3\,|\psi''(\varsigma)|^q$, $\vartheta_2'=|\psi''(\vartheta)|^q$, by $1+3^{\frac{1}{q}}\leq 4$, the desired outcome can be achieved straightforwardly. □

Remark 3. In the particular case when ρ tends to 1 in Theorem 3, we get

$$\left|\frac{1}{3}\left(2\psi(\varsigma)-\psi\left(\frac{\varsigma+\vartheta}{2}\right)+2\psi(\vartheta)\right)-\frac{1}{\vartheta-\varsigma}\int_{\varsigma}^{\vartheta}\psi(\varkappa)\mathrm{d}\varkappa\right|$$

$$\leq\frac{\vartheta-\varsigma}{12}\left(\frac{2^{2p+2}-1}{3p+3}\right)^{\frac{1}{p}}\left[\left(\frac{3\,|\psi'(\varsigma)|^q+|\psi'(\vartheta)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{|\psi'(\varsigma)|^q+3\,|\psi'(\vartheta)|^q}{4}\right)^{\frac{1}{q}}\right]$$

$$\leq\frac{\vartheta-\varsigma}{12}\left(\frac{2^{2p+4}-4}{3p+3}\right)^{\frac{1}{p}}\left(\left|\psi'(\varsigma)\right|+\left|\psi'(\vartheta)\right|\right),$$

which was proved by Budak et al. in [11].

Corollary 3. *According to Theorem 3, as* ρ *approaches* 0*, we have the following particular case:*

$$\left|\frac{1}{3}\left(2\psi'(\varsigma)-\psi'\left(\frac{\varsigma+\vartheta}{2}\right)+2\psi'(\vartheta)\right)-\frac{2\left(\psi(\vartheta)-\psi(\varsigma)\right)}{\vartheta-\varsigma}\right.$$

$$
\begin{aligned}
&-\frac{4}{(\vartheta-\varsigma)^2}\left(\int_{\varsigma}^{\frac{\varsigma+\vartheta}{2}}\psi(\varkappa)\mathrm{d}\varkappa-\int_{\frac{\varsigma+\vartheta}{2}}^{\vartheta}\psi(\varkappa)\mathrm{d}\varkappa\right)\Bigg| \\
&\leq \frac{(\vartheta-\varsigma)}{2}\left(\int_0^1\left(\frac{\mathfrak{s}^2}{2}+\frac{1}{6}\right)^p\mathrm{d}\mathfrak{s}\right)^{\frac{1}{p}} \\
&\quad\cdot\left[\left(\frac{|\psi''(\varsigma)|^q+3|\psi''(\vartheta)|^q}{4}\right)^{1/q}+\left(\frac{3|\psi''(\varsigma)|^q+|\psi''(\vartheta)|^q}{4}\right)^{1/q}\right] \\
&\leq \frac{(\vartheta-\varsigma)}{2}\left(4\int_0^1\left(\frac{\mathfrak{s}^2}{2}+\frac{1}{6}\right)^p\mathrm{d}\mathfrak{s}\right)^{\frac{1}{p}}\left(\left|\psi''(\varsigma)\right|+\left|\psi''(\vartheta)\right|\right).
\end{aligned}
$$

3. Milne-type inequalities for bounded mappings involving proportional Caputo-hybrid operator

Our task in this section is to obtain some Milne-type inequalities for bounded mappings with the help of a proportional Caputo-hybrid operator.

Theorem 4. *Suppose that the conditions of Lemma 2 hold. If there exist $m,n,M,N\in\mathbb{R}$ such that $m\leq\psi'(\mathfrak{s})\leq M$ and $n\leq\psi''(\mathfrak{s})\leq N$ for all $\mathfrak{s}\in[\varsigma,\vartheta]$, then we have*

$$
\begin{aligned}
&\left|\frac{\rho^2(\vartheta-\varsigma)^\rho 2^{2-2\rho}}{3}\left(2\psi(\varsigma)-\psi\left(\frac{\varsigma+\vartheta}{2}\right)+2\psi(\vartheta)\right)\right. \\
&\quad+\frac{(1-\rho)(\vartheta-\varsigma)^{1-\rho}}{3}\left(2\psi'(\varsigma)-\psi'\left(\frac{\varsigma+\vartheta}{2}\right)+2\psi'(\vartheta)\right) \\
&\quad\left.-\frac{2^{2-2\rho}\Gamma(1-\rho)}{(\vartheta-\varsigma)^{-\rho+1}}\left[{}^{PC}_{\varsigma^+}D^\rho_{\left(\frac{\varsigma+\vartheta}{2}\right)}\psi\left(\frac{\varsigma+\vartheta}{2}\right)+{}^{PC}_{\vartheta^-}D^\rho_{\left(\frac{\varsigma+\vartheta}{2}\right)}\psi\left(\frac{\varsigma+\vartheta}{2}\right)\right]\right| \\
&\leq \frac{5\rho^2(\vartheta-\varsigma)^{\rho+1}2^{1-2\rho}}{12}(M-m)+\frac{(1-\rho)(\vartheta-\varsigma)^{2-\rho}}{2}\left(\frac{1}{2(3-2\rho)}+\frac{1}{6}\right)(N-n).
\end{aligned}
$$

Proof. In the identity in Lemma 2, adding and subtracting $\frac{m+M}{2}$ at the point where the ψ' functions appear within the integral, and similarly adding and subtracting $\frac{n+N}{2}$ at the point where the ψ'' functions appear, support us in obtaining

$$
\begin{aligned}
&\frac{\rho^2(\vartheta-\varsigma)^\rho 2^{2-2\rho}}{3}\left(2\psi(\varsigma)-\psi\left(\frac{\varsigma+\vartheta}{2}\right)+2\psi(\vartheta)\right) \\
&\quad+\frac{(1-\rho)(\vartheta-\varsigma)^{1-\rho}}{3}\left(2\psi'(\varsigma)-\psi'\left(\frac{\varsigma+\vartheta}{2}\right)+2\psi'(\vartheta)\right) \\
&\quad-\frac{2^{2-2\rho}\Gamma(1-\rho)}{(\vartheta-\varsigma)^{-\rho+1}}\left[{}^{PC}_{\varsigma^+}D^\rho_{\left(\frac{\varsigma+\vartheta}{2}\right)}\psi\left(\frac{\varsigma+\vartheta}{2}\right)+{}^{PC}_{\vartheta^-}D^\rho_{\left(\frac{\varsigma+\vartheta}{2}\right)}\psi\left(\frac{\varsigma+\vartheta}{2}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \rho^2(\vartheta-\varsigma)^{\rho+1}2^{1-2\rho}\left\{\int_0^1\left(\frac{\mathfrak{s}}{2}+\frac{1}{6}\right)\left[\psi'\left(\frac{1-\mathfrak{s}}{2}\varsigma+\frac{1+\mathfrak{s}}{2}\vartheta\right)-\frac{m+M}{2}\right]d\mathfrak{s}\right.\\
&\quad\left.+\int_0^1\left(\frac{\mathfrak{s}}{2}+\frac{1}{6}\right)\left[\frac{m+M}{2}-\psi'\left(\frac{1+\mathfrak{s}}{2}\varsigma+\frac{1-\mathfrak{s}}{2}\vartheta\right)\right]d\mathfrak{s}\right\}\\
&\quad+\frac{(1-\rho)(\vartheta-\varsigma)^{2-\rho}}{2}\left\{\int_0^1\left(\frac{\mathfrak{s}^{2-2\rho}}{2}+\frac{1}{6}\right)\left[\psi''\left(\frac{1-\mathfrak{s}}{2}\varsigma+\frac{1+\mathfrak{s}}{2}\vartheta\right)-\frac{n+N}{2}\right]d\mathfrak{s}\right.\\
&\quad\left.+\int_0^1\left(\frac{\mathfrak{s}^{2-2\rho}}{2}+\frac{1}{6}\right)\left[\frac{n+N}{2}-\psi''\left(\frac{1+\mathfrak{s}}{2}\varsigma+\frac{1-\mathfrak{s}}{2}\vartheta\right)\right]d\mathfrak{s}\right\}. \qquad (3.1)
\end{aligned}
$$

After determining absolute value of (3.1), we get

$$
\begin{aligned}
&\left|\frac{\rho^2(\vartheta-\varsigma)^{\rho}2^{2-2\rho}}{3}\left(2\psi(\varsigma)-\psi\left(\frac{\varsigma+\vartheta}{2}\right)+2\psi(\vartheta)\right)\right.\\
&\quad+\frac{(1-\rho)(\vartheta-\varsigma)^{1-\rho}}{3}\left(2\psi'(\varsigma)-\psi'\left(\frac{\varsigma+\vartheta}{2}\right)+2\psi'(\vartheta)\right)\\
&\quad\left.-\frac{2^{2-2\rho}\Gamma(1-\rho)}{(\vartheta-\varsigma)^{-\rho+1}}\left[{}^{PC}_{\varsigma^+}D^{\rho}_{\left(\frac{\varsigma+\vartheta}{2}\right)}\psi\left(\frac{\varsigma+\vartheta}{2}\right)+{}^{PC}_{\vartheta^-}D^{\rho}_{\left(\frac{\varsigma+\vartheta}{2}\right)}\psi\left(\frac{\varsigma+\vartheta}{2}\right)\right]\right|\\
&\le\rho^2(\vartheta-\varsigma)^{\rho+1}2^{1-2\rho}\left\{\int_0^1\left(\frac{\mathfrak{s}}{2}+\frac{1}{6}\right)\left|\psi'\left(\frac{1-\mathfrak{s}}{2}\varsigma+\frac{1+\mathfrak{s}}{2}\vartheta\right)-\frac{m+M}{2}\right|d\mathfrak{s}\right.\\
&\quad\left.+\int_0^1\left(\frac{\mathfrak{s}}{2}+\frac{1}{6}\right)\left|\frac{m+M}{2}-\psi'\left(\frac{1+\mathfrak{s}}{2}\varsigma+\frac{1-\mathfrak{s}}{2}\vartheta\right)\right|d\mathfrak{s}\right\}\\
&\quad+\frac{(1-\rho)(\vartheta-\varsigma)^{2-\rho}}{2}\left\{\int_0^1\left(\frac{\mathfrak{s}^{2-2\rho}}{2}+\frac{1}{6}\right)\left|\psi''\left(\frac{1-\mathfrak{s}}{2}\varsigma+\frac{1+\mathfrak{s}}{2}\vartheta\right)-\frac{n+N}{2}\right|d\mathfrak{s}\right.\\
&\quad\left.+\int_0^1\left(\frac{\mathfrak{s}^{2-2\rho}}{2}+\frac{1}{6}\right)\left|\frac{n+N}{2}-\psi''\left(\frac{1+\mathfrak{s}}{2}\varsigma+\frac{1-\mathfrak{s}}{2}\vartheta\right)\right|d\mathfrak{s}\right\}.
\end{aligned}
$$

Because of $m\le\psi'(\mathfrak{s})\le M$ and $n\le\psi''(\mathfrak{s})\le N$ for all $\mathfrak{s}\in[\varsigma,\vartheta]$, we obtain

$$
\begin{aligned}
\left|\psi'\left(\frac{1-\mathfrak{s}}{2}\varsigma+\frac{1+\mathfrak{s}}{2}\vartheta\right)-\frac{m+M}{2}\right|&\le\frac{M-m}{2},\\
\left|\frac{m+M}{2}-\psi'\left(\frac{1+\mathfrak{s}}{2}\varsigma+\frac{1-\mathfrak{s}}{2}\vartheta\right)\right|&\le\frac{M-m}{2},
\end{aligned}
$$

and

$$\left|\psi''\left(\frac{1-\mathfrak{s}}{2}\varsigma+\frac{1+\mathfrak{s}}{2}\vartheta\right)-\frac{n+N}{2}\right|\leq\frac{N-n}{2},$$
$$\left|\frac{n+N}{2}-\psi''\left(\frac{1+\mathfrak{s}}{2}\varsigma+\frac{1-\mathfrak{s}}{2}\vartheta\right)\right|\leq\frac{N-n}{2}.$$

Thus, we have

$$\begin{aligned}&\left|\frac{\rho^2(\vartheta-\varsigma)^\rho 2^{2-2\rho}}{3}\left(2\psi(\varsigma)-\psi\left(\frac{\varsigma+\vartheta}{2}\right)+2\psi(\vartheta)\right)\right.\\&\quad+\frac{(1-\rho)(\vartheta-\varsigma)^{1-\rho}}{3}\left(2\psi'(\varsigma)-\psi'\left(\frac{\varsigma+\vartheta}{2}\right)+2\psi'(\vartheta)\right)\\&\quad\left.-\frac{2^{2-2\rho}\Gamma(1-\rho)}{(\vartheta-\varsigma)^{-\rho+1}}\left[{}^{PC}_{\varsigma^+}D^\rho_{\left(\frac{\varsigma+\vartheta}{2}\right)}\psi\left(\frac{\varsigma+\vartheta}{2}\right)+{}^{PC}_{\vartheta^-}D^\rho_{\left(\frac{\varsigma+\vartheta}{2}\right)}\psi\left(\frac{\varsigma+\vartheta}{2}\right)\right]\right|\\&\leq\frac{5\rho^2(\vartheta-\varsigma)^{\rho+1}2^{1-2\rho}}{12}(M-m)+\frac{(1-\rho)(\vartheta-\varsigma)^{2-\rho}}{2}\left(\frac{1}{2(3-2\rho)}+\frac{1}{6}\right)(N-n).\end{aligned}$$

□

Corollary 4. *As* ρ *tends to* 0 *in Theorem 4, we obtain the following specific case:*

$$\begin{aligned}&\left|\frac{1}{3}\left(2\psi'(\varsigma)-\psi'\left(\frac{\varsigma+\vartheta}{2}\right)+2\psi'(\vartheta)\right)-\frac{2\left(\psi(\vartheta)-\psi(\varsigma)\right)}{\vartheta-\varsigma}\right.\\&\quad\left.-\frac{4}{(\vartheta-\varsigma)^2}\left(\int_\varsigma^{\frac{\varsigma+\vartheta}{2}}\psi(\varkappa)\mathrm{d}\varkappa-\int_{\frac{\varsigma+\vartheta}{2}}^{\vartheta}\psi(\varkappa)\mathrm{d}\varkappa\right)\right|\leq\frac{\vartheta-\varsigma}{6}(N-n).\end{aligned}$$

Corollary 5. *Under the assumptions of Theorem 4, if there exist* $M,N\in\mathbb{R}^+$ *such that* $|\psi'(\mathfrak{s})|\leq M$ *and* $|\psi''(\mathfrak{s})|\leq N$ *for all* $\mathfrak{s}\in[\varsigma,\vartheta]$*, then we have*

$$\begin{aligned}&\left|\frac{\rho^2(\vartheta-\varsigma)^\rho 2^{2-2\rho}}{3}\left(2\psi(\varsigma)-\psi\left(\frac{\varsigma+\vartheta}{2}\right)+2\psi(\vartheta)\right)\right.\\&\quad+\frac{(1-\rho)(\vartheta-\varsigma)^{1-\rho}}{3}\left(2\psi'(\varsigma)-\psi'\left(\frac{\varsigma+\vartheta}{2}\right)+2\psi'(\vartheta)\right)\\&\quad\left.-\frac{2^{2-2\rho}\Gamma(1-\rho)}{(\vartheta-\varsigma)^{-\rho+1}}\left[{}^{PC}_{\varsigma^+}D^\rho_{\left(\frac{\varsigma+\vartheta}{2}\right)}\psi\left(\frac{\varsigma+\vartheta}{2}\right)+{}^{PC}_{\vartheta^-}D^\rho_{\left(\frac{\varsigma+\vartheta}{2}\right)}\psi\left(\frac{\varsigma+\vartheta}{2}\right)\right]\right|\\&\leq\frac{5\rho^2(\vartheta-\varsigma)^{\rho+1}2^{1-2\rho}}{6}M+\frac{(1-\rho)(\vartheta-\varsigma)^{2-\rho}}{2}\left(\frac{1}{3-2\rho}+\frac{1}{3}\right)N.\end{aligned}$$

Remark 4. Theorem 4 states that in the specific condition when ρ approaches to 1, we get

$$\left|\frac{1}{3}\left(2\psi(\varsigma)-\psi\left(\frac{\varsigma+\vartheta}{2}\right)+2\psi(\vartheta)\right)-\frac{1}{\vartheta-\varsigma}\int_{\varsigma}^{\vartheta}\psi(\varkappa)\mathrm{d}\varkappa\right|\leq\frac{5(\vartheta-\varsigma)}{24}(M-m),$$

which was shown by Budak et al. in [11].

Remark 5. In the particular case when ρ tends to 1 in Conclusion 5, we have

$$\left|\frac{1}{3}\left(2\psi(\varsigma)-\psi\left(\frac{\varsigma+\vartheta}{2}\right)+2\psi(\vartheta)\right)-\frac{1}{\vartheta-\varsigma}\int_{\varsigma}^{\vartheta}\psi(\varkappa)\mathrm{d}\varkappa\right|\leq\frac{5(\vartheta-\varsigma)}{12}M,$$

which was demonstrated by Alomari and Liu in [3].

4. Milne-type inequalities for mappings of bounded variation involving proportional Caputo-hybrid operator

Here, we present some Milne-type inequalities for proportional Caputo-hybrid operators using mappings of bounded variation.

Theorem 5. *Let $\psi,\psi'\colon[\varsigma,\vartheta]\to\mathbb{R}$ be a mapping of bounded variation on $[\varsigma,\vartheta]$. Then, we get*

$$\begin{aligned}&\left|\frac{\rho^2(\vartheta-\varsigma)^{\rho}2^{2-2\rho}}{3}\left(2\psi(\varsigma)-\psi\left(\frac{\varsigma+\vartheta}{2}\right)+2\psi(\vartheta)\right)\right.\\&\quad+\frac{(1-\rho)(\vartheta-\varsigma)^{1-\rho}}{3}\left(2\psi'(\varsigma)-\psi'\left(\frac{\varsigma+\vartheta}{2}\right)+2\psi'(\vartheta)\right)\\&\quad\left.-\frac{2^{2-2\rho}\Gamma(1-\rho)}{(\vartheta-\varsigma)^{-\rho+1}}\left[{}^{PC}_{\varsigma^+}D^{\rho}_{\left(\frac{\varsigma+\vartheta}{2}\right)}\psi\left(\frac{\varsigma+\vartheta}{2}\right)+{}^{PC}_{\vartheta^-}D^{\rho}_{\left(\frac{\varsigma+\vartheta}{2}\right)}\psi\left(\frac{\varsigma+\vartheta}{2}\right)\right]\right|\\&\leq\frac{2^{3-2\rho}\rho^2}{3(\vartheta-\varsigma)^{-\rho}}\bigvee_{\varsigma}^{\vartheta}(\psi)+\frac{2(1-\rho)}{3(\vartheta-\varsigma)^{-1+\rho}}\bigvee_{\varsigma}^{\vartheta}(\psi').\end{aligned}$$

Here $\bigvee_{\varsigma}^{\vartheta}(\psi)$ and $\bigvee_{\varsigma}^{\vartheta}(\psi')$ denote the total variations of ψ and ψ' on $[\varsigma,\vartheta]$, respectively.

Proof. Consider the mappings $K,L_{\rho}\colon[\varsigma,\vartheta]\to\mathbb{R}$ given by

$$K(\mathfrak{s})=\begin{cases}-\left(\frac{\varsigma+\vartheta}{2}-\mathfrak{s}\right)-\frac{(\vartheta-\varsigma)}{6}, & \varsigma\leq\mathfrak{s}\leq\frac{\varsigma+\vartheta}{2},\\ \left(\mathfrak{s}-\frac{\varsigma+\vartheta}{2}\right)+\frac{(\vartheta-\varsigma)}{6}, & \frac{\varsigma+\vartheta}{2}\leq\mathfrak{s}\leq\vartheta,\end{cases}$$

and

$$L_\rho(\mathfrak{s}) = \begin{cases} -\left(\frac{\varsigma+\vartheta}{2}-\mathfrak{s}\right)^{2-2\rho} - \frac{(\vartheta-\varsigma)^{2-2\rho}}{3.2^{2-2\rho}}, & \varsigma \le \mathfrak{s} \le \frac{\varsigma+\vartheta}{2}, \\ \left(\mathfrak{s}-\frac{\varsigma+\vartheta}{2}\right)^{2-2\rho} + \frac{(\vartheta-\varsigma)^{2-2\rho}}{3.2^{2-2\rho}}, & \frac{\varsigma+\vartheta}{2} \le \mathfrak{s} \le \vartheta. \end{cases}$$

From integration by parts it follows that

$$\begin{aligned} \int_\varsigma^\vartheta K(\mathfrak{s})\mathrm{d}\psi(\mathfrak{s}) &= -\int_\varsigma^{\frac{\varsigma+\vartheta}{2}} \left(\left(\frac{\varsigma+\vartheta}{2}-\mathfrak{s}\right)+\frac{(\vartheta-\varsigma)}{6}\right)\mathrm{d}\psi(\mathfrak{s}) \\ &\quad + \int_{\frac{\varsigma+\vartheta}{2}}^\vartheta \left(\left(\mathfrak{s}-\frac{\varsigma+\vartheta}{2}\right)+\frac{(\vartheta-\varsigma)}{6}\right)\mathrm{d}\psi(\mathfrak{s}) \\ &= -\frac{(\vartheta-\varsigma)}{6}\psi\left(\frac{\varsigma+\vartheta}{2}\right) + \frac{2(\vartheta-\varsigma)}{3}\psi(\varsigma) - \int_\varsigma^{\frac{\varsigma+\vartheta}{2}} \psi(\mathfrak{s})\mathrm{d}\mathfrak{s} \\ &\quad + \frac{2(\vartheta-\varsigma)}{3}\psi(\vartheta) - \frac{(\vartheta-\varsigma)}{6}\psi\left(\frac{\varsigma+\vartheta}{2}\right) - \int_{\frac{\varsigma+\vartheta}{2}}^\vartheta \psi(\mathfrak{s})\,\mathrm{d}\mathfrak{s} \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} \int_\varsigma^\vartheta L_\rho(\mathfrak{s})\mathrm{d}\psi'(\mathfrak{s}) &= -\int_\varsigma^{\frac{\varsigma+\vartheta}{2}} \left(\left(\frac{\varsigma+\vartheta}{2}-\mathfrak{s}\right)^{2-2\rho}+\frac{(\vartheta-\varsigma)^{2-2\rho}}{3.2^{2-2\rho}}\right)\mathrm{d}\psi'(\mathfrak{s}) \\ &\quad + \int_{\frac{\varsigma+\vartheta}{2}}^\vartheta \left(\left(\mathfrak{s}-\frac{\varsigma+\vartheta}{2}\right)^{2-2\rho}+\frac{(\vartheta-\varsigma)^{2-2\rho}}{3.2^{2-2\rho}}\right)\mathrm{d}\psi'(\mathfrak{s}) \\ &= -\frac{(\vartheta-\varsigma)^{2-2\rho}}{3.2^{2-2\rho}}\psi'\left(\frac{\varsigma+\vartheta}{2}\right) + \frac{4(\vartheta-\varsigma)^{2-2\rho}}{3.2^{2-2\rho}}\psi'(\varsigma) \\ &\quad - (2-2\rho)\int_\varsigma^{\frac{\varsigma+\vartheta}{2}} \left(\frac{\varsigma+\vartheta}{2}-\mathfrak{s}\right)^{1-2\rho}\psi'(\mathfrak{s})\mathrm{d}\mathfrak{s} \\ &\quad + \frac{4(\vartheta-\varsigma)^{2-2\rho}}{3.2^{2-2\rho}}\psi'(\vartheta) - \frac{(\vartheta-\varsigma)^{2-2\rho}}{3.2^{2-2\rho}}\psi'\left(\frac{\varsigma+\vartheta}{2}\right) \end{aligned}$$

$$-(2-2\rho)\int_{\frac{\varsigma+\vartheta}{2}}^{\vartheta}\left(\mathfrak{s}-\frac{\varsigma+\vartheta}{2}\right)^{1-2\rho}\psi'(\mathfrak{s})\mathrm{d}\mathfrak{s}. \tag{4.2}$$

Through the process of multiplying (4.1) by $2^{2-2\rho}\rho^2(\vartheta-\varsigma)^{-1+\rho}$ and (4.2) by $2^{1-2\rho}(1-\rho)(\vartheta-\varsigma)^{-1+\rho}$, and adding them side by side, we achieve the following result:

$$\begin{aligned}
&\frac{2^{2-2\rho}\rho^2}{(\vartheta-\varsigma)^{1-\rho}}\int_{\varsigma}^{\vartheta}K(\mathfrak{s})\mathrm{d}\psi(\mathfrak{s})+\frac{2^{1-2\rho}(1-\rho)}{(\vartheta-\varsigma)^{1-\rho}}\int_{\varsigma}^{\vartheta}L_\rho(\mathfrak{s})\mathrm{d}\psi'(\mathfrak{s})\\
&=\frac{\rho^2(\vartheta-\varsigma)^\rho 2^{2-2\rho}}{3}\left(2\psi(\varsigma)-\psi\left(\frac{\varsigma+\vartheta}{2}\right)+2\psi(\vartheta)\right)\\
&\quad+\frac{(1-\rho)(\vartheta-\varsigma)^{1-\rho}}{3}\left(2\psi'(\varsigma)-\psi'\left(\frac{\varsigma+\vartheta}{2}\right)+2\psi'(\vartheta)\right)\\
&\quad-\frac{2^{2-2\rho}\Gamma(1-\rho)}{(\vartheta-\varsigma)^{-\rho+1}}\left[{}^{PC}_{\varsigma^+}D^\rho_{\left(\frac{\varsigma+\vartheta}{2}\right)}\psi\left(\frac{\varsigma+\vartheta}{2}\right)+{}^{PC}_{\vartheta^-}D^\rho_{\left(\frac{\varsigma+\vartheta}{2}\right)}\psi\left(\frac{\varsigma+\vartheta}{2}\right)\right].
\end{aligned}$$

It is commonly known that if $g,h\colon[\varsigma,\vartheta]\to\mathbb{R}$ are in the sense that g is continuous on $[\varsigma,\vartheta]$ and h is of bounded variation on $[\varsigma,\vartheta]$, then $\int_{\varsigma}^{\vartheta}g(\mathfrak{s})\mathrm{d}h(\mathfrak{s})$ exists and

$$\left|\int_{\varsigma}^{\vartheta}g(\mathfrak{s})\mathrm{d}h(\mathfrak{s})\right|\le\sup_{\mathfrak{s}\in[\varsigma,\vartheta]}|g(\mathfrak{s})|\bigvee_{\varsigma}^{\vartheta}(h). \tag{4.3}$$

Thus, making use of (4.3), we find

$$\begin{aligned}
&\left|\frac{\rho^2(\vartheta-\varsigma)^\rho 2^{2-2\rho}}{3}\left(2\psi(\varsigma)-\psi\left(\frac{\varsigma+\vartheta}{2}\right)+2\psi(\vartheta)\right)\right.\\
&\quad+\frac{(1-\rho)(\vartheta-\varsigma)^{1-\rho}}{3}\left(2\psi'(\varsigma)-\psi'\left(\frac{\varsigma+\vartheta}{2}\right)+2\psi'(\vartheta)\right)\\
&\quad\left.-\frac{2^{2-2\rho}\Gamma(1-\rho)}{(\vartheta-\varsigma)^{-\rho+1}}\left[{}^{PC}_{\varsigma^+}D^\rho_{\left(\frac{\varsigma+\vartheta}{2}\right)}\psi\left(\frac{\varsigma+\vartheta}{2}\right)+{}^{PC}_{\vartheta^-}D^\rho_{\left(\frac{\varsigma+\vartheta}{2}\right)}\psi\left(\frac{\varsigma+\vartheta}{2}\right)\right]\right|\\
&\le\frac{2^{2-2\rho}\rho^2}{(\vartheta-\varsigma)^{1-\rho}}\left[\left|\int_{\varsigma}^{\frac{\varsigma+\vartheta}{2}}\left(\left(\frac{\varsigma+\vartheta}{2}-\mathfrak{s}\right)+\frac{(\vartheta-\varsigma)}{6}\right)\mathrm{d}\psi(\mathfrak{s})\right|\right.\\
&\quad\left.+\left|\int_{\frac{\varsigma+\vartheta}{2}}^{\vartheta}\left(\left(\mathfrak{s}-\frac{\varsigma+\vartheta}{2}\right)+\frac{(\vartheta-\varsigma)}{6}\right)\mathrm{d}\psi(\mathfrak{s})\right|\right]
\end{aligned}$$

$$
\begin{aligned}
&+\frac{2^{1-2\rho}(1-\rho)}{(\vartheta-\varsigma)^{1-\rho}}\left[\left|\int_{\varsigma}^{\frac{\varsigma+\vartheta}{2}}\left(\left(\frac{\varsigma+\vartheta}{2}-\mathfrak{s}\right)^{2-2\rho}+\frac{(\vartheta-\varsigma)^{2-2\rho}}{3.2^{2-2\rho}}\right)\mathrm{d}\psi'(\mathfrak{s})\right|\right.\\
&\left.+\left|\int_{\frac{\varsigma+\vartheta}{2}}^{\vartheta}\left(\left(\mathfrak{s}-\frac{\varsigma+\vartheta}{2}\right)^{2-2\rho}+\frac{(\vartheta-\varsigma)^{2-2\rho}}{3.2^{2-2\rho}}\right)\mathrm{d}\psi'(\mathfrak{s})\right|\right]\\
&\leq\frac{2^{2-2\rho}\rho^2}{(\vartheta-\varsigma)^{1-\rho}}\left[\sup_{\mathfrak{s}\in[\varsigma,\frac{\varsigma+\vartheta}{2}]}\left|\left(\frac{\varsigma+\vartheta}{2}-\mathfrak{s}\right)+\frac{(\vartheta-\varsigma)}{6}\right|\bigvee_{\varsigma}^{\frac{\varsigma+\vartheta}{2}}(\psi)\right.\\
&\left.+\sup_{\mathfrak{s}\in[\frac{\varsigma+\vartheta}{2},\vartheta]}\left|\left(\mathfrak{s}-\frac{\varsigma+\vartheta}{2}\right)+\frac{(\vartheta-\varsigma)}{6}\right|\bigvee_{\frac{\varsigma+\vartheta}{2}}^{\vartheta}(\psi)\right]\\
&+\frac{2^{1-2\rho}(1-\rho)}{(\vartheta-\varsigma)^{1-\rho}}\left[\sup_{\mathfrak{s}\in[\varsigma,\frac{\varsigma+\vartheta}{2}]}\left|\left(\frac{\varsigma+\vartheta}{2}-\mathfrak{s}\right)^{2-2\rho}+\frac{(\vartheta-\varsigma)^{2-2\rho}}{3.2^{2-2\rho}}\right|\bigvee_{\varsigma}^{\frac{\varsigma+\vartheta}{2}}(\psi')\right.\\
&\left.+\sup_{\mathfrak{s}\in[\frac{\varsigma+\vartheta}{2},\vartheta]}\left|\left(\mathfrak{s}-\frac{\varsigma+\vartheta}{2}\right)^{2-2\rho}+\frac{(\vartheta-\varsigma)^{2-2\rho}}{3.2^{2-2\rho}}\right|\bigvee_{\frac{\varsigma+\vartheta}{2}}^{\vartheta}(\psi')\right]\\
&=\frac{2^{2-2\rho}\rho^2}{(\vartheta-\varsigma)^{1-\rho}}\left[\frac{2(\vartheta-\varsigma)}{3}\bigvee_{\varsigma}^{\frac{\varsigma+\vartheta}{2}}(\psi)+\frac{2(\vartheta-\varsigma)}{3}\bigvee_{\frac{\varsigma+\vartheta}{2}}^{\vartheta}(\psi)\right]\\
&+\frac{2^{1-2\rho}(1-\rho)}{(\vartheta-\varsigma)^{1-\rho}}\left[\frac{4(\vartheta-\varsigma)^{2-2\rho}}{3.2^{2-2\rho}}\bigvee_{\varsigma}^{\frac{\varsigma+\vartheta}{2}}(\psi')+\frac{4(\vartheta-\varsigma)^{2-2\rho}}{3.2^{2-2\rho}}\bigvee_{\frac{\varsigma+\vartheta}{2}}^{\vartheta}(\psi')\right]\\
&=\frac{2^{3-2\rho}\rho^2}{3(\vartheta-\varsigma)^{-\rho}}\bigvee_{\varsigma}^{\vartheta}(\psi)+\frac{2(1-\rho)}{3(\vartheta-\varsigma)^{-1+\rho}}\bigvee_{\varsigma}^{\vartheta}(\psi').
\end{aligned}
$$

□

Remark 6. In the specific case when ρ goes to 1 in Theorem 5, we have

$$
\left|\frac{1}{3}\left(2\psi(\varsigma)-\psi\left(\frac{\varsigma+\vartheta}{2}\right)+2\psi(\vartheta)\right)-\frac{1}{\vartheta-\varsigma}\int_{\varsigma}^{\vartheta}\psi(\varkappa)\mathrm{d}\varkappa\right|\leq\frac{2}{3}\bigvee_{\varsigma}^{\vartheta}(\psi),
$$

which is identical to Theorem 6 offered by Budak et al. [11] under the assumption that $\rho=1$.

Corollary 6. *In the particular circumstance where* ρ *approaches* 0 *in Theorem 5, we obtain*

$$\left|\frac{1}{3}\left(2\psi'(\varsigma)-\psi'\left(\frac{\varsigma+\vartheta}{2}\right)+2\psi'(\vartheta)\right)-\frac{2\left(\psi(\vartheta)-\psi(\varsigma)\right)}{\vartheta-\varsigma}\right.$$
$$\left.-\frac{4}{(\vartheta-\varsigma)^2}\left(\int_{\varsigma}^{\frac{\varsigma+\vartheta}{2}}\psi(\varkappa)\mathrm{d}\varkappa-\int_{\frac{\varsigma+\vartheta}{2}}^{\vartheta}\psi(\varkappa)\mathrm{d}\varkappa\right)\right|\leq\frac{2}{3}\bigvee_{\varsigma}^{\vartheta}(\psi').$$

5. Conclusion

This study aims to establish novel Milne-type integral inequalities through the utilization of proportional Caputo hybrid operator for twice-differentiable convex mappings. In order to achieve this goal, we begin by proving a new integral identity of the Milne-type associated with proportional Caputo-hybrid operator. Then, we establish a fractional Milne-type inequality for twice-differentiable convex mappings. Moreover, we obtain some Milne-type inequalities for bounded mappings and mappings of bounded variation. The findings of this study are more applicable than in the traditional calculus since previously determined bounds are the special case of our findings for $\rho\to 1$ and our findings hold true across the entire interval given for ρ. With the proportional Caputo-hybrid operator, this is the first paper dedicated to Milne-type inequalities. Therefore, we hope that readers will be motivated to look into this subject further by our approaches and findings. In future work, one can investigate analogous inequalities for different fractional integrals and derive novel Milne-type inequalities by employing various forms of convexity.

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STABILITY OF CONFORMABLE NON-AUTONOMOUS NON-INSTANTANEOUS IMPULSIVE DIFFERENTIAL EQUATIONS

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Abstract. This paper focus on a type of conformable non-autonomous non-instantaneous impulsive equation. Firstly, we give a kind of non-autonomous conformable Cauchy matrix to present the solution of linear and nonlinear systems. Then, we investigate the asymptotic stability of linear homogeneous problem and the exponential stability of linear perturbed problem. Also, one presents the solution of nonlinear problem and verifies its Ulam-Hyers-Rassias stability.

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Keywords: asymptotical stability, non-autonomous, Ulam-Hyers-Rassias stability, conformable derivative

1. INTRODUCTION

Since the raise of impulsive equations, many papers based on the impulsive effects are published [5–7]. With the deepening of the research, Hernández and O'Regan firstly introduce the non-instantaneous impulsive differential equations [8] which can describe the processes in pharmacotherapy systems. Inspired by this work, the basic theory such as stability, periodicity, controllability are studied in the [3, 4, 9, 13, 15, 17, 18, 20].

There are many papers studying the non-instantaneous impulsive differential equations without non-autonomous effects. The behavior of the autonomous system depends only on the current state. However, in many systems, the state or behavior depends not only on the current state, but also on external factors. Hence, the non-autonomous non-instantaneous impulsive differential equations are worthy to consider. Also, motivated by the raise of conformable derivative in [10] and the relating works [1, 11, 12, 14, 19], we present the following problems:

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(i) Conformable linear homogeneous problem:

$$\left\{\begin{array}{ll} \mathfrak{D}_{\varsigma}^{\mathcal{U}}\mathcal{B}(t) & = \mathcal{A}(t)\mathcal{B}(t),\ t\in(s_l,t_{l+1}],\ l\in N_0,\\ \mathcal{B}(t_l^+) & = \mathcal{C}_l(t_l^+)\mathcal{B}(t_l^-),\ l\in N,\\ \mathcal{B}(t) & = \mathcal{C}_l(t)\mathcal{B}(t_l^-),\ t\in(t_l,s_l],\ l\in N,\\ \mathcal{B}(s_l^+) & = \mathcal{B}(s_l^-),\ l\in N,\\ \mathcal{B}(u) & = \mathcal{B}_u\in\mathbb{R}^n, \end{array}\right. \tag{1.1}$$

for $N_0=\{0,1,2,\cdots,w\}$ and $N=\{1,2,\cdots,w\}$. One sets t_l as the impulsive point and $s_0=\mathcal{U}<t_1<s_1\cdots<s_i<t_{i+1}<\cdots<t_w=\mathcal{V}$. Suppose that $\mathcal{A}(\cdot):\bigcup_{l=0}^{\infty}(s_l,t_{l+1}]\to\mathbb{R}^{n\times n}$ and $\mathcal{C}_l(\cdot):\bigcup_{l=1}^{\infty}(t_l,s_l]\to\mathbb{R}^{n\times n}$ are $n\times n$ matrices.

(ii) Conformable linear nonhomogeneous problem:

$$\left\{\begin{array}{ll} \mathfrak{D}_{\varsigma}^{\mathcal{U}}\mathcal{B}(t) & = \mathcal{A}(t)\mathcal{B}(t)+\mathcal{J}(t)\mathcal{B}(t),\ t\in(s_l,t_{l+1}],\ l\in N_0,\\ \mathcal{B}(t_l^+) & = \mathcal{C}_l(t_l^+)\mathcal{B}(t_l^-)+\mathcal{M}_l\mathcal{B}(t_l^-),\ l\in N,\\ \mathcal{B}(t) & = \mathcal{C}_l(t)\mathcal{B}(t_l^-)+\mathcal{M}_l\mathcal{B}(t_l^-),\ t\in(t_l,s_l],\ l\in N,\\ \mathcal{B}(s_l^+) & = \mathcal{B}(s_l^-),\ l\in N,\\ \mathcal{B}(u) & = \mathcal{B}_u\in\mathbb{R}^n, \end{array}\right. \tag{1.2}$$

where $\mathcal{J}(\cdot):\bigcup_{l=0}^{\infty}(s_l,t_{l+1}]\to\mathbb{R}^{n\times n}$ and $\mathcal{M}_l\in\mathbb{R}^{n\times n}$.

(iii) Conformable nonlinear non-autonomous problem:

$$\left\{\begin{array}{ll} \mathfrak{D}_{\varsigma}^{\mathcal{U}}\mathcal{B}(t) & = \mathcal{A}(t)\mathcal{B}(t)+z(t,\mathcal{B}(t)),\ t\in(s_l,t_{l+1}],\ l\in N_0,\\ \mathcal{B}(t_l^+) & = \mathcal{C}_l(t_l^+)\mathcal{B}(t_l^-)+\mathcal{L}_l,\ l\in N,\\ \mathcal{B}(t) & = \mathcal{C}_l(t)\mathcal{B}(t_l^-)+\mathcal{L}_l,\ t\in(t_l,s_l],\ l\in N,\\ \mathcal{B}(s_l^+) & = \mathcal{B}(s_l^-),\ l\in N,\\ \mathcal{B}(u) & = \mathcal{B}_u\in\mathbb{R}^n, \end{array}\right. \tag{1.3}$$

where $z(\cdot,\mathcal{B}(\cdot)):\bigcup_{l=0}^{\infty}(s_l,t_{l+1}]\times\mathbb{R}^n\to\mathbb{R}^n$ and $\mathcal{L}_l\in\mathbb{R}^n$.

Section 2 presents the basic theories and introduces the notation of non-autonomous conformable Cauchy matrix to present the solution of linear homogeneous problem. Next, we derive the norm estimation about $\Gamma(t,\mathcal{U})$, even obtain many sufficient conditions about linear equations which are asymptotically stable. In Section 3, we use the method of constant variation to present the solution of (1.2) and obtain its asymptotical stability. Then on this basis, using the classical contracting mapping principle, we discuss existence and uniqueness of solution about (1.3). After

introducing the notation of non-instantaneous impulsive differential equations Ulam-Hyers-Rassias stability, the sufficient conditions of Ulam-Hyers-Rassias stability of nonlinear non-instantaneous impulsive equations are also derived.

2. PRELIMINARY

Set $\mathbb{U}=[\mathcal{U},\mathcal{V}]$ and $PC(\mathbb{U},\mathbb{R}^n):=\{\mathcal{B}:\mathbb{U}\to\mathbb{R}^n:\mathcal{B}\in C\big((t_l,t_{l+1}],\mathbb{R}^n\big),\ l=0,1,\cdots,$ there exists $\mathcal{B}(t_l^-)$ and $\mathcal{B}(t_l^+),\ l=1,2,\cdots,$ with $\mathcal{B}(t_l^-)=\mathcal{B}(t_l)\}$, where $C\big((t_l,t_{l+1}],\mathbb{R}^n\big)$ is the space of all continuous functions from $(t_l,t_{l+1}]$ into $\mathbb{R}^n$ endowed with the Chebyshev *PC*-norm $\|\mathcal{B}\|_{PC}:=\sup\{\|\mathcal{B}(t)\|:t\in\mathbb{U}\}$. For $\zeta=(\zeta_1,\cdots,\zeta_n)^\top\in\mathbb{R}^n$, its vector norm is $\|\zeta\|=\sum_{l=1}^{n}|\zeta_l|$. For $\alpha\in\mathbb{R}^{n\times n}$, its matrix norm is $\|\alpha\|=\max_{\|\mathcal{B}\|=1}\|\alpha\mathcal{B}\|$.

Definition 1. (see [2, Definition 2.1]) The conformable derivative of $\mathcal{B}:\mathbb{U}\to\mathbb{R}$ is

$$\mathfrak{D}_{\varsigma}^{\mathcal{U}}\mathcal{B}(t)=\lim_{\varepsilon\to0}\frac{\mathcal{B}(t+\varepsilon(t-\mathcal{U})^{1-\varsigma})-\mathcal{B}(t)}{\varepsilon},\ t\geq\mathcal{U},\ 0<\varsigma<1,$$
$$\mathfrak{D}_{\varsigma}^{\mathcal{U}}\mathcal{B}(\mathcal{U})=\lim_{t\to\mathcal{U}^+}\mathfrak{D}_{\varsigma}^{\mathcal{U}}\mathcal{B}(t).$$

Remark 1. The conformable derivative $\mathfrak{D}_{\varsigma}^{\mathcal{U}}\mathcal{B}(t)$ exists iff $\mathcal{B}$ is differentiable at t and $\mathfrak{D}_{\varsigma}^{\mathcal{U}}\mathcal{B}(t)=(t-\mathcal{U})^{1-\varsigma}\mathcal{B}'(t)$ for $t>\mathcal{U}$.

Definition 2. (see [2, Notation]) The conformable integral of a function $\mathcal{B}:\mathbb{U}\to\mathbb{R}$ is

$$\mathfrak{I}_{\varsigma}^{\mathcal{U}}\mathcal{B}(t)=\int_{\mathcal{U}}^{t}\mathcal{B}(s)\mathcal{B}_{\varsigma}(s,\mathcal{U})=\int_{\mathcal{U}}^{t}(s-\mathcal{U})^{\varsigma-1}\mathcal{B}(s)ds,\ t\geq\mathcal{U},\ 0<\varsigma<1,$$

if $\mathcal{U}=0$, then we write $\mathcal{B}_{\varsigma}(s,\mathcal{U})$ as $\mathcal{B}_{\varsigma}(s)$.

Lemma 1. *A solution $\mathcal{B}\in C(\mathbb{U},\mathbb{R}^n)$ of*

$$\begin{cases}\mathfrak{D}_{\varsigma}^{\mathcal{U}}\mathcal{B}(t) &= \mathcal{A}(t)\mathcal{B}(t),\ 0<\varsigma<1,\\ \mathcal{B}(s) &= \mathcal{B}_s,\ t>s\geq\mathcal{U}.\end{cases}$$

is

$$\mathcal{B}(t)=\Xi(t,s)\mathcal{B}_s,$$

where $\Xi(\cdot,\cdot)$ is the Cauchy matrix for $s_l<s\leq t\leq t_{l+1}$.

Set $\|\Xi(t,s)\|\leq e^{\int_s^t\|\mathcal{A}(\theta)\|(\theta-\mathcal{U})^{\varsigma-1}d\theta},\ t,s\in(s_l,t_{l+1}]$ and $\xi_l=\max_{t\in(s_l,t_{l+1}]}\|\mathcal{A}(t)\|$.

Theorem 1. *Set $\mathcal{B}(t,s,\mathcal{B}_s)$ as the solution of (1.1) with $\mathcal{B}(s)=\mathcal{B}_s$. Then*

$$\mathcal{B}(t):=\mathcal{B}(t,s,\mathcal{B}_s)=\Gamma(t,s)\mathcal{B}_s,\ \mathcal{U}\leq s\leq t,$$

where $\Gamma(\cdot,\cdot)$ *is*

$$\Gamma(t,s)=\begin{cases}\Xi(t,s),\ t,s\in(s_l,t_{l+1}],\ l\in N_0;\\ C_l(t)I_l^{-1}(s),\ t,s\in(t_l,s_l],\ l\in N;\\ C_{\chi(\mathcal{U},t)}(t)\prod\limits_{l=\chi(\mathcal{U},s)+2}^{\chi(\mathcal{U},t)}[\Xi(t_l,s_{l-1})C_{l-1}(s_{l-1}^-)]\Xi(t_{\chi(\mathcal{U},s)+1},s),\\ s\in(s_{\chi(\mathcal{U},s)}\le t_{\chi(\mathcal{U},s)+1}],\ t\in(t_{\chi(\mathcal{U},t)},s_{\chi(\mathcal{U},t)}];\\ \Xi(t,s_{\chi(\mathcal{U},t)})C_{\chi(\mathcal{U},t)}(s_{\chi(\mathcal{U},t)}^-)\prod\limits_{l=\chi(\mathcal{U},s)+2}^{\chi(\mathcal{U},t)}[\Xi(t_l,s_{l-1})C_{l-1}(s_{l-1}^-)]\Xi(t_{\chi(\mathcal{U},s)+1},s),\\ s\in(s_{\chi(\mathcal{U},s)}\le t_{\chi(\mathcal{U},s)+1}],\ t\in(s_{\chi(\mathcal{U},t)},t_{\chi(\mathcal{U},t)+1}];\\ \Xi(t,s_{\chi(\mathcal{U},t)})C_{\chi(\mathcal{U},t)}(s_{\chi(\mathcal{U},t)}^-)\prod\limits_{l=\chi(\mathcal{U},s)+1}^{\chi(\mathcal{U},t)}[\Xi(t_l,s_{l-1})C_{l-1}(s_{l-1}^-)]I_{\chi(\mathcal{U},s)}^{-1}(s),\\ s\in(t_{\chi(\mathcal{U},s)},s_{\chi(\mathcal{U},s)}],\ t\in(s_{\chi(\mathcal{U},t)},t_{\chi(\mathcal{U},t)+1}];\\ C_{\chi(\mathcal{U},t)}(t)\prod\limits_{l=\chi(\mathcal{U},s)+1}^{\chi(\mathcal{U},t)}[\Xi(t_l,s_{l-1})C_{l-1}(s_{l-1}^-)]I_{\chi(\mathcal{U},s)}^{-1}(s),\\ s\in(t_{\chi(\mathcal{U},s)},s_{\chi(\mathcal{U},s)}],\ t\in(t_{\chi(\mathcal{U},t)},s_{\chi(\mathcal{U},t)}].\end{cases}$$

in which $\chi(\mathcal{U},t)$ *is the number of the impulsive points between* $\mathcal{U}$ *and* t. *When* $\chi(\mathcal{U},t)=\chi(\mathcal{U},s)$, *we have* $\prod\limits_{l=\chi(\mathcal{U},s)}^{\chi(\mathcal{U},t)-1}=E$, *where* E *is the identity matrix.*

In particular with $s=\mathcal{U}$,

$$\mathcal{B}(t):=\mathcal{B}(t,\mathcal{U},\mathcal{B}_{\mathcal{U}})=\Gamma(t,\mathcal{U})\mathcal{B}_{\mathcal{U}} \tag{2.1}$$

$$=\begin{cases}\Xi(t,\mathcal{U})\mathcal{B}_{\mathcal{U}},\ t\in(s_0,t_1];\\ C_{\chi(\mathcal{U},t)}(t)\prod\limits_{l=2}^{\chi(\mathcal{U},t)}[\Xi(t_l,s_{l-1})C_{l-1}(s_{l-1}^-)]\Xi(t_1,\mathcal{U})\mathcal{B}_{\mathcal{U}},\\ t\in(t_{\chi(\mathcal{U},t)},s_{\chi(\mathcal{U},t)}];\\ \Xi(t,s_{\chi(\mathcal{U},t)})C_{\chi(\mathcal{U},t)}(s_{\chi(\mathcal{U},t)}^-)\prod\limits_{l=2}^{\chi(\mathcal{U},t)}[\Xi(t_l,s_{l-1})C_{l-1}(s_{l-1}^-)]\Xi(t_1,\mathcal{U})\mathcal{B}_{\mathcal{U}},\\ t\in(s_{\chi(\mathcal{U},t)},t_{\chi(\mathcal{U},t)+1}].\end{cases}$$

Proof. Case 1: $\chi(\mathcal{U},t)=\chi(\mathcal{U},s)$.

(i) Consider any $t,s\in(s_l,t_{l+1}]$, $l\in N_0$.

If $t\in(\mathcal{U},t_1]$,

$$\mathcal{B}(t)=\Xi(t,\mathcal{U})\mathcal{B}_{\mathcal{U}},$$
$$\mathcal{B}(s)=\Xi(s,\mathcal{U})\mathcal{B}_{\mathcal{U}}.$$

Next, for $t,s\in(s_1,t_2]$,

$$\mathcal{B}(t)=\Xi(t,s_1)\mathcal{B}_{s_1},$$

$$\mathcal{B}(s) = \Xi(s, s_1)\mathcal{B}_{s_1}.$$

Summarizing, we obtain that

$$\Gamma(t,s) = \Xi(t,s).$$

(ii) Consider any $t, s \in (t_l, s_l]$, $l \in N$,

$$\mathcal{B}(t) = \mathcal{C}_l(t)\mathcal{B}(t_l^-),$$
$$\mathcal{B}(s) = \mathcal{C}_l(s)\mathcal{B}(t_l^-),$$

so $\Gamma(t,s) = \mathcal{C}_l(t)\mathcal{C}_l^{-1}(s)$.

(iii) Consider any $s \in (t_{\chi(\mathcal{U},t)}, s_{\chi(\mathcal{U},t)}]$ and any $t \in (s_{\chi(\mathcal{U},t)}, t_{\chi(\mathcal{U},t)+1}]$,

$$\begin{aligned}\mathcal{B}(t) &= \Xi(t, s_{\chi(\mathcal{U},t)})\mathcal{B}(s^+_{\chi(\mathcal{U},t)}) \\ &= \Xi(t, s_{\chi(\mathcal{U},t)})\mathcal{C}_{\chi(\mathcal{U},t)}(s_{\chi(\mathcal{U},t)})\mathcal{C}^{-1}_{\chi(\mathcal{U},t)}(s)\mathcal{B}(s),\end{aligned}$$

so

$$\Gamma(t,s) = \Xi(t, s_{\chi(\mathcal{U},t)})\mathcal{C}_{\chi(\mathcal{U},t)}(s_{\chi(\mathcal{U},t)})\mathcal{C}^{-1}_{\chi(\mathcal{U},t)}(s).$$

Case 2: $\chi(\mathcal{U},t) = \chi(\mathcal{U},s) + 1$.

(i) Consider any $s \in (s_{\chi(\mathcal{U},s)}, t_{\chi(\mathcal{U},s)+1}]$ and any $t \in (t_{\chi(\mathcal{U},t)}, s_{\chi(\mathcal{U},t)}]$,

$$\mathcal{B}(t) = \mathcal{C}_{\chi(\mathcal{U},t)}(t)\mathcal{B}(t^-_{\chi(\mathcal{U},t)}) = \mathcal{C}_{\chi(\mathcal{U},t)}(t)\Xi(t_{\chi(\mathcal{U},t)}, s)\mathcal{B}(s),$$

so

$$\Gamma(t,s) = \mathcal{C}_{\chi(\mathcal{U},t)}(t)\Xi(t_{\chi(\mathcal{U},t)}, s).$$

(ii) Consider any $s \in (s_{\chi(\mathcal{U},s)}, t_{\chi(\mathcal{U},s)+1}]$ and any $t \in (s_{\chi(\mathcal{U},t)}, t_{\chi(\mathcal{U},t)+1}]$,

$$\begin{aligned}\mathcal{B}(t) &= \Xi(t, s_{\chi(\mathcal{U},t)})\mathcal{B}(s^+_{\chi(\mathcal{U},t)}) \\ &= \Xi(t, s_{\chi(\mathcal{U},t)})\mathcal{C}_{\chi(\mathcal{U},t)}(s^-_{\chi(\mathcal{U},t)})\mathcal{B}(t^-_{\chi(\mathcal{U},t)}) \\ &= \Xi(t, s_{\chi(\mathcal{U},t)})\mathcal{C}_{\chi(\mathcal{U},t)}(s^-_{\chi(\mathcal{U},t)})\Xi(s_{\chi(\mathcal{U},t)}, s)\mathcal{B}(s),\end{aligned}$$

so

$$\Gamma(t,s) = \Xi(t, s_{\chi(\mathcal{U},t)})\mathcal{C}_{\chi(\mathcal{U},t)}(s^-_{\chi(\mathcal{U},t)})\Xi(s_{\chi(\mathcal{U},t)}, s).$$

(iii) Consider any $s \in (t_{\chi(\mathcal{U},s)}, s_{\chi(\mathcal{U},s)}]$ and any $t \in (t_{\chi(\mathcal{U},t)}, s_{\chi(\mathcal{U},t)}]$,

$$\begin{aligned}\mathcal{B}(t) &= \mathcal{C}_{\chi(\mathcal{U},t)}(t)\mathcal{B}(t^-_{\chi(\mathcal{U},t)}) \\ &= \mathcal{C}_{\chi(\mathcal{U},t)}(t)\Xi(t_{\chi(\mathcal{U},t)}, s_{\chi(\mathcal{U},s)})\mathcal{B}(s^+_{\chi(\mathcal{U},s)}) \\ &= \mathcal{C}_{\chi(\mathcal{U},t)}(t)\Xi(t_{\chi(\mathcal{U},t)}, s_{\chi(\mathcal{U},s)})\mathcal{C}_{\chi(\mathcal{U},s)}(s_{\chi(\mathcal{U},s)})\mathcal{C}^{-1}_{\chi(\mathcal{U},s)}(s)\mathcal{B}(s),\end{aligned}$$

so

$$\Gamma(t,s) = \mathcal{C}_{\chi(\mathcal{U},t)}(t)\Xi(t_{\chi(\mathcal{U},t)}, s_{\chi(\mathcal{U},s)})\mathcal{C}_{\chi(\mathcal{U},s)}(s_{\chi(\mathcal{U},s)})\mathcal{C}^{-1}_{\chi(\mathcal{U},s)}(s).$$

(iv) Consider any $s \in (t_{\chi(\mathcal{U},s)}, s_{\chi(\mathcal{U},s)}]$ and any $t \in (s_{\chi(\mathcal{U},t)}, t_{\chi(\mathcal{U},t)+1}]$,

$$\mathcal{B}(t) = \Xi(t, s_{\chi(\mathcal{U},t)})\mathcal{B}(s^+_{\chi(\mathcal{U},t)})$$

$$= \Xi(t, s_{\chi(\mathcal{U},t)}) \mathcal{C}_{\chi(\mathcal{U},t)}(s^-_{\chi(\mathcal{U},t)}) \mathcal{B}(t^-_{\chi(\mathcal{U},t)})$$
$$= \Xi(t, s_{\chi(\mathcal{U},t)}) \mathcal{C}_{\chi(\mathcal{U},t)}(s^-_{\chi(\mathcal{U},t)}) \Xi(t_{\chi(\mathcal{U},t)}, s_{\chi(\mathcal{U},s)}) \mathcal{B}(s^+_{\chi(\mathcal{U},s)})$$
$$= \Xi(t, s_{\chi(\mathcal{U},t)}) \mathcal{C}_l(s^-_{\chi(\mathcal{U},t)}) \Xi(t_{\chi(\mathcal{U},t)}, s_{\chi(\mathcal{U},s)}) \mathcal{C}_{\chi(\mathcal{U},s)}(s_{\chi(\mathcal{U},s)}) \mathcal{C}^{-1}_{\chi(\mathcal{U},s)}(s) \mathcal{B}(s),$$

so

$$\Gamma(t,s) = \Xi(t, s_{\chi(\mathcal{U},t)}) \mathcal{C}_{\chi(\mathcal{U},t)}(s^-_{\chi(\mathcal{U},t)}) \Xi(t_{\chi(\mathcal{U},t)}, s_{\chi(\mathcal{U},s)}) \mathcal{C}_{\chi(\mathcal{U},s)}(s_{\chi(\mathcal{U},s)}) \mathcal{C}^{-1}_{\chi(\mathcal{U},s)}(s).$$

Case 3: $\chi(\mathcal{U},t) = \chi(\mathcal{U},s) + 2$.

(i) Consider any $s \in (s_{\chi(\mathcal{U},s)}, t_{\chi(\mathcal{U},s)+1}]$ and any $t \in (t_{\chi(\mathcal{U},t)}, s_{\chi(\mathcal{U},t)}]$,

$$\mathcal{B}(t) = \mathcal{C}_{\chi(\mathcal{U},t)}(t) \mathcal{B}(t^-_{\chi(\mathcal{U},t)})$$
$$= \mathcal{C}_{\chi(\mathcal{U},t)}(t) \Xi(t_{\chi(\mathcal{U},t)}, s_{\chi(\mathcal{U},s)+1}) \mathcal{B}(s^+_{\chi(\mathcal{U},s)+1})$$
$$= \mathcal{C}_{\chi(\mathcal{U},t)}(t) \Xi(t_{\chi(\mathcal{U},t)}, s_{\chi(\mathcal{U},s)+1}) \mathcal{C}_{s_{\chi(\mathcal{U},s)+1}}(s^-_{\chi(\mathcal{U},s)+1}) \mathcal{B}(t^-_{\chi(\mathcal{U},s)+1})$$
$$= \mathcal{C}_{\chi(\mathcal{U},t)}(t) \Xi(t_{\chi(\mathcal{U},t)}, s_{\chi(\mathcal{U},s)+1}) \mathcal{C}_{s_{\chi(\mathcal{U},s)+1}}(s^-_{\chi(\mathcal{U},s)+1}) \Xi(t_{\chi(\mathcal{U},s)+1}, s) \mathcal{B}(s),$$

so

$$\Gamma(t,s) = \mathcal{C}_{\chi(\mathcal{U},t)}(t) \Xi(t_{\chi(\mathcal{U},t)}, s_{\chi(\mathcal{U},s)+1}) \mathcal{C}_{s_{\chi(\mathcal{U},s)+1}}(s^-_{\chi(\mathcal{U},s)+1}) \Xi(t_{\chi(\mathcal{U},s)+1}, s).$$

(ii) Consider any $s \in (s_{\chi(\mathcal{U},s)}, t_{\chi(\mathcal{U},s)+1}]$ and any $t \in (s_{\chi(\mathcal{U},t)}, t_{\chi(\mathcal{U},t)+1}]$,

$$\mathcal{B}(t) = \Xi(t, s_{\chi(\mathcal{U},t)}) \mathcal{B}(s^+_{\chi(\mathcal{U},t)})$$
$$= \Xi(t, s_{\chi(\mathcal{U},t)}) \mathcal{C}_{s_{\chi(\mathcal{U},t)}}(s^-_{\chi(\mathcal{U},t)}) \mathcal{B}(t^-_{\chi(\mathcal{U},t)})$$
$$= \Xi(t, s_{\chi(\mathcal{U},t)}) \mathcal{C}_{s_{\chi(\mathcal{U},t)}}(s^-_{\chi(\mathcal{U},t)}) \Xi(t_{\chi(\mathcal{U},t)}, s_{\chi(\mathcal{U},t)-1}) \mathcal{B}(s^+_{\chi(\mathcal{U},t)-1})$$
$$= \Xi(t, s_{\chi(\mathcal{U},t)}) \mathcal{C}_{s_{\chi(\mathcal{U},t)}}(s^-_{\chi(\mathcal{U},t)}) \Xi(t_{\chi(\mathcal{U},t)}, s_{\chi(\mathcal{U},t)-1})$$
$$\times \mathcal{C}_{\chi(\mathcal{U},t)-1}(s^-_{\chi(\mathcal{U},s)+1}) \mathcal{B}(t^-_{\chi(\mathcal{U},s)+1})$$
$$= \Xi(t, s_{\chi(\mathcal{U},t)}) \mathcal{C}_{s_{\chi(\mathcal{U},t)}}(s^-_{\chi(\mathcal{U},t)}) \Xi(t_{\chi(\mathcal{U},t)}, s_{\chi(\mathcal{U},t)-1})$$
$$\times \mathcal{C}_{\chi(\mathcal{U},t)-1}(s^-_{\chi(\mathcal{U},s)+1}) \Xi(t_{\chi(\mathcal{U},s)+1}, s) \mathcal{B}(s),$$

so

$$\Gamma(t,s) = \Xi(t, s_{\chi(\mathcal{U},t)}) \mathcal{C}_{s_{\chi(\mathcal{U},t)}}(s^-_{\chi(\mathcal{U},t)}) \Xi(t_{\chi(\mathcal{U},t)}, s_{\chi(\mathcal{U},t)-1})$$
$$\times \mathcal{C}_{\chi(\mathcal{U},t)-1}(s^-_{\chi(\mathcal{U},s)+1}) \Xi(t_{\chi(\mathcal{U},s)+1}, s).$$

(iii) Consider any $s \in (t_{\chi(\mathcal{U},s)}, s_{\chi(\mathcal{U},s)}]$ and any $t \in (t_{\chi(\mathcal{U},t)}, s_{\chi(\mathcal{U},t)}]$,

$$\mathcal{B}(t) = \mathcal{C}_{\chi(\mathcal{U},t)}(t) \mathcal{B}(t^-_{\chi(\mathcal{U},t)})$$
$$= \mathcal{C}_{\chi(\mathcal{U},t)}(t) \Xi(t_{\chi(\mathcal{U},t)}, s_{\chi(\mathcal{U},s)+1}) \mathcal{B}(s^+_{\chi(\mathcal{U},s)+1})$$
$$= \mathcal{C}_{\chi(\mathcal{U},t)}(t) \Xi(t_{\chi(\mathcal{U},t)}, s_{\chi(\mathcal{U},s)+1}) \mathcal{C}_{\chi(\mathcal{U},s)+1}(s^-_{\chi(\mathcal{U},s)+1}) \mathcal{B}(t^-_{\chi(\mathcal{U},s)+1})$$
$$= \mathcal{C}_{\chi(\mathcal{U},t)}(t) \Xi(t_{\chi(\mathcal{U},t)}, s_{\chi(\mathcal{U},s)+1}) \mathcal{C}_{\chi(\mathcal{U},s)+1}(s^-_{\chi(\mathcal{U},s)+1}) \Xi(t_{\chi(\mathcal{U},s)+1}, s_{\chi(\mathcal{U},s)})$$

$$\begin{aligned}
&\times \mathcal{B}(s^+_{\chi(\mathcal{U},s)})\\
&= \mathcal{C}_{\chi(\mathcal{U},t)}(t)\Xi(t_{\chi(\mathcal{U},t)}, s_{\chi(\mathcal{U},s)+1})\mathcal{C}_{\chi(\mathcal{U},s)+1}(s^-_{\chi(\mathcal{U},s)+1})\Xi(t_{\chi(\mathcal{U},s)+1}, s_{\chi(\mathcal{U},s)})\\
&\quad \times \mathcal{C}_{\chi(\mathcal{U},s)}(s^-_{\chi(\mathcal{U},s)})\mathcal{C}^{-1}_{\chi(\mathcal{U},s)}(s)\mathcal{B}(s),
\end{aligned}$$

so

$$\begin{aligned}
\Gamma(t,s) = \mathcal{C}_{\chi(\mathcal{U},t)}(t)\Xi(t_{\chi(\mathcal{U},t)}, s_{\chi(\mathcal{U},s)+1})\mathcal{C}_{\chi(\mathcal{U},s)+1}(s^-_{\chi(\mathcal{U},s)+1})\Xi(t_{\chi(\mathcal{U},s)+1}, s_{\chi(\mathcal{U},s)})\\
\times \mathcal{C}_{\chi(\mathcal{U},s)}(s^-_{\chi(\mathcal{U},s)})\mathcal{C}^{-1}_{\chi(\mathcal{U},s)}(s).
\end{aligned}$$

(iv) Consider any $s \in (t_{\chi(\mathcal{U},s)}, s_{\chi(\mathcal{U},s)}]$ and any $t \in (s_{\chi(\mathcal{U},t)}, t_{\chi(\mathcal{U},t)+1}]$,

$$\begin{aligned}
\mathcal{B}(t) &= \Xi(t, s_{\chi(\mathcal{U},t)})\mathcal{B}(s^+_{\chi(\mathcal{U},t)})\\
&= \Xi(t, s_{\chi(\mathcal{U},t)})\mathcal{C}_{\chi(\mathcal{U},t)}(s^-_{\chi(\mathcal{U},t)})\mathcal{B}(t^-_{\chi(\mathcal{U},t)})\\
&= \Xi(t, s_{\chi(\mathcal{U},t)})\mathcal{C}_{\chi(\mathcal{U},t)}(s^-_{\chi(\mathcal{U},t)})\Xi(t_{\chi(\mathcal{U},t)}, s_{\chi(\mathcal{U},s)+1})\mathcal{B}(s^+_{\chi(\mathcal{U},s)+1})\\
&= \Xi(t, s_{\chi(\mathcal{U},t)})\mathcal{C}_{\chi(\mathcal{U},t)}(s^-_{\chi(\mathcal{U},t)})\Xi(t_{\chi(\mathcal{U},t)}, s_{\chi(\mathcal{U},s)+1})\mathcal{C}_{\chi(\mathcal{U},s)+1}(s^-_{\chi(\mathcal{U},s)+1})\\
&\quad \times \mathcal{B}(t^-_{\chi(\mathcal{U},s)+1})\\
&= \Xi(t, s_{\chi(\mathcal{U},t)})\mathcal{C}_{\chi(\mathcal{U},t)}(s^-_{\chi(\mathcal{U},t)})\Xi(t_{\chi(\mathcal{U},t)}, s_{\chi(\mathcal{U},s)+1})\mathcal{C}_{\chi(\mathcal{U},s)+1}(s^-_{\chi(\mathcal{U},s)+1})\\
&\quad \times \Xi(t_{\chi(\mathcal{U},s)+1}, s_{\chi(\mathcal{U},s)})\mathcal{B}(s^+_{\chi(\mathcal{U},s)})\\
&= \Xi(t, s_{\chi(\mathcal{U},t)})\mathcal{C}_{\chi(\mathcal{U},t)}(s^-_{\chi(\mathcal{U},t)})\Xi(t_{\chi(\mathcal{U},t)}, s_{\chi(\mathcal{U},s)+1})\mathcal{C}_{\chi(\mathcal{U},s)+1}(s^-_{\chi(\mathcal{U},s)+1})\\
&\quad \times \Xi(t_{\chi(\mathcal{U},s)+1}, s_{\chi(\mathcal{U},s)})\mathcal{C}_{\chi(\mathcal{U},s)}(s^-_{\chi(\mathcal{U},s)})\mathcal{C}^{-1}_{\chi(\mathcal{U},s)}(s)\mathcal{B}(s),
\end{aligned}$$

so

$$\begin{aligned}
\Gamma(t,s) = \mathcal{C}_{\chi(\mathcal{U},t)}(s^-_{\chi(\mathcal{U},t)})\Xi(t_{\chi(\mathcal{U},t)}, s_{\chi(\mathcal{U},s)+1})\mathcal{C}_{\chi(\mathcal{U},s)+1}(s^-_{\chi(\mathcal{U},s)+1})\\
\times \Xi(t_{\chi(\mathcal{U},s)+1}, s_{\chi(\mathcal{U},s)})\mathcal{C}_{\chi(\mathcal{U},s)}(s^-_{\chi(\mathcal{U},s)})\mathcal{C}^{-1}_{\chi(\mathcal{U},s)}(s).
\end{aligned}$$

Case 4: General kinds of $\chi(\mathcal{U},t)$ and $\chi(\mathcal{U},s)$.

(i) Consider any $s \in (t_{\chi(\mathcal{U},s)}, s_{\chi(\mathcal{U},s)}]$ and any $t \in (s_{\chi(\mathcal{U},t)}, t_{\chi(\mathcal{U},t)+1}]$,

$$\Gamma(t,s) = \Xi(t, s_{\chi(\mathcal{U},t)})\mathcal{C}_{\chi(\mathcal{U},t)}(s^-_{\chi(\mathcal{U},t)}) \prod_{l=\chi(\mathcal{U},s)+1}^{\chi(\mathcal{U},t)} [\Xi(t_l, s_{l-1})\mathcal{C}_{l-1}(s^-_{l-1})]\mathcal{C}^{-1}_{\chi(\mathcal{U},s)}(s).$$

(ii) Consider any $s \in (s_{\chi(\mathcal{U},s)}, t_{\chi(\mathcal{U},s)+1}]$ and any $t \in (t_{\chi(\mathcal{U},t)}, s_{\chi(\mathcal{U},t)}]$,

$$\Gamma(t,s) = \mathcal{C}_{\chi(\mathcal{U},t)}(t) \prod_{l=\chi(\mathcal{U},s)+2}^{\chi(\mathcal{U},t)} [\Xi(t_l, s_{l-1})\mathcal{C}_{l-1}(s^-_{l-1})]\Xi(t_{\chi(\mathcal{U},s)+1}, s).$$

(iii) Consider any $s \in (s_{\chi(\mathcal{U},s)}, t_{\chi(\mathcal{U},s)+1}]$ and any $t \in (s_{\chi(\mathcal{U},t)}, t_{\chi(\mathcal{U},t)+1}]$,

$$\Gamma(t,s) = \Xi(t, s_{\chi(\mathcal{U},t)}) C_{\chi(\mathcal{U},t)}(s^-_{\chi(\mathcal{U},t)}) \prod_{l=\chi(\mathcal{U},s)+2}^{\chi(\mathcal{U},t)} [\Xi(t_l, s_{l-1}) C_{l-1}(s^-_{l-1})] \Xi(t_{\chi(\mathcal{U},s)+1}, s).$$

(iv) Consider any $s \in (t_{\chi(\mathcal{U},s)}, s_{\chi(\mathcal{U},s)}]$ and any $t \in (t_{\chi(\mathcal{U},t)}, s_{\chi(\mathcal{U},t)}]$,

$$\Gamma(t,s) = C_{\chi(\mathcal{U},t)}(t) \prod_{l=\chi(\mathcal{U},s)+1}^{\chi(\mathcal{U},t)} [\Xi(t_l, s_{l-1}) C_{l-1}(s^-_{l-1})] C^{-1}_{\chi(\mathcal{U},s)}(s).$$

Thus $\Gamma(t,s)$ can be written in the form:

$$\Gamma(t,s) = \begin{cases} \Xi(t,s),\ t,s \in (s_l, t_{l+1}],\ l \in N_0; \\ C_l(t) C_l^{-1}(s),\ t,s \in (t_l, s_l],\ l \in N; \\ C_{\chi(\mathcal{U},t)}(t) \prod\limits_{l=\chi(\mathcal{U},s)+2}^{\chi(\mathcal{U},t)} [\Xi(t_l, s_{l-1}) C_{l-1}(s^-_{l-1})] \Xi(t_{\chi(\mathcal{U},s)+1}, s), \\ s \in (s_{\chi(\mathcal{U},s)} \le t_{\chi(\mathcal{U},s)+1}],\ t \in (t_{\chi(\mathcal{U},t)}, s_{\chi(\mathcal{U},t)}]; \\ \Xi(t, s_{\chi(\mathcal{U},t)}) C_{\chi(\mathcal{U},t)}(s^-_{\chi(\mathcal{U},t)}) \prod\limits_{l=\chi(\mathcal{U},s)+2}^{\chi(\mathcal{U},t)} [\Xi(t_l, s_{l-1}) C_{l-1}(s^-_{l-1})] \Xi(t_{\chi(\mathcal{U},s)+1}, s), \\ s \in (s_{\chi(\mathcal{U},s)} \le t_{\chi(\mathcal{U},s)+1}],\ t \in (s_{\chi(\mathcal{U},t)}, t_{\chi(\mathcal{U},t)+1}]; \\ \Xi(t, s_{\chi(\mathcal{U},t)}) C_{\chi(\mathcal{U},t)}(s^-_{\chi(\mathcal{U},t)}) \prod\limits_{l=\chi(\mathcal{U},s)+1}^{\chi(\mathcal{U},t)} [\Xi(t_l, s_{l-1}) C_{l-1}(s^-_{l-1})] C^{-1}_{\chi(\mathcal{U},s)}(s), \\ s \in (t_{\chi(\mathcal{U},s)}, s_{\chi(\mathcal{U},s)}],\ t \in (s_{\chi(\mathcal{U},t)}, t_{\chi(\mathcal{U},t)+1}]; \\ C_{\chi(\mathcal{U},t)}(t) \prod\limits_{l=\chi(\mathcal{U},s)+1}^{\chi(\mathcal{U},t)} [\Xi(t_l, s_{l-1}) C_{l-1}(s^-_{l-1})] C^{-1}_{\chi(\mathcal{U},s)}(s), \\ s \in (t_{\chi(\mathcal{U},s)}, s_{\chi(\mathcal{U},s)}],\ t \in (t_{\chi(\mathcal{U},t)}, s_{\chi(\mathcal{U},t)}], \end{cases}$$

and $\Gamma(\cdot,\cdot)$ is called non-autonomous conformable Cauchy matrix. □

Definition 3. (1.1) is exponentially stable if there are $c_1 \ge 1$ and $c_2 > 0$ satisfying

$$\|\Gamma(t,s)\| \le c_1 e^{-c_2(t-s)},\ \mathcal{U} \le s \le t.$$

Lemma 2. *(see [16]) Let a nonnegative piecewise continuous function $\mathcal{B}(t)$ satisfies the inequality*

$$\mathcal{B}(t) \le B + \int_{\mathcal{U}}^{t} \nu(s)\mathcal{B}(s)ds + \sum_{\mathcal{U}<t_l<t} \alpha_l \mathcal{B}(t_l^-),\ l = 0,1,\cdots,m,$$

where $B \ge 0$, α_l, $\nu(t) > 0$, t_l are the first kind discontinuity points of the function $\mathcal{B}(t)$. Then the following estimate holds:

$$\mathcal{B}(t) \le B \prod_{\mathcal{U}<t_l<t} (1+\alpha_l) e^{\int_{\mathcal{U}}^{t} \nu(s)ds}.$$

3. Linear Homogeneous Problem

In this section, we present the stability of linear homogeneous problem.
Let $\xi = \max_{l\in N_0} \xi_l$.

Lemma 3. *For any $t \in (s_{\chi(\mathcal{U},t)}, t_{\chi(\mathcal{U},t)+1}]$, there is*

$$\|\Gamma(t,\mathcal{U})\| \le e^{\frac{\xi}{\varsigma}[((t-\mathcal{U})^\varsigma-(s_{\chi(\mathcal{U},t)}-\mathcal{U})^\varsigma)+\sum_{l=0}^{\chi(\mathcal{U},t)-1}((t_{l+1}-\mathcal{U})^\varsigma-(s_l-\mathcal{U})^\varsigma)]}\prod_{l=1}^{\chi(\mathcal{U},t)}\|C_l(s_l^-)\|.$$

and for any $t \in (t_{\chi(\mathcal{U},t)}, s_{\chi(\mathcal{U},t)}]$, there is

$$\|\Gamma(t,\mathcal{U})\| \le e^{\frac{\xi}{\varsigma}\sum_{l=0}^{\chi(\mathcal{U},t)-1}((t_{l+1}-\mathcal{U})^\varsigma-(s_l-\mathcal{U})^\varsigma)}\|\mathcal{C}_{\chi(\mathcal{U},t)}(t)\|\prod_{l=1}^{\chi(\mathcal{U},t)-1}\|C_l(s_l^-)\|.$$

Proof. By (2.1), for any $t \in (s_{\chi(\mathcal{U},t)}, t_{\chi(\mathcal{U},t)+1}]$,

$$\begin{aligned}
\|\Gamma(t,\mathcal{U})\| &= \|\Xi(t,s_{\chi(\mathcal{U},t)})\mathcal{C}_{\chi(\mathcal{U},t)}(s^-_{\chi(\mathcal{U},t)})\prod_{l=2}^{\chi(\mathcal{U},t)}[\Xi(t_l,s_{l-1})C_{l-1}(s_{l-1}^-)]\Xi(t_1,\mathcal{U})\| \\
&\le e^{\frac{\xi_{\chi(\mathcal{U},t)}}{\varsigma}((t-\mathcal{U})^\varsigma-(s_{\chi(\mathcal{U},t)}-\mathcal{U})^\varsigma)}\|\mathcal{C}_{\chi(\mathcal{U},t)}(s^-_{\chi(\mathcal{U},t)})\|\prod_{l=2}^{\chi(\mathcal{U},t)}[e^{\frac{\xi_l}{\varsigma}((t_l-\mathcal{U})^\varsigma-(s_{l-1}-\mathcal{U})^\varsigma)} \\
&\quad\times\|C_{l-1}(s_{l-1}^-)\|]e^{\frac{\xi_0}{\varsigma}(t_1-\mathcal{U})^\varsigma} \\
&\le e^{\frac{\xi}{\varsigma}[((t-\mathcal{U})^\varsigma-(s_{\chi(\mathcal{U},t)}-\mathcal{U})^\varsigma)+\sum_{l=0}^{\chi(\mathcal{U},t)-1}((t_{l+1}-\mathcal{U})^\varsigma-(s_l-\mathcal{U})^\varsigma)]}\prod_{l=1}^{\chi(\mathcal{U},t)}\|C_l(s_l^-)\|.
\end{aligned}$$

For any $t \in (t_{\chi(\mathcal{U},t)}, s_{\chi(\mathcal{U},t)}]$,

$$\begin{aligned}
\|\Gamma(t,\mathcal{U})\| &= \|\mathcal{C}_{\chi(\mathcal{U},t)}(t)\prod_{l=2}^{\chi(\mathcal{U},t)}[\Xi(t_l,s_{l-1})C_{l-1}(s_{l-1}^-)]\Xi(t_1,\mathcal{U})\| \\
&\le \|\mathcal{C}_{\chi(\mathcal{U},t)}(t)\|\prod_{l=2}^{\chi(\mathcal{U},t)}[e^{\frac{\xi_l}{\varsigma}((t_l-\mathcal{U})^\varsigma-(s_{l-1}-\mathcal{U})^\varsigma)}\|C_{l-1}(s_{l-1}^-)\|]e^{\frac{\xi_1}{\varsigma}(t_1-\mathcal{U})^\varsigma} \\
&\le e^{\xi\sum_{l=0}^{\chi(\mathcal{U},t)-1}((t_{l+1}-\mathcal{U})^\varsigma-(s_l-\mathcal{U})^\varsigma)}\|\mathcal{C}_{\chi(\mathcal{U},t)}(t)\|\prod_{l=1}^{\chi(\mathcal{U},t)-1}\|C_l(s_l^-)\|.
\end{aligned}$$

□

Definition 4. The solution $\mathcal{B}(t)$ of (1.1) is locally asymptotically stable if there exists $\delta > 0$ such that $\|\mathcal{B}_\mathcal{U}\| < \delta$, it holds

$$\lim_{t\to\infty}\|\mathcal{B}(t,\mathcal{U},\mathcal{B}_\mathcal{U})\| = 0.$$

If δ is arbitrary, then $\mathcal{B}(t)$ is globally asymptotically stable.

For the next theorem, we assume that s_l and t_{l+1} satisfy

$$(t_{l+1}-\mathcal{U})^{\varsigma}-(s_l-\mathcal{U})^{\varsigma}\le\eta<\infty,\ l=0,1,2,\cdots. \tag{3.1}$$

and define $\zeta=\sup\limits_{l\ge1}\max\limits_{t\in(t_l,s_l]}\|C_l(t)\|$.

Theorem 2. *Assume that* (3.1) *holds. If*

$$\frac{\eta}{\varsigma}+\frac{1}{\xi}\ln\zeta<0, \tag{3.2}$$

then (1.1) *is asymptotically stable.*

Proof. Combining Lemma 3 with (3.1), one has

$$\begin{aligned}\|\Gamma(t,\mathcal{U})\|&\le e^{\frac{\xi}{\varsigma}[((t-\mathcal{U})^{\varsigma}-(s_{\chi(\mathcal{U},t)}-\mathcal{U})^{\varsigma})+\sum\limits_{l=0}^{\chi(\mathcal{U},t)-1}((t_{l+1}-\mathcal{U})^{\varsigma}-(s_l-\mathcal{U})^{\varsigma})]}\zeta^{\chi(\mathcal{U},t)}\\&\le e^{\frac{\xi}{\varsigma}[((t-\mathcal{U})^{\varsigma}-(s_{\chi(\mathcal{U},t)}-\mathcal{U})^{\varsigma})+\sum\limits_{l=0}^{\chi(\mathcal{U},t)-1}\eta]}\zeta^{\chi(\mathcal{U},t)}\\&\le e^{\frac{\xi\eta}{\varsigma}}e^{\chi(\mathcal{U},t)(\frac{\xi\eta}{\varsigma}+\ln\zeta)}.\end{aligned}$$

By (3.2), there is

$$e^{\xi(\frac{\eta}{\varsigma}+\frac{1}{\xi}\ln\zeta)}<1,$$

so

$$\lim_{t\to\infty}\|\Gamma(t,\mathcal{U})\|\le e^{\frac{\xi\eta}{\varsigma}}e^{\chi(\mathcal{U},t)(\frac{\xi\eta}{\varsigma}+\ln\zeta)}\to0.$$

□

Theorem 3. *Suppose that*

$$\lim_{t\to\infty}\frac{\chi(\mathcal{U},t)}{((t-\mathcal{U})^{\varsigma}-(s_{\chi(\mathcal{U},t)}-\mathcal{U})^{\varsigma})+\sum\limits_{l=0}^{\chi(\mathcal{U},t)-1}((t_{l+1}-\mathcal{U})^{\varsigma}-(s_l-\mathcal{U})^{\varsigma})}:=\rho<\infty. \tag{3.3}$$

If $\vartheta=\frac{\xi}{\varsigma}+\rho\ln\zeta<0$, *then* (1.1) *is asymptotically stable.*

Proof.

$$\begin{aligned}\|\Gamma(t,\mathcal{U})\|&\le e^{\frac{\xi}{\varsigma}[(t-\mathcal{U})^{\varsigma}-(s_{\chi(\mathcal{U},t)}-\mathcal{U})^{\varsigma})+\sum\limits_{l=0}^{\chi(\mathcal{U},t)-1}((t_{l+1}-\mathcal{U})^{\varsigma}-(s_l-\mathcal{U})^{\varsigma})]}\zeta^{\chi(\mathcal{U},t)}\\&\le e^{\vartheta[((t-\mathcal{U})^{\varsigma}-(s_{\chi(\mathcal{U},t)}-\mathcal{U})^{\varsigma})+\sum_{l=0}^{\chi(\mathcal{U},t)-1}((t_{l+1}-\mathcal{U})^{\varsigma}-(s_l-\mathcal{U})^{\varsigma})]}\\&\quad\times e^{\ln\zeta\left(\chi(\mathcal{U},t)-\rho[((t-\mathcal{U})^{\varsigma}-(s_{\chi(\mathcal{U},t)}-\mathcal{U})^{\varsigma})+\sum_{l=0}^{\chi(\mathcal{U},t)-1}((t_{l+1}-\mathcal{U})^{\varsigma}-(s_l-\mathcal{U})^{\varsigma})]\right)}.\end{aligned}$$

Using (3.3), for any enough large t, one has

$$\left|\frac{\chi(\mathcal{U},t)}{((t-\mathcal{U})^{\varsigma}-(s_{\chi(\mathcal{U},t)}-\mathcal{U})^{\varsigma})+\sum\limits_{l=0}^{\chi(\mathcal{U},t)-1}((t_{l+1}-\mathcal{U})^{\varsigma}-(s_l-\mathcal{U})^{\varsigma})}-\rho\right|\le-\frac{\vartheta}{2\ln\zeta}.$$

Then

$$
\begin{aligned}
&e^{\ln\zeta\left(\chi(\mathcal{U},t)-\rho[((t-\mathcal{U})^{\varsigma}-(s_{\chi(\mathcal{U},t)}-\mathcal{U})^{\varsigma})+\sum_{l=0}^{\chi(\mathcal{U},t)-1}((t_{l+1}-\mathcal{U})^{\varsigma}-(s_l-\mathcal{U})^{\varsigma})]\right)}\\
&= e^{\ln\zeta[((t-\mathcal{U})^{\varsigma}-(s_{\chi(\mathcal{U},t)}-\mathcal{U})^{\varsigma})+\sum_{l=0}^{\chi(\mathcal{U},t)-1}((t_{l+1}-\mathcal{U})^{\varsigma}-(s_l-\mathcal{U})^{\varsigma})]\left(\frac{\chi(\mathcal{U},t)}{((t-\mathcal{U})^{\varsigma}-(s_{\chi(\mathcal{U},t)}-\mathcal{U})^{\varsigma})+\sum_{l=0}^{\chi(\mathcal{U},t)-1}((t_{l+1}-\mathcal{U})^{\varsigma}-(s_l-\mathcal{U})^{\varsigma})}-\rho\right)}\\
&\le e^{-\frac{\vartheta}{2}[((t-\mathcal{U})^{\varsigma}-(s_{\chi(\mathcal{U},t)}-\mathcal{U})^{\varsigma})+\sum_{l=0}^{\chi(\mathcal{U},t)-1}((t_{l+1}-\mathcal{U})^{\varsigma}-(s_l-\mathcal{U})^{\varsigma})]}.
\end{aligned}
$$

Hence,

$$
\lim_{t\to\infty}\|\Gamma(t,\mathcal{U})\| \le e^{\frac{\vartheta}{2}[((t-\mathcal{U})^{\varsigma}-(s_{\chi(\mathcal{U},t)}-\mathcal{U})^{\varsigma})+\sum_{l=0}^{\chi(\mathcal{U},t)-1}((t_{l+1}-\mathcal{U})^{\varsigma}-(s_l-\mathcal{U})^{\varsigma})]} \to 0.
$$

□

4. Linear Perturbed Problem

This section considers the stability of (1.2) by $\Gamma(\cdot,\cdot)$.

We firstly study the following problem:

$$
\begin{cases}
\mathfrak{D}_{\varsigma}^{\mathcal{U}}\mathcal{B}(t) = \mathcal{A}(t)\mathcal{B}(t) + j(t),\ t\in(s_l,t_{l+1}],\ l\in N_0,\\
\mathcal{B}(t_l^{+}) = \mathcal{C}_l(t)\mathcal{B}(t_l^{-}) + m_l,\ l\in N,\\
\mathcal{B}(t) = \mathcal{C}_l(t)\mathcal{B}(t_l^{-}) + m_l,\ t\in(t_l,s_l],\ l\in N,\\
\mathcal{B}(s_l^{+}) = \mathcal{B}(s_l^{-}),\ l\in N,
\end{cases}
\tag{4.1}
$$

where $\mathcal{A}(t),\ \mathcal{C}_l(t),\ s_l,\ t_l$ are defined in (1.1), $m_l\in\mathbb{R}^n$ and $j\in C(\mathbb{U},\mathbb{R}^n)$.

Theorem 4. *A solution $\mathcal{B}\in PC(\mathbb{U},\mathbb{R}^n)$ of (4.1) with $\mathcal{B}(\mathcal{U})=\mathcal{B}_{\mathcal{U}}\in\mathbb{R}^n$ is*

$$
\begin{aligned}
\mathcal{B}(t) = \Gamma(t,\mathcal{U})\mathcal{B}_{\mathcal{U}} &+ \sum_{l=0}^{\chi(\mathcal{U},t)-1}\int_{s_l}^{t_{l+1}}\Gamma(t,s)j(s)(s-\mathcal{U})^{\varsigma-1}ds\\
&+\int_{s_{\chi(\mathcal{U},t)}}^{t}\Gamma(t,s)j(s)(s-\mathcal{U})^{\varsigma-1}ds + \sum_{l=1}^{\chi(\mathcal{U},t)}\Gamma(t,s_l)m_l.
\end{aligned}
$$

Proof. When $t\in[s_0,t_1]$, we set

$$
\mathfrak{D}_{\varsigma}^{\mathcal{U}}\mathcal{B}(t) = \mathcal{A}(t)\mathcal{B}(t),
\tag{4.2}
$$

and Lemma 1 implies that the solution of (4.2) is

$$
\mathcal{B}(t) = \Gamma(t,\mathcal{U})\mathcal{B}_{\mathcal{U}}.
$$

Let $\mathcal{B}_{\mathcal{U}} = \mathcal{B}_{\mathcal{U}}(t)$,

$$
\begin{aligned}
\mathfrak{D}_{\varsigma}^{\mathcal{U}}\mathcal{B}(t) &= \mathfrak{D}_{\varsigma}^{\mathcal{U}}\Gamma(t,\mathcal{U})\mathcal{B}_{\mathcal{U}}(t) + \Gamma(t,\mathcal{U})\mathfrak{D}_{\varsigma}^{\mathcal{U}}\mathcal{B}_{\mathcal{U}}(t)\\
&= \mathcal{A}(t)\mathcal{B}(t) + \Gamma(t,\mathcal{U})(t-\mathcal{U})^{1-\varsigma}\mathcal{B}'_{\mathcal{U}}(t)\\
&= \mathcal{A}(t)\mathcal{B}(t) + j(t),
\end{aligned}
$$

and then

$$
\mathcal{B}'_{\mathcal{U}}(t) = \Gamma^{-1}(t,\mathcal{U})j(t)(t-\mathcal{U})^{\varsigma-1},
$$

$$\mathcal{B}_{\mathcal{U}}(t)=\int_{\mathcal{U}}^{t}\Gamma^{-1}(s,\mathcal{U})j(s)(s-\mathcal{U})^{\varsigma-1}ds+\mathcal{B}_{\mathcal{U}}.$$

Then

$$\begin{aligned}\mathcal{B}(t)&=\Gamma(t,\mathcal{U})\left[\int_{\mathcal{U}}^{t}\Gamma^{-1}(s,\mathcal{U})j(s)(s-\mathcal{U})^{\varsigma-1}ds+\mathcal{B}_{\mathcal{U}}\right]\\&=\Gamma(t,\mathcal{U})\mathcal{B}_{\mathcal{U}}+\int_{\mathcal{U}}^{t}\Gamma(t,s)j(s)(s-\mathcal{U})^{\varsigma-1}ds.\end{aligned}$$

When $t\in(t_1,s_1]$,

$$\begin{aligned}\mathcal{B}(t)&=\mathcal{C}_1(t)\mathcal{B}(t_1^-)+m_1\\&=\mathcal{C}_1(t)\Gamma(t_1,\mathcal{U})\mathcal{B}_{\mathcal{U}}+\mathcal{C}_1(t)\int_{\mathcal{U}}^{t_1}\Gamma(t_1,s)j(s)(s-\mathcal{U})^{\varsigma-1}ds+m_1.\end{aligned}$$

When $t\in(s_1,t_2]$,

$$\begin{aligned}\mathcal{B}(t)&=\Gamma(t,s_1)\mathcal{B}(s_1)+\int_{s_1}^{t}\Gamma(t,s)j(s)(s-\mathcal{U})^{\varsigma-1}ds\\&=\Gamma(t,s_1)\mathcal{C}_1(s_1)\Gamma(t_1,\mathcal{U})\mathcal{B}_{\mathcal{U}}+\Gamma(t,s_1)\mathcal{C}_1(s_1)\int_{\mathcal{U}}^{t_1}\Gamma(t_1,s)j(s)(s-\mathcal{U})^{\varsigma-1}ds\\&\quad+\Gamma(t,s_1)m_1+\int_{s_1}^{t}\Gamma(t,s)j(s)(s-\mathcal{U})^{\varsigma-1}ds\\&=\Gamma(t,\mathcal{U})\mathcal{B}_{\mathcal{U}}+\int_{\mathcal{U}}^{t_1}\Gamma(t,s)j(s)(s-\mathcal{U})^{\varsigma-1}ds+\int_{s_1}^{t}\Gamma(t,s)j(s)(s-\mathcal{U})^{\varsigma-1}ds\\&\quad+\Gamma(t,s_1)m_1.\end{aligned}$$

If the following equality holds:

$$\begin{aligned}\mathcal{B}(t)=\Gamma(t,\mathcal{U})\mathcal{B}_{\mathcal{U}}&+\sum_{l=0}^{\chi(\mathcal{U},t)-1}\int_{s_l}^{t_{l+1}}\Gamma(t,s)j(s)(s-\mathcal{U})^{\varsigma-1}ds\\&+\int_{s_{\chi(\mathcal{U},t)}}^{t}\Gamma(t,s)j(s)(s-\mathcal{U})^{\varsigma-1}ds+\sum_{l=1}^{\chi(\mathcal{U},t)}\Gamma(t,s_l)m_l,\end{aligned}$$

for $t\in(s_{\chi(\mathcal{U},t)},t_{\chi(\mathcal{U},t)+1}]$, then

$$\begin{aligned}\mathcal{B}(t)&=\mathcal{C}_l(t)\mathcal{B}(t_{\chi(\mathcal{U},t)+1}^-)+m_{\chi(\mathcal{U},t)+1}\\&=\mathcal{C}_l(t)\left[\Gamma(t_{\chi(\mathcal{U},t)+1},\mathcal{U})\mathcal{B}_{\mathcal{U}}+\sum_{l=0}^{\chi(\mathcal{U},t)-1}\int_{s_l}^{t_{l+1}}\Gamma(t_{\chi(\mathcal{U},t)+1},s)j(s)(s-\mathcal{U})^{\varsigma-1}ds\right.\\&\quad\left.+\int_{s_{\chi(\mathcal{U},t)}}^{t_{\chi(\mathcal{U},t)+1}}\Gamma(t_{\chi(\mathcal{U},t)+1},s)j(s)(s-\mathcal{U})^{\varsigma-1}ds+\sum_{l=1}^{\chi(\mathcal{U},t)}\Gamma(t_{\chi(\mathcal{U},t)+1},s_l)m_l\right]\\&\quad+m_{\chi(\mathcal{U},t)+1},\qquad t\in(t_{\chi(\mathcal{U},t)+1},s_{\chi(\mathcal{U},t)+1}].\end{aligned}$$

Hence, for $t \in (s_{\chi(\mathcal{U},t)+1}, t_{\chi(\mathcal{U},t)+2}]$,

$$\begin{aligned}
\mathcal{B}(t) &= \Gamma(t, s_{\chi(\mathcal{U},t)+1})\mathcal{B}(s_{\chi(\mathcal{U},t)+1}) + \int_{s_{\chi(\mathcal{U},t)+1}}^{t} \Gamma(t,s) j(s)(s-\mathcal{U})^{\varsigma-1} ds \\
&= \Gamma(t, s_{\chi(\mathcal{U},t)+1}) C_l(s_{\chi(\mathcal{U},t)+1}) \Gamma(t_{\chi(\mathcal{U},t)+1}, \mathcal{U})\mathcal{B}_{\mathcal{U}} \\
&\quad + \Gamma(t, s_{\chi(\mathcal{U},t)+1}) C_l(s_{\chi(\mathcal{U},t)+1}) \sum_{l=0}^{\chi(\mathcal{U},t)-1} \int_{s_l}^{t_{l+1}} \Gamma(t_{\chi(\mathcal{U},t)+1}, s) j(s)(s-\mathcal{U})^{\varsigma-1} ds \\
&\quad + \Gamma(t, s_{\chi(\mathcal{U},t)+1}) C_l(s_{\chi(\mathcal{U},t)+1}) \int_{s_{\chi(\mathcal{U},t)}}^{t_{\chi(\mathcal{U},t)+1}} \Gamma(t_{\chi(\mathcal{U},t)+1}, s) j(s)(s-\mathcal{U})^{\varsigma-1} ds \\
&\quad + \Gamma(t, s_{\chi(\mathcal{U},t)+1}) C_l(s_{\chi(\mathcal{U},t)+1}) \sum_{l=1}^{\chi(\mathcal{U},t)} \Gamma(t_{\chi(\mathcal{U},t)+1}, s_l) m_l \\
&\quad + \Gamma(t, s_{\chi(\mathcal{U},t)+1}) m_{\chi(\mathcal{U},t)+1} + \int_{s_{\chi(\mathcal{U},t)+1}}^{t} \Gamma(t,s) j(s)(s-\mathcal{U})^{\varsigma-1} ds \\
&= \Gamma(t, \mathcal{U})\mathcal{B}_{\mathcal{U}} + \sum_{l=0}^{\chi(\mathcal{U},t)} \int_{s_l}^{t_{l+1}} \Gamma(t,s) j(s)(s-\mathcal{U})^{\varsigma-1} ds \\
&\quad + \int_{s_{\chi(\mathcal{U},t)+1}}^{t} \Gamma(t,s) j(s)(s-\mathcal{U})^{\varsigma-1} ds + \sum_{l=1}^{\chi(\mathcal{U},t)+1} \Gamma(t, s_l) m_l.
\end{aligned}$$

Using mathematical induction, the solution of (4.1) is:

$$\begin{aligned}
\mathcal{B}(t) = \Gamma(t, \mathcal{U})\mathcal{B}_{\mathcal{U}} &+ \sum_{l=0}^{\chi(\mathcal{U},t)-1} \int_{s_l}^{t_{l+1}} \Gamma(t,s) j(s)(s-\mathcal{U})^{\varsigma-1} ds \\
&+ \int_{s_{\chi(\mathcal{U},t)}}^{t} \Gamma(t,s) j(s)(s-\mathcal{U})^{\varsigma-1} ds + \sum_{l=1}^{\chi(\mathcal{U},t)} \Gamma(t, s_l) m_l.
\end{aligned}$$

□

Then, the solution of (1.2) is

$$\begin{aligned}
\mathcal{B}(t) = \Gamma(t, \mathcal{U})\mathcal{B}_{\mathcal{U}} &+ \sum_{l=0}^{\chi(\mathcal{U},t)-1} \int_{s_l}^{t_{l+1}} \Gamma(t,s) \mathcal{J}(s)\mathcal{B}(s)(s-\mathcal{U})^{\varsigma-1} ds \\
&+ \int_{s_{\chi(\mathcal{U},t)}}^{t} \Gamma(t,s) \mathcal{J}(s)\mathcal{B}(s)(s-\mathcal{U})^{\varsigma-1} ds + \sum_{l=1}^{\chi(\mathcal{U},t)} \Gamma(t, s_l)\mathcal{M}_l \mathcal{B}(t_l^-).
\end{aligned} \tag{4.3}$$

Theorem 5. *Assume that* (1.1) *is exponentially stable. If*

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{\chi(\mathcal{U},t)}{t} = \varpi < \infty, \ \sup_{t\in\mathbb{U}} \frac{(t-\mathcal{U})^{\varsigma}}{\varsigma t} = \rho < \infty, \\
&\limsup_{t\to\infty} \|\mathcal{J}(t)\| < \delta, \ \limsup_{l\to\infty} \|\mathcal{M}_l\| < \delta,
\end{aligned}$$

in which ϖ , ρ *and* δ *satisfy*

$$Upsilon = c_2 - c_1\delta\rho - \varpi\ln(1+J\delta) > 0,$$

then (1.2) *is exponentially stable.*

Proof. Since the exponential stability of (1.1), one has $c_1 \geq 1$ and $c_2 > 0$ with

$$\|\Gamma(t,s)\| \leq c_1 e^{-c_2(t-s)}, \mathcal{U} \leq s \leq t.$$

When $t \in [\mathcal{U}, t_{l_{\mathcal{U}}+1}]$, $l_{\mathcal{U}} \in \mathbb{N}$, $J_{\mathcal{U}} = \max\limits_{t\in[\mathcal{U},t_{l_{\mathcal{U}}+1}]} \|\mathcal{J}(t)\|$, $\mathcal{M}_{\mathcal{U}} = \max\limits_{l\in\{1,2,\cdots l_{\mathcal{U}}\}} \|\mathcal{M}_l\|$, (4.3) gives

$$\|\mathcal{B}(t)\| \leq c_1\|\mathcal{B}_{\mathcal{U}}\| + c_1 J_{\mathcal{U}} \int_{\mathcal{U}}^{t} \|\mathcal{B}(s)\|(s-\mathcal{U})^{\varsigma-1}ds + c_1\mathcal{M}_{\mathcal{U}} \sum_{l=1}^{\chi(\mathcal{U},t)} \|\mathcal{B}(t_l^-)\|.$$

With Lemma 2, we yield

$$\begin{aligned}\|\mathcal{B}(t)\| &\leq c_1\|\mathcal{B}_{\mathcal{U}}\|(1+c_1\mathcal{M}_{\mathcal{U}})^{\chi(\mathcal{U},t)} e^{c_1 J_{\mathcal{U}} \frac{(t-\mathcal{U})^{\varsigma}}{\varsigma}} \\ &\leq c_1\|\mathcal{B}_{\mathcal{U}}\|(1+c_1\mathcal{M}_{\mathcal{U}})^{\chi(\mathcal{U},t)} e^{c_1 J_{\mathcal{U}} \frac{(t_{l_{\mathcal{U}}+1}-\mathcal{U})^{\varsigma}}{\varsigma}} = C_{\mathcal{U}}\|\mathcal{B}_{\mathcal{U}}\|,\end{aligned}$$

in which

$$C_{\mathcal{U}} = c_1(1+c_1\mathcal{M}_{\mathcal{U}})^{\chi(\mathcal{U},t)} e^{c_1 J_{\mathcal{U}} \frac{(t_{l_{\mathcal{U}}+1}-\mathcal{U})^{\varsigma}}{\varsigma}}.$$

For $\mathcal{J}(t)\| \leq \delta$ and $\|\mathcal{M}_l\| \leq \delta$ with any $t > t_{l_{\mathcal{U}}}$ and $\chi(\mathcal{U},t) > l_{\mathcal{U}}$, taking the norm of (4.3), there is

$$\begin{aligned}e^{c_2 t}\|\mathcal{B}(t)\| &\leq (c_1+\bar{c}_1)\|\mathcal{B}_{\mathcal{U}}\| + c_1\delta \sum_{l=l_{\mathcal{U}}}^{\chi(\mathcal{U},t)-1} \int_{s_l}^{t_{l+1}} e^{c_2 s}\|\mathcal{B}(s)\|(s-\mathcal{U})^{\varsigma-1}ds \\ &\quad + c_1\delta\int_{s_{\chi(\mathcal{U},t)}}^{t} e^{c_2 s}\|\mathcal{B}(s)\|(s-\mathcal{U})^{\varsigma-1}ds + c_1\delta\sum_{l=l_{\mathcal{U}}}^{\chi(\mathcal{U},t)} e^{c_2 s_l}\|\mathcal{B}(t_l^-)\| \\ &\leq \widetilde{c}_1\|\mathcal{B}_{\mathcal{U}}\| + c_1\delta\int_{\mathcal{U}}^{t} e^{c_2 s}\|\mathcal{B}(s)\|(s-\mathcal{U})^{\varsigma-1}ds + c_1\delta\sum_{l=1}^{\chi(\mathcal{U},t)} e^{c_2 s_l}\|\mathcal{B}(t_l^-)\|,\end{aligned}$$

where $\bar{c}_1 = c_1 C_{\mathcal{U}} \sum\limits_{l=0}^{l_{\mathcal{U}}-1} \int_{s_l}^{t_{l+1}} e^{c_2 s}\|\mathcal{J}(s)\|(s-\mathcal{U})^{\varsigma-1}ds + c_1 C_{\mathcal{U}} \sum\limits_{l=0}^{l_{\mathcal{U}}} e^{c_2 s_l}\|\mathcal{M}_l\|$, $\widetilde{c}_1 = c_1 + \bar{c}_1$. Summarizing, we have

$$e^{c_2 t}\|\mathcal{B}(t)\| \leq c\|\mathcal{B}_{\mathcal{U}}\| + c_1\delta\int_{\mathcal{U}}^{t} e^{c_2 s}\|\mathcal{B}(s)\|(s-\mathcal{U})^{\varsigma-1}ds + c_1\delta\sum_{l=1}^{\chi(\mathcal{U},t)} e^{c_2 s_l}\|\mathcal{B}(t_l^-)\|,$$

for any $t \geq \mathcal{U}$, in which $c = \max\{\widetilde{c}_1, C_{\mathcal{U}} e^{c_2 t_{l_{\mathcal{U}}}}\}$.

According to Lemma 2, we obtain

$$e^{c_2 t}\|\mathcal{B}(t)\| \leq c(1+c_1\delta)^{\chi(\mathcal{U},t)} e^{c_1\delta\frac{(t-\mathcal{U})^{\varsigma}}{\varsigma}}\|\mathcal{B}_{\mathcal{U}}\|,$$

so

$$\|\mathcal{B}(t)\| \leq ce^{-\left(c_2-c_1\delta\frac{(t-\mathcal{U})^{\varsigma}}{\varsigma t}-\frac{\chi(\mathcal{U},t)}{t}\ln(1+c_1\delta)\right)t}\|\mathcal{B}_{\mathcal{U}}\|.$$

Due to

$$\begin{aligned}\liminf_{t\to\infty}\left(c_2-c_1\delta\frac{(t-\mathcal{U})^{\varsigma}}{\varsigma t}-\frac{\chi(\mathcal{U},t)}{t}\ln(1+c_1\delta)\right)\\ \geq c_2-c_1\delta\rho-\varpi\ln(1+c_1\delta)=\Upsilon>0,\end{aligned}$$

we can find $t_1>\mathcal{U}$ and

$$c_2-c_1\delta\frac{(t-\mathcal{U})^{\varsigma}}{\varsigma t}-\frac{\chi(\mathcal{U},t)}{t}\ln(1+c_1\delta)\geq\frac{\Upsilon}{2},$$

for $t\geq t_1$. Hence, when $t\geq t_1$, we obtain

$$\|\mathcal{B}(t)\| \leq ce^{-\frac{\Upsilon}{2}t}\|\mathcal{B}_{\mathcal{U}}\|.$$

□

5. Nonlinear Problem

This section presents the Ulam-Hyers-Rassias stability of (1.3).
The solution of (1.3) is:

$$\begin{aligned}\mathcal{B}(t)=\Gamma(t,\mathcal{U})\mathcal{B}_{\mathcal{U}}+\sum_{l=0}^{\chi(\mathcal{U},t)-1}\int_{s_l}^{t_{l+1}}\Gamma(t,s)z(s,\mathcal{B}(s))(s-\mathcal{U})^{\varsigma-1}ds\\ +\int_{s_{\chi(\mathcal{U},t)}}^{t}\Gamma(t,s)z(s,\mathcal{B}(s))(s-\mathcal{U})^{\varsigma-1}ds+\sum_{l=1}^{\chi(\mathcal{U},t)}\Gamma(t,s_l)\mathcal{L}_l.\end{aligned}\tag{5.1}$$

Then, we present the condition: (H_1) $z\in C(\mathbb{U}\times\mathbb{R}^n,\mathbb{R}^n)$ and there is a $\mathcal{L}_z>0$ satisfying:

$$\|z(t,\mathcal{B}_1)-z(t,\mathcal{B}_2)\|\leq\mathcal{L}_z\|\mathcal{B}_1-\mathcal{B}_2\|,\ t\in\mathbb{U},\ \mathcal{B}_1,\ \mathcal{B}_2\in\mathbb{R}^n.$$

Theorem 6. *Assume that (H_1) holds. If system (1.1) is exponentially stable and*

$$\mathcal{L}_z c_1\frac{(\mathcal{V}-\mathcal{U})^{\varsigma}}{\varsigma}<1,$$

then (1.3) has a unique solution.

Proof. We set a mapping $\mathcal{P}:PC(\mathbb{U},\mathbb{R}^n)\to PC(\mathbb{U},\mathbb{R}^n)$,

$$\begin{aligned}(\mathcal{P}\mathcal{B})(t)=\Gamma(t,\mathcal{U})\mathcal{B}_{\mathcal{U}}+\sum_{l=0}^{\chi(\mathcal{U},t)-1}\int_{s_l}^{t_{l+1}}\Gamma(t,s)z(s,\mathcal{B}(s))(s-\mathcal{U})^{\varsigma-1}ds\\ +\int_{s_{\chi(\mathcal{U},t)}}^{t}\Gamma(t,s)z(s,\mathcal{B}(s))(s-\mathcal{U})^{\varsigma-1}ds+\sum_{l=1}^{\chi(\mathcal{U},t)}\Gamma(t,s_l)\mathcal{L}_l,\end{aligned}$$

and

$$\begin{aligned}
&\|(\mathcal{P}\mathcal{B})(t)-(\mathcal{P}\overline{\mathcal{B}})(t)\| \\
&\qquad\le \sum_{l=0}^{\chi(\mathcal{U},t)-1}\int_{s_l}^{t_{l+1}}\left\|\Gamma(t,s)(s-\mathcal{U})^{\varsigma-1}\left[z(s,\mathcal{B}(s))-z(s,\overline{\mathcal{B}}(s))\right]\right\|ds \\
&\qquad\quad +\int_{s_{\chi(\mathcal{U},t)}}^{t}\left\|\Gamma(t,s)(s-\mathcal{U})^{\varsigma-1}\left[z(s,\mathcal{B}(s))-z(s,\overline{\mathcal{B}}(s))\right]\right\|ds \\
&\qquad\le \mathcal{L}_z\Bigg(\sum_{l=0}^{\chi(\mathcal{U},t)-1}\int_{s_l}^{t_{l+1}}c_1e^{-c_2(t-s)}(s-\mathcal{U})^{\varsigma-1}ds \\
&\qquad\quad +\int_{s_{\chi(\mathcal{U},t)}}^{t}c_1e^{-c_2(t-s)}(s-\mathcal{U})^{\varsigma-1}ds\Bigg)\|\mathcal{B}-\overline{\mathcal{B}}\|_{PC} \\
&\qquad\le \mathcal{L}_z\left(\int_{\mathcal{U}}^{t}c_1(s-\mathcal{U})^{\varsigma-1}ds\right)\|\mathcal{B}-\overline{\mathcal{B}}\|_{PC} \\
&\qquad\le \mathcal{L}_zc_1\frac{(\mathcal{V}-\mathcal{U})^{\varsigma}}{\varsigma}\|\mathcal{B}-\overline{\mathcal{B}}\|_{PC}, \qquad (5.2)
\end{aligned}$$

for $\forall\ \mathcal{B},\ \overline{\mathcal{B}}\in PC(\mathbb{U},\mathbb{R}^n)$.

Then (5.2) implies that

$$\|(\mathcal{P}\mathcal{B})(t)-(\mathcal{P}\overline{\mathcal{B}})(t)\|\le\mathcal{L}_{\mathcal{P}}\|\mathcal{B}-\overline{\mathcal{B}}\|_{PC},$$

in which

$$\mathcal{L}_{\mathcal{P}}:=\mathcal{L}_zc_1\frac{(\mathcal{V}-\mathcal{U})^{\varsigma}}{\varsigma},$$

so

$$\|(\mathcal{P}\mathcal{B})-(\mathcal{P}\overline{\mathcal{B}})\|_{PC}\le\mathcal{L}_{\mathcal{P}}\|\mathcal{B}-\overline{\mathcal{B}}\|_{PC}.$$

Hence, (1.3) has a unique solution. □

Next, we refer to Ulam-Hyers-Rassias stability of (1.3).

Set $\varepsilon>0$, $\kappa>0$ and $\mu\in PC(\mathbb{U},\mathbb{R}_+)$ be a nondecreasing function, and

$$\begin{cases}
\|\mathfrak{D}_{\varsigma}^{\mathcal{U}}\widetilde{\mathcal{B}}(t)-\mathcal{A}(t)\widetilde{\mathcal{B}}(t)-z(t,\widetilde{\mathcal{B}}(t))\|\le\varepsilon\mu(t),\ t\in(s_l,t_{l+1}],\ l\in N_0,\\
\|\widetilde{\mathcal{B}}(t_l^+)-\mathcal{C}_l(t_l^+)\mathcal{B}(t_l^-)-\mathcal{L}_l\|\le\varepsilon\kappa,\ l\in N,\\
\|\widetilde{\mathcal{B}}(t)-\mathcal{C}_l(t)\mathcal{B}(t_l^-)-\mathcal{L}_l\|\le\varepsilon\kappa,\ t\in(t_l,s_l],\ l\in N.
\end{cases} \qquad (5.3)$$

Set

$$D:=PC^1(\mathbb{U},\mathbb{R}^n)\bigcap_{l\in\mathbb{N}}C^1\left((s_l,t_{l+1}],\mathbb{R}^n\right).$$

Definition 5. We set a solution $\mathcal{B}\in D$ of (1.3) is Ulam-Hyers-Rassias stable if for each $\varepsilon>0$ and for each function $\widetilde{\mathcal{B}}\in D$ satisfying (5.3) with $\widetilde{\mathcal{B}}(\mathcal{U})=\mathcal{B}_{\mathcal{U}}$, there is a

$c_{z,N,\mu} > 0$ satisfying

$$\|\widetilde{\mathcal{B}}(t) - \mathcal{B}(t)\| \le c_{z,N,\mu}\varepsilon\big(\mu(t) + \kappa\big),\ t \in \mathbb{U}.$$

Remark 2. (5.3) has a solution $\widetilde{\mathcal{B}} \in D$ iff there exists $G \in \bigcup_{l\in N_0} C^1\big((s_l, t_{l+1}], \mathbb{R}^n\big)$ and $g \in \bigcup_{l\in N} C\big((t_l, s_l], \mathbb{R}^n\big)$ satisfying

(i) $\|G(t)\| \le \varepsilon\mu(t),\ t \in \bigcup_{l\in N_0}(s_l, t_{l+1}],\ \|g(t)\| \le \varepsilon\kappa,\ t \in \bigcup_{l\in N}(t_l, s_l]$;

(ii) $\mathfrak{D}_{\varsigma}^{\mathcal{U}}\widetilde{\mathcal{B}}(t) = \mathcal{A}(t)\widetilde{\mathcal{B}}(t) + z(t, \widetilde{\mathcal{B}}(t)) + G(t),\ t \in (s_l, t_{l+1}],\ l \in N_0$;

(iii) $\widetilde{\mathcal{B}}(t_l^+) = \mathcal{C}_l(t_l^+)\mathcal{B}(t_l^-) + \mathcal{L}_l + g(t_l),\ l \in N$;

(iv) $\widetilde{\mathcal{B}}(t) = \mathcal{C}_l(t)\mathcal{B}(t_l^-) + \mathcal{L}_l + g(t),\ t \in (t_l, s_l],\ l \in N$.

We suppose that the following conditions hold:

(H_2) $\omega := \sup_{t\in\mathbb{U}} \sum_{l=1}^{\chi(\mathcal{U},t)} e^{-c_2(t_{\chi(\mathcal{U},t)+1} - s_l)} < \infty$ holds.

(H_3) There is a $\psi > 0$ satisfying

$$\int_{\mathcal{U}}^{t} e^{-c_2(t-s)}\mu(s)(s-\mathcal{U})^{\varsigma-1}ds \le \psi\mu(t),\ t \in \mathbb{U}.$$

Theorem 7. *Suppose that* (H_1), (H_2) *and* (H_3) *hold. If* (1.1) *is exponentially stable and*

$$c_1\mathcal{L}_z \max\Big\{\psi, \kappa\frac{(\mathcal{V}-\mathcal{U})^{\varsigma}}{\varsigma}\Big\} < 1 \tag{5.4}$$

is satisfied, then (1.3) *is Ulam-Hyers-Rassias stable.*

Proof. Let $\mathcal{B}(t)$ be a solution of (1.3) and $\widetilde{\mathcal{B}} \in D$ be a solution of (5.3) with $\widetilde{\mathcal{B}}(\mathcal{U}) = \mathcal{B}_{\mathcal{U}}$.

Consider

$$\begin{cases} \mathfrak{D}_{\varsigma}^{\mathcal{U}}\widetilde{\mathcal{B}}(t) = \mathcal{A}(t)\widetilde{\mathcal{B}}(t) + z(t, \widetilde{\mathcal{B}}(t)) + G(t),\ t \in (s_l, t_{l+1}],\ l \in N_0, \\ \widetilde{\mathcal{B}}(t_l^+) = \mathcal{C}_l(t_l^+)\mathcal{B}(t_l^-) + \mathcal{L}_l + g(t_l),\ l \in N, \\ \widetilde{\mathcal{B}}(t) = \mathcal{C}_l(t)\mathcal{B}(t_l^-) + \mathcal{L}_l + g(t),\ t \in (t_l, s_l],\ l \in N. \end{cases} \tag{5.5}$$

The solution $\widetilde{\mathcal{B}}(t)$ of (5.5) with $\widetilde{\mathcal{B}}(\mathcal{U}) = \mathcal{B}_{\mathcal{U}}$ has the following expression:

$$\begin{aligned} \widetilde{\mathcal{B}}(t) = \Gamma(t,\mathcal{U})\mathcal{B}_{\mathcal{U}} &+ \sum_{l=0}^{\chi(\mathcal{U},t)-1} \int_{s_l}^{t_{l+1}} \Gamma(t,s)\big[z(s,\widetilde{\mathcal{B}}(s)) + G(s)\big](s-\mathcal{U})^{\varsigma-1}ds \\ &+ \int_{s_{\chi(\mathcal{U},t)}}^{t} \Gamma(t,s)\big[z(s,\widetilde{\mathcal{B}}(s)) + G(s)\big](s-\mathcal{U})^{\varsigma-1}ds \\ &+ \sum_{l=1}^{\chi(\mathcal{U},t)} \Gamma(t,s_l)\big(\mathcal{L}_l + g(s_l)\big) = (\mathcal{P}\widetilde{\mathcal{B}})(t) + \widetilde{f}(t), \end{aligned}$$

in which

$$\widetilde{f}(t)=\sum_{l=0}^{\chi(\mathcal{U},t)-1}\int_{s_l}^{t_{l+1}}\Gamma(t,s)G(s)(s-\mathcal{U})^{\varsigma-1}ds+\int_{s_{(\mathcal{U},t)}}^{t}\Gamma(t,s)G(s)(s-\mathcal{U})^{\varsigma-1}ds+\sum_{l=1}^{\chi(\mathcal{U},t)}\Gamma(t,s_l)g(s_l).$$

Combining (H_2) with (H_3), there is

$$\begin{aligned}\|\widetilde{f}(t)\|&\le\sum_{l=0}^{\chi(\mathcal{U},t)-1}\int_{s_l}^{t_{l+1}}\|\Gamma(t,s)\|\|G(s)\|(s-\mathcal{U})^{\varsigma-1}ds\\&\quad+\int_{s_{(\mathcal{U},t)}}^{t}\|\Gamma(t,s)\|\|G(s)\|(s-\mathcal{U})^{\varsigma-1}ds+\sum_{l=1}^{\chi(\mathcal{U},t)}\|\Gamma(t,s_l)\|\|g(s_l)\|\\&\le\varepsilon c_1\left(\int_{\mathcal{U}}^{t}e^{-c_2(t-s)}\mu(s)(s-\mathcal{U})^{\varsigma-1}ds+\sum_{l=1}^{\chi(\mathcal{U},t)}e^{-c_2(t-s_l)}\kappa\right)\\&\le\varepsilon c_1\left(\psi\mu(t)+\sum_{l=1}^{\chi(\mathcal{U},t)-1}e^{-c_2(t_{\chi(\mathcal{U},t)}-s_l)}\kappa+\kappa\right)\\&\le\varepsilon c_1\max\{1+\omega,\psi\}\big(\mu(t)+\kappa\big),\ t\in(t_{\chi(\mathcal{U},t)},t_{\chi(\mathcal{U},t)+1}].\end{aligned}\tag{5.6}$$

Using (5.1) and (5.6), we get

$$\begin{aligned}\|\widetilde{\mathcal{B}}(t)-\mathcal{B}(t)\|&\le\|\widetilde{f}(t)\|+\|(\mathcal{P}\widetilde{\mathcal{B}})(t)-(\mathcal{P}\mathcal{B})(t)\|\\&\le\varepsilon c_1\max\{1+\omega,\psi\}\big(\mu(t)+\kappa\big)\\&\quad+c_1\mathcal{L}_z\int_{\mathcal{U}}^{t}e^{-c_2(t-s)}\|\widetilde{\mathcal{B}}(s)-\mathcal{B}(s)\|(s-\mathcal{U})^{\varsigma-1}ds.\end{aligned}\tag{5.7}$$

Set

$$h(\mathcal{V}):=\sup_{t\in\mathbb{U}}\frac{\|\widetilde{\mathcal{B}}(t)-\mathcal{B}(t)\|}{\mu(t)+\kappa},$$

and combining (5.7) with (H_3), one has

$$\begin{aligned}\|\widetilde{\mathcal{B}}(t)-\mathcal{B}(t)\|&\le\varepsilon c_1\max\{1+\omega,\psi\}\big(\mu(t)+\kappa\big)\\&\quad+c_1\mathcal{L}_z h(\mathcal{V})\int_{\mathcal{U}}^{t}e^{-c_2(t-s)}\big(\mu(t)+\kappa\big)(s-\mathcal{U})^{\varsigma-1}ds\\&\le\varepsilon c_1\max\{1+\omega,\psi\}\big(\mu(t)+\kappa\big)\\&\quad+c_1\mathcal{L}_z h(\mathcal{V})\left[\psi\mu(t)+\int_{\mathcal{U}}^{t}\kappa e^{-c_2(t-s)}(s-\mathcal{U})^{\varsigma-1}ds\right]\\&\le\varepsilon c_1\max\{1+\omega,\psi\}\big(\mu(t)+\kappa\big)\end{aligned}$$

$$
\begin{aligned}
&+c_1\mathcal{L}_z h(\mathcal{V})\left[\psi\mu(t)+\kappa\frac{(\mathcal{V}-\mathcal{U})^{\varsigma}}{\varsigma}\right]\\
&\le \varepsilon c_1\max\{1+\omega,\psi\}\big(\mu(t)+\kappa\big)\\
&\quad +c_1\mathcal{L}_z h(\mathcal{V})\max\left\{\psi,\kappa\frac{(\mathcal{V}-\mathcal{U})^{\varsigma}}{\varsigma}\right\}\big(\mu(t)+\kappa\big),\ t\in\mathbb{U}. \qquad (5.8)
\end{aligned}
$$

(5.8) implies

$$
h(\mathcal{V})\le \varepsilon c_1\max\{1+\omega,\psi\}+c_1\mathcal{L}_z h(\mathcal{V})\max\left\{\psi,\kappa\frac{(\mathcal{V}-\mathcal{U})^{\varsigma}}{\varsigma}\right\}.
$$

Then,

$$
h(\mathcal{V})\le \varepsilon\frac{c_1\max\{1+\omega,\psi\}}{1-c_1\mathcal{L}_z\max\left\{\psi,\kappa\frac{(\mathcal{V}-\mathcal{U})^{\varsigma}}{\varsigma}\right\}},
$$

so

$$
\begin{aligned}
\|\widetilde{\mathcal{B}}(t)-\mathcal{B}(t)\|\le & \quad \varepsilon\frac{c_1\max\{1+\omega,\psi\}}{1-c_1\mathcal{L}_z\max\left\{\psi,\kappa\frac{(\mathcal{V}-\mathcal{U})^{\varsigma}}{\varsigma}\right\}}\big(\mu(t)+\kappa\big)\\
&\le c_{z,N,\mu}\varepsilon\big(\mu(t)+\kappa\big),\ t\in\mathbb{U}, \qquad (5.9)
\end{aligned}
$$

and $c_{z,N,\mu}=\frac{c_1\max\{1+\omega,\psi\}}{1-c_1\mathcal{L}_z\max\left\{\psi,\kappa\frac{(\mathcal{V}-\mathcal{U})^{\varsigma}}{\varsigma}\right\}}>0$.

Since (5.9) is satisfied for any $\mathcal{V}>0$, (1.3) is Ulam-Hyers-Rassias stable. □

Example 1. We set $s_0=\mathcal{U}=\frac{3}{2}$, $t_1=2$, $\mathcal{V}=20$, $\varsigma=\frac{1}{2}$, $s_l=l+\frac{3}{2}$, $t_{l+1}=l+2$, $l=1,2,\cdots$, and

$$
\mathcal{A}(t)=\begin{pmatrix}\frac{1}{2}(t-s_l) & 0\\ 0 & \frac{1}{4}(t-s_l)\end{pmatrix},\qquad \mathcal{C}_l(t)=\begin{pmatrix}\frac{1}{20}+\frac{1}{20}\frac{t-t_l}{s_l-t_l} & 0\\ 0 & \frac{1}{20}+\frac{1}{20}\frac{t-t_l}{s_l-t_l}\end{pmatrix},
$$

$$
\mathcal{B}_{\mathcal{U}}=\begin{pmatrix}1\\0\end{pmatrix},\qquad z(t,\mathcal{B}(t))=\begin{pmatrix}\frac{\cos t}{e^7}(\mathcal{B}_1(t)+1)\\ \frac{\cos t}{e^7}\mathcal{B}_2(t)\end{pmatrix},\qquad \mathcal{L}_l=\begin{pmatrix}1\\0\end{pmatrix}.
$$

Now $\chi(\mathcal{U},t)=\lceil t\rceil-2$. For $s_{\chi(\mathcal{U},t)}<t\le t_{\chi(\mathcal{U},t)+1}$ and $s_{\chi(\mathcal{U},s)}<s\le t_{\chi(\mathcal{U},s)+1}$, $\chi(\mathcal{U},t)>\chi(\mathcal{U},s)$, there is

$$
\begin{aligned}
\Gamma(t,s)\le & \begin{pmatrix}e^{(t-1)^{\frac{1}{2}}-(s_{\chi(\mathcal{U},t)}-1)^{\frac{1}{2}}} & 0\\ 0 & e^{\frac{1}{2}(t-1)^{\frac{1}{2}}-\frac{1}{2}(s_{\chi(\mathcal{U},t)}-1)^{\frac{1}{2}}}\end{pmatrix}\begin{pmatrix}\frac{1}{10} & 0\\ 0 & \frac{1}{10}\end{pmatrix}\\
&\times\prod_{l=\chi(\mathcal{U},s)+2}^{\chi(\mathcal{U},t)}\begin{pmatrix}\frac{1}{10}e^{\frac{1}{4}} & 0\\ 0 & \frac{1}{10}e^{\frac{1}{8}}\end{pmatrix}\\
&\times\begin{pmatrix}e^{(t_{\chi(\mathcal{U},s)+1}-1)^{\frac{1}{2}}-(s-1)^{\frac{1}{2}}} & 0\\ 0 & e^{\frac{1}{2}(t_{\chi(\mathcal{U},s)+1}-1)^{\frac{1}{2}}-\frac{1}{2}(s-1)^{\frac{1}{2}}}\end{pmatrix}
\end{aligned}
$$

$$= \begin{pmatrix} e^{\frac{1}{2}(t-\lceil t\rceil+\frac{1}{2})} & 0 \\ 0 & e^{\frac{1}{4}(t-\lceil t\rceil+\frac{1}{2})} \end{pmatrix} \begin{pmatrix} \frac{1}{10}e^{\frac{1}{4}} & 0 \\ 0 & \frac{1}{10}e^{\frac{1}{8}} \end{pmatrix}^{\lceil t\rceil-\lceil s\rceil-1} \times \begin{pmatrix} e^{\frac{1}{2}(\lceil s\rceil-s)} & 0 \\ 0 & e^{\frac{1}{4}(\lceil s\rceil-s)} \end{pmatrix}.$$

Hence

$$\|\Gamma(t,s)\| \le 10e^{\frac{1}{2}}(\frac{1}{10}e^{\frac{1}{4}})^{\lceil t\rceil-\lceil s\rceil} = 10e^{\frac{1}{2}}(e^{\frac{1}{4}-\ln 10})^{\lceil t\rceil-\lceil s\rceil} \le e^{\ln 10+\frac{1}{2}}e^{-(\ln 10-\frac{1}{4})(t-s)}.$$

The system is exponentially stable with $c_2 = \ln 10 - \frac{1}{4}$ and $c_1 = e^{\ln 10+\frac{1}{2}}$.

Note

$$\sup_{t\in\mathbb{U}} \sum_{l=1}^{\chi(\mathcal{U},t)} e^{-(\ln 10-\frac{1}{4})(\chi(\mathcal{U},t)+2-l-\frac{3}{2})} \le \frac{e^{\frac{\frac{1}{4}-\ln 10}{2}}}{1-e^{\frac{1}{4}-\ln 10}},$$

so (H_2) holds and $\omega \le \frac{e^{\frac{\frac{1}{4}-\ln 10}{2}}}{1-e^{\frac{1}{4}-\ln 10}}$.

Let $\mu(t) = t^{\frac{3}{4}}, \kappa = 1$, and we have

$$\int_{\frac{3}{2}}^{t} e^{-(\ln 10-\frac{1}{4})(t-s)}s^{\frac{3}{2}}s^{-\frac{1}{2}}ds = \int_{\frac{3}{2}}^{t} e^{(\ln 10-\frac{1}{4})(s-t)}sds \le \frac{t^{\frac{1}{4}}}{\ln 10-\frac{1}{4}} \le \frac{t^{\frac{3}{4}}}{\ln 10-\frac{1}{4}},$$

so (H_3) holds with $\psi = \frac{1}{\ln 10-\frac{1}{4}}$.

Then,

$$\mathcal{L}_z\kappa\frac{(\mathcal{V}-\mathcal{U})^{\varsigma}}{\varsigma} = \frac{2\sqrt{20}}{e^7} < 1,$$

so (1.3) has a unique solution from Theorem 6.

Also,

$$\mathcal{L}_z \max\left\{\psi, \kappa\frac{(\mathcal{V}-\mathcal{U})^{\varsigma}}{\varsigma}\right\} < 1,$$

and (5.4) are satisfied.

Theorem 7 implies that (1.3) is Ulam-Hyers-Rassias stable with

$$c_{z,N,\mu} = \frac{c_1\max\{1+\omega,\psi\}}{1-c_1\mathcal{L}_z\max\left\{\psi,\kappa\frac{(\mathcal{V}-\mathcal{U})^{\varsigma}}{\varsigma}\right\}} = \frac{e^{\ln 10+\frac{1}{2}}(\frac{e^{\frac{\frac{1}{4}-\ln 10}{2}}}{1-e^{\frac{1}{4}-\ln 10}}+1)}{1-\frac{2\sqrt{20}e^{\ln 10+\frac{1}{2}}}{e^7}} = 26.87.$$

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AMPLY ESSENTIAL SUPPLEMENTED LATTICES

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Abstract. In this work, amply essential supplemented (briefly, amply e-supplemented) lattices are defined and some properties of these lattices are investigated. All lattices are complete modular lattices with the greatest element 1 and the smallest element 0 in this work. Let L be a lattice. If $b/0$ is essential supplemented for every $b \in L$, then L is amply essential supplemented. Let L be an amply e-supplemented lattice, b be a supplement of a in L and $b \trianglelefteq L$. Then $b/0$ is amply e-supplemented. Here also $x/0$ is amply supplemented for every e-supplement element x in L. Let L be a lattice. Then L is amply essential supplemented if and only if $a \wedge b$ has a supplement in $b/0$ for every $a \trianglelefteq L$ and $b \in L$ with $1 = a \vee b$. Let L be a lattice. If $a \wedge b$ lies above an e-supplement element in L for every $a, b \in L$ with $a \trianglelefteq L$ and $1 = a \vee b$, then L is amply e-supplemented.

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1. Introduction

In this paper, every lattice is a complete modular lattice with the smallest element 0 and the greatest element 1. Let L be a lattice, $x, y \in L$ and $x \leq y$. A sublattice $\{a \in L | x \leq a \leq y\}$ is called a *quotient* sublattice of L and denoted by y/x. An element a of a lattice L is called a *complement* of b in L if $a \vee b = 1$ and $a \wedge b = 0$, in this case we denote $1 = a \oplus b$ (here we also call a and b are *direct summands* of L). L is said to be *complemented* if each element of L has at least one complement in L. An element x of L is said to be *small* or *superfluous* in L and denoted by $x \ll L$ if $y = 1$ for every $y \in L$ such that $x \vee y = 1$. Let L be a lattice and $k \in L$. If $t = 0$ for every $t \in L$ with $k \wedge t = 0$, then k is called an *essential* element of L and denoted by $k \trianglelefteq L$. The meet of all maximal $(\neq 1)$ elements of a lattice L is called the *radical* of L and denoted by $r(L)$. If $a \ll L$, then $a \leq r(L)$ holds. An element a of L is called a *supplement* of b in L if it is minimal for $1 = b \vee a$. a is a supplement of b in a lattice L if and only if $1 = b \vee a$ and $b \wedge a \ll a/0$. A lattice L is said to be *supplemented* if every element of L has a supplement in L. If every essential element of L has a supplement in L, then L

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is called an *essential supplemented* (briefly, *e-supplemented*) lattice. We say that an element y of L *lies above* an element x of L if $x \leq y$ and $y \ll 1/x$. An element a of L is called a *weak supplement* of b in L if $a \vee b = 1$ and $a \wedge b \ll L$. A lattice L is said to be *weakly supplemented*, if every element of L has a weak supplement in L. Let L be a lattice. L is said to be *hollow* if every element distinct from 1 is superfluous in L. If every essential element of L with distinct from 1 is small in L or L have no essential elements with distinct from 1, then L is called an *e-hollow* lattice. L is said to be *local* if L has the greatest element ($\neq 1$). We say an element $x \in L$ has *ample supplements* in L if for every $y \in L$ with $x \vee y = 1$, x has a supplement z in L with $z \leq y$. L is said to be *amply supplemented* if every element of L has ample supplements in L. Let L be a lattice. It is defined β_* relation on the elements of L by $a\beta_* b$ with $a, b \in L$ if and only if for each $t \in L$ such that $a \vee t = 1$ then $b \vee t = 1$ and for each $k \in L$ such that $b \vee k = 1$ then $a \vee k = 1$.

More information about (amply) supplemented lattices are in [1,2,5,8]. More results about (amply) supplemented modules are in [7, 13–15]. More information about essential supplemented lattices are in [12]. More information about (amply) essential supplemented modules are in [10, 11]. The definition of β_* relation on lattices and some properties of this relation are in [9]. This relation is a generalization of β^* relation on modules. The definition of β^* relation on modules and some properties of this relation are in [4].

Lemma 1. *Let L be a lattice and $a, b, c \in L$ with $a \leq b$. If c is a supplement of b in L, then $a \vee c$ is a supplement of b in $1/a$.*

Proof. See [3, Lemma 1]. □

Lemma 2. *Let a be a supplement of b in L and $x, y \in a/0$. Then y is a supplement of x in $a/0$ if and only if y is a supplement of $b \vee x$ in L.*

Proof. See [8, Lemma 11]. □

Lemma 3. *Let L be a lattice and $a, b, c \in L$. If $1 = a \vee b$ and $c \leq b$, then c is a supplement of a in L if and only if c is a supplement of $a \wedge b$ in $b/0$.*

Proof. ($\Longrightarrow$) Since c is a supplement of a in L, $1 = a \vee c$ and $a \wedge c \ll c/0$. Since $1 = a \vee c$ and $c \leq b$, by modularity, $b = b \wedge 1 = b \wedge (a \vee c) = (a \wedge b) \vee c$. On the other hand, $a \wedge b \wedge c = a \wedge c \ll c/0$. Hence c is a supplement of $a \wedge b$ in $b/0$.

($\Longleftarrow$) Since c is a supplement of $a \wedge b$ in $b/0$, $b = (a \wedge b) \vee c$ and $a \wedge c = a \wedge b \wedge c \ll c/0$. Since $1 = a \vee b$, $1 = a \vee b = a \vee (a \wedge b) \vee c = a \vee c$. On the other hand, $a \wedge c \ll c/0$. Hence c is a supplement of a in L. □

Corollary 1. *Let L be a lattice and $a \in L$. Then a has ample supplements in L if and only if $a \wedge b$ has a supplement in $b/0$ for every $b \in L$ with $1 = a \vee b$.*

Proof. ($\Longrightarrow$) Let $1 = a \vee b$ with $b \in L$. Since a has ample supplements in L, then a has a supplement c in L with $c \leq b$. By Lemma 3, c is a supplement of $a \wedge b$ in $b/0$. Hence $a \wedge b$ has a supplement in $b/0$.

($\Longleftarrow$) Let $1 = a \vee b$ with $b \in L$. By hypothesis, $a \wedge b$ has a supplement c in $b/0$. By Lemma 3, c is a supplement of a in L. Moreover, $c \leq b$. Hence a has ample supplements in L. □

Corollary 2. *Let L be a lattice and $a \in L$. If $b/0$ is supplemented for every $b \in L$ with $1 = a \vee b$, then a has ample supplements in L.*

Proof. Clear from Corollary 1. □

Proposition 1. *Let L be a lattice. Then L is amply supplemented if and only if $a \wedge b$ has a supplement in $b/0$ for every $a, b \in L$ with $1 = a \vee b$.*

Proof. Clear from Corollary 1. □

Corollary 3. *Let L be a lattice. If $b/0$ is supplemented for every $b \in L$, then L is amply supplemented.*

Proof. Clear from Proposition 1. □

Corollary 4. *Let L be a lattice. If $a \wedge b$ is a direct summand of $b/0$ for every $a, b \in L$ with $1 = a \vee b$, then L is amply supplemented.*

Proof. Clear from Proposition 1. □

Lemma 4. *Let b be a supplement of a in L. If $a \vee x$ has ample supplements in L for every $x \in b/0$, then $b/0$ is amply supplemented.*

Proof. Let $x \in b/0$ and $b = x \vee y$ with $y \in b/0$. Then $1 = a \vee b = a \vee x \vee y$. By hypothesis, $a \vee x$ has ample supplements in L. Then $a \vee x$ has a supplement z in L with $z \leq y$. Since b is a supplement of a in L and $x, z \in b/0$, by Lemma 2, z is a supplement of x in $b/0$. Moreover, $z \leq y$. Hence $b/0$ is amply supplemented. □

Corollary 5. *Let b be a supplement of a in L. If $a \vee x$ has ample supplements in L for every $x \in L$, then $b/0$ is amply supplemented.*

Proof. Clear from Lemma 4. □

Corollary 6. *Let b be a supplement of a in L. If x has ample supplements in L for every $x \in 1/a$, then $b/0$ is amply supplemented.*

Proof. Clear from Corollary 5. □

2. Amply essential supplemented lattices

Definition 1. Let L be a lattice. If every essential element of L has ample supplements in L, then L is called an amply essential supplemented (briefly, amply e-supplemented) lattice. (See also [6]).

Proposition 2. *Every amply e-supplemented lattice is e-supplemented.*

Proof. Let L be an amply e-supplemented lattice and $a \trianglelefteq L$. Since L is amply e-supplemented, a has ample supplements in L. Since $1 = a \vee 1$, a has a supplement x in L with $x \leq 1$. Hence L is e-supplemented. □

Corollary 7. *Let L be an amply essential supplemented lattice. Then $1/r(L)$ does not have any essential element with distinct from* 1.

Proof. Since L is amply essential supplemented, by Proposition 2, L is essential supplemented. Then by [12, Proposition 4], $1/r(L)$ does not have any essential element with distinct from 1. □

Proposition 3. *Let L be an amply e-supplemented lattice. If every element of L with distinct from* 0 *is essential in L, then L is amply supplemented.*

Proof. Clear from definitions. □

Proposition 4. *Every e-hollow lattice is amply e-supplemented.*

Proof. Let L be an e-hollow lattice, $a \trianglelefteq L$ and $1 = a \vee b$ with $b \in L$. If $a = 1$, then 0 is a supplement of a in L. Moreover, $0 \leq b$. Let $a \neq 1$. Since L is e-hollow, $a \ll L$ and $b = 1$. Here 1 is a supplement of a in L with $1 \leq b$. Hence a has ample supplements in L and L is amply e-supplemented. □

Proposition 5. *Let L be a finite lattice. Then L is amply e-supplemented.*

Proof. Clear from definitions. □

Proposition 6. *Let L be a lattice. If $b/0$ is essential supplemented for every $b \in L$, then L is amply essential supplemented.*

Proof. Let $a \trianglelefteq L$ and $1 = a \vee b$ with $b \in L$. Since $a \trianglelefteq L$, $a \wedge b \trianglelefteq b/0$. Since $b/0$ is essential supplemented, $a \wedge b$ has a supplement c in $b/0$. By Lemma 3, c is a supplement of a in L. Moreover, $c \leq b$. Hence L is amply essential supplemented. □

Proposition 7. *Let L be an amply e-supplemented lattice, b be a supplement of a in L and $b \trianglelefteq L$. Then $b/0$ is amply e-supplemented.*

Proof. Let $x \trianglelefteq b/0$ and $b = x \vee y$ with $y \in b/0$. Then $1 = a \vee b = a \vee x \vee y$. Since $x \trianglelefteq b/0$ and $b \trianglelefteq L$, $x \trianglelefteq L$. Then $a \vee x \trianglelefteq L$ and since L is amply e-supplemented, $a \vee x$ has a supplement z in L with $z \leq y$. By Lemma 2, z is a supplement of x in $b/0$. Hence $b/0$ is amply e-supplemented. □

Lemma 5. *Let L be an amply e-supplemented lattice and $x \in L$. Then $1/x$ is also amply e-supplemented.*

Proof. Let a be any essential element of $1/x$. Then $a \trianglelefteq L$ and since L is amply e-supplemented, a has ample supplements in L. Let $1 = a \vee b$ with $b \in 1/x$. Since $1 = a \vee b$, a has a supplement y in L with $y \leq b$. By Lemma 1, $x \vee y$ is a supplement of a in $1/x$. Moreover, $x \vee y \leq b$. Hence a has ample supplements in $1/a$ and $1/a$ is amply e-supplemented. □

Corollary 8. *Let L be an amply e-supplemented lattice. Then $x/0$ is amply e-supplemented for every direct summand x of L.*

Proof. Since x is a direct summand of L, there exists $y \in L$ such that $1 = x \oplus y$. By Lemma 5, $1/y$ is amply e-supplemented. Then by $\frac{1}{y} = \frac{x \vee y}{y} \cong \frac{x}{x \wedge y} = \frac{x}{0}$, $x/0$ is also amply e-supplemented. □

Proposition 8. *Let L be a lattice. Then L is amply essential supplemented if and only if $a \wedge b$ has a supplement in $b/0$ for every $a \trianglelefteq L$ and $b \in L$ with $1 = a \vee b$.*

Proof. Clear from Corollary 1. □

Proposition 9. *Let b be a supplement of a in L. If $a \vee x$ has ample supplements in L for every $x \trianglelefteq b/0$, then $b/0$ is amply e-supplemented.*

Proof. Similar to proof of Lemma 4. □

Definition 2. Let L be a lattice and $a \in L$. If a is a supplement of an essential element in L, then a is called an e-supplement element in L. (See also [12]).

Lemma 6. *Let L be an amply e-supplemented lattice. Then $x/0$ is amply supplemented for every e-supplement element x in L.*

Proof. Since x is an e-supplement element in L, there exists an element $a \trianglelefteq L$ such that x is a supplement of a in L. Let $y \in x/0$ and $x = y \vee z$ with $z \in x/0$. Then $1 = a \vee x = a \vee y \vee z$. Since $a \trianglelefteq L$, $a \vee y \trianglelefteq L$ and since L is amply e-supplemented, $a \vee y$ has a supplement t in L with $t \leq z$. Since x is a supplement of a in L and $y, t \in x/0$, by Lemma 2, t is a supplement of y in $x/0$. Moreover, $t \leq z$. Hence $x/0$ is amply supplemented. □

Corollary 9. *Let L be an amply e-supplemented lattice. Then $x/0$ is amply e-supplemented for every e-supplement element x in L.*

Proof. Clear from Lemma 6. □

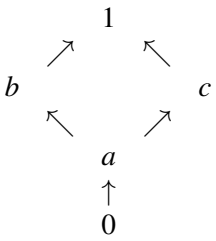
Proposition 10. *Let L be a lattice. If $a \wedge b$ lies above an e-supplement element in L for every $a, b \in L$ with $a \trianglelefteq L$ and $1 = a \vee b$, then L is amply e-supplemented.*

Proof. Let $a \trianglelefteq L$ and $1 = a \vee b$ with $b \in L$. By hypothesis, $a \wedge b$ lies above an e-supplement element x in L. Since x is an e-supplement element in L, there exists an element $y \trianglelefteq L$ such that x is a supplement of y in L. Here $1 = x \vee y$ and since $x \leq a \wedge b \leq b$, $1 = x \vee y = b \vee y$. By hypothesis, $b \wedge y$ lies above an e-supplement element c in L. Since $x \leq a \wedge b$, $1 = x \vee y = (a \wedge b) \vee y$ and by modularity, $b = b \wedge 1 = b \wedge ((a \wedge b) \vee y) = (a \wedge b) \vee (b \wedge y)$ and $1 = a \vee b = a \vee (a \wedge b) \vee (b \wedge y) = a \vee (b \wedge y)$. Since $b \wedge y$ lies above c in L, by [8, Lemma 5], $(b \wedge y)\, \beta_* c$ in L and since $1 = a \vee (b \wedge y)$, $1 = a \vee c$. Since x is a supplement of y in L, y is a weak supplement of x in L and since $a \wedge b$ lies above x in L, by [8, Lemma 15], y is a weak supplement of $a \wedge b$ in L. Here $a \wedge b \wedge y \ll L$ and since $c \leq b \wedge y$, $a \wedge c \ll L$. Since $a \wedge c \ll L$ and c is

a supplement element in L, by [8, Lemma 10], $a \wedge c \ll c/0$. Hence c is a supplement of a in L. Moreover, $c \leq b$. Hence L is amply e-supplemented. □

The converse of Proposition 4 is not true in general.

Example 1. Consider the lattice $L = \{0, a, b, c, 1\}$ given by the following diagram.



Then L is amply e-supplemented but not e-hollow.

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ON COFINITELY (D_{12}^*)-MODULES

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Abstract. In this paper, concepts of (cofinitely) (D_{12}^*)-modules which are a proper generalization of concept of $\oplus_\delta$-supplemented modules are studied. We say that M is a (D_{12}^*)-module if for every submodule A of M, there exists a direct summand B of M and an epimorphism $f : B \to \frac{M}{A}$ such that $\ker(f) \ll_\delta B$. The module M is called cofinitely (D_{12}^*)-module if for every cofinite submodule A of M, there exists a direct summand B of M and an epimorphism $f : B \to \frac{M}{A}$ such that $\ker(f) \ll_\delta B$. In this paper, various properties of these modules are given. In addition, a new characterization of δ-semiperfect rings is given using cofinitely (D_{12}^*)-modules.

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1. INTRODUCTION

Throughout this paper, all rings are associative with identity and all modules are unitary right modules, unless otherwise specified. Let R be such a ring and M be such a module. By the notation $X \leq M$, we mean that X is a submodule of M. A submodule X of M is called *small* in M if $M \neq X + Y$ for any proper submodule Y of M, denoted by $X \ll M$, and we denote that $\mathrm{Rad}(M)$, the sum of all small submodules of M. Dual to this concept, a submodule X of M is called *essential* in M, denoted by $X \trianglelefteq M$, if the intersection of X is non-zero with the other submodules of M, except for $\{0\}$. It is known that the set $Z(M) = \{m \in M \mid \mathrm{Ann}(m) \trianglelefteq R\}$ is the singular submodule of M. The module M is called *singular* in case $Z(M) = M$. A submodule X of M is called *cofinite* whenever $\frac{M}{X}$ is finitely generated. A supplement submodule T of X in M is minimal element of the set $\{Y \leq M \mid M = X + Y\}$ that equivalents $M = X + T$ and $X \cap T \ll T$. A module M is called *supplemented* if every submodule of M has a supplement in M [19]. A module M is called *cofinitely supplemented* if every cofinite submodule of M has a supplement in M [3]. A generalization of supplement submodule is defined as a Rad-supplement submodule (according to [18], a generalized supplement submodule). For a module M and a submodule A of M,

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a submodule B of M is called a *Rad-supplement* of A if $M = A + B$ and $A \cap B \subseteq \mathrm{Rad}(B)$. An R-module M is called *GS-module* (or briefly Rad-supplemented) if each submodule of M has a Rad-supplement in M. A module M is called $\oplus$-*cofinitely radical supplemented* (according to [8] generalized $\oplus$-*cofinitely supplemented*) if every cofinite submodule of M has a Rad-supplement that is a direct summand of M. In [15] , it is used a cgs$^{\oplus}$-module.

Small submodules are generalized to δ-small submodules in [20]. By [20], a submodule A of M is called δ-*small* in M (denoted by $A \ll_{\delta} M$) if for any submodule B of M with $\frac{M}{B}$ is singular, $M = A + B$ implies that $M = B$. The sum of δ-small submodules of a module M is denoted by $\delta(M)$. It is easy to see that every small submodule of a module M is δ-small in M, so $\mathrm{Rad}(M) \subseteq \delta(M)$ and $\mathrm{Rad}(M) = \delta(M)$ if M is singular. Also any non-singular semisimple submodule of M is δ-small in M and any δ-small submodules of a singular module are small submodules. For more detailed discussion on δ-small submodules we refer to [20].

Let A be a submodule of a module M. A submodule B of M is called a δ-*supplement* of A in M provided that $M = A + B$ and $M \neq A + X$ for any proper submodule X of B with $\frac{B}{X}$ singular; or equivalently, $M = A + B$ and $A \cap B \ll_{\delta} B$ in [7].The module M is called δ-*supplemented* if every submodule of M has a δ-supplement in M by [7]. Some properties of this modules class are investigated in [16]. Also, M is called $\oplus - \delta$-*supplemented (or* $\oplus_{\delta}$-supplemented) if every submodule of M has a δ-supplement which is a direct summand of M in [12]. According to [13], an R-module M is called $\oplus$-*cofinitely* δ-*supplemented* (or $\oplus - cof_{\delta}$-supplemented) if every cofinite submodule of M has a δ-supplement that is a direct summand of M. A module M is called δ-*lifting*, if for every submodule A of M there exists a direct summand K of M with $K \subseteq A$ and $\frac{A}{K} \ll_{\delta} \frac{M}{K}$. Equivalently, for any $A \leq M$, there exists a decomposition $M = K \oplus B$ such that $K \leq A$ and $A \cap B \ll_{\delta} B$ by [7].

(D_{12})-modules are generalized to $\oplus$-supplemented modules. To addition cofinitely (D_{12})-modules as a generalization of cofinitely $\oplus$- supplemented modules are introduced in [1,5] and [17], respectively. M is called a (D_{12})-module if for every submodule A of M, there exists a direct summand B of M and an epimorphism $f\colon B \to \frac{M}{A}$ such that $\ker(f) \ll B$. M is called a *cofinitely* (D_{12})-*module* if for every cofinite submodule A of M, there exists a direct summand B of M and an epimorphism $f\colon \frac{M}{B} \to \frac{M}{A}$ such that $\ker(f) \ll \frac{M}{B}$. Similarly, (cofinitely) Rad$-D_{12}$-modules are studied and some features are obtained in [6] and [11].

In this paper, inspired from the definitions given above, we introduce the concept of (D_{12}^{*}) and cofinitely (D_{12}^{*})-modules, as follows. We say that M is a (D_{12}^{*})-module if for every submodule A of M, there exists a direct summand B of M and an epimorphism $f\colon B \to \frac{M}{A}$ such that $\ker(f) \ll_{\delta} B$ and M is a cofinitely (D_{12}^{*})-module if for every cofinite submodule A of M, there exists a direct summand B of M and an epimorphism $f\colon B \to \frac{M}{A}$ such that $\ker(f) \ll_{\delta} B$. We give some results related with these concepts. We give an example which is a cofinitely (D_{12}^{*})-module but not a cofinitely

(D_{12})-module. We have given a new characterization of δ-semiperfect rings using cofinitely (D_{12}^*)-modules and we have shown that every free right $R-$ module over a δ-perfect ring R is (D_{12}^*)-module. By the definitions given above, we can get the following implication on modules:

$$\begin{array}{ccc} \delta-\text{lifting module} & \Longrightarrow & \text{cofinitely } \delta-\text{lifting module} \\ \Downarrow & & \Downarrow \\ \oplus_\delta-\text{supplemented module} & \Longrightarrow & \text{cofinitely } \oplus_\delta-\text{supplemented module} \\ \Downarrow & & \Downarrow \\ (D_{12}^*)-\text{module} & \Longrightarrow & \text{cofinitely } (D_{12}^*)-\text{module} \end{array}$$

2. (COFINITELY) (D_{12}^*)-MODULES

Definition 1. *M is called a (D_{12}^*)-module if for every submodule A of M, there exists a direct summand B of M and an epimorphism $f\colon B \to \frac{M}{A}$ provided that $\ker(f) \ll_\delta B$.*

Example 1. For $n>1$ consider the left $\mathbb{Z}$-module $M=\mathbb{Z}_{p^n}$ where p is an arbitrary prime integer. Since M is local, it is clear that M is a (D_{12}^*)-module. So $\mathbb{Z}$-module $\mathbb{Z}_4$, $\mathbb{Z}_8$ and $\mathbb{Z}_{p^\infty}$ are (D_{12}^*)-modules.

Proposition 1. *Let M be a $\oplus_\delta$-supplemented module. Then M is a (D_{12}^*)-module.*

Proof. Suppose that A be a submodule of M. There exist direct summands B and B_1 of M such that $M=A+B=B\oplus B_1$ and $A\cap B \ll_\delta B$ as M is a $\oplus_\delta$-supplemented module. From here, we have the epimorphism $\alpha\colon B\to \frac{M}{A}$, $\alpha(b)=b+A$ for every $b\in B$. Note that $\ker(\alpha)=B\cap A \ll_\delta B$. Finally M is a (D_{12}^*)-module. □

Corollary 1. *Let M be a $\oplus_\delta$-supplemented module and A be a submodule of M such that $\frac{M}{A}$ is projective. Then A is a (D_{12}^*)-module.*

Proof. By Theorem 2.7 in [12], A is a $\oplus_\delta$-supplemented module. If we use Proposition 1, then we get that A is a (D_{12}^*)-module. □

The notion of $I-\oplus$-supplemented modules are introduced in [14], where I is an ideal of R. A module M is called $I-\oplus-$*supplemented* if for every submodule A of M, there exists a direct summand B of M such that $M=A+B$, $A\cap B\subseteq IB$ and $A\cap B\ll_\delta B$. $I-\oplus$-supplemented modules are characterized in [14]. It is clear that every $I-\oplus$-supplemented module is $\oplus_\delta$-supplemented, for every ideal I of R.

Corollary 2. *Let M be an $I-\oplus$-supplemented module. Then M is a (D_{12}^*)-module.*

Example 2. Let R be a discrete valuation ring with maximal ideal m and any ideal I of R. By Proposition 3.7 in [14], ${}_RR$ is $I-\oplus$-supplemented module if and only if $I=m$ and $I=R$. So ${}_RR$ is a (D_{12}^*)-module.

Recall from [10] that a submodule $A \leq M$ is called *weak* δ-*supplement of a submodule* B of M if $M = A + B$ and $A \cap B \ll_\delta M$. The module M is called *weakly* δ-*supplemented if every submodule* A of M has a *weak* δ-*supplement*.

Recall from [19] that a module M is called *refinable* if for, every submodules A, $B \leq M$ with $M = A + B$, there exists a direct summand B_1 of M with $B_1 \leq B$ and $M = B_1 + B$.

Proposition 2. *Let M be a weakly δ-supplemented refinable module. Then M is a (D_{12}^*)-module.*

Proof. Suppose that $A \leq M$. Since M is *weakly* δ- *supplemented*, there exists a submodule B of M such that $M = A + B$ and $A \cap B \ll_\delta M$. Since M is a refinable module, then there is a direct summand A_1 of M such that $M = A_1 + A$ and $A_1 \leq B$. If we consider the natural epimorphism $\psi\colon A_1 \to \frac{A_1}{A_1 \cap A}$, we have $\ker(\psi) = A_1 \cap A$. As $A_1 \leq B$, $A_1 \cap A \leq B \cap A \ll_\delta M$. Since there exists an isomorphism $\theta\colon \frac{A_1}{A_1 \cap A} \to \frac{M}{A}$, say $f = \theta\psi\colon A_1 \to \frac{M}{A}$. Here $\ker(f) = \ker(\theta\psi) = \psi^{-1}(\ker\theta) = \psi^{-1}(0) = \ker\psi = A_1 \cap A$. From here $A_1 \cap A \ll_\delta A_1$ because A_1 is a direct summand of M. Therefore M is a (D_{12}^*)-module. □

Recall from [12] that a module M is called δ-*radical* if $\delta(M) = M$ and the sum of all δ-radical submodules of the module M is denoted by $P_\delta(M)$, that is, $P_\delta(M) = \{U \leq M |\ \delta(U) = U\}$. It is clear that, for any submodule M, $P_\delta(M)$ is the largest δ-*radical* submodule of M.

Proposition 3. *Let M be a (D_{12}^*)-module. If $P_\delta(M)$ is a direct summand of M, then $P_\delta(M)$ is a D_{12}^*-module.*

Proof. Since $P_\delta(M)$ is a direct summand of M, there exists a submodule A of M such that $M = P_\delta(M) \oplus A$. By the hypothesis, there exists a direct summand B of M and an epimorphism $\varphi\colon B \to \frac{M}{T \oplus A}$ such that $\ker(\varphi) \ll_\delta B$ for any submodule T of $P_\delta(M)$. Note that $\frac{M}{T \oplus A} \cong \frac{P_\delta(M)}{T}$. Hence $\delta\left(\frac{B}{\ker(\varphi)}\right) = \frac{B}{\ker(\varphi)}$. We have $\delta(B) = B$ because $\ker(\varphi) \ll_\delta B$ and so $B \leq P_\delta(M)$. □

Theorem 1. *Let $M = M_1 \oplus M_2$. Then M_2 is a (D_{12}^*)-module if and only if for every submodule A of M containing M_1, there exists a direct summand B of M_2 and an epimorphism $f\colon M \to \frac{M}{A}$ such that B is a δ-supplement of $\ker(f)$ in M.*

Proof.

($\Rightarrow$)**:** Assume that M_2 is a (D_{12}^*)-module and A is a submodule of M with $M_1 \leq A$. Consider the submodule $A \cap M_2$ of M_2. Since M_2 is a (D_{12}^*)-module, there exists a direct summand B of M_2 and an epimorphism $g\colon B \to \frac{M_2}{A \cap M_2}$ such that $\ker(g) \ll_\delta B$. On the other hand, we have $M = A + M_2$ and for any submodule B_1 of M, $M = B \oplus B_1$ because B is a direct summand of M_2. Consider the projection map $h\colon M \to B$ and the isomorphism $\mu\colon \frac{M_2}{A \cap M_2} \to \frac{M}{A}$ defined by

$\mu(m_2+A\cap M_2)=m_2+A$. Thus $f=\mu\circ g\circ h\colon M\to\frac{M}{A}$ is an epimorphism. It is clear that $\ker(f)=\ker(g)+B_1=A+B_1$. Hence $M=B+\ker(f)$. From here, $B\cap\ker(f)=B\cap A=\ker(g)\ll_\delta B$. Finally, B is a δ-supplement of $\ker(f)$ in M which is a direct summand of M.

($\Leftarrow$)**:** Conversely, suppose that every submodule of M containing M_1 has the stated property. Suppose that X be a submodule of M_2. Note that $X\oplus M_1\leq M$. By the hypothesis, there exists a direct summand B of M_2 and an epimorphism $f\colon M\to\frac{M}{X\oplus M_1}$ such that $M=B+\ker(f)$ and $B\cap\ker(f)\ll_\delta B$. Assume that $g\colon B\to\frac{M}{X\oplus M_1}$ be the restriction of f to B. We can take the isomorphism $h\colon\frac{M}{X\oplus M_1}\to\frac{M_2}{X}$ defined by $h(m_1+m_2+(X\oplus M_1))=m_2+X$. From here, $\mu=hog\colon B\to\frac{M_2}{X}$ is an epimorphism. Clearly,

$$\begin{aligned}\ker(\mu)=\ker(h\circ g)&=\{b\in B|\,h(g(b))=\{X\}\}\\&=\{b\in B|\,g(b)=\frac{X\oplus M_1}{X\oplus M_1}\}=\{b\in B|\,b\in\ker(g)\}.\end{aligned}$$

Since $\ker(g)=\ker(\mu)=B\cap\ker(f)\ll_\delta B$. Finally, M_2 is a (D_{12}^*)-module. □

Recall from [19] that a submodule X of M is called *fully invariant* if $\varphi(X)$ is contained in X for every R-endomorphism φ of M. In [4], a module M is called a *duo module* if every submodule of M is fully invariant.

Theorem 2. *Let M_i be a (D_{12}^*)-module for every $i\in I$ and $M=\oplus_{i\in I}M_i$. If M is a duo module, then M is a (D_{12}^*)-module.*

Proof. Suppose that $A\leq M$. Since M is a duo module, we have $A=\oplus_{i\in I}(A\cap M_i)$ by Lemma 2.1 in [4]. Now, we consider the submodule $A\cap M_i$ of M_i for every $i\in I$. As M_i is a (D_{12}^*)-module, we have a direct summand B_i of M_i and an epimorphism $\varphi_i\colon\oplus_{i\in I}B_i\to\oplus_{i\in I}\left(\frac{M_i}{A\cap M_i}\right)\cong\frac{\oplus_{i\in I}M_i}{\oplus_{i\in I}(A\cap M_i)}$ by $b_{i_1}+b_{i_2}+\cdots+b_{i_n}\to\varphi_{i_1}(b_{i_1})+\varphi_{i_2}(b_{i_2})+\cdots+\varphi_{i_n}(b_{i_n})$ with $b_{i_j}\in B_{i_j}$ for every $j=1,2,\ldots,n$. It is so easy to check that φ is an epimorphism with $\ker(\oplus_{i\in I}\varphi_i)=\oplus_{i\in I}\ker(\varphi_i)$ and $\oplus_{i\in I}B_i$ is a direct summand of M. For every $i\in I$, since $\ker(\varphi_i)\ll_\delta B_i$, we can get that $\oplus_{i\in I}\ker(\varphi_i)\ll_\delta\oplus_{i\in I}B_i$. Hence, M is a (D_{12}^*)-module. □

Definition 2. M is called a cofinitely (D_{12}^*)-module if for every cofinite submodule A of M, there exists a direct summand B of M and an epimorphism $f\colon B\to\frac{M}{A}$ such that $\ker(f)\ll_\delta B$.

Now, we give examples which are (D_{12}^*)-modules but not a (D_{12})-modules. Since ${}_RR$ is finitely generated, the module in the following example is (cofinitely) (D_{12}^*)-module but not (cofinitely) (D_{12})-module.

Example 3.

(i) (See [2, Example 3.10]) Let S be a field, $U = \begin{bmatrix} S & S \\ 0 & S \end{bmatrix}$ and

$$R = \{(x_1, x_2, \ldots, x_n, x, x, \ldots) | \, n \in \mathbb{N}, x_i \in M_2(S)\},$$

with component-wise operations. The Jacobson radical $\mathrm{Rad}(R) = 0$ and R is not regular ring, hence R is not semiperfect. Since

$$\delta(R) = \{(x_1, x_2, \ldots, x_n, x, x, \ldots) | \, n \in \mathbb{N}, x_i \in M_2(S), x \in V\},$$

where $V = \begin{bmatrix} 0 & S \\ 0 & 0 \end{bmatrix}$, R is δ-semiperfect. From here, R as a right R-module is (D_{12}^*)-module by Theorem 8. But ${}_RR$ is not (D_{12})-module by Theorem 4.7 in [5].

(ii) (See [15, Example 2.2]) Let M be a uniform module and $S = \mathrm{End}(M)$. Consider the projective S-module P with $\dim(P) = (1,0)$. So P is a indecomposable w-local module. Since $\dim(P) = (1,0)$, P is not finitely generated. By [15], P is a cgs$^{\oplus}$-module but not cofinitely supplemented. Therefore P is a $\oplus$-*cofinitely* δ-*supplemented* module. It follows from that P is a cofinitely (D_{12}^*)-module. Since P is not $\oplus$-cofinitely supplemented, P is not a cofinitely (D_{12})-module by [6, Example 2.4].

Proposition 4. *Let M be a cofinitely $\oplus_\delta$-supplemented module. Then M is a cofinitely (D_{12}^*)-module.*

Proof. The proof can be made similar to Proposition 1. □

Theorem 3. *Let M be a quasi-projective module.*

(i) *If M is a (D_{12}^*)-module, then M is $\oplus_\delta$-supplemented.*
(ii) *If M is a cofinitely (D_{12}^*)-module, then M is cofinitely $\oplus_\delta$-supplemented.*

Proof.

(i) For a submodule A of M, by the hypothesis there exists a direct summand B of M and an epimorphism $\varphi: B \to \frac{M}{A}$ such that $\ker(\varphi) \ll_\delta B$. Now, we consider $\alpha: M \to \frac{M}{A}$ be the natural epimorphism. Since M is quasi-projective, we get that the following commutative diagram for the homomorphism $\beta: M \to B$ with $\alpha = \varphi \circ \beta$:

$$\begin{array}{ccccc} & & & & M \\ & & & \overset{\beta}{\swarrow} & \downarrow \alpha \\ B & \longrightarrow & \frac{M}{A} & \longrightarrow & 0 \end{array}$$

From here $\beta(M) = B$ because $(\varphi \circ \beta)(M) = \alpha(M) = \frac{M}{A}$ and $\ker(\varphi) \ll_\delta B$. And so β is an epimorphism. On the other hand, β splits as B is M-projective. Therefore, there is a direct summand of B_1 of M such that $\beta\downarrow_{B_1}: B_1 \cong B$ and so $\alpha\downarrow_{B_1}$ is an epimorphism. Hence $M = B_1 + A$ and $B_1 \cap A = \ker(\alpha\downarrow_{B_1}) \ll_\delta$

M. It follows from $B_1 \cap A \ll_\delta B_1$ because B_1 is a direct summand of M. Finally M is $\oplus_\delta$-supplemented.

(ii) The proof can be made similar to (i).

□

Recall from [20] that an epimorphism $f: P \to M$ is called δ-*cover* if $\ker(f) \ll_\delta P$ and a cover f is called a *projective* δ-*cover* if P is a projective module. A module M is called δ-*semiperfect* if every factor module of M has a projective δ-cover. A ring R is called δ-*perfect* (δ-semiperfect) if every (finitely generated) right (or left) R-module has a projective δ-cover.

Theorem 4. *Let M be a projective module. Then M is δ-semiperfect if and only if M is a (D_{12}^*)-module.*

Proof.

($\Rightarrow$)**:** Assume that M is a projective δ-*semiperfect* module. By Lemma 2.4 in [9] and Proposition 1, M is $\oplus_\delta$-supplemented and so M is a (D_{12}^*)-module.

($\Leftarrow$)**:** This proof is made using Theorem 3 (ii) and Lemma 2.4 in [9], respectively.

□

Corollary 3. *Let R be a δ-perfect ring. Then every free right R-module is a (D_{12}^*)-module.*

Theorem 5. *Let $M = M_1 \oplus M_2$. Then M_2 is a cofinitely (D_{12}^*)-module if and only if for every cofinite submodule A of M containing M_1, there exists a direct summand B of M_2 and an epimorphism $f: M \to \frac{M}{A}$ such that B is a δ-supplement of $\ker(f)$ in M.*

Proof. The proof can be made similar to Theorem 1. □

Theorem 6. *Let $\{M_i\}_{i\in I}$ be any family cofinitely (D_{12}^*)-module and $M = \oplus_{i\in I} M_i$. If every cofinite submodule of M is fully invariant, then M is a cofinitely (D_{12}^*)-module.*

Proof. Let A be a cofinite submodule of M. As A is fully invariant, we can write $A = \oplus_{i\in I}(A \cap M_i)$. Since $\frac{M}{A} \cong \oplus_{i\in I} \frac{M_i}{A\cap M_i}$, $A \cap M_i$ is a cofinite submodule of M_i for every $i \in I$. The rest of the proof by Theorem 2. □

Recall from [19] that an R-module M has the *summand sum property* (SSP) if the sum of two direct of M is again a direct summand of M.

Proposition 5. *Let M be a (cofinitely) (D_{12}^*)-module with the property (SSP) and A be direct summand of M. Then $\frac{M}{A}$ is a (cofinitely) (D_{12}^*)-module.*

Proof. Suppose that $\frac{B}{A}$ be a submodule of $\frac{M}{A}$. Then B is a submodule of M. As M is (cofinitely) (D_{12}^*)-module, then there exists a direct summand T of M and an epimorphism $f\colon T \to \frac{M}{B}$ with $\ker(f) \ll_\delta T$. As M has the property (SSP), both A and T are direct summand of M . Hence, there exists a submodule Y of M such that $M = (A+T) \oplus Y$. Since the property $\frac{T+A}{A} \cap \frac{Y+A}{A} \subseteq \frac{[Y\cap(T+A)]+[A\cap(T+A+Y)]}{A} = \frac{A}{A}$, we have $\frac{M}{A} = \frac{T+A}{A} \oplus \frac{Y+A}{A}$. From here, we obtain that $\frac{\frac{M}{A}}{\frac{B}{A}} \cong \frac{M}{B}$. So we can define the homomorphism $g\colon \frac{T+A}{A} \to \frac{M}{B}$ by $t+a+A = t+A \to g(t)$ with $t \in T$, $a \in A$. It is clear that g is an epimorphism with $\ker(g) \ll_\delta \frac{T+A}{A}$ and $\frac{T+A}{A}$ is a direct summand of $\frac{M}{A}$. Finally, $\frac{M}{A}$ is a (cofinitely) (D_{12}^*)-module. □

Theorem 7. *Let M be a (cofinitely) (D_{12}^*)-module. If A is a fully invariant submodule of M, then $\frac{M}{A}$ is a (cofinitely) (D_{12}^*)-module.*

Proof. Let $\frac{B}{A}$ be a (cofinite) submodule of $\frac{M}{A}$. Then B is a (cofinite) submodule of M. Since M is a (cofinitely) (D_{12}^*)-module, there exists a direct summand T of M and an epimorphism $f\colon T \to \frac{M}{B}$ with $\ker(f) \ll_\delta T$. From here, we can write $M = T \oplus T_1$ for every submodule T_1 of M. As A is a fully invariant submodule of M, $A = (A\cap T) \oplus (A\cap T_1)$. Note that $\frac{M}{A} = \frac{A+T}{A} \oplus \frac{A+T_1}{A}$. It follows from $\frac{\frac{M}{A}}{\frac{B}{A}} \cong \frac{M}{B}$ that, we can define the homomorphism $g\colon \frac{T+A}{A} \to \frac{M}{B}$ by $t+A \to g(t+A) = f(t)$ with $t \in T$. Therefore g is an epimorphism with $\ker(g) \ll_\delta \frac{T+A}{A}$. Therefore, $\frac{M}{A}$ is a (cofinitely) (D_{12}^*)-module. □

Theorem 8. *R is a δ-semiperfect ring if and only if every free right R-module is cofinitely (D_{12}^*)-module.*

Proof.

- $(\Rightarrow)$**:** Let R be a δ-semiperfect ring. By Lemma 3.5 in [12], every free right R-module is $\oplus_\delta$-cofinitely supplemented. Then every free right R-module is cofinitely (D_{12}^*)-module by Proposition 4.
- $(\Leftarrow)$**:** As every free right R-module is a cofinitely (D_{12}^*)-module, it is $\oplus_\delta$-cofinitely supplemented by Theorem 3 (ii). If we use Lemma 3.5 in [12], then we get that R is δ-semiperfect.

□

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LS-SVR FOR THE FRACTIONAL FOURTH-ORDER PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. Fractional fourth-order partial differential equations find various applications such as image denoising, electrostatics, and geometric modeling. In this paper, we propose a new supervised machine learning algorithm for such problems. To do so, we develop a simulation method based on least squares support vector regression (LS-SVR). A polynomial kernel is used to approximate the solution in the training process by considering an inverse viewpoint to the residual function. For minimizing the loss function, we use the Petrov-Galerkin method for the constraints in the proposed LS-SVR. We study the stability and convergence of the method by providing some numerical examples and illustrating the error behavior as the degree of the solution increases.

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1. INTRODUCTION

Fractional partial differential equations (FPDEs) are used to model a variety of physical phenomena in fluid dynamics, material science, and chemical engineering such as anomalous diffusion, wave propagation, and viscoelasticity. Especially, fourth-order FPDEs play an important role in the study of phase transitions and chaotic phenomena in complex systems. Furthermore, they can be used to represent the dynamics of some nonlinear systems, such as the long time behavior for population models. By modeling physical phenomena with FPDEs, it is possible to obtain a more accurate description of the system, which can improve the prediction and decision-making. On the other hand, FPDEs are typically more difficult to solve than traditional partial differential equations, so they require more specific numerical methods to handle the singularity in the kernel of the integral in the fractional derivative [2, 31, 34, 36].

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Classic numerical methods have been implemented for solving fourth-order FPDEs, including the finite difference [10, 41], finite element [8, 18, 19], finite volume [3, 9, 30], and the Spectral method [11, 15, 17]. The choice of method depends on the specific problem and desired accuracy. The Finite difference is widely used and known for its simplicity, as it involves discretizing the equation and solving the resulting algebraic system. Finite element is more accurate and efficient for complex geometries, while finite volume suits non-uniform grids. The Spectral methods are highly accurate but computationally expensive.

On the other hand, machine learning (ML), neural networks, and deep learning algorithms have become increasingly popular for a wide range of problems in recent years, for instance in pattern recognition, recommender systems, health care, theorem proving, cybersecurity, financial services, and more. ML algorithms use some training data to recognize the objects in groups, detect similarities and give the target in classification, clustering, and regression problems [21].

Recently, ML techniques have been developed to approximate the solution of ordinary, partial, and integral equations (IEs). For instance, Mall et al. [26] presented Chebyshev neural networks for solving Lane-Emden type equations. Lu et al. [25] used deep learning to solve some PDEs numerically. Jafarian et al. [25] used neural networks to solve IEs and Mall et al. [15] used radial basis function neural networks to solve elliptic PDEs. These works show the efficiency of ML for solving complex mathematical problems, especially differential and integral equations.

Support Vector Machines (SVM) is a well-known learning algorithm that was first introduced by Cortes and Vapnik in 1995 for classification problems [6]. It was initially designed as a binary classification algorithm. The SVM model has since been further refined and is now used for a variety of classification and regression tasks. The original goal of the SVM algorithm was to find a maximum margin hyperplane that could separate two sets of data points in a high-dimensional space using kernel functions [39].

Support Vector Regression (SVR) was first proposed by Cortes and Vapnik in 1995 as a revision of SVM for regression problems. The main idea behind SVR is to find a function or hyperplane that best fits the data while keeping the error within a certain margin, called the epsilon-insensitive tube, a range around the true target values where errors are considered acceptable and do not contribute to the loss function [38]. Least Squares Support Vector Machines (LS-SVM) is an extension of the SVM algorithm introduced by Suykens et al. that transforms the problem into a convex optimization task with equality constraints, so it is faster for large datasets and complex regression problems. A revision of this algorithm for regression tasks, LS-SVR has been also extended for solving ODEs [25, 27], PDEs [1, 28], IEs [12, 32, 35], and differential equations on unbounded domains [5, 24].

The superior performance of these methods has motivated us to develop an ML method for solving fourth-order FPDEs. To present the problem and the method, we need some notations.

In this article, the fourth-order FPDE with Caputo time fractional derivative is considered as

$$\partial_t^{\alpha} u(x,t) + \partial_x^4 u(x,t) + \eta u(x,t) = f(x,t), \qquad x \in [0,1], \;\; t \in [0,T] \tag{1.1}$$

subject to the following initial condition

$$u(x,0) = g(x), \qquad x \in [0,1] \tag{1.2}$$

and boundary conditions

$$u(0,t) = 0, u(1,t) = 0, u_{xx}(0,t) = 0, u_{xx}(1,t) = 0, \qquad t \in [0,T] \tag{1.3}$$

where parameter $0 < \alpha \leq 1$ describes the order of the fractional time derivatives in the Caputo sense given in Eq (1.4), $u(x,t)$ is the unknown function which represents the concentration of the drug at the location x in the tissue at time t, in biological and drug release applications, T is the final time at a given period, η is a nonnegative constant, and $f(x,t)$ is the given source function [16, 31].

The fractional derivative has been defined in many different ways, Caputo and Riemann-Liouville definitions are frequently used in modeling and applications [4, 29]. Caputo's derivative seems to be more suitable to model some physical phenomena, especially in anomalous diffusion in crowded environments, because it allows boundary and initial conditions to be included in the formulation of the problem. The Caputo time-fractional derivative of order $0 < \alpha \leq 1$ denoted by $\partial_t^{\alpha} u(x,t)$ in Eq (1.1), is defined by

$$\partial_t^{\alpha} u(x,t) = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\frac{\partial u}{\partial t}(x,s)}{(t-s)^{\alpha}} ds, & 0 < \alpha < 1, \\ \partial_t u(x,t), & \alpha = 1. \end{cases} \tag{1.4}$$

The novelty of this work is to develop an LS-SVR algorithm with a polynomial kernel based on Petrov-Galerkin for the numerical solution of Eq (1.1)-(1.3).

The rest of the paper is organized as follows. Some preliminaries and notations for polynomial spaces, orthogonal kernels, and SVMs are presented in Section 2. Least squares support vector regression for a general differential operator is then presented in Section 3. Section 4 is devoted to the development of a LS-SVR algorithm for the fractional FPDEs (1.1)-(1.3). Furthermore, numerical examples are provided in Section 5 to demonstrate the accuracy of the proposed method.

2. PRELIMINARIES

In this section, we collect some results on the orthogonal polynomials and the least squares support vector regression required later in the formulation of the proposed method.

2.1. *functional spaces and function approximation*

We begin by introducing some notations about functional spaces that will be adopted in the subsequent discussion [13, 21, 24].

Let $\Omega \subset \mathbb{R}^n$, $C(\Omega)$ and $C^k(\Omega)$ denote the set of all continuous functions and functions with continuous derivative up to order k on Ω, respectively. L^2 be the set of square integrable functions on Ω such that

$$\int_\Omega |f|^2 d\mu < \infty,$$

with the standard inner product

$$\langle f, g\rangle = \int_\Omega f g d\mu.$$

Definition 1. [21] Let $\Omega \subset \mathbb{R}^n$ be an open set, m be a positive integer and $1 \leq p \leq \infty$. The sobolev space $W^{m,p}(\Omega)$ is defined by

$$W^{m,p}(\Omega) = \{u \in L^p(\Omega) \mid D^\alpha u \in L^p(\Omega), \ \ \forall |\alpha| \leq m\},$$

endowed with the norm $\|\cdot\|_{m,p,\Omega}$ as

$$\|u\|_{m,p,\Omega} = \Big(\sum_{|\alpha|\leq m} \|D^\alpha u\|^p_{L^p(\Omega)} \Big)^{\frac{1}{p}}$$

if $1 \leq p < \infty$ and

$$\|u\|_{m,p,\Omega} = \max_{|\alpha|\leq m} \|D^\alpha u\|_{L^\infty(\Omega)}.$$

Let $w(x)$ be a positive weight function on $\Omega = [0,1]$. We use the weighted Sobolev spaces $H^k_w(\Omega) = W^{k,2}(\Omega)$ whose norm is denoted by $\|\cdot\|_{k,w}$. To accommodate homogeneous boundary conditions, we introduce the following definition:

$$H^2_{0,w}(\Omega) = \{v \in H^2_w(\Omega) : v(0) = v(1) = v'(0) = v'(1) = 0\},$$

where $p = \infty$.

Function approximation can be achieved through various methods such as least squares approximation, interpolation, and fitting. Low degree polynomials are the most widely used functions in approximation theory, due to the easy computations as well as smoothness properties. Moreover, they provide a guaranteed convergence for approximating continuous functions on compact sets as stated by the Weierstrass theorem.

Orthogonal bases allow us to transform the complicated functional equations into algebraic equations that are computationally efficient. As the most common orthogonal functions, the Jacobi polynomials $J_n^{\alpha,\beta}(x)$ are introduced as the eigenfunctions $u(x)$ of Sturm-Liouville differential equation [20, 37],

$$-(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d}{dx}\left((1-x)^{\alpha+1}(1+x)^{\beta+1}\frac{d}{dx}u(x)\right) \tag{2.1}$$

$$= (x^2-1)\frac{d^2}{dx^2}u(x) + (\alpha-\beta+(\alpha+\beta+2)x)\frac{d}{dx}u(x),$$

with $\alpha,\beta > -1$. When $\alpha = \beta = 0$, the Jacobi polynomials reduce to a widely used, the Legendre polynomials $P_n(x)$, $n = 0,1,\cdots$. These polynomials are orthogonal as expressed by the inner product

$$\langle P_n(x), P_m(x)\rangle = \frac{2}{2n+1}\delta_{mn},$$

where δ_{mn} denotes the Kronecker delta and $\langle\cdot,\cdot\rangle$ being the standard inner product of two functions on the interval $[-1,1]$. The completeness and orthogonality of these polynomials are further discussed within the framework of Sturm-Liouville theory. Eq (2.1) is simplified for the Legendre polynomials as

$$\frac{d}{dx}((1-x^2)\frac{d}{dx}P_n(x)) + \lambda_n P_n(x) = 0,$$

with the eigenvalues $\lambda_n = n(n+1)$. An explicit formula is given by [37]

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx}(x^2-1)^n, \qquad n \geq 0.$$

However, it requires higher order derivatives. A simple recursive formula is given by

$$P_0(x) = 1, \quad P_1(x) = x,$$
$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x), \qquad n \geq 1.$$

These polynomials enjoy some computational advantages like the following symmetry and boundedness relations

$$P_n(-x) = (-1)^n P_n(x), \quad n \geq 0, \tag{2.2}$$
$$|P_n(x)| \leq 1, \qquad \forall x \in [-1,1], \tag{2.3}$$

where relation (2.2) reduces the computations and (2.3) keeps the error propagation small.

2.2. *Support vector regression*

Let us consider a given training set $\{(\mathbf{x}_i, y_i)\}_{i=1}^n$ with $\mathbf{x}_i \in \mathbb{R}^d$ as the training input data and $y_i \in \mathbb{R}$ as the corresponding output. In ε-SVR, the goal is to find a function $u(x)$ that has at most ε deviation from the obtained targets y_i for all the training data, and at the same time is as flat as possible [38, 40]. The unknown solution $u(x)$ is expanded in terms of a weighted summation over some basis functions

$$u(x) = \mathbf{w}^T\varphi(x) + b.$$

To find $\mathbf{w}$ and b, the following primal optimization problem is solved [7]

$$\min_{\mathbf{w},b,\varepsilon} \quad \frac{1}{2}\mathbf{w}^T\mathbf{w} + \frac{\gamma}{2}\varepsilon^T\varepsilon \tag{2.4}$$
$$\text{s.t.} \quad y_i = \mathbf{w}^T\varphi(x_i) + b + \varepsilon_i, \quad i = 1,\cdots,n.$$

where $\mathbf{w} = [w_1, \cdots, w_d]^T$ is a weight vector in $\mathbb{R}^d$ and $b \in \mathbb{R}$ is the bias term, γ is the regularization parameter that can be related to the Tikhonov regularization, $\varepsilon = [\varepsilon_1, \cdots, \varepsilon_n]^T$ is error vector where ε_i is the error of the ith training point. The vector valued function $\varphi(x) \in \mathbb{R}^d$ can be a linear or nonlinear function that maps an input space $\mathbb{R}^d$ into a higher dimensional feature space. The following theorem gives the approximate LS-SVR solution by solving a linear system of equations with explicitly given matrices. For the support vector machines, we consider dual space and use the Lagrangian primal method. To find the optimum of the constraint function in the dual space, we have to use the Karush-Kuhn-Tucker (KKT) conditions (Karush 1939; Kuhn and Tucker 1951; Fletcher 1987).

Theorem 1. *The solution of quadratic programming (2.4) is uniquely given by the following linear system,*

$$\left[\begin{array}{c|c} 0 & \mathbf{1}_n^T \\ \hline \mathbf{1}_n & \Omega + \frac{I}{\gamma} \end{array}\right] \left[\begin{array}{c} b \\ \hline \alpha \end{array}\right] = \left[\begin{array}{c} 0 \\ \hline \mathbf{y} \end{array}\right] \tag{2.5}$$

where I is the identity matrix, Ω a positive definite matrix, $\Omega_{ij} = \varphi(x_i)^T \varphi(x_j)$ for $i, j = 1, \cdots, n$, $\mathbf{y} = [y_1, \cdots, y_n]^T$, $\mathbf{1}_n = [1, \cdots, 1]^T$ and the dual variables $\alpha = [\alpha_1, \cdots, \alpha_n]^T$. Then, for the kernel $K(x, x_i) = \varphi(x)^T \varphi(x_i)$, we get the solution as

$$u(x) = \sum_{i=1}^{n} \alpha_i K(x, x_i) + b. \tag{2.6}$$

Proof. The proof is straightforward and can be found in [39]. □

In the primal form of LS-SVR, we apply kernel ridge regression to find an unknown function $u(x)$ so we can interpret this as a single hidden layer feed-forward network extended by the L^2-regularization technique. Such a network can be seen in Figure 1.

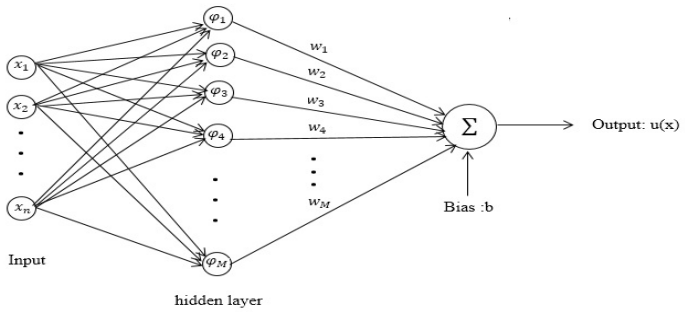

FIGURE 1. The function approximation model used in LS-SVR as a network, $u(x) = \mathbf{w}^T \varphi(x) + b$.

3. LS-SVR FORMULATION FOR A GENERAL DIFFERENTIAL OPERATOR

LS-SVR is a powerful and efficient method for solving functional equations. In this method, a set of input data points are mapped into a higher dimensional space, then solves a linear regression problem to make predictions for the new data points which use the least squares estimate to minimize the cost function.

We first present LS-SVR formulation for a general functional equation. Consider $\mathcal{L}u = f$ on a spatial domain Ω with some boundary or initial conditions $Bu(x_i) = u_i$ where $\mathcal{L}$ may be a differential, integral or integro differential operator. To solve this problem by a learning algorithm, we expand the unknown solution $u(x)$ in terms of a weighted summation over some basis functions as:

$$u(x) \simeq \tilde{u}(x) = \mathbf{w}^T \varphi(x), \tag{3.1}$$

where $\varphi = [\varphi_1, \cdots, \varphi_M]^T$ are basis functions, $\mathbf{w} = [w_1, \cdots, w_M]^T$ are unknown coefficients vector. To determine the unknown weight coefficients w_j in Eq (3.1), we consider the following form of optimization problem associated with Eq (3.1)

$$\begin{aligned} \min_{\mathbf{w},\varepsilon} \quad & \frac{1}{2}\mathbf{w}^T\mathbf{w} + \frac{\gamma}{2}\varepsilon^T\varepsilon \\ \text{s.t.} \quad & \left\langle w^T \mathcal{L}\phi(x), \psi_i \right\rangle - \langle f(x), \psi_i \rangle = \varepsilon_i, \quad i = 1, \cdots, n, \end{aligned} \tag{3.2}$$

where n is the number of training data point x_i in Ω, ε_i is the redidual at ith training point, γ is the regularization parameter and ψ_i the test functions. By using the Lagrangian multipliers the problem is solved in dual space leading to a linear or nonlinear system of algebraic equations based on the linearity of operator $\mathcal{L}$.

Remark 1. In this formulation, we assumed a steady state problem so we only have conditions on space. These conditions always can be assumed to be homogeneous without loss of generality. The nonhomogeneous problems are first converted to homogeneous ones. The basis functions are assumed to satisfy the homogeneous boundary conditions.

Theorem 2. *The solution of problem* (3.2) *is uniquely given by the following linear system,*

$$\left(\Omega + \frac{1}{\gamma} I\right) \alpha = \mathbf{y}, \tag{3.3}$$

where $\Omega_{ik} = \sum_{k=0}^{n} \left\langle \mathcal{L}\phi_j, \psi_i \right\rangle \left\langle \mathcal{L}\phi_j, \psi_k \right\rangle$ *for* $k, i = 0, \cdots, n$, $\mathbf{y}_i = \langle f, \psi_i \rangle$ *and* $\alpha = [\alpha_1, \cdots, \alpha_n]^T$ *are unknown Lagrange multipliers and*

$$u(x) = \sum_{i=1}^{n} \alpha_i K(x, x_i).$$

Proof. The proof is straightforward and can be found in [33]. □

Remark 2. Note that the system (3.3) is simpler than the linear system (2.5) for known data which was in a block matrix. This is due to the inclusion of bias in the basis functions.

4. Solving fourth-order FPDEs with LS-SVR

In this section, we present the LS-SVR for fourth-order FPDE (1.1)-(1.3) using an orthogonal polynomial kernel. We first discretize the problem in time by using L^1 discretization for the Caputo derivative $\partial_t^\alpha u$ [14, 22, 23].

Let $t_0 = 0 < t_1 < \cdots < t_N = T$ be a partition of $[0,T]$. By the Caputo derivative (1.4), in time for function $u(x,t)$ at $t = t_{n+1}, n = 0,1,\ldots$ is given by:

$$\partial_t^\alpha u\Big|_{t=t_{n+1}} = \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} \frac{\frac{\partial u}{\partial t}(x,s)}{(t_{n+1}-s)^\alpha} ds$$

by approximating $\frac{\partial u}{\partial t}(x,s)$ with a finite difference on the end points, taking $\tau = t_{j+1} - t_j$, and computing the integrals exactly, we have:

$$\partial_t^\alpha u\Big|_{t=t_{n+1}} \approx \frac{1}{\tau\Gamma(2-\alpha)} \sum_{j=0}^{n} (u^{j+1} - u^j)\Big[(n+1-j)^{1-\alpha} - (n-j)^{1-\alpha}\Big]. \tag{4.1}$$

Assume $a_j^\alpha = (n+1-j)^{1-\alpha} - (n-j)^{1-\alpha}$. Also, if we take $a_{-1}^\alpha = 0$ and $a_{n+1}^\alpha = 0$, Eq (4.1) can be rewritten as:

$$\partial_t^\alpha u\Big|_{t=t_{n+1}} \approx \frac{1}{\tau^\alpha\Gamma(2-\alpha)} \sum_{j=0}^{n+1} (a_{j-1}^\alpha - a_j^\alpha) u^j \tag{4.2}$$

By taking $\mu_\tau^\alpha := \frac{1}{\tau^\alpha\Gamma(2-\alpha)}$, $b_j^\alpha := a_{j-1}^\alpha - a_j^\alpha$ and $b_{n+1}^\alpha = 1$ in Eq (4.2), we get:

$$\partial_t^\alpha u\Big|_{t=t_{n+1}} \approx \mu_\tau^\alpha u^{n+1} + \mu_\tau^\alpha \sum_{j=0}^{n} b_j^\alpha u^j. \tag{4.3}$$

This is an discrete approximation of fractional derivative as a weighted sum of function values from t_0 to t_{n+1}.

Now, for the fourth-order FPDE (1.1)-(1.3), suppose $u^0, u^1, \cdots, u^n$ are known and the aim is to find an approximation for $u(x,t)$ at $t = t_{n+1}$. In order to solve the problem approximately with LS-SVR, we apply Crank-Nicolson scheme for $\theta = \frac{1}{2}$ to Eq (4.3), so we get:

$$\mu u^{n+1} + \sum_{j=0}^{n} b_j u^j + \frac{1}{2}\left(\partial_x^4 u^{n+1} + \eta u^{n+1} - f^{n+1}\right) + \frac{1}{2}\left(\partial_x^4 u^n + \eta u^n - f^n\right) = 0. \tag{4.4}$$

Let us assume that the general approximate solution to Eq (4.4) in $t = t_{n+1}$ is of the following form with the weights trained during the learning process

$$u_M^{n+1}(x) \simeq \sum_{j=0}^{M} w_j^{n+1} \varphi_j(x) = \mathbf{w}^T \varphi(x), \tag{4.5}$$

where $M+1$ is the number of basis functions, $\varphi = [\tilde{\varphi}_0, \cdots, \tilde{\varphi}_M]^T$ is the vector of orthogonal basis function that we considered to be modal shifted Legendre polynomials as:

$$\begin{aligned} &\tilde{\varphi}_i(x) = P_i(x) + a_i P_{i+1}(x) + b_i P_{i+2}(x) + c_i P_{i+3}(x) + d_i P_{i+4}(x), \\ &\varphi_i(x) = \tilde{\varphi}_i(2x-1), \qquad i = 0, 1, \cdots, M, \end{aligned} \tag{4.6}$$

where the weights of the polynomials are enforced such that the homogeneous conditions are satisfied, a_i, b_i, c_i and d_i are updated for any M by imposing $\varphi_i(0) = 0$, $\varphi_i(1) = 0$, $\varphi_i'(0) = 0$ and $\varphi_i'(1) = 0$, and $\mathbf{w} = [w_0, \cdots, w_M]^T$ are unknown weights to be determined in the learning process as described below. In Eq (4.5), for the sake of simplicity, we omit the superscript in $u_M^{n+1}(x)$ and w_j^{n+1}. So by replacing (4.5) in (4.4) we have:

$$\mu u^{n+1} + \frac{1}{2}\partial_x^4 u^{n+1} + \frac{1}{2}\eta u^{n+1} = \frac{1}{2}(f^{n+1} - f^n) - \frac{1}{2}(\partial_x^4 u^n + \eta u^n) - \sum_{j=0}^{n} b_j u^j. \tag{4.7}$$

This is written as

$$\mathcal{L} u^{n+1}(x) = F^n(x) \qquad n = 0, 1, \cdots, N, \tag{4.8}$$

where $\mathcal{L}$ is the operator on the left and $F^n(x)$ is the right side of Eq (4.7).

Remark 3. The weights w_j in Eq (4.5) are updated at each time step by (4.14) and (4.19).

The variational formulation for Eq (4.7) is given by

$$a(u, v) = \langle F^n, v \rangle, \qquad \forall v \in H_{0,w}^2(\Omega), \tag{4.9}$$

where

$$a(u, v) := \mu \langle u^{n+1}, v \rangle + \frac{1}{2} \langle \partial_x^2 u, \partial_x^2 v \rangle + \frac{1}{2} \eta \langle u^{n+1}, v \rangle, \quad \forall v \in H_{0,w}^2(\Omega),$$

Since $a(u, v)$ is a continuous coercive bilinear form in $H_{0,w}^2(\Omega) \times H_{0,w}^2(\Omega)$ and $\langle F^n, v \rangle$ is a bounded linear function, the Lax-Milgram Lemma [30], guarantees the existence of solution for Eq (4.9).

For the method to be practical, we choose a finite dimensional subspace of $H_{0,w}^2(\Omega)$ as $W_{0,M}(\Omega)$, including the polynomials with the degree at most M satisfying the

boundary conditions (1.3) and seek the approximate solution in $W_{0,M}(\Omega)$ as Eq (4.5) such that

$$\mu\left\langle u_M^{n+1}, v_M\right\rangle + \frac{1}{2}\left\langle \partial_x^2 u_M^{n+1}, \partial_x^2 v_M\right\rangle + \frac{1}{2}\eta\left\langle u_M^{n+1}, v_M\right\rangle = \left\langle F^n, v_M\right\rangle, \quad \forall v_M \in W_{0,M}(\Omega). \tag{4.10}$$

Note that if in (4.10), we set $v_M = \delta(x - x_i)$ for some learning points on the spatial interval, then the primal form of LS-SVR for Eq (4.8) is equivalent to

$$\begin{aligned} \min_{\mathbf{w},\varepsilon} \quad & \frac{1}{2}\mathbf{w}^T\mathbf{w} + \frac{\gamma}{2}\varepsilon^T\varepsilon \\ \text{s.t.} \quad & \mathcal{L}u^{n+1}(x_i) - F^n(x_i) = \varepsilon_i, \quad i = 1, \cdots, n \end{aligned} \tag{4.11}$$

with $\mathcal{L}u = \mu u + \frac{1}{2}\partial_x^4 u + \frac{1}{2}\eta u$. This is Collocation LS-SVR for the fourth-order FPDE (1.1)-(1.3). This is well established, however, to take advantage of the orthogonality of Legendre kernel, we propose an alternative which results in sparse matrices and less computational cost.

The alternative idea is to apply the orthogonality of the kernel to change the minimization problem i.e. $v_M = \varphi_i$ in (4.10) as follows:

$$\begin{aligned} \min_{\mathbf{w},\varepsilon} \quad & \frac{1}{2}\mathbf{w}^T\mathbf{w} + \frac{\gamma}{2}\varepsilon^T\varepsilon \\ \text{s.t.} \quad & a(u_M^{n+1}, \varphi_i) - \langle F^n, \varphi_i\rangle = \varepsilon_i, \quad i = 0, 1, \cdots, M. \end{aligned} \tag{4.12}$$

This is Galerkin LS-SVR for the fourth-order FPDE (1.1)-(1.3).

Theorem 3. *The solution of problem* (4.11) *can be obtained uniquely by the following linear system:*

$$\left(\Omega - \frac{1}{\gamma}I\right)\alpha = \mathbf{y}, \tag{4.13}$$

where $\Omega_{ki} := \sum_{j=0}^{n}\left\langle \mathcal{L}\varphi_j(x), \tilde{\varphi}_i(x)\right\rangle\left\langle \mathcal{L}\varphi_j(x), \tilde{\varphi}_k(x)\right\rangle$ *for* $i, j = 0, \cdots, M$, $\mathbf{y} = \left[\langle F^n, \tilde{\varphi}_0(x)\rangle, \cdots, \langle F^n, \tilde{\varphi}_M(x)\rangle\right]^T$ *and* $\alpha = [\alpha_0, \cdots, \alpha_M]^T$ *are unknown Lagrange multipliers.*

Proof. Introducing Lagrange multipliers $\alpha_i, i = 0, 1, \cdots, M$, the Lagrangian function of problem (4.11) written as:

$$L(\mathbf{w}, \varepsilon, \alpha) = \frac{1}{2}\mathbf{w}^T\mathbf{w} + \frac{\gamma}{2}\varepsilon^T\varepsilon - \sum_{i=0}^{M}\alpha_i\left[\left\langle \mathcal{L}u^{n+1}(x), \tilde{\varphi}_i(x)\right\rangle - \langle F^n(x), \tilde{\varphi}_i(x)\rangle - \varepsilon_i\right].$$

Setting the partial derivatives equal to zero, the following sets of equations are obtained:

$$\frac{\partial L}{\partial w_k} = 0 \Rightarrow w_k = \sum_{i=0}^{M}\alpha_k\left\langle \mathcal{L}\varphi_k(x), \tilde{\varphi}_i(x)\right\rangle \qquad k = 0, 1, \ldots, M \tag{4.14}$$

$$\frac{\partial L}{\partial \varepsilon_k} = 0 \Rightarrow \gamma\varepsilon_k - \alpha_k = 0 \Rightarrow \varepsilon_k = \frac{1}{\gamma}\alpha_k \qquad k = 0, 1, \ldots, M \tag{4.15}$$

$$\frac{\partial L}{\partial \alpha_k} = 0 \Rightarrow \left\langle \mathcal{L}u^{n+1}(x), \tilde{\varphi}_k(x) \right\rangle - \left\langle F^n(x), \tilde{\varphi}_k(x) \right\rangle - \varepsilon_k = 0 \qquad k = 0, 1, \ldots, M \tag{4.16}$$

By using (4.5) and (4.15) in $\frac{\partial L}{\partial \alpha_k} = 0$, we get:

$$\sum_{j=0}^{n} w_j \left\langle \mathcal{L}\varphi_j(x), \tilde{\varphi}_k(x) \right\rangle - \frac{1}{\gamma}\alpha_k = \left\langle F^n(x), \tilde{\varphi}_k(x) \right\rangle \quad k = 0, 1, \ldots, M. \tag{4.17}$$

Now if we replace the weights w_j from (4.14) in Eq (4.17), we have:

$$\sum_{i=0}^{M} \left(\sum_{j=0}^{n} \langle \mathcal{L}\varphi_j(x), \tilde{\varphi}_i(x) \rangle \langle \mathcal{L}\varphi_j(x), \tilde{\varphi}_k(x) \rangle \right) \alpha_i - \frac{1}{\gamma}\alpha_k = \langle F^n(x), \tilde{\varphi}_k(x) \rangle \quad k = 0, 1, \ldots, M. \tag{4.18}$$

We assume $\Omega_{ki} := \sum_{j=0}^{n} \left\langle \mathcal{L}\varphi_j(x), \tilde{\varphi}_i(x) \right\rangle \left\langle \mathcal{L}\varphi_j(x), \tilde{\varphi}_k(x) \right\rangle$ in (4.18), so we get:

$$\sum_{i=0}^{M} \Omega_{ki}\alpha_i - \frac{1}{\gamma}\alpha_k = \langle F^n(x), \tilde{\varphi}_k(x) \rangle \quad \text{for } k = 0, 1, \ldots, M$$

or equivalently,

$$\sum_{j=0}^{M} \Omega_{ij}\alpha_j - \frac{1}{\gamma}\alpha_i = \langle F^n(x), \tilde{\varphi}_i(x) \rangle \quad \text{for } i = 0, 1, \ldots, M, \tag{4.19}$$

which gives

$$u_M^{n+1} = \sum_{i=1}^{M} \alpha_i K(x, x_i), \tag{4.20}$$

as the appriximation solution at $t = t_{n+1}$. By solving the system of equations (4.19) we find α and then by (4.14) we can calculate unknown coefficients w_j at each time step. The matrices present in the structure Ω are sparse matrices due to the utilization of orthogonal polynomials. □

Remark 4. It is worth noting that the coefficient matrix $(\Omega - \frac{1}{\gamma}I)$ remains constant for all time steps of the proposed algorithm, so a single efficient decomposition makes it available for all t_n.

Remark 5. Choosing the polynomial kernel in (4.20) in terms of shifted Legendre polynomials in the modal form (4.6), leads to well-conditioned matrices as illustrated in Figure 2. This figure also depicts a least-squares curve fitting with a second-degree polynomial and verifies the low-degree polynomial behaviour of the conditioning of the matrix. For example, for $\alpha = 0.5$ and $\eta = 1$ we get the second-degree polynomial $34.12x^2 - 4.03 \times 10^2 + 1.39 \times 10^3$. Moreover, the matrix is well-structured i.e., displaying a banded pattern as depicted in Figure 3. Additionally, The matrix is symmetric, positive definite, and diagonal dominant.

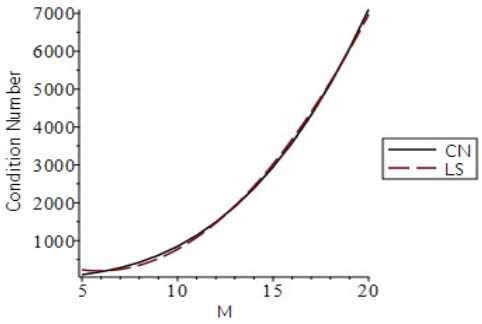

FIGURE 2. The condition number(CN) for the coefficient matrix and a comparison with a polynomial least squares(LS) fitting.

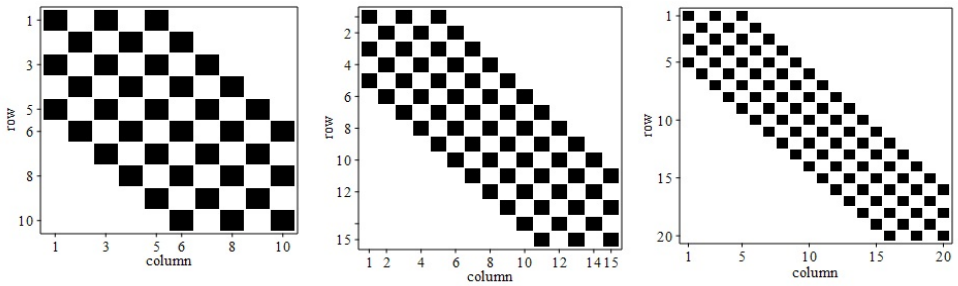

FIGURE 3. Band structure of the coefficient matrix for some numbers of basis functions $M = 10, 15, 20$, respectively.

5. NUMERICAL EXPERIMENTS

In this section, we provide some numerical simulation of the proposed method for the fourth-order FPDE (1.1)-(1.3). The errors at the desired final time $t = T$ are computed approximately as

$$L^2 \simeq \sqrt{\sum_{i=1}^{\mathcal{N}} \mid u(x_i,t_N) - u^N(x_i) \mid^2 \frac{1}{\mathcal{N}}} \tag{5.1}$$

where $u(x,t)$ and $u^N(x)$ are the exact and approximated solution respectivly. We set $\mathcal{N} = 100$ in all numerical results. In addition to the runtime and convergence of the proposed method, a measure of the order of convergence is also included in the numerical results. It is computed approximately as

$$EOC_i = \frac{\ln(e_{i+1}/e_i)}{\ln(M_i/M_{i+1})} \tag{5.2}$$

where e_i given by (5.1) corresponding to the $M = M_i$. The runtime required for each iteration is likewise tabulated to characterize the computational efficiency, convergence, and practical applicability of our approach.

The computational runtimes reported correspond to the CPU time in seconds required to execute $M_i, i = 1,2,\cdots$ time steps to reach the final solution at the final time $t = T$. The numerical algorithms were implemented in Maple 2022 and executed on a laptop with a Core i7 processor and 8GB of RAM. We also set $\eta = 1$.

Example 1. As the first example, we consider the problem (1.1)-(1.3) with $f(x,t)$ associated to the exact solution $u(x) = t^{3+\alpha}\sin(2\pi x)$ as given in [16]. The goal is to check the convergence behavior of the proposed method both in spatial and temporal domains with different numbers of training data and several α. In our investigation of spatial convergence, we have chosen the value of $N = 200$. Table 1 reports the L^2 errors (5.1) and runtimes for fractional orders $\alpha = 0.1, 0.25, 0.5, 0.75, 0.9$. Note that, α near 0 and 1 are also added to see how the performance of the method is affected by limiting values of α. The experimental order of convergence (EOC) (5.2), is also reported in this table. The spatial convergence of the proposed scheme is reported in Table 2. The results show the exponential convergence of the proposed Crank-Nicolson and LS-SVR. This type of convergence for different values of α is well illustrated in Figure 4. in which The logarithmic scale for the error is linear i.e. the error is of exponential convergence.

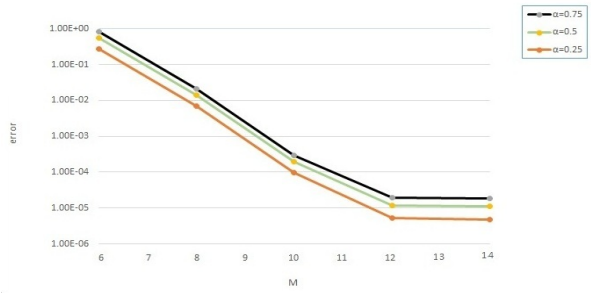

FIGURE 4. The spectral convergence of the proposed method for some fractional orders

Example 2. As the second example, we consider the problem (1.1)-(1.3) with $f(x,t)$ and the exact solution as $u = (t^{\alpha+2}+1)\cos(2\pi x)$ as given in [41]. We set $N = 100$ when investigating the convergence in space. Note that N must be large enough to be able to ignore the effects of temporal errors on spatial convergence. Table 3 reports the L^2 errors (5.1) and runtimes for fractional orders $\alpha = 0.1, 0.25, 0.5, 0.75, 0.9$ and The spatial convergence of the proposed scheme is reported in Table 4. Also, convergence for different values of α is well illustrated in Figure 5.

TABLE 1. The error and temporal order of convergence

	$\alpha = 0.1$		$\alpha = 0.25$		$\alpha = 0.5$		$\alpha = 0.75$		$\alpha = 0.9$	
M	Error	EOC	Error	EOC	Error	EOC	Error	EOC	Error	EOC
6	2.77E-01		2.77E-01		2.77E-01		2.77E-01		2.77E-01	
8	7.06E-03	12.76	7.06E-03	12.76	7.06E-03	12.76	7.06E-03	12.76	7.06E-03	12.76
10	9.65E-05	19.24	9.67E-05	19.23	9.72E-05	19.20	9.75E-05	19.19	9.69E-05	19.22
12	4.35E-06	17.00	5.12E-06	16.12	6.70E-06	14.68	7.62E-06	13.98	5.54E-06	15.69
14	3.84E-06	0.81	4.64E-06	0.64	6.25E-06	0.44	7.19E-06	0.37	5.07E-06	0.57

TABLE 2. The spatial convergence of the proposed scheme for $M = 16$

	$\alpha = 0.25$		$\alpha = 0.5$		$\alpha = 0.75$	
N	Error	EOC	Error	EOC	Error	EOC
40	2.29E-05		2.95E-05		3.07E-05	
80	1.16E-05	0.99	1.52E-05	0.95	1.66E-05	0.88
160	5.81E-06	0.99	7.79E-06	0.97	8.85E-06	0.91
320	2.92E-06	0.99	3.96E-06	0.98	4.65E-06	0.93
640	1.46E-06	1.00	2.00E-06	0.98	2.42E-06	0.94

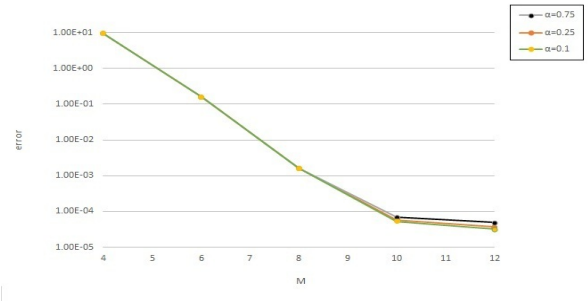

FIGURE 5. The spectral convergence of the proposed method for some fractional orders

TABLE 3. The error, temporal order of convergence, and CPU time in seconds

		$\alpha = 0.1$		$\alpha = 0.25$		$\alpha = 0.5$		$\alpha = 0.75$		$\alpha = 0.9$	
M	time	Error	EOC	Error	EOC	Error	EOC	Error	EOC	Error	EOC
4	16.31	9.45E+00		9.45E+00		9.44E+00		9.43E+00		9.42E+00	
6	28.44	1.60E-01	10.06	1.60E-01	10.06	1.60E-01	10.06	1.60E-01	10.06	1.59E-01	10.06
8	48.38	1.61E-03	15.98	1.61E-03	15.98	1.61E-03	15.97	1.61E-03	15.97	1.60E-03	15.97
10	75.61	5.29E-05	15.31	5.75E-05	14.94	6.62E-05	14.31	6.90E-05	14.13	5.36E-05	15.25
12	115.47	3.21E-05	2.74	3.71E-05	2.40	4.65E-05	1.94	4.95E-05	1.82E	3.29E-05	2.67

TABLE 4. The spatial convergence of the proposed scheme for $M = 16$

	$\alpha = 0.25$		$\alpha = 0.5$		$\alpha = 0.75$	
N	Error	EOC	Error	EOC	Error	EOC
40	9.46E-05		1.15E-04		1.16E-04	
80	4.76E-05	0.99	5.90E-05	0.96	6.20E-05	0.90
160	2.39E-05	0.99	3.00E-05	0.98	3.27E-05	0.92
320	1.20E-05	1.00	1.52E-05	0.98	1.71E-05	0.94
640	5.99E-06	1.00	7.67E-06	0.99	8.83E-06	0.95
1280	3.00E-06	1.00	3.86E-06	0.99	4.54E-06	0.96

6. CONCLUSION

In this work, we propose a supervised ML algorithm, least squares support vector regression for the fractional fourth-order PDEs. To enhance computational efficiency, we employ orthogonal polynomial kernels. Through various test problems, we demonstrate that our proposed approach achieves exponential convergence.

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ON THE BALANCING-BILINEAR SYSTEM OF DIFFERENCE EQUATIONS: SOLUTIONS VIA SOME NUMBER SEQUENCES AND GLOBAL STABILITY

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Abstract. In the current paper, we present a new class of systems called the Balancing-bilinear system of difference equations to investigate some theoretical proprieties. In this class of systems, the coefficients are allowed to depend on the Balancing sequence. One feature of this new class of systems is theoretically solvable in closed form. For this purpose, our interest is focused firstly to show that the solutions of this system are also related to Balancing numbers. As a consequence, we find that these solutions are also related to some number sequences (Pell numbers, Pell-Lucas numbers and Lucas-Balancing numbers). Secondly, we invest in the essential problems related to the Balancing-bilinear system, i.e. the global stability of positive solutions.

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Keywords: system of difference equations, balancing numbers, Pell numbers, Lucas-Balancing numbers, Binet formula, stability

1. INTRODUCTION

Despite the development of difference equations and systems, the class of standard bilinear difference equations and systems introduced by (Adamović [2], Boole [3], Brand [4], Jordan [23], Krechmar [32], Mitrinović and Keckić [33] and references therein) remains very popular in many areas of science, mathematical, and economic domains. These equations and systems are employed to model the temporal evolution of variables such as exchange rates, especially when the values of these variables are only observed at discrete time intervals. In econometric applications, difference equations are often modeled with stochastic terms [8–15, 17, 21]. The standard problem is to find formulas for their solutions in closed-form, in this context, can be found several classical solvable nonlinear difference equations and systems [1, 5–7, 16, 18–20, 24–31], such that the general solutions via the Fibonacci sequence [19, 34], and/or Pell sequence [19]. Here we continue the investigation of solvability of a bilinear system of difference equations but the coefficients depend on

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the Balancing sequence, as well as the general solutions are given via the Balancing sequence and some other sequences. For this purpose, we consider the following m-dimensional system of difference equations with Balancing-coefficients,

$$v_{n+1}^{(i)} = \frac{B_{k+2} - B_{k+1} v_{n-p}^{(i+1) \bmod m}}{B_{k+3} - B_{k+2} v_{n-p}^{(i+1) \bmod m}},\ n,k,p,m \in \mathbb{N}_0,\ i \in \{1,\ldots,m\}, \tag{1.1}$$

and the initial values $v_{-j}^{(i)}$, $i \in \{1,\ldots,m\}$, $j \in \{0,1,\ldots,p\}$.

2. Main results

To solve system (1.1) we require to utilize the following lemmas.

Lemma 1. *Let $(B_n, n \geq 0)$ the Balancing sequence, defined as follows*

$$B_{n+1} = 6B_n - B_{n-1}, n \geq 1,$$

with initial conditions $B_0 = 0$ and $B_1 = 1$. The following Binet formula of the Balancing numbers gives $B_n = \dfrac{a^n - b^n}{a-b}$ where $a = 3+2\sqrt{2}$ and $b = 3-2\sqrt{2}$. Moreover, the Lucas-Balancing sequence has the same homogeneous linear difference equation with constant coefficients as the Balancing sequence,

$$C_{n+1} = 6C_n - C_{n-1}, n \geq 1,$$

with distinct initial conditions $C_0 = 3$, $C_1 = 17$ and the closed-form expressions for the Lucas-Balancing numbers are $2C_n = a^n + b^n$. So, we have some important relations for $n, m \in \mathbb{N}$,

- $B_m B_{n+1} - B_{m-1} B_n = B_{m+n}$,
- $B_{n-1} B_{n+1} - B_n^2 = -1$,
- $B_{n-1} - B_{n+1} = 2C_n$,
- $B_{m(n+1)} B_{m+1} - B_{mn} = B_m B_{m(n+1)+1}$,
- $B_{m(n+1)} - B_{m+1} B_{mn} = -B_{mn-1} B_m$,
- $B_{m(n+1)+1} - B_{m+1} B_{mn+1} = -B_{mn} B_m$.

Moreover, the sequence $(B_n / B_{n+1}, n \geq 1)$ converges to $a^{-1} = b$.

Lemma 2. *Consider the homogeneous linear difference equation with constant coefficients*

$$u_{n+1} - 2C_m u_n + u_{n-1} = 0,\ n \geq 0, \tag{2.1}$$

with initial conditions $u_0, u_{-1} \in \mathbb{R}^*$. Then,

$$\forall n \geq 0,\ u_n = B_{m(n+1)} B_m^{-1} u_0 - B_{mn} B_m^{-1} u_{-1},$$

where $(B_n, n \geq 0)$ is the Balancing sequence and $(C_n, n \geq 0)$ is the Lucas-Balancing sequence.

Proof. Difference equation (2.1) is ordinarily solved by using the following characteristic polynomial

$$\lambda^2 - 2C_m\lambda + 1 = \lambda^2 - (a^m + b^m)\lambda + a^m b^m = (\lambda - a^m)(\lambda - b^m) = 0,$$

roots of this equation are $\lambda_1 = a^m$ and $\lambda_2 = b^m$. These roots are linked the roots of the Balancing number sequence. Then the closed form of general solution of equation (2.1) is

$$\forall n \geq -1,\ u_n = c_1 a^{mn} + c_2 b^{mn},$$

where u_0, u_{-1} are initial values such that

$$\begin{cases} u_0 = c_1 + c_2, \\ u_{-1} = \dfrac{c_1}{a^m} + \dfrac{c_2}{b^m}, \end{cases}$$

and we have

$$c_1 = \frac{a^m u_0 - u_{-1}}{a^m - b^m}, \quad c_2 = \frac{u_{-1} - b^m u_0}{a^m - b^m}.$$

After some calculations, we get

$$\begin{aligned} u_n &= \frac{a^m u_0 - u_{-1}}{a^m - b^m} a^{mn} + \frac{u_{-1} - b^m u_0}{a^m - b^m} b^{mn} \\ &= \frac{\left(a^{m(n+1)} - b^{m(n+1)}\right) u_0 - \left(a^{mn} - b^{mn}\right) u_{-1}}{a^m - b^m} \\ &= \frac{B_{m(n+1)} u_0 - B_{mn} u_{-1}}{B_m}. \end{aligned}$$

The lemma is proved. □

Lemma 3. *Consider the following rational difference equation,*

$$v_{n+1} = \frac{B_m - B_{m-1} v_n}{B_{m+1} - B_m v_n},\ n \geq 0. \tag{2.2}$$

Then,

$$\forall n \geq 1,\ v_n = \frac{-B_{mn} + B_{mn-1} v_0}{B_{mn+1} - B_{mn} v_0}.$$

Proof. Using the change of variables $w_n = B_{m+1} - B_m v_n$, we can write (2.2) as

$$w_{n+1} = \frac{\left(B_{m+1} - B_{m-1}\right) w_n - \left(B_m^2 - B_{m-1} B_{m+1}\right)}{w_n} = \frac{2C_m w_n - 1}{w_n},\ n \geq 0, \tag{2.3}$$

and $w_n = \dfrac{u_n}{u_{n-1}}$, we get $u_{n+1} - 2C_m u_n + u_{n-1} = 0,\ n \geq 0$, by Lemma 2, the closed form of general solution of the equation (2.3) is

$$w_n = \frac{B_{m(n+1)} u_0 - B_{mn} u_{-1}}{B_{mn} u_0 - B_{m(n-1)} u_{-1}} = \frac{B_{m(n+1)} w_0 - B_{mn}}{B_{mn} w_0 - B_{m(n-1)}},\ n \geq 1,$$

then,

$$\begin{aligned} v_n &= B_m^{-1}\left(w_n - B_{m+1}\right) \\ &= B_m^{-1}\left(\frac{B_{m(n+1)}w_0 - B_{mn}}{B_{mn}w_0 - B_{m(n-1)}} - B_{m+1}\right) \\ &= B_m^{-1}\left(\frac{\left(B_{m(n+1)}B_{m+1} - B_{mn}\right) - B_{m(n+1)}B_m v_0}{\left(B_{mn}B_{m+1} - B_{m(n-1)}\right) - B_{mn}B_m v_0} - B_{m+1}\right) \\ &= B_m^{-1}\left(\frac{B_{m(n+1)+1} - B_{m(n+1)}v_0}{B_{mn+1} - B_{mn}v_0} - B_{m+1}\right) \\ &= \frac{-B_{mn} + B_{mn-1}v_0}{B_{mn+1} - B_{mn}v_0},\ n \geq 1. \end{aligned}$$

The lemma is proved. □

2.1. *On the system (2.4)*

In this subsection, we consider the following system of difference equations of 1st-order

$$v_{n+1}^{(i)} = \frac{B_{k+2} - B_{k+1}v_n^{(i+1)\bmod m}}{B_{k+3} - B_{k+2}v_n^{(i+1)\bmod m}},\ n,k,m \in \mathbb{N}_0,\ i \in \{1,\ldots,m\}. \tag{2.4}$$

Now, using the last difference equation in (2.4), we get

$$v_{n+1}^{(m-1)} = \frac{B_{k+2} - B_{k+1}v_n^{(m)}}{B_{k+3} - B_{k+2}v_n^{(m)}} = \frac{B_{2k+4} - B_{2k+3}v_{n-1}^{(1)}}{B_{2k+5} - B_{2k+4}v_{n-1}^{(1)}},\ n \geq 1,$$

similarly, we get

$$v_{n+1}^{(m-2)} = \frac{B_{k+2} - B_{k+1}v_n^{(m-1)}}{B_{k+3} - B_{k+2}v_n^{(m-1)}} = \frac{B_{3k+6} - B_{3k+5}v_{n-1}^{(1)}}{B_{3k+7} - B_{3k+6}v_{n-1}^{(1)}},\ n \geq 2,$$

and recursively for the above, we can get

$$v_{n+1}^{(1)} = \frac{B_{m(k+2)} - B_{m(k+2)-1}v_{n-(m-1)}^{(1)}}{B_{m(k+2)+1} - B_{m(k+2)}v_{n-(m-1)}^{(1)}},\ n \geq m-1.$$

System (2.4) can be written as the following rational difference equation of mth-order

$$v_{n+1} = \frac{B_{m_k} - B_{m_k-1}v_{n-(m-1)}}{B_{m_k+1} - B_{m_k}v_{n-(m-1)}},\ n \geq m-1. \tag{2.5}$$

where $m_k = m(k+2)$. Let $v_{n,h} = v_{mn+h}$, $h \in \{0,1,\ldots,m-1\}$. For this, we have

$$v_{n+1,h} = \frac{B_{m_k} - B_{m_k-1}v_{n,h}}{B_{m_k+1} - B_{m_k}v_{n,h}},\ n \geq 0,\ h \in \{0,1,\ldots,m-1\}.$$

By Lemma 3, the closed form of the general solution of equation (2.5) is easily obtained in the following corollary.

Corollary 1. *Let* $\{v_n, n \geq 0\}$ *be a solution of equation (2.5). Then*

$$\forall n \geq m-1,\ v_{mn+h} = \frac{-B_{m_k n} + B_{m_k n-1} v_h}{B_{m_k n+1} - B_{m_k n} v_h},\ h \in \{0,1,..,m-1\},$$

where $(B_n, n \geq 0)$ *is the Balancing sequence.*

Through the above discussion, we can introduce the following theorem.

Theorem 1. *Let* $\left\{v_n^{(1)}, v_n^{(2)}, \ldots, v_n^{(m)}, n \geq 0\right\}$ *be a solution of system (2.4). Then,*

$$v_{mn+h}^{(i)} = \frac{-B_{m_k n+h_k} + B_{m_k n+h_k-1} v_0^{(h+i) \bmod (m)}}{B_{m_k n+h_k+1} - B_{m_k n+h_k} v_0^{(h+i) \bmod (m)}},\ h \in \{0,1,\ldots,m-1\},\ i \in \{1,\ldots,m\},$$

where $(B_n, n \geq 0)$ *is the Balancing sequence.*

Proof. From Corollary 1, we have

$$\forall n \geq m-1,\ v_{mn+h}^{(1)} = \frac{-B_{m_k n} + B_{m_k n-1} v_h^{(1)}}{B_{m_k n+1} - B_{m_k n} v_h^{(1)}},\ h \in \{0,1,\ldots,m-1\},$$

and by system (2.4), we get

$$v_h^{(1)} = \frac{B_{h_k} - B_{h_k-1} v_0^{(h+1) \bmod (m)}}{B_{h_k+1} - B_{h_k} v_0^{(h+1) \bmod (m)}},\ h \in \{0,1,\ldots,m-1\}.$$

Now, using Lemma 2, we obtain

$$\begin{aligned} v_{mn+h}^{(1)} &= \frac{\left(B_{m_k n-1} B_{h_k} - B_{m_k n} B_{h_k+1}\right) - \left(B_{m_k n-1} B_{h_k-1} - B_{m_k n} B_{h_k}\right) v_0^{(h+1) \bmod (m)}}{-\left(B_{m_k n} B_{h_k} - B_{m_k n+1} B_{h_k+1}\right) + \left(B_{m_k n} B_{h_k-1} - B_{m_k n+1} B_{h_k}\right) v_0^{(h+1) \bmod (m)}} \\ &= \frac{-B_{m_k n+h_k} + B_{m_k n+h_k-1} v_0^{(h+1) \bmod (m)}}{B_{m_k n+h_k+1} - B_{m_k n+h_k} v_0^{(h+1) \bmod (m)}}, h \in \{0,1,\ldots,m-1\}. \end{aligned}$$

The theorem is proved. □

2.2. *On the system (1.1)*

In this article, we study system (1.1), which is an extension of system (2.4). Therefore, system (1.1) can be written as follows

$$v_{(p+1)(n+1)-s}^{(i)} = \frac{B_{k+2} - B_{k+1} v_{(p+1)n-s}^{(i+1) \bmod m}}{B_{k+3} - B_{k+2} v_{(p+1)n-s}^{(i+1) \bmod m}} \tag{2.6}$$

for $s \in \{0,1,\ldots,p\}$, $i \in \{1,\ldots,m\}$ and $n \in \mathbb{N}$. Now, using the following notation,

$$v_{n,s}^{(i)} = v_{(p+1)n-s}^{(i)}, s \in \{0,1,\ldots,p\},\ i \in \{1,\ldots,m\},$$

we can get $(p+1)$-systems similar to system (2.4),

$$v_{n+1,s}^{(i)} = \frac{B_{k+2} - B_{k+1} v_{n,s}^{(i+1) \bmod m}}{B_{k+3} - B_{k+2} v_{n,s}^{(i+1) \bmod m}},\ i \in \{1,\ldots,m\},\ n \in \mathbb{N}_0 \tag{2.7}$$

for $s \in \{0,1,\ldots,p\}$. Through the above discussion, we can introduce the following theorem.

Theorem 2. *Let* $\left\{v_n^{(1)}, v_n^{(2)}, \ldots, v_n^{(m)}, n \geq 0\right\}$ *be a solution of system (1.1). Then, for* $s \in \{0,1,\ldots,p\}$

$$v_{(p+1)(mn+h)-s}^{(i)} = \frac{-B_{m_k n+h_k} + B_{m_k n+h_k-1} v_{-s}^{(h+i) \bmod (m)}}{B_{m_k n+h_k+1} - B_{m_k n+h_k} v_{-s}^{(h+i) \bmod (m)}},$$

$h \in \{0,1,\ldots,m-1\}$, $i \in \{1,\ldots,m\}$, *where* $(B_n, n \geq 0)$ *is the Balancing sequence.*

Proof. Let $\left\{v_{n,s}^{(1)}, v_{n;s}^{(2)}, \ldots, v_{n;s}^{(m)}, n \geq 0, s \in \{0,1,\ldots,p\}\right\}$ be a solution of systems (2.7) with initial values $v_{0,s}^{(i)}$, $i \in \{1,\ldots,m\}$, $s \in \{0,1,\ldots,p\}$. Using Theorem 1, we obtain for $s \in \{0,1,\ldots,p\}$,

$$v_{mn+h,s}^{(i)} = \frac{-B_{m_k n+h_k} + B_{m_k n+h_k-1} v_{0,s}^{(h+i) \bmod (m)}}{B_{m_k n+h_k+1} - B_{m_k n+h_k} v_{0;s}^{(h+i) \bmod (m)}},\ h \in \{0,1,\ldots,m-1\},\ i \in \{1,\ldots,m\}.$$

Returning to the original notation, we obtain

$$v_{(p+1)(mn+h)-s}^{(i)} = \frac{-B_{m_k n+h_k} + B_{m_k n+h_k-1} v_{-s}^{(h+i) \bmod (m)}}{B_{m_k n+h_k+1} - B_{m_k n+h_k} v_{-s}^{(h+i) \bmod (m)}},$$

for $h \in \{0,1,\ldots,m-1\}$, $s \in \{0,1,\ldots,p\}$, $i \in \{1,\ldots,m\}$. □

Based on Theorem 2, we can express the solutions of system (1.1) using some sequences.

Corollary 2. *Let* $\left\{v_n^{(1)}, v_n^{(2)}, \ldots, v_n^{(m)}, n \geq 0\right\}$ *be a solution of system (1.1). Then, for* $s \in \{0,1,\ldots,p\}$,

$$v_{(p+1)(mn+h)-s}^{(i)} = \frac{-P_{2(m_k n+h_k)} + P_{2(m_k n+h_k-1)} v_{-s}^{(h+i) \bmod (m)}}{P_{2(m_k n+h_k+1)} - P_{2(m_k n+h_k)} v_{-s}^{(h+i) \bmod (m)}},$$

where $h \in \{0,1,\ldots,m-1\}$, $i \in \{1,\ldots,m\}$ *and* $(P_n, n \geq 0)$ *is the Pell sequence.*

Proof. We see that it suffices to remark $a = a_1^2\left(1+\sqrt{2}\right)^2$ and $b = b_1^2$, where $a_1 = 1+\sqrt{2}$ and $b_1 = 1-\sqrt{2}$, we get

$$B_n = \frac{a^n - b^n}{a-b} = \frac{a_1^{2n} - b_1^{2n}}{(a_1-b_1)(a_1+b_1)} = \frac{1}{2}P_{2n} \text{ (see [22])}.$$

□

Corollary 3. *Let* $\left\{v_n^{(1)}, v_n^{(2)}, \ldots, v_n^{(m)}, n \geq 0\right\}$ *be a solution of system (1.1). Then, for* $s \in \{0,1,\ldots,p\}$,

$$v^{(i)}_{(p+1)(mn+h)-s} = \frac{-P_{(m_kn+h_k)}Q_{(m_kn+h_k)} + P_{(m_kn+h_k-1)}Q_{(m_kn+h_k-1)}v_{-s}^{(h+i) \bmod (m)}}{P_{(m_kn+h_k+1)}Q_{(m_kn+h_k+1)} - P_{(m_kn+h_k)}Q_{(m_kn+h_k)}v_{-s}^{(h+i) \bmod (m)}},$$

where $h \in \{0,1,\ldots,m-1\}$, $i \in \{1,\ldots,m\}$, $(P_n, n \geq 0)$ *is the Pell sequence and* $(Q_n, n \geq 0)$ *is the Pell-Lucas sequence.*

Proof. We see that it suffices to remark

$$B_n = \frac{a_1^n - b_1^n}{a_1 - b_1}\frac{a_1^n + b_1^n}{a_1 + b_1} = \frac{1}{2}P_nQ_n \text{ (see [22])}.$$

□

Corollary 4. *Let* $\left\{v_n^{(1)}, v_n^{(2)}, \ldots, v_n^{(m)}, n \geq 0\right\}$ *be a solution of system (1.1). Then, for* $s \in \{0,1,\ldots,p\}$,

$$v^{(i)}_{(p+1)(mn+h)-s} = \frac{-C_{m_kn+h_k+1} + C_{m_kn+h_k-1} + \left(C_{m_kn+h_k} - C_{m_kn+h_k-2}\right)v_{-s}^{(h+i) \bmod (m)}}{C_{m_kn+h_k+2} - C_{m_kn+h_k} - \left(C_{m_kn+h_k+1} - C_{m_kn+h_k-1}\right)v_{-s}^{(h+i) \bmod (m)}},$$

where $h \in \{0,1,\ldots,m-1\}$, $i \in \{1,\ldots,m\}$ *and* $(C_n, n \geq 0)$ *is the Lucas-Balancing sequence.*

Proof. We see that it suffices to remark $16B_n = C_{n+1} - C_{n-1}$ (see [22]). □

3. Global stability of positive solutions of (1.1)

In the following, we will study the global stability character of the solutions of system (1.1). Obviously, the unique positive equilibrium of system (1.1) is

$$E = \left(\overline{v}^{(1)}, \overline{v}^{(2)}, \ldots, \overline{v}^{(m)}\right) = -b\underline{1}'_{(m)},$$

where $\underline{1}_{(m)}$ denotes the vector of order $m \times 1$ whose entries are ones. Let the functions $f_i \colon (0,+\infty)^{m(p+1)} \to (0,+\infty)$, $i \in \{1,\ldots,m\}$ defined by

$$f_i\left(\left(\underline{x}_{0:p}^{(1)}\right)', \left(\underline{x}_{0:p}^{(2)}\right)', \ldots, \left(\underline{x}_{0:p}^{(m)}\right)'\right) = \frac{B_{k+2} - B_{k+1}x_p^{(i+1) \bmod m}}{B_{k+3} - B_{k+2}x_p^{(i+1) \bmod m}},\ i \in \{1,\ldots,m\},$$

where $\underline{x}_{0:m} = (x_0, x_1, \ldots, x_m)'$. Now, it is usually useful to linearize system (1.1) around the equilibrium point E in order to facilitate its study. For this purpose, introduce the vectors

$$\underline{X}'_n := \left(\left(\underline{X}_n^{(1)}\right)', \left(\underline{X}_n^{(2)}\right)', \ldots, \left(\underline{X}_n^{(m)}\right)' \right)$$

where $\underline{X}_n^{(i)} = \left(x_n^{(i)}, x_{n-1}^{(i)}, \ldots, x_{n-p}^{(i)}\right)$, $i \in \{1, \ldots, m\}$. With these notations, we obtain the following representation

$$\underline{X}_{n+1} = \Delta_p \underline{X}_n, \tag{3.1}$$

where $\Delta_p =$

$$\begin{pmatrix}
\underline{O}'_{(p-1)} & 0 & \underline{O}'_{(p-1)} & \frac{1}{(B_{k+3}+bB_{k+2})^2} & \cdots & O_{(p-1)} & 0 & O_{(p-1)} & 0 \\
I_{(p-1)} & \underline{O}_{(p-1)} & O_{(p-1)} & \underline{O}_{(p-1)} & \cdots & O_{(p-1)} & \underline{O}_{(p-1)} & O_{(p-1)} & \underline{O}_{(p-1)} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
O_{(p-1)} & 0 & \underline{O}'_{(p-1)} & 0 & \cdots & \underline{O}'_{(p-1)} & 0 & \underline{O}'_{(p-1)} & \frac{1}{(B_{k+3}+bB_{k+2})^2} \\
O_{(p-1)} & \underline{O}_{(p-1)} & I_{p-1} & \underline{O}_{(p-1)} & \cdots & I_{(p-1)} & \underline{O}_{(p-1)} & O_{(p-1)} & \underline{O}_{(p-1)} \\
\underline{O}'_{(p-1)} & \frac{1}{(B_{k+3}+bB_{k+2})^2} & O_{(p-1)} & 0 & \cdots & O_{(p-1)} & 0 & \underline{O}'_{(p-1)} & 0 \\
O_{(p-1)} & \underline{O}_{(p-1)} & O_{(p-1)} & \underline{O}_{(p-1)} & \cdots & O_{(p-1)} & \underline{O}_{(p-1)} & I_{(p-1)} & \underline{O}_{(p-1)}
\end{pmatrix},$$

where $O_{(k,l)}$ denotes the matrix of order $k \times l$ whose entries are zeros, for simplicity, and we set $O_{(k)} := O_{(k,k)}$ and $\underline{O}_{(k)} := O_{(k,1)}$ and $I_{(m)}$ is the $m \times m$ identity matrix. We summarize the above discussion in the following theorem.

Theorem 3. *The positive equilibrium point E is locally asymptotically stable.*

Proof. After some priliminary calculations, the characteristic polynomial of Δ_p is

$$P_{\Delta_p}(\lambda) = \det\left(\Delta_p - \lambda I_{(m(p+1))}\right) = (-1)^{m(p+1)}\,\Omega_1(\lambda) + (-1)^p\,\Omega_{2,k}(\lambda),$$

where $\Omega_1(\lambda) = \lambda^{m(p+1)}$ and $\Omega_{2,k}(\lambda) = \dfrac{1}{(B_{k+3} + B_{k+2}b)^{2m}}$, then $|\Omega_{2,k}(\lambda)| < |\Omega_1(\lambda)|$, $\forall \lambda\colon\ |\lambda| = 1$. So, according to Rouche's Theorem, all zeros of $\Omega_1(\lambda) - \Omega_{2,k}(\lambda) = 0$, lie in the unit disc $|\lambda| < 1$. Thus, the positive equilibrium point E is locally asymptotically stable. □

Corollary 5. *For every well defined solution of system (1.1), we have*

$$\lim v_n^{(i)} = -b,\ i \in \{1, \ldots, m\}.$$

Proof. From Theorem 2, we have

$$\begin{aligned}
\lim v^{(i)}_{(p+1)(mn+h)-h} &= \lim \frac{-B_{m_k n+h_k} + B_{m_k n+h_k-1} v_{-s}^{(h+i) \bmod (m)}}{B_{m_k n+h_k+1} - B_{m_k n+h_k} v_{-s}^{(h+i) \bmod (m)}},\ h \in \{0, 1, \ldots, m-1\} \\
&= \lim \frac{-1 + \frac{B_{m_k n+h_k-1}}{B_{m_k n+h_k}} v_{-s}^{(h+i) \bmod (m)}}{\frac{B_{m_k n+h_k+1}}{B_{m_k n+h_k}} - v_{-s}^{(h+i) \bmod (m)}}
\end{aligned}$$

$$= \frac{-1+bv_{-j}^{(h+i)\bmod(m)}}{a-v_{-j}^{(h+i)\bmod(m)}}$$

for $i \in \{1,\dots,m\}$, hence

$$\begin{aligned}
\lim v_{(p+1)(mn+h)-h}^{(i)} &= \frac{-1+\left(3-2\sqrt{2}\right)v_{-j}^{(h+i)\bmod(m)}}{\left(3+2\sqrt{2}\right)-v_{-j}^{(h+i)\bmod(m)}} \\
&= \frac{\left(\left(-1+3v_{-j}^{(h+i)\bmod(m)}\right)-2\sqrt{2}v_{-j}^{(h+i)\bmod(m)}\right)\left(\left(3-v_{-j}^{(h+i)\bmod(m)}\right)-2\sqrt{2}\right)}{\left(\left(3-v_{-j}^{(h+i)\bmod(m)}\right)+2\sqrt{2}\right)\left(\left(3-v_{-j}^{(h+i)\bmod(m)}\right)-2\sqrt{2}\right)} \\
&= \frac{-3\left(\left(3-v_{-j}^{(h+i)\bmod(m)}\right)^2-8\right)-2\sqrt{2}\left(8-\left(3-v_{-j}^{(h+i)\bmod(m)}\right)^2\right)}{\left(\left(3-v_{-j}^{(h+i)\bmod(m)}\right)^2-8\right)} \\
&= -3+2\sqrt{2} \\
&= -b,\ i \in \{1,\dots,m\}.
\end{aligned}$$

□

The following Corollary is an immediate consequence of Theorem 3 and Corollary 5.

Corollary 6. *The unique positive equilibrium point E is globally asymptotically stable.*

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HERMITE-HADAMARD TYPE INEQUALITIES FOR THE RIGHT RIEMANN-LIOUVILLE FRACTIONAL INTEGRALS WITH VARIABLE ORDER

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Abstract. In this paper, we prove significant results related to the Hermite-Hadamard inequality for convex functions in the framework of Riemann-Liouville fractional integrals with variable order. Two novel equalities are proved to have new fractional trapezoid and midpoint-type inequalities for the differentiable convex functions. This work provides a computational analysis of these inequalities, with numerical examples for convex functions. Additionally, it is demonstrated that the newly identified inequalities represent a generalisation of the comparable inequalities in the literature.

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1. INTRODUCTION

Convexity is a basic math concept important for learning function geometry and finding the optimal solutions. It has roots in ancient cultures; however, to the best of our knowledge, in the late 19th century, the German mathematician Karl Hermann Amandus Schwarz made a revolutionary contribution by introducing the notion of convex functions [9]. It can be used to describe real-world phenomena in optimization, economics, and physics because it has both theoretical and practical applications [11, 22].

The Hermite-Hadamard (H-H) inequality is a key concept in convex function theory and was independently established by two renowned mathematicians, C. Hermite and J. Hadamard [7, 17]. The inequality is stated as follows: If a function $\psi\colon \mathbb{I} \to \mathbb{R}$ is a convex on the interval $\mathbb{I}$, then for any $\mu, \upsilon \in \mathbb{I}$ with $\mu \neq \upsilon$, we have the following

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double inequality:

$$\psi\left(\frac{\mu+\upsilon}{2}\right) \leq \frac{1}{\upsilon-\mu}\int_{\mu}^{\upsilon}\psi(z)dz \leq \frac{\psi(\mu)+\psi(\upsilon)}{2}. \tag{1.1}$$

If a function is convex, the H-H inequality states that the average of the function values at the endpoints of an interval, weighted by their respective distances from the midpoint, is greater than or equal to the value of the function at the midpoint. This inequality finds extensive utility across various domains of mathematics and economics, where convexity also plays a significant role. The fractional integral variants of the inequality (1.1) given by Sarikaya et al. [21] using the integral identity as follows:

$$\psi\left(\frac{\mu+\upsilon}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(\upsilon-\mu)^{\alpha}}\left[J^{\alpha}_{\mu^{+}}\psi(\upsilon)+J^{\alpha}_{\upsilon^{-}}\psi(\mu)\right] \leq \frac{\psi(\mu)+\psi(\upsilon)}{2}, \tag{1.2}$$

with $\alpha>0$. It is obvious that the inequality (1.2) turn into (1.1) as $\alpha=1$.

In order to establish various bounds for the right-hand side of the inequality (1.1), Dragomir and Agarwal [8] initially derive a specific identity.

Lemma 1. *Let $\psi\colon \mathbb{I}^{\circ}\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on $\mathbb{I}^{\circ}, \mu,\upsilon\in\mathbb{I}^{\circ}$ with $\mu<\upsilon$. If $\psi'\in L[\mu,\upsilon]$, then the following equality holds:*

$$\frac{1}{\upsilon-\mu}\int_{\mu}^{\upsilon}\psi(z)\mathrm{d}z-\frac{\psi(\mu)+\psi(\upsilon)}{2}=\frac{\upsilon-\mu}{2}\int_{0}^{1}(1-2\tau)\psi'((1-\tau)\mu+\tau\upsilon)\mathrm{d}\tau. \tag{1.3}$$

Kirmaci [15], introduces a particular identity and then uses it to establish a variety of bounds for the left-hand side of the inequality (1.1).

Lemma 2. *If $\psi\colon \mathbb{I}^{\circ}\to\mathbb{R}$ is differentiable on $\mathbb{I}^{\circ}$ and $\psi'\in L[\mu,\upsilon]$, then we obtain*

$$\begin{aligned}&\psi\left(\frac{\mu+\upsilon}{2}\right)-\frac{1}{\upsilon-\mu}\int_{\mu}^{\upsilon}\psi(z)\mathrm{d}z \\ &=(\upsilon-\mu)\left[\int_{0}^{1/2}\tau\psi'((1-\tau)\mu+\tau\upsilon)\mathrm{d}\tau+\int_{1/2}^{1}(\tau-1)\psi'((1-\tau)\mu+\tau\upsilon)\mathrm{d}\tau\right], \forall\,\mu,\upsilon\in\mathbb{I}^{\circ}.\end{aligned} \tag{1.4}$$

In 1998, Dragomir and Agarwal [8] presented two inequalities applicable to differentiable convex functions. They established a link with the well-known H-H integral inequality that holds for convex functions and addressed error estimation concerns related to the trapezoidal formula. Aamir et al. [3] have presented two new fractional H-H type inequalities for convex functions. They also derived new Simpson's type inequalities for differentiable convex functions and proved two more Simpson's type inequalities using power mean and Hölder's inequalities. Xie et al. [24] have established H-H type inequalities for Riemann-Liouville fractional integrals and derived some new Simpson's and Ostrowski's type inequalities for differentiable convex mapping. Abbas et al. [1] have presented H-H type inequalities for exponentially sub-additive functions by utilizing Riemann-Liouville fractional integrals. Budak et al. [5] have derived certain q-H-H type inequalities and build a new q-H-H type

inequalities using the product of two coordinate convex functions. You et al. [25] have introduced a new version of H-H-Mercer inequalities for harmonically convex functions, and discusses special means for addressing newly discovered inequalities. Jleli and Sameet [14] have established new inequalities of the H-H type, these inequalities involve fractional integrals in relation to another function. Such fractional integrals are generalisations of the Riemann-Liouville and Hadamard fractional integrals. Iscan has formulated the H-H-Fejér type inequality for fractional integrals. As well as derived an integral identity for H-H-Fejér type integral, which is associated with fractional integrals [13]. By using fractional Riemann-Liouville integrals for functions whose absolute values of the second derivatives are convex, Bayraktar has defined the identity and derived several generalised inequalities of the H-H type [4]. Kunt et al. [16] presented a novel method for establishing new fractional H-H type inequalities using only the right Riemann-Liouville fractional integral. They also proved two additional equalities for fractional trapezoid and midpoint-type inequalities for differentiable convex functions. For an in-depth analysis of the various applications arising from the H-H inequality, readers are encouraged to explore references [2, 6, 12, 19, 23].

Definition 1 ([20])**.** A function ψ defined on the interval $[\mu,\upsilon]$, where $\mu \leq \upsilon$, and assuming $\psi \in L_1[\mu,\upsilon]$, the Riemann-Liouville fractional integral of order $\alpha > 0$ with $\mu \geq 0$ is defined as follows:

$$\mathfrak{J}^{\alpha}_{\mu+}\psi(\varkappa) = \frac{1}{\Gamma(\alpha)}\int_{\mu}^{\varkappa}(\varkappa-\tau)^{\alpha-1}\psi(\tau)\mathrm{d}\tau, \quad \varkappa > \mu$$

and

$$\mathfrak{J}^{\alpha}_{\upsilon-}\psi(\varkappa) = \frac{1}{\Gamma(\alpha)}\int_{\varkappa}^{\upsilon}(\tau-\varkappa)^{\alpha-1}\psi(\tau)\mathrm{d}\tau, \quad \varkappa < \upsilon.$$

Here, $\Gamma(\alpha)$ is the well-known Gamma function and $\mathfrak{J}^{0}_{\mu+}\psi(\varkappa) = \mathfrak{J}^{0}_{\upsilon-}\psi(\varkappa) = \psi(\varkappa)$.

Definition 2 ([10])**.** Let $0 < \alpha(\varkappa) < 1$ for all $\varkappa \in [\mu,\upsilon]$ and $\psi \in L_1[\mu,\upsilon]$. Then the left and right Riemann-Liouville integrals of variable fractional order $\alpha(\varkappa)$ defined by

$$\mathfrak{L}^{\alpha(\varkappa)}_{\mu+}\psi(\varkappa) = \frac{1}{\Gamma[\alpha(\varkappa)]}\int_{\mu}^{\varkappa}(\varkappa-\tau)^{\alpha(\varkappa)-1}\psi(\tau)\mathrm{d}\tau, \quad \Re\alpha(x) > 0$$

and

$$\mathfrak{L}^{\alpha(\varkappa)}_{\upsilon-}\psi(\varkappa) = \frac{1}{\Gamma[\alpha(\varkappa)]}\int_{\varkappa}^{\upsilon}(\tau-\varkappa)^{\alpha(\varkappa)-1}\psi(\tau)\mathrm{d}\tau, \quad \Re\alpha(\varkappa) > 0.$$

The classical Riemann-Liouville fractional integral has a fixed constant order of integration α, while the variable order definition allows the order of integration $\alpha(\varkappa)$ to vary as a function of another variable $\varkappa$. In this case $\alpha(\varkappa)$ is depend on the integral domain, resulting in different values at distinct points within this domain. This dynamic interplay between α and the position within the interval provides increased

flexibility and paves the way for graphical analysis, ultimately enhancing our understanding of the applicability of newly introduced generalized inequalities. The variability of the function $\alpha(\varkappa)$ is greatly impacted by the integral domain, resulting in a wide range of values for α over the domain. Consequently, the interplay between the integration order and the place inside the interval offers a degree of versatility that is absent in fixed-order fractional integrals.

The main goal of this study is to generalize H-H inequality by using Riemann-Liouville fractional integral with variable order $\alpha(\varkappa)$, depending upon the function and interval point, which vary simultaneously. Our method precisely explains how functions behave at specific points within intervals. Additionally, we introduce two novel identities applicable to differentiable convex functions, which give fractional trapezoid and midpoint-type inequalities.

2. RESULTS

The following lemma is required to prove the next result.

Lemma 3. *For any integrable mapping $\psi\colon [\mu,\upsilon]\to\mathbb{R}$ and let f be a convex mapping, the following inequality is true:*

$$\frac{\alpha(\mu)}{(\upsilon-\mu)^{\alpha(\mu)}}\int_{\mu}^{\upsilon}(\varkappa-\mu)^{\alpha(\mu)-1}f(\psi(\varkappa))\mathrm{d}\varkappa\geq f\left(\frac{\alpha(\mu)}{(\upsilon-\mu)^{\alpha(\mu)}}\int_{\mu}^{\upsilon}(\varkappa-\mu)^{\alpha(\mu)-1}\psi(\varkappa)\mathrm{d}\varkappa\right) \tag{2.1}$$

with $\alpha(\mu)>0$.

Proof. Since f is a convex mapping, therefore for every point $(\varkappa_0, f(\varkappa_0))$ on the graph of f there is a line $y=\beta(\varkappa-\varkappa_0)+f(\varkappa_0)$ such that $f(\varkappa)\geq\beta(\varkappa-\varkappa_0)+f(\varkappa_0)$ for all $\varkappa$ in the domain of f. Now, we assume that

$$x_0=\frac{\alpha(\mu)}{(\upsilon-\mu)^{\alpha(\mu)}}\int_{\mu}^{\upsilon}(\varkappa-\mu)^{\alpha(\mu)-1}\psi(\varkappa)\mathrm{d}\varkappa$$

and integrate the inequality

$$f(\psi(\varkappa))\geq\beta(\psi(\varkappa)-\varkappa_0)+f(\varkappa_0),$$

$$\begin{aligned}\int_{\mu}^{\upsilon}(\varkappa-\mu)^{\alpha(\mu)-1}f(\psi(\varkappa))\mathrm{d}\varkappa\geq{}&\beta\int_{\mu}^{\upsilon}(\varkappa-\mu)^{\alpha(\mu)-1}f(\psi(\varkappa))\mathrm{d}\varkappa\\&-\beta\varkappa_0\int_{\mu}^{\upsilon}(\varkappa-\mu)^{\alpha(\mu)-1}\mathrm{d}\varkappa\\&+f(\varkappa_0)\int_{\mu}^{\upsilon}(\varkappa-\mu)^{\alpha(\mu)-1}\mathrm{d}\varkappa.\end{aligned}$$

By utilizing the value of $\varkappa_0$, we observe that

$$\int_{\mu}^{\upsilon}(\varkappa-\mu)^{\alpha(\mu)-1}f(\psi(\varkappa))\mathrm{d}\varkappa\geq\beta(\varkappa_0-\varkappa_0)\frac{(\upsilon-\mu)^{\alpha(\mu)}}{\alpha(\mu)}+\frac{(\upsilon-\mu)^{\alpha(\mu)}}{\alpha(\mu)}f(\varkappa_0).$$

Thus,

$$\frac{\alpha(\mu)}{(\upsilon-\mu)^{\alpha(\mu)}}\int_{\mu}^{\upsilon}(\varkappa-\mu)^{\alpha(\mu)-1}f(\psi(\varkappa))\mathrm{d}\varkappa \geq f\left(\frac{\alpha(\mu)}{(\upsilon-\mu)^{\alpha(\mu)}}\int_{\mu}^{\upsilon}(\varkappa-\mu)^{\alpha(\mu)-1}\psi(\varkappa)\mathrm{d}\varkappa\right)$$

and the proof is completed. □

Theorem 1. *Let* $\psi\colon [\mu,\upsilon]\to\mathbb{R}$ *be a positive function with* $0\leq\mu<\upsilon$, $\alpha(\mu)>0$ *and* $\psi\in L_1[\mu,\upsilon]$. *If* ψ *is a convex function on* $[\mu,\upsilon]$, *then the following inequalities for fractional integrals hold:*

$$\psi\left(\frac{\alpha(\mu)\upsilon+\mu}{\alpha(\mu)+1}\right)\leq\frac{\Gamma(\alpha(\mu)+1)}{(\upsilon-\mu)^{\alpha(\mu)}}\mathbb{J}_{\upsilon^-}^{\alpha(\mu)}\psi(\mu)\leqslant\frac{\alpha(\mu)\psi(\upsilon)+\psi(\mu)}{\alpha(\mu)+1}. \tag{2.2}$$

Proof. Since ψ is convex, therefore by using Lemma 3, we have

$$\psi\left(\frac{\alpha(\mu)}{(\upsilon-\mu)^{\alpha(\mu)}}\int_{\mu}^{\upsilon}(\varkappa-\mu)^{\alpha(\mu)-1}\varkappa\mathrm{d}\varkappa\right)\leq\frac{\alpha(\mu)}{(\upsilon-\mu)^{\alpha(\mu)}}\int_{\mu}^{\upsilon}(\varkappa-\mu)^{\alpha(\mu)-1}\psi(\varkappa)\mathrm{d}\varkappa. \tag{2.3}$$

One can easily observe that

$$\psi\left(\frac{\alpha(\mu)\upsilon+\mu}{\alpha(\mu)+1}\right)\leq\frac{\Gamma(\alpha(\mu)+1)}{(\upsilon-\mu)^{\alpha(\mu)}}\mathbb{J}_{\upsilon^-}^{\alpha(\mu)}\psi(\mu). \tag{2.4}$$

Thus the first inequality of (2.2) is proved.

To prove the second inequality, we again use the convexity and we have

$$\psi(\varkappa)\leq\psi(\upsilon)+\frac{\psi(\upsilon)-\psi(\mu)}{\upsilon-\mu}(\varkappa-\upsilon). \tag{2.5}$$

Multiplying both sides of (2.5) by $\frac{1}{\Gamma(\alpha(\mu))}(\varkappa-\mu)^{\alpha(\mu)-1}$, then integrating the resulting inequality with respect to $\varkappa$ over $[0,1]$, we obtain

$$\mathbb{J}_{\upsilon^-}^{\alpha(\mu)}\psi(\mu)\leq\frac{(\upsilon-\mu)^{\alpha(\mu)}}{\Gamma(\alpha(\mu)+1)}\left[\psi(\upsilon)-\frac{\psi(\upsilon)-\psi(\mu)}{\alpha(\mu)+1}\right],$$

after simplification, we have

$$\frac{\Gamma(\alpha(\mu)+1)}{(\upsilon-\mu)^{\alpha(\mu)}}\mathbb{J}_{\upsilon^-}^{\alpha(\mu)}\psi(\varkappa)\leq\frac{\alpha(\mu)\psi(\upsilon)+\psi(\mu)}{\alpha(\mu)+1}, \tag{2.6}$$

by adding equations (2.4) and (2.6), we get the desired inequality (2.2). □

Remark 1. If we assign $\alpha(\varkappa)=1$, in Theorem 1, then we find inequality (1.1).

Remark 2. If we choose $\alpha(\varkappa)=\beta$ (constant), then we have Theorem 2.1 proved by Kunt et al. in [16].

Lemma 4. *Let $\psi\colon \mathbb{I}^\circ \to \mathbb{R}$ be a differentiable mapping on $\mathbb{I}^\circ$, $\mu,\upsilon \in \mathbb{I}^\circ$ with $\mu<\upsilon$ and $\alpha(\mu)>0$. If $\psi' \in L[\mu,\upsilon]$, then the following equality for the right Riemann-Liouville fractional integral holds:*

$$\begin{aligned}&\frac{\Gamma(\alpha(\mu)+1)}{(\upsilon-\mu)^{\alpha(\mu)}}\mathbb{J}^{\alpha(\mu)}_{\upsilon-}\psi(\mu)-\frac{\alpha(\mu)\psi(\upsilon)+\psi(\mu)}{\alpha(\mu)+1}\\&=\frac{\upsilon-\mu}{\alpha(\mu)+1}\int_0^1\left[1-(\alpha(\mu)+1)\tau^{\alpha(\mu)}\right]\psi'((1-\tau)\mu+\tau\upsilon)\mathrm{d}\tau.\end{aligned}\tag{2.7}$$

Proof. If we apply the integration by parts to the right hand side of (2.7), we have

$$\begin{aligned}&\frac{\upsilon-\mu}{\alpha(\mu)+1}\int_0^1\left[1-(\alpha(\mu)+1)\tau^{\alpha(\mu)}\right]\psi'((1-\tau)\mu+\tau\upsilon)\mathrm{d}\tau\\&=(\upsilon-\mu)\left[\frac{1}{\alpha(\mu)+1}\int_0^1\psi'((1-\tau)\mu+\tau\upsilon)\mathrm{d}\tau-\int_0^1\tau^{\alpha(\mu)}\psi'((1-\tau)\mu+\tau\upsilon)\mathrm{d}\tau\right]\\&=\left[\frac{\psi(\mu)-\psi(\upsilon)}{\alpha(\mu)+1}+\psi(\upsilon)-\alpha(\mu)\int_0^1\tau^{\alpha(\mu)-1}\psi((1-\tau)\mu+\tau\upsilon)\mathrm{d}\tau\right]\\&=\frac{\Gamma(\alpha(\mu)+1)}{(\upsilon-\mu)^{\alpha(\mu)}}\mathbb{J}^{\alpha(\mu)}_{\upsilon-}\psi(\mu)-\frac{\alpha(\mu)\psi(\upsilon)+\psi(\mu)}{\alpha(\mu)+1}.\end{aligned}$$

This completes the proof. □

Remark 3. If we take $\alpha(\varkappa)=1$, in Lemma 4, then we obtain Lemma 1.

Remark 4. If we choose $\alpha(\varkappa)=\beta$ (constant), then we will get [16, Lemma 3.1].

Lemma 5. *Let $\psi\colon \mathbb{I}^\circ \to \mathbb{R}$ be a differentiable mapping on $\mathbb{I}^\circ, \mu,\upsilon \in \mathbb{I}^\circ$ with $\mu<\upsilon$ and $\alpha(\mu)>0$. If $\psi' \in L[\mu,\upsilon]$, then the following equality for the right Riemann-Liouville fractional integral holds:*

$$\begin{aligned}&\psi\left(\frac{\alpha(\mu)\upsilon+\mu}{\alpha(\mu)+1}\right)-\frac{\Gamma(\alpha(\mu)+1)}{(\upsilon-\mu)^{\alpha(\mu)}}\mathbb{J}^{\alpha(\mu)}_{\upsilon-}\psi(\mu)\\&=(\upsilon-\mu)\left[\int_0^{\frac{\alpha(\mu)}{\alpha(\mu)+1}}\tau^{\alpha(\mu)}\psi'((1-\tau)\mu+\tau\upsilon)\mathrm{d}\tau+\int_{\frac{\alpha(\mu)}{\alpha(\mu)+1}}^1\left(\tau^{\alpha(\mu)}-1\right)\psi'((1-\tau)\mu+\tau\upsilon)\mathrm{d}\tau\right].\end{aligned}\tag{2.8}$$

Proof. If we apply the integration by parts to the right hand side of (2.8), we find

$$\begin{aligned}&(\upsilon-\mu)\left[\int_0^{\frac{\alpha(\mu)}{\alpha(\mu)+1}}\tau^{\alpha(\mu)}\psi'((1-\tau)\mu+\tau\upsilon)\mathrm{d}\tau+\int_{\frac{\alpha(\mu)}{\alpha(\mu)+1}}^1\left(\tau^{\alpha(\mu)}-1\right)\psi'((1-\tau)\mu+\tau\upsilon)\mathrm{d}\tau\right]\\&=(\upsilon-\mu)\left[\int_0^1\tau^{\alpha(\mu)}\psi'((1-\tau)\mu+\tau\upsilon)\mathrm{d}\tau-\int_{\frac{\alpha(\mu)}{\alpha(\mu)+1}}^1\psi'((1-\tau)\mu+\tau\upsilon)\mathrm{d}\tau\right]\\&=\psi\left(\frac{\alpha(\mu)\upsilon+\mu}{\alpha(\mu)+1}\right)-\frac{\Gamma(\alpha(\mu)+1)}{(\upsilon-\mu)^{\alpha(\nu)}}\mathbb{J}^{\alpha(\mu)}_{\upsilon-}\psi(\mu).\end{aligned}$$

This completes the proof. □

Remark 5. If we take $\alpha(\varkappa)=1$, in Lemma 5, it becomes Lemma 2.

Remark 6. If we assign $\alpha(\varkappa)=\beta$ (constant), then we will get [16, Lemma 3.2].

3. Midpoint and trapezoid inequalities

By using Lemma 4 and Lemma 5, we will obtain some new right Riemann-Liouville fractional trapezoid and midpoint-type inequalities in this section.

Theorem 2. *Let $\psi\colon \mathbb{I}^\circ \to \mathbb{R}$ be a differentiable mapping on $\mathbb{I}^\circ, \mu, \upsilon \in \mathbb{I}^\circ$ with $\mu<\upsilon$ and $\alpha(\mu)>0$. If $|\psi'|$ is convex on $[\mu,\upsilon]$, then the following right Riemann-Liouville fractional integral inequality holds:*

$$\left|\frac{\Gamma(\alpha(\mu)+1)}{(\upsilon-\mu)^{\alpha(\mu)}}\mathbb{J}^{\alpha(\mu)}_{\upsilon-}\psi(\mu)-\frac{\alpha(\mu)\psi(\upsilon)+\psi(\mu)}{\alpha(\mu)+1}\right|$$
$$\leq \frac{\upsilon-\mu}{\alpha(\mu)+1}\left[\mathbb{A}_1(\alpha(\mu))\left|\psi'(\mu)\right|+\mathbb{A}_2(\alpha(\mu))\left|\psi'(\upsilon)\right|+\mathbb{A}_3(\alpha(\mu))\left|\psi'(\mu)\right|+\mathbb{A}_4(\alpha(\mu))\left|\psi'(\upsilon)\right|\right],$$

where

$$\mathbb{A}_1(\alpha(\mu))=\frac{\alpha(\mu)\left(2(\alpha(\mu)+2)(\alpha(\mu)+1)^{\frac{1}{\alpha(\mu)}-1}-1\right)}{2(\alpha(\mu)+2)(\alpha(\mu)+1)^{\frac{2}{\alpha(\mu)}}},$$
$$\mathbb{A}_2(\alpha(\mu))=\frac{\alpha(\mu)}{2(\alpha(\mu)+2)(\alpha(\mu)+1)^{\frac{2}{\alpha(\mu)}}},$$
$$\mathbb{A}_3(\alpha(\mu))=\frac{\alpha(\mu)\left(2(\alpha(\mu)+2)(\alpha(\mu)+1)^{\frac{1}{\alpha(\mu)}-1}-1-(\alpha(\mu)+1)^{\frac{2}{\alpha(\mu)}}\right)}{2(\alpha(\mu)+2)(\alpha(\mu)+1)^{\frac{2}{\alpha(\mu)}}},$$
$$\mathbb{A}_4(\alpha(\mu))=\frac{\alpha(\mu)\left(1+(\alpha(\mu)+1)^{\frac{2}{\alpha(\mu)}}\right)}{2(\alpha(\mu)+2)(\alpha(\mu)+1)^{\frac{2}{\alpha(\mu)}}}.$$

Proof. Using Lemma 4 and the convexity of $|\psi'|$, we acquire

$$\left|\frac{\Gamma(\alpha(\mu)+1)}{(\upsilon-\mu)^{\alpha(\mu)}}\mathbb{J}^{\alpha(\mu)}_{\upsilon-}\psi(\mu)-\frac{\alpha(\mu)\psi(\upsilon)+\psi(\mu)}{\alpha(\mu)+1}\right|$$
$$\leq \frac{\upsilon-\mu}{\alpha(\mu)+1}\int_0^1\left|1-(\alpha(\mu)+1)\tau^{\alpha(\mu)}\right|\left|\psi'((1-\tau)\mu+\tau\upsilon)\right|\mathrm{d}\tau$$
$$\leq \frac{\upsilon-\mu}{\alpha(\mu)+1}\left[\int_0^{\frac{1}{\sqrt[\alpha(\mu)]{\alpha(\mu)+1}}}\left(1-(\alpha(\mu)+1)\tau^{\alpha(\mu)}\right)\left[(1-\tau)\left|\psi'(\mu)\right|+\tau\left|\psi'(\upsilon)\right|\right]\mathrm{d}\tau\right.$$
$$\left.+\int_{\frac{1}{\sqrt[\alpha(\mu)]{\alpha(\mu)}}}^1\left((\alpha(\mu)+1)\tau^{\alpha(\mu)}-1\right)\left[(1-\tau)\left|\psi'(\mu)\right|+\tau\left|\psi'(\upsilon)\right|\right]\mathrm{d}\tau\right]$$
$$\leq \frac{\upsilon-\mu}{\alpha(\mu)+1}\left[\mathbb{A}_1(\alpha(\mu))\left|\psi'(\mu)\right|+\mathbb{A}_2(\alpha(\mu))\left|\psi'(\upsilon)\right|+\mathbb{A}_3(\alpha(\mu))\left|\psi'(\mu)\right|+\mathbb{A}_4(\alpha(\mu))\left|\psi'(\upsilon)\right|\right].$$

Finally, the proof of Theorem 2 is completed. □

Remark 7. If we take $\alpha(\varkappa)=1$, in Theorem 2, then we obtain [8, Theorem 2.2].

Remark 8. If we choose $\alpha(\varkappa)=\beta$ (constant), then we find Theorem 4.1 in [16].

Theorem 3. *Let $\psi\colon \mathbb{I}^{\circ}\to\mathbb{R}$ be a differentiable mapping on $\mathbb{I}^{\circ},\mu,\upsilon\in\mathbb{I}^{\circ}$ with $\mu<\upsilon$. If $|\psi'|^{q}$ is convex on $[\mu,\upsilon]$ for $q\geq 1$, then the following right Riemann-Liouville fractional integral inequality holds:*

$$\left|\frac{\Gamma(\alpha(\mu)+1)}{(\upsilon-\mu)^{\alpha(\mu)}}\mathbb{J}^{\alpha(\mu)}_{\upsilon-}\psi(\mu)-\frac{\alpha(\mu)\psi(\upsilon)+\psi(\mu)}{\alpha(\mu)+1}\right|\leq\frac{\upsilon-\mu}{\alpha(\mu)+1}\left(\frac{2\alpha(\mu)}{(\alpha(\mu)+1)^{1+\frac{1}{\alpha(\mu)}}}\right)^{1-\frac{1}{q}}$$

$$\times\left(\mathbb{A}_1(\alpha(\mu))\left|\psi'(\mu)\right|^{q}+\mathbb{A}_2(\alpha(\mu))\left|\psi'(\upsilon)\right|^{q}+\mathbb{A}_3(\alpha(\mu))\left|\psi'(\mu)\right|^{q}+\mathbb{A}_4(\alpha(\mu))\left|\psi'(\upsilon)\right|^{q}\right)^{\frac{1}{q}},$$

where $\mathbb{A}_1(\alpha(\mu))-\mathbb{A}_4(\alpha(\mu))$ are the same as in Theorem 2 and $\alpha(\mu)>0$.

Proof. Using Lemma 4, power mean inequality and the convexity of $|\psi'|^{q}$, we establish

$$\begin{aligned}
&\left|\frac{\Gamma(\alpha(\mu)+1)}{(\upsilon-\mu)^{\alpha(\mu)}}\mathbb{J}^{\alpha(\mu)}_{\upsilon-}\psi(\mu)-\frac{\alpha(\mu)\psi(\upsilon)+\psi(\mu)}{\alpha(\mu)+1}\right|\\
&\leq\frac{\upsilon-\mu}{\alpha(\mu)+1}\int_0^1\left|1-(\alpha(\mu)+1)\tau^{\alpha(\mu)}\right|\left|\psi'((1-\tau)\mu+\tau\upsilon)\right|\mathrm{d}\tau\\
&\leq\frac{\upsilon-\mu}{\alpha(\mu)+1}\left[\left(\int_0^1\left|1-(\alpha(\mu)+1)\tau^{\alpha(\mu)}\right|\mathrm{d}\tau\right)^{1-\frac{1}{q}}\right.\\
&\quad\left.\times\left(\int_0^1\left|1-(\alpha(\mu)+1)\tau^{\alpha(\mu)}\right|\left|\psi'((1-\tau)\mu+\tau\upsilon)\right|^{q}\,\mathrm{d}\tau\right)^{\frac{1}{q}}\right]\\
&\leq\frac{\upsilon-\mu}{\alpha(\mu)+1}\left(\frac{2\alpha(\mu)}{(\alpha(\mu)+1)^{1+\frac{1}{\alpha(\mu)}}}\right)^{1-\frac{1}{q}}\\
&\quad\times\left(\mathbb{A}_1(\alpha(\mu))\left|\psi'(\mu)\right|^{q}+\mathbb{A}_2(\alpha(\mu))\left|\psi'(\upsilon)\right|^{q}+\mathbb{A}_3(\alpha(\mu))\left|\psi'(\mu)\right|^{q}+\mathbb{A}_4(\alpha(\mu))\left|\psi'(\upsilon)\right|^{q}\right)^{\frac{1}{q}}.
\end{aligned}$$

Finally, the proof of Theorem 3 is accomplished. □

Remark 9. If we assign $\alpha(\varkappa)=1$, in Theorem 3, then we have Theorem 1 in [18].

Remark 10. If we choose $\alpha(\varkappa)=\beta$ (constant), then we find Theorem 4.2 in [16].

Theorem 4. *Let $\psi\colon \mathbb{I}^{\circ}\to\mathbb{R}$ be a differentiable mapping on $\mathbb{I}^{\circ},\mu,\upsilon\in\mathbb{I}^{\circ}$ with $\mu<\upsilon$. If $|\psi'|^{q}$ is convex on $[\mu,\upsilon]$ for $q>1$, then the following right Riemann-Liouville fractional integral inequality holds:*

$$\begin{aligned}
&\left|\frac{\Gamma(\alpha(\mu)+1)}{(\upsilon-\mu)^{\alpha(\mu)}}\mathbb{J}^{\alpha(\mu)}_{\upsilon-}\psi(\mu)-\frac{\alpha(\mu)\psi(\upsilon)+\psi(\mu)}{\alpha(\mu)+1}\right|\\
&\leq\frac{\upsilon-\mu}{\alpha(\mu)+1}\left(\mathbb{A}_5(\alpha(\mu),p)+\mathbb{A}_6(\alpha(\mu),p)\right)^{\frac{1}{p}}\left(\frac{|\psi'(\mu)|^{q}+|\psi'(\upsilon)|^{q}}{2}\right)^{\frac{1}{q}},
\end{aligned}$$

where

$$\mathbb{A}_5(\alpha(\mu),p)=\int_0^{\frac{1}{\sqrt[\alpha(\mu)]{\alpha(\mu)+1}}}\left(1-(\alpha(\mu)+1)\tau^{\alpha(\mu)}\right)^p\,\mathrm{d}\tau,$$

$$\mathbb{A}_6(\alpha(\mu),p)=\int_{\frac{1}{\sqrt[\alpha(\mu)]{\alpha(\mu)+1}}}^1\left((\alpha(\mu)+1)\tau^{\alpha(\mu)}-1\right)^p\,\mathrm{d}\tau,$$

with $\frac{1}{p}+\frac{1}{q}=1$ *and* $\alpha(\mu)>0$.

Proof. Using Lemma 4, Hölder inequality and the convexity of $|\psi'|^q$, we obtain

$$\begin{aligned}
&\left|\frac{\Gamma(\alpha(\mu)+1)}{(\upsilon-\mu)^{\alpha(\mu)}}\mathbb{J}^{\alpha(\mu)}_{\upsilon-}\psi(\mu)-\frac{\alpha(\mu)\psi(\upsilon)+\psi(\mu)}{\alpha(\mu)+1}\right|\\
&\le\frac{\upsilon-\mu}{\alpha(\mu)+1}\int_0^1\left|1-(\alpha(\mu)+1)\tau^{\alpha(\mu)}\right|\left|\psi'((1-\tau)\mu+\tau\upsilon)\right|\mathrm{d}\tau\\
&\le\frac{\upsilon-\mu}{\alpha(\mu)+1}\left[\left(\int_0^1\left|1-(\alpha(\mu)+1)\tau^{\alpha(\mu)}\right|^p\,\mathrm{d}\tau\right)^{\frac{1}{p}}\left(\int_0^1\left|\psi'((1-\tau)\mu+\tau\upsilon)\right|^q\,\mathrm{d}\tau\right)^{\frac{1}{q}}\right]\\
&\le\frac{\upsilon-\mu}{\alpha(\mu)+1}\left(\int_0^{\frac{1}{\sqrt[\alpha(\mu)]{\alpha(\mu)+1}}}\left(1-(\alpha(\mu)+1)\tau^{\alpha(\mu)}\right)^p\,\mathrm{d}\tau\right.\\
&\quad\left.+\int_{\frac{1}{\sqrt[\alpha(\mu)]{\alpha(\mu)+1}}}^1\left((\alpha(\mu)+1)\tau^{\alpha(\mu)}-1\right)^p\,\mathrm{d}\tau\right)^{\frac{1}{p}}\times\left(\int_0^1\left[\tau\left|\psi'(\mu)\right|^q+(1-\tau)\left|\psi'(\upsilon)\right|^q\right]\mathrm{d}\tau\right)^{\frac{1}{q}}\\
&\le\frac{\upsilon-\mu}{\alpha(\mu)+1}\left[(\mathbb{A}_5(\alpha(\mu),p)+\mathbb{A}_6(\alpha(\mu),p))^{\frac{1}{p}}\left(\frac{|\psi'(\mu)|^q+|\psi'(\upsilon)|^q}{2}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Thus, the proof of Theorem 4 is finished. □

Remark 11. If we assign $\alpha(\varkappa)=1$, then we obtain Theorem 2.3 in [8].

Remark 12. If we choose $\alpha(\varkappa)=\beta$ (constant), then we have Theorem 4.3 in [16].

Theorem 5. *Let* $\psi\colon\mathbb{I}^\circ\to\mathbb{R}$ *be a differentiable mapping on* $\mathbb{I}^\circ,\mu,\upsilon\in\mathbb{I}^\circ$ *with* $\mu<\upsilon$ *and with* $\alpha(\mu)>0$. *If* $|\psi'|$ *is convex on* $[\mu,\upsilon]$, *then the following right Riemann-Liouville fractional integral inequality holds:*

$$\begin{aligned}
&\left|\psi\left(\frac{\alpha(\mu)\upsilon+\mu}{\alpha(\mu)+1}\right)-\frac{\Gamma(\alpha(\mu)+1)}{(\upsilon-\mu)^{\alpha(\mu)}}\mathbb{J}^{\alpha(\mu)}_{\upsilon-}\psi(\mu)\right|\\
&\le(\upsilon-\mu)\left[\mathbb{A}_7(\alpha(\mu))\left|\psi'(\mu)\right|+\mathbb{A}_8(\alpha(\mu))\left|\psi'(\upsilon)\right|+\mathbb{A}_9(\alpha(\mu))\left|\psi'(\mu)\right|+\mathbb{A}_{10}(\alpha(\mu))\left|\psi'(\upsilon)\right|\right],
\end{aligned}$$

where

$$\mathbb{A}_7(\alpha(\mu))=\frac{2\left(\frac{\alpha(\mu)}{\alpha(\mu)+1}\right)^{\alpha(\mu)+1}}{(\alpha(\mu)+1)(\alpha(\mu)+2)},\quad\mathbb{A}_8(\alpha(\mu))=\frac{\alpha(\mu)\left(\frac{\alpha(\mu)}{\alpha(\mu)+1}\right)^{\alpha(\mu)+1}}{(\alpha(\mu)+1)(\alpha(\mu)+2)},$$

$$\mathbb{A}_9(\alpha(\mu))=\frac{4(\alpha(\mu))^{\alpha(\mu)+1}-\alpha(\mu)(\alpha(\mu)+1)^{\alpha(\mu)}}{2(\alpha(\mu)+1)^{\alpha(\mu)+2}(\alpha(\mu)+2)},$$

$$\mathbb{A}_{10}(\alpha(\mu)) = \frac{\alpha(\mu)\left(2(\alpha(\mu))^{\alpha(\mu)} + (\alpha(\mu)+1)^{\alpha(\mu)}\right)}{2(\alpha(\mu)+1)^{\alpha(\mu)+2}(\alpha(\mu)+2)}.$$

Proof. Using Lemma 5 and the convexity of $|\psi'|$, we acquire

$$\begin{aligned}
&\left|\psi\left(\frac{\alpha(\mu)\upsilon+\mu}{\alpha(\mu)+1}\right) - \frac{\Gamma(\alpha(\mu)+1)}{(\upsilon-\mu)^{\alpha(\mu)}}\mathbb{J}^{\alpha(\mu)}_{\upsilon-}\psi(\mu)\right| \\
&\leq (\upsilon-\mu)\left[\int_0^{\frac{\alpha(\mu)}{\alpha(\mu)+1}} \tau^{\alpha(\mu)}\left|\psi'((1-\tau)\mu+\tau\upsilon)\right|d\tau + \int_{\frac{\alpha(\mu)}{\alpha(\mu)+1}}^1 \left(1-\tau^{\alpha(\mu)}\right)\left|\psi'((1-\tau)\mu+\tau\upsilon)\right|d\tau\right] \\
&\leq (\upsilon-\mu)\left[\int_0^{\frac{\alpha(\mu)}{\alpha(\mu)+1}} \tau^{\alpha(\mu)}\left[(1-\tau)\left|\psi'(\mu)\right|\right.\right. \\
&\quad \left.\left.+\tau\left|\psi'(\upsilon)\right|\right]d\tau + \int_{\frac{\alpha(\mu)}{\alpha(\mu)+1}}^1 \left(1-\tau^{\alpha(\mu)}\right)\left[(1-\tau)\left|\psi'(\mu)\right|+\tau\left|\psi'(\upsilon)\right|\right]d\tau\right] \\
&\leq (\upsilon-\mu)\left[\frac{\alpha(\mu)\left(\frac{\alpha(\mu)}{\alpha(\mu)+1}\right)^{\alpha(\mu)+1}}{(\alpha(\mu)+1)(\alpha(\mu)+2)}\left|\psi'(\mu)\right| + \frac{2\left(\frac{\alpha(\mu)}{\alpha(\mu)+1}\right)^{\alpha(\mu)+1}}{(\alpha(\mu)+1)(\alpha(\mu)+2)}\left|\psi'(\upsilon)\right|\right. \\
&\quad \left.+\frac{\alpha(\mu)\left(2\alpha^{\alpha(\mu)}+(\alpha(\mu)+1)^{\alpha(\mu)}\right)}{2(\alpha(\mu)+1)^{\alpha(\mu)+2}(\alpha(\mu)+2)}\left|\psi'(\mu)\right| + \frac{4\alpha(\mu)^{\alpha(\mu)+1}-\alpha(\mu)(\alpha(\mu)+1)^{\alpha(\mu)}}{2(\alpha(\mu)+1)^{\alpha(\mu)+2}(\alpha(\mu)+2)}\left|\psi'(\upsilon)\right|\right] \\
&\leq (\upsilon-\mu)\left[\mathbb{A}_7(\alpha(\mu))\left|\psi'(\mu)\right| + \mathbb{A}_8(\alpha\mu)\left|\psi'(\upsilon)\right| + \mathbb{A}_9(\alpha(\mu))\left|\psi'(\mu)\right| + \mathbb{A}_{10}(\alpha(\mu))\left|\psi'(\upsilon)\right|\right].
\end{aligned}$$

Thus, the proof of Theorem 5 is finished. □

Remark 13. If we choose $\alpha(\varkappa) = 1$, then we obtain Theorem 2.2 in [15].

Remark 14. If we choose $\alpha(\varkappa) = \beta$ (constant), then we get Theorem 4.4 in [16].

Theorem 6. *Let $\psi\colon \mathbb{I}^\circ \to \mathbb{R}$ be a differentiable mapping on $\mathbb{I}^\circ, \mu, \upsilon \in \mathbb{I}^\circ$ with $\mu < \upsilon$. If $|\psi'|^q$ is convex on $[\mu,\upsilon]$ for $q \geq 1$, then the following right Riemann-Liouville fractional integral inequality holds:*

$$\begin{aligned}
&\left|\psi\left(\frac{\alpha(\mu)\upsilon+\mu}{\alpha(\mu)+1}\right) - \frac{\Gamma(\alpha(\mu)+1)}{(\upsilon-\mu)^{\alpha(\mu)}}\mathbb{J}^{\alpha(\mu)}_{\upsilon-}\psi(\mu)\right| \leq (\upsilon-\mu)\left(\frac{(\alpha(\mu))^{\alpha(\mu)+1}}{(\alpha(\mu)+1)^{\alpha(\mu)+2}}\right)^{1-\frac{1}{q}} \times \\
&\left[\left(\mathbb{A}_7(\alpha(\mu))\left|\psi'(\mu)\right|^q + \mathbb{A}_8(\alpha(\mu))\left|\psi'(\upsilon)\right|^q\right)^{\frac{1}{q}} + \left(\mathbb{A}_9(\alpha(\mu))\left|\psi'(\mu)\right|^q + \mathbb{A}_{10}(\alpha(\mu))\left|\psi'(\upsilon)\right|^q\right)^{\frac{1}{q}}\right],
\end{aligned}$$

where $\mathbb{A}_7(\alpha(\mu)) - \mathbb{A}_{10}(\alpha(\mu))$ are the same as in Theorem 5 and $\alpha(\mu) > 0$.

Proof. Using Lemma 5, power mean inequality and the convexity of $|\psi'|^q$, we have

$$\left|\psi\left(\frac{\alpha(\mu)\upsilon+\mu}{\alpha(\mu)+1}\right) - \frac{\Gamma(\alpha(\mu)+1)}{(\upsilon-\mu)^{\alpha(\mu)}}\mathbb{J}^{\alpha(\mu)}_{\upsilon-}\psi\mu\right|$$

$$\leq (\upsilon-\mu)\left[\int_0^{\frac{\alpha(\mu)}{\alpha(\mu)+1}}\tau^{\alpha(\mu)}\left|\psi'((1-\tau)\mu+\tau\upsilon)\right|d\tau+\int_{\frac{\alpha(\mu)}{\alpha(\mu)+1}}^1\left(1-\tau^{\alpha(\mu)}\right)\left|\psi'((1-\tau)\mu+\tau\upsilon)\right|d\tau\right]$$

$$\leq (\upsilon-\mu)\left[\left(\int_0^{\frac{\alpha(\mu)}{\alpha(\mu)+1}}\tau^{\alpha(\mu)}d\tau\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{\alpha(\mu)}{\alpha(\mu)+1}}\tau^{\alpha(\mu)}\left|\psi'((1-\tau)\mu+\tau\upsilon)\right|^q\,d\tau\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_{\frac{\alpha(\mu)}{\alpha(\mu)+1}}^1\left(1-\tau^{\alpha(\mu)}\right)d\tau\right)^{1-\frac{1}{q}}\left(\int_{\frac{\alpha(\mu)}{\alpha(\mu)+1}}^1\left(1-\tau^{\alpha(\mu)}\right)\left|\psi'((1-\tau)\mu+\tau\upsilon)\right|^q\,d\tau\right)^{\frac{1}{q}}\right]$$

$$\leq (\upsilon-\mu)\left(\frac{\alpha(\mu)^{\alpha(\mu)+1}}{(\alpha(\mu)+1)^{\alpha(\mu)+2}}\right)^{1-\frac{1}{q}}\left[\left(\int_0^{\frac{\alpha(\mu)}{\alpha(\mu)+1}}\tau^{\alpha(\mu)}\left[(1-\tau)\left|\psi'(\mu)\right|^q+\tau\left|\psi'(\upsilon)\right|^q\right]d\tau\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_{\frac{\alpha(\mu)}{\alpha(\mu)+1}}^1\left(1-\tau^{\alpha(\mu)}\right)\left[(1-\tau)\left|\psi'(\mu)\right|^q+\tau\left|\psi'(\upsilon)\right|^q\right]d\tau\right)^{\frac{1}{q}}\right]$$

$$\leq (\upsilon-\mu)\left(\frac{(\alpha(\mu))^{\alpha(\mu)+1}}{(\alpha(\mu)+1)^{\alpha(\mu)+2}}\right)^{1-\frac{1}{q}}\times$$

$$\left[\left(\mathbb{A}_7(\alpha(\mu))\left|\psi'(\mu)\right|^q+\mathbb{A}_8(\alpha(\mu))\left|\psi'(\upsilon)\right|^q\right)^{\frac{1}{q}}+\left(\mathbb{A}_9(\alpha(\mu))\left|\psi'(\mu)\right|^q+\mathbb{A}_{10}(\alpha(\mu))\left|\psi'(\upsilon)\right|^q\right)^{\frac{1}{q}}\right].$$

Hence, the proof of Theorem 6 is completed. □

Remark 15. If we take $\alpha(\varkappa)=1$ in Theorem 6, we obtain the following midpoint-type inequality

$$\left|\psi\left(\frac{\mu+\upsilon}{2}\right)-\frac{1}{\upsilon-\mu}\int_\mu^\upsilon\psi(z)dz\right|$$

$$\leq\frac{\upsilon-\mu}{8}\left[\left(\frac{\left|\psi'(\mu)\right|^q+2\left|\psi'(\upsilon)\right|^q}{3}\right)^{\frac{1}{q}}+\left(\frac{2\left|\psi'(\mu)\right|^q+\left|\psi'(\upsilon)\right|^q}{3}\right)^{\frac{1}{q}}\right].$$

Remark 16. If we choose $\alpha(\varkappa)=\beta$ (constant), then we have Theorem 4.5 in [16].

Theorem 7. *Let $\psi\times\mathbb{I}^\circ\to\mathbb{R}$ be a differentiable mapping on $\mathbb{I}^\circ,\mu,\upsilon\in\mathbb{I}^\circ$, $\mu<\upsilon$, $\alpha(\mu)>0$ with $\frac{1}{p}+\frac{1}{q}=1$. If $|\psi'|^q$ is convex on $[\mu,\upsilon]$ for $q>1$, then the following right Riemann-Liouville fractional integral inequality holds:*

$$\left|\psi\left(\frac{\alpha(\mu)\upsilon+\mu}{\alpha(\mu)+1}\right)-\frac{\Gamma(\alpha(\mu)+1)}{(\upsilon-\mu)^{\alpha(\mu)}}\mathbb{J}_{\upsilon-}^{\alpha(\mu)}\psi(\mu)\right|$$

$$\leq(\upsilon-\mu)\left[\mathbb{A}_{11}^{\frac{1}{p}}(\alpha(\mu),p)\left(\frac{(\alpha(\mu))^2}{2(\alpha(\mu)+1)^2}\left|\psi'(\mu)\right|^q+\frac{\alpha(\mu)^2+2\alpha(\mu)}{2(\alpha(\mu)+1)^2}\left|\psi'(\upsilon)\right|^q\right)^{\frac{1}{q}}\right.$$

$$+\mathbb{A}_{12}^{\frac{1}{p}}(\alpha(\mu),p)\left(\frac{2\alpha(\mu)+1}{2(\alpha(\mu)+1)^2}\left|\psi'(\mu)\right|^q+\frac{1}{2(\alpha(\mu)+1)^2}\left|\psi'(\upsilon)\right|^q\right)^{\frac{1}{q}}\Bigg],$$

where

$$\mathbb{A}_{11}(\alpha(\mu),p)=\int_0^{\frac{\alpha(\mu)}{\alpha(\mu)+1}}\tau^{\alpha(\mu)p}\,\mathrm{d}\tau,\quad \mathbb{A}_{12}(\alpha(\mu),p)=\int_{\frac{\alpha(\mu)}{\alpha(\mu)+1}}^1\left(1-\tau^{\alpha(\mu)}\right)^p\,\mathrm{d}\tau.$$

Proof. Using Lemma 5, Hölder inequality and the convexity of $|\psi'|^q$, we obtain

$$\begin{aligned}
&\left|\psi\left(\frac{\alpha(\mu)\upsilon+\mu}{\alpha(\mu)+1}\right)-\frac{\Gamma(\alpha(\mu)+1)}{(\upsilon-\mu)^{\alpha(\mu)}}\mathbb{J}_{\upsilon-}^{\alpha(\mu)}\psi(\mu)\right|\\
&\le(\upsilon-\mu)\left[\int_0^{\frac{\alpha(\mu)}{\alpha(\mu)+1}}\tau^{\alpha(\mu)}\left|\psi'((1-\tau)\mu+\tau\upsilon)\right|\mathrm{d}\tau+\int_{\frac{\alpha(\mu)}{\alpha(\mu)+1}}^1\left(1-\tau^{\alpha(\mu)}\right)\left|\psi'((1-\tau)\mu+\tau\upsilon)\right|\mathrm{d}\tau\right]\\
&\le(\upsilon-\mu)\left[\left(\int_0^{\frac{\alpha(\mu)}{\alpha(\mu)+1}}\tau^{\alpha(\mu)p}\,\mathrm{d}\tau\right)^{\frac{1}{p}}\left(\int_0^{\frac{\alpha(\mu)}{\alpha(\mu)+1}}\left|\psi'((1-\tau)\mu+\tau\upsilon)\right|^q\,\mathrm{d}\tau\right)^{\frac{1}{q}}\right.\\
&\quad\left.+\left(\int_{\frac{\alpha(\mu)}{\alpha(\mu)+1}}^1\left(1-\tau^{\alpha(\mu)}\right)^p\,\mathrm{d}\tau\right)^{\frac{1}{p}}\left(\int_{\frac{\alpha(\mu)}{\alpha(\mu)+1}}^1\left|\psi'((1-\tau)\mu+\tau\upsilon)\right|^q\,\mathrm{d}\tau\right)^{\frac{1}{q}}\right]\\
&\le(\upsilon-\mu)\left[\mathbb{A}_{11}^{\frac{1}{p}}(\alpha(\mu),p)\left(\frac{(\alpha(\mu))^2}{2(\alpha(\mu)+1)^2}\left|\psi'(\mu)\right|^q+\frac{(\alpha(\mu))^2+2\alpha(\mu)}{2(\alpha(\mu)+1)^2}\left|\psi'(\upsilon)\right|^q\right)^{\frac{1}{q}}\right.\\
&\quad\left.+\mathbb{A}_{12}^{\frac{1}{p}}(\alpha(\mu),p)\left(\frac{2\alpha(\mu)+1}{2(\alpha(\mu)+1)^2}\left|\psi'(\mu)\right|^q+\frac{1}{2(\alpha(\mu)+1)^2}\left|\psi'(\upsilon)\right|^q\right)^{\frac{1}{q}}\right].
\end{aligned}$$

This completes the proof. □

Remark 17. In Theorem 7, when $\alpha(\varkappa)=1$ is chosen, we get Theorem 2.3 in [15].

Remark 18. If we choose $\alpha(\varkappa)=\beta$ (constant), then we have Theorem 4.6 in [16].

4. Numerical Examples

In this section, we represent some examples of variable order functions.

Example 1. We define a convex mapping $\psi(\varkappa)=\varkappa^2$. Then, from inequality (2.2) for $\alpha(\varkappa)=\sin\varkappa$, $\mu=1$ and $\upsilon=2$, we have

$$\psi\left(\frac{(\sin(1))2+1}{\sin(1)+1}\right)=2\frac{37087}{100000},\tag{4.1}$$

$$\frac{\Gamma(\sin(1)+1)}{(2-1)^{\sin(1)}}\mathbb{J}_{2-}^{\sin(1)}\psi(1)=2\frac{4201}{20000}\tag{4.2}$$

and

$$\frac{\sin(1)\psi(2)+\psi(1)}{\sin(1)+1}=2\frac{767}{6250}. \tag{4.3}$$

From (4.1), (4.2) and (4.3), we attain

$$2\frac{37807}{100000}<2\frac{4201}{20000}<2\frac{767}{6250}.$$

Which shows that the inequality (2.2) is valid for convex functions.

Example 2. We define a convex mapping $\psi(\varkappa)=\varkappa^2$. Then, from inequality (2.2) for $\alpha(\varkappa)=\cos\varkappa, \mu=0$ and $\upsilon=1$, we have

$$\psi\left(\frac{(\cos(0))1+0}{\cos(0)+1}\right)=\frac{1}{4}, \tag{4.4}$$

$$\frac{\Gamma(\cos(0)+1)}{(1-0)^{\cos(0)}}\mathbb{J}_{1^-}^{\cos(0)}\psi(0)=\frac{1}{3} \tag{4.5}$$

and

$$\frac{\cos(0)\psi(1)+\psi(0)}{\cos(0)+1}=\frac{1}{2}. \tag{4.6}$$

From (4.4), (4.5) and (4.6), we achieve

$$\frac{1}{4}<\frac{1}{3}<\frac{1}{2}.$$

Which shows that the inequality (2.2) is valid for convex functions.

Remark 19. Both Examples 1 and 2 demonstrate the validity of the inequalities stated in Theorem 1. This conclusion is further emphasized by a comparative study, as shown in Figures 1 and 2.

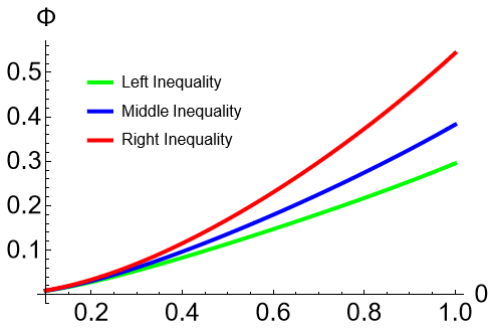

FIGURE 1. In 2D-Plot when μ lies between 1 and 2 and υ is fixed.

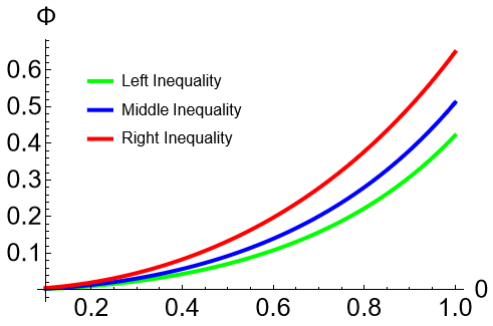

FIGURE 2. In 2D-Plot when μ lies between 0 and 1 and υ is fixed.

Remark 20. When we see 2-D plot as 'μ' and 'υ' lies between 1 and 2 we can clearly see from Figure 1, inequality stated in Theorem 1 is satisfied.

Remark 21. When we see 2-D plot as 'μ' and 'υ' lies between 0 and 1, we can clearly see from Figure 2, inequality stated in Theorem 1 is satisfied.

CONCLUSION

This research has contributed novel findings to the realm of H-H inequalities, particularly in their application to convex functions and fractional integrals with variable order. We have established new H-H type inequalities through an examination of convex functions and fractional integrals, as well as identified innovative fractional trapezoid and midpoint-type inequalities. Furthermore, we have provided numerical examples and graphical analysis of these newly established inequalities. Future research endeavors may focus on expanding the scope of these inequalities and exploring their implications in diverse mathematical disciplines. Additionally, researchers may apply the Riemann-Liouville fractional integral with variable order for different types of inequalities and explore these results in a real-world scenario.

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BKN RINGS: A STUDY OF NONZERO HOMOMORPHISMS BETWEEN MODULES AND SOME APPLICATIONS TO HOMOLOGICAL ALGEBRA

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Abstract. In this paper, we study BKN rings, which are rings that satisfy the property $\mathrm{Hom}_R(M,N) \neq 0$ for all nonzero left (right) R-modules M and N. Our aim is to provide a comprehensive overview of BKN rings and their connections to other types of rings, such as retractable and completely coretractable rings, commutative rings, and their connections to injective R-modules and non-trivial torsion theories. We prove several results, including the left-right symmetry of the BKN property and its preservation under various ring operations. Furthermore, we explore the relationship between BKN rings and the tensor product of modules. Our study provides insight into the comprehensive understanding of BKN rings, providing information about the relation between BKN rings and the Tor, Ext, and ext functors.

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Keywords: BKN ring, left local, left semiartinian, left MAX

1. INTRODUCTION

In this article, all rings are assumed to be associative with identity. The category of left (right) unitary modules over the ring R is denoted by R-Mod (Mod-R), and when the side of an R-module is not specified, it will be considered as an R-bimodule.

The property $\mathrm{Hom}_R(M,N) \neq 0$ for certain R-modules M and N has been the subject of various studies. For example, in [12], *retractable rings* were examined. These are rings where nonzero homomorphisms exist between modules and their nonzero submodules. Dual to this concept, *coretractable rings* were investigated in [3]. In [9], it was proved that a fully right-bounded right Noetherian ring R (*FBN ring*) has the property that $\mathrm{Hom}_R(M,N) \neq 0$ for any nonzero submodule N of a nonzero right R-module M if and only if $\mathrm{Hom}_R(M,R/P) \neq 0$ for every associated prime ideal P of M. Additionally, nonzero homomorphisms between nonzero injective (projective)

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modules have been recently studied in [6] and [13]. The general case of nonzero homomorphisms between arbitrary nonzero modules M and N has been characterized in [5] by Bican, Kepka, and Němec. Inspired by this characterization, a ring R is referred to as a *left* (or *right*) *BKN ring* if $\mathrm{Hom}_R(M,N) \neq 0$ for all nonzero left (or right) R-modules M and N. A BKN ring is a ring that is both a left and right BKN ring. This term was first introduced in 2017 by Rincón-Mejía et al. in [2]. They established, among other interesting properties, that a ring R is a finite direct product of left BKN rings if and only if the lattices of R-tors (hereditary torsion classes), R-nat (natural classes), and R-conat (conatural classes) coincide. Moreover, it was shown in [1] that if R is a nonsemisimple BKN ring, then the operator $\phi_Z\colon R\text{-nat} \to R\text{-nat}$, defined by $\phi_Z(\mathcal{C}) = \{M \in R\text{-Mod} | Z(M) \in \mathcal{C}\}$ (where $Z(M)$ denotes the singular submodule of M), is a homomorphism of boolean algebras.

This paper is organized as follows: Section 2 provides a comprehensive overview of left BKN rings and their connections to injective left R-modules, retractable and completely coretractable rings, and other types of rings. In this section, several equivalent properties of left BKN rings are established. For instance, in Theorem 2, we prove that a ring R is a left BKN ring if and only if it is a left MAX ring and every nonzero injective left R-module is a cogenerator. This, in turn, is equivalent to there being no nontrivial torsion theories for the category of left R-modules. Furthermore, in Theorem 3, we establish that the property of being a BKN ring is left-right symmetric. In addition, we present some results regarding the preservation of the BKN property under various ring operations. For example, in Theorem 4, we prove that if a unitary ring homomorphism from a BKN ring R to another ring S is such that the Change of Rings functor is full, then S is also a BKN ring. We also examine the relationship between BKN rings and the tensor product of modules, showing in Proposition 2 that if R is a BKN ring, then the tensor product of any nonzero right R-module L with any nonzero R-module M is nonzero.

Finally, in Section 3, our focus is to determine conditions under which $\mathrm{Tor}_n^R(M,N)$, $\mathrm{Ext}_R^n(N,M)$, and $\mathrm{ext}_R^n(M,N)$ are zero. Here, N represents a fixed nonzero R-module, while M is a nonzero right (or left) R-module, and R is a BKN ring. Also in Theorem 7, we establish that $_\otimes_R N$ reflects epimorphisms if and only if $M \otimes_R N \neq 0$ for each nonzero right R-module M. Dual results are obtained for the functors $\mathrm{Hom}_R(N,_)$ and $\mathrm{Hom}_R(_,N)$.

2. BKN RINGS

2.1. *Preliminary results*

Our main goal is to investigate the properties and implications of left and right BKN rings. As mentioned earlier, a characterization of this kind of ring appears in [5]. For the sake of clarity, we include a proof of this result, which can be also be found in [1].

Recall that a ring R is *left local* if there is a unique (up to isomorphism) nonzero simple left module in R-**Mod**; *left semiartinian* if each nonzero left R-module M contains an essential socle, or equivalently, contains a left simple module; *left MAX* if every left module has a simple quotient.

Theorem 1. *The ring R is left BKN if and only if R is left local, left semiartinian, and left MAX.*

Proof. Suppose that R is a left BKN ring. In this case, any nonzero homomorphism between two left simple modules must be an isomorphism. Therefore, R is left local. Furthermore, if M is a nonzero left R-module and S is a simple left R-module, then any nonzero homomorphisms $S \longrightarrow M$ and $M \longrightarrow S$ must be a monomorphism and an epimorphism, respectively. It follows that R is left semiartinian and left MAX.

Conversely, let M and N be nonzero left R-modules. If S is a left simple R-module, then there exists an epimorphism $M \xrightarrow{\alpha} S$ and a monomorphism $S \xrightarrow{\beta} N$. Hence $0 \neq \beta\alpha \in \mathrm{Hom}_R(M,N)$. □

The following result is a consequence of the definitions and other known results on ring and module theory.

Proposition 1. *Over a left BKN ring R, every nonzero injective left R-module is a cogenerator.*

2.2. *Retractable and completely coretractable rings*

Recall that a ring R is *left mod-retractable* if $\mathrm{Hom}_R(M,N) \neq 0$ for each nonzero left R-module M and each nonzero left R-submodule N of M.

Theorem 2. *The following properties of a ring R are equivalent:*

(i) *R is left mod-retractable, and each nonzero injective left R-module is a cogenerator,*
(ii) *R is left MAX, and each nonzero injective left R-module is a cogenerator,*
(iii) *R is a left BKN ring,*
(iv) *There are no nontrivial torsion theories for the category of left R-modules.*

Proof. (i)⇒(ii). It follows from [11, Theorem 3.3].

(ii)⇒(iii) Let M,N be nonzero left R-modules. Since R is left MAX there exists a simple left R-module S and a nonzero epimorphism $f\colon M \to S$. Since $E(N)$ is a cogenerator, $\mathrm{Hom}_R(S,E(N)) \neq 0$. This entails S embeds in N. The composition of f with this embedding yields a nonzero homomorphism from M to N, as required.

(iii)⇒(iv). Let $(\mathcal{T},\mathcal{F})$ be a torsion theory for R-**Mod** and suppose that $0 \neq M \in \mathcal{T}$ and $N \in \mathcal{F}$. Then $\mathrm{Hom}_R(M,N) = 0$, and since R is left BKN, this implies that $N = 0$. Therefore, $\mathcal{F} = 0$ and $\mathcal{T}$ is the class of all left R-modules.

(iv)⇒(i). Let E be a nonzero injective left R-module. Then E cogenerates the torsion theory $(0,\mathcal{F})$, where $\mathcal{F}$ is the class of all left R-modules. This shows that E

is certainly an injective cogenerator. Finally, note that the only two torsion theories for R-**Mod** are hereditary. It follows from [11, Theorem 3.5] that R is left mod-retractable. □

Following [3], a ring R is *left completely coretractable* (*left CC* for short) if, for each nonzero left module M and each proper left R-submodule N, there exists a nonzero homomorphism $M/N \to M$. Hence, we have the following theorem:

Theorem 3. *The following statements are equivalent for a ring R:*

(i) *R is a left BKN ring,*
(ii) *R is a left CC and a left local ring,*
(iii) *R is a right BKN ring,*
(iv) *R is a right CC and a right local ring.*

Proof. (i)⇒(ii) and (iii)⇒(iv) are clear.

(iv)⇒ (iii). According to [3, Theorem 3.10], R is a two-sided perfect ring. Consequently, it is right semiartinian as stated in [10, Proposition 5.1, pag. 189], and it is also a right MAX ring based on [4, Theorem 28.4]. Therefore, applying Theorem 1, we conclude that R is a right BKN ring. Similarly (ii)⇒(i).

(i)⇒(iii). At this stage, we can suppose that R is a two-sided perfect ring and consequently R is a right MAX and semiartinian ring. Also, R/J is a semisimple ring with only one isomorphism class of left simple R/J-modules, with J being the Jacobson radical. By the Wedderburn-Artin Theorem, we conclude that there is only one isomorphism class of right simple R/J-modules. As every right simple R-module is annihilated by J, we obtain the desired result. In a similar manner, we obtain (iii)⇒(i). □

By conditions (i) and (iii) in the above theorem, left BKN rings are also right BKN rings, and vice versa. Therefore, we can simply refer to such rings as BKN rings. Consequently, Theorem 2 will hold when the phrase "left BKN" is replaced by "right BKN" in condition (iii) of that theorem. Moreover, Theorem 2 will hold if the word "left" is randomly replaced by "right" in conditions (i) through (iv).

2.3. *Morita equivalent BKN rings*

Given two unitary rings R and S, a ring homomorphism $\rho\colon R \to S$ is called *unitary* if ρ maps the unit element of R to the unit element of S. To every unitary ring homomorphism $\rho\colon R \to S$, the well-known *Change of Rings functor* (see [7]) $\mathbf{F}_\rho\colon S$-**Mod** $\to R$-**Mod** is defined as follows: For a left S-module M, let the additive group of $\mathbf{F}_\rho(M)$ be equal to the underlying abelian group M, and the structure of a left R-module is defined by

$$rm = \rho(r)m, \qquad m \in M,\ r \in R.$$

Let $f\colon M \to N$ be a homomorphism of left S-modules. Then $\mathbf{F}_\rho(f)(rm) = f(\rho(r)m) = \rho(r)f(m) = rf(m)$. Thus, every S-homomorphism is an R-homomorphism. It follows as a consequence that $\mathrm{Hom}_S(M,N) \leq \mathrm{Hom}_R(M,N)$.

Finally, if ρ is surjective, then we have that $\mathrm{Hom}_S(M,N) = \mathrm{Hom}_R(M,N)$.

Theorem 4. *If $\rho\colon R \to S$ is a unitary ring homomorphism with $\boldsymbol{F}_\rho$ being a full functor and R being a BKN ring, then S is also a BKN ring.*

Proof. If M and N are nonzero left S-modules, then $\mathbf{F}_\rho(M)$ and $\mathbf{F}_\rho(N)$ are nonzero left R-modules. Since R is a left BKN ring, there exists a nonzero homomorphism $f\colon \mathbf{F}_\rho(M) \to \mathbf{F}_\rho(N)$. By hypothesis, there exists $g \in \mathrm{Hom}_S(M,N)$ such that $\mathbf{F}_\rho(g) = f$. Since $f \neq 0$, then g must be a nonzero homomorphism. □

Corollary 1. *If $\rho\colon R \to S$ is a surjective unitary ring homomorphism with R being a BKN ring, then S is also a BKN ring.*

Proof. Since for every left S-modules M and N, we have $\mathrm{Hom}_S(M,N) = \mathrm{Hom}_R(M,N)$, the Change of Rings functor $\mathbf{F}_\rho$ is full and, by Theorem 4, we are done. □

Recall that a left R-module M is a *progenerator* if it is finitely generated, projective, and a generator. With this in mind, we have the following theorem:

Theorem 5. *In the situation of Corollary 1, if $Q = \boldsymbol{F}_\rho(S)$ is a projective left R-module, then R and S are Morita equivalent BKN rings.*

Proof. Suppose that Q is a projective left R-module. Since R is a BKN ring, $\mathrm{Hom}_R(Q,C) \neq 0$ for each nonzero left R-module C. It follows from [10, Proposition 6.3. Pag. 93] that Q is a generator module. After noting that Q is finitely generated by 1_S, we obtain that Q is actually a progenerator module. Now, we know that $\mathrm{End}_R(Q)$ and $\mathrm{End}_S(S)$ are the same as abelian groups. Moreover, we claim that they are isomorphic rings. Indeed, the assignment $\alpha \mapsto \mathbf{F}_\rho(\alpha)$ is a ring isomorphism from $\mathrm{End}_S(S)$ onto $\mathrm{End}_R(Q)$. But $\mathrm{End}_S(S) \cong S$ by [7, Lemma 3.7.3]. The result now follows from [4, Corollary 22.4]. □

Corollary 2. *In the situation of Corollary 1 if $Q = \boldsymbol{F}_\rho(S)$ and there exists a R-homomorphism $\varphi\colon Q \to R$ with $\varphi(1_S) = 1_R$, then R and S are Morita equivalent BKN rings.*

Proof. Each element of Q has the form $x = r \cdot 1_S$ for some $r \in R$. Hence,

$$\varphi(x) \cdot 1_S = \varphi(r \cdot 1_S) \cdot 1_S = (r\varphi(1_S)) \cdot 1_S = (r \cdot 1_R) \cdot 1_S = r \cdot 1_S = x.$$

According to [10, Proposition 6.3, p.p. 20], Q is a projective left R-module, and the proof is concluded by virtue of the preceding theorem. □

2.4. *BKN rings and Tensor product*

Given two rings R and S and modules L_R, ${}_RM_S$, N_S, from the well-known natural isomorphism

$$\mathrm{Hom}_S(L\otimes_R M,N)\cong \mathrm{Hom}_R(L,\mathrm{Hom}_S(M,N)), \tag{2.1}$$

the following proposition is obtained:

Proposition 2. *If R is a BKN ring, then for any nonzero modules L_R and ${}_RM_R$, $L\otimes_R M\neq 0$.*

Proof. This can be proven by taking $N=M$ and $R=S$ in formula (2.1). □

Definition 1. A nonzero right S-module N is *a right anti-terminal* S-module whenever $\mathrm{Hom}_S(M,N)\neq 0$ for all nonzero right S-modules M.

Note that every right injective cogenerator S-module is an anti-terminal right S-module. Also, if $L\otimes_R M$ is nonzero for all nonzero modules L_R, and ${}_RM_S$, then the following proposition shows that $\mathrm{Hom}_S(M,N)$ must be an anti-terminal right S-module.

Proposition 3. *If R and S are rings such that $L\otimes_R M\neq 0$ for all nonzero modules L_R and ${}_RM_S$, and if N is an anti-terminal right S-module, then $\mathrm{Hom}_S(M,N)$ is an anti-terminal right R-module for every nonzero $(R$-$S)$-bimodule M.*

Proof. Let L be a nonzero right R-module. Since $L\otimes_R M\neq 0$ and N is anti-terminal, we obtain $\mathrm{Hom}_S(L\otimes_R M,N)\neq 0$. Hence, the right-hand side of (2.1) is nonzero, and this establishes the result. □

2.5. *Commutative BKN rings*

Since the notions semiartinian and mod-retractable coincide for commutative rings ([11, Theorem 3.10]), together with Theorem 2, we obtain the following:

Proposition 4. *The following conditions are equivalent for a commutative ring R:*

(i) *R is mod-retractable and each nonzero injective R-module is a cogenerator,*
(ii) *R is semiartinian and each nonzero injective R-module is a cogenerator,*
(iii) *R is MAX and each nonzero injective R-module is a cogenerator,*
(iv) *R is a BKN ring,*
(v) *There are no nontrivial torsion theories.*

Theorem 6. *The following sentences are equivalent for a commutative ring R:*

(i) *If L and M are nonzero R-modules, then $L\otimes_R M\neq 0$,*
(ii) *R is a local ring with a unique maximal ideal J, such that for any R-module M, $M=JM$ implies $M=0$,*
(iii) *R is a BKN ring.*

Proof. (i)⇒(ii). Suppose that $L \otimes_R M \neq 0$ whenever L and M are nonzero R-modules. Since $R/I_1 \otimes_R R/I_2 \cong R/(I_1 + I_2)$, the sum of two proper ideals is a proper ideal, and therefore R is a local ring with a unique maximal ideal J. Let M be an R-module such that $M = JM$. Since $JM \otimes_R R/J = 0$, we must have that $M = 0$.

(ii)⇒(iii). Notice that as R is a local ring, by virtue of Theorem 1, it suffices to show that R is semiartinian and MAX. By [4, Lemma 28.3], J is a T-nilpotent ideal. Since R/J is semisimple (by [4, Theorem 28.4]), we can conclude that R is a perfect ring. Furthermore, by part (c) of [4, Theorem 28.4], R is a MAX ring. Finally, [10, Proposition 5.1, p.p. 189] implies that R is semiartinian.

(iii)⇒(i). Let R be a BKN ring, and let L and M be nonzero R-modules. Using Proposition 2 we see that $L \otimes_R M \neq 0$. □

3. BKN RINGS AND THE FUNCTORS TOR, EXT, AND EXT

In this section, we establish a relationship between BKN rings and the functors Tor, Ext and ext. Our goal is to determine when $\mathrm{Tor}_n^R(M,N)$, $\mathrm{Ext}_R^n(N,M)$, and $\mathrm{ext}_R^n(M,N)$ are nonzero for an arbitrary (but fixed) nonzero R-module N, provided M is a nonzero right (left) R-module and R is a BKN ring.

First, let us recall that an additive covariant (respectively, contravariant) functor T between abelian categories *reflects epimorphisms* if the exactness of $TM \longrightarrow TK \longrightarrow 0$ (respectively, $TK \longrightarrow TM \longrightarrow 0$) implies the exactness of $M \longrightarrow K \longrightarrow 0$ (respectively, $0 \longrightarrow M \longrightarrow K$). A similar definition is valid if we replace "epimorphisms" with "monomorphisms".

Definition 2. An additive covariant (contravariant) functor T between abelian categories is *anti projective* if for every exact nonsplit short sequence $0 \longrightarrow K \longrightarrow L \longrightarrow M \longrightarrow 0$, the exactness of $0 \longrightarrow TK \longrightarrow TL \longrightarrow TM \longrightarrow 0$ (respectively, $0 \longrightarrow TM \longrightarrow TL \longrightarrow TK \longrightarrow 0$) implies that $TK = 0$. The functor is *anti injective* if for every nonsplit exact sequence $0 \longrightarrow M \longrightarrow L \longrightarrow K \longrightarrow 0$, the exactness of $0 \longrightarrow TM \longrightarrow TL \longrightarrow TK \longrightarrow 0$ (respectively, $0 \longrightarrow TK \longrightarrow TL \longrightarrow TM \longrightarrow 0$) implies that $TK = 0$.

Notice that if $\mathbf{C}$ is an abelian category where every short exact sequence splits, then any covariant (contravariant) additive functor having $\mathbf{C}$ as the source is anti projective (anti injective). Examples of these sorts of categories are *spectral categories* (see [10, Pag. 128]). A Grothendieck category whose objects are all projectives (and consequently injectives) is a spectral category (*e.g.*, the category R-**Mod**, for R a semisimple ring).

3.1. *The functor Tor*

Here, we study the relationship between BKN rings and the functor Tor. We begin with the following theorem:

Theorem 7. *Let R be a ring and N be a nonzero left R-module. The functor $_\otimes_R N$ reflects epimorphisms if and only if $M\otimes_R N\neq 0$ for each nonzero right R-module M.*

Proof. Let M be a right R-module, and consider any short exact sequence

$$0\longrightarrow X\xrightarrow{\alpha} Y\xrightarrow{\beta} M\longrightarrow 0$$

in Mod-R of the form shown (such short exact sequences clearly exist). Applying the right exact functor $_\otimes_R N$, yields the right exact sequence

$$X\otimes_R N\xrightarrow{\alpha\otimes 1_N} Y\otimes_R N\xrightarrow{\beta\otimes 1_N} M\otimes_R N\longrightarrow 0.$$

Suppose $M\otimes_R N=0$. Then $\alpha\otimes 1_N$ is an epimorphism. Since $_\otimes_R N$ reflects epimorphisms, α is an epimorphism, whence $\beta=0$ and $M=0$.

For the converse, let $\alpha\colon A\to B$ be an arbitrary homomorphism of right R-modules and suppose that $\alpha\otimes 1_N\colon A\otimes_R N\to B\otimes_R N$ is an epimorphism. Applying $_\otimes_R N$ to the exact sequence

$$A\xrightarrow{\alpha} B\xrightarrow{\pi}\operatorname{Coker}\alpha\longrightarrow 0$$

we obtain:

$$A\otimes_R N\xrightarrow{\alpha\otimes 1_N} B\otimes_R N\xrightarrow{\pi\otimes 1_N}\operatorname{Coker}\alpha\otimes_R N\longrightarrow 0.$$

Since $\alpha\otimes 1_N$ is an epimorphism, $\pi\otimes 1_N=0$, whence $\operatorname{Coker}\alpha\otimes_R N=0$. By hypothesis, this entails $\operatorname{Coker}\alpha=0$ which is to say α is an epimorphism. We conclude that $_\otimes_R N$ reflects epimorphisms. □

Note that there are plenty of modules satisfying the hypothesis of the above theorem. For example, any free nonzero R-module N satisfies the property that the functor $_\otimes_R N$ reflects epimorphisms. Additionally, as we will see in the following theorem, every nonzero module over a BKN ring also satisfies this property.

For the next result, we need to introduce the notion of an "*anti flat module*".

Definition 3. An R-module N is *anti flat* if the functor $_\otimes_R N$ is anti projective.

Theorem 8. *Let R be a BKN ring and N a nonzero anti flat R-module. Then $\operatorname{Tor}_1^R(M,N)=0$ if and only if M is a projective right R-module. Moreover, for a projective resolution of M:*

$$\boldsymbol{P}=\longrightarrow P_2\xrightarrow{d_2} P_1\xrightarrow{d_1} P_0\xrightarrow{\varepsilon} M\longrightarrow 0,$$

define $K_0=\operatorname{Ker}\varepsilon$, and define $K_n=\operatorname{Ker}d_n$ for all $n\geq 1$. Then $\operatorname{Tor}_{n+1}^R(M,N)=0$ if and only if K_{n-1} is projective.

Proof. Let M be a nonzero right R-module, and consider any short exact sequence in Mod-R of the form

$$0\longrightarrow X\xrightarrow{\alpha} Y\xrightarrow{\beta} M\longrightarrow 0. \tag{3.1}$$

Applying the functor $_\otimes_R N$, yields the (right exact) sequence:

$$0 \dashrightarrow X\otimes_R N \xrightarrow{\alpha\otimes 1_N} Y\otimes_R N \xrightarrow{\beta\otimes 1_N} M\otimes_R N \longrightarrow 0. \tag{3.2}$$

Now suppose that M is projective. This implies that Sequence 3.1 splits, so that Sequence 3.2 is exact (all additive functors are exact on split sequences). It follows that $\mathrm{Ker}(\alpha\otimes 1_N) = \mathrm{Tor}_1^R(M,N) = 0$, as required.

Conversely, let us suppose that $\mathrm{Tor}_1^R(M,N) = \mathrm{Ker}(\alpha\otimes 1_N) = 0$. It follows that Sequence 3.2 is exact. If Sequence 3.1 is non-splitting, then it follows from the anti-flat property for N, that $X\otimes_R N = 0$. This, in turn, implies $X = 0$ by the BKN property (Proposition 2). This contradicts the fact that Sequence 3.1 is non-splitting. Since every short exact sequence of the form shown in 3.1 splits, we may conclude that M is projective, as required.

For the second part, the first part gives $\mathrm{Tor}_1^R(K_{n-1},N) = 0$ if and only if K_{n-1} is projective. On the other hand, [8, Corollary 6.23], establishes that $\mathrm{Tor}_{n+1}^R(M,N) \cong \mathrm{Tor}_1^R(K_{n-1},N)$ for all $n \geq 1$. The result now follows easily. □

3.2. *BKN rings and functors Ext and ext*

The proofs of the following theorems concerning the functor Ext are essentially duals of the proofs we have given for the functor Tor, and so they will be omitted (see [8, Sections 6.2.3] for required prior results). In this regard, we note that BKN rings possess the necessary symmetry insofar as $\mathrm{Hom}_R(M,N)$ and $M\otimes_R N$ are nonzero for all nonzero modules ${}_RM_R$ and ${}_RN$.

Theorem 9. *Let R be a ring and N a left nonzero R-module. Then the functor $\mathrm{Hom}_R(N,_)$ reflects monomorphisms if and only if $\mathrm{Hom}_R(N,M) \neq 0$ for each nonzero left R-module M.*

Definition 4. A left R-module N is *domain anti injective* whenever the functor $\mathrm{Hom}_R(N,_)$ is anti injective.

Theorem 10. *Let R be a BKN ring and N a nonzero domain anti injective left R-module. Then $\mathrm{Ext}_R^1(N,M) = 0$ if and only if M is an injective left R-module. Moreover, for an injective resolution of M:*

$$\boldsymbol{E} = 0 \longrightarrow M \xrightarrow{\eta} E_0 \xrightarrow{d_0} E_1 \xrightarrow{d_1} E_2 \xrightarrow{d_2} E_3 \longrightarrow,$$

define $V_0 = \mathrm{Im}\,\eta$ and for all $n \geq 1$ define $V_n = \mathrm{Im}\,d_{n-1}$. Then $\mathrm{Ext}_R^{n+1}(N,M) = 0$ if and only if V_{n-1} is injective.

Likewise, the proof of Theorems 11 to 12 concerning the functor ext are similar to those of Section 3.1, so they will also be omitted (see [8, Section 6.2.4] for the necessary details).

Theorem 11. *Let R be a ring and N a nonzero left R-module. Then the contravariant functor $\mathrm{Hom}_R(_,N)$ reflects monomorphisms if and only if $\mathrm{Hom}_R(M,N) \neq 0$ for each nonzero left R-module M.*

Definition 5. The left R-module N is *codomain anti projective* if the functor $\mathrm{Hom}_R(_,N)$ is anti projective.

Theorem 12. *Let R be a BKN ring and N a codomain anti projective left R-module. Then, $\mathrm{ext}^1_R(M,N) \neq 0$ if and only if M is a nonprojective left R-module. Moreover, for a projective resolution of M:*

$$\boldsymbol{P} = \longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} M \longrightarrow 0,$$

define $K_0 = \mathrm{Ker}\,\varepsilon$ and $K_n = \mathrm{Ker}\,d_n$ for all $n \geq 1$. Then, $\mathrm{ext}^{n+1}_R(M,N) \neq 0$ if and only if K_{n-1} is nonprojective.

Question 1. Does there exist a nonsemisimple (BKN) ring R and a nonzero anti flat (domain anti injective, codomain anti projective) R-module N?

Question 2. What are the consequences and implications for the ring R or in the category R-**Mod** if there were no such modules?

Question 3. Does $L \otimes_R M \neq 0$ for all nonzero L and M imply R is BKN for the noncommutative case?

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SOME NEWTON'S INEQUALITIES ON CO-ORDINATES FOR RIEMANN–LIOUVILLE FRACTIONAL INTEGRALS

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Abstract. In this paper, we give an identity involving Riemann–Liouville fractional integrals and partially differentiable function of two variables. Then, we use the newly established identity and prove some new Newton's type inequalities for differentiable co-ordinated convex functions. We also recapture some existing inequalities and obtain some new inequalities in the special cases of newly established results. Furthermore, we present an example with graphs to illustrate the obtained main result.

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1. INTRODUCTION

The Hermite–Hadamard and Simpson's type inequalities have been studied widely in recent years. Simpson's inequalities are inequalities that are created from Simpson's following rules:

i. Simpson's quadrature formula (Simpson's 1/3 rule):

$$\int_a^b f(x)\,\mathrm{d}x \approx \frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right].$$

ii. Simpson's second formula or Newton-Cotes quadrature formula (Simpson's 3/8 rule):

$$\int_a^b f(x)\,\mathrm{d}x \approx \frac{b-a}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right].$$

Simpson's second rule has the rule of three-point Newton-Cotes quadrature, hence evaluations for three steps quadratic kernel are sometimes called Newton-type results. These results are known as Newton-type inequalities in the literature. Many mathematicians have been investigated to Newton-type inequalities. For instance, Gao and Shi investigated Newton's type inequalities to the case of functions whose second

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derivatives are convex in the paper [4]. Erden et al. established some error estimates of Newton-type quadrature formula by bounded variation and Lipschitzian mappings in [3]. Newton's type inequalities for quantum differentiable convex mappings were given in the paper [1]. Luangboon et al. established Newton-type inequalities by post-quantum integrals in the paper [11]. In addition, Noor et al. gave Newton's type inequalities based on harmonic convex and p-harmonic convex functions in [13] and [14], respectively. The reader is referred to [7,8,12,21] and the references therein for more information and unexplained subjects about Newton's type inequalities including convex differentiable functions.

Fractional calculus has grown in popularity in recent years. Because of the significance of fractional calculus, mathematicians have investigated different fractional integral inequalities. Riemann-Lioville fractional integrals are the most commonly used integrals among fractional integrals. Sarikaya et al. presented some inequalities of Hermite–Hadamard type and trapezoid type by using the first time generalised the Riemann–Liouville fractional integrals in [18]. Ostrowski-type inequalities using Riemann–Liouville fractional for differentiable functions are investigated in the paper [20]. İşcan and Wu proved Hermite–Hadamard type inequalities to the case of harmonic convexity in the paper [9]. Sarikaya and Yıldırım presented new Hermite–Hadamard type inequalities and midpoint type inequalities via Riemann–Liouville fractional integrals in [19]. Chen and Huang investigated to $1/3$ rule type inequality for s-convex functions involving Riemann–Liouville fractional integrals. Moreover, several fractional Simpson type inequalities were investigated to functions whose second derivatives in absolute value are convex in the paper [6]. The reader is referred to [2,15,16,22] and the references therein for further information about Simpson type inequalities and varied properties of Riemann–Liouville fractional integrals and various fractional integral operators.

2. Preliminaries

In this section, we present the basic definitions and facts about the Riemann–Liouville integrals for one and two variables.

Definition 1 ([5, 10])**.** Let us consider that $f \in L_1[a,b]$ is a function and $\alpha > 0$ and $a \geq 0$. Then, Riemann–Liouville fractional integrals $J^{\alpha}_{a+} f(x)$ and $J^{\alpha}_{b-} f(x)$ are defined as follows:

$$J^{\alpha}_{a+} f(x) = \frac{1}{\Gamma(a)} \int_a^x (x-t)^{\alpha-1} f(t)\,\mathrm{d}t, \quad x > a$$

and

$$J^{\alpha}_{b-} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)\,\mathrm{d}t \quad x < b.$$

Here, Γ is the well-known Gamma function.

Definition 2 ([17]). Let $f \in L_1(\Delta = [a,b] \times [c,d])$ be a function and $\alpha, \beta > 0$ and $a, c \geq 0$. Then, Riemann–Liouville fractional integrals $J^{\alpha,\beta}_{a+,c+}, J^{\alpha,\beta}_{a+,d-}, +J^{\alpha,\beta}_{b-,c+}$ and $J^{\alpha,\beta}_{b-,d-}$ on co-ordinates are described as follows:

$$J^{\alpha,\beta}_{a+,c+} f(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x \int_c^y (x-t)^{\alpha-1} (y-s)^{\beta-1} f(t,s)\,\mathrm{d}s\mathrm{d}t, \quad x > a,\ y > c,$$

$$J^{\alpha,\beta}_{a+,d-} f(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x \int_y^d (x-t)^{\alpha-1} (s-y)^{\beta-1} f(t,s)\,\mathrm{d}s\mathrm{d}t, \quad x > a,\ y > d,$$

$$J^{\alpha,\beta}_{b-,c+} f(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_x^b \int_c^y (t-x)^{\alpha-1} (y-s)^{\beta-1} f(t,s)\,\mathrm{d}s\mathrm{d}t, \quad x < b,\ y > c,$$

$$J^{\alpha,\beta}_{b-,d-} f(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_x^b \int_y^d (t-x)^{\alpha-1} (s-y)^{\beta-1} f(t,s)\,\mathrm{d}s\mathrm{d}t, \quad x < b,\ y < d,$$

where Γ is the well-known Gamma function.

3. An identity

In this section, we prove an identity which help us to obtain the main results of this paper.

Lemma 1. *If $f\colon \Delta \to \mathbb{R}$ is a partially differentiable function on Δ°, then the following Riemann–Liouville fractional integrals identity hold:*

$$\Theta^{\alpha,\beta}(a,b;c,d) = \frac{(b-a)(d-c)}{4} \sum_{i=1}^{9} I_i, \tag{3.1}$$

where

$$\begin{aligned}
\Theta^{\alpha,\beta}(a,b;c,d) &:= \frac{[f(a,c)+f(a,d)+f(b,c)+f(b,d)]}{64} \\
&+ \frac{3}{64}\left[f\left(a,\frac{2c+d}{3}\right)+f\left(a,\frac{c+2d}{3}\right)+f\left(b,\frac{2c+d}{3}\right)+f\left(b,\frac{c+2d}{3}\right)\right] \\
&+ \frac{3}{64}\left[f\left(\frac{2a+b}{3},c\right)+f\left(\frac{2a+b}{3},d\right)+f\left(\frac{a+2b}{3},c\right)+f\left(\frac{a+2b}{3},d\right)\right] \\
&+ \frac{9}{64}\left[f\left(\frac{2a+b}{3},\frac{2c+d}{3}\right)+f\left(\frac{2a+b}{3},\frac{c+2d}{3}\right)+f\left(\frac{a+2b}{3},\frac{2c+d}{3}\right)\right. \\
&\left.+f\left(\frac{a+2b}{3},\frac{c+2d}{3}\right)\right] - \frac{3\Gamma(\alpha+1)}{16(b-a)^{\alpha}}\left[J^{\alpha}_{b-} f\left(a,\frac{2c+d}{3}\right)+J^{\alpha}_{b-} f\left(a,\frac{c+2d}{3}\right)\right.
\end{aligned} \tag{3.2}$$

$$+J^{\alpha}_{a+}f\left(b,\frac{2c+d}{3}\right)+J^{\alpha}_{a+}f\left(b,\frac{c+2d}{3}\right)\Big]-\frac{\Gamma(\alpha+1)}{16(b-a)^{\alpha}}\left[J^{\alpha}_{b-}f(a,c)+J^{\alpha}_{b-}f(a,d)\right.$$
$$\left.+J^{\alpha}_{a+}f(b,c)+J^{\alpha}_{a+}f(b,d)\right]-\frac{3\Gamma(\beta+1)}{16(d-c)^{\beta}}\left[J^{\beta}_{d-}f\left(\frac{2a+b}{3},c\right)+J^{\beta}_{d-}f\left(\frac{a+2b}{3},c\right)\right.$$
$$\left.+J^{\beta}_{c+}f\left(\frac{2a+b}{3},d\right)+J^{\beta}_{c+}f\left(\frac{a+2b}{3},d\right)\right]-\frac{\Gamma(\beta+1)}{16(d-c)^{\beta}}\left[J^{\beta}_{d-}f(a,c)+J^{\beta}_{c+}f(a,d)\right.$$
$$\left.+J^{\beta}_{d-}f(b,c)+J^{\beta}_{c+}f(b,d)\right]+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^{\alpha}(d-c)^{\beta}}\left[J^{\alpha,\beta}_{b-,d-}f(a,c)+J^{\alpha,\beta}_{b-,c+}f(a,d)\right.$$
$$\left.+J^{\alpha,\beta}_{a+,d-}f(b,c)+J^{\alpha,\beta}_{a+,c+}f(b,d)\right],$$

$$\begin{cases} I_1=\int_0^{\frac{1}{3}}\int_0^{\frac{1}{3}}\left(t^{\alpha}-\frac{1}{8}\right)\left(s^{\beta}-\frac{1}{8}\right)\Omega(t,s)\,\mathrm{d}s\mathrm{d}t, & I_6=\int_{\frac{1}{3}}^{\frac{2}{3}}\int_{\frac{2}{3}}^{1}\left(t^{\alpha}-\frac{1}{2}\right)\left(s^{\beta}-\frac{7}{8}\right)\Omega(t,s)\,\mathrm{d}s\mathrm{d}t,\\ I_2=\int_0^{\frac{1}{3}}\int_{\frac{1}{3}}^{\frac{2}{3}}\left(t^{\alpha}-\frac{1}{8}\right)\left(s^{\beta}-\frac{1}{2}\right)\Omega(t,s)\,\mathrm{d}s\mathrm{d}t, & I_7=\int_{\frac{2}{3}}^{1}\int_0^{\frac{1}{3}}\left(t^{\alpha}-\frac{7}{8}\right)\left(s^{\beta}-\frac{1}{8}\right)\Omega(t,s)\,\mathrm{d}s\mathrm{d}t,\\ I_3=\int_0^{\frac{1}{3}}\int_{\frac{2}{3}}^{1}\left(t^{\alpha}-\frac{1}{8}\right)\left(s^{\beta}-\frac{7}{8}\right)\Omega(t,s)\,\mathrm{d}s\mathrm{d}t, & I_8=\int_{\frac{2}{3}}^{1}\int_{\frac{1}{3}}^{\frac{2}{3}}\left(t^{\alpha}-\frac{7}{8}\right)\left(s^{\beta}-\frac{1}{2}\right)\Omega(t,s)\,\mathrm{d}s\mathrm{d}t,\\ I_4=\int_{\frac{1}{3}}^{\frac{2}{3}}\int_0^{\frac{1}{3}}\left(t^{\alpha}-\frac{1}{2}\right)\left(s^{\beta}-\frac{1}{8}\right)\Omega(t,s)\,\mathrm{d}s\mathrm{d}t, & I_9=\int_{\frac{2}{3}}^{1}\int_{\frac{2}{3}}^{1}\left(t^{\alpha}-\frac{7}{8}\right)\left(s^{\beta}-\frac{7}{8}\right)\Omega(t,s)\,\mathrm{d}s\mathrm{d}t,\\ I_5=\int_{\frac{1}{3}}^{\frac{2}{3}}\int_{\frac{1}{3}}^{\frac{2}{3}}\left(t^{\alpha}-\frac{1}{2}\right)\left(s^{\beta}-\frac{1}{2}\right)\Omega(t,s)\,\mathrm{d}s\mathrm{d}t, \end{cases}$$

and

$$\Omega(t,s)=\frac{\partial^2}{\partial s\partial t}f(tb+(1-t)a,sd+(1-s)c)-\frac{\partial^2}{\partial s\partial t}f(tb+(1-t)a,sc+(1-s)d)$$
$$-\frac{\partial^2}{\partial s\partial t}f(ta+(1-t)b,sd+(1-s)c)+\frac{\partial^2}{\partial s\partial t}f(ta+(1-t)b,sc+(1-s)d).$$

Proof. From the fact of the fundamental rules of integration by parts, we obtain

$$I_1=\frac{1}{(b-a)(d-c)}\left\{\left(\frac{1}{3^{\alpha}}-\frac{1}{8}\right)\left(\frac{1}{3^{\beta}}-\frac{1}{8}\right)\left[f\left(\frac{2a+b}{3},\frac{2c+d}{3}\right)\right.\right.$$
$$\left.+f\left(\frac{2a+b}{3},\frac{c+2d}{3}\right)+f\left(\frac{a+2b}{3},\frac{2c+d}{3}\right)+f\left(\frac{a+2b}{3},\frac{c+2d}{3}\right)\right]$$
$$+\frac{1}{8}\left(\frac{1}{3^{\beta}}-\frac{1}{8}\right)\left[f\left(a,\frac{2c+d}{3}\right)+f\left(a,\frac{c+2d}{3}\right)+f\left(b,\frac{2c+d}{3}\right)+f\left(b,\frac{c+2d}{3}\right)\right]$$
$$+\frac{1}{8}\left(\frac{1}{3^{\alpha}}-\frac{1}{8}\right)\left[f\left(\frac{2a+b}{3},c\right)+f\left(\frac{2a+b}{3},d\right)+f\left(\frac{a+2b}{3},c\right)+f\left(\frac{a+2b}{3},d\right)\right]$$

$$+\frac{1}{64}\left[f(a,c)+f(a,d)+f(b,c)+f(b,d)\right]$$
$$-\alpha\left(\frac{1}{3^{\beta}}-\frac{1}{8}\right)\int_0^{\frac{1}{3}}t^{\alpha-1}\left[f\left(tb+(1-t)a,\frac{2c+d}{3}\right)+f\left(tb+(1-t)a,\frac{c+2d}{3}\right)\right.$$
$$\left.+f\left(ta+(1-t)b,\frac{2c+d}{3}\right)+f\left(ta+(1-t)b,\frac{c+2d}{3}\right)\right]\mathrm{d}t$$
$$-\frac{\alpha}{8}\int_0^{\frac{1}{3}}t^{\alpha-1}\left[f(tb+(1-t)a,c)+f(tb+(1-t)a,d)+f(ta+(1-t)b,c)\right.$$
$$\left.+f(ta+(1-t)b,d)\right]\mathrm{d}t-\beta\left(\frac{1}{3^{\alpha}}-\frac{1}{8}\right)\int_0^{\frac{1}{3}}s^{\beta-1}\left[f\left(\frac{2a+b}{3},sd+(1-s)c\right)\right.$$
$$\left.+f\left(\frac{2a+b}{3},sc+(1-s)d\right)+f\left(\frac{a+2b}{3},sd+(1-s)c\right)+f\left(\frac{a+2b}{3},sc+(1-s)d\right)\right]\mathrm{d}s$$
$$-\frac{\beta}{8}\int_0^{\frac{1}{3}}s^{\beta-1}\left[f(a,sd+(1-s)c)+f(a,sc+(1-s)d)+f(b,sd+(1-s)c)\right.$$
$$\left.+f(b,sc+(1-s)d)\right]\mathrm{d}s$$
$$+\alpha\beta\int_0^{\frac{1}{3}}\int_0^{\frac{1}{3}}t^{\alpha-1}s^{\beta-1}\left[f(tb+(1-t)a,sd+(1-s)c)+f(tb+(1-t)a,sc+(1-s)d)\right.$$
$$\left.+f(ta+(1-t)b,sd+(1-s)c)+f(ta+(1-t)b,sc+(1-s)d)\right]\mathrm{d}s\mathrm{d}t\Bigg\}. \tag{3.3}$$

Similarly, we get

$$I_2=\frac{1}{(b-a)(d-c)}\left\{\left(\frac{2^{\beta}-1}{3^{\beta}}\right)\left(\frac{1}{3^{\alpha}}-\frac{1}{8}\right)\left[f\left(\frac{2a+b}{3},\frac{2c+d}{3}\right)\right.\right.$$
$$\left.+f\left(\frac{2a+b}{3},\frac{c+2d}{3}\right)+f\left(\frac{a+2b}{3},\frac{2c+d}{3}\right)+f\left(\frac{a+2b}{3},\frac{c+2d}{3}\right)\right]$$
$$+\frac{1}{8}\left(\frac{2^{\beta}-1}{3^{\beta}}\right)\left[f\left(a,\frac{2c+d}{3}\right)+f\left(a,\frac{c+2d}{3}\right)+f\left(b,\frac{2c+d}{3}\right)+f\left(b,\frac{c+2d}{3}\right)\right]$$
$$+\alpha\left(\frac{1-2^{\beta}}{3^{\beta}}\right)\int_0^{\frac{1}{3}}t^{\alpha-1}\left[f\left(tb+(1-t)a,\frac{2c+d}{3}\right)+f\left(tb+(1-t)a,\frac{c+2d}{3}\right)\right.$$
$$\left.+f\left(ta+(1-t)b,\frac{2c+d}{3}\right)+f\left(ta+(1-t)b,\frac{c+2d}{3}\right)\right]\mathrm{d}t$$

$$
\begin{aligned}
&-\beta\left(\frac{1}{3^{\alpha}}-\frac{1}{8}\right)\int_{\frac{1}{3}}^{\frac{2}{3}} s^{\beta-1}\left[f\left(\frac{2a+b}{3},sd+(1-s)c\right)+f\left(\frac{2a+b}{3},sc+(1-s)d\right)\right.\\
&\left.+f\left(\frac{a+2b}{3},sd+(1-s)c\right)+f\left(\frac{a+2b}{3},sc+(1-s)d\right)\right]\mathrm{d}s\\
&-\frac{\beta}{8}\int_{\frac{1}{3}}^{\frac{2}{3}} s^{\beta-1}\left[f(a,sd+(1-s)c)+f(a,sc+(1-s)d)+f(b,sd+(1-s)c)\right.\\
&\left.+f(b,sc+(1-s)d)\right]\mathrm{d}s\\
&+\alpha\beta\int_{0}^{\frac{1}{3}}\int_{\frac{1}{3}}^{\frac{2}{3}} t^{\alpha-1}s^{\beta-1}\left[f(tb+(1-t)a,sd+(1-s)c)+f(tb+(1-t)a,sc+(1-s)d)\right.\\
&\left.+f(ta+(1-t)b,sd+(1-s)c)+f(ta+(1-t)b,sc+(1-s)d)\right]\mathrm{d}s\mathrm{d}t\Bigg\},
\end{aligned}
\tag{3.4}
$$

$$
\begin{aligned}
I_3=\frac{1}{(b-a)(d-c)}&\left\{\left(\frac{7}{8}-\left(\frac{2}{3}\right)^{\beta}\right)\left(\frac{1}{3^{\alpha}}-\frac{1}{8}\right)\left[f\left(\frac{2a+b}{3},\frac{2c+d}{3}\right)\right.\right.\\
&\left.+f\left(\frac{2a+b}{3},\frac{c+2d}{3}\right)+f\left(\frac{a+2b}{3},\frac{2c+d}{3}\right)+f\left(\frac{a+2b}{3},\frac{c+2d}{3}\right)\right]\\
&+\frac{1}{8}\left(\frac{7}{8}-\left(\frac{2}{3}\right)^{\beta}\right)\left[f\left(a,\frac{2c+d}{3}\right)+f\left(a,\frac{c+2d}{3}\right)+f\left(b,\frac{2c+d}{3}\right)+f\left(b,\frac{c+2d}{3}\right)\right]\\
&+\frac{1}{8}\left(\frac{1}{3^{\alpha}}-\frac{1}{8}\right)\left[f\left(\frac{2a+b}{3},c\right)+f\left(\frac{2a+b}{3},d\right)+f\left(\frac{a+2b}{3},c\right)+f\left(\frac{a+2b}{3},d\right)\right]\\
&+\frac{1}{64}\left[f(a,c)+f(a,d)+f(b,c)+f(b,d)\right]\\
&+\alpha\left(\left(\frac{2}{3}\right)^{\beta}-\frac{7}{8}\right)\int_{0}^{\frac{1}{3}} t^{\alpha-1}\left[f\left(tb+(1-t)a,\frac{2c+d}{3}\right)+f\left(tb+(1-t)a,\frac{c+2d}{3}\right)\right.\\
&\left.+f\left(ta+(1-t)b,\frac{2c+d}{3}\right)+f\left(ta+(1-t)b,\frac{c+2d}{3}\right)\right]\mathrm{d}t\\
&-\frac{\alpha}{8}\int_{0}^{\frac{1}{3}} t^{\alpha-1}\left[f(tb+(1-t)a,d)+f(tb+(1-t)a,c)+f(ta+(1-t)b,d)\right.\\
&\left.+f(ta+(1-t)b,c)\right]\mathrm{d}t
\end{aligned}
$$

$$
\begin{aligned}
&-\beta\left(\frac{1}{3^{\alpha}}-\frac{1}{8}\right)\int_{\frac{2}{3}}^{1}s^{\beta-1}\left[f\left(\frac{2a+b}{3},sd+(1-s)c\right)+f\left(\frac{2a+b}{3},sc+(1-s)d\right)\right.\\
&\left.+f\left(\frac{a+2b}{3},sd+(1-s)c\right)+f\left(\frac{a+2b}{3},sc+(1-s)d\right)\right]\mathrm{d}s\\
&-\frac{\beta}{8}\int_{\frac{2}{3}}^{1}s^{\beta-1}\left[f(a,sd+(1-s)c)+f(a,sc+(1-s)d)+f(b,sd+(1-s)c)\right.\\
&\left.+f(b,sc+(1-s)d)\right]\mathrm{d}s\\
&+\alpha\beta\int_{0}^{\frac{1}{3}}\int_{\frac{2}{3}}^{1}t^{\alpha-1}s^{\beta-1}\left[f(tb+(1-t)a,sd+(1-s)c)+f(tb+(1-t)a,sc+(1-s)d)\right.\\
&\left.+f(ta+(1-t)b,sd+(1-s)c)+f(ta+(1-t)b,sc+(1-s)d)\right]\mathrm{d}s\mathrm{d}t\Bigg\},
\end{aligned}
\tag{3.5}
$$

$$
\begin{aligned}
I_4=\frac{1}{(b-a)(d-c)}\Bigg\{&\left(\frac{2^{\alpha}-1}{3^{\alpha}}\right)\left(\frac{1}{3^{\beta}}-\frac{1}{8}\right)\left[f\left(\frac{2a+b}{3},\frac{2c+d}{3}\right)+f\left(\frac{2a+b}{3},\frac{c+2d}{3}\right)\right.\\
&\left.+f\left(\frac{a+2b}{3},\frac{2c+d}{3}\right)+f\left(\frac{a+2b}{3},\frac{c+2d}{3}\right)\right]\\
&+\frac{1}{8}\left(\frac{2^{\alpha}-1}{3^{\alpha}}\right)\left[f\left(\frac{2a+b}{3},c\right)+f\left(\frac{2a+b}{3},d\right)+f\left(\frac{a+2b}{3},c\right)+f\left(\frac{a+2b}{3},d\right)\right]\\
&-\alpha\left(\frac{1}{3^{\beta}}-\frac{1}{8}\right)\int_{\frac{1}{3}}^{\frac{2}{3}}t^{\alpha-1}\left[f\left(tb+(1-t)a,\frac{2c+d}{3}\right)+f\left(tb+(1-t)a,\frac{c+2d}{3}\right)\right.\\
&\left.+f\left(ta+(1-t)b,\frac{2c+d}{3}\right)+f\left(ta+(1-t)b,\frac{c+2d}{3}\right)\right]\mathrm{d}t\\
&-\frac{\alpha}{8}\int_{\frac{1}{3}}^{\frac{2}{3}}t^{\alpha-1}\left[f(tb+(1-t)a,d)+f(tb+(1-t)a,c)+f(ta+(1-t)b,d)\right.\\
&\left.+f(ta+(1-t)b,c)\right]\mathrm{d}t\\
&-\beta\left(\frac{2^{\alpha}-1}{3^{\alpha}}\right)\int_{0}^{\frac{1}{3}}s^{\beta-1}\left[f\left(\frac{2a+b}{3},sd+(1-s)c\right)+f\left(\frac{2a+b}{3},sc+(1-s)d\right)\right.\\
&\left.+f\left(\frac{a+2b}{3},sd+(1-s)c\right)+f\left(\frac{a+2b}{3},sc+(1-s)d\right)\right]\mathrm{d}s
\end{aligned}
$$

$$+\alpha\beta\int_{\frac{1}{3}}^{\frac{2}{3}}\int_{0}^{\frac{1}{3}}t^{\alpha-1}s^{\beta-1}\left[f\left(tb+(1-t)a,sd+(1-s)c\right)+f\left(tb+(1-t)a,sc+(1-s)d\right)\right.$$

$$\left.+f\left(ta+(1-t)b,sd+(1-s)c\right)+f\left(ta+(1-t)b,sc+(1-s)d\right)\right]\mathrm{d}s\mathrm{d}t\Bigg\}, \tag{3.6}$$

$$I_5=\frac{1}{(b-a)(d-c)}\Bigg\{\left(\frac{2^{\alpha}-1}{3^{\alpha}}\right)\left(\frac{2^{\beta}-1}{3^{\beta}}\right)\left[f\left(\frac{2a+b}{3},\frac{2c+d}{3}\right)+f\left(\frac{2a+b}{3},\frac{c+2d}{3}\right)\right.$$

$$\left.+f\left(\frac{a+2b}{3},\frac{2c+d}{3}\right)+f\left(\frac{a+2b}{3},\frac{c+2d}{3}\right)\right]$$

$$-\alpha\left(\frac{2^{\beta}-1}{3^{\beta}}\right)\int_{\frac{1}{3}}^{\frac{2}{3}}t^{\alpha-1}\left[f\left(tb+(1-t)a,\frac{2c+d}{3}\right)+f\left(tb+(1-t)a,\frac{c+2d}{3}\right)\right.$$

$$\left.+f\left(ta+(1-t)b,\frac{2c+d}{3}\right)+f\left(ta+(1-t)b,\frac{c+2d}{3}\right)\right]\mathrm{d}t$$

$$-\beta\left(\frac{2^{\alpha}-1}{3^{\alpha}}\right)\int_{\frac{1}{3}}^{\frac{2}{3}}s^{\beta-1}\left[f\left(\frac{2a+b}{3},sd+(1-s)c\right)+f\left(\frac{2a+b}{3},sc+(1-s)d\right)\right.$$

$$\left.+f\left(\frac{a+2b}{3},sd+(1-s)c\right)+f\left(\frac{a+2b}{3},sc+(1-s)d\right)\right]\mathrm{d}s$$

$$+\alpha\beta\int_{\frac{1}{3}}^{\frac{2}{3}}\int_{\frac{1}{3}}^{\frac{2}{3}}t^{\alpha-1}s^{\beta-1}\left[f\left(tb+(1-t)a,sd+(1-s)c\right)+f\left(tb+(1-t)a,sc+(1-s)d\right)\right.$$

$$\left.+f\left(ta+(1-t)b,sd+(1-s)c\right)+f\left(ta+(1-t)b,sc+(1-s)d\right)\right]\mathrm{d}s\mathrm{d}t\Bigg\}, \tag{3.7}$$

$$I_6=\frac{1}{(b-a)(d-c)}\Bigg\{-\left(\frac{2^{\alpha}-1}{3^{\alpha}}\right)\left(\left(\frac{2}{3}\right)^{\beta}-\frac{7}{8}\right)\left[f\left(\frac{2a+b}{3},\frac{2c+d}{3}\right)\right.$$

$$\left.+f\left(\frac{2a+b}{3},\frac{c+2d}{3}\right)+f\left(\frac{a+2b}{3},\frac{2c+d}{3}\right)+f\left(\frac{a+2b}{3},\frac{c+2d}{3}\right)\right]$$

$$+\frac{1}{8}\left(\frac{2^{\alpha}-1}{3^{\alpha}}\right)\left[f\left(\frac{2a+b}{3},c\right)+f\left(\frac{2a+b}{3},d\right)+f\left(\frac{a+2b}{3},c\right)+f\left(\frac{a+2b}{3},d\right)\right]$$

$$-\frac{\alpha}{8}\int_{\frac{1}{3}}^{\frac{2}{3}}t^{\alpha-1}\left[f\left(tb+(1-t)a,d\right)+f\left(tb+(1-t)a,c\right)+f\left(ta+(1-t)b,d\right)\right.$$

$$\left.+f\left(ta+(1-t)b,c\right)\right]\mathrm{d}t$$

$$+\alpha\left(\left(\frac{2}{3}\right)^{\beta}-\frac{7}{8}\right)\int_{\frac{1}{3}}^{\frac{2}{3}}t^{\alpha-1}\left[f\left(tb+(1-t)a,\frac{2c+d}{3}\right)+f\left(tb+(1-t)a,\frac{c+2d}{3}\right)\right.$$
$$\left.+f\left(ta+(1-t)b,\frac{2c+d}{3}\right)+f\left(ta+(1-t)b,\frac{c+2d}{3}\right)\right]\mathrm{d}t$$
$$-\beta\left(\frac{2^{\alpha}-1}{3^{\alpha}}\right)\int_{\frac{2}{3}}^{1}s^{\beta-1}\left[f\left(\frac{2a+b}{3},sd+(1-s)c\right)+f\left(\frac{2a+b}{3},sc+(1-s)d\right)\right.$$
$$\left.+f\left(\frac{a+2b}{3},sd+(1-s)c\right)+f\left(\frac{a+2b}{3},sc+(1-s)d\right)\right]\mathrm{d}s$$
$$+\alpha\beta\int_{\frac{1}{3}}^{\frac{2}{3}}\int_{\frac{2}{3}}^{1}t^{\alpha-1}s^{\beta-1}\left[f\left(tb+(1-t)a,sd+(1-s)c\right)+f\left(tb+(1-t)a,sc+(1-s)d\right)\right.$$
$$\left.+f\left(ta+(1-t)b,sd+(1-s)c\right)+f\left(ta+(1-t)b,sc+(1-s)d\right)\right]\mathrm{d}s\mathrm{d}t\Bigg\}, \tag{3.8}$$

$$I_7=\frac{1}{(b-a)(d-c)}\Bigg\{-\left(\frac{1}{3^{\beta}}-\frac{1}{8}\right)\left(\left(\frac{2}{3}\right)^{\alpha}-\frac{7}{8}\right)\left[f\left(\frac{2a+b}{3},\frac{2c+d}{3}\right)\right.$$
$$\left.+f\left(\frac{2a+b}{3},\frac{c+2d}{3}\right)+f\left(\frac{a+2b}{3},\frac{2c+d}{3}\right)+f\left(\frac{a+2b}{3},\frac{c+2d}{3}\right)\right]$$
$$+\frac{1}{8}\left(\frac{1}{3^{\beta}}-\frac{1}{8}\right)\left[f\left(a,\frac{2c+d}{3}\right)+f\left(a,\frac{c+2d}{3}\right)+f\left(b,\frac{2c+d}{3}\right)+f\left(b,\frac{c+2d}{3}\right)\right]$$
$$-\frac{1}{8}\left(\left(\frac{2}{3}\right)^{\alpha}-\frac{7}{8}\right)\left[f\left(\frac{2a+b}{3},c\right)+f\left(\frac{2a+b}{3},d\right)+f\left(\frac{a+2b}{3},c\right)+f\left(\frac{a+2b}{3},d\right)\right]$$
$$+\frac{1}{64}\left[f(a,c)+f(a,d)+f(b,c)+f(b,d)\right]$$
$$-\frac{\alpha}{8}\int_{\frac{2}{3}}^{1}t^{\alpha-1}\left[f\left(tb+(1-t)a,d\right)+f\left(tb+(1-t)a,c\right)+f\left(ta+(1-t)b,d\right)\right.$$
$$\left.+f\left(ta+(1-t)b,c\right)\right]\mathrm{d}t$$
$$-\alpha\left(\frac{1}{3^{\beta}}-\frac{1}{8}\right)\int_{\frac{2}{3}}^{1}t^{\alpha-1}\left[f\left(tb+(1-t)a,\frac{2c+d}{3}\right)+f\left(tb+(1-t)a,\frac{c+2d}{3}\right)\right.$$
$$\left.+f\left(ta+(1-t)b,\frac{2c+d}{3}\right)+f\left(ta+(1-t)b,\frac{c+2d}{3}\right)\right]\mathrm{d}t$$
$$+\beta\left(\left(\frac{2}{3}\right)^{\alpha}-\frac{7}{8}\right)\int_{0}^{\frac{1}{3}}s^{\beta-1}\left[f\left(\frac{2a+b}{3},sd+(1-s)c\right)+f\left(\frac{2a+b}{3},sc+(1-s)d\right)\right.$$

$$+f\left(\frac{a+2b}{3},sd+(1-s)c\right)+f\left(\frac{a+2b}{3},sc+(1-s)d\right)\Bigg]\mathrm{d}s$$
$$-\frac{\beta}{8}\int_0^{\frac{1}{3}}s^{\beta-1}\left[f(a,sd+(1-s)c)+f(a,sc+(1-s)d)+f(b,sd+(1-s)c)\right.$$
$$\left.+f(b,sc+(1-s)d)\right]\mathrm{d}s$$
$$+\alpha\beta\int_{\frac{2}{3}}^1\int_0^{\frac{1}{3}}t^{\alpha-1}s^{\beta-1}\left[f(tb+(1-t)a,sd+(1-s)c)+f(tb+(1-t)a,sc+(1-s)d)\right.$$
$$\left.+f(ta+(1-t)b,sd+(1-s)c)+f(ta+(1-t)b,sc+(1-s)d)\right]\mathrm{d}s\mathrm{d}t\Bigg\}, \tag{3.9}$$

$$I_8=\frac{1}{(b-a)(d-c)}\Bigg\{-\left(\frac{2^\beta-1}{3^\beta}\right)\left(\left(\frac{2}{3}\right)^\alpha-\frac{7}{8}\right)\left[f\left(\frac{2a+b}{3},\frac{2c+d}{3}\right)\right.$$
$$\left.+f\left(\frac{2a+b}{3},\frac{c+2d}{3}\right)+f\left(\frac{a+2b}{3},\frac{2c+d}{3}\right)+f\left(\frac{a+2b}{3},\frac{c+2d}{3}\right)\right]$$
$$+\frac{1}{8}\left(\frac{2^\beta-1}{3^\beta}\right)\left[f\left(a,\frac{2c+d}{3}\right)+f\left(a,\frac{c+2d}{3}\right)+f\left(b,\frac{2c+d}{3}\right)+f\left(b,\frac{c+2d}{3}\right)\right]$$
$$-\alpha\left(\frac{2^\beta-1}{3^\beta}\right)\int_{\frac{2}{3}}^1 t^{\alpha-1}\left[f\left(tb+(1-t)a,\frac{2c+d}{3}\right)+f\left(tb+(1-t)a,\frac{c+2d}{3}\right)\right.$$
$$\left.+f\left(ta+(1-t)b,\frac{2c+d}{3}\right)+f\left(ta+(1-t)b,\frac{c+2d}{3}\right)\right]\mathrm{d}t$$
$$+\beta\left(\left(\frac{2}{3}\right)^\alpha-\frac{7}{8}\right)\int_{\frac{1}{3}}^{\frac{2}{3}}s^{\beta-1}\left[f\left(\frac{2a+b}{3},sd+(1-s)c\right)+f\left(\frac{2a+b}{3},sc+(1-s)d\right)\right.$$
$$\left.+f\left(\frac{a+2b}{3},sd+(1-s)c\right)+f\left(\frac{a+2b}{3},sc+(1-s)d\right)\right]\mathrm{d}s$$
$$-\frac{\beta}{8}\int_{\frac{1}{3}}^{\frac{2}{3}}s^{\beta-1}\left[f(a,sd+(1-s)c)+f(a,sc+(1-s)d)+f(b,sd+(1-s)c)\right.$$
$$\left.+f(b,sc+(1-s)d)\right]\mathrm{d}s$$
$$+\alpha\beta\int_{\frac{2}{3}}^1\int_{\frac{1}{3}}^{\frac{2}{3}}t^{\alpha-1}s^{\beta-1}\left[f(tb+(1-t)a,sd+(1-s)c)+f(tb+(1-t)a,sc+(1-s)d)\right.$$

$$+f(ta+(1-t)b,sd+(1-s)c)+f(ta+(1-t)b,sc+(1-s)d)]\,\mathrm{d}s\mathrm{d}t\Bigg\} \tag{3.10}$$

and

$$\begin{aligned}
I_9=\frac{1}{(b-a)(d-c)}\Bigg\{&\left(\left(\frac{2}{3}\right)^{\beta}-\frac{7}{8}\right)\left(\left(\frac{2}{3}\right)^{\alpha}-\frac{7}{8}\right)\left[f\left(\frac{2a+b}{3},\frac{2c+d}{3}\right)\right.\\
&\left.+f\left(\frac{2a+b}{3},\frac{c+2d}{3}\right)+f\left(\frac{a+2b}{3},\frac{2c+d}{3}\right)+f\left(\frac{a+2b}{3},\frac{c+2d}{3}\right)\right]\\
&-\frac{1}{8}\left(\left(\frac{2}{3}\right)^{\beta}-\frac{7}{8}\right)\left[f\left(a,\frac{2c+d}{3}\right)+f\left(a,\frac{c+2d}{3}\right)+f\left(b,\frac{2c+d}{3}\right)+f\left(b,\frac{c+2d}{3}\right)\right]\\
&-\frac{1}{8}\left(\left(\frac{2}{3}\right)^{\alpha}-\frac{7}{8}\right)\left[f\left(\frac{2a+b}{3},c\right)+f\left(\frac{2a+b}{3},d\right)+f\left(\frac{a+2b}{3},c\right)+f\left(\frac{a+2b}{3},d\right)\right]\\
&+\frac{1}{64}\left[f(a,c)+f(a,d)+f(b,c)+f(b,d)\right]\\
&+\alpha\left(\left(\frac{2}{3}\right)^{\beta}-\frac{7}{8}\right)\int_{\frac{2}{3}}^{1}t^{\alpha-1}\left[f\left(tb+(1-t)a,\frac{2c+d}{3}\right)+f\left(tb+(1-t)a,\frac{c+2d}{3}\right)\right.\\
&\left.+f\left(ta+(1-t)b,\frac{2c+d}{3}\right)+f\left(ta+(1-t)b,\frac{c+2d}{3}\right)\right]\mathrm{d}t\\
&-\frac{\alpha}{8}\int_{\frac{2}{3}}^{1}t^{\alpha-1}\left[f(tb+(1-t)a,d)+f(tb+(1-t)a,c)+f(ta+(1-t)b,d)\right.\\
&\left.+f(ta+(1-t)b,c)\right]\mathrm{d}t\\
&+\beta\left(\left(\frac{2}{3}\right)^{\alpha}-\frac{7}{8}\right)\int_{\frac{2}{3}}^{1}s^{\beta-1}\left[f\left(\frac{2a+b}{3},sd+(1-s)c\right)+f\left(\frac{2a+b}{3},sc+(1-s)d\right)\right.\\
&\left.+f\left(\frac{a+2b}{3},sd+(1-s)c\right)+f\left(\frac{a+2b}{3},sc+(1-s)d\right)\right]\mathrm{d}s\\
&-\frac{\beta}{8}\int_{\frac{1}{3}}^{\frac{2}{3}}s^{\beta-1}\left[f(a,sd+(1-s)c)+f(a,sc+(1-s)d)+f(b,sd+(1-s)c)\right.\\
&\left.+f(b,sc+(1-s)d)\right]\mathrm{d}s\\
&+\alpha\beta\int_{\frac{1}{3}}^{\frac{2}{3}}\int_{\frac{1}{3}}^{\frac{2}{3}}t^{\alpha-1}s^{\beta-1}\left[f(tb+(1-t)a,sd+(1-s)c)+f(tb+(1-t)a,sc+(1-s)d)\right.\\
&\left.+f(ta+(1-t)b,sd+(1-s)c)+f(ta+(1-t)b,sc+(1-s)d)\right]\mathrm{d}s\mathrm{d}t\Bigg\}.
\end{aligned} \tag{3.11}$$

By adding the equalities (3.3)–(3.11) and by using the change of the variable, it can be rewritten as follows:

$$\begin{aligned}\sum_{i=1}^{9} I_i = \frac{1}{(b-a)(d-c)} \Bigg\{ & \frac{1}{16}\left[f(a,c)+f(a,d)+f(b,c)+f(b,d)\right] \\ & + \frac{3}{16}\left[f\left(a,\frac{2c+d}{3}\right)+f\left(a,\frac{c+2d}{3}\right)+f\left(b,\frac{2c+d}{3}\right)+f\left(b,\frac{c+2d}{3}\right)\right] \\ & + \frac{3}{16}\left[f\left(\frac{2a+b}{3},c\right)+f\left(\frac{2a+b}{3},d\right)+f\left(\frac{a+2b}{3},c\right)+f\left(\frac{a+2b}{3},d\right)\right] \\ & + \frac{9}{16}\left[f\left(\frac{2a+b}{3},\frac{2c+d}{3}\right)+f\left(\frac{2a+b}{3},\frac{c+2d}{3}\right)+f\left(\frac{a+2b}{3},\frac{2c+d}{3}\right)\right. \\ & \left. + f\left(\frac{a+2b}{3},\frac{c+2d}{3}\right)\right] - \frac{3\Gamma(\alpha+1)}{4(b-a)^{\alpha}}\left[J^{\alpha}_{b-}f\left(a,\frac{2c+d}{3}\right)+J^{\alpha}_{b-}f\left(a,\frac{c+2d}{3}\right)\right. \\ & \left. + J^{\alpha}_{a+}f\left(b,\frac{2c+d}{3}\right)+J^{\alpha}_{a+}f\left(b,\frac{c+2d}{3}\right)\right] \\ & - \frac{\Gamma(\alpha+1)}{4(b-a)^{\alpha}}\left[J^{\alpha}_{b-}f(a,c)+J^{\alpha}_{b-}f(a,d)+J^{\alpha}_{a+}f(b,c)+J^{\alpha}_{a+}f(b,d)\right] \\ & - \frac{3\Gamma(\beta+1)}{4(d-c)^{\beta}}\left[J^{\beta}_{d-}f\left(\frac{2a+b}{3},c\right)+J^{\beta}_{d-}f\left(\frac{a+2b}{3},c\right)+J^{\beta}_{c+}f\left(\frac{2a+b}{3},d\right)\right. \\ & \left. + J^{\beta}_{c+}f\left(\frac{a+2b}{3},d\right)\right] - \frac{\Gamma(\beta+1)}{4(d-c)^{\beta}}\left[J^{\beta}_{d-}f(a,c)+J^{\beta}_{c+}f(a,d)+J^{\beta}_{d-}f(b,c)+J^{\beta}_{c+}f(b,d)\right] \\ & + \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{(b-a)^{\alpha}(d-c)^{\beta}}\left[J^{\alpha,\beta}_{b-,d-}f(a,c)+J^{\alpha,\beta}_{b-,c+}f(a,d)+J^{\alpha,\beta}_{a+,d-}f(b,c)+J^{\alpha,\beta}_{a+,c+}f(b,d)\right]\Bigg\}. \end{aligned} \tag{3.12}$$

Multiplying the both sides of equality (3.12) by $\frac{(b-a)(d-c)}{4}$, equality (3.1) is obtained. This finishes the proof of Lemma 1. □

4. Newton inequalities for Riemann–Liouville fractional integrals

In this section, we will give fractional Newton-type inequalities in the case of differentiable co-ordinated convex function.

Theorem 1. *Let us note that the conditions of Lemma 1 are valid. If* $\left|\frac{\partial^2 f}{\partial s\partial t}\right|$ *is co-ordinated convex function, then we obtain the following inequality*

$$\begin{aligned}\left|\Theta^{\alpha,\beta}(a,b;c,d)\right| \leq{}& \frac{(b-a)(d-c)}{4}\left[\varphi_1(\alpha)+\varphi_2(\alpha)+\varphi_3(\alpha)\right]\left[\varphi_1(\beta)+\varphi_2(\beta)+\varphi_3(\beta)\right] \\ & \times\left[\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|+\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|+\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|+\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|\right],\end{aligned} \tag{4.1}$$

where

$$\begin{cases} \varphi_1(\eta) = \int_0^{\frac{1}{3}} \left| t^\eta - \frac{1}{8} \right| dt = \frac{2\eta}{\eta+1} \left(\frac{1}{8}\right)^{1+\frac{1}{\eta}} + \frac{1}{(\eta+1)3^{\eta+1}} - \frac{1}{24}, \\ \varphi_2(\eta) = \int_{\frac{1}{3}}^{\frac{2}{3}} \left| t^\eta - \frac{1}{2} \right| dt = \frac{2\eta}{\eta+1} \left(\frac{1}{2}\right)^{1+\frac{1}{\eta}} + \frac{1+2^{\eta+1}}{(\eta+1)3^{\eta+1}} - \frac{1}{2}, \\ \varphi_3(\eta) = \int_{\frac{2}{3}}^{1} \left| t^\eta - \frac{7}{8} \right| dt = \frac{2\eta}{\eta+1} \left(\frac{7}{8}\right)^{1+\frac{1}{\eta}} + \frac{2^{\eta+1}+3^{\eta+1}}{(\eta+1)3^{\eta+1}} - \frac{35}{24}, \end{cases}$$

and $\Theta^{\alpha,\beta}(a,b;c,d)$ *is given in equality* (3.2).

Proof. If we take absolute value of (3.1), then the following inequality holds:

$$\left|\Theta^{\alpha,\beta}(a,b;c,d)\right| \leq \frac{(b-a)(d-c)}{4} \sum_{k=1}^{9} |I_k|. \tag{4.2}$$

From the fact that $\left|\frac{\partial^2 f}{\partial s \partial t}\right|$ is co-ordinated convex function, we have

$$\begin{aligned} |I_1| \leq & \int_0^{\frac{1}{3}} \int_0^{\frac{1}{3}} \left| t^\alpha - \frac{1}{8} \right| \left| s^\beta - \frac{1}{8} \right| \left| \frac{\partial^2}{\partial s \partial t} f(tb+(1-t)a, sd+(1-s)c) \right. \\ & - \frac{\partial^2}{\partial s \partial t} f(tb+(1-t)a, sc+(1-s)d) - \frac{\partial^2}{\partial s \partial t} f(ta+(1-t)b, sd+(1-s)c) \\ & \left. + \frac{\partial^2}{\partial s \partial t} f(ta+(1-t)b, sc+(1-s)d) \right| ds dt \\ \leq & \int_0^{\frac{1}{3}} \int_0^{\frac{1}{3}} \left| t^\alpha - \frac{1}{8} \right| \left| s^\beta - \frac{1}{8} \right| \left[\left| \frac{\partial^2 f}{\partial s \partial t}(a,c) \right| + \left| \frac{\partial^2 f}{\partial s \partial t}(a,d) \right| + \left| \frac{\partial^2 f}{\partial s \partial t}(b,c) \right| + \left| \frac{\partial^2 f}{\partial s \partial t}(b,d) \right| \right] ds dt \\ = & \varphi_1(\alpha) \varphi_1(\beta) \left[\left| \frac{\partial^2 f}{\partial s \partial t}(a,c) \right| + \left| \frac{\partial^2 f}{\partial s \partial t}(a,d) \right| + \left| \frac{\partial^2 f}{\partial s \partial t}(b,c) \right| + \left| \frac{\partial^2 f}{\partial s \partial t}(b,d) \right| \right]. \end{aligned} \tag{4.3}$$

Similarly, we state the following inequalities

$$|I_2| \leq \varphi_1(\alpha) \varphi_2(\beta) \left[\left| \frac{\partial^2 f}{\partial s \partial t}(a,c) \right| + \left| \frac{\partial^2 f}{\partial s \partial t}(a,d) \right| + \left| \frac{\partial^2 f}{\partial s \partial t}(b,c) \right| + \left| \frac{\partial^2 f}{\partial s \partial t}(b,d) \right| \right], \tag{4.4}$$

$$|I_3| \leq \varphi_1(\alpha) \varphi_3(\beta) \left[\left| \frac{\partial^2 f}{\partial s \partial t}(a,c) \right| + \left| \frac{\partial^2 f}{\partial s \partial t}(a,d) \right| + \left| \frac{\partial^2 f}{\partial s \partial t}(b,c) \right| + \left| \frac{\partial^2 f}{\partial s \partial t}(b,d) \right| \right], \tag{4.5}$$

$$|I_4| \leq \varphi_2(\alpha) \varphi_1(\beta) \left[\left| \frac{\partial^2 f}{\partial s \partial t}(a,c) \right| + \left| \frac{\partial^2 f}{\partial s \partial t}(a,d) \right| + \left| \frac{\partial^2 f}{\partial s \partial t}(b,c) \right| + \left| \frac{\partial^2 f}{\partial s \partial t}(b,d) \right| \right], \tag{4.6}$$

$$|I_5| \leq \varphi_2(\alpha)\varphi_2(\beta)\left[\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right| + \left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right| + \left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right| + \left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|\right], \tag{4.7}$$

$$|I_6| \leq \varphi_2(\alpha)\varphi_3(\beta)\left[\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right| + \left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right| + \left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right| + \left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|\right], \tag{4.8}$$

$$|I_7| \leq \varphi_3(\alpha)\varphi_1(\beta)\left[\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right| + \left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right| + \left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right| + \left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|\right], \tag{4.9}$$

$$|I_8| \leq \varphi_3(\alpha)\varphi_2(\beta)\left[\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right| + \left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right| + \left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right| + \left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|\right], \tag{4.10}$$

and

$$|I_9| \leq \varphi_3(\alpha)\varphi_3(\beta)\left[\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right| + \left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right| + \left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right| + \left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|\right]. \tag{4.11}$$

Hence, we reach the required inequality by adding from (4.3) to (4.11) and then multiplying by $\frac{(b-a)(d-c)}{4}$. □

Example 1. Let us consider that the function $f\colon [0,1]\times[0,1]\to\mathbb{R}$, $f(t,s)=t^2s^2$ such that $\left|\frac{\partial^2 f}{\partial s\partial t}\right|$ is co-ordinated convex function on $[0,1]\times[0,1]$. From the facts of assumptions, the following inequality holds:

$$\left|\frac{1}{9}-\frac{1}{6}\left[\frac{\alpha^2+\alpha+2}{(\alpha+1)(\alpha+2)}+\frac{\beta^2+\beta+2}{(\beta+1)(\beta+2)}\right]+\frac{[\alpha(\alpha+1)+2][\beta(\beta+1)+2]}{4(\alpha+1)(\alpha+2)(\beta+1)(\beta+2)}\right|$$
$$\leq [\varphi_1(\alpha)+\varphi_2(\alpha)+\varphi_3(\alpha)][\varphi_1(\beta)+\varphi_2(\beta)+\varphi_3(\beta)],$$

where φ_1, φ_2 and φ_3 functions are as given in Theorem 1.

With easy computations, we obtain the following equalities

$$\frac{[f(a,c)+f(a,d)+f(b,c)+f(b,d)]}{64}=\frac{1}{64},$$

$$\frac{3}{64}\left[f\left(a,\frac{2c+d}{3}\right)+f\left(a,\frac{c+2d}{3}\right)+f\left(b,\frac{2c+d}{3}\right)+f\left(b,\frac{c+2d}{3}\right)\right]=\frac{5}{192},$$

$$\frac{3}{64}\left[f\left(\frac{2a+b}{3},c\right)+f\left(\frac{2a+b}{3},d\right)+f\left(\frac{a+2b}{3},c\right)+f\left(\frac{a+2b}{3},d\right)\right]=\frac{5}{192},$$

$$\frac{9}{64}\left[f\left(\frac{2a+b}{3},\frac{2c+d}{3}\right)+f\left(\frac{2a+b}{3},\frac{c+2d}{3}\right)+f\left(\frac{a+2b}{3},\frac{2c+d}{3}\right)\right.$$

$$+f\left(\frac{a+2b}{3},\frac{c+2d}{3}\right)\Bigg]=\frac{25}{576},$$

and

$$\frac{3\Gamma(\alpha+1)}{16(b-a)^{\alpha}}\left[J^{\alpha}_{b-}f\left(a,\frac{2c+d}{3}\right)+J^{\alpha}_{b-}f\left(a,\frac{c+2d}{3}\right)+J^{\alpha}_{a+}f\left(b,\frac{2c+d}{3}\right)\right.$$
$$\left.+J^{\alpha}_{a+}f\left(b,\frac{c+2d}{3}\right)\right]=\frac{5\left(\alpha^2+\alpha+2\right)}{48(\alpha+2)(\alpha+1)}.$$

From the facts of the definitions of Riemann–Liouville fractional integrals, we obtain

$$\frac{\Gamma(\alpha+1)}{16(b-a)^{\alpha}}\left[J^{\alpha}_{b-}f(a,c)+J^{\alpha}_{b-}f(a,d)+J^{\alpha}_{a+}f(b,c)+J^{\alpha}_{a+}f(b,d)\right]=\frac{\left(\alpha^2+\alpha+2\right)}{16(\alpha+2)(\alpha+1)},$$

$$\frac{3\Gamma(\beta+1)}{16(d-c)^{\beta}}\left[J^{\beta}_{d-}f\left(\frac{2a+b}{3},c\right)+J^{\beta}_{d-}f\left(\frac{a+2b}{3},c\right)+J^{\beta}_{c+}f\left(\frac{2a+b}{3},d\right)\right.$$
$$\left.+J^{\beta}_{c+}f\left(\frac{a+2b}{3},d\right)\right]=\frac{5\left(\beta^2+\beta+2\right)}{48(\beta+2)(\beta+1)},$$

$$\frac{\Gamma(\beta+1)}{16(d-c)^{\beta}}\left[J^{\beta}_{d-}f(a,c)+J^{\beta}_{c+}f(a,d)+J^{\beta}_{d-}f(b,c)+J^{\beta}_{c+}f(b,d)\right]=\frac{\left(\beta^2+\beta+2\right)}{16(\beta+2)(\beta+1)},$$

and

$$\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^{\alpha}(d-c)^{\beta}}\left[J^{\alpha,\beta}_{b-,d-}f(a,c)+J^{\alpha,\beta}_{b-,c+}f(a,d)+J^{\alpha,\beta}_{a+,d-}f(b,c)+J^{\alpha,\beta}_{a+,c+}f(b,d)\right]$$
$$=\frac{5\left(\beta^2+\beta+2\right)}{48(\beta+2)(\beta+1)}.$$

The left hand side of the inequality (4.1) is

$$\left|\frac{1}{9}-\frac{1}{6}\left[\frac{\alpha^2+\alpha+2}{(\alpha+1)(\alpha+2)}+\frac{\beta^2+\beta+2}{(\beta+1)(\beta+2)}\right]+\frac{[\alpha(\alpha+1)+2][\beta(\beta+1)+2]}{4(\alpha+1)(\alpha+2)(\beta+1)(\beta+2)}\right|:=\Psi_1$$

and the right hand side of the inequality (4.1) is

$$[\varphi_1(\alpha)+\varphi_2(\alpha)+\varphi_3(\alpha)][\varphi_1(\beta)+\varphi_2(\beta)+\varphi_3(\beta)]:=\Psi_2.$$

As you can see in Figure 1 above, the result of our appropriate choices in Example 1 is satisfied in the inequality (4.1).

Remark 1. If we choose $\alpha=\beta=1$ in Theorem 1, then we get

$$\left|\Theta^{1,1}(a,b;c,d)\right|$$
$$\leq(b-a)(d-c)\frac{625}{331776}\left[\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|+\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|+\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|+\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|\right],$$

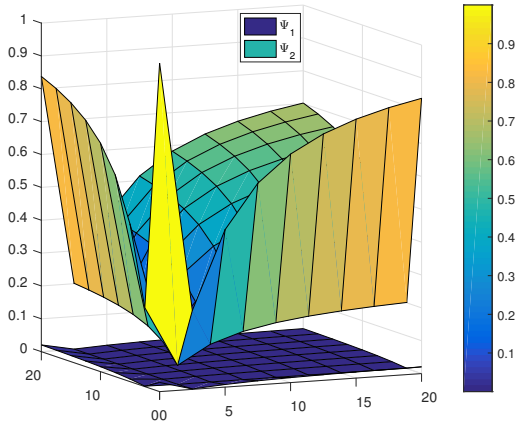

FIGURE 1. Graph for the result of Example 1 computed and plotted in MATLAB program.

which is given by Iftikhar et al. in [12] and $\Theta^{1,1}(a,b;c,d)$ equals to

$$\begin{aligned}
\Theta^{1,1}(a,b;c,d) &:= \frac{[f(a,c)+f(a,d)+f(b,c)+f(b,d)]}{64} \\
&+\frac{3}{64}\left[f\left(a,\frac{2c+d}{3}\right)+f\left(a,\frac{c+2d}{3}\right)+f\left(b,\frac{2c+d}{3}\right)+f\left(b,\frac{c+2d}{3}\right)\right. \\
&\left.+f\left(\frac{2a+b}{3},c\right)+f\left(\frac{a+2b}{3},c\right)+\left(\frac{2a+b}{3},d\right)+f\left(\frac{a+2b}{3},d\right)\right] \\
&+\frac{9}{64}\left[f\left(\frac{2a+b}{3},\frac{2c+d}{3}\right)+f\left(\frac{2a+b}{3},\frac{c+2d}{3}\right)\right. \\
&\left.+f\left(\frac{2a+b}{3},\frac{2c+d}{3}\right)+f\left(\frac{a+2b}{3},\frac{c+2d}{3}\right)\right]-\frac{1}{8(b-a)}\left[\int_a^b f(x,c)\,\mathrm{d}x\right. \\
&\left.+\int_a^b f(x,d)\,\mathrm{d}x+3\int_a^b f\left(x,\frac{2c+d}{3}\right)\mathrm{d}x+3\int_a^b f\left(x,\frac{c+2d}{3}\right)\mathrm{d}x\right] \\
&-\frac{1}{8(d-c)}\left[\int_c^d f(a,y)\,\mathrm{d}y+\int_c^d f(b,y)\,\mathrm{d}y+3\int_c^d f\left(\frac{2a+b}{3},y\right)\mathrm{d}y\right. \\
&\left.+3\int_c^d f\left(\frac{a+2b}{3},y\right)\mathrm{d}y\right]+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,\mathrm{d}y\mathrm{d}x.
\end{aligned} \tag{4.12}$$

Theorem 2. *Suppose that the conditions of Lemma 1 are valid. Suppose also that* $\left|\frac{\partial^2 f}{\partial s\partial t}\right|^q$, *is a co-ordinated convex function for* $q>1$. *Then, we obtain the following Newton inequality:*

$$\begin{aligned}
\left|\Theta^{\alpha,\beta}(a,b;c,d)\right| &\leq \frac{(b-a)(d-c)}{4}\Bigg\{(\varphi_4(\alpha,p)+\varphi_6(\alpha,p))(\varphi_4(\beta,p)+\varphi_6(\beta,p)) \\
&\times\Bigg[\left(\frac{\left|\frac{\partial^2}{\partial s\partial t}f(b,d)\right|^q+5\left|\frac{\partial^2}{\partial s\partial t}f(b,c)\right|^q+5\left|\frac{\partial^2}{\partial s\partial t}f(a,d)\right|^q+25\left|\frac{\partial^2}{\partial s\partial t}f(a,c)\right|^q}{324}\right)^{\frac{1}{q}} \\
&+\left(\frac{5\left|\frac{\partial^2}{\partial s\partial t}f(b,d)\right|^q+\left|\frac{\partial^2}{\partial s\partial t}f(b,c)\right|^q+25\left|\frac{\partial^2}{\partial s\partial t}f(a,d)\right|^q+5\left|\frac{\partial^2}{\partial s\partial t}f(a,c)\right|^q}{324}\right)^{\frac{1}{q}} \\
&+\left(\frac{5\left|\frac{\partial^2}{\partial s\partial t}f(b,d)\right|^q+25\left|\frac{\partial^2}{\partial s\partial t}f(b,c)\right|^q+\left|\frac{\partial^2}{\partial s\partial t}f(a,d)\right|^q+5\left|\frac{\partial^2}{\partial s\partial t}f(a,c)\right|^q}{324}\right)^{\frac{1}{q}} \\
&+\left(\frac{25\left|\frac{\partial^2}{\partial s\partial t}f(b,d)\right|^q+5\left|\frac{\partial^2}{\partial s\partial t}f(b,c)\right|^q+5\left|\frac{\partial^2}{\partial s\partial t}f(a,d)\right|^q+\left|\frac{\partial^2}{\partial s\partial t}f(a,c)\right|^q}{324}\right)^{\frac{1}{q}}\Bigg] \\
&+2[\varphi_4(\alpha,p)+\varphi_6(\alpha,p)]\varphi_5(\beta,p) \\
&\times\Bigg[\left(\frac{\left|\frac{\partial^2}{\partial s\partial t}f(b,d)\right|^q+\left|\frac{\partial^2}{\partial s\partial t}f(b,c)\right|^q+5\left|\frac{\partial^2}{\partial s\partial t}f(a,d)\right|^q+5\left|\frac{\partial^2}{\partial s\partial t}f(a,c)\right|^q}{108}\right)^{\frac{1}{q}} \\
&+\left(\frac{5\left|\frac{\partial^2}{\partial s\partial t}f(b,d)\right|^q+5\left|\frac{\partial^2}{\partial s\partial t}f(b,c)\right|^q+\left|\frac{\partial^2}{\partial s\partial t}f(a,d)\right|^q+\left|\frac{\partial^2}{\partial s\partial t}f(a,c)\right|^q}{108}\right)^{\frac{1}{q}}\Bigg] \\
&+2\varphi_5(\alpha,p)[\varphi_4(\beta,p)+\varphi_6(\beta,p)] \\
&\times\Bigg[\left(\frac{\left|\frac{\partial^2}{\partial s\partial t}f(b,d)\right|^q+5\left|\frac{\partial^2}{\partial s\partial t}f(b,c)\right|^q+\left|\frac{\partial^2}{\partial s\partial t}f(a,d)\right|^q+5\left|\frac{\partial^2}{\partial s\partial t}f(a,c)\right|^q}{108}\right)^{\frac{1}{q}}
\end{aligned}$$

$$+\left(\frac{5\left|\frac{\partial^2}{\partial s\partial t}f(b,d)\right|^q+\left|\frac{\partial^2}{\partial s\partial t}f(b,c)\right|^q+5\left|\frac{\partial^2}{\partial s\partial t}f(a,d)\right|^q+\left|\frac{\partial^2}{\partial s\partial t}f(a,c)\right|^q}{108}\right)^{\frac{1}{q}}\Bigg]$$

$$+4\varphi_5(\alpha,p)\varphi_5(\beta,p)$$

$$\left(\frac{\left|\frac{\partial^2}{\partial s\partial t}\times f(b,d)\right|^q+\left|\frac{\partial^2}{\partial s\partial t}f(b,c)\right|^q+\left|\frac{\partial^2}{\partial s\partial t}f(a,d)\right|^q+\left|\frac{\partial^2}{\partial s\partial t}f(a,c)\right|^q}{36}\right)^{\frac{1}{q}}\Bigg\}.$$

Here $q^{-1}+p^{-1}=1$, $\Theta^{\alpha,\beta}(a,b;c,d)$ *is given by* (3.2)*, and*

$$\begin{cases}\varphi_4(\eta,p)=\left(\int_0^{\frac{1}{3}}\left|t^\eta-\frac{1}{8}\right|^p\mathrm{d}t\right)^{\frac{1}{p}},\\ \varphi_5(\eta,p)=\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|t^\eta-\frac{1}{2}\right|^p\mathrm{d}t\right)^{\frac{1}{p}},\\ \varphi_6(\eta,p)=\left(\int_{\frac{2}{3}}^{1}\left|t^\eta-\frac{7}{8}\right|^p\mathrm{d}t\right)^{\frac{1}{p}}.\end{cases}$$

Proof. From the facts of the Hölder inequality and co-ordinated convexity of $\left|\frac{\partial^2 f}{\partial s\partial t}(t,s)\right|^q$, the following inequalities hold:

$$|I_1|\le\int_0^{\frac{1}{3}}\int_0^{\frac{1}{3}}\left|t^\alpha-\frac{1}{8}\right|\left|s^\beta-\frac{1}{8}\right|\left|\frac{\partial^2}{\partial s\partial t}f(tb+(1-t)a,sd+(1-s)c)\right|\mathrm{d}s\mathrm{d}t$$

$$+\int_0^{\frac{1}{3}}\int_0^{\frac{1}{3}}\left|t^\alpha-\frac{1}{8}\right|\left|s^\beta-\frac{1}{8}\right|\left|\frac{\partial^2}{\partial s\partial t}f(tb+(1-t)a,sc+(1-s)d)\right|\mathrm{d}s\mathrm{d}t$$

$$+\int_0^{\frac{1}{3}}\int_0^{\frac{1}{3}}\left|t^\alpha-\frac{1}{8}\right|\left|s^\beta-\frac{1}{8}\right|\left|\frac{\partial^2}{\partial s\partial t}f(ta+(1-t)b,sd+(1-s)c)\right|\mathrm{d}s\mathrm{d}t$$

$$+\int_0^{\frac{1}{3}}\int_0^{\frac{1}{3}}\left|t^\alpha-\frac{1}{8}\right|\left|s^\beta-\frac{1}{8}\right|\left|\frac{\partial^2}{\partial s\partial t}f(ta+(1-t)b,sc+(1-s)d)\right|\mathrm{d}s\mathrm{d}t$$

$$
\begin{aligned}
&\leq \left(\int_0^{\frac{1}{3}} \int_0^{\frac{1}{3}} \left| t^{\alpha} - \frac{1}{8} \right|^{p} \left| s^{\beta} - \frac{1}{8} \right|^{p} \mathrm{d}s\mathrm{d}t \right)^{\frac{1}{p}} \\
&\quad \times \left[\left(\int_0^{\frac{1}{3}} \int_0^{\frac{1}{3}} \left| \frac{\partial^2}{\partial s \partial t} f\left(tb + (1-t)a, sd + (1-s)c\right) \right|^{q} \mathrm{d}s\mathrm{d}t \right)^{\frac{1}{q}} \right. \\
&\quad + \left(\int_0^{\frac{1}{3}} \int_0^{\frac{1}{3}} \left| \frac{\partial^2}{\partial s \partial t} f\left(tb + (1-t)a, sc + (1-s)d\right) \right|^{q} \mathrm{d}s\mathrm{d}t \right)^{\frac{1}{q}} \\
&\quad + \left(\int_0^{\frac{1}{3}} \int_0^{\frac{1}{3}} \left| \frac{\partial^2}{\partial s \partial t} f\left(ta + (1-t)b, sd + (1-s)c\right) \right|^{q} \mathrm{d}s\mathrm{d}t \right)^{\frac{1}{q}} \\
&\quad \left. + \left(\int_0^{\frac{1}{3}} \int_0^{\frac{1}{3}} \left| \frac{\partial^2}{\partial s \partial t} f\left(ta + (1-t)b, sc + (1-s)d\right) \right|^{q} \mathrm{d}s\mathrm{d}t \right)^{\frac{1}{q}} \right] \\
&\leq \varphi_4(\alpha, p)\, \varphi_4(\beta, p) \\
&\quad \times \left[\left(\frac{\left| \frac{\partial^2}{\partial s \partial t} f(b,d) \right|^{q} + 5 \left| \frac{\partial^2}{\partial s \partial t} f(b,c) \right|^{q} + 5 \left| \frac{\partial^2}{\partial s \partial t} f(a,d) \right|^{q} + 25 \left| \frac{\partial^2}{\partial s \partial t} f(a,c) \right|^{q}}{324} \right)^{\frac{1}{q}} \right. \\
&\quad + \left(\frac{5 \left| \frac{\partial^2}{\partial s \partial t} f(b,d) \right|^{q} + \left| \frac{\partial^2}{\partial s \partial t} f(b,c) \right|^{q} + 25 \left| \frac{\partial^2}{\partial s \partial t} f(a,d) \right|^{q} + 5 \left| \frac{\partial^2}{\partial s \partial t} f(a,c) \right|^{q}}{324} \right)^{\frac{1}{q}} \\
&\quad + \left(\frac{5 \left| \frac{\partial^2}{\partial s \partial t} f(b,d) \right|^{q} + 25 \left| \frac{\partial^2}{\partial s \partial t} f(b,c) \right|^{q} + \left| \frac{\partial^2}{\partial s \partial t} f(a,d) \right|^{q} + 5 \left| \frac{\partial^2}{\partial s \partial t} f(a,c) \right|^{q}}{324} \right)^{\frac{1}{q}} \\
&\quad \left. + \left(\frac{25 \left| \frac{\partial^2}{\partial s \partial t} f(b,d) \right|^{q} + 5 \left| \frac{\partial^2}{\partial s \partial t} f(b,c) \right|^{q} + 5 \left| \frac{\partial^2}{\partial s \partial t} f(a,d) \right|^{q} + \left| \frac{\partial^2}{\partial s \partial t} f(a,c) \right|^{q}}{324} \right)^{\frac{1}{q}} \right].
\end{aligned}
$$

Similarly,

$$|I_2| \leq 2\varphi_4(\alpha, p)\, \varphi_5(\beta, p)$$

$$\times \left[\left(\frac{\left|\frac{\partial^2}{\partial s\partial t} f(b,d)\right|^q + \left|\frac{\partial^2}{\partial s\partial t} f(b,c)\right|^q + 5\left|\frac{\partial^2}{\partial s\partial t} f(a,d)\right|^q + 5\left|\frac{\partial^2}{\partial s\partial t} f(a,c)\right|^q}{108} \right)^{\frac{1}{q}} \right.$$

$$\left. + \left(\frac{5\left|\frac{\partial^2}{\partial s\partial t} f(b,d)\right|^q + 5\left|\frac{\partial^2}{\partial s\partial t} f(b,c)\right|^q + \left|\frac{\partial^2}{\partial s\partial t} f(a,d)\right|^q + \left|\frac{\partial^2}{\partial s\partial t} f(a,c)\right|^q}{108} \right)^{\frac{1}{q}} \right],$$

$$|I_3| \leq \varphi_4(\alpha,p)\,\varphi_6(\beta,p)$$

$$\times \left[\left(\frac{5\left|\frac{\partial^2}{\partial s\partial t} f(b,d)\right|^q + \left|\frac{\partial^2}{\partial s\partial t} f(b,c)\right|^q + 25\left|\frac{\partial^2}{\partial s\partial t} f(a,d)\right|^q + 5\left|\frac{\partial^2}{\partial s\partial t} f(a,c)\right|^q}{324} \right)^{\frac{1}{q}} \right.$$

$$+ \left(\frac{\left|\frac{\partial^2}{\partial s\partial t} f(b,d)\right|^q + 5\left|\frac{\partial^2}{\partial s\partial t} f(b,c)\right|^q + 5\left|\frac{\partial^2}{\partial s\partial t} f(a,d)\right|^q + 25\left|\frac{\partial^2}{\partial s\partial t} f(a,c)\right|^q}{324} \right)^{\frac{1}{q}}$$

$$+ \left(\frac{25\left|\frac{\partial^2}{\partial s\partial t} f(b,d)\right|^q + 5\left|\frac{\partial^2}{\partial s\partial t} f(b,c)\right|^q + 5\left|\frac{\partial^2}{\partial s\partial t} f(a,d)\right|^q + \left|\frac{\partial^2}{\partial s\partial t} f(a,c)\right|^q}{324} \right)^{\frac{1}{q}}$$

$$\left. + \left(\frac{5\left|\frac{\partial^2}{\partial s\partial t} f(b,d)\right|^q + 25\left|\frac{\partial^2}{\partial s\partial t} f(b,c)\right|^q + \left|\frac{\partial^2}{\partial s\partial t} f(a,d)\right|^q + 5\left|\frac{\partial^2}{\partial s\partial t} f(a,c)\right|^q}{324} \right)^{\frac{1}{q}} \right],$$

$$|I_4| \leq 2\varphi_5(\alpha,p)\,\varphi_4(\beta,p)$$

$$\times \left[\left(\frac{\left|\frac{\partial^2}{\partial s\partial t} f(b,d)\right|^q + 5\left|\frac{\partial^2}{\partial s\partial t} f(b,c)\right|^q + \left|\frac{\partial^2}{\partial s\partial t} f(a,d)\right|^q + 5\left|\frac{\partial^2}{\partial s\partial t} f(a,c)\right|^q}{108} \right)^{\frac{1}{q}} \right.$$

$$\left. + \left(\frac{5\left|\frac{\partial^2}{\partial s\partial t} f(b,d)\right|^q + \left|\frac{\partial^2}{\partial s\partial t} f(b,c)\right|^q + 5\left|\frac{\partial^2}{\partial s\partial t} f(a,d)\right|^q + \left|\frac{\partial^2}{\partial s\partial t} f(a,c)\right|^q}{108} \right)^{\frac{1}{q}} \right],$$

$$|I_5| \leq 4\varphi_5(\alpha,p)\,\varphi_5(\beta,p)$$

$$\times \left(\frac{\left|\frac{\partial^2}{\partial s\partial t} f(b,d)\right|^q + \left|\frac{\partial^2}{\partial s\partial t} f(b,c)\right|^q + \left|\frac{\partial^2}{\partial s\partial t} f(a,d)\right|^q + \left|\frac{\partial^2}{\partial s\partial t} f(a,c)\right|^q}{36} \right)^{\frac{1}{q}},$$

$$|I_6| \leq 2\varphi_5(\alpha,p)\,\varphi_6(\beta,p)$$
$$\times \left[\left(\frac{5\left|\frac{\partial^2}{\partial s\partial t} f(b,d)\right|^q + \left|\frac{\partial^2}{\partial s\partial t} f(b,c)\right|^q + 5\left|\frac{\partial^2}{\partial s\partial t} f(a,d)\right|^q + \left|\frac{\partial^2}{\partial s\partial t} f(a,c)\right|^q}{108} \right)^{\frac{1}{q}} \right.$$
$$\left. + \left(\frac{\left|\frac{\partial^2}{\partial s\partial t} f(b,d)\right|^q + 5\left|\frac{\partial^2}{\partial s\partial t} f(b,c)\right|^q + \left|\frac{\partial^2}{\partial s\partial t} f(a,d)\right|^q + 5\left|\frac{\partial^2}{\partial s\partial t} f(a,c)\right|^q}{108} \right)^{\frac{1}{q}} \right],$$

$$|I_7| \leq \varphi_6(\alpha,p)\,\varphi_4(\beta,p)$$
$$\times \left[\left(\frac{5\left|\frac{\partial^2}{\partial s\partial t} f(b,d)\right|^q + 25\left|\frac{\partial^2}{\partial s\partial t} f(b,c)\right|^q + \left|\frac{\partial^2}{\partial s\partial t} f(a,d)\right|^q + 5\left|\frac{\partial^2}{\partial s\partial t} f(a,c)\right|^q}{324} \right)^{\frac{1}{q}} \right.$$
$$+ \left(\frac{25\left|\frac{\partial^2}{\partial s\partial t} f(b,d)\right|^q + 5\left|\frac{\partial^2}{\partial s\partial t} f(b,c)\right|^q + 5\left|\frac{\partial^2}{\partial s\partial t} f(a,d)\right|^q + \left|\frac{\partial^2}{\partial s\partial t} f(a,c)\right|^q}{324} \right)^{\frac{1}{q}}$$
$$+ \left(\frac{\left|\frac{\partial^2}{\partial s\partial t} f(b,d)\right|^q + 5\left|\frac{\partial^2}{\partial s\partial t} f(b,c)\right|^q + 5\left|\frac{\partial^2}{\partial s\partial t} f(a,d)\right|^q + 25\left|\frac{\partial^2}{\partial s\partial t} f(a,c)\right|^q}{324} \right)^{\frac{1}{q}}$$
$$\left. + \left(\frac{5\left|\frac{\partial^2}{\partial s\partial t} f(b,d)\right|^q + \left|\frac{\partial^2}{\partial s\partial t} f(b,c)\right|^q + 25\left|\frac{\partial^2}{\partial s\partial t} f(a,d)\right|^q + 5\left|\frac{\partial^2}{\partial s\partial t} f(a,c)\right|^q}{324} \right)^{\frac{1}{q}} \right],$$

$$|I_8| \leq 2\varphi_6(\alpha,p)\,\varphi_5(\beta,p)$$
$$\times \left[\left(\frac{5\left|\frac{\partial^2}{\partial s\partial t} f(b,d)\right|^q + 5\left|\frac{\partial^2}{\partial s\partial t} f(b,c)\right|^q + \left|\frac{\partial^2}{\partial s\partial t} f(a,d)\right|^q + \left|\frac{\partial^2}{\partial s\partial t} f(a,c)\right|^q}{108} \right)^{\frac{1}{q}} \right.$$

$$+\left(\frac{\left|\frac{\partial^2}{\partial s\partial t}f(b,d)\right|^q+\left|\frac{\partial^2}{\partial s\partial t}f(b,c)\right|^q+5\left|\frac{\partial^2}{\partial s\partial t}f(a,d)\right|^q+5\left|\frac{\partial^2}{\partial s\partial t}f(a,c)\right|^q}{108}\right)^{\frac{1}{q}}\Bigg],$$

and

$$\begin{aligned}|I_9| &\leq \varphi_6(\alpha,p)\,\varphi_6(\beta,p)\\ &\times\Bigg[\left(\frac{25\left|\frac{\partial^2}{\partial s\partial t}f(b,d)\right|^q+5\left|\frac{\partial^2}{\partial s\partial t}f(b,c)\right|^q+5\left|\frac{\partial^2}{\partial s\partial t}f(a,d)\right|^q+\left|\frac{\partial^2}{\partial s\partial t}f(a,c)\right|^q}{324}\right)^{\frac{1}{q}}\\ &+\left(\frac{5\left|\frac{\partial^2}{\partial s\partial t}f(b,d)\right|^q+25\left|\frac{\partial^2}{\partial s\partial t}f(b,c)\right|^q+\left|\frac{\partial^2}{\partial s\partial t}f(a,d)\right|^q+5\left|\frac{\partial^2}{\partial s\partial t}f(a,c)\right|^q}{324}\right)^{\frac{1}{q}}\\ &+\left(\frac{5\left|\frac{\partial^2}{\partial s\partial t}f(b,d)\right|^q+\left|\frac{\partial^2}{\partial s\partial t}f(b,c)\right|^q+25\left|\frac{\partial^2}{\partial s\partial t}f(a,d)\right|^q+5\left|\frac{\partial^2}{\partial s\partial t}f(a,c)\right|^q}{324}\right)^{\frac{1}{q}}\\ &+\left(\frac{\left|\frac{\partial^2}{\partial s\partial t}f(b,d)\right|^q+5\left|\frac{\partial^2}{\partial s\partial t}f(b,c)\right|^q+5\left|\frac{\partial^2}{\partial s\partial t}f(a,d)\right|^q+25\left|\frac{\partial^2}{\partial s\partial t}f(a,c)\right|^q}{324}\right)^{\frac{1}{q}}\Bigg].\end{aligned}$$

This finishes the proof of Theorem 2. □

Corollary 1. *Let us consider* $\alpha=\beta=1$ *in Theorem 2. Then, it follows*

$$\begin{aligned}\left|\Theta^{1,1}(a,b;c,d)\right| &\leq \frac{(b-a)(d-c)}{4}\Bigg\{4\left(\frac{1}{p+1}\left(\left(\frac{1}{8}\right)^{p+1}+\left(\frac{5}{24}\right)^{p+1}\right)\right)^{\frac{2}{p}}\\ &\times\Bigg[\left(\frac{\left|\frac{\partial^2}{\partial s\partial t}f(b,d)\right|^q+5\left|\frac{\partial^2}{\partial s\partial t}f(b,c)\right|^q+5\left|\frac{\partial^2}{\partial s\partial t}f(a,d)\right|^q+25\left|\frac{\partial^2}{\partial s\partial t}f(a,c)\right|^q}{324}\right)^{\frac{1}{q}}\\ &+\left(\frac{5\left|\frac{\partial^2}{\partial s\partial t}f(b,d)\right|^q+\left|\frac{\partial^2}{\partial s\partial t}f(b,c)\right|^q+25\left|\frac{\partial^2}{\partial s\partial t}f(a,d)\right|^q+5\left|\frac{\partial^2}{\partial s\partial t}f(a,c)\right|^q}{324}\right)^{\frac{1}{q}}\\ &+\left(\frac{5\left|\frac{\partial^2}{\partial s\partial t}f(b,d)\right|^q+25\left|\frac{\partial^2}{\partial s\partial t}f(b,c)\right|^q+\left|\frac{\partial^2}{\partial s\partial t}f(a,d)\right|^q+5\left|\frac{\partial^2}{\partial s\partial t}f(a,c)\right|^q}{324}\right)^{\frac{1}{q}}\end{aligned}$$

$$+\left(\frac{25\left|\frac{\partial^2}{\partial s\partial t}f(b,d)\right|^q+5\left|\frac{\partial^2}{\partial s\partial t}f(b,c)\right|^q+5\left|\frac{\partial^2}{\partial s\partial t}f(a,d)\right|^q+\left|\frac{\partial^2}{\partial s\partial t}f(a,c)\right|^q}{324}\right)^{\frac{1}{q}}\Bigg]$$

$$+2\left[2\left(\frac{1}{p+1}\left(\left(\frac{1}{8}\right)^{p+1}+\left(\frac{5}{24}\right)^{p+1}\right)\right)^{\frac{1}{p}}\right]\left(\frac{2}{p+1}\left(\frac{1}{6}\right)^{p+1}\right)^{\frac{1}{p}}$$

$$\times\Bigg[\left(\frac{\left|\frac{\partial^2}{\partial s\partial t}f(b,d)\right|^q+\left|\frac{\partial^2}{\partial s\partial t}f(b,c)\right|^q+5\left|\frac{\partial^2}{\partial s\partial t}f(a,d)\right|^q+5\left|\frac{\partial^2}{\partial s\partial t}f(a,c)\right|^q}{108}\right)^{\frac{1}{q}}$$

$$+\left(\frac{5\left|\frac{\partial^2}{\partial s\partial t}f(b,d)\right|^q+5\left|\frac{\partial^2}{\partial s\partial t}f(b,c)\right|^q+\left|\frac{\partial^2}{\partial s\partial t}f(a,d)\right|^q+\left|\frac{\partial^2}{\partial s\partial t}f(a,c)\right|^q}{108}\right)^{\frac{1}{q}}$$

$$+\left(\frac{\left|\frac{\partial^2}{\partial s\partial t}f(b,d)\right|^q+5\left|\frac{\partial^2}{\partial s\partial t}f(b,c)\right|^q+\left|\frac{\partial^2}{\partial s\partial t}f(a,d)\right|^q+5\left|\frac{\partial^2}{\partial s\partial t}f(a,c)\right|^q}{108}\right)^{\frac{1}{q}}$$

$$+\left(\frac{5\left|\frac{\partial^2}{\partial s\partial t}f(b,d)\right|^q+\left|\frac{\partial^2}{\partial s\partial t}f(b,c)\right|^q+5\left|\frac{\partial^2}{\partial s\partial t}f(a,d)\right|^q+\left|\frac{\partial^2}{\partial s\partial t}f(a,c)\right|^q}{108}\right)^{\frac{1}{q}}\Bigg]$$

$$+4\left(\frac{2}{p+1}\left(\frac{1}{6}\right)^{p+1}\right)^{\frac{2}{p}}$$

$$\times\left(\frac{\left|\frac{\partial^2}{\partial s\partial t}f(b,d)\right|^q+\left|\frac{\partial^2}{\partial s\partial t}f(b,c)\right|^q+\left|\frac{\partial^2}{\partial s\partial t}f(a,d)\right|^q+\left|\frac{\partial^2}{\partial s\partial t}f(a,c)\right|^q}{36}\right)^{\frac{1}{q}}\Bigg\}.$$

Here, $\Theta^{1,1}(a,b;c,d)$ *is given by* (4.12).

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A REMARK ON MIDPOINT-TYPE INEQUALITIES FOR CONFORMABLE FRACTIONAL INTEGRALS

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Abstract. This paper proves an equality for the case of differentiable functions involving the conformable fractional integrals. By using this equality, we establish new midpoint-type inequalities via convex functions with the aid of the conformable fractional integrals. Some significant inequalities are obtained by taking advantage of the convexity, the Hölder inequality, and the power mean inequality. Moreover, we give example using graph in order to show that our main results are correct. Furthermore, our results generalize known results from the literature by special choices.

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Keywords: midpoint type inequality, fractional conformable integrals, fractional conformable derivatives, fractional calculus, convex function

1. INTRODUCTION

The convex theory is an available way to solve a large number of problems from different branches of mathematics. One of the most famous inequalities for the case of convex functions is the Hermite-Hadamard-type inequality. Thus, the Hermite-Hadamard-type inequalities have been considered seriously by many researchers in the last century. In addition, Riemann-Liouville fractional integrals, conformable fractional integrals, and many types of fractional integrals have been investigated to these type of inequalities.

Fractional calculus is a field of mathematics that expands the traditional derivative and integral ideas to non-integer orders. In recent decades, it has piqued the curiosity of mathematicians, physicists, and engineers [3, 24]. Fractional derivatives are also used to model a wide range of mathematical biology, as well as chemical processes and engineering problems [4, 9]. Fractional integral operators and their applicability in a variety of scientific disciplines have been concentrated. Using the derivative's

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fundamental limit formulation, a newly well-behaved straightforward fractional derivative known as the conformable derivative is improved in paper [16]. Some important requirements that cannot be fulfilled by the Riemann-Liouville and Caputo definitions are fulfilled by the conformable derivative. Nevertheless, in paper [1] the author proved that the conformable approach in [16] cannot yield good results when compared to the Caputo definition for specific functions. This flaw in the conformable definition was avoided by some extensions of the conformable approach [12,25].

A large number of mathematicians have focused on acquired trapezoid and midpoint type inequalities that give bounds via the right-hand side and left-hand side of the Hermite–Hadamard inequality, respectively. Dragomir and Agarwal first established trapezoid inequalities for convex functions in [8], while Kirmacı first proved inequalities of midpoint-type for convex functions in [18]. Moreover, Qaisar and Hussain in [21] investigated several generalized midpoint-type inequalities. Sarikaya et al. and Iqbal et al. presented some fractional midpoint and trapezoid-type inequalities for the case of convex mappings in papers [20] and [22], respectively. Furthermore, some Hermite–Hadamard-type inequalities for the fractional integrals are given in paper [13]. In addition to these, many mathematicians have considered twice differentiable functions to get many important inequalities. For instance, Barani et al. proved inequalities for twice differentiable convex mappings which are connected with Hadamard's inequality in papers [5, 6]. Some generalized fractional integral inequalities of midpoint-type and trapezoid-type for the case of twice differentiable convex functions are obtained in paper [19]. In [23], authors obtained some new inequalities of the Simpson and the Hermite–Hadamard-type for functions whose absolute values of derivatives are convex. Some recent results connected with fractional integral inequalities, see [7, 10] and the references cited therein.

With the aid of the continuing studies and mentioned papers above, we will establish some midpoint-type inequalities for differentiable convex function involving conformable fractional integrals. The entire form of the study takes the form of four sections including introduction. In Section 2, the basic definitions of Riemann-Liouville integrals and conformable integrals will be explained for building our main results. In Section 3, an identity will be established for the case of differentiable convex functions including the conformable fractional integrals. By using this equality, we prove some midpoint-type inequalities by convex functions with the aid of the conformable fractional integrals. Moreover, we present example using graph in order to show that our main results are correct. Furthermore, we also give some corollaries and remarks.

2. PRELIMINARIES

For building our main results, this section discusses the fundamental definitions of Riemann-Liouville integrals and conformable integrals, which are well known in the literature, are given as follows:

Definition 1. For $0 < x, y < \infty$ and $x, y \in \mathbb{R}$, the gamma function, beta function, and incomplete beta function are described

$$\Gamma(x) := \int_0^\infty \mu^{x-1} e^{-\mu} d\mu, \qquad \mathcal{B}(x,y) := \int_0^1 \mu^{x-1} (1-\mu)^{y-1} d\mu,$$

and

$$\mathcal{B}(x,y,r) := \int_0^r \mu^{x-1} (1-\mu)^{y-1} d\mu,$$

respectively.

In paper [17], Kilbas et al. defined fractional integrals, also called Riemann-Liouville integrals as follows:

Definition 2 ([17, Page 71])**.** For $F \in L_1[\sigma, \delta]$, the Riemann-Liouville integrals $J^{\beta}_{\sigma+}F(x)$ and $J^{\beta}_{\delta-}F(x)$ of order $\beta > 0$ are given by

$$J^{\beta}_{\sigma+}F(x) = \frac{1}{\Gamma(\beta)} \int_\sigma^x (x-\mu)^{\beta-1} F(\mu) d\mu, \qquad x > \sigma \tag{2.1}$$

and

$$J^{\beta}_{\delta-}F(x) = \frac{1}{\Gamma(\beta)} \int_x^\delta (\mu-x)^{\beta-1} F(\mu) d\mu, \qquad x < \delta, \tag{2.2}$$

respectively. The Riemann-Liouville integrals reduce to the classical integrals for the case of $\beta = 1$.

In paper [15], Jarad et al. presented the fractional conformable integral operators. They also provided certain characteristics and relationships between these operators and several other fractional operators in the literature. The fractional conformable integral operators are defined as follows:

Definition 3 ([15, Page 5])**.** For $F \in L_1[\sigma, \delta]$, the fractional conformable integral operators ${}^{\beta}\mathcal{J}^{\alpha}_{\sigma+}F(x)$ and ${}^{\beta}\mathcal{J}^{\alpha}_{\delta-}F(x)$ of order $\beta \in C$, $Re(\beta) > 0$ and $\alpha \in (0,1]$ are given by

$${}^{\beta}\mathcal{J}^{\alpha}_{\sigma+}F(x) = \frac{1}{\Gamma(\beta)} \int_\sigma^x \left(\frac{(x-\sigma)^{\alpha} - (\mu-\sigma)^{\alpha}}{\alpha} \right)^{\beta-1} \frac{F(\mu)}{(\mu-\sigma)^{1-\alpha}} d\mu, \qquad \mu > \sigma \tag{2.3}$$

and

$${}^{\beta}\mathcal{J}^{\alpha}_{\delta-}F(x) = \frac{1}{\Gamma(\beta)} \int_x^\delta \left(\frac{(\delta-x)^{\alpha} - (\delta-\mu)^{\alpha}}{\alpha} \right)^{\beta-1} \frac{F(\mu)}{(\delta-\mu)^{1-\alpha}} d\mu, \qquad \mu < \delta, \tag{2.4}$$

respectively.

If we consider $\sigma = 0$ and $\alpha = 1$, then the fractional integral in (2.3) reduces to the Riemann-Liouville fractional integral in (2.1). Furthermore, the fractional integral in (2.4) coincides with the Riemann-Liouville fractional integral in (2.2) when $\delta = 0$ and $\alpha = 1$. Some recent results connected with fractional integral inequalities, see [2, 14] and the references cited therein.

3. Main Results

In this section, midpoint-type inequalities are created for differentiable convex functions by using the conformable fractional integrals. Let us first set up the following identity for obtaining midpoint-type inequalities.

Lemma 1. *Note that* $F\colon [\sigma,\delta] \to \mathbb{R}$ *is a differentiable function on* (σ,δ) *such that* $F' \in L_1[\sigma,\delta]$. *Then, the following equality holds:*

$$F\left(\frac{\sigma+\delta}{2}\right) - \frac{\alpha^{\beta}\Gamma(\beta+1)}{2(\delta-\sigma)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}^{\alpha}_{\delta-}F(\sigma) + {}^{\beta}\mathcal{J}^{\alpha}_{\sigma+}F(\delta)\right] \tag{3.1}$$
$$= \frac{(\delta-\sigma)\alpha^{\beta}}{2}\left\{A_1 - A_2 - A_3 + A_4\right\}.$$

Here,

$$\begin{cases} A_1 = \int_0^{\frac{1}{2}} \left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta} F'(\mu\delta + (1-\mu)\sigma)\,d\mu, \\ A_2 = \int_0^{\frac{1}{2}} \left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta} F'(\mu\sigma + (1-\mu)\delta)\,d\mu, \\ A_3 = \int_{\frac{1}{2}}^{1} \left[\frac{1}{\alpha^{\beta}} - \left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right] F'(\mu\delta + (1-\mu)\sigma)\,d\mu, \\ A_4 = \int_{\frac{1}{2}}^{1} \left[\frac{1}{\alpha^{\beta}} - \left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right] F'(\mu\sigma + (1-\mu)\delta)\,d\mu. \end{cases}$$

Proof. From the fact of the integrating by parts, we obtain

$$A_1 = \int_0^{\frac{1}{2}} \left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta} F'(\mu\delta + (1-\mu)\sigma)\,d\mu \tag{3.2}$$
$$= \frac{1}{\delta-\sigma}\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta} F(\mu\delta + (1-\mu)\sigma)\Big|_0^{\frac{1}{2}}$$
$$- \frac{\beta}{\delta-\sigma}\int_0^{\frac{1}{2}} \left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta-1} (1-\mu)^{\alpha-1} F(\mu\delta + (1-\mu)\sigma)\,d\mu$$

$$= \frac{1}{\delta-\sigma}\left(\frac{1-\left(\frac{1}{2}\right)^{\alpha}}{\alpha}\right)^{\beta} \mathrm{F}\left(\frac{\sigma+\delta}{2}\right)$$
$$- \frac{\beta}{\delta-\sigma}\int_0^{\frac{1}{2}}\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta-1}(1-\mu)^{\alpha-1}\mathrm{F}(\mu\delta+(1-\mu)\sigma)\,\mathrm{d}\mu.$$

Similar to foregoing process, we have

$$A_2 = -\frac{1}{\delta-\sigma}\left(\frac{1-\left(\frac{1}{2}\right)^{\alpha}}{\alpha}\right)^{\beta} \mathrm{F}\left(\frac{\sigma+\delta}{2}\right)$$
$$+ \frac{\beta}{\delta-\sigma}\int_0^{\frac{1}{2}}\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta-1}(1-\mu)^{\alpha-1}\mathrm{F}(\mu\sigma+(1-\mu)\delta)\,\mathrm{d}\mu,$$

$$A_3 = \frac{1}{\delta-\sigma}\left[\left(\frac{1-\left(\frac{1}{2}\right)^{\alpha}}{\alpha}\right)^{\beta} - \frac{1}{\alpha^{\beta}}\right]\mathrm{F}\left(\frac{\sigma+\delta}{2}\right)$$
$$+ \frac{\beta}{\delta-\sigma}\int_{\frac{1}{2}}^1\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta-1}(1-\mu)^{\alpha-1}\mathrm{F}(\mu\delta+(1-\mu)\sigma)\,\mathrm{d}\mu,$$

and

$$A_4 = \frac{1}{\delta-\sigma}\left[\frac{1}{\alpha^{\beta}} - \left(\frac{1-\left(\frac{1}{2}\right)^{\alpha}}{\alpha}\right)^{\beta}\right]\mathrm{F}\left(\frac{\sigma+\delta}{2}\right) \tag{3.3}$$
$$- \frac{\beta}{\delta-\sigma}\int_{\frac{1}{2}}^1\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta-1}(1-\mu)^{\alpha-1}\mathrm{F}(\mu\sigma+(1-\mu)\delta)\,\mathrm{d}\mu.$$

If we collect from (3.2) to (3.3), then we get

$$A_1 - A_2 - A_3 + A_4 = \frac{2}{(\delta-\sigma)\,\alpha^{\beta}}\mathrm{F}\left(\frac{\sigma+\delta}{2}\right) \tag{3.4}$$
$$- \frac{\beta}{\delta-\sigma}\int_0^1\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta-1}(1-\mu)^{\alpha-1}\mathrm{F}(\mu\delta+(1-\mu)\sigma)\,\mathrm{d}\mu$$
$$- \frac{\beta}{\delta-\sigma}\int_0^1\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta-1}(1-\mu)^{\alpha-1}\mathrm{F}(\mu\sigma+(1-\mu)\delta)\,\mathrm{d}\mu.$$

If we use change of variables in (3.4), then the equality (3.4) is converted as follows:

$$\begin{aligned}
&A_1 - A_2 - A_3 + A_4 \\
&= \frac{2}{(\delta-\sigma)\alpha^{\beta}} \mathrm{F}\left(\frac{\sigma+\delta}{2}\right) \\
&\quad - \left(\frac{1}{\delta-\sigma}\right)^{\alpha\beta+1} \frac{\Gamma(\beta+1)}{\Gamma(\beta)} \int_{\sigma}^{\delta} \left(\frac{(\delta-\sigma)^{\alpha} - (\delta-x)^{\alpha}}{\alpha}\right)^{\beta-1} \frac{\mathrm{F}(x)}{(\delta-x)^{1-\alpha}} dx \\
&\quad - \left(\frac{1}{\delta-\sigma}\right)^{\alpha\beta+1} \frac{\Gamma(\beta+1)}{\Gamma(\beta)} \int_{\sigma}^{\delta} \left(\frac{(\delta-\sigma)^{\alpha} - (x-\sigma)^{\alpha}}{\alpha}\right)^{\beta-1} \frac{\mathrm{F}(x)}{(x-\sigma)^{1-\alpha}} dx \\
&= \frac{2}{(\delta-\sigma)\alpha^{\beta}} \mathrm{F}\left(\frac{\sigma+\delta}{2}\right) - \frac{\Gamma(\beta+1)}{(\delta-\sigma)^{\alpha\beta+1}} \left[{}^{\beta}\mathcal{J}^{\alpha}_{\delta-}\mathrm{F}(\sigma) + {}^{\beta}\mathcal{J}^{\alpha}_{\sigma+}\mathrm{F}(\delta)\right].
\end{aligned} \tag{3.5}$$

If the equality (3.5) is multiplied by $\frac{(\delta-\sigma)\alpha^{\beta}}{2}$, then the proof of Lemma 1 is completed. □

Theorem 1. *Let us note that* $\mathrm{F}\colon [\sigma,\delta] \to \mathbb{R}$ *is a differentiable function on* (σ,δ) *such that* $\mathrm{F}' \in L_1[\sigma,\delta]$ *and* $|\mathrm{F}'|$ *is convex on* $[\sigma,\delta]$. *Then, the following inequality holds:*

$$\begin{aligned}
&\left|\mathrm{F}\left(\frac{\sigma+\delta}{2}\right) - \frac{\alpha^{\beta}\Gamma(\beta+1)}{2(\delta-\sigma)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}^{\alpha}_{\delta-}\mathrm{F}(\sigma) + {}^{\beta}\mathcal{J}^{\alpha}_{\sigma+}\mathrm{F}(\delta)\right]\right| \\
&\leq \frac{(\delta-\sigma)\alpha^{\beta}}{2}\left(\varphi_1(\alpha,\beta) + \varphi_2(\alpha,\beta)\right)\left(\left|\mathrm{F}'(\sigma)\right| + \left|\mathrm{F}'(\delta)\right|\right).
\end{aligned} \tag{3.6}$$

Here,

$$\begin{cases}
\varphi_1(\alpha,\beta) = \int_0^{\frac{1}{2}} \left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta} \mathrm{d}\mu \\
= \frac{1}{\alpha^{\beta+1}}\left[\mathcal{B}\left(\frac{1}{\alpha},\beta+1\right) - \mathscr{B}\left(\frac{1}{\alpha},\beta+1,\left(\frac{1}{2}\right)^{\alpha}\right)\right], \\
\varphi_2(\alpha,\beta) = \int_{\frac{1}{2}}^{1}\left[\frac{1}{\alpha^{\beta}} - \left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right]\mathrm{d}\mu = \frac{1}{2\alpha^{\beta}} - \frac{1}{\alpha^{\beta+1}}\mathscr{B}\left(\frac{1}{\alpha},\beta+1,\left(\frac{1}{2}\right)^{\alpha}\right),
\end{cases} \tag{3.7}$$

where $\mathcal{B}$ *and* $\mathscr{B}$ *denote the beta function and incomplete beta function, respectively.*

Proof. If we take the absolute value of both sides of (3.1), then we obtain

$$\left|\mathrm{F}\left(\frac{\sigma+\delta}{2}\right) - \frac{\alpha^{\beta}\Gamma(\beta+1)}{2(\delta-\sigma)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}^{\alpha}_{\delta-}\mathrm{F}(\sigma) + {}^{\beta}\mathcal{J}^{\alpha}_{\sigma+}\mathrm{F}(\delta)\right]\right| \tag{3.8}$$

$$\leq \frac{(\delta-\sigma)\alpha^{\beta}}{2}\left\{\int_0^{\frac{1}{2}}\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\left|F'(\mu\delta+(1-\mu)\sigma)\right|d\mu\right.$$
$$+\int_0^{\frac{1}{2}}\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\left|F'(\mu\sigma+(1-\mu)\delta)\right|d\mu$$
$$+\int_{\frac{1}{2}}^{1}\left[\frac{1}{\alpha^{\beta}}-\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right]\left|F'(\mu\delta+(1-\mu)\sigma)\right|d\mu$$
$$\left.+\int_{\frac{1}{2}}^{1}\left[\frac{1}{\alpha^{\beta}}-\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right]\left|F'(\mu\sigma+(1-\mu)\delta)\right|d\mu\right\}.$$

It is known that $|F'|$ is convex on $[\sigma,\delta]$. It follows

$$\left|F\left(\frac{\sigma+\delta}{2}\right)-\frac{\alpha^{\beta}\Gamma(\beta+1)}{2(\delta-\sigma)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}^{\alpha}_{\delta-}F(\sigma)+{}^{\beta}\mathcal{J}^{\alpha}_{\sigma+}F(\delta)\right]\right|$$
$$\leq \frac{(\delta-\sigma)\alpha^{\beta}}{2}$$
$$\times\left\{\int_0^{\frac{1}{2}}\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right.$$
$$\times\left(\mu\left|F'(\delta)\right|+(1-\mu)\left|F'(\sigma)\right|+\mu\left|F'(\sigma)\right|+(1-\mu)\left|F'(\delta)\right|\right)d\mu$$
$$+\int_{\frac{1}{2}}^{1}\left[\frac{1}{\alpha^{\beta}}-\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right]$$
$$\left.\times\left(\mu\left|F'(\delta)\right|+(1-\mu)\left|F'(\sigma)\right|+\mu\left|F'(\sigma)\right|+(1-\mu)\left|F'(\delta)\right|\right)d\mu\right\}$$
$$=\frac{(\delta-\sigma)\alpha^{\beta}}{2}\left(\int_0^{\frac{1}{2}}\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}d\mu+\int_{\frac{1}{2}}^{1}\left[\frac{1}{\alpha^{\beta}}-\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right]d\mu\right)$$
$$\times\left(\left|F'(\sigma)\right|+\left|F'(\delta)\right|\right).$$

This completes the proof of Theorem 1. □

Example 1. Let $F\colon[0,1]\to\mathbb{R}$ be defined by $F(x)=x^2$. From Theorem 1 with $\beta\in(0,10)$ and $\alpha\in(0,1)$, the left side of (3.6) becomes

$$\left|F\left(\frac{\sigma+\delta}{2}\right)-\frac{\alpha^{\beta}\Gamma(\beta+1)}{2(\delta-\sigma)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}^{\alpha}_{\delta-}F(\sigma)+{}^{\beta}\mathcal{J}^{\alpha}_{\sigma+}F(\delta)\right]\right| \tag{3.9}$$
$$=\left|\beta\left[\mathcal{B}\left(\frac{1}{\alpha}+1,\beta\right)-\mathcal{B}\left(\frac{2}{\alpha}+1,\beta\right)\right]-\frac{1}{4}\right|$$

and the right side of (3.6) becomes

$$\frac{(\delta-\sigma)\alpha^{\beta}}{2}\left(\varphi_1(\alpha,\beta)+\varphi_2(\alpha,\beta)\right)\left(\left|F'(\sigma)\right|+\left|F'(\delta)\right|\right) \tag{3.10}$$
$$=\frac{1}{2}+\frac{1}{\alpha}\left[\mathcal{B}\left(\frac{1}{\alpha},\beta+1\right)-2\mathscr{B}\left(\frac{1}{\alpha},\beta+1,\left(\frac{1}{2}\right)^{\alpha}\right)\right].$$

After calculating (3.9) and (3.10), the graph is shown as in Figure 1 using the MATLAB software.

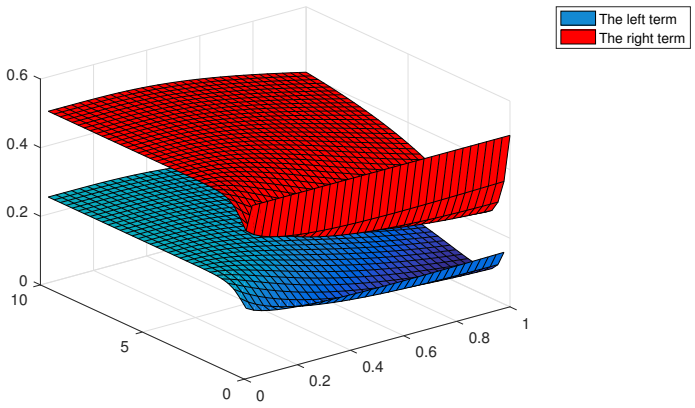

FIGURE 1. Plot illustration for Theorem 1.

Hence, we can see that the inequality (3.6) is valid in Figure 1.

Remark 1. If we set $\alpha=1$ in Theorem 1, then Theorem 1 reduces to [11, Theorem 2.1].

Remark 2. For $\alpha=1$ and $\beta=1$ in Theorem 1, Theorem 1 becomes to [18, Theorem 2.2].

Theorem 2. *Note that* $F:[\sigma,\delta]\to\mathbb{R}$ *is a differentiable function on* (σ,δ) *so that* $F'\in L_1[\sigma,\delta]$. *Note also that* $|F'|^q$ *is convex on* $[\sigma,\delta]$ *with* $q>1$. *Then, the following*

inequalities hold:

$$\begin{aligned}
&\left|\mathrm{F}\left(\frac{\sigma+\delta}{2}\right)-\frac{\alpha^{\beta}\Gamma(\beta+1)}{2(\delta-\sigma)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}^{\alpha}_{\delta-}\mathrm{F}(\sigma)+{}^{\beta}\mathcal{J}^{\alpha}_{\sigma+}\mathrm{F}(\delta)\right]\right| \\
&\leq \frac{(\delta-\sigma)\alpha^{\beta}}{2^{1+\frac{1}{q}}}\left(\left(\psi_1^{\alpha,\beta}(p)\right)^{\frac{1}{p}}+\left(\psi_2^{\alpha,\beta}(p)\right)^{\frac{1}{p}}\right) \\
&\quad\times\left[\left(\frac{|\mathrm{F}'(\delta)|^q+3|\mathrm{F}'(\sigma)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{|\mathrm{F}'(\sigma)|^q+3|\mathrm{F}'(\delta)|^q}{4}\right)^{\frac{1}{q}}\right] \\
&\leq \frac{(\delta-\sigma)\alpha^{\beta}}{2^{1+\frac{1}{q}}}\left(\left(4\psi_1^{\alpha,\beta}(p)\right)^{\frac{1}{p}}+\left(4\psi_2^{\alpha,\beta}(p)\right)^{\frac{1}{p}}\right)\left[\left|\mathrm{F}'(\sigma)\right|+\left|\mathrm{F}'(\delta)\right|\right].
\end{aligned}$$

Here, $\frac{1}{p}+\frac{1}{q}=1$ *and*

$$\begin{cases}
\psi_1^{\alpha,\beta}(p)=\int\limits_0^{\frac{1}{2}}\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta p}\mathrm{d}\mu, \\
\psi_2^{\alpha,\beta}(p)=\int\limits_{\frac{1}{2}}^{1}\left[\frac{1}{\alpha^{\beta}}-\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right]^{p}\mathrm{d}\mu.
\end{cases}$$

Proof. If we use Hölder inequality in (3.8), then we have

$$\begin{aligned}
&\left|\mathrm{F}\left(\frac{\sigma+\delta}{2}\right)-\frac{\alpha^{\beta}\Gamma(\beta+1)}{2(\delta-\sigma)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}^{\alpha}_{\delta-}\mathrm{F}(\sigma)+{}^{\beta}\mathcal{J}^{\alpha}_{\sigma+}\mathrm{F}(\delta)\right]\right| \\
&\leq \frac{(\delta-\sigma)\alpha^{\beta}}{2} \\
&\quad\times\left\{\left(\int_0^{\frac{1}{2}}\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta p}\mathrm{d}\mu\right)^{\frac{1}{p}}\right. \\
&\quad\times\left[\left(\int_0^{\frac{1}{2}}\left|\mathrm{F}'(\mu\delta+(1-\mu)\sigma)\right|^q\mathrm{d}\mu\right)^{\frac{1}{q}}+\left(\int_0^{\frac{1}{2}}\left|\mathrm{F}'(\mu\sigma+(1-\mu)\delta)\right|^q\mathrm{d}\mu\right)^{\frac{1}{q}}\right] \\
&\quad\left.+\left(\int_{\frac{1}{2}}^{1}\left[\frac{1}{\alpha^{\beta}}-\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right]^{p}\mathrm{d}\mu\right)^{\frac{1}{p}}\right.
\end{aligned}$$

$$\times \left[\left(\int_{\frac{1}{2}}^{1} \left| F'\left(\mu\delta + (1-\mu)\sigma\right)\right|^q d\mu \right)^{\frac{1}{q}} + \left(\int_{\frac{1}{2}}^{1} \left| F'\left(\mu\sigma + (1-\mu)\delta\right)\right|^q d\mu \right)^{\frac{1}{q}} \right] \Bigg\}.$$

From the fact that $|F'|^q$ is convex on $[\sigma,\delta]$, then it yields

$$\begin{aligned}
&\left| F\left(\frac{\sigma+\delta}{2}\right) - \frac{\alpha^\beta \Gamma(\beta+1)}{2(\delta-\sigma)^{\alpha\beta}} \left[{}^{\beta}\mathcal{J}^{\alpha}_{\delta-} F(\sigma) + {}^{\beta}\mathcal{J}^{\alpha}_{\sigma+} F(\delta) \right] \right| \\
&\leq \frac{(\delta-\sigma)\alpha^\beta}{2} \Bigg\{ \left(\int_0^{\frac{1}{2}} \left(\frac{1-(1-\mu)^\alpha}{\alpha} \right)^{\beta p} d\mu \right)^{\frac{1}{p}} \\
&\quad \times \left[\left(\int_0^{\frac{1}{2}} \left(\mu \left| F'(\delta)\right|^q + (1-\mu)\left| F'(\sigma)\right|^q \right) d\mu \right)^{\frac{1}{q}} \right. \\
&\qquad \left. + \left(\int_0^{\frac{1}{2}} \left(\mu \left| F'(\sigma)\right|^q + (1-\mu)\left| F'(\delta)\right|^q \right) d\mu \right)^{\frac{1}{q}} \right] \\
&\quad + \left(\int_{\frac{1}{2}}^{1} \left[\frac{1}{\alpha^\beta} - \left(\frac{1-(1-\mu)^\alpha}{\alpha} \right)^{\beta} \right]^p d\mu \right)^{\frac{1}{p}} \\
&\quad \times \left[\left(\int_{\frac{1}{2}}^{1} \left(\mu \left| F'(\delta)\right|^q + (1-\mu)\left| F'(\sigma)\right|^q \right) d\mu \right)^{\frac{1}{q}} \right. \\
&\qquad \left. + \left(\int_{\frac{1}{2}}^{1} \left(\mu \left| F'(\sigma)\right|^q + (1-\mu)\left| F'(\delta)\right|^q \right) d\mu \right)^{\frac{1}{q}} \right] \Bigg\} \\
&= \frac{(\delta-\sigma)\alpha^\beta}{2^{1+\frac{1}{q}}} \left(\left(\psi_1^{\alpha,\beta}(p) \right)^{\frac{1}{p}} + \left(\psi_2^{\alpha,\beta}(p) \right)^{\frac{1}{p}} \right) \\
&\quad \times \left[\left(\frac{\left| F'(\delta)\right|^q + 3\left| F'(\sigma)\right|^q}{4} \right)^{\frac{1}{q}} + \left(\frac{\left| F'(\sigma)\right|^q + 3\left| F'(\delta)\right|^q}{4} \right)^{\frac{1}{q}} \right].
\end{aligned}$$

The second inequality of Theorem 2 can be obtained immediately by letting $\varpi_1 = 3\left|F'(\sigma)\right|^q$, $\rho_1 = \left|F'(\delta)\right|^q$, $\varpi_2 = \left|F'(\sigma)\right|^q$ and $\rho_2 = 3\left|F'(\delta)\right|^q$ and applying the inequality:

$$\sum_{k=1}^{n}(\varpi_k+\rho_k)^s \le \sum_{k=1}^{n}\varpi_k^s + \sum_{k=1}^{n}\rho_k^s, \quad 0 \le s < 1.$$

Thus, the proof of Theorem 2 is finished. □

Corollary 1. *If we set* $\alpha = 1$ *in Theorem 2, then we derive*

$$\begin{aligned}
&\left|F\left(\frac{\sigma+\delta}{2}\right) - \frac{\Gamma(\beta+1)}{2(\delta-\sigma)^{\beta}}\left[J^{\beta}_{\delta-}F(\sigma) + J^{\beta}_{\sigma+}F(\delta)\right]\right| \\
&\le \frac{\delta-\sigma}{2^{1+\frac{1}{q}}} \\
&\quad\times\left(\left(\frac{1}{(\beta p+1)2^{\beta p+1}}\right)^{\frac{1}{p}} + \left(\frac{1}{\beta}\left[\mathcal{B}\left(\frac{1}{\beta},p+1\right) - \mathscr{B}\left(\frac{1}{\beta},p+1,\left(\frac{1}{2}\right)^{\beta}\right)\right]\right)^{\frac{1}{p}}\right) \\
&\quad\times\left[\left(\frac{\left|F'(\delta)\right|^q + 3\left|F'(\sigma)\right|^q}{4}\right)^{\frac{1}{q}} + \left(\frac{\left|F'(\sigma)\right|^q + 3\left|F'(\delta)\right|^q}{4}\right)^{\frac{1}{q}}\right] \\
&\le \frac{\delta-\sigma}{2^{1+\frac{1}{q}}}\left(\left(\frac{4}{(\beta p+1)2^{\beta p+1}}\right)^{\frac{1}{p}}\right. \\
&\quad\left.+\left(\frac{4}{\beta}\left[\mathcal{B}\left(\frac{1}{\beta},p+1\right) - \mathscr{B}\left(\frac{1}{\beta},p+1,\left(\frac{1}{2}\right)^{\beta}\right)\right]\right)^{\frac{1}{p}}\right)\left[\left|F'(\sigma)\right| + \left|F'(\delta)\right|\right].
\end{aligned}$$

Remark 3. If we take $\alpha = 1$ and $\beta = 1$ in Theorem 2, we have

$$\begin{aligned}
&\left|F\left(\frac{\sigma+\delta}{2}\right) - \frac{1}{\delta-\sigma}\int_{\sigma}^{\delta}F(x)\,dx\right| \\
&\le \frac{\delta-\sigma}{4}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\left(\frac{\left|F'(\delta)\right|^q + 3\left|F'(\sigma)\right|^q}{4}\right)^{\frac{1}{q}} + \left(\frac{\left|F'(\sigma)\right|^q + 3\left|F'(\delta)\right|^q}{4}\right)^{\frac{1}{q}}\right] \\
&\le \frac{\delta-\sigma}{4}\left(\frac{4}{p+1}\right)^{\frac{1}{p}}\left[\left|F'(\sigma)\right| + \left|F'(\delta)\right|\right],
\end{aligned}$$

which is given in [18, Theorem 2.4].

Theorem 3. *Suppose that* $F\colon [\sigma,\delta] \to \mathbb{R}$ *is a differentiable function on* (σ,δ) *such that* $F' \in L_1([\sigma,\delta])$. *Suppose also that* $\left|F'\right|^q$ *is convex on* $[\sigma,\delta]$ *with* $q \ge 1$. *Then, the*

following inequality holds:

$$\begin{aligned}
&\left|F\left(\frac{\sigma+\delta}{2}\right)-\frac{2^{\alpha\beta-1}\alpha^{\beta}\Gamma(\beta+1)}{(\delta-\sigma)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}^{\alpha}_{\delta-}F\left(\frac{\sigma+\delta}{2}\right)+{}^{\beta}\mathcal{J}^{\alpha}_{\sigma+}F\left(\frac{\sigma+\delta}{2}\right)\right]\right| \\
&\leq \frac{(\delta-\sigma)\alpha^{\beta}}{2} \\
&\quad\times\left\{(\varphi_1(\alpha,\beta))^{1-\frac{1}{q}}\right. \\
&\qquad\times\left[\left(\varphi_3(\alpha,\beta)\left|F'(\delta)\right|^q+(\varphi_1(\alpha,\beta)-\varphi_3(\alpha,\beta))\left|F'(\sigma)\right|^q\right)^{\frac{1}{q}}\right. \\
&\qquad\quad\left.+\left(\varphi_3(\alpha,\beta)\left|F'(\sigma)\right|^q+(\varphi_1(\alpha,\beta)-\varphi_3(\alpha,\beta))\left|F'(\delta)\right|^q\right)^{\frac{1}{q}}\right] \\
&\qquad+(\varphi_2(\alpha,\beta))^{1-\frac{1}{q}} \\
&\qquad\times\left[\left(\varphi_4(\alpha,\beta)\left|F'(\delta)\right|^q+(\varphi_2(\alpha,\beta)-\varphi_4(\alpha,\beta))\left|F'(\sigma)\right|^q\right)^{\frac{1}{q}}\right. \\
&\qquad\quad\left.\left.+\left(\varphi_4(\alpha,\beta)\left|F'(\sigma)\right|^q+(\varphi_2(\alpha,\beta)-\varphi_4(\alpha,\beta))\left|F'(\delta)\right|^q\right)^{\frac{1}{q}}\right]\right\}.
\end{aligned}$$

Here, $\varphi_1(\alpha,\beta)$, $\varphi_2(\alpha,\beta)$ *are defined in* (3.7) *and*

$$\begin{cases}
\varphi_3(\alpha,\beta)=\int_0^{\frac{1}{2}}\mu\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}d\mu=\frac{1}{\alpha^{\beta+1}}\left[\mathcal{B}\left(\frac{1}{\alpha},\beta+1\right)-\mathscr{B}\left(\frac{1}{\alpha},\beta+1,\left(\frac{1}{2}\right)^{\alpha}\right)\right. \\
\left.-\mathcal{B}\left(\frac{2}{\alpha},\beta+1\right)+\mathscr{B}\left(\frac{2}{\alpha},\beta+1,\left(\frac{1}{2}\right)^{\alpha}\right)\right], \\
\varphi_4(\alpha,\beta)=\int_{\frac{1}{2}}^{1}\mu\left[\frac{1}{\alpha^{\beta}}-\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right]d\mu \\
=\frac{3}{8\alpha^{\beta}}+\frac{1}{\alpha^{\beta+1}}\left[\mathscr{B}\left(\frac{2}{\alpha},\beta+1,\left(\frac{1}{2}\right)^{\alpha}\right)-\mathscr{B}\left(\frac{1}{\alpha},\beta+1,\left(\frac{1}{2}\right)^{\alpha}\right)\right],
\end{cases}$$

where $\mathcal{B}$ *and* $\mathscr{B}$ *denote the beta function and incomplete beta function, respectively.*

Proof. Let us apply power-mean inequality in (3.8). Then, we have

$$\begin{aligned}
&\left|F\left(\frac{\sigma+\delta}{2}\right)-\frac{\alpha^{\beta}\Gamma(\beta+1)}{2(\delta-\sigma)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}^{\alpha}_{\delta-}F(\sigma)+{}^{\beta}\mathcal{J}^{\alpha}_{\sigma+}F(\delta)\right]\right| \\
&\leq \frac{(\delta-\sigma)\alpha^{\beta}}{2} \\
&\quad\times\left\{\left(\int_0^{\frac{1}{2}}\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}d\mu\right)^{1-\frac{1}{q}}\right.
\end{aligned}$$

$$\times\left[\left(\int_0^{\frac{1}{2}}\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\left|F'(\mu\delta+(1-\mu)\sigma)\right|^{q}d\mu\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_0^{\frac{1}{2}}\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\left|F'(\mu\sigma+(1-\mu)\delta)\right|^{q}d\mu\right)^{\frac{1}{q}}\right]$$
$$+\left(\int_{\frac{1}{2}}^{1}\left[\frac{1}{\alpha^{\beta}}-\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right]d\mu\right)^{1-\frac{1}{q}}$$
$$\times\left[\left(\int_{\frac{1}{2}}^{1}\left[\frac{1}{\alpha^{\beta}}-\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right]\left|F'(\mu\delta+(1-\mu)\sigma)\right|^{q}d\mu\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_{\frac{1}{2}}^{1}\left[\frac{1}{\alpha^{\beta}}-\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right]\left|F'(\mu\sigma+(1-\mu)\delta)\right|^{q}d\mu\right)^{\frac{1}{q}}\right]\Bigg\}.$$

Since $|F'|^{q}$ is convex on $[\sigma,\delta]$, we obtain

$$\left|F\left(\frac{\sigma+\delta}{2}\right)-\frac{\alpha^{\beta}\Gamma(\beta+1)}{2(\delta-\sigma)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}^{\alpha}_{\delta-}F(\sigma)+{}^{\beta}\mathcal{J}^{\alpha}_{\sigma+}F(\delta)\right]\right|$$
$$\leq\frac{(\delta-\sigma)\alpha^{\beta}}{2}$$
$$\times\left\{\left(\int_0^{\frac{1}{2}}\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}d\mu\right)^{1-\frac{1}{q}}\right.$$
$$\times\left[\left(\int_0^{\frac{1}{2}}\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\left(\mu\left|F'(\delta)\right|^{q}+(1-\mu)\left|F'(\sigma)\right|^{q}\right)d\mu\right)^{\frac{1}{q}}\right.$$
$$\left.\left.+\left(\int_0^{\frac{1}{2}}\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\left(\mu\left|F'(\sigma)\right|^{q}+(1-\mu)\left|F'(\delta)\right|^{q}\right)d\mu\right)^{\frac{1}{q}}\right]\right.$$

$$+\left(\int_{\frac{1}{2}}^{1}\left[\frac{1}{\alpha^{\beta}}-\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right]d\mu\right)^{1-\frac{1}{q}}$$
$$\times\left[\left(\int_{\frac{1}{2}}^{1}\left[\frac{1}{\alpha^{\beta}}-\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right]\left(\mu\left|F'(\delta)\right|^{q}+(1-\mu)\left|F'(\sigma)\right|^{q}\right)d\mu\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_{\frac{1}{2}}^{1}\left[\frac{1}{\alpha^{\beta}}-\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}\right]\left(\mu\left|F'(\sigma)\right|^{q}+(1-\mu)\left|F'(\delta)\right|^{q}\right)d\mu\right)^{\frac{1}{q}}\right]\Bigg\},$$

which completes the proof of Theorem 3. □

Corollary 2. *If we set* $\alpha=1$ *in Theorem 3, then we acquire*

$$\left|F\left(\frac{\sigma+\delta}{2}\right)-\frac{2^{\beta-1}\Gamma(\beta+1)}{(\delta-\sigma)^{\beta}}\left[J^{\beta}_{\delta-}F(\sigma)+J^{\beta}_{\sigma+}F(\delta)\right]\right|$$
$$\leq\frac{\delta-\sigma}{2}$$
$$\times\left\{\left(\frac{1}{(\beta+1)2^{\beta+1}}\right)\right.$$
$$\times\left[\left(\frac{(\beta+1)\left|F'(\delta)\right|^{q}}{2(\beta+2)}+\frac{(\beta+3)\left|F'(\sigma)\right|^{q}}{2(\beta+2)}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{(\beta+1)\left|F'(\sigma)\right|^{q}}{2(\beta+2)}+\frac{(\beta+3)\left|F'(\delta)\right|^{q}}{2(\beta+2)}\right)^{\frac{1}{q}}\right]+\left(\frac{2^{\beta}(\beta-1)+1}{(\beta+1)2^{\beta+1}}\right)^{1-\frac{1}{q}}$$
$$\times\left[\left(\frac{2^{\beta-1}(3\beta-2)+1}{(\beta+2)2^{\beta+2}}\left|F'(\delta)\right|^{q}+\frac{2^{\beta-1}\left(\beta^{2}+3\beta-6\right)+\beta+3}{(\beta+1)(\beta+2)2^{\beta+2}}\left|F'(\sigma)\right|^{q}\right)^{\frac{1}{q}}\right.$$
$$+\left(\frac{2^{\beta-1}(3\beta-2)+1}{(\beta+2)2^{\beta+2}}\left|F'(\sigma)\right|^{q}\right.$$
$$\left.\left.\left.+\frac{2^{\beta-1}\left(\beta^{2}+3\beta-6\right)+\beta+3}{(\beta+1)(\beta+2)2^{\beta+2}}\left|F'(\delta)\right|^{q}\right)^{\frac{1}{q}}\right]\right\}.$$

Remark 4. If we take $\alpha = 1$ and $\beta = 1$ in Theorem 3, then we obtain

$$\left| \mathrm{F}\left(\frac{\sigma+\delta}{2}\right) - \frac{1}{\delta-\sigma}\int_{\sigma}^{\delta} \mathrm{F}(x)\,dx \right| \leq \frac{\delta-\sigma}{8}\left[\left(\frac{|\mathrm{F}'(\delta)|^q + 2|\mathrm{F}'(\sigma)|^q}{3}\right)^{\frac{1}{q}} + \left(\frac{|\mathrm{F}'(\sigma)|^q + 2|\mathrm{F}'(\delta)|^q}{3}\right)^{\frac{1}{q}}\right],$$

which is given in [13].

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ESTIMATES ON SCHIPPERS' HIGHER-ORDER SCHWARZIAN DERIVATIVES OF A SUBCLASS OF UNIVALENT ANALYTIC FUNCTIONS

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Abstract. In the theory of univalent analytic functions, the properties and characteristics of convex functions and starlike functions are widely studied, such as the estimates on Schippers' higher-order Schwarzian derivatives at zero. In this paper, we consider the bounds for Schippers' higher-order Schwarzian derivatives of $f(z)$ at $z=0$ when the function f belongs to the class of a kind of combination of convex functions and starlike functions, which consists of analytic functions f in the open unit disk $\mathbb{D}$ with the normalized conditions given by

$$f(0)=f'(0)-1=0$$

and satisfying the following inequality:

$$\Re\left(\frac{1}{2}\frac{zf'(z)}{f(z)}+\frac{1}{2}\left(1+\frac{zf''(z)}{f'(z)}\right)\right)>0 \qquad (z\in\mathbb{D}).$$

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Keywords: analytic functions, univalent functions, starlike functions, convex functions, higher-order Schwarzian derivatives, estimates and bounds, Janowski function class

1. Introduction and the main result

We denote by $\mathbb{D}$ the open unit disk given by

$$\mathbb{D}=\{z\colon z\in\mathbb{C} \quad \text{and} \quad |z|<1\}.$$

Let $\mathcal{A}$ be the class of all analytic functions f in $\mathbb{D}$ with

$$f(0)=f'(0)-1=0.$$

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Hence, $f \in \mathcal{A}$ can be expanded in the following form (see, for details, [1] and [4])

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots . \tag{1.1}$$

The subclass of $\mathcal{A}$ consisting of univalent functions is denoted by $\mathcal{S}$.

The Schwarzian derivative of a locally univalent function f is defined by

$$S_f = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2 .$$

It is well known that S_f plays an important role in the study of univalent functions. Nehari [16] proved that, if $f \in \mathcal{S}$, then

$$\|S_f\| = \sup_{z\in\mathbb{D}} |S_f(z)|(1-|z|^2)^2 \leq 6.$$

For some subclasses of $\mathcal{S}$, the upper bounds for $\|S_f\|$ have been widely studied (see, for example, [8] and [17]).

Owing to the importance of the Schwarzian derivatives, several different higher-order Schwarzian derivatives are given (see, for details, [5, 6, 17, 20]). Among these different higher-order Schwarzian derivatives, the following kind defined by Schippers was also widely studied.

Let f be locally univalent analytic in $\mathbb{D}$. Schippers [17] introduced a higher-order Schwarzian derivative $\sigma_n(f)$ as follows:

$$\sigma_{n+1}(f) = \sigma_n'(f) - (n-1)\sigma_n(f)\frac{f''}{f'} \qquad (n \geqq 3),$$

where $\sigma_3(f) = S_f$. The norm $\|\sigma_{n+1}(f)\|$ of $\sigma_{n+1}(f)$ is defined as follows:

$$\|\sigma_{n+1}(f)\| = \sup_{z\in\mathbb{D}} |\sigma_{n+1}(f)(z)|(1-|z|^2)^n \qquad (n \geqq 2).$$

The sharp upper bound for $\|\sigma_n(f)\|$ of $f \in \mathcal{S}$ was given by Schippers [17]. Analogously, for subclasses of $\mathcal{S}$, the upper bounds for $|\sigma_n(f)(0)|$ $(n = 3,4,5)$ have also been investigated. For instance, Dorff and Szynal [3] derived the bounds of $\sigma_n(f)(0)$ for convex functions. Cho *et al.* [2] investigated the bounds of $\sigma_n(f)(0)$ $(n = 3,4,5)$ in general forms of these classes consisting of the Janowski classes satisfying the following subordination conditions (see also [2] and [19]):

$$\left\{ f\colon f \in \mathcal{S} \quad \text{and} \quad \frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz} \right\}$$

and

$$\left\{ f\colon f \in \mathcal{S} \quad \text{and} \quad 1 + \frac{zf''(z)}{f'(z)} \prec \frac{1+Az}{1+Bz} \right\},$$

where $-1 \leq B < A \leq 1$, which generalized the results in [3]. Recently, Kumar et al. [10] studied the bounds on the first three consecutive higher-order Schwarzian

derivatives for the class given by

$$\left\{f\colon f\in\mathcal{S} \quad \text{and} \quad \frac{zf'(z)}{f(z)} \prec e^{e^z-1}\right\}.$$

More recently, for some subclasses of close-to-convex functions, the upper bounds for $|\sigma_4(f)(0)|$ were considered by Hu et al. (see [7]).

In order to introduce the aims of this paper, we first recall the definitions of starlike and convex functions as follows. A function $f\in\mathcal{S}$ is said to be starlike if

$$\Re\left(\frac{zf'(z)}{f(z)}\right)>0 \qquad (z\in\mathbb{D}). \tag{1.2}$$

A function $f\in\mathcal{S}$ is said to be convex if

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right)>0 \qquad (z\in\mathbb{D}). \tag{1.3}$$

Let $\mathcal{S}^*$ and $\mathcal{C}$ be the classes of starlike functions and convex functions, respectively. A well-known generalization of these classes is the class $\mathcal{M}_\alpha$ of α-convex functions, which was given by Mocanu [15]. It is shown in Mocanu [15] that $\mathcal{M}_\alpha\subset\mathcal{S}^*$ for all real α. Moreover, if $\alpha\geqq 1$, then $\mathcal{M}_\alpha\subset\mathcal{C}$ (see, for details, [15]). A great deal of attention has been given to problems concerning functions in the class $\mathcal{M}_\alpha$, particularly relating to the coefficient problems (see [9, 13, 14, 18, 21]).

Let $f\in\mathcal{A}$ be given by (1.1) and let $f\in\mathcal{M}_{\frac{1}{2}}$ if

$$\Re\left(\frac{1}{2}\frac{zf'(z)}{f(z)}+\frac{1}{2}\left(1+\frac{zf''(z)}{f'(z)}\right)\right)>0 \qquad (z\in\mathbb{D}).$$

We see from [15] that $\mathcal{M}_{\frac{1}{2}}\subset S^*$. Motivated by the above-cited papers [2, 3, 7, 10, 17], the following question arises naturally.

Question. *Can we give the upper bounds on* $|\sigma_3(f)(0)|$ *and* $|\sigma_4(f)(0)|$ *if* $f\in\mathcal{M}_{\frac{1}{2}}$*?*

In this paper, we investigate the bounds for $|\sigma_3(f)(0)|$ and $|\sigma_4(f)(0)|$ of analytic functions f in $\mathbb{D}$ at zero if $f\in\mathcal{M}_{\frac{1}{2}}$. We will also give a positive answer to the above question.

We now state our result as follows.

Theorem 1. *If* $f\in\mathcal{M}_{\frac{1}{2}}$, *then*

$$|\sigma_3(f)(0)|\leq 3 \tag{1.4}$$

and

$$|\sigma_4(f)(0)|\leq\frac{32}{5}\sqrt{\frac{18}{11}}. \tag{1.5}$$

The inequality (1.4) *is sharp.*

The paper is organized as follows. In Section 2, we give the preliminaries and the proof of Theorem 1. In the concluding section (Section 3), we present our concluding remarks and observations.

2. Proof of Theorem 1

In order to prove Theorem 1, we need each of the following lemmas.

Lemma 1 ([4]). *Let $\mathcal{P}$ be the class of analytic functions p in $\mathbb{D}$ with positive real part and given by*

$$p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots . \tag{2.1}$$

If $p(z) \in \mathcal{P}$, then the following sharp inequality holds true:

$$|c_k| \leq 2 \qquad (k \geqq 1).$$

Lemma 2 ([11, 12]). *If $p \in \mathcal{P}$ with $c_1 > 0$, then*

$$2c_2 = c_1^2 + \zeta(4 - c_1^2)$$

and

$$4c_3 = c_1^3 + 2c_1(4 - c_1^2)\zeta - c_1(4 - c_1^2)\zeta^2 + 2(4 - c_1^2)(1 - |\zeta|^2)\eta$$

for some ζ and η such that

$$\max\{|\zeta|, |\eta|\} \leq 1.$$

We next prove our main result (Theorem 1 above).

Proof of Theorem 1. Let the function $f \in \mathcal{M}_{\frac{1}{2}}$ be of the form (1.1). Then, by applying (2.1), we can write

$$\frac{1}{2}\frac{zf'(z)}{f(z)} + \frac{1}{2}\left(1 + \frac{zf''(z)}{f'(z)}\right) = p(z), \tag{2.2}$$

where $p \in \mathcal{P}$. Equating the coefficients in (2.2), we have

$$a_2 = \frac{2c_1}{3}, \tag{2.3}$$

$$a_3 = \frac{10c_1^2 + 9c_2}{36} \tag{2.4}$$

and

$$a_4 = \frac{33c_1^3 + \frac{189}{2}c_1c_2 + 54c_3}{405}. \tag{2.5}$$

We first estimate the upper bound for $|\sigma_3(f)(0)|$ if $f \in \mathcal{M}_{\frac{1}{2}}$. Since $\mathcal{M}_\alpha$ and $\sigma_3(f)(0)$ are rotationally invariant, by Lemma 1, we assume that $c_1 \in [0, 2]$. Moreover, by Lemma 2, and the equations (2.3) and (2.4), we have

$$\sigma_3(f)(0) = 6(a_3 - a_2^2) = -c_1^2 + \frac{3}{2}c_2 = \frac{1}{4}\left[-c_1^2 + 3\left(4 - c_1^2\right)\zeta\right], \tag{2.6}$$

where $|\zeta| \le 1$.

Let $x = |\zeta|$. Then, by (2.6), we have

$$|\sigma_3(f)(0)| \le \frac{1}{4}\left[-c_1^2 + 3(4-c_1^2)\right] \le 3 \tag{2.7}$$

for $f \in \mathcal{M}_{\frac{1}{2}}$. The equality in (2.7) holds true for

$$c_1 = 0, \quad c_2 = 2 \quad \text{and} \quad \zeta = 1.$$

Therefore, the equality in (2.7) holds true for

$$\frac{1}{2}\frac{zf'(z)}{f(z)} + \frac{1}{2}\left(1 + \frac{zf''(z)}{f'(z)}\right) = \frac{1+z^2}{1-z^2}.$$

We next estimate the upper bound for $|\sigma_4(f)(0)|$ if $f \in \mathcal{M}_{\frac{1}{2}}$. Indeed, noting that $\mathcal{M}_{\frac{1}{2}}$ and $\sigma_4(f)(0)$ are rotationally invariant, by Lemma 1, we assume that

$$c_1 =: t \in [0,2].$$

Furthermore, by Lemma 2, and the equations (2.3), (2.4) and (2.5), we have

$$\begin{aligned}\sigma_4(f)(0) &= 24(a_4 - 3a_3a_2 + 2a_2^3)\\ &= \frac{64}{45}\left(2c_1^3 - \frac{9}{2}c_1c_2 + \frac{9}{4}c_3\right)\\ &= \frac{16}{45}\left(\frac{5}{4}c_1^3 - \frac{9}{2}(4-c_1^2)c_1\zeta - \frac{9}{4}(4-c_1^2)c_1\zeta^2 + \frac{9}{2}(4-c_1^2)(1-|\zeta|^2)\eta\right),\end{aligned} \tag{2.8}$$

where

$$\max\{|\zeta|, |\eta|\} \le 1.$$

Let $x = |\zeta|$. Then, by (2.8), we have

$$\begin{aligned}|\sigma_4(f)(0)| &\le \frac{16}{45}\left(\frac{5}{4}c_1^3 + \frac{9}{2}(4-c_1^2)c_1x + \frac{9}{4}(4-c_1^2)c_1x^2 + \frac{9}{2}(4-c_1^2)(1-x^2)\right)\\ &= \frac{16}{45}T(t,x),\end{aligned}$$

where

$$(t,x) \in \Delta := [0,2] \times [0,1]$$

and

$$T(t,x) = \frac{5}{4}t^3 + \frac{9}{2}(4-t^2)tx + \frac{9}{4}(4-t^2)tx^2 + \frac{9}{2}(4-t^2)(1-x^2). \tag{2.9}$$

In order to complete the proof of Theorem 1, we consider five cases as follows.

Case 1. On the vertices of Δ, we have

$$T(0,0) = 18, \quad T(0,1) = 0 \quad \text{and} \quad T(2,0) = T(2,1) = 10.$$

Case 2. On the side $x=0$, the function $T(t,x)$ becomes

$$T(t,0)=\frac{5}{4}t^3+\frac{9}{2}(4-t^2),$$

which, upon differentiating with respect to t, yields

$$T'(t,0)=\frac{15}{4}t^2-9t,$$

so that $t_0=\frac{12}{5}>2$ is the root of $T'(t,0)=0$. Consequently, we have

$$T'(t,0)<0 \qquad \big(t\in(0,2)\big).$$

It follows that

$$T(t,0)\leq T(0,0)=18.$$

Case 3. On the side $x=1$, the function $T(t,x)$ becomes

$$T(t,1)=\frac{5}{4}t^3+\frac{9}{2}(4-t^2)t+\frac{9}{4}(4-t^2)t.$$

Therefore, we have

$$T'(t,1)=\frac{-33}{2}t^2+27.$$

A short calculation shows that the root of $T'(t,1)=0$ is given by

$$t_0=\sqrt{\frac{18}{11}}<2.$$

It follows that

$$T(t,1)\leq T(t_0,1)=-\frac{11}{2}t_0^3+27t_0=18\sqrt{\frac{18}{11}}.$$

Case 4. It is easy to show that

$$T(2,x)=10 \qquad \text{and} \qquad T(0,x)\leq 18.$$

Case 5. It now remains to consider the interior of Δ. We first solve the following equations:

$$T_t(t,x)=\frac{15}{4}t^2-2t\left(\frac{9}{2}tx+\frac{9}{4}\left[(t-2)x^2+2\right]\right)+(4-t^2)\left(\frac{9}{2}x+\frac{9}{4}x^2\right)=0 \quad (2.10)$$

and

$$T_x(t,x)=\frac{9}{2}(4-t^2)(t+tx-2x)=0. \quad (2.11)$$

Secondly, we will prove that that the equations (2.10) and (2.11) have no critical points in the interior of Δ. We thus find from (2.11) that

$$x=\frac{t}{2-t}. \quad (2.12)$$

So, putting (2.12) into (2.10), we get

$$\rho(t) = \frac{15}{4}t^2 - 2t\left(\frac{9}{2}t\frac{t}{2-t} + \frac{9}{4}\left[(t-2)\left(\frac{t}{2-t}\right)^2 + 2\right]\right) + (4-t^2)\left(\frac{9}{2}\frac{t}{2-t} + \frac{9}{4}\left(\frac{t}{2-t}\right)^2\right). \tag{2.13}$$

Calculations show that $\rho(t) = 0$ has no root in $t \in (0,2)$. Consequently, the equations (2.10) and (2.11) have no critical points in the interior of Δ.

Upon summarizing all of cases which we have considered above, we have the following inequality:

$$|\sigma_4(f)(0)| \leq \frac{32}{5}\sqrt{\frac{18}{11}}.$$

The proof of Theorem 1 is thus completed. □

3. Concluding remarks and observations

The present investigation is motivated essentially by the fact that, in the theory of analytic and univalent functions, the properties and characteristics of various subclasses of such analytic functions as (for example) convex functions, starlike functions and close-to-convex functions are widely- and extensively-studied. Specifically, there is a great deal of literature on the bounds for the Taylor-Maclaurin coefficients and the estimates on Schippers' higher-order Schwarzian derivatives at zero. Here, in this article, we have successfully derived the bounds for the Schippers' higher-order Schwarzian derivatives of $f(z)$ at $z = 0$ when the function f belongs to the class of a kind of combination of convex functions and starlike functions, which consists of analytic functions f in the open unit disk $\mathbb{D}$ with the normalized conditions given by

$$f(0) = f'(0) - 1 = 0$$

and satisfying the following inequality:

$$\Re\left(\frac{1}{2}\frac{zf'(z)}{f(z)} + \frac{1}{2}\left(1 + \frac{zf''(z)}{f'(z)}\right)\right) > 0 \qquad (z \in \mathbb{D}).$$

We have also discussed relevant connections of the results, which we have presented here, with those available in earlier works on the subject.

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NEW QUANTUM BOUNDARIES FOR q-TRAPEZOIDAL AND q-MIDPOINT TYPES INEQUALITIES FOR (η_1,η_2)-CONVEX FUNCTIONS

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Abstract. Quantum calculus, also known as q-calculus, extends classical calculus with a deformation parameter q. The aim of this paper is to present new integral identities of q-trapezoidal and q-midpoint types for first-order q-differentiable functions. These identities are fundamental for establishing novel q-trapezoidal and q-midpoint type integral inequalities that can be applied to functions characterized by their first quantum derivatives and their absolute values being (η_1,η_2)-convex functions. To achieve this goal, the paper employs various mathematical tools, including the q-power mean inequality and q-Hölder's inequality. These tools are essential for deriving and demonstrating the proposed inequalities, which play a crucial role in understanding and characterizing the behavior of functions within the framework of quantum calculus.

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1. INTRODUCTION

Calculus, a branch of mathematics, plays a crucial role in the study of derivatives and integrals. However, the concept of classical calculus has been expanded and modified to create a new calculus called quantum calculus. Quantum calculus, also known as q-calculus, encompasses the study of calculus without limitations. It includes q-fractional calculus, q-integral calculus, and q-transform analysis, which have proven to be valuable tools in various fields of mathematics and physics. The development of quantum calculus can be attributed to early scholars like Euler and Jacobi. However, in recent decades, significant advancements and progress have been made, as evidenced in the works of Jackson and others [12]. Jackson was among the pioneers who defined the q-analogues of derivative and integral operators and explored their applications. Quantum integral inequalities, a vital aspect of this calculus, have been shown to be more valuable and informative than their classical counterparts.

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The study of quantum calculus has led to the proposal and detailed discussion of new generalizations of classical concepts. For instance, Tariboon and Ntouyas [28] introduced quantum calculus ideas and derived various q-analogues of conventional mathematical objects on finite intervals. This inspiration has resulted in numerous innovative findings published in scientific literature, where quantum analogs of classical mathematical achievements have been explored. Overall, the emergence and development of quantum calculus have broadened the scope of calculus, providing powerful tools and concepts to tackle complex problems in mathematics and physics, for more details see in [4, 7, 13–16, 18] and the references cited therein.

Inequalities play a crucial role in the mathematical analysis of dynamic systems, control theory, and optimization problems. Sun et al. [21,30,32] employed inequalities in the sliding mode control of fuzzy Markov jump systems, ensuring stability and robustness in control mechanisms. Gong et al. [8] used inequalities to study boundedness and compactness of Cauchy-type integrals in weighted Morrey spaces, providing insights into integral operator behavior. Additionally, Bohner et al. [3] explored oscillation criteria for dynamic equations, applying inequalities to establish sufficient conditions for oscillatory or non-oscillatory behavior in nonlinear systems. Zhao et al. [42] and Zhang et al. [36] extended this to boundary value problems, demonstrating how inequalities can determine solvability and oscillation in fractional and neutral delay dynamic equations. Across these studies, inequalities are used as fundamental tools to analyze the behavior of complex systems, ensuring control, stability, and efficiency. Qiu et al. [19, 29] extend this analysis to neutral delay dynamic and fractional differential equations, offering solutions to boundary value problems and highlighting the impact of damping and perturbations. Zhang et al. [24, 37] address sliding mode control in the presence of stochastic disturbances and cyber-attacks, reinforcing the relevance of robust control strategies. Together, these works illustrate the deep interplay between inequalities, oscillation theory, and control methods in modern mathematical and applied research. The selected references explore various aspects of supply chain management, focusing on fuzzy control and robust strategies under uncertain conditions, which are areas where quantum calculus can offer valuable insights. Zhang et al. work covers models that address supply chain disruptions [41] and lead-time variations [38,39] by utilizing fuzzy and robust control approaches to ensure stable supply chain operations. These studies are particularly relevant to quantum calculus due to its capability to model non-linear dynamics and discrete-time phenomena in complex systems. Quantum calculus can contribute to refining these models by offering tools to analyze small-scale variations in system parameters, essential for precise control in dynamic and uncertain supply chains, as demonstrated in their discrete switched model for dynamic networks [40]. Integrating quantum calculus into these frameworks could enhance decision-making accuracy, potentially improving stability and resilience in supply chain networks [20, 23].

In recent times, the classical notion of convexity has undergone extensions and generalizations. Hanson, as cited in [10], introduced the concept of invexity, which stands as a significant generalization of convexity. Mohan and Weir, as mentioned in [27,34,35], have made significant contributions by establishing a novel class of functions called preinvexity. This introduction has opened up new avenues for exploring convexity from a different perspective. The research on invexity has considerably heightened its importance in optimization, as documented in [6, 17]. This concept serves as a further generalization of the traditional notion of convexity, allowing for more comprehensive and versatile mathematical frameworks. The development and study of these extended notions of convexity, such as invexity and preinvexity, have enhanced our understanding of mathematical inequalities and their practical applications in diverse fields, including optimization and scientific modeling. These advancements provide valuable tools for addressing complex problems and exploring new realms of mathematical analysis. Moreover, these studies highlight the importance of symmetry, integrability, and resonance in analyzing dynamical systems, employing advanced methods like center manifold analysis and quantum calculus for discrete modeling [22, 25, 26]. This collection of research contributes to a structured framework for understanding complex dynamical behaviors, with significant implications for mathematical physics and applied mathematics.

Suppose that function $\mathcal{H}\colon J \subset \mathbb{R} \to \mathbb{R}$ be convex on $[\delta_1, \delta_2]$, where $\delta_1, \delta_2 \in J$ with $\delta_1 < \delta_2$.

The inequality expressed in Equation (1.1) is defined as follows,

$$\mathcal{H}\left(\frac{\delta_1+\delta_2}{2}\right) \leq \frac{1}{\delta_2-\delta_1}\int_{\delta_1}^{\delta_2} \mathcal{H}(\delta)\,\mathrm{d}\delta \leq \frac{\mathcal{H}(\delta_1)+\mathcal{H}(\delta_2)}{2}. \tag{1.1}$$

This inequality, denoted as Equation (1.1), was initially introduced by C. Hermite [11] and further explored by J. Hadamard [9] in 1893. It is worth noting that both of these inequalities are valid in the opposite direction if the function $\mathcal{H}$ is concave. The significance of the Hermit–Hadamard inequality cannot be overstated, as it plays a crucial role in various domains of modern mathematics, including numerical analysis, functional analysis, and mathematical analysis. Its validity and implications have been extensively studied and documented in the literature. Researchers and scholars in these fields consider this inequality to be of great importance due to its wide-ranging applications and its ability to provide valuable insights into the properties of functions and related mathematical phenomena.

2. Preliminaries

Mond introduced the notion of invex sets in his paper

Definition 1 ([27, Definition 2.1])**.** An invex set is a generalization of a convex set. Formally, a set $S \subseteq \mathbb{R}^n$ is said to be invex with respect to a vector-valued function $\eta\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ if for every pair of points $\delta_1, \delta_2 \in S$, the following condition holds,

$$\delta_1 + \tau\eta(\delta_2, \delta_1) \in S \qquad \forall \tau \in [0,1].$$

This definition is a generalization of convexity, where the function η governs the direction of the combination of points. In the case where $\eta(\delta_2, \delta_1) = \delta_2 - \delta_1$, the invex set reduces to the usual convex set.

Weir has been introduced the definition of preinvex mapping

Definition 2 ([34, Definition 2.1])**.** A real-valued function $\mathcal{H}\colon S \to \mathbb{R}$, where $S \subseteq \mathbb{R}^n$ is an invex set, is called preinvex with respect to a function $\eta\colon S \times S \to \mathbb{R}^n$ if for all $\delta_1, \delta_2 \in S$ and $\tau \in [0,1]$, the following inequality holds,

$$\mathcal{H}(\delta_1 + \tau\eta(\delta_2, \delta_1)) \leq (1-\tau)\mathcal{H}(\delta_1) + \tau\mathcal{H}(\delta_2).$$

This is a generalization of the convexity property. If $\eta(\delta_2, \delta_1) = \delta_2 - \delta_1$, then the preinvex function reduces to a convex function. The function η determines how the combination of points δ_1 and δ_2 is taken, and the inequality ensures that the function values respect this invex combination. In this paper, we employ continuous bifunctions denoted as $\eta(.,.)$ which operate on pairs of real vectors from $\mathbb{R}^n$ and produce vectors in $\mathbb{R}^n$.

Definition 3 ([5, Definition 1.2])**.** Let $I \subseteq \mathbb{R}$ be an invex set with respect to $\eta_1 : I \times I \to \mathbb{R}$. Consider $\mathcal{H}\colon I \to \mathbb{R}$ and $\eta_2\colon \mathcal{H}(I) \times \mathcal{H}(I) \to \mathbb{R}$. The function $\mathcal{H}$ is said to be (η_1, η_2)-convax function

$$\mathcal{H}(\delta_2 + \tau\eta(\delta_1, \delta_2)) \leq \mathcal{H}(\delta_2) + \tau\eta(\mathcal{H}(\delta_1), \mathcal{H}(\delta_2))$$

for all $\delta_1, \delta_2 \in I$, $\tau \in [0,1]$.

2.1. *q-Derivative and q-Integral*

In this part, we provide a detailed presentation of the necessary definitions for q-Calculus, along with significant quantum integral inequalities for Hermite-Hadamard on both the left and right sides of the intervals.

Throughout this entire paper, we utilize the constants $0 < q < 1$.

The integers denoted as $[n]_q$ are referred to as q-integers and can be expressed as follows,

$$[n]_q = 1 + q + q^2 + \cdots + q^n - 1 = \frac{1-q^n}{1-q} \qquad \text{for } n = 1, 2, \ldots;$$
$$[n]_q = n \qquad \text{for } n = 1.$$

The symbols $[n]_q!$ and $\begin{bmatrix} n \\ i \end{bmatrix}_q !$ represent the q-factorial and q-series, respectively, and are expressed as follows,

$$[n]_q! = \prod_{i=1}^{n} [i]_q, \quad n \geq 1, \quad [0]_q! = 1,$$

$$\left[\begin{array}{c} n \\ i \end{array} \right]_q ! = \frac{[n]_q !}{[n-i]_q ! \, [i]_q !}.$$

Reverend Frank Hilton Jackson emerged as a prominent figure during the early twentieth century, leaving a lasting impact on the advancement of the classical concept of function derivatives at specific points. His invaluable contributions played a pivotal role in simplifying the exploration of ordinary calculus and number theory, making these subjects more accessible to scholars. Jackson's exceptional accomplishments encompassed groundbreaking studies, including the one documented in [12], as well as the formulation of q-analogues that extended the applicability of key principles in these domains.

To this end, Jackson introduced the notion of the q-derivative, symbolized as $D_q \mathcal{H}(\varpi)$. This derivative is defined through the following expression

$$D_q \mathcal{H}(\varpi) = \frac{\mathcal{H}(\varpi) - \mathcal{H}(q\varpi)}{(1-q)\varpi}, \qquad \varpi \neq 0. \tag{2.1}$$

Jackson's pioneering work led to the development of a classic approach, which we shall now outline.

$$\int_0^{\delta_2} \mathcal{H}(\varpi) \, \mathrm{d}_q \varpi = (1-q)\delta_2 \sum_{n=0}^{\infty} q^n \mathcal{H}(\delta_2 q^n),$$

subject to the condition that the corresponding series converges absolutely.

The q-Jackson integral, defined within the interval $[\delta_1, \delta_2]$, is expressed as follows,

$$\int_{\delta_1}^{\delta_2} \mathcal{H}(\varpi) \, \mathrm{d}_q \varpi = \int_0^{\delta_2} \mathcal{H}(\varpi) \, \mathrm{d}_q \varpi - \int_0^{\delta_1} \mathcal{H}(\varpi) \, \mathrm{d}_q \varpi.$$

As the value of q approaches 1^-, the outcomes in number theory, deduction, and ordinary integration can be represented by polynomial expressions in a real variable q.

Definition 4 ([28, Definition 2.1])**.** We consider an arbitrary function

$$\mathcal{H}: \ [\delta_1, \delta_2] \to \mathbb{R}.$$

Then q_{δ_1}-derivative of $\mathcal{H}$ at $\varpi \in [\delta_1, \delta_2]$ is defined as follows,

$${}_{\delta_1} D_q \mathcal{H}(\varpi) = \frac{\mathcal{H}(\varpi) - \mathcal{H}(q\varpi + (1-q)\delta_1)}{(1-q)(\varpi - \delta_1)}, \qquad \varpi \neq \delta_1. \tag{2.2}$$

As arbitrary function $\mathcal{H}$ from $[\delta_1, \delta_2]$ to $\mathbb{R}$, so for $\varpi = \delta_1$, we define ${}_{\delta_1} D_q \mathcal{H}(\delta_1) = \lim_{\varpi \to \delta_1} {}_{\delta_1} D_q \mathcal{H}(\varpi)$ if it exists and is finite.

Remark 1. An important observation to make is that when $\delta_1 = 0$ in equation (2.2), we arrive at a similar form of the q-derivative as defined in equation (2.1).

The subsequent lemma assumes a pivotal role in the computation of q_{δ_1}-derivatives.

Lemma 1 ([28, Lemma 2.1]). *Taking* $\xi \in \mathbb{R}$, *we have*

$$ {}_{\delta_1}D_q (x-\delta_1)^{\xi} = \left(\frac{1-q^{\xi}}{1-q} \right) (x-\delta_1)^{\xi-1}. $$

Definition 5 ([28, Definition 2.3]). We consider an arbitrary function

$$ \mathcal{H}\colon [\delta_1, \delta_2] \to \mathbb{R}, $$

then the q_{δ_1}-definite integral on $\varpi \in [\delta_1, \delta_2]$ is described as follows,

$$ \int_{\delta_1}^{\varpi} \mathcal{H}(\varpi)\ {}_{\delta_1}\mathrm{d}_q\varpi = (1-q)(\varpi-\delta_1) \sum_{n=0}^{\infty} q^n \mathcal{H}\left(q^n\varpi + (1-q^n)\delta_1\right), \qquad \varpi \in [\delta_1, \delta_2]. \tag{2.3} $$

If $\delta_1 = 0$, then (2.3) becomes

$$ \int_{0}^{\varpi} \mathcal{H}(\varpi)\ {}_{0}\mathrm{d}_q\varpi = (1-q)\varpi \sum_{n=0}^{\infty} q^n \mathcal{H}(q^n\varpi), \qquad \varpi \in [\delta_1, \delta_2]. $$

The following properties are of great importance in the field of quantum calculus

Theorem 1 ([28, Theorem 2.1]). *Let* $\mathcal{H}\colon I \to \mathbb{R}$ *be a continuous function. Then,*

(1) ${}_{\delta_1}D_q \int_{\delta_1}^{x} \mathcal{H}(\tau)\ {}_{\delta_1}\mathrm{d}_q\tau = \mathcal{H}(x)$;

(2) $\int_{\chi}^{x} {}_{\delta_1}D_q\mathcal{H}(\tau)\ {}_{\delta_1}\mathrm{d}_q\tau = \mathcal{H}(x) - \mathcal{H}(\chi)$, $\chi \in (\delta_1, x)$.

The following results are valuable when it comes to evaluating q_{δ_1}-integrals

Lemma 2 ([28, Lemma 2.2]). *Suppose that* $\zeta \in \mathbb{R}\setminus\{-1\}$, *then*

$$ \int_{\delta_1}^{\delta} (\tau-\delta_1)^{\zeta}\ {}_{\delta_1}\mathrm{d}_q\tau = \left(\frac{1-q}{1-q^{\zeta+1}} \right) (\delta-\delta_1)^{\zeta+1}. $$

Alp et al. introduced and established the q_{δ_1}-Hermite-Hadamard inequalities for convexity, which can be defined as follows,

Theorem 2 ([1, Theorem 6]). *Suppose that differentiable function* $\mathcal{H}\colon [\delta_1, \delta_2] \to \mathbb{R}$ *be convex on the interval* $[\delta_1, \delta_2]$. *In this case, the* q_{δ_1}*-Hermite-Hadamard inequalities can be stated as follows,*

$$ \mathcal{H}\left(\frac{q\delta_1 + \delta_2}{[2]_q} \right) \le \frac{1}{\delta_2 - \delta_1} \int_{\delta_1}^{\delta_2} \mathcal{H}(\varpi)\ {}_{\delta_1}\mathrm{d}_q\varpi \le \frac{q\mathcal{H}(\delta_1) + \mathcal{H}(\delta_2)}{[2]_q}. $$

In contrast, Bermudo et al. presented a novel characterization of q^{δ_2}-derivatives, q^{δ_2}-integrals, and the associated q^{δ_2}-Hermite-Hadamard inequalities in their study [2].

Definition 6 ([2, Definition 4]). Let $\mathcal{H}:\ [\delta_1,\delta_2]\to\mathbb{R}$ be an arbitrary function. The q^{δ_2}-derivative of $\mathcal{H}$ at $\varpi\in[\delta_1,\delta_2]$ is defined as follows,

$$ {}^{\delta_2}D_q\mathcal{H}(\varpi)=\frac{\mathcal{H}(q\varpi+(1-q)\delta_2)-\mathcal{H}(\varpi)}{(1-q)(\delta_2-\varpi)},\qquad \varpi\neq\delta_2. $$

Taking $\delta_2=1$ then we get

$$ {}^{1}D_q\mathcal{H}(\varpi)=\frac{\mathcal{H}(q\varpi+(1-q))-\mathcal{H}(\varpi)}{(1-q)(1-\varpi)},\qquad \varpi\neq 1. $$

As $\mathcal{H}$ is an arbitrary function from $[\delta_1,\delta_2]$ to $\mathbb{R}$, so for $\varpi=\delta_2$, we define

$$ {}^{\delta_2}D_q\mathcal{H}(\delta_2)=\lim_{\varpi\to\delta_2}{}^{\delta_2}D_q\mathcal{H}(\varpi) $$

if it exists and is finite.

Definition 7 ([2, Definition 6]). Let $\mathcal{H}:\ [\delta_1,\delta_2]\to\mathbb{R}$ be an arbitrary function. The q^{δ_2}-definite integral on $[\delta_1,\delta_2]$ is defined as follows,

$$ \int_{\varpi}^{\delta_2}\mathcal{H}(\varpi)\ {}^{\delta_2}\mathrm{d}_q\varpi=(1-q)(\delta_2-\varpi)\sum_{n=0}^{\infty}q^n\mathcal{H}(q^n\varpi+(1-q^n)\delta_2),\qquad \varpi\in[\delta_1,\delta_2]. \tag{2.4} $$

If $\delta_2=1$, then (2.4) becomes

$$ \int_{\varpi}^{1}\mathcal{H}(\varpi)\ {}^{1}\mathrm{d}_q\varpi=(1-q)(1-\varpi)\sum_{n=0}^{\infty}q^n\mathcal{H}(q^n\varpi+(1-q^n)),\qquad \varpi\in[\delta_1,\delta_2]. $$

Theorem 3 ([2, Theorem 12]). *Let $\mathcal{H}:\ [\delta_1,\delta_2]\to\mathbb{R}$ be a convex functions on $[\delta_1,\delta_2]$. In this case, the q^{δ_2}-Hermite-Hadamard inequalities can be stated as follows,*

$$ \mathcal{H}\left(\frac{\delta_1+q\delta_2}{[2]_q}\right)\leq\frac{1}{\delta_2-\delta_1}\int_{\delta_1}^{\delta_2}\mathcal{H}(\varpi)\ {}^{\delta_2}\mathrm{d}_q\varpi\leq\frac{\mathcal{H}(\delta_1)+q\mathcal{H}(\delta_2)}{[2]_q}. $$

Based on Theorem 2 and Theorem 3, we can derive the following inequalities

Lemma 3 ([31, Lemma 1]). *Suppose we have two continuous functions, $\mathcal{H}$ and G, defined on the interval $[\delta_1,\delta_2]$ and whose values are real numbers. Under these conditions, the following equality is valid,*

$$ \int_0^c G(\varpi)\ {}^{\delta_2}D_q\mathcal{H}(\varpi\delta_1+(1-\varpi)\delta_2)\,\mathrm{d}_q\varpi $$

$$= \frac{1}{\delta_2-\delta_1}\int_0^c D_q\, G(\varpi)\mathcal{H}\left(q\varpi\delta_1+(1-q\varpi)\delta_2\right)\mathrm{d}_q\varpi - \left.\frac{G(\varpi)\mathcal{H}\left(\varpi\delta_1+(1-\varpi)\delta_2\right)}{\delta_2-\delta_1}\right|_0^c.$$

Lemma 4 ([33, Lemma 2.1])**.** *If we examine continuous functions $\mathcal{H}$ and G that are defined on the interval $[\delta_1,\delta_2]$ and whose outputs are real numbers, we can conclude that the following equality holds true,*

$$\begin{aligned}&\int_0^c G(\varpi)\ {}_{\delta_1}D_q\mathcal{H}\left(\varpi\delta_2+(1-\varpi)\delta_1\right)\mathrm{d}_q\varpi\\ &= \left.\frac{G(\varpi)\mathcal{H}\left(\varpi\delta_2+(1-\varpi)\delta_1\right)}{\delta_2-\delta_1}\right|_0^c - \frac{1}{\delta_2-\delta_1}\int_0^c D_qG(\varpi)\mathcal{H}\left(q\varpi\delta_2+(1-q\varpi)\delta_1\right)\mathrm{d}_q\varpi.\end{aligned}$$

3. q-TRAPEZOIDAL INEQUALITIES

Lemma 5. *Suppose that $I\subset\mathbb{R}$ is an invex set with respect to η_1. If for $v,\delta\in I$ with $\eta_1>0$ If $\mathcal{H}\colon [v,v+\eta_1(\delta,v)],[\delta,\delta+\eta_1(v,\delta)]\subset\mathbb{R}\to\mathbb{R}$ is q-differentiable such that ${}_vD_q\mathcal{H}$ and ${}^{\delta}D_q\mathcal{H}$ are integrable and continuous on $[v,v+\eta_1(\delta,v)],[\delta,\delta+\eta_1(v,\delta)]$, then*

$$\begin{aligned}\Phi(v,\delta) =:& \frac{\mathcal{H}(v+\eta_1(\delta,v))+\mathcal{H}(\delta+\eta_1(v,\delta))}{2}\\ &-\frac{1}{2\eta_1(\delta,v)}\left[\int_v^{v+\eta_1(\delta,v)}\mathcal{H}(x)\ {}_v\mathrm{d}_qx+\int_{\delta+\eta_1(v,\delta)}^{\delta}\mathcal{H}(x)\ {}^{\delta}\mathrm{d}_qx\right]\\ &=\frac{\eta_1(\delta,v)}{2}\left[\int_0^1 q\tau\ {}_vD_q\mathcal{H}(v+\tau\eta_1(\delta,v))\ \mathrm{d}_q\tau\right.\\ &\qquad\left.+\int_0^1 q\tau\ {}^{\delta}D_q\mathcal{H}(\delta+\tau\eta_1(v,\delta))\ \mathrm{d}_q\tau\right].\end{aligned}\tag{3.1}$$

Proof. According to Lemma 4, we perform the computation

$$\begin{aligned}I_1 &= \int_0^1 q\tau\ {}_vD_q\mathcal{H}(v+\tau\eta_1(\delta,v))\ \mathrm{d}_q\tau\\ &= q\tau\frac{\mathcal{H}(v+\tau\eta_1(\delta,v))}{\eta_1(\delta,v)}\Big|_0^1-\frac{q}{\eta_1(\delta,v)}\int_0^1\mathcal{H}(v+\tau\eta_1(\delta,v))\ \mathrm{d}_q\tau\\ &= q\frac{\mathcal{H}(v+\eta_1(\delta,v))}{\eta_1(\delta,v)}-\frac{q}{\eta_1(\delta,v)}\\ &\quad\times\left\{\frac{1-q}{q}\sum_{n=0}^{\infty}q^n\mathcal{H}\left(v+\tau\eta_1q^n(\delta,v)\right)-\frac{(1-q)}{q}\mathcal{H}(v+\eta_1(\delta,v))\right\}\end{aligned}\tag{3.2}$$

$$\eta_1(\delta,v)I_1=\mathcal{H}(v+\eta_1(\delta,v))-\frac{1}{\eta_1(\delta,v)}\int_v^{v+\eta_1(\delta,v)}\mathcal{H}(x)\ {}_v\mathrm{d}_qx.$$

Likewise, by applying Lemma 3, we obtain

$$\begin{aligned} I_2 &= \int_0^1 q\tau\ {}^{\delta}D_q\mathcal{H}(\delta+\tau\eta_1(v,\delta))\ \mathrm{d}_q\tau \\ &= -q\tau\frac{\mathcal{H}(\delta+\tau\eta_1(v,\delta))}{\eta_1(\delta,v)}\Big|_0^1+\frac{q}{\eta_1(\delta,v)}\int_0^1\mathcal{H}(\delta+\tau\eta_1(v,\delta))\ \mathrm{d}_q\tau \\ &= -q\frac{\mathcal{H}(\delta+\eta_1(v,\delta))}{\eta_1(\delta,v)}+\frac{q}{\eta_1(\delta,v)} \\ &\quad\times\left\{\frac{1-q}{q}\sum_{n=0}^{\infty}q^n\mathcal{H}\left(\delta+\tau\eta_1q^n(v,\delta)\right)-\frac{(1-q)}{q}\mathcal{H}(\delta+\eta_1(v,\delta))\right\} \end{aligned} \tag{3.3}$$

$$\eta_1(\delta,v)I_2=\frac{1}{\eta_1(\delta,v)}\int_{\delta+\eta_1(v,\delta)}^{\delta}\mathcal{H}(x)\ {}^{\delta}\mathrm{d}_qx-\mathcal{H}(\delta+\eta_1(v,\delta)).$$

To obtain the desired identity (3.1), we need to combine equations (3.2) and (3.3). □

Theorem 4. *Based on the assumptions in Lemma 5, the following inequality holds if both* $\left|{}_vD_q\mathcal{H}\right|$ *and* $\left|{}^{\delta}D_q\mathcal{H}\right|$ *are* (η_1,η_2)*-convex functions*

$$\begin{aligned} |\Phi(v,\delta)|\le\frac{q\eta_1(\delta,v)}{2}\Bigg[&\frac{\left|{}_vD_q\mathcal{H}(v)\right|+\left|{}^{\delta}D_q\mathcal{H}(\delta)\right|}{[2]_q} \\ &+\frac{\eta_2(|{}_vD_q\mathcal{H}(\delta)|,|{}_vD_q\mathcal{H}(v)|)}{[3]_q} \\ &+\frac{\eta_2(|{}^{\delta}D_q\mathcal{H}(v)|,|{}^{\delta}D_q\mathcal{H}(\delta)|)}{[3]_q}\Bigg]. \end{aligned}$$

Proof. Using Lemma 5 and the (η_1,η_2)-convexity of $\left|{}_vD_q\mathcal{H}\right|$ and $\left|{}^{\delta}D_q\mathcal{H}\right|$, we obtain the following result:

$$\begin{aligned} |\Phi(v,\delta)| &\le \frac{\eta_1(\delta,v)}{2}\Bigg[\int_0^1 q\tau\ \left|{}_vD_q\mathcal{H}(v+\tau\eta_1(\delta,v))\right|\ \mathrm{d}_q\tau \\ &\qquad+\int_0^1 q\tau\ \left|{}^{\delta}D_q\mathcal{H}(\delta+\tau\eta_1(v,\delta))\right|\ \mathrm{d}_q\tau\Bigg] \\ &\le \frac{\eta_1(\delta,v)}{2}\Bigg[\int_0^1 q\tau\left\{(|{}_vD_q\mathcal{H}(v)|+\tau\eta_2(|{}_vD_q\mathcal{H}(\delta)|,|{}_vD_q\mathcal{H}(v)|))\right\}\ \mathrm{d}_q\tau \\ &\qquad+\int_0^1 q\tau\left\{(|{}^{\delta}D_q\mathcal{H}(\delta)|+\tau\eta_2(|{}^{\delta}D_q\mathcal{H}(v)|,|{}^{\delta}D_q\mathcal{H}(\delta)|))\right\}\ \mathrm{d}_q\tau \\ &= \frac{\eta_1(\delta,v)}{2}\Bigg[\left|{}_vD_q\mathcal{H}(v)\right|\int_0^1 q\tau\ \mathrm{d}_q\tau+\eta_2(|{}_vD_q\mathcal{H}(\delta)|,|{}_vD_q\mathcal{H}(v)|)\int_0^1 q\tau^2\ \mathrm{d}_q\tau \end{aligned}$$

$$
\begin{aligned}
&+\left|{}^{\delta}D_q\mathcal{H}(\delta)\right|\int_0^1 q\tau\,\mathrm{d}_q\tau \\
&\quad+\eta_2(|{}^{\delta}D_q\mathcal{H}(v)|,|{}^{\delta}D_q\mathcal{H}(\delta)|)\int_0^1 q\tau^2\,\mathrm{d}_q\tau\Bigg] \\
&=\frac{\eta_1(\delta,v)}{2}\left[\left|{}_vD_q\mathcal{H}(v)\right|\frac{q}{[2]_q}+\eta_2(|{}_vD_q\mathcal{H}(\delta)|,|{}_vD_q\mathcal{H}(v)|)\frac{q}{[3]_q}\right. \\
&\quad\left.+\left|{}^{\delta}D_q\mathcal{H}(\delta)\right|\frac{q}{[2]_q}+\eta_2(|{}^{\delta}D_q\mathcal{H}(v)|,|{}^{\delta}D_q\mathcal{H}(\delta)|)\frac{q}{[3]_q}\right] \\
&=\frac{\eta_1(\delta,v)}{2}\left[\left(\left|{}_vD_q\mathcal{H}(v)\right|+\left|{}^{\delta}D_q\mathcal{H}(\delta)\right|\right)\frac{q}{[2]_q}\right. \\
&\quad+((\eta_2(|{}_vD_q\mathcal{H}(\delta)|,|{}_vD_q\mathcal{H}(v)|) \\
&\quad\left.+\eta_2(|{}^{\delta}D_q\mathcal{H}(v)|,|{}^{\delta}D_q\mathcal{H}(\delta)|)\right)\frac{q}{[3]_q}\Bigg)\Bigg].
\end{aligned}
$$

This concludes the proof. □

Theorem 5. *In the context of Lemma 5 assumptions, the following inequality holds if* $\left|{}_vD_q\mathcal{H}\right|^{p_1}$ *and* $\left|{}^{\delta}D_q\mathcal{H}\right|^{p_1}$ *with* $p_1\geq 1$ *are* (η_1,η_2)*-convex functions*

$$
\begin{aligned}
|\Phi(v,\delta)|\leq\frac{q\eta_1(\delta,v)}{2[2]_q}&\left[\left(\left|{}_vD_q\mathcal{H}(v)\right|^{p_1}\right.\right. \\
&\left.+\frac{[2]_q\eta_2(|{}_vD_q\mathcal{H}(\delta)|^{p_1},|{}_vD_q\mathcal{H}(v)|^{p_1})}{[3]_q}\right)^{\frac{1}{p_1}} \\
&+\left(\left|{}^{\delta}D_q\mathcal{H}(\delta)\right|^{p_1}\right. \\
&\left.\left.+\frac{[2]_q\eta_2(|{}^{\delta}D_q\mathcal{H}(v)|^{p_1},|{}^{\delta}D_q\mathcal{H}(\delta)|^{p_1})}{[3]_q}\right)^{\frac{1}{p_1}}\right].
\end{aligned}
$$

Proof. Using the power mean inequality along with Lemma 5, we get:

$$
\begin{aligned}
|\Phi(v,\delta)|&\leq\frac{\eta_1(\delta,v)}{2}\left[\int_0^1 q\tau\,\left|{}_vD_q\mathcal{H}(v+\tau\eta_1(\delta,v))\right|\,\mathrm{d}_q\tau\right. \\
&\quad\left.+\int_0^1 q\tau\,\left|{}^{\delta}D_q\mathcal{H}(\delta+\tau\eta_1(v,\delta))\right|\,\mathrm{d}_q\tau\right] \\
&\leq\frac{\eta_1(\delta,v)}{2}\left[\left(\int_0^1 q\tau\mathrm{d}_q\tau\right)^{1-\frac{1}{p_1}}\left(\int_0^1 q\tau\,\left|{}_vD_q\mathcal{H}(v+\tau\eta_1(\delta,v))\right|^{p_1}\mathrm{d}_q\tau\right)^{\frac{1}{p_1}}\right. \\
&\quad\left.+\left(\int_0^1 q\tau\,\mathrm{d}_q\tau\right)^{1-\frac{1}{p_1}}\right.
\end{aligned}
$$

$$\times \left(\int_0^1 q\tau \left| {}^{\delta}D_q\mathcal{H}(\delta+\tau\eta_1(v,\delta)) \right|^{p_1} \mathrm{d}_q\tau \right)^{\frac{1}{p_1}} \Bigg].$$

By using Lemma 5 and the (η_1,η_2)-convexity of $\left|{}_vD_q\mathcal{H}\right|^{p_1}$ and $\left|{}^{\delta}D_q\mathcal{H}\right|^{p_1}$, we derive:

$$\begin{aligned}
|\Phi(v,\delta)| \leq \frac{\eta_1(\delta,v)}{2} \Bigg[& \left(\int_0^1 q\tau \, \mathrm{d}_q\tau \right)^{1-\frac{1}{p_1}} \left(\int_0^1 q\tau \left\{ (|{}_vD_q\mathcal{H}(v)|^{p_1} \right.\right. \\
& \left.\left. +\tau\eta_2(|{}_vD_q\mathcal{H}(\delta)|^{p_1}, |{}_vD_q\mathcal{H}(v)|^{p_1})) \right\} \mathrm{d}_q\tau \right)^{\frac{1}{p_1}} \\
& + \left(\int_0^1 q\tau \, \mathrm{d}_q\tau \right)^{1-\frac{1}{p_1}} \left(\int_0^1 q\tau \left\{ (|{}^{\delta}D_q\mathcal{H}(\delta)|^{p_1} \right.\right. \\
& \left.\left. +\tau\eta_2(|{}^{\delta}D_q\mathcal{H}(v)|^{p_1}, |{}^{\delta}D_q\mathcal{H}(\delta)|^{p_1})) \right\} \mathrm{d}_q\tau \right)^{\frac{1}{p_1}} \Bigg] \\
= \frac{q\eta_1(\delta,v)}{2[2]_q} \Bigg[& \left(\left|{}_vD_q\mathcal{H}(v)\right|^{p_1} \right. \\
& \left. + \frac{[2]_q\eta_2(|{}_vD_q\mathcal{H}(\delta)|^{p_1}, |{}_vD_q\mathcal{H}(v)|^{p_1})}{[3]_q} \right)^{\frac{1}{p_1}} \\
& + \left(\left|{}^{\delta}D_q\mathcal{H}(\delta)\right|^{p_1} \right. \\
& \left. + \frac{[2]_q\eta_2(|{}^{\delta}D_q\mathcal{H}(v)|^{p_1}, |{}^{\delta}D_q\mathcal{H}(\delta)|^{p_1})}{[3]_q} \right)^{\frac{1}{p_1}} \Bigg].
\end{aligned}$$

This concludes the proof. □

Theorem 6. *In the context of Lemma 5 assumptions, the following inequality holds if $\left|{}_vD_q\mathcal{H}\right|^{p_1}$ and $\left|{}^{\delta}D_q\mathcal{H}\right|^{p_1}$ with $p_1 > 1$ are (η_1,η_2)-convex functions*

$$\begin{aligned}
|\Phi(v,\delta)| \leq \frac{q\eta_1(\delta,v)}{2} \left(\frac{1}{[r_1+1]_q} \right)^{\frac{1}{r_1}} \Bigg[& \left(\left|{}_vD_q\mathcal{H}(v)\right|^{p_1} \right. \\
& \left. + \frac{\eta_2(|{}_vD_q\mathcal{H}(\delta)|^{p_1}, |{}_vD_q\mathcal{H}(v)|^{p_1})}{[2]_q} \right)^{\frac{1}{p_1}} \\
& + \left(\left|{}^{\delta}D_q\mathcal{H}(\delta)\right|^{p_1} \right. \\
& \left. + \frac{\eta_2(|{}^{\delta}D_q\mathcal{H}(v)|^{p_1}, |{}^{\delta}D_q\mathcal{H}(\delta)|^{p_1})}{[2]_q} \right)^{\frac{1}{p_1}} \Bigg],
\end{aligned}$$

where $r_1^{-1} + p_1^{-1} = 1$.

Proof. The Hölder inequality, combined with Lemma 5, provides

$$\begin{aligned}|\Phi(v,\delta)| &\leq \frac{\eta_1(\delta,v)}{2}\left[\int_0^1 q\tau\ \left|{}_vD_q\mathcal{H}(v+\tau\eta_1(\delta,v))\right|\ \mathrm{d}_q\tau\right.\\ &\qquad\left.+\int_0^1 q\tau\ \left|{}^{\delta}D_q\mathcal{H}(\delta+\tau\eta_1(v,\delta))\right|\ \mathrm{d}_q\tau\right]\\ &\leq \frac{\eta_1(\delta,v)}{2}\left[\left(\int_0^1 (q\tau)^{r_1}\mathrm{d}_q\tau\right)^{\frac{1}{r_1}}\left(\int_0^1 \left|{}_vD_q\mathcal{H}(v+\tau\eta_1(\delta,v))\right|^{p_1}\mathrm{d}_q\tau\right)^{\frac{1}{p_1}}\right.\\ &\qquad\left.+\left(\int_0^1 (q\tau)^{r_1}\mathrm{d}_q\tau\right)^{\frac{1}{r_1}}\left(\int_0^1 \left|{}^{\delta}D_q\mathcal{H}(\delta+\tau\eta_1(v,\delta))\right|^{p_1}\mathrm{d}_q\tau\right)^{\frac{1}{p_1}}\right].\end{aligned}$$

By applying Lemma 5 and utilizing the (η_1,η_2)-convexity of $\left|{}_vD_q\mathcal{H}\right|^{p_1}$ and $\left|{}^{\delta}D_q\mathcal{H}\right|^{p_1}$, we derive

$$\begin{aligned}|\Phi(v,\delta)| \leq \frac{q\eta_1(\delta,v)}{2}&\left[\left(\int_0^1 (\tau)^{r_1}\mathrm{d}_q\tau\right)^{\frac{1}{r_1}}\left(\int_0^1 \left\{(|{}_vD_q\mathcal{H}(v)|^{p_1}\right.\right.\right.\\ &\left.+\tau\eta_2(|{}_vD_q\mathcal{H}(\delta)|^{p_1},|{}_vD_q\mathcal{H}(v)|^{p_1}))\right\}\ \mathrm{d}_q\tau\Big)^{\frac{1}{p_1}}\\ &+\left(\int_0^1 (\tau)^{r_1}\mathrm{d}_q\tau\right)^{\frac{1}{r_1}}\left(\int_0^1 \left\{(|{}^{\delta}D_q\mathcal{H}(\delta)|^{p_1}\right.\right.\\ &\left.\left.\left.+\tau\eta_2(|{}^{\delta}D_q\mathcal{H}(v)|^{p_1},|{}^{\delta}D_q\mathcal{H}(\delta)|^{p_1}))\right\}\ \mathrm{d}_q\tau\right)^{\frac{1}{p_1}}\right].\end{aligned}$$

After applying the definition of the q-integral, we obtain the desired result. □

4. q-MIDPOINT INEQUALITIES

This section contains the proof of many correct estimates for the inequality (1.1), which is relevant to (η_1,η_2)-convex functions that are differentiable.

Lemma 6. *Suppose that $I\subset\mathbb{R}$ is an invex set with respect to η_1. If for $v,\delta\in I$ with $\eta_1>0$ If $\mathcal{H}\colon [v,v+\eta_1(\delta,v)],[\delta,\delta+\eta_1(v,\delta)]\subset\mathbb{R}\to\mathbb{R}$ is q-differentiable such that ${}_vD_q\mathcal{H}$ and ${}^{\delta}D_q\mathcal{H}$ are integrable and continuous on $[v,v+\eta_1(\delta,v)],[\delta,\delta+\eta_1(v,\delta)]$, then*

$$\begin{aligned}\Psi(v,\delta) =:\ &\mathcal{H}\left(\frac{(v+\eta_1(\delta,v))+(\delta+\eta_1(v,\delta))}{2}\right)\\ &-\frac{1}{2\eta_1(\delta,v)}\left[\int_v^{v+\eta_1(\delta,v)}\mathcal{H}(x)\ {}_v\mathrm{d}_qx\right.\\ &\left.+\int_{\delta+\eta_1(v,\delta)}^{\delta}\mathcal{H}(x)\ {}^{\delta}\mathrm{d}_qx\right]\end{aligned}$$

$$= \frac{\eta_1(\delta,v)}{2}\left[\int_0^{\frac{1}{2}} q\tau\ {}_vD_q\mathcal{H}(v+\tau\eta_1(\delta,v))\ \mathrm{d}_q\tau \right.$$
$$+\int_{\frac{1}{2}}^1 (q\tau-1)\ {}_vD_q\mathcal{H}(v+\tau\eta_1(\delta,v))\ \mathrm{d}_q\tau$$
$$+\int_0^{\frac{1}{2}} (-q\tau)\ {}^{\delta}D_q\mathcal{H}(\delta+\tau\eta_1(v,\delta))\ \mathrm{d}_q\tau$$
$$\left.+\int_{\frac{1}{2}}^1 (1-q\tau)\ {}^{\delta}D_q\mathcal{H}(\delta+\tau\eta_1(v,\delta))\ \mathrm{d}_q\tau\right].$$

Proof. The proof can be easily shown using the same method as in Lemma 5. □

Theorem 7. *In the context of Lemma 6 assumptions, the following inequality holds if* $\left|{}^{\delta}D_q\mathcal{H}\right|$ *and* $\left|{}_vD_q\mathcal{H}\right|$ *are* (η_1,η_2)*-convex functions*

$$|\Psi(v,\delta)| \le \frac{\eta_1(\delta,v)}{2}\left[|{}_vD_q\mathcal{H}(v)|\frac{q}{4[2]_q}\right.$$
$$+\eta_2(|{}_vD_q\mathcal{H}(\delta)|,|{}_vD_q\mathcal{H}(v)|)\frac{q}{8[3]_q}$$
$$+|{}_vD_q\mathcal{H}(v)|\left(\frac{q-2}{4[2]_q}\right)$$
$$+\eta_2(|{}_vD_q\mathcal{H}(\delta)|,|{}_vD_q\mathcal{H}(v)|)\left(\frac{-6+q^2+q}{8[2]_q[3]_q}\right)$$
$$+|{}^{\delta}D_q\mathcal{H}(\delta)|\left(\frac{q^2}{4[2]_q}\right)$$
$$+\eta_2(|{}^{\delta}D_q\mathcal{H}(v)|,|{}^{\delta}D_q\mathcal{H}(\delta)|)\left(\frac{q^3}{8[3]_q}\right)$$
$$+|{}^{\delta}D_q\mathcal{H}(\delta)|\left(\frac{6-3q^2-2q}{4[2]_q}\right)$$
$$+\eta_2(|{}^{\delta}D_q\mathcal{H}(v)|,|{}^{\delta}D_q\mathcal{H}(\delta)|)$$
$$\left.\times\left(\frac{7q+9q^3-2q^2-8q^4-2}{8[2]_q[3]_q}\right)\right].$$

Proof. By applying Lemma 6

$$\Psi(v,\delta) \le \frac{\eta_1(\delta,v)}{2}\left[\int_0^{\frac{1}{2}} q\tau\ |{}_vD_q\mathcal{H}(v+\tau\eta_1(\delta,v))|\ \mathrm{d}_q\tau\right.$$
$$\left.+\int_{\frac{1}{2}}^1 (q\tau-1)\ |{}_vD_q\mathcal{H}(v+\tau\eta_1(\delta,v))|\ \mathrm{d}_q\tau\right.$$

$$+\int_0^{\frac{1}{2}}(-q\tau)\ |{}^{\delta}D_q\mathcal{H}(\delta+\tau\eta_1(v,\delta))|\ \mathrm{d}_q\tau$$
$$+\int_{\frac{1}{2}}^{1}(1-q\tau)\ |{}^{\delta}D_q\mathcal{H}(\delta+\tau\eta_1(v,\delta))|\ \mathrm{d}_q\tau\Bigg].$$

Utilizing the (η_1,η_2)-convexity of $\left|{}_vD_q\mathcal{H}\right|$ and $\left|{}^{\delta}D_q\mathcal{H}\right|$, we derive

$$\leq \frac{\eta_1(\delta,v)}{2}\Bigg[|{}_vD_q\mathcal{H}(v)|\int_0^{\frac{1}{2}}q\tau\ \mathrm{d}_q\tau$$
$$+\eta_2(|{}_vD_q\mathcal{H}(\delta)|,|{}_vD_q\mathcal{H}(v)|)\int_0^{\frac{1}{2}}q\tau^2\ \mathrm{d}_q\tau$$
$$+|{}_vD_q\mathcal{H}(v)|\int_{\frac{1}{2}}^{1}(q\tau-1)\ \mathrm{d}_q\tau$$
$$+\eta_2(|{}_vD_q\mathcal{H}(\delta)|,|{}_vD_q\mathcal{H}(v)|)\int_{\frac{1}{2}}^{1}(q\tau-1)\tau\ \mathrm{d}_q\tau$$
$$+|{}^{\delta}D_q\mathcal{H}(\delta)|\int_0^{\frac{1}{2}}(-q\tau)\mathrm{d}_q\tau$$
$$+\eta_2(|{}^{\delta}D_q\mathcal{H}(v)|,|{}^{\delta}D_q\mathcal{H}(\delta)|)\int_0^{\frac{1}{2}}(-q\tau^2)\mathrm{d}_q\tau$$
$$+|{}^{\delta}D_q\mathcal{H}(\delta)|\int_{\frac{1}{2}}^{1}(1-q\tau)\ \mathrm{d}_q\tau$$
$$+\eta_2(|{}^{\delta}D_q\mathcal{H}(v)|,|{}^{\delta}D_q\mathcal{H}(\delta)|)\int_{\frac{1}{2}}^{1}(1-q\tau)\tau\ \mathrm{d}_q\tau\Bigg].$$

After applying the definition of the q-integral, we obtain the desired result. □

Theorem 8. *In the context of Lemma 6 assumptions, the following inequality holds if* $\left|{}^{\delta}D_q\mathcal{H}\right|^{p_1}$ *and* $\left|{}_vD_q\mathcal{H}\right|^{p_1}, p_1\geq 1$ *are* (η_1,η_2)*-convex functions*

$$|\Psi(v,\delta)|\leq\frac{\eta_1(\delta,v)}{2}\Bigg[\left(\frac{q}{4[2]_q}\right)^{1-\frac{1}{p_1}}\Bigg(|{}_vD_q\mathcal{H}(v)|^{p_1}\frac{q}{4[2]_q}$$
$$+\eta_2(|{}_vD_q\mathcal{H}(\delta)|^{p_1},|{}_vD_q\mathcal{H}(v)|^{p_1})\frac{q}{8[3]_q}\Bigg)^{\frac{1}{p_1}}$$
$$+\left(\frac{q-2}{4[2]_q}\right)^{1-\frac{1}{p_1}}\Bigg(|{}_vD_q\mathcal{H}(v)|^{p_1}\left(\frac{q-2}{4[2]_q}\right)$$
$$+\eta_2(|{}_vD_q\mathcal{H}(\delta)|^{p_1},|{}_vD_q\mathcal{H}(v)|^{p_1})\left(\frac{-6+q^2+q}{8[2]_q[3]_q}\right)\Bigg)^{\frac{1}{p_1}}$$

$$+\left(\frac{q^2}{4[2]_q}\right)^{1-\frac{1}{p_1}}\left(|{}^{\delta}D_q\mathcal{H}(\delta)|^{p_1}\frac{q^2}{4[2]_q}\right.$$
$$\left.+\eta_2(|{}^{\delta}D_q\mathcal{H}(v)|^{p_1},|{}^{\delta}D_q\mathcal{H}(\delta)|^{p_1})\frac{q^3}{8[3]_q}\right)^{\frac{1}{p_1}}$$
$$+\left(\frac{6-3q^2-2q}{4[2]_q}\right)^{1-\frac{1}{p_1}}\left(|{}^{\delta}D_q\mathcal{H}(\delta)|^{p_1}\left(\frac{6-3q^2-2q}{4[2]_q}\right)\right.$$
$$+\eta_2(|{}^{\delta}D_q\mathcal{H}(v)|^{p_1},|{}^{\delta}D_q\mathcal{H}(\delta)|^{p_1})$$
$$\left.\left.\times\left(\frac{7q+9q^3-2q^2-8q^4-2}{8[2]_q[3]_q}\right)\right)^{\frac{1}{p_1}}\right].$$

Proof. The power mean inequality, in conjunction with Lemma 6 and utilizing the (η_1,η_2)-convexity of $\left|{}_vD_q\mathcal{H}\right|^{p_1}$ and $\left|{}^{\delta}D_q\mathcal{H}\right|^{p_1}$, we derive

$$=\frac{\eta_1(\delta,v)}{2}\left[\left(\int_0^{\frac{1}{2}}q\tau\,\mathrm{d}_q\tau\right)^{1-\frac{1}{p_1}}\left(|{}_vD_q\mathcal{H}(v)|^{p_1}\int_0^{\frac{1}{2}}q\tau\,\mathrm{d}_q\tau\right.\right.$$
$$\left.+\eta_2(|{}_vD_q\mathcal{H}(\delta)|^{p_1},|{}_vD_q\mathcal{H}(v)|^{p_1})\int_0^{\frac{1}{2}}q\tau^2\,\mathrm{d}_q\tau\right)^{\frac{1}{p_1}}$$
$$+\left(\int_{\frac{1}{2}}^1(q\tau-1)\,\mathrm{d}_q\tau\right)^{1-\frac{1}{p_1}}\left(|{}_vD_q\mathcal{H}(v)|^{p_1}\int_{\frac{1}{2}}^1(q\tau-1)\,\mathrm{d}_q\tau\right.$$
$$\left.+\eta_2(|{}_vD_q\mathcal{H}(\delta)|^{p_1},|{}_vD_q\mathcal{H}(v)|^{p_1})\int_{\frac{1}{2}}^1(q\tau-1)\tau\,\mathrm{d}_q\tau\right)^{\frac{1}{p_1}}$$
$$+\left(\int_0^{\frac{1}{2}}(-q\tau)\,\mathrm{d}_q\tau\right)^{1-\frac{1}{p_1}}\left(|{}^{\delta}D_q\mathcal{H}(\delta)|^{p_1}\int_0^{\frac{1}{2}}(-q\tau)\mathrm{d}_q\tau\right.$$
$$\left.+\eta_2(|{}^{\delta}D_q\mathcal{H}(v)|^{p_1},|{}^{\delta}D_q\mathcal{H}(\delta)|^{p_1})\int_0^{\frac{1}{2}}(-q\tau^2)\mathrm{d}_q\tau\right)^{\frac{1}{p_1}}$$
$$+\left(\int_{\frac{1}{2}}^1(1-q\tau)\,\mathrm{d}_q\tau\right)^{1-\frac{1}{p_1}}\left(|{}^{\delta}D_q\mathcal{H}(\delta)|^{p_1}\int_{\frac{1}{2}}^1(1-q\tau)\,\mathrm{d}_q\tau\right.$$
$$\left.\left.+\eta_2(|{}^{\delta}D_q\mathcal{H}(v)|^{p_1},|{}^{\delta}D_q\mathcal{H}(\delta)|^{p_1})\int_{\frac{1}{2}}^1(1-q\tau)\tau\,\mathrm{d}_q\tau\right)^{\frac{1}{p_1}}\right].$$

After applying the definition of the q-integral, we obtain the desired result. □

Theorem 9. *In the context of Lemma 6 assumptions, the following inequality holds if* $\left|{}^{\delta}D_q\mathcal{H}\right|^{p_1}$ *and* $\left|{}_{v}D_q\mathcal{H}\right|^{p_1}, p_1>1$ *are* (η_1,η_2)*-convex functions*

$$
\begin{aligned}
|\Psi(v,\delta)| \leq \frac{\eta_1(\delta,v)}{2}\Bigg[& q\left(\frac{1}{2^{r_1+1}\,[r_1+1]_q}\right)^{\frac{1}{r_1}}\Bigg(\left|{}_{v}D_q\mathcal{H}(v)\right|^{p_1}\left(\frac{q}{4[2]_q}\right)\\
&+\eta_2\left(\left|{}_{v}D_q\mathcal{H}(\delta)\right|^{p_1},\left|{}_{v}D_q\mathcal{H}(v)\right|^{p_1}\right)\left(\frac{q}{8[3]_q}\right)\Bigg)^{\frac{1}{p_1}}\\
&+\left(\int_{\frac{1}{2}}^{1}(1-q\tau)^{r_1}\,\mathrm{d}_q\tau\right)^{\frac{1}{r_1}}\Bigg(\left|{}_{v}D_q\mathcal{H}(v)\right|^{p_1}\left(\frac{q-2}{4[2]_q}\right)\\
&+\eta_2\left(\left|{}_{v}D_q\mathcal{H}(\delta)\right|^{p_1},\left|{}_{v}D_q\mathcal{H}(v)\right|^{p_1}\right)\left(\frac{-6+q^2+q}{8[2]_q[3]_q}\right)\Bigg)^{\frac{1}{p_1}}\\
&+q\left(\frac{1}{2^{r_1+1}\,[r_1+1]_q}\right)^{\frac{1}{r_1}}\Bigg(\left|{}^{\delta}D_q\mathcal{H}(\delta)\right|^{p_1}\left(\frac{q^2}{4[2]_q}\right)\\
&+\eta_2\left(\left|{}^{\delta}D_q\mathcal{H}(v)\right|^{p_1},\left|{}^{\delta}D_q\mathcal{H}(\delta)\right|^{p_1}\right)\left(\frac{q^3}{8[3]_q}\right)\Bigg)^{\frac{1}{p_1}}\\
&+\left(\int_{\frac{1}{2}}^{1}(1-q\tau)^{r_1}\,\mathrm{d}_q\tau\right)^{\frac{1}{r_1}}\Bigg(\left|{}^{\delta}D_q\mathcal{H}(\delta)\right|^{p_1}\left(\frac{6-3q^2-2q}{4[2]_q}\right)\\
&+\eta_2\left(\left|{}^{\delta}D_q\mathcal{H}(v)\right|^{p_1},\left|{}^{\delta}D_q\mathcal{H}(\delta)\right|^{p_1}\right)\\
&\times\left(\frac{7q+9q^3-2q^2-8q^4-2}{8[2]_q[3]_q}\right)\Bigg)^{\frac{1}{p_1}}\Bigg],
\end{aligned}
$$

where $p_1^{-1}+r_1^{-1}=1$.

Proof. The Hölder inequality, combined with Lemma 6 and utilizing the (η_1,η_2)-convexity of $\left|{}_{v}D_q\mathcal{H}\right|^{p_1}$ and $\left|{}^{\delta}D_q\mathcal{H}\right|^{p_1}$, we derive

$$
\begin{aligned}
|\Psi(v,\delta)| \leq \frac{\eta_1(\delta,v)}{2}\Bigg[& \left(\int_0^{\frac{1}{2}}(q\tau)^{r_1}\,\mathrm{d}_q\tau\right)^{\frac{1}{r_1}}\Bigg(\left|{}_{v}D_q\mathcal{H}(v)\right|^{p_1}\int_0^{\frac{1}{2}}q\tau\,\mathrm{d}_q\tau\\
&+\eta_2\left(\left|{}_{v}D_q\mathcal{H}(\delta)\right|^{p_1},\left|{}_{v}D_q\mathcal{H}(v)\right|^{p_1}\right)\int_0^{\frac{1}{2}}q\tau^2\,\mathrm{d}_q\tau\Bigg)^{\frac{1}{p_1}}\\
&+\left(\int_{\frac{1}{2}}^{1}(q\tau-1)^{r_1}\,\mathrm{d}_q\tau\right)^{1-\frac{1}{r_1}}\Bigg(\left|{}_{v}D_q\mathcal{H}(v)\right|^{p_1}\int_{\frac{1}{2}}^{1}(q\tau-1)\,\mathrm{d}_q\tau
\end{aligned}
$$

$$+\eta_2(|_vD_q\mathcal{H}(\delta)|^{p_1},|_vD_q\mathcal{H}(v)|^{p_1})\int_{\frac{1}{2}}^{1}(q\tau-1)\tau\,\mathrm{d}_q\tau\Bigg)^{\frac{1}{p_1}}$$

$$+\left(\int_0^{\frac{1}{2}}(-q\tau)^{r_1}\,\mathrm{d}_q\tau\right)^{1-\frac{1}{r_1}}\Bigg(|^{\delta}D_q\mathcal{H}(\delta)|^{p_1}\int_0^{\frac{1}{2}}(-q\tau)\mathrm{d}_q\tau$$

$$+\eta_2(|^{\delta}D_q\mathcal{H}(v)|^{p_1},|^{\delta}D_q\mathcal{H}(\delta)|^{p_1})\int_0^{\frac{1}{2}}(-q\tau^2)\mathrm{d}_q\tau\Bigg)^{\frac{1}{p_1}}$$

$$+\left(\int_{\frac{1}{2}}^{1}(1-q\tau)^{r_1}\,\mathrm{d}_q\tau\right)^{1-\frac{1}{r_1}}\Bigg(|^{\delta}D_q\mathcal{H}(\delta)|^{p_1}\int_{\frac{1}{2}}^{1}(1-q\tau)\,\mathrm{d}_q\tau$$

$$+\eta_2(|^{\delta}D_q\mathcal{H}(v)|^{p_1},|^{\delta}D_q\mathcal{H}(\delta)|^{p_1})\int_{\frac{1}{2}}^{1}(1-q\tau)\tau\,\mathrm{d}_q\tau\Bigg)^{\frac{1}{p_1}}\Bigg].$$

After applying the definition of the q-integral, we obtain the desired result. □

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OSCILLATION THEOREMS FOR PARTIAL DYNAMIC EQUATIONS WITH TIME-SCALE SOLUTIONS

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Abstract. For the even-order nonlinear functional dynamic equations with unbounded neutral coefficients on time scales, we will establish some oscillation criteria in this paper.

$$a(t)\Delta_x u(x,t) = \frac{\partial}{\Delta t}\left(r(t)\left(\frac{\partial u}{\Delta t}(x,t)\right)\right) + q(t)u^{\Delta}(x,t) + p(x,t)f\left(t,x,u(x,t)\right),$$

for all $(x,t) \in \Omega \times \mathcal{J}_{t_0}$, utilizing either the Neumann boundary condition

$$\nabla_x u(x,t) = 0 \quad \text{for all } (x,t) \in \Gamma \times \mathcal{J}_{t_0},$$

or the Dirichlet boundary condition

$$u(x,t) = 0 \quad \text{for all } (x,t) \in \Gamma \times \mathcal{J}_{t_0}.$$

We assume throughout this paper that the given time scale $\mathbb{T}$ is unbounded above and $\mathcal{J}_{t_0} = [t_0,\infty) \cap \mathbb{T}$. Here, Ω is a bounded domain in $\mathbb{R}^n$ with piecewise smooth boundary $\Gamma = \partial\Omega$, $\frac{\partial u}{\Delta t}$ is the partial dynamic operator with respect to t, Δ_x is laplacian operator on Ω, and N is the unit exterior normal vector to Γ.The purpose of this study is to introduce some fresh, necessary conditions for the oscillatory solution of partial dynamic equations on time scales.

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Keywords: neutral differential equation, oscillation, partial dynamic equations, time scales

1. INTRODUCTION

Stefan Hilger [4] invented the time scale calculus in 1988. It integrates and advances both continuous and discrete analytic theories. A large portion of time scale calculus is summarized and organized in the works of Bohner and Peterson, [1, 2], on the subject of time scale, or measure chain. The study of oscillatory events in technology, the natural and social sciences, and other fields is connected to the theory of oscillations, a significant subfield of the applied theory of dynamic equations. Additionally, [3, 5, 6] have examined the oscillatory behavior of dynamic equations on time scales. We point the reader to the articles in the references because there has

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been relatively little research on the oscillation of solutions to these kinds of partial dynamic equations on time scales.

We consider the partial dynamic equation.

$$a(t)\Delta_x u(x,t) = \frac{\partial}{\Delta t}\left(r(t)\left(\frac{\partial u}{\Delta t}(x,t)\right)\right) + q(t)\frac{\partial u}{\Delta t}(x,t) + p(x,t)f(t,x,u(x,t)), \quad (1.1)$$

for all $(x,t) \in \Omega \times \mathcal{J}_{t_0}$, with either the Neumann boundary condition

$$\nabla_x u(x,t) = 0 \quad \text{for all } (x,t) \in \Gamma \times \mathcal{J}_{t_0}, \quad (1.2)$$

or the Dirichlet boundary condition

$$u(x,t) = 0 \quad \text{for all } (x,t) \in \Gamma \times \mathcal{J}_{t_0}. \quad (1.3)$$

We assume throughout this paper that the given time scale $\mathbb{T}$ is unbounded above and $\mathcal{J}_{t_0} = [t_0,\infty) \cap \mathbb{T}$. Here, Ω is a bounded domain in $\mathbb{R}^n$ with piecewise smooth boundary $\Gamma = \partial\Omega$, Δ_t is the partial dynamic operator with respect to t, Δ_x is a laplacian operator on, and N is the unit exterior normal vector to. Under the following presumptions, the equations (1.1)-(1.2) will be examined.

(C_1) The function $f\colon \mathcal{J}_{t_0} \times \Omega \times \mathbb{R} \to \mathbb{R}$ such that

$$f \in \mathcal{C}(\mathcal{J}_{t_0} \times \Omega \times \mathbb{R}, \mathbb{R}), \quad uf(t,x,u) > 0 \quad \text{for all } (t,x,u) \in \mathcal{J}_{t_0} \times \Omega \times \mathbb{R} - \{0\}$$

and there is $b \in \mathcal{C}(\mathcal{J}_{t_0}, [0,\infty))$ such that

$$\frac{f(t,x,u)}{u} \geq b(t) \quad \text{for all } (t,x,u) \in \mathcal{J}_{t_0} \times \Omega \times \mathbb{R} - \{0\}. \quad (1.4)$$

(C_2) The functions $r,q,a\colon \mathcal{J}_{t_0} \to [0,\infty)$ and $p\colon \mathcal{J}_{t_0} \times \Omega \to \mathbb{R}$, such as

$$r,q,a \in \mathcal{C}(\mathcal{J}_{t_0}, [0,\infty)), \quad p \in \mathcal{C}(\mathcal{J}_{t_0} \times \Omega, \mathbb{R}), \quad r(t) - \mu(t)q(t) \neq 0 \quad \text{for all } t \in \mathcal{J}_{t_0}.$$

In order to satisfy (1.1) on $\mathcal{J}_{t_0}$, a nontrivial real-valued function $u \in \mathcal{C}(\mathcal{J}_{t_0} \times \Omega, \mathbb{R})$ must satisfy the following conditions: $\frac{\partial u}{\Delta t}$, $\left(r\left(\frac{\partial u}{\Delta t}\right)\right)^{\Delta}$, $\partial_{x_i} u$, $\partial_{x_i,y_i} u \in \mathcal{C}(\mathcal{J}_{t_0} \times \Omega, \mathbb{R})$, for $i,j \in \{1,2,\ldots,n\}$. We shall not investigate any solutions that disappear in some neighborhood of infinity.

If there is a $t_1 \in \mathcal{J}_{t_0}$ such that $u > 0$ and $\int_\Omega u dx > 0$, (resp. $u < 0$ and $\int_\Omega u dx < 0$) for all $t \in \mathcal{J}_{t_1}$, then the function $u\colon \mathcal{J}_{t_0} \times \Omega \to \mathbb{R}$ is said to be ultimately positive (eventually negative). If a solution u to equation (1.1) is neither ultimately positive nor ultimately negative, it is said to be oscillatory; otherwise, it is said to be nonoscillatory. If all of the solutions to equation (1.1) are oscillatory, the equation is said to be oscillatory. There haven't been any oscillatory findings for (1.1) to date. This paper's goal is to provide some oscillation criteria for this.

2. PRELIMINARIES

An arbitrary nonempty closed subset of real numbers is referred to as a temporal scale. The ideas of differential equations and difference equations are combined in the theory of dynamic equations. We assume that the reader is already aware with the fundamental ideas behind dynamic equations' time scales math. Otherwise, the majority of the information required to understand this article may be found in the books [1] and [2].

If a function $u\colon \mathbb{T}\to\mathbb{R}$ is continuous at each right-dense point and has a left-sided limit at each point, it is said to be rd-continuous, and we write $u\in C_{rd}(\mathbb{T})=C_{rd}(\mathbb{T},\mathbb{R})$.

Let $\mathbb{T}$ be a time scale, and let $u\colon \mathbb{T}\to\mathbb{R}$ be a real valued function. Then, given $t\in\mathbb{T}^k$, we define $u^{\Delta}(t)$, if one exists, as the integer such that for all $\varepsilon>0$, a neighborhood $\mathcal{V}$ of t such that for all $s\in\mathcal{V}$,

$$\left|u^{\sigma}(t)-u(s)-u^{\Delta}(t)\left(\sigma(t)-s\right)\right|\leq\varepsilon\left|\sigma(t)-s\right|.$$

where the definition of the forward jump operator $\sigma\colon \mathbb{T}\to\mathbb{T}$is defined by

$$\sigma(t)=\inf\{s\in\mathbb{T}\colon s>t\}\quad \text{for any } t\in\mathbb{T}.$$

If $u^{\Delta}(t)$ exists for every $t\in\mathbb{T}^k=\mathbb{T}-\{\sup\mathbb{T}\}$, then we may claim that u is delta differentiable on $\mathbb{T}$.

The Cauchy integral of u is defined as for $a,b\in\mathbb{T}$ and a differentiable function u, then

$$\int_a^b u^{\Delta}(s)\,\Delta s=u(b)-u(a).$$

Regressive function $p\colon \mathbb{T}\to\mathbb{R}$ is defined as follows

$$1+\mu(t)\,p(t)\neq 0\quad \text{for all } t\in\mathbb{T}^{\kappa},$$

where the graininess function $\mu\colon \mathbb{T}\to[0,+\infty)$ is defined by $\mu(t)=\sigma(t)-t$. The set of all regressive and rd-continuous functions $p\colon \mathbb{T}\to\mathbb{R}$ will be denoted by $\mathcal{R}=\mathcal{R}(\mathbb{T},\mathbb{R})$.

For $p\in\mathcal{R}(\mathbb{T},\mathbb{R})$ and $t_0,t\in\mathbb{T}$, the generalized exponential function is defined by

$$e_p(t,t_0)=\exp\left(\int_{t_0}^{t}\xi_{\mu(s)}\left(p(s)\right)\Delta s\right),\quad \xi_h(z)=\begin{cases}\frac{1}{z}\ln(1+hz), & h\neq 0;\\ z, & h=0.\end{cases}$$

Then $e_p(t,t_0)$ is a nonzero real valued function, and is the unique solution of the IVP

$$\begin{cases}u^{\Delta}(t)=p(t)\,v(t) & \text{for all } t\in\mathbb{T}^k,\\ v(t_0)=1.\end{cases}$$

If $p,q\in\mathcal{R}(\mathbb{T},\mathbb{R})$, then the function $p\oplus q$ defined by

$$(p\oplus q)(t)=p(t)+q(t)+\mu(t)\,p(t)\,q(t)\quad \text{for all } t\in\mathbb{T}^k$$

and the function $\ominus p$ defined by

$$\ominus p(t) = \frac{-p(t)}{1+\mu(t)\,p(t)} \quad \text{for all } t \in \mathbb{T}^k.$$

The following result is used frequently in the remainder of this paper, let $p,q \in \mathcal{R}(\mathbb{T},\mathbb{R})$, and $t,t_0 \in \mathbb{T}^k$, we have

(i) $e_{p\oplus q}(t,t_0) = e_p(t,t_0)\,e_q(t,t_0)$,
(ii) $e_p^{\sigma}(t,t_0) = e_p(\sigma(t),t_0) = (1+\mu(t)\,p(t))\,e_p(t,t_0)$,
(iii) $\frac{1}{e_p(t,t_0)} = e_{\ominus p}(t,t_0)$.

3. Oscillation with Neumann boundary condition

We create certain necessary criteria in this section that ensure any solution u of (1.1)-(1.2) oscillates on $\mathcal{J}_{t_0}$. We present the key findings and note them for simplicity.

$$P(t) := \min\{p(t,x) : x \in \Omega\} > 0 \quad \text{for all } t \in \mathcal{J}_{t_0}. \tag{3.1}$$

Theorem 1. *Assume that (C_1) to (C_2) are true. If $\delta \in C^1_{rd}(\mathcal{J}_{t_0},\mathbb{R})$ is a positive function that exists, then*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \frac{e_{\ominus \frac{q(t)}{r(t)}}(s,t)}{r(s)}\Delta s = \infty, \tag{3.2}$$

and

$$\limsup_{t\to\infty} \int_{t_0}^{t} \Lambda_\delta(s,t_0)\,\Delta s = \infty, \tag{3.3}$$

where

$$\Lambda_\delta(t,t_0) := \delta(t)\,e_{\ominus\left[-\frac{q}{r}\right]}(\sigma(t),t_0)\,b(t)\,P(t) - e_{\ominus\left[-\frac{q}{r}\right]}(t,t_0)\,\frac{r(t)\left[\delta^{\Delta}(t)\right]^2}{4\delta(t)}.$$

Any (1.1)-(1.2) solution is then oscillatory.

Proof. If (1.1)-(1.2) has a non-oscillatory solution u on $\mathcal{J}_{t_0}$, then this is assumed. Without sacrificing generality, we can assume that there is $t_1 \in \mathcal{J}_{t_0}$ such that $u(t,x) \geq 0$ and $v(t) = \int_\Omega u(t,x)\,dx > 0$, for $t \in \mathcal{J}_{t_0}$. Since identical justifications can be provided for the cases $u(t,x) \geq 0$ and $v(t) = \int_\Omega u(t,x)\,dx < 0$, for $t \in \mathcal{J}_{t_0}$. Let u be an eventual solution of (1.1)-(1.2) that is positive. Green's theorem and (1.2) provide the following results:

$$\int_\Omega \Delta_x u(x,t)dx = \int_\Gamma \nabla_x u(x,t).dS = 0.$$

Integrating (1.1) with respect to x over Ω, we obtain

$$\int_\Omega \frac{\partial}{\Delta t}\left(r(t)\left(\frac{\partial u}{\Delta t}(x,t)\right)\right)dx + q(t)\int_\Omega \frac{\partial u}{\Delta t}(x,t)dx + \int_\Omega p(x,t)f(t,x,u(x,t))\,dx = 0.$$

From (1.4), we have

$$\int_\Omega \frac{\partial}{\Delta t}\left(r(t)\left(\frac{\partial u}{\Delta t}(x,t)\right)\right)\mathrm{d}x + q(t)\int_\Omega \frac{\partial u}{\Delta t}(x,t)\mathrm{d}x + P(t)b(t)\int_\Omega u(x,t)\,\mathrm{d}x \le 0.$$

Using Theorem [1, Theorem 1.117], we obtain

$$\frac{\partial}{\Delta t}\left(r(t)\left(\frac{\partial}{\Delta t}\int_\Omega u(x,t)\,\mathrm{d}x\right)\right) + q(t)\frac{\partial}{\Delta t}\int_\Omega u(x,t)\mathrm{d}x + b(t)P(t)\int_\Omega u(x,t)\mathrm{d}x \le 0.$$

Let $v(t) = \int_\Omega u(x,t)\,\mathrm{d}x$, for all $t \in \mathcal{J}_{t_0}$, It implies that.

$$\left(r(t)v^\Delta(t)\right)^\Delta + q(t)v^\Delta(t) + b(t)P(t)v(t) \le 0 \quad \text{for all } t \in \mathcal{J}_{t_1}. \tag{3.4}$$

Let $\alpha(t) = \ominus - \frac{q(t)}{r(t)}$, for $t \in \mathcal{J}_{t_0}$, but, we also have

$$\begin{aligned}
\left(e_{\alpha(t)}(t,t_0)r(t)v^\Delta(t)\right)^\Delta &= e^\sigma_{\alpha(t)}(t,t_0)\left(r(t)v^\Delta(t)\right)^\Delta + \alpha(t)e_{\alpha(t)}(t,t_0)r(t)v^\Delta(t)\\
&= e^\sigma_{\alpha(t)}(t,t_0)\left[\left(r(t)v^\Delta(t)\right)^\Delta + \oplus\alpha(t)r(t)v^\Delta(t)\right]\\
&= e^\sigma_{\alpha(t)}(t,t_0)\left[\left(r(t)v^\Delta(t)\right)^\Delta + q(t)v^\Delta(t)\right]\\
&\le -e^\sigma_{\alpha(t)}(t,t_0)b(t)P(t)v(t) \quad \text{for all } t \in \mathcal{J}_{t_1}.
\end{aligned}$$

Thus, $e_{\alpha(t)}(t,t_0)r(t)v^\Delta(t)$ is a nonincreasing function on $\mathcal{J}_{t_0}$. We now claim that $e_{\alpha(t)}(t,t_0)r(t)v^\Delta(t) > 0$, for $t \in \mathcal{J}_{t_1}$. If not, then there exists $t_2 \in \mathcal{J}_{t_1}$, such that

$$e_{\alpha(t)}(t,t_0)r(t)v^\Delta(t) \le -e_{\alpha(t)}(t_2,t_0)b(t_2)v^\Delta(t_2) = -c < 0 \quad \text{for all } t \in \mathcal{J}_{t_2}.$$

It means that

$$v^\Delta(t) \le -c\frac{e_{\ominus\alpha(t)}(t,t_0)}{r(t)} \quad \text{for all } t \in \mathcal{J}_{t_2}.$$

Integrating from t_2 to t, we get

$$v(t) \le v(t_2) - c\int_{t_2}^t \frac{e_{\ominus\alpha(t)}(s,t_0)}{r(s)}\Delta s \to -\infty \quad \text{as } t \to \infty,$$

which defies the assertion tha $v(t) > 0$ for $t \in \mathcal{J}_{t_2}$ exists.

By using the generalized Riccati substitution, define the function w,

$$w(t) = \delta(t)e_{\alpha(t)}(t,t_0)r(t)\frac{v^\Delta(t)}{v(t)} \quad \text{for all } t \in \mathcal{J}_{t_2}. \tag{3.5}$$

Then $w(t) > 0$ for $t \in \mathcal{J}_{t_2}$. Differentiating (3.5) and using (3.4), we have

$$\begin{aligned}
w^\Delta(t) &= \frac{\delta^\Delta(t)}{\delta(t)}w(t) + \frac{\delta(t)}{v(t)}\left(e_{\alpha(t)}(t,t_0)r(t)v^\Delta(t)\right)^\Delta - \delta(t)e_{\alpha(t)}(t,t_0)r(t)\frac{\left(v^\Delta(t)\right)^2}{v(t)v^\sigma(t)}\\
&\le -\delta(t)e^\sigma_{\alpha(t)}(t,t_0)b(t)P(t) + \frac{\delta^\Delta(t)}{\delta(t)}w(t) - \frac{w(t)}{\delta(t)e_{\alpha(t)}(t,t_0)r(t)}.
\end{aligned}$$

Using the inequality, $Bx - Ax^2 \leq \frac{B^2}{4A}$, we have

$$w^{\Delta}(t) \leq -\delta(t) e^{\sigma}_{\alpha(t)}(t,t_0) b(t) P(t) + \frac{1}{4} e_{\alpha(t)}(t,t_0) r(t) \frac{\left[\delta^{\Delta}(t)\right]^2}{\delta(t)}.$$

Integrating the last inequality from t_2 to t and taking the limit supremum on both sides, yields

$$\limsup_{t\to\infty} \int_{t_2}^{t} \delta(s) e^{\sigma}_{\alpha(t)}(s,t_0) b(s) P(s) - \frac{1}{4} e_{\alpha(t)}(s,t_0) r(s) \frac{\left[\delta^{\Delta}(s)\right]^2}{\delta(s)} \Delta s \leq w(t_2),$$

which is in opposition to equation (3.3). The theorem has been fully proven. □

Theorem 2. *Assume that conditions* (C_1) *through* (C_2) *are met, in which case, for all suitably large* $t_1 \in \mathcal{I}_{t_0}$,

$$\int_{t}^{\infty} \frac{1}{\pi(s,t_1)} \left(\int_{t_1}^{s} b(\tau) P(\tau) \Delta\tau \right) \Delta s > 1 \quad \text{for all } t \in \mathcal{I}_{t_1}, \tag{3.6}$$

where

$$\pi(t,t_1) = r(t) + e_{\ominus\left[-\frac{q}{r}\right]}(t,t_0) r(t) \int_{t_1}^{t} e_{-\frac{q}{r}}(s,s_0) \frac{q(s)}{r(s)} \Delta s \quad \text{for all } t \in \mathcal{I}_{t_1}. \tag{3.7}$$

If there exists a positive function $\delta \in \mathcal{C}^1_{rd}(\mathcal{I}_{t_0}, \mathbb{R})$, *such that* (3.3) *holds. Then any solution of* (1.1)-(1.2) *is oscillatory.*

Proof. If (1.1)-(1.2) has a non-oscillatory solution u on $\mathcal{I}_{t_0}$, then this is assumed. Without sacrificing generality, we can assume that there is $t_1 \in \mathcal{I}_{t_0}$ such that $u(t,x) \geq 0$ and $v(t) = \int_{\Omega} u(t,x)\,dx > 0$, for $t \in \mathcal{I}_{t_1}$. Since similar justifications can be provided for the cases where $u(t,x) \geq 0$ and $v(t) = \int_{\Omega} u(t,x)\,dx < 0$, for $t \in \mathcal{I}_{t_1}$. Let u be an eventual solution of (1.1)-(1.2) that is positive. Inequality (3.4) must be satisfied for all $t \in \mathcal{I}_{t_1}$ such that only one of the two following cases occurs, just like in the proof of Theorem 1.

Case 1: $v^{\Delta}(t) < 0$ and $\left(e_{\alpha(t)}(t,t_0) r(t) v^{\Delta}(t)\right)^{\Delta} < 0$, for $t \in \mathcal{I}_{t_1}$. By integrating the inequality (3.4) from time t_1 to t, we arrive at

$$r(t)v^{\Delta}(t) + \int_{t_1}^{t} q(s) v^{\Delta}(s) \Delta s + \int_{t_1}^{t} b(s) P(s) v(s) \Delta s \leq r(t_1) v^{\Delta}(t_1) \quad \text{for } t \in \mathcal{I}_{t_1}.$$

Since $v < 0$ and $e_{\alpha(t)}(.,t_0) r v^{\Delta}$are decreasing on $\mathcal{I}_{t_1}$, we obtain

$$\left[r(t) + e_{\alpha(t)}(t,t_0) r(t) \int_{t_1}^{t} \frac{q(s)}{e_{\alpha(t)}(s,s_0) r(s)} \Delta s \right] v^{\Delta}(t) \leq -v(t) \int_{t_1}^{t} b(s) P(s) \Delta s.$$

It means that

$$v^{\Delta}(t) \leq -\frac{v(t)}{\pi(t,t_1)} \int_{t_1}^{t} b(s) P(s) \Delta s \quad \text{for all } t \in \mathcal{I}_{t_1}.$$

Integrating the above inequality over $[t,\infty)$, we obtain

$$\lim_{t\to\infty} v(t) \le v(t) - \int_t^\infty \frac{v(s)}{\pi(s,t_1)} \left(\int_{t_1}^s b(\tau) P(\tau) \Delta\tau \right) \Delta s.$$

Since v is positive and nonincreasing on $\mathcal{I}_{t_1}$, then $\lim_{t\to\infty} v(t)$ exists and is positive, we have

$$0 \le v(t) \left(1 - \int_t^\infty \frac{1}{\pi(s,t_1)} \left(\int_{t_1}^s b(\tau) P(\tau) \Delta\tau \right) \Delta s \right) \quad \text{for all } t \in \mathcal{I}_{t_1}.$$

Since $v(t) > 0$, for $t \in \mathcal{I}_{t_1}$, what implies

$$\int_t^\infty \frac{1}{\pi(s,t_1)} \left(\int_{t_1}^s b(\tau) P(\tau) \Delta\tau \right) \Delta s \le 1 \quad \text{for all } t \in \mathcal{I}_{t_1},$$

which is in conflict (3.6).

Case 2: $v^\Delta(t) > 0$ and $\left(e_{\alpha(t)}(t,t_0) r(t) v^\Delta(t)\right)^\Delta < 0$, for $t \in \mathcal{I}_{t_1}$. The proof in this instance is identical to that of Theorem 1, so it is omitted. The evidence is now complete.

□

4. Oscillation with Dirichlet boundary condition

In this section, we establish some sufficient conditions which guarantee that every solution u of (1.1)-(1.3) oscillates on $\mathcal{I}_{t_0}$.

Theorem 3. *Assume that conditions* $(C_1)-(C_2)$ *and* (3.2) *holds. If there exists a positive function* $\theta \in C^1_{rd}(\mathcal{I}_{t_0},\mathbb{R})$ *and a constant* $\lambda > 0$*, such that*

$$\limsup_{t\to\infty} \int_{t_0}^t \gamma_\theta(s,t_0) \Delta s = \infty. \tag{4.1}$$

where

$$\gamma_\theta(t,t_0) := \theta(t) e_{\ominus\left[-\frac{q}{r}\right]}(\sigma(t),t_0) \left[b(t) P(t) + \lambda a(t) \right] - e_{\ominus\left[-\frac{q}{r}\right]}(t,t_0) \frac{r(t)\left[\theta^\Delta(t)\right]^2}{4\theta(t)},$$

and P *is defined in* (3.1).

Then any solution of (1.1)-(1.3) *is oscillatory.*

Proof. Suppose that (1.1)-(1.3) has a nonoscillatory solution u on $\mathcal{I}_{t_0}$. We may assume without loss of generality that there exists $t_1 \in \mathcal{I}_{t_0}$ such that $u(t,x) \ge 0$ and $v(t) = \int_\Omega u(t,x)\,dx > 0$, for $t \in \mathcal{I}_{t_0}$. Since similar arguments can be made, for the case $u(t,x) \ge 0$ and $v(t) = \int_\Omega u(t,x)\,dx > 0$, for $t \in \mathcal{I}_{t_0}$. Let u be an eventually positive solution of (1.1)-(1.3). Let $\psi\colon \Omega \to \mathbb{R}$ be a function defined by

$$\psi(x) := \sum_{i=1}^{i=n} e^{\sqrt{\lambda} x_i} \quad \text{for all } x \in \Omega.$$

By Green's theorem and (1.3), we have

$$\int_\Omega \psi(x)\,\Delta_x u(x,t)\,dx = -\int_\Omega u(x,t)\,\Delta\psi(x)\,dx + \int_\Gamma u(x,t)\,\nabla\psi(x)\,ds$$
$$= -\int_\Omega u(x,t)\,\Delta\psi(x)\,dx = -\lambda\int_\Omega u(x,t)\,\psi(x)\,dx.$$

We multiply (1.1) by ψ and integrate with respect to x over Ω and get

$$\left(r(t)\left(v^\Delta(t)\right)\right)^\Delta + q(t)v^\Delta(t) + \left(b(t)P(t) + \lambda a(t)\right)v(t) \le 0 \quad \text{for all } t \in \mathcal{J}_{t_1}. \tag{4.2}$$

The rest of proof is similar to that of Theorem 1. □

Theorem 4. *Assume that conditions $(C_1)-(C_2)$ holds, such that for all sufficiently large $t_1 \in \mathcal{J}_{t_0}$. If there exists a constant $\beta > 0$, such that*

$$\int_t^\infty \frac{1}{\pi(s,t_1)}\left(\int_{t_1}^s b(\tau)P(\tau) + \beta a(\tau)\Delta\tau\right)\Delta s > 1 \quad \text{for all } t \in \mathcal{J}_{t_1},$$

where $\pi(.,t_1)$ is defined in (3.7) and P is defined in (3.1).

If there exists a positive function $\theta \in C^1_{rd}(\mathcal{J}_{t_0},\mathbb{R})$ and a constant $\lambda > 0$, such that (4.1) holds. Then any solution of (1.1)-(1.2) is oscillatory.

Proof. Suppose that (1.1)-(1.3) has a nonoscillatory solution u on $\mathcal{J}_{t_0}$. We may assume without loss of generality that there exists $t_1 \in \mathcal{J}_{t_0}$ such that $u(t,x) \ge 0$ and $v(t) = \int_\Omega u(t,x)\,dx > 0$, for $t \in \mathcal{J}_{t_0}$. Since similar arguments can be made, for the case $u(t,x) \ge 0$ and $v(t) = \int_\Omega u(t,x)\,dx > 0$, for $t \in \mathcal{J}_{t_0}$. Let u be an eventually positive solution of (1.1)-(1.3). As in the proof of Theorem 3, inequality (4.2) is satisfied for all $t \in \mathcal{J}_{t_1}$, such that only one of the following two cases happens. Case 1. $v^\Delta(t) > 0$ and $\left(e_{\alpha(t)}(t,t_0)\,r(t)\,v^\Delta(t)\right)^\Delta < 0$, for $t \in \mathcal{J}_{t_1}$. The proof, in this case, is the same as that of Theorem 3. Case 2. $v^\Delta(t) < 0$ and $\left(e_{\alpha(t)}(t,t_0)\,r(t)\,v^\Delta(t)\right)^\Delta < 0$, for $t \in \mathcal{J}_{t_1}$. Integrating the inequality (4.2) over $[t_1,t)$, we obtain

$$r(t)v^\Delta(t) + \int_{t_1}^t q(s)v^\Delta(s)\Delta s + \int_1^t \left(b(s)P(s) + \lambda a(s)\right)v(s)\,\Delta s \le 0 \quad \text{for all } t \in \mathcal{J}_{t_1}.$$

The rest of proof is similar to that of Theorem 2. This completes the proof. □

5. Examples

We provide an example to demonstrate our major finding in this section.

Example 1. Consider the Neumann boundary problem

$$\begin{cases} \partial_t^2 u(x,t) + \partial_t u(x,t) + (\|x\| + t)\,u(x,t) = t\Delta_x u(x,t) & \text{for } (x,t) \in B_\rho(0,d) \times \mathbb{R}^+, \\ \nabla_x u(x,t) = 0 & \text{for all } (x,t) \in S(0,d) \times \mathbb{R}^+, \end{cases}$$

with $d > 0$ and $B_\rho(0,d) = \{x \in \mathbb{R}^n : \|x\| < d\}$. Here, $\Omega = B_\rho(0,d)$, $\Gamma = S(0,\rho)$, $\mathbb{T} = \mathbb{R}$, $r(t) = q(t) = 1$, $p(t,x) = \|x\| + t$, $a(t) = t$, and $f(t,x,u) = u$. Then $b(t) =$

t, $P(t)=t$, $e_{\ominus-\frac{q(t)}{r(t)}}(s,t)=e^{t-s}$ and the hypotheses (C_1)-(C_2), (3.2) are true. Let $\delta(t)=1$, for $t\in\mathbb{R}^+$. Therefore, (1) is valid. Equation (1) is oscillatory according to Theorem1.

Example 2. Consider the Dirichlet boundary problem

$$\begin{cases}\Delta_t\left(t.\Delta_t u(x,t)\right)+\frac{t}{2}\Delta_t u(x,t)+u(x,t)=t\Delta_x u(x,t) & \text{for } (x,t)\in\Omega\times\mathbb{Z}^+,\\ u(x,t)=0 & \text{for } (x,t)\in\Gamma\times\mathbb{Z}^+.\end{cases}$$

In this case, Ω is a bounded domain in $\mathbb{R}^n$ with piecewise smooth boundary $\Gamma=\partial\Omega$. Here, $\mathbb{T}=\mathbb{R}$, $r(t)=t$, $q(t)=\frac{t}{2}$, $p(t,x)=1$, $a(t)=t$, and $f(t,x,u)=u$. Then $b(t)=t$, $P(t)=1$, and

$$e_{\ominus-\frac{q(t)}{r(t)}}(s,t)=e_1(s,t)=2^{t-s}\quad\text{for all } t,s\in\mathbb{Z}^+.$$

Then, (3.2) is maintained. By substituting in (4.1) and setting $\theta(t)=1$, for $t\in\mathbb{Z}^+$, we get

$$\limsup_{t\to\infty}\int_{t_0}^{t}2^{s+1}\left[s^2+\lambda s\right]\Delta s=+\infty\quad\text{for all }\lambda>0.$$

Therefore, (4.1) is valid. Equation (2) is oscillatory according to Theorem 3.

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INCORPORATING VIGILANCE AND HARVESTING INTO A DISCRETE PREDATOR-PREY SYSTEM WITH IMPRECISE PARAMETER ESTIMATES

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Abstract. In this paper, we consider a discrete predator-prey system with imprecise biological parameters, and consider the harvesting of prey and predator. We first study the existence and local stability conditions of the ecological equilibrium points of the system. Secondly, the possible bioeconomic equilibrium points of the system are analyzed. Thirdly, the optimal harvesting strategy and the optimal equilibrium solution are obtained by using the Pontryagin maximum principle. Finally, the feasibility of the theoretical results is verified by numerical simulation, and the influence of various parameters in the model on the population density of prey and predator are analyzed.

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1. INTRODUCTION

As a renewable natural resource, biological resources can evolve and maintain a stable number by means of growth and reproduction under natural and artificial conditions. If it is used and managed reasonably, it will be inexhaustible. Let $x(t)$ and $y(t)$ denote the population densities of prey and predator, respectively, and a general predator-prey system with harvesting can be written as:

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = rx\left(1-\frac{x}{K}\right) - f(x)y - q_1E_1x, \\ \frac{\mathrm{d}y}{\mathrm{d}t} = ef(x)y - dy - q_2E_2y, \end{cases} \tag{1.1}$$

where r is the prey's intrinsic growth rate and K is the environmental carrying capacity of the prey. d is the predator's natural death rate. e denotes the conversion

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efficiency from prey biomass to predator biomass. q_1, q_2 (>0) are the catchability coefficient of two species. E_1 and E_2 are the harvesting effort for prey and predator populations, respectively. $f(x)$ represents the functional response of the predator. Many scholars have studied the harvesting problem of various population systems on the basis of system (1.1), and have achieved many results in the fields of fish resource management, pest control and so on [4, 6].

In the survival strategy of the prey, the vigilance effect helps the prey to avoid the attack of the predator. The prey can determine when the attack occurs based on high vigilance, so as to take necessary actions, such as escape, hiding, and maintaining a safe distance [7, 14]. We assume that vigilance will have a negative impact on the predation rate, that is, as vigilance increases, the predation rate will decrease, and vice versa. Therefore, the above system (1.1) in the presence of vigilance takes the following form:

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = rx\left(1-v-\frac{x}{K}\right) - \frac{\beta xy}{l+\alpha v} - q_1E_1x, \\ \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{axy}{l+\alpha v} - dy - q_2E_2y, \end{cases} \tag{1.2}$$

where $f(x) = \beta x$, β is the predation rate of the predator, $a = e\beta$ stands for the efficiency rate with which captured prey are converted to new predators, $\frac{1}{l}$ is the lethality of the predator, v is the level of prey vigilance, and α denotes the effectiveness of vigilance.

In the above discussion, all researchers have developed models based on the assumption that biological parameters are precisely known, but in real life the situation is different. Biological parameters may not be fixed, and may change for several reasons. For example, lack of sufficient information data and experimental errors. Therefore, the model with inaccurate biological parameters is more reasonable. In 2013, Pal et al. [12] first proposed the concept of interval number and used it in a Lotka-Volterra prey-predator capture model with uncertain parameters. They investigated the existence of biological and bionomic equilibrium points, the local stability of biological equilibrium points, the optimal harvesting policy and the solution in the interior equilibrium. On the basis of model (1.2), we establish a predator-prey system with vigilance effect and inaccurate biological parameters,

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = \bar{r}x\left(1-v-\frac{x}{K}\right) - \frac{\bar{\beta}xy}{l+\alpha v} - q_1E_1x, \\ \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\bar{a}xy}{l+\alpha v} - \bar{d}y - q_2E_2y, \end{cases} \tag{1.3}$$

where $\bar{r} \in [r_1, r_2]$, $\bar{\beta} \in [\beta_1, \beta_2]$, $\bar{a} \in [a_1, a_2]$, $\bar{d} \in [d_1, d_2]$ and $r_1 > 0$, $r_2 > 0$, $\beta_1 > 0$, $\beta_2 > 0$, $c_1 > 0$, $c_2 > 0$, $d_1 > 0$, $d_2 > 0$.

Based on Theorem 1 in [12], we easily obtain that the above system (1.3) is equivalent to the system

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = r_1^{1-k}r_2^kx - r_2^{1-k}r_1^kx\left(v+\frac{x}{K}\right) - \beta_2^{1-k}\beta_1^k\frac{xy}{l+\alpha v} - q_1E_1x, \\ \frac{\mathrm{d}y}{\mathrm{d}t} = a_1^{1-k}a_2^k\frac{xy}{l+\alpha v} - d_2^{1-k}d_1^ky - q_2E_2y, \end{cases} \tag{1.4}$$

where $k \in [0,1]$.

However, in recent years, an increasing number of experts have discovered that the discrete model is more realistic than the continuous model for populations with a short lifetime, non-overlapping generations, or a limited number of populations. At the same time, discrete systems have more dynamic phenomena than continuous systems [1–3, 5, 8–10, 13]. As far as we know, the discrete form of system (1.4) has not been studied. Therefore, we discretize system (1.4) by Euler difference method and obtain the following discrete-time system:

$$\begin{cases} x_{t+1} = (1 + r_1^{1-k} r_2^k) x_t - r_2^{1-k} r_1^k x_t \left(v + \frac{x_t}{K}\right) - \beta_2^{1-k} \beta_1^k \frac{x_t y_t}{l + \alpha v} - q_1 E_1 x_t, \\ y_{t+1} = (1 - d_2^{1-k} d_1^k) y_t + a_1^{1-k} a_2^k \frac{x_t y_t}{l + \alpha v} - q_2 E_2 y_t. \end{cases} \tag{1.5}$$

The rest of this paper is organized as follows. In Section 2, the existence and stability of the equilibrium points of system (1.5) are analyzed in detail. Also, the existence of bionomic equilibriums of system (1.5) is discussed in Section 3. Furthermore, we study the optimal harvesting policy for system (1.5) in Section 4. In Section 5, we give three numerical examples and two tables to substantiate our analytical results. Finally, some conclusions are made in the last section.

2. The existence and stability of fixed points

The equilibrium points of system (1.5) satisfy the following equations:

$$\begin{cases} r_1^{1-k} r_2^k x - r_2^{1-k} r_1^k x \left(v + \frac{x}{K}\right) - \beta_2^{1-k} \beta_1^k \frac{xy}{l + \alpha v} - q_1 E_1 x = 0, \\ a_1^{1-k} a_2^k \frac{xy}{l + \alpha v} - d_2^{1-k} d_1^k y - q_2 E_2 y = 0. \end{cases}$$

By calculating we can get the equilibrium points of the system as follows :

(1) Trivial equilibrium: $A_0 = (0,0)$.

(2) Semi-trivial equilibrium $A_1 = (\Theta, 0)$ exists if $r_1^{1-k} r_2^k - r_2^{1-k} r_1^k v - q_1 E_1 > 0$, where $\Theta = \frac{K(r_1^{1-k} r_2^k - r_2^{1-k} r_1^k v - q_1 E_1)}{r_2^{1-k} r_1^k}$.

(3) Interior equilibrium: $A^* = (x^*, y^*)$, where

$$x^* = \frac{(l + \alpha v)(d_2^{1-k} d_1^k + q_2 E_2)}{a_1^{1-k} a_2^k}, y^* = \frac{(l + \alpha v)[r_1^{1-k} r_2^k - r_2^{1-k} r_1^k \left(v + \frac{x^*}{K}\right) - q_1 E_1]}{\beta_2^{1-k} \beta_1^k}$$

exists if

$$r_1^{1-k} r_2^k - r_2^{1-k} r_1^k \left(v + \frac{x^*}{K}\right) - q_1 E_1 > 0.$$

Assuming that λ_1 and λ_2 are the eigenvalues of system (1.5) at the equilibrium point (x, y), there is the following definition:

Definition 1 ([15, Definition 2.1]). The equilibrium point (x, y) is called

(1) sink point if $|\lambda_1| < 1$ and $|\lambda_2| < 1$, and it is locally asymptotically stable;

(2) source point if $|\lambda_1| > 1$ and $|\lambda_2| > 1$, and it is locally unstable;

(3) saddle point if either ($|\lambda_1|>1$ and $|\lambda_2|<1$) or ($|\lambda_1|<1$ and $|\lambda_2|>1$);
(4) non-hyperbolic point if either $|\lambda_1|=1$ or $|\lambda_2|=1$.

Jacobian matrix can be evaluated at $A_0(0,0)$ as

$$J(E_0)=\begin{pmatrix}1+r_1^{1-k}r_2^k-r_2^{1-k}r_1^k v-q_1E_1 & 0\\ 0 & 1-d_2^{1-k}d_1^k-q_2E_2\end{pmatrix}. \tag{2.1}$$

The eigenvalues of the Jacobian are $\lambda_1=1+r_1^{1-k}r_2^k-r_2^{1-k}r_1^k v-q_1E_1$ and $\lambda_2=1-d_2^{1-k}d_1^k-q_2E_2$ at trivial equilibrium point $A_0(0,0)$. So, we can get the following conclusions.

Lemma 1.

(1) $A_0(0,0)$ *is a sink if*
$$0<r_2^{1-k}r_1^k v+q_1E_1-r_1^{1-k}r_2^k<2 \text{ and } 0<d_2^{1-k}d_1^k+q_2E_2<2.$$
(2) $A_0(0,0)$ *is a saddle if one of the following conditions holds:*
(II-1) $r_1^{1-k}r_2^k-r_2^{1-k}r_1^k v-q_1E_1>0$ *and* $0<d_2^{1-k}d_1^k+q_2E_2<2$;
(II-2) $2+r_1^{1-k}r_2^k-r_2^{1-k}r_1^k v-q_1E_1<0$ *and* $0<d_2^{1-k}d_1^k+q_2E_2<2$;
(II-3) $0<r_2^{1-k}r_1^k v+q_1E_1-r_1^{1-k}r_2^k<2$ *and* $2-d_2^{1-k}d_1^k-q_2E_2<0$.
(3) $A_0(0,0)$ *is a source if one of the following conditions holds:*
(III-1) $r_1^{1-k}r_2^k-r_2^{1-k}r_1^k v-q_1E_1>0$ *and* $2-d_2^{1-k}d_1^k-q_2E_2<0$;
(III-2) $2+r_1^{1-k}r_2^k-r_2^{1-k}r_1^k v-q_1E_1<0$ *and* $2-d_2^{1-k}d_1^k-q_2E_2<0$.

Proof.

(1): It can be obtained from (2.1) that the eigenvalues of the system at the equilibrium point A_0 are $\lambda_1=1+r_1^{1-k}r_2^k-r_2^{1-k}r_1^k v-q_1E_1$ and $\lambda_2=1-d_2^{1-k}d_1^k-q_2E_2$. It can be seen from Definition 1 that when $\lambda_1<1$ and $\lambda_2<1$, the equilibrium point is the sink. Therefore, when $0<r_2^{1-k}r_1^k v+q_1E_1-r_1^{1-k}r_2^k<2$ and $0<d_2^{1-k}d_1^k+q_2E_2<2$, the equilibrium point A_0 is a sink point and is locally asymptotically stable.
(2) and (3): These can be proved similarly.

□

Jacobian matrix can be evaluated at $A_1(\Theta,0)$ as

$$J(E_K)=\begin{pmatrix}1+r_1^{1-k}r_2^k-r_2^{1-k}r_1^k\left(v+\frac{2\Theta}{K}\right)-q_1E_1 & \frac{\beta_2^{1-k}\beta_1^k\Theta}{l+\alpha v}\\ 0 & 1+\frac{a_1^{1-k}a_2^k}{l+\alpha v}-d_2^{1-k}d_1^k-q_2E_2\end{pmatrix}. \tag{2.2}$$

Lemma 2. *The characteristic roots at the boundary equilibrium point* $A_1(\Theta,0)$ *are* $\lambda_1=1+r_1^{1-k}r_2^k-r_2^{1-k}r_1^k\left(v+\frac{2\Theta}{K}\right)-q_1E_1, \lambda_2=1+\frac{a_1^{1-k}a_2^k}{l+\alpha v}-d_2^{1-k}d_1^k-q_2E_2$, *then*

(1) *A_1 is a sink if*

$$0 < r_2^{1-k} r_1^k \left(v + \frac{2\Theta}{K} \right) + q_1 E_1 - r_1^{1-k} r_2^k < 2 \text{ and } 0 < d_2^{1-k} d_1^k + q_2 E_2 - \frac{a_1^{1-k} a_2^k}{l + \alpha v} < 2.$$

(2) *A_1 is a saddle if one of the following conditions holds:*

(II-1) $0 < r_2^{1-k} r_1^k \left(v + \frac{2\Theta}{K} \right) + q_1 E_1 - r_1^{1-k} r_2^k < 2$ *and* $\frac{a_1^{1-k} a_2^k}{l+\alpha v} - d_2^{1-k} d_1^k - q_2 E_2 > 0$;

(II-2) $0 < r_2^{1-k} r_1^k \left(v + \frac{2\Theta}{K} \right) + q_1 E_1 - r_1^{1-k} r_2^k < 2$ *and* $2 + \frac{a_1^{1-k} a_2^k}{l+\alpha v} - d_2^{1-k} d_1^k - q_2 E_2 < 0$;

(II-3) $r_1^{1-k} r_2^k - r_2^{1-k} r_1^k \left(v + \frac{2\Theta}{K} \right) - q_1 E_1 > 0$ *and* $0 < d_2^{1-k} d_1^k + q_2 E_2 - \frac{a_1^{1-k} a_2^k}{l+\alpha v} < 2$;

(II-4) $2 + r_1^{1-k} r_2^k - r_2^{1-k} r_1^k \left(v + \frac{2\Theta}{K} \right) - q_1 E_1 < 0$ *and* $0 < d_2^{1-k} d_1^k + q_2 E_2 - \frac{a_1^{1-k} a_2^k}{l+\alpha v} < 2$.

(3) *A_1 is a source if one of the following conditions holds:*

(III-1) $r_1^{1-k} r_2^k - r_2^{1-k} r_1^k \left(v + \frac{2\Theta}{K} \right) - q_1 E_1 > 0$ *and* $\frac{a_1^{1-k} a_2^k}{l+\alpha v} - d_2^{1-k} d_1^k - q_2 E_2 > 0$;

(III-2) $r_1^{1-k} r_2^k - r_2^{1-k} r_1^k \left(v + \frac{2\Theta}{K} \right) - q_1 E_1 > 0$ *and* $2 + \frac{a_1^{1-k} a_2^k}{l+\alpha v} - d_2^{1-k} d_1^k - q_2 E_2 < 0$;

(III-3) $2 + r_1^{1-k} r_2^k - r_2^{1-k} r_1^k \left(v + \frac{2\Theta}{K} \right) - q_1 E_1 < 0$ *and* $\frac{a_1^{1-k} a_2^k}{l+\alpha v} - d_2^{1-k} d_1^k - q_2 E_2 > 0$;

(III-4) $2 + r_1^{1-k} r_2^k - r_2^{1-k} r_1^k \left(v + \frac{2\Theta}{K} \right) - q_1 E_1 < 0$ *and* $2 + \frac{a_1^{1-k} a_2^k}{l+\alpha v} - d_2^{1-k} d_1^k - q_2 E_2 < 0$.

Proof.

(1): It can be obtained from (2.2) that the eigenvalues of the system at the equilibrium point A_1 are $\lambda_1 = 1 + r_1^{1-k} r_2^k - r_2^{1-k} r_1^k \left(v + \frac{2\Theta}{K} \right) - q_1 E_1$ and $\lambda_2 = 1 + \frac{a_1^{1-k} a_2^k}{l+\alpha v} - d_2^{1-k} d_1^k - q_2 E_2$. It can be seen from Definition 1 that when $\lambda_1 < 1$ and $\lambda_2 < 1$, the equilibrium point is the sink. Therefore, when $0 < r_2^{1-k} r_1^k \left(v + \frac{2\Theta}{K} \right) + q_1 E_1 - r_1^{1-k} r_2^k < 2$ and $0 < d_2^{1-k} d_1^k + q_2 E_2 - \frac{a_1^{1-k} a_2^k}{l+\alpha v} < 2$, the equilibrium point A_1 is a sink point and is locally asymptotically stable.

(2) and (3): These can be proved similarly.

□

$J(x,y)$ evaluated at the positive equilibrium point $A^* = (x^*, y^*)$ is

$$J(E^*) = \begin{pmatrix} 1 - \frac{x^*}{K} r_2^{1-k} r_1^k & -\frac{\beta_2^{1-k} \beta_1^k x^*}{l+\alpha v} \\ \frac{a_1^{1-k} a_2^k y^*}{l+\alpha v} & 1 \end{pmatrix}. \tag{2.3}$$

Let

$$T = 2 - \frac{x^*}{K} r_2^{1-k} r_1^k,$$

$$D = 1 - \frac{x^*}{K} r_2^{1-k} r_1^k + \frac{a_1^{1-k} a_2^k \beta_2^{1-k} \beta_1^k x^* y^*}{(l+\alpha v)^2}.$$

Then, the characteristic equation corresponding to matrix (2.3) is:

$$\lambda^2 - T\lambda + D = 0.$$

Lemma 3 ([11, Lemma 2.2]). *Assume that $F(\lambda) = \lambda^2 - T\lambda + D$, and $F(1) > 0$ with λ_1, λ_2 are roots of $F(\lambda) = 0$. Then the following results hold true:*

(1) $|\lambda_1| < 1$ *and* $|\lambda_2| < 1$ *if and only if* $F(-1) > 0$ *and* $D < 1$;
(2) $|\lambda_1| < 1$ *and* $|\lambda_2| > 1$, *or* $|\lambda_1| > 1$ *and* $|\lambda_2| < 1$ *if and only if* $F(-1) < 0$;
(3) $|\lambda_1| > 1$ *and* $|\lambda_2| > 1$ *if and only if* $F(-1) > 0$ *and* $D > 1$;
(4) $\lambda_1 = 1$ *and* $\lambda_2 \neq 1$ *if and only if* $F(-1) = 0$ *and* $D \neq 0, 2$;
(5) λ_1 *and* λ_2 *are complex and* $|\lambda_1| = 1$ *and* $|\lambda_2| = 1$ *if and only if* $T^2 - 4D < 0$ *and* $D = 1$

In summary, we have the following theorem.

Theorem 1. *System (1.5) at the equilibrium point $A^* = (x^*, y^*)$ is local asymptotically stable when the condition*

$$\frac{a_1^{1-k} a_2^k \beta_2^{1-k} \beta_1^k x^* y^*}{(l+\alpha v)^2} < \frac{x^*}{K} r_2^{1-k} r_1^k < \frac{a_1^{1-k} a_2^k \beta_2^{1-k} \beta_1^k x^* y^*}{2(l+\alpha v)^2} + 2 \tag{2.4}$$

holds.

Proof. According to Definition 1 and Lemma 3 (1), when $D < 1$ and $F(-1) > 0$ hold, the equilibrium point A^* is the sink point and is locally asymptotically stable. It can be seen from formula (2.3) that

$$T = 2 - \frac{x^*}{K} r_2^{1-k} r_1^k, \quad D = 1 - \frac{x^*}{K} r_2^{1-k} r_1^k + \frac{a_1^{1-k} a_2^k \beta_2^{1-k} \beta_1^k x^* y^*}{(l+\alpha v)^2},$$

so the equilibrium point A^* is locally asymptotically stable when condition (2.4) is established. □

3. Bionomic equilibrium

If the total revenue (TR) from selling harvested organisms is equal to the total cost (TC) of harvesting effort, an economic equilibrium can be obtained. To discuss the bionomic equilibrium of the imprecise prey-predator system, we consider the parameters such as

c_1=fishing cost per unit effort for prey species;
c_2=fishing cost per unit effort for predator species;
p_1=price per unit biomass of the prey;
p_2=price per unit biomass of the predator.
Net income N is expressed as:

$$N = (p_1 q_1 x - c_1) E_1 + (p_2 q_2 y - c_2) E_2 = N_1 + N_2,$$

where N_1 and N_2 represent the net revenues for the prey and predator species, respectively. The bionomic equilibrium $(x_\infty, y_\infty, E_{1\infty}, E_{2\infty})$ is given by the following simultaneous equations

$$\begin{cases} (1+r_1^{1-k}r_2^k)x - r_2^{1-k}r_1^k x\left(v+\frac{x}{K}\right) - \beta_2^{1-k}\beta_1^k \frac{xy}{l+\alpha v} - q_1E_1x = 0, \\ (1-d_2^{1-k}d_1^k)y + a_1^{1-k}a_2^k \frac{xy}{l+\alpha v} - q_2E_2y = 0, \\ (p_1q_1x - c_1)E_1 + (p_2q_2y - c_2)E_2 = 0. \end{cases} \tag{3.1}$$

We discuss the bionomic equilibrium of system (3.1) in the following four cases.

Case I. If $c_1 > p_1q_1x$, that is, the cost of harvesting the prey x is greater than the income from selling them, then the harvesting of x needs to be stopped, and the harvesting of the predator y can continue. So $E_1 = 0$ and $c_2 < p_2q_2y$. We calculate that $y_\infty = \frac{c_2}{p_2q_2}$ and $(x_\infty, E_{2\infty})$ will be any point on the line

$$\frac{a_1^{1-k}a_2^k x}{l+\alpha v} = d_2^{1-k}d_1^k + q_2E_2$$

in the first quadrant of the xE_2-plane.

Case II. If $c_2 > p_2q_2y$, that is, the cost of harvesting the predator y is greater than the income from selling them, then the harvesting of y needs to be stopped, and the harvesting of the prey x can continue. So $E_2 = 0$ and $c_1 < p_1q_1x$. We calculate that $x_\infty = \frac{c_1}{p_1q_1}$ and $(y_\infty, E_{1\infty})$ will be any point on the line

$$\frac{\beta_2^{1-k}\beta_1^k y}{l+\alpha v} + q_1E_1 = r_1^{1-k}r_2^k - r_2^{1-k}r_1^k\left(v+\frac{x_\infty}{K}\right)$$

in the first quadrant of the yE_1-plane if $r_1^{1-k}r_2^k > r_2^{1-k}r_1^k\left(v+\frac{x_\infty}{K}\right)$.

Case III. If $c_1 > p_1q_1x$ and $c_2 > p_2q_2y$, that is, the cost of harvesting the prey x and the predator y is greater than the income from selling them, then the harvesting of x and y needs to be stopped. So $E_1 = 0$ and $E_2 = 0$.

Case IV. If $c_1 < p_1q_1x$ and $c_2 < p_2q_2y$, that is, both the prey x and the predator y can be harvested. This yields that $x_\infty = \frac{c_1}{p_1q_1}$ and $y_\infty = \frac{c_2}{p_2q_2}$. We have

$$E_{1\infty} = \frac{1}{q_1}\left[1 + r_1^{1-k}r_2^k - r_2^{1-k}r_1^k(v+\frac{x}{K}) - \beta_2^{1-k}\beta_1^k\frac{y_\infty}{l+\alpha v}\right]$$

and

$$E_{2\infty} = \frac{1}{q_2}\left[1 - d_2^{1-k}d_1^k + a_1^{1-k}a_2^k\frac{x_\infty}{l+\alpha v}\right].$$

We easily know that $E_{1\infty} > 0$ and $E_{2\infty} > 0$ provided that

$$1 + r_1^{1-k}r_2^k > r_2^{1-k}r_1^k\left(v+\frac{x}{K}\right) + \beta_2^{1-k}\beta_1^k\frac{y_\infty}{l+\alpha v} \tag{3.2}$$

and

$$1 + a_1^{1-k}a_2^k\frac{x_\infty}{l+\alpha v} > d_2^{1-k}d_1^k. \tag{3.3}$$

In summary, we have the following theorem.

Theorem 2.

(1) *The trivial bionomic equilibrium* $(x_\infty, y_\infty, 0, E_{2\infty})$ *exists, where* $y_\infty = \frac{c_2}{p_2 q_2}$ *and* $(x_\infty, E_{2\infty})$ *will be any point on the line*

$$\frac{a_1^{1-k} a_2^k x}{l+\alpha v} = d_2^{1-k} d_1^k + q_2 E_2$$

in the first quadrant of the xE_2*-plane.*

(2) *The trivial bionomic equilibrium* $(x_\infty, y_\infty, E_{1\infty}, 0)$ *exists, where* $x_\infty = \frac{c_1}{p_1 q_1}$ *and* $(y_\infty, E_{1\infty})$ *will be any point on the line*

$$\frac{\beta_2^{1-k}\beta_1^k y}{l+\alpha v} + q_1 E_1 = r_1^{1-k} r_2^k - r_2^{1-k} r_1^k \left(v + \frac{x_\infty}{K}\right)$$

in the first quadrant of the yE_1*-plane if* $r_1^{1-k} r_2^k > r_2^{1-k} r_1^k \left(v + \frac{x_\infty}{K}\right)$.

(3) *The nontrivial bionomic equilibrium* $(x_\infty, y_\infty, E_{1\infty}, E_{2\infty})$ *exists if* (3.2) *and* (3.3) *hold, where* $x_\infty = \frac{c_1}{p_1 q_1}$, $y_\infty = \frac{c_2}{p_2 q_2}$,

$$E_{1\infty} = \frac{1}{q_1}\left[1 + r_1^{1-k} r_2^k - r_2^{1-k} r_1^k \left(v + \frac{x}{K}\right) - \beta_2^{1-k}\beta_1^k \frac{y_\infty}{l+\alpha v}\right]$$

and

$$E_{2\infty} = \frac{1}{q_2}\left[1 - d_2^{1-k} d_1^k + a_1^{1-k} a_2^k \frac{x_\infty}{l+\alpha v}\right].$$

4. Optimal harvesting policy

In this section, we will study the optimal harvesting strategy of system (1.5) in three cases.

Case I. $E_1 = 0$ and $E_2 \neq 0$. That is, only the predator y is harvested. Then system (1.5) becomes:

$$\begin{cases} x_{t+1} = (1 + r_1^{1-k} r_2^k) x_t - r_2^{1-k} r_1^k x_t \left(v + \frac{x_t}{K}\right) - \beta_2^{1-k}\beta_1^k \frac{x_t y_t}{l+\alpha v}, \\ y_{t+1} = (1 - d_2^{1-k} d_1^k) y_t + a_1^{1-k} a_2^k \frac{x_t y_t}{l+\alpha v} - q_2 E_2 y_t. \end{cases}$$

We define the present value of the net revenue function as

$$J = \max \sum_{t=1}^{k} e^{-\delta t} (p_2 q_2 y - c_2) E_2(t)$$

We first construct the following Hamiltonian function:

$$H_t = \lambda_{1(t+1)} \left[(1 + r_1^{1-k} r_2^k) x_t - r_2^{1-k} r_1^k x_t \left(v + \frac{x_t}{K}\right) - \beta_2^{1-k}\beta_1^k \frac{x_t y_t}{l+\alpha v} - x_{t+1}\right]$$
$$+ \lambda_{2(t+1)} \left[(1 - d_2^{1-k} d_1^k) y_t + a_1^{1-k} a_2^k \frac{x_t y_t}{l+\alpha v} - q_2 E_2 y_t - y_{t+1}\right]$$

$$+e^{-\delta t}(p_2q_2y-c_2)E_2(t),$$

where λ_1 and λ_2 are adjoint variables. By Pontryagin's maximum principle, we obtain

$$\lambda_{1(t+1)} = -\frac{\partial H_t}{\partial x_t} = -\lambda_{1(t+1)}\left[r_1^{1-k}r_2^k - r_2^{1-k}r_1^k\left(v+\frac{2x_t}{K}\right) - \beta_2^{1-k}\beta_1^k\frac{y_t}{l+\alpha v}\right] - \lambda_{2(t+1)}\cdot\frac{a_1^{1-k}a_2^k y}{l+\alpha v}, \tag{4.1}$$

$$\lambda_{2(t+1)} = -\frac{\partial H_t}{\partial y_t} = -e^{\delta t}p_2q_2E_2 + \lambda_{1(t+1)}\cdot\beta_2^{1-k}\beta_1^k\frac{x_t}{l+\alpha v} - \lambda_{2(t+1)}\left[a_1^{1-k}a_2^k\frac{x_t}{l+\alpha v} - d_2^{1-k}d_1^k - q_2E_2\right], \tag{4.2}$$

$$\frac{\partial H_t}{\partial E_2} = e^{-\delta t}(p_2q_2y-c_2) - q_2y\lambda_{2(t+1)} = 0. \tag{4.3}$$

From (4.3) we have $\lambda_{2(t+1)} = \frac{1}{q_2y}e^{-\delta t}(p_2q_2y-c_2)$. Using $\lambda_{2(t+1)}$ in (4.1) we get $\lambda_{1(t+1)}$. Now substituting $\lambda_{1(t+1)}$ and $\lambda_{2(t+1)}$ in (4.2) we get the value of the control variable as E_2. Thus the optimal harvesting is obtained as E_2^* at the optimal population level (x_*, y_*).

Case II. $E_1 \neq 0$ and $E_2 = 0$. That is, only the prey x is harvested. Then system (1.5) becomes:

$$\begin{cases} x_{t+1} = (1+r_1^{1-k}r_2^k)x_t - r_2^{1-k}r_1^k x_t\left(v+\frac{x_t}{K}\right) - \beta_2^{1-k}\beta_1^k\frac{x_ty_t}{l+\alpha v} - q_1E_1x_t, \\ y_{t+1} = (1-d_2^{1-k}d_1^k)y_t + a_1^{1-k}a_2^k\frac{x_ty_t}{l+\alpha v}. \end{cases}$$

We define the present value of the net revenue function as

$$J = \max\sum_{t=1}^{k} e^{-\delta t}(p_1q_1x-c_1)E_1(t).$$

We first construct the following Hamiltonian function:

$$H_t = \lambda_{1(t+1)}\left[(1+r_1^{1-k}r_2^k)x_t - r_2^{1-k}r_1^k x_t\left(v+\frac{x_t}{K}\right) - \beta_2^{1-k}\beta_1^k\frac{x_ty_t}{l+\alpha v} - q_1E_1x_t - x_{t+1}\right] + \lambda_{2(t+1)}\left[(1-d_2^{1-k}d_1^k)y_t + a_1^{1-k}a_2^k\frac{x_ty_t}{l+\alpha v} - y_{t+1}\right] + e^{-\delta t}(p_1q_1x-c_1)E_1(t),$$

where λ_1 and λ_2 are adjoint variables. By Pontryagin's maximum principle, we obtain

$$\lambda_{1(t+1)} = -\frac{\partial H_t}{\partial x_t} = -\lambda_{1(t+1)}\left[r_1^{1-k}r_2^k - r_2^{1-k}r_1^k\left(v+\frac{2x_t}{K}\right) - \beta_2^{1-k}\beta_1^k\frac{y_t}{l+\alpha v} - q_1E_1\right]$$

$$-\lambda_{2(t+1)}\cdot\frac{c_1^{1-k}c_2^k y}{l+\alpha v}-e^{\delta t}p_1q_1E_1, \tag{4.4}$$

$$\lambda_{2(t+1)}=-\frac{\partial H_t}{\partial y_t}=\lambda_{1(t+1)}\cdot\beta_2^{1-k}\beta_1^k\frac{x_t}{l+\alpha v}-\lambda_{2(t+1)}\left[a_1^{1-k}a_2^k\frac{x_t}{l+\alpha v}-d_2^{1-k}d_1^k\right], \tag{4.5}$$

$$\frac{\partial H_t}{\partial E_1}=e^{-\delta t}(p_1q_1x-c_1)-q_1x\lambda_{1(t+1)}=0. \tag{4.6}$$

From (4.6) we have $\lambda_{1(t+1)}=\frac{1}{q_1x}e^{-\delta t}(p_1q_1x-c_1)$. Using $\lambda_{1(t+1)}$ in (4.5) we get $\lambda_{2(t+1)}$. Now substituting $\lambda_{1(t+1)}$ and $\lambda_{2(t+1)}$ in (4.4) we get the value of the control variable as E_1. Thus the optimal harvesting is obtained as E_1^* at the optimal population level (x_*,y_*).

Case III. $E_1\neq 0$ and $E_2\neq 0$.

$$\begin{cases}x_{t+1}=(1+r_1^{1-k}r_2^k)x_t-r_2^{1-k}r_1^kx_t\left(v+\frac{x_t}{K}\right)-\beta_2^{1-k}\beta_1^k\frac{x_ty_t}{l+\alpha v}-q_1E_1x_t,\\ y_{t+1}=(1-d_2^{1-k}d_1^k)y_t+a_1^{1-k}a_2^k\frac{x_ty_t}{l+\alpha v}-q_2E_2y_t.\end{cases}$$

We define the present value of the net revenue function as

$$J=\max\sum_{t=1}^{k}e^{-\delta t}\left[(p_1q_1x-c_1)E_1(t)+(p_2q_2x-c_2)E_2(t)\right].$$

We first construct the following Hamiltonian function:

$$\begin{aligned}H_t=&\lambda_{1(t+1)}\left[(1+r_1^{1-k}r_2^k)x_t-r_2^{1-k}r_1^kx_t\left(v+\frac{x_t}{K}\right)-\beta_2^{1-k}\beta_1^k\frac{x_ty_t}{l+\alpha v}-q_1E_1x_t-x_{t+1}\right]\\&+\lambda_{2(t+1)}\left[(1-d_2^{1-k}d_1^k)y_t+a_1^{1-k}a_2^k\frac{x_ty_t}{l+\alpha v}-q_2E_2y_t-y_{t+1}\right]\\&+e^{-\delta t}\left[(p_1q_1x-c_1)E_1(t)+(p_2q_2x-c_2)E_2(t)\right],\end{aligned}$$

where λ_1 and λ_2 are adjoint variables. By Pontryagin's maximum principle, we obtain

$$\begin{aligned}\lambda_{1(t+1)}=-\frac{\partial H_t}{\partial x_t}=&-\lambda_{1(t+1)}\left[r_1^{1-k}r_2^k-r_2^{1-k}r_1^k\left(v+\frac{2x_t}{K}\right)-\beta_2^{1-k}\beta_1^k\frac{y_t}{l+\alpha v}-q_1E_1\right]\\&-e^{\delta t}p_1q_1E_1-\lambda_{2(t+1)}\cdot\frac{c_1^{1-k}c_2^k y}{l+\alpha v},\end{aligned} \tag{4.7}$$

$$\begin{aligned}\lambda_{2(t+1)}=-\frac{\partial H_t}{\partial y_t}=&\lambda_{1(t+1)}\cdot\beta_2^{1-k}\beta_1^k\frac{x_t}{l+\alpha v}-\lambda_{2(t+1)}\left[a_1^{1-k}a_2^k\frac{x_t}{l+\alpha v}-d_2^{1-k}d_1^k-q_2E_2\right]\\&-e^{\delta t}p_2q_2E_2+,\end{aligned} \tag{4.8}$$

$$\frac{\partial H_t}{\partial E_1}=e^{-\delta t}(p_1q_1x-c_1)-q_1x\lambda_{1(t+1)}=0, \tag{4.9}$$

$$\frac{\partial H_t}{\partial E_2} = e^{-\delta t}(p_2 q_2 x - c_2) - q_2 y \lambda_{2(t+1)} = 0. \tag{4.10}$$

From (4.9) and (4.10) we have

$$\lambda_{1(t+1)} = \frac{1}{q_1 x} e^{-\delta t}(p_1 q_1 x - c_1), \quad \lambda_{2(t+1)} = \frac{1}{q_2 y} e^{-\delta t}(p_2 q_2 x - c_2).$$

Now substituting $\lambda_{1(t+1)}$ and $\lambda_{2(t+1)}$ in (4.7) and (4.8) we get the value of the control variable as E_1 and E_2. Thus the optimal harvesting is obtained as E_1^* and E_2^* at the optimal population level (x_*, y_*).

5. Numerical Simulations

In this section, we will analyze the effects of imprecise parameter and harvest on ecological dynamics through numerical simulation.

Example 1. Consider the following system with imprecise parameters

$$\begin{cases} x_{t+1} = (1 + 2^{1-k} 2.2^k) x_t - 2.2^{1-k} 2^k x_t \left(v + \frac{x_t}{4}\right) - 1.6^{1-k} 1.4^k \frac{x_t y_t}{2+0.6v} - 0.2 E_1 x_t, \\ y_{t+1} = (1 - 0.2^{1-k} 0.1^k) y_t + 1.2^{1-k} 1.4^k \frac{x_t y_t}{2+0.6v} - 0.1 E_2 y_t, \end{cases} \tag{5.1}$$

where $E_1 = E_2 = 0$ and v is bifurcation parameter.

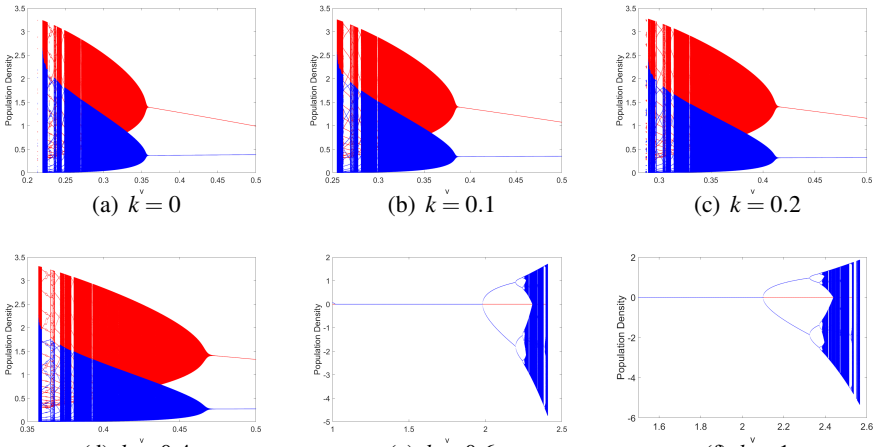
Fig 5 is the bifurcation diagram of system (5.1). We obtain the dynamic behavior of the system when k takes different values. Firstly, we find that when $k = 0$, the Neimark-Sacker bifurcation will occur in system (5.1), and as k increases, the critical value of bifurcation will gradually increase. Combined with Fig 5, it can be seen that when $v = 0.45$, the oscillation time of the solution will be extended with the increase of k, and the periodic solution will be gradually generated. When $k = 0.6$ and $k = 1$, system (5.1) will produce flip bifurcation. However, it can be seen from the image that the system will produce negative values. This is because the discrete system (1.5) is obtained by the continuous system (1.4) through the Euler difference method. Because the random values of the parameters in system (1.5) do not guarantee that the population size is always positive, so the Euler difference method has some limitations.

Example 2. Consider the following system

$$\begin{cases} x_{t+1} = x_t + r x_t \left(1 - v - \frac{x_t}{K}\right) - \frac{\beta x_t y_t}{l + \alpha v} - q_1 E_1 x_t, \\ y_{t+1} = y_t + \frac{a x_t y_t}{l + \alpha v} - d y_t - q_2 E_2 y_t, \end{cases}$$

where $E_1 = E_2 = 0$ and v is bifurcation parameter.

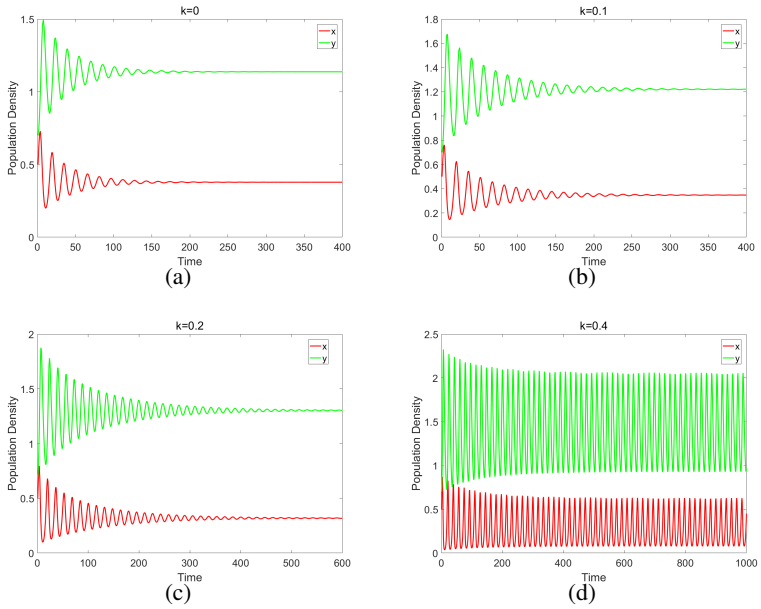
It is easy to find from Fig 5 that when the harvesting effort E_2 of the predator remains unchanged, the harvesting effort E_1 of the prey gradually increases, the number of prey populations gradually decreases and tends to extinction, and the number



(a) $k = 0$ (b) $k = 0.1$ (c) $k = 0.2$

(d) $k = 0.4$ (e) $k = 0.6$ (f) $k = 1$

FIGURE 1. The bifurcation diagram of system (5.1) when k takes different values. Red represents prey x and blue represents predator y.



(a) (b) (c) (d)

FIGURE 2. When $v = 0.45$, the time sequence diagram corresponding to Fig 5 (a, b, c, d).

of predator populations also gradually decreases and tends to extinction. This phenomenon is caused by the strict dependence of predators on the number of prey populations. When E_1 is small and remains unchanged, as E_2 increases, the number of predator populations gradually decreases. For prey, the reduction of natural enemies will gradually increase the number of their own populations. When E_1 is large, the number of prey population is small, and the predator population is extinct, so the harvest of predator E_2 will not affect the number of prey population.

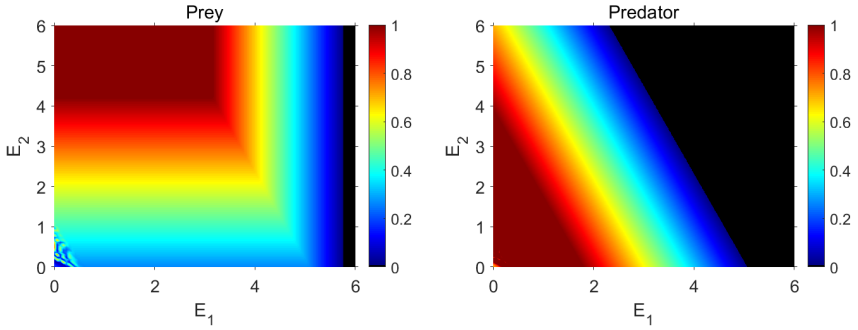


FIGURE 3. Figure shows the variation of densities of all populations in $E_1 - E_2$ space.

In Fig 5, we will analyze the effects of prey growth rate r and vigilance effect v on the population density of prey and predator. By observing the image, it can be found that when r remains unchanged and v gradually increases, the number of prey gradually decreases and the region becomes extinct, and the number of predator populations gradually decreases and the region becomes extinct. This is due to the increase in the vigilance effect of the prey on the predator, which reduces the number of prey foraging, and the resources obtained cannot meet the reproduction of their own populations, thus the number of predator populations gradually reduces and eventually becomes extinction. The reproduction of predators is strictly dependent on the prey population. Therefore, the decrease of prey population will lead to the decrease of predator population and extinction. When v remains unchanged and the growth rate of the prey gradually increases, the number of predators in the population gradually increases. For the prey, the birth rate increases and the number of natural enemies increases, so that the number of its own population will not change much.

It can be found from Fig 5 that when the predation rate β remains unchanged and the natural mortality d of the predator gradually increases, the population density of the predator gradually decreases and tends to be extinct, so that the population density of the prey gradually increases. When the natural mortality d of the predator remains unchanged and the predation rate β increases, it is found that the number of predator populations decreases gradually, while the number of prey populations

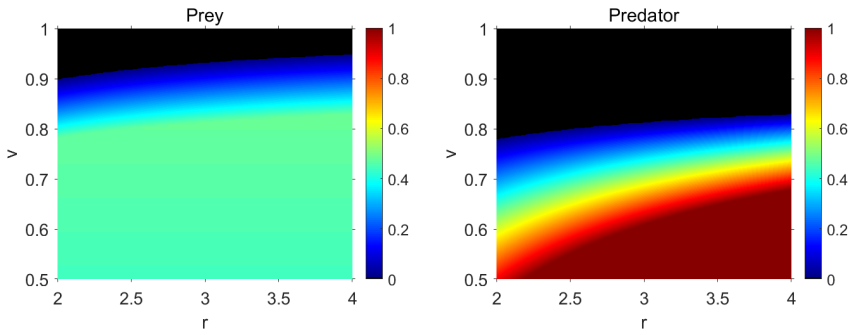


FIGURE 4. Figure shows the variation of densities of all populations in $r - v$ space.

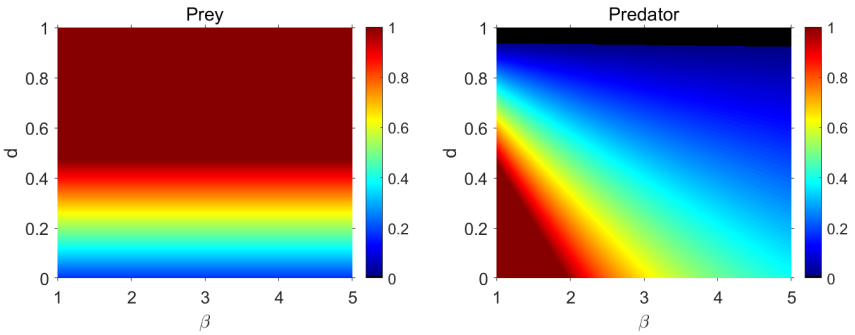


FIGURE 5. Figure shows the variation of densities of all populations in $\beta - d$ space.

remains almost unchanged. In ecology, this is because when the predation rate increases, the number of prey populations gradually decreases, so that the number of predator populations gradually decreases, and the reduction of natural enemies will increase the number of prey populations, so that the number of prey populations will not change much.

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(CO)LOCALIZATION IN (n-)COTILTING COMODULES

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Abstract. The purpose of this article is to study the localization and colocalization for cotilting comodules and n-cotilting comodules. In this paper, firstly we give the definition of (co)localization for preenvelopes and special preenvelopes. And secondly, we apply the theory of (co)localization to cotilting comodules. Following this result, we investigate the (co)localization for n-cotilting comodules.

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Keywords: (pre)envelop, cotilting comodule, (co)localization

1. INTRODUCTION

The (co)localization is developed continuously from many aspects, which plays an important role in the theory of coalgebras. One of the most famous procedure is the localization in rings, which is a systematic method of adding multiplicative inverses to a ring. Following the localization for rings, Năstăsescu and Torrecillas [15] developed a theory of localization for coalgebras. Concretely, if C is a coalgebra over a field K, and $\mathcal{T}$ is a dense subcategory or a Serre class of the category $\mathcal{M}^C$ of right C-comodules, Năstăsescu and Torrecillas considered the quotient category $\mathcal{M}^C/\mathcal{T}$ and the canonical functor $T\colon \mathcal{M}^C \to \mathcal{M}^C/\mathcal{T}$. What's more, Năstăsescu and Torrecillas considered the colocalizing subcategory $\mathcal{T}$, where the functor T has a left adjoint H, rather than considering the localizing subcategory $\mathcal{T}$, where $\mathcal{T}$ is closed under arbitrary direct sums (or equivalently, T has a right adjoint S). In addition, Gabriel described the localization abstractly in abelian and Grothendieck category in [9]. Later, Navarro developed the ideas of Gabriel in the categories of comodules (Grothendieck categories of finite type) in [10, 11]. The main reason for this is that the category $\mathcal{M}^C$ of right comodules over a coalgebra C is a locally finite Grothendieck category in which the theory of localization can be applied. The key-point of

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the theory lies in the fact that the quotient category becomes a comodule category. In other words, a quotient category $\mathcal{M}^C/\mathcal{T}$ is a category of comodules $\mathcal{M}^D$ for certain coalgebra D, where C is a coalgebra, and $\mathcal{T}$ is a localizing subcategory of a Grothendieck category. Indeed, it is better understood than in the case of modules over arbitrary algebras.

The notions of (pre)envelope and (n-)cotilting are well-established and of significant importance in general category theory, particularly within the context of Abelian categories. These concepts have proven to be fundamental in understanding the structural and homological properties of various mathematical objects. Similarly, (n-)cotilting theory in Abelian categories has led to a deeper understanding of the duality between projective and injective objects. Given the profound impact of these concepts in Abelian categories, it is natural to extend their application to other categorical settings where similar structural insights might be gained. One such setting is the category of comodules over a coalgebra, known as the comodule category.

The representation theory of coalgebra has been developed by some scholars, see [1–5, 7, 8, 10–25]. The fifth open question raised by Simson in [16] is to develop cotilting theory in the categories of comodules, which is an important part of the representation theory of coalgebras. Therefore it is rather natural to think about the following question:

Question 1. How to apply the theory of (co)localization in comodule categories to the (pre)envelope and n-cotilting comodule?

We present our main result as follows.

Theorem 1. *Let K be a field, C be a K-coalgebra and $e \in C^*$ be an idempotent, then we have:*

(1) *A C-comodule homomorphism $\phi \in \mathrm{Com}_C(M,X)$ is an $\mathcal{F}$-preenvelope of M if and only if the right eCe-comodule homomorphism $e\phi \in \mathrm{Com}_{eCe}(eM,eX)$ is an $e\mathcal{F}$-preenvelope of eM for each idempotent $e \in C^*$.*
(2) *A C-comodule homomorphism $\phi \in \mathrm{Com}_C(M,X)$ is an $\mathcal{F}$-envelope of M if and only if the right eCe-comodule homomorphism $e\phi \in \mathrm{Com}_{eCe}(eM,eX)$ is an $e\mathcal{F}$-envelope of eM for each idempotent $e \in C^*$.*
(3) *A C-comodule homomorphism $\phi \in \mathrm{Com}_C(M,X)$ is a special $\mathcal{F}$-preenvelope of M if and only if the right eCe-comodule homomorphism $e\phi \in \mathrm{Com}_{eCe}(eM, eX)$ is a special $e\mathcal{F}$-preenvelope of eM for each idempotent $e \in C^*$.*

Theorem 2. *Let K be a field, C be a K-coalgebra, and $e \in C^*$ be an idempotent defining a perfect localization. Suppose that $X \cong Ce$ is a quasi-finite injective cogenerator, then M is an n-cotilting eCe-comodule if and only if $H(M)$ is an n-cotilting C-comodule.*

2. PRELIMINARIES

Let (C,Δ,ε) be a coalgebra over a field K, where Δ is the comultiplication map $\Delta\colon C\to C\otimes C$ and $\varepsilon\colon C\to K$ is the counit map. We denote by $\mathcal{M}^C$ the category of all right C-comodules, where C is a coalgebra over a field K. Unless otherwise stated, all comodules are right comodules throughtout this paper. Give a comodule class,

$$^{\perp}\mathcal{F}=\{X\in \text{Comod-}C \mid \text{for any } F\in\mathcal{F}, \text{have } \operatorname{Ext}^1_C(X,F)=0\}.$$

Following [11], a full subcategory $\mathcal{T}$ of $\mathcal{M}^C$ is called dense if for every exact sequence $0\to M'\to M\to M''\to 0$ in $\mathcal{M}^C$, the comodule M lies in $\mathcal{T}$ if and only if each of the comodules M' and M'' lies in $\mathcal{T}$. For any dense subcategory $\mathcal{T}$ of $\mathcal{M}^C$, there exists an abelian category $\mathcal{M}^C/\mathcal{T}$ and an exact functor $T\colon \mathcal{M}^C\to\mathcal{M}^C/\mathcal{T}$ such that $T(M)=0$, for each $M\in\mathcal{T}$, satisfying the following universal property: for any exact functor $F\colon \mathcal{M}^C\to\mathcal{C}$ such that $F(M)=0$ for each $M\in\mathcal{T}$, there exists a unique functor $\overline{F}\colon \mathcal{M}^C/\mathcal{T}\to\mathcal{C}$ verifying that $F=\overline{F}T$, where $\mathcal{C}$ is an arbitrary abelian category. The category $\mathcal{M}^C/\mathcal{T}$ is called the quotient category of $\mathcal{M}^C$ with respect to $\mathcal{T}$, and T is known as the quotient functor.

A dense subcategory $\mathcal{T}$ of $\mathcal{M}^C$ is said to be localizing if the quotient functor $T\colon \mathcal{M}^C\to\mathcal{M}^C/\mathcal{T}$ has a right adjoint functor $S\colon \mathcal{M}^C/\mathcal{T}\to\mathcal{M}^C$, called the section functor. If the section functor S is exact, a localizing subcategory $\mathcal{T}$ is called perfect localizing. $\mathcal{T}$ is said to be colocalizing if T has a left adjoint functor $H\colon \mathcal{M}^C/\mathcal{T}\to\mathcal{M}^C$, called the colocalizing functor. $\mathcal{T}$ is called a perfect colocalizing subcategory if the colocalizing functor H is exact.

Consider the dual algebra $C^*=\operatorname{Hom}_K(C,K)$ with the convolution product $f*g(C)=\sum f(C_1)g(C_2)$, where $f,g\in C^*$. The counit $\varepsilon\colon C\to K$ of C is the identity element of the algebra C^*. In [6, 11, 24], a localizing subcategory of $\mathcal{M}^C$ was described by means of idempotent $e\in C^*$. Furthermore, it was proved that the quotient category is the category of right comodules over the coalgebra eCe, where e is an idempotent associated to the localizing subcategory. The coalgebra structure of eCe is given by

$$\triangle_{eCe}(ece)=\sum_{(c)} ec_{(1)}e\otimes ec_{(2)}e \quad\text{and}\quad \varepsilon_{eCe}(ece)=e(c),$$

for any $c\in C$, where $\triangle_C(c)=\sum_{(c)} c_{(1)}\otimes c_{(2)}$ is the Sweedler's notation in [21]. If M is a right C-comodule, then eM has a natural structure of right eCe-comodule given by

$$\rho(em)=\sum_{(m)} em_{(0)}\otimes em_{(1)}e, \quad\text{where}\quad \rho_M(m)=\sum_{(m)} m_{(0)}\otimes m_{(1)},$$

for any $m\in M$ by the Sweedler's notation in [21]. Throughout we denote by $\mathcal{T}_e$ the localizing subcategory associated to e for an idempotent $e\in C^*$ for any coalgebra C over a field K.

The following two lemmas are the properties of the (co)localizing functor which are from (cf.[11],[15] and [9]).

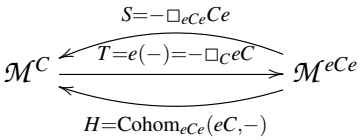
Lemma 1. *Let $\mathcal{T}$ be a dense subcategory of the category of right comodules $\mathcal{M}^C$ over a coalgebra C. Then the following statements hold.*

(1) *T is exact.*
(2) *If $\mathcal{T}$ is localizing, then the section functor S is left exact and the equivalence $TS \simeq 1_{\mathcal{M}^C/\mathcal{T}}$ holds.*
(3) *If $\mathcal{T}$ is colocalizing, then the colocalizing functor H is right exact and the equivalence $TH \simeq 1_{\mathcal{M}^C/\mathcal{T}}$ holds.*

Lemma 2. *Let C be a coalgebra, and e be an idempotent in C^*. Then the following statements hold.*

(1) *The quotient functor $T\colon \mathcal{M}^C \to \mathcal{M}^{eCe}$ is naturally equivalent to the functor $e(-)$. The quotient functor T is also equivalent to the cotensor functor $-\square_C eC$.*
(2) *The section functor $S\colon \mathcal{M}^{eCe} \to \mathcal{M}^C$ is naturally equivalent to the cotensor functor $-\square_{eCe}Ce$.*
(3) *If $\mathcal{T}$ is a colocalizing subcategory of $\mathcal{M}^C$, then the colocalizing functor $H\colon \mathcal{M}^{eCe} \to \mathcal{M}^C$ is naturally equivalent to the functor* $\mathrm{Cohom}_{eCe}(eC,-)$.

For the convenience of the reader, the functors can be presented as shown in the following diagram:



For the convenience of writing, we use $T\cdot$ instead of $T(\cdot)$.

3. (Co)localization in (pre)envelopes

In this section, we give the concepts of (pre)envelopes and special preenvelopes for comodules and apply the theory of (co)localization to them.

Definition 1. Assume that $\mathcal{F}$ is a class of comodules in $\mathcal{M}^C$ and $M \in \mathcal{M}^C$. We call $\phi \in \mathrm{Com}_C(M,X)$ with $X \in \mathcal{F}$ an $\mathcal{F}$-preenvelope of M if it satisfies that $\mathrm{Com}_C(\phi,F)\colon \mathrm{Com}_C(X,F) \to \mathrm{Com}_C(M,F)$ is surjective for every $F \in \mathcal{F}$.

Remark 1. Let $\phi \in \mathrm{Com}_C(M,X)$ be an $\mathcal{F}$-preenvelope of M

(i) ϕ is called an $\mathcal{F}$-envelope of M, if $g\phi = \phi$ and $g \in \mathrm{End}(X)$, then g is an automorphism of X;
(ii) ϕ is said to be special, if $\phi \in \mathrm{Com}_C(M,X)$ is injective and $\mathrm{Coker}\phi \in {}^{\perp}\mathcal{F}$.

A comodule class $\mathcal{F} \subseteq \mathcal{M}^C$ is called a preenvelope (envelope) class if every comodule in $\mathcal{M}^C$ has an $\mathcal{F}$-preenvelope ($\mathcal{F}$-envelope).

Proposition 1. *Let C be a K-coalgebra, $e \in C^*$ be an idempotent, $\phi \in \mathrm{Com}_C(M,X)$ be a right C-comodule homomorphism.*

(1) *If $\phi \in \mathrm{Com}_C(M,X)$ is an $\mathcal{F}$-preenvelope of M, then right eCe-comodule homomorphism $e\phi \in \mathrm{Com}_{eCe}(eM,eX)$ is an $e\mathcal{F}$-preenvelope of eM.*

(2) *If $\phi \in \mathrm{Com}_C(M,X)$ is an $\mathcal{F}$-envelope of M, then the right eCe-comodule homomorphism $e\phi \in \mathrm{Com}_{eCe}(eM,eX)$ is an $e\mathcal{F}$-envelope of eM.*

(3) *If $\phi \in \mathrm{Com}_C(M,X)$ is a special $\mathcal{F}$-preenvelope of M, then the right eCe-comodule homomorphism $e\phi \in \mathrm{Com}_{eCe}(eM,eX)$ is a special $e\mathcal{F}$-preenvelope of eM.*

Proof.

(1) Since the right C-comodule homomorphism $\phi \in \mathrm{Com}_C(M,X)$ is an $\mathcal{F}$-preenvelope of M, based on Definition 1, we can deduce that

$$\phi^* = \mathrm{Com}_C(\phi,F)\colon \mathrm{Com}_C(X,F) \to \mathrm{Com}_C(M,F)$$

is surjective for each $F \in \mathcal{F}$, i.e., $\mathrm{Com}_C(X,F) \xrightarrow{\phi^*} \mathrm{Com}_C(M,F) \to 0$ in $\mathcal{M}^C$ is exact. Now, for any $\alpha \in \mathrm{Com}_{eCe}(eM, eF)$, where $F \in \mathcal{F}$, there exists an $f \in \mathrm{Com}_C(M,F)$ such that $e(f) = \alpha$ since $e(-)$ is full. Consequently, there is a $g \in \mathrm{Com}_C(X,F)$ such that $f = g\phi$ since ϕ is an $\mathcal{F}$-preenvelope of M. Thereby, we can determine that $e(f) = e(g\phi) = e(g).e(\phi)$, i.e., $\alpha = e(g).e(\phi)$. Hence,$(e\phi)^*$ is surjective, i.e. $\mathrm{Com}_{eCe}(eX,eF) \xrightarrow{(e\phi)^*} \mathrm{Com}_{eCe}(eM,eF) \to 0$ in $\mathcal{M}^{eCe}$ is exact, where $eF \in e\mathcal{F}$. This shows that the eCe-comodule homomorphism $e\phi$ is an $e\mathcal{F}$-preenvelope of eM.

(2) Because the right C-comodule homomorphism $\phi \in \mathrm{Com}_C(M,X)$ is an $\mathcal{F}$-envelope of M, according to Remark 1(i), it can be inferred that $\phi \in \mathrm{Com}_C(M,X)$ is primarily an $\mathcal{F}$-preenvelope. Furthermore, if $g\phi = \phi$ and $g \in \mathrm{End}(X)$, then g is an automorphism of X. From Proposition 1(1), we know that $e\phi$ is an $e\mathcal{F}$-preenvelope of eM. Furthermore, since the C-comodule endomorphism g is an automorphism of X, we can deduce from [12, Lemma 3.7(3)] that for any eCe-comodule endomorphism $eg\colon eX \to eX$, eg is an automorphism of eX. Based on the above two aspects, therefore, it can be concluded that the eCe-comodule homomorphism $e\phi$ is an $e\mathcal{F}$-envelope of eM.

(3) Because the C-comodule homomorphism $\phi \in \mathrm{Com}_C(M,X)$ is a special $\mathcal{F}$-preenvelope of M, according to Remark 1(ii), it can be inferred that $\phi \in \mathrm{Com}_C(M,X)$ is primarily an $\mathcal{F}$-preenvelope. Furthermore, we infer that $\phi \in \mathrm{Com}_C(M,X)$ is injective and $\mathrm{Coker}\phi \in {}^{\perp}\mathcal{F}$. From Proposition 1(1), we know that $e\phi$ is an $e\mathcal{F}$-preenvelope of eM. Let $N = \mathrm{Coker}\phi$, then we get the exact

sequence $0 \to M \xrightarrow{\phi} X \to N \to 0$ in $\mathcal{M}^C$. Since T is an exact functor, there is the short exact sequence $0 \to eM \xrightarrow{e\phi} eX \to eN \to 0$ in $\mathcal{M}^{eCe}$. As Coker$\phi \in {}^{\perp}\mathcal{F}$, it holds that $\mathrm{Ext}^1_C(N,F) = 0$ for any $F \in \mathcal{F}$. From [12, Lemma 3.5], it follows that $e\mathrm{Ext}^1_C(N,F) = 0$, i.e., $\mathrm{Ext}^1_{eCe}(eN,eF) = 0$. This shows that $eN = \mathrm{Coker}(e\phi) \in {}^{\perp}(e\mathcal{F})$. Consequently, the eCe-comodule homomorphism $e\phi$ is a special $e\mathcal{F}$-preenvelope.

□

Corollary 1. *Suppose that C is a coalgebra and $\mathcal{F} \subseteq \mathcal{M}^C$ is a (pre)envelope class, then $e\mathcal{F} \subseteq \mathcal{M}^{eCe}$ is also a (pre)envelope class.*

Proof. It can be proved by Proposition 1. □

Proposition 2. *Let k be a field, C be a K-coalgebra, $\phi \in \mathrm{Com}_{eCe}(M,X)$ be a right eCe-comodule homomorphism and $e \in C^*$ be an idempotent which defines a perfect localization.*

(1) *If $\phi \in \mathrm{Com}_{eCe}(M,X)$ is an $\mathcal{F}$-preenvelope of M, then the right C-comodule homomorphism $S(\phi) \in \mathrm{Com}_C(S(M),S(X))$ is an $S(\mathcal{F})$-preenvelope of $S(M)$.*

(2) *If $\phi \in \mathrm{Com}_{eCe}(M,\ X)$ is an $\mathcal{F}$-envelope of M, then the right C-comodule homomorphism $S(\phi) \in \mathrm{Com}_C(S(M),\ S(X))$ is an $S(\mathcal{F})$-envelope of $S(M)$.*

(3) *If $\phi \in \mathrm{Com}_{eCe}(M,X)$ is a special $\mathcal{F}$-preenvelope of M, then the right C-comodule homomorphism $S(\phi) \in \mathrm{Com}_C(S(M),S(X))$ is a special $S(\mathcal{F})$-preenvelope of $S(M)$.*

Proof.

(1) Since the right eCe-comodule homomorphism $\phi \in \mathrm{Com}_{eCe}(M,X)$ is an $\mathcal{F}$-preenvelope of M, we infer that

$$\phi^* = \mathrm{Com}_{eCe}(\phi,F)\colon \mathrm{Com}_{eCe}(X,F) \xrightarrow{\phi^*} \mathrm{Com}_{eCe}(M,F)$$

is surjective for each $F \in \mathcal{F}$, i.e., $\mathrm{Com}_{eCe}(X,F) \xrightarrow{\phi^*} \mathrm{Com}_{eCe}(M,\ F) \to 0$ in $\mathcal{M}^{eCe}$ is exact. Now, for any $\alpha \in \mathrm{Com}_C(S(M),S(F))$, where $F \in \mathcal{F}$, there exists an $f \in \mathrm{Com}_{eCe}(M,F)$ such that $S(f) = \alpha$ since S is full. In consequence, there is a $g \in \mathrm{Com}_{eCe}(X,F)$ such that $f = g\phi$ since ϕ is an $\mathcal{F}$-preenvelope of M.Consequently, we have $S(f) = S(g)S(\phi)$, i.e., $\alpha = S(g)S(\phi)$. Thus, $(S(\phi))^*$ is surjective, that is, $\mathrm{Com}_C(S(X),S(F)) \xrightarrow{(S(\phi))^*} \mathrm{Com}_C(S(M),S(F)) \to 0$ in $\mathcal{M}^C$ is exact, where $S(F) \in S(\mathcal{F})$. This shows that the C-comodule homomorphism $S(\phi)$ is an $S(\mathcal{F})$-preenvelope of $S(M)$.

(2) Since $\phi \in \mathrm{Com}_{eCe}(M,X)$ is an $\mathcal{F}$-envelope of M, according to Remark 1(i), it can be inferred that $\phi \in \mathrm{Com}_{eCe}(M,X)$ is mainly an $\mathcal{F}$-preenvelope. Additionally, since $g\phi = \phi$ and $g \in \mathrm{End}_{eCe}(X)$, we derive that the eCe-comodule endomorphism g is an automorphism of X. From Proposition 2(1), we know that $S(\phi)$ is an $S(\mathcal{F})$-preenvelope of $S(M)$. And also since the eCe-comodule

endomorphism $g \in \mathrm{Com}_{eCe}(X,X)$ is an automorphism of X, by [12, Lemma 3.8(4)], it follows that the C-comodule endomorphism $S(g) = \mathrm{Com}_C(S(X), S(X))$ is also an automorphism of $S(X)$. Taking into account the above two aspects, we can conclude that C-comodule homomorphism $S(\phi)$ is an $S(\mathcal{F})$-envelope of $S(M)$.

(3) Since $\phi \in \mathrm{Com}_{eCe}(M,X)$ is a special $\mathcal{F}$-preenvelope of M, it follows that $\phi \in \mathrm{Com}_{eCe}(M,X)$ is an $\mathcal{F}$-preenvelope, and furthermore, $\phi \in \mathrm{Com}_{eCe}(M,X)$ is injective and $\mathrm{Coker}\phi \in {}^{\perp}\mathcal{F}$. From Proposition 2(1), it can be concluded that $S(\phi)$ is an $S(\mathcal{F})$-preenvelope of $S(M)$. By [12, Lemma 3.8(2)], the C-comodule homomorphism $S(\phi) \in \mathrm{Com}_C(S(M),S(X))$ is injective. Let $N = \mathrm{Coker}\phi$, then we have the exact sequence $0 \to M \xrightarrow{\phi} X \to N \to 0$ in $\mathcal{M}^{eCe}$. Since $\mathrm{Coker}\phi \in {}^{\perp}\mathcal{F}$, it follows that $\mathrm{Ext}^1_{eCe}(N,F) = 0$ for any $F \in \mathcal{F}$. Since S is an exact functor, there is the short exact sequence $0 \to S(M) \xrightarrow{S(\phi)} S(X) \to S(N) \to 0$ in $\mathcal{M}^C$. We obtain the following commutative diagram (denote $S(\cdot)$ by $S\cdot$ for convenience)

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Com}_{eCe}(N,F) & \longrightarrow & \mathrm{Com}_{eCe}(X,F) & \longrightarrow & \mathrm{Com}_{eCe}(M,F) & \longrightarrow & 0 \\
 & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \\
0 & \longrightarrow & \mathrm{Com}_C(SN,SF) & \longrightarrow & \mathrm{Com}_C(SX,SF) & \longrightarrow & \mathrm{Com}_C(SM,SF) & \longrightarrow & \mathrm{Ext}^1_C(SN,SF)
\end{array}$$

where $S(F) \in S(\mathcal{F})$, because S is a fully faithful functor. This shows that $\mathrm{Ext}^1_C(S(N), S(F)) = 0$ and $S(N) \in {}^{\perp}S(\mathcal{F})$. Consequently, $S(N) = \mathrm{Coker}(S(\phi)) \in {}^{\perp}S(\mathcal{F})$. Based on the above two aspects, we can conclude that the C-comodule homomorphism $S(\phi)$ is a special $S(\mathcal{F})$-preenvelope.

□

Proposition 3. *Let C be a coalgebra, $\phi \in \mathrm{Com}_{eCe}(M,X)$ be a right eCe-comodule homomorphism and $e \in C^*$ be an idempotent which defines a perfect colocalization.*

(1) *If $\phi \in \mathrm{Com}_{eCe}(M,X)$ is an $\mathcal{F}$-preenvelope of M, then the right C-comodule homomorphism $H(\phi) \in \mathrm{Com}_C(H(M),H(X))$ is an $H(\mathcal{F})$-preenvelope of $H(M)$.*

(2) *If $\phi \in \mathrm{Com}_{eCe}(M,X)$ is an $\mathcal{F}$-envelope of M, then the right C-comodule homomorphism $H(\phi) \in \mathrm{Com}_C(H(M), H(X))$ is an $H(\mathcal{F})$-envelope of $H(M)$.*

(3) *If $\phi \in \mathrm{Com}_{eCe}(M,X)$ is a special $\mathcal{F}$-preenvelope of M, then the right C-comodule homomorphism $H(\phi) \in \mathrm{Com}_C(H(M),H(X))$ is a special $H(\mathcal{F})$-preenvelope of $H(M)$.*

Proof.

(1) Since the right eCe-comodule homomorphism $\phi \in \mathrm{Com}_{eCe}(M,X)$ is an $\mathcal{F}$-preenvelope of M, we infer that

$$\phi^* = \mathrm{Com}_{eCe}(\phi,F)\colon \mathrm{Com}_{eCe}(X,F) \xrightarrow{\phi^*} \mathrm{Com}_{eCe}(M,F)$$

is surjective for each $F \in \mathcal{F}$, i.e., $\mathrm{Com}_{eCe}(X,F) \xrightarrow{\phi^*} \mathrm{Com}_{eCe}(M,\ F) \to 0$ in $\mathcal{M}^{eCe}$ is exact. Now, for any $\alpha \in \mathrm{Com}_C(H(M),H(F))$, where $F \in \mathcal{F}$, there exists an $f \in \mathrm{Com}_{eCe}(M,F)$ such that $H(f) = \alpha$ since H is full. Consequently, there exists a $g \in \mathrm{Com}_{eCe}(X,F)$ such that $f = g\phi$ since ϕ is an $\mathcal{F}$-preenvelope of M. Thus, we can conclude that $H(f) = H(g)H(\phi)$, i.e., $\alpha = H(g) \cdot H(\phi)$. As a result, it follows that $(H(\phi))^*$ is surjective, i.e.

$$\mathrm{Com}_C(H(X),H(F)) \xrightarrow{(H(\phi))^*} \mathrm{Com}_C(H(M),H(F)) \to 0$$

in $\mathcal{M}^C$ is exact, where $H(F) \in H(\mathcal{F})$. This shows that the C-comodule homomorphism $H(\phi)$ is an $H(\mathcal{F})$-preenvelope of $H(M)$.

(2) Since $\phi \in \mathrm{Com}_{eCe}(M,X)$ is an $\mathcal{F}$-envelope of M, firstly, $\phi \in \mathrm{Com}_{eCe}(M,X)$ is an $\mathcal{F}$-preenvelope, and secondly, with $g\phi = \phi$ and $g \in \mathrm{End}_{\mathrm{eCe}}(X)$, we deduced that the eCe-comodule endomorphism g is an automorphism of X. As a result of Proposition 3(1), we can assert that the C-comodule homomorphism $H(\phi)$ is an $H(\mathcal{F})$-preenvelope of $H(M)$. Additionally, since the eCe-comodule endomorphism $g \in \mathrm{Com}_{eCe}(X,X)$ is an automorphism of X, [12, Lemmma 3.10(4)] allows us to infer that the C-comodule endomorphism $H(g) = \mathrm{Com}_C(H(X),H(X))$ is also an automorphism of $H(X)$. With the two aspects in mind, we can assert that the C-comodule homomorphism $H(\phi)$ is an $H(\mathcal{F})$-envelope of $H(M)$.

(3) Considering $\phi \in \mathrm{Com}_{eCe}(M,X)$ as a special $\mathcal{F}$-preenvelope of M, it is clear that $\phi \in \mathrm{Com}_{eCe}(M,X)$ is an $\mathcal{F}$-preenvelope at first, and then it is observed that $\phi \in \mathrm{Com}_{eCe}(M,X)$ is injective and $\mathrm{Coker}\phi \in {}^{\perp}\mathcal{F}$. From the perspective of Proposition 3(1), it is evident that the C-comodule homomorphism $H(\phi)$ is an $H(\mathcal{F})$-preenvelope of $H(M)$. From [12, Lemma 3.10(2)], it follows that the C-comodule homomorphism $H(\phi) \in \mathrm{Com}_C(H(M),H(X))$ is injective. Then there is an exact sequence $0 \to M \xrightarrow{\phi} X \to N \to 0$ in $\mathcal{M}^{eCe}$, where $N = \mathrm{Coker}\phi$. It follows that $\mathrm{Ext}^1_{eCe}(N,F) = 0$ for any $F \in \mathcal{F}$, since $\mathrm{Coker}\phi \in {}^{\perp}\mathcal{F}$. Since H is an exact functor, there is the following exact sequence $0 \to H(M) \xrightarrow{H(\phi)} H(X) \to H(N) \to 0$ in $\mathcal{M}^C$. Then there is the following commutative diagram (denote $H(\cdot)$ by $H\cdot$ for convenience)

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Com}_{eCe}(N,F) & \longrightarrow & \mathrm{Com}_{eCe}(X,F) & \longrightarrow & \mathrm{Com}_{eCe}(M,F) & \longrightarrow & 0 \\
 & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\
0 & \longrightarrow & \mathrm{Com}_C(HN,HF) & \longrightarrow & \mathrm{Com}_C(HX,HF) & \longrightarrow & \mathrm{Com}_C(HM,HF) & \longrightarrow & \mathrm{Ext}^1_C(HN,HF)
\end{array}$$

where $HF \in H\mathcal{F}$, because H is a fully faithful functor. This shows that $\mathrm{Ext}^1_C(H(N),H(F)) = 0$ and $H(N) \in {}^{\perp}H(\mathcal{F})$. As a consequence, $H(N) =$

$\mathrm{Coker}(H(\phi)) \in {}^{\perp}H(\mathcal{F})$. Given the information from the two aspects, we conclude that the C-comodule homomorphism $H(\phi)$ is a special $H(\mathcal{F})$-preenvelope.

□

Theorem 3. *Let K be a field, C be a K-coalgebra and $e \in C^*$ be an idempotent, then we have:*

(1) *A C-comodule homomorphism $\phi \in \mathrm{Com}_C(M,X)$ is an $\mathcal{F}$-preenvelope of M if and only if the right eCe-comodule homomorphism $e\phi \in \mathrm{Com}_{eCe}(eM,eX)$ is an $e\mathcal{F}$-preenvelope of eM for each idempotent $e \in C^*$.*

(2) *A C-comodule homomorphism $\phi \in \mathrm{Com}_C(M,X)$ is an $\mathcal{F}$-envelope of M if and only if the right eCe-comodule homomorphism $e\phi \in \mathrm{Com}_{eCe}(eM,eX)$ is an $e\mathcal{F}$-envelope of eM for each idempotent $e \in C^*$.*

(3) *A C-comodule homomorphism $\phi \in \mathrm{Com}_C(M,X)$ is a special $\mathcal{F}$-preenvelope of M if and only if the right eCe-comodule homomorphism $e\phi \in \mathrm{Com}_{eCe}(eM, eX)$ is a special $e\mathcal{F}$-preenvelope of eM for each idempotent $e \in C^*$.*

Proof.

(1) The necessity is obtained from Proposition 1(1).

Sufficiency. If the right eCe-comodule homomorphism $e\phi \in \mathrm{Com}_{eCe}(eM, eX)$ is an $e\mathcal{F}$-preenvelope of eM, then

$$\mathrm{Com}_{eCe}(eX,eF) \xrightarrow{(e\phi)^*} \mathrm{Com}_{eCe}(eM,eF) \to 0$$

in $\mathcal{M}^{eCe}$ is exact. Consequently, for any $\alpha\colon eM \to eF$, there exists a $\beta\colon eX \to eF$ such that $\alpha = \beta \cdot e\phi$. Since $e(-)$ is full, there must be $f\colon M \to F$ and $g\colon X \to F$ such that $e(f) = \alpha$, $e(g) = \beta$. Therefore, it can be inferred that $e(f) = e(g)e(\phi)$, which means $e(f) = e(g\phi)$. Since $e(-)$ is full, it can be obtained that $f = g\phi$. Therefore, $\mathrm{Com}_C(X,F) \xrightarrow{\phi^*} \mathrm{Com}_C(M,F) \to 0$ in $\mathcal{M}^C$ is exact. This indicates that the C-comodule homomorphism $\phi \in \mathrm{Com}_C(M,X)$ is an $\mathcal{F}$-preenvelope of M.

(2) According to Proposition 1(2), the necessity is obvious.

Sufficiency. Suppose that the right eCe-comodule homomorphism $e\phi \in \mathrm{Com}_{eCe}(eM,eX)$ is an $e\mathcal{F}$-envelope of eM. First of all, the eCe-comodule homomorphism $e\phi \in \mathrm{Com}_{eCe}(eM,eX)$ is an $e\mathcal{F}$-preenvelope of eM. From (1), it follows that the C-comodule homomorphism $\phi \in \mathrm{Com}_C(M,X)$ is an $\mathcal{F}$-preenvelope of M. Furthermore, with $(eg)(e\phi) = (e\phi)$ and $eg \in \mathrm{End}_{eCe}(eX, eX)$, we deduce that the eCe-comodule endomorphism eg is an automorphism of eX. By [12, Lemma 3.7(3)], we can derive that the C-comodule endomorphism g is an automorphism of X from $\phi g = \phi$ and $g \in \mathrm{End}_C(X)$. Consequently, the C-comodule homomorphism $\phi \in \mathrm{Com}_C(M,X)$ is an $\mathcal{F}$-envelope of M.

(3) The necessity follows from Proposition 1(3).

Sufficiency. If the right eCe-comodule homomorphism $e\phi \in \mathrm{Com}_{eCe}(eM, eX)$ is a special $e\mathcal{F}$-preenvelope of eM, then $e\phi$ is injective and $\mathrm{Coker}(e\phi) \in {}^{\perp}(e\mathcal{F})$, where $\phi \in \mathrm{Com}_C(M,X)$. From [12, Lemma 3.7(1)], it follows that ϕ is injective. Let $\mathrm{Coker}(e\phi) = eN$, then there is the exact sequence $0 \to eM \to eX \to eN \to 0$ in $\mathcal{M}^{eCe}$. Take $eF \in e\mathcal{F}$, we get $\mathrm{Ext}^1_{\mathrm{eCe}}(eN, eF) = e\mathrm{Ext}^1_{\mathrm{C}}(N,F) = 0$, since $eN \in {}^{\perp}(e\mathcal{F})$. It follows from [12, Lemma 3.5] that $\mathrm{Ext}^1_{\mathrm{C}}(N,F) = 0$, where $F \in \mathcal{F}$. This shows that $N = \mathrm{Coker}\phi \in {}^{\perp}\mathcal{F}$, where $\phi \in \mathrm{Com}_C(M,X)$. As a consequence, the C-comodule homomorphism $\phi \in \mathrm{Com}_C(M,X)$ is a special $\mathcal{F}$-preenvelope of M.

□

Theorem 4. *Let C be a coalgebra and $e \in C^*$ be an idempotent defining a localization, then we have:*

(1) *An eCe-comodule homomorphism $\phi \in \mathrm{Com}_{eCe}(M,X)$ is an $\mathcal{F}$-preenvelope of M if and only if the right C-comodule homomorphism $S(\phi) \in \mathrm{Com}_C(S(M), S(X))$ is an $S(\mathcal{F})$-preenvelope of $S(M)$.*
(2) *An eCe-comodule homomorphism $\phi \in \mathrm{Com}_{eCe}(M,X)$ is an $\mathcal{F}$-envelope of M if and only if the right C-comodule homomorphism $S(\phi) \in \mathrm{Com}_C(S(M), S(X))$ is an $S(\mathcal{F})$-envelope of $S(M)$.*
(3) *An eCe-comodule homomorphism $\phi \in \mathrm{Com}_{eCe}(M,X)$ is a special $\mathcal{F}$-preenvelope of M if and only if the right C-comodule homomorphism $S(\phi) \in \mathrm{Com}_C(S(M), S(X))$ is a special $S(\mathcal{F})$-preenvelope of $S(M)$.*

Proof.

(1) The necessity follows from Proposition 2(1).

Sufficiency. If the right C-comodule homomorphism $S(\phi) \in \mathrm{Com}_C(S(M), S(X))$ is an $S(\mathcal{F})$-preenvelope of $S(M)$, then the eCe-comodule homomorphism $T(S(\phi))$ is a $TS(\mathcal{F})$-preenvelope of $T(S(M))$ by Proposition 1(1). By Lemma 1(2) $TS \simeq 1_{\mathcal{M}^C/\mathcal{T}}$, we infer that the eCe-comodule homomorphism $T(S(\phi)) \cong \phi$ is an $\mathcal{F}$-preenvelope of M.

(2) The necessity is obtained from Proposition 2(2).

Sufficiency. If the right C-comodule homomorphism $S(\phi) \in \mathrm{Com}_C(S(M), S(X))$ is an $S(\mathcal{F})$-envelope of $S(M)$, then the eCe-comodule homomorphism $T(S(\phi))$ is a $TS(\mathcal{F})$-envelope of $T(S(M))$ by Proposition 1(2). By Lemma 1(2) $TS \simeq 1_{\mathcal{M}^C/\mathcal{T}}$, the eCe-comodule homomorphism $T(S(\phi)) \cong \phi$ is an $\mathcal{F}$-envelope of M.

(3) By Proposition 2(3), the necessity is clear.

Sufficiency. If the right C-comodule homomorphism $S(\phi) \in \mathrm{Com}_C(S(M), S(X))$ is a special $S(\mathcal{F})$-preenvelope of $S(M)$, then the eCe-comodule homomorphism $T(S(\phi))$ is a special $TS(\mathcal{F})$-preenvelope of $T(S(M))$ by Proposition 1(3). By Lemma 1(2) $TS \simeq 1_{\mathcal{M}^C/\mathcal{T}}$, we infer that the eCe-comodule homomorphism $T(S(\phi)) \cong \phi$ is a special $\mathcal{F}$-preenvelope of M.

□

Theorem 5. *Let C be a coalgebra and $e \in C^*$ be an idempotent which defines a colocalization, then we have:*

(1) *An eCe-comodule homomorphism $\phi \in \mathrm{Com}_{eCe}(M,X)$ is an $\mathcal{F}$-preenvelope of M if and only if the right C-comodule homomorphism $H(\phi) \in \mathrm{Com}_C(H(M), H(X))$ is an $H(\mathcal{F})$-preenvelope of $H(M)$.*

(2) *An eCe-comodule homomorphism $\phi \in \mathrm{Com}_{eCe}(M,X)$ is an $\mathcal{F}$-envelope of M if and only if the right C-comodule homomorphism $H(\phi) \in \mathrm{Com}_C(H(M), H(X))$ is an $H(\mathcal{F})$-envelope of $H(M)$.*

(3) *An eCe-comodule homomorphism $\phi \in \mathrm{Com}_{eCe}(M,X)$ is a special $\mathcal{F}$-preenvelope of M if and only if the right C-comodule homomorphism $H(\phi) \in \mathrm{Com}_C(H(M), H(X))$ is a special $H(\mathcal{F})$-preenvelope of $H(M)$.*

Proof.

(1) The necessity is obtained from Proposition 3(1).
Sufficiency. If the right C-comodule homomorphism $H(\phi) \in \mathrm{Com}_C(H(M), H(X))$ is an $H(\mathcal{F})$-preenvelope of $H(M)$, then the eCe-comodule homomorphism $T(H(\phi))$ is a $TH(\mathcal{F})$-preenvelope of $T(H(M))$ by Proposition 1(1). By Lemma 1(3) $TH \simeq 1_{\mathcal{M}^C/\mathcal{T}}$, the eCe-comodule homomorphism $T(H(\phi)) \cong \phi$ is an $\mathcal{F}$-preenvelope of M.

(2) The necessity follows from Proposition 3(2).
Sufficiency. If the right C-comodule homomorphism $H(\phi) \in \mathrm{Com}_C(H(M), H(X))$ is an $H(\mathcal{F})$-envelope of $H(M)$, then the eCe-comodule homomorphism $T(H(\phi))$ is a $TH(\mathcal{F})$-envelope of $T(H(M))$ by Proposition 1(2). By Lemma 1(3) $TH \simeq 1_{\mathcal{M}^C/\mathcal{T}}$, the eCe-comodule homomorphism $T(H(\phi)) \cong \phi$ is an $\mathcal{F}$-envelope of M.

(3) The necessity is obtained from Proposition 3(3).
Sufficiency. If the right C-comodule homomorphism $H(\phi) \in \mathrm{Com}_C(H(M), H(X))$ is a special $H(\mathcal{F})$-preenvelope of $H(M)$, then the eCe-comodule homomorphism $T(H(\phi))$ is a special $TH(\mathcal{F})$-preenvelope of $T(H(M))$ by Proposition 1(3). From Lemma 1(3) $TH \simeq 1_{\mathcal{M}^C/\mathcal{T}}$, it follows that the eCe-comodule homomorphism $T(H(\phi)) \cong \phi$ is a special $\mathcal{F}$-preenvelope of M.

□

4. (Co)localization in cotilting comodules

This section is devoted to applying the theory of (co)localization to cotilting comodules.

Definition 2. A comodule M is called a cotilting comodule if the following three conditions hold:

(1) $\mathrm{proj.dim}(M) \leq 1$;
(2) $\mathrm{Ext}^1_C(M, M^{(X)}) = 0$ for any cardinal X;
(3) there exists an exact sequence $0 \to W \to M_0 \to M_1 \to 0$, where W is a projective generator in $\mathcal{M}^C$, and $M_i \in \mathrm{Add}\, T$ for $i = 0, 1$.

Proposition 4. *Let C be a semiperfect coalgebra and $e \in C^*$ be an idempotent. Assume that M is a cotilting C-comodule and $X \cong Ce$ is a quasi-finite injective cogenerator, then $T(M)$ is a cotilting eCe-comodule.*

Proof. The proof is similar to that of Proposition 6 in Section 5. □

Proposition 5. *Let C be a K-coalgebra and $e \in C^*$ be an idempotent defining a perfect colocalization. If M is a cotilting eCe-comodule, then $H(M)$ is a cotilting C-comodule.*

Proof. The proof is similar to that of Proposition 7 in Section 5. □

Theorem 6. *Assume that C is a K-coalgebra, idempotent $e \in C^*$ defines a perfect localization and $X \cong Ce$ is a quasi-finite injective cogenerator, then M is a cotilting eCe-comodule if and only if $H(M)$ is a cotilting C-comodule.*

Proof. The proof is similar to that of Theorem 7 in Section 5. □

5. (Co)localization in n-cotilting comodules

This section is devoted to studying the theory of the (co)localization for n-cotilting comodules.

Definition 3. A comodule M is called an n-cotilting comodule if the following three conditions hold:

(1) $\mathrm{proj.dim}(M) \leq n$;
(2) $\mathrm{Ext}^i_C(M, M^{(X)}) = 0$ for each $i > 0$ and for any cardinal X;
(3) there exists a long exact sequence $0 \to W \to M_0 \to M_1 \to \ldots \to M_n \to 0$, where W is a projective generator in $\mathcal{M}^C$ and $M_i \in \mathrm{Add}\, M$, for every $0 \leq i \leq n$.

Proposition 6. *Let C be a semi-perfect coalgebra and $e \in C^*$ be an idempotent. If M is an n-cotilting C-comodule and $X \cong Ce$ is a quasi-finite injective cogenerator, then $T(M)$ is an n-cotilting eCe-comodule.*

Proof. By [24, Theorem 1.13], we infer that $X \cong Ce$ is a quasi-finite injective cogenerator if and only if the functor T is an equivalence.

(1) By the hypothesis, we have proj.dim$(M) \leq n$, i.e., there is an exact sequence $0 \to P_n \to \ldots \to P_1 \to P_0 \to M \to 0$ in $\mathcal{M}^C$, where P_i's, for $0 \leq i \leq n$, are projective C-comodules. Since T is an equivalence, it follows that $0 \to T(P_n) \to \ldots \to T(P_1) \to T(P_0) \to T(M) \to 0$ is exact, where $T(P_i)'$s, for $0 \leq i \leq n$, are projective right eCe-comodules. It implies that proj.dim$(T(M)) \leq n$.

(2) Take an exact sequence $0 \to K \to P \xrightarrow{\phi} M \to 0$ in $\mathcal{M}^C$, where P is a projective C-comodule and $K = \text{Ker}\phi$. By the hypothesis, we infer that $\text{Ext}^i{}_C(M, M^{(X)}) = 0$, for each $i > 0$. Since T is an equivalence, it follows that $T(\text{Ext}^i_C(M, M^{(X)})) = \text{Ext}^i_{eCe}(TM, (TM)^{(X)}) = 0$, for every $i \geq 0$. As a consequence, we obtain $\text{Ext}^i_{eCe}(TM, (TM)^{(X)}) = 0$.

(3) From the assumption, we get the exact sequence $0 \to W \to M_0 \to M_1 \to \ldots \to M_n \to 0$ in $\mathcal{M}^C$, where W is a projective generator in $\mathcal{M}^C$ and C-comodules $M_i \in \text{Add } M$. Since T is an equivalence, there is the short exact sequence $0 \to T(W) \to T(M_0) \to T(M_1) \to \ldots \to T(M_n) \to 0$ in $\mathcal{M}^{eCe}$, where eCe-comodules $T(M_i) \in \text{Add } T(M)$.

□

Proposition 7. *Let C be a K-coalgebra and $e \in C^*$ be an idempotent defining a perfect colocalization. If M is an n-cotilting eCe-comodule, then $H(M)$ is an n-cotilting C-comodule.*

Proof.

(1) By the hypothesis, we infer that proj.dim$(M) \leq n$, i.e., there is an exact sequence $0 \to P_n \to \ldots \to P_1 \to P_0 \to M \to 0$ in $\mathcal{M}^{eCe}$, where $P_i's$ are projective eCe-comodules. Since T is an exact functor, H preserves projective comodules. It follows that there is the short exact sequence $0 \to H(P_n) \to \ldots \to H(P_1) \to H(P_0) \to H(M) \to 0$ in $\mathcal{M}^C$, where $H(P_i)'s$ are projective right C-comodules, for $0 \leq i \leq n$. It implies that proj.dim$(H(M)) \leq n$.

(2) Take an exact sequence $0 \to K \to P \xrightarrow{\delta} M \to 0$ in $\mathcal{M}^{eCe}$, where P is an projective eCe-comodule and $K = \text{Ker}\delta$. By the hypothesis, we infer that $\text{Ext}^i_{eCe}(M, M^{(X)}) = 0$. Since H is a fully faithful functor, there is the following commutative diagram, see Diagram 1 (denote $H(\cdot)$ by $H\cdot$ for convenience) where $H(M^{(X)}) = (HM)^{(X)}$ because H preserves coproducts of comodules. Since H is a fully faithful functor, it follows that $H(\text{Ext}^i_{eCe}(M, M^{(X)})) = \text{Ext}^i_C(HM, H(M^{(X)})) = 0$, for each $i \geq 0$. As a consequence, we get $\text{Ext}^i_C(HM, (HM)^{(X)}) = 0$.

(3) According to the assumption, it infers that $0 \to W \to M_0 \to M_1 \to \ldots \to M_n \to 0$, where W is a projective generator in $\mathcal{M}^{eCe}$ and eCe-comodules $M_i \in \text{Add } M$. Since H is an exact functor and preserves coproducts of comodules,

there is the short exact sequence $0 \to H(W) \to H(M_0) \to H(M_1) \to \ldots \to H(M_n) \to 0$ in $\mathcal{M}^C$, where C-comodules $H(M_i) \in \text{Add}\, H(M)$.

□

Theorem 7. *Let K be a field, C be a K-coalgebra, and $e \in C^*$ be an idempotent defining a perfect localization. Suppose that $X \cong Ce$ is a quasi-finite injective cogenerator, then M is an n-cotilting eCe-comodule if and only if $H(M)$ is an n-cotilting C-comodule.*

Proof. Necessity. It follows from Proposition 7.

Sufficiency. It follows from [24, Theorem 1.13] that the functor T is an equivalence if and only if $X \cong Ce$ is a quasi-finite injective cogenerator.

(1) According to the assumption, we obtain $\text{proj.dim} H(M) \leq n$, i.e., there is an exact sequence $0 \to Q_n \to \ldots \to Q_1 \to Q_0 \to H(M) \to 0$ in $\mathcal{M}^C$, where Q_i's are projective C-comodules, for each $0 \leq i \leq n$. Since T is an equivalence, it follows that $0 \to T(Q_n) \to \ldots \to T(Q_1) \to T(Q_0) \to TH(M) \to 0$ in $\mathcal{M}^{eCe}$ is exact, where $T(P_i)'$s, for every $0 \leq i \leq n$, are projective right eCe-comodules. By Lemma 1(3) $TH \simeq 1_{\mathcal{M}/\mathcal{T}}$, we obtain the short exact sequence $0 \to T(Q_n) \to \ldots \to T(Q_1) \to T(Q_0) \to M \to 0$ in $\mathcal{M}^{eCe}$, where $T(Q_i)'$s are projective eCe-comodule. Consequently, we obtain $\text{proj.dim}(M) \leq n$.

(2) Take an exact sequence $0 \to H(K) \to H(P) \xrightarrow{\lambda} H(M) \to 0$ in $\mathcal{M}^C$, where $H(P)$ is an projective C-comodule and $H(K) = \text{Ker}\lambda$. According to the given conditions, we get $\text{Ext}^1_C(H(M), (HM)^{(X)}) = 0$. Since T is an equivalence, there is the following commutative diagram, see Diagram 2 (denote $H(\cdot)$ by $H\cdot$ for convenience) where $T((HM)^{(X)}) = TH(M^{(X)})$ because H preserves coproducts. Due to the equivalent functor T, we infer that

$$T(\text{Ext}^i_C(HM, (HM)^{(X)})) = \text{Ext}^i_{eCe}(THM, T((HM)^{(X)})) = 0.$$

As a consequence, $\text{Ext}^i_{eCe}(M, M^{(X)}) = 0$.

(3) According to the assumption, there is the short exact sequence $0 \to H(W) \to H(M_0) \to H(M_1) \to \ldots \to H(M_n) \to 0$ in $\mathcal{M}^C$, where $H(W)$ is a projective generator in $\mathcal{M}^C$ and C-comodules $H(M_i) \in \text{Add}\, H(M)$. Since T is an equivalence, we obtain the short exact sequence $0 \to T(H(W)) \to T(H(M_0)) \to T(H(M_1)) \to \ldots \to T(H(M_n)) \to 0$ in $\mathcal{M}^{eCe}$, where $TH(M_i) \in \text{Add}\, TH(M)$. By Lemma 1(3) $TH \simeq 1_{\mathcal{M}^C/\mathcal{T}}$, it infers that there is the short exact sequence $0 \to W \to M_0 \to M_1 \to \ldots \to M_n \to 0$ in $\mathcal{M}^{eCe}$, where W is a projective generator in $\mathcal{M}^{eCe}$ and eCe-comodules $M_i \in \text{Add}\, M$.

□

Remark 2. When n in Definition 3 takes the value of 1, the result is consistent with Definition 2 in Section 4. Thus, it can be seen that Section 4 is a special case

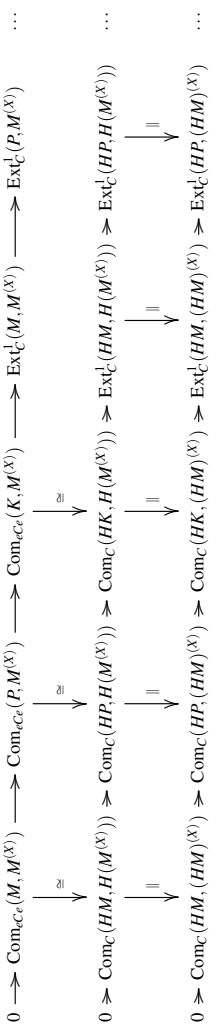


Diagram 1

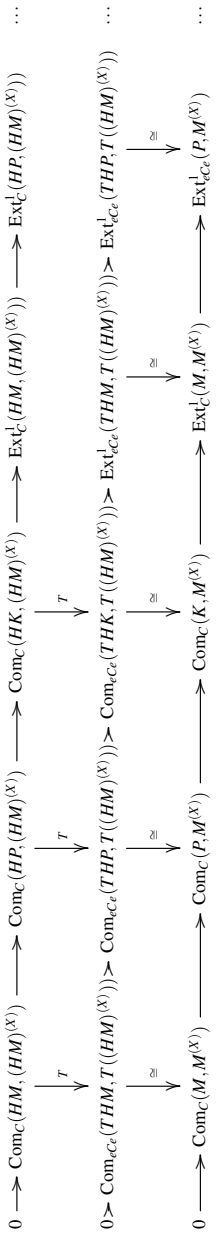


Diagram 2

of Section 5 when $n = 1$. In view of this, we omit the proofs of Proposition 4, Proposition 5, and Theorem 6 in Section 4.

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EXISTENCE AND STABILITY OF SOLUTIONS FOR CAPUTO FRACTIONAL DIFFERENTIAL EQUATION THROUGH BIELECKI METRIC

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Abstract. In this paper, we have shown that, by considering the Caputo fractional differential equation and Bielecki norm, the existence and uniqueness of solution for Cauchy problem is established using Banach fixed point theorem and further demonstrate the existence of solution without the Lipschitz condition utilizing the Schauder fixed point theorem. Furthermore, under the condition of existence and uniqueness of solution for Caputo fractional differential equation, we present some results about the error estimation, local approximation of solutions and the Ulam-Hyers-Rassias stability. Finally, we provide a numerical example to verify the theoretical results about error estimation.

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1. INTRODUCTION

Fractional calculus has significantly attracted attention in theories [7,12,19,21,22] and applications [10,13,14], owing to their accuracy and effectiveness in describing memory processes, power law phenomena and others. However, solving fractional differential equations still faces substantial challenges. Therefore, many scholars have focused on two main aspects of solving these equations. On the one hand, authors have focused on utilizing numerical and analytical approaches to derive approximate solutions for fractional differential equations, which include Picard's iterative method [2], Adams-type predictor–corrector method [5], radial basis functions [6], variational iteration method [20], wavelet method [1] and others. On the other hand,some scholars have applied fixed point theorem to analyze diverse types

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of fractional differential equations and demonstrate the existence and uniqueness of solutions under different definition of fractional derivative [3,4,8,18].

Ulam-Hyers stability, being one of the most active research topics in differential equations, has acquired a series of results. Li et al. [8] demonstrate the existence and uniqueness of solution and Ulam-Hyers and Ulam-Hyers-Rassias stability for conformable fractional differential equation on finite and infinite time intervals. Liu et al. [9] studied the Ulam-Hyers stability for the linear Caputo-Fabrizio fractional differential equations with the Mittag-Leffler kernel. Wang et al. [16] prove the Ulam-Hyers stability and Ulam-Hyers-Rassias stability using Laplace transform method in terms of Caputo linear fractional differential equations.

Most papers are concerned with the existence and uniqueness of solution for the Caputo fractional differential equation in Banach space, where the solution is given in the sense of all continuous functions with the maximum norm. Thus, motivated by [8,11,17], the objective of this paper is to consider the following Cauchy problem of Caputo type fractional differential equation:

$$\begin{cases} {}^{c}\mathrm{D}^{\alpha}y = f(x,y), \\ y(0) = y_0, \end{cases} \quad x \in I = [0,a], \tag{1.1}$$

where $a > 0$, $y \in \mathbb{R}^m$, the operator ${}^{c}\mathrm{D}^{\alpha}$ represents the Caputo fractional derivative of order α, $\alpha \in (0,1)$. The function $f : I \times \mathbb{R}^m \to \mathbb{R}^m$ is continuous.

In the following, by using the Bielecki norm, we show the existence and uniqueness of solution to Caputo fractional differential equation via the Banach fixed point theorem and Schauder fixed point theorem. Further, under the condition of the existence and uniqueness of solution, we present the results about error estimation, local approximation of solutions and Ulam-Hyers-Rassias stability.

2. Preliminaries

Throughout this paper, we use $\mathbb{R}^m$ to denote m-dimensional Euclidean space, where the norm is defined as $\|\xi\| = \left(\sum_{i=1}^{m} |\xi_i|^2\right)^{\frac{1}{2}}$ and $\xi \in \mathbb{R}^m$. Moreover, let $C([a,b])$ be Banach space consisting of all continuous functions from $[a,b]$ into $\mathbb{R}$ endowed with the Bielecki norm $\|u\|_{\lambda} = \max_{t\in[a,b]} e^{-\lambda t}\|u(t)\|$, where $0 < a < b < \infty$ and $\lambda > 0$.

Definition 1 (see [7])**.** The fractional integral of order $\alpha > 0$ with the lower limit zero for a function y is defined as

$$\mathrm{I}^{\alpha}y(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} y(s)ds,$$

provided the right side is point-wise defined on $[0,\infty)$, where $\Gamma(\cdot)$ is the gamma function.

Definition 2 (see [7]). For $\alpha > 0$ and $y : (0,\infty) \to \mathbb{R}$, the Caputo fractional derivative of y is given by

$$^{c}\mathrm{D}^{\alpha}y(x) = \frac{1}{\Gamma(n-\alpha)}\int_0^x (x-s)^{n-\alpha-1}y^{(n)}(s)ds,$$

where $n \in \mathbb{N}$ such that $n-1 < \alpha < n$.

Lemma 1 (see [7]). *The solution of Cauchy problem (1.1) is equivalent to the solution of the second kind of Volterra integral equation*

$$y(x) = y_0 + \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}f(s,y(s))ds. \tag{2.1}$$

Definition 3 (see [15]). If for each function y satisfying

$$|^{c}\mathrm{D}^{\alpha}y(x) - f(x,y)| \le \varphi(x), \quad x \in [a,b],$$

there is a solution $\hat{y}(x)$ of the Caputo fractional differential equation (1.1) and a constant $C > 0$ such that

$$|y(x) - \hat{y}(x)| \le C\varphi(x), \quad x \in [a,b],$$

where $\varphi : [a,b] \to (0,+\infty)$ is a nonnegative continuous function, then we say that the Caputo fractional differential equation (1.1) has the Ulam-Hyers-Rassias stability.

3. Results for existence of solution

To get our conclusions, we need to introduce the following assumptions.

A1 Let $f : G_0 \subset I \times \mathbb{R}^m \to \mathbb{R}^m$ be continuous function and satisfy equation (1.1), where

$$G_0 = \{(x,y) \in I \times \mathbb{R}^m : 0 \le x \le a, \|y - y_0\| \le b\},$$

$a > 0$, $b > 0$ and $y_0 \in \mathbb{R}^m$. Then there exists a $L > 0$ such that

$$\|f(x,y_1) - f(x,y_2)\| \le L\|y_1 - y_2\|, \quad \forall y_1, y_2 \in \mathbb{R}^m.$$

A2 Let $g : G_0 \to \mathbb{R}^m$ be continuous function. Then there exists a monotone function $p : I \to \mathbb{R}^+$ such that

$$\|f(x,y) - g(x,y)\| \le p(x), \quad \forall (x,y) \in I \times \mathbb{R}^m.$$

A3 Let $\Phi, \Psi_n : G_0 \to \mathbb{R}^m$ be continuous functions and $n \in \mathbb{N}^+$. Φ and Ψ_n are Lipschitz with respect to second argument, i.e.

$$\|\Phi(x,y_1) - \Phi(x,y_2)\| \le L_1\|y_1 - y_2\|,$$

and

$$\|\Psi_n(x,y_1) - \Psi_n(x,y_2)\| \le L_2\|y_1 - y_2\|,$$

where $L_1, L_2 > 0$.

A4 The sequence $(\Psi_n)_{n\ge 1}$ uniformly converges to Ψ over the domain G_0.

Let IVP_n and IVP be notations of following Caputo fractional differential equations, i.e.

$$(IVP_n)\begin{cases} {}^c\mathrm{D}^\alpha y = \Psi_n(x,y)+\Phi(x,y), \\ y(0)=y_0, \end{cases} \quad n\in\mathbb{N}^+,$$

and

$$(IVP)\begin{cases} {}^c\mathrm{D}^\alpha y = \Psi(x,y)+\Phi(x,y), \\ y(0)=y_0. \end{cases}$$

Next, we first present the existence and uniqueness of solution to the Cauchy problem for Caputo fractional differential equation in the sense of Bielecki norm.

Theorem 1. *Assume that* [A1] *is true. Then the equation* (1.1) *possess a unique solution defined on* $|x|\le h_0$, *where* $h_0=\min\left\{a,\left[\frac{b\Gamma(\alpha+1)}{M}\right]^{\frac{1}{\alpha}}\right\}$ *and* $M=\sup\limits_{(x,y)\in G_0}\|f(x,y)\|$.

Proof. According to Lemma 1, we know that the solution of equation (1.1) is equivalent to the solution of following Volterra integral equation

$$y(x)=y_0+\frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}f(s,y(s))ds, \quad |x|\le h_0.$$

Here, we choose the Banach space consisting of all continuous functions on $|x|\le h_0$. The initial value problem equation (1.1) for the case where $x\in[0,h_0]:=I_0$ are discussed. Then we define the operator $T:C(I_0)\to C(I_0)$,

$$(Ty)(x):=y_0+\frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}f(s,y(s))ds.$$

By direct calculation, we have

$$\begin{aligned}\|Ty-y_0\|_\lambda &\le \frac{1}{\Gamma(\alpha)}\max_{x\in I_0}e^{-\lambda x}\left\|\int_0^x (x-s)^{\alpha-1}f(s,y(s))ds\right\| \\ &\le \frac{M}{\Gamma(\alpha)}\max_{x\in I_0}e^{-\lambda x}\left|\int_0^x (x-s)^{\alpha-1}ds\right| \\ &\le \frac{Mh_0^\alpha}{\Gamma(\alpha+1)}\max_{x\in I_0}e^{-\lambda x}\le\frac{Mh_0^\alpha}{\Gamma(\alpha+1)}\le b.\end{aligned}$$

For all $y_1,y_2\in C(I_0)$, we obtain

$$\begin{aligned}\|Ty_1(x)-Ty_2(x)\| &= \left\|\frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}(f(s,y_1(s))-f(s,y_2(s))ds\right\| \\ &\le \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\|f(s,y_1(s))-f(s,y_2(s)\|\,ds \\ &\le \frac{1}{\Gamma(\alpha)}\int_0^x L(x-s)^{\alpha-1}\|y_1(s)-y_2(s)\|\,ds.\end{aligned}$$

Further, we get

$$\begin{aligned}\|Ty_1 - Ty_2\|_\lambda &= \frac{L}{\Gamma(\alpha)} \max_{x\in I_0} e^{-\lambda x} \int_0^x e^{\lambda s}(x-s)^{\alpha-1} e^{-\lambda s} \|y_1(s) - y_2(s)\| ds \\ &\leq \frac{L\|y_1 - y_2\|_\lambda}{\Gamma(\alpha)} \max_{x\in I_0} e^{-\lambda x} \int_0^x e^{\lambda s}(x-s)^{\alpha-1} ds \qquad (3.1) \\ &\leq \frac{L\|y_1 - y_2\|_\lambda}{\lambda^\alpha} := \beta \|y_1 - y_2\|_\lambda, \qquad (3.2)\end{aligned}$$

since

$$\begin{aligned}\int_0^x e^{\lambda s}(x-s)^{\alpha-1} ds &= e^{\lambda x} \int_0^x t^{\alpha-1} e^{-\lambda t} dt = e^{\lambda x}\lambda^{-\alpha} \int_0^{\lambda x} r^{\alpha-1} e^{-r} dr \\ &\leq e^{\lambda x}\lambda^{-\alpha} \int_0^\infty r^{\alpha-1} e^{-r} dr = \frac{e^{\lambda x}\Gamma(\alpha)}{\lambda^\alpha}.\end{aligned}$$

Here, we choose the appropriate λ such that $0 < \beta < 1$. Thus the operator T is a contraction via the Bielecki norm $\|\cdot\|_\lambda$ on $C(I_0)$, which demonstrate the equation (1.1) possess a unique solution defined on I_0. □

In the following, we shall use the Schauder fixed point theorem to demonstrate the existence of solution for Caputo fractional differential equation without the Lipschitz condition.

Theorem 2. *Assume that $f(\cdot,\cdot)$ is a continuous function defined on G_0. Then, the equation (1.1) has at least one solution.*

Proof. Similar operator T can be established in Theorem 1 and we will divide our demonstration into several procedures.

Step 1: T is continuous.

Let u_n be a sequence such that $u_n \to u$ in $C(I)$. For all $x \in I$, we obtain that

$$\begin{aligned}\|Tu_n(x) - Tu(x)\| &= \frac{1}{\Gamma(\alpha)} \left\| \int_0^x (x-s)^{\alpha-1} (f(s,u_n(s)) - f(s,u(s))) ds \right\| \\ &\leq \frac{1}{\Gamma(\alpha)} \left\| \int_0^x (x-s)^{\alpha-1} e^{\lambda s} e^{-\lambda s} (f(s,u_n(s)) - f(s,u(s))) ds \right\| \\ &\leq \frac{Le^{\lambda x}\|f(\cdot,u_n(\cdot)) - f(\cdot,u(\cdot))\|_\lambda}{\lambda^\alpha}.\end{aligned}$$

Further, we have

$$\|Tu_n - Tu\|_\lambda \leq \frac{L\|f(\cdot,u_n(\cdot)) - f(\cdot,u(\cdot))\|_\lambda}{\lambda^\alpha} \to 0$$

since $\|f(\cdot,u_n(\cdot)) - f(\cdot,u(\cdot))\|_\lambda \to 0$. This shows the fact that operator T is continuous.

Step 2: T maps bounded sets into bounded sets of $C(I)$.

$$\begin{aligned}\|Tu(x)\| &\leq \|y_0\| + \frac{1}{\Gamma(\alpha)}\left\|\int_0^x (x-s)^{\alpha-1} f(s,u(s))ds\right\| \\ &\leq \|y_0\| + \frac{M}{\Gamma(\alpha)}\left|\int_0^x (x-s)^{\alpha-1} ds\right| \leq \|y_0\| + \frac{Ma^\alpha}{\Gamma(\alpha+1)},\end{aligned}$$

where $M = \sup_{(x,y)\in G_0} \|f(x,y(x))\|$. Further, we have

$$\|Tu\|_\lambda \leq \|y_0\| + \frac{Ma^\alpha}{\Gamma(\alpha+1)}.$$

Step 3: T maps bounded sets into equicontinuous sets in $C(I)$.

Let $x_1, x_2 \in I$, with $0 < x_1 < x_2 < h$. Thus, we have following results

$$\begin{aligned}\|Tu(x_1) - Tu(x_2)\| = \frac{1}{\Gamma(\alpha)}\Bigg\|&\int_0^{x_1} (x_1-s)^{\alpha-1} f(s,u(s))ds \\ &- \int_0^{x_2} (x_2-s)^{\alpha-1} f(s,u(s))ds\Bigg\| \\ \leq \frac{M}{\Gamma(\alpha+1)}&\left|x_2^\alpha - x_1^\alpha\right|.\end{aligned}$$

It is obvious that the right-hand side of above inequality tends to zero as $x_1 \to x_2$. Thus, the operator T is equicontinuous.

By combined with Step $1-3$, the operator T is completely continuous with the Arzelà-Ascoli theorem. Further, based on the Schauder fixed point theorem, the operator T has a fixed point corresponding to a solution for (1.1). The proof is finished. □

Under the condition of the Bielecki metric, we present the corollary about the prior and posterior error estimate by using the Picard iteration.

Corollary 1. *Let $(C(I), \|\cdot\|_\lambda)$ be a complete metric space and assume that [A1] holds. Then the prior and posterior error estimate are present as follows*

$$d(y_n, y^*) = \frac{L^n}{\lambda^{n\alpha-1}(\lambda^\alpha - L)} d(y_0, y_1), \quad \forall n \geq 0,\ y \in C(I),$$

and

$$d(y_n, y^*) = \frac{L}{\lambda^\alpha - L} d(y_{n-1}, y_n),$$

where $d(x,y) = \|x-y\|_\lambda$, $\forall x, y \in C(I)$, and y^* is a unique fixed point.

Proof. By Lemma 1, we construct the Picard iteration

$$y_n(x) = y_0 + \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1} f(s, y_{n-1}(s))ds,$$

and define the operator $S : C(I) \to C(I)$ by

$$(Sy)(x) = y_0 + \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} f(s, y(s)) ds.$$

Notice that, by (3.1), $y_n = S(y_{n-1})$, we have

$$d(y_1, y_2) = d(Sy_0, Sy_1) \leq \beta d(y_0, y_1).$$

Thus, we deduce that

$$d(y_n, y_{n+1}) \leq \beta^n d(y_0, y_1),$$

and

$$\begin{aligned} d(y_n, y_{n+p}) &\leq \sum_{k=n}^{n+p-1} d(y_k, y_{k+1}) \leq \sum_{k=n}^{n+p-1} \beta^k d(y_0, y_1) \\ &\leq \frac{\beta^n}{1-\beta} d(y_0, y_1), \quad \forall n, p \in \mathbb{N}, p > 0. \end{aligned}$$

Finally, as $p \to \infty$, we obtain that

$$d(y_n, y^*) = \lim_{p \to \infty} d(y_n, y_{n+p}) \leq \frac{\beta^n}{1-\beta} d(y_0, y_1) = \frac{L^n}{\lambda^{n\alpha-1}(\lambda^\alpha - L)} d(y_0, y_1), \quad \forall n \geq 0.$$

Let us notice that by (3.1), we have

$$d(y_{n+1}, y_n) \leq \beta d(y_n, y_{n-1}),$$

and by induction,

$$d(y_{n+k-1}, y_{n+k}) \leq \beta^k d(y_n, y_{n-1}).$$

Then we get

$$d(y_n, y_{n+p}) \leq (\beta + \beta^2 + \cdots + \beta^p) d(y_{n-1}, y_n) \leq \frac{\beta}{1-\beta} d(y_{n-1}, y_n).$$

By letting $p \to \infty$, we get

$$d(y_n, y_{n+p}) \leq \frac{\beta}{1-\beta} d(y_{n-1}, y_n) = \frac{L}{\lambda^\alpha - L} d(y_{n-1}, y_n).$$

□

Remark 1.

(i) The result of prior and posterior estimate indicate that the approximation error of the nth iteration starting from the initial value y_0 is determined by the Lipschitz coefficient, the parameter λ, and the initial displacement.

(ii) The posterior estimate provides a direct stopping criterion for Picard iteration approximation of the fixed point, while the prior estimate indirectly gives a stopping criterion.

(iii) The Picard iteration shows the rate of convergence is linear.

(iv) In most cases, the definition of (3.1) is locally satisfied on the whole space $C(I)$.

Then we present the sufficient condition to the error estimation of solutions for Cauchy problem of Caputo fractional differential equations.

Theorem 3. *Assume that* [A1] *and* [A2] *are satisfied, then we have the error estimation*

$$\|u-v\|_\lambda \le \frac{h_1^\alpha \lambda^\alpha P}{\Gamma(\alpha+1)(\lambda^\alpha - L)}, \quad 0 \le x \le h_1,$$

where $0 < h_1 \le \min\left\{a, \left[\frac{b\Gamma(\alpha+1)}{M_1}\right]^{\frac{1}{\alpha}}\right\}$, $P = \max\limits_{x\in[0,h_1]} |p(x)|$, $M_1 = \max\{\|f(x,y)\|, \|g(x,y)\|\}$ *and* u, v *are solutions satisfying equation* (1.1) *with respect to* f, g.

Proof. By Lemma 1, by using the equivalent Volterra integral equations (2.1), we define the operators $A, B : C(I) \to C(I)$ by

$$(Au)(x) := y_0 + \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1} f(s,u(s))ds,$$
$$(Bv)(x) := y_0 + \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1} g(s,v(s))ds.$$

Draw on the Theorem 1, the existence of a solution is guaranteed by the continuous conditions on f and g. The domain G_0 turns into G_1, i.e.

$$G_1 = \{(x,y) \in I_1 \times \mathbb{R}^m : 0 \le x \le h_1, \|y - y_0\| \le b\},$$

where $h_1 = \min\left\{a, \left[\frac{b\Gamma(\alpha+1)}{M_1}\right]^{\frac{1}{\alpha}}\right\}$. We can write the initial value problems of Caputo fractional differential equations as fixed points $u = Au, v = Bv, u, v \in C(I_1)$. By using the assumption [A2], we have following inequalities

$$\begin{aligned}\|Ay(x) - By(x)\| &= \left\| \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}(f(s,y(s)) - g(s,y(s)))ds \right\| \\ &\le \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\|f(s,y(s)) - g(s,y(s))\|ds \\ &\le \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1} p(s)ds.\end{aligned}$$

Further, we get

$$\begin{aligned}\|Ay - By\|_\lambda &= \frac{1}{\Gamma(\alpha)}\max_{x\in I_1} e^{-\lambda x}\int_0^x (x-s)^{\alpha-1}p(s)ds \\ &\le \frac{P}{\Gamma(\alpha+1)}\max_{x\in I_1} e^{-\lambda x}x^\alpha \le \frac{h_1^\alpha P}{\Gamma(\alpha+1)},\end{aligned} \tag{3.3}$$

where $P = \max_{x\in I_1} |p(x)|$. By (3.1), we have

$$\|Ay_1 - Ay_2\|_\lambda \leq \frac{L\|y_1 - y_2\|_\lambda}{\lambda^\alpha} := \beta \|y_1 - y_2\|_\lambda . \tag{3.4}$$

The choice of β is similar to Theorem 1. Then we combine (3.3) and (3.4) to obtain following results

$$\begin{aligned}\|u - v\|_\lambda = \|Au - Bv\|_\lambda &\leq \|Au - Av\|_\lambda + \|Av - Bv\|_\lambda \\ &\leq \frac{L\|u - v\|_\lambda}{\lambda^\alpha} + \frac{h_1^\alpha P}{\Gamma(\alpha+1)} \leq \frac{h_1^\alpha \lambda^\alpha P}{\Gamma(\alpha+1)(\lambda^\alpha - L)}.\end{aligned}$$

□

By using above approach, we investigate the local approximation of solutions for a class of Cauchy problems based on IVP_n and IVP.

Theorem 4. *Assume that* $[A3]$ *and* $[A4]$ *are satisfied, then the Cauchy problem of Caputo fractional differential equation* (IVP) *has (locally) a unique solution.*

Proof. Under the condition of Theorem 1, the Cauchy problem (IVP) has at least one solution. Further, we can deduce from $[A4]$ that sequence $(\Psi_n)_{n\geq 1}$ is uniformly bounded such that

$$\|\Psi_n\| \leq M_2, \quad \|\Psi\| \leq M_2, \quad \forall n \geq 1,$$

for some $M_2 > 0$. Let u_n and u be solutions with respect to IVP_n and IVP. It is shown that the problem (IVP_n) possess a unique solution u_n, which defined on

$$u_n : (0, h_2) \to \mathbb{R},$$

where $h_2 = \min\left\{a, \left[\frac{b\Gamma(\alpha+1)}{M_2}\right]^{\frac{1}{\alpha}}\right\}$. The domain G_0 turns into G_2, and we keep the same notations in Theorem 3, i.e.

$$G_2 = \{(x, y) \in I_2 \times \mathbb{R}^m : 0 \leq x \leq h_2,\ \|y - y_0\| \leq b\},$$

and

$$f(x, y) = \Psi_n(x, y) + \Phi(x, y), \quad g(x, y) = \Psi(x, y) + \Phi(x, y).$$

Further, by using the assumption $[A3]$, we derive the results from f and g, i.e.

$$\begin{aligned}\|f(x, y_1) - f(x, y_2)\| &= \|\Phi(x, y_1) + \Psi_n(x, y_1) - \Phi(x, y_2) - \Psi_n(x, y_2)\| \\ &\leq (L_1 + L_2)\|y_1 - y_2\|,\end{aligned}$$

where $(x, y_1), (x, y_2) \in G_2$, and

$$\|f(x, y) - g(x, y)\| = \|\Psi_n(x, y) - \Phi(x, y)\|.$$

Within the framework of Theorem 3, we obtain following results between IVP_n and IVP

$$\begin{aligned}\|Au_n - Au\|_\lambda &= \frac{(L_1+L_2)}{\Gamma(\alpha)} \max_{x\in I_2} e^{-\lambda x} \int_0^x e^{\lambda s}(x-s)^{\alpha-1} e^{-\lambda s} \|u_n(s)-u(s)\|\, ds \\ &\le \frac{(L_1+L_2)\,\|u_n-u\|_\lambda}{\Gamma(\alpha)} \max_{x\in I_2} e^{-\lambda x} \int_0^x e^{\lambda s}(x-s)^{\alpha-1} ds \\ &\le \frac{(L_1+L_2)\,\|u_n-u\|_\lambda}{\lambda^\alpha},\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}\|Au - Bu\|_\lambda &= \frac{1}{\Gamma(\alpha)} \max_{x\in I_2} e^{-\lambda x} \int_0^x e^{\lambda s}(x-s)^{\alpha-1} e^{-\lambda s} \|\Psi_n - \Psi\|\, ds \\ &\le \frac{\|\Psi_n-\Psi\|_\lambda}{\Gamma(\alpha)} \max_{x\in I_2} e^{-\lambda x} \int_0^x e^{\lambda s}(x-s)^{\alpha-1} ds \le \frac{\|\Psi_n-\Psi\|_\lambda}{\lambda^\alpha}.\end{aligned} \tag{3.6}$$

By virtue of (3.5) and (3.6), we obtain that

$$\|u_n - u\|_\lambda \le \frac{\|\Psi_n-\Psi\|_\lambda}{\lambda^\alpha - (L_1+L_2)}, \quad \forall x \in (0, h_2).$$

Finally, as $n \to \infty$, we obtain uniform convergence that

$$u = \lim_{n\to\infty} u_n,$$

which demonstrates the uniqueness of u. □

4. Results for Ulam-Hyers-Rassias stability

Theorem 5. *Assume that [A1] and [A2] are satisfied. If $y \in C(I)$ is such that*

$$\|{}^c\mathrm{D}^\alpha y(x) - f(x,y)\| \le \varphi(x).$$

Then there is a unique solution $\hat{y} \in C(I)$ satisfying equation (1.1) and

$$\|y(x) - \hat{y}(x)\|_\lambda \le \frac{\lambda^\alpha}{\lambda^\alpha - L}\varphi(x), \quad x \in I. \tag{4.1}$$

This implies that equation (1.1) exhibits Hyers-Ulam-Rassias stability under the aforementioned condition.

Proof. Applying the Lemma 1, the equivalent integral equation for (1.1) is constructed and define the same operator T in Theorem 1. Further, for any $u, v \in C(I)$, we have

$$\|Tu - Tv\|_\lambda \le \frac{L}{\lambda^\alpha}\|u-v\|_\lambda.$$

By choosing large enough parameter λ, then the operator T is strictly contractive in $C(I)$. Thus, the equation (1.1) is Ulam-Hyers-Rassis stability. Further, by the Bielecki metric in $C(I)$, we have

$$d(Tu,Tv) \leq \frac{L}{\lambda^{\alpha}} d(u,v),$$

for any $u.v \in C(I)$. Based on the above inequality, we obtain

$$d(y,\hat{y}) \leq d(y,Ty) + d(Ty,\hat{y}) = d(y,Ty) + \frac{L}{\lambda^{\alpha}} d(y,\hat{y}).$$

Further, we have following results

$$d(y,\hat{y}) \leq \frac{\lambda^{\alpha}}{\lambda^{\alpha} - L} d(y,Ty).$$

Thus, the result of (4.1) is obtained. □

Remark 2. If $\varphi(\cdot)$ is a constant function in Theorem 5, we say that the equation (1.1) has the Ulam-Hyers stability.

5. An example

We consider the Cauchy problems of following Caputo fractional differential equations

$$\begin{cases} {}^{c}\mathrm{D}^{\alpha} y = x, \\ y(0) = 0, \end{cases} \quad \textit{and} \quad \begin{cases} {}^{c}\mathrm{D}^{\alpha} y = x - y, \\ y(0) = 0, \end{cases} \tag{5.1}$$

where the domain is defined by

$$G = \left\{ (x,y) \in \mathbb{R} \times \mathbb{R} : 0 \leq x \leq 1, 0 \leq y \leq \frac{1}{2} \right\}.$$

We denote

$$f(x,y) = x, \quad g(x,y) = x - y,$$

and let u and v be solutions with respect to f and g. The fractional differential equations (5.1) satisfy the following conditions

$$\lambda = 10000, \quad \alpha = \frac{1}{2}, \quad L = \frac{1}{2}, \quad h = \min\left\{1, \frac{\pi}{16}\right\}.$$

Then we obtain the difference with respect to f and g,

$$|f(x,y) - g(x,y)| = |y| \leq \frac{1}{2}.$$

Setting $p(x) = \frac{1}{2}$ and combining with Theorem 3, we observe that

$$\|u - v\|_{\lambda} \leq 0.2512562.$$

The numerical results for the equations (5.1) are presented in Figure 1.

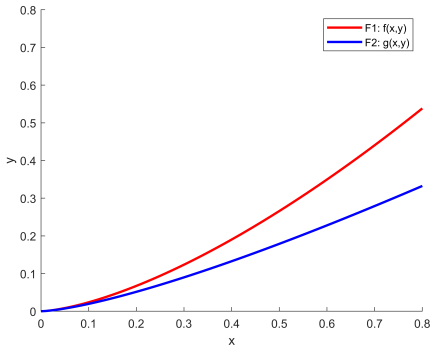

FIGURE 1. Error estimation of numerical result by (5.1).

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$\overrightarrow{p}(\cdot)-$ANISOTROPIC DEGENERATE ELLIPTIC EQUATIONS HAVING NATURAL EXPONENTIAL FUNCTION

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Abstract. In this work, we establish existence results for distributional solutions of anisotropic nonlinear elliptic equations with variable growth, L^1-data, and degenerate coercivity involving natural exponential function, where the nonlinear anisotropic differential operator is well defined between $\overrightarrow{p}(x)-$anisotropic Sobolev space $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$ and its dual, yet it lacks coercivity as $|u|$ becomes large, i.e., when $e^{|u|}$ is large.

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1. INTRODUCTION

In this paper we will prove the existence of distributional solutions to a class of anisotropic nonlinear elliptic equations with variable exponents

$$\begin{aligned} -\sum_{i=1}^{N} D_i\left(\frac{a(x,u)|D_iu|^{p_i(x)-2}D_iu}{1+e^{|u|}}\right)+u\sum_{i=1}^{N}|u|^{p_i(x)-2}=f, \quad &\text{in } \Omega,\\ u=0, \quad &\text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where Ω is a bounded open subset of $\mathbb{R}^N$ ($N\geq 2$) with Lipschitz boundary $\partial\Omega$, f is a function in $L^1(\Omega)$, and $a\colon \Omega\times\mathbb{R}\to\mathbb{R}^+$ is a Carathéodory function, such that

$$0<\alpha\leq a(x,s)\leq\beta,\ \forall x\in\Omega,\ \forall s\in\mathbb{R}, \tag{1.2}$$

where α and β are positive constants.

The nonlinear anisotropic operator A defined between the space $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$ (will be mentioned later in Section 2) and its dual by

$$A(u)=-\sum_{i=1}^{N} D_i\left(\frac{a(x,u)|D_iu|^{p_i(x)-2}D_iu}{1+e^{|u|}}\right),$$

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it fails to be coercive if $|u|$ is large (i.e., $e^{|u|}$ is large). This difficulty arises in proving the existence of solutions to the approximate problems, as well as when taking the limit, and classical existence theorems cannot be applied. The existence and regularity results for distributional solutions of degenerate anisotropic nonlinear elliptic problems with variable exponents given (for example, but not limited to) in the works [10–12, 14, 20].

In our proof we establish uniform anisotropic estimates on the sequence of solutions (u_n) and its derivative, where we have proven the a.e. convergence of the partial derivatives $D_i u_n$, $i = 1, \dots, N$, which can be turned into strong L^1-convergence. Equipped with this convergence we pass to the limit in the strong $L^1(\Omega)$ sense in

$$\frac{a_n(x,u_n)|D_i u_n|^{p_i(x)-2}D_i u_n}{1+e^{|u_n|}} \quad \text{and} \quad u_n \sum_{i=1}^{N} |u_n|^{p_i(x)-2},$$

finally conclude that the approximate solutions u_n converge to the solution of (1.1).

The work has been organized in the following form: Section 2 for some mathematical preliminaries, where we remind the p(x)-anisotropic Lebesgue-Sobolev spaces. The main theorem and its proof are in Section 3.

2. Mathematical preliminaries

In this section we recall the Lebesgue-Sobolev spaces with variable exponent in both cases isotropic and anisotropic, with mentioning some of its properties and facts (see [4, 5, 8]).

Let Ω be a bounded open subset of $\mathbb{R}^N$ $(N \geq 2)$, we denote

$$\mathcal{C}_+(\overline{\Omega}) = \{\text{continuous function } p(\cdot)\colon \overline{\Omega} \longmapsto \mathbb{R} \text{ such that } 1 < p^- \leq p^+ < \infty\},$$

where

$$p^+ = \max_{x\in\overline{\Omega}} p(x) \quad \text{and} \quad p^- = \min_{x\in\overline{\Omega}} p(x).$$

We define the Lebesgue space with variable exponent $L^{p(\cdot)}(\Omega)$ by

$$L^{p(\cdot)}(\Omega) := \{\text{measurable functions } u\colon \Omega \mapsto \mathbb{R}; \rho_{p(\cdot)}(u) < \infty\},$$

where

$$\rho_{p(\cdot)}(u) := \int_\Omega |u(x)|^{p(x)} dx \quad \text{the convex modular.}$$

The space $L^{p(\cdot)}(\Omega)$ equipped with the norm

$$u \mapsto \|u\|_{p(\cdot)} := \|u\|_{L^{p(\cdot)}(\Omega)} = \inf\left\{\lambda > 0 \mid \rho_{p(\cdot)}(u/\lambda) \leq 1\right\}$$

becomes a reflexive Banach space. $L^{p'(\cdot)}(\Omega)$ symbolizes to the dual of $L^{p(\cdot)}(\Omega)$ where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$.

$\forall u \in L^{p(\cdot)}(\Omega)$, $\forall v \in L^{p'(\cdot)}(\Omega)$ the Hölder type inequality

$$\left|\int_\Omega uv\,dx\right| \leq \left(\frac{1}{p^-}+\frac{1}{p'^-}\right)\|u\|_{p(\cdot)}\|v\|_{p'(\cdot)} \leq 2\|u\|_{p(\cdot)}\|v\|_{p'(\cdot)}$$

holds true.

The variable exponents Sobolev space $W^{1,p(\cdot)}(\Omega)$ is defined as follows:

$$W^{1,p(\cdot)}(\Omega) := \left\{u \in L^{p(\cdot)}(\Omega) : |Du| \in L^{p(\cdot)}(\Omega)\right\},$$

it becomes a Banach space when equipped with the norm

$$u \mapsto \|u\|_{W^{1,p(\cdot)}(\Omega)} := \|Du\|_{p(\cdot)}. \tag{2.1}$$

We define also the Banach space $W_0^{1,p(\cdot)}(\Omega)$ by

$$W_0^{1,p(\cdot)}(\Omega) := \overline{C_0^\infty(\Omega)}^{W^{1,p(\cdot)}(\Omega)},$$

endowed with the norm (2.1). Moreover, it is reflexive and separable if $p(\cdot) \in \mathcal{C}_+(\overline{\Omega})$.

The following lemma will be used later.

Lemma 1 ([4, 5]). *If (u_n), $u \in L^{p(\cdot)}(\Omega)$, then the following relations hold:*

(i) $\|u\|_{p(\cdot)} < 1$ (*respectively* $= 1, > 1$) $\Longleftrightarrow \rho_{p(\cdot)}(u) < 1$ (*respectively* $= 1, > 1$),

(ii) $\min\left(\rho_{p(\cdot)}(u)^{\frac{1}{p^+}}, \rho_{p(\cdot)}(u)^{\frac{1}{p^-}}\right) \leq \|u\|_{p(\cdot)} \leq \max\left(\rho_{p(\cdot)}(u)^{\frac{1}{p^+}}, \rho_{p(\cdot)}(u)^{\frac{1}{p^-}}\right)$,

(iii) $\min\left(\|u\|_{p(\cdot)}^{p^-}, \|u\|_{p(\cdot)}^{p^+}\right) \leq \rho_{p(\cdot)}(u) \leq \max\left(\|u\|_{p(\cdot)}^{p^-}, \|u\|_{p(\cdot)}^{p^+}\right)$,

(iv) $\|u\|_{p(\cdot)} \leq \rho_{p(\cdot)}(u) + 1$,

(v) $\|u_n - u\|_{p(\cdot)} \to 0 \Longleftrightarrow \rho_{p(\cdot)}(u_n - u) \to 0$.

Now, we present the anisotropic Sobolev space with variable exponent which is used for the study of problem (1.1).

Let $p_i(\cdot)\colon \overline{\Omega} \to [1, +\infty)$ for all $i = 1, \ldots, N$ be continuous functions, and we set for all $x \in \overline{\Omega}$

$$\overrightarrow{p}(x) = (p_1(x), \ldots, p_N(x)), \quad p_+(x) = \max_{1\leq i\leq N} p_i(x), \quad p_-(x) = \min_{1\leq i\leq N} p_i(x),$$

$$p_+^+ = \max_{x\in\overline{\Omega}} p_+(x), \quad p_-^- = \min_{x\in\overline{\Omega}} p_-(x),$$

$$\frac{1}{\overline{p}(x)} = \frac{1}{N}\sum_{i=1}^N \frac{1}{p_i(x)}, \quad \overline{p}^\star(x) = \begin{cases} \frac{N\overline{p}(x)}{N-\overline{p}(x)}, & \text{for } \overline{p}(x) < N, \\ +\infty, & \text{for } \overline{p}(x) \geq N. \end{cases}$$

The anisotropic variable exponent Sobolev space $W^{1,\overrightarrow{p}(\cdot)}(\Omega)$ is defined as follows:

$$W^{1,\overrightarrow{p}(\cdot)}(\Omega) = \left\{u \in L^{p_+(\cdot)}(\Omega), D_i u \in L^{p_i(\cdot)}(\Omega),\ i = 1, \ldots, N\right\},$$

which is Banach space with respect to the norm

$$\|u\|_{\overrightarrow{p}(\cdot)} = \|u\|_{p_+(\cdot)} + \sum_{i=1}^{N} \|D_i u\|_{p_i(\cdot)}.$$

We define the spaces $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ and $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$ as follows:

$$W_0^{1,\overrightarrow{p}(\cdot)}(\Omega) = \overline{C_0^\infty(\Omega)}^{W^{1,\overrightarrow{p}(\cdot)}(\Omega)}, \quad \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega) = W^{1,\overrightarrow{p}(\cdot)}(\Omega) \cap W_0^{1,1}(\Omega).$$

Remark 1 ([7]). If Ω is a bounded open set with Lipschitz boundary $\partial\Omega$, then

$$\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega) = \left\{ u \in W^{1,\overrightarrow{p}(\cdot)}(\Omega),\ u_{|\partial\Omega} = 0 \right\},$$

where, $u_{|\partial\Omega}$ denotes the trace on $\partial\Omega$ of u in $W^{1,1}(\Omega)$.

We have the following embedding results.

Lemma 2 ([7, 8]). *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and $\overrightarrow{p}(\cdot) \in (\mathcal{C}_+(\overline{\Omega}))^N$. If $r \in \mathcal{C}_+(\overline{\Omega})$ and $\forall x \in \overline{\Omega}$, $r(x) < \max(p_+(x),\ \overline{p}^\star(x))$. Then the embedding*

$$\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega) \hookrightarrow L^{r(\cdot)}(\Omega) \text{ is compact.} \tag{2.2}$$

Lemma 3 ([7, 8]). *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and $\overrightarrow{p}(\cdot) \in (\mathcal{C}_+(\overline{\Omega}))^N$. Suppose that*

$$\forall x \in \overline{\Omega},\ p_+(x) < \overline{p}^\star(x). \tag{2.3}$$

Then the following Poincaré-type inequality holds

$$\|u\|_{p_+(\cdot)} \le C \sum_{i=1}^{N} \|D_i u\|_{p_i(\cdot)},\ \forall u \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega), \tag{2.4}$$

where C is a positive constant independent of u.

Remark 2. If (2.3) holds, then (2.4) implies that

$$u \mapsto \sum_{i=1}^{N} \|D_i u\|_{p_i(\cdot)}$$

is an equivalent norm on $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$.

For any $t > 0$, the standard scalar truncation function $T_t : \mathbb{R} \to \mathbb{R}$ (at height $t > 0$) is defined as

$$T_t(r) := \begin{cases} r, & \text{if } |r| \le t, \\ \frac{r}{|r|} t, & \text{if } |r| > t. \end{cases} \tag{2.5}$$

This function will be used to derive a priori estimates for our approximate solutions. We also need its derivative (see [20]):

$$(DT_t)(r) = \begin{cases} 1, & |r| < t, \\ 0, & |r| > t. \end{cases} \tag{2.6}$$

We need further the following function defined for $s \in \mathbb{R}$ by

$$G_t(s) = \begin{cases} 0, & if\ |s| \le t, \\ s-t, & if\ s > t, \quad t > 0, \\ s+t, & if\ s < -t, \end{cases}$$

as a test function in the approximate weak formulation.

3. Statement of results

Definition 1. We say that u is a distributional solution for problem (1.1) if $u \in W_0^{1,1}(\Omega)$, and for all $\varphi \in C_c^\infty(\Omega)$,

$$\int_\Omega \sum_{i=1}^N \frac{a(x,u)|D_i u|^{p_i(x)-2} D_i u}{1+e^{|u|}} D_i\varphi \, dx + \int_\Omega u \sum_{i=1}^N |u|^{p_i(x)-2}\varphi \, dx = \int_\Omega f(x)\varphi \, dx.$$

Our main result is the following.

Theorem 1. *Let $p_i(\cdot) > 1$, $i = 1,\dots,N$, are continuous functions on Ω such that (2.3) holds and $\overline{p} < N$. Assume that f is in $L^1(\Omega)$, and $a\colon \Omega \times \mathbb{R} \to \mathbb{R}^+$ is a Carathéodory function satisfying (1.2). Then the problem (1.1) has at least one solution $u \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$ in the sense of distributions.*

3.1. *Approximate solutions*

We are going to prove the existence of solution to problem (1.1). We define

$$f_n(x) = \frac{f(x)}{1+\frac{|f(x)|}{n}}, \quad a_n(x,u) = \frac{a(x,u)}{1+\frac{a(x,u)}{n}}, \quad n \in \mathbf{N}^*. \tag{3.1}$$

We first notice that since $\Theta(x) = \frac{x}{1+\frac{x}{n}}$ is increasing and $a_n(x,u) = \Theta(a(x,u))$, (1.2) implies that

$$\frac{\alpha}{1+\alpha} \le a_n \le \beta. \tag{3.2}$$

Lemma 4. *Let $p_i(\cdot) > 1$, $i = 1,\dots,N$, are continuous functions on Ω such that $\overline{p} < N$, and (2.3) holds. Assume that f is in $L^1(\Omega)$, and $a\colon \Omega \times \mathbb{R} \to \mathbb{R}^+$ is a Carathéodory function satisfying (1.2). Then, there exists at least one weak solution $u_n \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$ to the approximated problems*

$$\begin{aligned} -\sum_{i=1}^N D_i\left(\frac{a_n(x,u_n)|D_i u_n|^{p_i(x)-2} D_i u_n}{1+e^{|T_n(u_n)|}}\right) + u_n \sum_{i=1}^N |u_n|^{p_i(x)-2} &= f_n, \quad \text{in } \Omega, \\ u_n &= 0, \quad \text{on } \partial\Omega, \end{aligned} \tag{3.3}$$

in the sense that

$$\sum_{i=1}^N \int_\Omega \frac{a_n(x,u_n)|D_i u_n|^{p_i(x)-2} D_i u_n}{1+e^{|T_n(u_n)|}} D_i\varphi \, dx + \int_\Omega u_n \sum_{i=1}^N |u_n|^{p_i(x)-2}\varphi \, dx = \int_\Omega f_n\varphi \, dx \tag{3.4}$$

for every $\varphi \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega) \cap L^{\infty}(\Omega)$.

Proof. Consider the problem

$$-\sum_{i=1}^{N} D_i\left(\frac{a_n(x,u_{n_k})|D_iu_{n_k}|^{p_i(x)-2}D_iu_{n_k}}{1+e^{|T_n(u_{n_k})|}}\right)+\sum_{i=1}^{N} T_k\left(u_{n_k}|u_{n_k}|^{p_i(x)-2}\right)=f_n, \quad \text{in } \Omega,$$
$$u_{n_k}=0, \quad \text{on } \partial\Omega. \tag{3.5}$$

Let's prove that the operator defined between $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$ and its dual space by

$$\mathbf{A}\colon u_{n_k} \mapsto \left(v \mapsto \sum_{i=1}^{N}\int_{\Omega}\frac{a_n(x,u_{n_k})|D_iu_{n_k}|^{p_i(x)-2}D_iu_{n_k}}{1+e^{|T_n(u_{n_k})|}}D_iv\,\mathrm{d}x+\sum_{i=1}^{N}\int_{\Omega}T_k\left(u_{n_k}|u_{n_k}|^{p_i(x)-2}\right)v\,\mathrm{d}x\right),$$

is bounded, pseudomonotone and coercive.

First of all we put $\mathbf{A}=\mathbf{B}+\mathbf{C}$, with $\mathbf{B},\mathbf{C}\colon \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega) \to \left(\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)\right)^{*}$ defined by

$$\langle \mathbf{B}(u_{n_k}),v\rangle = \sum_{i=1}^{N}\int_{\Omega}\frac{a_n(x,u_{n_k})|D_iu_{n_k}|^{p_i(x)-2}D_iu_{n_k}}{1+e^{|T_n(u_{n_k})|}}D_iv\,\mathrm{d}x,$$
$$\langle \mathbf{C}(u_{n_k}),v\rangle = \sum_{i=1}^{N}\int_{\Omega}T_k\left(u_{n_k}|u_{n_k}|^{p_i(x)-2}\right)v\,\mathrm{d}x.$$

Here $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$ and its dual.

From (3.2), the fact that $1+e^{|T_n(u_{n_k})|}\geq 2$, Lemma 1, the fact that $\frac{1}{p_-^-}\leq 1$, Remark (2) (due (2.3)) and Hölder inequality, we obtain

$$\begin{aligned}
|\langle \mathbf{B}(u_{n_k}),v\rangle| &\leq \frac{\beta}{2}n\sum_{i=1}^{N}\int_{\Omega}|D_iu_{n_k}|^{p_i(x)-1}|D_iv|\,\mathrm{d}x\\
&\leq \beta n\sum_{i=1}^{N}\left\||D_iu_{n_k}|^{p_i(x)-1}\right\|_{p_i'(\cdot)}\|D_iv\|_{p_i(\cdot)}\\
&\leq \beta n\sum_{i=1}^{N}\left(1+\int_{\Omega}|D_iu_{n_k}|^{p_i(x)}\,\mathrm{d}x\right)^{\frac{1}{p_-^-}}\sum_{i=1}^{N}\|D_iv\|_{p_i(\cdot)}\\
&\leq \beta n\sum_{i=1}^{N}\left(1+\int_{\Omega}|D_iu_{n_k}|^{p_i(x)}\,\mathrm{d}x\right)\sum_{i=1}^{N}\|D_iv\|_{p_i(\cdot)}\\
&\leq \beta n\sum_{i=1}^{N}\left(2+\|D_iu_{n_k}\|_{p_i(\cdot)}^{p_+^+}\right)\sum_{i=1}^{N}\|D_iv\|_{p_i(\cdot)}\\
&\leq \beta n\left(2N+\|u_{n_k}\|_{\overrightarrow{p}(\cdot)}^{p_+^+}\right)\|v\|_{\overrightarrow{p}(\cdot)},
\end{aligned}$$

which implies the boundedness of $\mathbf{B}$.

On the other hand, thanks to the Hölder inequality and Lemma 1, we have

$$\begin{aligned}\forall u_{n_k}, v \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega) : |\langle \mathbf{C}(u_{n_k}), v\rangle| &\leq \sum_{i=1}^{N} \int_\Omega |T_k\left(u_{n_k}|u_{n_k}|^{p_i(x)-2}\right)||v|\,\mathrm{d}x \\ &\leq 2\sum_{i=1}^{N} \|T_k\left(u_{n_k}|u_{n_k}|^{p_i(x)-2}\right)\|_{p'_+(\cdot)}\|v\|_{p_+(\cdot)} \\ &\leq C\big(1+\int_\Omega k^{p'_+(x)}\,\mathrm{d}x\big)^{1/(p'_+)^-}\|v\|_{\overrightarrow{p}(\cdot)} \\ &\leq C\big(1+k^{(p'_+)^+}|\Omega|\big)^{1/(p'_+)^-}\|v\|_{\overrightarrow{p}(\cdot)}.\end{aligned}$$

Hence $\mathbf{C}$ is bounded.

Through (3.2), the fact that $1+e^{|T_n(u_{n_k})|} \leq e^n$, the fact that $T_k\left(u_{n_k}|u_{n_k}|^{p_i(x)-2}\right)u_{n_k} \geq 0$, and Lemma 1, we have

$$\begin{aligned}\frac{\langle \mathbf{A}(u_{n_k}), u_{n_k}\rangle}{\|u_{n_k}\|_{\overrightarrow{p}(\cdot)}} &\geq \frac{\alpha}{n(1+\alpha)}\frac{\sum_{i=1}^{N}\int_\Omega |D_i u_{n_k}|^{p_i(x)}\,\mathrm{d}x}{\|u_{n_k}\|_{\overrightarrow{p}(\cdot)}} \\ &\geq \frac{\alpha}{n(1+\alpha)}\frac{\sum_{i=1}^{N}\|D_i u_{n_k}\|^{p_-^-}_{p_i(\cdot)}-N}{\|u_{n_k}\|_{\overrightarrow{p}(\cdot)}} \\ &\geq \frac{\alpha}{n(1+\alpha)}\frac{\left(\frac{1}{N}\sum_{i=1}^{N}\|D_i u_{n_k}\|_{p_i(\cdot)}\right)^{p_-^-}-N}{\|u_{n_k}\|_{\overrightarrow{p}(\cdot)}} \\ &= \frac{\alpha}{n(1+\alpha)N^{p_-^-}}\|u_{n_k}\|^{p_-^- -1}_{\overrightarrow{p}(\cdot)} - \frac{N}{\|u_{n_k}\|_{\overrightarrow{p}(\cdot)}}.\end{aligned}$$

This implies that $\mathbf{A}$ is coercive.

Let us prove the pseudo-monotonicity of $\mathbf{A}$. Let $(u_{n_k})_k$ be a sequence in $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$ such that

$$\begin{cases} u_{n_k} \rightharpoonup u_n \text{ in } \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)|, \\ \limsup\limits_{k\to\infty}\langle \mathbf{A}(u_{n_k}), u_{n_k}-u_n\rangle \leq 0. \end{cases}$$

We have to prove that

$$\liminf_{k\to\infty}\langle \mathbf{A}(u_{n_k}), u_{n_k}-v\rangle \geq \langle \mathbf{A}(u_n), u_n-v\rangle, \quad \forall v \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)|.$$

Here we need the following known inequalities, that hold for all two real vectors ξ, ξ' $((\xi,\xi') \neq (0,0))$ and a real $p>1$:

$$(|\xi|^{p-2}\xi - |\xi'|^{p-2}\xi')(\xi-\xi') \geq \begin{cases} 2^{2-p}|\xi-\xi'|^p, & \text{if } p \geq 2, \\ (p-1)\frac{|\xi-\xi'|^2}{(|\xi|+|\xi'|)^{2-p}}, & \text{if } 1<p<2. \end{cases} \tag{3.6}$$

Like in [14, 19] with using (3.6) we can easily prove the required result. Then, the main Theorem on pseudo-monotone operators (Theorem 27.A in [22] and see

also [3, 6, 13]) applies and ensures the existence of at least one weak solution $u_{n_k} \in \mathring{W}^{1,\vec{p}(\cdot)}(\Omega)$ to problem (3.5) in the sense that

$$\sum_{i=1}^{N}\int_{\Omega}\frac{a_n(x,u_{n_k})|D_iu_{n_k}|^{p_i(x)-2}D_iu_{n_k}}{1+e^{|T_n(u_{n_k})|}}D_i\varphi\,dx+\sum_{i=1}^{N}\int_{\Omega}T_k\left(u_{n_k}|u_{n_k}|^{p_i(x)-2}\right)\varphi\,dx = \int_{\Omega}f_n\varphi\,dx,\quad \forall\varphi\in\mathring{W}^{1,\vec{p}(\cdot)}(\Omega). \tag{3.7}$$

Now, we choose $\varphi = u_{n_k}$ as a test function in (3.7) and using (3.2), the fact that $1+e^{|T_n(u_{n_k})|}\le 1+e^n$, and after dropping the nonegative term (due $T_k\left(u_{n_k}|u_{n_k}|^{p_i(x)-2}\right)u_{n_k}\ge 0$), we obtain

$$\frac{\alpha}{(1+\alpha)(1+e^n)}\sum_{i=1}^{N}\int_{\Omega}|D_iu_{n_k}|^{p_i(x)}\,dx\le n\int_{\Omega}|u_{n_k}|\,dx. \tag{3.8}$$

Using the fact that $|D_iu_{n_k}|^{p_-}\le 1+|D_iu_{n_k}|^{p_i(x)}$, $i=1,\dots,N$, Poincaré inequality and Young's inequality for all $\varepsilon>0$, we obtain

$$\begin{aligned}\sum_{i=1}^{N}\int_{\Omega}|D_iu_{n_k}|^{p_i(x)}\,dx &\le \frac{(1+\alpha)n(1+e^n)}{\alpha}\int_{\Omega}|u_{n_k}|\,dx\\ &\le \frac{(1+\alpha)n(1+e^n)}{\alpha}\left(\varepsilon\int_{\Omega}|u_{n_k}|^{p_-}\,dx+C(\varepsilon)\right)\\ &\le \frac{(1+\alpha)n(1+e^n)}{\alpha}\left(\varepsilon c\sum_{i=1}^{N}\int_{\Omega}|D_iu_{n_k}|^{p_-}\,dx+C(\varepsilon)\right)\\ &\le \frac{(1+\alpha)n(1+e^n)}{\alpha}\left(\varepsilon c\left(N|\Omega|+\sum_{i=1}^{N}\int_{\Omega}|D_iu_{n_k}|^{p_i(x)}\,dx\right)+C(\varepsilon)\right).\end{aligned} \tag{3.9}$$

Choosing $\varepsilon=\frac{\alpha}{2c(1+\alpha)n(1+e^n)}$, we obtain

$$\sum_{i=1}^{N}\int_{\Omega}|D_iu_{n_k}|^{p_i(x)}\,dx\le c(n). \tag{3.10}$$

On the other hand, by (iii) of Lemma 1, we get

$$1+\int_{\Omega}|D_iu_{n_k}|^{p_i(x)}\,dx\ge\left\|D_iu_{n_k}\right\|_{p_i(x)}^{p_-}.$$

Then, we conclude

$$N+\sum_{i=1}^{N}\int_{\Omega}|D_iu_{n_k}|^{p_i(x)}\,dx\ge\sum_{i=1}^{N}\left\|D_iu_{n_k}\right\|_{p_i(x)}^{p_-}.$$

This implies

$$\sum_{i=1}^{N}\int_{\Omega}|D_iu_{n_k}|^{p_i(x)}\,\mathrm{d}x\geq\left(\frac{1}{N}\sum_{i=1}^{N}\left\|D_iu_{n_k}\right\|_{p_i(x)}\right)^{p^-_-}-N.$$

Since (2.3) holds, Remark 2 gives us $\sum\limits_{i=1}^{N}\left\|D_iu_{n_k}\right\|_{p_i(x)}=\left\|u_{n_k}\right\|_{\overrightarrow{p}(\cdot)}$, thus the last inequality is equivalent to

$$\sum_{i=1}^{N}\int_{\Omega}|D_iu_{n_k}|^{p_i(x)}\,\mathrm{d}x\geq\left(\frac{\left\|u_{n_k}\right\|_{\overrightarrow{p}(\cdot)}}{N}\right)^{p^-_-}-N. \tag{3.11}$$

By combining (3.10) and (3.11), we conclude

$$\left(\frac{\left\|u_{n_k}\right\|_{\overrightarrow{p}(\cdot)}}{N}\right)^{p^-_-}-N\leq c(n),$$

implying that

$$\left\|u_{n_k}\right\|_{\overrightarrow{p}(\cdot)}\leq C(n). \tag{3.12}$$

Relation (3.12) guarantees the existence of a subsequence (still denoted by u_{n_k}) and $u_n\in\mathring{W}^{1,\vec{p}(\cdot)}(\Omega)$ such that

$$u_{n_k}\rightharpoonup u_n\ \text{ weakly in }\mathring{W}^{1,\vec{p}(\cdot)}(\Omega)\text{ and a.e. in }\Omega. \tag{3.13}$$

Taking $T_t(u_{n_k})$ as a test function in (3.7), we obtain

$$\sum_{i=1}^{N}\int_{\Omega}\frac{a_n(x,u_{n_k})|D_iu_{n_k}|^{p_i(x)-2}D_iu_{n_k}}{1+e^{|T_n(u_{n_k})|}}D_iT_t(u_{n_k})\,\mathrm{d}x+\sum_{i=1}^{N}\int_{\Omega}T_k\left(u_{n_k}|u_{n_k}|^{p_i(x)-2}\right)T_t(u_{n_k})\,\mathrm{d}x$$
$$=\int_{\Omega}f_nT_t(u_{n_k})\,\mathrm{d}x.$$

From (2.6), (3.2) and the fact that $1+e^{|T_n(u_{n_k})|}\leq 1+e^n$, we get

$$\frac{\alpha}{(1+\alpha)(1+e^n)}\sum_{i=1}^{N}\int_{\{|u_{n_k}|<t\}}|D_iu_{n_k}|^{p_i(x)}\,\mathrm{d}x\leq\int_{\Omega}|f_n||T_t(u_{n_k})|\,\mathrm{d}x$$
$$+\sum_{i=1}^{N}\int_{\Omega}\left|T_k\left(u_{n_k}|u_{n_k}|^{p_i(x)-2}\right)\right||T_t(u_{n_k})|\,\mathrm{d}x.$$

Since $\int_\Omega |D_i(T_t(u_{n_k}))|^{p_i(x)}\,dx = \int_{\{|u_{n_k}|<t\}} |D_i u_{n_k}|^{p_i(x)}\,dx$, $|T_t(u_{n_k})| \le t$, $|f_n| \le |f|$ (due (3.1)), then by Young's inequality we obtain for any $\varepsilon > 0$

$$c(n)\sum_{i=1}^{N}\int_\Omega |D_i(T_t(u_{n_k}))|^{p_i(x)}\,dx \le t\|f\|_{L^1(\Omega)} + C(\varepsilon)\sum_{i=1}^{N}\int_\Omega \left|T_k\left(u_{n_k}|u_{n_k}|^{p_i(x)-2}\right)\right|^{p_i'(x)}\,dx + \varepsilon\sum_{i=1}^{N}\int_\Omega |T_t(u_{n_k})|^{p_i(x)}\,dx. \tag{3.14}$$

Since $|T_t(u_{n_k})|^{p_i(x)} \le 1 + t^{p_+^+}$, and $T_k\left(u_{n_k}|u_{n_k}|^{p_i(x)-2}\right) \in L^{p_i'(x)}(\Omega)$ (due $u_{n_k}|u_{n_k}|^{p_i(x)-2} \in L^{p_i'(x)}(\Omega)$), we deduce for all $t > 0$ that

$$T_t(u_{n_k}) \rightharpoonup T_t(u_n) \ \text{ weakly in } \mathring{W}^{1,\vec{p}(\cdot)}(\Omega) \text{ and a.e. in } \Omega. \tag{3.15}$$

Now, let's prove that for all $i = 1,\dots,N$

$$D_i u_{n_k} \to D_i u_n, \text{ a.e. in } \Omega. \tag{3.16}$$

For that let us define for all $i = 1,\dots,N$ and $t > 0$ fixed

$$I_{i,n,k}^t(x) = \left(|D_i(T_t(u_{n_k}))|^{p_i(x)-2}D_i(T_t(u_{n_k})) - D_i(T_t(u_n))|^{p_i(x)-2}D_i(T_t(u_n))\right)\left(D_i(T_t(u_{n_k})) - D_i(T_t(u_n))\right).$$

For $0 < \theta < 1$, $0 < h < t$, and all $i = 1,\dots,N$, we can get

$$\begin{aligned}\int_\Omega \left(I_{i,n,k}^t(x)\right)^\theta dx &= \int_{\{|T_t(u_{n_k})-T_t(u_n)|>h\}} \left(I_{i,n,k}^t(x)\right)^\theta dx + \int_{\{|T_t(u_{n_k})-T_t(u_n)|\le h\}} \left(I_{i,n,k}^t(x)\right)^\theta dx \\ &\le \left(\int_\Omega I_{i,n,k}^t(x)\,dx\right)^\theta \left|\{|T_t(u_{n_k}) - T_t(u_n)| > h\}\right|^{1-\theta} \\ &\quad + \left(\int_{\{|T_t(u_{n_k})-T_t(u_n)|\le h\}} I_{i,n,k}^t(x)\,dx\right)^\theta |\Omega|^{1-\theta}.\end{aligned} \tag{3.17}$$

Then, we can write

$$\int_\Omega \left(I_{i,n,k}^t(x)\right)^\theta dx \le J_1 + J_2, \tag{3.18}$$

where

$$J_1 = \left(\int_\Omega I_{i,n,k}^t(x)\,dx\right)^\theta \left|\{|T_t(u_{n_k}) - T_t(u_n)| > h\}\right|^{1-\theta},$$

$$J_2 = \left(\int_{\{|T_t(u_{n_k})-T_t(u_n)|\le h\}} I_{i,n,k}^t(x)\,dx\right)^\theta |\Omega|^{1-\theta}.$$

For every fixed h, thanks to (3.15), and the convergence in measure of $T_t(u_{n_k})$, we can get

$$\lim_{k\to+\infty} J_1 = 0. \tag{3.19}$$

Now, choosing $T_h(u_{n_k} - T_t(u_n))$ (with $0 < h < t$) as a test function in (3.7), we obtain

$$\begin{aligned}
&\sum_{i=1}^{N} \int_\Omega \frac{a_n(x,u_{n_k})}{1+e^{|T_n(u_{n_k})|}} |D_i(T_t(u_{n_k}))|^{p_i(x)-2} D_i(T_t(u_{n_k})) D_i T_h\big(T_t(u_{n_k}) - T_t(u_n)\big)\,\mathrm{d}x \\
&- \sum_{i=1}^{N} \int_{\{|u_{n_k} - T_t(u_n)|<h\}} \frac{a_n(x,u_{n_k})}{1+e^{|T_n(u_{n_k})|}} |D_i(G_t(u_{n_k}))|^{p_i(x)-2} D_i(G_t(u_{n_k})) D_i\big(T_t(u_n)\big)\,\mathrm{d}x \\
&+ \sum_{i=1}^{N} \int_\Omega T_k\left(u_{n_k}|u_{n_k}|^{p_i(x)-2}\right) T_h(u_{n_k} - T_t(u_n))\,\mathrm{d}x \le ch. \qquad (3.20)
\end{aligned}$$

By (3.2), we get

$$\begin{aligned}
&\sum_{i=1}^{N} \int_\Omega |D_i(T_t(u_{n_k}))|^{p_i(x)-2} D_i(T_t(u_{n_k})) D_i T_h\big(T_t(u_{n_k}) - T_t(u_n)\big)\,\mathrm{d}x \\
&\le c_1(n)h + c_2(n) \sum_{i=1}^{N} \int_{\{|u_{n_k} - T_t(u_n)|<h\}} |D_i(G_t(u_{n_k}))|^{p_i(x)-2} D_i(G_t(u_{n_k})) D_i\big(T_t(u_n)\big)\,\mathrm{d}x \\
&- c_3(n) \sum_{i=1}^{N} \int_\Omega T_k\left(u_{n_k}|u_{n_k}|^{p_i(x)-2}\right) T_h(u_{n_k} - T_t(u_n))\,\mathrm{d}x. \qquad (3.21)
\end{aligned}$$

Then, by using (3.21) and the fact that $I^t_{i,n}(x) \ge 0$ (since (3.6)), we deduce

$$\begin{aligned}
0 &\le \sum_{i=1}^{N} \int_{\{|T_t(u_{n_k}) - T_t(u_n)|\le h\}} I^t_{i,n,k}(x)\,\mathrm{d}x \\
&= \sum_{i=1}^{N} \int_\Omega \big(|D_i(T_t(u_{n_k}))|^{p_i(x)-2} D_i(T_t(u_{n_k})) \\
&\qquad - |D_i(T_t(u_n))|^{p_i(x)-2} D_i(T_t(u_n))\big) D_i T_h\big(T_t(u_{n_k}) - T_t(u_n)\big)\,\mathrm{d}x \\
&\le c_1(n)h - c_3(n) \sum_{i=1}^{N} \int_\Omega T_k\left(u_{n_k}|u_{n_k}|^{p_i(x)-2}\right) T_h(u_{n_k} - T_t(u_n))\,\mathrm{d}x \\
&\quad + c_2(n) \sum_{i=1}^{N} \int_{\{|u_{n_k}|>t\}\cap\{|u_{n_k} - T_t(u_n)|<h\}} |D_i(u_{n_k})|^{p_i(x)-2} D_i(u_{n_k}) D_i\big(T_t(u_n)\big)\,\mathrm{d}x \\
&\quad - \sum_{i=1}^{N} \int_\Omega |D_i(T_t(u_n))|^{p_i(x)-2} D_i(T_t(u_n)) D_i T_h\big(T_t(u_{n_k}) - T_t(u_n)\big)\,\mathrm{d}x. \qquad (3.22)
\end{aligned}$$

Since $\{|u_{n_k} - T_t(u_n)| < h\} \subset \{|u_{n_k}| \le h + t\} \subset \{|u_{n_k}| \le 2t\}$, the boundedness of both $\big\{|D_i(T_{2t}(u_{n_k}))|^{p_i(x)-2} D_i(T_{2t}(u_{n_k}))\big\}_k$, $\big\{T_k\left(u_{n_k}|u_{n_k}|^{p_i(x)-2}\right)\big\}_k$ in $L^{p_i'(x)}(\Omega)$, and the use of (3.15), we can pass to the limit in (3.22) when $k \to +\infty$. So, we get

$$\limsup_{k\to+\infty} \sum_{i=1}^{N} \int_{\{|T_t(u_{n_k}) - T_t(u_n)|\le h\}} I^t_{i,n,k}(x)\,\mathrm{d}x \le c_1(n)h - c_3(n) \sum_{i=1}^{N} \int_\Omega \nu T_h(G_t(u_n))\,\mathrm{d}x$$

$$+c_2(n)\sum_{i=1}^{N}\int_{\{t<|u_n|<t+h\}}\tau_t D_i\big(T_t(u_n)\big), \tag{3.23}$$

where $\tau_t, \nu \in L^{p_i'(x)}(\Omega)$ are the weak limits of $\big\{|D_i(T_{2t}(u_{n_k}))|^{p_i(x)-2}D_i(T_{2t}(u_{n_k}))\big\}_k$ and $\big\{T_k\left(u_{n_k}|u_{n_k}|^{p_i(x)-2}\right)\big\}_k$ respectively.

After letting $h \to 0$ in (3.23), we obtain

$$\lim_{k\to+\infty} J_2 = 0. \tag{3.24}$$

We combine (3.18), (3.19), (3.24) and recalling (3.6), we get

$$\lim_{k\to+\infty}\int_\Omega \left(I^t_{i,n,k}(x)\right)^\theta \,dx = 0. \tag{3.25}$$

From (3.25) we deduce, like in [1], that for every $t>0$ and every $i=1,\dots,N$

$$D_i(T_t(u_{n_k})) \to D_i(T_t(u_n)), \text{ almost everywhere in } \overline{\Omega}. \tag{3.26}$$

And through the results obtained in [21], we can get (3.16).

By (3.16) we have

$$|D_iu_{n_k}|^{p_i(x)-2}D_iu_{n_k} \rightharpoonup |D_iu_n|^{p_i(x)-2}D_iu_n \text{ weakly in } L^{p_i'(\cdot)}(\Omega).$$

From (3.13) and Lebesgue's dominated convergence theorem, we obtain

$$\frac{a_n(x,u_{n_k})}{1+e^{|T_n(u_{n_k})|}}D_i\varphi \to \frac{a_n(x,u_n)}{1+e^{|T_n(u_n)|}}D_i\varphi \text{ strongly in } L^{p_i(\cdot)}(\Omega),\ i=1,\dots,N.$$

We will now prove the equi-integrability of $\left\{\sum_{i=1}^{N} T_k\left(u_{n_k}|u_{n_k}|^{p_i(x)-2}\right)\right\}_k$ in $L^1(\Omega)$. First, let's prove that

$$\sum_{i=1}^{N}\int_{|u_{n_k}|>t}|T_k\left(u_{n_k}|u_{n_k}|^{p_i(x)-2}\right)|\,dx \le \int_{|u_{n_k}|>t}|f_n|\,dx. \tag{3.27}$$

For that, we can choose $\phi_l(u_{n_k})$ as a test function in (3.7), where $(\phi_l)_l$ is a sequence of real smooth increasing functions with $\phi_l' \in L^\infty(\Omega)$, $\phi_l(0)=0$ and it converges to the function $\phi_t, t>0$ defined by

$$\phi_t(s) = \begin{cases} 1, & if\ s \ge t,\\ 0, & if\ |s|<t, \quad t>0,\\ -1, & if\ s\le -t. \end{cases}$$

Through this choice and following the technique used in [9], we can easily get (3.27). Let $E\subset\Omega$ be any measurable set, using (3.27), we have

$$\sum_{i=1}^{N}\int_E |T_k(|u_{n_k}|^{p_i(x)-2}u_{n_k})|\,dx = \sum_{i=1}^{N}\int_{E\cap\{|u_{n_k}|\le t\}}|T_k(|u_{n_k}|^{p_i(x)-2}u_{n_k})|\,dx$$

$$+\sum_{i=1}^{N}\int_{E\cap\{|u_{n_k}|>t\}}|T_k(|u_{n_k}|^{p_i(x)-2}u_{n_k})|\,\mathrm{d}x$$
$$\leq N(1+t^{p_+^+})|E|+\int_{\{|u_{n_k}|>t\}}|f_n|\,\mathrm{d}x.$$

Then we deduce the desired result.

From (3.13), we obtain

$$\sum_{i=1}^{N}T_k(|u_{n_k}|^{p_i(x)-2}u_{n_k})\to\sum_{i=1}^{N}|u_n|^{p_i(x)-2}u_n \text{ a.e. in } \Omega. \tag{3.28}$$

Since (3.28), Vitali's theorem implies that

$$\sum_{i=1}^{N}T_k(|u_{n_k}|^{p_i(x)-2}u_{n_k})\to\sum_{i=1}^{N}|u_n|^{p_i(x)-2}u_n \quad \text{in } L^1(\Omega).$$

Therefore, we can get (3.4) by passing to the limit in (3.7). □

3.1.1. *A priori estimates*

Lemma 5. *Let f, a and p_i, $i=1,\dots,N$ be restricted as in Theorem 1. Then*

$$u_n \text{ is bounded in } \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega), \tag{3.29}$$

where u_n the weak solution to the problem (3.3).

Proof. By choosing $\varphi=u_n$ as a test function in (3.4) and using the same way as in proving (3.12), we can get (3.29). □

Lemma 6. *There exists a subsequence (still denoted (u_n)) such that, for all $i=1,\dots,N$*

$$D_iu_n\to D_iu \text{ a.e. in } \Omega, \tag{3.30}$$

where u is the weak limit of the sequence (u_n) in $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$.

Proof. From (3.29) we conclude that, there exists a function $u\in\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$ and a subsequence (still denoted by (u_n)) such that

$$u_n\rightharpoonup u \quad \text{weakly in } \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega). \tag{3.31}$$

In the same way of proof as in [2] or in [14–19] we can easily get that

$$\lim_{n\to+\infty}\sum_{i=1}^{N}\int_{\Omega}\left(|D_iu_n|^{p_i(x)-2}D_iu_n-|D_iu|^{p_i(x)-2}D_iu\right)(D_iu_n-D_iu)\,\mathrm{d}x=0. \tag{3.32}$$

Through (3.6) we conclude that for all $i=1,\dots,N$

$$\left(|D_iu_n|^{p_i(x)-2}D_iu_n-|D_iu|^{p_i(x)-2}D_iu\right)(D_iu_n-D_iu)>0. \tag{3.33}$$

Then, (3.33) and (3.32) gives us for all $i = 1, \dots, N$

$$\left(|D_i u_n|^{p_i(x)-2} D_i u_n - |D_i u|^{p_i(x)-2} D_i u\right)(D_i u_n - D_i u) \to 0, \quad \text{strongly in } L^1(\Omega). \tag{3.34}$$

Extracting a subsequence (still denoted by (u_n)), we have for all $i = 1, \dots, N$

$$\left(|D_i u_n|^{p_i(x)-2} D_i u_n - |D_i u|^{p_i(x)-2} D_i u\right)(D_i u_n - D_i u) \to 0 \quad \text{a.e. in } \Omega. \tag{3.35}$$

Then there exists a subset Γ of Ω of Lebesgue zero measure, such that, for $x \in \Omega \backslash \Gamma$

$$|D_i u(x)| < \infty \text{ and}$$
$$\left(|D_i u_n(x)|^{p_i(x)-2} D_i u_n(x) - |D_i u(x)|^{p_i(x)-2} D_i u(x)\right)(D_i u_n(x) - D_i u(x)) \to 0.$$

From (3.35), we have

$$\left(|D_i u_n|^{p_i(x)-2} D_i u_n - |D_i u|^{p_i(x)-2} D_i u\right)(D_i u_n - D_i u) \leq h(x)$$

for some function h.

Let us prove that there exists a function g such that

$$\left|D_i u_n(x)\right| \leq g(x). \tag{3.36}$$

By inequality (3.6), we obtain

$$h(x) \geq \begin{cases} c\left((|D_i u_n| - |D_i u|)^{p_-^-} - 1\right), & \text{if } p_i(x) \geq 2, \\ c'\left(\frac{|D_i u_n| - |D_i u|}{1 + |D_i u_n| + |D_i u|}\right)^2, & \text{if } 1 < p_i(x) < 2 \end{cases}$$

and this implies (3.36).
We are going to prove that

$$D_i u_n(x) \to D_i u(x) \text{ in } \Omega \backslash \Gamma. \tag{3.37}$$

Assume by contradiction that there exists $x_0 \in \Omega \backslash \Gamma$ such that $D_i u_n(x_0)$ does not converge to $D_i u(x_0)$. The Bolzano-Weierstrass theorem implies that $D_i u_n(x_0) \to b$, $b \in \mathbb{R}$, up to a subsequence. Passing to the limit in

$$\left(\left(|D_i u_n(x_0)|^{p_i(x_0)-2} D_i u_n(x_0) - |D_i u x_0)|^{p_i(x_0)-2} D_i u(x_0)\right)(D_i u_n(x_0) - D_i u(x_0)),\right.$$

we get

$$\left(|b|^{p_i(x_0)-2} b - |D_i u(x_0)|^{p_i(x_0)-2} D_i u(x_0)\right)(b - D_i u(x_0)) = 0,$$

which yields $b = D_i u(x_0)$ using inequality (3.6). Through this we get (3.30). □

3.2. *Proof of Theorem 1*

From (3.30) and the boundedness of $\{|D_iu_n|^{p_i(x)-2}D_iu_n\}$ in $L^{p_i'(\cdot)}(\Omega)$, we can get, for all $i=1,\dots,N$

$$|D_iu_n|^{p_i(x)-2}D_iu_n \rightharpoonup |D_iu|^{p_i(x)-2}D_iu \text{ weakly in } L^{p_i'(\cdot)}(\Omega). \tag{3.38}$$

From (3.31), (3.2), and the fact that $0<\frac{1}{1+e^{|u_n|}}\leq\frac{1}{2}$ we can apply the Lebesgue's dominated convergence theorem to obtain

$$\frac{a_n(x,u_n)}{1+e^{|u_n|}}D_i\varphi\to\frac{a(x,u)}{1+e^{|u|}}D_i\varphi \quad \text{strongly in } L^{p_i(\cdot)}(\Omega),\ 1\leq i\leq N.$$

Now, from (3.31), we conclude that

$$u_n\sum_{i=1}^{N}|u_n|^{p_i(x)-2}\to u\sum_{i=1}^{N}|u|^{p_i(x)-2} \quad \text{a.e. in } \Omega. \tag{3.39}$$

With similar arguments as in the proof of (3.27), we obtain

$$\int_{|u_n|>t}|u_n\sum_{i=1}^{N}|u_n|^{p_i(x)-2}|\,\mathrm{d}x\leq\int_{|u_n|>t}|f_n|\,\mathrm{d}x. \tag{3.40}$$

For any measurable set $E\subset\Omega$, by (3.40) we have

$$\begin{aligned}\int_E|u_n\sum_{i=1}^{N}|u_n|^{p_i(x)-2}|\,\mathrm{d}x&=\int_{E\cap\{|u_n|\leq t\}}|u_n\sum_{i=1}^{N}|u_n|^{p_i(x)-2}|\,\mathrm{d}x\\&\quad+\int_{E\cap\{|u_n|>t\}}|u_n\sum_{i=1}^{N}|u_n|^{p_i(x)-2}|\,\mathrm{d}x\\&\leq N(1+t^{p_+^+})|E|+\int_{\{|u_n|>t\}}|f_n|\,\mathrm{d}x.\end{aligned}$$

Then, we conclude the equi-integrability of $\left(u_n\sum_{i=1}^{N}|u_n|^{p_i(x)-2}\right)$ in $L^1(\Omega)$, and since (3.39), Vitali's theorem implies that

$$u_n\sum_{i=1}^{N}|u_n|^{p_i(x)-2}\to u\sum_{i=1}^{N}|u|^{p_i(x)-2} \quad \text{strongly in } L^1(\Omega). \tag{3.41}$$

So, we can easily pass to the limit in the weak formulation (3.4). This proves Theorem 1.

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WEAKLY REGULAR RESIDUATED MAPPINGS

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Abstract. For residual functions, two regularity notions are known in the literature, one is weak regularity introduced by Blyth and Janowitz, and the other is a closely related condition introduced by Nicholson et al., which we call N-regularity. In our paper, we provide several characterizations of weakly regular residuated maps defined in complete lattices and of N-regular residuated maps in complemented modular lattices. The relations between the mentioned notions and some related conditions (such as Jordan conditions J1 and J2 on residuated maps) are also investigated. We provide some significant examples of weakly regular residuated maps and of N-regular residuated maps.

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1. INTRODUCTION

Let A and B be two partially ordered sets. The mapping $f: A \to B$ is called *residuated* [1] if there exists a mapping $f^*: B \to A$ such that for all $x \in A$ and all $y \in B$

$$f(x) \leq y \iff x \leq f^*(y).$$

In this case the mapping f^*, called the *residual* of f, is uniquely determined. In fact, $f^*(y) = \bigvee\{z \in A \mid f(z) \leq y\}$. The set of all residuated mappings from A to B is denoted by $\mathrm{Res}(A,B)$. Let now L be a complete lattice. We use the notations $\mathrm{Res}(L) = \mathrm{Res}(L,L)$ and $\mathrm{Res}^*(L) = \{f^* | f \in \mathrm{Res}(L)\}$. It is well-known that $\mathrm{Res}(L)$ and $\mathrm{Res}^*(L)$ consist of complete join-endomorphisms and complete meet-endomorphisms of L, respectively, and they are complete lattices with respect to the pointwise defined order relation [5].

Residuated maps are applied in several areas of Algebra, Logic, Formal Languages, etc. The study of their lattice was initiated by G.N. Reney and E.T. Schreiner [7] (some basic results are going back to G. Grätzer and E.T. Schmidt [4]). For instance, they independently proved that for a completely distributive lattice L the

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atoms of Res(L) are mappings with 2-element range (see our Example 3(d)). Their results were extended by Z. Shmuely [8]. In their seminal book [2] S. Blyth and M.F. Janowitz, initiated a systematic study of residuated maps and their monoid [2]. The concept of weakly regular residuated maps was introduced also in [2], and it turned out to be an extremely useful tool in characterizing modular lattices. Weakly regular residuated maps occur in several fundamental algebraic areas, for instance, linear transformations of a vector space always induce weakly regular residuated maps on its subspace lattice, as it is shown in Example 1(c). K. Kaarli and S. Radeleczki in [5] characterized those regular residuated maps which are atoms in Res(L); from their result it follows that in several important lattice classes the atoms of Res(L) are exactly the weakly regular maps. Another notion of weak regularity was introduced by G. E. Nicholson at al. in [6] for modular lattices. In a pioneering paper, J. Szigeti [9] introduced the Jordan conditions J1, J2 strongly related with weak regularity, and using them he proved the existence of the so-called n-nilpotent Jordan base for the join-endomorphisms of a submodule lattice. These examples show the importance to examine the relationship between the mentioned natural concepts. In our paper we provide several characterizations of weakly regular residuated maps. Then, by using these results, the relations between the above conditions are investigated.

2. Preliminaries

A *complete join-endomorphis* of L is a mapping $f: L \to L$ satisfying

$$f\left(\bigvee_{t\in T} x_t\right) = \bigvee_{t\in T} f(x_t)$$

for any system of elements $x_t \in L, t \in T$. Notice that this definition yields also $f(0) = 0$. A *complete meet-endomorphism* of L is defined dually. It is worth noting that any complete join-endomorphism of a complete lattice L is a residuated map on L, and any complete meet-endomorphism is a residual map on L. A complete $\vee$-semilattice congruence of a complete lattice L is an equivalence relation $\theta \subseteq L \times L$ preserved by arbitrary joins. We note that for any $f \in \mathrm{Res}(L)$, the equivalence

$$\mathrm{Ker}f = \{(x,y) \in L^2 \mid f(x) = f(y)\}$$

is a complete $\vee$-semilattice congruence, and $[a]_{\mathrm{Ker}f}$ stands for the equivalence class of the element $a \in L$. For any elements $a, b \in L$ we will use the notations $\downarrow a = \{x \in L | x \leq a\}$, $\uparrow b = \{x \in L | x \geq b\}$. The set $\downarrow a$ is called the principal ideal belonging to a, and $\uparrow b$ is called the principal filter of b. Since $[0]_{\mathrm{Ker}f}$ is a principal ideal, it has a greatest element k_f, and hence $[0]_{\mathrm{Ker}f} = \downarrow k_f$. It is clear, that for any $x, y \in L$, $x \vee k_f = y \vee k_f$ implies $f(x) = f(y)$, i.e. $[x]_{\mathrm{Ker}f} = [y]_{\mathrm{Ker}f}$.

Definition 1. A mapping $f \in \mathrm{Res}(L)$ is called *range-closed* if $f(L) = [0, f(1)]$ and *dually range-closed* if $f^*(L) = [f^*(0), 1]$. A mapping which is both range-closed and dually range-closed is called *weakly regular*. [2]

In view of [1] a residuated mapping f defined on a complete lattice L is dually range-closed, if and only if $x \vee f^*(0) = f^*(f(x))$, for all $x \in L$. Observe that $f^*(0) = \bigvee\{x \in L \mid f(x) \leq 0\} = \bigvee[0]_{\mathrm{Ker}f} = k_f$ and $f^*(f(x)) = \bigvee\{z \in L \mid f(z) \leq f(x)\}$

Lemma 1. *The following assertions are equivalent:*

(i) $f \in \mathrm{Res}(L)$ *is dually range-closed.*
(ii) *For any* $x, y \in L$,

$$[x]_{\mathrm{Ker}f} = [y]_{\mathrm{Ker}f} \Leftrightarrow x \vee k_f = y \vee k_f \tag{2.1}$$

Proof.

(i)⇒(ii): Assume that (i) holds. Let $[x]_{\mathrm{Ker}f} = [y]_{\mathrm{Ker}f}$ for some $x, y \in L$. Then $f(x) = f(y)$ implies $f^*(f(x)) = f^*(f(y))$. Now, in view of [1] we obtain $x \vee k_f = x \vee f^*(0) = f^*(f(x)) = f^*(f(y)) = y \vee f^*(0) = y \vee k_f$.

As $x \vee k_f = y \vee k_f$ obviously yields $f(x) = f(y)$, i.e. $[x]_{\mathrm{Ker}f} = [y]_{\mathrm{Ker}f}$, our result means that (ii) is satisfied.

(ii)⇒(i): Suppose that (ii) holds and let us prove that $x \vee f^*(0) = f^*(f(x))$. Clearly, $f(x \vee f^*(0)) = f(x \vee k_f) = f(x)$, hence $x \vee f^*(0) \leq f^*(f(x))$. Conversely, $f(f^*(f(x))) = f(x)$ implies $[f^*(f(x))]_{\mathrm{Ker}f} = [x]_{\mathrm{Ker}f}$, and now, by using (ii) we get $f^*(f(x)) \vee k_f = x \vee k_f = x \vee f^*(0)$. This yields $f^*(f(x)) \leq x \vee f^*(0)$, and hence $x \vee f^*(0) = f^*(f(x))$. In view of [1] this implies (i).

□

Corollary 1. *A map* $f \in \mathrm{Res}(L)$ *is weakly regular if and only if* $f(L) = [0, f(1)]$ *and we have* $[x]_{\mathrm{Ker}f} = [y]_{\mathrm{Ker}f} \Leftrightarrow x \vee k_f = y \vee k_f$, *for any* $x, y \in L$.

An other characterization of weakly regular residuated mappings can be found in [2] (see also [5]) In view of [2] Theorem 13.2 and the remark after it we obtain:

Proposition 1. *Let* L *be a complete lattice and* $f\colon L \to L$. *Then the following are equivalent:*

(i) $f \in \mathrm{Res}(L)$ *is weakly regular;*
(ii) *There are* $a, b \in L$ *and a lattice isomorphism* $i\colon \uparrow a \to \downarrow b$ *such that for all* $x \in L$ $f(x) = i(x \vee a)$ *holds.*

Remark 1. Clearly, we have $i(1) = b$, $i(a) = 0$ and $f(x) = i(x)$, for all $x \geq a$ in Proposition 1(ii). Observe, that $f^*(y) = i^{-1}(y)$, for any $y \leq b$, and $a = k_f$ also hold. Indeed, $f^*(y) = \bigvee\{z \in L \mid i(z \vee a) \leq y\} = \bigvee\{t \in \uparrow a \mid i(t) \leq y\} = \bigvee\{t \in \uparrow a \mid t \leq i^{-1}(y)\} = i^{-1}(y)$. Hence $k_f = f^*(0) = i^{-1}(0) = a$.

Now we use the above characterizations to exhibit some significant examples for weakly residuated mappings on complete lattices.

Example 1.

(a) By Corollary 1, any automorphism f of L is a weakly regular residuated map, because it is a complete join-endomorphism with $f(L) = [0,1] = L$ and $k_f = 0$. We have $[x]_{\mathrm{Ker}f} = [y]_{\mathrm{Ker}f} \Leftrightarrow x = y \Leftrightarrow x \vee k_f = y \vee k_f$ for any $x, y \in L$.
(b) Let L be a modular lattice and $s,t \in L$ complements each of other. If we substitute in Proposition 1(ii) $a := t$ and $i(x) := x \wedge s$, then the map $p_{t,s} = i(x \vee a) = (x \vee t) \wedge s$ is called a "*projection*" (see e.g [5]). Since L is modular, the map $i\colon [t,1] \to [0,s]$ is a lattice isomorphism. Thus any projection $p_{t,s}$ is a weakly regular residuated mapping of L, according to Proposition 1.
(c) Let $(V,+,K)$ be a vector space over a field K and $L\colon V \to V$ a linear mapping. Then L induces a weakly regular residuated mapping $f_L\colon \mathrm{Sub}(V) \to \mathrm{Sub}(V)$, $f_L(W) = L(W) := \{L(w) \mid w \in W\}$ of the subspace lattice $(\mathrm{Sub}(V), \subseteq)$, according to [1], Example 4.2 (see also [5]).

We say that a mapping $f\colon L \to L$ is *idempontent* if $f^2 = f$. It is easy to check that on a modular lattice L any projection $p_{t,s} = (x \vee t) \wedge s$ is idempotent.

Remark 2. Let L be a complete modular lattice. In view of [[1], Exercise 4.3] any idempotent weakly regular $f \in \mathrm{Res}(L)$ has the form $f(x) = (x \vee f^*(0)) \wedge f(1)$, $x \in L$, moreover, $f^*(0) = k_f$ and $f(1)$ are complements each of other, according to [[1], Theorem 6.23]. Therefore, f equals to a projection $p_{k_f, f(1)}$. This means that idempotent weakly regular mappings in $\mathrm{Res}(L)$ and the projections of a (complete) modular lattice L coincide.

3. Notions related with weak regularity

For a complemented modular lattice we have another notion of "weak regularity" introduced by G. E. Nicholson and his coauthors, which we will call N-regular. ([6])

A join $\bigvee\{x_i \in L_i | i \in I\}$ in a complete lattice L is called *irredundant*, if $x_j \not\leq \bigvee\{x_i | i \in I \setminus \{j\}\}$ holds for each index $j \in I$. We recall that in a complemented modular lattice L of a finite height any element $x \in L \setminus \{0\}$ is equal to an irredundant join of atoms and the *number* of the atoms in this irredundant join is called the *dimension* of the element x, denoted by $d(x)$. We denote the set of atoms of our lattice L by A. It is worth to mention that in the case of a finite dimensional vector space $(V,+,K)$ and of a subspace $X \in \mathrm{Sub}(V)$, $\mathrm{d}(X)$ equals to its usual (linear algebraic) dimension.

Definition 2. A join-endomorphism f of a complemented modular lattice L of finite height is said to be *N-regular* (according to Nicholson) if it has the following properties:

(a) $d(f(x)) \leq d(x)$ for all $x \in L$, and
(b) $d(f(x)) < d(x)$ implies that there exists an atom $a \leq x$ such that $f(a) = 0$.

Proposition 2 ([6, Proposition 3.7])**.** *Let f be a join-endomorphism on a complemented modular lattice of a finite height. Then f is N-regular if and only if the following conditions are satisfied:*

(a) *f maps atoms into atoms or 0.*
(b) *For any nonzero elements $x,y \in L$, $x \wedge y = 0$ and $f(x) \leq f(y)$ imply that there exists an element $z \in L$ such that $0 < z \leq x \vee y$ and $f(z) = 0$.*

We note that all residuated maps presented in Example 1 in the particular case of a complemented modular lattice L of finite height are also N-regular maps. In spite of the fact that the two notions are very closed, they are not equivalent, as our Theorem 1 will show. To prove it, we will need the following lemma:

Lemma 2. *Let L be a complete modular and complemented lattice. Then any weakly regular residuated mapping $f\colon L \to L$ has the form $f = l \circ p_{k_f,s_f}$, where s_f is a complement of k_f, and $l\colon [0,s_f] \to [0,f(1)]$ is a lattice isomorphism.*

Proof. Assume that f is a weakly regular residuated mapping. Let s_f be a complement of k_f and i the isomorphism defined in Proposition 1(ii). Observe that $i\colon [k_f,1] \to [0,f(1)]$ is the restriction of f to $[k_f,1]$, according to Remark 2 and consider the projection $p_{k_f,s_f}\colon L \to [0,s_f]$, $p_{k_f,s_f}(x) = (x \vee k_f) \wedge s_f$.

Because L is modular, $\varphi\colon [0,s_f] \to [k_f,1]$, $\varphi(x) = x \vee k_f$ is a lattice isomorphism (having as inverse $\varphi^{-1}(x) = x \wedge s_f$, $x \in [k_f,1]$), and the map $l := i \circ \varphi$, $l\colon [0,s_f] \to [0,f(1)]$ is also a lattice isomorphism. It is clear that $\varphi(p_{k_f,s_f}(x)) = \varphi(\varphi^{-1}(x \vee k_f)) = x \vee k_f$, hence $\left(i \circ \varphi \circ p_{k_f,s_f}\right)(x) = i(x \vee k_f) = f(x)$. This implies $l \circ p_{k_f,s_f} = f$. □

Theorem 1. *Let L be a complemented modular lattice of a finite height and $f\colon L \longrightarrow L$ a residuated map. Then the following assertions are equivalent:*

(1) *f is weakly regular,*
(2) *f is N-regular and for any atom $p \leq f(1)$ there exists an atom $a_p \in L$ such that $p = f(a_p)$.*

Proof.

(1)⇒(2): If f is weakly-regular, then $f\colon L \to [0,f(1)]$ is surjective, and hence for any atom $p \leq f(1)$ there exists an $x \in L \setminus \{0\}$ with $p = f(x)$. Clearly, we have $0 \leq f(a) \leq p$, for any atom $a \leq x$. Since $x = \bigvee\{a \in A | a \leq x\}$, the assumption $f(a) = 0$, for all atoms $a \leq x$ would imply $f(x) = 0$, a contradiction. Thus there exists an $a_p \in A$, $a_p \leq x$, with $f(a_p) \neq 0$, i.e. with $f(a_p) = p$. This proves the second part of (2). In order to prove that f is N-regular we will show that conditions (a) and (b) of Proposition 2 are satisfied. Since f is weakly-regular and L is modular and complemented, in view of Lemma 2, f has the form $f = l \circ p_{k_f,s_f}$, where s_f is a complement of k_f denoted as far as follows by $\overline{k_f}$, and $l\colon [0,s_f] \to [0,f(1)]$ is a lattice isomorphism. Now let a be an atom of L. Then the projection p_{k_f,s_f} maps a into an atom or 0. Because an isomorphism $l\colon [0,s_f] \to [0,f(1)]$ maps atoms into atoms and 0 into 0, condition (a) holds. In order to prove (b), take any $x,y \in L$ with $x \wedge y = 0$ and assume that $f(x) \leq f(y)$. Then $l \circ p_{k_f,s_f}(x) \leq l \circ p_{k_f,s_f}(y)$

implies $p_{k_f,s_f}(x) \leq p_{k_f,s_f}(y)$, i.e. $(x \vee k_f) \wedge \overline{k_f} \leq (y \vee k_f) \wedge \overline{k_f}$. Since the mapping $\mu\colon y \mapsto y \wedge \overline{k_f}$ is an order-isomorphism between the intervals $[k_f, 1]$ and $[0, \overline{k_f}]$ (having as inverse the map $\lambda\colon x \mapsto x \vee k_f$), we obtain $x \vee k_f \leq y \vee k_f$, and hence $y \vee k_f = x \vee y \vee k_f$. We claim that now $(x \vee y) \wedge k_f = 0$ is not possible. Indeed, $(x \vee y) \wedge k_f = 0$ would yield also $y \wedge k_f = 0$, and hence we would obtain that $\{0, y, x \vee y, k_f, x \vee y \vee k_f\}$ is a sublattice of L isomorphic to N_5. This is not possible, because L modular. Thus we get $z = (x \vee y) \wedge k_f \neq 0$. Since $z \leq x \vee y$, and because $z \leq k_f$ implies $f(z) = 0$, condition (b) holds, and in view of Proposition 2 we get (2).

(2)⇒(1): Let $y \leq f(1)$. Since $y = \bigvee\{a \in A | a \leq y\}$, and because by (2) for any atom $a \leq y$, there exists an atom $a' \in A$ with $f(a') = a$, we deduce $y = \bigvee\{f(a') | f(a') \leq y\} = f(\bigvee\{a' \in A | f(a') \leq y\})$, and this proves that $f\colon L \to [0, f(1)]$ is surjective. Thus f is range-closed. In order to prove that f is dually range-closed, it is enough to show relation (2.1). By the way of contradiction, suppose that (2.1) fails, i.e. let $x, y \in L$ with $f(x) = f(y)$ and assume $x \vee k_f \neq y \vee k_f$. Then $x \vee k_f < x \vee y \vee k_f$, hence we can choose an atom $a \in L$ such that $a \not\leq x \vee k_f$, but $a \leq x \vee y \vee k_f$. Since L is also relatively complemented, there exists a relative complement u of the element k_f in the interval $[0, x \vee k_f]$, i.e. an element $u \leq x \vee k_f$ such that

$u \vee k_f = x \vee k_f$ and $u \wedge k_f = 0$.

Since $a \not\leq x \vee k_f$, we have $a \not\leq u$, whence $a \wedge u = 0$. On the other hand, we obtain:

$f(u) = f(u) \vee f(k_f) = f(u \vee k_f) = f(x \vee k_f) = f(x) \vee f(k_f) = f(x)$, and

$f(a) \leq f(x \vee y \vee k_f) = f(x) \vee f(y) \vee f(k_f) = f(x)$,

Since, in view of (2), f is N-regular, we can apply condition (b) from Proposition 2, and this implies the existence of a nonzero element $z \leq a \vee u$ with $f(z) = 0$. Now let β an atom of z. Then $\beta \leq a \vee u$ and $f(\beta) = 0$, i.e. $\beta \leq k_f$. Because $u \wedge k_f = 0$, we have also $u \wedge \beta = 0$. Since L is modular, we get $u \prec u \vee \beta \leq u \vee (a \vee u) = u \vee a$. As $a \wedge u = 0$, we have also $u \prec u \vee a$, and hence we obtain $u \vee \beta = u \vee a$. This implies $a \leq u \vee \beta \leq u \vee k_f = x \vee k_f$ - a contradiction to the selection of a. Thus relation (2.1) holds, and hence f is weakly regular.

□

Corollary 2. *Let L be a complemented modular lattice of a finite height and let $f\colon L \to L$ be a residuated map. Then f is weakly regular if and only if f maps atoms into atoms or 0, for any atom $a \leq f(1)$ there is an atom $a' \in L$ with $a = f(a')$, and for any $x, y \in L \setminus \{0\}$ with $x \wedge y = 0$ and $f(x) \leq f(y)$ there exists an atom $p \leq x \vee y$ such that $f(p) = 0$.*

Now, we can provide a simple example for a residuated mapping which is N-regular only, and it is not weakly regular.

Example 2. Let L stand for the direct product of lattices $M_3 = \{0,a,b,c,1\}$ and $M_2 = \{0,p,q,1\}$ on Figure 2.

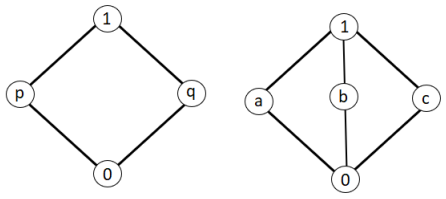

FIGURE 1. Lattices M_2 and M_3

Then clearly $L = M_3 \times M_2$ is a finite modular atomistic lattice with the atoms $(a,0),(b,0),(c,0),(0,p)$ and $(0,q)$. Since L is a geometric lattice, it is complemented, too, and in addition, it has the finite height 4.

Now, define a mapping $f\colon L \to L$ as follows:

$$f(x,y) = \begin{cases} (0,0), \text{ if } y = 0; \\ (a,0), \text{ if } y = p; \\ (c,0), \text{ if } y = q; \\ (1,0), \text{ if } y = 1. \end{cases}$$

It is clear that $f(x,y) = f(0,y)$, and that $y_1 \leq y_2$ yields $f(0,y_1) \leq f(0,y_2)$, therefore f is order-preserving. We will prove that f it is a join-endomorphism, i.e. $f((x_1,y_1) \vee (x_2,y_2)) = f(x_1,y_1) \vee f(x_2,y_2)$. This is clear, whenever $y_1 \leq y_2$. Assume now that $y_1,y_2 \in M_2$ are incomparable. Then either $y_1 = p$, $y_2 = q$ or vice-versa. In both case $f((x_1,y_1) \vee (x_2,y_2)) = f(x_1 \vee x_2, p \vee q) = f(x_1 \vee x_2, 1) = (1,0) = (a,0) \vee (c,0) = f(x_1,y_1) \vee f(x_2,y_2)$.

Thus f is a residuated mapping, because L is a finite lattice. Let us observe also that $f(a,0) = f(b,0) = f(c,0) = (0,0)$, and $f(0,p) = (a,0)$, $f(0,q) = (c,0)$, i.e. f maps atoms into atoms or 0, which is the first condition in Proposition 2.

Now, suppose that for some $u = (x_1,y_1) \neq (0,0)$ and $v = (x_2,y_2) \neq (0,0)$ we have $u \wedge v = 0$ and $f(u) \leq f(v)$. Then $x_1 \wedge x_2 = 0$, $y_1 \wedge y_2 = 0$ and $y_1 \leq y_2$. The latter two relations imply $y_1 = 0$, hence we obtain $f(u) = f(x_1,0) = 0$. Since $0 \neq u \leq u \vee v$, this result means that the conditions in Proposition 2 are satisfied, therefore, f is an N-regular residuated mapping.

Observe that f is not weakly regular, because do not satisfies the second statement in Theorem 1(2). Indeed, we have $f(1_L) = f(1,1) = (1,0)$, however the atom $(b,0) \leq f(1_L)$ is not the image of any atom of L.

In [9] two other conditions (called Jordan conditions) imposed on complete join-endomorphisms $f\colon L \to L$, i.e. on residuated mappings on L are discussed:

(J1) For any choice of the elements $x \leq y$ with $f(x) = f(y)$ there is an element $u \in L$ such that $y = x \vee u$ and $f(u) = 0$.

(J2) For any $x \in L$ the mapping $f\colon [0,x] \to [0, f(x)]$ is surjective.

Condition (J2) means that the image under f of every principal ideal $x\downarrow$ is a principal ideal $f(x)\downarrow$; residuated mappings with this property in [1] are called *totally range-closed*. Clearly, any totally range-closed residuated mapping is range-closed. In view of [[1], Thm 6.21], $f \in \mathrm{Res}(L)$ is totally range-closed if and only if

$$f(f^*(x) \wedge y) = x \wedge f(y) \tag{3.1}$$

holds for all $x, y \in L$. As a consequence of this result we obtain

Corollary 3. *Let L be a complete modular lattice. Then any weakly regular residuated mapping $f\colon L \to L$ is totally range-closed.*

Proof. Let $f \in \mathrm{Res}(L)$. In view of Proposition 1, $f(x) = i(x \vee a)$, where $a = k_f$ and $i\colon \uparrow a \to \downarrow b$ is a lattice isomorphism with $b := f(1)$. Then, in view of Remark 2, $f^*(x) = i^{-1}(x)$, for any $x \leq b$. Hence $f(f^*(x) \wedge y) = f(i^{-1}(x) \wedge y) = i((i^{-1}(x) \wedge y) \vee a)$. Since $a \leq i^{-1}(x)$ and L is modular, we get $(i^{-1}(x) \wedge y) \vee a = i^{-1}(x) \wedge (y \vee a)$. Thus we obtain $f(f^*(x) \wedge y) = i(i^{-1}(x) \wedge (y \vee a)) = i(i^{-1}(x)) \wedge i(y \vee a) = x \wedge f(y)$. Thus in virtue of (3.1) f is totally range-closed. □

Condition (J1) seems to be similar to our condition (2.1), see the following

Proposition 3. *Condition (J1) implies condition (2.1) from Lemma 1. If L is a modular lattice, then (J1) and (2.1) are equivalent.*

Proof. Let L be a complete lattice, and assume that $f\colon L \to L$ satisfies (J1). First, take any $x, y \in L$ with $[x]_{\mathrm{Ker}f} = [y]_{\mathrm{Ker}f}$. Then $(x,y) \in \mathrm{Ker}f$, i.e. $f(x) = f(y)$. Hence $f(x) = f(x) \vee f(y) = f(x \vee y)$. As $x \leq x \vee y$, in view of (J1) there exists an element $u \in L$ with $f(u) = 0$ and such that $x \vee y = x \vee u$. Clearly, $u \leq k_f$. Similarly, $f(y) = f(x \vee y)$ and $y \leq x \vee y$ by (J1) imply $x \vee y = y \vee v$ for some $v \in L$ with $v \leq k_f$. Since $x \vee u = y \vee v$ and $u, v \leq k_f$, we obtain $x \vee k_f = x \vee u \vee k_f = y \vee v \vee k_f = y \vee k_f$. Conversely, for any $x, y \in L$, $x \vee k_f = y \vee k_f$ always yields $f(x) = f(y)$, i.e. $[x]_{\mathrm{Ker}f} = [y]_{\mathrm{Ker}f}$. Thus we proved that (J1) implies the equivalence $[x]_{\mathrm{Ker}f} = [y]_{\mathrm{Ker}f} \Leftrightarrow x \vee k_f = y \vee k_f$, for any $x, y \in L$, i.e. it implies condition (2.1).

Suppose now that f satisfies (2.1), and furthermore, that L is modular. Take any $x, y \in L$, $x \leq y$ with $f(x) = f(y)$. Then $[x]_{\mathrm{Ker}f} = [y]_{\mathrm{Ker}f}$ implies $x \vee k_f = y \vee k_f$, according to (2.1). Since L is modular, $x \leq y$ implies $y = (y \vee k_f) \wedge y = (x \vee k_f) \wedge y =$

$x \vee (k_f \wedge y)$. Now for $u := k_f \wedge y$ we obtain $f(u) \leq f(k_f) = 0$, and $y = x \vee u$. Thus (J1) holds. Hence (2.1) is equivalent to (J1), whenever L is modular. □

Since in view of Lemma 1, condition (2.1) is equivalent to the fact that the residuated mapping $f\colon L \to L$ is dually range-closed, using Proposition 3 and Corollary 3 we obtain:

Corollary 4. *Let L be a complete modular lattice and $f\colon L \to L$ a residuated mapping. Then*

(i) *f is dually range-closed if and only it if satisfies condition (J1).*
(ii) *f is weakly regular if and only it satisfies conditions (J1) and (J2)*

Example 3.

(a) Obviously, any automorphism f of L is totally range-closed, and in view of Example 1(a) is also a weakly regular residuated mapping. Hence, it satisfies conditions (J1) and (J2).
(b) In view of [9], any residuated mapping f defined in Example 1(c) by a linear transformation $L\colon V \to V$ on the subspace lattice $\mathrm{Sub}(V)$ of a vector space $(V,+,K)$ satisfies conditions (J1) and (J2).
(c) Projections, being weakly regular maps, satisfy conditions (J1) and (J2) in case of a modular lattice L.
(d) Finally, let L be a complete lattice and p a dual atom and q an atom in L. In view of [5], the mapping

$$e_{p,q} := \begin{cases} 0, & \text{if } x \leq p; \\ q, & \text{if } x \not\leq p. \end{cases}$$

is a weakly regular residuated mapping of L, which is an atom in the lattice $\mathrm{Res}(L)$. Thus by Lemma 1, $e_{p,q}$ satisfies condition (2.1). It is obvious that $e_{p,q}$ also satisfies (J2), i.e. it is totally range-closed. However, it is easy to see that $e_{p,q}$, fails to satisfy (J1), in general. Indeed, let $L = \{0,p,r,q,1\}$ be an N_5 lattice where $p < r$ and p,q are atoms of L and r,q are dual atoms of L. Then $\ker(e_{p,q}) = \{0,p\}$. If $e_{p,q}$ would satisfy (J1), then $r < 1$ and $e_{p,q}(r) = e_{p,q}(1) = q$, would imply $1 = r \vee u$, for some $u \in \ker(e_{p,q})$. Clearly, this is not possible (because $r \vee 0 = r \vee p = r \neq 1$).

In [9] several properties of residuated mappings sharing properties (J1) and (J2) are proved. For instance, it is shown that for any $f_1, f_2 \in \mathrm{Res}(L)$ satisfying conditions (J1) and (J2), $f_1 \circ f_2 \in \mathrm{Res}(L)$ also satisfies (J1) and (J2). An important result of [9] with applications in module theory was showing the existence of a so-called n-nilpotent Jordan normal base for any n-nilpotent residuated mapping $f\colon L \to L$ defined on a geometric lattice L. We note that a map $f\colon L \to L$ is n-nilpotent, if $f^n = 0$ for some integer $n \geq 1$, but $f^{n-1} \neq 0$. Any linear transformation $L\colon \mathbb{R}^p \to \mathbb{R}^p$, $L(\bar{x}) = A\bar{x}$, with a square matrix A of order $p \geq 1$ is n-nilpotent, if and only if n is the least

positive integer with the property that A^n equals the 0 matrix. (i.e. the matrix A is *n-nilpotent*). For example, $L\colon \mathbb{R}^3 \to \mathbb{R}^3$, $L(x_1,x_2,x_3) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ is a 3-nilpotent linear transformation.

As a consequence of the first mentioned result of [9] we obtain:

Corollary 5. *Let L be a finite modular lattice L and $f\colon L \to L$ a weakly regular residuated mapping on it. Then $f\colon L \to L$ is either nilpotent or there exists an integer $n \geq 1$ such that the power $f^n\colon L \to L$ is a (nonzero) projection.*

Proof. If our condition hold, then f satisfies conditions (J1) and (J2), according to Corollary 4. In view of [9], for any integer $n \geq 1$ $f^n\colon L \to L$ is also a residuated mapping satisfying conditions (J1) and (J2), i.e. it is a weakly regular residuated mapping. Because L is a finite set, the semigroup formed by the powers $f,\ldots,f^n,\ldots$ of f is finite, hence in view of [3], it contains an idempotent element g, where $g = f^k$, for some integer $k \geq 1$. If $g = 0$, then f is nilpotent. If $g \neq 0$, then in view of Remark 1 the map $g = f^k$ is a nonzero projection on L. □

Remark 3. We note that the converse of Corollary 5 is not true in general. For instance, the residuated mapping $f\colon L \to L$ defined in Example 2, (where $L = M_3 \times M_2$) is 2-nilpotent, because $f^2(x,y) = (0,0) = 0_L$, for all $(x,y) \in L$, however f is not weakly regular, as it is shown in Example 2. Next, in order to examine the case when a power of residuated map is a projection, consider the lattice $L' := L \times M_2$, i.e. $L' = M_3 \times M_2 \times M_2$, and the mapping $g\colon L' \to L'$, $g(x,y,z) = (f(x,y),z)$. Then L' is a modular atomistic lattice with the atoms $(a,0,0)$, $(b,0,0)$, $(c,0,0)$, $(0,p,0)$, $(0,q,0)$, $(0,0,p)$ and $(0,0,q)$ (see Figure 1). It is easy to check that g is a residuated map. Because $f^2(x,y) = (0,0)$, we get $g^2(x,y,z) = (f^2(x,y),z) = (0,0,z)$, hence g is a projection (corresponding to the pair $s = (1,1,0), t = (0,0,1)$). However, g is not weakly regular, because do not satisfies the second condition in Theorem 1(2). Indeed, $g(1_{L'}) = g(1,1,1) = (1,0,1)$, but the atom $(b,0,0) \leq g(1_{L'})$ is not the image of any atom of L'. (We have $g(a,0,0) = g(b,0,0) = g(c,0,0) = (0,0,0)$, $g(0,p,0) = (a,0,0)$, $g(0,q,0) = (c,0,0)$ and $g(0,0,p) = (0,0,p)$, $g(0,0,q) = (0,0,q)$.)

4. Conclusions

We have shown that the notion of weak regularity for residuated mappings is stronger than the notion of N-regulatiy, even for complemented modular lattices. We also proved that a residuated mapping satisfying the Jordan conditions J1 and J2 is weakly regular and totaly-range closed. Therefore, Jordan condition J1+J2 in general is stronger than the weak regularity condition, but according to our Corollary 4, for complete modular lattices the two conditions are equivalent. The benefit of the residuated maps satisfying the previous Jordan condition lies in the fact that composing them we obtain residuated maps with the same property. As further work, we

would like to characterize those residuated maps $f\colon L \to L$ which have the property that f^n $(n \geq 1)$ is weakly regular, and to examine for any atomistic modular lattice L whether there are N-regular but not weakly regular atoms in the lattice $\mathrm{Res}(L)$.

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UNIFORM BOUNDS OF POPOVICIU'S DIFFERENCE VIA WEIGHTED HADAMARD INEQUALITY WITH APPLICATIONS

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Abstract. We consider differences coming from Popoviciu's inequality and give upper and lower bounds by employing weighted Hermite-Hadamard inequality along with the approximations of Fink's two point formula. We testify this scenario by utilizing the theory of n-times differentiable convex functions. Our results hold for all $n \geq 2$ and we provide explicit examples to show the correctness of the bounds obtained for special cases. Last but not least, we provide applications in information theory by providing new uniform estimations of the generalized Csiszár divergence, Rényi-divergence, Shannon-entropy, Kullback-Leibler divergence, χ^2-divergence, Bhattacharyya distance, Zipf and Hybrid Zipf-Mandelbrot entropies.

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1. INTRODUCTION

We will start with the definition of convex functions that show a vital role in several areas of mathematics.

Definition 1 ([19, Definition 1.1])**.** The mapping $f\colon [a,b] \subseteq \mathbb{R} \to \mathbb{R}$, is said to be convex if the following inequality holds

$$f(tq_1 + (1-t)q_2) \leq tf(q_1) + (1-t)f(q_2)$$

for all $q_1, q_2 \in [a,b]$ and $t \in [0,1]$. We say that f is concave if $(-f)$ is convex.

There is a nice connection between mathematical inequalities and the theory of convex functions (see the papers [2, 3, 10]). Like Jensen's inequality, Popoviciu's inequality also characterizes convex functions. Nowadays, Popoviciu's inequality is also a topic of interest for many mathematicians since last fifty years.

In 1965, Tiberiu Popoviciu (see [21]) gave the following characterization of convex function.

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Theorem 1. *Let* $f\colon [\rho_1,\rho_2] \longrightarrow \mathbb{R}$ *is convex. Then, for each* $z_1,z_2,z_3 \in [\rho_1,\rho_2]$ *and all* $q_1,q_2,q_3 > 0$*, the following result holds*

$$\begin{aligned}(q_1+q_2+q_3)f\left(\frac{q_1z_1+q_2z_2+q_3z_3}{q_1+q_2+q_3}\right)-(q_2+q_3)f\left(\frac{q_2z_2+q_3z_3}{q_2+q_3}\right)\\ -(q_3+q_1)f\left(\frac{q_3z_3+q_1z_1}{q_3+q_1}\right)-(q_1+q_2)f\left(\frac{q_1z_1+q_2z_2}{q_1+q_2}\right)\\ +q_1f(z_1)+q_2f(z_2)+q_3f(z_3)\geq 0.\end{aligned}$$

The ability to maximise and/or minimise a particular function (or specific real numbers) without requiring derivatives is one of Popoviciu inequality's most useful applications, making this kind of inequality a crucial tool in approximations and optimizations. Another important application is the generalization of several well-known classical inequalities. For instance, Popoviciu's inequality is a beautiful generalization of Hlawka's inequality [16] that makes use of the geometry concept of convexity.

Without the aid of Hlawka's inequality, Popoviciu's inequality could not have been extended to many variables since the authors were motivated to create a higher dimensional [4] counterpart of Popoviciu's inequality based on his characterization. Popoviciu's inequality has several intriguing generalizations and analogues with some ramifications, which may be found in [12].

Nowadays, a convex function axiom that T. Popoviciu proved in [21] is the subject of extensive research (see [19] and references with in). Recent developments in Popoviciu's inequality are introduced by M. V. Mihai in 2016 (see [15]). The integral form of Popoviciu's inequality was presented by C. P. Niculescu in 2009 (see [17]). Popoviciu's inequality in [18] was further developed by C. P. Niculescu in 2006. Recently, Butt et al. provide a few recent updates and enhancements to Popoviciu's inequality in [1, 5, 7, 8].

The following is Vasić and Stanković shape of Popoviciu's inequality given in [23] (see also [19] page 173).

Theorem 2. *Consider* $m,r \in \mathbb{N}$ *where* $m \geq 3$ *and* $2 \leq r \leq m-1$ *also* $[\rho_1,\rho_2] \subset \longrightarrow \mathbb{R}$, $z=(z_1,\ldots,z_m) \in [\rho_1,\rho_2]^m$, $q=(q_1,\ldots,q_m)$ *be positive* m*-tuple in such a way that* $\sum_{i=1}^{m} q_i = 1$. *If* $f\colon [\rho_1,\rho_2] \to \mathbb{R}$ *is a convex function, then*

$$\frac{1}{C_{r-1}^{m-1}}\sum_{1\leq i_1<\ldots<i_r\leq m}\left(\sum_{j=1}^{r}q_{i_j}\right)f\left(\frac{\sum_{j=1}^{r}q_{i_j}z_{i_j}}{\sum_{j=1}^{r}q_{i_j}}\right)\leq\frac{m-r}{m-1}\sum_{i=1}^{m}q_if(z_i)+\frac{r-1}{m-1}f\left(\sum_{i=1}^{m}q_iz_i\right),$$

or

$$P_r^m(z,q;f)\leq\frac{m-r}{m-1}P_1^m(z,q;f)+\frac{r-1}{m-1}P_m^m(z,q;f), \tag{1.1}$$

where

$$P_{\mathfrak{r}}^{\mathfrak{m}}(\mathfrak{z},\mathfrak{q};\mathfrak{f})=P_{\mathfrak{r}}^{\mathfrak{m}}(\mathfrak{z},\mathfrak{q};\mathfrak{f}(\mathfrak{z})):=\frac{1}{C_{\mathfrak{r}-1}^{\mathfrak{m}-1}}\sum_{1\leq i_1<\ldots<i_{\mathfrak{r}}\leq\mathfrak{m}}\left(\sum_{j=1}^{\mathfrak{r}}\mathfrak{q}_{i_j}\right)\mathfrak{f}\left(\frac{\sum\limits_{j=1}^{\mathfrak{r}}\mathfrak{q}_{i_j}\mathfrak{z}_{i_j}}{\sum\limits_{j=1}^{\mathfrak{r}}\mathfrak{q}_{i_j}}\right),$$

is the linear w.r.t. $\mathfrak{f}$.

From inequality (1.1) we can write

$$\mathbf{F}(\mathfrak{z},\mathfrak{q};\mathfrak{f}):=\frac{\mathfrak{m}-\mathfrak{r}}{\mathfrak{m}-1}P_1^{\mathfrak{m}}(\mathfrak{z},\mathfrak{q};\mathfrak{f})+\frac{\mathfrak{r}-1}{\mathfrak{m}-1}P_{\mathfrak{m}}^{\mathfrak{m}}(\mathfrak{z},\mathfrak{q};\mathfrak{f})-P_{\mathfrak{r}}^{\mathfrak{m}}(\mathfrak{z},\mathfrak{q};\mathfrak{f}). \tag{1.2}$$

In this research article we generalized Popoviciu's inequality by utilizing Fink's identity along with weighted Hermite Hadamard inequality, then for obtained results we give interesting applications in information theory.

The following theorem is proved by A. M. Fink:

Theorem 3. *Let* $n\in\mathbb{N}$, $\mathfrak{f}\colon I\to\mathbb{R}$ *be such that* $\mathfrak{f}^{(n-1)}$ *is absolutely continuous,* $I\subset\mathbb{R}$ *an open interval,* $\rho_1,\rho_2\in I$, $\rho_1<\rho_2$. *Then, the following identity holds*

$$\begin{aligned}\mathfrak{f}(\mathfrak{z})&=\frac{n}{\rho_2-\rho_1}\int_{\rho_1}^{\rho_2}\mathfrak{f}(r)\,\mathrm{d}r\\&\quad-\sum_{w=1}^{n-1}\left(\frac{n-w}{w!}\right)\left(\frac{\mathfrak{f}^{(w-1)}(\rho_1)(\mathfrak{z}-\rho_1)^w-\mathfrak{f}^{(w-1)}(\rho_2)(\mathfrak{z}-\rho_2)^w}{\rho_2-\rho_1}\right)\\&\quad+\frac{1}{(n-1)!(\rho_2-\rho_1)}\int_{\rho_1}^{\rho_2}(\mathfrak{z}-r)^{n-1}w^{[\rho_1,\rho_2]}(r,\mathfrak{z})\mathfrak{f}^{(n)}(r)\,\mathrm{d}r,\end{aligned}$$

where

$$R^{[\rho_1,\rho_2]}(r,\mathfrak{z})=\begin{cases}r-\rho_1, & \rho_1\leq r\leq\mathfrak{z}\leq\rho_2;\\ r-\rho_2, & \rho_1\leq\mathfrak{z}<r\leq\rho_2.\end{cases} \tag{1.3}$$

Butt et al. gave in [1] the following generalized identity for Popoviciu inequality under the effect of Fink's identity:

Theorem 4. *Let all the assumptions of Theorem 3 be valid and let* $\mathfrak{m},\mathfrak{r}\in N$, $\mathfrak{m}\geq 3$, $2\leq\mathfrak{r}\leq\mathfrak{m}-1$, $[\rho_1,\rho_2]\subset\longrightarrow\mathbb{R}$, $\mathfrak{z}=(\mathfrak{z}_1,\mathfrak{z}_2,\ldots,\mathfrak{z}_{\mathfrak{m}})\in[\rho_1,\rho_2]^{\mathfrak{m}}$ $\mathfrak{q}=(\mathfrak{q}_1,\mathfrak{q}_2,\ldots,\mathfrak{q}_{\mathfrak{m}})$ *be a real* $\mathfrak{m}$*-tuple such that* $\sum_{j=1}^{\mathfrak{r}}\mathfrak{q}_{i_j}\neq 0$ *for any* $1\leq i_1<\cdots<i_{\mathfrak{r}}\leq\mathfrak{m}$ *and* $\sum_{i=1}^{\mathfrak{m}}\mathfrak{q}_i=1$. *Also let* $\frac{\sum\limits_{j=1}^{\mathfrak{r}}\mathfrak{q}_{i_j}\mathfrak{z}_{i_j}}{\sum\limits_{j=1}^{\mathfrak{r}}\mathfrak{q}_{i_j}}\in[\rho_1,\rho_2]$ *for any* $1\leq i_1<\cdots<i_{\mathfrak{r}}\leq\mathfrak{m}$ *with* $R^{[\rho_1,\rho_2]}(r,\mathfrak{z})$ *be the same as defined in* (1.3) *then, we have the following identity:*

$$\mathbf{F}(z,q;\mathfrak{f}(z)) = \sum_{w=2}^{n-1}\left(\frac{n-w}{w!(\rho_2-\rho_1)}\right) \times \left\{f^{(w-1)}(\rho_2)\mathbf{F}(z,q;(z-\rho_2)^w) - f^{(w-1)}(\rho_1)\mathbf{F}(z,\mathbf{q};(z-\rho_1)^w)\right\} + \frac{1}{(n-1)!(\rho_2-\rho_1)}\int_{\rho_1}^{\rho_2}\mathbf{F}\left(z,q;(z-r)^{n-1}R^{[\rho_1,\rho_2]}(r,z)\right)\mathfrak{f}^{(n)}(r)\,dr. \tag{1.4}$$

The weighted Hermite-Hadamard inequality given in the following theorem (see [24]).

Theorem 5. *If* $\mathfrak{f}\colon [\rho_1,\rho_2]\to\mathbb{R}$ *is convex and* $p\colon [\rho_1,\rho_2]\to\mathbb{R}$ *is of a constant sign on* $[\rho_1,\rho_2]$*, then we have*

$$\mathfrak{f}(\lambda) \le \frac{1}{P(\rho_2)}\int_{\rho_1}^{\rho_2} p(t)\mathfrak{f}(t)\,dt \le \frac{\rho_2-\lambda}{\rho_2-\rho_1}\mathfrak{f}(\rho_1) + \frac{\lambda-\rho_1}{\rho_2-\rho_1}\mathfrak{f}(\rho_2),$$

where

$$P(\rho_2) = \int_{\rho_1}^{\rho_2} p(t)\,dt, \tag{1.5}$$

and

$$\lambda = \frac{1}{P(\rho_2)}\int_{\rho_1}^{\rho_2} t\,p(t)\,dt. \tag{1.6}$$

We can write above inequality as

$$P\mathfrak{f}(\lambda) \le \int_{\rho_1}^{\rho_2} p(t)\mathfrak{f}(t)\,dt \le P\left(\frac{\rho_2-\lambda}{\rho_2-\rho_1}\mathfrak{f}(\rho_1) + \frac{\lambda-\rho_1}{\rho_2-\rho_1}\mathfrak{f}(\rho_2)\right). \tag{1.7}$$

In this research article, we will formulate upper and lower bounds for Popoviciu's difference by utilizing Fink's identity along with weighted Hermite Hadamard inequality. We will also give results for a particular case when $n=2$, then for obtained results we will give applications in information theory.

2. Main Results

Theorem 6. *Under the assumptions of Theorem 4, let for all* $n\ge 2$, $\mathfrak{f}^{(n)}(\cdot)$ *(n-th derivative of* $\mathfrak{f}$*) is convex and*

$$\mathbf{F}\left(z,q;(z-r)^{n-1}R^{[\rho_1,\rho_2]}(r,z)\right) > 0, \qquad r\in[\rho_1,\rho_2] \tag{2.1}$$

then the following inequalities are valid

$$\frac{P_1(n)}{(n-1)!(\rho_2-\rho_1)}\mathfrak{f}^{(n)}(\lambda_1(n)) \le \mathbf{F}(z,q;\mathfrak{f}(z)) - \sum_{w=2}^{n-1}\left(\frac{n-w}{w!(\rho_2-\rho_1)}\right)$$

$$\times\left(\mathfrak{f}^{(w-1)}(\rho_2)\,\mathbf{F}(z,q;(z-\rho_2)^w)-\mathfrak{f}^{(w-1)}(\rho_1)\,\mathbf{F}(z,q;(z-\rho_1)^w)\right)$$
$$\leq\frac{P_1(n)}{(n-1)!\,(\rho_2-\rho_1)}\left(\frac{\rho_2-\lambda_1(n)}{\rho_2-\rho_1}\mathfrak{f}^{(n)}(\rho_1)+\frac{\lambda_1(n)-\rho_1}{\rho_2-\rho_1}\mathfrak{f}^{(n)}(\rho_2)\right). \tag{2.2}$$

Where

$$P_1(n)=\int_{\rho_1}^{\rho_2}\mathbf{F}\left(z,q;(z-r)^{n-1}R^{[\rho_1,\rho_2]}(r,z)\right)dr$$
$$=\mathbf{F}\left(z,q;\left(\frac{(z-\rho_1)^{n+1}-(z-\rho_2)^{n+1}}{n(n+1)}\right)\right),$$

or

$$P_1(n)=(n-1)!\,(\rho_2-\rho_1)\left[\mathbf{F}\left(z,q;\frac{z^n}{n!}\right)-\sum_{w=2}^{n-1}\frac{(n-w)}{n!(n-w+1)(\rho_2-\rho_1)}\right.$$
$$\left.\times\binom{n}{w}\left\{\rho_2^{(n-w+1)}\mathbf{F}(z,q;(z-\rho_2)^w)-\rho_1^{(n-w+1)}\mathbf{F}(z,q;(z-\rho_1)^w)\right\}\right],$$

and

$$\lambda_1(n)=\frac{\int_{\rho_1}^{\rho_2}r\mathbf{F}\left(z,q;(z-r)^{n-1}R^{[\rho_1,\rho_2]}(r,z)\right)dr}{\int_{\rho_1}^{\rho_2}\mathbf{F}\left(z,q;(z-r)^{n-1}R^{[\rho_1,\rho_2]}(r,z)\right)dr}$$
$$=\mathbf{F}\left(z,\mathbf{q};\left(\frac{(n\rho_1+2z)(z-\rho_1)^{n+1}-(n\rho_2+2z)(z-\rho_2)^{n+1}}{n(n+1)(n+2)P_1(n)}\right)\right),$$

or

$$\lambda_1(n)=\frac{(n-1)!(\rho_2-\rho_1)}{P_1(n)}$$
$$\times\left[\mathbf{F}\left(z,q;\frac{z^{n+1}}{(n+1)!}\right)-\sum_{w=2}^{n-1}\left(\frac{n-w}{(n+1)!(n-w+2)(\rho_2-\rho_1)}\right)\right.$$
$$\left.\times\binom{n+1}{w}\left(\rho_2^{(n-w+2)}\mathbf{F}(z,q;(z-\rho_2)^w)-\rho_1^{(n-w+2)}\mathbf{F}(z,q;(z-\rho_1)^w)\right)\right].$$

Proof. As $\mathfrak{f}^{(n)}$ is assumed to be convex for all $n\geq 2$, thus by applying weighted Hermite Hadamard inequality (1.7) for the convex function $\mathfrak{f}^{(n)}$ with positive weight $p(r)=\mathbf{F}\left(z,q;(z-r)^{n-1}R^{[\rho_1,\rho_2]}(r,z)\right)$, we get

$$\frac{P_1(n)}{(n-1)!\,(\rho_2-\rho_1)}\mathfrak{f}^{(n)}(\lambda_1(n))$$
$$\leq\frac{1}{(n-1)!\,(\rho_2-\rho_1)}\int_{\rho_1}^{\rho_2}\mathbf{F}\left(z,q;(z-r)^{n-1}R^{[\rho_1,\rho_2]}(r,z)\right)\mathfrak{f}^{(n)}(r)\,dr$$

$$\leq \frac{P_1(n)}{(n-1)!\,(\rho_2-\rho_1)}\left(\frac{\rho_2-\lambda_1(n)}{\rho_2-\rho_1}\mathfrak{f}^{(n)}(\rho_1)+\frac{\lambda_1(n)-\rho_1}{\rho_2-\rho_1}\mathfrak{f}^{(n)}(\rho_2)\right). \tag{2.3}$$

Thus, by substituting the value of

$$\frac{1}{(n-1)!\,(\rho_2-\rho_1)}\int_{\rho_1}^{\rho_2}\mathbf{F}\left(\mathrm{z},\mathfrak{q};(\mathrm{z}-r)^{n-1}R^{[\rho_1,\rho_2]}(r,\mathrm{z})\right)\mathfrak{f}^{(n)}(r)\,\mathrm{d}r$$

from (1.4) in (2.3), we get required result.

Now, we find the value of $P_1(n)$ by utilizing the kernel function (1.3) defined in the remainder term of Fink's identity. By replacing z with z_i in (1.3) we get,

$$R^{[\rho_1,\rho_2]}(r,\mathrm{z}_i)=\begin{cases}r-\rho_1, & \rho_1\leq r\leq \mathrm{z}_i\leq\rho_2;\\ r-\rho_2, & \rho_1\leq \mathrm{z}_i<r\leq\rho_2.\end{cases}$$

Now we calculate $\int_{\rho_1}^{\rho_2}\mathbf{F}\left(\mathrm{z},\mathfrak{q};(\mathrm{z}-r)^{n-1}R^{[\rho_1,\rho_2]}(r,\mathrm{z})\right)\mathrm{d}r$ as by comparing (1.5), we have

$$\begin{aligned}P_1(n)&=\int_{\rho_1}^{\rho_2}\mathbf{F}\left(\mathrm{z},\mathfrak{q};(\mathrm{z}-r)^{n-1}R^{[\rho_1,\rho_2]}(r,\mathrm{z})\right)\mathrm{d}r \qquad (2.4)\\ &=\int_{\rho_1}^{\mathrm{z}}\mathbf{F}\left(\mathrm{z},\mathfrak{q};(\mathrm{z}-r)^{n-1}(r-\rho_1)\right)\mathrm{d}r+\int_{\mathrm{z}}^{\rho_2}\mathbf{F}\left(\mathrm{z},\mathfrak{q};(\mathrm{z}-r)^{n-1}(r-\rho_2)\right)\mathrm{d}r\\ &=I_1+I_2-I_3.\end{aligned}$$

It is pertinent to mention that the difference in last equality is obtained by using the explicit form of our functional as given in (1.2), where

$$I_1=\frac{\mathfrak{m}-\mathfrak{r}}{\mathfrak{m}-1}\sum_{i=1}^{\mathfrak{m}}\mathfrak{q}_i\left[\int_{\rho_1}^{\mathrm{z}_i}(\mathrm{z}_i-r)^{n-1}(r-\rho_1)\,\mathrm{d}r+\int_{\mathrm{z}_i}^{\rho_2}(\mathrm{z}_i-r)^{n-1}(r-\rho_2)\,\mathrm{d}r\right],$$

$$\begin{aligned}I_2=\frac{\mathfrak{r}-1}{\mathfrak{m}-1}\Bigg[&\int_{\rho_1}^{\sum_{i=1}^{\mathfrak{m}}\mathfrak{q}_i\mathrm{z}_i}\left(\sum_{i=1}^{\mathfrak{m}}\mathfrak{q}_i\mathrm{z}_i-r\right)^{n-1}(r-\rho_1)\,\mathrm{d}r\\ &+\int_{\sum_{i=1}^{\mathfrak{m}}\mathfrak{q}_i\mathrm{z}_i}^{\rho_2}\left(\sum_{i=1}^{\mathfrak{m}}\mathfrak{q}_i\mathrm{z}_i-r\right)^{n-1}(r-\rho_2)\,\mathrm{d}r\Bigg],\end{aligned}$$

and

$$\begin{aligned}I_3=&\frac{1}{C_{\mathfrak{r}-1}^{\mathfrak{m}-1}}\sum_{1\leq i_1<\ldots<i_{\mathfrak{r}}\leq\mathfrak{m}}\left(\sum_{j=1}^{\mathfrak{r}}\mathfrak{q}_{i_j}\right)\\ &\times\left[\int_{\rho_1}^{\frac{\sum_{j=1}^{\mathfrak{r}}\mathfrak{q}_{i_j}\mathrm{z}_{i_j}}{\sum_{j=1}^{\mathfrak{r}}\mathfrak{q}_{i_j}}}\left(\frac{\sum_{j=1}^{\mathfrak{r}}\mathfrak{q}_{i_j}\mathrm{z}_{i_j}}{\sum_{j=1}^{\mathfrak{r}}\mathfrak{q}_{i_j}}-r\right)^{n-1}(r-\delta_1)\,\mathrm{d}r\right.\end{aligned}$$

$$+\int_{\frac{\sum_{j=1}^{\mathfrak{r}} q_{i_j} z_{i_j}}{\sum_{j=1}^{\mathfrak{r}} q_{i_j}}}^{\rho_2} \left(\frac{\sum_{j=1}^{\mathfrak{r}} q_{i_j} z_{i_j}}{\sum_{j=1}^{\mathfrak{r}} q_{i_j}} - r \right)^{n-1} (r-\rho_2)\, dr \Bigg].$$

First we calculate I_1 as,

$$\begin{aligned} I_1 &= \frac{\mathfrak{m}-\mathfrak{r}}{\mathfrak{m}-1} \sum_{i=1}^{\mathfrak{m}} q_i \left[\int_{\rho_1}^{z_i} (z_i - r)^{n-1} (r-\rho_1)\, dr + \int_{z_i}^{\rho_2} (z_i - r)^{n-1} (r-\rho_2)\, dr \right] \\ &= \frac{\mathfrak{m}-\mathfrak{r}}{\mathfrak{m}-1} \sum_{i=1}^{\mathfrak{m}} q_i \left[\frac{(\rho_1 - r)(z_i - r)^n}{n} - \frac{(z_i - r)^{n+1}}{n(n+1)} \Bigg|_{\rho_1}^{z_i} \right. \\ &\qquad \left. + \frac{(\rho_2 - r)(z_i - r)^n}{n} - \frac{(z_i - r)^{n+1}}{n(n+1)} \Bigg|_{z_i}^{\rho_2} \right] \\ &= \frac{\mathfrak{m}-\mathfrak{r}}{\mathfrak{m}-1} \sum_{i=1}^{\mathfrak{m}} q_i \left[0 + \frac{(z_i - \rho_1)^{n+1}}{n(n+1)} - \frac{(z_i - \rho_2)^{n+1}}{n(n+1)} - 0 \right] \\ &= \frac{\mathfrak{m}-\mathfrak{r}}{\mathfrak{m}-1} \sum_{i=1}^{\mathfrak{m}} q_i \left[\frac{(z_i - \rho_1)^{n+1} - (z_i - \rho_2)^{n+1}}{n(n+1)} \right]. \end{aligned}$$

Similarly, we can evaluate

$$I_2 = \frac{\mathfrak{r}-1}{\mathfrak{m}-1} \left[\frac{\left(\sum_{i=1}^{\mathfrak{m}} q_i z_i - \rho_1 \right)^{n+1} - \left(\sum_{i=1}^{\mathfrak{m}} q_i z_i - \rho_2 \right)^{n+1}}{n(n+1)} \right],$$

and

$$I_3 = \frac{1}{C_{\mathfrak{r}-1}^{\mathfrak{m}-1}} \sum_{1 \le i_1 < \ldots < i_{\mathfrak{r}} \le \mathfrak{m}} \left(\sum_{j=1}^{\mathfrak{r}} q_{i_j} \right) \left[\frac{\left(\frac{\sum_{j=1}^{\mathfrak{r}} q_{i_j} z_{i_j}}{\sum_{j=1}^{\mathfrak{r}} q_{i_j}} - \rho_1 \right)^{n+1} - \left(\frac{\sum_{j=1}^{\mathfrak{r}} q_{i_j} z_{i_j}}{\sum_{j=1}^{\mathfrak{r}} q_{i_j}} - \rho_2 \right)^{n+1}}{n(n+1)} \right].$$

By taking difference $I_1 + I_2 - I_3$ and rewriting it in compact functional form, we get

$$I_1 + I_2 - I_3 = \frac{\mathfrak{m}-\mathfrak{r}}{\mathfrak{m}-1} \sum_{i=1}^{\mathfrak{m}} q_i \left[\frac{(z_i - \rho_1)^{n+1} - (z_i - \rho_2)^{n+1}}{n(n+1)} \right]$$

$$+\frac{\mathfrak{r}-1}{\mathfrak{m}-1}\left[\frac{\left(\sum_{i=1}^{\mathfrak{m}}\mathfrak{q}_i\mathfrak{z}_i-\rho_1\right)^{n+1}-\left(\sum_{i=1}^{\mathfrak{m}}\mathfrak{q}_i\mathfrak{z}_i-\rho_2\right)^{n+1}}{n(n+1)}\right]$$

$$-\frac{1}{C_{\mathfrak{r}-1}^{\mathfrak{m}-1}}\sum_{1\le i_1<\ldots<i_{\mathfrak{r}}\le\mathfrak{m}}\left(\sum_{j=1}^{\mathfrak{r}}\mathfrak{q}_{i_j}\right)\left[\frac{\left(\frac{\sum_{j=1}^{\mathfrak{r}}\mathfrak{q}_{i_j}\mathfrak{z}_{i_j}}{\sum_{j=1}^{\mathfrak{r}}\mathfrak{q}_{i_j}}-\rho_1\right)^{n+1}-\left(\frac{\sum_{j=1}^{\mathfrak{r}}\mathfrak{q}_{i_j}\mathfrak{z}_{i_j}}{\sum_{j=1}^{\mathfrak{r}}\mathfrak{q}_{i_j}}-\rho_2\right)^{n+1}}{n(n+1)}\right]$$

$$=\mathbf{F}\left(\mathfrak{z},\mathfrak{q};\left(\frac{(\mathfrak{z}-\rho_1)^{n+1}-(\mathfrak{z}-\rho_2)^{n+1}}{n(n+1)}\right)\right).$$

Method 2 for computing $P_1(n)$: If we choose $\mathfrak{f}(\mathfrak{z})=\frac{\mathfrak{z}^n}{n!}$ in (1.4), then we obtain

$$\mathbf{F}\left(\mathfrak{z},\mathbf{q};\frac{\mathfrak{z}^n}{n!}\right)-\frac{1}{n!}\sum_{w=2}^{n-1}\left(\frac{n-w}{w!(\rho_2-\rho_1)}\right)(n(n-1)\cdots(n-w+2))$$
$$\times\left\{{\rho_2}^{(n-w+1)}\mathbf{F}\left(\mathfrak{z},\mathbf{q};(\mathfrak{z}-\rho_2)^w\right)-\rho_1^{(n-w+1)}\mathbf{F}\left(\mathfrak{z},\mathbf{q};(\mathfrak{z}-\rho_1)^w\right)\right\}$$
$$=\frac{1}{(n-1)!\,(\rho_2-\rho_1)}\int_{\rho_1}^{\rho_2}\mathbf{F}\left(\mathfrak{z},\mathbf{q};(\mathfrak{z}-r)^{n-1}R^{[\rho_1,\rho_2]}(r,\mathfrak{z})\right)\mathrm{d}r. \tag{2.5}$$

Comparing (2.4) and (2.5) leads to

$$\frac{P_1(n)}{(n-1)!\,(\rho_2-\rho_1)}=\mathbf{F}\left(\mathfrak{z},\mathbf{q};\frac{\mathfrak{z}^n}{n!}\right)-\frac{1}{n!}\sum_{w=2}^{n-1}\left(\frac{n-w}{w!(\rho_2-\rho_1)}\right)(n(n-1)\cdots(n-w+2))$$
$$\times\left\{{\rho_2}^{(n-w+1)}\mathbf{F}\left(\mathfrak{z},\mathbf{q};(\mathfrak{z}-\rho_2)^w\right)-\rho_1^{(n-w+1)}\mathbf{F}\left(\mathfrak{z},\mathbf{q};(\mathfrak{z}-\rho_1)^w\right)\right\}.$$

Now by multiplying above result with $(n-1)!\,(\rho_2-\rho_1)$ and after simple calculations, we get

$$P_1(n)=(n-1)!\,(\rho_2-\rho_1)\left[\mathbf{F}\left(\mathfrak{z},\mathbf{q};\frac{\mathfrak{z}^n}{n!}\right)-\sum_{w=2}^{n-1}\left(\frac{n-w}{n!(n-w+1)(\rho_2-\rho_1)}\right)\right.$$
$$\left.\times\binom{n}{w}\left\{{\rho_2}^{(n-w+1)}\mathbf{F}\left(\mathfrak{z},\mathbf{q};(\mathfrak{z}-\rho_2)^w\right)-\rho_1^{(n-w+1)}\mathbf{F}\left(\mathfrak{z},\mathbf{q};(\mathfrak{z}-\rho_1)^w\right)\right\}\right].$$

Here we use the fact that

$$\frac{n(n-1)\cdots(n-w+2)}{w!}=\frac{1}{(n-w+1)}\binom{n}{w}.$$

Now, we have to calculate the value of $\lambda_1(n)$, which can be obtained directly in our case by comparing with (1.6) as

$$\lambda_1(n) = \frac{\int_{\rho_1}^{\rho_2} r\mathbf{F}\left(z,q;(z-r)^{n-1}R^{[\rho_1,\rho_2]}(r,z)\right)dr}{\int_{\rho_1}^{\rho_2}\mathbf{F}\left(z,q;(z-r)^{n-1}R^{[\rho_1,\rho_2]}(r,z)\right)dr} = \frac{1}{P_1(n)}\int_{\rho_1}^{\rho_2} r\mathbf{F}\left(z,q;(z-r)^{n-1}R^{[\rho_1,\rho_2]}(r,z)\right)dr$$

For this, we first calculate $\int_{\rho_1}^{\rho_2} r\mathbf{F}\left(z,q;(z-r)^{n-1}R^{[\rho_1,\rho_2]}(r,z)\right)dr$ as under by using integration by parts:

$$\begin{aligned}&\int_{\rho_1}^{\rho_2} r\mathbf{F}\left(z,q;(z-r)^{n-1}R^{[\rho_1,\rho_2]}(r,z)\right)dr \qquad (2.6)\\ &= \int_{\rho_1}^{z} r\mathbf{F}\left(z,q;(z-r)^{n-1}(r-\rho_1)\right)dr + \int_{z}^{\rho_2} r\mathbf{F}\left(z,q;(z-r)^{n-1}(r-\rho_2)\right)dr\\ &= I_4 + I_5 - I_6\end{aligned}$$

Again, we use the explicit form of our functional as given in (1.2) and for simplifications of integrals , we use the following notations

$$I_4 = \frac{m-r}{m-1}\sum_{i=1}^{m} q_i\left[\int_{\rho_1}^{z_i}(z_i-r)^{n-1}\left(r^2-\rho_1 r\right)dr + \int_{z_i}^{\rho_2}(z_i-r)^{n-1}\left(r^2-\rho_2 r\right)dr\right],$$

$$I_5 = \frac{r-1}{m-1}\left[\int_{\rho_1}^{\sum_{i=1}^{m} q_i z_i}\left(\sum_{i=1}^{m} q_i z_i - r\right)^{n-1}\left(r^2-\rho_1 r\right)dr + \int_{\sum_{i=1}^{m} q_i z_i}^{\rho_2}\left(\sum_{i=1}^{m} q_i z_i - r\right)^{n-1}\left(r^2-\rho_2 r\right)dr\right],$$

and

$$I_6 = \frac{1}{C_{r-1}^{m-1}}\sum_{1\le i_1<\ldots<i_r\le m}\left(\sum_{j=1}^{r} q_{i_j}\right)\left[\int_{\rho_1}^{\frac{\sum_{j=1}^{r} q_{i_j}z_{i_j}}{\sum_{j=1}^{r} q_{i_j}}}\left(\frac{\sum_{j=1}^{r} q_{i_j}z_{i_j}}{\sum_{j=1}^{r} q_{i_j}} - r\right)^{n-1}\left(r^2-\rho_1 r\right)dr + \int_{\frac{\sum_{j=1}^{r} q_{i_j}z_{i_j}}{\sum_{j=1}^{r} q_{i_j}}}^{\rho_2}\left(\frac{\sum_{j=1}^{r} q_{i_j}z_{i_j}}{\sum_{j=1}^{r} q_{i_j}} - r\right)^{n-1}\left(r^2-\rho_2 r\right)dr\right]$$

$$= I_4 + I_5 - I_6.$$

First we calculate the integrals in I_4 as

$$\int_{\rho_1}^{z_i} (z_i - r)^{n-1} \left(r^2 - \rho_1 r\right) \mathrm{d}r = \frac{1}{n(n+1)} \left[\rho_1 (z_i - \rho_1)^{n+1} + \frac{2(z_i - \rho_1)^{n+2}}{n+2}\right],$$

and

$$\int_{z_i}^{\rho_2} (z_i - r)^{n-1} \left(r^2 - \rho_2 r\right) \mathrm{d}r = \frac{1}{n(n+1)} \left[-\rho_2 (z_i - \rho_2)^{n+1} - \frac{2(z_i - \rho_2)^{n+2}}{n+2}\right].$$

Hence, we can get

$$I_4 = \frac{\mathfrak{m} - \mathfrak{r}}{\mathfrak{m} - 1} \sum_{i=1}^{\mathfrak{m}} q_i \left[\frac{(n\rho_1 + 2z_i)(z_i - \rho_1)^{n+1} - (n\rho_2 + 2z_i)(z_i - \rho_2)^{n+1}}{n(n+1)(n+2)}\right].$$

Similarly, we have

$$I_5 = \frac{\mathfrak{r} - 1}{\mathfrak{m} - 1} \times \left[\frac{\left(n\rho_1 + 2\sum\limits_{i=1}^{\mathfrak{m}} q_i z_i\right)\left(\sum\limits_{i=1}^{\mathfrak{m}} q_i z_i - \rho_1\right)^{n+1} - \left(n\rho_2 + 2\sum\limits_{i=1}^{\mathfrak{m}} q_i z_i\right)\left(\sum\limits_{i=1}^{\mathfrak{m}} q_i z_i - \rho_2\right)^{n+1}}{n(n+1)(n+2)}\right],$$

and

$$I_6 = \frac{1}{C_{\mathfrak{r}-1}^{\mathfrak{m}-1}} \sum_{1 \le i_1 < \ldots < i_{\mathfrak{r}} \le \mathfrak{m}} \left(\sum_{j=1}^{\mathfrak{r}} q_{i_j}\right) \times \left[\frac{\left(n\rho_1 + 2\frac{\sum\limits_{j=1}^{\mathfrak{r}} q_{i_j} z_{i_j}}{\sum\limits_{j=1}^{\mathfrak{r}} q_{i_j}}\right)\left(\frac{\sum\limits_{j=1}^{\mathfrak{r}} q_{i_j} z_{i_j}}{\sum\limits_{j=1}^{\mathfrak{r}} q_{i_j}} - \rho_1\right)^{n+1}}{n(n+1)(n+2)} - \frac{\left(n\rho_2 + 2\frac{\sum\limits_{j=1}^{\mathfrak{r}} q_{i_j} z_{i_j}}{\sum\limits_{j=1}^{\mathfrak{r}} q_{i_j}}\right)\left(\frac{\sum\limits_{j=1}^{\mathfrak{r}} q_{i_j} z_{i_j}}{\sum\limits_{j=1}^{\mathfrak{r}} q_{i_j}} - \rho_2\right)^{n+1}}{n(n+1)(n+2)}\right].$$

Now, by taking the difference $I_4 + I_5 - I_6$, we get

$$I_4 + I_5 - I_6 = \frac{\mathfrak{m} - \mathfrak{r}}{\mathfrak{m} - 1}$$

$$\times \sum_{i=1}^{\mathfrak{m}} \mathfrak{q}_i \left[\frac{(n\rho_1+2z_i)(z_i-\rho_1)^{n+1}-(n\rho_2+2z_i)(z_i-\rho_2)^{n+1}}{n(n+1)(n+2)P_1(n)} \right] + \frac{\mathfrak{r}-1}{\mathfrak{m}-1}$$

$$\times \left[\frac{\left(n\rho_1+2\sum\limits_{i=1}^{\mathfrak{m}} \mathfrak{q}_i z_i\right)\left(\sum\limits_{i=1}^{\mathfrak{m}} \mathfrak{q}_i z_i-\rho_1\right)^{n+1}-\left(n\rho_2+2\sum\limits_{i=1}^{\mathfrak{m}} \mathfrak{q}_i z_i\right)\left(\sum\limits_{i=1}^{\mathfrak{m}} \mathfrak{q}_i z_i-\rho_2\right)^{n+1}}{n(n+1)(n+2)P_1(n)} \right]$$

$$- \frac{1}{C_{\mathfrak{r}-1}^{\mathfrak{m}-1}} \sum_{1\le i_1<\ldots<i_{\mathfrak{r}}\le \mathfrak{m}} \left(\sum_{j=1}^{\mathfrak{r}} \mathfrak{q}_{i_j}\right) \left[\frac{\left(n\rho_1+2\frac{\sum\limits_{j=1}^{\mathfrak{r}} \mathfrak{q}_{i_j} z_{i_j}}{\sum\limits_{j=1}^{\mathfrak{r}} \mathfrak{q}_{i_j}}\right)\left(\frac{\sum\limits_{j=1}^{\mathfrak{r}} \mathfrak{q}_{i_j} z_{i_j}}{\sum\limits_{j=1}^{\mathfrak{r}} \mathfrak{q}_{i_j}}-\rho_1\right)^{n+1}}{n(n+1)(n+2)P_1(n)} \right.$$

$$\left. - \frac{\left(n\rho_2+2\frac{\sum\limits_{j=1}^{\mathfrak{r}} \mathfrak{q}_{i_j} z_{i_j}}{\sum\limits_{j=1}^{\mathfrak{r}} \mathfrak{q}_{i_j}}\right)\left(\frac{\sum\limits_{j=1}^{\mathfrak{r}} \mathfrak{q}_{i_j} z_{i_j}}{\sum\limits_{j=1}^{\mathfrak{r}} \mathfrak{q}_{i_j}}-\rho_2\right)^{n+1}}{n(n+1)(n+2)P_1(n)} \right]$$

$$= \mathbf{F}\left(z, \mathfrak{q}; \left(\frac{(n\rho_1+2z)(z-\rho_1)^{n+1}-(n\rho_2+2z)(z-\rho_2)^{n+1}}{n(n+1)(n+2)P_1(n)}\right)\right).$$

Method 2 for computing $\lambda_1(n)$: Now we will find the value of $\lambda_1(n)$ by employing identity (1.4) for $\mathfrak{f}(z)=\frac{z^{n+1}}{(n+1)!}$ as:

$$\mathbf{F}\left(z,\mathfrak{q};\frac{z^{n+1}}{(n+1)!}\right)-\frac{1}{(n+1)!}\sum_{w=2}^{n-1}\left(\frac{n-w}{w!(\rho_2-\rho_1)}\right)((n+1)n\cdots(n-w+3))$$

$$\times\left\{\rho_2^{(n-w+2)}\mathbf{F}\left(z,\mathfrak{q};(z-\rho_2)^w\right)-\rho_1^{(n-w+2)}\mathbf{F}\left(z,\mathfrak{q};(z-\rho_1)^w\right)\right\}$$

$$=\frac{1}{(n-1)!(\rho_2-\rho_1)}\int_{\rho_1}^{\rho_2} r\mathbf{F}\left(z,\mathfrak{q};(z-r)^{n-1}R^{[\rho_1,\rho_2]}(r,z)\right)dr. \tag{2.7}$$

By comparing (2.6) and (2.7), we get

$$\frac{P_1(n)}{(n-1)!(\rho_2-\rho_1)}\lambda_1(n)=\mathbf{F}\left(z,\mathfrak{q};\frac{z^{n+1}}{(n+1)!}\right)-\frac{1}{(n+1)!}\sum_{w=2}^{n-1}\left(\frac{n-w}{w!(\rho_2-\rho_1)}\right)$$

$$\times((n+1)n\cdots(n-w+3))$$

$$\times\left\{\rho_2^{(n-w+2)}\mathbf{F}\left(z,\mathfrak{q};(z-\rho_2)^w\right)-\rho_1^{(n-w+2)}\mathbf{F}\left(z,\mathfrak{q};(z-\rho_1)^w\right)\right\},$$

after simple calculations, we get

$$\begin{aligned}\lambda_1(n) = {} & \frac{(n-1)!(\rho_2-\rho_1)}{P_1(n)} \\ & \times \left[\mathbf{F}\left(z,q;\frac{z^{n+1}}{(n+1)!}\right) - \sum_{w=2}^{n-1}\left(\frac{n-w}{(n+1)!(n-w+2)(\rho_2-\rho_1)}\right)\right. \\ & \times \binom{n+1}{w}\left(\rho_2^{(n-w+2)}\mathbf{F}\left(z,q;(z-\rho_2)^w\right)\right. \\ & \left.\left. -\rho_1^{(n-w+2)}\mathbf{F}\left(z,q;(z-\rho_1)^w\right)\right)\right].\end{aligned}$$

Here we use the fact that

$$\frac{(n+1)n(n-1)\cdots(n-w+3)}{w!} = \frac{1}{(n-w+2)}\binom{n+1}{w}.$$

□

The following corollary is the special case of Theorem 6

Corollary 1. *Considering the assumptions of Theorem 6, then for a particular case when $n = 2$, if $\mathfrak{f}''(\cdot)$ (twice derivative of $\mathfrak{f}$) is convex and the weight function condition (2.1) is satisfied, then the following result holds:*

$$\begin{aligned}\frac{P_1(2)}{(\rho_2-\rho_1)}\mathfrak{f}''(\lambda_1(2)) &\le \mathbf{F}(z,q;\mathfrak{f}(z)) \\ &\le \frac{P_1(2)}{(\rho_2-\rho_1)}\left(\frac{\rho_2-\lambda_1(2)}{\rho_2-\rho_1}\mathfrak{f}''(\rho_1) + \frac{\lambda_1(2)-\rho_1}{\rho_2-\rho_1}\mathfrak{f}''(\rho_2)\right).\end{aligned}$$

Where

$$\begin{aligned}P_1(2) &= \int_{\rho_1}^{\rho_2}\mathbf{F}\left(z,q;(z-r)R^{[\rho_1,\rho_2]}(r,z)\right)dr \\ &= \mathbf{F}\left(z,q;\left(\frac{(z-\rho_1)^3-(z-\rho_2)^3}{6}\right)\right),\end{aligned}$$

or

$$P_1(2) = (\rho_2-\rho_1)\mathbf{F}\left(z,q;\frac{z^2}{2!}\right),$$

and

$$\begin{aligned}\lambda_1(2) &= \frac{\int_{\rho_1}^{\rho_2} r\mathbf{F}\left(z,q;(z-r)R^{[\rho_1,\rho_2]}(r,z)\right)dr}{\int_{\rho_1}^{\rho_2}\mathbf{F}\left(z,q;(z-r)R^{[\rho_1,\rho_2]}(r,z)\right)dr} \\ &= \mathbf{F}\left(z,q;\left(\frac{(2\rho_1+2z)(z-\rho_1)^3-(2\rho_2+2z)(z-\rho_2)^3}{24P_1(2)}\right)\right),\end{aligned}$$

or

$$\lambda_1(2) = (\rho_2 - \rho_1)P_1(2)\mathbf{F}\left(z, q; \frac{z^3}{3!}\right).$$

The following are the examples related to our main results given in Theorem 6:

Example 1. For even $(n \geq 4)$ the function $g(z) = (z-r)^{n-1} R^{[\rho_1,\rho_2]}(r,z)$ in Fink identity (1.4) is convex. We start with the case when $n = 4$, first we satisfy the following condition:

$$\mathbf{F}\left(z, q; (z-r)^3 R^{[\rho_1,\rho_2]}(r,z)\right) > 0, \quad r \in [\rho_1, \rho_2]$$

Now taking $[\rho_1, \rho_2] = [0.1, 1]$ and $z_1 = 0.2, z_2 = 0.3, z_3 = 0.4$ and $r = 0.35$. Since it is true for any positive p_i, here we choose $q_1 = q_2 = q_3 = \frac{1}{3}$.

We can rewrite the above functional for $\mathfrak{m} = 3$ and $\mathfrak{r} = 2$ as:

$$\begin{aligned}
&\mathbf{F}\left(z, q; (z-r)^3 R^{[a,b]}(r,z)\right) \\
&= q_1(z_1-r)^3(r-b) + q_2(z_2-r)^3(r-b) + q_3(z_3-r)^3(r-a) \\
&\quad + (q_1+q_2+q_3)\left(\frac{q_1z_1+q_2z_2+q_3z_3}{q_1+q_2+q_3} - r\right)^3 (r-b) \\
&\quad - \left[(q_2+q_3)\left(\frac{q_2z_2+q_3z_3}{q_2+q_3} - r\right)^3 (r-a) + (q_3+q_1)\left(\frac{q_3z_3+q_1z_1}{q_3+q_1} - r\right)^3 (r-b)\right. \\
&\quad \left. + (q_1+q_2)\left(\frac{q_1z_1+q_2z_2}{q_1+q_2} - r\right)^3 (r-b)\right].
\end{aligned}$$

By substituting the values and after simple calculations we get

$$= 0.0003625 > 0$$

For the function $f(z) = z^6$ where $f'(z) = 6z^5$, $f''(z) = 30z^4$, $f'''(z) = 120z^3$, $f''''(z) = 360z^2$. Now by writing (2.2) for $n = 4$, we get

$$\begin{aligned}
&\frac{P_1(4)}{(3)!\,(0.9)} f^{(4)}(\lambda_1(4)) \\
&\leq \mathbf{F}(z, q; f(z)) \\
&\quad - \left(\frac{1}{(0.9)}\right)\left(f'(1)\mathbf{F}\left(z, q; (z-1)^2\right) - f'(0.1)\mathbf{F}\left(z, q; (z-0.1)^2\right)\right) \\
&\quad - \left(\frac{1}{3!(0.9)}\right)\left(f''(1)\mathbf{F}\left(z, q; (z-1)^3\right) - f''(0.1)\mathbf{F}\left(z, q; (z-0.1)^3\right)\right) \\
&\leq \frac{P_1(4)}{(3)!\,(0.9)}\left(\frac{1-\lambda_1(4)}{0.9} f^{(4)}(0.1) + \frac{\lambda_1(4)-0.1}{0.9} f^{(4)}(1)\right), \qquad (2.8)
\end{aligned}$$

now by substituting the values, we get

$$0.015936 \leq 0.017155 \leq 0.023741.$$

Where

$$P_1(4) = \mathbf{F}\left(z, q; \left(\frac{(z-0.1)^5 - (z-1)^5}{20}\right)\right) = 0.000598,$$

and

$$\lambda_1(4) = \mathbf{F}\left(z, q; \left(\frac{(4(0.1)+2z)(z-0.1)^5 - (4+2z)(z-1)^5}{120P_1(4)}\right)\right) = 0.632173.$$

The following example is same as above example but we calculate $P_1(n)$ and $\lambda_1(n)$ with second method which give us same results.

Example 2. Assuming all the values taken in example 1 but here we calculate $P_1(n)$ and $\lambda_1(n)$ as given below:

$$\begin{aligned}
P_1(4) = (3)!\,(0.9) \Bigg[& \mathbf{F}\left(z, q; \frac{z^4}{4!}\right) \\
& - \frac{1}{4!}\left(\frac{1}{(0.9)}\right) \\
& \quad \times \left\{ \frac{d}{dz}(z^4)|_{z=1}\mathbf{F}\left(z, q; (z-1)^2\right) - \frac{d}{dz}(z^4)|_{z=0.1}\mathbf{F}\left(z, q; (z-0.1)^2\right) \right\} \\
& - \frac{1}{4!}\left(\frac{1}{3!(0.9)}\right) \\
& \quad \times \left\{ \frac{d^2}{dz^2}(z^4)|_{z=1}\mathbf{F}\left(z, q; (z-1)^3\right) - \frac{d^2}{dz^2}(z^4)|_{z=0.1}\mathbf{F}\left(z, q; (z-0.1)^3\right) \right\} \Bigg] \\
= 0.000598,
\end{aligned}$$

and

$$\begin{aligned}
\lambda_1(4) = \frac{(3)!\,(0.9)}{P_1(4)} \Bigg[& \mathbf{F}\left(z, q; \frac{z^5}{5!}\right) \\
& - \frac{1}{(5)!}\left(\frac{1}{(0.9)}\right) \\
& \quad \times \left\{ \frac{d}{dz}(z^5)|_{z=1}\mathbf{F}\left(z, q; (z-1)^2\right) - \frac{d}{dz}(z^5)|_{z=0.1}\mathbf{F}\left(z, q; (z-0.1)^2\right) \right\} \\
& - \frac{1}{(5)!}\left(\frac{1}{3!(0.9)}\right) \\
& \quad \times \left\{ \frac{d^2}{dz^2}(z^5)|_{z=1}\mathbf{F}\left(z, q; (z-1)^3\right) \frac{d^2}{dz^2}(z^5)|_{z=0.1}\mathbf{F}\left(z, q; (z-0.1)^3\right) \right\} \Bigg] \\
= 0.632560.
\end{aligned}$$

Now by substituting $P_1(4)$, $\lambda_1(4)$ and all other values in (2.8), we get

$$0.015952 \leq 0.017155 \leq 0.023753$$

3. Applications in Information Theory

Mathematically, the techniques and ideas that control message transmission through communication systems are covered by information theory. When Claude Shannon, an American electrical engineer, published a ground breaking article that transformed communication, the field received a significant boost. Using an Abel-Gontscharoff interpolating polynomial, Butt et al. in [9] have provided some interesting bounds for Shannon, relative, and Mandelbrot entropies.

First, we present some fundamental definitions and results concerned with this section. Csiszár in [11] presented the following divergence functional:

Definition 2 (Csiszár divergence)**.** Let $I=[\rho_1,\rho_2]\subset\mathbb{R}$ and $\mathfrak{f}\colon I\to\mathbb{R}$ be a function, then for $\mathfrak{q}=(\mathfrak{q}_1,\ldots,\mathfrak{q}_\mathfrak{m})\in\mathbb{R}^\mathfrak{m}$ and $\mathfrak{p}=(\mathfrak{p}_1,\ldots,\mathfrak{p}_\mathfrak{m})\in\mathbb{R}_+^\mathfrak{m}$ such that $\frac{\mathfrak{q}_i}{\mathfrak{p}_i}\in I(i=1,\ldots,\mathfrak{m})$, the Csiszár $\mathfrak{f}$-divergence functional is defined as:

$$\bar{D}_c(\mathfrak{q},\mathfrak{p})=\sum_{i=1}^{\mathfrak{m}}\mathfrak{p}_i\mathfrak{f}\left(\frac{\mathfrak{q}_i}{\mathfrak{p}_i}\right)$$

Definition 3 (Kullback-Leibler divergence)**.** Let $\mathfrak{q}$ and $\mathfrak{p}$ be two p.p.d, then Kullback-Leibler divergence is defined as

$$D_{kl}(\mathfrak{q},\mathfrak{p})=\sum_{i=1}^{\mathfrak{m}}\mathfrak{q}_i\ln\frac{\mathfrak{q}_i}{\mathfrak{p}_i}.$$

Definition 4 (Rényi-divergence)**.** Let $\mathfrak{q}$ and $\mathfrak{p}$ be two positive probability distributions (p.p.d) and μ be a real number which is nonnegative such that $\mu\neq 1$, the Rényi-divergence is defined as

$$D_{re}(\mathfrak{q},\mathfrak{p})=\frac{1}{\mu-1}\ln\left(\sum_{i=1}^{\mathfrak{m}}\mathfrak{q}_i^{\mu}\mathfrak{p}_i^{(1-\mu)}\right).$$

Definition 5 (Shannon-entropy)**.** The Shannon-entropy (information divergence) for a p.p.d $\mathfrak{p}$ is given as:

$$S(\mathfrak{p})=-\sum_{i=1}^{\mathfrak{m}}\mathfrak{p}_i\ln\mathfrak{p}_i. \tag{3.1}$$

Theorem 7. *Let the assumptions of Theorem 6 are valid and also let* $\mathfrak{q}=(\mathfrak{q}_1,\ldots,\mathfrak{q}_\mathfrak{m})\in\mathbb{R}^\mathfrak{m}$ *and* $\mathfrak{p}=(\mathfrak{p}_1,\ldots,\mathfrak{p}_\mathfrak{m})\in\mathbb{R}_+^\mathfrak{m}$ *such that* $\frac{\sum_{i=1}^{\mathfrak{m}}\mathfrak{q}_i}{\sum_{i=1}^{\mathfrak{m}}\mathfrak{p}_i}$, $\frac{\mathfrak{q}_i}{\mathfrak{p}_i}\in[\rho_1,\rho_2]$ *for* $i=$

$1,\dots,\mathfrak{m}$, *then*

$$\begin{aligned}
&\frac{P_1(n)}{(n-1)!\,(\rho_2-\rho_1)}\mathfrak{f}^{(n)}(\lambda_1(n))\\
&\leq \frac{\mathfrak{m}-\mathfrak{r}}{\mathfrak{m}-1}\bar{D}_c(\mathfrak{q},\mathfrak{p})+\frac{\mathfrak{r}-1}{\mathfrak{m}-1}\mathfrak{f}\left(\sum_{i=1}^{\mathfrak{m}}\mathfrak{q}_i\right)-\frac{1}{C_{\mathfrak{r}-1}^{\mathfrak{m}-1}}\sum_{1\leq i_1<\ldots<i_{\mathfrak{r}}\leq\mathfrak{m}}\left(\sum_{j=1}^{\mathfrak{r}}\mathfrak{p}_{i_j}\right)\mathfrak{f}\left(\frac{\sum_{j=1}^{\mathfrak{r}}\mathfrak{q}_{i_j}}{\sum_{j=1}^{\mathfrak{r}}\mathfrak{p}_{i_j}}\right)\\
&\quad-\sum_{w=2}^{n-1}\left(\frac{n-w}{w!(\rho_2-\rho_1)}\right)\\
&\qquad\times\left\{\mathfrak{f}^{(w-1)}(\rho_2)\mathbf{F}\left(\mathfrak{q},\frac{\mathfrak{q}_i}{\mathfrak{p}_i};(\mathfrak{z}-\rho_2)^w\right)-\mathfrak{f}^{(w-1)}(\rho_1)\mathbf{F}\left(\mathfrak{q},\frac{\mathfrak{q}_i}{\mathfrak{p}_i};(\mathfrak{z}-\rho_1)^w\right)\right\}\\
&\leq \frac{P_1(n)}{(n-1)!\,(\rho_2-\rho_1)}\left(\frac{\rho_2-\lambda_1(n)}{\rho_2-\rho_1}\mathfrak{f}^{(n)}(\rho_1)+\frac{\lambda_1(n)-\rho_1}{\rho_2-\rho_1}\mathfrak{f}^{(n)}(\rho_2)\right). \qquad (3.2)
\end{aligned}$$

Proof. First we rewrite the equivalent form of (2.2) as

$$\begin{aligned}
&\frac{P_1(n)}{(n-1)!\,(\rho_2-\rho_1)}\mathfrak{f}^{(n)}(\lambda_1(n))\\
&\leq \frac{\mathfrak{m}-\mathfrak{r}}{\mathfrak{m}-1}\sum_{i=1}^{\mathfrak{m}}\mathfrak{q}_i\mathfrak{f}(\mathfrak{z}_i)+\frac{\mathfrak{r}-1}{\mathfrak{m}-1}\mathfrak{f}\left(\sum_{i=1}^{\mathfrak{m}}\mathfrak{q}_i\mathfrak{z}_i\right)\\
&\quad-\frac{1}{C_{\mathfrak{r}-1}^{\mathfrak{m}-1}}\sum_{1\leq i_1<\ldots<i_{\mathfrak{r}}\leq\mathfrak{m}}\left(\sum_{j=1}^{\mathfrak{r}}\mathfrak{q}_{i_j}\right)\mathfrak{f}\left(\frac{\sum_{j=1}^{\mathfrak{r}}\mathfrak{q}_{i_j}\mathfrak{z}_{i_j}}{\sum_{j=1}^{\mathfrak{r}}\mathfrak{q}_{i_j}}\right)-\sum_{w=2}^{n-1}\left(\frac{n-w}{w!(\rho_2-\rho_1)}\right)\\
&\qquad\times\left\{\mathfrak{f}^{(w-1)}(\rho_2)\mathbf{F}(\mathfrak{q},\mathfrak{z}_i;(\mathfrak{z}-\rho_2)^w)-\mathfrak{f}^{(w-1)}(\rho_1)\mathbf{F}(\mathfrak{q},\mathfrak{z}_i;(\mathfrak{z}-\rho_1)^w)\right\}\\
&\leq \frac{P_1(n)}{(n-1)!\,(\rho_2-\rho_1)}\left(\frac{\rho_2-\lambda_1(n)}{\rho_2-\rho_1}\mathfrak{f}^{(n)}(\rho_1)+\frac{\lambda_1(n)-\rho_1}{\rho_2-\rho_1}\mathfrak{f}^{(n)}(\rho_2)\right),
\end{aligned}$$

now replace $\mathfrak{z}_i$ by $\frac{\mathfrak{q}_i}{\mathfrak{p}_i}$ and $\mathfrak{q}_i$ by $\mathfrak{p}_i$ after simple calculations we get the required result. □

Remark 1. Under the assumptions of Theorem 7, then for $n=2$ we have the following result

$$\frac{P_1(2)}{(\rho_2-\rho_1)}\mathfrak{f}''(\lambda_1(2))\leq\frac{\mathfrak{m}-\mathfrak{r}}{\mathfrak{m}-1}\bar{D}_c(\mathfrak{q},\mathfrak{p})+\frac{\mathfrak{r}-1}{\mathfrak{m}-1}\mathfrak{f}\left(\sum_{i=1}^{\mathfrak{m}}\mathfrak{q}_i\right)$$

$$-\frac{1}{C_{\mathfrak{r}-1}^{\mathfrak{m}-1}}\sum_{1\le i_1<\ldots<i_{\mathfrak{r}}\le\mathfrak{m}}\left(\sum_{j=1}^{\mathfrak{r}}\mathfrak{p}_{i_j}\right)\mathfrak{f}\left(\frac{\sum_{j=1}^{\mathfrak{r}}\mathfrak{q}_{i_j}}{\sum_{j=1}^{\mathfrak{r}}\mathfrak{p}_{i_j}}\right)$$
$$\le\frac{P_1(2)}{(\rho_2-\rho_1)}\left(\frac{\rho_2-\lambda_1(2)}{\rho_2-\rho_1}\mathfrak{f}''(\rho_1)+\frac{\lambda_1(2)-\rho_1}{\rho_2-\rho_1}\mathfrak{f}''(\rho_2)\right).$$

Corollary 2. *Let* $[\rho_1,\rho_2]$ *be a subset of the set of positive real numbers and* $\mathfrak{q}$, $\mathfrak{p}$ *be two p.p.d such that* $\frac{\mathfrak{q}_i}{\mathfrak{p}_i}\in[\rho_1,\rho_2]$ *for* $i=1,\ldots,\mathfrak{m}$, *then*

$$\begin{aligned}
&\frac{(-1)^nP_1(n)}{(n-1)(\rho_2-\rho_1)(\lambda_1(n))^{n-1}}\\
&\le\frac{\mathfrak{m}-\mathfrak{r}}{\mathfrak{m}-1}D_{kl}(\mathfrak{q},\mathfrak{p})+\frac{\mathfrak{r}-1}{\mathfrak{m}-1}\sum_{i=1}^{\mathfrak{m}}\mathfrak{q}_i\ln\left(\sum_{i=1}^{\mathfrak{m}}\mathfrak{q}_i\right)\\
&\quad-\frac{1}{C_{\mathfrak{r}-1}^{\mathfrak{m}-1}}\sum_{1\le i_1<\ldots<i_{\mathfrak{r}}\le\mathfrak{m}}\left(\sum_{j=1}^{\mathfrak{r}}\mathfrak{q}_{i_j}\right)\ln\left(\frac{\sum_{j=1}^{\mathfrak{r}}\mathfrak{q}_{i_j}}{\sum_{j=1}^{\mathfrak{r}}\mathfrak{p}_{i_j}}\right)-\sum_{w=2}^{n-1}\left(\frac{n-w}{w!(\rho_2-\rho_1)}\right)\\
&\quad\times\left\{(-1)^{w-1}\frac{(w-3)!}{{\rho_2}^{w-2}}\mathbf{F}\left(\mathfrak{q},\frac{\mathfrak{q}_i}{\mathfrak{p}_i};(z-\rho_2)^w\right)\right.\\
&\quad\left.-(-1)^{w-1}\frac{(w-3)!}{{\rho_1}^{w-2}}\mathbf{F}\left(\mathfrak{q},\frac{\mathfrak{q}_i}{\mathfrak{p}_i};(z-\rho_1)^w\right)\right\}\\
&\le\frac{P_1(n)}{(n-1)(\rho_2-\rho_1)}\left(\frac{(-1)^n(\rho_2-\lambda_1(n))}{{\rho_1}^{n-1}(\rho_2-\rho_1)}+\frac{(-1)^n(\lambda_1(n)-\rho_1)}{{\rho_2}^{n-1}(\rho_2-\rho_1)}\right).
\end{aligned}\tag{3.3}$$

Proof. Let $\mathfrak{f}(z)=z\ln z$, $z\in[\rho_1,\rho_2]$ then for even n, $\mathfrak{f}^{(n)}(z)=(-1)^n\frac{(n-2)!}{z^{n-1}}>0$ and $\mathfrak{f}^{(n+2)}(z)=(-1)^{n+2}\frac{n!}{z^{n+1}}>0$. It follows that $\mathfrak{f}^{(n)}$ and $\mathfrak{f}^{(n+2)}$ are convex functions, therefore using (3.2) for $\mathfrak{f}(z)=z\ln z$, we derive (3.3).

□

Remark 2. Under the assumptions of Corollary 2, then for $n=2$ we have the following result

$$\begin{aligned}
\frac{P_1(2)}{(\rho_2-\rho_1)(\lambda_1(2))}&\le\frac{\mathfrak{m}-\mathfrak{r}}{\mathfrak{m}-1}D_{kl}(\mathfrak{q},\mathfrak{p})+\frac{\mathfrak{r}-1}{\mathfrak{m}-1}\sum_{i=1}^{\mathfrak{m}}\mathfrak{q}_i\ln\left(\sum_{i=1}^{\mathfrak{m}}\mathfrak{q}_i\right)\\
&\quad-\frac{1}{C_{\mathfrak{r}-1}^{\mathfrak{m}-1}}\sum_{1\le i_1<\ldots<i_{\mathfrak{r}}\le\mathfrak{m}}\left(\sum_{j=1}^{\mathfrak{r}}\mathfrak{q}_{i_j}\right)\ln\left(\frac{\sum_{j=1}^{\mathfrak{r}}\mathfrak{q}_{i_j}}{\sum_{j=1}^{\mathfrak{r}}\mathfrak{p}_{i_j}}\right)
\end{aligned}$$

$$\leq \frac{P_1(2)}{(\rho_2-\rho_1)}\left(\frac{(\rho_2-\lambda_1(2))}{\rho_1(\rho_2-\rho_1)}+\frac{(\lambda_1(2)-\rho_1)}{\rho_2(\rho_2-\rho_1)}\right).$$

Corollary 3. *Let* $[\rho_1,\rho_2]\subseteq\mathbb{R}^+$. *Also let* $\mathfrak{q}$, $\mathfrak{p}$ *be two p.p.d and* $\mu>1$ *such that* $\sum_{i=1}^{\mathfrak{m}} \mathfrak{q}_i(\frac{\mathfrak{q}_i}{\mathfrak{p}_i})^{\mu}$, $(\frac{\mathfrak{q}_i}{\mathfrak{p}_i})^{\mu-1}\in[\rho_1,\rho_2]$ *for* $i=1,\ldots,\mathfrak{m}$, *then*

$$\begin{aligned}
&\frac{(-1)^n P_1(n)}{(\mu-1)\,(\rho_2-\rho_1)\,(\lambda_1(n))^n}\\
&\leq \frac{\mathfrak{m}-\mathfrak{r}}{\mathfrak{m}-1}\sum_{i=1}^{\mathfrak{m}}\mathfrak{q}_i\left(\frac{-1}{\mu-1}\ln\left(\frac{\mathfrak{q}_i}{\mathfrak{p}_i}\right)^{\mu-1}\right)-\frac{\mathfrak{r}-1}{\mathfrak{m}-1}D_{re}(\mathfrak{q},\mathfrak{p})\\
&\quad+\frac{1}{C_{\mathfrak{r}-1}^{\mathfrak{m}-1}}\sum_{1\leq i_1<\ldots<i_{\mathfrak{r}}\leq\mathfrak{m}}\left(\sum_{j=1}^{\mathfrak{r}}\mathfrak{q}_{i_j}\right)\ln\left(\frac{\sum_{j=1}^{\mathfrak{r}}\mathfrak{q}_{i_j}^{\mu}\mathfrak{p}_{i_j}{}^{1-\mu}}{\sum_{j=1}^{\mathfrak{r}}\mathfrak{q}_{i_j}}\right)\\
&\quad-\sum_{w=2}^{n-1}\left(\frac{n-w}{w(w-1)(\mu-1)(\rho_2-\rho_1)}\right)\\
&\qquad\times\left\{\left(\frac{(-1)^{w-1}}{\rho_2^{w-1}}\right)\mathbf{F}\left(\mathfrak{q},\left(\frac{\mathfrak{q}_i}{\mathfrak{p}_i}\right)^{\mu-1};(z-\rho_2)^w\right)\right.\\
&\qquad\left.-\left(\frac{(-1)^{w-1}}{\rho_1^{w-1}}\right)\mathbf{F}\left(\mathfrak{q},\left(\frac{\mathfrak{q}_i}{\mathfrak{p}_i}\right)^{\mu-1};(z-\rho_1)^w\right)\right\}\\
&\leq \frac{P_1(n)}{(\mu-1)\,(\rho_2-\rho_1)}\left(\frac{(-1)^n(\rho_2-\lambda_1(n))}{\rho_1^n(\rho_2-\rho_1)}+\frac{(-1)^n(\lambda_1(n)-\rho_1)}{\rho_2^n(\rho_2-\rho_1)}\right). \qquad (3.4)
\end{aligned}$$

Proof. Let $z\in[\rho_1,\rho_2]$, $\mathfrak{f}(z)=-\frac{1}{\mu-1}\ln(z)$ then for even n, $\mathfrak{f}^{(n)}(z)=(-1)^n\frac{(n-1)!}{(\mu-1)z^n}>0$ and $\mathfrak{f}^{(n+2)}(z)=(-1)^{(n+2)}\frac{(n+1)!}{(\mu-1)z^{n+2}}>0$. Which grantee that the functions $\mathfrak{f}^{(n)}$ and $\mathfrak{f}^{(n+2)}$ are convex, now using (2.2) for $\mathfrak{f}(z)=-\frac{1}{\mu-1}\ln(z)$ and $z_i=(\frac{\mathfrak{q}_i}{\mathfrak{p}_i})^{\mu-1}$, we derive (3.4). □

Remark 3. Under the assumptions of Corollary 3, then for $n=2$ we have the following result

$$\begin{aligned}
&\frac{P_1(2)}{(\mu-1)\,(\rho_2-\rho_1)\,(\lambda_1(2))^2}\\
&\leq \frac{\mathfrak{m}-\mathfrak{r}}{\mathfrak{m}-1}\sum_{i=1}^{\mathfrak{m}}\mathfrak{q}_i\left(\frac{-1}{\mu-1}\ln\left(\frac{\mathfrak{q}_i}{\mathfrak{p}_i}\right)^{\mu-1}\right)-\frac{\mathfrak{r}-1}{\mathfrak{m}-1}D_{re}(\mathfrak{q},\mathfrak{p})
\end{aligned}$$

$$+\frac{1}{C_{\mathfrak{r}-1}^{\mathfrak{m}-1}}\sum_{1\le i_1<\ldots<i_{\mathfrak{r}}\le\mathfrak{m}}\left(\sum_{j=1}^{\mathfrak{r}}\mathfrak{q}_{i_j}\right)\ln\left(\frac{\sum_{j=1}^{\mathfrak{r}}\mathfrak{q}_{i_j}^{\mu}\mathfrak{p}_{i_j}{}^{1-\mu}}{\sum_{j=1}^{\mathfrak{r}}\mathfrak{q}_{i_j}}\right)$$
$$\le\frac{P_1(n)}{(\mu-1)(\rho_2-\rho_1)}\left(\frac{(\rho_2-\lambda_1(2))}{\rho_1^2(\rho_2-\rho_1)}+\frac{(\lambda_1(2)-\rho_1)}{\rho_2^2(\rho_2-\rho_1)}\right).$$

Corollary 4. *Let* $[\rho_1,\rho_2]\subseteq\mathbb{R}^+$ *and* $\mathfrak{p}$ *be a p.p.d such that* $\frac{1}{\mathfrak{p}_i}\in[\rho_1,\rho_2]$ *for* $i=1,\ldots,\mathfrak{m}$, *then*

$$\begin{aligned}
&\frac{(-1)^nP_1(n)}{(\rho_2-\rho_1)(\lambda_1(n))^n}\\
&\le\frac{\mathfrak{m}-\mathfrak{r}}{\mathfrak{m}-1}S(\mathfrak{p})-\frac{\mathfrak{r}-1}{\mathfrak{m}-1}\ln(\mathfrak{m})-\frac{1}{C_{\mathfrak{r}-1}^{\mathfrak{m}-1}}\sum_{1\le i_1<\ldots<i_{\mathfrak{r}}\le\mathfrak{m}}\left(\sum_{j=1}^{\mathfrak{r}}\mathfrak{p}_{i_j}\right)\ln\left(\sum_{j=1}^{\mathfrak{r}}\mathfrak{p}_{i_j}\right)\\
&\quad-\sum_{w=2}^{n-1}\left(\frac{n-w}{w(w-1)(\rho_2-\rho_1)}\right)\\
&\qquad\times\left\{\left(\frac{(-1)^{w-1}}{\rho_2^{w-1}}\right)\mathbf{F}\left(\mathfrak{q},\frac{1}{\mathfrak{p}_i};(z-\rho_2)^w\right)-\left(\frac{(-1)^{w-1}}{\rho_1^{w-1}}\right)\mathbf{F}\left(\mathfrak{q},\frac{1}{\mathfrak{p}_i};(z-\rho_1)^w\right)\right\}\\
&\le\frac{P_1(n)}{(\rho_2-\rho_1)}\left(\frac{(-1)^n(\rho_2-\lambda_1(n))}{\rho_1^n(\rho_2-\rho_1)}+\frac{(-1)^n(\lambda_1(n)-\rho_1)}{\rho_2^n(\rho_2-\rho_1)}\right).
\end{aligned}\tag{3.5}$$

Proof. Let $\mathfrak{f}(z)=-\ln(z)$, $z\in[\rho_1,\rho_2]$ then for even n, $\mathfrak{f}^{(n)}(z)=(-1)^n\frac{(n-1)!}{z^n}>0$ and $\mathfrak{f}^{(n+2)}(z)=(-1)^{n+2}\frac{(n+1)!}{z^{n+2}}>0$. It follows that the functions $\mathfrak{f}^{(n)}$ and $\mathfrak{f}^{(n+2)}$ are convex, therefore using (3.2) for $\mathfrak{f}(z)=-ln(z)$, and $(\mathfrak{q}_1,\ldots,\mathfrak{q}_{\mathfrak{m}})=(1,1,\ldots,1)$, we derive (3.5).

□

Remark 4. Under the assumptions of Corollary 4, then for $n=2$ we have the following result

$$\begin{aligned}
\frac{P_1(2)}{(\rho_2-\rho_1)(\lambda_1(2))^2}&\le\frac{\mathfrak{m}-\mathfrak{r}}{\mathfrak{m}-1}S(\mathfrak{p})-\frac{\mathfrak{r}-1}{\mathfrak{m}-1}\ln(\mathfrak{m})\\
&\quad-\frac{1}{C_{\mathfrak{r}-1}^{\mathfrak{m}-1}}\sum_{1\le i_1<\ldots<i_{\mathfrak{r}}\le\mathfrak{m}}\left(\sum_{j=1}^{\mathfrak{r}}\mathfrak{p}_{i_j}\right)\ln\left(\sum_{j=1}^{\mathfrak{r}}\mathfrak{p}_{i_j}\right)\\
&\le\frac{P_1(2)}{(\rho_2-\rho_1)}\left(\frac{(\rho_2-\lambda_1(2))}{\rho_1^2(\rho_2-\rho_1)}+\frac{(\lambda_1(2)-\rho_1)}{\rho_2^2(\rho_2-\rho_1)}\right).
\end{aligned}$$

4. Results for Zipf and hybrid Zipf-Mandelbrot entropy

One of the basic laws in information science is Zipf's law (see [20, 22]). It has many interesting applications in linguistics. Let c,d be the real numbers such that $c \in [0,\infty)$ and $d \in (0,\infty)$ and N be natural numbers, then the Zipf-Mandelbrot entropy can be given as:

$$Z_M(H,c,d) = \frac{d}{H^N_{c,d}} \sum_{i=1}^{N} \frac{\ln(i+c)}{(i+c)^d} + \ln(H^N_{c,d}) \tag{4.1}$$

where

$$H^N_{c,d} = \sum_{\sigma=1}^{N} \frac{1}{(\sigma+c)^d}.$$

Consider

$$\mathfrak{p}_i = \mathfrak{f}(i;N,c,d) = \frac{1}{((i+c)^d H^N_{c,d})}, \tag{4.2}$$

where (4.2) is a discrete probability distribution known as Zipf-Mandelbrot law. There are many applications of Zipf-Mandelbrot law in linguistic and information sciences. One can find recent results about Zipf-Mandelbrot in (see [6, 13, 14]).

Theorem 8. *Let $\mathfrak{p}$ be a discrete probability distribution given in (4.2) with parameters $0 \le c < \infty$, $0 < d < \infty$ and N be natural numbers, we have the following result*

$$\begin{aligned}
&\frac{(-1)^n P_1(n)}{(\rho_2-\rho_1)(\lambda_1(n))^n} \\
&\le \frac{\mathfrak{m}-\mathfrak{r}}{\mathfrak{m}-1} Z_M(H,c,d) = S(\mathfrak{p}) - \frac{\mathfrak{r}-1}{\mathfrak{m}-1}\ln(\mathfrak{m}) \\
&\quad - \frac{1}{C^{\mathfrak{m}-1}_{\mathfrak{r}-1}} \sum_{1\le i_1<\ldots<i_{\mathfrak{r}}\le \mathfrak{m}} \left(\sum_{j=1}^{\mathfrak{r}} \mathfrak{p}_{i_j}\right) \ln\left(\sum_{j=1}^{\mathfrak{r}} \mathfrak{p}_{i_j}\right) \\
&\quad - \sum_{w=2}^{n-1} \left(\frac{n-w}{w(w-1)(\rho_2-\rho_1)}\right) \\
&\qquad \times \left\{ \left(\frac{(-1)^{w-1}}{\rho_2^{w-1}}\right) \mathbf{F}\left(\mathfrak{q}, ((i+c)^d H^N_{c,d}); (\mathfrak{z}-\rho_2)^w\right) \right. \\
&\qquad \left. - \left(\frac{(-1)^{w-1}}{\rho_1^{w-1}}\right) \mathbf{F}\left(\mathfrak{q}, ((i+c)^d H^N_{c,d}); (\mathfrak{z}-\rho_1)^w\right) \right\} \\
&\le \frac{P_1(n)}{(\rho_2-\rho_1)} \left(\frac{(-1)^n(\rho_2-\lambda_1(n))}{\rho_1^n(\rho_2-\rho_1)} + \frac{(-1)^n(\lambda_1(n)-\rho_1)}{\rho_2^n(\rho_2-\rho_1)}\right).
\end{aligned}$$

Proof. We can obtain Mandelbrot entropy (4.1) from Shannon entropy by substituting $\mathfrak{p}_i$ in (3.1).

$$\begin{aligned} S(\mathfrak{p}) &= -\sum_{i=1}^{N} \mathfrak{p}_i \ln \mathfrak{p}_i \\ &= -\sum_{i=1}^{N} \frac{1}{((i+c)^d H_{c,d}^N)} \ln \frac{1}{((i+c)^d H_{c,d}^N)} \\ &= \frac{d}{(H_{c,d}^N)} \sum_{i=1}^{N} \frac{\ln(i+c)}{(i+c)^d} + \ln(H_{c,d}^N). \end{aligned}$$

Finally, substituting this $\mathfrak{p}_i = \frac{1}{((i+c)^d H_{c,d}^N)}$ in (3.5), we get the required result. □

Remark 5. Under the assumptions of Theorem 8, then for $n = 2$ we have the following result

$$\begin{aligned} \frac{P_1(2)}{(\rho_2-\rho_1)(\lambda_1(2))^2} &\leq \frac{\mathfrak{m}-\mathfrak{r}}{\mathfrak{m}-1} Z_M(H,c,d) = S(\mathfrak{p}) \\ &\quad - \frac{\mathfrak{r}-1}{\mathfrak{m}-1} \ln(\mathfrak{m}) - \frac{1}{C_{\mathfrak{r}-1}^{\mathfrak{m}-1}} \sum_{1\leq i_1<\ldots<i_{\mathfrak{r}}\leq \mathfrak{m}} \left(\sum_{j=1}^{\mathfrak{r}} \mathfrak{p}_{i_j} \right) \ln \left(\sum_{j=1}^{\mathfrak{r}} \mathfrak{p}_{i_j} \right) \\ &\leq \frac{P_1(2)}{(\rho_2-\rho_1)} \left(\frac{(\rho_2-\lambda_1(2))}{\rho_1^2(\rho_2-\rho_1)} + \frac{(\lambda_1(2)-\rho_1)}{\rho_2^2(\rho_2-\rho_1)} \right). \end{aligned}$$

Corollary 5. *Let Zipf-Mandelbrot law defined as* $\mathfrak{q}$ *and* $\mathfrak{p}$ *with parameters* $c_1, c_2 \geq 0$, $d_1, d_2 > 0$, *let* $H_{c_1,d_1}^N = \sum_{\sigma=1}^{m} \frac{1}{(\sigma+c_1)^{d_1}}$ *and* $H_{c_2,d_2}^N = \sum_{\sigma=1}^{m} \frac{1}{(\sigma+c_2)^{d_2}}$. *Now using* $\mathfrak{p}_i = \frac{1}{(i+c_1)^{d_1} H_{c_1,d_1}^N}$ *and* $\mathfrak{q}_i = \frac{1}{(i+c_2)^{d_2} H_{c_2,d_2}^N}$ *in* (3.3), *then the following holds*

$$\begin{aligned} \frac{(-1)^n P_1(n)}{(n-1)(\rho_2-\rho_1)(\lambda_1(n))^{n-1}} &\leq \frac{\mathfrak{m}-\mathfrak{r}}{\mathfrak{m}-1} D_{kl}(\mathfrak{q},\mathfrak{p}) + \frac{\mathfrak{r}-1}{\mathfrak{m}-1} \sum_{i=1}^{\mathfrak{m}} \mathfrak{q}_i \ln \left(\sum_{i=1}^{\mathfrak{m}} \mathfrak{q}_i \right) \\ &- \frac{1}{C_{\mathfrak{r}-1}^{\mathfrak{m}-1}} \sum_{1\leq i_1<\ldots<i_{\mathfrak{r}}\leq \mathfrak{m}} \left(\sum_{j=1}^{\mathfrak{r}} \mathfrak{q}_{i_j} \right) \ln \left(\frac{\sum_{j=1}^{\mathfrak{r}} \mathfrak{q}_{i_j}}{\sum_{j=1}^{\mathfrak{r}} \mathfrak{p}_{i_j}} \right) - \sum_{w=2}^{n-1} \left(\frac{n-w}{w!(\rho_2-\rho_1)} \right) \\ &\times \left\{ (-1)^{w-1} \frac{(w-3)!}{{\rho_2}^{w-2}} \mathbf{F} \left(\mathfrak{q}, \frac{(i+c_1)^{d_1} H_{c_1,d_1}^N}{(i+c_2)^{d_2} H_{c_2,d_2}^N}; (z-\rho_2)^w \right) \right. \\ &\left. -(-1)^{w-1} \frac{(w-3)!}{{\rho_1}^{w-2}} \mathbf{F} \left(\mathfrak{q}, \frac{(i+c_1)^{d_1} H_{c_1,d_1}^N}{(i+c_2)^{d_2} H_{c_2,d_2}^N}; (z-\rho_1)^w \right) \right\} \end{aligned}$$

$$\leq \frac{P_1(n)}{(n-1)(\rho_2-\rho_1)}\left(\frac{(-1)^n(\rho_2-\lambda_1(n))}{{\rho_1}^{n-1}(\rho_2-\rho_1)}+\frac{(-1)^n(\lambda_1(n)-\rho_1)}{{\rho_2}^{n-1}(\rho_2-\rho_1)}\right).$$

Remark 6. Under the assumption of Corollary 5, then for $n=2$ we have the following result

$$\begin{aligned}\frac{P_1(2)}{(\rho_2-\rho_1)(\lambda_1(2))} &\leq \frac{\mathfrak{m}-\mathfrak{r}}{\mathfrak{m}-1}D_{kl}(\mathfrak{q},\mathfrak{p})+\frac{\mathfrak{r}-1}{\mathfrak{m}-1}\sum_{i=1}^{\mathfrak{m}}\mathfrak{q}_i\ln\left(\sum_{i=1}^{\mathfrak{m}}\mathfrak{q}_i\right)\\ &\quad-\frac{1}{C_{\mathfrak{r}-1}^{\mathfrak{m}-1}}\sum_{1\leq i_1<\ldots<i_{\mathfrak{r}}\leq\mathfrak{m}}\left(\sum_{j=1}^{\mathfrak{r}}\mathfrak{q}_{i_j}\right)\ln\left(\frac{\sum_{j=1}^{\mathfrak{r}}\mathfrak{q}_{i_j}}{\sum_{j=1}^{\mathfrak{r}}\mathfrak{p}_{i_j}}\right)\\ &\leq \frac{P_1(2)}{(\rho_2-\rho_1)}\left(\frac{(\rho_2-\lambda_1(2))}{\rho_1(\rho_2-\rho_1)}+\frac{(\lambda_1(2)-\rho_1)}{\rho_2(\rho_2-\rho_1)}\right).\end{aligned}$$

Now we will give result for Hybrid zipf-Mandelbrot entropy.

Hybrid Zipf-Mandelbrot entropy is the generalization of Zipf-Mandelbrot entropy. Consider N be natural numbers, $0\leq c<\infty$ and $0<d,\omega<\infty$.

Hybrid Zipf-Mandelbrot entropy is defined as:

$$\widehat{Z}_M(H^*,c,d,\omega)=\frac{1}{H^*_{c,d,\omega}}\sum_{i=1}^{N}\frac{\omega^i}{(i+c)^s}\ln\left(\frac{(i+c)^s}{\omega^i}\right)+\ln(H^*_{c,d,\omega}). \tag{4.3}$$

Where

$$H^*_{c,d,\omega}=\sum_{i=1}^{N}\frac{\omega^i}{(i+c)^d}$$

Consider

$$\mathfrak{p}_i=f(i;N,c,d,\omega)=\frac{\omega^i}{(i+c)^d H^*_{c,d,\omega}}. \tag{4.4}$$

is known as Hybrid Zipf-Mandelbrot law.

Theorem 9. *Let* $\mathbf{p}$ *be a discrete probability distribution given in* (4.4) *with parameters* $0\leq c<\infty$, $0<d,\omega<\infty$ *and* N *be natural numbers, then we have*

$$\begin{aligned}&\frac{(-1)^n P_1(n)}{(\rho_2-\rho_1)(\lambda_1(n))^n}\leq\frac{\mathfrak{m}-\mathfrak{r}}{\mathfrak{m}-1}S(\mathfrak{p})-\frac{\mathfrak{r}-1}{\mathfrak{m}-1}\ln(\mathfrak{m})\\ &-\frac{1}{C_{\mathfrak{r}-1}^{\mathfrak{m}-1}}\sum_{1\leq i_1<\ldots<i_{\mathfrak{r}}\leq\mathfrak{m}}\left(\sum_{j=1}^{\mathfrak{r}}\mathfrak{p}_{i_j}\right)\ln\left(\sum_{j=1}^{\mathfrak{r}}\mathfrak{p}_{i_j}\right)-\sum_{w=2}^{n-1}\left(\frac{n-w}{w(w-1)(\rho_2-\rho_1)}\right)\\ &\times\left\{\left(\frac{(-1)^{w-1}}{\rho_2^{w-1}}\right)\mathbf{F}\left(\mathfrak{q},\frac{(i+c)^d H^*_{c,d,\omega}}{\omega^i};(z-\rho_2)^w\right)\right.\end{aligned}$$

$$-\left(\frac{(-1)^{w-1}}{\rho_1^{w-1}}\right)\mathbf{F}\left(\mathfrak{q},\frac{(i+c)^d H^*_{c,d,\omega}}{\omega^i};(\mathfrak{z}-\rho_1)^w\right)\Bigg\}$$
$$\leq \frac{P_1(n)}{(\rho_2-\rho_1)}\left(\frac{(-1)^n(\rho_2-\lambda_1(n))}{\rho_1^n(\rho_2-\rho_1)}+\frac{(-1)^n(\lambda_1(n)-\rho_1)}{\rho_2^n(\rho_2-\rho_1)}\right). \tag{4.5}$$

Proof. We can get Hybrid Zipf-Mandelbrot entropy (4.3) by substituting $\mathfrak{p}_i$ defined in (4.4) from Shannon entropy (3.1).

$$\begin{aligned}
S(\mathbf{p}) &= -\sum_{i=1}^{N}\mathfrak{p}_i\ln\mathfrak{p}_i\\
&= -\sum_{i=1}^{N}\frac{\omega^i}{(i+c)^d H^*_{c,d,\omega}}\ln\frac{\omega^i}{(i+c)^d H^*_{c,d,\omega}}\\
&= \frac{-1}{H^*_{c,d,\omega}}\sum_{i=1}^{N}\frac{\omega^i}{(i+c)^d}\left[\ln\left(\frac{\omega^i}{(i+c)^d}\right)+\ln\left(\frac{1}{H^*_{c,d,\omega}}\right)\right]\\
&= \frac{1}{H^*_{c,d,\omega}}\sum_{i=1}^{N}\frac{\omega^i}{(i+c)^d}\left[\ln\left(\frac{(i+c)^d}{\omega^i}\right)+\ln\left(H^*_{c,d,\omega}\right)\right]\\
&= \frac{1}{H^*_{c,d,\omega}}\sum_{i=1}^{N}\frac{\omega^i}{(i+c)^d}\ln\left(\frac{(i+c)^d}{\omega^i}\right)+\ln\left(H^*_{c,d,\omega}\right).
\end{aligned}$$

Finally, substituting this $\mathfrak{p}_i=\frac{\omega^i}{(i+c)^d H^*_{c,d,\omega}}$ in (3.5), after simple calculations we get (4.5). □

Remark 7. Under the assumption of Theorem 9, then for $n=2$ we have the following result

$$\begin{aligned}
\frac{P_1(2)}{(\rho_2-\rho_1)(\lambda_1(2))} &\leq \frac{\mathfrak{m}-\mathfrak{r}}{\mathfrak{m}-1}S(\mathfrak{p})-\frac{\mathfrak{r}-1}{\mathfrak{m}-1}\ln(\mathfrak{m})\\
&\quad -\frac{1}{C^{\mathfrak{m}-1}_{\mathfrak{r}-1}}\sum_{1\leq i_1<\ldots<i_{\mathfrak{r}}\leq\mathfrak{m}}\left(\sum_{j=1}^{\mathfrak{r}}\mathfrak{p}_{i_j}\right)\ln\left(\sum_{j=1}^{\mathfrak{r}}\mathfrak{p}_{i_j}\right)\\
&\leq \frac{P_1(2)}{(\rho_2-\rho_1)}\left(\frac{(\rho_2-\lambda_1(2))}{\rho_1^2(\rho_2-\rho_1)}+\frac{(\lambda_1(2)-\rho_1)}{\rho_2^2(\rho_2-\rho_1)}\right).
\end{aligned}$$

Corollary 6. *Let Hybrid Zipf-Mandelbrot law defined as* $\mathbf{q}$ *and* $\mathbf{p}$ *with parameters* $c_1,c_2\geq 0$, $\omega_1,\omega_2,d_1,d_2>0$. *Now using* $\mathfrak{p}_i=\dfrac{\omega_1^i}{(i+c_1)^{d_1}H^*_{c_1,d_1,\omega_1}}$ *and* $\mathfrak{q}_i=$

$\frac{\omega_2^i}{(i+c_2)^{d_2}H^*_{c_2,d_2,\omega_2}}$ *in* (3.3)*, then the following holds*

$$\frac{(-1)^n P_1(n)}{(n-1)(\rho_2-\rho_1)(\lambda_1(n))^{n-1}} \le \frac{\mathfrak{m}-\mathfrak{r}}{\mathfrak{m}-1} D_{kl}(\mathfrak{q},\mathfrak{p}) + \frac{\mathfrak{r}-1}{\mathfrak{m}-1}\sum_{i=1}^{\mathfrak{m}} \mathfrak{q}_i \ln\left(\sum_{i=1}^{\mathfrak{m}} \mathfrak{q}_i\right)$$

$$-\frac{1}{C_{\mathfrak{r}-1}^{\mathfrak{m}-1}} \sum_{1\le i_1<\ldots<i_{\mathfrak{r}}\le \mathfrak{m}} \left(\sum_{j=1}^{\mathfrak{r}} \mathfrak{q}_{i_j}\right) \ln\left(\frac{\sum_{j=1}^{\mathfrak{r}} \mathfrak{q}_{i_j}}{\sum_{j=1}^{\mathfrak{r}} \mathfrak{p}_{i_j}}\right) - \sum_{w=2}^{n-1}\left(\frac{n-w}{w!(\rho_2-\rho_1)}\right)$$

$$\times\left\{(-1)^{w-1}\frac{(w-3)!}{{\rho_2}^{w-2}}\mathbf{F}\left(\mathfrak{q}, \frac{\omega_2^i(i+c_1)^{d_1}H^*_{c_1,d_1,\omega_1}}{\omega_1^i(i+c_2)^{d_2}H^*_{c_2,d_2,\omega_2}}; (\mathfrak{z}-\rho_2)^w\right)\right.$$

$$\left.-(-1)^{w-1}\frac{(w-3)!}{{\rho_1}^{w-2}}\mathbf{F}\left(\mathfrak{q}, \frac{\omega_2^i(i+c_1)^{d_1}H^*_{c_1,d_1,\omega_1}}{\omega_1^i(i+c_2)^{d_2}H^*_{c_2,d_2,\omega_2}}; (\mathfrak{z}-\rho_1)^w\right)\right\}$$

$$\le \frac{P_1(n)}{(n-1)(\rho_2-\rho_1)}\left(\frac{(-1)^n(\rho_2-\lambda_1(n))}{{\rho_1}^{n-1}(\rho_2-\rho_1)} + \frac{(-1)^n(\lambda_1(n)-\rho_1)}{{\rho_2}^{n-1}(\rho_2-\rho_1)}\right).$$

Remark 8. Under the assumption of Corollary 6, then for $n=2$ we have the following result

$$\frac{P_1(2)}{(\rho_2-\rho_1)(\lambda_1(2))} \le \frac{\mathfrak{m}-\mathfrak{r}}{\mathfrak{m}-1} D_{kl}(\mathfrak{q},\mathfrak{p}) + \frac{\mathfrak{r}-1}{\mathfrak{m}-1}\sum_{i=1}^{\mathfrak{m}} \mathfrak{q}_i \ln\left(\sum_{i=1}^{\mathfrak{m}} \mathfrak{q}_i\right)$$

$$-\frac{1}{C_{\mathfrak{r}-1}^{\mathfrak{m}-1}} \sum_{1\le i_1<\ldots<i_{\mathfrak{r}}\le \mathfrak{m}} \left(\sum_{j=1}^{\mathfrak{r}} \mathfrak{q}_{i_j}\right) \ln\left(\frac{\sum_{j=1}^{\mathfrak{r}} \mathfrak{q}_{i_j}}{\sum_{j=1}^{\mathfrak{r}} \mathfrak{p}_{i_j}}\right)$$

$$\le \frac{P_1(2)}{(\rho_2-\rho_1)}\left(\frac{(\rho_2-\lambda_1(2))}{\rho_1(\rho_2-\rho_1)} + \frac{(\lambda_1(2)-\rho_1)}{\rho_2(\rho_2-\rho_1)}\right).$$

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ON THE WEIGHTED MIDPOINT TYPE INEQUALITIES FOR CONVEX FUNCTIONS

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Abstract. The main object of this paper is to present some weighted midpoint type inequalities for convex functions.

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1. INTRODUCTION

The usefulness of inequalities involving convex functions is realized from the very beginning and is now widely acknowledged as one of the prime driving forces behind the development of several modern branches of mathematics and has been given considerable attention. Some famous results for such estimations consist of Hermite-Hadamard, trapezoid, midpoint, Simpson or Jensen inequalities etc.

Let $f: I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping defined on the interval I of real numbers and $a,b \in I$, with $a < b$. The following double inequalities are well known in the literature as the Hermite-Hadamard inequalities [5]:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

The most well-known inequalities related to the integral mean of a convex function are the Hermite-Hadamard inequalities. It gives an estimate from both sides of the mean value of a convex function and also ensure the integrability of a convex function. It is also a matter of great interest and one has to note that some of the classical inequalities for means can be obtained from Hadamard's inequality under the utility of peculiar convex functions. These inequalities for convex functions play a crucial role in analysis and as well as in other areas of pure and applied mathematics. The absolute value of the difference of the second part of the (1.1) inequalities is known as the trapezoidal inequality in the literature and was given by Dragomir and Agarwal

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in 1998 [2]. Then, in 2004, the absolute value of the difference of the first part of the (1.1) inequalities, known as the midpoint inequality by Kirmaci, was given in [7]. Thus, these two important inequalities have attracted the attention of many readers to date, and many studies have been carried out for different types of convex functions. For recent results and generalizations concerning Hermite-Hadamard's inequalities see [1, 3, 6, 9–11, 13–18] and the references given therein.

The most well-known inequalities related to the integral mean of a convex function are the Hermite-Hadamard inequalities or its weighted versions, the so-called Hermite-Hadamard-Fejér inequalities (see, [8–12]). In [4], Fejér gave a weighted generalization of the inequalities (1.1) as the following:

Theorem 1. *Let $f:[a,b]\to\mathbb{R}$ be a convex function, then the inequalities*

$$f\left(\frac{a+b}{2}\right)\int_a^b w(x)\mathrm{d}x \le \frac{1}{b-a}\int_a^b f(x)w(x)\mathrm{d}x \le \frac{f(a)+f(b)}{2}\int_a^b w(x)\mathrm{d}x$$

hold, where $w:[a,b]\to\mathbb{R}$ is non-negative, integrable, and symmetric about $x=\frac{a+b}{2}$ (i.e. $w(x)=w(a+b-x)$).

In [7], Kirmaci proved the following results connected to the left part of (1.1). In [7] some inequalities of Hermite-Hadamard type for differentiable convex mappings were proved using the following lemma.

Lemma 1. *Let $f:I^\circ\subset\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I°, $a,b\in I^\circ$ with $a<b$. If $f'\in L([a,b])$, then we have*

$$\begin{aligned}&\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x - f\left(\frac{a+b}{2}\right)\\ &= (b-a)\left[\int_0^{\frac{1}{2}} t f'(ta+(1-t)b)\mathrm{d}t + \int_{\frac{1}{2}}^1 (t-1)\,f'(ta+(1-t)b)\mathrm{d}t\right].\end{aligned}$$

The purpose of this paper is to establish new Hermite-Hadamard-Fejér type inequalities involving convex functions. Using functions whose first derivatives absolute values are convex, we obtained new weighted midpoint inequalities that are connected to the celebrated Hermite-Hadamard-Fejér type which cover the previously published results.

2. Weighted midpoint inequalities

Lemma 2. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable function on (a,b) and $g:[a,b]\to\mathbb{R}$ be continuous and symmetric to $\frac{a+b}{2}$ with $a<b$. If $f'\in L[a,b]$, then the following equality holds:*

$$f\left(\frac{a+b}{2}\right)\frac{1}{(b-a)}\int_a^b g(x)\,\mathrm{d}x - \frac{1}{(b-a)}\int_a^b g(x)\,f(x)\,\mathrm{d}x \tag{2.1}$$

$$= \frac{(b-a)}{4}\left\{\int_0^1\left(\int_t^1 g\left(bs+(1-s)\frac{a+b}{2}\right)\mathrm{d}s\right)f'\left(at+(1-t)\frac{a+b}{2}\right)\mathrm{d}t\right.$$
$$\left.-\int_0^1\left(\int_t^1 g\left(as+(1-s)\frac{a+b}{2}\right)\mathrm{d}s\right)f'\left(bt+(1-t)\frac{a+b}{2}\right)\mathrm{d}t\right\}.$$

Proof. Here, we apply integration by parts in integrals of right part of (2.1), and by using the change of the variable, we have

$$\int_0^1\left(\int_t^1 g\left(bs+(1-s)\frac{a+b}{2}\right)\mathrm{d}s\right)f'\left(at+(1-t)\frac{a+b}{2}\right)\mathrm{d}t \tag{2.2}$$

$$= \frac{2}{a-b}\left(\int_t^1 g\left(bs+(1-s)\frac{a+b}{2}\right)\mathrm{d}s\right)f\left(at+(1-t)\frac{a+b}{2}\right)\Bigg|_0^1$$
$$+\frac{2}{a-b}\int_0^1 g\left(bt+(1-t)\frac{a+b}{2}\right)f\left(at+(1-t)\frac{a+b}{2}\right)\mathrm{d}t$$
$$= \frac{4}{(b-a)^2}\left(\int_{\frac{a+b}{2}}^b g(x)\,\mathrm{d}s\right)f\left(\frac{a+b}{2}\right)-\frac{4}{(b-a)^2}\int_a^{\frac{a+b}{2}} g(a+b-x)\,f(x)\,\mathrm{d}x$$

and

$$\int_0^1\left(\int_t^1 g\left(as+(1-s)\frac{a+b}{2}\right)\mathrm{d}s\right)f'\left(bt+(1-t)\frac{a+b}{2}\right)\mathrm{d}t \tag{2.3}$$

$$= \frac{2}{b-a}\left(\int_t^1 g\left(as+(1-s)\frac{a+b}{2}\right)\mathrm{d}s\right)f\left(bt+(1-t)\frac{a+b}{2}\right)\Bigg|_0^1$$
$$+\frac{2}{b-a}\int_0^1 g\left(at+(1-t)\frac{a+b}{2}\right)f\left(bt+(1-t)\frac{a+b}{2}\right)\mathrm{d}t$$
$$= -\frac{4}{(b-a)^2}\left(\int_a^{\frac{a+b}{2}} g(x)\,\mathrm{d}s\right)f\left(\frac{a+b}{2}\right)+\frac{4}{(b-a)^2}\int_{\frac{a+b}{2}}^b g(a+b-x)\,f(x)\,\mathrm{d}x.$$

If we subtract (2.2) from (2.3), use the symmetry of g and multiply by $\frac{(b-a)}{4}$, we obtain the proof of (2.1). □

Remark 1. With the notations in Lemma 2, if we choose $g(x)=1$, we have

$$f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,dx=\frac{1}{(b-a)}\left\{\int_a^{\frac{a+b}{2}}(x-a)f'(x)\,dx+\int_{\frac{a+b}{2}}^b(x-b)f'(x)\,dx\right\},$$

which is proved by Kirmaci in [7].

Now, we extend some estimates of the left hand side of a Hermite-Hadamard-Fejér type inequalities for functions whose first derivatives absolute values are convex as follows.

Theorem 2. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b), $g:[a,b]\to\mathbb{R}$ be continuous and symmetric to $\frac{a+b}{2}$ with $a<b$ and $f'\in L[a,b]$. If $|f'|$ is convex on $[a,b]$, then the following inequalities hold:*

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)\frac{1}{(b-a)}\int_a^b g(x)\,dx-\frac{1}{(b-a)}\int_a^b g(x)f(x)\,dx\right| \qquad (2.4)\\
&\le\frac{1}{(b-a)^2}\left[\left|f'(a)\right|+\left|f'(b)\right|\right]\int_a^{\frac{a+b}{2}}|g(x)|\left(\frac{a+b}{2}-x\right)^2dx\\
&\quad+\frac{1}{2}\left|f'\left(\frac{a+b}{2}\right)\right|\int_a^{\frac{a+b}{2}}|g(x)|\left(1-4\left(\frac{x-a}{b-a}\right)^2\right)dx\\
&\le\left[\left|f'(a)\right|+\left|f'(b)\right|\right]\left\{\frac{1}{(b-a)^2}\int_a^{\frac{a+b}{2}}|g(x)|\left(\frac{a+b}{2}-x\right)^2dx\right.\\
&\qquad\left.+\frac{1}{4}\int_a^{\frac{a+b}{2}}|g(x)|\left(1-4\left(\frac{x-a}{b-a}\right)^2\right)dx\right\},
\end{aligned}$$

where

$$\left|f'\left(\frac{a+b}{2}\right)\right|\le\frac{|f'(a)|+|f'(b)|}{2}.$$

Proof. Using Lemma 2, by change of the order of integration and the convexity of $|f'|$, we find that

$$\begin{aligned}
&\left| f\left(\frac{a+b}{2}\right)\frac{1}{(b-a)}\int_a^b g(x)\,\mathrm{d}x - \frac{1}{(b-a)}\int_a^b g(x)f(x)\,\mathrm{d}x \right| \\
&\leq \frac{(b-a)}{4}\left\{ \int_0^1\int_t^1 \left| g\left(bs+(1-s)\frac{a+b}{2}\right)\right| \left| f'\left(at+(1-t)\frac{a+b}{2}\right)\right| \mathrm{d}s\mathrm{d}t \right. \\
&\qquad \left. + \int_0^1\int_t^1 \left| g\left(as+(1-s)\frac{a+b}{2}\right)\right| \left| f'\left(bt+(1-t)\frac{a+b}{2}\right)\right| \mathrm{d}s\mathrm{d}t \right\} \\
&\leq \frac{(b-a)}{4}\int_0^1\int_0^s \left| g\left(bs+(1-s)\frac{a+b}{2}\right)\right| \left[(1-t)\left|f'\left(\frac{a+b}{2}\right)\right| + t\left|f'(a)\right|\right] \mathrm{d}t\mathrm{d}s \\
&\quad + \frac{(b-a)}{4}\int_0^1\int_0^s \left| g\left(as+(1-s)\frac{a+b}{2}\right)\right| \left[(1-t)\left|f'\left(\frac{a+b}{2}\right)\right| + t\left|f'(b)\right|\right] \mathrm{d}t\mathrm{d}s \\
&= \frac{(b-a)}{8}\int_0^1 \left| g\left(bs+(1-s)\frac{a+b}{2}\right)\right| \left[(1-(1-s)^2)\left|f'\left(\frac{a+b}{2}\right)\right| + s^2\left|f'(a)\right|\right] \mathrm{d}s \\
&\quad + \frac{(b-a)}{8}\int_0^1 \left| g\left(as+(1-s)\frac{a+b}{2}\right)\right| \left[(1-(1-s)^2)\left|f'\left(\frac{a+b}{2}\right)\right| + s^2\left|f'(b)\right|\right] \mathrm{d}s.
\end{aligned}$$

By using the change of the variable it follows that

$$\begin{aligned}
&\left| f\left(\frac{a+b}{2}\right)\frac{1}{(b-a)}\int_a^b g(x)\,\mathrm{d}x - \frac{1}{(b-a)}\int_a^b g(x)f(x)\,\mathrm{d}x \right| \\
&\leq \frac{1}{(b-a)^2}\left|f'(a)\right| \int_{\frac{a+b}{2}}^b |g(x)| \left(x-\frac{a+b}{2}\right)^2 \mathrm{d}x \\
&\quad + \frac{1}{(b-a)^2}\left|f'(b)\right| \int_a^{\frac{a+b}{2}} |g(x)| \left(x-\frac{a+b}{2}\right)^2 \mathrm{d}x \\
&\quad + \frac{1}{4}\left|f'\left(\frac{a+b}{2}\right)\right| \int_{\frac{a+b}{2}}^b \left[|g(x)|+|g(a+b-x)|\right] \left(1-\frac{4}{(b-a)^2}(b-x)^2\right) \mathrm{d}x.
\end{aligned}$$

Since $g(x)$ is symmetric to $x=\frac{a+b}{2}$, for $g(x)=g(a+b-x)$, we write

$$\int_{\frac{a+b}{2}}^{b}|g(x)|\left(x-\frac{a+b}{2}\right)^{2}\mathrm{d}x=\int_{a}^{\frac{a+b}{2}}|g(x)|\left(\frac{a+b}{2}-x\right)^{2}\mathrm{d}x$$

and

$$\int_{\frac{a+b}{2}}^{b}|g(x)|\left(1-4\left(\frac{b-x}{b-a}\right)^{2}\right)\mathrm{d}x=\int_{a}^{\frac{a+b}{2}}|g(x)|\left(1-4\left(\frac{x-a}{b-a}\right)^{2}\right)\mathrm{d}x,$$

which completes the proof of (2.4). □

Remark 2. If in Theorem 2, we get $g(x)=1$, then inequality (2.4) becomes

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{(b-a)}\int_{a}^{b}f(x)\,\mathrm{d}x\right|\leq\frac{(b-a)^{2}}{4}\left[\frac{|f'(a)|+|f'(b)|}{2}\right],$$

which is proved by Kirmaci in [7].

Theorem 3. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b), $g:[a,b]\to\mathbb{R}$ be continuous and symmetric to $\frac{a+b}{2}$ with $a<b$ and $f'\in L[a,b]$. If $|f'|^{q}$ is convex on $[a,b]$ for some $q>1$, then the following inequality holds:*

$$\left|f\left(\frac{a+b}{2}\right)\frac{1}{(b-a)}\int_{a}^{b}g(x)\,\mathrm{d}x-\frac{1}{(b-a)}\int_{a}^{b}g(x)f(x)\,\mathrm{d}x\right| \tag{2.5}$$

$$\leq\frac{(b-a)^{1-\frac{2}{p}}}{2^{2-\frac{2}{p}}}\left(\int_{a}^{\frac{a+b}{2}}\left(\frac{a+b}{2}-x\right)|g(x)|^{p}\,\mathrm{d}x\right)^{\frac{1}{p}}$$

$$\times\left[\left(\frac{2|f'(a)|^{q}+|f'(b)|^{q}}{6}\right)^{\frac{1}{q}}+\left(\frac{|f'(a)|^{q}+2|f'(b)|^{q}}{6}\right)^{\frac{1}{q}}\right],$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Using Lemma 2, Hölder's inequality and the convexity of $|f'|^{q}$, we find that

$$\left|f\left(\frac{a+b}{2}\right)\frac{1}{(b-a)}\int_{a}^{b}g(x)\,\mathrm{d}x-\frac{1}{(b-a)}\int_{a}^{b}g(x)f(x)\,\mathrm{d}x\right|$$

$$\leq\frac{(b-a)}{4}\left\{\int_{0}^{1}\int_{t}^{1}\left|g\left(bs+(1-s)\frac{a+b}{2}\right)\right|\left|f'\left(at+(1-t)\frac{a+b}{2}\right)\right|\mathrm{d}s\mathrm{d}t\right.$$

$$
\begin{aligned}
&+\int_0^1\int_t^1\left|g\left(as+(1-s)\frac{a+b}{2}\right)\right|\left|f'\left(bt+(1-t)\frac{a+b}{2}\right)\right|\mathrm{d}s\mathrm{d}t\Bigg\}\\
&\leq\frac{(b-a)}{4}\left\{\left(\int_0^1\int_t^1\left|g\left(bs+(1-s)\frac{a+b}{2}\right)\right|^p\mathrm{d}s\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_0^1\int_t^1\left|f'\left(at+(1-t)\frac{a+b}{2}\right)\right|^q\mathrm{d}s\mathrm{d}t\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\int_0^1\int_t^1\left|g\left(as+(1-s)\frac{a+b}{2}\right)\right|^p\mathrm{d}s\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_0^1\int_t^1\left|f'\left(bt+(1-t)\frac{a+b}{2}\right)\right|^q\mathrm{d}s\mathrm{d}t\right)^{\frac{1}{q}}\right\}\\
&\leq\frac{(b-a)}{4}\left(\int_0^1\int_0^s\left|g\left(bs+(1-s)\frac{a+b}{2}\right)\right|^p\mathrm{d}t\mathrm{d}s\right)^{\frac{1}{p}}\\
&\times\left(\int_0^1\left[(1-t)^2\left|f'\left(\frac{a+b}{2}\right)\right|^q+t(1-t)\left|f'(a)\right|^q\right]\mathrm{d}t\right)^{\frac{1}{q}}\\
&+\frac{(b-a)}{4}\left(\int_0^1\int_0^s\left|g\left(as+(1-s)\frac{a+b}{2}\right)\right|^p\mathrm{d}t\mathrm{d}s\right)^{\frac{1}{p}}\\
&\times\left(\int_0^1\left[(1-t)^2\left|f'\left(\frac{a+b}{2}\right)\right|^q+t(1-t)\left|f'(b)\right|^q\right]\mathrm{d}t\right)^{\frac{1}{q}}\\
&=\frac{(b-a)}{4}\left(\int_0^1 s\left|g\left(bs+(1-s)\frac{a+b}{2}\right)\right|^p\mathrm{d}s\right)^{\frac{1}{p}}\left(\frac{1}{3}\left|f'\left(\frac{a+b}{2}\right)\right|^q+\frac{1}{6}\left|f'(a)\right|^q\right)^{\frac{1}{q}}\\
&+\frac{(b-a)}{4}\left(\int_0^1 s\left|g\left(as+(1-s)\frac{a+b}{2}\right)\right|^p\mathrm{d}s\right)^{\frac{1}{p}}\left(\frac{1}{3}\left|f'\left(\frac{a+b}{2}\right)\right|^q+\frac{1}{6}\left|f'(b)\right|^q\right)^{\frac{1}{q}}\\
&\leq\frac{(b-a)}{4}\left(\int_0^1 s\left|g\left(bs+(1-s)\frac{a+b}{2}\right)\right|^p\mathrm{d}s\right)^{\frac{1}{p}}\left(\frac{2\left|f'(a)\right|^q+\left|f'(b)\right|^q}{6}\right)^{\frac{1}{q}}\\
&+\frac{(b-a)}{4}\left(\int_0^1 s\left|g\left(as+(1-s)\frac{a+b}{2}\right)\right|^p\mathrm{d}s\right)^{\frac{1}{p}}\left(\frac{\left|f'(a)\right|^q+2\left|f'(b)\right|^q}{6}\right)^{\frac{1}{q}}.
\end{aligned}
$$

By using the change of the variable

$$
\left|f\left(\frac{a+b}{2}\right)\frac{1}{(b-a)}\int_a^b g(x)\,\mathrm{d}x-\frac{1}{(b-a)}\int_a^b g(x)f(x)\,\mathrm{d}x\right|
$$

$$\leq \frac{(b-a)}{4}\frac{2^{\frac{2}{p}}}{(b-a)^{\frac{2}{p}}}\left(\int_{\frac{a+b}{2}}^{b}\left(x-\frac{a+b}{2}\right)|g(x)|^{p}\,dx\right)^{\frac{1}{p}}\left(\frac{2|f'(a)|^{q}+|f'(b)|^{q}}{6}\right)^{\frac{1}{q}}$$

$$+\frac{(b-a)}{4}\frac{2^{\frac{2}{p}}}{(b-a)^{\frac{2}{p}}}\left(\int_{a}^{\frac{a+b}{2}}\left(\frac{a+b}{2}-x\right)|g(x)|^{p}\,dx\right)^{\frac{1}{p}}\left(\frac{|f'(a)|^{q}+2|f'(b)|^{q}}{6}\right)^{\frac{1}{q}}.$$

Since $g(x)$ is symmetric to $x=\frac{a+b}{2}$, for $g(x)=g(a+b-x)$, we write

$$\int_{\frac{a+b}{2}}^{b}\left(x-\frac{a+b}{2}\right)|g(x)|^{p}\,dx=\int_{a}^{\frac{a+b}{2}}\left(\frac{a+b}{2}-x\right)|g(x)|^{p}\,dx,$$

which completes the proof of (2.5). □

Corollary 1. *With the notations in Theorem 3, we get $g(x)=1$, then (2.5) becomes the following inequality*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_{a}^{b}f(x)\,dx\right|$$

$$\leq \frac{(b-a)}{2^{2+\frac{1}{p}}}\left[\left(\frac{2|f'(a)|^{q}+|f'(b)|^{q}}{6}\right)^{\frac{1}{q}}+\left(\frac{|f'(a)|^{q}+2|f'(b)|^{q}}{6}\right)^{\frac{1}{q}}\right].$$

3. Applications to special means

For arbitrary positive real numbers a,b $(a\neq b)$, we shall consider the following special means:

(M1) The arithmetic mean:

$$A=A(a,b):=\frac{a+b}{2},\quad a,b\geq 0.$$

(M2) The geometric mean:

$$G=G(a,b):=\sqrt{ab},\quad a,b\geq 0.$$

(M3) The logarithmic mean:

$$L=L(a,b):=\begin{cases}a, & \text{if } a=b,\\ \frac{b-a}{\ln b-\ln a}, & \text{if } a\neq b,\end{cases}\qquad a,b>0.$$

(M4) The $p-$logarithmic mean:

$$L_p = L_p(a,b) := \begin{cases} \left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}}, & \text{if} \quad a \neq b, \\ a, & \text{if} \quad a = b, \end{cases} \qquad p \in \mathbb{R} \setminus \{-1,0\};\ a,b>0.$$

It is well known that L_p is monotonic non-decreasing over $p \in \mathbb{R}$ with $L_{-1} := L$. In particular, we have the following inequality

$$L \leq A.$$

Now, using the results of Section 2, some new inequalities is derived for the above means.

(1) Let $f\colon [a,b] \to \mathbb{R}$, $(0<a<b)$, $f(x)=x^s$, $s \geq 1$ and $g(x) = \left(x - \frac{a+b}{2}\right)^2$. Then,

$$\frac{1}{b-a}\int_a^b g(x)f(x)\mathrm{d}x = L_{s+2}^{s+2}(a,b) - 2A(a,b)L_{s+1}^{s+1}(a,b) + A^2(a,b)L_s^s(a,b),$$

$$\int_a^b \left(x - \frac{a+b}{2}\right)^2 \mathrm{d}x = \frac{(b-a)^3}{12},$$

$$f\left(\frac{a+b}{2}\right) = A^s(a,b).$$

It is easy to verify that the function $|f'(x)|^q = s^q x^{(s-1)q}$ is convex on $[a,b]$.

(a) From Theorem 2, we obtain

$$\left|A^s(a,b)\frac{(b-a)^2}{12} - L_{s+2}^{s+2}(a,b) + 2A(a,b)L_{s+1}^{s+1}(a,b) - A^2(a,b)L_s^s(a,b)\right|$$

$$\leq sA(a^{s-1},b^{s-1})\frac{(b-a)^3}{5.2^5} + sA^{s-1}(a,b)\frac{3(b-a)^3}{5.2^4} \leq sA(a^{s-1},b^{s-1})\frac{(b-a)^3}{5.2^3}.$$

(b) From Theorem 3, we have

$$\left|A^s(a,b)\frac{(b-a)^2}{12} - L_{s+2}^{s+2}(a,b) + 2A(a,b)L_{s+1}^{s+1}(a,b) - A^2(a,b)L_s^s(a,b)\right|$$

$$\leq s\frac{(b-a)^{2+\frac{2}{p}}}{8(p+2)^{\frac{1}{p}}}\left[\left(\frac{2a^{(s-1)q}+b^{(s-1)q}}{6}\right)^{\frac{1}{q}} + \left(\frac{a^{(s-1)q}+2b^{(s-1)q}}{6}\right)^{\frac{1}{q}}\right]$$

$$= s\frac{(b-a)^{2+\frac{2}{p}}}{8(p+2)^{\frac{1}{p}}}$$

$$\times \left[\left(\frac{1}{3}A\left(a^{(s-1)q},b^{(s-1)q}\right) + \frac{a^{(s-1)q}}{6}\right)^{\frac{1}{q}} + \left(\frac{1}{3}A\left(a^{(s-1)q},b^{(s-1)q}\right) + \frac{b^{(s-1)q}}{6}\right)^{\frac{1}{q}}\right],$$

where $q>1$ and $\frac{1}{p}+\frac{1}{q}=1$.

(2) Let $f\colon [a,b]\subseteq[0,\infty)\to\mathbb{R}$, $(0<a<b)$, $f(x)=\dfrac{1}{x^s}$, $s\in(0,1]$. Then,

$$\frac{1}{b-a}\int_a^b g(x)f(x)\mathrm{d}x = L_{2-s}^{2-s}(a,b)-2A(a,b)L_{1-s}^{1-s}(a,b)+A^2(a,b)L_{-s}^{-s}(a,b),$$

$$f\left(\frac{a+b}{2}\right)=A^{-s}(a,b).$$

(a) From Theorem 2, we obtain

$$\left|A^{-s}(a,b)\frac{(b-a)^2}{12}-L_{2-s}^{2-s}(a,b)+2A(a,b)L_{1-s}^{1-s}(a,b)-A^2(a,b)L_{-s}^{-s}(a,b)\right|$$

$$\leq s(s+1)\frac{L_s^s(a,b)}{G^{2(s+1)}(a,b)}\frac{(b-a)^4}{5.2^5}+s\frac{3(b-a)^3}{A^{s+1}(a,b)\cdot 5.2^4}$$

$$\leq s(s+1)\frac{L_s^s(a,b)}{G^{2(s+1)}(a,b)}\frac{(b-a)^4}{2^6}.$$

(b) From Theorem 3, we have

$$\left|A^{-s}(a,b)\frac{(b-a)^2}{12}-L_{2-s}^{2-s}(a,b)+2A(a,b)L_{1-s}^{1-s}(a,b)-A^2(a,b)L_{-s}^{-s}(a,b)\right|$$

$$\leq\frac{s(b-a)}{(p+1)^{1/p}2^{1+\frac{1}{p}}(s+1)^{1/q}}\left[m\left\{2^{-(s+1)}a^{-q(s+1)}+\left(1-2^{-(s+1)}\right)b^{-q(s+1)}\right\}^{(p-1)/p}\right.$$

$$\left.+n\left\{\left(1-2^{-(s+1)}\right)a^{-q(s+1)}+2^{-(s+1)}b^{-q(s+1)}\right\}^{(p-1)/p}\right],$$

where $q>1$ and $\frac{1}{p}+\frac{1}{q}=1$.

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SOME PROPERTIES OF AMPLY r-SUPPLEMENTED MODULES

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Abstract. In this work, amply r-supplemented modules are defined and some properties of these modules are investigated. It is proved that every factor module and every homomorphic image of an amply r-supplemented module are amply r-supplemented. Let M be a $\pi-$projective and r-supplemented module. Then M is amply r-supplemented. Let M be an $R-$module. If every submodule of M is r-supplemented, then M is amply r-supplemented. Let R be any ring. Then every $R-$module is r-supplemented if and only if every $R-$module is amply r-supplemented. Let R be any ring. Then ${}_RR$ is (amply) r-supplemented if and only if every finitely generated $R-$module is (amply) r-supplemented.

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1. INTRODUCTION

Throughout this paper all rings will be associative with identity and all modules will be unital left modules.

Let R be a ring and M be an $R-$module. We will denote a submodule N of M by $N \leq M$. Let M be an $R-$module and $N \leq M$. If $L = M$ for every submodule L of M such that $M = N + L$, then N is called a *small* submodule of M and denoted by $N \ll M$. Let M be an $R-$module. M is called a *hollow* module if every proper submodule of M is small in M. M is said to be *local* if M has the largest submodule, i.e. a proper submodule which contains all other proper submodules. Let M be an $R-$module and $U, V \leq M$. If $M = U + V$ and V is minimal with respect to this property, or equivalently, $M = U + V$ and $U \cap V \ll V$, then V is called a *supplement* of U in M. M is called a *supplemented* module if every submodule of M has a supplement in M. Let M be an $R-$module and $U \leq M$. If for every $V \leq M$ such that $M = U + V$, U has a supplement V' with $V' \leq V$, we say U has *ample supplements* in M. If every submodule of M has ample supplements in M, then M is called an *amply supplemented* module. The intersection of all maximal submodules of an $R-$module M is called the *radical* of M and denoted by $RadM$. If M have no maximal submodules, then we denote $RadM = M$.

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Let M be an $R-$module and $U,V \leq M$. If $M = U+V$ and $U \cap V \leq RadV$, then V is called a *generalized* (*radical*) *supplement* (briefly, *Rad-supplement*) of U in M. M is said to be *generalized* (*radical*) *supplemented* (briefly, *Rad-supplemented*) if every submodule of M has a Rad-supplement in M. Let M be an $R-$module and $K \leq M$. If $K \ll RadM$, then K is called an *r-small* submodule of M and denoted by $K \ll_r M$. Let M be an $R-$module and $U,V \leq M$. If $M = U+V$ and $U \cap V \ll_r V$, then V is called an *r-supplement* of U in M. If every submodule of M has an r-supplement in M, then M is called an *r-supplemented* module. Let M be an $R-$module and $K \leq V \leq M$. We say V *lies above* K in M if $V/K \ll M/K$. Let M be an $R-$module. M is said to be $\pi-$*projective* if for every $U,V \leq M$ with $M = U+V$, there exists an $R-$module homomorphism $f: M \to M$ such that $f(M) \leq U$ and $(1-f)(M) \leq V$.

More informations about (amply) supplemented modules are in [1, 2, 5, 7, 8, 13]. More results about Rad-supplemented modules are in [12]. More informations of $\pi-$projective modules are in [6, 13]. The definition of r-small submodules and some properties of them are in [3, 4]. The definition of r-supplemented modules and some properties of them are in [9, 11].

Lemma 1. *Let M be an $R-$module. The following assertions hold.*

(*i*) *If $K \ll_r M$, then $K \ll M$.*
(*ii*) *If $L \ll_r M$ and $K \leq L$, then $K \ll_r M$.*
(*iii*) *If $K \ll_r L \leq M$, then $K \ll_r M$.*
(*iv*) *If $K_i \ll_r L_i \leq M$ for $i = 1,2,...,n$, then $K_1+K_2+...+K_n \ll_r L_1+L_2+...+L_n$.*
(*v*) *If $K_i \ll_r M$ for $i = 1,2,...,n$, then $K_1+K_2+...+K_n \ll_r M$.*
(*vi*) *If $K \ll_r M$, then $(K+L)/L \ll_r M/L$ for every $L \leq M$.*
(*vii*) *If $K \ll M$ and $RadM$ is a supplement submodule in M, then $K \ll_r M$.*
(*viii*) *Let N be an $R-$module and $f: M \to N$ be an $R-$module homomorphism. If $K \ll_r M$, then $f(K) \ll_r f(M)$.*
(*ix*) *$Rad(RadM) = \sum_{K \ll_r M} K$.*

Proof. See [3, 4]. □

2. Amply *r*-supplemented modules

Definition 1. Let M be an $R-$module and $U \leq M$. If for every $V \leq M$ such that $M = U+V$, U has an r-supplement V' with $V' \leq V$, we say U has *ample r-supplements* in M. If every submodule of M has ample r-supplements in M, then M is called an *amply r-supplemented* module. (See also [10]).

Clearly, every amply r-supplemented module is r-supplemented. We can also see that every amply r-supplemented module is amply supplemented, but the converse of this statement is not true in general (See Lemma 8 and Example 1).

Proposition 1. *Let M be an $R-$module and $M = U+V$ with $U,V \leq M$. If U and V have ample r-supplements in M, then $U \cap V$ has ample r-supplements in M.*

Proof. Let $M = U \cap V + T$ with $T \leq M$. Since $M = U \cap V + T$, we can see that $M = U + V \cap T = V + U \cap T$. Since $M = U + V \cap T$ and U has ample r-supplements in M, U has an r-supplement X in M with $X \leq V \cap T$. Here $M = U + X$ and $U \cap X \ll_r X$. Since $M = V + U \cap T$ and V has ample r-supplements in M, V has an r-supplement Y in M with $Y \leq U \cap T$. Here $M = V + Y$ and $V \cap Y \ll_r Y$. Since $M = U + X$ and $X \leq V$, by Modular Law, $V = M \cap V = (U + X) \cap V = U \cap V + X$. Then $M = V + Y = U \cap V + X + Y$ and by Lemma 1, $U \cap V \cap (X + Y) = U \cap (X + V \cap Y) = U \cap X + V \cap Y \ll_r X + Y$. Hence $X + Y$ is an r-supplement of $U \cap V$ in M. Moreover, $X + Y \leq T$. Thus $U \cap V$ has ample r-supplements in M. □

Proposition 2. *Let M be an amply r-supplemented module. Then $M/Rad\,(RadM)$ is semisimple.*

Proof. Since M is amply r-supplemented, then M is r-supplemented. Then by [11, Proposition 2], $M/Rad\,(RadM)$ is semisimple. □

Lemma 2. *Let M be an amply r-supplemented module. Then every factor module of M is amply r-supplemented.*

Proof. Let $\frac{M}{K}$ be any factor module of M. Let $\frac{U}{K} \leq \frac{M}{K}$ and $\frac{M}{K} = \frac{U}{K} + \frac{V}{K}$. Then $M = U + V$. Since M is amply r-supplemented, U has an r-supplement X in M with $X \leq V$. Since $K \leq U$, by [11, Lemma 9], $\frac{X+K}{K}$ is an r-supplement of $\frac{U}{K}$ in $\frac{M}{K}$. Moreover, $\frac{X+K}{K} \leq \frac{V}{K}$. Hence $\frac{M}{K}$ is amply r-supplemented. □

Corollary 1. *Every homomorphic image of an amply r-supplemented module is amply r-supplemented.*

Proof. Clear from Lemma 2. □

Lemma 3. *Let M be a hollow module and $RadM = M$. Then M is amply r-supplemented.*

Proof. Clear from definitions. □

Lemma 4. *Let M be a $\pi-$projective and r-supplemented module. Then M is amply r-supplemented.*

Proof. Let $U \leq M$, $M = U + V$ and X be an r-supplement of U in M. Here $M = U + X$ and $U \cap X \ll_r X$. Since M is $\pi-$projective and $M = U + V$, there exists an $R-$module homomorphism $f : M \to M$ such that $f(M) \subset V$ and $(1 - f)(M) \subset U$. So, we have $M = f(M) + (1 - f)(M) = f(U) + f(X) + U = U + f(X)$. Since $U \cap X \ll_r X$, by Lemma 1, $f(U \cap X) \ll_r f(X)$. Suppose that $a \in U \cap f(X)$. Since $a \in f(X)$, then there exists $x \in X$ such that $a = f(x)$. Since $a = f(x) = f(x) - x + x = x - (1 - f)(x)$ and $(1 - f)(x) \in U$, we have $x = a + (1 - f)(x) \in U$. Thus $x \in U \cap X$ and so, $a = f(x) \in f(U \cap X)$. Therefore we have $U \cap f(X) \leq f(U \cap X) \ll_r f(X)$. This means that $f(X)$ is an r-supplement of U in M. Moreover, $f(X) \subset V$. Therefore M is amply r-supplemented. □

Corollary 2. *If M is a projective and r-supplemented module, then M is an amply r-supplemented module.*

Proof. Clear from Lemma 4. □

Lemma 5. *Let Λ be a finite index set and $\{M_\lambda\}_\Lambda$ be a family of projective $R-$modules. If M_λ is r-supplemented for every $\lambda \in \Lambda$, then $\underset{\lambda\in\Lambda}{\oplus} M_\lambda$ is amply r-supplemented.*

Proof. Since M_λ is r-supplemented for every $\lambda \in \Lambda$, by [11, Corollary 8], $\underset{\lambda\in\Lambda}{\oplus} M_\lambda$ is r-supplemented. Since M_λ is projective for every $\lambda \in \Lambda$, by [13, 18.1], $\underset{\lambda\in\Lambda}{\oplus} M_\lambda$ is projective. Since $\underset{\lambda\in\Lambda}{\oplus} M_\lambda$ is projective and r-supplemented, by Corollary 2, $\underset{\lambda\in\Lambda}{\oplus} M_\lambda$ is amply r-supplemented. □

Corollary 3. *Let M be a projective $R-$module. If M is r-supplemented, then $M^{(\Lambda)}$ is amply r-supplemented for every finite index set Λ.*

Proof. Clear from Lemma 5. □

Corollary 4. *Let M be a projective $R-$module. If M is r-supplemented, then every finitely $M-$generated $R-$module is amply r-supplemented.*

Proof. Let N be a finitely $M-$generated $R-$module. Then there exist a finite index set Λ and an $R-$module epimorphism $f\colon M^{(\Lambda)} \to N$. Since M is projective and r-supplemented, by Lemma 5, $M^{(\Lambda)}$ is amply r-supplemented. Then by Corollary 1, N is amply r-supplemented. □

Lemma 6. *Let M be an $R-$module. If every submodule of M is r-supplemented, then M is amply r-supplemented.*

Proof. Let $U \leq M$ and $M = U + V$ with $V \leq M$. By hypothesis, V is r-supplemented. Then $U \cap V$ has an r-supplement X in V. By this, $V = U \cap V + X$ and $U \cap X = U \cap V \cap X \ll_r X$. Then $M = U + V = U + U \cap V + X = U + X$ and $U \cap X \ll_r X$. Hence M is amply r-supplemented. □

Lemma 7. *Let R be any ring. Then every $R-$module is r-supplemented if and only if every $R-$module is amply r-supplemented.*

Proof. ($\Longrightarrow$) Let M be any $R-$module. Since every $R-$module is r-supplemented, every submodule of M is r-supplemented. Then by Lemma 6, M is amply r-supplemented.
($\Longleftarrow$) Clear. □

Proposition 3. *Let R be a ring. The following assertions are equivalent.*

(*i*) ${}_RR$ *is r-supplemented.*
(*ii*) ${}_RR$ *is amply r-supplemented.*

(iii) *Every finitely generated $R-$module is r-supplemented.*
(iv) *Every finitely generated $R-$module is amply r-supplemented.*

Proof. $(i) \Longleftrightarrow (ii)$ Clear from Corollary 2, since ${}_RR$ is projective.
$(i) \Longleftrightarrow (iii)$ Clear from [11, Proposition 3].
$(iii) \Longrightarrow (iv)$ Let M be a finitely generated $R-$module. Then there exist a finite index set Λ and an $R-$module epimorphism $f: R^{(\Lambda)} \to M$. Since every finitely generated $R-$module is r-supplemented, $R^{(\Lambda)}$ is r-supplemented. Since ${}_RR$ is projective, by [13, 18.1], $R^{(\Lambda)}$ is also projective. Then by Corollary 2, $R^{(\Lambda)}$ is amply r-supplemented. Since $f: R^{(\Lambda)} \to M$ is an $R-$module epimorphism, by Corollary 1, M is also amply r-supplemented.
$(iv) \Longrightarrow (i)$ Clear. □

Lemma 8. *Let M be a local module with $RadM \neq 0$. Then M is not amply r-supplemented, but M is amply supplemented.*

Proof. Clearly we can see that M is amply supplemented. By [11, Lemma 12], M is not r-supplemented. Because of this M is not amply r-supplemented. □

Example 1. Consider the $\mathbb{Z}-$module $\mathbb{Z}_8$. Since $\mathbb{Z}_8$ is local and $Rad\mathbb{Z}_8 = \{\overline{0}, \overline{2}, \overline{4}, \overline{6}\} \neq 0$, by Lemma 8, $\mathbb{Z}_8$ is amply supplemented, but not amply r-supplemented.

Example 2. Consider the $\mathbb{Z}-$module $\mathbb{Z}_{p^\infty}$. Since $\mathbb{Z}_{p^\infty}$ is hollow and $Rad\mathbb{Z}_{p^\infty} = \mathbb{Z}_{p^\infty}$, by Lemma 3, $\mathbb{Z}_{p^\infty}$ is amply r-supplemented.

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OSCILLATORY PROPERTIES OF FOURTH-ORDER NEUTRAL DIFFERENTIAL EQUATIONS USING NEW MONOTONIC PROPERTIES

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Abstract. The authors investigate the asymptotic behavior of solutions of the fourth-order half-linear neutral differential equation

$$\left(\alpha(t)\left((\mu(t)+d(t)\mu(a(t)))'''\right)^b\right)'+m(t)\mu^b(\delta(t))=0 \tag{e}$$

without assuming $\alpha'(t)\geq 0$. By using a linearization method and deriving some new monotonic properties of the nonoscillatory solutions, they analyze the oscillatory behavior of solutions of (e). They use two different techniques, namely, a comparison with second-order delay differential inequalities, and a generalization of very effective Koplatadze's method. They illustrate the improvements over known results by providing specific examples.

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1. INTRODUCTION

We are concerned with the oscillatory behavior of solutions of a class of fourth-order half-linear neutral differential equations of the type

$$\left(\alpha(t)\left(\eta'''(t)\right)^b\right)'+m(t)\mu^b(\delta(t))=0,\quad t\geq t_0\geq 0, \tag{E}$$

where

$$\eta(t)=\mu(t)+d(t)\mu(a(t)),$$

subject to the conditions:

(D1) $\alpha\in C'([t_0,\infty),\mathbb{R})$, $\alpha(t)>0$, $d, m\in C([t_0,\infty),\mathbb{R})$, $m(t)>0$, and

$$0\leq d(t)\leq d_0<1;$$

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(D2) $a, \delta \in C([t_0,\infty),\mathbb{R})$, $a(t) \le t$, $\delta(t) \le t$, $\delta'(t) > 0$, and

$$\lim_{t\to\infty} a(t) = \lim_{t\to\infty} \delta(t) = \infty;$$

(D3) b is a ratio of odd positive integers.

Let $t^* = \min\{\min_{t\ge t_0} a(t),\ \min_{t\ge t_0} \delta(t)\}$. By a *solution* of (E), we mean a function $\mu \in C([t^*,\infty),\mathbb{R})$ such that $\alpha(\eta'')^b \in C'[t^*,\infty),\mathbb{R})$, and μ satisfies (E) on $[t_0,\infty)$. We consider only those solutions of (E) that are continuable, that is, they satisfy $\sup\{|\mu(t)| : t \ge T\} > 0$ for all $T \ge t_0$, and we tacitly assume that (E) possess such solutions. A solution $\mu(t)$ of (E) is said to be *oscillatory* if it has infinitely many zeros on $[t_\mu,\infty)$ for some $t_\mu \ge t_0$; otherwise, it is called *nonoscillatory*. Equation (E) itself is called oscillatory if all its solutions are oscillatory.

Fourth-order functional differential equations are often encountered in mathematical models of various physical, biological, and chemical phenomena, for example, see [4,10,14]. In mechanics, engineering, economics and population dynamics, questions related to the existence of oscillatory and nonoscillatory solutions play a significant role; see [13], for details.

In recent decades, several researchers have studied the oscillatory behavior of solutions to various classes of fourth order functional differential equations. See for instance, the monographs [2, 16], the papers [1, 3–13, 17, 18], and the references cited therein. Under the conditions

$$\int_{t_o}^{\infty} \alpha^{-\frac{1}{b}}(t)\mathrm{d}t = \infty, \tag{1.1}$$

or

$$\int_{t_o}^{\infty} \alpha^{-\frac{1}{b}}(t)\mathrm{d}t < \infty, \tag{1.2}$$

with

$$\alpha'(t) \ge 0 \tag{1.3}$$

and

$$\eta(t) \ge \theta t\eta'(t), \quad \theta \in (0,1), \tag{1.4}$$

the authors in [5, 8–10, 12, 15–17, 20–23, 25] established several sufficient conditions that guarantee that all solutions of (E) are oscillatory.

Our aim of this paper is to analyze the oscillatory properties of solutions of (E) in the case where condition (1.1) is satisfied without the restrictive assumption (1.3) and the relation (1.4) which contains the arbitrary constant θ. Therefore, the results obtained in this paper improve and generalize other oscillation criteria reported for (E) or its special cases. Our oscillation results will be obtained by using some new monotonic properties of positive solutions and the linearization of (E). Examples are provided to show the significance of our results over previously known ones.

2. MAIN RESULTS

We begin with a lemma that determines the signs of the derivatives of the positive solutions of (E); it can be obtained from a direct application of [2, Lemma 2.2.1] to (E).

Lemma 1. *Let $\mu(t)$ be a positive solution of (E). Then the companion function $\eta(t)$ is positive and satisfies either*

$$\eta(t) \in E_1 = \{\eta(t) : \eta'(t) > 0,\ \eta''(t) < 0,\ \eta'''(t) > 0,\ \left(\alpha(t)(\eta'''(t))^b\right)' < 0\}$$

or

$$\eta(t) \in E_3 = \{\eta(t) : \eta'(t) > 0,\ \eta''(t) > 0,\ \eta'''(t) > 0,\ \left(\alpha(t)(\eta'''(t))^b\right)' < 0\},$$

eventually.

For convenience, we introduce the following notation:

$$\Omega(t) = \int_{t_*}^{t} \alpha^{-\frac{1}{b}}(s)\mathrm{d}s, \quad \Omega_1(t) = \int_{t_*}^{t} (t-s)\alpha^{-\frac{1}{b}}(s)\mathrm{d}s, \quad \Omega_2(t) = \int_{t_*}^{t} \frac{(t-s)^2}{2}\alpha^{-\frac{1}{b}}(s)\mathrm{d}s,$$

$$M(t) = m(t)(1-d(\delta(t)))^b, \quad M_1(t) = \int_t^{\infty} \left(\frac{1}{\alpha(s)}\int_s^{\infty} M(s_1)\mathrm{d}s_1\right)^{\frac{1}{b}}\mathrm{d}s,$$

where $t_* \geq t_0$.

The next three lemmas are concerned with the asymptotic behavior of the companion function $\eta(t)$.

Lemma 2. *Assume that $\mu(t)$ is an eventually positive solution of (E). Then $\eta(t)$ eventually satisfies the inequality*

$$\left(\alpha(t)(\eta'''(t))^b\right)' + M(t)\eta^b(\delta(t)) \leq 0. \tag{2.1}$$

Proof. Let μ be an eventually positive solution of (E). Then there exists $t_1 \geq t_0$ such that $\mu(a(t)) > 0$ and $\mu(\delta(t) > 0$ for all $t \geq t_1$. From the definition of $\eta(t)$, it is easy to see that $\eta(t) > 0$ for all $t \geq t_1$, and in view of Lemma 1, we have $\eta'(t) > 0$ for all $t \geq t_1$ in both cases E_1 and E_3.

Now

$$\mu(t) = \eta(t) - d(t)\mu(a(t)) \geq \eta(t) - d(t)\eta(a(t)) \geq (1-d(t))\eta(t),$$

and using this in (E) gives

$$\left(\alpha(t)(\eta'''(t))^b\right)' + M(t)\eta^b(\delta(t)) \leq 0.$$

This completes the proof. □

Note that from Lemma 1 and inequality (2.1), it is easy to see that the condition

$$\int_{t_0}^{\infty} M(t)\mathrm{d}t = \infty$$

implies the oscillation of (E), and so in the remainder of this paper, we assume that the opposite condition is satisfied, that is,

$$\int_{t_0}^{\infty} M(t)\mathrm{d}t < \infty.$$

Lemma 3. *Let $\mu(t)$ be a positive solution of* (E) *and assume that*

$$\int_{t_0}^{\infty} \delta(s)M_1(s)\mathrm{d}s = \infty. \tag{2.2}$$

If $\eta \in E_1$, then

$$\frac{\eta(t)}{t} \text{ is decreasing and } \lim_{t\to\infty} \frac{\eta(t)}{t} = 0. \tag{2.3}$$

Proof. Let $\mu(t)$ be a positive solution of (E). Then we see that $\eta(t)$ is positive and by Lemma 2, it satisfies (2.1). Since $\eta(t) \in E_1$, integrating (2.1) from t to ∞, we have

$$\eta'''(t) \geq \eta(\delta(t)) \left(\frac{1}{\alpha(t)} \int_t^{\infty} M(s)\mathrm{d}s \right)^{\frac{1}{b}}.$$

Integrating again from t to ∞ gives

$$\eta''(t) + M_1(t)\eta(\delta(t)) \leq 0. \tag{2.4}$$

From the last inequality, we obtain

$$-\left(t^2 \left(\frac{\eta(t)}{t} \right)' \right)' \geq tM_1(t)\eta(\delta(t)). \tag{2.5}$$

Letting $k = \eta(t_0) - t_0\eta'(t_0)$ and integrating (2.5) from t_0 to t yields

$$\begin{aligned} -t^2 \left(\frac{\eta(t)}{t} \right)' &\geq k + \int_{t_0}^{t} sM_1(s)\eta(\delta(s))\mathrm{d}s \\ &\geq k + \eta(\delta(t_0)) \int_{t_0}^{t} sM_1(s)\mathrm{d}s \to \infty \text{ as } t \to \infty \end{aligned} \tag{2.6}$$

by (2.2). Hence,

$$\left(\frac{\eta(t)}{t} \right)' < 0,$$

eventually, and so $\frac{\eta(t)}{t}$ is decreasing, which also implies $k > 0$ for t large enough.

Now suppose that

$$\lim_{t\to\infty} \frac{\eta(t)}{t} = l > 0.$$

Using (2.6), we see that $z(t) = \frac{\eta(t)}{t}$ satisfies

$$z'(t) + \frac{1}{t^2}\int_{t_0}^{t} s\delta(s)M_1(s)z(\delta(s))\mathrm{d}s \le -\frac{k}{t^2} < 0. \tag{2.7}$$

Integrating (2.7) from t_0 to ∞ and then integrating by parts, we obtain

$$z(t_0) - l \ge l\int_{t_0}^{\infty}\frac{1}{x^2}\left(\int_{t_0}^{x} s\delta(s)M_1(s)\mathrm{d}s\right)\mathrm{d}x = l\int_{t_0}^{\infty}\delta(s)M_1(s)\mathrm{d}s,$$

which contradicts (2.2) and so we conclude that $l = 0$. This completes the proof of the lemma. □

Next, we set

$$\alpha_1(t) = \frac{1}{t^2}\int_{t_0}^{t} s\delta(s)M_1(s)\mathrm{d}s, \quad \beta_1'(t) = \alpha_1(t), \quad \text{and} \quad \gamma_1(t) = e^{\beta_1(t)}.$$

Lemma 4. *Let (2.2) hold. If $\eta \in E_1$, then $z(t) = \frac{\eta(t)}{t}$ satisfies*

$$z(t)\gamma_1(t) \text{ is decreasing}. \tag{2.8}$$

Proof. Let $\eta \in E_1$; then $\eta(t)$ is a positive solution of (2.1) and $z(t)$ satisfies

$$z'(t) + \alpha_1(t)z(t) \le 0$$

by (2.7). Therefore,

$$(z(t)\gamma_1(t))' \le 0,$$

which proves the lemma. □

The next two lemmas give conditions under which the class E_1 is empty.

Lemma 5. *Let (2.2) hold. If*

$$\liminf_{t\to\infty}\int_{\delta(t)}^{t}\frac{\gamma_1(\delta(x))}{x^2}\left(\int_{t_0}^{x}\frac{s\delta(s)M_1(s)}{\gamma_1(\delta(s))}\mathrm{d}s\right)\mathrm{d}x > \frac{1}{e}, \tag{2.9}$$

then the class E_1 is empty.

Proof. Assume, to the contrary, that $\eta(t) \in E_1$. It follows from (2.7) that

$$0 \ge z'(t) + \frac{1}{t^2}\left(\int_{t_0}^{t} s\delta(s)M_1(s)z(\delta(s))\mathrm{d}s\right),$$

which in view of (2.8) implies that $z(t)$ is a positive solution of the differential inequality

$$z'(t) + \left(\frac{\gamma_1(\delta(t))}{t^2}\int_{t_0}^{t}\frac{s\delta(s)M_1(s)}{\gamma_1(\delta(s))}\mathrm{d}s\right)z(\delta(t)) \le 0.$$

But by [16, Theorem 2.1.1], and (2.9), we see that the last inequality has no positive solution. This contradiction completes the proof. □

Lemma 6. *Let (2.2) hold. If*

$$\liminf_{t\to\infty} t\int_t^\infty \frac{M_1(s)\delta(s)\gamma_1(s)}{s\gamma_1(\delta(s))}\mathrm{d}s > \frac{1}{4}, \tag{2.10}$$

then the class E_1 is empty.

Proof. Assume that $\eta(t)\in E_1$. Then, from Lemma 3, we have

$$\eta''(t)+M_1(t)\eta(\delta(t))\le 0. \tag{2.11}$$

Using (2.8) in (2.11) gives

$$\eta''(t)+M_1(t)\frac{\delta(t)}{t}\frac{\gamma_1(t)}{\gamma_1(\delta(t))}\eta(t)\le 0.$$

But by [15, Lemma 1], we see that $\eta(t)$ is a positive solution of the differential equation

$$\eta''(t)+M_1(t)\frac{\delta(t)}{t}\frac{\gamma_1(t)}{\gamma_1(\delta(t))}\eta(t)=0, \tag{2.12}$$

and in view of [15, Theorem 3] and (2.10), $\eta(t)$ is an oscillatory solution of (2.12). This contradiction proves the lemma. □

Next, we recall the following result from [25],

Lemma 7. *Let (2.2) hold. Then every positive solution $\eta(t)$ of (2.11) satisfies*

$$\eta(t)\ge t\int_t^\infty M_1(s)\eta(\delta(s))\mathrm{d}s+\int_{t_0}^t sM_1(s)\eta(\delta(s))\mathrm{d}s. \tag{2.13}$$

We have one more result that shows the class E_1 is empty. Set $\gamma_0(t)=\frac{\gamma_1(t)}{t}$.

Lemma 8. *Let (2.2) hold. If*

$$\limsup_{t\to\infty}\Big\{\delta(t)\int_t^\infty M_1(s)\mathrm{d}s+\gamma_1(\delta(t))\int_{\delta(t)}^t\frac{M_1(s)}{\gamma_0(\delta(s))}\mathrm{d}s+\gamma_0(\delta(t))\int_{t_0}^{\delta(t)}\frac{sM_1(s)}{\gamma_0(\delta(s))}\mathrm{d}s\Big\}>1, \tag{2.14}$$

then the class E_1 is empty.

Proof. Assume, to the contrary, that $\eta(t)\in E_1$. Then, as in the proof of from Lemma 6, we obtain equation (2.11). It follows from (2.13) that

$$\eta(\delta(t))\ge\delta(t)\int_t^\infty M_1(s)\eta(\delta(s))\mathrm{d}s+\delta(t)\int_{\delta(t)}^t M_1(s)\eta(\delta(s))\mathrm{d}s+\int_{t_0}^{\delta(t)} sM_1(s)\eta(\delta(s))\mathrm{d}s.$$

Taking into account that $\delta(t)\le t$, $\eta(\delta(t))$ is increasing and $z(t)\gamma_1(t)=\frac{\eta(t)}{t}\gamma_1(t)$ is decreasing, we obtain

$$\eta(\delta(t))\ge\eta(\delta(t))\Big\{\delta(t)\int_t^\infty M_1(s)\mathrm{d}s+\gamma_1(\delta(t))\int_{\delta(t)}^t\frac{M_1(s)}{\gamma_0(\delta(s))}\mathrm{d}s+\gamma_0(\delta(t))\int_{t_0}^{\delta(t)}\frac{sM_1(s)}{\gamma_0(\delta(s))}\mathrm{d}s\Big\},$$

where $\gamma_0(t)=\frac{\gamma_1(t)}{t}$. This contradicts (2.14), so the class E_1 is empty, and this completes the proof. □

Next, we turn our attention to the behavior of solutions in the class E_3.

Lemma 9. *Let $\eta(t) \in E_3$. Then,*

$$\eta(t) \geq \frac{\Omega_2(t)}{\Omega(t)}\eta''(t). \tag{2.15}$$

Proof. Since $\eta \in E_3$, we see that

$$\eta''(t) = \eta''(t_1) + \int_{t_1}^{t} \frac{\alpha^{\frac{1}{b}}(s)\eta'''(s)}{\alpha^{\frac{1}{b}}(s)} ds \geq \Omega(t)\alpha^{\frac{1}{b}}(t)\eta'''(t), \tag{2.16}$$

and so

$$\left(\frac{\eta''(t)}{\Omega(t)}\right)' = \frac{\Omega(t)\alpha^{\frac{1}{b}}(t)\eta'''(t) - \eta''(t)}{\alpha^{\frac{1}{b}}(t)\Omega^2(t)} < 0.$$

Hence, $\frac{\eta''(t)}{\Omega(t)}$ is decreasing and

$$\eta'(t) = \eta'(t_1) + \int_{t_1}^{t} \frac{\Omega(s)\eta''(s)}{\Omega(s)} ds \geq \Omega_1(t)\frac{\eta''(t)}{\Omega(t)}. \tag{2.17}$$

From (2.17), we have

$$\left(\frac{\eta'(t)}{\Omega_1(t)}\right)' = \frac{\Omega_1(t)\eta''(t) - \Omega(t)\eta'(t)}{\Omega_1^2(t)} < 0,$$

so $\frac{\eta'(t)}{\Omega_1(t)}$ is decreasing. On the other hand,

$$\eta(t) = \eta(t_1) + \int_{t_1}^{t} \Omega_1(s)\frac{\eta'(s)}{\Omega_1(s)} ds \geq \Omega_2(t)\frac{\eta'(t)}{\Omega_1(t)}. \tag{2.18}$$

Using (2.17) in (2.18), we obtain (2.15). This completes the proof. □

Lemma 10. *Let $b \geq 1$. Assume that $\mu(t)$ is a positive solution of (E) and $\eta(t) \in E_3$. Then $w(t) = \eta''(t)$ is a positive solution of the second-order differential equation*

$$\left(\alpha^{\frac{1}{b}}(t)w'(t)\right)' + \frac{1}{b}M_2(t)w(\delta(t)) = 0, \tag{2.19}$$

where $M_2(t) = M(t)\frac{\Omega_2^b(\delta(t))}{\Omega(\delta(t))}$.

Proof. Let $\mu(t)$ be an eventually positive solution of (E). Then there exists $t_1 \geq t_0$ such that $\mu(a(t)) > 0$ and $\mu(\delta(t)) > 0$ for all $t \geq t_1$. Then $\eta(t) > 0$ and satisfies inequality (2.1). Now

$$\left(\left(\alpha^{\frac{1}{b}}(t)\eta'''(t)\right)^b\right)' = b\left(\alpha^{\frac{1}{b}}(t)\eta'''(t)\right)^{b-1}\left(\alpha^{\frac{1}{b}}(t)\eta'''(t)\right)'$$

and using (2.1), we obtain

$$\left(\alpha^{\frac{1}{b}}(t)\eta'''(t)\right)' + \frac{1}{b}\left(\alpha^{\frac{1}{b}}(t)\eta'''(t)\right)^{1-b} M(t)\eta^b(\delta(t)) \leq 0. \tag{2.20}$$

From (2.16)–(2.18), we have

$$\eta(t) \geq \Omega_2(t)\alpha^{\frac{1}{b}}(t)\eta'''(t)$$

and so

$$\eta(\delta(t)) \geq \Omega_2(\delta(t))\alpha^{\frac{1}{b}}(\delta(t))\eta'''(\delta(t)) \geq \Omega_2(\delta(t))\alpha^{\frac{1}{b}}(t)\eta'''(t), \tag{2.21}$$

since $\alpha^{\frac{1}{b}}(t)\eta'''(t)$ is positive and decreasing. Also, since $b \geq 1$, from (2.21)

$$\left(\alpha^{\frac{1}{b}}(t)\eta'''(t)\right)^{1-b} \geq \Omega_2^{b-1}(\delta(t))\eta^{1-b}(\delta(t)).$$

Using the last inequality in (2.20) yields

$$\left(\alpha^{\frac{1}{b}}(t)\eta'''(t)\right)' + \frac{1}{b}M(t)\Omega_2^{b-1}(\delta(t))\eta(\delta(t)) \leq 0. \tag{2.22}$$

Let $w(t) = \eta''(t)$ and using (2.15) in (2.22), we obtain that $w(t)$ is a positive solution of the differential inequality

$$\left(\alpha^{\frac{1}{b}}(t)w'(t)\right)' + \frac{1}{b}M_2(t)w(\delta(t)) \leq 0.$$

But by [15, Lemma1 1], equation (2.19) also has a positive solution. This proves the lemma. □

Lemma 11. *Let $b \geq 1$ and $\eta \in E_3$. If*

$$\int_{t_0}^{\infty} \Omega(\delta(s))M_2(s)\mathrm{d}s = \infty, \tag{2.23}$$

then every positive solution $w(t)$ of (2.19) satisfies

$$\frac{w(t)}{\Omega(t)} \text{ is decreasing and } \lim_{t\to\infty}\frac{w(t)}{\Omega(t)} = 0. \tag{2.24}$$

Proof. Let $w(t) = \eta''(t)$; then from the proof of Lemma 9, $\frac{w(t)}{\Omega(t)}$ is decreasing. From (2.19), it is easy to see that

$$-\left(\alpha^{\frac{1}{b}}(t)\Omega^2(t)\left(\frac{w(t)}{\Omega(t)}\right)'\right)' = \frac{\Omega(t)}{b}M_2(t)w(\delta(t)). \tag{2.25}$$

Set $k = w(t_0) - \Omega(t_0)\alpha^{\frac{1}{b}}(t_0)w'(t_0)$; then integrating (2.25) from t_0 to t gives

$$\begin{aligned} -\alpha^{\frac{1}{b}}(t)\Omega^2(t)\left(\frac{w(t)}{\Omega(t)}\right)' &= k + \frac{1}{b}\int_{t_0}^{t}\Omega(s)M_2(s)w(\delta(s))\mathrm{d}s \\ &\geq k + \frac{w(\delta(t_0))}{b}\int_{t_0}^{t}\Omega(s)M_2(s)\mathrm{d}s \to \infty \text{ as } t\to\infty, \end{aligned} \tag{2.26}$$

by (2.23). Hence, $\left(\frac{w(t)}{\Omega(t)}\right)' < 0$ eventually, and we conclude that $\frac{w(t)}{\Omega(t)}$ is decreasing, which also implies $k > 0$ for t_0 large enough.

Now assume that

$$\lim_{t\to\infty}\frac{w(t)}{\Omega(t)}=l>0.$$

The function $p(t)=\frac{w(t)}{\Omega(t)}\geq 0$, and in view of (2.26), satisfies

$$p'(t)+\frac{1}{b\alpha^{\frac{1}{b}}(t)\Omega^2(t)}\int_{t_0}^{t}\Omega(s)\Omega(\delta(s))M_2(s)p(\delta(s))\mathrm{d}s=-\frac{k}{\alpha^{\frac{1}{b}}(t)\Omega^2(t)}<0. \quad (2.27)$$

Integrating (2.27) from t_0 to ∞ and then using integration by parts, gives

$$\begin{aligned}p(t_0)-l&\geq\frac{l}{b}\int_{t_0}^{\infty}\frac{1}{\alpha^{\frac{1}{b}}(s)\Omega^2(s)}\left(\int_{t_0}^{s}\Omega(x)\Omega(\delta(x))M_2(x)\mathrm{d}x\right)\mathrm{d}s\\&=\frac{l}{b}\int_{t_0}^{\infty}\Omega(\delta(s))M_2(s)\mathrm{d}s,\end{aligned}$$

which contradicts (2.23), and so $l=0$. This proves the lemma. □

Next, we introduce the notation

$$\alpha_2(t)=\frac{1}{b\alpha^{\frac{1}{b}}(t)\Omega^2(t)}\int_{t_0}^{t}\Omega(s)\Omega(\delta(s))M_2(s)\mathrm{d}s,\quad \beta_2'(t)=\alpha_2(t),\ \text{and}\ \ \gamma_2(t)=e^{\beta_2(t)}.$$

Lemma 12. *Let $b\geq 1$, $\eta\in E_3$ and (2.23) hold. Then for every positive solution $w(t)$ of (2.19), the function $p(t)=\frac{w(t)}{\Omega(t)}$ satisfies*

$$p(t)\gamma_2(t)\ \textit{is decreasing}. \quad (2.28)$$

Proof. From (2.27), the function $p(t)$ satisfies

$$p'(t)+\alpha_2(t)p(t)\leq 0.$$

Therefore,

$$(p(t)\gamma_2(t))'\leq 0,$$

which proves the result. □

In our next four lemmas we give conditions to ensure that the class E_3 is empty.

Lemma 13. *Let $b\geq 1$ and (2.23) hold. If*

$$\liminf_{t\to\infty}\int_{\delta(t)}^{t}\left(\frac{\gamma_2(\delta(x))}{\alpha^{\frac{1}{b}}(x)\Omega^2(x)}\int_{t_0}^{x}\frac{\Omega(s)\Omega(\delta(s))M_2(s)}{\gamma_2(\delta(s))}\mathrm{d}s\right)\mathrm{d}x>\frac{b}{e}, \quad (2.29)$$

then the class E_3 is empty.

Proof. Assume there exists $\eta(t)\in E_3$. It follows from (2.27) that

$$0\geq p'(t)+\frac{1}{b\alpha^{\frac{1}{b}}(t)\Omega^2(t)}\int_{t_0}^{t}\Omega(s)\Omega(\delta(s))M_2(s)p(\delta(s))\mathrm{d}s,$$

which in view of (2.28) implies that $p(t)$ is a positive solution of the differential inequality

$$p'(t)+\left(\frac{\gamma_2(\delta(t))}{b\alpha^{\frac{1}{b}}(t)\Omega^2(t)}\int_{t_0}^{t}\frac{\Omega(s)\Omega(\delta(s))M_2(s)}{\gamma_2(\delta(s))}\mathrm{d}s\right)p(\delta(t))\le 0.$$

But by [16, Theorem 2.1.1] and (2.29), we see that the last inequality has no positive solutions. This contradiction completes the proof of the lemma. □

Lemma 14. *Let $b\ge 1$ and (2.23) hold. If*

$$\liminf_{t\to\infty}\Omega(t)\int_t^{\infty}M_3(s)\mathrm{d}s>\frac{b}{4}, \tag{2.30}$$

where $M_3(t)=M_2(t)\frac{\Omega(\delta(t))}{\Omega(t)}\frac{\gamma_2(t)}{\gamma_2(\delta(t))}$, then the class E_3 is empty.

Proof. Assume that $\eta(t)\in E_3$. Then $w(t)=\eta''(t)$ satisfies (2.19), that is,

$$\left(\alpha^{\frac{1}{b}}(t)w'(t)\right)'+\frac{1}{b}M_2(t)w(\delta(t))=0,$$

by Lemma 10. By Lemma 12, the function $p(t)\gamma_2(t)=\frac{w(t)}{\Omega(t)}\gamma_2(t)$ is decreasing; using this in (2.19) yields

$$\left(\alpha^{\frac{1}{b}}(t)w'(t)\right)'+\frac{1}{b}M_2(t)\frac{\Omega(\delta(t))}{\gamma_2(\delta(t))}\frac{\gamma_2(t)}{\Omega(t)}w(t)\le 0.$$

But by [21, Lemma 1], $w(t)$ is a positive solution of the differential equation

$$\left(\alpha^{\frac{1}{b}}(t)w'(t)\right)'+\frac{1}{b}M_3(t)w(t)=0. \tag{2.31}$$

In view of [11, Theorem 3] and condition (2.30), equation (2.31) is oscillatory. This contradiction proves the lemma. □

Lemma 15. *Let $b\ge 1$ and (2.23) hold. If*

$$\limsup_{t\to\infty}\left\{\frac{\gamma_2(\delta(t))}{\Omega(\delta(t))}\int_{t_0}^{\delta(t)}\frac{\Omega(s)\Omega(\delta(s))M_2(s)}{\gamma_2(\delta(s))}\mathrm{d}s+\gamma_2(\delta(t))\int_{\delta(t)}^{t}\frac{\Omega(\delta(s))M_2(s)}{\gamma_2(\delta(s))}\mathrm{d}s+\Omega(\delta(t))\int_t^{\infty}M_2(s)\mathrm{d}s\right\}>b, \tag{2.32}$$

then the class E_3 is empty.

Proof. Assume that $\eta(t)\in E_3$. Then $\eta''(t)=w(t)$ satisfies (2.19), and integrating it from t to ∞ yields

$$w'(t)\ge\frac{1}{b}\frac{1}{\alpha^{\frac{1}{b}}(t)}\int_t^{\infty}M_2(s)w(\delta(s))\mathrm{d}s.$$

Integrating from $t_1 \geq t_0$ to t, gives

$$w(t) \geq \frac{1}{b}\int_{t_1}^{t} \frac{1}{\alpha^{\frac{1}{b}}(s)}\left(\int_s^{\infty} M_2(x)w(\delta(x))\mathrm{d}x\right)\mathrm{d}s,$$

and integrating by parts, we obtain

$$bw(t) \geq \Omega(t)\int_t^{\infty} M_2(s)w(\delta(s))\mathrm{d}s + \int_{t_1}^{t} \Omega(s)M_2(s)w(\delta(s))\mathrm{d}s.$$

Hence,

$$\begin{aligned} bw(\delta(t)) \geq{}& \Omega(\delta(t))\int_{\delta(t)}^{t} M_2(s)w(\delta(s))\mathrm{d}s + \Omega(\delta(t))\int_t^{\infty} M_2(s)w(\delta(s))\mathrm{d}s \\ &+ \int_{t_1}^{\delta(t)} \Omega(s)M_2(s)w(\delta(s))\mathrm{d}s. \end{aligned}$$

Taking into account that $w(t)$ is increasing, $\frac{w(t)}{\Omega(t)}\gamma_2(t)$ is decreasing, $\delta(t) \leq t$, and $\delta(t)$ is nondecreasing, we have

$$\begin{aligned} bw(\delta(t)) \geq w(\delta(t))\Bigg\{&\gamma_2(\delta(t))\int_{\delta(t)}^{t} \frac{\Omega(\delta(s))M_2(s)}{\gamma_2(\delta(s))}\mathrm{d}s \\ &+ \Omega(\delta(t))\int_t^{\infty} M_2(s)\mathrm{d}s + \frac{\gamma_2(\delta(t))}{\Omega(\delta(t))}\int_{t_1}^{\delta(t)} \Omega(s)\frac{\Omega(\delta(s))M_2(s)}{\gamma_2(\delta(s)))}\mathrm{d}s\Bigg\}. \end{aligned}$$

This contradicts (2.32) and completes the proof of the lemma. □

Without assuming $b \geq 1$, we obtain the following result.

Lemma 16. *Assume that μ is an eventually positive solution of (E). If*

$$\liminf_{t\to\infty} \Omega^b(\delta(t))\int_t^{\infty} M_4(s)\mathrm{d}s > \frac{b^b}{(b+1)^{(b+1)}}, \tag{2.33}$$

where $M_4(t) = M(t)\frac{\Omega_2^b(\delta(t))}{\Omega^b(\delta(t))}$, then the class E_3 is empty.

Proof. Assume, to the contrary, that $\eta(t) \in E_3$. From Lemma 2, we have (2.1). Now, using (2.15) from Lemma 9 in (2.1), we have

$$(\alpha(t)(\eta'''(t))^b)' + M_4(t)(\eta''(\delta(t)))^b \leq 0.$$

Let $w(t) = \eta''(t) > 0$. Then $w(t)$ is a positive solution of the differential inequality

$$(\alpha(t)(w'(t))^b)' + M_4(t)w^b(\delta(t)) \leq 0,$$

and by [15, Lemma 1], the differential equation

$$(\alpha(t)(w'(t))^b)' + M_4(t)w^b(\delta(t)) = 0$$

also has a positive solution. But [15, Theorem 3], and condition (2.33) imply that $w(t)$ is oscillatory. This contradiction completes the proof. □

Now we are ready to state and prove our oscillation theorems. Using Lemmas 5, 6, and 8 combined with Lemmas 13, 14 and 15, we can formulate several oscillation criteria for equation (E).

Theorem 1. *Let $b \geq 1$ and conditions (2.2) and (2.23) hold. If conditions (2.9) and (2.29) hold, then (E) is oscillatory.*

Proof. Let $\mu(t)$ be a positive solution of (E). Then there exists $t_1 \geq t_0$ such that $\mu(a(t)) > 0$ and $\mu(\delta(t)) > 0$ for all $t \geq t_1$. Then by Lemma 1, the function $\eta(t)$ is positive and either $\eta(t) \in E_1$ or $\eta(t) \in E_3$ for all $t \geq t_1$.

First assume that $\eta(t) \in E_1$. Then, by Lemma 5 we get a contradiction. So next assume $\eta(t) \in E_3$. Then by Lemma 13, we again have a contradiction. This completes the proof of the theorem. □

Theorem 2. *Let $b \geq 1$ and conditions (2.2) and (2.23) hold. If conditions (2.10) and (2.30) hold, then (E) is oscillatory.*

Proof. Let $\mu(t)$ be a positive solution of (E). Then either $\eta(t) \in E_1$ or $\eta(t) \in E_3$ for all $t \geq t_1$ for some $t_1 \geq t_0$.

First assume that $\eta(t) \in E_1$. Then by Lemma 6, we are led to a contradiction. Next, assume that $\eta(t) \in E_3$. Then by Lemma 14, we again obtain a contradiction. This completes the proof. □

Theorem 3. *Let $b \geq 1$ and conditions (2.2) and (2.23) hold. If conditions (2.14) and (2.32) are satisfied, then (E) is oscillatory.*

Proof. Let $\mu(t)$ be a positive solution of (E); then $\eta(t) \in E_1$ or $\eta(t) \in E_3$ for all $t \geq t_1 \geq t_0$. If $\eta(t) \in E_1$, then Lemma 8 gives a contradiction, and if $\eta(t) \in E_3$, then Lemma 15 yields a contradiction. This proves the theorem. □

Similar to those above, we can easily prove the following theorems.

Theorem 4. *Let $b \geq 1$ and conditions (2.2), (2.9), and (2.23) hold. If condition (2.30) or (2.32) is satisfied, then (E) is oscillatory.*

Theorem 5. *Let $b \geq 1$ and conditions (2.2), (2.10), and (2.23) hold. If condition (2.29) or (2.32) is satisfied, then (E) is oscillatory.*

Theorem 6. *Let $b \geq 1$ and conditions (2.2), (2.14), and (2.23) hold. If condition (2.29) or (2.30) is satisfied, then (E) is oscillatory.*

In our final theorem, we obtain a oscillation criterion without assuming that $b \geq 1$ or that (2.23) holds.

Theorem 7. *Let conditions (2.2) and (2.33) hold. If either condition (2.9), or (2.10), or (2.14) holds, then equation (E) is oscillatory.*

Proof. The proof follows from Lemma 16 along with either Lemma 5, or 6, or 8. This completes the proof of the theorem. □

3. EXAMPLES

In this section, we provide four examples to show how our results are improvements over previously known ones. Note that the covering assumptions (D1)–(D3) hold for all of these examples.

Example 1. Consider the fourth-order delay differential equation

$$\mu^{(4)}(t)+\frac{q_0}{t^4}\mu\left(\frac{9t}{10}\right)=0,\quad t\geq 1, \tag{3.1}$$

where $q_0>0$ is a constant. Comparing this to equation (E), we see that $\alpha(t)=1$, $d(t)=0$, $b=1$, $m(t)=\frac{q_0}{t^4}$, and $\delta(t)=\frac{9t}{10}$. Simple calculations show that $M(t)=\frac{q_0}{t^4}$, $M_1(t)=\frac{q_0}{6t^2}$, $\Omega(t)\approx t$, $\Omega_1(t)\approx\frac{t^2}{2}$, $\Omega_2(t)\approx\frac{t^3}{6}$, and $M_2(t)=\frac{27q_0}{200t^2}$. Conditions (2.2) and (2.23) become

$$\int_1^\infty \frac{9}{10}s\frac{q_0}{6s^2}\mathrm{d}s=\frac{3q_0}{20}\int_1^\infty\frac{1}{s}\mathrm{d}s=\infty$$

and

$$\int_1^\infty \frac{9s}{10}\frac{27}{200}\frac{q_0}{s^2}\mathrm{d}s=\frac{243}{2000}q_0\int_1^\infty\frac{1}{s}\mathrm{d}s=\infty,$$

respectively, so they both hold. Also, we see that $\gamma_1(t)=t^{\frac{3q_0}{20}}$ and $\gamma_2=t^{\frac{243q_0}{2000}}$. Condition (2.10) is satisfied if $q_0>3$, and condition (2.30) is satisfied if $q_0\geq 4$. Hence, by Theorem 2, equation (3.1) is oscillatory if $q_0\geq 4$.

Note that [6, Corrolary 2.2] shows that (3.1) is oscillatory if $q_0>6.17$. Hence, Theorem 2 improves Corollary 2.2 in [6]. Also, as indicated in [6], Theorem 2 would also improve the results in [12, 24].

Example 2. Consider the fourth-order neutral delay equation

$$\left(\frac{1}{t}(\eta'''(t)\right)'+\frac{q_0}{t^5}\mu\left(\frac{t}{2}\right)=0,\quad t\geq 1, \tag{3.2}$$

where $\eta(t)=\mu(t)+\frac{1}{2}\mu\left(\frac{t}{3}\right)$, $b=1$, and $q_0>0$. By comparing (E) to (3.2), we see that $\alpha(t)=\frac{1}{t}$, $a(t)=\frac{t}{3}$, $d(t)=\frac{1}{2}$, $\delta(t)=\frac{t}{2}$, and $m(t)=\frac{q_0}{t^5}$. In addition, $\Omega(t)\approx\frac{t^2}{2}$, $\Omega_1(t)\approx\frac{t^3}{6}$, $\Omega_2(t)\approx\frac{t^4}{24}$, $M(t)=\frac{q_0}{2t^5}$, $M_1(t)=\frac{q_0}{24t^2}$, and $M_2(t)=\frac{q_0}{96t^3}$. Conditions (2.2) and (2.23) are satisfied since they become

$$\int_1^\infty \frac{s}{2}\frac{q_0}{24s^2}\mathrm{d}s=\frac{q_0}{48}\int_1^\infty\frac{1}{s}\mathrm{d}s=\infty$$

and

$$\int_1^\infty \frac{s^2}{8}\frac{q_0}{96s^3}\mathrm{d}s=\frac{q_0}{768}\int_1^\infty\frac{1}{s}\mathrm{d}s=\infty.$$

Furthermore, it is easy to see that $\gamma_1(t)=t^{\frac{q_0}{48}}$ and $\gamma_2(t)=t^{\frac{q_0}{768}}$. Conditions (2.10) and (2.30) become $q_0(2)^{\frac{q_0}{48}}>12$ and $q_0(2)^{\frac{q_0}{768}}>384$, respectively. Hence for $q_0\geq 297$, all conditions of Theorem 2 are satisfied. Thus, equation (3.2) is oscillatory if $q_0\geq 297$.

Note that none of the results reported in [1, 5, 6, 9, 11, 17, 25] apply to equation (3.2) since $\alpha'(t) < 0$ for all $t \geq 1$. Hence, our results are new and more applicable than those mentioned above.

Example 3. Consider the fourth-order equation

$$\eta^{(4)}(t) + \frac{q_0}{t^4}\mu(\lambda t) = 0, \quad t \geq 1, \tag{3.3}$$

where $\eta(t) = \mu(t) + d_0\mu(\theta t)$, θ, λ, $d_0 \in (0,1)$, and $q_0 > 0$ is a constant. Here we have $\alpha(t) = 1$, $b = 1$, $d(t) = d_0$, $\delta(t) = \lambda t$, $a(t) = \theta t$, and $m(t) = \frac{q_0}{t^4}$. Calculation shows that $M(t) = \frac{q_0(1-d_0)}{t^4}$, $M_1(t) = \frac{q_0(1-d_0)}{6t^2}$, $\Omega(t) \approx t$, $\Omega_1(t) \approx \frac{t^2}{2}$, $\Omega_2(t) \approx \frac{t^3}{6}$, and $M_2(t) = \frac{q_0(1-d_0)\lambda^2}{6t^2}$. Conditions (2.2) and (2.23) are easily seen to be satisfied. We also have $\gamma_1(t) = t^{q_0(1-d_0)\frac{\lambda}{6}}$ and $\gamma_2(t) = t^{q_0(1-d_0)\frac{\lambda^3}{6}}$.

For $d_0 = \lambda = \frac{1}{2}$, we see that (2.9) holds if $q_0 > 8.3218$ and the (2.29) is satisfied if $q_0 > 30$. Hence, by Theorem 1, equation (3.3) is oscillatory if $q_0 > 30$, but [8, Theorem 4], shows that (3.3) is oscillatory provided $q_0 > 72$. Therefore, Theorem 1 improves Theorem 4 of [8].

For $d_0 = 0$ and $\lambda = \frac{1}{2}$, we see that condition (2.10) holds if $q_0 \geq 3$ and (2.30) holds if $q_0 \geq 12$. Hence, by Theorem 2, equation (3.3) with $d_0 = 0$ and $\lambda = \frac{1}{2}$ is oscillatory if $q_0 > 12$. However, by [21, Theorem 2], we see that the same equation (3.3) is oscillatory if $q_0 > 36$. Thus, our Theorem 2 improves Theorem 2 of [21].

Example 4. Consider the equation

$$\left(t(\mu'''(t))^3\right)' + \frac{q_0}{t^9}\mu^3\left(\frac{t}{2}\right) = 0, \quad t \geq 1, \tag{3.4}$$

where $q_0 > 0$ is a constant. We have $\alpha(t) = t$, $d(t) = 0$, $b = 3$, $m(t) = \frac{q_0}{t^9}$, and $\delta(t) = \frac{t}{2}$. In this case, $\Omega(t) \approx \frac{3}{2}t^{\frac{2}{3}}$, $\Omega_1(t) \approx \frac{9}{10}t^{\frac{5}{3}}$, $\Omega_2(t) \approx \frac{27}{80}t^{\frac{8}{3}}$, and $M_1(t) = q_0^{\frac{1}{3}}/4t^2$. Condition (2.2) becomes

$$\int_1^\infty \frac{t}{8}\frac{q_0^{\frac{1}{3}}}{t^2}\mathrm{d}t = \frac{q_0^{\frac{1}{3}}}{8}\int_1^\infty \frac{1}{t}\mathrm{d}t = \infty,$$

that is, (2.2) holds. Condition (2.33) is satisfied if $q_0 > 70.2332$. Furthermore, $\gamma_1(t) = t^{\frac{q_0^{\frac{1}{3}}}{8}}$ and (2.10) is satisfied if $q_0 > 8$. Hence, by Theorem 7, equation (3.4) is oscillatory if $q_0 > 70.2332$.

Note that [19, Theorem 3.1] gives oscillation of (3.4) provided $q_0 > 5832$. Therefore, Theorem 7 significantly improves Theorem 3.1 in [19].

4. Conclusion

In this paper, we obtained new oscillation criteria by first obtaining some new monotonic properties of the positive solutions of the equation. Our results are applicable to a broader class of fourth-order neutral type differential equations than

previously covered in the literature. Thus, the results obtained here generalize and improve many previously known results. This is demonstrated by some particular examples.

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AN ANALYTICAL APPROACH FOR SOLVING SYSTEM OF RICCATI EQUATIONS

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Abstract. In this paper, we investigate the solution of fractional system of Riccati equations. This system is important since it appears in several applications in science such as control theory. We use the operational matrix method to solve this system. The block-pulse operational matrices will initially assist in the reduction of the nonlinear fractional order Riccati-differential problem into an algebraic system. Benefits of this approach include inexpensive setup costs for the equations without the use of projection techniques like Galerkin, collocation, etc. In addition, we prove the convergence of the approximate solution using operational matrix method to the exact solution. Finally, we present two examples to provide numerical evidences to the efficiency of numerical approach used in this paper. We notice that the error is close to zero. Also, the approximate solutions are convergent to the exact solutions for different values of fractional derivative. Moreover, the approximate solutions approach to the solution of system of first order when the derivative order approaches to one.

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Keywords: Riccati equation, fractional derivative, operational matrices

1. INTRODUCTION

Day after day the fractional order calculus is gradually replacing the classical calculus. Due to their complexity compared to their integer order counterparts, fractional models necessitate more sophisticated computations, both analytically and numerically. Numerous researchers from around the world have recently investigated various aspects of fractional models using theoretical and numerical methods. Taking the Caputo fractional derivative as an example, Shah et al. [12] studied the Cauchy type dynamical system under piece-wise equations with fractional order Caputo derivative. A discrete analogue for the composition of the fractional integral and Caputo derivative has been proved by Płociniczak [11]. Fractional derivatives with non-singular kernels are also discussed by several researchers. For example, Baleanu et

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al. [2] used this type of derivative to find a new model in optimal control of a tumor-immune surveillance. Also, Caputo and Fabrizio [5] and Atangana and Baleanu [1] found more definitions for fractional derivatives with non-singular kernels. One of the most interesting nonlinear differential equations is the Riccati equation. The Riccati equation is applied in both physics and other fields of mathematics, such as algebraic geometry and conformal mapping theory. It also shows up in lots of real-world problems. Many researchers have been interested in various methods for determining estimate solutions to the Riccati differential equations. Recently, a variety of numerical techniques have been established to solve the fractional differential equations. Fathy and Abdelgaber [6] have been discussed and used the Galerkin method to provide fractional order semi-analytical solutions to the quadratic Riccati and Bagley-Torvik differential equations. Also, they compared the proposed method with other methods to prove the efficiency and reliability of Galerkin method. Khashan and Syam [8] used the the fractional series method to solve a special type of Riccati equation. In addition, Kashkari and Syam [7] solved this type of equations using the the implicit hybrid block method. Operational matrix method is one of the useful tools to use in solving fractional differential equations. Baleanu et al. [3] used Bernstein operational matrices to solve fractional differential equations using the Riemann-Liouville derivative. More researchers used different types of operation matrices to solve fractional initial value problems such as Bhrawy et al. [4] and Shiri and Baleanu [13]. A set of nonlinear Riccati-differential equations can be solved by using the operational block-pulse functions. The block-pulse operational will initially assist in the reduction of the nonlinear fractional order Riccati-differential problem into an algebraic system. Benefits of this approach include inexpensive setup costs for the equations without the use of projection techniques like Galerkin, collocation, etc. In this article, we study the following system of Riccati equations

$$D^{\gamma}G(t) = \Lambda_{21} + \Lambda_{22}G(t) - G(t)\Lambda_{11} - G(t)\Lambda_{12}G(t) \tag{1.1}$$

with

$$G(0) = Q, \tag{1.2}$$

where $\Lambda_{11}, \Lambda_{12}, \Lambda_{21}$, and Λ_{22} are $1\times 1, 1\times k_2, k_2\times 1$, and $k_2\times k_2$ real matrices, respectively. Also, $G(t)$ and G_0 are of size $k_2\times 1$ matrices. We should note that $\Lambda_{ij}, 1\leq i,j\leq 2$, are matrices depend on the variable t in most of the cases.

This article will be organized as follows. In the next section, we give the main definitions and theorems which we use in this article. In Section 3, we derive the operational matrices which we need to solve the system of Riccati equations. Then, we discuss the convergence analysis in Section 4 and some numerical results will be given in Section 5. Finally, we give some conclusions and discussion for our results in the last section.

2. PRELIMINARIES

Two important concepts which we use in our paper are fractional derivative and block pulse functions (BPF). For this reason, we start this section by the definition of the Caputo derivative and its properties [10, 13].

Definition 1 ([14, 15])**.** Let $\iota \in (0,1)$ be a positive real number and λ be a real number. Then,

$$D^{\iota}\alpha(\lambda) = \frac{1}{\Gamma(1-\iota)} \int_0^{\lambda} (\lambda - q)^{-\iota} \alpha'(q) \mathrm{d}q \tag{2.1}$$

is called the Caputo derivative of order $\iota \in (0,1)$. The fractional integral operator is given as

$$I^{\iota}\alpha(\lambda) = \frac{1}{\Gamma(\iota)} \int_0^{\lambda} (\lambda - q)^{\iota-1} \alpha(q) \mathrm{d}q. \tag{2.2}$$

Let $\xi = \lambda q$. Simple calculations yield to

$$D^{\iota}q^{\mu} = \begin{cases} 0, & \mu < \iota, \mu \in \{0,1,2,\ldots\}, \\ \frac{\Gamma(\mu+1)}{\Gamma(\mu-\iota+1)} q^{\mu-\iota}, & \text{otherwise.} \end{cases}, \iota > 0. \tag{2.3}$$

The relations between the two operators in Definition 1 are given as follows:

$$D^{\iota}I^{\iota}\alpha(q) = \alpha(q) \tag{2.4}$$

and

$$I^{\iota}D^{\iota}\alpha(q) = \alpha(q) - \alpha(0). \tag{2.5}$$

The second concept which we need is the BPF which is given by the following definition.

Definition 2 ([9, 14, 15])**.** Let L be a natural number and η be a real number. Then, for $l \in \{0,1,\ldots,L-1\}$, the lth BPF is given as

$$\varsigma_l(q) = \begin{cases} 1, & lz \le q < (l+1)z, \\ 0, & q \in [0,\eta) - [lz,(l+1)z) \end{cases} \tag{2.6}$$

with $z = \frac{\eta}{L}$.

We can write these functions in vector form as

$$\varsigma(q) = \begin{pmatrix} \varsigma_0(q) \\ \varsigma_1(q) \\ \ldots \\ \varsigma_{L-1}(q) \end{pmatrix}. \tag{2.7}$$

Here are two important properties of the BPF which are given as

$$\varsigma_{l_1}(q)\varsigma_{l_2}(q) = \begin{cases} \varsigma_{l_1}(q), & l_1 = l_2, \\ 0, & l_1 \neq l_2 \end{cases} \tag{2.8}$$

and

$$\int_0^{\eta} \varsigma_{l_1}(q)\varsigma_{l_2}(q)\mathrm{d}q = \begin{cases} z, & l_1 = l_2, \\ 0, & l_1 \neq l_2 \end{cases} \tag{2.9}$$

for $0 \leq l_1, l_2 \leq L-1$. The third property is the completeness which is given as follows.

Theorem 1. *If α is square integrable function on $[0,\eta)$, then*

$$\alpha(q) = \sum_{l_1=0}^{\infty} \alpha_{l_1}\varsigma_{l_1}(q), \tag{2.10}$$

where

$$\alpha_{l_1} = \frac{1}{z}\int_{l_1 z}^{(l_1+1)z} \alpha(q)\mathrm{d}q. \tag{2.11}$$

Proof. The result will follow directly from equations (2.8) and (2.9) after multiplying equation (2.10) by $\varsigma_{l_2}(q)$. □

If we take the first L terms of equations (2.10), then we can approximate $\alpha(q)$ as

$$\alpha(q) \approx \alpha\varsigma(q), \tag{2.12}$$

where $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_{L-1})$.

3. Operational matrices for system of Riccati equations

Operational matrices are the main components of the proposed method. Hence, we start by deriving them in this section. Let us approximate the coefficient matrices and the function $G(t)$ of equation (1.1) in terms of the BPFs as

$$\Lambda_{ij}(t) = \sum_{l_1=0}^{L-1} \Lambda_{ij,l_1}\varsigma_{l_1}(t), \tag{3.1}$$

$$G(t) = \sum_{l_1=0}^{L-1} G_{l_1}\varsigma_{l_1}(t), \tag{3.2}$$

where Λ_{ij,l_1} are $1 \times k_2$ matrices and G_{l_1} are $k_2 \times 1$ matrices for $i, j = 1, 2$. Since

$$\varsigma_{l_1}(0) = \begin{cases} 1, & l_1 = 0, \\ 0, & l_1 \neq 0, \end{cases}$$

then $G_0 = G(0) = Q$. Then, system (1.1)-(1.2) becomes

$$D^{\gamma}G(t) = \sum_{l_1=0}^{L-1} \Lambda_{21,l_1}\varsigma_{l_1}(t) + \left(\sum_{l_1=0}^{L-1} \Lambda_{22,l_1}\varsigma_{l_1}(t)\right)\left(\sum_{l_1=0}^{L-1} G_{l_1}\varsigma_{l_1}(t)\right) - \left(\sum_{l_1=0}^{L-1} G_{l_1}\varsigma_{l_1}(t)\right)\left(\sum_{l_1=0}^{L-1} \Lambda_{11,l_1}\varsigma_{l_1}(t)\right)$$

$$-\left(\sum_{l_1=0}^{L-1} G_{l_1}\varsigma_{l_1}(t)\right)\left(\sum_{l_1=0}^{L-1}\Lambda_{12,l_1}\varsigma_{l_1}(t)\right)\left(\sum_{l_1=0}^{L-1} G_{l_1}\varsigma_{l_1}(t)\right),$$

which can be written using equation (2.8) as

$$D^{\gamma}G(t)=\sum_{l_1=0}^{L-1}\Lambda_{21,l_1}\varsigma_{l_1}(t)+\sum_{l_1=0}^{L-1}\Lambda_{22,l_1}G_{l_1}\varsigma_{l_1}(t)$$
$$-\sum_{l_1=0}^{L-1}G_{l_1}\Lambda_{11,l_1}\varsigma_{l_1}(t)-\sum_{l_1=0}^{L-1}G_{l_1}\Lambda_{12,l_1}G_{l_1}\varsigma_{l_1}(t).$$

Using equation (2.5), we get

$$G(t)-G(0)=\sum_{l_1=0}^{L-1}\left(\Lambda_{21,l_1}+\Lambda_{22,l_1}G_{l_1}-G_{l_1}\Lambda_{11,l_1}-G_{l_1}\Lambda_{12,l_1}G_{l_1}\right)I^{\gamma}\varsigma_{l_1}(t). \tag{3.3}$$

For any $t\in[0,\eta)$, there exists non-negative integer l_1 such that $t\in[l_1z,(l_1+1)z)$. Thus,

$$\varsigma_{l_2}(t)=\begin{cases}1, & l_2=l_1,\\ 0, & l_2\neq l_1.\end{cases}$$

Then,

$$\sum_{l_1=0}^{L-1}\varsigma_{l_1}(t)=1. \tag{3.4}$$

Now, $G(0)=Q=G_0$ and can be written using equations (2.8) and (2.9) as

$$G(0)=G_0\sum_{l_1=0}^{L-1}\varsigma_{l_1}(t)=G_0. \tag{3.5}$$

Hence, equations (3.3) and (3.5) imply that

$$\sum_{l_1=0}^{L-1}\left(G_{l_1}-G_0I_{k_1}\right)\varsigma_{l_1}(t)=\sum_{l_1=0}^{L-1}\left(\Lambda_{21,l_1}+\Lambda_{22,l_1}G_{l_1}-G_{l_1}\Lambda_{11,l_1}-G_{l_1}\Lambda_{12,l_1}G_{l_1}\right)I^{\gamma}\varsigma_{l_1}(t). \tag{3.6}$$

Let

$$A_{11}=\begin{pmatrix}\Lambda_{11,0}\\ \Lambda_{11,1}\\ \cdots\\ \Lambda_{11,L-1}\end{pmatrix},\ A_{12}=\begin{pmatrix}\Lambda_{12,0}\\ \Lambda_{12,1}\\ \cdots\\ \Lambda_{12,L-1}\end{pmatrix},\ A_{21}=\begin{pmatrix}\Lambda_{21,0} & 0 & 0 & \ldots & 0\\ 0 & \Lambda_{21,1} & 0 & \ldots & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & 0 & \ddots & 0\\ 0 & 0 & 0 & \ldots & \Lambda_{21,L-1}\end{pmatrix},$$

$$A_{22}=\begin{pmatrix}\Lambda_{22,0}\\ \Lambda_{22,1}\\ \dots\\ \Lambda_{22,L-1}\end{pmatrix},\ \Omega(t)=\begin{pmatrix}\varsigma_0(t)\\ \varsigma_1(t)\\ \dots\\ \varsigma_{L-1}(t)\end{pmatrix},\ G=\begin{pmatrix}G_0\\ G_1\\ \dots\\ G_{L-1}\end{pmatrix},\ G_0=\begin{pmatrix}G_0\\ G_0\\ \dots\\ G_0\end{pmatrix}.$$

We define the operator $*$ on the matrix of block matrices as follows

$$A^*=\begin{pmatrix}A_0\\ A_1\\ \dots\\ A_{L-1}\end{pmatrix}^*=\begin{pmatrix}A_0 & A_1 & \dots & A_{L-1}\end{pmatrix}. \tag{3.7}$$

Then, equations (3.2) and (3.5) imply that

$$G(t)=G^*\Omega(t), G_0=G_0^*\Omega(t). \tag{3.8}$$

Therefore, we can rewrite equation (3.6) as

$$(G^*-G_0^*)\,\Omega(t)=(A_{21}^*+A_{22}^*G-G^*A_{11}-G^*A_{12}G)\,I^\gamma\Omega(t). \tag{3.9}$$

To complete the derivation of the proposed method, we need to compute the operational matrix of the operator $I^\gamma\omega(t)$ which is given in the following theorem.

Theorem 2. *The operational matrix of the operator $I^\gamma\Omega$ is*

$$\Pi=\frac{z^\alpha}{\Gamma(\gamma+2)}\begin{pmatrix}1 & \eta_1 & \eta_2 & \dots & \eta_{L-2} & \eta_{L-1}\\ 0 & 1 & \eta_1 & \dots & \eta_{L-3} & \eta_{L-2}\\ 0 & 0 & 1 & \dots & \eta_{L-4} & \eta_{L-3}\\ \dots & \dots & \dots & \ddots & \ddots & \dots\\ 0 & 0 & 0 & \dots & 1 & \eta_1\\ 0 & 0 & 0 & \dots & 0 & 1\end{pmatrix}, \tag{3.10}$$

where $\eta_r=(r+1)^{\gamma+1}-2r^{\gamma+1}+(r-1)^{\gamma+1}$, $r=1,2,...,L-1$.

Proof. Let $r\in\{0,1,...,L-1\}$. Then,

$$I^\gamma\varsigma_r(t)=\frac{1}{\Gamma(\gamma)}\int_0^t(t-s)^{\gamma-1}\varsigma_r(s)ds=\begin{cases}0, & t<rz,\\ \frac{(t-rz)^\gamma}{\Gamma(\gamma+1)}, & rz\le t<(r+1)z,\\ \frac{(t-rz)^\gamma-(t-rz-z)^\gamma}{\Gamma(\gamma+1)}, & (r+1)z\le t<\eta.\end{cases}$$

If

$$I^\gamma\varsigma_r(t)=\sum_{l1=0}^{L-1}a_{l_1,r}\varsigma_r(t),$$

then

$$a_{l_1,r}=\frac{1}{z}\int_0^\eta\left(I^\gamma\varsigma_{l_1}(s)\right)\varsigma_r(s)\mathrm{d}s=\frac{1}{z}\int_{rz}^{(r+1)z}\left(I^\gamma\varsigma_{l_1}(s)\right)\mathrm{d}s$$

$$= \begin{cases} \frac{z^{\gamma}}{\Gamma(\gamma+2)}, & 0 \le l_1 = r \le L-1, \\ \frac{z^{\gamma}\left(r-l_1+1\right)^{\gamma+1}-2(r-l_1)^{\gamma+1}+(r-l_1-1)^{\gamma+1}\right)}{\Gamma(\gamma+2)}, & 0 \le l_1 < r \le L-1, \\ 0, & 0 \le r < l_1 \le L-1, \end{cases}$$

which completes the proof of the theorem. □

Now, equations (3.9) and (3.10) imply that

$$(G^* - G_0^*)\,\Omega(t) = (A_{21}^* + A_{22}^* G - G^* A_{11} - G^* A_{12} G)\,\Pi\Omega(t). \tag{3.11}$$

Since $\{\varsigma_0(t), \varsigma_1(t), \ldots, \varsigma_{L-1}(t)\}$ are linearly independent, then

$$(A_{21}^* + A_{22}^* G - G^* A_{11} - G^* A_{12} G)\,\Pi + G_0^* - G^* = 0. \tag{3.12}$$

We use Mathematica to solve the algebraic system (3.12).

4. Convergence Analysis

Let $G\colon [0,\eta) \longrightarrow \Re^{k_2\times 1}$ be a square integrable function with $\eta \ge 1$. Define the following norm

$$\|G\| = \sqrt{\int_0^{\eta} \|G\|_E^2 \mathrm{d}t}, \tag{4.1}$$

where $\|G\|_E$ is the Euclidean norm. Using equation (3.2), we have

$$G_L(t) = \sum_{l_1=0}^{L-1} G_{l_1}\varsigma_{l_1}(t) \tag{4.2}$$

For the first theorem, we want to prove that the mean squared error reaches a minimum when G_{l_1} is given by

$$G_{l_1} = \frac{1}{z}\int_0^{\eta} G(s)\varsigma_{l_1}(s)\mathrm{d}s. \tag{4.3}$$

Theorem 3. *Let G be a square integrable on $[0,\eta)$. Then, the least squares error*

$$E(G_0, G_1, \ldots, G_{L-1}) = \int_0^{\eta} \|G(s) - G_L(s)\|_E^2 \mathrm{d}s$$

achieves its minimum value at G_{l_1} given by formula (4.3) for $l_1 = 0, 1, \ldots, L-1$.

Proof. For any $l_1 \in \{0, 1, \ldots, L-1\}$, we have

$$\frac{\partial E}{\partial G_{l_1}} = -2\int_0^{\eta} (G(s) - G_L(s))\varsigma_{l_1}(s)\mathrm{d}s = -2\left(\int_0^{\eta} G(s)\varsigma_{l_1}(s)\mathrm{d}s - zG_{l_1}\right) = 0.$$

Hence,

$$G_{l_1} = \frac{1}{z}\int_0^{\eta} G(s)\varsigma_{l_1}(s)\mathrm{d}s.$$

Equations (2.8) and (2.9) imply that

$$\frac{\partial^2 E}{\partial G_{l_1}\partial G_{l_2}} = 2I_{k_2}\int_0^{\eta} \varsigma_{l_1}(s)\varsigma_{l_2}(s)\mathrm{d}s = \begin{cases} 2zI_{k_2}, & l_1 = l_2 \\ 0, & l_1 \neq l_2 \end{cases}, 0 \le l_1, l_2 \le L-1.$$

Thus, if $0 \le l_1 \le L-1$, we have

$$\begin{vmatrix} \frac{\partial^2 E}{\partial G_0^2} & \frac{\partial^2 E}{\partial G_0 \partial G_1} & \cdots & \frac{\partial^2 E}{\partial G_0 \partial G_{l_1}} \\ \frac{\partial^2 E}{\partial G_1 \partial G_0} & \frac{\partial^2 E}{\partial G_1^2} & \cdots & \frac{\partial^2 E}{\partial G_1 \partial G_{l_1}} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial^2 E}{\partial G_{l_1} \partial G_0} & \frac{\partial^2 E}{\partial G_{l_1} \partial G_1} & \cdots & \frac{\partial^2 E}{\partial G_{l_1}^2} \end{vmatrix} = \begin{vmatrix} 2zI_{k_2} & 0 & \cdots & 0 \\ 0 & 2zI_{k_2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 2zI_{k_2} \end{vmatrix} = (2z)^{l_1+1} > 0.$$

Thus, E achieves its minimum value at G_{l_1} given by formula (4.3) for $l_1 = 0, 1, \ldots, L-1$. □

Theorem 4. *If $G(s)$ is bounded continuous function in $[0,\eta)$, then $\{G_{l_1}(s)\}$ converges pointwise to $G(s)$ on $[0,\eta)$. In addition,*

$$\int_0^{\eta} \|G(s)\|_E^2 \mathrm{d}s = \sum_{l_1=0}^{\infty} \|G_{l_1}\|_E^2 \|\varsigma_{l_1}\|^2.$$

Proof. Theorem 2 implies that

$$G_{l_1} = \frac{L}{\eta}\int_{\frac{l_1\eta}{L}}^{\frac{(l_1+1)\eta}{L}} G(s)\mathrm{d}s, l_1 = 0, 1, \ldots, L-1. \tag{4.4}$$

For $l_1 = 0$, partition the interval $[0, \frac{\eta}{L}]$ into j uniform subintervals of length a. Then,

$$G_0 = \frac{L}{\eta}\int_0^{\frac{\eta}{L}} G(s)\mathrm{d}s = \frac{L}{\eta}\lim_{a\to 0}\sum_{l_1=0}^{j-1} G(s_{l_1})a = \frac{L}{\eta}\lim_{a\to 0}\sum_{l_1=0}^{j-1} G(l_1 a)\frac{\eta}{jL} = \lim_{a\to 0}\sum_{l_1=0}^{j-1}\frac{G(l_1 a)}{j}.$$

The continuity of the function G yields to

$$\lim_{L\to\infty} G_0 = \lim_{L\to\infty}\lim_{a\to 0}\sum_{l_1=0}^{j-1}\frac{G(l_1 a)}{j} = \lim_{a\to 0}\sum_{l_1=0}^{j-1}\frac{G(\lim_{L\to\infty}\frac{l_1\eta}{Lj})}{j} = \lim_{a\to 0}\sum_{l_1=0}^{j-1}\frac{G(0)}{j} = G(0).$$

Similarly, if $s_{l_1} \in [0,\eta)$, then

$$\lim_{L\to\infty} G_{l_1} = G(s_{l_1}).$$

Hence, $\{G_L(s)\}$ converges pointwise to $G(s)$ on $[0,\eta)$. Therefore,

$$\int_0^{\eta} \|G(s)\|_E^2 \mathrm{d}s = \lim_{L\to\infty}\int_0^{\eta} \|G_L\|_E^2 \mathrm{d}s = \lim_{L\to\infty}\sum_{l_1=0}^{L-1} \|G_{l_1}\|_E^2 \int_0^{\eta} \varsigma_{l_1}(s)^2 \mathrm{d}s$$

$$= \sum_{l_1=0}^{\infty} \|G_{l_1}\|_E^2 \|\varsigma_{l_1}\|^2.$$

□

In the next theorem, the order of the mean square error is investigated.

Theorem 5. *If $\xi > 0$ and $G(t)$ is differentiable function on $[0,\eta)$ with*

$$\|G'(t)\|_E \leq \xi, t \in [0,\eta), \tag{4.5}$$

then

$$\|G - G_L\|_E^2 \leq \alpha z^2. \tag{4.6}$$

Proof. If $s_{l_1} = l_1 z$, then the mean value theorem for integrals gives us

$$G_L(s) = G_{l_1}, s \in [s_{l_1}, s_{l_1+1}) \tag{4.7}$$

$$= \frac{1}{z}\int_{s_{l_1}}^{s_{l_1+1}} G(s)\mathrm{d}s \tag{4.8}$$

$$= G(\zeta_{l_1}), \zeta_{l_1} \in [s_{l_1}, s_{l_1+1}), l_1 = 0,1,\ldots,L-1. \tag{4.9}$$

Mean value theorem for integrals implies that

$$\|G(t) - G_L(t)\|^2 = \int_0^{\eta} \|G(s) - G_L(s)\|_E^2 \mathrm{d}s = \sum_{l_1=0}^{L-1} \int_{s_{l_1}}^{s_{l_1+1}} \|G(s) - G_L(s)\|_E^2 \mathrm{d}s$$

$$= \sum_{l_1=0}^{L-1} \int_{s_{l_1}}^{s_{l_1+1}} \|G(s) - G(\zeta_{l_1})\|_E^2 \mathrm{d}s = z\sum_{l_1=0}^{L-1} \|G(\omega_{l_1}) - G(\zeta_{l_1})\|_E^2,$$

where $\omega_{l_1}, \zeta_{l_1} \in [s_{l_1}, s_{l_1+1}), l_1 = 0,1,\ldots,L-1$. Therefore, mean value theorem yields to

$$\|G(t) - G_L(t)\|^2 \leq z\xi^2 \sum_{l_1=0}^{L-1} (\omega_{l_1} - \zeta_{l_1})^2 \leq z\xi^2 \sum_{l_1=0}^{L-1} z^2 = \alpha z^2,$$

where $\alpha = \xi^2\eta$. □

In the next theorem we study the pointwise convergence of the product of two sequences.

Theorem 6. *Let $G_1(t), G_2(t)$, and $G_3(t)$ be continuous bounded functions in $[0,\eta)$ such that G_1G_2 and $G_1G_2G_3$ are well-defined. Then,*

1: *$\{G_{1,L}(t)G_{2,L}(t)\}$ converges pointwise to $G_1(t)G_2(t)$,*
2: *$\{G_{1,L}(t)G_{2,L}(t)G_{3,L}(t)\}$ converges pointwise to $G_1(t)G_2(t)G_3(t)$.*

Proof. Let $G_1(t)$ and $G_2(t)$ be continuous bounded functions in $[0,\eta)$. Then, there exists $\xi_1, \xi_2 \in \mathbb{R}$ such that

$$\|G_1(t)\| \leq \xi_1, \|G_2(t)\| \leq \xi_2. \tag{4.10}$$

Let $0 < \alpha \leq \{3\xi_1\xi_2\}$. Theorem 4 implies that $\{G_{1,L}(t)\}$ and $\{G_{2,L}(t)\}$ converges pointwise to $G_1(t)$ and $G_2(t)$, respectively. Thus, there exists positive integer $M(\alpha)$ such that

$$\|G_{1,L}(t) - G_1(t)\| < \frac{\alpha}{3\xi_2}, \|G_{2,L}(t) - G_2(t)\| < \frac{\alpha}{3\xi_1}, L \geq M(\alpha).$$

Then, for $L \geq M(\alpha)$, we have

$$\begin{aligned}\|G_{1,L}(t)G_{2,L}(t) - G_1(t)G_2(t)\| &\leq \|(G_{1,L}(t) - G_1(t))(G_{2,L}(t) - G_2(t))\| \\ &\quad + \|G_1(t)(G_{2,L}(t) - G_2(t))\| \\ &\quad + \|G_2(t)(G_{1,L}(t) - G_1(t))\| \\ &\leq \frac{\alpha^2}{9\xi_1\xi_2} + \frac{\xi_1\alpha}{3\xi_1} + \frac{\xi_2\alpha}{3\xi_2} \leq \alpha. \end{aligned} \tag{4.11}$$

Hence, $\{G_{1,L}(t)G_{2,L}(t)\}$ converges pointwise to $G_1(t)G_2(t)$. Part (2) follows directly from the first part. □

Now we prove the convergence theorem for problem (1.1)-(1.2).

Theorem 7. *Let $G(t)$ and $G_L(t)$ be the exact and approximate solutions of equations (1.1) and (1.2). If $G(t)$ is continuous bounded function in $[0,\eta)$, then, $\{G_L(t)\}$ converges pointwise to $G(t)$.*

Proof. Let $G(t)$ and $G_L(t)$ be the exact and approximate solutions of equations (1.1) and (1.2), respectively. Then,

$$\begin{aligned} D^{\gamma}G(t) &= \Lambda_{21} + \Lambda_{22}G(t) - G(t)\Lambda_{11} - G(t)\Lambda_{12}G(t), \\ D^{\gamma}G_L(t) &= \Lambda_{21} + \Lambda_{22}G_L(t) - G_L(t)\Lambda_{11} - G_L(t)\Lambda_{12}G_L(t). \end{aligned}$$

By taking the fractional integral operator for both sides of the last two equations, we get

$$\begin{aligned} G(t) - Q &= \frac{1}{\Gamma(\gamma)}\int_0^t (\Lambda_{21} + \Lambda_{22}G(s) - G(s)\Lambda_{11} - G(s)\Lambda_{12}G(s))\,\mathrm{d}s, \\ G_L(t) - Q &= \frac{1}{\Gamma(\gamma)}\int_0^t (\Lambda_{21} + \Lambda_{22}G_L(s) - G_L(s)\Lambda_{11} - G_L(s)\Lambda_{12}G_L(s))\,\mathrm{d}s. \end{aligned}$$

Then,

$$\begin{aligned} G(t) - G_L(t) = \frac{1}{\Gamma(\gamma)}\int_0^t &(\Lambda_{22}(G(s) - G_L(s)) - (G(s) - G_L(s))\Lambda_{11} \\ &- (G(s)\Lambda_{12}G(s) - G_L(s)\Lambda_{12}G_L(s)))\mathrm{d}s. \end{aligned}$$

Then,

$$\begin{aligned} \|G - G_L\|_E &\leq \frac{1}{\Gamma(\gamma)}\int_0^t (\|G - G_L\|_E(\|\Lambda_{22}\|_E + \|\Lambda_{11}\|_E) + \|G\Lambda_{12}G - G_L\Lambda_{12}G_L\|_E)\,\mathrm{d}s \\ &= \frac{\eta}{\Gamma(\gamma)}(\|G - G_L\|_E(\|\Lambda_{22}\|_E + \|\Lambda_{11}\|_E) + \|G\Lambda_{12}G - G_L\Lambda_{12}G_L\|_E). \end{aligned}$$

We can simplify the last equation as

$$\|G - G_L\|_E \leq \frac{\upsilon\eta}{\Gamma(\gamma)}(\|G\Lambda_{12}G - G_L\Lambda_{12}G_L\|_E),$$

where $\upsilon = \frac{\eta}{\Gamma(\gamma)-\eta\|\Lambda_{22}\|_E-\eta\|\Lambda_{11}\|_E}$. Using the second part of Theorem 6, for any $\xi > 0$, there exists $N(\xi) \in \mathbb{N}$ such that

$$\|G\Lambda_{12}G - G_L\Lambda_{12}G_L\|_E \leq \xi\frac{\Gamma(\gamma)}{\upsilon\eta}, L \geq N(\xi).$$

Therefore, $\|G - G_L\|_E \leq \xi$, which gives the result of the theorem. □

5. Numerical results

In this section, two examples will be given to show the efficiency of the operational matrix method. In this section, we use $L = 20$.

Example 1. Consider the following system of Riccati equations

$$D^\gamma G(t) = \Lambda_{21} + \Lambda_{22}G(t) - G(t)\Lambda_{11} - G(t)\Lambda_{12}G(t), G(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

where $h_1(t) = 2t^\gamma - 2t^{2\gamma} + 1, h_2(t) = t^\gamma - 4t^{2\gamma} + t^{3\gamma} - 1, \Lambda_{22} = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}, \Lambda_{11} = 2,$

$$\Lambda_{21} = \begin{pmatrix} 2\Gamma(\gamma+1) - \frac{2\Gamma(2\gamma+1)}{\Gamma(\gamma+1)}t^\gamma + h_1^2(t) + 2h_1(t)h_2(t) - h_2(t) \\ \frac{\Gamma(3\gamma+1)}{\Gamma(2\gamma+1)}t^{2\gamma} - \frac{4\Gamma(2\gamma+1)}{\Gamma(\gamma+1)}t^\gamma + \Gamma(\gamma+1) + 2h_2^2(t) + h_1(t)h_2(t) - 3h_1(t) \end{pmatrix},$$

$\Lambda_{12} = \begin{pmatrix} 1 & 2 \end{pmatrix}$. Then, the exact solution is $G(t) = \begin{pmatrix} 2t^\gamma - 2t^{2\gamma} + 1 \\ t^\gamma - 4t^{2\gamma} + t^{3\gamma} - 1 \end{pmatrix}$.

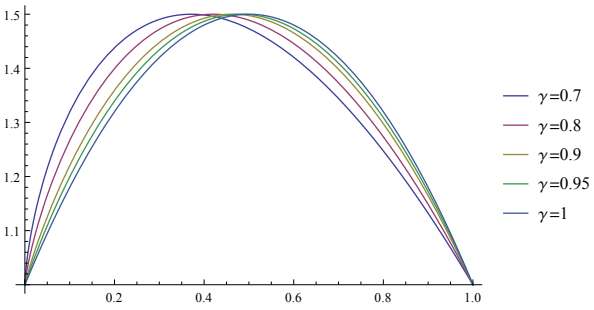

FIGURE 1. The approximate solution $G_1(t)$ for different values of γ

Then, the approximate solutions of $G_1(t)$ and $G_2(t)$ for different values of γ are given in Figures 1 and 2, respectively.

Let us define the L_2-error as follows:

$$\phi(\gamma) = \sqrt{\int_0^1 \|G(t) - G_{app}(t)\|_E^2 \mathrm{d}t}.$$

Then, the L_2-error for $\gamma = 0.5, 0.6, 0.7, 0.8., 0.9$ and 1 are reported in Table 1.

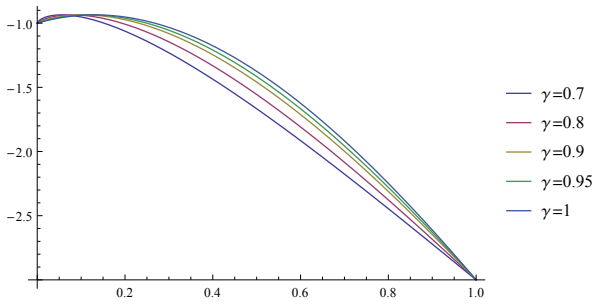

FIGURE 2. The approximate solution $G_2(t)$ for different values of γ

γ	$\phi(\gamma)$
0.5	$1.23*10^{-13}$
0.6	$2.34*10^{-13}$
0.7	$2.87*10^{-13}$
0.8	$3.16*10^{-13}$
0.9	$3.30*10^{-13}$
1	$3.98*10^{-13}$

TABLE 1. The L_2-error for different values of γ for Example 1

Example 2. Consider the following system of Riccati equations

$$D^{\gamma}G(t) = \Lambda_{21} + \Lambda_{22}G(t) - G(t)\Lambda_{11} - G(t)\Lambda_{12}G(t), G(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

where $\Lambda_{22} = \begin{pmatrix} t^{\gamma} & 0 \\ 0 & t^{\gamma} \end{pmatrix}, \Lambda_{11} = t^{2\gamma}, \Lambda_{12} = \begin{pmatrix} -t^{\gamma} & t^{\gamma} \end{pmatrix},$

$$\Lambda_{21} = \begin{pmatrix} \frac{\Gamma(3\gamma+1)}{\Gamma(2\gamma+1)} t^{2\gamma} - t^{7\gamma} + 2t^{5\gamma} \\ \Gamma(\gamma+1) - t^{5\gamma} - t^{4\gamma} + 2t^{3\gamma} + 2t^{2\gamma} + t^{\gamma} - 1 \end{pmatrix}.$$

Then, the exact solution is $G(t) = \begin{pmatrix} t^{3\gamma} \\ t^{\gamma}+1 \end{pmatrix}$. Then, the approximate solutions of $G_1(t)$ and $G_2(t)$ for different values of γ are given in Figures 3 and 4, respectively. Now, the L_2-error for $\gamma = 0.5, 0.6, 0.7, 0.8., 0.9$ and 1 are reported in Table 2.

6. CONCLUSION AND CLOSING REMARKS

In this paper, we investigate the solution of the fractional system of Riccati equations. This system is important since it appears in several applications in science such as algebraic geometry, conformal mapping theory, and control theory. We use the operational matrix method to solve this system. The block-pulse operational

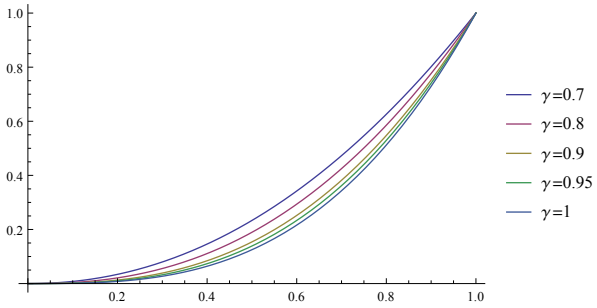

FIGURE 3. The approximate solution $G_1(t)$ for different values of γ

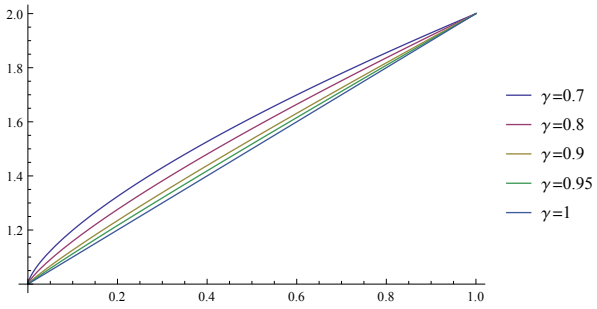

FIGURE 4. The approximate solution $G_2(t)$ for different values of γ

γ	$\phi(\gamma)$
0.5	$5.61 * 10^{-14}$
0.6	$5.69 * 10^{-14}$
0.7	$5.93 * 10^{-14}$
0.8	$6.01 * 10^{-14}$
0.9	$6.21 * 10^{-14}$
1	$6.40 * 10^{-14}$

TABLE 2. The L_2 error for different values of γ for Example 2

matrices will initially assist in the reduction of the nonlinear fractional order Riccati-differential problem into an algebraic system. Benefits of this approach include inexpensive setup costs for the equations without the use of projection techniques like Galerkin, collocation, etc. In addition, we prove the convergence of the approximate solution using the operational matrix method to the exact solution. We arrive to this conclusion after proving a series of theorems in Section four. Finally, we present

two examples to provide the reader with numerical evidences to the efficiency of numerical approach used in this paper. Based on these two examples, we notice the following.

1: The error is within 10^{-13} in Example 1 and within 10^{-14} in Example 2 as in Tables 1 and 2, respectively.

2: We notice that the approximate solutions are convergent to the exact solutions for different values of γ as reported in Tables 1 and 2.

3: We notice that the approximate solutions approach to the solution when $\gamma = 1$ as γ approaches to one as in Figures 1-4.

4: The operational matrix method works very efficiently for this type of problems. It is a useful tool to solve nonlinear fractional equations. We recommend this numerical approach for other problems in physics and engineering such as a system of integro-differential equations.

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NEW GENERALIZED CONTRACTION MAPPINGS AND FIXED POINT RESULTS IN FUZZY METRIC SPACES

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Abstract. In this paper, we introduce the notions of fuzzy α-$\mathcal{H}$-contractions and fuzzy α-$\mathcal{H}$-Φ-contractions via fuzzy α-admissible mappings, within the framework of complete fuzzy metric spaces $(\mathcal{F}.\mathcal{M}.\mathcal{S})$ as defined by Georg and Veermani. We present several fixed point theorems for this class of contractions and provide practical examples to illustrate our results.

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1. INTRODUCTION

Fixed point theory is a fundamental area of research in nonlinear functional analysis, with wide-ranging applications in various fields such as physics, mathematics, economics, quantum mechanics, and other domains. It serves as a powerful tool for solving a diverse range of problems and holds significant potential for practical applications. (see e.g. [2, 4, 5, 11, 12, 19, 28]), engineering sciences (see e.g. [17, 22]) and various domains. The most famous theoretical development in the chronology of the fixed point theory, is the famous principle of contraction due to the Polish mathematician Banach, [21] (BCP), this theorem has been developed and generalized with several methods, the reader can see for example [15, 16, 24] and the references therein. Kaleva and Selikkala were the first to define the distance between two objects in a $\mathcal{F}.\mathcal{M}.\mathcal{S}$ in [14], using a non-negative, upper semi-continuous fuzzy number that is both convex and normal. Since then, various researchers have explored the properties of $\mathcal{F}.\mathcal{M}.\mathcal{S}$, including Felbin [6], George et al. [7], Gregory [10], and Hadzic [13], and applied these results to practical problems in the fuzzy domain. In 1988, Grabic [9] introduced a contraction compatible with the structure of $\mathcal{F}.\mathcal{M}.\mathcal{S}$, followed by D. Mihet [18], who generalized it to the ψ-contraction. Meanwhile,

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Wardowski introduced the concept of fuzzy $\mathcal{H}$-contractive mapping in [29], proving existence and uniqueness of fixed points in $\mathcal{M}_F$-complete $\mathcal{F}.\mathcal{M}.\mathcal{S}$. Shukla extended this concept to fuzzy $\mathcal{H}$-weak contraction mappings in [27], applying it to some fixed point theorems attributed to Wardowski. Beg et al. defined fuzzy α-$\mathcal{H}$-contraction in [3], demonstrating fixed point results with regard to existence and uniqueness in fuzzy $\mathcal{M}_F$-complete metric spaces. For more information, the reader can refer to [1, 20, 23, 26] and the references therein.

Building on the existing works in this field, we introduce new concepts of fuzzy $\mathcal{H}$-contractive, $\mathcal{H}$-weak contraction, α-$\mathcal{H}$-contraction, and α-$\mathcal{H}$-ψ-contraction mappings within the framework of $\mathcal{F}.\mathcal{M}.\mathcal{S}$. In this work, we establish various results pertaining to the existence and uniqueness of fixed points for these mappings in $\mathcal{F}.\mathcal{M}.\mathcal{S}$. Additionally, we provide practical examples to illustrate these results on existence and uniqueness. The findings presented in this study contribute to the advancement and consolidation of different contraction results that are well-known in the literature of fixed point theory on $\mathcal{F}.\mathcal{M}.\mathcal{S}$.

2. Preliminaries

During this article, we will be working only the notion of fuzzy metric in the Georg and Veermani sense, that is given as follows:

Definition 1 ([25]). The couple $([0,1],*)$ is said to be a continuous t-norm if $*\colon [0,1]^2 \to [0,1]$ is a binary operation and $([0,1],*)$ is a topological commutative monoid with unit 1 and $\eta \leq \gamma$ and $\beta \leq \lambda$ implies $\eta * \beta \leq \gamma * \lambda \quad (\eta,\beta,\gamma,\lambda \in [0,1])$.

Definition 2 ([7]). Let Σ is an arbitrary set, $*$ is a continuous t-norm and $\mathcal{M}_F$ is a fuzzy set on $\Sigma^2 \times \mathbb{R}^+$ satisfying the following conditions:

(C1) $\mathcal{M}_F(\sigma,\rho,t) > 0$ for all $\sigma,\rho \in \Sigma$;

(C2) $\mathcal{M}_F(\sigma,\rho,t) = 1$, if and only if $\sigma = \rho$, for every $t > 0$;

(C3) $\mathcal{M}_F(\sigma,\rho,t) = \mathcal{M}_F(\rho,\sigma,t)$;

(C4) $\mathcal{M}_F(\sigma,\delta,t) * \mathcal{M}_F(\rho,\delta,s) \leq \mathcal{M}_F(\sigma,\rho,t+s)$;

(C5) $t \mapsto \mathcal{M}_F(\sigma,\rho,t)$ is continuous from $[0,\infty)$ to $[0,1]$ for every $\sigma,\rho \in \Sigma$ and $s,t > 0$.

Then, the 3-uplet $(\Sigma,\mathcal{M}_F,*)$ is called a $\mathcal{F}.\mathcal{M}.\mathcal{S}$.

Definition 3 ([7]). Let $(\Sigma,\mathcal{M}_F,*)$ be a $\mathcal{F}.\mathcal{M}.\mathcal{S}$. The open ball of centre $\sigma \in \Sigma$ and radius ρ, $0 < \rho < 1$, will be denoted by $B(\sigma,\rho,t) = \{\rho \in \Sigma,\ \mathcal{M}_F(\sigma,\rho,t) > 1-\rho\}$. The family $\{B(\sigma,\rho,t)/\sigma \in \Sigma,\ 0 < \rho < 1,\ t > 0\}$ is a basis of neighborhood's for a separate topology on Σ, that can induced by $\mathcal{M}_F$ (fuzzy metric).

In the sequel $(\Sigma, \mathcal{M}_F, *)$ will be a $\mathcal{F}.\mathcal{M}.\mathcal{S}$ in the sense of George and Veeramani (see [7]).

Definition 4 ([7])**.** A sequence $(\sigma_n)_{n\in\mathbb{N}}$ is called $\mathcal{M}_F$-Cauchy in the metric space $(\Sigma, \mathcal{M}_F, *)$, if for every $\varepsilon \in]0,1[$ and $t > 0$, there is $m_0 \in \mathbb{N}$ such that $\mathcal{M}_F(\sigma_n, \sigma_m, t) > 1-\varepsilon$ for all $m, n \geq m_0$.

A $\mathcal{F}.\mathcal{M}.\mathcal{S}$ is complete, when every $\mathcal{M}_F$-Cauchy sequence is convergent.

Theorem 1 ([7])**.** *A sequence $(\sigma_n)_{n\in\mathbb{N}}$ in a $\mathcal{F}.\mathcal{M}.\mathcal{S}$ $(\Sigma, \mathcal{M}_F, *)$ converges to σ if and only if* $\lim_{n\to\infty} \mathcal{M}_F(\sigma_n, \sigma, t) = 1$.

Remark 1.

(1) In the setting of $\mathcal{F}.\mathcal{M}.\mathcal{S}$ $(\Sigma, \mathcal{M}_F, *)$, the limit of any convergent sequence is unique.
(2) As in the classical case, the two notions of convergent sequence and $\mathcal{M}_F$-Cauchy sequence are equivalent.
(3) For all $\sigma, \rho \in \Sigma$, the mapping $\mathcal{M}_F(\sigma, \rho, .)$ is nondecreasing on $(0, \infty)$.
(4) In the space $(\Sigma, \mathcal{M}_F, *)$, the mapping $\mathcal{M}_F$ is continuous on $\Sigma \times \Sigma \times (0, \infty)$.

Definition 5 ([9])**.** Let $(\Sigma, \mathcal{M}_F, *)$ be a $\mathcal{F}.\mathcal{M}.\mathcal{S}$. A self-mapping $Q\colon \Sigma \to \Sigma$ is said to be fuzzy contractive if there is $\lambda > 1$ such that

$$\mathcal{M}_F(Q\sigma, Q\rho, t) \geq \mathcal{M}_F(\sigma, \rho, \lambda t) \text{ for all } \sigma, \rho \in \Sigma \text{ and } t > 0.$$

Definition 6 ([8])**.** Let $(\Sigma, \mathcal{M}_F, *)$ be a $\mathcal{F}.\mathcal{M}.\mathcal{S}$. A self-mapping $Q\colon \Sigma \to \Sigma$ is said to be fuzzy α-admissible if there is a function $\alpha\colon \Sigma \times \Sigma \times (0, \infty) \to (0, \infty)$ such that

$$\alpha(\sigma, \rho, t) \geq 1, \text{ implies that } \alpha(Q\sigma, Q\rho, t) \geq 1 \text{ for all } \sigma, \rho \in \Sigma \text{ and } t > 0.$$

3. Main Results

In the rest of this paper, we extend certain results concerning the notion of contraction, in $\mathcal{F}.\mathcal{M}.\mathcal{S}$.

Definition 7 ([9])**.** Let $(\Sigma, \mathcal{M}_F, *)$ be a $\mathcal{F}.\mathcal{M}.\mathcal{S}$. A self-mapping $Q\colon \Sigma \to \Sigma$ is said to be fuzzy contractive, if there is $\lambda > 1$ such that

$$\mathcal{M}_F(Q\sigma, Q\rho, t) \geq \mathcal{M}_F(\sigma, \rho, \lambda t) \text{ for all } \sigma, \rho \in \Sigma \text{ and } t > 0.$$

We are going to introduce our first contraction named fuzzy $\mathcal{H}$-contraction, let $\mathcal{H} = \{\eta\colon (0,1] \to [0, \infty) / (\mathcal{H}_1) - (\mathcal{H}_3) \text{ hold }\}$, where

$(\mathcal{H}_1)$ η is continuous bijective;
$(\mathcal{H}_2)$ η is strictly decreasing;
$(\mathcal{H}_3)$ $\eta(s) = 0$ if and only if $s = 1$.

Definition 8. Let $(\Sigma, \mathcal{M}_F, *)$ be a $\mathcal{F}.\mathcal{M}.\mathcal{S}$, a mapping $Q\colon \Sigma \to \Sigma$ is called fuzzy $\mathcal{H}$-contractive if there are $\eta \in \mathcal{H}$ and $\lambda > 1$ such that

$$\eta(\mathcal{M}_F(Q\sigma, Q\rho, t)) \leq \eta(\mathcal{M}_F(\sigma, \rho, \lambda t)), \text{ for all } \sigma, \rho \in \Sigma, t > 0. \tag{3.1}$$

Proposition 1. *Every fuzzy $\mathcal{H}$-contraction mapping $Q\colon \Sigma \to \Sigma$ is continuous.*

Proof. Let $(\sigma_n)_{n\in\mathbb{N}} \subset \Sigma$, with

$$\lim_{n\to+\infty} \mathcal{M}_F(\sigma_n,\sigma,t) = 1 \text{ and } \mathcal{M}_F(Q\sigma_n, Q\sigma,t) < 1 \text{ for all } n \in \mathbb{N},$$

for some $\sigma \in \Sigma$. According to the strictly decreasing of η and (3.1), we can obtain

$$\eta(\mathcal{M}_F(Q\sigma_n, Q\sigma,t)) \leq \eta(\mathcal{M}_F(\sigma_n,\sigma,\lambda t)) < \eta(\mathcal{M}_F(\sigma_n,\sigma,t)),$$

which implies that

$$\mathcal{M}_F(Q\sigma_n, Q\sigma,t) > \mathcal{M}_F(\sigma_n,\sigma,t).$$

Hence,

$$\lim_{n\to+\infty} \mathcal{M}_F(Q\sigma_n, Q\sigma,t) = 1.$$

Therefore, Q is continuous. □

The theorem which will come is our first contribution in this framework.

Theorem 2. *Let $(\Sigma, \mathcal{M}_F, *)$ be an $\mathcal{M}_F$-complete $\mathcal{F}.\mathcal{M}.\mathcal{S}$. Assume that $Q\colon \Sigma \to \Sigma$ is a fuzzy $\mathcal{H}$-contraction mapping. Then, Q possesses a fixed point, and moreover it is unique.*

Proof. Let $\sigma_0 \in \Sigma$, $\sigma_n = Q^n\sigma_0$. By using the contraction given in condition (3.1), we can obtain

$$0 < \eta(\mathcal{M}_F(Q\sigma_{n-1}, Q\sigma_n,t)) \leq \eta(\mathcal{M}_F(\sigma_{n-1},\sigma_n,\lambda t))$$

and by induction, we have

$$\eta(\mathcal{M}_F(Q\sigma_{n-1}, Q\sigma_n,t)) \leq \eta(\mathcal{M}_F(\sigma_0,\sigma_1,\lambda^n t)).$$

Since $\lim_{n\to+\infty} \mathcal{M}_F(\sigma_0,\sigma_1,t) = 1$ and η is continuous, then

$$\lim_{n\to+\infty} \mathcal{M}_F(Q\sigma_{n-1}, Q\sigma_n,t) = 0,$$
$$\lim_{n\to+\infty} \eta(\mathcal{M}_F(Q\sigma_{n-1}, Q\sigma_n,t)) = 0$$

and thus

$$\lim_{n\to+\infty} \mathcal{M}_F(Q\sigma_{n-1}, Q\sigma_n,t) = 1. \tag{3.2}$$

Now, let us prove that $(\sigma_n)_{n\in\mathbb{N}}$ is an $\mathcal{M}_F$-Cauchy sequence. Assume in the contrast that $(\sigma_n)_{n\in\mathbb{N}}$ is not $\mathcal{M}_F$-Cauchy, which implies that there are $t_0 > 0$, $\varepsilon \in (0,1)$ and two subsequences (σ_{m_k}), (σ_{n_k}) of $(\sigma_n)_{n\in\mathbb{N}}$ such that $m_k > n_k \geq k$ for all $k \in \mathbb{N}$ and

$$\mathcal{M}_F(\sigma_{m_k},\sigma_{n_k},t_0) \leq 1-\varepsilon, \tag{3.3}$$

since the mapping $\mathcal{M}_F(\sigma,\rho,.)$ is nondecreasing on $(0,\infty)$ for all $\sigma,\rho \in \Sigma$, we obtain

$$\mathcal{M}_F(\sigma_{m_k},\sigma_{n_k},\frac{t_0}{2}) \leq 1-\varepsilon. \tag{3.4}$$

The smallest integer exceeding m_k and satisfying inequality (3.4) will be noted n_k, so thus, we obtain

$$\mathcal{M}_F(\sigma_{m_k-1},\sigma_{n_k},\frac{t_0}{2})>1-\varepsilon. \tag{3.5}$$

By the contractive inequality given in (3.1) we take $\sigma=\sigma_{m_k-1}$, $\rho=\rho_{n_k-1}$ and $t=t_0$, it follows

$$\eta(\mathcal{M}_F(\sigma_{m_k},\sigma_{n_k},t_0))\leq\eta(\mathcal{M}_F(\sigma_{m_k-1},\sigma_{n_k-1},\lambda t_0))<\eta(\mathcal{M}_F(\sigma_{m_k-1},\sigma_{n_k-1},t_0)).$$

Thanks to the strictly decreasing of η, we have

$$\mathcal{M}_F(\sigma_{m_k},\sigma_{n_k},t_0)>\mathcal{M}_F(\sigma_{m_k-1},\sigma_{n_k-1},t_0). \tag{3.6}$$

We use (3.3) and (3.5) in (3.6) and using (C4), we can write

$$\begin{aligned}1-\varepsilon&\geq\mathcal{M}_F(\sigma_{m_k},\sigma_{n_k},t_0)\\&>\mathcal{M}_F(\sigma_{m_k-1},\sigma_{n_k-1},t_0)\\&>\mathcal{M}_F(\sigma_{m_k-1},\sigma_{n_k},\frac{t_0}{2})*\mathcal{M}_F(\sigma_{n_k},\sigma_{n_k-1},\frac{t_0}{2})\\&>(1-\varepsilon)*\mathcal{M}_F(\sigma_{n_k},\sigma_{n_k-1},\frac{t_0}{2}).\end{aligned}$$

By passing to the limit as k tends to $+\infty$ and use (3.2), we can deduce

$$\lim_{k\to\infty}\mathcal{M}_F(\sigma_{m_k},\sigma_{n_k},t_0)=1-\varepsilon$$

and

$$\lim_{k\to\infty}\mathcal{M}_F(\sigma_{m_k-1},\sigma_{n_k-1},t_0)=1-\varepsilon.$$

On the other side, we have

$$\begin{aligned}\eta(\mathcal{M}_F(\sigma_{m_k},\sigma_{n_k},t_0))&\leq\eta(\mathcal{M}_F(\sigma_{m_k-1},\sigma_{n_k-1},\lambda t_0))\\&\leq\eta(\mathcal{M}_F(\sigma_{m_k-2},\sigma_{n_k-2},\lambda^2 t_0))\\&\vdots\\&\leq\eta(\mathcal{M}_F(\sigma_0,\sigma_{n_1},\lambda^{n_k}t_0)).\end{aligned}$$

Since η is continuous, we get

$$\eta(\lim_{k\to\infty}\mathcal{M}_F(\sigma_{m_k},\sigma_{n_k},t_0))\leq\eta(\lim_{k\to\infty}\mathcal{M}_F(\sigma_0,\sigma_1,\lambda^{n_k}t_0)),$$

this implies

$$\eta(1)\geq\eta(1-\varepsilon),$$

that constitutes is a contradiction, and thus (σ_n) is an $\mathcal{M}_F$-Cauchy sequence in Σ. The completeness of $(\Sigma,\mathcal{M}_F,*)$, implies that there is $\sigma\in\Sigma$ satisfying

$$\lim_{n\to\infty}\mathcal{M}_F(\sigma_n,\sigma,t)=1,t>0.$$

From Proposition 1, we have $\lim_{n\to\infty} \mathcal{M}_F(Q\sigma_n, Q\sigma, t) = 1$. According to the uniqueness of the limit, we obtain $Q\sigma = \sigma$. Now let us prove the uniqueness. Let σ_1 and σ_2 $(\sigma_1 \neq \sigma_2)$ such that $Q\sigma_1 = \sigma_1$ and $Q\sigma_2 = \sigma_2$, then we have

$$\eta(\mathcal{M}_F(Q\sigma_1, Q\sigma_2, t_0)) \leq \eta(\mathcal{M}_F(\sigma_1, \sigma_2, \lambda t_0)) < \eta(\mathcal{M}_F(\sigma_1, \sigma_2, t_0)),$$

which is a contradiction. □

Example 1. Consider the following fuzzy set

$$\mathcal{M}_F : \mathbb{R}^2 \times \mathbb{R}^+ \to [0,1]; \quad (\sigma, \rho, t) \mapsto \mathcal{M}_F(\sigma, \rho, t) = \exp\{-\frac{|\sigma - \rho|}{t}\}.$$

It is clear that $(\mathbb{R}, \mathcal{M}_F, *)$ is an $\mathcal{M}_F$-complete $\mathcal{F}.\mathcal{M}.\mathcal{S}$ with the product t-norm $*$ $(b * d = bd)$, $b, d \in [0,1]$. We define a self mapping Q by

$$Q_F : \mathbb{R} \to \mathbb{R}; \quad \sigma \mapsto Q\sigma = \frac{\sigma}{5},$$

and a function η in $\mathcal{H}$ given by

$$\eta : (0,1] \to \mathbb{R}^+; \quad \beta \mapsto \eta(\beta) = -\ln(\beta).$$

For each $t > 0$, $\sigma, \rho \in \mathbb{R}$, we have

$$\begin{aligned}
\eta\big(\mathcal{M}_F(Q\sigma, Q\rho, t)\big) &= \eta\big(\exp\{-\frac{|Q\sigma - Q\rho|}{t}\}\big) \\
&\leq \ln\big(\exp\{\frac{|\sigma - \rho|}{5t}\}\big) \\
&\leq -\ln\big(\exp\{-\frac{|\sigma - \rho|}{2t}\}\big) \\
&\leq \eta\big(\mathcal{M}_F(\sigma, \rho, \lambda t),
\end{aligned}$$

where $\lambda = 2 > 1$. Then, Q is a fuzzy $\mathcal{H}$-contractive mapping. So, from Theorem 2, Q admits 0 as a fixed point.

Definition 9. A mapping $Q : \Sigma \to \Sigma$ over a fuzzy metric space $(\Sigma, \mathcal{M}_F, *)$, is called fuzzy $\mathcal{H}$-weak contractive if there are $\eta \in \mathcal{H}$ and $\lambda > 1$ such that

$$\eta(\mathcal{M}_F(Q\sigma, Q\rho, t)) \leq \mathcal{N}(\sigma, \rho, \lambda t), \tag{3.7}$$

where $\mathcal{N}(\sigma, \rho, \lambda t) = \max\big\{\eta(\mathcal{M}_F(\sigma, \rho, t)), \eta(\mathcal{M}_F(\sigma, Q\sigma, t)), \eta(\mathcal{M}_F(\rho, Q\rho, t))\big\}$.

Theorem 3. *Let $(\Sigma, \mathcal{M}_F, *)$ be an $\mathcal{M}_F$-complete $\mathcal{F}.\mathcal{M}.\mathcal{S}$. Assume that $Q : \Sigma \to \Sigma$ is a fuzzy $\mathcal{H}$-weak contractive mapping, then Q admits a unique fixed point.*

Proof. Let us consider σ_0 in Σ, define a sequence $(\sigma_n)_{n\in\mathbb{N}}$ by

$$\sigma_n = Q\sigma_{n-1}.$$

We have

$$\eta(\mathcal{M}_F(\sigma_n, \sigma_{n+1}, t)) = \eta(\mathcal{M}_F(Q\sigma_{n-1}, Q\sigma_n, t))$$

$$\begin{aligned}
&\leq \mathcal{N}(\sigma_{n-1},\sigma_n,\lambda t)\\
&\leq \max\left\{\eta(\mathcal{M}_F(\sigma_{n-1},\sigma_n,\lambda t)),\eta(\mathcal{M}_F(\sigma_{n-1},Q\sigma_{n-1},\lambda t)),\eta(\mathcal{M}_F(\sigma_n,Q\sigma_n,t))\right\}\\
&= \max\left\{\eta(\mathcal{M}_F(\sigma_{n-1},\sigma_n,\lambda t)),\eta(\mathcal{M}_F(\sigma_n,\sigma_{n+1},t))\right\}.
\end{aligned}$$

If for example $\eta(\mathcal{M}_F(\sigma_{n-1},\sigma_n,\lambda t)) \leq \eta(\mathcal{M}_F(\sigma_n,\sigma_{n+1},\lambda t))$, then

$$\eta(\mathcal{M}_F(\sigma_n,\sigma_{n+1},t)) \leq \eta(\mathcal{M}_F(\sigma_n,\sigma_{n+1},\lambda t)).$$

Since $\lambda t > t$ and η is strictly decreasing, what is absurd. Hence,

$$\eta(\mathcal{M}_F(\sigma_n,\sigma_{n+1},t)) \leq \eta(\mathcal{M}_F(\sigma_{n-1},\sigma_n,\lambda t)).$$

Then, we find that

$$\begin{aligned}
0 &< \eta(\mathcal{M}_F(\sigma_n,\sigma_{n+1},t))\\
&\leq \mathcal{N}(\sigma_{n-1},\sigma_n,\lambda t)\\
&\leq \eta(\mathcal{M}_F(\sigma_{n-1},\sigma_n,\lambda t))\\
&\leq \mathcal{N}(\mathcal{M}_F(\sigma_{n-2},\sigma_{n-1},\lambda^2 t)),
\end{aligned}$$

and by induction, we obtain

$$0 < \eta(\mathcal{M}_F(\sigma_n,\sigma_{n+1},t)) \leq \eta(\mathcal{M}_F(\sigma_0,\sigma_1,\lambda^n t)).$$

So,

$$\lim_{n\to+\infty} \eta(\mathcal{M}_F(\sigma_n,\sigma_{n+1},t)) = 0.$$

Which means that

$$\lim_{n\to+\infty} (\mathcal{M}_F(\sigma_n,\sigma_{n+1},t)) = 1. \tag{3.8}$$

Now, we prove by contradiction that $(\sigma_n)_{n\in\mathbb{N}}$ is $\mathcal{M}_F$-Cauchy. Assume that $(\sigma_n)_{n\in\mathbb{N}}$ is not $\mathcal{M}_F$-Cauchy, so there are $\varepsilon \in (0,1)$, $t_0 > 0$ and two subsequences $(\sigma_{n_k})_{k\in\mathbb{N}}$, $(\sigma_{m_k})_{k\in\mathbb{N}}$ of $(\sigma_n)_{n\in\mathbb{N}}$, such that $m_k \geq n_k$ for all $k \in \mathbb{N}$ and

$$\mathcal{M}_F(\sigma_{m_k},\sigma_{n_k},t_0) \leq 1-\varepsilon. \tag{3.9}$$

Then, we have

$$\mathcal{M}_F(\sigma_{m_k},\sigma_{n_k},\frac{t_0}{2}) \leq \mathcal{M}_F(\sigma_{m_k},\sigma_{n_k},t_0) \leq 1-\varepsilon. \tag{3.10}$$

The smallest integer exceeding m_k and satisfying inequality (3.4) will be denoted by n_k. So, we can have

$$\mathcal{M}_F(\sigma_{m_k-1},\sigma_{n_k},\frac{t_0}{2}) > 1-t.$$

From inequality (3.7), we get

$$\eta(\mathcal{M}_F(\sigma_{m_k},\sigma_{n_k},t_0)) = \eta(\mathcal{M}_F(Q\sigma_{m_k-1},Q\sigma_{n_k-1},t_0) \leq \mathcal{N}(\sigma_{m_k-1},\sigma_{n_k-1},\lambda t_0), \tag{3.11}$$

where

$\mathcal{N}(\sigma_{m_k-1},\sigma_{n_k-1},\lambda t_0)$

$$= \max\{\eta(\mathcal{M}_F(\sigma_{m_k-1},\sigma_{n_k-1},\lambda t_0)),\eta(\mathcal{M}_F(\sigma_{m_k-1},\sigma_{m_k},\lambda t_0)),\eta(\mathcal{M}_F(\sigma_{n_k-1},\sigma_{n_k},\lambda t_0))\}.$$

If $\mathcal{N}(\sigma_{m_k-1},\sigma_{n_k-1},\lambda t_0) = \eta(\mathcal{M}_F(\sigma_{m_k-1},\sigma_{m_k},\lambda t_0))$, then (3.11) becomes

$$\eta(\mathcal{M}_F(\sigma_{m_k},\sigma_{m_k},t_0)) \leq \eta(\mathcal{M}_F(\sigma_{m_k-1},\sigma_{m_k},\lambda t_0)). \tag{3.12}$$

Passing to the limit as k tends to $+\infty$ in (3.12) and by using (3.8), Remark 1 and taking into consideration the fact that η is continuous, we obtain

$$\eta\Big(\lim_{k\to\infty}\mathcal{M}_F(\sigma_{m_k},\sigma_{n_k},t_0)\Big) = 0.$$

Therefore, $\lim_{k\to\infty}\mathcal{M}_F(\sigma_{m_k},\sigma_{n_k},t_0) = 1$, which contradicts the formula (3.9). In the same way, if we consider $\mathcal{N}(\sigma_{m_k-1},\sigma_{n_k-1},t_0) = \eta(\mathcal{M}_F(\sigma_{m_k-1},\sigma_{n_k},t_0))$, then we come to an absurdity again. We must have

$$\mathcal{N}(\sigma_{m_k-1},\sigma_{n_k-1},t_0) = \eta(\mathcal{M}_F(\sigma_{m_k-1},\sigma_{n_k-1},t_0))$$

and thus (3.11) gives

$$\eta\Big(\mathcal{M}_F(\sigma_{m_k},\sigma_{n_k},t_0)\Big) \leq \eta\Big(\mathcal{M}_F(\sigma_{m_k},\sigma_{n_k},\lambda t_0)\Big) < \eta\Big(\mathcal{M}_F(\sigma_{m_k-1},\sigma_{n_k-1},t_0)\Big). \tag{3.13}$$

Thanks to the monotony of η

$$\mathcal{M}_F(\sigma_{m_k},\sigma_{n_k},t_0) > \mathcal{M}_F(\sigma_{m_k-1},\sigma_{n_k-1},t_0). \tag{3.14}$$

By using (3.14) and (3.9)

$$\begin{aligned}
1-\varepsilon &\geq \mathcal{M}_F(\sigma_{m_k},\sigma_{n_k},t_0)\\
&> \mathcal{M}_F(\sigma_{m_k-1},\sigma_{n_k-1},t_0)\\
&\geq \mathcal{M}_F(\sigma_{m_k-1},\sigma_{n_k},\frac{t_0}{2}) * \mathcal{M}_F(\sigma_{n_k},\sigma_{n_k-1},\frac{t_0}{2})\\
&> (1-\varepsilon) * \mathcal{M}_F(\sigma_{n_k},\sigma_{n_k-1},\frac{t_0}{2}),
\end{aligned}$$

which yields by letting $k\to\infty$ and from (3.8) and (C3) that

$$\lim_{k\to\infty}\mathcal{M}_F(\sigma_{m_k-1},\sigma_{n_k-1},t_0) = 1-\varepsilon = \lim_{k\to\infty}\mathcal{M}_F(\sigma_{m_k},\sigma_{n_k},t_0).$$

Now, using (3.13), (3.8), the continuity of η and taking $k\to\infty$, we get

$$\eta(1) > \eta(1-\varepsilon),$$

which gives an absurdity, and thus $(\sigma_n)_{n\in\mathbb{N}}$ is $\mathcal{M}_F$-Cauchy in Σ. The completeness of $(\Sigma,\mathcal{M}_F,*)$ implies that there is $\sigma\in\Sigma$ such that

$$\lim_{k\to\infty}\mathcal{M}_F(\sigma_n,\sigma,t) = 1, \quad \text{for all } t>0.$$

Next, we prove that σ is a fixed point of Q. Assume that there is $t_1>0$, such that $\mathcal{M}_F(\sigma,Q\sigma,t_1)<1$, then $\eta(\mathcal{M}_F(\sigma,Q\sigma,t_1))>0$. Also, as Q is $\mathcal{H}$-weak contraction mapping in fuzzy setting, we have

$$\eta((\mathcal{M}_F(\sigma_{n+1},Q\sigma,t_1)) = \eta(\mathcal{M}_F(Q\sigma_n,Q\sigma,t_1)) \leq \mathcal{N}(\sigma_n,\sigma,\lambda t_1),$$

with $\mathcal{N}(\sigma_n,\sigma,t_1)=\max\{\eta(\mathcal{M}_F(\sigma_n,\sigma,t_1)),\eta(\mathcal{M}_F(\sigma_n,\sigma_{n+1},t_1)),\eta(\mathcal{M}_F(\sigma,Q\sigma,t_1))\}$. According to the continuity of η, we have

$$\lim_{n\to\infty}\eta(\mathcal{M}_F(\sigma_n,\sigma,t_1))=\lim_{n\to\infty}\eta(\mathcal{M}_F(\sigma_n,\sigma_{n+1},t_1))=0$$

for every $t>0$, and thus

$$\lim_{n\to\infty}\mathcal{N}(\sigma_n,x,t_1)=\max\{0,0,\mathcal{M}_F(x,Q\sigma,t_1)\}=\eta(\mathcal{M}_F(x,Q\sigma,t_1)).$$

If we tend n to infinity, we get

$$\eta(\mathcal{M}_F(\sigma,Q\sigma,t_1))\le\eta(\mathcal{M}_F(\sigma,Q\sigma,\lambda t_1))<\eta(\mathcal{M}_F(\sigma,Q\sigma,t_1)),\quad \text{as } \lambda>1,$$

which is a contradiction. Hence, we have $\mathcal{M}_F(\sigma,Q\sigma,t)=1$, $t>0$, then σ is a fixed point of Q. To show the uniqueness of σ, let σ_1,σ_2 be two different fixed points of Q. So, using (3.7), we get

$$\begin{aligned}\eta(\mathcal{M}_F(\sigma_1,\sigma_2,t))&=\eta(\mathcal{M}_F(Q\sigma_1,Q\sigma_2,t))\\&\le\max\left\{\eta(\mathcal{M}_F(\sigma_1,\sigma_2,\lambda t)),\eta(\mathcal{M}_F(\sigma_1,Q\sigma_1,\lambda t)),\eta(\mathcal{M}_F(\sigma_2,Q\sigma_2,\lambda t))\right\}\\&=\eta(\mathcal{M}_F(\sigma_1,\sigma_2,\lambda t)).\end{aligned}\tag{3.15}$$

If $\mathcal{M}_F(\sigma_1,\sigma_2,t)<1$, then $\eta(\mathcal{M}_F(\sigma_1,\sigma_2,t))>0$ and hence (3.15) becomes

$$\eta(\mathcal{M}_F(\sigma_1,\sigma_2,t))\le\eta(\mathcal{M}_F(\sigma_1,\sigma_2,\lambda t))<\eta(\mathcal{M}_F(\sigma_1,\sigma_2,t)),$$

that is a contradiction. Therefore, $\mathcal{M}_F(\sigma_1,\sigma_2,t)=1$, it yields $\sigma_1=\sigma_2$, which is the desired result. □

Definition 10. Let $(\Sigma,\mathcal{M}_F,*)$ be a $\mathcal{F}.\mathcal{M}.\mathcal{S}$. A mapping $Q\colon\Sigma\to\Sigma$ is called fuzzy α-$\mathcal{H}$-contractive if there exist $\eta\in\mathcal{H}$ and $\lambda>1$ such that

$$\eta\big(\mathcal{M}_F(Q\sigma,Q\rho,\frac{t}{\alpha(\sigma,\rho,t)})\big)\le\eta(\mathcal{M}_F(\sigma,\rho,\lambda t)),\ \text{for every } \rho,\sigma\in\Sigma,t>0.\tag{3.16}$$

Theorem 4. *Let $Q\colon\Sigma\to\Sigma$ is a mapping as described in Definition 10. We further assume that it satisfies conditions (i)-(iv), where*

(i) *there is $\sigma_0\in\Sigma$ with $\alpha(\sigma_0,Q\sigma_0,t)\ge1$, $t\ge0$;*
(ii) *Q is fuzzy α-admissible;*
(iii) *for each subsequence $(\sigma_{n_k})_{k\in\mathbb{N}}\subset(\sigma_n)_{n\in\mathbb{N}}=(Q^n\sigma_0)_{n\in\mathbb{N}}$, we have*

$$\alpha(\sigma_{n_k},\sigma_{n_l},t)\ge1,\quad \text{where } l,k\in\mathbb{N},\ l<k \text{ and } t>0;$$

(iv) *if $(\sigma_n)_{n\in\mathbb{N}}\subset\Sigma$ with $\alpha(\sigma_n,\sigma_{n+1},t)\ge1$, and $\lim_{n\to\infty}\sigma_n=\sigma$, then*

$$\alpha(\sigma_n,\sigma_{n+1},t)\ge1,\ \text{for all } t>0,n\in\mathbb{N}.$$

Then, Q possesses a unique fixed point.

Proof. From (i), there is $\sigma_0 \in \Sigma$ with $\alpha(\sigma_0, Q\sigma_0, t) \geq 1,\ t \geq 0$. We define a sequence $(\sigma_n)_{n\in\mathbb{N}}$ in Σ by $\sigma_n = Q\sigma_{n-1}$, for all $n \in \mathbb{N}$. Since Q is fuzzy α-admissible, then we have

$$\alpha(\sigma_0, \sigma_1, t) = \alpha(\sigma_0, Q\sigma_0, t) \geq 1,$$

implying

$$\alpha(Q\sigma_0, Q\sigma_1, t) = \alpha(\sigma_1, \sigma_2, t) \geq 1,\ t > 0.$$

By induction, we get

$$\alpha(\sigma_n, \sigma_{n+1}, t) \geq 1, \quad t > 0,\ n \in \mathbb{N}. \tag{3.17}$$

Since Q is a fuzzy α-$\mathcal{H}$-contractive mapping and by using (3.17), we get

$$\begin{aligned}
0 &\leq \eta\big(\mathcal{M}_F(\sigma_n, \sigma_{n+1}, t)\big) \\
&= \eta\big(\mathcal{M}_F(Q\sigma_{n-1}, Q\sigma_n, t)\big) \\
&\leq \eta\big(\mathcal{M}_F(Q\sigma_{n-1}, Q\sigma_n, \frac{t}{\alpha(\sigma_{n-1}, \sigma_n, t)})\big) \\
&\leq \eta\big(\mathcal{M}_F(\sigma_{n-1}, \sigma_n, \lambda t)\big) \\
&= \eta\big(\mathcal{M}_F(Q\sigma_{n-2}, Q\sigma_{n-1}, \lambda t)\big) \\
&\leq \eta\big(\mathcal{M}_F(Q\sigma_{n-2}, Q\sigma_{n-1}, \lambda\frac{t}{\alpha(\sigma_{n-2}, \sigma_{n-1}, t)})\big) \\
&\leq \eta\big(\mathcal{M}_F(\sigma_{n-2}, \sigma_{n-1}, \lambda^2 t)\big) \\
&\ \ \vdots \\
&\leq \eta\big(\mathcal{M}_F(\sigma_{n-2}, \sigma_{n-1}, \lambda^n t)\big).
\end{aligned} \tag{3.18}$$

Taking the limit as n tends to $+\infty$ in (3.18), we can get

$$\lim_{n\to\infty} \eta\big(\mathcal{M}_F(\sigma_n, \sigma_{n+1}, t)\big) = 0.$$

And thus $\lim_{n\to\infty} \mathcal{M}_F(\sigma_n, \sigma_{n+1}, t) = 1$.

In the sequel, we show by contradiction that $(\sigma_n)_{n\in\mathbb{N}}$ is $\mathcal{M}_F$-Cauchy. If $(\sigma_n)_{n\in\mathbb{N}}$ is not $\mathcal{M}_F$-Cauchy, then there exist $\varepsilon \in (0,1)$, $t_0 > 0$ and two subsequences $(\sigma_{n_k})_{k\in\mathbb{N}}$, $(\sigma_{m_k})_{k\in\mathbb{N}}$ of $(\sigma_n)_{n\in\mathbb{N}}$, such that

$$m_k > n_k \geq k \text{ and } \mathcal{M}_F(\sigma_{m_k}, \sigma_{n_k}, t_0) \leq 1 - \varepsilon \text{ for every } k \in \mathbb{N}.$$

From (2) in Remark 1, we have

$$\mathcal{M}_F(\sigma_{m_k}, \sigma_{n_k}, \frac{t_0}{2}) \leq 1 - \varepsilon. \tag{3.19}$$

The smallest integer exceeding m_k and satisfying inequality (3.19) will be assumed equal n_k, then we can write

$$\mathcal{M}_F(\sigma_{m_k-1},\sigma_{n_k},\frac{t_0}{2})>1-\varepsilon.$$

By inequality (3.16), (2) in Remark 1 and from condition (iii), we obtain

$$\begin{aligned}\eta\big(\mathcal{M}_F(\sigma_{m_k-1},\sigma_{n_k},t_0)\big)&<\eta\big(\mathcal{M}_F(\sigma_{m_k-1},\sigma_{n_k},\frac{t_0}{\alpha(\sigma_{m_k-1},\sigma_{n_k},t_0)})\big)\\&\leq\eta\big(\mathcal{M}_F(\sigma_{m_k-1},\sigma_{n_k},\lambda t_0)\big)\\&\leq\eta\big(\mathcal{M}_F(\sigma_{m_k-1},\sigma_{n_k},\lambda t_0)\big).\end{aligned}$$

It remains to show Chauchyness that it can be proved as in Theorem 2. Now, since $(\Sigma,\mathcal{M}_F,*)$ is $\mathcal{M}_F$-complete, then there exists $\sigma\in\Sigma$ such that

$$\lim_{n\to\infty}\mathcal{M}_F(\sigma_n,x,t)=1,\quad t>0. \tag{3.20}$$

By condition (ii) and (3.16), we have

$$0\leq\eta\big(\mathcal{M}_F(Q\sigma_n,Q\sigma,t)\big)\leq\eta\big(\mathcal{M}_F(Q\sigma_n,Q\sigma,\frac{t}{\alpha(Q\sigma_n,Q\sigma,t)})\big)\leq\eta\big(\mathcal{M}_F(\sigma_n,x,\lambda t)\big).$$

By passing to the limit, letting n tend to infinity in the last inequality, using (3.20) and the fact that η is continuous, we conclude

$$\lim_{n\to\infty}\eta\big(\mathcal{M}_F(\sigma_{n+1},Q\sigma,t)\big)=0,\quad t>0.$$

That implies

$$\lim_{n\to\infty}\mathcal{M}_F(\sigma_{n+1},Q\sigma,t)=1,\quad t>0.$$

By the fact that the limit is unique, we get $Q\sigma=\sigma$, that is, σ is a fixed point of Q. □

Definition 11. Let $(\Sigma,\mathcal{M}_F,*)$ be a $\mathcal{F}.\mathcal{M}.\mathcal{S}$. A mapping $Q\colon\Sigma\to\Sigma$ is said to be fuzzy α-$\mathcal{H}$-weak contractive, if there exist $\eta\in\mathcal{H}$ and $\lambda>1$ such that

$$\eta\big(\mathcal{M}_F(Q\sigma,Q\rho,\frac{t}{\alpha(\sigma,\rho,t)})\big)\leq\eta(\mathcal{N}(\sigma,\rho,\lambda t)),\ \text{for all}\ \sigma,\rho\in\Sigma,\ t>0.$$

where $\mathcal{N}(\sigma,\rho,t)=\max\big\{\eta(\mathcal{M}_F(\sigma,\rho,t)),\eta(\mathcal{M}_F(x,Q\sigma,t)),\eta(\mathcal{M}_F(y,Q\rho,t))\big\}$.

Theorem 5. *Let $Q\colon\Sigma\to\Sigma$ is a mapping as described in Definition 11, such that*

(i) *there is $\sigma_0\in\Sigma$ with $\alpha(\sigma_0,Q\sigma_0,t)\geq1,\quad t\geq0$;*

(ii) *Q is fuzzy α-admissible;*

(iii) *for each subsequence $(\sigma_{n_k})_{k\in\mathbb{N}}\subset(\sigma_n)_{n\in\mathbb{N}}=(Q^n\sigma_0)_{n\in\mathbb{N}}$, we have*

$$\alpha(\sigma_{n_k},\sigma_{n_l},t)\geq1,\quad \text{where } l<k,\ \ l,k\in\mathbb{N},\ \text{and } t>0;$$

(iv) *if $(\sigma_n)_{n\in\mathbb{N}}\subset\Sigma$ with $\alpha(\sigma_n,\sigma_{n+1},t)\geq1$, and $\lim_{n\to\infty}\sigma_n=\sigma$, then*

$$\alpha(\sigma_n,\sigma_{n+1},t)\geq1,\ t>0,\ n\in\mathbb{N}.$$

Then, Q possesses a unique fixed point.

Proof. By (i), there is $\sigma_0 \in \Sigma$ with $\alpha(\sigma_0, Q\sigma_0, t) \geq 1$, $t \geq 0$. We define a sequence $(\sigma_n)_{n\in\mathbb{N}}$ in Σ by $\sigma_n = Q\sigma_{n-1}$, $\forall n \in \mathbb{N}$. Since Q is fuzzy α-admissible, then

$$\alpha(\sigma_0,\sigma_1,t) = \alpha(\sigma_0, Q\sigma_0, t) \geq 1 \text{ implies } \alpha(Q\sigma_0, Q\sigma_1, t) = \alpha(\sigma_1,\sigma_2,t) \geq 1,\ t > 0.$$

By induction, we get

$$\alpha(\sigma_n,\sigma_{n+1},t) \geq 1, \quad t > 0,\ n \in \mathbb{N}. \tag{3.21}$$

Since Q is a fuzzy α-$\mathcal{H}$-weak contractive and thanks to (3.21), we get

$$\begin{aligned}\eta\big(\mathcal{M}_F(\sigma_n,\sigma_{n+1},t)\big) &= \eta\big(\mathcal{M}_F(Q\sigma_{n-1}, Q\sigma_n, t)\big)\\ &\leq \eta\big(\mathcal{M}_F(Q\sigma_{n-1}, Q\sigma_n, \frac{t}{\alpha(\sigma_{n-1},\sigma_n,t)})\big)\\ &\leq \eta\big(\mathcal{N}(\sigma_{n-1},\sigma_n,\lambda t)\big),\end{aligned}$$

where $\mathcal{N}(\sigma,\rho,t) = \max\big\{\eta(\mathcal{M}_F(\sigma,\rho,t)), \eta(\mathcal{M}_F(x, Q\sigma, t)), \eta(\mathcal{M}_F(y, Q\rho, t))\big\}$. □

We can show the Chauchyness of the sequence as in Theorem 3.

Let $\Phi = \Big\{\psi\colon [0,1] \to [0,1] /\ (\Phi_1), (\Phi_2) \text{ hold } \Big\}$, where

(Φ_1) ψ is nondecreasing;
(Φ_2) $\psi(t) > t$, for all $t \in (0,1)$.

Definition 12. Let $(\Sigma, \mathcal{M}_F, *)$ be a $\mathcal{F}.\mathcal{M}.\mathcal{S}$. A mapping $Q\colon \Sigma \to \Sigma$ is called fuzzy α-$\mathcal{H}$-Φ-contractive, if there are $\eta \in \mathcal{H}$, $\psi \in \Phi$ and $\lambda > 1$ such that

$$\eta\big(\mathcal{M}_F(Q\sigma, Q\rho, \frac{t}{\alpha(\sigma,\rho,t)})\big) \leq \eta\big(\psi(\mathcal{M}_F(\sigma,\rho,\lambda t))\big), \text{ for all } \sigma,\rho \in \Sigma, t > 0.$$

Theorem 6. *Let $Q\colon \Sigma \to \Sigma$ be a fuzzy α-$\mathcal{H}$-Φ-contraction mapping over the space $(\Sigma, \mathcal{M}_F, *)$, such that*

(i) *there is $\sigma_0 \in \Sigma$ with $\alpha(\sigma_0, Q\sigma_0, t) \geq 1$, $t \geq 0$;*
(ii) *Q is fuzzy α-admissible;*
(iii) *for each subsequence $(\sigma_{n_k})_{k\in\mathbb{N}} \subset (\sigma_n)_{n\in\mathbb{N}} = (Q^n\sigma_0)_{n\in\mathbb{N}}$, we have*

$$\alpha(\sigma_{n_k},\sigma_{n_l},t) \geq 1, \quad \text{where } l < k,\ l,k \in \mathbb{N}, \text{ and } t > 0;$$

(iv) *if $(\sigma_n)_{n\in\mathbb{N}} \subset \Sigma$ with $\alpha(\sigma_n,\sigma_{n+1},t) \geq 1$, and $\lim_{n\to\infty} \sigma_n = \sigma$, then*

$$\alpha(\sigma_n,\sigma_{n+1},t) \geq 1, \text{ for all } t > 0, n \in \mathbb{N}.$$

Then, Q possesses a unique fixed point.

Proof. From condition (i), there is $\sigma_0 \in X$ with $\alpha(\sigma_0, Q\sigma_0, t) \geq 1$, $t \geq 0$. Define the sequence $(\sigma_n)_{n\in\mathbb{N}}$ in Σ by $\sigma_n = Q\sigma_{n-1}$, for all $n \in \mathbb{N}$. If there exist $n_0 \in \mathbb{N}$ such

that $\sigma_{n_0} = \sigma_{n_0+1}$, then $v = \sigma_0$ is a fixed point of Q. So, we can suppose that $\sigma_n \neq \sigma_{n+1}$ for every $n \in \mathbb{N}$. Since Q is fuzzy α-admissible, then the following implication holds:

$$\alpha(\sigma_0,\sigma_1,t) = \alpha(\sigma_0, Q\sigma_0, t) \geq 1, \text{ implies } \alpha(Q\sigma_0, Q\sigma_1, t) = \alpha(\sigma_1,\sigma_2,t) \geq 1,\ t > 0.$$

Inductively, we get

$$\alpha(\sigma_n,\sigma_{n+1},t) \geq 1, \quad t > 0,\ n \in \mathbb{N}. \tag{3.22}$$

Since Q is a fuzzy α-$\mathcal{H}$-Φ-contractive mapping and by (3.22), we get

$$\begin{aligned}
\eta\big(\mathcal{M}_F(\sigma_n,\sigma_{n+1},t)\big) &= \eta\big(\mathcal{M}_F(Q\sigma_{n-1}, Q\sigma_n, t)\big) \\
&\leq \eta\big(\mathcal{M}_F(Q\sigma_{n-1}, Q\sigma_n, \frac{t}{\alpha(\sigma_{n-1},\sigma_n,t)})\big) \\
&\leq \eta\big(\psi\big(\mathcal{M}_F(\sigma_{n-1},\sigma_n,\lambda t)\big)\big) \\
&\leq \eta\big(\psi\big(\mathcal{M}_F(\sigma_{n-1},\sigma_n,t)\big)\big) \\
&\leq \eta\big(\mathcal{M}_F(\sigma_{n-1},\sigma_n,t)\big).
\end{aligned} \tag{3.23}$$

Then,

$$\mathcal{M}_F(\sigma_n,\sigma_{n+1},t) \geq \psi\big(\mathcal{M}_F(\sigma_{n-1},\sigma_n,t)\big) \text{ for all } t > 0. \tag{3.24}$$

And thus,

$$\mathcal{M}_F(\sigma_n,\sigma_{n+1},t) \geq \mathcal{M}_F(\sigma_{n-1},\sigma_n,t). \tag{3.25}$$

Then, the sequence $\{\mathcal{M}_F(\sigma_n,\sigma_{n+1},t)\}$ is nondescreasing. So, $\{\mathcal{M}_F(\sigma_n,\sigma_{n+1},t)\}$ is convergent and denote

$$\lim_{n\to+\infty} \mathcal{M}_F(\sigma_n,\sigma_{n+1},t) = r \in (0,1].$$

By formula (3.24) and the continuity of ψ, it follows

$$r \geq \psi(r).$$

This implies $r = 1$, and thus

$$\lim_{n\to+\infty} \mathcal{M}_F(\sigma_n,\sigma_{n+1},t) = 1, \quad \text{for all } t > 0. \tag{3.26}$$

If $(\sigma_n)_{n\in\mathbb{N}}$ is not $\mathcal{M}_F$-Cauchy, there exist $t_0 > 0$ and $\varepsilon \in (0,1)$ such that for all $k \in \mathbb{N}$ there are $m_k, n_k \in \mathbb{N}$ with $m_k > n_k \geq k$ and

$$\mathcal{M}_F(\sigma_{m_k},\sigma_{n_n},t_0) \leq 1-\varepsilon. \tag{3.27}$$

If n_k will be assumed to be the least integer exceeding m_k satisfying (3.27), then

$$\mathcal{M}_F(\sigma_{m_k-1},\sigma_{n_n},t_0) > 1-\varepsilon.$$

From (3.25), (3.27) and (C4), the following estimations hold:

$$\begin{aligned}
1-\varepsilon &\geq \mathcal{M}_F(\sigma_{m_k},\sigma_{n_k},t_0) \\
&\geq \mathcal{M}_F(\sigma_{m_k-1},\sigma_{n_k-1},t_0)
\end{aligned}$$

$$\geq \mathcal{M}_F(\sigma_{m_k-1}, \sigma_{n_k}, \frac{t_0}{2}) * \mathcal{M}_F(\sigma_{n_k}, \sigma_{n_k-1}, \frac{t_0}{2})$$
$$\geq (1-\varepsilon) * \mathcal{M}_F(\sigma_{n_k}, \sigma_{n_k-1}, \frac{t_0}{2}),$$

which by passing to the limit as k tends to $+\infty$ and using (3.26) with (C3) yields

$$\lim_{k\to\infty} \mathcal{M}_F(\sigma_{m_k-1}, \sigma_{n_k-1}, t_0) = 1-\varepsilon = \lim_{k\to\infty} \mathcal{M}_F(\sigma_{m_k}, \sigma_{n_k}, t_0).$$

On the other hand, since Q is a fuzzy α-$\mathcal{H}$-Φ-contractive mapping and thanks to (3.22) we have

$$\begin{aligned}
\eta\big(\mathcal{M}_F(\sigma_{m_k}, \sigma_{n_k}, t)\big) &= \eta\big(\mathcal{M}_F(Q\sigma_{m_k-1}, Q\sigma_{n_k-1}, t)\big) \\
&\leq \eta\Big(\mathcal{M}_F(Q\sigma_{m_k-1}, Q\sigma_{n_k-1}, \frac{t}{\alpha(Q\sigma_{m_k-1}, Q\sigma_{n_k-1}, t)})\Big) \\
&\leq \eta\big(\psi\big(\mathcal{M}_F(\sigma_{m_k-1}, \sigma_{n_k-1}, \lambda t)\big)\big) \\
&\leq \eta\big(\mathcal{M}_F(\sigma_{m_k-1}, \sigma_{n_k-1}, \lambda t)\big) \\
&\quad \vdots \\
&\leq \eta\big(\mathcal{M}_F(\sigma_1, \sigma_0, \lambda^{n_k} t)\big).
\end{aligned}$$

Since η is continuous, we get

$$\eta(\lim_{k\to\infty} \mathcal{M}_F(\sigma_{m_k}, \sigma_{n_k}, t_0)) \leq \eta(\lim_{k\to\infty} \mathcal{M}_F(\sigma_0, \sigma_1, \lambda^{n_k} t_0)).$$

This implies $\eta(1-\varepsilon) \leq \eta(1)$, which gives an absurdity, and thus $(\sigma_n)_{n\in\mathbb{N}}$ is an $\mathcal{M}_F$-Cauchy in Σ. Now, since $(\Sigma, \mathcal{M}_F, *)$ is a $\mathcal{M}_F$-complete, then there is $\sigma \in \Sigma$ such that

$$\lim_{n\to\infty} \mathcal{M}_F(\sigma_n, \sigma, t) = 1, \quad t > 0.$$

By using (3.23), we can get

$$\lim_{n\to\infty} \mathcal{M}_F(\sigma_{n+1}, Q\sigma, t) = 1, \quad t > 0.$$

Since the limit is unique, we conclude that $Q\sigma = \sigma$, that is, σ is a fixed point of Q. To show the uniqueness of σ, let σ_1, σ_2 be two different fixed points of Q. Then using (3.7), we have

$$\begin{aligned}
\eta(\mathcal{M}_F(\sigma_1, \sigma_2, t)) &= \eta(\mathcal{M}_F(Q\sigma_1, Q\sigma_2, t)) \\
&\leq \eta\Big(\mathcal{M}_F(Q\sigma_1, Q\sigma_2, \frac{t}{\alpha(\sigma_1, \sigma_0, t)})\Big) \\
&\leq \eta\big(\psi\big(\mathcal{M}_F(\sigma_1, \sigma_0, \lambda t)\big)\big) \\
&\leq \eta(\mathcal{M}_F(\sigma_1, \sigma_2, \lambda t)) \\
&\leq \eta(\mathcal{M}_F(\sigma_1, \sigma_2, t)). \qquad (3.28)
\end{aligned}$$

If $\mathcal{M}_F(\sigma_1,\sigma_2,t) < 1$, then $\eta(\mathcal{M}_F(\sigma_1,\sigma_2,t)) > 0$ and hence (3.28) we have

$$\eta(\mathcal{M}_F(\sigma_1,\sigma_2,t)) \leq \eta(\mathcal{M}_F(\sigma_1,\sigma_2,\lambda t)) < \eta(\mathcal{M}_F(\sigma_1,\sigma_2,t)),$$

which is a contradiction. Therefore, $\mathcal{M}_F(\sigma_1,\sigma_2,t) = 1$ yielding $\sigma_1 = \sigma_2$, which completes the proof. □

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SOME NEW INTEGRAL INEQUALITIES OF HERMITE–HADAMARD TYPE FOR FUNCTIONS WHOSE THIRD DERIVATIVES ARE (α, s, m)-CONVEX

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Abstract. In the paper, the authors establish several new integral inequalities of the Hermite–Hadamard type for functions whose third derivatives are (α, s, m)-convex.

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1. A BRIEF REVIEW

We first recall definitions of several convex functions.

Definition 1. A function $h\colon J \subseteq \mathbb{R} = (-\infty, \infty) \to \mathbb{R}$ is said to be convex if

$$h(\vartheta\varrho + (1-\vartheta)\rho) \le \vartheta h(\varrho) + (1-\vartheta)h(\rho)$$

for all $\varrho, \rho \in J$ and $\vartheta \in [0,1]$.

In the papers [2, 10], the concept of s-convex functions was innovated as follows.

Definition 2 ([2, p. 13] and [10, p. 100])**.** Let $s \in (0,1]$ be a real number. A function $h\colon \mathbb{R}_0 = [0,\infty) \to \mathbb{R}_0$ is said to be s-convex (in the second sense) if

$$h(\vartheta\varrho + (1-\vartheta)\rho) \le \vartheta^s h(\varrho) + (1-\vartheta)^s h(\rho)$$

for all $\varrho, \rho \in J$ and $\vartheta \in [0,1]$.

In the conference paper [26], the concept of m-convex functions was innovated as follows.

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Definition 3 ([26]). Let $h\colon [0,b]\to\mathbb{R}$ for $b>0$ and let $m\in(0,1]$. If the inequality

$$h(\vartheta\varrho+m(1-\vartheta)\rho)\le\vartheta h(\varrho)+m(1-\vartheta)h(\rho)$$

is valid for all $\varrho,\rho\in[0,b]$ and $\vartheta\in[0,1]$, then we call that h is an m-convex function on $[0,b]$.

Moreover, we recite the following several definitions of various convex functions.

Definition 4 ([31, Definition 2.1]). A function $h\colon J\subseteq\mathbb{R}\to\mathbb{R}$ is said to be extended s-convex for some fixed $s\in[-1,1]$ if the inequality

$$h(\vartheta\varrho+(1-\vartheta)\rho)\le\vartheta^s h(\varrho)+(1-\vartheta)^s h(\rho)$$

holds for all $\varrho,\rho\in J$ and $\vartheta\in(0,1)$.

Definition 5 ([29, Definition 2.1]). For some $s\in[-1,1]$ and $\alpha\in(0,1]$, a function $h\colon J\subseteq\mathbb{R}\to\mathbb{R}$ is said to be (α,s)-convex if

$$h(\vartheta\varrho+(1-\vartheta)\rho)\le\vartheta^{\alpha s}h(\varrho)+(1-\vartheta^\alpha)^s h(\rho)$$

for all $\varrho,\rho\in J$ and $\vartheta\in(0,1)$.

Definition 6 ([29, Definition 2.2]). For some $s\in[-1,1]$ and $(\alpha,m)\in(0,1]^2$, a function $h\colon [0,b]\to\mathbb{R}$ is said to be (α,s,m)-convex if

$$h(\vartheta\varrho+m(1-\vartheta)\rho)\le\vartheta^{\alpha s}h(\varrho)+m(1-\vartheta^\alpha)^s h(\rho)$$

for all $\varrho,\rho\in[0,b]$ and $\vartheta\in(0,1)$.

To establish integral inequalities of the Hermite–Hadamard type is a very active research topic. We would like to recite the following integral inequalities of the Hermite–Hadamard type.

Theorem 1 ([4, Theorem 2.2]). *Let $h\colon J^\circ\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on the interior J° of J and let $\varrho,\rho\in J^\circ$ with $\varrho<\rho$. If $|h'|$ is convex on $[\varrho,\rho]$, then*

$$\left|\frac{h(\varrho)+h(\rho)}{2}-\frac{1}{\rho-\varrho}\int_\varrho^\rho h(x)\,\mathrm{d}x\right|\le\frac{(\rho-\varrho)(|h'(\varrho)|+|h'(\rho)|)}{8}.$$

Theorem 2 ([21, Theorems 1 and 2]). *Let $h\colon J\subseteq\mathbb{R}\to\mathbb{R}$ be differentiable on J° and let $\varrho,\rho\in J$ such that $\varrho<\rho$. If $|h'|^q$ is convex on $[\varrho,\rho]$ and $q\ge1$, then*

$$\left|\frac{h(\varrho)+h(\rho)}{2}-\frac{1}{\rho-\varrho}\int_\varrho^\rho h(x)\,\mathrm{d}x\right|\le\frac{\rho-\varrho}{4}\left(\frac{|h'(\varrho)|^q+|h'(\rho)|^q}{2}\right)^{1/q}$$

and

$$\left|h\left(\frac{\varrho+\rho}{2}\right)-\frac{1}{\rho-\varrho}\int_\varrho^\rho h(x)\,\mathrm{d}x\right|\le\frac{\rho-\varrho}{4}\left(\frac{|h'(\varrho)|^q+|h'(\rho)|^q}{2}\right)^{1/q}.$$

Theorem 3 ([5, Remark 2]). *Let $h\colon \mathbb{R}_0 = [0,\infty) \to \mathbb{R}$ be m-convex and $m \in (0,1]$. If $h \in L_1([\varrho,\rho])$ for $0 \le \varrho < \rho < \infty$, then*

$$\frac{1}{\rho-\varrho}\int_{\varrho}^{\rho} h(x)\,\mathrm{d}x \le \min\left\{\frac{h(\varrho)+mh(\rho/m)}{2}, \frac{mh(\varrho/m)+h(\rho)}{2}\right\}.$$

Theorem 4 ([13, Theorem 1]). *Let $h\colon J \subseteq \mathbb{R}_0 \to \mathbb{R}$ be differentiable on J° and let $\varrho,\rho \in J$ with $\varrho < \rho$. If $|h'|^q$ is s-convex on $[\varrho,\rho]$ for some fixed $s \in (0,1]$ and $q \ge 1$, then*

$$\left|\frac{h(\varrho)+h(\rho)}{2} - \frac{1}{\rho-\varrho}\int_{\varrho}^{\rho} h(x)\,\mathrm{d}x\right| \le \frac{\rho-\varrho}{2}\left(\frac{1}{2}\right)^{1-1/q}\left[\frac{2+1/2^s}{(s+1)(s+2)}\right]^{1/q}\left[|h'(\varrho)|^q + |h'(\rho)|^q\right]^{1/q}.$$

Theorem 5 ([11, Theorem 4]). *Let $h\colon J \subseteq \mathbb{R}_0 \to \mathbb{R}$ be differentiable on J°, let $\varrho,\rho \in J$ such that $\varrho < \rho$, and let $h' \in L[\varrho,\rho]$. If $|h'|^q$ is s-convex on $[\varrho,\rho]$ for some fixed $s \in (0,1]$ and $q > 1$, then*

$$\begin{aligned}\left|h\left(\frac{\varrho+\rho}{2}\right) - \frac{1}{\rho-\varrho}\int_{\varrho}^{\rho} h(x)\,\mathrm{d}x\right| &\le \frac{\rho-\varrho}{4}\left[\frac{1}{(s+1)(s+2)}\right]^{1/q}\left(\frac{1}{2}\right)^{1/p}\left\{\left[|h'(\varrho)|^q + (s+1)\left|h'\left(\frac{\varrho+\rho}{2}\right)\right|^q\right]^{1/q}\right.\\ &\quad\left. + \left[|h'(\rho)|^q + (s+1)\left|h'\left(\frac{\varrho+\rho}{2}\right)\right|^q\right]^{1/q}\right\}, \quad (1.1)\end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Remark 1. The inequality (1.1) is a corrected version of [11, Theorem 4].
By the way, the identity in [11, Lemma 3] should also be corrected as

$$f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x = \frac{b-a}{4}\int_0^1\left[(1-t)f'\left(ta+(1-t)\frac{a+b}{2}\right) - tf'\left(t\frac{a+b}{2}+(1-t)b\right)\right]\mathrm{d}t.$$

Consequently, the whole body of the paper [11] should be rewritten and modified.

Theorem 6 ([25, Theorem 7]). *Let $h\colon J \subseteq \mathbb{R}_0 \to \mathbb{R}$ be differentiable on J°, $\varrho,\rho \in J$ with $\varrho < \rho$, and $h' \in L[\varrho,\rho]$. If $|h'|$ is s-convex on $[\varrho,\rho]$ for some $s \in (0,1]$ and $p > 1$, then*

$$\left|\frac{1}{6}\left[h(\varrho) + 4h\left(\frac{\varrho+\rho}{2}\right) + h(\rho)\right] - \frac{1}{\rho-\varrho}\int_{\varrho}^{\rho} h(x)\,\mathrm{d}x\right|$$

$$\le \frac{(s-4)6^{s+1}+2\times5^{s+2}-2\times3^{s+2}+2}{6^{s+2}(s+1)(s+2)}(\rho-\varrho)\big(|h'(\varrho)|+|h'(\rho)|\big),$$

where $\frac{1}{p}+\frac{1}{q}=1$.

For more conclusions on this topic, please refer to the papers [1, 6, 8, 9, 12, 14, 15, 17–20, 22–24, 27, 28, 30, 32–34] and closely related references therein.

The main purpose of this paper is to establish some new integral inequalities of the Hermite–Hadamard type for (α,s,m)-convex functions.

2. A LEMMA

For attaining the aim of this paper, we need the following lemma.

Lemma 1 ([3, Lemma 2.1])**.** *Let $h\colon J\subseteq\mathbb{R}\to\mathbb{R}$ be a three-time differentiable mapping on J° and let $\varrho,\rho\in J^\circ$ such that $\varrho<\rho$. If $h'''\in L_1([\varrho,\rho])$, then*

$$\begin{aligned}h\left(\frac{\varrho+\rho}{2}\right)-\frac{1}{\rho-\varrho}\int_\varrho^\rho h(x)\,\mathrm{d}x+\frac{\rho-\varrho}{24}\big[h'(\rho)-h'(\varrho)\big]\\ =\frac{(\rho-\varrho)^3}{96}\left[\int_0^1\vartheta(1-\vartheta)(2-\vartheta)h'''\left(\frac{1-\vartheta}{2}\varrho+\frac{1+\vartheta}{2}\rho\right)\mathrm{d}\vartheta\right.\\ \left.-\int_0^1\vartheta(1-\vartheta)(2-\vartheta)h'''\left(\frac{1+\vartheta}{2}\varrho+\frac{1-\vartheta}{2}\rho\right)\mathrm{d}\vartheta\right].\end{aligned}$$

3. INTEGRAL INEQUALITIES OF HERMITE–HADAMARD TYPE

Now we start out to establish some integral inequalities of the Hermite–Hadamard type for (α,s,m)-convex functions.

Theorem 7. *For $(\alpha,m)\in(0,1]^2$, $s\in(-1,1]$, and $\rho>\varrho>0$, let $f\colon\left(0,\frac{\rho}{m}\right]\to\mathbb{R}$ be a three-time differentiable function and $f'''\in L_1\left(\left[\varrho,\frac{\rho}{m}\right]\right)$. If $|f'''|^q$ is an (α,s,m)-convex function on $\left(0,\frac{\rho}{m}\right]$ for $q\ge1$, then*

$$\begin{aligned}&\left|f\left(\frac{\varrho+\rho}{2}\right)-\frac{1}{\rho-\varrho}\int_\varrho^\rho h(x)\,\mathrm{d}x+\frac{\rho-\varrho}{24}[f'(\rho)-f'(\varrho)]\right|\\ &\le\frac{(\rho-\varrho)^3}{96}\left(\frac{1}{4}\right)^{1-1/q}\left\{\left[\frac{2^{1-\alpha s}}{(\alpha s+2)(\alpha s+4)}|f'''(\varrho)|^q+mQ(\alpha,s)\left|f'''\left(\frac{\rho}{m}\right)\right|^q\right]^{1/q}\right.\\ &\qquad\left.+\left[W(\alpha,s)|f'''(\varrho)|^q+mR(\alpha,s)\left|f'''\left(\frac{\rho}{m}\right)\right|^q\right]^{1/q}\right\},\end{aligned}$$

where $\Gamma(x)$, $B(x,y)$, and ${}_2F_1(c,d;e;z)$ are the Euler gamma function, the Euler beta function, and the Gauss hypergeometric functions, which can be defined respectively

by

$$\Gamma(x)=\int_0^1 z^{x-1}\,\mathrm{e}^{-z}\,\mathrm{d}z,\quad B(x,y)=\int_0^1 z^{x-1}(1-z)^{y-1}\,\mathrm{d}z,$$

and

$${}_2F_1(c,d;e;z)=\frac{\Gamma(e)}{\Gamma(d)\Gamma(e-d)}\int_0^1 t^{d-1}(1-t)^{e-d-1}(1-zt)^{-c}\,\mathrm{d}t$$

for $x,y>0$, $e>d>0$, $|z|<1$, *and* $c\in\mathbb{R}$, *and*

$$Q(\alpha,s)=\frac{2^{1-\alpha s}(\alpha s+3)(\alpha s+9)+4[(\alpha s)^2+7\alpha s-12]}{(\alpha s+1)(\alpha s+2)(\alpha s+3)(\alpha s+4)},$$

$$W(\alpha,s)=\frac{\alpha}{8}{}_2F_1\left(\frac{2}{\alpha},-s;\frac{\alpha+2}{\alpha};\frac{1}{2^\alpha}\right)-\frac{\alpha}{8}{}_2F_1\left(\frac{4}{\alpha},-s;\frac{\alpha+4}{\alpha};\frac{1}{2^\alpha}\right),$$

$$R(\alpha,s)=\frac{4}{\alpha}\left[B\left(\frac{1}{\alpha},s+1\right)-B\left(\frac{2}{\alpha},s+1\right)+\frac{\alpha}{8}{}_2F_1\left(\frac{2}{\alpha},-s;\frac{\alpha+2}{\alpha};\frac{1}{2^\alpha}\right)-\frac{\alpha}{8}{}_2F_1\left(\frac{4}{\alpha},-s;\frac{\alpha+4}{\alpha};\frac{1}{2^\alpha}\right)\right].$$

Proof. Using Lemma 1 and Hölder's integral inequality [7], we obtain

$$\begin{aligned}
&\left|h\left(\frac{\varrho+\rho}{2}\right)-\frac{1}{\rho-\varrho}\int_\varrho^\rho h(x)\,\mathrm{d}x+\frac{\rho-\varrho}{24}[h'(\rho)-h'(\varrho)]\right|\\
&\le\frac{(\rho-\varrho)^3}{96}\left\{\int_0^1\vartheta(1-\vartheta)(2-\vartheta)\left|h'''\left(\frac{1-\vartheta}{2}\varrho+\frac{1+\vartheta}{2}\rho\right)\right|\mathrm{d}\vartheta\right.\\
&\qquad\left.+\int_0^1\vartheta(1-\vartheta)(2-\vartheta)\left|h'''\left(\frac{1+\vartheta}{2}\varrho+\frac{1-\vartheta}{2}\rho\right)\right|\mathrm{d}\vartheta\right\}\\
&\le\frac{(\rho-\varrho)^3}{96}\left[\int_0^1\vartheta(1-\vartheta)(2-\vartheta)\,\mathrm{d}\vartheta\right]^{1-1/q}\\
&\quad\times\left\{\left[\int_0^1\vartheta(1-\vartheta)(2-\vartheta)\left|h'''\left(\frac{1-\vartheta}{2}\varrho+\frac{1+\vartheta}{2}\rho\right)\right|^q\mathrm{d}\vartheta\right]^{1/q}\right.\\
&\quad\left.+\left[\int_0^1\vartheta(1-\vartheta)(2-\vartheta)\left|h'''\left(\frac{1+\vartheta}{2}\varrho+\frac{1-\vartheta}{2}\rho\right)\right|^q\mathrm{d}\vartheta\right]^{1/q}\right\}.
\end{aligned}\tag{3.1}$$

By the (α,s,m)-convexity of $|h'''|^q$, we have

$$\begin{aligned}
&\int_0^1\vartheta(1-\vartheta)(2-\vartheta)\left|h'''\left(\frac{1-\vartheta}{2}\varrho+\frac{1+\vartheta}{2}\rho\right)\right|^q\mathrm{d}\vartheta\\
&\le\int_0^1\vartheta(1-\vartheta)(2-\vartheta)\left(\frac{1-\vartheta}{2}\right)^{\alpha s}|h'''(\varrho)|^q\,\mathrm{d}\vartheta\\
&\quad+m\int_0^1\vartheta(1-\vartheta)(2-\vartheta)\left[1-\left(\frac{1-\vartheta}{2}\right)^\alpha\right]^s\left|h'''\left(\frac{\rho}{m}\right)\right|^q\mathrm{d}\vartheta
\end{aligned}\tag{3.2}$$

and

$$\begin{aligned}&\int_0^1 \vartheta(1-\vartheta)(2-\vartheta)\left|h'''\left(\frac{1+\vartheta}{2}\varrho+\frac{1-\vartheta}{2}\rho\right)\right|\mathrm{d}\vartheta\\&\le\int_0^1 \vartheta(1-\vartheta)(2-\vartheta)\left(\frac{1+\vartheta}{2}\right)^{\alpha s}|h'''(\varrho)|^q\,\mathrm{d}\vartheta\\&\quad+m\int_0^1 \vartheta(1-\vartheta)(2-\vartheta)\left[1-\left(\frac{1+\vartheta}{2}\right)^{\alpha}\right]^s\left|h'''\left(\frac{\rho}{m}\right)\right|^q\mathrm{d}\vartheta.\end{aligned}\tag{3.3}$$

Straightforward computation gives

$$\begin{aligned}\int_0^1 \vartheta(1-\vartheta)(2-\vartheta)\left(\frac{1-\vartheta}{2}\right)^{\alpha s}\mathrm{d}\vartheta&=\frac{2^{1-\alpha s}}{(\alpha s+2)(\alpha s+4)},\\\int_0^1 \vartheta(1-\vartheta)(2-\vartheta)\left(\frac{1+\vartheta}{2}\right)^{\alpha s}\mathrm{d}\vartheta&=Q(\alpha,s),\\\int_0^1 \vartheta(1-\vartheta)(2-\vartheta)\left[1-\left(\frac{1-\vartheta}{2}\right)^{\alpha}\right]^s\mathrm{d}\vartheta&=W(\alpha,s),\end{aligned}\tag{3.4}$$

and

$$\begin{aligned}\int_0^1 \vartheta(1-\vartheta)(2-\vartheta)\left[1-\left(\frac{1+\vartheta}{2}\right)^{\alpha}\right]^s\mathrm{d}\vartheta&\le\int_0^1(1-\vartheta)\left[1-\left(\frac{1+\vartheta}{2}\right)^{\alpha}\right]^s\mathrm{d}\vartheta\\&=R(\alpha,s),\end{aligned}\tag{3.5}$$

where

$$\vartheta(1-\vartheta)(2-\vartheta)\le(1-\vartheta)\left[\frac{\vartheta+(2-\vartheta)}{2}\right]^2=1-\vartheta$$

for $\vartheta\in[0,1]$. Substituting those in (3.2), (3.3), (3.4), (3.5) into (3.1) gives

$$\begin{aligned}&\left|h\left(\frac{\varrho+\rho}{2}\right)-\frac{1}{\rho-\varrho}\int_\varrho^\rho h(x)\,\mathrm{d}x+\frac{\rho-\varrho}{24}[h'(\rho)-h'(\varrho)]\right|\\&\le\frac{(\rho-\varrho)^3}{96}\left(\frac{1}{4}\right)^{1-1/q}\left\{\left[\frac{2^{1-\alpha s}}{(\alpha s+2)(\alpha s+4)}|h'''(\varrho)|^q+m|h'''(\rho)|^qQ(\alpha,s)\right]^{1/q}\right.\\&\qquad\left.+\left[|h'''(\varrho)|^qW(\alpha,s)+m|h'''(\rho)|^qR(\alpha,s)\right]^{1/q}\right\}.\end{aligned}$$

The proof of Theorem 7 is thus proved. □

Theorem 8. *For $(\alpha,m)\in(0,1]^2$, $s\in(-1,1]$, and $\rho>\varrho>0$, let $h\colon\left(0,\frac{\rho}{m}\right]\to\mathbb{R}$ be a three-time differentiable function and $h'''\in L_1\left(\left[\varrho,\frac{\rho}{m}\right]\right)$. If $|h'''|^q$ is an (α,s,m)-convex function on $\left(0,\frac{\rho}{m}\right]$ for $q>1$, then*

$$\left|h\left(\frac{\varrho+\rho}{2}\right)-\frac{1}{\rho-\varrho}\int_\varrho^\rho h(x)\,\mathrm{d}x+\frac{\rho-\varrho}{24}[h'(\rho)-h'(\varrho)]\right|$$

$$
\begin{aligned}
&\le \frac{(\rho-\varrho)^3}{96}\left[\frac{3}{4}B\left(\frac{2q-1}{2(q-1)},\frac{3q-2}{q-1}\right)\right]^{1-1/q}\\
&\quad\times\left\{\left[\frac{2^{-\alpha s}}{\alpha s+1}|h'''(\varrho)|^q+mB\left(\frac{1}{2^\alpha};\frac{1}{\alpha},s+1\right)\left|h'''\left(\frac{\rho}{m}\right)\right|^q\right]^{1/q}\right.\\
&\quad+\left[\frac{2-2^{-\alpha s}}{\alpha s+1}|h'''(\varrho)|^q+m\left(B\left(\frac{1}{\alpha},s+1\right)\right.\right.\\
&\quad\left.\left.\left.-B\left(\frac{1}{2^\alpha};\frac{1}{\alpha},s+1\right)\right)\left|h'''\left(\frac{\rho}{m}\right)\right|^q\right]^{1/q}\right\},
\end{aligned}
\tag{3.6}
$$

where

$$
B(z;\varrho,\rho)=\int_0^z u^{\varrho-1}(1-u)^{\rho-1}\,\mathrm{d}u,\quad 0\le z\le 1
$$

is the incomplete beta function with $B(\varrho;\varrho,\rho)=B(\varrho,\rho)$.

Proof. By Lemma 1 and Hölder's integral inequality [16], we have

$$
\begin{aligned}
&\left|h\left(\frac{\varrho+\rho}{2}\right)-\frac{1}{\rho-\varrho}\int_\varrho^\rho h(x)\,\mathrm{d}x+\frac{\rho-\varrho}{24}[h'(\rho)-h'(\varrho)]\right|\\
&\le\frac{(\rho-\varrho)^3}{96}\left\{\int_0^1\vartheta(1-\vartheta)(2-\vartheta)\left|h'''\left(\frac{1-\vartheta}{2}\varrho+\frac{1+\vartheta}{2}\rho\right)\right|\mathrm{d}\vartheta\right.\\
&\qquad\left.+\int_0^1\vartheta(1-\vartheta)(2-\vartheta)\left|h'''\left(\frac{1+\vartheta}{2}\varrho+\frac{1-\vartheta}{2}\rho\right)\right|\mathrm{d}\vartheta\right\}\\
&\le\frac{(\rho-\varrho)^3}{96}\left(\int_0^1[\vartheta(1-\vartheta)(2-\vartheta)]^{q/(q-1)}\,\mathrm{d}\vartheta\right)^{1-1/q}\\
&\quad\times\left\{\left[\int_0^1\left|h'''\left(\frac{1-\vartheta}{2}\varrho+\frac{1+\vartheta}{2}\rho\right)\right|^q\mathrm{d}\vartheta\right]^{1/q}\right.\\
&\qquad\left.+\left[\int_0^1\left|h'''\left(\frac{1+\vartheta}{2}\varrho+\frac{1-\vartheta}{2}\rho\right)\right|^q\mathrm{d}\vartheta\right]^{1/q}\right\}.
\end{aligned}
\tag{3.7}
$$

Utilizing the (α,s,m)-convexity of $|h'''|^q$, we acquire

$$
\begin{aligned}
&\int_0^1\left|h'''\left(\frac{1-\vartheta}{2}\varrho+\frac{1+\vartheta}{2}\rho\right)\right|^q\mathrm{d}\vartheta\\
&\le\int_0^1\left(\frac{1-\vartheta}{2}\right)^{\alpha s}|h'''(\varrho)|^q\,\mathrm{d}\vartheta+m\int_0^1\left[1-\left(\frac{1-\vartheta}{2}\right)^\alpha\right]^s\left|h'''\left(\frac{\rho}{m}\right)\right|^q\mathrm{d}\vartheta\\
&=\frac{2^{-\alpha s}}{\alpha s+1}|h'''(\varrho)|^q+mB\left(\frac{1}{2^\alpha};\frac{1}{\alpha},s+1\right)\left|h'''\left(\frac{\rho}{m}\right)\right|^q
\end{aligned}
\tag{3.8}
$$

and

$$\begin{aligned}
&\int_0^1 \left|h'''\left(\frac{1+\vartheta}{2}\varrho+\frac{1-\vartheta}{2}\rho\right)\right|\mathrm{d}\vartheta \\
&\le \int_0^1 \left(\frac{1+\vartheta}{2}\right)^{\alpha s}|h'''(\varrho)|^q\,\mathrm{d}\vartheta + m\int_0^1\left[1-\left(\frac{1+\vartheta}{2}\right)^{\alpha}\right]^s\left|h'''\left(\frac{\rho}{m}\right)\right|^q\mathrm{d}\vartheta \qquad (3.9)\\
&= \frac{2-2^{-\alpha s}}{\alpha s+1}|h'''(\varrho)|^q + m\left(B\left(\frac{1}{\alpha},s+1\right)-B\left(\frac{1}{2^\alpha};\frac{1}{\alpha},s+1\right)\right)\left|h'''\left(\frac{\rho}{m}\right)\right|^q.
\end{aligned}$$

It is not difficult to see that

$$\int_0^1 [\vartheta(1-\vartheta)(2-\vartheta)]^{q/(q-1)}\,\mathrm{d}\vartheta = \frac{3}{4}B\left(\frac{2q-1}{2(q-1)},\frac{3q-2}{q-1}\right). \qquad (3.10)$$

Substituting (3.8), (3.9), and (3.10) into the inequality (3.7) leads to the inequality (3.6). The proof of Theorem 8 is thus proved. □

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AN EXISTENCE RESULT FOR PARABOLIC EQUATION OF KIRCHHOFF TYPE BY TOPOLOGICAL DEGREE METHOD

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Abstract. In the present paper, we will study the existence of at least one weak solution for the nonlinear parabolic initial boundary value problem associated with the following equation of Kirchoff type

$$\frac{\partial u}{\partial t} - \mathcal{M}\left(\int_\Omega (\mathcal{B}(x,t,\nabla u) + \frac{1}{\theta}|\nabla u|^\theta)dx\right) \operatorname{div}\left(b(x,t,\nabla u) + |\nabla u|^{\theta-2}\nabla u\right) = f(x,t) - g(x,t,u,\nabla u).$$

By using the Topological degree theory for operators of the type $\mathtt{L}+\mathtt{S}+\mathtt{C}$, where $\mathtt{L}$ is a maximal monotone map, $\mathtt{S}$ is bounded demicontinuous map of class (S_+) and $\mathtt{C}$ be compact and belongs to Γ^τ_σ (i.e there exist $\tau,\sigma \geq 0$ such that $\|\mathtt{C}x\| \leq \tau\|x\| + \sigma$). Our focus of the study is centered on this problem in space $L^\theta(0,T,W^{1,\theta}(\Omega))$, where $\theta \geq 2$ and Ω is a bounded open domain in $\mathbb{R}^N$.

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1. INTRODUCTION

Throughout the paper, $\Omega \subset \mathbb{R}^N$ $(N \geqslant 2)$ designantes a bounded open subset with Lipschitz boundary $\partial\Omega$, and we denote by Q the cylinder $\Omega \times (0,T)$ that $\Gamma = \partial\Omega \times (0,T)$ is its lateral surface, where $T > 0$ is a fixing time.

The objective of this article, is to study the existence of at least one weak solution of the the following parabolic equations of Kirchhoff type,

$$(\mathcal{P}) \begin{cases} \frac{\partial u}{\partial t} - \mathcal{M}\left(\mathcal{E}(u)\right) \operatorname{div}\left(b(x,t,\nabla u) + |\nabla u|^{\theta-2}\nabla u\right) = f(x,t) - g(x,t,u,\nabla u) & \text{in } Q; \\ u(x,0) = u_0(x) & \text{in } \Omega; \\ u(x,t) = 0 & \text{on } \Gamma; \end{cases}$$

where

$$\mathcal{E}(u) = \int_\Omega (\mathcal{B}(x,t,\nabla u) + \frac{1}{\theta}|\nabla u|^\theta)dx$$

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the terme $-\mathrm{div}b(x,t,\nabla u)$ is a Leray-Lions operator acting from $\mathcal{V}$ to its dual $\mathcal{V}^*$. which is coercive, such that

$$\mathcal{V} = L^{\theta}(0,T,W_0^{1,\theta}(\Omega)) \text{ and } \mathcal{V}^* = L^{\theta'}(0,T,W^{-1,\theta}(\Omega)) \quad (\theta \geq 2),$$

where $\frac{1}{\theta} + \frac{1}{\theta'} = 1$, the Kirchhoff type function $\mathcal{M}\colon \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is continuous, the right-hand side f is assumed to belong to $\mathcal{V}^*$, and the Carathéodory function $g\colon Q \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ satisfy some conditions which will be stated later. Problem $(\mathcal{P})$ is related to the stationary version of the Kirchhoff equation

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{\rho_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx \right) \frac{\partial^2 u}{\partial x^2} = 0$$

introduced in 1883 by Kirchhoff see [14] for mor details, where the parameters h, E, ρ, L, ρ_0, are all constants which respectively have some physical meaning, this equation is an extension of the classical d'Alembert's wave equation. In recent years, much interest has grown on Kirchhoff type problems see([4, 10–12, 20, 22, 23, 25]). This interest arises from their contributions to the modeling of many physical and biological phenomena. We refer the readers to [2, 5, 13, 16–18], for some interesting results and further references.

In [25] S. Zediri, R. Guefaifia and S. Boulaaras studied the existence of positive solutions for $p(x)$-Kirchhoff parabolic systems of the form

$$\begin{cases} \frac{\partial u}{\partial t} - M\left(\int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} \mathrm{d}x \right) \Delta_{p(x)} u = \lambda^{p(x)} [\lambda_1 f(v) + m u_1 h(u)] & \text{in } Q; \\ \frac{\partial u}{\partial t} - M\left(\int_\Omega \frac{1}{p(x)} |\nabla v|^{p(x)} \mathrm{d}x \right) \Delta_{p(x)} v = \lambda^{p(x)} [\lambda_2 g(v) + \mu_2 \tau(u)] & \text{in } Q; \\ u(x,0) = \phi(x) & \text{in } \Omega; \\ u = v = 0 & \text{on } \partial\Omega \times (0,T). \end{cases}$$

$M(t)$ is a continuous function, the technical approach is bassed on the sub-super-solutions method.

In [12] Y. Fu, and M. Xiang, proved the d the local existence of weak solutions for the following parabolic initial boundary problem for nonlocal parabolic equations

$$\begin{cases} \frac{\partial u}{\partial t} - \left[a + b \left(\int_\Omega \frac{|\nabla u|^{p(x,t)}}{p(x,t)} \mathrm{d}x \right)^{r(t)} \right] \mathrm{div} \left(|\nabla u|^{p(x,t)-2} \nabla u \right) & \text{in } Q, \\ + |u|^{p(x,t)-2} u = f(x,t,u), & \\ u(x,0) = u_0(x), & \text{in } \Omega, \end{cases}$$

where a, b are tow positive constants. The authors proved the local existence of the weak solution of the above problem, using the Galerkin approximation method.

Motivated by the previous results, and the work of Assfaw to show the existence of at least one weak solutions of $(\mathcal{P})$ in $\mathcal{V}$ by using the Topological degree theory for operators of the type type $\mathtt{L} + \mathtt{S} + \mathtt{C}$, where $\mathtt{L}\colon D(\mathtt{L}) \subset \mathtt{V} \longrightarrow \mathcal{V}^*$ is a linear densely

defined maximal monotone map, $\mathtt{S}$ defined from $\mathcal{V}$ to its dual $\mathcal{V}^*$ is a bounded map of type (S_+) and $\mathtt{C}\colon D(\mathtt{C}) \subset \mathcal{V} \longrightarrow \mathcal{V}^*$ is compact with $D(\mathtt{L}) \subset D(\mathtt{C})$ such that $\mathtt{C}$ belongs to Γ_σ^τ. For more information on the history of this theory and it applications, see for example [1, 6–9, 15, 19, 24].

The paper is organized as follows. In Section 2, we state some mathematical preliminaries about the functional framework where we will treat our problem and the degree theory for operator of the type $\mathtt{L}+\mathtt{S}+\mathtt{C}$. In Section 3, we give some hypotheses and we will prove our main result.

2. Preliminaries

In this part, we briefly review some basic knowledge of the functional framework required to investigate the problem $(\mathcal{P})$, as well as the fundamental definitions and theorems of topological degree theory that are relevant to our goal.

Let $\mathcal{D} \subseteq \mathbb{R}^N$ be a bounded open set with smooth boundary. Let $\theta \geq 2$ and $\theta' = \frac{\theta}{\theta-1}$. We will denote by $L^\theta(\mathcal{D})$ the Banach space of all measurable functions ϕ defined in $\mathcal{D}$ such that

$$\|\phi\|_{L^\theta(\mathcal{D})} = \left(\int_{\mathcal{D}} |\phi(x)|^\theta dx\right)^{1/\theta} < \infty.$$

We define the functional space $W_0^{1,\theta}(\mathcal{D})$ as the closure of $C_0^\infty(\mathcal{D})$ in the Sobolev space

$$W^{1,\theta}(\mathcal{D}) = \left\{\phi \in L^\theta(\mathcal{D})\colon \nabla\phi \in L^\theta(\mathcal{D})\right\},$$

with respect to the norm

$$\|\phi\|_{1,\theta} = \left(\|\phi\|^\theta_{L^\theta(\mathcal{D})} + \|\nabla\phi\|^\theta_{L^\theta(\mathcal{D})}\right)^{1/\theta}.$$

According to the Poincaré inequality, the norm $\|\cdot\|_{1,\theta}$ on $W_0^{1,\theta}(\mathcal{D})$ is equivalent to the norm $\|\cdot\|_{W_0^{1,\theta}(\mathcal{D})}$ setting by

$$\|\phi\|_{W_0^{1,\theta}(\mathcal{D})} = \|\nabla\phi\|_{L^\theta(\mathcal{D})} \text{ for } \phi \in W_0^{1,\theta}(\mathcal{D}).$$

Remember that the Sobolev space $W_0^{1,\theta}(\mathcal{D})$ is a uniformly convex Banach space and the embedding $W_0^{1,\theta}(\mathcal{D}) \hookrightarrow\hookrightarrow L^\theta(\mathcal{D})$ is compact (see [26]).

Following that, we consider the functional space

$$\mathcal{V} := L^\theta(0,T;W_0^{1,\theta}(\mathcal{D})), \quad (T>0),$$

that is a separable and reflexive Banach space with the norm

$$|\phi|_{\mathcal{V}} = \left(\int_0^T \|\phi\|^\theta_{W_0^{1,\theta}(\mathcal{D})} dt\right)^{1/\theta}$$

or, by the Poincaré inequality, the equivalent norm in $\mathcal{V}$ given by

$$\|\phi\|_{\mathcal{V}} = \left(\int_0^T \|\nabla \phi\|^{\theta}_{L^{\theta}(\mathcal{D})} dt \right)^{1/\theta}.$$

2.1. *Topological degree theory for $L+S+C$ operators with $D(L) \subset D(C)$*

In this subsection, we give some results and properties from thedegree theory for $L+S+C$ operators of generalized (S_+) type in real reflexive Banach. In what follows, let X be a real separable reflexive Banach space with dual X^* and with continuous dual pairing $\langle \cdot, \cdot \rangle$ and given a nonempty subset Ω of X, and $\rightharpoonup$ represents the weak convergence.

Let T from X to 2^{X^*} be a multi-values mapping. We denote by $Gr(T)$ the graph of T, i.e.

$$Gr(T) = \{(u,v) \in X \times X^* : \ v \in T(u)\}.$$

Definition 1. The multi-values mapping T is called

(1) monotone, if for each pair of elements $(\eta_1, \theta_1), (\eta_2, \theta_2)$ in $Gr(T)$, we have the inequality
$$\langle \theta_1 - \theta_2, \eta_1 - \eta_2 \rangle \geq 0.$$
(2) maximal monotone, if it is monotone and maximal in the sense of graph inclusion among monotone multi-values mappings from X to 2^{X^*}. An equivalent version of the last clause is that for any $(\eta_0, \theta_0) \in X \times X^*$ for which $\langle \theta_0 - \theta, \eta_0 - \eta \rangle \geq 0$, for all $(\eta, \theta) \in Gr(T)$, we have $(\eta_0, \theta_0) \in Gr(T)$.

Let Y be another real Banach space.

Definition 2. Let Y be another real Banach space. A mapping $F : D(F) \subset X \to Y$ is said to be

(1) Demicontinuous, if for each sequence $(u_n) \subset \Omega$, $u_n \to u$ implies $F(u_n) \rightharpoonup F(u)$.
(2) Of type (S_+), if for any sequence $(u_n) \subset D(F)$ with $u_n \rightharpoonup u$ and $\limsup_{n\to\infty} \langle Fu_n, u_n - u \rangle \leq 0$, we have $u_n \to u$.
(3) Belongs to Γ^{τ}_{σ}, if there exist $\sigma \geq 0$ and $\tau \geq 0$ such that
$$\|Fx\|_Y \leq \tau \|x\|_X + \sigma.$$

Lemma 1 ([6, Lemma 7])**.** *Let G be a nonempty, bounded, and open subset of X. Let $L : D(L) \subset X \longrightarrow 2^{X^*}$ be maximal monotone, $L : \subset X \longrightarrow 2^{X^*}$ be bounded and of type (S_+) and $C : D(C) \subset X \longrightarrow 2^{X^*}$ be compact and $D(L) \subseteq D(C)$ such that C belongs to class Γ^{τ}_{σ}. Assume, further, that $\phi^* \notin (L+S+C)(D(L) \cap \partial G)$. Then there exists $r_0 \geq 0$ such that $d(L_r + S + CJ_r, G, \phi^*)$ s well-defined and independent of $r \in (0, r_0]$.*

According to Lemma 1, the associated degree mapping is established as follows:

Definition 3. Let G be a nonempty, bounded, and open subset of X, $L\colon D(L)\subset X\longrightarrow 2^{X^*}$ be maximal monotone, $L\colon \subset X\longrightarrow 2^{X^*}$ be bounded and of type S_+ and $C\colon D(C)\subset X\longrightarrow 2^{X^*}$ be compact and $D(L)\subseteq D(C)$ such that C belongs to class Γ^{τ}_{σ}. Assume, further, that $f^*\notin (L+S+C)(D(L)\cap\partial G)$. Then the degree mapping d of $L+S+C$ at $\phi^*\in X^*$ with respect to G is defined by

$$d(L+S+C,G,\phi^*)=\lim_{r\longrightarrow 0^+} d_{S_+}(L_r+S+CJ_r,G,\phi^*),$$

where d_{S_+} is the degree mapping for multivalued bounded operators of type (S_+) from [19].

Theorem 1 ([6, Theorem 9])**.** *Let G be a nonempty, bounded, and open subset of X, $L\colon D(L)\subset X\longrightarrow 2^{X^*}$ be maximal monotone, $S\colon \subset X\longrightarrow 2^{X^*}$ be bounded and of type (S_+) and $C\colon D(C)\subset X\longrightarrow 2^{X^*}$ be compact and $D(L)\subseteq D(C)$ such that C belongs to class Γ^{τ}_{σ}. Then following properties hold:*

(1) *(Normalization) $d(J,G,0)=1$ if $0\in G$ and $d(J,G,0)=0$ if $0\notin\overline{G}$.*

(2) *(Additivity) Let G_1 and G_2 are two disjoint open subsets of G such that $\phi^*\notin (L+S+C)\left((\overline{G}\backslash(G_1\cup G_2))\cap D(L)\right)$ then, we have*

$$d((L+S+C),G,\phi^*)=d((L+S+C),G_1,\phi^*)+d((L+S+C),G_2,\phi^*)\ .$$

(3) *(Existence) If $\phi^*\notin (L+S+C)(D(L)\cap\partial G)$ and $d(L+S+C,G,\phi^*)\neq 0$ then $\phi^*\in (L+S+C)(D(L)\cap G)$.*

(4) *(Translation invariance) Let $\phi^*\notin (L+S+C)(D(L)\cap\partial G)$. Then we have*

$$d(L+S+C-\phi^*,G,0)=d(L+S+C,G,\phi^*).$$

(5) *Let $H(t,x)=Lx+t(S_1x+Cx)+(1-t)S_2x$, $(t,x)\in[0,1]$, where $S_i\colon X\longrightarrow 2^{X^*}$ $(i=1,2)$ is bounded and of type (S_+) and $0\notin H(t,D(L)\cap\partial G)$ for all $t\in[0,t]$. Then $d(H(t,\cdot),G,0)$ is independent of $t\in[0,1]$.*

Theorem 2 ([6, Theorem 12])**.** *Let $L\colon D(L)\subset X\longrightarrow 2^{X^*}$ be maximal monotone with $0\in L(0)$, $S\colon \subset X\longrightarrow 2^{X^*}$ be bounded pseudomonotone and $C\colon D(C)\subset X\longrightarrow 2^{X^*}$ be compact and $D(L)\subseteq D(C)$ such that C belongs to class Γ^{τ}_{σ}. Let $\phi^*\in X^*$. Assume further, that there exists $R>0$ such that*

$$\langle Lx+Sx+Cx-\phi^*,x\rangle>0$$

for all $x\in D(L)\cap\partial B_{R(0)}$. Then $\phi^*\in\overline{(L+S+C)(D(L)\cap\partial B_R(0))}$. Furthermore $R\overline{(L+S+C)}=X^*$ provided that $L+S+C$ is coercive.

3. Basic assumptions and main result

Throughout this paper, we use the following assumptions: $b(x,t,\xi)\colon Q\times\mathbb{R}^N\longrightarrow\mathbb{R}^N$ is a Carathéodory, and a continuous derivative with respect to ξ of the continuous mapping $\mathcal{B}(x,t,\xi)\colon Q\times\mathbb{R}^N\longrightarrow\mathbb{R}$. for all $(x,t)\in Q$ and all $\xi,\xi'\in\mathbb{R}^N$, with $(\xi\neq\xi')$.

(h_1) $\mathcal{B}(x,t,0)=0$ and $b(x,t,\xi)=\nabla_\xi\mathcal{B}(x,t,\xi)$,

(h_2) $\alpha|\xi|^\theta \le b(x,t,\xi)\cdot\xi \le \theta\mathcal{B}(x,t,\xi)$,
(h_3) $|b(x,t,\xi)| \le k(x,t)+\beta\,|\xi|^{\theta-1}$,
(h_4) $[b(x,t,\xi)-b(x,t,\xi')]\cdot(\xi-\xi') > 0$,

where α, β are some real positive number and $k(x,t)$ is a positive function in $L^{\theta'}(Q)$.

(h_5) $g\colon Q\times\mathbb{R}\times\mathbb{R}^N \longrightarrow \mathbb{R}^N$ is Carathéodory function and there exist $\rho>0$ and $\gamma\in L^{\theta'}(Q)$ such that

$$|g(x,t,\xi,\eta)| \le \rho(|\xi|^{\theta-1}+|\eta|^{\theta-1})+\gamma(x,t),$$
$$g(x,t,\xi,\eta)\cdot\xi \ge |\xi|^\theta,$$

for all $(x,t)\in Q$, $\xi\in\mathbb{R}$ and $\eta\in\mathbb{R}^N$ with $\eta\neq\xi$.

In order to obtain the existence of weak solutions, the authors always assume that the Kirchhoff function $\mathcal{M}\colon \mathbb{R}^+\to\mathbb{R}^+$ is continuous and non-decreasing function, and satisfies the following conditions:

(M_0) there exist two positive constant m_0 and m_1, such that,

$$m_0 \le \mathcal{M}(t) \le m_1,$$

for all $t\in[0,+\infty[$.

Let G be a nonempty, bounded, a

Lemma 2 ([3, Lemma 2.10])**.** *Assume that (h_2)-(h_4) hold, let $(u_n)_n$ be a sequence in $L^\theta\left(0,T,W_0^{1,\theta}(\Omega)\right)$ such that $u_n \rightharpoonup u$ weakly in $L^\theta\left(0,T,W_0^{1,\theta}(\Omega)\right)$ and*

$$\int_Q [b(x,t,\nabla u_n)-b(x,t,\nabla u)]\,\nabla(u_n-u)\mathrm{d}x \longrightarrow 0.$$

Then $u_n\longrightarrow u$ strongly in $L^\theta\left(0,T,W_0^{1,\theta}(\Omega)\right)$.

Let us consider the following functional

$$\mathcal{T}(u)=\int_0^T \widehat{\mathcal{M}}\left(\int_\Omega\left(\mathcal{B}(x,\nabla u)+\frac{1}{\theta}|\nabla u|^\theta\right)\mathrm{d}x\right)\mathrm{d}t \quad \text{for all } u\in\mathcal{V},$$

where $\widehat{\mathcal{M}}\colon [0,+\infty[\longrightarrow[0,+\infty[$ be the primitive of the function $\mathcal{M}$, defned by

$$\widehat{\mathcal{M}}(n)=\int_0^n \mathcal{M}(\xi)d\xi.$$

It is well known that $\mathcal{T}$ is well defined and continuously Gâteaux differentiable whose Gâteaux derivatives at point $u\in\mathcal{V}$ is the functional $\mathtt{S}(u)\in\mathcal{V}^*$ setting by

$$\langle \mathtt{S}u,\varphi\rangle = \int_0^T\left\{\mathcal{M}\left(\int_\Omega(\mathcal{B}(x,t,\nabla u)+\frac{1}{\theta}|\nabla u|^\theta)\mathrm{d}x\right)\right.$$
$$\left.\times\left[\int_\Omega b(x,t,\nabla u)\nabla\varphi\mathrm{d}x+\int_\Omega|\nabla u|^{\theta-2}\nabla u\nabla\varphi\mathrm{d}x\right]\right\}\mathrm{d}t$$

for all $\varphi \in \mathcal{V}$.

Lemma 3. *Suppose that the assumption (h_2)-(h_4) and (M_0) hold, then*

- $\mathtt{S}$ *is continuous and bounded mapping.*
- *the mapping* $\mathtt{S}$ *is of class* (S_+).

Proof.

- Given that $\mathcal{T}$ is continuously Gâteaux differentiable and whose Gâteaux derivatives at point $u \in \mathcal{V}$, is $\mathtt{S}u \in \mathcal{V}^*$ with

$$\langle \mathcal{T}'u, \varphi \rangle = \langle \mathtt{S}u, \varphi \rangle, \quad \forall \varphi \in \mathcal{V}.$$

Therefore $\mathtt{S}$ is the Fréchet derivative of $\mathcal{T}$. So we can conclude that the operator $\mathtt{S}$ is continuous.

Now, we will prove that $\mathtt{S}$ is bounded.

$$\begin{aligned}
|\langle \mathtt{S}u, \varphi \rangle| &= \left| \int_0^T \left\{ \mathcal{M}\left(\int_\Omega \mathcal{B}(x,t,\nabla u) + \frac{1}{\theta}|\nabla u|^\theta \right) dx \right) \right. \\
&\quad \left. \times \left(\int_\Omega b(x,t,\nabla u)\nabla\varphi \mathrm{d}x + \int_\Omega |\nabla u|^{\theta-2}\nabla u \nabla\varphi \mathrm{d}x \right\} \mathrm{d}t \right| \\
&\leq \int_0^T m_1 \times \left(\int_\Omega |b(x,t,\nabla u)| \cdot |\nabla\varphi| \mathrm{d}x + \int_\Omega |\nabla u|^{\theta-1} \cdot |\nabla\varphi| \mathrm{d}x \right) \mathrm{d}t \\
&\leq m_1 \left(\int_0^T \int_\Omega |b(x,t,\nabla u)| \cdot |\nabla\varphi| \mathrm{d}x\, dt + \int_0^T \left(\int_\Omega |\nabla u|^{\theta-1} \cdot |\nabla\varphi| \mathrm{d}x \right) \mathrm{d}t \right) \\
&\leq 2m_1 \int_0^T \left(\|b(x,t,\nabla u)\|_{L^{\theta'}(\Omega)} \cdot \|\nabla\varphi\|_{L^\theta(\Omega)} \right) dt \\
&\quad + \int_0^T \|\nabla u\|_{L^\theta(\Omega)}^{\frac{\theta}{\theta'}} \cdot \|\nabla\varphi\|_{L^\theta(\Omega)} \mathrm{d}t \\
&\leq Cst \int_0^T \|b(x,t,\nabla u)\|_{L^{\theta'}(\Omega)} \|\varphi\|_{1,\theta} dt + Cst \int_0^T \|u\|_{L^\theta(\Omega)}^{\frac{\theta}{\theta'}} \|\varphi\|_{1,\theta}\, dt.
\end{aligned}$$

From the growth condition (h_2), we can easily show that $\|b(x,t,\nabla u)\|_{L^{\theta'}(\Omega)}$ is bounded for all $u \in W_0^{1,\theta}(\Omega)$. Then

$$|\langle \mathtt{S}u, \varphi \rangle| \leq C_1 \int_0^T \|\varphi\|_{1,\theta} + C_2 \int_0^T \|\varphi\|_{1,\theta}.$$

By the continuous embedding $\mathcal{V} \hookrightarrow L^1\left(0,T,W^{1,\theta}(\Omega)\right)$, we concludes that

$$|\langle \mathtt{S}u, \varphi \rangle| \leq \text{Const } \|\varphi\|_{\mathcal{V}}.$$

Which means that the operator $\mathtt{S}$ is bounded.

- Next, we verify that the operator $\mathtt{S}$ is of type (S_+).

Assume that $(u_n)_n \subset \mathcal{V}$ and

$$\begin{cases} u_n \rightharpoonup u \quad \text{in } \mathcal{V}; \\ \limsup_{n\to\infty} \langle \mathtt{S}u_n, u_n - u\rangle \leq 0. \end{cases}$$

We will show that $u_n \to u$ in $\mathcal{V}$.

On the one hand, since $u_n \rightharpoonup u$ in $\mathcal{V}$, so $(u_n)_n$ is a bounded sequence in $\mathcal{V}$ and since $\mathcal{V}$ embeds compactly in $L^\theta(Q)$, then there exist a subsequence still denoted by $(u_n)_n$ such that $u_n \to u$ in $L^\theta(Q)$.

On the other hand, we have

$$\limsup_{n\to\infty} \langle \mathtt{S}u_n, u_n - u\rangle = \limsup_{n\to\infty} \langle \mathtt{S}u_n - \mathtt{S}u,\ u_n - u\rangle = \lim_{n\to\infty} \langle \mathtt{S}u_n - \mathtt{S}u,\ u_n - u\rangle \leq 0.$$

Then,

$$\begin{aligned} &\lim_{n\to\infty} \langle \mathtt{S}u_n - \mathtt{S}u,\ u_n - u\rangle \\ &= \lim_{n\to\infty} \left(\int_0^T \mathcal{M}\left(\mathcal{E}(u_n)\right) \left[\int_\Omega b(x,t,\nabla u_n)\nabla(u_n - u)\mathrm{d}x \right.\right. \\ &\qquad\qquad \left.\left. + \int_\Omega |\nabla u_n|^{\theta-2}\nabla u_n \nabla(u_n - u)\mathrm{d}x \right] \mathrm{d}t \right. \\ &\quad - \left. \int_0^T \mathcal{M}\left(\mathcal{E}(u)\right) \left[\int_\Omega b(x,t,\nabla u)\nabla(u_n - u)\mathrm{d}x \right.\right. \\ &\qquad\qquad \left.\left. + \int_\Omega |\nabla u|^{\theta-2}\nabla u \nabla(u_n - u)\mathrm{d}x \right] \mathrm{d}t \right) \leq 0. \end{aligned}$$

Or by (A_1) we have for any $x \in \Omega$ and $\xi \in \mathbb{R}^n$

$$\mathcal{B}(x,\xi) = \int_0^1 \frac{d}{ds}\mathcal{B}(x,s\xi)ds = \int_0^1 b(x,t,s\xi)\xi ds,$$

by combining (h_3), Fubini's theorem and Young's inequality we have

$$\begin{aligned} \int_\Omega \mathcal{B}(x,t,\nabla u)\mathrm{d}x &= \int_\Omega \int_0^1 b(x,t,s\nabla u_n)\nabla u ds\mathrm{d}x \\ &\leq \int_0^1 \left[C_{\theta'} \int_\Omega |b(x,t,s\nabla u)|^{\theta'}\,\mathrm{d}x + C_\theta \int_\Omega |\nabla u|^\theta\, dx \right] \mathrm{d}s \\ &\leq C_1 + C' \int_0^1 \int_\Omega |s\nabla u|^\theta \mathrm{d}x\mathrm{d}s + C_\theta \|u\|^\theta_{L^\theta(\Omega)} \\ &\leq C_1 + C_2 \int_\Omega |\nabla u|^\theta \mathrm{d}x + C_\theta \|u\|^\theta_{1,\theta} \\ &\leq C\|u\|^\theta_{1,\theta}. \end{aligned} \tag{3.1}$$

by (3.1), Then $\int_\Omega (\mathcal{B}(x,t,\nabla u_n)\mathrm{d}x$ is bounded.

As $\mathcal{M}$ is continuous, up to a subsequence there is $k \geq 0$ by

$$\mathcal{M}(\mathcal{E}(u_n)) \longrightarrow \mathcal{M}(k) \geq m_0 \quad \text{as } n \to \infty.$$

In addition by appliying the assumption (M_0), we get

$$\begin{aligned}\lim_{n\to\infty} m_0 \Bigg(\int_0^T \Bigg[\int_\Omega & b(x,t,\nabla u_n)\nabla(u_n-u)\mathrm{d}x \\ & + \int_\Omega |\nabla u_n|^{\theta-2}\nabla u_n \nabla(u_n-u)\,dx \Bigg] \mathrm{d}t \\ & - \int_0^T \Bigg[\int_\Omega b(x,t,\nabla u)\nabla(u_n-u)\,dx \\ & + \int_\Omega |\nabla u|^{\theta-2}\nabla u\nabla(u_n-u)\mathrm{d}x \Bigg] \mathrm{d}t \Bigg) \leq 0.\end{aligned}$$

Using the compact embedding $\mathcal{V} \hookrightarrow\hookrightarrow L^\theta(Q)$, we have

$$\lim_{n\to\infty} \int_\Omega |\nabla u_n|^{\theta-2}\nabla u_n(\nabla u_n - \nabla u)\mathrm{d}x = 0, \qquad \lim_{n\to\infty} \int_\Omega |\nabla u|^{\theta-2}\nabla u(\nabla u_n - \nabla u)\mathrm{d}x = 0.$$

Since $m_0 \geq 0$ then, we have

$$\lim_{n\to\infty} \left(\int_0^T \int_\Omega b(x,t,\nabla u_n)\nabla(u_n-u)\mathrm{d}x\mathrm{d}t - \int_0^T \int_\Omega b(x,t,\nabla u)\nabla(u_n-u)\,dx\mathrm{d}t \right) \leq 0,$$

which means

$$\lim_{n\to\infty} \int_0^T \int_\Omega [b(x,t,\nabla u_n) - b(x,t,\nabla u)](\nabla u_n - \nabla u)\mathrm{d}x\mathrm{d}t \leq 0, \tag{3.2}$$

By combining (3.2) and (h_4) we deduce that

$$\lim_{n\to\infty} \int_0^T \int_\Omega [b(x,t,\nabla u_n) - b(x,t,\nabla u)](\nabla u_n - \nabla u)\mathrm{d}x\mathrm{d}t = 0,$$

In light of Lemma 2, we obtain

$$u_n \longrightarrow u \quad \text{in } \mathcal{V},$$

which implies that $\mathtt{S}$ is of type (S_+). □

Lemma 4. *Suppose that the assumption (h_5) hold, then the mapping $\mathtt{C}\colon D(\mathtt{C}) \subset \mathcal{V} \longrightarrow \mathcal{V}^*$ defined by*

$$\langle \mathtt{C}u, \varphi \rangle = \int_Q g(x,t,u,\nabla u)\varphi \mathrm{d}x\mathrm{d}t, \text{ and } D(\mathtt{C}) = \{v \in \mathcal{V} : \ v' \in \mathcal{V}^*\},$$

is compact and belong to the class Γ^τ_σ.

Proof. We first show that $\mathtt{C}$ is compact.

It is know that $D(\mathtt{C})$ is compactly embedded in $L^\theta(Q)$, because $W^{1,\theta}(\Omega)$ is compactly embedded in $L^\theta(\Omega)$ (see Proposition 1.3 in [21]), as a result $\mathtt{C}$ is completly continuous, that is $\mathtt{C}$ is compact operator.

Next, we prove that $\mathtt{C}$ belongs to Γ_σ^τ. By apliying condition (h_4), we get

$$\begin{aligned}
|\langle \mathtt{C}u,\varphi\rangle| &= \left|\int_0^T\left(\int_\Omega g(x,t,u,\nabla u)\varphi\,dx\right)\mathrm{d}t\right| \\
&= \int_0^T\left(\int_\Omega |g(x,t,u,\nabla u)|\,|\varphi|\mathrm{d}x\right)\mathrm{d}t \\
&\le \int_0^T\left[\rho\int_\Omega\left(|u|^{\theta-1}+|\nabla u|^{\theta-1}\right)|\varphi|\,dx\right]\mathrm{d}t+\int_Q|\gamma(x,t)||\varphi|\mathrm{d}x\mathrm{d}t \\
&\le \rho\int_0^T\|u\|_{L^\theta(\Omega)}^{\frac{\theta}{\theta'}}\|\varphi\|_{L^\theta(\Omega)}\mathrm{d}t+\rho\int_0^T\|\nabla u\|_{L^\theta(\Omega)}^{\frac{\theta}{\theta'}}\|\varphi\|_{L^\theta(\Omega)}\mathrm{d}t \\
&\quad+\int_0^T\|\gamma(x,t)\|_{L^{\theta'}(\Omega)}\|\varphi\|_{L^\theta(\Omega)}\mathrm{d}t \\
&\le 2\rho\int_0^T\|\nabla u\|_{L^\theta(\Omega)}^{\frac{\theta}{\theta'}}\|\varphi\|_{L^\theta(\Omega)}+\int_0^T\|\gamma(x,t)\|_{L^{\theta'}(\Omega)}\|\varphi\|_{L^\theta(\Omega)}\mathrm{d}t \\
&\le 2\rho\int_0^T\|\nabla u\|_{1,\theta}^{\theta-1}\|\varphi\|_{1,\theta}+\int_0^T\|\gamma(x,t)\|_{L^{\theta'}(\Omega)}\|\varphi\|_{1,\theta}\mathrm{d}t \\
&\le 2\rho\|u\|_{\mathcal{V}}^{\theta-1}\|\varphi\|_{\mathcal{V}}+\|\gamma\|_{L^{\theta'}(Q)}\|\varphi\|_{\mathcal{V}} \\
&\le Cst\|u\|_{\mathcal{V}}\|\varphi\|_{\mathcal{V}}+\|\gamma\|_{L'^{\theta}(Q)}\|\varphi\|_{\mathcal{V}} \quad \text{for all } u,\varphi\in\mathcal{V}.
\end{aligned}$$

Consequently, taking supremum overall $\varphi\in\mathcal{V}$ with $\|\varphi\|_{\mathcal{V}}\le 1$, we conclude that

$$\|\mathtt{C}u\|_{\mathcal{V}^*}\le\tau\|u\|_{\mathcal{V}}+\sigma \quad \forall\, u\in\mathcal{V}.$$

Where $\tau=Cst\ge 0$ and $\sigma=\|\gamma\|_{L^{\theta'}(Q)}\ge 0$, that is $\mathtt{C}$ belong to the class Γ_σ^τ. □

Now, let us consider the following operator $\mathtt{L}$ defined from the subset $D(\mathtt{L})$ of $\mathcal{V}$ into its dual $\mathcal{V}^*$, such that

$$D(\mathtt{L})=\left\{\varphi\in\mathcal{V}:\ \varphi'\in\mathcal{V}^*,\ \varphi(0)=0\right\},\ \text{that is } D(\mathtt{L})\subset D(\mathtt{C}),$$

by

$$\langle \mathtt{L}u,\varphi\rangle=-\int_Q u\varphi_t\mathrm{d}x\mathrm{d}t,\quad \text{for all } u\in D(\mathtt{L}),\ \varphi\in\mathcal{V}.$$

Consequently, the operator $\mathtt{L}$ is generated by $\partial/\partial t$ by means of the relation

$$\langle \mathtt{L}u,\varphi\rangle=\int_0^T\langle u'(t),\varphi(t)\rangle\mathrm{d}t,\ \text{ for all } u\in D(\mathtt{L}),\ \varphi\in\mathcal{V}.$$

Lemma 5 ([26, Theorem 32.L]). $\mathtt{L}$ *is linear, densely defined and maximal monotone map.*

Our main result is the following existence theorem:

Theorem 3. *Let $f \in \mathcal{V}^*$, $u_0 \in L^2(\Omega)$ and assume that the hypothesis $(h_1)-(h_5)$ and (M_0) hold. Then, there exists at least one weak solution $u \in D(\mathtt{L})$ of problem $(\mathcal{P})$ in the following sense*

$$-\int_Q u\varphi_t \mathrm{d}x\mathrm{d}t + \int_0^T \left\{ \mathcal{M}(\mathcal{E}(u)) \left[\int_\Omega b(x,t,\nabla u)\nabla\varphi |\, dx + \int_\Omega |\nabla u|^{\theta-2}\nabla u \nabla\varphi \mathrm{d}x \right] \right\} \mathrm{d}t$$
$$+ \int_Q g(x,t,u,\nabla u)\varphi \, dx\mathrm{d}t = \int_0^T \langle f, \varphi \rangle \, dt,$$

for all $\varphi \in \mathcal{V}$.

Proof. On the one hand, from the Lemma 5, the operator

$$\mathtt{L}\colon D(\mathtt{L}) \subset \mathcal{V} \longrightarrow \mathcal{V},$$
$$\langle \mathtt{L}u, \varphi \rangle_{\mathcal{V}} = \int_0^T \langle u'(t), \varphi(t) \rangle \, dt, \ \text{ for all } u \in D\mathtt{L}),\ \varphi \in \mathcal{V}.$$

is a densely defined maximal monotone operator.

By the monotonicity of $\mathtt{L}$ we have

$$\langle \mathtt{L}u, u \rangle \geq 0 \ \text{ for all } u \in D(\mathtt{L}),$$

then we obtain

$$\begin{aligned} \langle \mathtt{L}u + \mathtt{S}u + \mathtt{C}u, u \rangle &\geq \langle \mathtt{S}u + \mathtt{C}u, u \rangle \qquad (3.3) \\ &= \int_0^T \left\{ \mathcal{M}\left(\int_\Omega A(x,t,\nabla u) + \frac{1}{p}|\nabla u|^p \mathrm{d}x \right) \int_\Omega a(x,t,\nabla u)\nabla u \mathrm{d}x \right. \\ &\qquad \left. + \int_\Omega |\nabla u|^p \mathrm{d}x \right\} \mathrm{d}t \\ &\qquad + \int_Q g(x,t,u,\nabla u) u \mathrm{d}x\mathrm{d}t, \end{aligned}$$

by using th assumptions (h_2), (M_0) and (h_4), we get

$$\begin{aligned} \langle \mathtt{L}u + \mathtt{S}u + \mathtt{C}u, u \rangle &\geq \int_0^T m_0 \int_\Omega b(x,t,\nabla u)\nabla u \mathrm{d}x\mathrm{d}t + \int_0^T m_0 \int_\Omega |\nabla u|^p \mathrm{d}x\mathrm{d}t \\ &\quad + \int_Q g(x,t,u,\nabla u) u \mathrm{d}x\mathrm{d}t \\ &\geq m_0 \int_Q b(x,t,\nabla u)\nabla u \mathrm{d}x\mathrm{d}t + m_0 \int_Q |\nabla u|^\theta \mathrm{d}x\mathrm{d}t + \int_Q |u|^\theta \mathrm{d}x\mathrm{d}t \\ &\geq m_0 \alpha \int_Q |\nabla u|^\theta \mathrm{d}x\mathrm{d}t + m_0 \int_Q |\nabla u|^\theta \mathrm{d}x\mathrm{d}t + \int_Q |u|^\theta \mathrm{d}x\mathrm{d}t \\ &\geq C_{min} \int_Q |\nabla u|^\theta \mathrm{d}x\mathrm{d}t + C_{min} \int_Q |\nabla u|^\theta \mathrm{d}x\mathrm{d}t + \int_Q |u|^\theta \mathrm{d}x\mathrm{d}t \\ &\geq C_{min} \int_0^T \left(\int_\Omega \left(|u|^\theta + |\nabla u|^\theta \right) dx \right) \mathrm{d}t \end{aligned}$$

$$\geq C_{min}\|u\|_{\mathcal{V}}^{\theta},$$

for all $u \in \mathcal{V}$.

Since the right side of the above inequality (3.3) tends to ∞ as $\|u\|_{\mathcal{V}} \to \infty$, then for each $f \in \mathcal{V}^*$ there exists $R = R(f)$ such that

$$\langle \mathtt{L}u + \mathtt{S}u + \mathtt{C}u - f, u\rangle > 0,$$

for all $u \in \mathtt{B}_R(0) \cap D(\mathtt{L})$.

By appliying Lemma 2, we infer that the equation

$$\mathtt{L}u + \mathtt{S}u + \mathtt{C}u = f,$$

is solvable in $D(\mathtt{L})$. That is the problem $(\mathcal{P})$ admits at least one-weak solution.

Which implies that the problem $(\mathcal{P})$ admits at least one weak solution. This ends the proof. □

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NO COMPRESSION OF SWITCHING MOMENTS AND STABILITY FOR FRACTIONAL DIFFERENTIAL EQUATIONS WITH VARIABLE STRUCTURE AND IMPULSES

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Abstract. In this paper, we investigate the switching moments have no compression point and stability of the zero solution for fractional differential equations with variable structure and impulses. In order to study properties of the solution, we seek some sufficient conditions, under which the switching moments do not have a compression point. Then several sufficient conditions for stability, uniform stability, uniform asymptotically stability of the zero solution for the systems investigated by Lyapunov functions. Finally, two examples are given to illustrate our results.

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Keywords: fractional differential equations, variable structure, impulses, compression point, stability, Lyapunov functions

1. INTRODUCTION

Fractional calculus extend the classical integer order calculus to arbitrary order case, playing a crucial part in several research areas, such as engineering, physics, finance and signal analysis. There are many important monographs on fractional differential equations and their applications, we recommend the reader to read [11–13, 17, 22, 26, 27, 30] and reference therein.

Fractional differential equations with fixed moments of impulses have been regarded as an interesting and challenging subfield of impulsive differential equations, because fractional order derivatives display a global character and highlight the memory accumulated by the long-term effects throughout the entire evolution process, including some fixed impulsive moments. This is much different from the same problem arises from the classical integer order differential equations with fixed moments of impulses, which has local character and can be solved by dividing the whole time span into several subintervals related to fixed moments of impulses, using a step-by-step methodology.

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The main trends to offer fractional differential equations with fixed moments of impulses by defining the lower limit for the fractional order derivatives. Consider the following initial problem for fractional order differential equations with fixed moments of impulses on the initial condition and impulsive moments:

$$\begin{cases} {}^{c}D^{\alpha}_{t_i,t}x(t) = g(t),\ t \neq t_i,\ \alpha \in (0,1), \\ x(t_i+0) = x(t_i) + I_i(x(t_i)),\ \ i = 1,2,\cdots, \\ x(t_0) = x_0, \end{cases} \tag{1.1}$$

where ${}^{c}D^{\alpha}_{t_i,t}$ denotes *the classical Caputo fractional derivative of the order* α *(see Definition 3) via changing the lower limit associated with fixed moments of impulses* t_i, $0 \leq t_0 < t_1 < t_2 < \cdots$ and $t_i \to \infty$ as $i \to \infty$, $g : R^+ \to R^n$ is a given function.

For this case, Agarwal et al. [3,5] introduced the concepts of solution and derived the new representation of piecewise continuous solution of (1.1) as follows:

$$\begin{aligned} x(t;t_0,x_0) = x_0 &+ \sum_{t_0<t_i<t} I_i(x(t_i)) \\ &+ \frac{1}{\Gamma(\alpha)}\left(\sum_{t_0<t_i<t}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}g(s)ds + \int_{t_i}^{t}(t-s)^{\alpha-1}g(s)ds\right) \end{aligned} \tag{1.2}$$

for $t \in (t_i, t_{i+1}]$, $i = 1,2,\cdots$. The main idea to derive the representation of (1.2) is motivated by the framework of seeking the formula of solution to integer order impulsive differential equations. There are many works on the existence, stability of solution and controllability, see[7, 8, 14–16, 28, 29]. One of the main properties in the qualitative theory of differential equations is stability of solution. Stability enables us to compare the behavior of solution starting at different points, and Lyapunov approach is one of the methods for studying stability of fractional order systems [2, 4, 24, 25]. Stability for the zero solution of fractional nonlinear equations is studied in [6, 10], which requires differentiability of the applied Lyapunov functions.

In the real world life there are many processes and phenomena that are characterized by rapid changes in their state, so called impulses. The study of processes with a short-term rapid change is a subject of several sciences, and classical instantaneous impulses are suitable to show some evolution processes in engineering, physics, biology, medical science, etc, see [1, 18, 23]. Moreover, in the study of discontinuous processes using mathematical modeling in the form of differential equations with impulsive effects, as a rule, is required.

The differential equations with variable structure without impulsive were originated in the works by T.Vogel [19], in these systems there is a change in the right hand sides when the trajectory reaches give set defined, primarily applied in the control theory.

In this paper, we discuss the following initial problem for fractional order differential equations with variable structure and impulses:

$$\begin{cases} {}^{c}D^{\alpha}_{t_{i-1},t}x(t) = f_i(t,x),\ \varphi_i(x(t)) \neq 0,\ t_{i-1} < t < t_i,\ \alpha \in (0,1),\ i = 1,2,\cdots, \\ x(t_i+0) = x(t_i) + I_i(x(t_i)),\ \varphi_i(x(t_i)) = 0,\ \ i = 1,2,\cdots, \\ x(t_0) = x_0, \end{cases} \tag{1.3}$$

where ${}^{c}D^{\alpha}_{t_{i-1},t}$ denotes the classical Caputo fractional derivative of the order α via changing the lower limit associated with fixed moments of impulses t_{i-1}. The functions $f_i : R^+ \times D \to R^n$, D is a nonempty domain in R^n, $t_i \in R^+$, $0 \leq t_0 < t_1 < t_2 < \cdots$, $\varphi_i : D \to R$, $I_i : D \to R^n$, $Id + I_i : D \to D$ and $x_0 \in D$. The identity in R^n is denoted by Id. $x(t_i+0) = \lim_{\varepsilon \to 0+} x(t_i+\varepsilon)$ represent the right limit of $x(t)$ at $t = t_i$.

At the moments $t_1, t_2, \cdots$, the right hand side of the system is changing and the impulsive effect takes place, these moments are called switching moments. We set $\Phi_i = \{x \in D; \varphi_i(x) = 0\},\ i = 1,2,\cdots$ and $\Phi = \bigcup_{i=1,2,\cdots} \Phi_i$. If the switching moments have a compression point, then the solution not continuable at the right hand side of this point. In this case, a number of important properties (stability, monotony, periodicity equivalence, etc.) of the solution can not be studied. Therefore, we need find the switching moments do not have a compression point in sufficient conditions, lay the foundations for studying the properties of solution to the equations.

Since the concepts of solution and existence results have been reported in [21] on finite time interval, one can adopt the same methods [20] to obtain the existence of solution of (1.3) on infinite time interval $[t_0, \infty)$ by using contracting mapping principle. Thus, we follows the concepts in [9] and focus on the unlimited moments of switching of (1.3) in the present paper. In order to discuss the stability, uniform stability, uniform asymptotically stability of the zero solution for the system investigated by Lyapunov functions.

The representation of the solution to the problem (1.3) is denoted by $x(\cdot; x_0) \in PC([t_0, \infty), R^n)$ as follows:

$$\begin{aligned} x(t;x_0) &= x(t;x_0, f_1, \varphi_1, I_1, f_2, \varphi_2, I_2, \cdots) \\ &= \begin{cases} x(t;x_0,f_1), & t_0 < t < t_1, \\ x(t;x_0,f_1,\varphi_1,I_1,f_2), & t_1 < t < t_2, \\ \vdots & \\ x(t;x_0,f_1,\varphi_1,I_1,f_2,\varphi_2,I_2,\cdots,f_i,\varphi_i,I_i,f_{i+1}), & t_i < t < t_{i+1}, \\ \vdots & \end{cases} \end{aligned}$$

The differential equations with variable structures and impulses have been explored about integer order, the breadth of research into fractional differential equations with similar attributes is notably less extensive.

Note that the properties for various aspects of the dynamic impulsive mathematical models from the epidemiology and pharmacokinetics are connected with unlimited moments of switching for fractional differential equations with variable structure and

impulses. In [9], Dishlieva and Dishliev initially study the properties in respect of integer order differential equations with variable structure and impulses. To fix this gap, we discuss the characteristics in respect of fractional differential equations with variable structure and impulses, which is an extension for integer order system.

The rest of this paper is organized as follows. In Section 2, we collect some concepts of fractional calculus and Lyapunov functions, give the definitions of stability. In Section 3, we present main results for our original fractional problem with variable structure and impulses. Some examples are given in final section to demonstrate the application of our main results.

2. PRELIMINARIES

In this section, we introduce the notations, definitions and preliminary facts which are used throughout this paper.

We define $PC(R^+,R^n) := \{x : R^+ \to R^n : x \in C((t_k,t_{k+1}],R^n),\ k = 0,1,\cdots$ and there exist $x(t_k^-)$ and $x(t_k^+), k = 1,2,\cdots$, with $x(t_k) = x(t_k^-)\}$ with the norm $\|x\|_{PC} := \sup\{\|x(t)\| : t \in R^+\}$.

The Euclidean distance between non-empty sets $X,Y \subset R^n$ is

$$\rho(X,Y) = \inf\{\|x-y\|; x \in X, y \in Y\}.$$

Define $\Delta = \bigcap_{i=1,2,\cdots}\{x \in D : V_i(t,x) \le \omega(\sigma)\}$, for $t \ge t_0$, where the $\sigma > 0$ is a constant such that $B_\sigma(0) = \{x \in R^n, \|x\| < \sigma\}$.

Let us recall the following known definitions of fractional calculus. For more details see [4, 12].

Definition 1. The fractional integral of order $\alpha \in (0,1)$ with the lower limit t_0 for an integrable function $f : [t_0,\infty) \to R^n$ is defined as

$$I^\alpha_{t_0,t} f(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^t \frac{f(s)}{(t-s)^{1-\alpha}} ds,\ t > t_0,$$

where $\Gamma(\cdot)$ is the Gamma function.

Definition 2. The Riemann-Liouville derivative of order $\alpha \in (0,1)$ with the lower limit t_0 for an integrable function $f : [t_0,\infty) \to R^n$ can be written as

$${}^L D^\alpha_{t_0,t} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_{t_0}^t \frac{f(s)}{(t-s)^\alpha} ds,\ t > t_0.$$

Definition 3. The classical Caputo derivative of order $\alpha \in (0,1)$ with the lower limit t_0 for a differentiable function $f : [t_0,\infty) \to R^n$ be defined by

$${}^C D^\alpha_{t_0,t} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^t (t-s)^{-\alpha} f'(s) ds,\ t > t_0.$$

Definition 4. The Grünwald-Letnikov fractional derivative of order $\alpha \in (0,1)$ with the lower limit t_0 for a function $f : [t_0, \infty) \to R^n$ is given by

$$ {}^{GL}D^{\alpha}_{t_0,t} f(t) = \lim_{h \to 0} \frac{1}{h^{\alpha}} \sum_{r=0}^{[\frac{t-t_0}{h}]} (-1)^r \binom{\alpha}{r} f(t - rh),\ t > t_0, $$

and the Grünwald-Letnikov fractional Dini derivative by

$$ {}^{GL}_{t_0,t}D^{\alpha}_{+} f(t) = \limsup_{h \to 0^+} \frac{1}{h^{\alpha}} \sum_{r=0}^{[\frac{t-t_0}{h}]} (-1)^r \binom{\alpha}{r} f(t - rh),\ t > t_0, $$

where $\binom{\alpha}{r} = \frac{\alpha(\alpha-1)\cdots(\alpha-r+1)}{r!}$ and $[\frac{t-t_0}{h}]$ denotes the integer part of the fraction $\frac{t-t_0}{h}$.

Definition 5. The generalized Caputo derivative of order $\alpha \in (0,1)$ with the lower limit t_0 for an integrable function $f : [t_0, \infty) \to R^n$ be defined by

$$ {}^{c}D^{\alpha}_{t_0,t} f(t) = {}^{L}D^{\alpha}_{t_0,t} \big[f(t) - f(t_0) \big],\ t > t_0. $$

If the function f is sufficiently smooth, then ${}^{L}D^{\alpha}_{t_0,t} f(t) = {}^{GL}D^{\alpha}_{t_0,t} f(t)$ for $t > t_0$. Also, according to Definition 5, we have

$$ {}^{c}D^{\alpha}_{t_0,t} f(t) = {}^{L}D^{\alpha}_{t_0,t} \big[f(t) - f(t_0) \big] = {}^{GL}D^{\alpha}_{t_0,t} \big[f(t) - f(t_0) \big],\ t > t_0. $$

From the relation between Caputo fractional derivative and Grünwald-Letnikov fractional derivative, using the Grünwald-Letnikov fractional Dini derivative, we define the Caputo fractional Dini derivative of a function as

$$ \begin{aligned} {}^{c}_{t_0,t}D^{\alpha}_{+} f(t) &= {}^{GL}_{t_0,t}D^{\alpha}_{+} \big[f(t) - f(t_0) \big] \\ &= \limsup_{h \to 0^+} \frac{1}{h^{\alpha}} [f(t) - f(t_0) - \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1} \binom{\alpha}{r} (f(t - rh) - f(t_0))]. \end{aligned} \tag{2.1} $$

Remark 1. If $f \in C^{\alpha}([t_0, T], R^n)$, then ${}^{c}_{t_0,t}D^{\alpha}_{+} f(t) = {}^{c}D^{\alpha}_{t_0,t} f(t)$.

Definition 6. The zero solution of the problem (1.3) is said to be

- stable if $\forall t_0 \in R^+, \forall \varepsilon > 0, \exists \delta = \delta(t_0, \varepsilon) > 0$, such that for $\forall x_0 \in D$, $\|x_0\| < \delta$, then $\|x(t; x_0)\| < \varepsilon$ for $t \geq t_0$.
- uniformly stable if $\forall \varepsilon > 0, \exists \delta = \delta(\varepsilon) > 0$, such that for $t_0 \in R^+, x_0 \in D$ with $\|x_0\| < \delta$, then $\|x(t; x_0)\| < \varepsilon$ for $t \geq t_0$.
- uniformly attractive if for $\lambda > 0$, $\forall \varepsilon > 0$, $\exists T = T(\varepsilon) > 0$ such that for $t_0 \in R^+, x_0 \in D$ with $\|x_0\| < \lambda$, then $\|x(t; x_0)\| < \varepsilon$ for $t \geq t_0 + T$.
- uniformly asymptotically stable if the zero solution is uniformly stable and uniformly attractive.

Definition 7. The piecewise continuous Lyapunov functions is given:

$$ \{ V_i; V_i : R^+ \times D \to R^+,\ i = 1, 2, \cdots \}, $$

corresponding to the problem (1.3), if $V_i \in C[R^+ \times D\backslash\Phi_i, R^+]$, $i = 1,2,\cdots$, and for every point $(t, x_{\Phi_i}) \in R^+ \times \Phi_i$, $i = 1,2,\cdots$, there exist limits:

$$\lim_{x \to x_{\Phi_i}, \Phi_i(x)<0} V_i(t,x) = V_i(t, x_{\Phi_i} - 0) = V_i(t, x_{\Phi_i}),\ for\ (t,x) \in R^+ \times D,$$

and

$$\lim_{x \to x_{\Phi_i}, \Phi_i(x)>0} V_i(t,x) = V_i(t, x_{\Phi_i} + 0),\ for\ (t,x) \in R^+ \times D,$$

implies

$$V_i(t, x_{\Phi_i} - 0) = V_i(t, x_{\Phi_i}), for (t, x_{\Phi_i}) \in R^+ \times \Phi_i,\ i = 1,2,\cdots.$$

Definition 8. Let $\{V_i;\ i = 1,2,\cdots\}$ is piecewise continuous Lyapunov functions, then for $\forall (t,x) \in R^+ \times D\backslash\Phi_i$, $i = 1,2,\cdots$, we define the generalized Caputo fractional Dini derivative of Lyapunov functions V_i at the point (t,x) of the problem (1.3):

$$\begin{aligned} {}^{C}_{(1.3)}D^{\alpha}_{+}V_i(t,x) = \limsup_{h \to 0^+} \frac{1}{h^{\alpha}} \Bigg[& V_i(t,x) - V_i(t_0,x_0) \\ & - \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1} \binom{\alpha}{r} (V_i(t - rh, x - h^{\alpha} f(t,x)) - V_i(t_0,x_0)) \Bigg], \end{aligned}$$

where $x, x_0 \in D$, for $\forall t \in R^+, \exists\, h_1 > 0$, such that $t - h \in R^+, x - h^{\alpha} f \in D$, for $0 < h \leq h_1$.

3. Main results

3.1. *No Compression of Switching Moments*

The main aim of the section is to find some sufficient conditions which ensure that the switching moments for the problem (1.3) have no compression point.

In the following, we list the assumptions:

- H$_1$ The functions $f_i : R^+ \times D \to R^n, I_i : D \to R^n, i = 1,2,\cdots$ are continuous.
- H$_2$ There exist positive constants L_i such that $\|f_i(t,x)\| \leq L_i$, for any $(t,x) \in R^+ \times D$, $i = 1,2,\cdots$.
- H$_3$ For any point $(t_0,x_0) \in R^+ \times D$ and for each $i = 1,2,\cdots$, then the problem (1.3) without impulses has a unique solution in interval $[t_0, \infty)$. That is, the following fractional Cauchy problem

$$\begin{cases} {}^{c}D^{\alpha}_{t_{i-1},t} x(t) = f_i(t,x),\ \alpha \in (0,1), \\ x(t_0) = x_0, \end{cases}$$

  has a unique solution $x_i(t; x_0)$.
- H$_4$ There exist positive constants K_i, $i = 1,2,\cdots$, such that

$$|\varphi_i(x^*) - \varphi_i(x)| \leq K_i \|x^* - x\|,\ i = 1,2,\cdots$$

  for each $x^*, x \in D$.

H_5 There exist positive constants C_i such that $|\varphi_{i+1}((Id+I_i)(x))| = |\varphi_{i+1}(x+I_i(x))| \geq C_i$, for any $x \in \Phi_i$, $i = 1,2,\cdots$.

H_6 There exist positive constants K_{φ_i}, $i = 1,2,\cdots$, such that

$$|\varphi_i(x)| \leq K_{\varphi_i}\rho(x,\Phi_i),\ i = 1,2,\cdots$$

for each $x \in D$.

H_7 There exist positive constants C_{N_i} and domains N_i, $\Phi_i \subset N_i \subset D$, such that

$$|\varphi_i(x)| \geq C_{N_i},\ i = 1,2,\cdots$$

for $x \in \partial N_i \cap D$.

H_8 For $x \in \Phi_i$, it is satisfied $x + I_i(x) \in D\backslash N_{i+1}$, $i = 1,2,\cdots$.

Remark 2. The proofs for the existence and uniqueness of the solution to problem (1.3) are trivial. In fact, by repeating the similar procedure in [20, Theorem 4.1], we can verify that the problem (1.3) has a unique solution $x \in PC([t_0,\infty),R^n)$ under the standard assumed conditions.

The solution of the problem (1.3) is denoted by $x(t;x_0)$. For $t_{i-1} < t < t_i$, the solution of the problem (1.3) coincides with the solution $x_i(t;x_{i-1}^+)$ of the problem

$$\begin{cases} {}^cD^{\alpha}_{t_{i-1},t}x(t) = f_i(t,x),\ \alpha \in (0,1), \\ x(t_{i-1}) = x_{i-1}^+, \end{cases}$$

where, we set $\varphi_i(x(t;x_0)) = \varphi_i(x_i(t;x_{i-1}^+)) > 0$, for $t = t_i$, we have $\varphi_i(x(t_i;x_0)) = \varphi_i(x_i(t_i;x_{i-1}^+)) = 0$, denote $x_i = x(t_i;x_0)$, then we define the equality $x(t_i+0) = x(t_i) + I_i(x(t_i;x_0)) = x(t_i) + I_i(x_i) = x_i^+$. Especially, $0 = t_0 \leq t < t_1$, we set $x_0 = x_0^+$.

In the arbitrary interval of continuity (t_{i-1},t_i), the solution of the problem (1.3) starts at the point $x_{i-1}^+ \in D$. The switching moment t_i is the first type of discontinuous point of the function $x(t;x_0)$, and the right hand side of system coincides with function $f_{i+1}(t,x)$ in the next interval of continuity of the solution. See Fig.1, showing the trajectory of the problem considered.

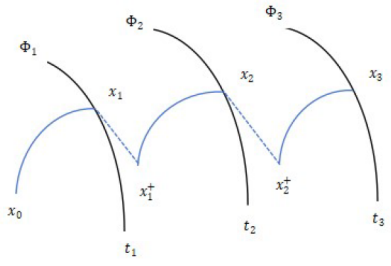

Fig. 1

Theorem 1. *Assume that the conditions* $[H_1]-[H_5]$ *hold. If the series*

$$\sum_{i=1}^{\infty}\frac{\Gamma(\alpha+1)C_i}{L_{i+1}K_{i+1}}$$

is divergent, then the switching moments for problem (1.3) *have no compression point.*

Proof. We will evaluate the difference between t_i and t_{i+1} for each $i=1,2,\cdots$.

For the interval $t\in[t_0,t_1)$, we obtain

$$\begin{aligned}\|x(t_1;x_0)-x(t_0;x_0)\| &= \|x_1(t_1;x_0)-x_0\| \le \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t_1}(t_1-s)^{\alpha-1}\|f_1(s,x_1(s;x_0))\|ds\\ &\le \frac{L_1}{\Gamma(\alpha+1)}(t_1-t_0)^{\alpha}.\end{aligned}$$

Note that

$$\begin{aligned}(t_1-t_0)^{\alpha} &\ge \frac{\Gamma(\alpha+1)}{L_1}\|x(t_1;x_0)-x(t_0;x_0)\| \\ &\ge \frac{\Gamma(\alpha+1)}{L_1K_1}|\varphi_1(x(t_1;x_0))-\varphi_1(x(t_0;x_0))|.\end{aligned} \tag{3.1}$$

In the interval $t\in[t_0,t_1)$, $\varphi_1(x(t_0;x_0))>0$, for $t=t_1$, $\varphi_1(x(t_1;x_0))=0$. We get $(t_1-t_0)^{\alpha}\ge 0$.

Next, we consider the interval (t_1,t_2), we have

$$\begin{aligned}\|x(t_2;x_0)-x(t_1+0;x_0)\| &= \left\|x_2(t_2;x_1^+)-x_2(t_1;x_1^+)\right\| \\ &\le \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\|f_2(s,x_2(s;x_1^+))\|ds\\ &\le \frac{L_2}{\Gamma(\alpha+1)}(t_2-t_1)^{\alpha}.\end{aligned}$$

Note that the conditions $[H_4]$ and $[H_5]$, we get

$$\begin{aligned}(t_2-t_1)^{\alpha} &\ge \frac{\Gamma(\alpha+1)}{L_2}\|x(t_2;x_0)-x(t_1+0;x_0)\| \\ &\ge \frac{\Gamma(\alpha+1)}{L_2K_2}|\varphi_2(x(t_2;x_0))-\varphi_2(x(t_1+0;x_0))| \ge \frac{\Gamma(\alpha+1)}{L_2K_2}|\varphi_2(x(t_1+0;x_0))| \\ &= \frac{\Gamma(\alpha+1)}{L_2K_2}|\varphi_2(x_1+I_1(x_1))| \ge \frac{C_1}{L_2K_2}\Gamma(\alpha+1).\end{aligned} \tag{3.2}$$

According to the discussion, for (t_i,t_{i+1}), we have

$$\begin{aligned}\|x(t_{i+1};x_0)-x(t_i+0;x_0)\| &= \|x_{i+1}(t_{i+1};x_i^+)-x_{i+1}(t_i;x_i^+)\| \\ &\le \frac{1}{\Gamma(\alpha)}\int_{t_i}^{t_{i+1}}(t_{i+1}-s)^{\alpha-1}\|f_{i+1}(s,x_{i+1}(s;x_i^+))\|ds\end{aligned}$$

$$\leq \frac{L_{i+1}}{\Gamma(\alpha+1)}(t_{i+1}-t_i)^{\alpha}.$$

Then

$$\begin{aligned}(t_{i+1}-t_i)^{\alpha} &\geq \frac{\Gamma(\alpha+1)}{L_{i+1}}\|x(t_{i+1};x_0)-x(t_i+0;x_0)\| \\ &\geq \frac{\Gamma(\alpha+1)}{L_{i+1}K_{i+1}}|\varphi_{i+1}(x(t_{i+1};x_0))-\varphi_{i+1}(x(t_i+0;x_0))| \\ &\geq \frac{\Gamma(\alpha+1)}{L_{i+1}K_{i+1}}|\varphi_{i+1}(x(t_i+0;x_0))| \\ &= \frac{\Gamma(\alpha+1)}{L_{i+1}K_{i+1}}|\varphi_{i+1}(x_i+I_i(x_i))| \geq \frac{C_i}{L_{i+1}K_{i+1}}\Gamma(\alpha+1).\end{aligned} \tag{3.3}$$

It is fulfilled

$$\begin{aligned}\lim_{i\to\infty} t_i &= \lim_{i\to\infty}\Big((t_i-t_{i-1})+(t_{i-1}-t_{i-2})+\cdots+(t_2-t_1)+(t_1-t_0)\Big)+t_0 \\ &\geq \lim_{i\to\infty}\Big((t_i-t_{i-1})^{\alpha}+(t_{i-1}-t_{i-2})^{\alpha}+\cdots+(t_2-t_1)^{\alpha}+(t_1-t_0)^{\alpha}\Big) \\ &\geq \lim_{i\to\infty}\Big(\frac{C_{i-1}}{L_iK_i}\Gamma(\alpha+1)+\frac{C_{i-2}}{L_{i-1}K_{i-1}}\Gamma(\alpha+1)+\cdots+\frac{C_1}{L_2K_2}\Gamma(\alpha+1)\Big) \\ &= \sum_{i=1}^{\infty}\frac{\Gamma(\alpha+1)C_i}{L_{i+1}K_{i+1}}=\infty.\end{aligned} \tag{3.4}$$

Therefore, the switching moments do not have a compression point. The proof is completed. □

Theorem 2. *Assume that the conditions* $[H_1]-[H_3]$, $[H_5]-[H_6]$ *hold. If the series*

$$\sum_{i=1}^{\infty}\frac{\Gamma(\alpha+1)C_{i-1}}{L_iK_{\varphi_i}}$$

is divergent, then the switching moments for problem (1.3) *have no compression point.*

Proof. We will consider the difference $t_{i+1}-t_i$ for each interval.

For $t\in[t_0,t_1)$, from (3.1), we get

$$\begin{aligned}(t_1-t_0)^{\alpha} &\geq \frac{\Gamma(\alpha+1)}{L_1}\|x(t_1;x_0)-x(t_0;x_0)\| \\ &\geq \frac{\Gamma(\alpha+1)}{L_1}\rho(x(t_0;x_0),\Phi_1) \geq \frac{\Gamma(\alpha+1)}{L_1K_{\varphi_1}}|\varphi_1(x(t_0;x_0))|.\end{aligned}$$

For $t\in[t_0,t_1)$, $\varphi_1(x(t_0;x_0))>0$, we get $(t_1-t_0)^{\alpha}\geq 0$.

From (3.2), we get

$$(t_2-t_1)^{\alpha}\geq\frac{\Gamma(\alpha+1)}{L_2}\|x(t_2;x_0)-x(t_1+0;x_0)\|\geq\frac{\Gamma(\alpha+1)}{L_2}\rho(x(t_1+0;x_0),\Phi_2)$$

$$\geq \frac{\Gamma(\alpha+1)}{L_2 K_{\varphi_2}}|\varphi_2(x(t_1+0;x_0))| = \frac{\Gamma(\alpha+1)}{L_2 K_{\varphi_2}}|\varphi_2(x_1+I_1(x_1))|$$
$$\geq \frac{C_1}{L_2 K_{\varphi_2}}\Gamma(\alpha+1).$$

From (3.3), we get

$$(t_{i+1}-t_i)^\alpha \geq \frac{\Gamma(\alpha+1)}{L_{i+1}}\|x(t_{i+1};x_0)-x(t_i+0;x_0)\| \geq \frac{\Gamma(\alpha+1)}{L_{i+1}}\rho(x(t_i+0;x_0),\Phi_{i+1})$$
$$\geq \frac{\Gamma(\alpha+1)}{L_{i+1}K_{\varphi_{i+1}}}|\varphi_{i+1}(x(t_i+0;x_0))| = \frac{\Gamma(\alpha+1)}{L_{i+1}K_{\varphi_{i+1}}}|\varphi_{i+1}(x_i+I_i(x_i))|$$
$$\geq \frac{C_i}{L_{i+1}K_{\varphi_{i+1}}}\Gamma(\alpha+1).$$

Analogously to (3.4), we obtain

$$\lim_{i\to\infty} t_i \geq \sum_{i=1}^{\infty}\frac{\Gamma(\alpha+1)C_{i-1}}{L_i K_{\varphi_i}} = \infty.$$

The theorem is proved. □

Theorem 3. *Assume that the conditions* $[H_1]-[H_3]$, $[H_6]-[H_8]$ *hold. If the series*

$$\sum_{i=1}^{\infty}\frac{\Gamma(\alpha+1)C_{N_i}}{L_i K_{\varphi_i}}$$

is divergent, then the switching moments for problem (1.3) *have no compression point.*

Proof. We consider the point $x_i^+ = x(t_i+0;x_0) = x(t_i)+I_i(x(t_i;x_0)) = x(t_i)+I_i(x_i)$. Since $x_i \in \Phi_i$ and under condition $[H_8]$, we have $x_i^+ \in D\backslash N_{i+1}$. We go into the solution $x_{i+1}(t;x_i^+)$ of initial value problem

$$\begin{cases} {}^cD^{\alpha}_{t_i,t}x(t) = f_{i+1}(t,x),\ \alpha\in(0,1), \\ x(t_i) = x_i^+. \end{cases}$$

Since

$$x_{i+1}(t;x_i^+) = x_i^+ + \frac{1}{\Gamma(\alpha)}\int_{t_i}^{t}(t-s)^{\alpha-1}f_{i+1}(s,x_{i+1}(s;x_i^+))ds,$$

then $x_{i+1}(t_i;x_i^+) = x_i^+ \in D\backslash N_{i+1}$, $x_{i+1}(t_{i+1};x_i^+) = x(t_{i+1};x_0) = x_{i+1} \in \Phi_{i+1} \subset N_{i+1}$.

According to the continuity of the curve $x_{i+1}(t;x_i^+)$, then $\exists\ t_{\partial N_{i+1}}$, for $t_i < t_{\partial N_{i+1}} < t_{i+1}$, such that

$$x_{i+1}(t_{\partial N_{i+1}};x_i^+) = x(t_{\partial N_{i+1}};x_0) \in \partial N_{i+1}\cap D.$$

By the condition $[H_7]$, we obtain

$$|\varphi_{i+1}(x(t_{\partial N_{i+1}};x_0))| \geq C_{N_{i+1}}.$$

On the other hand, we have

$$\|x(t_{i+1};x_0)-x(t_{\partial N_{i+1}};x_0)\| \le \frac{L_{i+1}}{\Gamma(\alpha+1)}(t_{i+1}-t_{\partial N_{i+1}})^{\alpha},$$

then

$$\begin{aligned}(t_{i+1}-t_{\partial N_{i+1}})^{\alpha} &\ge \frac{\Gamma(\alpha+1)}{L_{i+1}}\|x(t_{i+1};x_0)-x(t_{\partial N_{i+1}};x_0)\| \\ &\ge \frac{\Gamma(\alpha+1)}{L_{i+1}}\rho(x(t_{\partial N_{i+1}};x_0),\Phi_{i+1}) \ge \frac{\Gamma(\alpha+1)}{L_{i+1}K_{\varphi_{i+1}}}|\varphi_{i+1}(x(t_{\partial N_{i+1}};x_0))| \\ &\ge \Gamma(\alpha+1)\frac{C_{N_{i+1}}}{L_{i+1}K_{\varphi_{i+1}}}.\end{aligned}$$

From the estimate above, we find

$$(t_{i+1}-t_i)^{\alpha} > (t_{i+1}-t_{\partial N_{i+1}})^{\alpha} \ge \Gamma(\alpha+1)\frac{C_{N_{i+1}}}{L_{i+1}K_{\varphi_{i+1}}},\ \ i=1,2,\cdots$$

From (3.4), we obtain

$$\lim_{i\to\infty} t_i \ge \sum_{i=1}^{\infty}\frac{\Gamma(\alpha+1)C_{N_i}}{L_iK_{\varphi_i}} = \infty.$$

The theorem is proved. □

3.2. *Stability*

In the section, we will find some sufficient conditions for stability of the zero solution for the problem (1.3).

We will use the following conditions:

$\Lambda = \{\omega \in C(R^+,R^+),\ \omega \text{ is strictly increasing and } \omega(0)=0\}$.

H$_9$ $0\in D$ and $f_i(t,0)=0, t\in R^+, i=1,2,\cdots,\ \varphi_i(0)=\varphi_i(0,0,\cdots,0)\ne 0, i=1,2,\cdots$.

H$_{10}$ There exists a sequence of functions $V_i\in C(R^+\times D,R^+)$ such that $V_i(t,0)=0,\ i=1,2,\cdots$.

H$_{11}$ $\omega(\|x\|)\le V_i(t,x)$, for $(t,x)\in R^+\times D,\ i=1,2,\cdots$, where $\omega\in\Lambda$;

H$_{11'}$ $\omega(\|x\|)\le V_i(t,x)\le\mu(\|x\|)$, for $(t,x)\in R^+\times D,\ i=1,2,\cdots$, where $\omega,\mu\in\Lambda$;

H$_{12}$ $V_{i+1}(t+0,x+I_i(x))=V_{i+1}(t,x+I_i(x))\le V_i(t,x)$, for $(t,x)\in R^+\times\Phi_i,\ i=1,2,\cdots$;

H$_{13}$ ${}^{C}_{(1.3)}D^{\alpha}_{+}V_i(t,x)\le 0$, for $(t,x)\in R^+\times D\backslash\Phi_i,\ i=1,2,\cdots$;

H$_{13'}$ ${}^{C}_{(1.3)}D^{\alpha}_{+}V_i(t,x)\le -\kappa(\|x\|)$, for $(t,x)\in R^+\times D\backslash\Phi_i,\ i=1,2,\cdots$, where $\kappa\in\Lambda$.

Remark 3. The conditions $[H_1]$ and $[H_9]$ are fulfilled. Then the problem (1.3) possesses a zero solution, i.e. $x(t;0)=0$, for $0\le t_0\le t<\infty$.

Lemma 1. *(see[10]). Let* ${}^{c}D^{\alpha}_{t_0,t}x(t)\le {}^{c}D^{\alpha}_{t_0,t}y(t)$, *where* $\alpha\in(0,1)$, *and* $x(t_0)=y(t_0)$, *then* $x(t)\le y(t)$.

Theorem 4. *Assume that the conditions* $[H_1]-[H_5]$, $[H_9]-[H_{13}]$ *hold. Then the zero solution of the problem* (1.3) *is stable.*

Proof. Obviously, the problem (1.3) has a zero solution for $t \in [t_0,\infty)$.

Since $V_1(t_0,0)=0$, set $\varepsilon>0$, $\exists\delta(t_0,\varepsilon)>0$ such that $V_1(t_0,x)<\omega(\varepsilon)$, for $\|x\|<\delta$. For $x_0 \in B_\delta(0)$, then $V_1(t_0,x_0)<\omega(\varepsilon)$.

We say the problem (1.3) have infinity many switching moments, i.e. $t_0<t_1<t_2<t_3<\cdots$.

For $t_{i-1}<t<t_i$, the solution of the problem (1.3) is continuous and coincides with the solution of the problem without impulses.

$$\begin{cases} {}^cD^{\alpha}_{t_{i-1},t}x(t)=f_i(t,x), \\ x(t_{i-1}+0)=x^+_{i-1}. \end{cases}$$

By conditions $[H_{11}]-[H_{13}]$, we have

$$\omega(\|x(t;x_0)\|)\le V_i(t,x(t;x_0))\le V_i(t_{i-1}+0,x^+_{i-1})\le V_1(t_0,x_0)<\omega(\varepsilon).$$

Since $\omega\in\Lambda$, from the last inequality, we get

$$\|x(t;x_0)\|<\varepsilon, \quad \text{for } t_{i-1}<t<t_i,\ i=1,2,\cdots.$$

Apply (3.4), we have $\lim_{i\to\infty}t_i=\infty$, use the equalities $x(t_i-0;x_0)=x(t_i;x_0)$, we obtain that

$$\|x(t;x_0)\|<\varepsilon, \quad \text{for } t_0\le t<\infty.$$

The theorem is proved. □

Theorem 5. *Assume that the conditions* $[H_1]-[H_5]$, $[H_9]-[H_{10}]$, $[H_{11'}]$, $[H_{12}]-[H_{13}]$ *hold. Then the zero solution of the problem* (1.3) *is uniformly stable.*

Proof. For $\forall t>t_0$, $t\neq t_i$, $i=1,2,\cdots$. Set k be the biggest number and $t>t_k$. Let $\varepsilon>0$, $\exists\delta=\delta(\varepsilon)$ satisfied the inequality $\mu(\delta)<\omega(\varepsilon)$. For $x_0\in B_\delta(0)$, then

$$\omega(\|x(t;x_0)\|)\le V_k(t,x(t;x_0))\le V_1(t_0,x_0)\le\mu(\|x_0\|)\le\mu(\delta)<\omega(\varepsilon).$$

Based on $\omega\in\Lambda$ and Theorem 1, we obtain that

$$\|x(t;x_0)\|\le\varepsilon, \quad for \ \ t_0\le t<\infty.$$

The theorem is proved. □

Theorem 6. *Assume that the conditions* $[H_1]-[H_5]$, $[H_9]-[H_{10}]$, $[H_{11'}]$, $[H_{12}]$, $[H_{13'}]$ *hold. Then the zero solution of the problem* (1.3) *is uniformly asymptotically stable.*

Proof. From Theorem 5, the zero solution of the problem (1.3) is uniformly stable.

For $x_0\in\Delta$, we get $V_k(t,x(t;x_0))\le\omega(\sigma)$. Let a constant $\lambda>0$ such that $\mu(\lambda)<\omega(\sigma)$, then for $x_0\in B_\lambda(0)$, it is fulfilled

$$V_1(t_0,x_0)\le\mu(\|x_0\|)\le\mu(\lambda)<\omega(\sigma).$$

It means that the solution $x(t;x_0)$ of problem (1.3) is defined for $t\in[t_0,\infty)$.

Let $\varepsilon > 0$, choose the constant $\eta = \eta(\varepsilon) > 0$ such that $\mu(\eta) < \omega(\varepsilon)$. Set the constant T satisfies the inequality

$$T > \sqrt[\alpha]{\alpha\Gamma(\alpha)\frac{\mu(\sigma)}{\kappa(\eta)}}.$$

We now prove that $\|x(t;x_0)\| < \varepsilon$, for $t \geq t_0 + T$. Assume $\|x(t;x_0)\| \geq \eta$, for $t \in [t_0, t_0 + T]$.

From the conditions ${}^{C}_{(1.3)}D^{\alpha}_{+}V_i(t,x) \leq -\kappa(\|x\|)$ and $V_{i+1}(t+0, x+I_i(x)) \leq V_i(t,x)$, then for $t \in [t_0, T]$, we obtain

$$V_i(t, x(t;x_0)) \leq V_1(t_0,x_0) - \frac{1}{\Gamma(\alpha)}\left(\sum_{m=0}^{i-1}\int_{t_m}^{t_{m+1}}(t_{m+1}-s)^{\alpha-1}\kappa(\|x(s,x_0)\|)ds + \int_{t_i}^{t} t-s)^{\alpha-1}\kappa(\|x(s,x_0)\|)ds\right).$$

Then from the conditions and inequality $A^{\alpha} + B^{\alpha} \geq (A+B)^{\alpha}$ for $A, B > 0$, we get successively

$$\begin{aligned}
\omega(\|x(t;x_0)\|) &\leq V_k(t, x(t;x_0)) \\
&\leq V_k(t_k + 0, x(t_k+0;x_0)) - \frac{1}{\Gamma(\alpha)}\int_{t_k+0}^{t}(t-s)^{\alpha-1}\kappa(\|x(s,x_0)\|)ds \\
&\leq V_k(t_k, x(t_k-0;x_0) + I_{k-1}(x(t_k-0;x_0))) \\
&\quad - \frac{1}{\Gamma(\alpha)}\int_{t_k+0}^{t}(t-s)^{\alpha-1}\kappa(\|x(s,x_0)\|)ds \\
&\leq V_{k-1}(t_{k-1}+0, x(t_{k-1}+0;x_0)) \\
&\quad - \frac{1}{\Gamma(\alpha)}\int_{t_{k-1}+0}^{t_k}(t_k-s)^{\alpha-1}\kappa(\|x(s,x_0)\|)ds \\
&\quad - \frac{1}{\Gamma(\alpha)}\int_{t_k+0}^{t}(t-s)^{\alpha-1}\kappa(\|x(s,x_0)\|)ds \\
&\vdots \\
&\leq V_1(t_0,x_0) - \frac{1}{\Gamma(\alpha)}\left(\sum_{m=0}^{k-1}\int_{t_m}^{t_{m+1}}(t_{m+1}-s)^{\alpha-1}\kappa(\|x(s,x_0)\|)ds \right. \\
&\quad \left. + \int_{t_k}^{t}(t-s)^{\alpha-1}\kappa(\|x(s,x_0)\|)ds\right) \\
&\leq \mu(\|x_0\|) - \frac{\kappa(\eta)}{\Gamma(\alpha)}\left(\sum_{m=0}^{k-1}\int_{t_m}^{t_{m+1}}(t_{m+1}-s)^{\alpha-1}ds + \int_{t_k}^{t}(t-s)^{\alpha-1}ds\right) \\
&\leq \mu(\sigma) - \frac{\kappa(\eta)}{\alpha\Gamma(\alpha)}\left((t_1-t_0)^{\alpha} + \sum_{m=1}^{k-1}(t_{m+1}-t_1)^{\alpha} + (t-t_k)^{\alpha}\right)
\end{aligned}$$

$$\leq \mu(\sigma) - \frac{\kappa(\eta)}{\alpha\Gamma(\alpha)}\left((t_1 - t_0) + \sum_{m=1}^{k-1}(t_{m+1} - t_1) + (t - t_k)\right)^{\alpha}$$
$$\leq \mu(\sigma) - \frac{\kappa(\eta)}{\alpha\Gamma(\alpha)}(t - t_0)^{\alpha} < 0.$$

The above contradiction proves there exists $t^* \in [t_0, t_0 + T]$ such that $\|x(t^*; x_0)\| < \eta$.

Let $t^* \leq t_0 + T \leq t$, satisfies the inequality $t_{k-p-1} < t^* < t_{k-p} < \cdots < t_{k-1} < t_k < t$, and k is the biggest number. In accordance with $[H_{11'}]$, $[H_{12}]$, $[H_{13'}]$, then

$$\omega(\|x(t; x_0)\|) \leq V_k(t, x(t; x_0)) \leq V_{k-p-1}(t^*, x(t^*; x_0)) \leq \mu(\|x(t^*; x_0)\|) = \mu(\eta) < \omega(\varepsilon).$$

Since $\omega \in \Lambda$, we have $\|x(t; x_0)\| < \varepsilon$, for $t \geq t_0 + T$.

The theorem is proved. □

4. Examples

In this section, two examples are given to illustrate our above theoretical results respectively.

Example 1. Consider the following initial problem for fractional order differential equations

$$\begin{cases} {}^{c}D^{\frac{1}{2}}_{t_{i-1},t} W(t) = -PW(t), \ \varphi_i(W(t)) = W_{\max i} - W(t) > 0, \ t_{i-1} < t < t_i, \ P \geq 0, \\ W(t_i + 0) = W(t_i) + \frac{D_i}{i^2} W(t_i), \varphi_i(W(t_i)) = W_{\max i} - W(t_i) = 0, \ D_i > 0, \ i = 1, 2, \ldots \\ W(t_0) = W_0. \end{cases} \tag{4.1}$$

Set $\alpha = \frac{1}{2}$, define $f_i(W) = f(W) = -PW(t)$, $I_i(W) = \frac{D_i}{i^2} W(t)$, $i = 1, 2 \cdots$. Obviously, the function $f : D \to R^n$, $I_i : D \to R^n$ is continuous for each $W \in D$. We have $|f(W)| \leq P \sup_{W_i, W_j \in D} |W_i - W_j| = L_i = L$. Thus, $[H_1] - [H_2]$ hold.

Consider the system associated with (4.1) without impulse

$$\begin{cases} {}^{c}D^{\frac{1}{2}}_{t_0,t} W(t) = -PW(t), \ P \geq 0, \\ W(t_0) = W_0. \end{cases} \tag{4.2}$$

Obviously, (4.2) has a unique solution $W(t) = E_{\frac{1}{2}}(-P(t - t_0)^{\frac{1}{2}}) W_0$ in interval $[t_0, \infty)$. Thus, $[H_3]$ holds.

$W_{\max i} > 0$, $i = 1, 2 \cdots$, are the upper barrier constants that are specific for each interval of continuity of the solution. It is fulfilled $W_{\max i} \leq \sup_{W_i, W_j \in D} |W_i - W_j|$. The functions $\varphi_i(W) = W_{\max i} - W$, $\varphi_i : D \to R$, $i = 1, 2 \cdots$. Lipschitz constants for condition $[H_4]$ is $K_i = 1$.

Furthermore, we assume that $I_i(W) = \frac{D_i}{i^2} W(t) = \frac{1}{2} W_{\max_{i+1}} - W_{\max i}$, $i = 1, 2 \cdots$. Then $|\varphi_{i+1}((Id + I_i)(x))| = \frac{1}{2} W_{\max_{i+1}} = C_i$, for any $W \in \Phi_i$. The $\Phi_i = \{W_{\max i}\}$, $i =$

$1,2,\cdots$ each one of the switching sets consists of one point. Thereby, it is shown that $[H_5]$ is valid. Then the problem (4.1) have no compression point.

Example 2. Consider the following initial problem for fractional order differential equations

$$\begin{cases} {}^{c}D^{\frac{1}{2}}_{t_{i-1},t}x(t) = \dfrac{-\frac{1}{\sqrt{t\pi}}+\frac{1}{2}\sqrt{t}E_{2,\frac{3}{2}}(-t^2)}{2-\sin t}x, \ \ \varphi_i(x(t)) = x_{\max i} - x(t) > 0, \ \ t_{i-1} < t < t_i, \\ x(t+0) = J_i(t)x(t), \varphi_i(x(t)) = x_{\max i} - x(t) = 0, \ t = t_i, \ i = 1,2,\cdots, \\ x(t_0) = x_0, \end{cases} \tag{4.3}$$

where $f_i(t,x) = f(t,x) = \dfrac{-\frac{1}{\sqrt{t\pi}}+\frac{1}{2}\sqrt{t}E_{2,\frac{3}{2}}(-t^2)}{2-\sin t}x$, $E_{\alpha,\beta}$ is the Mittag-Leffler function defined by $E_{\alpha,\beta}(z) = \frac{z^k}{\Gamma(\alpha k+\beta)}$, $\sup_{t=t_i}|J_i(t)| \le 1, \ i = 1,2,\cdots$.

Let $V_i(t,x) = V(t,x) = (2-\sin t)x^2$. The inequality $\|x\| \le V(t,x) \le 2\|x\|$ is satisfied for any $x \in R$. Consider $\lim_{H\to\infty}\sum_{r=0}^{H}(-1)^r\binom{\alpha}{r} = 0$, where H is a natural number and $\lim_{h\to 0^+}[\frac{t-t_0}{h}] = \infty$, then

$$\lim_{h\to 0^+}\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^r\binom{\alpha}{r} = -1,$$

and

$$\limsup_{h\to 0^+}\frac{1}{h^\alpha}\sum_{r=0}^{[\frac{t-t_0}{h}]}(-1)^r\binom{\alpha}{r} = D_0^\alpha(1) = \frac{1}{(t-t_0)^\alpha\Gamma(1-\alpha)}.$$

Now apply generalized Caputo fractional Dini derivative to the considered Lyapunov function, we obtain

$$\begin{aligned} &{}^{C}_{(4.3)}D^\alpha_+V(t,x;t_0,x_0) \\ &\quad = \limsup_{h\to 0^+}\frac{1}{h^\alpha}\Bigg((2-\sin t)x^2 - (2-\sin t_0)x_0^2 \\ &\qquad - \sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1}\binom{\alpha}{r}\big((2-\sin(t-rh))(x-h^\alpha f(t,x))^2 - (2-\sin t_0)x_0^2\big)\Bigg) \\ &\quad = -\frac{(2-\sin t_0)x_0^2}{(t-t_0)^\alpha\Gamma(1-\alpha)} + \limsup_{h\to 0^+}\frac{1}{h^\alpha}\Bigg\{x^2\sum_{r=0}^{[\frac{t-t_0}{h}]}(-1)^r\binom{\alpha}{r}(2-\sin(t-rh)) \\ &\qquad - 2xh^\alpha f(t,x)\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^r\binom{\alpha}{r}(2-\sin(t-rh)) \\ &\qquad + h^{2\alpha}f^2(t,x)\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^r\binom{\alpha}{r}(2-\sin(t-rh))\Bigg\} \end{aligned}$$

$$
\begin{aligned}
&= -\frac{2x_0^2}{\sqrt{t\pi}} + x^2 \limsup_{h\to 0^+} \frac{1}{h^\alpha} \sum_{r=0}^{[\frac{t-t_0}{h}]} (-1)^r \binom{\alpha}{r} (2-\sin(t-rh)) \\
&\quad - 2xf(t,x) \limsup_{h\to 0^+} \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^r \binom{\alpha}{r} (2-\sin(t-rh)) \\
&\quad + f^2(t,x) \limsup_{h\to 0^+} h^\alpha \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^r \binom{\alpha}{r} (2-\sin(t-rh)) \Big\} \\
&= -\frac{2x_0^2}{\sqrt{t\pi}} + x^2 D_0^\alpha (2-\sin t) + 2x(2-\sin t) f(t,x) = -\frac{2x_0^2}{\sqrt{t\pi}} \le 0.
\end{aligned}
$$

We get $V(t,J_i(t)x) = (2-\sin t)(J_i(t)x)^2 \le (2-\sin t)x^2 = V(t,x)$, for $t = t_i$, $i = 1,2,\cdots$. Therefore the zero solution of the problem (4.3) is uniformly stable.

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ON THE SOLVABILITY IN CLOSED-FORM OF A RATIONAL THREE-DIMENSIONAL SYSTEM OF DIFFERENCE EQUATIONS

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Abstract. In this paper, we are interested in the solvability in closed-form of the following three-dimensional system of nonlinear difference equations

$$x_{n+1} = \frac{a_1 x_{n-k} y_{n-k}}{a x_{n-k} + b y_{n-k} + c z_{n-k}},$$
$$y_{n+1} = \frac{a_2 y_{n-k} z_{n-k}}{a x_{n-k} + b y_{n-k} + c z_{n-k}},$$
$$z_{n+1} = \frac{a_3 x_{n-k} z_{n-k}}{a x_{n-k} + b y_{n-k} + c z_{n-k}},$$

where $n, k \in \mathbb{N}_0$, the parameters a_1, a_2, a_3, a, b, c are real numbers, and the initial values $x_{-k}, \ldots, x_0, y_{-k}, \ldots, y_0, z_{-k}, \ldots, z_0$, are non-zero real numbers. Firstly, we establish some preliminaries results for the general case, then we solve in a closed form, via some change of variables, our system in the two particular cases $k = 0$ and $k = 1$.

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Keywords: systems of difference equations, closed-form of the solutions

1. Introduction and preliminary results

A huge number of real-world phenomena in various domains of sciences are described using different models of non-linear difference equations and systems. These models find applications in biology, where they describe population dynamics, disease spread, and predator-prey interactions; in economics, where they explain market fluctuations, economic growth, and financial forecasting; and in engineering, where they are used for control systems, feedback mechanisms, and signal processing. This practical significance has motivated researchers to work on this subject, as shown by the following contributions [3, 5, 7, 12, 13].

To provide a tool that allows to understand the behavior of the corresponding phenomena, many papers, investigated the solvability of some difference equations and

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systems, for some examples of solvable models, we can consult the following references [1, 4, 6, 8–11].

In [2], Elsayed et al. derived explicit solution expressions for the following three dimensional system of difference equations

$$x_{n+1} = \frac{a_1 y_{n-1} z_{n-1}}{x_{n-1} + y_{n-1} + z_{n-1}},$$
$$y_{n+1} = \frac{a_2 x_{n-1} z_{n-1}}{x_{n-1} + y_{n-1} + z_{n-1}},$$
$$z_{n+1} = \frac{a_3 x_{n-1} y_{n-1}}{x_{n-1} + y_{n-1} + z_{n-1}}.$$

Among others, and as extension of the previous system, Zabat et al. in [14], solved in closed form the more general system

$$x_{n+1} = \frac{a_1 y_{n-1} z_{n-1}}{a x_{n-1} + b y_{n-1} + c z_{n-1}},$$
$$y_{n+1} = \frac{a_2 x_{n-1} z_{n-1}}{a x_{n-1} + b y_{n-1} + c z_{n-1}},$$
$$z_{n+1} = \frac{a_3 x_{n-1} y_{n-1}}{a x_{n-1} + b y_{n-1} + c z_{n-1}}.$$

As a contribution to this area of research, and motivated by our investigation in [14], we consider here the following rational three-dimensional system of difference equations defined by

$$\begin{aligned} x_{n+1} &= \frac{a_1 x_{n-k} y_{n-k}}{a x_{n-k} + b y_{n-k} + c z_{n-k}}, \\ y_{n+1} &= \frac{a_2 y_{n-k} z_{n-k}}{a x_{n-k} + b y_{n-k} + c z_{n-k}}, \\ z_{n+1} &= \frac{a_3 x_{n-k} z_{n-k}}{a x_{n-k} + b y_{n-k} + c z_{n-k}}, \end{aligned} \tag{1.1}$$

where $n, k \in \mathbb{N}_0$, the parameters a_1, a_2, a_3, a, b, c are real numbers, and the initial values $x_{-k}, \ldots, x_0, y_{-k}, \ldots, y_0, z_{-k}, \ldots, z_0$, are non-zero real numbers.

We will show that System (1.1) is transformable to a $6k+6$-periodic system of difference equations, and as consequence System (1.1) is reduced to three independents linear difference equations of order $k+1$. We will investigate the solvability of System (1.1) in details in the two cases $k = 0$ and $k = 1$. The interested readers can use the technique developed here to deal with the cases $k = 2, 3, \ldots$.

Definition 1. By a well defined-solution $(x_n, y_n, z_n)_{n \geq -k}$ of System (1.1), we mean a solution such that

$$a x_{n-k} + b y_{n-k} + c z_{n-k} \neq 0, \qquad n \in \mathbb{N}_0.$$

Lemma 1. *Let $(x_n, y_n, z_n)_{n \geq -k}$ be a well defined solution of System (1.1). Then,*

$$x_n . y_n . z_n \neq 0, \qquad n = -k, -k+1, \ldots.$$

Proof. Assume for example that there exists $n_0 \geq -k$ such that $x_{n_0} = 0$. It is not hard to see that $x_{n_0+k+1} = 0$ and $z_{n_0+k+1} = 0$, and consequently, we get $x_{n_0+2k+2} = 0, y_{n_0+2k+2} = 0$ and $z_{n_0+2k+2} = 0$. From these last equalities, it follows that $x_{n_0+3k+3} = \frac{0}{0}, y_{n_0+3k+3} = \frac{0}{0}$ and $z_{n_0+3k+3} = \frac{0}{0}$. That is the terms $x_{n_0+3k+3}, y_{n_0+3k+3}, z_{n_0+3k+3}$ are undefined. □

From here on, by a solution of (1.1), we mean a well-defined solution.

Lemma 2. *Using the following change of variables*

$$u_n = \frac{x_n}{y_n}, \quad v_n = \frac{y_n}{z_n}, \quad w_n = \frac{z_n}{x_n}, \quad n = -k, -k+1, \ldots, \tag{1.2}$$

System (1.1)*, is transformable to the following system*

$$u_{n+1} = \frac{a_1}{a_2 w_{n-k}}, \quad v_{n+1} = \frac{a_2}{a_3 u_{n-k}}, \quad w_{n+1} = \frac{a_3}{a_1 v_{n-k}}, \quad n = 0, 1, \ldots, \tag{1.3}$$

and, we get System (1.1) *reduced to the three independents linear difference equations*

$$x_{n+1} = \frac{a_1}{a u_{n-k} + c u_{n-k} w_{n-k} + b} x_{n-k}, \qquad n \in \mathbb{N}_0, \tag{1.4}$$

$$y_{n+1} = \frac{a_2}{a u_{n-k} v_{n-k} + b v_{n-k} + c} y_{n-k}, \qquad n \in \mathbb{N}_0, \tag{1.5}$$

$$z_{n+1} = \frac{a_3}{b v_{n-k} w_{n-k} + c w_{n-k} + a} z_{n-k}, \qquad n \in \mathbb{N}_0. \tag{1.6}$$

Proof. From (1.1), we get

$$\frac{x_{n+1}}{y_{n+1}} = \frac{a_1 x_{n-k}}{a_2 z_{n-k}}, \quad \frac{y_{n+1}}{z_{n+1}} = \frac{a_2 y_{n-k}}{a_3 x_{n-k}}, \quad \frac{z_{n+1}}{x_{n+1}} = \frac{a_3 z_{n-k}}{a_1 y_{n-k}},$$

so, by the change of variables (1.2), we get

$$u_{n+1} = \frac{a_1}{a_2 w_{n-k}}, \quad v_{n+1} = \frac{a_2}{a_3 u_{n-k}}, \quad w_{n+1} = \frac{a_3}{a_1 v_{n-k}}.$$

The formulae (1.4), (1.5) and (1.6), follows directly from (1.1) and (1.2). □

To solve (1.1), we need to solve System (1.3) and equations (1.4), (1.5) and (1.6). In the following result, we show that every solution of System (1.3) is periodic with period $6k+6$.

Lemma 3. *Let* $(u_n, v_n, w_n)_{n \geq -k}$ *be a solution of System* (1.3)*, then*

$$u_{n+6k+6} = u_n, \quad v_{n+6k+6} = v_n, \quad w_{n+6k+6} = w_n, \quad n = -k, -k+1, \ldots,$$

that is the solution is periodic of period $6k+6$*.*

Proof. From (1.3), we have

$$u_{n+6k+6} = \frac{a_1}{a_2 w_{n+5k+5}}, \quad v_{n+6k+6} = \frac{a_2}{a_3 u_{n+5k+5}}, \quad w_{n+6k+6} = \frac{a_3}{a_1 v_{n+5k+5}}, \tag{1.7}$$

$$u_{n+5k+5} = \frac{a_1}{a_2 w_{n+4k+4}}, \quad v_{n+5k+5} = \frac{a_2}{a_3 u_{n+4k+4}}, \quad w_{n+5k+5} = \frac{a_3}{a_1 v_{n+4k+4}}, \tag{1.8}$$

$$u_{n+4k+4}=\frac{a_1}{a_2w_{n+3k+3}},\quad v_{n+4k+4}=\frac{a_2}{a_3u_{n+3k+3}},\quad w_{n+4k+4}=\frac{a_3}{a_1v_{n+3k+3}},\tag{1.9}$$

$$u_{n+3k+3}=\frac{a_1}{a_2w_{n+2k+2}},\quad v_{n+3k+3}=\frac{a_2}{a_3u_{n+2k+2}},\quad w_{n+3k+3}=\frac{a_3}{a_1v_{n+2k+2}},\tag{1.10}$$

$$u_{n+2k+2}=\frac{a_1}{a_2w_{n+k+1}},\quad v_{n+2k+2}=\frac{a_2}{a_3u_{n+1k+1}},\quad w_{n+2k+2}=\frac{a_3}{a_1v_{n+k+1}},\tag{1.11}$$

$$u_{n+k+1}=\frac{a_1}{a_2w_n},\qquad v_{n+k+1}=\frac{a_2}{a_3u_n},\qquad w_{n+k+1}=\frac{a_3}{a_1v_n}.\tag{1.12}$$

From (1.7)-(1.12), we get

$$u_{n+6k+6}=\frac{a_1}{a_2w_{n+5k+5}}=\frac{a_1^2}{a_2a_3}v_{n+4k+4}=\frac{a_1^2}{a_3^2u_{n+3k+3}}=\frac{a_1a_2}{a_3^2}w_{n+2k+2}$$
$$=\frac{a_2}{a_3v_{n+k+1}}=u_n.$$

Similarly, we prove that

$$v_{n+6k+6}=v_n,\quad w_{n+6k+6}=w_n.$$

□

Lemma 4. *Let $(x_n,y_n,z_n)_{n\geq -k}$ be a solution of System (1.1). Assume that the parameters a_1,a_2,a_3,a,b,c and the initial values $x_{-k},\ldots,x_0,y_{-k},\ldots,y_0,z_{-k},\ldots,z_0$ are positive and assume that $a_1<b,a_2<c,a_3<a$, then for every solution of (1.1), we have*

$$\lim_{n\to+\infty}(x_n,y_n,z_n)=(0,0,0).$$

Proof. Using the assumption on the parameters and the initial values, we get that the every solution of System (1.1) is positive, and it follows that

$$x_{n+1}\leq\frac{a_1}{b}x_{n-k},\quad y_{n+1}\leq\frac{a_2}{c}y_{n-k},\quad z_{n+1}\leq\frac{a_3}{a}z_{n-k},\quad n\in\mathbb{N}_0,$$

from which, we get for $i=1,2,\ldots,k+1$, that

$$x_{(k+1)n+i}\leq\left(\frac{a_1}{b}\right)^{n+1}x_{i-k-1},$$
$$y_{(k+1)n+i}\leq\left(\frac{a_2}{c}\right)^{n+1}y_{i-k-1},$$
$$z_{(k+1)n+i}\leq\left(\frac{a_3}{a}\right)^{n+1}z_{i-k-1}.$$

Using the fact that $a_1<b,a_2<c,a_3<a$, the desired result follows. □

In the following sections, we solve System (1.1) when $k=0$ and $k=1$. To do this, we need in each case to solve the corresponding system (1.3) and the corresponding equations (1.4), (1.5) and (1.6).

2. THE CASE $k = 0$.

In this case the system (1.1) takes the form

$$\begin{aligned} x_{n+1} &= \frac{a_1 x_n y_n}{a x_n + b y_n + c z_n}, \\ y_{n+1} &= \frac{a_2 y_n z_n}{a x_n + b y_n + c z_n}, \\ z_{n+1} &= \frac{a_3 x_n z_n}{a x_n + b y_n + c z_n}. \end{aligned} \tag{2.1}$$

For System (2.1), the corresponding system (1.3), will be

$$u_{n+1} = \frac{a_1}{a_2 w_n}, \quad v_{n+1} = \frac{a_2}{a_3 u_n}, \quad w_{n+1} = \frac{a_3}{a_1 v_n}.$$

From Lemma (3), we get that the sequences $(u_n)_{n\geq 0}, (v_n)_{n\geq 0}$ and $(w_n)_{n\geq 0}$ are periodic of period 6 and we have

$$u_{6n} = u_0, \qquad u_{6n+1} = \frac{a_1}{a_2 w_0}, \qquad u_{6n+2} = \frac{a_1^2}{a_2 a_3} v_0, \tag{2.2}$$

$$u_{6n+3} = \frac{a_1^2}{a_3^2 u_0}, \qquad u_{6n+4} = \frac{a_1 a_2}{a_3^2} w_0, \qquad u_{6n+5} = \frac{a_2}{a_3 v_0}, \tag{2.3}$$

$$v_{6n} = v_0, \qquad v_{6n+1} = \frac{a_2}{a_3 u_0}, \qquad v_{6n+2} = \frac{a_2^2}{a_1 a_3} w_0, \tag{2.4}$$

$$v_{6n+3} = \frac{a_2^2}{a_1^2 v_0}, \qquad v_{6n+4} = \frac{a_2 a_3}{a_1^2} u_0, \qquad v_{6n+5} = \frac{a_3}{a_1 w_0}, \tag{2.5}$$

$$w_{6n} = w_0, \qquad w_{6n+1} = \frac{a_3}{a_1 v_0}, \qquad w_{6n+2} = \frac{a_3^2}{a_1 a_2} u_0, \tag{2.6}$$

$$w_{6n+3} = \frac{a_3^2}{a_2^2 w_0}, \qquad w_{6n+4} = \frac{a_1 a_3}{a_2^2} v_0, \qquad w_{6n+5} = \frac{a_1}{a_2 u_0}. \tag{2.7}$$

Also for System (2.1), the corresponding equations (1.4), (1.5) and (1.6) will be

$$x_{n+1} = \frac{a_1}{a u_n + c u_n w_n + b} x_n, \qquad n \in \mathbb{N}_0, \tag{2.8}$$

$$y_{n+1} = \frac{a_2}{a u_n v_n + b v_n + c} y_n, \qquad n \in \mathbb{N}_0, \tag{2.9}$$

$$z_{n+1} = \frac{a_3}{b v_n w_n + c w_n + a} z_n, \qquad n \in \mathbb{N}_0. \tag{2.10}$$

Now, from (1.2), (2.2)-(2.7) and (2.8)-(2.10), we get

$$x_{6n+1} = \alpha_0 x_{6n}, \tag{2.11}$$

$$x_{6n+2} = \alpha_1 \alpha_0 x_{6n}, \tag{2.12}$$

$$x_{6n+3} = \alpha_2 \alpha_1 \alpha_0 x_{6n}, , \tag{2.13}$$

$$x_{6n+4} = \alpha_3\alpha_2\alpha_1\alpha_0 x_{6n}, \tag{2.14}$$
$$x_{6n+5} = \alpha_4\alpha_3\alpha_2\alpha_1\alpha_0 x_{6n}, \tag{2.15}$$
$$x_{6n+6} = \alpha_5\alpha_4\alpha_3\alpha_2\alpha_1\alpha_0 x_{6n}, \tag{2.16}$$

$$y_{6n+1} = \beta_0 y_{6n}, \tag{2.17}$$
$$y_{6n+2} = \beta_1\beta_0 y_{6n}, \tag{2.18}$$
$$y_{6n+3} = \beta_2\beta_1\beta_0 y_{6n}, \tag{2.19}$$
$$y_{6n+4} = \beta_3\beta_2\beta_1\beta_0 y_{6n}, \tag{2.20}$$
$$y_{6n+5} = \beta_4\beta_3\beta_2\beta_1\beta_0 y_{6n}, \tag{2.21}$$
$$y_{6n+6} = \beta_5\beta_4\beta_3\beta_2\beta_1\beta_0 y_{6n}, \tag{2.22}$$

$$z_{6n+1} = \gamma_0 z_{6n}, \tag{2.23}$$
$$z_{6n+2} = \gamma_1\gamma_0 z_{6n}, \tag{2.24}$$
$$z_{6n+3} = \gamma_2\gamma_1\gamma_0 z_{6n}, \tag{2.25}$$
$$z_{6n+4} = \gamma_3\gamma_2\gamma_1\gamma_0 z_{6n}, \tag{2.26}$$
$$z_{6n+5} = \gamma_4\gamma_3\gamma_2\gamma_1\gamma_0 z_{6n}, \tag{2.27}$$
$$z_{6n+6} = \gamma_5\gamma_4\gamma_3\gamma_2\gamma_1\gamma_0 z_{6n}, \tag{2.28}$$

where

$$\alpha_0 = \frac{a_1 y_0}{ax_0 + by_0 + cz_0}, \qquad \alpha_1 = \frac{a_1 a_2 y_0 z_0}{aa_1 x_0 y_0 + ba_2 y_0 z_0 + ca_3 x_0 z_0},$$
$$\alpha_2 = \frac{a_1 a_2^2 a_3 z_0}{aa_1^2 a_2 y_0 + ba_2^2 a_3 z_0 + ca_1 a_3^2 x_0}, \qquad \alpha_3 = \frac{a_1 a_2^2 a_3^2 x_0 z_0}{aa_1^2 a_2^2 y_0 z_0 + ba_2^2 a_3^2 x_0 z_0 + ca_1^2 a_3^2 x_0 y_0},$$
$$\alpha_4 = \frac{a_1 a_2 a_3^2 x_0}{aa_1 a_2^2 z_0 + ba_2 a_3^2 x_0 + ca_1^2 a_3 y_0}, \qquad \alpha_5 = \frac{a_1 a_3 x_0 y_0}{aa_2 x_0 z_0 + ba_3 x_0 y_0 + ca_1 y_0 z_0},$$
$$\beta_0 = \frac{a_2 z_0}{ax_0 + by_0 + cz_0}, \qquad \beta_1 = \frac{a_2 a_3 x_0 z_0}{aa_1 x_0 y_0 + ba_2 y_0 z_0 + ca_3 x_0 z_0},$$
$$\beta_2 = \frac{a_1 a_2 a_3^2 x_0}{aa_1^2 a_2 y_0 + ba_2^2 a_3 z_0 + ca_1 a_3^2 x_0}, \qquad \beta_3 = \frac{a_1^2 a_2 a_3^2 x_0 y_0}{aa_1^2 a_2^2 y_0 z_0 + ba_2^2 a_3^2 x_0 z_0 + ca_1^2 a_3^2 x_0 y_0},$$
$$\beta_4 = \frac{a_1^2 a_2 a_3 y_0}{aa_1 a_2^2 z_0 + ba_2 a_3^2 x_0 + ca_1^2 a_3 y_0}, \qquad \beta_5 = \frac{a_1 a_2 y_0 z_0}{aa_2 x_0 z_0 + ba_3 x_0 y_0 + ca_1 y_0 z_0},$$
$$\gamma_0 = \frac{a_3 x_0}{ax_0 + by_0 + cz_0}, \qquad \gamma_1 = \frac{a_1 a_3 x_0 y_0}{aa_1 x_0 y_0 + ba_2 y_0 z_0 + ca_3 x_0 z_0},$$
$$\gamma_2 = \frac{a_1^2 a_2 a_3 y_0}{aa_1^2 a_2 y_0 + ba_2^2 a_3 z_0 + ca_1 a_3^2 x_0}, \qquad \gamma_3 = \frac{a_1^2 a_2^2 a_3 y_0 z_0}{aa_1^2 a_2^2 y_0 z_0 + ba_2^2 a_3^2 x_0 z_0 + ca_1^2 a_3^2 x_0 y_0},$$
$$\gamma_4 = \frac{a_1 a_2^2 a_3 z_0}{aa_1 a_2^2 z_0 + ba_2 a_3^2 x_0 + ca_1^2 a_3 y_0}, \qquad \gamma_5 = \frac{a_2 a_3 x_0 z_0}{aa_2 x_0 z_0 + ba_3 x_0 y_0 + ca_1 y_0 z_0}.$$

Now, we will solve equations (2.11)-(2.28). We have $(x_{6n})_{n\geq 0}$ is the solution of the first-order linear homogenous difference equation given in (2.16), from which

$$x_{6n} = (\alpha_5\alpha_4\alpha_3\alpha_2\alpha_1\alpha_0)^n x_0. \tag{2.29}$$

Hence, from (2.29) and (2.11)-(2.15) it follows that

$$\begin{aligned}
x_{6n+1} &= \alpha_0 (\alpha_5\alpha_4\alpha_3\alpha_2\alpha_1\alpha_0)^n x_0,\\
x_{6n+2} &= \alpha_1\alpha_0 (\alpha_5\alpha_4\alpha_3\alpha_2\alpha_1\alpha_0)^n x_0,\\
x_{6n+3} &= \alpha_2\alpha_1\alpha_0 (\alpha_5\alpha_4\alpha_3\alpha_2\alpha_1\alpha_0)^n x_0,\\
x_{6n+4} &= \alpha_3\alpha_2\alpha_1\alpha_0 (\alpha_5\alpha_4\alpha_3\alpha_2\alpha_1\alpha_0)^n x_0,\\
x_{6n+5} &= \alpha_4\alpha_3\alpha_2\alpha_1\alpha_0 (\alpha_5\alpha_4\alpha_3\alpha_2\alpha_1\alpha_0)^n x_0.
\end{aligned}$$

Similarly, $(y_{6n})_{n\geq 0}$ is the solution of the first-order linear homogenous difference equation (2.22), and we have

$$y_{6n} = (\beta_0\beta_1\beta_2\beta_3\beta_4\beta_5)^n y_0. \tag{2.30}$$

Thus, from (2.30) and (2.17)-(2.21), we get

$$\begin{aligned}
y_{6n+1} &= \beta_0 (\beta_5\beta_4\beta_3\beta_2\beta_1\beta_0)^n y_0,\\
y_{6n+2} &= \beta_1\beta_0 (\beta_5\beta_4\beta_3\beta_2\beta_1\beta_0)^n y_0,\\
y_{6n+3} &= \beta_2\beta_1\beta_0 (\beta_5\beta_4\beta_3\beta_2\beta_1\beta_0)^n y_0,\\
y_{6n+4} &= \beta_3\beta_2\beta_1\beta_0 (\beta_5\beta_4\beta_3\beta_2\beta_1\beta_0)^n y_0,\\
y_{6n+5} &= \beta_4\beta_3\beta_2\beta_1\beta_0 (\beta_5\beta_4\beta_3\beta_2\beta_1\beta_0)^n y_0.
\end{aligned}$$

Also, $(z_{6n})_{n\geq 0}$ is the solution of the first-order linear homogenous difference equation (2.28), and we have

$$z_{6n} = (\gamma_0\gamma_1\gamma_2\gamma_3\gamma_4\gamma_5)^n z_0 \tag{2.31}$$

From (2.31) and (2.23)-(2.27), we obtain

$$\begin{aligned}
z_{6n+1} &= \gamma_0 (\gamma_5\gamma_4\gamma_3\gamma_2\gamma_1\gamma_0)^n z_0,\\
z_{6n+2} &= \gamma_1\gamma_0 (\gamma_5\gamma_4\gamma_3\gamma_2\gamma_1\gamma_0)^n z_0,\\
z_{6n+3} &= \gamma_2\gamma_1\gamma_0 (\gamma_5\gamma_4\gamma_3\gamma_2\gamma_1\gamma_0)^n z_0,\\
z_{6n+4} &= \gamma_3\gamma_2\gamma_1\gamma_0 (\gamma_5\gamma_4\gamma_3\gamma_2\gamma_1\gamma_0)^n z_0,\\
z_{6n+5} &= \gamma_4\gamma_3\gamma_2\gamma_1\gamma_0 (\gamma_5\gamma_4\gamma_3\gamma_2\gamma_1\gamma_0)^n z_0.
\end{aligned}$$

In summary, the formulas of well-defined solutions of System (2.1) are presented in the following result.

Theorem 1. *Consider System (2.1) and let $(x_n, y_n, z_n)_{n\geq 0}$ be a solution of (2.1). Then, for $n = 0, 1, \ldots$, we have*

$$x_{6n} = (\alpha_5\alpha_4\alpha_3\alpha_2\alpha_1\alpha_0)^n x_0, \qquad x_{6n+1} = \alpha_0(\alpha_5\alpha_4\alpha_3\alpha_2\alpha_1\alpha_0)^n x_0,$$
$$x_{6n+2} = \alpha_1\alpha_0(\alpha_5\alpha_4\alpha_3\alpha_2\alpha_1\alpha_0)^n x_0, \qquad x_{6n+3} = \alpha_2\alpha_1\alpha_0(\alpha_5\alpha_4\alpha_3\alpha_2\alpha_1\alpha_0)^n x_0,$$
$$x_{6n+4} = \alpha_3\alpha_2\alpha_1\alpha_0(\alpha_5\alpha_4\alpha_3\alpha_2\alpha_1\alpha_0)^n x_0, \qquad x_{6n+5} = \alpha_4\alpha_3\alpha_2\alpha_1\alpha_0 \times (\alpha_5\alpha_4\alpha_3\alpha_2\alpha_1\alpha_0)^n x_0,$$

$$y_{6n} = (\beta_5\beta_4\beta_3\beta_2\beta_1\beta_0)^n y_0, \qquad y_{6n+1} = \beta_0(\beta_5\beta_4\beta_3\beta_2\beta_1\beta_0)^n y_0,$$
$$y_{6n+2} = \beta_1\beta_0(\beta_5\beta_4\beta_3\beta_2\beta_1\beta_0)^n y_0, \qquad y_{6n+3} = \beta_2\beta_1\beta_0(\beta_5\beta_4\beta_3\beta_2\beta_1\beta_0)^n y_0,$$
$$y_{6n+4} = \beta_3\beta_2\beta_1\beta_0(\beta_5\beta_4\beta_3\beta_2\beta_1\beta_0)^n y_0, \qquad y_{6n+5} = \beta_4\beta_3\beta_2\beta_1\beta_0(\beta_5\beta_4\beta_3\beta_2\beta_1\beta_0)^n y_0,$$

$$z_{6n} = (\gamma_5\gamma_4\gamma_3\gamma_2\gamma_1\gamma_0)^n z_0, \qquad z_{6n+1} = \gamma_0(\gamma_5\gamma_4\gamma_3\gamma_2\gamma_1\gamma_0)^n z_0,$$
$$z_{6n+2} = \gamma_1\gamma_0(\gamma_5\gamma_4\gamma_3\gamma_2\gamma_1\gamma_0)^n z_0, \qquad z_{6n+3} = \gamma_2\gamma_1\gamma_0(\gamma_5\gamma_4\gamma_3\gamma_2\gamma_1\gamma_0)^n z_0,$$
$$z_{6n+4} = \gamma_3\gamma_2\gamma_1\gamma_0(\gamma_5\gamma_4\gamma_3\gamma_2\gamma_1\gamma_0)^n z_0, \qquad z_{6n+5} = \gamma_4\gamma_3\gamma_2\gamma_1\gamma_0(\gamma_5\gamma_4\gamma_3\gamma_2\gamma_1\gamma_0)^n z_0,$$

where

$$\alpha_0 = \frac{a_1 y_0}{ax_0 + by_0 + cz_0}, \qquad \alpha_1 = \frac{a_1 a_2 y_0 z_0}{aa_1 x_0 y_0 + ba_2 y_0 z_0 + ca_3 x_0 z_0},$$
$$\alpha_2 = \frac{a_1 a_2^2 a_3 z_0}{aa_1^2 a_2 y_0 + ba_2^2 a_3 z_0 + ca_1 a_3^2 x_0}, \qquad \alpha_3 = \frac{a_1 a_2^2 a_3^2 x_0 z_0}{aa_1^2 a_2^2 y_0 z_0 + ba_2^2 a_3^2 x_0 z_0 + ca_1^2 a_3^2 x_0 y_0},$$
$$\alpha_4 = \frac{a_1 a_2 a_3^2 x_0}{aa_1 a_2^2 z_0 + ba_2 a_3^2 x_0 + ca_1^2 a_3 y_0}, \qquad \alpha_5 = \frac{a_1 a_3 x_0 y_0}{aa_2 x_0 z_0 + ba_3 x_0 y_0 + ca_1 y_0 z_0},$$
$$\beta_0 = \frac{a_2 z_0}{ax_0 + by_0 + cz_0}, \qquad \beta_1 = \frac{a_2 a_3 x_0 z_0}{aa_1 x_0 y_0 + ba_2 y_0 z_0 + ca_3 x_0 z_0},$$
$$\beta_2 = \frac{a_1 a_2 a_3^2 x_0}{aa_1^2 a_2 y_0 + ba_2^2 a_3 z_0 + ca_1 a_3^2 x_0}, \qquad \beta_3 = \frac{a_1^2 a_2 a_3^2 x_0 y_0}{aa_1^2 a_2^2 y_0 z_0 + ba_2^2 a_3^2 x_0 z_0 + ca_1^2 a_3^2 x_0 y_0},$$
$$\beta_4 = \frac{a_1^2 a_2 a_3 y_0}{aa_1 a_2^2 z_0 + ba_2 a_3^2 x_0 + ca_1^2 a_3 y_0}, \qquad \beta_5 = \frac{a_1 a_2 y_0 z_0}{aa_2 x_0 z_0 + ba_3 x_0 y_0 + ca_1 y_0 z_0},$$
$$\gamma_0 = \frac{a_3 x_0}{ax_0 + by_0 + cz_0}, \qquad \gamma_1 = \frac{a_1 a_3 x_0 y_0}{aa_1 x_0 y_0 + ba_2 y_0 z_0 + ca_3 x_0 z_0},$$
$$\gamma_2 = \frac{a_1^2 a_2 a_3 y_0}{aa_1^2 a_2 y_0 + ba_2^2 a_3 z_0 + ca_1 a_3^2 x_0}, \qquad \gamma_3 = \frac{a_1^2 a_2^2 a_3 y_0 z_0}{aa_1^2 a_2^2 y_0 z_0 + ba_2^2 a_3^2 x_0 z_0 + ca_1^2 a_3^2 x_0 y_0},$$
$$\gamma_4 = \frac{a_1 a_2^2 a_3 z_0}{aa_1 a_2^2 z_0 + ba_2 a_3^2 x_0 + ca_1^2 a_3 y_0}, \qquad \gamma_5 = \frac{a_2 a_3 x_0 z_0}{aa_2 x_0 z_0 + ba_3 x_0 y_0 + ca_1 y_0 z_0}.$$

3. THE CASE $k = 1$

When $k = 1$, System (1.1) takes the form

$$\begin{aligned} x_{n+1} &= \frac{a_1 x_{n-1} y_{n-1}}{a x_{n-1} + b y_{n-1} + c z_{n-1}}, \\ y_{n+1} &= \frac{a_2 y_{n-1} z_{n-1}}{a x_{n-1} + b y_{n-1} + c z_{n-1}}, \\ z_{n+1} &= \frac{a_3 x_{n-1} z_{n-1}}{a x_{n-1} + b y_{n-1} + c z_{n-1}}, \end{aligned} \tag{3.1}$$

and the corresponding system (1.3) will be

$$u_{n+1} = \frac{a_1}{a_2 w_{n-1}}, \quad v_{n+1} = \frac{a_2}{a_3 u_{n-1}}, \quad w_{n+1} = \frac{a_3}{a_1 v_{n-1}}.$$

From Lemma 3, it follows that the sequences $(u_n)_{n\geq -1}, (v_n)_{n\geq -1}$ and $(w_n)_{n\geq -1}$ are periodic of period 12 and we have

$$u_{12n-1} = u_{-1}, \qquad u_{12n} = u_0, \qquad u_{12n+1} = \frac{a_1}{a_2 w_{-1}}, \tag{3.2}$$

$$u_{12n+2} = \frac{a_1}{a_2 w_0}, \qquad u_{12n+3} = \frac{a_1^2}{a_2 a_3} v_{-1}, \qquad u_{12n+4} = \frac{a_1^2}{a_2 a_3} v_0, \tag{3.3}$$

$$u_{12n+5} = \frac{a_1^2}{a_3^2 u_{-1}}, \qquad u_{12n+6} = \frac{a_1^2}{a_3^2 u_0}, \qquad u_{12n+7} = \frac{a_1 a_2}{a_3^2} w_{-1}, \tag{3.4}$$

$$u_{12n+8} = \frac{a_1 a_2}{a_3^2} w_0, \qquad u_{12n+9} = \frac{a_2}{a_3 v_{-1}}, \qquad u_{12n+10} = \frac{a_2}{a_3 v_0}, \tag{3.5}$$

$$v_{12n-1} = v_{-1}, \qquad v_{12n} = v_0, \qquad v_{12n+1} = \frac{a_2}{a_3 u_{-1}}, \tag{3.6}$$

$$v_{12n+2} = \frac{a_2}{a_3 u_0}, \qquad v_{12n+3} = \frac{a_2^2}{a_1 a_3} w_{-1}, \qquad v_{12n+4} = \frac{a_2^2}{a_1 a_3} w_0, \tag{3.7}$$

$$v_{12n+5} = \frac{a_2^2}{a_1^2 v_{-1}}, \qquad v_{12n+6} = \frac{a_2^2}{a_1^2 v_0}, \qquad v_{12n+7} = \frac{a_2 a_3}{a_1^2} w_{-1}, \tag{3.8}$$

$$v_{12n+8} = \frac{a_2 a_3}{a_1^2} w_0, \qquad v_{12n+9} = \frac{a_3}{a_1 w_{-1}}, \qquad v_{12n+10} = \frac{a_3}{a_1 w_0}, \tag{3.9}$$

$$w_{12n-1} = w_{-1}, \qquad w_{12n} = w_0, \qquad w_{12n+1} = \frac{a_3}{a_1 v_{-1}}, \tag{3.10}$$

$$w_{12n+2} = \frac{a_3}{a_1 v_0}, \qquad w_{12n+3} = \frac{a_3^2}{a_1 a_2} w_{-1}, \qquad w_{12n+4} = \frac{a_3^2}{a_1 a_2} u_0, \tag{3.11}$$

$$w_{12n+5} = \frac{a_3^2}{a_2^2 w_{-1}}, \qquad w_{12n+6} = \frac{a_3^2}{a_2^2 w_0}, \qquad w_{12n+7} = \frac{a_1 a_3}{a_2^2} v_{-1}, \tag{3.12}$$

$$w_{12n+8} = \frac{a_1 a_3}{a_2^2} v_0, \qquad w_{12n+9} = \frac{a_1}{a_2 w_{-1}}, \qquad w_{12n+10} = \frac{a_1}{a_2 w_0}. \tag{3.13}$$

For system (3.1), the corresponding equations (1.4)-(1.6) will be

$$x_{n+1} = \frac{a_1}{au_{n-1} + cu_{n-1}w_{n-1} + b} x_{n-1}, \qquad n \in \mathbb{N}_0, \tag{3.14}$$

$$y_{n+1} = \frac{a_2}{au_{n-1}v_{n-1} + bv_{n-1} + c} y_{n-1}, \qquad n \in \mathbb{N}_0, \tag{3.15}$$

$$z_{n+1} = \frac{a_3}{bv_{n-1}w_{n-1} + cw_{n-1} + a} z_{n-1}, \qquad n \in \mathbb{N}_0. \tag{3.16}$$

Now, from (1.2), (3.2)-(3.13) and the equations (3.14)-(3.16), we obtain

$$x_{12n+1} = \alpha_0 x_{12n-1}, \tag{3.17}$$
$$x_{12n+2} = \alpha_1 x_{12n}, \tag{3.18}$$
$$x_{12n+3} = \alpha_0\alpha_2 x_{12n-1}, \tag{3.19}$$
$$x_{12n+4} = \alpha_1\alpha_3 x_{12n}, \tag{3.20}$$
$$x_{12n+5} = \alpha_0\alpha_2\alpha_4 x_{12n-1}, \tag{3.21}$$
$$x_{12n+6} = \alpha_1\alpha_3\alpha_5 x_{12n}, \tag{3.22}$$
$$x_{12n+7} = \alpha_0\alpha_2\alpha_4\alpha_6 x_{12n-1}, \tag{3.23}$$
$$x_{12n+8} = \alpha_1\alpha_3\alpha_5\alpha_7 x_{12n}, \tag{3.24}$$
$$x_{12n+9} = \alpha_0\alpha_2\alpha_4\alpha_6\alpha_8 x_{12n-1}, \tag{3.25}$$
$$x_{12n+10} = \alpha_1\alpha_3\alpha_5\alpha_7\alpha_9 x_{12n}, \tag{3.26}$$
$$x_{12n+11} = \alpha_0\alpha_2\alpha_4\alpha_6\alpha_8\alpha_{10} x_{12n-1}, \tag{3.27}$$
$$x_{12n+12} = \alpha_1\alpha_3\alpha_5\alpha_7\alpha_9\alpha_{11} x_{12n}, \tag{3.28}$$

$$y_{12n+1} = \beta_0 y_{12n-1}, \tag{3.29}$$
$$y_{12n+2} = \beta_1 y_{12n}, \tag{3.30}$$
$$y_{12n+3} = \beta_0\beta_2 y_{12n-1}, \tag{3.31}$$
$$y_{12n+4} = \beta_3\beta_1 y_{12n}, \tag{3.32}$$
$$y_{12n+5} = \beta_0\beta_2\beta_4 y_{12n-1}, \tag{3.33}$$
$$y_{12n+6} = \beta_1\beta_3\beta_5 y_{12n}, \tag{3.34}$$
$$y_{12n+7} = \beta_0\beta_2\beta_4\beta_6 y_{12n-1}, \tag{3.35}$$
$$y_{12n+8} = \beta_1\beta_3\beta_5\beta_7 y_{12n}, \tag{3.36}$$
$$y_{12n+9} = \beta_0\beta_2\beta_4\beta_6\beta_8 y_{12n-1}, \tag{3.37}$$
$$y_{12n+10} = \beta_1\beta_3\beta_5\beta_7\beta_9 y_{12n}, \tag{3.38}$$
$$y_{12n+11} = \beta_0\beta_2\beta_4\beta_6\beta_8\beta_{10} y_{12n-1}, \tag{3.39}$$
$$y_{12n+12} = \beta_1\beta_3\beta_5\beta_7\beta_9\beta_{11} y_{12n}, \tag{3.40}$$

$$z_{12n+1} = \gamma_0 z_{12n-1}, \tag{3.41}$$
$$z_{12n+2} = \gamma_1 z_{12n}, \tag{3.42}$$
$$z_{12n+3} = \gamma_0\gamma_2 z_{12n-1}, \tag{3.43}$$
$$z_{12n+4} = \gamma_3\gamma_1 z_{12n}, \tag{3.44}$$
$$z_{12n+5} = \gamma_4\gamma_2\gamma_0 z_{12n-1}, \tag{3.45}$$
$$z_{12n+6} = \gamma_5\gamma_3\gamma_1 z_{12n}, \tag{3.46}$$
$$z_{12n+7} = \gamma_0\gamma_2\gamma_4\gamma_6 z_{12n-1}, \tag{3.47}$$
$$z_{12n+8} = \gamma_1\gamma_3\gamma_5\gamma_7 z_{12n}, \tag{3.48}$$
$$z_{12n+9} = \gamma_0\gamma_2\gamma_4\gamma_6\gamma_8 z_{12n-1}, \tag{3.49}$$
$$z_{12n+10} = \gamma_1\gamma_3\gamma_5\gamma_7\gamma_9 z_{12n}, \tag{3.50}$$
$$z_{12n+11} = \gamma_0\gamma_2\gamma_4\gamma_6\gamma_8\gamma_{10} z_{12n-1}, \tag{3.51}$$
$$z_{12n+12} = \gamma_1\gamma_3\gamma_5\gamma_7\gamma_9\gamma_{11} z_{12n}, \tag{3.52}$$

where

$$\alpha_0 = \frac{a_1 y_{-1}}{a x_{-1} + b y_{-1} + c z_{-1}},$$
$$\alpha_1 = \frac{a_1 y_0}{a x_0 + b y_0 + c z_0},$$
$$\alpha_2 = \frac{a_1 a_2 y_{-1} z_{-1}}{a a_1 x_{-1} y_{-1} + b a_2 y_{-1} z_{-1} + c a_3 x_{-1} z_{-1}},$$
$$\alpha_3 = \frac{a_1 a_2 y_0 z_0}{a a_1 x_0 y_0 + b a_2 y_0 z_0 + c a_3 x_0 z_0},$$
$$\alpha_4 = \frac{a_1 a_2^2 a_3 z_{-1}}{a a_1^2 a_2 y_{-1} + b a_2^2 a_3 z_{-1} + c a_1 a_3^2 x_{-1}},$$
$$\alpha_5 = \frac{a_1 a_2^2 a_3 z_0}{a a_1^2 a_2 y_0 + b a_2^2 a_3 z_0 + c a_1 a_3^2 x_0},$$
$$\alpha_6 = \frac{a_1 a_2^2 a_3^2 x_{-1} z_{-1}}{a a_1^2 a_2^2 y_{-1} z_{-1} + b a_2^2 a_3^2 x_{-1} z_{-1} + c a_1^2 a_3^2 x_{-1} y_{-1}},$$
$$\alpha_7 = \frac{a_1 a_2^2 a_3^2 x_0 z_0}{a a_1^2 a_2^2 y_0 z_0 + b a_2^2 a_3^2 x_0 z_0 + c a_1^2 a_3^2 x_0 y_0},$$
$$\alpha_8 = \frac{a_1 a_2 a_3^2 x_{-1}}{a a_1 a_2^2 z_{-1} + b a_2 a_3^2 x_{-1} + c a_1^2 a_3 y_{-1}},$$
$$\alpha_9 = \frac{a_1 a_2 a_3^2 x_0}{a a_1 a_2^2 z_0 + b a_2 a_3^2 x_0 + c a_1^2 a_3 y_0},$$
$$\alpha_{10} = \frac{a_1 a_3 x_{-1} y_{-1}}{a a_2 x_{-1} z_{-1} + b a_3 x_{-1} y_{-1} + c a_1 y_{-1} z_{-1}},$$

$$\alpha_{11} = \frac{a_1 a_3 x_0 y_0}{a a_2 x_0 z_0 + b a_3 x_0 y_0 + c a_1 y_0 z_0},$$

$$\beta_0 = \frac{a_2 z_{-1}}{a x_{-1} + b y_{-1} + c z_{-1}},$$
$$\beta_1 = \frac{a_2 z_0}{a x_0 + b y_0 + c z_0},$$
$$\beta_2 = \frac{a_2 a_3 x_{-1} z_{-1}}{a a_1 x_{-1} y_{-1} + b a_2 y_{-1} z_{-1} + c a_3 x_{-1} z_{-1}},$$
$$\beta_3 = \frac{a_2 a_3 x_0 z_0}{a a_1 x_0 y_0 + b a_2 y_0 z_0 + c a_3 x_0 z_0},$$
$$\beta_4 = \frac{a_1 a_2 a_3^2 x_{-1}}{a a_1^2 a_2 y_{-1} + b a_2^2 a_3 z_{-1} + c a_1 a_3^2 x_{-1}},$$
$$\beta_5 = \frac{a_1 a_2 a_3^2 x_0}{a a_1^2 a_2 y_0 + b a_2^2 a_3 z_0 + c a_1 a_3^2 x_0},$$
$$\beta_6 = \frac{a_1^2 a_2 a_3^2 x_{-1} y_{-1}}{a a_1^2 a_2^2 y_{-1} z_{-1} + b a_2^2 a_3^2 x_{-1} z_{-1} + c a_1^2 a_3^2 x_{-1} y_{-1}},$$
$$\beta_7 = \frac{a_1^2 a_2 a_3^2 x_0 y_0}{a a_1^2 a_2^2 y_0 z_0 + b a_2^2 a_3^2 x_0 z_0 + c a_1^2 a_3^2 x_0 y_0},$$
$$\beta_8 = \frac{a_1^2 a_2 a_3 y_{-1}}{a a_1 a_2^2 z_{-1} + b a_2 a_3^2 x_{-1} + c a_1^2 a_3 y_{-1}},$$
$$\beta_9 = \frac{a_1^2 a_2 a_3 y_0}{a a_1 a_2^2 z_0 + b a_2 a_3^2 x_0 + c a_1^2 a_3 y_0},$$
$$\beta_{10} = \frac{a_1 a_2 y_{-1} z_{-1}}{a a_2 x_{-1} z_{-1} + b a_3 x_{-1} y_{-1} + c a_1 y_{-1} z_{-1}},$$
$$\beta_{11} = \frac{a_1 a_2 y_0 z_0}{a a_2 x_0 z_0 + b a_3 x_0 y_0 + c a_1 y_0 z_0},$$

$$\gamma_0 = \frac{a_3 x_{-1}}{a x_{-1} + b y_{-1} + c z_{-1}},$$
$$\gamma_1 = \frac{a_3 x_0}{a x_0 + b y_0 + c z_0},$$
$$\gamma_2 = \frac{a_1 a_3 x_{-1} y_{-1}}{a a_1 x_{-1} y_{-1} + b a_2 y_{-1} z_{-1} + c a_3 x_{-1} z_{-1}},$$
$$\gamma_3 = \frac{a_1 a_3 x_0 y_0}{a a_1 x_0 y_0 + b a_2 y_0 z_0 + c a_3 x_0 z_0},$$
$$\gamma_4 = \frac{a_1^2 a_2 a_3 y_{-1}}{a a_1^2 a_2 y_{-1} + b a_2^2 a_3 z_{-1} + c a_1 a_3^2 x_{-1}},$$

$$\gamma_5 = \frac{a_1^2 a_2 a_3 y_0}{a a_1^2 a_2 y_0 + b a_2^2 a_3 z_0 + c a_1 a_3^2 x_0},$$
$$\gamma_6 = \frac{a_1^2 a_2^2 a_3 y_{-1} z_{-1}}{a a_1^2 a_2^2 y_{-1} z_{-1} + b a_2^2 a_3^2 x_{-1} z_{-1} + c a_1^2 a_3^2 x_{-1} y_{-1}},$$
$$\gamma_7 = \frac{a_1^2 a_2^2 a_3 y_0 z_0}{a a_1^2 a_2^2 y_0 z_0 + b a_2^2 a_3^2 x_0 z_0 + c a_1^2 a_3^2 x_0 y_0},$$
$$\gamma_8 = \frac{a_1 a_2^2 a_3 z_{-1}}{a a_1 a_2^2 z_{-1} + b a_2 a_3^2 x_{-1} + c a_1^2 a_3 y_{-1}},$$
$$\gamma_9 = \frac{a_1 a_2^2 a_3 z_0}{a a_1 a_2^2 z_0 + b a_2 a_3^2 x_0 + c a_1^2 a_3 y_0},$$
$$\gamma_{10} = \frac{a_2 a_3 x_{-1} z_{-1}}{a a_2 x_{-1} z_{-1} + b a_3 x_{-1} y_{-1} + c a_1 y_{-1} z_{-1}},$$
$$\gamma_{11} = \frac{a_2 a_3 z_0 x_0}{a a_2 x_0 z_0 + b a_3 x_0 y_0 + c a_1 y_0 z_0}.$$

Now, we will solve Equations (3.17)-(3.28). The equation (3.28), satisfies a first order homogenous difference equation, and its general solution is given by

$$x_{12n} = (\alpha_1 \alpha_3 \alpha_5 \alpha_7 \alpha_9 \alpha_{11})^n x_0, \tag{3.53}$$

The equation (3.27), can be written as

$$x_{12n+11} = \alpha_0 \alpha_2 \alpha_4 \alpha_6 \alpha_8 \alpha_{10} x_{12(n-1)+11},$$

which is also a first-order homogenous linear difference equation, and its general solution is

$$x_{12n-1} = (\alpha_0 \alpha_2 \alpha_4 \alpha_6 \alpha_8 \alpha_{10})^n x_{-1}. \tag{3.54}$$

It follows from (3.53) and (3.54) and the equations (3.17)-(3.27), that we have for all $n \in \mathbb{N}_0$:

$$\begin{aligned}
x_{12n-1} &= (\alpha_0 \alpha_2 \alpha_4 \alpha_6 \alpha_8 \alpha_{10})^n x_{-1}, \\
x_{12n} &= (\alpha_1 \alpha_3 \alpha_5 \alpha_7 \alpha_9 \alpha_{11})^n x_0, \\
x_{12n+1} &= \alpha_0 (\alpha_0 \alpha_2 \alpha_4 \alpha_6 \alpha_8 \alpha_{10})^n x_{-1}, \\
x_{12n+2} &= \alpha_1 (\alpha_1 \alpha_3 \alpha_5 \alpha_7 \alpha_9 \alpha_{11})^n x_0, \\
x_{12n+3} &= \alpha_0 \alpha_2 (\alpha_0 \alpha_2 \alpha_4 \alpha_6 \alpha_8 \alpha_{10})^n x_{-1}, \\
x_{12n+4} &= \alpha_1 \alpha_3 (\alpha_1 \alpha_3 \alpha_5 \alpha_7 \alpha_9 \alpha_{11})^n x_0, \\
x_{12n+5} &= \alpha_0 \alpha_2 \alpha_4 (\alpha_0 \alpha_2 \alpha_4 \alpha_6 \alpha_8 \alpha_{10})^n x_{-1}, \\
x_{12n+6} &= \alpha_1 \alpha_3 \alpha_5 (\alpha_1 \alpha_3 \alpha_5 \alpha_7 \alpha_9 \alpha_{11})^n x_0, \\
x_{12n+7} &= \alpha_0 \alpha_2 \alpha_4 \alpha_6 (\alpha_0 \alpha_2 \alpha_4 \alpha_6 \alpha_8 \alpha_{10})^n x_{-1}, \\
x_{12n+8} &= \alpha_1 \alpha_3 \alpha_5 \alpha_7 (\alpha_1 \alpha_3 \alpha_5 \alpha_7 \alpha_9 \alpha_{11})^n x_0,
\end{aligned}$$

$$x_{12n+9} = \alpha_0\alpha_2\alpha_4\alpha_6\alpha_8(\alpha_0\alpha_2\alpha_4\alpha_6\alpha_8\alpha_{10})^n x_{-1},$$
$$x_{12n+10} = \alpha_1\alpha_3\alpha_5\alpha_7\alpha_9(\alpha_1\alpha_3\alpha_5\alpha_7\alpha_9\alpha_{11})^n x_0,$$

Now, we move to the equations (3.29)-(3.40). Similarly, (3.40) and (3.39) are first-order linear homogeneous difference equations and their general solutions are as follows

$$y_{12n-1} = (\beta_0\beta_2\beta_4\beta_6\beta_8\beta_{10})^n y_{-1}, \tag{3.55}$$
$$y_{12n} = (\beta_1\beta_3\beta_5\beta_7\beta_9\beta_{11})^n y_0, \tag{3.56}$$

From (3.55) and (3.56) and the equations (3.29)-(3.38), we get that

$$y_{12n+1} = \beta_0(\beta_0\beta_2\beta_4\beta_6\beta_8\beta_{10})^n y_{-1},$$
$$y_{12n+2} = \beta_1(\beta_1\beta_3\beta_5\beta_7\beta_9\beta_{11})^n y_0,$$
$$y_{12n+3} = \beta_0\beta_2(\beta_0\beta_2\beta_4\beta_6\beta_8\beta_{10})^n y_{-1},$$
$$y_{12n+4} = \beta_1\beta_3(\beta_1\beta_3\beta_5\beta_7\beta_9\beta_{11})^n y_0,$$
$$y_{12n+5} = \beta_0\beta_2\beta_4(\beta_0\beta_2\beta_4\beta_6\beta_8\beta_{10})^n y_{-1},$$
$$y_{12n+6} = \beta_1\beta_3\beta_5(\beta_1\beta_3\beta_5\beta_7\beta_9\beta_{11})^n y_0,$$
$$y_{12n+7} = \beta_0\beta_2\beta_4\beta_6(\beta_0\beta_2\beta_4\beta_6\beta_8\beta_{10})^n y_{-1},$$
$$y_{12n+8} = \beta_1\beta_3\beta_5\beta_7(\beta_1\beta_3\beta_5\beta_7\beta_9\beta_{11})^n y_0,$$
$$y_{12n+9} = \beta_0\beta_2\beta_4\beta_6\beta_8(\beta_0\beta_2\beta_4\beta_6\beta_8\beta_{10})^n y_{-1},$$
$$y_{12n+10} = \beta_1\beta_3\beta_5\beta_7\beta_9(\beta_1\beta_3\beta_5\beta_7\beta_9\beta_{11})^n y_0,$$

To conclude, we solve the equations (3.41) and (3.52). We begin with the first-order linear homogeneous difference equations (3.51) and (3.52). We have,

$$z_{12n-1} = (\gamma_1\gamma_3\gamma_5\gamma_7\gamma_9\gamma_{11})^n z_{-1}, \tag{3.57}$$
$$z_{12n} = (\gamma_0\gamma_2\gamma_4\gamma_6\gamma_8\gamma_{10})^n z_0, \tag{3.58}$$

From (3.57) and (3.58), and the equations (3.41)-(3.50), we get

$$z_{12n+1} = \gamma_0(\gamma_1\gamma_3\gamma_5\gamma_7\gamma_9\gamma_{11})^n z_{-1},$$
$$z_{12n+2} = \gamma_1(\gamma_0\gamma_2\gamma_4\gamma_6\gamma_8\gamma_{10})^n z_0,$$
$$z_{12n+3} = \gamma_0\gamma_2(\gamma_1\gamma_3\gamma_5\gamma_7\gamma_9\gamma_{11})^n z_{-1},$$
$$z_{12n+4} = \gamma_1\gamma_3(\gamma_0\gamma_2\gamma_4\gamma_6\gamma_8\gamma_{10})^n z_0,$$
$$z_{12n+5} = \gamma_0\gamma_2\gamma_4(\gamma_1\gamma_3\gamma_5\gamma_7\gamma_9\gamma_{11})^n z_{-1},$$
$$z_{12n+6} = \gamma_1\gamma_3\gamma_5(\gamma_0\gamma_2\gamma_4\gamma_6\gamma_8\gamma_{10})^n z_0,$$
$$z_{12n+7} = \gamma_0\gamma_2\gamma_4\gamma_6(\gamma_1\gamma_3\gamma_5\gamma_7\gamma_9\gamma_{11})^n z_{-1},$$
$$z_{12n+8} = \gamma_1\gamma_3\gamma_5\gamma_7(\gamma_0\gamma_2\gamma_4\gamma_6\gamma_8\gamma_{10})^n z_0,$$
$$z_{12n+9} = \gamma_0\gamma_2\gamma_4\gamma_6\gamma_8(\gamma_1\gamma_3\gamma_5\gamma_7\gamma_9\gamma_{11})^n z_{-1},$$

$$z_{12n+10} = \gamma_1\gamma_3\gamma_5\gamma_7\gamma_9(\gamma_0\gamma_2\gamma_4\gamma_6\gamma_8\gamma_{10})^n z_0,$$

In summary, we have the following result.

Theorem 2. *Let $(x_n, y_n, z_n)_{n\geq -1}$ be a solution of System (3.1), then for $n \geq 0$, we have*

$$\begin{aligned}
x_{12n-1} &= (\alpha_0\alpha_2\alpha_4\alpha_6\alpha_8\alpha_{10})^n x_{-1},\\
x_{12n} &= (\alpha_1\alpha_3\alpha_5\alpha_7\alpha_9\alpha_{11})^n x_0,\\
x_{12n+1} &= \alpha_0(\alpha_0\alpha_2\alpha_4\alpha_6\alpha_8\alpha_{10})^n x_{-1},\\
x_{12n+2} &= \alpha_1(\alpha_1\alpha_3\alpha_5\alpha_7\alpha_9\alpha_{11})^n x_0,\\
x_{12n+3} &= \alpha_0\alpha_2(\alpha_0\alpha_2\alpha_4\alpha_6\alpha_8\alpha_{10})^n x_{-1},\\
x_{12n+4} &= \alpha_1\alpha_3(\alpha_1\alpha_3\alpha_5\alpha_7\alpha_9\alpha_{11})^n x_0,\\
x_{12n+5} &= \alpha_0\alpha_2\alpha_4(\alpha_0\alpha_2\alpha_4\alpha_6\alpha_8\alpha_{10})^n x_{-1},\\
x_{12n+6} &= \alpha_1\alpha_3\alpha_5(\alpha_1\alpha_3\alpha_5\alpha_7\alpha_9\alpha_{11})^n x_0,\\
x_{12n+7} &= \alpha_0\alpha_2\alpha_4\alpha_6(\alpha_0\alpha_2\alpha_4\alpha_6\alpha_8\alpha_{10})^n x_{-1},\\
x_{12n+8} &= \alpha_1\alpha_3\alpha_5\alpha_7(\alpha_1\alpha_3\alpha_5\alpha_7\alpha_9\alpha_{11})^n x_0,\\
x_{12n+9} &= \alpha_0\alpha_2\alpha_4\alpha_6\alpha_8(\alpha_0\alpha_2\alpha_4\alpha_6\alpha_8\alpha_{10})^n x_{-1},\\
x_{12n+10} &= \alpha_1\alpha_3\alpha_5\alpha_7\alpha_9(\alpha_1\alpha_3\alpha_5\alpha_7\alpha_9\alpha_{11})^n x_0,
\end{aligned}$$

$$\begin{aligned}
y_{12n-1} &= (\beta_0\beta_2\beta_4\beta_6\beta_8\beta_{10})^n y_{-1},\\
y_{12n} &= (\beta_1\beta_3\beta_5\beta_7\beta_9\beta_{11})^n y_0,\\
y_{12n+1} &= \beta_0(\beta_0\beta_2\beta_4\beta_6\beta_8\beta_{10})^n y_{-1},\\
y_{12n+2} &= \beta_1(\beta_1\beta_3\beta_5\beta_7\beta_9\beta_{11})^n y_0,\\
y_{12n+3} &= \beta_0\beta_2(\beta_0\beta_2\beta_4\beta_6\beta_8\beta_{10})^n y_{-1},\\
y_{12n+4} &= \beta_1\beta_3(\beta_1\beta_3\beta_5\beta_7\beta_9\beta_{11})^n y_0,\\
y_{12n+5} &= \beta_0\beta_2\beta_4(\beta_0\beta_2\beta_4\beta_6\beta_8\beta_{10})^n y_{-1},\\
y_{12n+6} &= \beta_1\beta_3\beta_5(\beta_1\beta_3\beta_5\beta_7\beta_9\beta_{11})^n y_0,\\
y_{12n+7} &= \beta_0\beta_2\beta_4\beta_6(\beta_0\beta_2\beta_4\beta_6\beta_8\beta_{10})^n y_{-1},\\
y_{12n+8} &= \beta_1\beta_3\beta_5\beta_7(\beta_1\beta_3\beta_5\beta_7\beta_9\beta_{11})^n y_0,\\
y_{12n+9} &= \beta_0\beta_2\beta_4\beta_6\beta_8(\beta_0\beta_2\beta_4\beta_6\beta_8\beta_{10})^n y_{-1},\\
y_{12n+10} &= \beta_1\beta_3\beta_5\beta_7\beta_9(\beta_1\beta_3\beta_5\beta_7\beta_9\beta_{11})^n y_0,
\end{aligned}$$

$$\begin{aligned}
z_{12n-1} &= (\gamma_1\gamma_3\gamma_5\gamma_7\gamma_9\gamma_{11})^n z_{-1},\\
z_{12n} &= (\gamma_0\gamma_2\gamma_4\gamma_6\gamma_8\gamma_{10})^n z_0,\\
z_{12n+1} &= \gamma_0(\gamma_1\gamma_3\gamma_5\gamma_7\gamma_9\gamma_{11})^n z_{-1},
\end{aligned}$$

$$
\begin{aligned}
z_{12n+2} &= \gamma_1(\gamma_0\gamma_2\gamma_4\gamma_6\gamma_8\gamma_{10})^n z_0,\\
z_{12n+3} &= \gamma_0\gamma_2(\gamma_1\gamma_3\gamma_5\gamma_7\gamma_9\gamma_{11})^n z_{-1},\\
z_{12n+4} &= \gamma_1\gamma_3(\gamma_0\gamma_2\gamma_4\gamma_6\gamma_8\gamma_{10})^n z_0,\\
z_{12n+5} &= \gamma_0\gamma_2\gamma_4(\gamma_1\gamma_3\gamma_5\gamma_7\gamma_9\gamma_{11})^n z_{-1},\\
z_{12n+6} &= \gamma_1\gamma_3\gamma_5(\gamma_0\gamma_2\gamma_4\gamma_6\gamma_8\gamma_{10})^n z_0,\\
z_{12n+7} &= \gamma_0\gamma_2\gamma_4\gamma_6(\gamma_1\gamma_3\gamma_5\gamma_7\gamma_9\gamma_{11})^n z_{-1},\\
z_{12n+8} &= \gamma_1\gamma_3\gamma_5\gamma_7(\gamma_0\gamma_2\gamma_4\gamma_6\gamma_8\gamma_{10})^n z_0,\\
z_{12n+9} &= \gamma_0\gamma_2\gamma_4\gamma_6\gamma_8(\gamma_1\gamma_3\gamma_5\gamma_7\gamma_9\gamma_{11})^n z_{-1},\\
z_{12n+10} &= \gamma_1\gamma_3\gamma_5\gamma_7\gamma_9(\gamma_0\gamma_2\gamma_4\gamma_6\gamma_8\gamma_{10})^n z_0,
\end{aligned}
$$

where

$$
\begin{aligned}
\alpha_0 &= \frac{a_1 y_{-1}}{a x_{-1} + b y_{-1} + c z_{-1}},\\
\alpha_1 &= \frac{a_1 y_0}{a x_0 + b y_0 + c z_0},\\
\alpha_2 &= \frac{a_1 a_2 y_{-1} z_{-1}}{a a_1 x_{-1} y_{-1} + b a_2 y_{-1} z_{-1} + c a_3 x_{-1} z_{-1}},\\
\alpha_3 &= \frac{a_1 a_2 y_0 z_0}{a a_1 x_0 y_0 + b a_2 y_0 z_0 + c a_3 x_0 z_0},\\
\alpha_4 &= \frac{a_1 a_2^2 a_3 z_{-1}}{a a_1^2 a_2 y_{-1} + b a_2^2 a_3 z_{-1} + c a_1 a_3^2 x_{-1}},\\
\alpha_5 &= \frac{a_1 a_2^2 a_3 z_0}{a a_1^2 a_2 y_0 + b a_2^2 a_3 z_0 + c a_1 a_3^2 x_0},\\
\alpha_6 &= \frac{a_1 a_2^2 a_3^2 x_{-1} z_{-1}}{a a_1^2 a_2^2 y_{-1} z_{-1} + b a_2^2 a_3^2 x_{-1} z_{-1} + c a_1^2 a_3^2 x_{-1} y_{-1}},\\
\alpha_7 &= \frac{a_1 a_2^2 a_3^2 x_0 z_0}{a a_1^2 a_2^2 y_0 z_0 + b a_2^2 a_3^2 x_0 z_0 + c a_1^2 a_3^2 x_0 y_0},\\
\alpha_8 &= \frac{a_1 a_2 a_3^2 x_{-1}}{a a_1 a_2^2 z_{-1} + b a_2 a_3^2 x_{-1} + c a_1^2 a_3 y_{-1}},\\
\alpha_9 &= \frac{a_1 a_2 a_3^2 x_0}{a a_1 a_2^2 z_0 + b a_2 a_3^2 x_0 + c a_1^2 a_3 y_0},\\
\alpha_{10} &= \frac{a_1 a_3 x_{-1} y_{-1}}{a a_2 x_{-1} z_{-1} + b a_3 x_{-1} y_{-1} + c a_1 y_{-1} z_{-1}},\\
\alpha_{11} &= \frac{a_1 a_3 x_0 y_0}{a a_2 x_0 z_0 + b a_3 x_0 y_0 + c a_1 y_0 z_0},
\end{aligned}
$$

$$\beta_0 = \frac{a_2 z_{-1}}{a x_{-1} + b y_{-1} + c z_{-1}},$$
$$\beta_1 = \frac{a_2 z_0}{a x_0 + b y_0 + c z_0},$$
$$\beta_2 = \frac{a_2 a_3 x_{-1} z_{-1}}{a a_1 x_{-1} y_{-1} + b a_2 y_{-1} z_{-1} + c a_3 x_{-1} z_{-1}},$$
$$\beta_3 = \frac{a_2 a_3 x_0 z_0}{a a_1 x_0 y_0 + b a_2 y_0 z_0 + c a_3 x_0 z_0},$$
$$\beta_4 = \frac{a_1 a_2 a_3^2 x_{-1}}{a a_1^2 a_2 y_{-1} + b a_2^2 a_3 z_{-1} + c a_1 a_3^2 x_{-1}},$$
$$\beta_5 = \frac{a_1 a_2 a_3^2 x_0}{a a_1^2 a_2 y_0 + b a_2^2 a_3 z_0 + c a_1 a_3^2 x_0},$$
$$\beta_6 = \frac{a_1^2 a_2 a_3^2 x_{-1} y_{-1}}{a a_1^2 a_2^2 y_{-1} z_{-1} + b a_2^2 a_3^2 x_{-1} z_{-1} + c a_1^2 a_3^2 x_{-1} y_{-1}},$$
$$\beta_7 = \frac{a_1^2 a_2 a_3^2 x_0 y_0}{a a_1^2 a_2^2 y_0 z_0 + b a_2^2 a_3^2 x_0 z_0 + c a_1^2 a_3^2 x_0 y_0},$$
$$\beta_8 = \frac{a_1^2 a_2 a_3 y_{-1}}{a a_1 a_2^2 z_{-1} + b a_2 a_3^2 x_{-1} + c a_1^2 a_3 y_{-1}},$$
$$\beta_9 = \frac{a_1^2 a_2 a_3 y_0}{a a_1 a_2^2 z_0 + b a_2 a_3^2 x_0 + c a_1^2 a_3 y_0},$$
$$\beta_{10} = \frac{a_1 a_2 y_{-1} z_{-1}}{a a_2 x_{-1} z_{-1} + b a_3 x_{-1} y_{-1} + c a_1 y_{-1} z_{-1}},$$
$$\beta_{11} = \frac{a_1 a_2 y_0 z_0}{a a_2 x_0 z_0 + b a_3 x_0 y_0 + c a_1 y_0 z_0},$$

$$\gamma_0 = \frac{a_3 x_{-1}}{a x_{-1} + b y_{-1} + c z_{-1}},$$
$$\gamma_1 = \frac{a_3 x_0}{a x_0 + b y_0 + c z_0},$$
$$\gamma_2 = \frac{a_1 a_3 x_{-1} y_{-1}}{a a_1 x_{-1} y_{-1} + b a_2 y_{-1} z_{-1} + c a_3 x_{-1} z_{-1}},$$
$$\gamma_3 = \frac{a_1 a_3 x_0 y_0}{a a_1 x_0 y_0 + b a_2 y_0 z_0 + c a_3 x_0 z_0},$$
$$\gamma_4 = \frac{a_1^2 a_2 a_3 y_{-1}}{a a_1^2 a_2 y_{-1} + b a_2^2 a_3 z_{-1} + c a_1 a_3^2 x_{-1}},$$
$$\gamma_5 = \frac{a_1^2 a_2 a_3 y_0}{a a_1^2 a_2 y_0 + b a_2^2 a_3 z_0 + c a_1 a_3^2 x_0},$$

$$\gamma_6 = \frac{a_1^2 a_2^2 a_3 y_{-1} z_{-1}}{a a_1^2 a_2^2 y_{-1} z_{-1} + b a_2^2 a_3^2 x_{-1} z_{-1} + c a_1^2 a_3^2 x_{-1} y_{-1}},$$

$$\gamma_7 = \frac{a_1^2 a_2^2 a_3 y_0 z_0}{a a_1^2 a_2^2 y_0 z_0 + b a_2^2 a_3^2 x_0 z_0 + c a_1^2 a_3^2 x_0 y_0},$$

$$\gamma_8 = \frac{a_1 a_2^2 a_3 z_{-1}}{a a_1 a_2^2 z_{-1} + b a_2 a_3^2 x_{-1} + c a_1^2 a_3 y_{-1}},$$

$$\gamma_9 = \frac{a_1 a_2^2 a_3 z_0}{a a_1 a_2^2 z_0 + b a_2 a_3^2 x_0 + c a_1^2 a_3 y_0},$$

$$\gamma_{10} = \frac{a_2 a_3 x_{-1} z_{-1}}{a a_2 x_{-1} z_{-1} + b a_3 x_{-1} y_{-1} + c a_1 y_{-1} z_{-1}},$$

$$\gamma_{11} = \frac{a_2 a_3 z_0 x_0}{a a_2 x_0 z_0 + b a_3 x_0 y_0 + c a_1 y_0 z_0}.$$

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