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CHARACTERIZATIONS OF TRACE CLASS SEMICLASSICAL FOURIER INTEGRAL OPERATORS

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Abstract. In this work we study a characterizations of trace class and nuclearity of semiclassical Fourier integral operators associated with a particular class of symbols $S_{1,0}^m(\mathbb{R}^{2n})$ on Lebesgue spaces.

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1. INTRODUCTION

A Fourier integral operator or FIO in short is a singular operator defined by

$$I_{a,\phi}u(x) = \int_{\mathbb{R}^n} e^{i\phi(x,\xi)} a(x,\xi) \hat{u}(\xi) d\xi, \quad u \in \mathcal{S}(\mathbb{R}^n),$$

where $\mathcal{S}(\mathbb{R}^n)$ is the Schwartz space, i.e. the space of all rapidly decreasing smooth functions. Moreover $\hat{u}$ denote the Fourier transformation of u (we also denote $\hat{u}$ via $\mathcal{F}(u)$). Additionally, $a(x,\xi)$ and $\phi(x,\xi)$ are smooth functions called respectively the amplitude and the phase function.

The theory of FIOs has a lengthy history of study. The regularity of these operators in functional spaces has been studied extensively and there is a huge body of evidence regarding regularity, such as in [8, 13].

According to the theory of FIO developed by Hörmander [13], the phase functions in $C^\infty(\mathbb{R}^n \times \mathbb{R}^n \setminus 0)$ homogenous of degree 1 in the frequency variable ξ and with non-vanishing determinant of the mixed Hessian matrix (i.e. $\phi \in \Phi(\mathbb{R}^{2n})$), while the symbols in $C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ such that

$$\left| \partial_\xi^\alpha \partial_x^\beta a(x, \xi) \right| \leq C_{\alpha,\beta} \langle \xi \rangle^{m-\rho|\alpha|+\delta|\beta|}$$

(i.e. $a(x, \xi) \in S_{\rho,\delta}^m(\mathbb{R}^{2n})$), was initiated in the classical paper of L. Hörmander [13]. Moreover, G. Eskin [9] (in the case $a \in S_{1,0}^0$) and Hörmander [13] (in the case $a \in$

$S_{\rho,1-\rho}^0$, $\frac{1}{2} < \rho \leq 1$) treated the local L^2 boundedness of FIO with non-degenerate phase functions.

Gala-Liu-Ragusa [10], Kheloufi-Ikassoulene [14], Ragusa [18–20] and Seeger-Sogge-Stein [21] have extended local L^2 -boundedness results to L^p regularity (for $1 \leq p \leq \infty$) since the 1970s, spurred by applications in microlocal analysis and hyperbolic partial differential equations.

In Asada-Fujiwara [7], the initial result was obtained using Fourier integral operators associated to symbols in $S_{0,0}^0$ and inhomogeneous phase functions.

This conclusion was extended to a general class of symbols in $S_{\rho,\delta}^m$ that met certain criteria, as shown in [3–5, 11, 12].

In the semiclassical case an h -Fourier integral operator has the following form

$$I_h(a, \phi)u(x) = (2\pi h)^{-n} \iint e^{ih^{-1}[\phi(x,\xi) - \langle y, \xi \rangle]} a(x, \xi) u(y) dy d\xi, \quad u \in \mathcal{S}(\mathbb{R}^n), \quad (1.1)$$

where $h \in]0, h_0]$ is a semiclassical parameter.

A fundamental principle of h -Fourier integral operator is that linear differential operators known as semi classical Schrodinger operators are used to represent quantum systems including atoms, molecules, solids, and, to a certain extent, nuclei and even stars. For a description of this theory, we point the reader toward the widely used text of Landau and Lifshitz [15]. The Schrodinger operator for a quantum system is the linear partial differential operator

$$H = -\frac{h^2}{2m}\Delta + V,$$

The Laplace operator Δ is the second-order differential operator that, in Cartesian coordinates on $\mathbb{R}^n$, is given by

$$\Delta = \sum_{k=0}^n \frac{\partial^2}{\partial x_i^2}.$$

The constant m is the reduced mass of the system, and the semiclassical parameter h is called Planck's constant. The real-valued function V is called the potential.

In comparison to the research of FIOs, there has been less activity for h -Fourier integral operators concerning the study of the corresponding r -nuclearity properties.

Many results of boundedness and compactness (on L^p , on Bezaov spaces and on Bessel potential spaces) are obtained for semiclassical Fourier integral operators with symbols introduced by Hörmander and non degenerate and homogeneous phase functions, see [1, 2].

The lack of r -nuclearity results for h -FIOs in semiclassical analysis literature prompted this study.

The aim of this work is to extend results obtained in [6] for a general class of semiclassical Fourier integral operators. Our investigation is to study characterizations of nuclear h -FIOs on Lebesgue spaces with the same conditions on the phase functions and on the symbols are kept.

Now let us go over this article's structure. We introduce the pertinent preliminaries and notations that will be used throughout the work in the second section.

Following that, we'll go over some fundamental theorems from the theory of r -nuclear operators and trace class operators (for $r = 1$), which will serve as the starting point for our research. Finally, the last section is devoted to prove our main result.

2. NOTATIONS AND PRELIMINARIES

We assume $n \in \mathbb{N}$ throughout the whole paper unless otherwise noted. In particular $n \neq 0$. For all $x, \xi \in \mathbb{R}^n$ and $A \subset \mathbb{R}^n$ we define

$$\langle x, \xi \rangle = \sum_{j=0}^n x_j \xi_j \quad \text{and} \quad \widehat{d_h \xi} = (2\pi h)^{-n} d\xi.$$

Moreover, let us recall weight and characteristic functions defined by

$$\langle \xi \rangle = \left(1 + |\xi|^2\right)^{1/2} \quad \text{and} \quad 1_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{else.} \end{cases}$$

Furthermore, we define the semiclassical Fourier transform $\mathcal{F}_h$ and its inverse $\mathcal{F}_h^{-1}$ by

$$\mathcal{F}_h f(\xi) = \int_{\mathbb{R}^n} e^{-ih^{-1}\langle x, \xi \rangle} f(x) dx \quad \text{and} \quad \mathcal{F}_h^{-1} f(x) = \int_{\mathbb{R}^n} e^{ih^{-1}\langle x, \xi \rangle} f(\xi) \widehat{d_h \xi},$$

where $f \in \mathcal{S}(\mathbb{R}^n)$.

Additionally we scale partial derivatives with respect to a variable $x \in \mathbb{R}^n$ with the factor $-i$ are denoted by

$$D_x^\alpha = (-i)^{|\alpha|} \partial_x^\alpha = (-i)^{|\alpha|} \partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n},$$

where $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ is a multi-index and $|\alpha| = \sum_{j=1}^n \alpha_j$ is the length of α .

We note that the dual space of a topological vector space E is denoted by E' .

Finally, let us denote by $B(x, r)$ the Euclidean ball centered at x with radius $r > 0$, and by $m(B(x, r))$ the Lebesgue measure of $B(x, r)$.

To present our main results, we will need the notion of trace class operators.

Definition 1. Let X and Y be complex Banach spaces and let $T: D(T) \subset X \rightarrow Y$ (where $D(T)$ is the domain of T) be a linear operator. Suppose that we can find sequences $\{x'_k\}_{k \geq 0}$ in the dual space X' and $\{y_k\}_{k \geq 0}$ in Y such that

$$\sum_{k=0}^{\infty} \|x'_k\|_{X'}^r \|y_k\|_Y^r < \infty,$$

and

$$Tx = \sum_{k=0}^{\infty} x'_k(x) y_k, \quad \forall x \in X.$$

Then we call $T: D(T) \subset X \rightarrow Y$ a r -nuclear operator.

The class of r -nuclear operators is usually endowed with the natural semi-norm

$$q_r(T) = \inf \left\{ \left(\sum_{k=0}^{\infty} \|x'_k\|_{X'}^r \|y_k\|_Y^r \right)^{1/r} ; T = \sum_{k=0}^{\infty} x'_k \otimes y_k \right\}.$$

Remark 1. When $r = 1$, q_1 is a norm and we obtain the ideal of trace class operators.

Proposition 1. *For the case of Hilbert spaces H , the set of r -nuclear operators is identical to with the Schatten-von Neumann class of order r .*

Proof. See Pietsch [16, 17]. □

The following theorem will be required to investigate the trace class properties of the h -Fourier integral operator.

Theorem 1. *Let (X_1, μ_1) and (X_2, μ_2) be two σ -finite measure spaces. Let us consider $p_1, p_2 \in [1, \infty[$, and let p'_1, p'_2 be conjugates of p_1, p_2 respectively (i.e. $\frac{1}{p_i} + \frac{1}{p'_i} = 1 \forall i \in \{1, 2\}$). An operator $T: L^{p_1}(X_1, \mu_1) \rightarrow L^{p_2}(X_2, \mu_2)$ is trace class if, and only if, there exist sequences $\{u_k\}_{k \geq 0}$ in $L^{p_2}(X_2, \mu_2)$, and $\{v_k\}_{k \geq 0}$ in $L^{p'_1}(X_1, \mu_1)$, such that*

$$\sum_{k=0}^{\infty} \|u_k\|_{L^{p_2}} \|v_k\|_{L^{p'_1}} < \infty,$$

and

$$Tf(x) = \int_{X_1} \sum_{k=0}^{\infty} u_k(x) v_k(y) f(y) dy, \quad \forall f \in L^{p_1}(X_1, \mu_1).$$

Proof. Let be T a trace class operator from L^{p_1} to L^{p_2} . Then there are sequences $\{u_k\}_{k \geq 0}$ in L^{p_2} and $\{v_k\}_{k \geq 0}$ in $L^{p'_1}$ so that

$$\sum_{k=0}^{\infty} \|u_k\|_{L^{p_2}} \|v_k\|_{L^{p'_1}} < \infty$$

and

$$Tf = \sum_{k=0}^{\infty} \langle f, v_k \rangle u_k.$$

Now

$$Tf = \sum_{k=0}^{\infty} \langle f, v_k \rangle u_k = \sum_{k=0}^{\infty} \left(\int_{X_1} f(y) v_k(y) dy \right) u_k,$$

where the sum converges in L^{p_2} norm. There exist two subsequences $\{\tilde{u}_k\}_{k \geq 0}$ and $\{\tilde{v}_k\}_{k \geq 0}$ of $\{u_k\}_{k \geq 0}$ and $\{v_k\}_{k \geq 0}$ respectively, so for all $x \in X_2$

$$Tf(x) = \sum_{k=0}^{\infty} \langle f, \tilde{v}_k \rangle \tilde{u}_k(x) = \sum_{k=0}^{\infty} \left(\int_{X_1} f(y) \tilde{v}_k(y) dy \right) \tilde{u}_k(x).$$

Since $\{\tilde{u}_k\}_{k \geq 0}$ and $\{\tilde{v}_k\}_{k \geq 0}$ satisfy

$$\sum_{k=0}^{\infty} \|\tilde{u}_k\|_{L^{p_2}} \|\tilde{v}_k\|_{L^{p'_1}} < \infty,$$

then

$$\begin{aligned} T f(x) &= \sum_{k=0}^{\infty} \langle f, \tilde{v}_k \rangle \tilde{u}_k(x) = \lim_{n \rightarrow \infty} \sum_{k=0}^n \left(\int_{X_1} f(y) \tilde{v}_k(y) dy \right) \tilde{u}_k(x) \\ &= \lim_{n \rightarrow \infty} \int_{X_1} \left(\sum_{k=0}^n f(y) \tilde{u}_k(x) \tilde{v}_k(y) \right) dy = \int_{X_1} \left(\sum_{k=0}^{\infty} \tilde{u}_k(x) \tilde{v}_k(y) \right) f(y) dy, \end{aligned}$$

for all $x \in X_2$. The proof of the "if" part is complete. Thus, let us consider the "only if" part.

We assume that there exists sequences $\{u_k\}_{k \geq 0}$ in L^{p_2} and $\{v_k\}_{k \geq 0}$ in $L^{p'_1}$ such that

$$\sum_{k=0}^{\infty} \|u_k\|_{L^{p_2}} \|v_k\|_{L^{p'_1}} < \infty.$$

So for all $f \in L^{p_1}$, we have

$$\begin{aligned} \int_{X_1} \left(\sum_{k=0}^{\infty} u_k(x) v_k(y) \right) f(y) dy &= \lim_{n \rightarrow \infty} \int_{X_1} \left(\sum_{k=0}^n f(y) u_k(x) v_k(y) \right) dy \\ &= \lim_{n \rightarrow \infty} \sum_{k=0}^n \left(\int_{X_1} f(y) v_k(y) dy \right) u_k(x) \\ &= \sum_{k=0}^{\infty} \left(\int_{X_1} f(y) v_k(y) dy \right) u_k(x) \\ &= \sum_{k=0}^{\infty} \langle f, v_k \rangle u_k(x) = T f(x). \end{aligned}$$

We put

$$g_n = \sum_{k=0}^n \langle f, \tilde{v}_k \rangle \tilde{u}_k,$$

then $\{g_n\}_{n \geq 0}$ is a sequence in L^{p_2} and

$$|g_n(x)| \leq h_n(x) \leq h(x),$$

where for all $x \in X_2$

$$h_n(x) = \|f\|_{L^{p_1}} \sum_{k=0}^n \|v_k\|_{L^{p'_1}} |u_k(x)|, \quad \forall n \in \mathbb{N}$$

and

$$h(x) = \lim_{n \rightarrow \infty} h_n(x).$$

Since we have

$$\|h_n\|_{L^{p_2}} \leq \|h\|_{L^{p_2}} \leq \|f\|_{L^{p_1}} \sum_{k=0}^{\infty} \|v_k\|_{L^{p'_1}} \|u_k\|_{L^{p_2}} < \infty,$$

then $h \in L^{p_2}$ is a limit of $\{h_n\}_{n \geq 0}$ of L^{p_2} -functions.

So by Lebesgue dominated convergence theorem we get that $\{g_n\}_{n \geq 0}$ converge to Tf in L^{p_2} . $\square$

Corollary 1. *When $p_1 = p_2$ and $\mu_1 = \mu_2 = \mu$, the trace of T is given by*

$$\text{Tr}(T) = \int_{X_1} \sum_{k=0}^{\infty} u_k(x) v_k(x) d\mu.$$

3. CHARACTERIZATION OF TRACE CLASS h -FOURIER INTEGRAL OPERATORS

In this section we prove our main result concerning the trace class property of the $I_h(a, \phi)$ defined as in (1.1). First let us observe that $I_h(a, \phi)$ is a integral operator of distribution kernel

$$k_h(x, \xi) = \int_{\mathbb{R}^n} e^{ih^{-1}[\phi(x, \xi) - \langle y, \xi \rangle]} a(x, \xi) \widehat{d_h \xi}, \quad \text{for all } x, \xi \in \mathbb{R}^n.$$

Theorem 2. *Let $p_1 \in [2, \infty[$ and $p_2 \in [1, \infty[$ and let $I_h(a, \phi)$ defined by*

$$I_h(a, \phi)f(x) = \iint e^{ih^{-1}[\phi(x, \xi) - \langle y, \xi \rangle]} a(x, \xi) f(y) dy \widehat{d_h \xi},$$

where $a \in S_{1,0}^m(\mathbb{R}^{2n})$ and $\phi \in \Phi(\mathbb{R}^{2n})$. Then $I_h(a, \phi): L^{p_1}(\mathbb{R}^n) \rightarrow L^{p_2}(\mathbb{R}^n)$ is a trace class operator if and only if the symbol a admits a decomposition of the form

$$a(x, \xi) = e^{-ih^{-1}\phi(x, \xi)} \sum_{k=0}^{\infty} u_{h,k}(x) \mathcal{F}_h^{-1} v_{h,k}(\xi) \quad (3.1)$$

where $\{u_{h,k}\}_{k \geq 0}$ and $\{v_{h,k}\}_{k \geq 0}$ are sequences of functions satisfying

$$\sum_{k=0}^{\infty} \|u_{h,k}\|_{L^{p_2}} \|v_{h,k}\|_{L^{p'_1}} < \infty. \quad (3.2)$$

Proof. Let us assume that $I_h(a, \phi): L^{p_1}(\mathbb{R}^n) \rightarrow L^{p_2}(\mathbb{R}^n)$ is a trace class operator. Then by using the theorem 1, there exist sequences $\{u_{h,k}\}_{k \geq 0}$ in L^{p_2} and $\{v_{h,k}\}_{k \geq 0}$ in $L^{p'_1}$ satisfying

$$I_h(a, \phi)\varphi(x) = \int_{\mathbb{R}^n} \left(\sum_{k=0}^{\infty} u_{h,k}(x) v_{h,k}(y) \right) \varphi(y) dy, \quad \forall \varphi \in L^{p_1}$$

with

$$\sum_{k=0}^{\infty} \|u_{h,k}\|_{L^{p_2}} \|v_{h,k}\|_{L^{p'_1}} < \infty.$$

Let us choose $\xi_0 \in \mathbb{R}^n$ and $r > 0$, then we define a useful function by

$$\lambda_{\xi_0, r}(x) = \frac{1_{B(\xi_0, r)}(x)}{m(B(\xi_0, r))}$$

By using the Hausdorff-Young inequality we have

$$\|\mathcal{F}_h^{-1}(\lambda_{\xi_0, r})\|_{L^{p_1}} \leq \|\lambda_{\xi_0, r}\|_{L^{p_1'}} = 1.$$

So, for every $\xi_0 \in \mathbb{R}^n$ and $r > 0$, the function $\mathcal{F}_h^{-1}(\lambda_{\xi_0, r}) \in L^{p_1}(\mathbb{R}^n)$, and we get,

$$I_h(a, \phi)(\mathcal{F}_h^{-1}(\lambda_{\xi_0, r}))(x) = \int_{\mathbb{R}^n} \left(\sum_{k=0}^{\infty} u_{h,k}(x) v_{h,k}(y) \right) \mathcal{F}_h^{-1}(\lambda_{\xi_0, r})(y) dy,$$

$$\lim_{r \rightarrow 0^+} \mathcal{F}_h^{-1}(\lambda_{\xi_0, r})(x) = \lim_{r \rightarrow 0^+} \frac{1}{m(B(\xi_0, r))} \int_{B(\xi_0, r)} e^{ih^{-1}\langle x, \xi \rangle} d\xi = e^{ih^{-1}\langle x, \xi_0 \rangle},$$

The dominated convergence theorem now gives us

$$\begin{aligned} \lim_{r \rightarrow 0^+} I_h(a, \phi) \mathcal{F}_h^{-1}(\lambda_{\xi_0, r})(x) &= I_h(a, \phi) \left(\lim_{r \rightarrow 0^+} \mathcal{F}_h^{-1}(\lambda_{\xi_0, r})(x) \right) \\ &= \int_{\mathbb{R}^n} \left(\sum_{k=0}^{\infty} u_{h,k}(x) v_{h,k}(y) \right) e^{ih^{-1}\langle y, \xi_0 \rangle} dy = \sum_{k=0}^{\infty} u_{h,k}(x) (\mathcal{F}_h^{-1} v_{h,k})(\xi_0). \end{aligned}$$

In fact, for all $x \in \mathbb{R}^n$,

$$\begin{aligned} &\left| \left(\sum_{k=0}^{\infty} u_{h,k}(x) v_{h,k}(y) \right) \mathcal{F}_h^{-1}(\lambda_{\xi_0, r})(y) \right| \\ &= \left| \left(\sum_{k=0}^{\infty} u_{h,k}(x) v_{h,k}(y) \right) \frac{1}{m(B(\xi_0, r))} \int_{B(\xi_0, r)} e^{ih^{-1}\langle x, \xi \rangle} d\xi \right| \\ &\leq \left| \sum_{k=0}^{\infty} u_{h,k}(x) v_{h,k}(y) \right| \leq |k_h(x, y)|. \end{aligned}$$

Now, by the dominated convergence theorem, we have

$$\begin{aligned} \lim_{r \rightarrow 0^+} I_h(a, \phi) (\mathcal{F}_h^{-1}(\lambda_{\xi_0, r}))(x) &= \lim_{r \rightarrow 0^+} \int_{\mathbb{R}^n} k_h(x, y) \mathcal{F}_h^{-1}(\lambda_{\xi_0, r})(y) dy \\ &= \int_{\mathbb{R}^n} k_h(x, y) \lim_{r \rightarrow 0^+} \mathcal{F}_h^{-1}(\lambda_{\xi_0, r})(y) dy = \int_{\mathbb{R}^n} e^{ih^{-1}\langle y, \xi_0 \rangle} dy \\ &= \lim_{l \rightarrow \infty} \int_{|y| \leq l} e^{ih^{-1}\langle y, \xi_0 \rangle} dy \\ &= \lim_{l \rightarrow \infty} \int_{\mathbb{R}^n} \left(\sum_{k=0}^{\infty} u_{h,k}(x) v_{h,k}(y) \right) e^{ih^{-1}\langle y, \xi_0 \rangle} 1_{B(0, l)}(y) dy \end{aligned}$$

$$\begin{aligned}
&= \lim_{l \rightarrow \infty} \int_{\mathbb{R}^n} \left(\lim_{m \rightarrow \infty} \sum_{k=0}^m u_{h,k}(x) v_{h,k}(y) \right) e^{ih^{-1}\langle y, \xi_0 \rangle} 1_{B(0,l)}(y) dy \\
&= \lim_{l \rightarrow \infty} \lim_{m \rightarrow \infty} \sum_{k=0}^m u_{h,k}(x) \int_{\mathbb{R}^n} v_{h,k}(y) e^{ih^{-1}\langle y, \xi_0 \rangle} 1_{B(0,l)}(y) dy \\
&= \lim_{l \rightarrow \infty} \lim_{m \rightarrow \infty} \sum_{k=0}^m u_{h,k}(x) \int_{|y| \leq l} v_{h,k}(y) e^{ih^{-1}\langle y, \xi_0 \rangle} dy \\
&= \sum_{k=0}^{\infty} u_{h,k}(x) \mathcal{F}_h^{-1} v_{h,k}(\xi_0).
\end{aligned}$$

On the other hand, if we compute $I_h(a, \phi) (\mathcal{F}_h^{-1}(\lambda_{\xi_0, r}))$, we have

$$\lim_{r \rightarrow 0^+} I_h(a, \phi) (\mathcal{F}_h^{-1}(\lambda_{\xi_0, r})) (x) = e^{ih^{-1}\phi(x, \xi_0)} a(x, \xi_0).$$

Consequently, we deduce the identity

$$e^{ih^{-1}\phi(x, \xi_0)} a(x, \xi_0) = \sum_{k=0}^{\infty} u_{h,k}(x) \mathcal{F}_h^{-1} v_{h,k}(\xi_0),$$

which in turn is equivalent to

$$a(x, \xi_0) = e^{-ih^{-1}\phi(x, \xi_0)} \sum_{k=0}^{\infty} u_{h,k}(x) \mathcal{F}_h^{-1} v_{h,k}(\xi_0).$$

We've now established the first half of the theorem. Now, if we suppose that the symbol a of $I_h(a, \phi)$ fulfills the decomposition formula (3.1) for fixed sequences $u_{h,k}$ in L^{p_2} and $v_{h,k}$ in $L^{p_1'}$ fulfilling (3.2), then from (1.1) we can write (in the sense of distributions)

$$\begin{aligned}
I_h(a, \phi) f(x) &= \int_{\mathbb{R}^n} e^{ih^{-1}\phi(x, \xi)} a(x, \xi) \mathcal{F}_h f(\xi) \widehat{d_h \xi} \\
&= \int_{\mathbb{R}^n} \sum_{k=0}^{\infty} u_{h,k}(x) \mathcal{F}_h^{-1} v_{h,k}(\xi) \mathcal{F}_h f(\xi) \widehat{d_h \xi} \\
&= \int_{\mathbb{R}^n} \sum_{k=0}^{\infty} u_{h,k}(x) \int_{\mathbb{R}^n} e^{ih^{-1}\langle y, \xi_0 \rangle} v_{h,k}(y) dy \mathcal{F}_h f(\xi) \widehat{d_h \xi} \\
&= \int_{\mathbb{R}^n} \left(\sum_{k=0}^{\infty} u_{h,k}(x) v_{h,k}(y) \right) \left(\int_{\mathbb{R}^n} e^{ih^{-1}\langle y, \xi_0 \rangle} v_{h,k}(y) \mathcal{F}_h f(\xi) \widehat{d_h \xi} \right) dy \\
&= \int_{\mathbb{R}^n} \left(\sum_{k=0}^{\infty} u_{h,k}(x) v_{h,k}(y) \right) f(y) dy.
\end{aligned}$$

So, by Theorem 1 we finish the proof. $\square$

Furthermore, as a result of Theorem 2, we gain the following result.

Corollary 2. *Let $I_h(a, \phi)$ be a h -Fourier integral operator. So, if $I_h(a, \phi): L^{p_1}(\mathbb{R}^n) \rightarrow L^{p_2}(\mathbb{R}^n)$ is a trace class operator, then $a \in L_x^{p_2} L_\xi^{p_1}(\mathbb{R}^{2n}) \cap L_\xi^{p_1} L_x^{p_2}(\mathbb{R}^{2n})$; this means that*

$$\|a\|_{L_\xi^{p_1} L_x^{p_2}} = \left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |a(x, \xi)|^{p_1} d\xi \right)^{\frac{p_2}{p_1}} dx \right)^{\frac{1}{p_2}} < \infty,$$

and

$$\|a\|_{L_x^{p_2} L_\xi^{p_1}} = \left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |a(x, \xi)|^{p_2} dx \right)^{\frac{p_1}{p_2}} d\xi \right)^{\frac{1}{p_1}} < \infty.$$

Proof. If $I_h(a, \phi): L^{p_1}(\mathbb{R}^n) \rightarrow L^{p_2}(\mathbb{R}^n)$ is a trace class operator, then Theorem 2 guarantees the decomposition

$$a(x, \xi) = e^{-ih^{-1}\phi(x, \xi)} \sum_{k=0}^{\infty} u_{h,k}(x) \mathcal{F}_h^{-1} v_{h,k}(\xi),$$

where $u_{h,k}(x)$ and $v_{h,k}(\xi)$ are sequences of functions satisfying

$$\sum_{k=0}^{\infty} \|u_{h,k}\|_{L^{p_2}} \|v_{h,k}\|_{L^{p_1'}} < \infty.$$

So, if we take the $L_x^{p_2}$ -norm, we have,

$$\begin{aligned} \|a(\cdot, \xi)\|_{L_x^{p_2}} &= \left\| e^{-ih^{-1}\phi(\cdot, \xi)} \sum_{k=0}^{\infty} u_{h,k}(\cdot) \mathcal{F}_h^{-1} v_{h,k}(\xi) \right\|_{L_x^{p_2}} \\ &= \left\| \sum_{k=0}^{\infty} u_{h,k}(x) \mathcal{F}_h^{-1} v_{h,k}(\xi) \right\|_{L_x^{p_2}} \leq \sum_{k=0}^{\infty} \|u_{h,k}\|_{L^{p_2}} |\mathcal{F}_h^{-1} v_{h,k}(\xi)|. \end{aligned}$$

Using the Hausdorff-Young inequality, we can now deduce that $\|\mathcal{F}_h^{-1} v_{h,k}\|_{L^{p_1}} \leq \|v_{h,k}\|_{L^{p_1'}}$. Consequently,

$$\begin{aligned} \|a(\cdot, \cdot)\|_{L_x^{p_2} L_\xi^{p_1}} &= \left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |a(x, \xi)|^{p_2} dx \right)^{\frac{p_1}{p_2}} d\xi \right)^{\frac{1}{p_1}} \leq \left\| \sum_{k=0}^{\infty} \|u_{h,k}\|_{L^{p_2}} |\mathcal{F}_h^{-1} v_{h,k}(\xi)| \right\|_{L_\xi^{p_1}} \\ &\leq \sum_{k=0}^{\infty} \|u_{h,k}\|_{L^{p_2}} \|\mathcal{F}_h^{-1} v_{h,k}\|_{L^{p_1}} \leq \sum_{k=0}^{\infty} \|u_{h,k}\|_{L^{p_2}} \|v_{h,k}\|_{L^{p_1'}} < \infty. \end{aligned}$$

In a similar vein, we can show that

$$\|a(\cdot, \cdot)\|_{L_\xi^{p_1} L_x^{p_2}} \leq \sum_{k=0}^{\infty} \|u_{h,k}\|_{L^{p_2}} \|v_{h,k}\|_{L^{p_1'}} < \infty.$$

As a result, we've completed the proof. $\square$

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WEYL'S CLASSIFICATION OF SINGULAR IMPULSIVE HAHN–STURM–LIOUVILLE EQUATIONS

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Abstract. In this work, Weyl's famous classification was made for singular impulsive Hahn–Sturm–Liouville equations.

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1. INTRODUCTION

Sturm–Liouville problems are one of the important problems in the theory of differential equations. When solving partial differential equations with the Fourier method, encountering such equations increases the importance of Sturm–Liouville problems. Such problems under various conditions are intensively investigated. Impulsive boundary conditions are one of them. There are many studies in the literature under such conditions ([2, 5–9, 12–17]). In 1910 Weyl published his famous work on Sturm–Liouville problems in the singular case [18]. With this study, he made the famous limit circle/limit point classification.

In 2018, Annaby et al. investigated the Hahn–Sturm–Liouville problems in the regular case by taking the Hahn derivative instead of the classical derivative in the Sturm–Liouville problems [4]. The Hahn derivative operator gives the opportunity to gather two different operators under one roof. These are the q -difference and forward difference operators. Recently in [16], the author proved a spectral expansion theorem by constructing the spectral function of the Hahn–Sturm–Liouville equation in the singular case under impulsive conditions.

In this study, the singular Hahn–Sturm–Liouville problems defined as

$$\left[-\frac{1}{q} D_{-\frac{\omega}{q}, \frac{1}{q}} D_{\omega, q} + r(\zeta) \right] y(\zeta) = \lambda y(\zeta), \quad \zeta \in (\omega_0, d) \cup (d, \infty),$$

are considered under impulsive conditions. Weyl's classification has been obtained.

2. PRELIMINARIES

In this context, we provide a concise overview of the Hahn calculus [3, 4, 10, 11]. Let $q \in (0, 1)$, $\omega_0 := \omega / (1 - q)$, $\omega > 0$, and let $\Psi : J \subset \mathbb{R} \rightarrow \mathbb{R}$ is a function such that $\omega_0 \in J$.

Definition 1 ([10], [11]). We define the Hahn derivative $D_{\omega,q}\Psi$ by

$$D_{\omega,q}\Psi(\zeta) = \begin{cases} \frac{\Psi(\omega+q\zeta) - \Psi(\zeta)}{\omega + (q-1)\zeta}, & \zeta \neq \omega_0, \\ \Psi'(\omega_0), & \zeta = \omega_0. \end{cases}$$

Definition 2 ([3]). Let $a, b, \omega_0 \in J$. We define the Hahn integral (ω, q -integral) by

$$\int_a^b \Psi(\zeta) D_{\omega,q}\zeta := \int_{\omega_0}^b \Psi(\zeta) d_{\omega,q}\zeta - \int_{\omega_0}^a \Psi(\zeta) d_{\omega,q}\zeta,$$

where

$$\int_{\omega_0}^{\zeta} \Psi(t) d_{\omega,q}t := ((1-q)\zeta - \omega) \sum_{n=0}^{\infty} q^n \Psi\left(\omega \frac{1-q^n}{1-q} + \zeta q^n\right), \quad \zeta \in J$$

provided that the series converges at $\zeta = a$ and $\zeta = b$.

3. MAIN RESULTS

Consider now the following impulsive boundary-value problem (BVP)

$$\Gamma y := \left[-\frac{1}{q} D_{-\frac{\omega}{q}, \frac{1}{q}} D_{\omega,q} + r(\zeta) \right] y(\zeta) = \lambda y(\zeta), \quad (3.1)$$

$$y(\omega_0) \cos \beta + D_{\omega,q}y(\omega_0) \sin \beta = 0, \quad (3.2)$$

$$y(d-) - k_1 y(d+) = 0, \quad (3.3)$$

$$D_{-\frac{\omega}{q}, \frac{1}{q}} y(d-) - k_2 D_{-\frac{\omega}{q}, \frac{1}{q}} y(d+) = 0, \quad (3.4)$$

$$y\left(\frac{1}{q^\varsigma}\right) \cos \delta + D_{\omega,q}y\left(\frac{1}{q^\varsigma}\right) \sin \delta = 0, \quad (3.5)$$

where $\zeta \in (\omega_0, d) \cup (d, q^\varsigma)$, $\beta, \delta \in \mathbb{R}$, $\lambda \in \mathbb{C}$, $\varsigma \in \mathbb{N} := \{1, 2, \dots\}$, $q^{-\varsigma} > d$, $k_1 k_2 = \eta > 0$; r is a real-valued, continuous function on $[\omega_0, d) \cup (d, \infty)$, one-sided limits $r(d\pm)$ exist. The same problem has been studied by Allahverdiev and Tuna without impulsive boundary conditions [1].

$H = L_{\omega,q}^2(\omega_0, d) + L_{\omega,q}^2(d, \infty)$ is a Hilbert space endowed with the following inner product

$$\langle y, z \rangle_H := \int_{\omega_0}^d y^{(1)} \overline{z^{(1)}} d_{\omega,q}\zeta + \eta \int_d^\infty y^{(2)} \overline{z^{(2)}} d_{\omega,q}\zeta,$$

where

$$y(\zeta) = \begin{cases} y^{(1)}(\zeta), & \zeta \in (\omega_0, d) \\ y^{(2)}(\zeta), & \zeta \in (d, \infty), \end{cases} \quad \text{and} \quad z(\zeta) = \begin{cases} z^{(1)}(\zeta), & \zeta \in (\omega_0, d) \\ z^{(2)}(\zeta), & \zeta \in (d, \infty). \end{cases}$$

Set

$$\mathcal{D}_{\max} = \left\{ y \in H : \begin{array}{l} y \text{ and } D_{-\frac{\omega}{q}, \frac{1}{q}} y \text{ are continuous at } \omega_0; \text{ one-sided} \\ \text{limits } y(d\pm) \text{ and } D_{-\frac{\omega}{q}, \frac{1}{q}} y(d\pm) \text{ exist and finite,} \\ y(d-) - k_1 y(d+) = 0, \\ D_{-\frac{\omega}{q}, \frac{1}{q}} y(d-) - k_2 D_{-\frac{\omega}{q}, \frac{1}{q}} \phi_1(d+, \lambda) = 0 \\ \text{and } \Gamma y \in H \end{array} \right\}.$$

Then the *maximal operator* $\mathcal{L}_{\max}$ on $\mathcal{D}_{\max}$ is defined by $\mathcal{L}_{\max} y = \Gamma y$, $y \in \mathcal{D}_{\max}$.

The ω, q -Green formula is given by

$$\begin{aligned} & \int_{\omega_0}^{\zeta} \left[(\Gamma y)(t) \overline{z(t)} - y(t) \overline{(\Gamma z)(t)} \right] d_{\omega, q} t \\ &= [y, z](\zeta) - [y, z](d+) + [y, z](d-) - [y, z](\omega_0), \end{aligned} \quad (3.6)$$

where $y, z \in \mathcal{D}_{\max}$, $\zeta \in (d, \infty)$ and

$$[y, z] := y \left(\overline{D_{-\frac{\omega}{q}, \frac{1}{q}} z} \right) - \left(D_{-\frac{\omega}{q}, \frac{1}{q}} y \right) \bar{z}.$$

Theorem 1. For any $\lambda \in \mathbb{C}$, Eq. (3.1) has a solution $\phi(\zeta, \lambda)$ satisfying conditions (3.2)-(3.4) which is an entire function of λ for every $\zeta \in [\omega_0, d) \cup (d, \infty)$.

Proof. According to a study by Annaby et al. [4], the following problem

$$\begin{aligned} & \left[-\frac{1}{q} D_{-\frac{\omega}{q}, \frac{1}{q}} D_{\omega, q} + r(\zeta) \right] y(\zeta) = \lambda y(\zeta), \quad \zeta \in [\omega_0, d), \\ & y(\omega_0) = -\sin \beta, \\ & D_{-\frac{\omega}{q}, \frac{1}{q}} y(\omega_0) = \cos \beta, \end{aligned}$$

has a unique solution $\phi^{(1)}(\zeta, \lambda)$ which is an entire function of λ .

Now let us consider the problem

$$\left[-\frac{1}{q} D_{-\frac{\omega}{q}, \frac{1}{q}} D_{\omega, q} + r(\zeta) \right] y(\zeta) = \lambda y(\zeta), \quad \zeta \in (d, \infty), \quad (3.7)$$

$$y(d+) = \frac{1}{k_1} \phi^{(1)}(d-, \lambda), \quad (3.8)$$

$$D_{-\frac{\omega}{q}, \frac{1}{q}} y(d+) = \frac{1}{k_2} D_{-\frac{\omega}{q}, \frac{1}{q}} \phi^{(1)}(d-, \lambda). \quad (3.9)$$

Let

$$u_n(\zeta, \lambda) = u_0(\zeta, \lambda) + q \int_d^{\zeta} \left(\frac{S_{\omega, q}(\sqrt{\lambda} \zeta; q)}{\sqrt{\lambda}} C_{\omega, q}(\sqrt{\lambda} q \gamma; q) \right) d_{\omega, q} \gamma \quad (3.10)$$

$$-C_{\omega,q}(\sqrt{\lambda}\zeta;q) \frac{S_{\omega,q}(\sqrt{\lambda}q\gamma;q)}{\sqrt{\lambda}} \Bigg) \times r(h(\gamma)) u_{n-1}(h(\gamma), \lambda) d_{\omega,q}\gamma,$$

where $h(\gamma) = q\gamma + \omega$,

$$u_0(\zeta, \lambda) = \frac{1}{k_1} \phi^{(1)}(d-, \lambda) + \frac{1}{k_2} (\zeta - d) D_{-\frac{\omega}{q}, \frac{1}{q}} \phi^{(1)}(d-, \lambda), \quad \zeta \in (d, \infty),$$

and the functions $\frac{S_{\omega,q}(\sqrt{\lambda}\zeta;q)}{\sqrt{\lambda}}$, $C_{\omega,q}(\sqrt{\lambda}q\gamma;q)$ are the fundamental solutions of ([4])

$$-\frac{1}{q} D_{-\frac{\omega}{q}, \frac{1}{q}} D_{\omega,q} y(\zeta) = \lambda y(\zeta). \quad (3.11)$$

It is obvious that u_n is the entire function.

The basic estimate

$$|u_{n+1}(\zeta, \lambda) - u_n(\zeta, \lambda)| \leq q^{n+1} \widetilde{\sigma}(\lambda) \frac{(A\sigma(\lambda) [\zeta(1-q) - \omega])^n}{(q; q)_n} \quad (3.12)$$

is proved by induction. For $n = 1$,

$$\begin{aligned} & |u_1(\zeta, \lambda) - u_0(\zeta, \lambda)| \\ & \leq \left| q \int_d^\zeta \begin{pmatrix} \frac{S_{\omega,q}(\sqrt{\lambda}\zeta;q)}{\sqrt{\lambda}} C_{\omega,q}(\sqrt{\lambda}q\gamma;q) \\ -C_{\omega,q}(\sqrt{\lambda}\zeta;q) \frac{S_{\omega,q}(\sqrt{\lambda}q\gamma;q)}{\sqrt{\lambda}} \end{pmatrix} r(h(\gamma)) u_0(h(\gamma), \lambda) d_{\omega,q}\gamma \right| \\ & \leq q\sigma(\lambda) A\widetilde{\sigma}(\lambda) \left| \int_{\omega_0}^\zeta d_{\omega,q}\gamma \right| = q\sigma(\lambda) A\widetilde{\sigma}(\lambda) \frac{[\zeta(1-q) - \omega]}{(1-q)}, \end{aligned}$$

where $\lambda \in \mathbb{C}$; $\sigma(\lambda)$, $\widetilde{\sigma}(\lambda)$ and A are positive numbers such that

$$\left| \begin{pmatrix} \frac{S_{\omega,q}(\sqrt{\lambda}\zeta;q)}{\sqrt{\lambda}} C_{\omega,q}(\sqrt{\lambda}q\gamma;q) \\ -C_{\omega,q}(\sqrt{\lambda}\zeta;q) \frac{S_{\omega,q}(\sqrt{\lambda}q\gamma;q)}{\sqrt{\lambda}} \end{pmatrix} \right| \leq \sigma(\lambda),$$

and $\max_{\zeta \in (d, a]} |r(\zeta)| = A$, $|u_0(\zeta, \lambda)| \leq \widetilde{\sigma}(\lambda)$, $\zeta \in (d, a]$.

Based on the induction hypothesis

$$\begin{aligned} & |u_n(\zeta, \lambda) - u_{n-1}(\zeta, \lambda)| \\ & \leq \left| q \int_d^\zeta \begin{pmatrix} \frac{S_{\omega,q}(\sqrt{\lambda}\zeta;q)}{\sqrt{\lambda}} C_{\omega,q}(\sqrt{\lambda}q\gamma;q) \\ -C_{\omega,q}(\sqrt{\lambda}\zeta;q) \frac{S_{\omega,q}(\sqrt{\lambda}q\gamma;q)}{\sqrt{\lambda}} \end{pmatrix} \right. \\ & \quad \left. \times r(h(\gamma)) [u_{n-1}(h(\gamma), \lambda) - u_{n-2}(h(\gamma), \lambda)] d_{\omega,q}\gamma \right| \end{aligned}$$

$$\leq q^n \widetilde{\sigma(\lambda)} \frac{(A\sigma(\lambda) [\zeta(1-q) - \omega])^{n-1}}{(q; q)_{n-1}}$$

The remainder of the argument is straightforward. Hence, the series

$$u_1(\zeta, \lambda) + \sum_{n=1}^{\infty} \{u_{n+1}(\zeta, \lambda) - u_n(\zeta, \lambda)\} \quad (3.13)$$

is uniformly convergent with respect to the variable ζ on $(d, a]$, due to the series

$$\sum_{n=1}^{\infty} q^{n+1} \widetilde{\sigma(\lambda)} \frac{(A\sigma(\lambda) [\zeta(1-q) - \omega])^n}{(q; q)_n}$$

is convergent.

Let

$$\phi^{(2)}(\zeta, \lambda) = u_1(\zeta, \lambda) + \sum_{n=1}^{\infty} \{u_{n+1}(\zeta, \lambda) - u_n(\zeta, \lambda)\}.$$

Then we find

$$\lim_{n \rightarrow \infty} u_n(\zeta, \lambda) = \phi^{(2)}(\zeta, \lambda).$$

(3.10) implies

$$\begin{aligned} & D_{\omega, q} u_{n+1}(\zeta, \lambda) - D_{\omega, q} u_n(\zeta, \lambda) \\ &= q \int_d^\zeta \left(\begin{aligned} & D_{\omega, q} \frac{S_{\omega, q}(\sqrt{\lambda} \zeta; q)}{\sqrt{\lambda}} C_{\omega, q}(\sqrt{\lambda} q \gamma; q) \\ & - D_{\omega, q} C_{\omega, q}(\sqrt{\lambda} \zeta; q) \frac{S_{\omega, q}(\sqrt{\lambda} q \gamma; q)}{\sqrt{\lambda}} \end{aligned} \right) \\ & \quad \times r(h(\gamma)) \begin{bmatrix} u_n(h(\gamma), \lambda) \\ -u_{n-1}(h(\gamma), \lambda) \end{bmatrix} d_{\omega, q} \gamma, \end{aligned}$$

and

$$\begin{aligned} & -\frac{1}{q} D_{-\frac{\omega}{q}, \frac{1}{q}} D_{\omega, q} u_{n+1}(\zeta, \lambda) + \frac{1}{q} D_{-\frac{\omega}{q}, \frac{1}{q}} D_{\omega, q} u_n(\zeta, \lambda) \\ &= q \int_d^\zeta \left(\begin{aligned} & -\frac{1}{q} D_{-\frac{\omega}{q}, \frac{1}{q}} D_{\omega, q} \frac{S_{\omega, q}(\sqrt{\lambda} \zeta; q)}{\sqrt{\lambda}} C_{\omega, q}(\sqrt{\lambda} q \gamma; q) \\ & + \frac{1}{q} D_{-\frac{\omega}{q}, \frac{1}{q}} D_{\omega, q} C_{\omega, q}(\sqrt{\lambda} \zeta; q) \frac{S_{\omega, q}(\sqrt{\lambda} q \gamma; q)}{\sqrt{\lambda}} \end{aligned} \right) \\ & \quad \times r(h(\gamma)) \begin{bmatrix} u_n(h(\gamma), \lambda) \\ -u_{n-1}(h(\gamma), \lambda) \end{bmatrix} d_{\omega, q} \gamma - r(\zeta) [u_n(\zeta, \lambda) - u_{n-1}(\zeta, \lambda)]. \end{aligned}$$

From (3.12), we conclude that the series

$$\sum_{n=1}^{\infty} (D_{\omega, q} u_{n+1}(\zeta, \lambda) - D_{\omega, q} u_n(\zeta, \lambda))$$

and

$$\sum_{n=1}^{\infty} \left(-\frac{1}{q} D_{-\frac{\omega}{q}, \frac{1}{q}} D_{\omega, q} u_{n+1}(\zeta, \lambda) + \frac{1}{q} D_{-\frac{\omega}{q}, \frac{1}{q}} D_{\omega, q} u_n(\zeta, \lambda) \right)$$

are uniformly convergent on (d, ∞) with respect to the variable ζ for every $\lambda \in \mathbb{C}$. From (3.11), we find

$$\begin{aligned} & -\frac{1}{q} D_{-\frac{\omega}{q}, \frac{1}{q}} D_{\omega, q} \phi^{(2)}(\zeta, \lambda) \\ &= \sum_{n=1}^{\infty} \left(-\frac{1}{q} D_{-\frac{\omega}{q}, \frac{1}{q}} D_{\omega, q} u_{n+1}(\zeta, \lambda) + \frac{1}{q} D_{-\frac{\omega}{q}, \frac{1}{q}} D_{\omega, q} u_n(\zeta, \lambda) \right) \\ &= (\lambda - v(\zeta)) \sum_{n=1}^{\infty} (u_n(\zeta, \lambda) - u_{n-1}(\zeta, \lambda)) = (\lambda - v(\zeta)) \phi^{(2)}(\zeta, \lambda). \end{aligned}$$

It follows easily that $\phi^{(2)}$ satisfies (3.8)-(3.9). Thus, we see that the function

$$\phi(\zeta, \lambda) = \begin{cases} \phi^{(1)}(\zeta, \lambda), & \zeta \in [0, d) \\ \phi^{(2)}(\zeta, \lambda), & \zeta \in (d, \infty) \end{cases} \quad (3.14)$$

satisfies the BVP (3.1)-(3.4). $\square$

A similar theorem can be proved.

Theorem 2. For any $\lambda \in \mathbb{C}$, Eq. (3.1) has a solution

$$\psi(\zeta, \lambda) = \begin{cases} \psi^{(1)}(\zeta, \lambda), & \zeta \in [\omega_0, d) \\ \psi^{(2)}(\zeta, \lambda), & \zeta \in (d, \infty) \end{cases} \quad (3.15)$$

satisfying conditions (3.3)-(3.5) which is an entire function of λ for every $\zeta \in [\omega_0, d) \cup (d, \infty)$.

Consider

$$\phi(\zeta, \lambda) = \begin{cases} \phi^{(1)}(\zeta), & \zeta \in [\omega_0, d) \\ \phi^{(2)}(\zeta), & \zeta \in (d, \infty) \end{cases}$$

and

$$\psi(\zeta, \lambda) = \begin{cases} \psi^{(1)}(\zeta), & \zeta \in [\omega_0, d) \\ \psi^{(2)}(\zeta), & \zeta \in (d, \infty) \end{cases}$$

as solutions of the BVP (3.1)-(3.3) satisfying

$$\begin{aligned} \phi^{(1)}(\omega_0, \lambda) &= \sin \beta, \quad \psi^{(1)}(\omega_0, \lambda) = \cos \beta, \\ D_{\omega, q} \phi^{(1)}(\omega_0, \lambda) &= -\cos \beta, \quad D_{\omega, q} \psi^{(1)}(\omega_0, \lambda) = \sin \beta, \end{aligned} \quad (3.16)$$

where $0 \leq \beta < \pi$.

Write

$$W_{\omega, q}^{(i)} := W_{\omega, q}(\psi^{(i)}, \phi^{(i)}) \quad (i = 1, 2),$$

where

$$W_{\omega, q}(y, z) := y D_{\omega, q} z - z D_{\omega, q} y. \quad (3.17)$$

An explicit calculation shows that $W_{\omega,q}^{(1)} = \eta W_{\omega,q}^{(2)}$. Let $W_{\omega,q} := W_{\omega,q}^{(1)} = \eta W_{\omega,q}^{(2)}$. It follows from (3.6) that

$$\int_{\omega_0}^{\zeta} \Gamma y \bar{z} d_{\omega,q} t - \int_{\omega_0}^{\zeta} y \bar{\Gamma z} d_{\omega,q} t = W_{\omega,q}(y, \bar{z})(h^{-1}(\zeta)) - W_{\omega,q}(y, \bar{z})(\omega_0), \quad (3.18)$$

where $h(\zeta) = q\zeta + \omega$ and $\zeta \in (\omega_0, d)$.

Theorem 3. *Let ϕ and ψ be two solutions of the BVP (3.1), (3.3), (3.4). Then, $W_{\omega,q}(\phi, \psi)$ is independent of ζ .*

Proof. Let $\lambda \in \mathbb{C}$ and $\zeta \in (\omega_0, d)$. By (3.6), we find

$$\begin{aligned} \int_{\omega_0}^{\zeta} \Gamma \phi \psi d_{\omega,q} t - \int_{\omega_0}^{\zeta} \phi \Gamma \psi d_{\omega,q} t &= (\lambda - \lambda) \int_{\omega_0}^{\zeta} \phi(t, \lambda) \psi(t, \lambda) d_{\omega,q} t \\ &= W_{\omega,q}(\phi, \psi)(h^{-1}(\zeta), \lambda) - W_{\omega,q}(\phi, \psi)(\omega_0, \lambda). \end{aligned} \quad (3.19)$$

since $\Gamma \phi = \lambda \phi$ and $\Gamma \psi = \lambda \psi$. Hence

$$W_{\omega,q}(\phi, \psi)(h^{-1}(\zeta), \lambda) = W_{\omega,q}(\phi, \psi)(\omega_0, \lambda).$$

Likewise, for $\zeta \in (d, \infty)$, we obtain

$$W_{\omega,q}(\phi, \psi)(h^{-1}(\zeta), \lambda) = \frac{1}{\eta} W_q(\phi, \psi)(\omega_0, \lambda).$$

□

Lemma 1. *Let $\phi(\zeta, \lambda)$ be a solution of the BVP (3.1), (3.3), (3.4) and let $\sigma = \text{Im} \lambda$, where $\lambda \in \mathbb{C}$. Then the following relations hold:*

1)

$$2i\sigma \int_{\omega_0}^b |\phi(\zeta, \lambda)|^2 d_{\omega,q} \zeta = W_{\omega,q}(\phi, \bar{\phi})(h^{-1}(b), \lambda) - W_{\omega,q}(\phi, \bar{\phi})(\omega_0, \lambda), \quad (3.20)$$

where $b \in (\omega_0, d)$.

2)

$$\begin{aligned} 2i\sigma \left[\int_{\omega_0}^d |\phi(\zeta, \lambda)|^2 d_{\omega,q} \zeta + \eta \int_d^b |\phi(\zeta, \lambda)|^2 d_{\omega,q} \zeta \right] \\ = W_{\omega,q}(\phi, \bar{\phi})(h^{-1}(b), \lambda) - W_{\omega,q}(\phi, \bar{\phi})(\omega_0, \lambda), \end{aligned} \quad (3.21)$$

where $b \in (d, \infty)$.

Proof. The proof is straightforward. □

The solution of (3.1) is given by

$$\vartheta(\zeta, \lambda) := \phi(\zeta, \lambda) + m\psi(\zeta, \lambda),$$

where $\zeta \in [\omega_0, d) \cup (d, \infty)$ and m is a constant. Then ϑ satisfies (3.5) if and only if

$$m = -\frac{\Xi \cos \delta + \Omega \sin \delta}{\Phi \cos \delta + \Upsilon \sin \delta}, \quad (3.22)$$

where

$$\begin{aligned} \Omega &= D_{\omega, q} \phi \left(\frac{1}{q^\varsigma}, \lambda \right), \quad \Upsilon = D_{\omega, q} \psi \left(\frac{1}{q^\varsigma}, \lambda \right), \\ \Xi &= \phi \left(\frac{1}{q^\varsigma}, \lambda \right), \quad \Phi = \psi \left(\frac{1}{q^\varsigma}, \lambda \right). \end{aligned}$$

From (3.22), we obtain

$$m = -\frac{\Xi z + \Omega}{\Phi z + \Upsilon}, \quad (3.23)$$

where $\cot \delta = z$. Thus, m describes a circle $C_{\frac{1}{q^\varsigma}}$ ($\varsigma \in \mathbb{N}$, $q^{-\varsigma} > d$). Now let's calculate the radius and center of this circle

Theorem 4. Let $\varsigma \in \mathbb{N}$, $q^{-\varsigma} > d$ and $\lambda \in \mathbb{C}$, where $\sigma = \text{Im } \lambda \neq 0$. Then, we have

$$\Theta_\varsigma(\lambda) = -\frac{W_{\omega, q}(\phi, \bar{\psi}) \left(\frac{1}{q^\varsigma}, \lambda \right)}{W_{\omega, q}(\psi, \bar{\phi}) \left(\frac{1}{q^\varsigma}, \lambda \right)}, \quad (3.24)$$

$$r_\varsigma(\lambda) = \left(2\sigma \int_{\omega_0}^d |\psi(\zeta, \lambda)|^2 d_{\omega, q} \zeta + 2\sigma \eta \int_d^{\frac{1}{q^\varsigma}} |\psi(\zeta, \lambda)|^2 d_{\omega, q} \zeta \right)^{-1}, \quad (3.25)$$

where Θ_ς is the center of the circle $C_{\frac{1}{q^\varsigma}}$ and r_ς is its radius.

Proof. Clearly, $\Theta_\varsigma(\lambda)$ is the symmetric point at ∞ . Set $m(\lambda, z') = \infty$, $m(\lambda, z'') = \Theta_\varsigma(\lambda)$. Then we conclude that $z' = \overline{z''}$. On the other hand,

$$m(\lambda, z') = \infty \Leftrightarrow z' = -\frac{D_{\omega, q} \psi \left(\frac{1}{q^\varsigma}, \lambda \right)}{\psi \left(\frac{1}{q^\varsigma}, \lambda \right)}.$$

Hence we find

$$\begin{aligned} \Theta_\varsigma(\lambda) &= m \left(\lambda, -\frac{D_{\omega, q} \psi \left(\frac{1}{q^\varsigma}, \lambda \right)}{\psi \left(\frac{1}{q^\varsigma}, \lambda \right)} \right) \\ &= -\frac{\phi \left(\frac{1}{q^\varsigma}, \lambda \right) \left(-\frac{D_{\omega, q} \psi \left(\frac{1}{q^\varsigma}, \lambda \right)}{\psi \left(\frac{1}{q^\varsigma}, \lambda \right)} \right) + D_{\omega, q} \phi \left(\frac{1}{q^\varsigma}, \lambda \right)}{\psi \left(\frac{1}{q^\varsigma}, \lambda \right) \left(-\frac{D_{\omega, q} \psi \left(\frac{1}{q^\varsigma}, \lambda \right)}{\psi \left(\frac{1}{q^\varsigma}, \lambda \right)} \right) + D_{\omega, q} \psi \left(\frac{1}{q^\varsigma}, \lambda \right)} \end{aligned}$$

$$\begin{aligned}
&= -\frac{\phi\left(\frac{1}{q^\varsigma}, \lambda\right) D_{\omega, q} \psi\left(\frac{1}{q^\varsigma}, \bar{\lambda}\right) - \psi\left(\frac{1}{q^\varsigma}, \bar{\lambda}\right) D_{\omega, q} \phi\left(\frac{1}{q^\varsigma}, \lambda\right)}{\psi\left(\frac{1}{q^\varsigma}, \lambda\right) D_{\omega, q} \psi\left(\frac{1}{q^\varsigma}, \bar{\lambda}\right) - \psi\left(\frac{1}{q^\varsigma}, \bar{\lambda}\right) D_{\omega, q} \psi\left(\frac{1}{q^\varsigma}, \lambda\right)} \\
&= -\frac{W_{\omega, q}(\phi, \bar{\psi})\left(\frac{1}{q^\varsigma}, \lambda\right)}{W_{\omega, q}(\psi, \bar{\psi})\left(\frac{1}{q^\varsigma}, \lambda\right)},
\end{aligned}$$

where $\varsigma \in \mathbb{N}$. Since $W_{\omega, q}(\phi, \psi)\left(\frac{1}{q^\varsigma}\right) = \frac{1}{\eta}$ ($\varsigma \in \mathbb{N}$, $q^{-\varsigma} > d$) we see that

$$\begin{aligned}
r_n(\lambda) &= \left| \frac{D_{\omega, q} \phi\left(\frac{1}{q^\varsigma}, \lambda\right)}{D_{\omega, q} \psi\left(\frac{1}{q^\varsigma}, \lambda\right)} - \frac{W_{\omega, q}(\phi, \bar{\psi})\left(\frac{1}{q^\varsigma}, \lambda\right)}{W_{\omega, q}(\psi, \bar{\psi})\left(\frac{1}{q^\varsigma}, \lambda\right)} \right| \\
&= \left| \frac{D_{\omega, q} \psi\left(\frac{1}{q^\varsigma}, \bar{\lambda}\right) W_{\omega, q}(\phi, \psi)\left(\frac{1}{q^\varsigma}, \lambda\right)}{D_{\omega, q} \psi\left(\frac{1}{q^\varsigma}, \lambda\right) W_{\omega, q}(\psi, \bar{\psi})\left(\frac{1}{q^\varsigma}, \lambda\right)} \right| \\
&= \left| \frac{W_{\omega, q}(\phi, \psi)\left(\frac{1}{q^\varsigma}, \lambda\right)}{W_{\omega, q}(\psi, \bar{\psi})\left(\frac{1}{q^\varsigma}, \lambda\right)} \right| = \frac{1}{\eta \left| W_{\omega, q}(\psi, \bar{\psi})\left(\frac{1}{q^\varsigma}, \lambda\right) \right|}.
\end{aligned}$$

By Lemma 1, we deduce that

$$W_{\omega, q}(\psi, \bar{\psi})\left(\frac{1}{q^\varsigma}, \lambda\right) = 2i\sigma \int_{\omega_0}^d |\psi(\zeta, \lambda)|^2 d_{\omega, q} \zeta + 2i\sigma\eta \int_d^{\frac{1}{q^\varsigma}} |\psi(\zeta, \lambda)|^2 d_{\omega, q} \zeta.$$

implies

$$\left| W_{\omega, q}(\psi, \bar{\psi})\left(\frac{1}{q^\varsigma}, \lambda\right) \right| = 2|\sigma| \int_{\omega_0}^d |\psi(\zeta, \lambda)|^2 d_{\omega, q} \zeta + 2|\sigma|\eta \int_d^{\frac{1}{q^\varsigma}} |\psi(\zeta, \lambda)|^2 d_{\omega, q} \zeta,$$

which proves the theorem. $\square$

Theorem 5. Let $\lambda = \rho + i\sigma$, where $\text{Im } \lambda = \sigma > 0$. Then the exterior of the circle $C_{\frac{1}{q^\varsigma}}$ ($\varsigma \in \mathbb{N}$) is mapped onto the upper half-plane of the z -plane.

Proof. By (3.21), we conclude that

$$\begin{aligned}
\text{Im} \left\{ \frac{D_{\omega, q} \psi\left(\frac{1}{q^\varsigma}, \lambda\right)}{\psi\left(\frac{1}{q^\varsigma}, \lambda\right)} \right\} &= \frac{1}{2}i \left\{ -\frac{D_{\omega, q} \psi\left(\frac{1}{q^\varsigma}, \lambda\right)}{\psi\left(\frac{1}{q^\varsigma}, \lambda\right)} + \frac{D_{\omega, q} \psi\left(\frac{1}{q^\varsigma}, \bar{\lambda}\right)}{\psi\left(\frac{1}{q^\varsigma}, \bar{\lambda}\right)} \right\} \\
&= \frac{1}{2}i \frac{W_{\omega, q}(\bar{\psi}, \psi)\left(\frac{1}{q^\varsigma}, \lambda\right)}{\left| \psi\left(\frac{1}{q^\varsigma}, \lambda\right) \right|^2} = -\frac{1}{2}i \frac{W_{\omega, q}(\psi, \bar{\psi})\left(\frac{1}{q^\varsigma}, \lambda\right)}{\left| \psi\left(\frac{1}{q^\varsigma}, \lambda\right) \right|^2}
\end{aligned}$$

$$= \frac{\sigma}{\left| \Psi\left(\frac{1}{q^\varsigma}, \lambda\right) \right|^2} \left(\int_{\omega_0}^d |\Psi(\zeta, \lambda)|^2 d_{\omega, q} \zeta + \eta \int_d^{\frac{1}{q^\varsigma}} |\Psi(\zeta, \lambda)|^2 d_{\omega, q} \zeta \right) > 0$$

where $\varsigma \in \mathbb{N}$. □

Theorem 6. Let $\varsigma \in \mathbb{N}$, $q^{-\varsigma} > d$ and $\text{Im} \lambda = \sigma > 0$. Then m lies on the circles $C_{\frac{1}{q^\varsigma}}$ if and only if

$$\int_{\omega_0}^d |\phi(\zeta, \lambda) + m\Psi(\zeta, \lambda)|^2 d_{\omega, q} \zeta + \eta \int_d^{\frac{1}{q^\varsigma}} |\phi(\zeta, \lambda) + m\Psi(\zeta, \lambda)|^2 d_{\omega, q} \zeta = \frac{\text{Im} m}{\sigma}, \quad (3.26)$$

and m is inside $C_{\frac{1}{q^\varsigma}} \Leftrightarrow$

$$\int_{\omega_0}^d |\phi(\zeta, \lambda) + m\Psi(\zeta, \lambda)|^2 d_{\omega, q} \zeta + \eta \int_d^{\frac{1}{q^\varsigma}} |\phi(\zeta, \lambda) + m\Psi(\zeta, \lambda)|^2 d_{\omega, q} \zeta < \frac{\text{Im} m}{\sigma}.$$

Proof. Let $m \in \mathbb{C}$. Then, we obtain

$$\begin{aligned} W_{\omega, q}(\phi + m\Psi, \overline{\phi + m\Psi})(\omega_0, \lambda) &= W_{\omega, q}(\phi, \overline{\phi})(\omega_0, \lambda) + mW_q(\Psi, \overline{\phi})(\omega_0, \lambda) \\ &\quad + \overline{m}W_{\omega, q}(\phi, \overline{\Psi})(\omega_0, \lambda) + |m|^2 W_{\omega, q}(\Psi, \overline{\Psi})(\omega_0, \lambda) \\ &= -m + \overline{m} = -2i \text{Im} m. \end{aligned}$$

By Lemma 1, we infer that

$$\begin{aligned} 2\sigma \int_{\omega_0}^d |\phi(\zeta, \lambda) + m\Psi(\zeta, \lambda)|^2 d_{\omega, q} \zeta + 2\sigma\eta \int_d^{\frac{1}{q^\varsigma}} |\phi(\zeta, \lambda) + m\Psi(\zeta, \lambda)|^2 d_{\omega, q} \zeta \\ = \frac{1}{i} \left(W_{\omega, q}(\phi + m\Psi, \phi + m\Psi) \left(\frac{1}{q^\varsigma}, \lambda \right) + 2i \text{Im} m \right), \end{aligned} \quad (3.27)$$

where $\varsigma \in \mathbb{N}$, $q^{-\varsigma} > d$. From Theorem 5, for $\sigma > 0$, m is inside $C_{\frac{1}{q^\varsigma}}$ if $\text{Im} z < 0$. (3.23) implies

$$m = -\frac{\Xi z + \Omega}{\Phi z + \Upsilon} \Leftrightarrow z = -\frac{\Omega + m\Upsilon}{\Xi + m\Phi}.$$

Hence

$$i(z - \bar{z}) = i \left\{ -\frac{\Omega + m\Upsilon}{\Xi + m\Phi} + \frac{\overline{\Omega} + \overline{m}\overline{\Upsilon}}{\overline{\Xi} + \overline{m}\overline{\Phi}} \right\} = i \frac{W_{\omega, q}(\phi + m\Psi, \overline{\phi + m\Psi}) \left(\frac{1}{q^\varsigma}, \lambda \right)}{|\Xi + m\Phi|^2},$$

where $\varsigma \in \mathbb{N}$, $q^{-\varsigma} > d$. Therefore

$$\text{Im} z < 0 \Leftrightarrow iW_q(\phi + m\Psi, \overline{\phi + m\Psi}) \left(\frac{1}{q^\varsigma}, \lambda \right) > 0, \quad (3.28)$$

where $\varsigma \in \mathbb{N}, q^{-\varsigma} > d$. By combining (3.27) and (3.28), we see that

$$\int_{\omega_0}^d |\phi(\varsigma, \lambda) + m\psi(\varsigma, \lambda)|^2 d_{\omega, q} \varsigma + \eta \int_d^{\frac{1}{q^\varsigma}} |\phi(\varsigma, \lambda) + m\psi(\varsigma, \lambda)|^2 d_{\omega, q} \varsigma < \frac{\text{Im } m}{\sigma}, \quad (3.29)$$

where $\varsigma \in \mathbb{N}, q^{-\varsigma} > d$. Since m is on $C_{\frac{1}{q^\varsigma}} \Leftrightarrow \text{Im } z = 0$, we infer that

$$W_{\omega, q}(\phi + m\psi, \overline{\phi + m\psi})\left(\frac{1}{q^\varsigma}, \lambda\right) = 0, \quad (3.30)$$

where $\varsigma \in \mathbb{N}, q^{-\varsigma} > d$. By virtue of (3.30) and (3.27), we get the desired result. $\square$

Theorem 7. Let $\text{Im } \lambda = \sigma > 0$ and let k be a point such that $d < \frac{1}{q^k} < \frac{1}{q^\varsigma}$. Then $C_{\frac{1}{q^\varsigma}} \subset C_{\frac{1}{q^k}}$.

Proof. By (3.27), we see that

$$\begin{aligned} & \int_{\omega_0}^d |\phi(\varsigma, \lambda) + m\psi(\varsigma, \lambda)|^2 d_{\omega, q} \varsigma + \eta \int_d^{\frac{1}{q^k}} |\phi(\varsigma, \lambda) + m\psi(\varsigma, \lambda)|^2 d_{\omega, q} \varsigma \\ & < \int_{\omega_0}^d |\phi(\varsigma, \lambda) + m\psi(\varsigma, \lambda)|^2 d_{\omega, q} \varsigma + \eta \int_d^{\frac{1}{q^\varsigma}} |\phi(\varsigma, \lambda) + m\psi(\varsigma, \lambda)|^2 d_{\omega, q} \varsigma < \frac{\text{Im } m}{\sigma}, \end{aligned}$$

which proves the theorem. $\square$

Corollary 1. As $\varsigma \rightarrow \infty$, the circles $C_{\frac{1}{q^\varsigma}}$ contract either to a circle or a point. These represent limit-circle and limit-point cases respectively.

Theorem 8. Let m be a point lying on or inside the limiting circle C_∞ , and $\Psi(\varsigma, \lambda) := \phi(\varsigma, \lambda) + m\psi(\varsigma, \lambda)$, $\text{Im } \lambda = \sigma > 0$, be the solution of (3.1). Then we have

$$\int_{\omega_0}^d |\Psi(\varsigma, \lambda)|^2 d_{\omega, q} \varsigma + \eta \int_d^\infty |\Psi(\varsigma, \lambda)|^2 d_{\omega, q} \varsigma < \infty,$$

i.e., $\Psi(\cdot, \lambda) \in H$.

Proof. Let $\varsigma \in \mathbb{N}(q^{-\varsigma} > d)$. Since

$$\int_{\omega_0}^d |\Psi(\varsigma, \lambda)|^2 d_{\omega, q} \varsigma + \eta \int_d^{\frac{1}{q^\varsigma}} |\Psi(\varsigma, \lambda)|^2 d_{\omega, q} \varsigma < \frac{\text{Im } m}{\sigma},$$

we arrive at

$$\int_{\omega_0}^d |\Psi(\varsigma, \lambda)|^2 d_{\omega, q} \varsigma + \eta \int_d^\infty |\Psi(\varsigma, \lambda)|^2 d_{\omega, q} \varsigma < \frac{\text{Im } m}{\sigma},$$

i.e., $\Psi(\cdot, \lambda) \in H$. $\square$

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EXISTENCE OF SOLUTIONS TO NON-LOCAL UNCERTAIN DIFFERENTIAL EQUATIONS UNDER THE ψ -CAPUTO FRACTIONAL DERIVATIVE

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Abstract. In this manuscript, we look into the existence of solutions to a class of uncertain differential equations with non-local derivatives. The method relies on the Krasnosel'skii fixed point theorem as well as a lengthened Schauder fixed point theorem that is applicable on fuzzy metric spaces. This theorems implies that the topic in question has a fuzzy solution specified on a certain interval. Our strategy requires considering a linked integral problem wherein the aforementioned tools apply. We'll wrap off with a physical incentive.

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Keywords: uncertain differential equations, Schauder fixed point theorem, Krasnosel'skii fixed point theorem, fuzzy metric spaces, semilinear space

1. INTRODUCTION

Uncertain valuation is a mathematical description that aims to minimize uncertainties in computational, mathematical, and practical uses. Due to a variety of factors, uncertainty has seemed in mathematical approaches and experimental observations. The uncertainty principle, for example, is crucial in quantum physics [10].

Nonlocal derivatives, while on the other side, have grown in prominence in recent years as a result of the attention and strength of their applicability in the creation of models for complicated systems in connection to several sectors of research and engineering. Certain well potential applications include the investigation of electromagnetism or control processes, diffusion processes via diverse media, biological challenges, and electronic systems. [2, 7, 13, 14, 17].

Agarwal et al. [3] developed the notion of solutions to uncertain differential equations having non-local derivatives of fraction form in 2010 (denoted as UFDEs). The equations provided in [3] have an unusual attitude because they mix the input of uncertainty and non-local impacts. This effort has resulted in the development of a novel

class of hybrid operators [2], which gather the characteristics of uncertain equations. Arshad and Lupulescu [5, 6] have extracted various conclusions about the existence and uniqueness concerns for UFDE by studying the Riemann-Liouville derivative.

The creation of novel extensions of fixed point findings is significant in the context of UFDEs; this truth is inspired by the distinctive peculiarities observed in the domains of fuzzy sets and functions. Agarwal et al. [3] for an intriguing illustration of an expansion of the Schauder fixed point theorem. The traditional fixed point finding is expanded to semi-linear spaces with the cancellation feature in their investigation. This plugin helps you to investigate the existence of UFDE solutions. The researchers in [12] used this concept to investigate UFDEs and investigate the availability of solutions. Another extended fixed point findings may be found in [15], in which the researchers established Krasnosel'skii's fixed point theorem in the setting of semi-linear Banach spaces of common class and used it to derive various existence results for UFDE's.

In this work, we look at a class of non-linear differential equations with uncertainty and non-local derivatives, which comprises a wider class of equations than [12]. The use of an extended Schauder fixed point theorem accessible in the setting of fuzzy metric spaces is the major mathematical instrument for the investigation. With this conclusion, We demonstrate that the problems of concern have at least one solution, assuming that some adequate criteria are met. The goal of this paper is to offer a finding that generalize the discoveries in [18] to the situation of uncertain differential equations. As a result, the article's reaching new is to show that there's at least one solution to a class of uncertain differential equations with non-local derivatives under the ψ -Caputo fractional derivative of order $\alpha \in (0, 1]$.

This article is organized as follows: after this introduction, we have presented some concepts related to fuzzy semilinear metric spaces, the ψ -Caputo fractional derivative and we provide all of the required tools that will be utilized in this study. The main results were then discussed in Section 3. Finally, we conclude our work with an illustrative example.

2. BASIC NOTIONS

To identify the topic of concern and the key outcomes, certain notions and relevant representations first be introduced.

In this paper, we indicate the space of fuzzy sets by $\mathbb{E}$. The α -cuts of an element $\phi \in \mathbb{E}$ are defined as:

$$[\phi]^\alpha := \{t \in \mathbb{R} \mid \phi(t) \geq \alpha\}, \quad \forall \alpha \in (0, 1].$$

The support of ϕ is given by:

$$[\phi]^0 := \text{cl}\{t \in \mathbb{R} \mid \phi(t) > 0\}.$$

It would also be useful to denote the α -cuts of ϕ as:

$$[\phi]^\alpha = [\underline{\phi}^\alpha, \bar{\phi}^\alpha].$$

Taking two elements $u, v \in \mathbb{E}$ and a scalar $r \in \mathbb{R}$, we compute the sum $u + v$ and multiplication by scalar ru by using formulas

$$[u + v]^\alpha = [u]^\alpha + [v]^\alpha, \text{ and } [ru]^\alpha = r[u]^\alpha, \quad \forall \alpha \in [0, 1].$$

As previously stated, these ideas are based on the classical summation of real intervals and the scalar multiplication of a real interval.

Due on the Pompeiu-Hausdorff distance between the α -cuts, the set E is simply supplied with a measure $D: E \times E \rightarrow \mathbb{R}^+ \cup \{0\}$, as shown in:

$$D(\phi, \varphi) := \sup_{\alpha \in [0, 1]} d_H([\phi]^\alpha, [\varphi]^\alpha) = \sup_{\alpha \in [0, 1]} \max \left\{ |\underline{\phi}^\alpha - \underline{\varphi}^\alpha|, |\bar{\phi}^\alpha - \bar{\varphi}^\alpha| \right\}, \quad \phi, \varphi \in \mathbb{E}.$$

The aforementioned distance possesses several notable properties, including translation invariance and positive homogeneity:

- $D(\phi + \Phi, \varphi + \Phi) = D(\phi, \varphi)$, for all $\phi, \varphi, \Phi \in E$,
- $D(r\phi, r\varphi) = |r|D(\phi, \varphi)$, for all $r \in \mathbb{R}$ and $\phi, \varphi \in E$,
- $D(\phi + \varphi, \mu + \nu) \leq D(\phi, \mu) + D(\varphi, \nu)$, for all $\phi, \varphi, \mu, \nu \in E$,
- $D(\gamma\phi, \kappa\varphi) = |\gamma - \kappa|D(\phi, \tilde{0})$, for all $\gamma, \kappa \geq 0$ and $\phi \in E$, and the space (E, D) is metrically complete.

Since the space E is not a vector space, traditional fixed point concepts in Banach spaces are not applicable to solving problems in this context. Therefore, we adopt the notion of semi-linear Banach spaces, which encompasses a framework that includes the set of all fuzzy sets. The concept of semi-linear spaces has been explored in previous academic works, such as [9].

We define U as a semi-linear metric space if it is equipped with a metric $d: U \times U \rightarrow \mathbb{R}^+$ satisfying:

- $d(\phi, \Phi) = d(\phi + \Xi, \Phi + \Xi)$ (translation invariance),
- $\Upsilon d(\phi, \Phi) = d(\Upsilon\phi, \Upsilon\Phi)$, for $\Upsilon \geq 0$ (positive homogeneity)

for all $\phi, \Phi, \Upsilon \in U$, where $\tilde{0}$ denotes the zero element in U . The norm $\|x\| := d(x, \tilde{0})$ can be defined on U . It is feasible to establish that addition and scalar multiplication are continuous operations in semi-linear metric spaces. If U is complete with respect to the metric d , it is referred to as a semi-linear Banach space.

For instance, since $\mathbb{E}$ is not a linear space, it cannot be a Banach space. The property that $\phi + \Phi = \varphi + \Phi$ implies $\phi = \varphi$ for all $\phi, \Phi, \varphi \in U$ indicates the cancellation property in the semi-linear space U .

Let $a > 0$. The space of all continuous fuzzy functions defined on $(0, a]$ is denoted by $C((0, a], \mathbb{E})$. Suppose $r \geq 0$. We define:

$$C_r([0, a], \mathbb{E}) = \{\Phi \in C((0, a], \mathbb{E}) \mid \Phi_r \in C([0, a], \mathbb{E})\},$$

where $\Phi_r(t) = \psi^r(t)\Phi(t)$ for $t \in (0, a]$. Clearly, $C_r([0, a], \mathbb{E})$ is a complete metric space with respect to the distance:

$$d_r(\Phi, \varphi) = \max_{t \in [0, a]} \psi^r(t) D(\Phi(t), \varphi(t)), \quad \Phi, \varphi \in C_r([0, a], \mathbb{E}).$$

We denote $d_r(u, \tilde{0})$ by $\|\Phi\|_r$, which is not a norm in the classical sense because $C_r([0, a], \mathbb{E})$ is not a vector space. It's important to note that $C_0([0, a], \mathbb{E}) = C([0, a], \mathbb{E})$.

We denote $\mathbb{E}^c$ as the set of fuzzy elements $\Phi \in \mathbb{E}$ where the mapping $\alpha \mapsto [\Phi]^\alpha$ is continuous with respect to the Hausdorff distance on $[0, 1]$. It is well-known that the metric space $(\mathbb{E}^c, D)$ is complete [11].

If the mappings take values in $\mathbb{E}^c$, we obtain the space $C_r([0, 1], \mathbb{E}^c)$ for $r \geq 0$.

Definition 1 ([11]). We say that M is compact-supported if there's a real compact subspace N with $[x]^0 \subseteq N, \forall x \in M$.

Definition 2 ([11]). Allow $M \subseteq E^c$. If, for each $\varepsilon > 0$, there's $\eta > 0$ such as

$$|\alpha - \beta| < \eta \text{ implies } D([x]^\alpha, [x]^\beta) < \varepsilon, \forall x \in M.$$

Consequently, at $\beta \in [0, 1]$, we state that M is level-equicontinuous.

We say that M is level-equicontinuous on $[0, 1]$ if the condition described in Definition 2 is true for each $\alpha \in [0, 1]$.

Theorem 1 ([11]). Assume that M is a compact supported subset of E^c . The set M is relatively compact in (E^c, D) iff it's level equicontinuous on $[0, 1]$.

Furthermore, we remember the semi-linear version of the classic Schauder fixed point theorem stated in [1].

Theorem 2. Assume U is a semi-linear Banach space with cancellation. Assume S is a nonempty, bounded, closed, convex subset of U . Consider $P: S \rightarrow S$ to be a compact map. Therefore, in S , P has (at least) one fixed point.

In [12] Khastan et al. used H-differences to demonstrate the following generalization of the Krasnosel'skii fixed point finding in the uncertain environment.

Theorem 3 (Krasnosel'skii fixed point theorem for fuzzy metric spaces). Allow $\mathcal{M}$ to be a non empty, closed and convex subset of $C(I, \mathbb{E}^c)$ and assume that Q and $\mathcal{H}$ map $\mathcal{M}$ into S and

- i) Q is continuous and compact,
- ii) $Q_t + \mathcal{H}s \in \mathcal{M}$, for every $t, s \in \mathcal{M}$,
- iii) $\mathcal{H}$ is a contraction mapping.

Then, there exists a fixed point for $Q + \mathcal{H}$ in $\mathcal{M}$, that is, there is $s \in \mathcal{M}$ for which $Q_s + \mathcal{H}s = s$.

Definition 3 ([16]). The Gamma function $\Gamma(z)$ is defined by integral

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt \quad (2.1)$$

which converges on the complex half-plane $\operatorname{Re}(z) > 0$.

Integrating by parts, we show that:

$$\Gamma(z+1) = z\Gamma(z), \quad \operatorname{Re}(z) > 0.$$

And in particular,

$$\begin{aligned} \forall n \in \mathbb{N}; \quad \Gamma(n+1) &= n!, \\ \frac{1}{\Gamma(-m)} &= 0, \quad (m = 0, 1, 2, \dots), \\ \Gamma(1) &= 1, \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}. \end{aligned}$$

Definition 4 ([4]). Assume $\alpha > 0, g \in L^1(J, \mathbb{R})$ and $\psi \in C^n(I, \mathbb{R})$ such that $\psi'(t) > 0 \forall t \in I$. The ψ -Riemann Liouville fractional integral of level α of the function g is written as

$$I_{0+}^{\alpha, \psi} g(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s) (\psi(t) - \psi(s))^{\alpha-1} g(s) ds. \quad (2.2)$$

Definition 5 ([4]). Assume $\alpha > 0, g \in C^{n-1}(I, \mathbb{R})$ and $\psi \in C^n(I, \mathbb{R})$ such that $\psi'(t) > 0 \forall t \in I$. The ψ -Caputo fractional derivative of order α of the function g is expressed as

$${}^c D_{0+}^{\alpha, \psi} g(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \psi'(s) (\psi(t) - \psi(s))^{n-\alpha-1} g_{\psi}^{[n]}(s) ds, \quad (2.3)$$

where

$$g_{\psi}^{[n]}(s) = \left(\frac{1}{\psi'(s)} \frac{d}{ds} \right)^n g(s) \text{ and } n = [\alpha] + 1$$

and $[\alpha]$ denotes the integer part of the real number α .

Remark 1. More specifically, if $\alpha \in]0, 1[$, so there is

$${}^c D_{0+}^{\alpha, \psi} g(s) = \frac{1}{\Gamma(\alpha)} \int_0^s (\psi(s) - \psi(t))^{\alpha-1} g'(t) dt$$

and

$${}^c D_{0+}^{\alpha, \psi} g(s) = I_{0+}^{1-\alpha, \psi} \left(\frac{g'(s)}{\psi'(s)} \right).$$

Proposition 1 ([4]). Allow $\alpha > 0$, if $f \in C^{n-1}(I, \mathbb{R})$, so we've

$$(1) \quad {}^c D_{0+}^{\alpha, \psi} I_{0+}^{\alpha, \psi} g(s) = g(s).$$

- (2) $I_{0+}^{\alpha, \Psi^C} D_{0+}^{\alpha, \Psi} g(s) = g(s) - \sum_{k=0}^{n-1} \frac{g_{\Psi}^{[k]}(0)}{k!} (\psi(s) - \psi(0))^k.$
- (3) $I_{0+}^{\alpha, \Psi}$ is linear, bounded from $C(I, \mathbb{R})$ to $C(I, \mathbb{R})$.

Proposition 2 ([4]). Allow $t > 0$ and $\alpha, \beta > 0$, therefore we get

- a) $I_{0+}^{\alpha, \Psi} (\psi(s) - \psi(0))^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta + \alpha)} (\psi(s) - \psi(0))^{\alpha + \beta - 1}.$
- b) ${}^C D_{0+}^{\alpha, \Psi} (\psi(s) - \psi(0))^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta - \alpha)} (\psi(s) - \psi(0))^{\alpha - \beta - 1}.$
- c) ${}^C D_{0+}^{\alpha, \Psi} (\psi(s) - \psi(0))^n = 0,$ for all $n \in \mathbb{N}.$

3. MAIN RESULTS

Considering the uncertain non-local fuzzy fractional differential equation:

$$\begin{cases} {}^C D_{0+}^{\alpha, \Psi} u = f(t, u) + g(t, u), & t \in I = [0, 1], \\ u(0) = \tilde{0} \in \mathbb{E}, \end{cases} \quad (3.1)$$

where the derivation order is $0 < \alpha < 1$, the fuzzy function $f: [0, 1] \times \mathbb{E}^c \rightarrow \mathbb{E}^c$ is continuous and compact, and $g: [0, 1] \times \mathbb{E}^c \rightarrow \mathbb{E}^c$ fulfills

$$D(g(t, u), g(t, v)) \leq LD(u, v), \quad u, v \in \mathbb{E}^c, \text{ for } L \geq 0.$$

Definition 6. A fuzzy function $u \in C((0, 1], \mathbb{E}) \cap L^1((0, 1], \mathbb{E})$ with continuous fractional derivative ${}^C D_{0+}^{\alpha, \Psi} u$ on $(0, 1]$ is a solution of (3.1) if

$${}^C D_{0+}^{\alpha, \Psi} u = f(t, u) + g(t, u) \quad \text{for all } t \in (0, 1]. \quad (3.2)$$

Remark 2. Using the findings from Section 2, considering a fuzzy integral equation that permits us to solve the problem (3.1). Very much so, if $u \in C([0, 1], \mathbb{E})$ is a solution to the fuzzy integral equation

$$u(t) = \frac{1}{\Gamma(\alpha)} \odot \int_0^t \Psi'(s) (\psi(t) \ominus \psi(s))^{\alpha-1} [f(s, u(s)) \oplus g(s, u(s))] ds,$$

and $f(t, u), g(t, u) \in C((0, 1], \mathbb{E}) \cap L^1((0, 1], \mathbb{E})$. So u is a solution to equation (3.1) as well.

3.1. Existence of solution via Schauder fixed point theorem

Lemma 1. If $u: [0, 1] \rightarrow \mathbb{E}$ is continuous, then u is bounded.

Proof. If u is continuous, the function $\underline{u}^\alpha$ and $\bar{u}^\alpha$ are continuous and bounded on $[0, 1]$. Then $D(u, \tilde{0}) \leq \max\{|\underline{u}^\alpha|, |\bar{u}^\alpha|\}$ is bounded. $\square$

Below we define the operator $\mathcal{M}: C([0, 1], \mathbb{E}^c) \rightarrow C([0, 1], \mathbb{E}^c)$ for $u \in C([0, 1], \mathbb{E}^c)$ and $0 \leq r < \alpha < 1$ by

$$\begin{aligned} [\mathcal{M}u](t) &= \frac{1}{\Gamma(\alpha)} \odot \int_0^t \Psi'(s)(\Psi(t) - \Psi(s))^{\alpha-1} u_r(s) ds \\ &= \frac{1}{\Gamma(\alpha)} \odot \int_0^t \Psi'(s)(\Psi(t) - \Psi(s))^{\alpha-1} \Psi^r(s) u(s) ds. \end{aligned}$$

And

$$\begin{aligned} \mathcal{N}: \Omega &\rightarrow C([0, 1], \mathbb{R}^c), \\ \mathcal{N}u(t) &= \Psi^{-r}(t) (f(t, u(t)) + g(t, u(t))), \quad t \in [0, 1], \end{aligned}$$

where $\Omega = \{u \in C([0, 1], \mathbb{E}^c) \mid d_0(u, \tilde{0}) \leq R_0\}$. The operator $\mathcal{N}$ is well-defined if $(t, u) \rightarrow \Psi^{-r}(t) (f(t, u(t)) + g(t, u(t)))$ is continuous on $[0, a] \times \mathbb{R}_F^c$ with values in $\mathbb{E}^c$.

Lemma 2. *On $C([0, 1], \mathbb{E}^c)$, the operator $\mathcal{M}$ is properly delineated and continuous.*

Proof. Firstly, we demonstrate that $\mathcal{M}$ is well defined. That is, for a fixed $u \in C([0, 1], \mathbb{E}^c)$, ensure that $\mathcal{M}u \in C([0, 1], \mathbb{E}^c)$. In reality, we demonstrate $\mathcal{M}u$ is uniformly continuous on $[0, 1]$. Allow $t_1, t_2 \in [0, 1], t_1 < t_2$, and let R such as

$$D(u(s), \tilde{0}) \leq R, \quad \forall s \in [0, 1].$$

Then

$$\begin{aligned} &D(\mathcal{M}u(t_1), \mathcal{M}u(t_2)) \\ &= \frac{1}{\Gamma(\alpha)} D\left(\int_0^{t_1} \Psi'(s)(\Psi(t_1) - \Psi(s))^{\alpha-1} \Psi^r(s) u(s) ds, \int_0^{t_2} \Psi'(s)(\Psi(t_2) - \Psi(s))^{\alpha-1} \Psi^r(s) u(s) ds\right) \\ &\leq \frac{1}{\Gamma(\alpha)} D\left(\int_0^{t_1} \Psi'(s)(\Psi(t_1) - \Psi(s))^{\alpha-1} \Psi^r(s) u(s) ds, \int_0^{t_1} \Psi'(s)(\Psi(t_2) - \Psi(s))^{\alpha-1} \Psi^r(s) u(s) ds\right) \\ &\quad + D\left(\int_{t_1}^{t_2} \Psi'(s)(\Psi(t_2) - \Psi(s))^{\alpha-1} \Psi^r(s) u(s) ds, \tilde{0}\right) \\ &\leq \frac{1}{\Gamma(\alpha)} \left[\int_0^{t_1} D(\Psi'(s)(\Psi(t_1) - \Psi(s))^{\alpha-1} \Psi^r(s) u(s), \Psi'(s)(\Psi(t_2) - \Psi(s))^{\alpha-1} \Psi^r(s) u(s)) ds \dots \right. \\ &\quad \left. + \int_{t_1}^{t_2} \Psi'(s)(\Psi(t_1) - \Psi(s))^{\alpha-1} \Psi^r(s) D(u(s), \tilde{0}) ds \right] \\ &\leq \frac{R}{\Gamma(\alpha)} \left[\int_0^{t_1} |\Psi'(s)((\Psi(t_1) - \Psi(s))^{\alpha-1} \Psi^r(s) - (\Psi(t_2) - \Psi(s))^{\alpha-1} \Psi^r(s))| ds \right. \\ &\quad \left. + \int_{t_1}^{t_2} \Psi'(s)(\Psi(t_2) - \Psi(s))^{\alpha-1} \Psi^r(s) ds \right] \\ &\leq R \frac{\Gamma(-r+1)}{\Gamma(-r+\alpha+1)} [|(\Psi^{\alpha-r}(t_1) - \Psi^{\alpha-r}(t_2))| + 2 |(\Psi(t_1) - \Psi(t_2))^{\alpha-r}|]. \end{aligned}$$

Applying the Mean Value Theorem to the functions $\psi^{\alpha-r}(t)$ and $\psi(t)$, the previous inequality becomes

$$\begin{aligned} & D(\mathcal{M}u(t_1), \mathcal{M}u(t_2)) \\ & \leq R \frac{\Gamma(-r+1)}{\Gamma(-r+\alpha+1)} \left[((\alpha-r)\psi'(c_1)\psi^{\alpha-r-1}(c_1)|t_1-t_2|) + 2(\psi'(c_2)|t_1-t_2|)^{\alpha-r} \right], \end{aligned}$$

where, $c_1, c_2 \in [0, 1]$, which implies that $D(\mathcal{M}u(t_1), \mathcal{M}u(t_2)) \rightarrow 0$ when $|t_1 - t_2| \rightarrow 0$.

Following that, we demonstrate the continuity of $\mathcal{M}$. Allow $u_n \rightarrow u$ as $n \rightarrow \infty$ in $C([0, 1], \mathbb{E}^c)$, That is, $d_0(u_n, u) \rightarrow 0$ as $n \rightarrow \infty$. Well there's

$$\begin{aligned} d_0(\mathcal{M}u_n, \mathcal{M}u) &= \sup_{t \in [0, 1]} D(\mathcal{M}u_n(t), \mathcal{M}u(t)) \\ &= \sup_{t \in [0, 1]} D\left(\frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} \psi^r(s) u_n(s) ds, \right. \\ &\quad \left. \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} \psi^r(s) u(s) ds\right) \\ &\leq \frac{1}{\Gamma(\alpha)} \sup_{t \in [0, 1]} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} \psi^r(s) D(u_n(s), u(s)) ds \\ &\leq \frac{1}{\Gamma(\alpha)} d_0(u_n, u) \sup_{t \in [0, 1]} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} \psi^r(s) ds \\ &\leq d_0(u_n, u) \sup_{t \in [0, 1]} \frac{\Gamma(-r+1)}{\Gamma(-r+\alpha+1)} \psi^{\alpha-r}(t) \\ &\leq \frac{\Gamma(-r+1)}{\Gamma(-r+\alpha+1)} d_0(u_n, u) \sup_{t \in [0, 1]} \psi^{\alpha-r}(t). \end{aligned}$$

Therefore $\mathcal{M}u_n \rightarrow \mathcal{M}u$ as $n \rightarrow \infty$ in $C([0, 1], \mathbb{E}^c)$. $\square$

Remark 3. If $G \subseteq C([0, 1], \mathbb{E}^c)$ is bounded, so $\mathcal{M}(G)$ is bounded in $C([0, 1], \mathbb{E}^c)$. Obviously, for $v \in G$ we obtain

$$\begin{aligned} D(\mathcal{M}v(t), \tilde{0}) &\leq \sup_{t \in [0, 1]} D(v(t), \tilde{0}) \frac{\Gamma(1-r)}{\Gamma(1-r+\alpha)} \psi^{\alpha-r}(t) \\ &\leq \frac{\Gamma(1-r)}{\Gamma(1-r+\alpha)} \sup_{t \in [0, 1]} \psi^{\alpha-r}(t) D(v(t), \tilde{0}). \end{aligned}$$

Lemma 3. If $G \subseteq C([0, 1], \mathbb{E}^c)$ is bounded, then $\mathcal{M}(G)$ is equicontinuous in $C([0, 1], \mathbb{E}^c)$.

Proof. Allow $d_0(u, \tilde{0}) \leq M$, for all $u \in G$ and $t_1, t_2 \in [0, 1], t_1 < t_2$. According to the previous calculations of Lemma 2, we get, $\forall u \in G$,

$$D(\mathcal{M}u(t_1), \mathcal{M}u(t_2)) \leq M \left[\frac{\Gamma(-r+1)}{\Gamma(-r+\alpha+1)} (\Psi^{\alpha-r}(t_1) - \Psi^{\alpha-r}(t_2)) + \frac{2}{\Gamma(\alpha)} \int_{t_1}^{t_2} \Psi'(s) (\Psi(t_2) - \Psi(s))^{\alpha-1} \Psi'(s) ds \right],$$

which tends to 0 as $|t_1 - t_2| \rightarrow 0$. As a result, $\mathcal{M}(G)$ is equicontinuous in $C([0, 1], \mathbb{E}^c)$. $\square$

Proposition 3. *If $G \subseteq C([0, 1], \mathbb{E}^c)$ in such a way that $\{v(s) \mid v \in G, s \in [0, 1]\}$ is compact supported in $\mathbb{E}^c$, therefore G is bounded.*

Proof. Obviously, there's a compact set K in $\mathbb{R}$ such as

$$\{[u(s)]^0 \mid u \in G, s \in [0, 1]\} \subseteq K.$$

In contrast, for $u \in G$,

$$\begin{aligned} d_0(u, \tilde{0}) &= \sup_{t \in [0, 1]} \sup_{\theta \in [0, 1]} d_H([u(t)]^\theta, \{0\}) \\ &= \sup_{t \in [0, 1]} \sup_{\theta \in [0, 1]} d_H([\underline{u}^\theta(t), \bar{u}^\theta(t)], \{0\}) \\ &= \sup_{t \in [0, 1]} \sup_{\theta \in [0, 1]} \max \{|\underline{u}^\theta(t)|, |\bar{u}^\theta(t)|\} \\ &\leq \sup_{t \in [0, 1]} \max \{|\underline{u}^0(t)|, |\bar{u}^0(t)|\} \\ &\leq \sup_{t \in [0, 1]} d_H([u(t)]^0, \{0\}) \\ &< +\infty. \end{aligned}$$

Well, $G \subseteq C([0, 1], \mathbb{E}^c)$ is bounded. $\square$

Lemma 4. *If $G \subseteq C([0, 1], \mathbb{E}^c)$ is like*

$$\{u(s) \mid u \in G, s \in [0, 1]\}$$

is compactly supported and level equicontinuous, so $\mathcal{M}(G)$ is relatively compact in $C([0, 1], \mathbb{E}^c)$.

Proof. We demonstrate that $\mathcal{M}(G)$ is a uniformly equicontinuous subset of $C([0, 1], \mathbb{E}^c)$ and that $\mathcal{M}(G)$ is relatively compact in $\mathbb{E}^c$ for all $t \in [0, 1]$, using the Arzelà-Ascoli theorem. Since $\{u(s) \mid u \in G, s \in [0, 1]\}$ has compact support, Proposition 3 implies that G is bounded. Additionally, by applying Lemma 3 and given that G is bounded, we conclude that $\mathcal{M}(G)$ is uniformly equicontinuous. Therefore, it

suffices to show that $\mathcal{M}(G)(t)$ is relatively compact in $\mathbb{E}^c$ for all $t \in [0, 1]$. According to Theorem 1, this is equivalent to proving that $\mathcal{M}(G)(t)$ is a subset of $\mathbb{E}^c$ with compact support and uniformly equicontinuous over $[0, 1]$ for all $t \in [0, 1]$. Given that $\{u(s) \mid u \in G, s \in [0, 1]\}$ has compact support, there exists a compact set $K \subseteq \mathbb{R}$ such that $[u(s)]^0 \subseteq K$ for each $s \in [0, 1]$ and $u \in G$. Therefore, for every $u \in G$ and $t \in [0, 1]$,

$$\begin{aligned} [\mathcal{M}(u)(t)]^0 &= \left[\frac{1}{\Gamma(\alpha)} \int_0^t \Psi'(s)(\Psi(t) - \Psi(s))^{\alpha-1} \Psi^r(s) u(s) ds \right]^0 \\ &= \frac{1}{\Gamma(\alpha)} \int_0^t \Psi'(s)(\Psi(t) - \Psi(s))^{\alpha-1} \Psi^r(s) [u(s)]^0 ds \\ &\subseteq \frac{K}{\Gamma(\alpha)} \int_0^t \Psi'(s)(\Psi(t) - \Psi(s))^{\alpha-1} \Psi^r(s) ds \\ &\subseteq \frac{\Gamma(-r+1) \sup_{t \in [0,1]} \Psi^{\alpha-r}(t)}{\Gamma(-r+\alpha+1)} K. \end{aligned}$$

Hence, $\mathcal{M}(G)(t)$ is compactly supported for any $t \in [0, 1]$.

Furthermore, to demonstrate level equicontinuity, we fix $t \in [0, 1]$ and $\varepsilon > 0$. If $w \in \mathcal{M}(G)$, we get for $u \in G$ such as $w = \mathcal{M}(u)(t)$,

$$[w]^\theta = \frac{1}{\Gamma(\alpha)} \int_0^t \Psi'(s)(\Psi(t) - \Psi(s))^{\alpha-1} \Psi^r(s) [u(s)]^\theta ds.$$

Then

$$d_H([w]^\theta, [w]^\beta) \leq \frac{1}{\Gamma(\alpha)} \int_0^t \Psi'(s)(\Psi(t) - \Psi(s))^{\alpha-1} \Psi^r(s) d_H([u(s)]^\theta, [u(s)]^\beta) ds.$$

Since $\{u(s) \mid u \in G, s \in [0, 1]\}$ is level equicontinuous, as well as for provided

$$\varepsilon \frac{\Gamma(-r+\alpha+1)}{2\lambda\Gamma(-r+1)} > 0$$

with $\lambda = \sup_{t \in [0,1]} \Psi^{\alpha-r}(t)$, there's $\delta > 0$ satisfies $|\theta - \beta| < \delta$, well

$$d_H([u(s)]^\theta, [u(s)]^\beta) < \frac{\varepsilon\Gamma(-r+\alpha+1)}{2\lambda\Gamma(-r+1)} \quad u \in G, \quad s \in [0, 1].$$

So

$$d_H([w]^\theta, [w]^\beta) \leq \frac{\varepsilon\Gamma(-r+\alpha+1)}{2\lambda\Gamma(-r+1)} \frac{\Gamma(-r+1)}{\Gamma(-r+\alpha+1)} \Psi^{\alpha-r}(t) \leq \frac{\varepsilon}{2} < \varepsilon.$$

Thus, $\mathcal{M}(G)(t)$ is level equicontinuous in $\mathbb{E}^c$ on $[0, 1]$, $\forall t \in [0, 1]$. $\square$

Define $f_{-r}(t, u) = \Psi^{-r}(t)f(t, u)$, $t \in [0, a]$. Let we now consider

$$S = \{x \in \mathbb{E}^c \mid D(x, \tilde{0}) \leq R\}.$$

As a consequence, we get the next result:

Lemma 5. Allow $f_{-r} + g_{-r}: [0, 1] \times \mathbb{E}^c \rightarrow \mathbb{E}^c$ to be uniformly continuous and bounded in $[0, 1] \times S$. Hence, the operator $\mathcal{N}$ is continuous and bounded in $C([0, a], \mathbb{E}^c)$.

Proof. Take $u_n, u \in \Omega$, if $u_n \rightarrow u$, as $n \rightarrow \infty$ in $C([0, 1], \mathbb{E}^c)$. Then, for $\varepsilon > 0$ by the uniform continuity of $f_{-r} + g_{-r}$ in $[0, 1] \times S$, there's $\delta > 0$ such as for $(t, u), (s, v) \in [0, 1] \times S$ we obtain

$$H((t, u), (s, v)) = |t - s| + D(u, v) < \delta,$$

which means that

$$D(\Psi^{-r}(t)(f(t, u) + g(t, u)), \Psi^{-r}(s)(f(s, v) + g(s, v))) < \varepsilon.$$

However, provided $\delta > 0$, since $u_n \rightarrow u$ as $n \rightarrow \infty$, in terms of the limit there's $n_0 \in \mathbb{N}$ in such a way that, for $n \geq n_0$, we gain

$$d_0(u_n, u) < \delta \quad \text{i.e.,} \quad \sup_{t \in [0, 1]} D(u_n(t), u(t)) < \delta.$$

In this case

$$H((t, u_n(t)), (t, u(t))) = D(u_n(t), u(t)) < \delta, \quad \forall t \in [0, 1], \forall n \geq n_0,$$

so that

$$D(f_{-r}(t, u_n(t)) + g_{-r}(t, u_n(t)), f_{-r}(t, u(t)) + g_{-r}(t, u(t))) < \varepsilon$$

and

$$\begin{aligned} d_0(\mathcal{N}u_n, \mathcal{N}u) &= \sup_{t \in [0, a]} D(\mathcal{N}u_n(t), \mathcal{N}u(t)) \\ &= \sup_{t \in [0, 1]} D(f_{-r}(t, u_n(t)) + g_{-r}(t, u_n(t)), f_{-r}(t, u(t)) + g_{-r}(t, u(t))) \\ &\leq \varepsilon. \end{aligned}$$

This demonstrates that $\mathcal{N}u_n \rightarrow \mathcal{N}u$ in $C([0, a], \mathbb{E}^c)$.

In contrast, let set B is bounded in Ω , so $\forall u \in B$ we have

$$d_0(u, \tilde{0}) \leq M, \quad \text{i.e.,} \quad \sup_{t \in [0, 1]} D(u(t), \tilde{0}) \leq M.$$

Then,

$$d_0(\mathcal{N}u, \tilde{0}) = \sup_{t \in [0, 1]} D(\mathcal{N}u(t), \tilde{0}) = \sup_{t \in [0, 1]} D(f_{-r}(t, u(t)) + g_{-r}(t, u(t)), \tilde{0}), \forall u \in B.$$

Since f_{-r} is bounded in $[0, a] \times S$ and g_{-r} is continuous, then there's a $K > 0$ such that $d_0(\mathcal{N}u, \tilde{0}) \leq K, \forall u \in B$.

Hence, $\mathcal{N}(B)$ is bounded. $\square$

Lemma 6. *If $\{(f_{-r} + g_{-r})(s, x) \mid (s, x) \in [0, a] \times S\}$ is compactly supported and level equicontinuous, then*

$$\{u(s) \mid u \in N(\Omega), s \in [0, a]\}$$

is relatively compact.

Proof. Because

$$\begin{aligned} \{(f_{-r} + g_{-r})(s, u(s)) \mid u \in \Omega, s \in [0, a]\} &= \{(\mathcal{N}u)(s) \mid u \in \Omega, s \in [0, a]\} \\ &= \{u(s) \mid u \in \mathcal{N}(\Omega), s \in [0, a]\} \end{aligned}$$

is compact supported and level equicontinuous, it is relatively compact. $\square$

Lemma 7. *Allow $f_{-r}(t, u)$ and $g_{-r}(t, u)$ to be a continuous functions on $[0, 1] \times S$. Thus, they are compact on $[0, 1] \times S$ iff the set*

$$\{\Psi^{-r}(t)(f(t, u) + g(t, u)) \mid t \in [0, 1], u \in S\}$$

is compact supported and level equicontinuous.

Proof. Firstly, consider $f_{-r}(t, x)$ and $g_{-r}(t, u)$ as two functions continuous and compact on $[0, 1] \times S$. Then by Theorem 1, we have

$$\{f_{-r}(t, u) + g_{-r}(t, u) \mid t \in [0, 1], u \in S\}$$

is compact supported and level equicontinuous.

Now let $\{f_{-r}(t, u) + g_{-r}(t, u) \mid t \in [0, 1], u \in S\}$ is compact supported and level equicontinuous. By Theorem 1 once again, it is relatively compact. Hence f_{-r} and g_{-r} are compact on $[0, 1] \times S$. because for any bounded set $B \subseteq [0, 1] \times S$, the set

$$\{(f_{-r} + g_{-r})(t, u) \mid (t, u) \in B\}$$

is relatively compact. $\square$

Definition 7. $\mathcal{A}: C([0, 1], \mathbb{E}) \rightarrow C([0, 1], \mathbb{E})$ is a bounded operator if for every bounded B in $C([0, 1], \mathbb{E})$, $\mathcal{A}(B)$ is bounded in $C([0, 1], \mathbb{E})$.

The local existence theorem for fuzzy fractional differential equations (3.1) is shown here.

Theorem 4. *Let $0 \leq r < \alpha < 1$ and let f and g be two continuous applications from $(0, 1] \times \mathbb{E}^c$ to $\mathbb{E}^c$. If $f_{-r} + g_{-r}$ is compact and uniformly continuous on $[0, 1] \times \mathbb{E}^c$. As a result, for a sufficient $0 < \xi \leq 1$, the fuzzy integral equation has at least one continuous solution specified on $[0, \xi]$.*

Proof. We simply need to examine the fuzzy integral equation, according to Remark 2:

$$u(t) = \frac{1}{\Gamma(\alpha)} \odot \int_0^t \Psi'(s)(\Psi(t) - \Psi(s))^{\alpha-1} [f(s, u(s)) \oplus g(s, u(s))] ds.$$

Let's take the set

$$\Omega = \{u \in C([0, 1], \mathbb{E}^c) \mid d_0(u, \tilde{0}) \leq R_0\}.$$

It is straightforward to demonstrate that Ω is a closed, bounded, convex subset of the semilinear Banach space $C([0, 1], \mathbb{E}^c)$. We define the operator $\mathcal{A}: \Omega \rightarrow C([0, 1], \mathbb{E}^c)$ as

$$(\mathcal{A}u)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} (f(s, u(s)) + g(t, u(s))) ds.$$

The operator $\mathcal{A}$ is declared to be continuous and compact. Clearly, the operator is the combination of the continuous and bounded operators $\mathcal{A} = \mathcal{M} \circ \mathcal{N}$, with

$$\mathcal{N}u(t) = \Psi^{-r}(t)(f(t, u(t)) + g(t, u(t))), \quad t \in [0, 1]$$

and

$$\mathcal{M}u(t) = \frac{1}{\Gamma(\alpha)} \odot \int_0^t \Psi'(s)(\Psi(t) - \Psi(s))^{\alpha-1} \Psi^r(s) u(s) ds, \quad u \in C([0, 1], \mathbb{E}^c).$$

The operator $\mathcal{A}$ is properly defined since it is the combination of $\mathcal{M}$ and $\mathcal{N}$. Because f_r and g_r are continuous and compact, $\{(f_{-r} + g_{-r})(t, u) \mid t \in [0, 1], u \in S\}$ is compact-supported and level-equicontinuous according to Lemma 8. Thus, according to Lemma 6, $\{u(s) \mid u \in N(\Omega), s \in [0, 1]\}$ is compact supported and level equicontinuous. As a result, according to Lemma 4, $\mathcal{M}(\mathcal{N}(\Omega))$ is relatively compact in $C([0, 1], \mathbb{E}^c)$. The operator $\mathcal{A}$ is then compact on Ω . Besides that, given $0 \leq t \leq \delta \leq 1$ such as $\Psi(t) \leq \delta$, we obtain

$$\begin{aligned} D(\mathcal{M}u(t), \tilde{0}) &\leq \sup_{t \in [0, \delta]} D(u(t), \tilde{0}) \frac{1}{\Gamma(\alpha)} \int_0^t \Psi'(s)(\Psi(t) - \Psi(s))^{\alpha-1} \Psi^r(s) ds \\ &\leq \frac{\Gamma(1-r)}{\Gamma(1-r+\alpha)} \Psi^{\alpha-r}(t) \|u\|_0 \\ &\leq \frac{\Gamma(1-r)}{\Gamma(1-r+\alpha)} \delta^{\alpha-r} \|u\|_0. \end{aligned}$$

Hence, we have

$$\|\mathcal{M}u\|_0 \leq \varepsilon \|u\|_0,$$

where we are able to reduce $\delta > 0$ to make $\varepsilon > 0$ as small as we wish. Furthermore define B_ρ as the field of the operator $\mathcal{A}$, where $B_\rho = \{u \in C([0, \delta], \mathbb{E}^c) : \|u\|_0 \leq \rho\}$, a convex, limited, and close subset of the full metric space $C([0, \delta], \mathbb{E}^c)$. For $\delta > 0$ small enough, we get

$$\mathcal{A}(B_\rho) \subseteq B_\rho.$$

Theorem 2 guarantees that the operator $\mathcal{A}$ has at least one fixed point. As a result, equation (3.1) has at least one solution u given on $[0, \delta]$, where $\delta > 0$ and $\delta \leq 1$. $\square$

Corollary 1. *Under the circumstances of Theorem 4 and with the assumption that $f(\cdot, u(\cdot)) \in L^1((0, 1), \mathbb{E}^c)$ and $g(\cdot, u(\cdot)) \in L^1((0, 1), \mathbb{E}^c)$, $\forall u \in C([0, 1], \mathbb{E}^c)$, the fuzzy fractional differential equation (3.1) must thus have at least one solution specified on $[0, \delta]$, seeking an appropriate $0 < \delta \leq 1$.*

Proof. Suppose $u \in C([0, \delta], \mathbb{E}^c)$ is a solution to the fuzzy equation

$$u(t) = I^\alpha (f(t, u(t)) + g(t, u(t)))$$

utilizing that $f : (0, \delta] \times \mathbb{E}^c \rightarrow \mathbb{E}^c$ and $g : (0, \delta] \times \mathbb{E}^c \rightarrow \mathbb{E}^c$ are continuous and $f(\cdot, u(\cdot)) \in L^1((0, \delta), \mathbb{E}^c)$ and $g(\cdot, u(\cdot)) \in L^1((0, \delta), \mathbb{E}^c)$, for each $u \in C([0, \delta], \mathbb{E}^c)$, hence it is obvious that $f(t, u(t)) + g(t, u(t)) \in C((0, \delta], \mathbb{E}) \cap L^1((0, \delta), \mathbb{E})$ and u is a solution to equation (3.1) in $[0, \delta]$. $\square$

3.2. Existence of solution via Krasnosel'skii fixed point theorem

Let $I_\delta := [0, \delta]$, where $\delta \leq 1$, and $r_0 > 0$ such as

$$R := \sup \{ D(\tilde{0}, f(t, x)) \mid t \in I_{\delta_0}, D(x, \tilde{0}) \leq r_0 \} < +\infty.$$

It is also feasible to pick δ to be minimal enough so that $\frac{R\Gamma(-r+1)}{\Gamma(-r+\alpha+1)}\psi^{\alpha-r}(\delta) \leq r_0$.

Define the set

$$\Phi := \{x \in C(I_\delta, \mathbb{E}^c) \mid D(\tilde{0}, x(t)) \leq r_0, \text{ for all } t \in I_\delta, \text{ and } x(0) = \tilde{0}\}.$$

We can easily confirm that the set $\Phi \subseteq C(I_\delta, \mathbb{E}^c)$ is bounded, closed, and convex. Additionally, $C(I_\delta, \mathbb{E}^c)$ is a semi-linear Banach space. Within the set Φ , we examine two mappings corresponding to the functions f and g . Specifically, consider the mapping $\mathcal{F} : \Phi \rightarrow C(I_\delta, \mathbb{E}^c)$, defined as follows:

$$(\mathcal{F}u)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} \psi^r(s) f(s, u(s)) ds, \text{ for } t \in I_\delta. \quad (3.3)$$

To simplify the method, we refine several notations related to the various constraints on the function f that arise due to the non-linearity of the equation.

We introduce the criterion (H) for the function $f \in C([0, 1] \times \mathbb{E}^c, \mathbb{E}^c)$, which involves the following condition: the inequality

$$D(f(t, u(t)), f(t, v(t))) \leq p(t)w(d_0(u, v))$$

holds for any pair of points $(t, u), (t, v) \in [0, a] \times C([0, 1], \mathbb{E}^c)$, where w is a real continuous function on $[0, \infty)$ with $w(0) = 0$, and $p : [0, 1] \rightarrow \mathbb{R}^+$ satisfies $I^\alpha p(t) < N$ for every $t \in [0, 1]$.

Lemma 8. *Consider f fulfill (H) on $[0, a] \times \mathbb{E}^c$. The following assumptions are thus met: Let f fulfill (H) on $[0, a] \times \mathbb{E}^c$; the following assumptions are thus verified:*

- i) $\mathcal{F}$ is well-defined on $(I_\delta, \mathbb{E}^c)$.
- ii) $\mathcal{F}$ is a continuous mapping on $C(I_\delta, \mathbb{E}^c)$.

Proof. To begin, we demonstrate that the map $\mathcal{F}$ is quite defined. For any $uin\Phi$, $(\mathcal{F}u)(0) = \tilde{0}$ is evident by construction. With a given $uin\Phi$, we prove that $\mathcal{F}u \in C(I_\delta, \mathbb{E}^c)$. Furthermore, we demonstrate that $\mathcal{F}$ is a uniformly continuous function on the interval I_δ . We choose a determined $t, t' \in I_\delta$, with $t < t'$. We derive these components

$$\begin{aligned}
 D((\mathcal{F}u)(t), (\mathcal{F}u)(t')) &= \frac{1}{\Gamma(\alpha)} D \left(\int_0^t \Psi'(s) (\Psi(t) - \Psi(s))^{\alpha-1} \Psi^r(s) f(s, u(s)) ds, \right. \\
 &\quad \left. \int_0^{t'} \Psi'(s) (\Psi(t') - \Psi(s))^{\alpha-1} \Psi^r(s) f(s, u(s)) ds \right) \\
 &= \frac{1}{\Gamma(\alpha)} D \left(\int_0^t \Psi'(s) (\Psi(t) - \Psi(s))^{\alpha-1} \Psi^r(s) f(s, u(s)) ds, \right. \\
 &\quad \int_0^t \Psi'(s) (\Psi(t') - \Psi(s))^{\alpha-1} \Psi^r(s) f(s, u(s)) ds \dots \\
 &\quad \left. + \int_t^{t'} \Psi'(s) (\Psi(t') - \Psi(s))^{\alpha-1} \Psi^r(s) f(s, u(s)) ds \right) \\
 &\leq \frac{1}{\Gamma(\alpha)} \left[D \left(\int_0^t \Psi'(s) (\Psi(t) - \Psi(s))^{\alpha-1} \Psi^r(s) f(s, u(s)) ds, \right. \right. \\
 &\quad \left. \int_0^t \Psi'(s) (\Psi(t') - \Psi(s))^{\alpha-1} \Psi^r(s) f(s, u(s)) ds \right) \dots \\
 &\quad \left. + D \left(\int_t^{t'} \Psi'(s) (\Psi(t') - \Psi(s))^{\alpha-1} \Psi^r(s) f(s, u(s)) ds, \tilde{0} \right) \right] \\
 &\leq \frac{R}{\Gamma(\alpha)} \left[\int_0^t \Psi'(s) |(\Psi(t) - \Psi(s))^{\alpha-1} - (\Psi(t') - \Psi(s))^{\alpha-1}| \Psi^r(s) ds \right. \\
 &\quad \left. + \int_t^{t'} \Psi'(s) |\Psi(t') - \Psi(s)|^{\alpha-1} \Psi^r(s) ds \right] \\
 &\leq \frac{R\Gamma(-r+1)}{\Gamma(-r+\alpha+1)} \left(2 |\Psi(t') - \Psi(t)|^{\alpha-r} + |\Psi^{\alpha-r}(t) - \Psi^{\alpha-r}(t')| \right).
 \end{aligned}$$

Therefore, $D((\mathcal{F}u)(t), (\mathcal{F}u)(t')) \rightarrow 0$, when $|t - t'|$ tends to 0, so $\mathcal{F}u$ is a continuous mapping on I_δ , for $u \in \Phi$. In addition, we have

$$\begin{aligned}
 D((\mathcal{F}u)(t), \tilde{0}) &\leq \frac{1}{\Gamma(\alpha)} \int_0^t \Psi'(s) (\Psi(t) - \Psi(s))^{\alpha-1} \Psi^r(s) D(f(s, u(s)), \tilde{0}) ds \\
 &\leq \frac{R\Gamma(-r+1)}{\Gamma(-r+\alpha+1)} \Psi^{\alpha-r}(\delta) \\
 &\leq r_0
 \end{aligned}$$

$\forall t \in I_\delta$. As a result, $\mathcal{F}$ is a self-mapping $\mathcal{F} : \Phi \rightarrow \Phi$.

We next demonstrate that $\mathcal{T}$ is a continuous map. Consider $y_n, y \in \Phi, n = 1, 2, \dots$ such as $y_n \xrightarrow[n \rightarrow \infty]{} y$, with convergence in the set $C([0, 1], \mathbb{E}^c)$, i.e, fulfill $d_0(y_n, y) \xrightarrow[n \rightarrow \infty]{} 0$.

As a result, we may derive that for any $t \in J := [0, 1]$,

$$\begin{aligned}
d_0(\mathcal{F}y_n, \mathcal{F}y) &= \sup_{t \in J} D((\mathcal{F}y_n)(t), (\mathcal{F}y)(t)) \\
&= \sup_{t \in I_{\delta_0}} D\left(\frac{1}{\Gamma(\alpha)} \int_0^t \Psi'(s)(\Psi(t) - \Psi(s))^{\alpha-1} \Psi^r(s) f(s, y_n(s)) ds, \right. \\
&\quad \left. \frac{1}{\Gamma(\alpha)} \int_0^t \Psi'(s)(\Psi(t) - \Psi(s))^{\alpha-1} \Psi^r(s) f(s, y(s)) ds\right) \\
&\leq \frac{1}{\Gamma(\alpha)} \sup_{t \in I_{\delta}} \int_0^t \Psi'(s)(\Psi(t) - \Psi(s))^{\alpha-1} \Psi^r(s) D(f(s, y_n(s)), f(s, y(s))) ds \\
&\leq \frac{1}{\Gamma(\alpha)} \sup_{t \in I_{\delta}} \int_0^t \Psi'(s)(\Psi(t) - \Psi(s))^{\alpha-1} \Psi^r(s) p(s) w(d_0(y_n, y)) ds \\
&\leq \sup_{t \in I_{\delta_0}} I^\alpha p(t) w(\rho_0(y_n, y)).
\end{aligned}$$

Because, according to the criteria in (H), $w(0) = 0$ and w is continuous on its domain $[0, \infty)$, it is obvious that $w(r) \xrightarrow{r \rightarrow 0^+} 0$. Due to $\rho_0(y_n, y) \xrightarrow{n \rightarrow \infty} 0$ and the estimation $I^\alpha p(t) < N$, we get to the conclusion that $\mathcal{F}y_n \rightarrow \mathcal{F}y$ as $n \rightarrow \infty$, i.e., $\mathcal{F}$ is continuous. $\square$

Theorem 5. Take a given $u \in (0, 1)$ and assume that the fuzzy function $f: [0, 1] \times \mathbb{E}^c \rightarrow \mathbb{E}^c$ is continuous and compact, fulfilling (H), and suppose that $g: [0, 1] \times \mathbb{E}^c \rightarrow \mathbb{E}^c$ fulfills (3.2). Given these constraints, the non-local UDE (3.1) has (at least) one solution specified on $[0, \delta]$. This is a continuous function, with δ being an acceptable positive integer $\delta < 1$.

Proof. Using Remark 2, we are restricted to investigate the uncertain integral equation

$$\begin{aligned}
u(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t \Psi'(s)(\Psi(t) - \Psi(s))^{\alpha-1} \Psi^r(s) f(s, u(s)) ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^t \Psi'(s)(\Psi(t) - \Psi(s))^{\alpha-1} \Psi^r(s) g(s, u(s)) ds.
\end{aligned}$$

The mapping $\mathcal{F}$ is precisely considered by Lemma 8, equation (3.3), as well as its continuity on the set $C(I_\delta, \mathbb{E}^c)$. The next section argues that $\mathcal{F}$ is a compact mapping. If we choose an arbitrary determined $u \in \Phi$ and let $t, t' \in I_\delta$ with $t \leq t'$, so

$$\begin{aligned}
D((\mathcal{F}u)(t), (\mathcal{F}u)(t')) &= \frac{1}{\Gamma(\alpha)} D\left(\int_0^t \Psi'(s)(\Psi(t) - \Psi(s))^{\alpha-1} \Psi^r(s) f(s, u(s)) ds, \right. \\
&\quad \left. \int_0^{t'} \Psi'(s)(\Psi(t') - \Psi(s))^{\alpha-1} \Psi^r(s) f(s, u(s)) ds\right)
\end{aligned}$$

$$\leq \frac{R\Gamma(-r+1)}{\Gamma(-r+\alpha+1)} \left(2(\Psi(t') - \Psi(t))^{\alpha-r} + \Psi^{\alpha-r}(t) - \Psi^{\alpha-r}(t') \right).$$

Thus means that $\mathcal{F}(\Phi)$ is equicontinuous in $C(I_\delta, \mathbb{E}^c)$.

Furthermore, we employ Theorem 1 to demonstrate the relative compactness of $\mathcal{F}(\Phi)(t)$ in $\mathbb{E}^c$, implying that $\mathcal{F}(\Phi)(t)$ is compact supported and level equicontinuous in $\mathbb{E}^c$.

Consider $t \in [0, \delta]$, we check that $\mathcal{F}(\Phi)(t) \subseteq \mathbb{E}^c$. For every $u \in \mathbb{T}(\Phi)(t)$, it is possible to write

$$u = \frac{1}{\Gamma(\alpha)} \int_0^t \Psi'(s)(\Psi(t) - \Psi(s))^{\alpha-1} \Psi^r(s) f(s, y(s)) ds, \text{ for some } y \in \Phi.$$

Given that f is a compact mapping, $f(I_\delta \times \Phi)$ is a relatively compact in $\mathbb{E}^c$. According to Theorem 1, $f(I_\delta \times \Phi)$ is a level equicontinuous. As a result, we can assert the existence of $\delta' > 0$ for any $\varepsilon > 0$.

$$D([f(s, y(s))]^{\alpha_1}, [f(s, y(s))]^{\alpha_2}) < \frac{\Gamma(-r+\alpha+1)}{\Psi^{\alpha-r}(\delta)\Gamma(-r+1)} \varepsilon, \forall (s, y) \in I_{\delta_0} \times \Phi,$$

provided that $|\alpha_1 - \alpha_2| < \delta$. Therefore, similarly to [1], for $|\alpha_1 - \alpha_2| < \delta$, we deduce

$$\begin{aligned} D([u]^{\alpha_1}, [u]^{\alpha_2}) &= D([\mathcal{F}(y)(t)]^{\alpha_1}, [\mathcal{F}(y)(t)]^{\alpha_2}) \\ &\leq \frac{1}{\Gamma(\alpha)} D \left(\left[\int_0^t \Psi'(s)(\Psi(t) - \Psi(s))^{\alpha-1} \Psi^r(s) f(s, y(s)) ds \right]^{\alpha_1}, \right. \\ &\quad \left. \left[\int_0^t \Psi'(s)(\Psi(t) - \Psi(s))^{\alpha-1} \Psi^r(s) f(s, y(s)) ds \right]^{\alpha_2} \right) \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^t \Psi'(s)(\Psi(t) - \Psi(s))^{\alpha-1} \Psi^r(s) D([f(s, y(s))]^{\alpha_1}, [f(s, y(s))]^{\alpha_2}) ds \\ &\leq \frac{\Psi^{\alpha-r}(\delta)\Gamma(-r+1)}{\Gamma(-r+\alpha+1)} D([f(s, y(s))]^{\alpha_1}, [f(s, y(s))]^{\alpha_2}) \leq \varepsilon. \end{aligned}$$

Thus, $\mathcal{F}(\Phi)(t)$ is level equicontinuous in $\mathbb{E}^c$. Consequently, given the relative compactness of $f(I_\delta \times \Phi)$, for any $(s, y) \in I_\delta \times \Phi$, there exists a compact set $M \subset \mathbb{R}^n$ such that $[f(s, y(s))]_0 \subseteq M$. Therefore, we obtain

$$\begin{aligned} &\left[\frac{1}{\Gamma(\alpha)} \int_0^t \Psi'(s)(\Psi(t) - \Psi(s))^{\alpha-1} \Psi^r(s) f(s, y(s)) ds \right]^0 \\ &= \frac{1}{\Gamma(\alpha)} \int_0^t \Psi'(s)(\Psi(t) - \Psi(s))^{\alpha-1} \Psi^r(s) [f(s, y(s))]^0 ds \\ &\subseteq \frac{M}{\Gamma(\alpha)} \int_0^t \Psi'(s)(\Psi(t) - \Psi(s))^{\alpha-1} \Psi^r(s) ds = \frac{\Psi^{\alpha-r}(t)\Gamma(-r+1)}{\Gamma(-r+\alpha+1)} M. \end{aligned}$$

As a consequence, as a subset of $\mathbb{E}^c$, $\mathcal{F}(\Phi)(t)$ is compact supported. Hence, the relative compactness of $\mathcal{F}(\Phi)$ in $C([0, \delta], \mathbb{E}^c)$ is inferred using the Arzelà-Ascoli principle.

On the other hand, we take into account the mapping $\tilde{\mathcal{F}}: \Phi \rightarrow C(I_\delta, \mathbb{E}^c)$ is described as

$$(\tilde{\mathcal{F}}u)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} \psi^r(s) g(s, u(s)) ds, \quad t \in I_{\delta_0}.$$

Therefore,

$$\begin{aligned} D((\tilde{\mathcal{F}}x)(t), (\tilde{\mathcal{F}}y)(t)) &\leq \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} \psi^r(s) D(g(s, x(s)), g(s, y(s))) ds \\ &\leq \frac{L}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} \psi^r(s) D(x(s), y(s)) ds \\ &\leq \frac{L\psi^{\alpha-r}(t)\Gamma(-r+1)}{\Gamma(-r+\alpha+1)} d_0(x, y) \\ &\leq \frac{L\psi^{\alpha-r}(\delta)\Gamma(-r+1)}{\Gamma(-r+\alpha+1)} d_0(x, y). \end{aligned}$$

As a result, if δ is short enough that $L\psi^{\alpha-r}(\delta)\Gamma(-r+1) < \Gamma(-r+\alpha+1)$, therefore the mapping $\tilde{\mathcal{F}}$ is contraction. As a result, invoking Theorem 3, the mapping $\mathcal{F} + \tilde{\mathcal{F}}$ has at least one fixed point in Φ . As a consequence, for a given $0 < \delta \leq a$, there's (at least) one fuzzy solution u for (3.1) on I_δ . $\square$

4. AN ILLUSTRATIVE EXAMPLE

The research has highlighted the practical use of arbitrary-order mathematics to physical issues. We concentrate on a particular model presented in [8] as an illustration of how our approach might be used.

$$\begin{cases} {}^C D_{0+}^{\alpha, \psi} u(t) = \eta v(t) \oplus \rho w(t)(x(t) \sin(\xi) \ominus v(t) \cos(\xi)) \ominus \frac{1}{T_2} u(t), \\ {}^C D_{0+}^{\alpha, \psi} v(t) = (-\eta)u(t) \ominus w(t) \oplus \rho w(t)(v(t) \sin(\xi) \oplus u(t) \cos(\xi)) \ominus \frac{1}{T_2} v(t), \\ {}^C D_{0+}^{\alpha, \psi} w(t) = v(t) \ominus \rho \sin(\xi) (u^2(t) \oplus v^2(t)) \ominus \frac{1}{T_1} (w(t) - 1), \end{cases} \quad (4.1)$$

where $\alpha \in (0, 1)$, $\eta = -0.4\pi$, $\rho = 30$, $\xi = 0.173$, $T_1 = 5$ and $T_2 = 2.5$.

The ψ -Caputo fractional derivative is used. In f nuclear magnetic resonance (NMR) difficulties, the spinning mechanism is disrupted by a sequence of short-duration radiofrequency (RF) pulses (generally under a millisecond), which is lengthy in relation to the frequency of resonance period (about 10 ns for 3 T magnetic resonance imaging (MRI) at 127 MHz), yet small in comparison to the T_1 and T_2 relaxation times (300 and 100 ms, respectively). Thus we has to be capable of expressing the Bloch equation with a derivative that we may switch on and off in a pulse- or step-like way. Under such conditions, the Caputo is clearly defined, but the Riemann-Liouville is not, and it also requires fractional order beginning conditions, which we cannot provide.

This system's initial equation relates to the framework under consideration. Alternatively, comparable results may be achieved for the space $C([0, 1], \mathbb{E} \times \mathbb{E} \times \mathbb{E})$ with noticeable modifications; thus, it might be feasible to recast issue (4.1) as a higher dimension extension of problem (3.1), by selecting

$$f(t, u, v, w) = (\eta v \oplus \rho w(x \sin(\xi) \ominus v \cos(\xi)), (-\eta)u \ominus w \oplus \rho w(v \sin(\xi) \oplus u \cos(\xi)), \\ v \ominus \rho \sin(\xi) (u^2 \oplus v^2))$$

and

$$g(t, u, v, w) = \left(\ominus \frac{1}{T_2} u, \ominus \frac{1}{T_2} v, \ominus \frac{1}{T_1} (w \ominus 1) \right)$$

which meet the relevant requirements if g is well-defined.

CONCLUSION

In this study, we investigated the existence of solutions to non-local uncertain differential equations including Caputo type fractional derivatives with regard to another function ψ . The formulation of mild solutions is provided. Our demonstrations of existence are based on the fixed point theorems of Schauder and Krasnosel'skii. An example is provided as an application to demonstrate the acquired results.

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PROPERTIES OF SECOND ORDER DIFFERENTIAL EQUATIONS WITH ADVANCED AND DELAY ARGUMENT

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Abstract. In the paper, we study the bounded and unbounded oscillation of the second-order differential equations with deviating argument of the form

$$y''(t) = p(t)y(\tau(t)).$$

We introduce new monotonicities for the nonoscillatory solutions and apply them to offer new criteria for elimination of certain types of solutions. The presented results will be supported by set of examples to confirm our achieved progress in the oscillation theory.

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1. INTRODUCTION

In this paper, we shall study the asymptotic and oscillation behavior of the solutions for half-linear second order differential equations with deviating argument

$$y''(t) = p(t)y(\tau(t)). \quad (E)$$

We shall assume that

- (H₁) $p(t) \in C^1([t_0, \infty))$, $p(t) > 0$,
(H₂) $\tau(t) \in C^1([t_0, \infty))$, $\tau'(t) > 0$, $\lim_{t \rightarrow \infty} \tau(t) = \infty$.

By a solution of Eq. (E) we mean a function $y(t) \in C^2([T_y, \infty))$, $T_y \geq t_0$ which satisfies Eq. (E) on $[T_y, \infty)$. We consider only those solutions $y(t)$ of (E) for which $\sup\{|y(t)| : t \geq T\} > 0$ for all $T \geq T_y$. We assume that (E) possesses such a solution. A solution of (E) is called oscillatory if it has arbitrarily large zeros on $[T_y, \infty)$ and otherwise it is called to be nonoscillatory. An equation itself is said to be oscillatory if all its solutions are oscillatory.

Due to linearity of (E) it is sufficient to deal only with positive solutions of (E). The problem of establishing oscillatory criteria for various types of differential equations has been a very active research area over the past decades (see [1–12]).

It is known that the equation

$$y''(t) = p(t)y(t)$$

always possesses both positive decreasing and positive increasing solution. The situation for equation with deviating argument (E) may be different. If we denote by N the set of all positive solutions of (E), then it has the following decomposition

$$N = N_0 \cup N_2,$$

where the class N_0 involves positive decreasing solutions while N_2 includes positive increasing ones.

Koplatadze and Chanturia [10] have shown that for $\tau(t) \leq t$ the condition

$$\limsup_{t \rightarrow \infty} \int_{\tau(t)}^t (s - \tau(t))p(s)ds > 1 \quad (1.1)$$

does not allow the presence of positive decreasing solutions. i.e. $N_0 = \emptyset$.

On the other hand, (E) does not possess positive increasing solutions if $\tau(t) \geq t$ and

$$\limsup_{t \rightarrow \infty} \int_t^{\tau(t)} (\tau(t) - s)p(s)ds > 1, \quad (1.2)$$

that is $N_2 = \emptyset$.

Both criteria (1.1) and (1.2) are based on the standard monotonicities of possible positive solutions. So, if we intend to refine on above mentioned criteria, we have to improve these monotonicities. And this is the first aim of this paper. The progress will be demonstrated via the following couple of differential equations

$$y''(t) = p_*^2 y(t \pm 2\tau_*)$$

and

$$y''(t) = \frac{p_*}{t^2} y(\lambda t).$$

Some attempts on this direction have been made by several authors (see e.g. [6–8]) included the present one [3–5]. But there is no one general monotonicity formula that would fully cover classes N_0 and N_2 of (E), respectively. There are only partial formulas separately distinguish between $p(t) = p_*$ and $p(t) = p_*/t^2$.

The second aim of this paper is to provide one general monotonicity formula for all possible functions $p(t)$. To achieve this, we introduce new technique for investigation of properties of nonoscillatory solutions. In generally, when establishing desired monotonicity, the authors always try to decrease the order of derivative, but we proceed in opposite direction and we increase order of derivative of nonoscillatory solution. The third goal of the paper is to imply new monotonicities to essentially improve criteria (1.1) and (1.2).

2. MAIN RESULTS

For positive increasing solutions of (E) we can improve the monotonicity as follows.

Lemma 1. *Let*

$$\int_{t_0}^{\infty} sp(s) ds = \infty. \quad (2.1)$$

Assume that $y(t) \in N_2$. Then

$$\frac{y(t)}{t} \text{ is increasing } \left(\text{i.e., } \frac{y(t)}{t} \uparrow \right). \quad (2.2)$$

Proof. Assume that $y(t)$ is positive increasing solution of (E). It is easy to see that

$$\left(t^2 \left(\frac{y(t)}{t} \right)' \right)' = tp(t)y(\tau(t)). \quad (2.3)$$

An integration of (2.3) from t_0 to t in view of (2.1) yields

$$\begin{aligned} t^2 \left(\frac{y(t)}{t} \right)' &= k + \int_{t_0}^t sp(s)y(\tau(s)) ds \\ &\geq k + y(\tau(t_0)) \int_{t_0}^t sp(s) ds \rightarrow \infty \quad \text{as } t \rightarrow \infty. \end{aligned}$$

Hence

$$t^2 \left(\frac{y(t)}{t} \right)' > 0$$

for all t large enough. □

To simplify our notation we introduce the following triple of functions

$$\alpha_2(t) = \frac{p'(t)}{p(t)} + \frac{\tau'(t)}{\tau(t)} \int_t^{\tau(t)} p(s)\tau(s) ds,$$

$$\beta_2'(t) = \alpha_2(t), \quad \gamma_2(t) = p(t)e^{-\beta_2(t)}.$$

Lemma 2. *Let (2.1) hold and $\tau(t) \geq t$. Assume that $y(t) \in N_2$. Then*

$$\gamma_2(t)y(\tau(t)) \uparrow.$$

Proof. Assume that (E) possesses a positive increasing solution $y(t)$, i.e. $y(t) \in N_2$. Thanks to assumptions $p(t), \tau(t) \in C^1([t_0, \infty])$, we can differentiate (E), which leads to

$$y'''(t) = p'(t)y(\tau(t)) + p(t)\tau'(t)y'(\tau(t)). \quad (2.4)$$

On the other hand, integrating (E) from t to $\tau(t)$ we are lead to

$$\begin{aligned} y'(\tau(t)) &= y'(t) + \int_t^{\tau(t)} p(s)\tau(s) \frac{y(\tau(s))}{\tau(s)} ds \\ &\geq \frac{y(\tau(t))}{\tau(t)} \int_t^{\tau(t)} p(s)\tau(s) ds, \end{aligned} \quad (2.5)$$

where we used (2.2). By combining inequalities (2.4) and (2.5), we conclude that

$$y'''(t) \geq y(\tau(t)) \left(p'(t) + p(t) \frac{\tau'(t)}{\tau(t)} \int_t^{\tau(t)} p(s)\tau(s) ds \right),$$

which in view of (E) means that

$$y'''(t) \geq \alpha_2(t)y''(t).$$

Consequently,

$$\left(e^{-\beta_2(t)} y''(t) \right)' \geq 0$$

and we conclude that $e^{-\beta_2(t)} y''(t)$ is increasing, which is in view of (E) equivalent to $\gamma_2(t)y(\tau(t))$ is increasing. $\square$

Remark 1. The new monotonicity presented in Lemma 2 is in closed form for all possible functions $p(t)$ and as compared with previous results it does not require to distinguish whether $p(t) = p_*$ or $p(t) = p_*/t^2$ or something else. It is useful to notice that technique used in the proof of Lemma 2 is new and unique.

Having established new monotonicity we are prepared to improve criterion (1.2).

Theorem 1. Let $\tau(t) \geq t$ and (2.1) hold. If

$$\limsup_{t \rightarrow \infty} \gamma_2(t) \int_t^{\tau(t)} \frac{p(s)(\tau(t) - s)}{\gamma_2(s)} ds > 1, \quad (2.6)$$

then $N_2 = \emptyset$ for (E).

Proof. We argue by contradiction. Assume that (E) possesses an eventually positive increasing solution $y(t)$. Integrating (E) from t to u and using the monotonicity of $\gamma_2(t)y(\tau(t))$, we obtain

$$y'(u) = y'(t) + \int_t^u p(s)y(\tau(s)) ds \geq \gamma_2(t)y(\tau(t)) \int_t^u \frac{p(s)}{\gamma_2(s)} ds.$$

Integrating once more from u to t , we get

$$y(u) \geq \gamma_2(t)y(\tau(t)) \int_t^u \int_t^x \frac{p(s)}{\gamma_2(s)} ds dx = \gamma_2(t)y(\tau(t)) \int_t^u \frac{p(s)(u-s)}{\gamma_2(s)} ds.$$

Setting $u = \tau(t)$, we have

$$y(\tau(t)) \geq y(\tau(t))\gamma_2(t) \int_t^{\tau(t)} \frac{p(s)(\tau(t) - s)}{\gamma_2(s)} ds,$$

which contradicts to condition (2.6) and we conclude, that class N_2 is empty. $\square$

The above theorem can be reformulated in term of unbounded oscillation.

Theorem 2. *Let $\tau(t) \geq t$ and (2.1), (2.6) hold. Then every unbounded solution of (E) is oscillatory.*

If $1/\gamma_2(t)$ is increasing, then it is easy to see that (1.2) implies (2.6). In the following couple of illustrative examples we will demonstrate the real progress.

Example 1. We consider advanced differential equation with constant coefficients

$$y''(t) = p_*^2 y(t + 2\tau_*), \quad p_* > 0, \quad \tau_* > 0. \quad (E_{x1})$$

It is easy to verify that criterion (1.2) guarantees $N_2 = \emptyset$ provided that

$$p_* \tau_* > \frac{1}{\sqrt{2}} \approx 0.707106.$$

On the other hand, simple calculation yields

$$\alpha_2(t) = 2p_*^2 \tau_* + 2p_*^2 \tau_*^2 \frac{1}{t + 2\tau_*}.$$

Thus

$$\beta_2(t) = 2p_*^2 \tau_* t + 2p_*^2 \tau_*^2 \ln(t + 2\tau_*)$$

and

$$\gamma_2(t) = p_*^2 \frac{e^{-2p_*^2 \tau_* t}}{(t + 2\tau_*)^{2p_*^2 \tau_*^2}}$$

and we conclude that $\gamma_2(t)y(t + 2\tau_0)$ is increasing. But then also

$$\tilde{\gamma}_2(t)y(t + 2\tau_*) \uparrow \quad \text{with } \tilde{\gamma}_2(t) = e^{-2p_*^2 \tau_* t}.$$

The reason for introducing function $\tilde{\gamma}_2(t)$ is the fact that it is more acceptable than $\gamma_2(t)$ because the term $(t + 2\tau_0)^{-2p_*^2 \tau_*^2}$ does not bring any progress for monotonicity of $y(t) \in N_2$.

Condition (2.6) with $\gamma_2(t) = \tilde{\gamma}_2(t)$ reduces to

$$e^{4p_*^2 \tau_*^2} > 8p_*^2 \tau_*^2 + 1. \quad (2.7)$$

By standard numerical methods we can verify that the inequality

$$e^x > 2x + 1, \quad x > 0$$

is satisfied for all $x > 1.256431$ and consequently (2.7) holds true provided that

$$p_* \tau_* > \sqrt{\frac{1.256431}{4}} \approx 0.560453.$$

Consequently, our progress is significant.

Example 2. We consider the advanced Euler differential equation

$$y''(t) = \frac{p_*}{t^2} y(\lambda t), \quad (E_{x2})$$

with $p_* > 0$ and $\lambda > 1$. According to Koplatadze and Chanturia's criterion (1.2), the class $N_2 = \emptyset$ for Eq. (E_{x2}) provided that

$$p_*(\lambda - 1 - \ln \lambda) > 1. \quad (2.8)$$

On the other hand, simple calculation yields

$$\alpha_2(t) = \frac{-2 + p_* \lambda \ln \lambda}{t}.$$

Therefore,

$$\beta_2(t) = (-2 + p_* \lambda \ln \lambda) \ln t \quad \text{and} \quad \gamma_2(t) = t^{-p_* \lambda \ln \lambda}.$$

Now, some necessary calculation yields that (2.6) takes the form

$$p_* \left(\frac{\lambda^{p_* \lambda \ln \lambda} - \lambda}{p_* \lambda \ln \lambda - 1} - \frac{\lambda^{p_* \lambda \ln \lambda} - 1}{p_* \lambda \ln \lambda} \right) > 1. \quad (2.9)$$

To see our progress let us set e.g. $\lambda = e$. Then (2.8) takes the form

$$p_* > \frac{1}{e-2} \approx 1.392211,$$

while (2.9) reduces to

$$\frac{(e^{p_* e} - e) p_* e - (e^{p_* e} - 1)(p_* e - 1)}{p_* e - 1} > e. \quad (2.10)$$

It easy to verify that the inequality

$$\frac{(e^x - e)x - (e^x - 1)(x - 1)}{x - 1} > e, \quad x > 1$$

holds for all $x > 1.911063$. Thus (2.10) is satisfied for

$$p_* > \frac{1.911063}{e} \approx 0.703041.$$

Our progress is outstanding.

The next considerations are intended to improve criterion (1.1). We define triple of functions as follows.

$$\begin{aligned} \alpha_0(t) &= -\frac{p'(t)}{p(t)} + \tau'(t) \int_{\tau(t)}^t p(s) ds, \\ \beta'_0(t) &= \alpha_0(t), \quad \gamma_0(t) = p(t)e^{\beta_0(t)}. \end{aligned} \quad (2.11)$$

Lemma 3. Let $\tau(t) \leq t$. Assume that $y(t) \in N_0$. Then

$$\gamma_0(t)y(\tau(t)) \downarrow.$$

Proof. Assume that (E) possesses a positive decreasing solution $y(t)$, i.e. $y(t) \in N_0$. Proceeding exactly as in the proof of Theorem 2 we are led to (2.4). On the other hand, an integration (E) from $\tau(t)$ to t yields

$$-y'(\tau(t)) = -y'(t) + \int_{\tau(t)}^t p(s)y(\tau(s)) \, ds \geq y(\tau(t)) \int_{\tau(t)}^t p(s) \, ds. \quad (2.12)$$

We conclude

$$y'''(t) \leq y(\tau(t)) \left(p'(t) - p(t)\tau'(t) \int_{\tau(t)}^t p(s) \, ds \right),$$

by combining inequalities (2.4) and (2.12). Taking (E) into account, we obtain

$$y'''(t) + \alpha_0(t)y''(t) \leq 0.$$

Consequently,

$$\left(e^{\beta_0(t)} y''(t) \right)' \leq 0$$

and one can see that $e^{\beta_0(t)} y''(t)$ is decreasing, which is in view of (E) equivalent to $\gamma_0(t)y(\tau(t))$ is decreasing. $\square$

Theorem 3. Let $\tau(t) \leq t$. If

$$\limsup_{t \rightarrow \infty} \gamma_0(t) \int_{\tau(t)}^t \frac{p(s)(s - \tau(t))}{\gamma_0(s)} \, ds > 1, \quad (2.13)$$

then $N_0 = \emptyset$ for (E).

Proof. We admit that $N_0 \neq \emptyset$ which means that (E) possesses an eventually positive decreasing solution $y(t)$. Integrating (E) from u to t and using the monotonicity of $\gamma_0(t)y(\tau(t))$, we obtain

$$-y'(u) = -y'(t) + \int_u^t p(s)y(\tau(s)) \, ds \geq \gamma_0(t)y(\tau(t)) \int_u^t \frac{p(s)}{\gamma_0(s)} \, ds.$$

Integrating once more from u to t , we have

$$y(u) \geq \gamma_0(t)y(\tau(t)) \int_u^t \int_x^t \frac{p(s)}{\gamma_0(s)} \, ds \, dx = \gamma_0(t)y(\tau(t)) \int_u^t \frac{p(s)(s - u)}{\gamma_0(s)} \, ds.$$

Setting $u = \tau(t)$, we have

$$y(\tau(t)) \geq y(\tau(t))\gamma_0(t) \int_{\tau(t)}^t \frac{p(s)(s - \tau(t))}{\gamma_0(s)} \, ds,$$

which contradicts to condition (2.13) and we deduce that class N_0 is empty. $\square$

We provide an alternative formulation of the previous theorem.

Theorem 4. Let $\tau(t) \leq t$ and (2.13) hold. Then every bounded solution of (E) is oscillatory.

We again support our results by couple of illustrative examples.

Example 3. We consider retarded differential equation with constant coefficients

$$y''(t) = p_*^2 y(t - 2\tau_*), \quad p_* > 0, \quad \tau_* > 0. \quad (E_{x3})$$

By (1.1) the class $N_0 = \emptyset$ provided that

$$p_* \tau_* > \frac{1}{\sqrt{2}} \approx 0.707106.$$

On the other hand,

$$\alpha_0(t) = 2p_*^2 \tau_*, \quad \beta_0(t) = 2p_*^2 \tau_* t, \quad \gamma_0(t) = p_*^2 e^{2p_*^2 \tau_* t}.$$

Consequently condition (2.13) reduces to

$$e^{4p_*^2 \tau_*^2} > 8p_*^2 \tau_*^2 + 1. \quad (2.14)$$

Proceeding exactly as in Example 1 we see that (2.14) holds if

$$p_* \tau_* > \sqrt{\frac{1.256431}{4}} \approx 0.560453.$$

Example 4. Consider delay Euler differential equation

$$y''(t) = \frac{p_*}{t^2} y(\lambda t), \quad (E_{x4})$$

with $p_* > 0$ and $\lambda \in (0, 1)$. By (1.1), the class $N_0 = \emptyset$ for Eq. (E_{x4}) provided that

$$p_*(\lambda - 1 - \ln \lambda) > 1$$

which for $\lambda = 1/e$ takes the form

$$p_* > e = 2.718281.$$

On the other hand,

$$\alpha_0(t) = \frac{2 + p_*(1 - \lambda)}{t}, \quad \beta_0(t) = (2 + p_*(1 - \lambda)) \ln t, \quad \gamma_0(t) = p_* t^{p_*(1 - \lambda)}.$$

Now, (2.13) takes the form

$$p_* \left(\frac{\lambda^{-p_*(1 - \lambda)} - 1}{p_*(1 - \lambda)} - \frac{\lambda^{-p_*(1 - \lambda)} - \lambda}{1 + p_*(1 - \lambda)} \right) > 1$$

which is equivalent to

$$\lambda^{-p_*(1 - \lambda)} > 2 + 2p_*(1 - \lambda) - 2\lambda p_*(1 - \lambda) - \lambda. \quad (2.15)$$

We denote $x = p_*(1 - \lambda)$ and set $\lambda = 1/e$. Then (2.15) becomes algebraic inequality

$$e^x > 2 + 2x - 2e^{-1}x - e^{-1}, \quad x > 0$$

which holds true for all $x > 1.110604$ and so we deduce that

$$p_* > \frac{1.110604}{1 - e^{-1}} \approx 1.756950.$$

Our progress is relevant.

3. EXTENSION

From our previous results it is natural to expect that there will be no nonoscillatory solution, or equivalently all solutions will be oscillatory for certain differential equations involving both advanced and delayed arguments. The purpose of this section is to show that this is indeed for

$$y''(t) = p(t)y(\tau(t)) + q(t)y(\sigma(t)), \quad (E^*)$$

where $p(t), q(t), \tau(t), \sigma(t)$ obey the corresponding conditions presented in (H_1) and (H_2) and $\gamma_0(t)$ is defined by (2.11) with $p(t)$ and $\tau(t)$ replaced by $q(t)$ and $\sigma(t)$, respectively.

Theorem 5. Let $\tau(t) \geq t$ and (2.1), (2.6) hold. Moreover, assume that $\sigma(t) \leq t$ and

$$\limsup_{t \rightarrow \infty} \gamma_0(t) \int_{\sigma(t)}^t \frac{q(s)(s - \sigma(t))}{\gamma_0(s)} ds > 1,$$

then (E^*) is oscillatory.

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MONTE CARLO METHOD AND GROUP ALGEBRAS

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Abstract. Let FG be the group algebra of a finite p -group G over a finite field F of characteristic p . Let $\otimes$ be an involution of FG and $V_{\otimes}(FG)$ the $\otimes$ -unitary subgroup of FG . The order of $V_{\otimes}(FG)$ is known when p is an odd prime, and $\otimes$ arises from G , however the case of two characteristic is a challenging problem. The RAMEGA package of GAP system contains implementations of functions based on random methods related to group algebras. In this paper we provide the theoretical background of some random functions of RAMEGA that related to the $*$ -unitary subgroup of FG , where $*$ is the canonical involution. We estimate the order of the $*$ -unitary subgroup of FG for non-abelian 2-groups of order 2^5 using Monte Carlo method. Furthermore, we verify the estimated orders for certain groups of order 2^5 .

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1. INTRODUCTION

Let FG be the group algebra of a finite p -group G over a finite field F of characteristic p . We denote by $V(FG)$ the normalized unit group of FG , that is,

$$V(FG) = \left\{ x = \sum_{g \in G} \alpha_g g \in FG \mid \chi(x) = \sum_{g \in G} \alpha_g = 1 \right\}, \quad (1.1)$$

where $\chi(x)$ is the augmentation map of FG (see [13, Chapters 2-3, p. 194-196]). As a consequence, the order of $V(FG)$ is equal to $|F|^{|G|-1}$ and it is easy to see that $V(FG)$ may be very large even for small group basis G . Therefore, studying the structure of the normalized group of units is a really difficult task. While random methods may serve as effective tools for various purposes, group algebras have yet to be explored using such techniques. Using the formula (1.1) it is easy to obtain a sequence of uniformly distributed random normalized units. In this paper, we propose some random functions and take advantage of them to study the structure of normalized units. The

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implementations of the functions can be found in our RAMEGA package [3] of the GAP computer algebra system.

While numerous authors have delved into the investigation of the structure of $V(FG)$, our understanding of its structure and that of its subgroups remains limited. For an overview of this topic, we refer the reader to [10] and [13].

Assume $\ast$ acts as an anti-automorphism on G . This anti-automorphism can be linearly extended to FG , providing an algebra involution of FG which we shall also denote as $\ast$. In this context, we say that the algebra involution $\ast$ arises from the group basis G . An example of such an involution arising from G is the canonical $\ast$ -involution, representing the linear extension of the anti-automorphism on G that maps each element of G to its inverse. An element $u \in V(FG)$ is called $\ast$ -unitary if $u^\ast = u^{-1}$, with respect to an involution $\ast$ of FG . The set comprising all unitary elements of $V(FG)$ forms a subgroup denoted as $V_\ast(FG)$ and is referred to as the $\ast$ -unitary subgroup. Interest in the unitary subgroups first appeared in algebraic topology and unitary K-theory, as evidenced by Novikov's paper [19], playing a pivotal role in the examination of the structure of $V(FG)$. For more details we refer the reader to [13]. This subgroup has proven immensely valuable in various studies, as highlighted in [1, 4, 5, 7, 8, 12, 14, 16, 17]. Consider a finite Galois extension L of F with Galois group G , where F is a finite field of characteristic two. Serre, in [20], unveiled an intriguing connection between the self-dual normal basis of L over F and the unitary subgroup of FG . This correlation emphasizes the timeliness and relevance of investigating unitary subgroups.

Although the order of the $\ast$ -unitary subgroup of FG , where p is an odd prime and $\ast$ arises from G is well known [6], the case with characteristic two is a challenging problem. Let $G\{2\}$ be the set of all elements of G having order 2 with the identity. In [11], Bovdi and Sakach gave a formula for the order of the unitary subgroup when G is a finite abelian 2-group, proving that

Proposition 1 (Theorem 2, [11]). *Let G be a finite abelian 2-group and F a finite field of characteristic two. Then*

$$|V_\ast(FG)| = |G^2\{2\}| \cdot |F|^{\frac{1}{2}(|G|+|G\{2\}|-1)}.$$

Computing the order of $V_\ast(FG)$ remained an open question, when G is a non-abelian 2-group and has been addressed by many authors ever since ([2, 9, 15, 21, 22]). In this case it turned out that the order of the $\ast$ -unitary subgroup of FG determines the order of G . Let $\xi(G)$ denote the center of the group G and $\xi(G)\{2\}$ denote the set of elements of order two in $\xi(G)$.

Proposition 2 ([6]). *Let G be a finite 2-group. If F is a finite field of characteristic two, then*

$$|V_\ast(FG)| = \Theta \cdot |F|^{\frac{1}{2}(|G|+|G\{2\}|-1)}$$

for some integer Θ . Moreover, if the set $T_c = \{g \in G \mid g^2 = c\}$ is commutative for some $c \in \xi(G)\{2\}$, then Θ does not depend on the field F .

The parameter Θ has been determined for many groups ([2, 15, 21, 22]), but no general formula has been found for that, except the case when G is an abelian 2-group.

In the next section we present some random methods corresponding to the order of the $\ast$ -unitary subgroup showing their probability theoretical background. These methods together with several others can be found in the RAMEGA package [3] of GAP. With the help of random methods the order of $V_*(FG)$ can be estimated within a reasonable time even for larger group algebras. Using Monte Carlo method we show that Θ can be estimated for all the groups G of order 2^5 . We also verify the estimated orders for certain groups of order 2^5 .

2. MONTE CARLO METHOD

Let FG be a group algebra of a finite p -group G over a finite field F of characteristic p . Every element of FG can be written as

$$x = \sum_{g \in G} \alpha_g,$$

where $\alpha \in F$. Therefore it is easy to generate a random element from FG with a uniform distribution.

RAMEGA [3] stands for RANdom MEthods in Group Algebras and includes several random methods for studying group algebras. There are also some functions available for the $\ast$ -unitary subgroup, such as *GetRandomNormalizedUnitaryUnit*, *RandomUnitarySubgroup* or *RandomUnitaryOrder*. The function *RandomUnitaryOrder* estimates the order of the $\ast$ -unitary subgroup of FG using Monte Carlo method. For odd primes the function returns the exact value of the order (see Theorem 1 in [6]), however general formula for the order is not known when the group basis G is a non-abelian 2-group. Therefore we deal with only the case when $|F| = 2^m$ for some m and G is a group of order 2^n for some n .

Let us randomly select a unit ξ from the normalized unit group $V(FG)$. The RAMEGA function *GetRandomUnit* can generate a random normalized unit with uniform distribution. Consider the experiment as success if the selected unit is unitary, that is $\xi\xi^* = 1$, otherwise it is failure. To be more precise let ξ be a random variables defined by the following way:

$$\xi = \begin{cases} 1 & \text{if } \xi \text{ is a unitary unit;} \\ 0 & \text{if } \xi \text{ is not a unitary unit.} \end{cases}$$

It is well-known that the distribution of ξ is Bernoulli with parameter q , where $q = \frac{|V_*(FG)|}{|V(FG)|}$. Let us denote by η the distribution of the number k of the Bernoulli trials needed to get one success. The probability $P(\eta = k)$ is equal to $(1 - q)^{k-1}q$ therefore η has geometric distribution and its mean $\mu = E(\eta) = \frac{1}{q}$ and its standard deviation is $\sigma = \frac{\sqrt{1-q}}{q}$. Thus $|V_*(FG)|$ can be estimated as $|V_*(FG)| = q \cdot |V(FG)| = \frac{|V(FG)|}{E(\eta)}$.

Since $|G| \leq |V_*(FG)| \leq |V(FG)| = |F|^{|G|-1} = 2^{m|G|-m}$ we conclude that $q = \frac{1}{2^{i_0}}$, where $0 \leq i_0 \leq m|G| - m - n$. The central limit theorem asserts that as the number of replications n increases, the standardized estimator $\frac{\hat{\mu} - \mu}{\frac{\sigma}{\sqrt{n}}}$ converges in distribution to the standard normal, where $\hat{\mu} = \frac{\eta_1 + \eta_2 + \dots + \eta_n}{n}$, $\mu = \frac{1}{q}$ and $\sigma = \sqrt{\mu(\mu - 1)}$. Therefore

$$\lim_{n \rightarrow \infty} P(|\hat{\mu} - \mu| \leq \frac{x\sigma}{\sqrt{n}}) = \Phi(x).$$

The algorithm works as a statistical test with null hypothesis that the mean $E(\frac{\hat{\mu} - \mu}{\frac{\sigma}{\sqrt{n}}})$ is zero. The test is at the Z percent confidence level if

$$P(|\hat{\mu} - \mu| \leq z_p \frac{\sigma}{\sqrt{n}}) = \frac{Z}{100},$$

where $z_p = \Phi^{-1}(\frac{Z}{100})$ and the corresponding confidence interval is $(-z_p \frac{\sigma}{\sqrt{n}}, z_p \frac{\sigma}{\sqrt{n}})$.

Consider n trials such that the i th random sample η_i is x_i . If n is large enough, then $|V_*(FG)|$ can be estimated by $\frac{n \cdot |V(FG)|}{x_1 + x_2 + \dots + x_n} = \frac{n \cdot |F|^{|G|-1}}{x_1 + x_2 + \dots + x_n}$. Since $|V_*(FG)|$ is a p -group $\frac{n \cdot |F|^{|G|-1}}{x_1 + x_2 + \dots + x_n}$ has to be round to the closest power of p .

The pseudocode of our algorithm can be seen in Algorithm 1.

Algorithm 1 Order of unitary subgroup by random way using geometric distribution

```

function RANDOMUNITARYORDER( $kg, n$ )                                ▷  $kg$  is the group algebra,  $n$  is the number of trials
   $mean \leftarrow 0$ 
   $trials \leftarrow []$                                                 ▷ empty list
   $counter \leftarrow 0$ 
   $grouporder \leftarrow \text{order of } G$ 
   $fieldsize \leftarrow \text{size of } F$ 
   $p \leftarrow \text{characteristic of } F$ 
  repeat
     $m \leftarrow 0$ 
    repeat
       $x \leftarrow \text{random normalized unit}$ 
       $m \leftarrow m + 1$ 
    until  $x$  is unitary
     $trials \leftarrow m$ ;
  until  $counter = n$ 
   $mean = \text{Sum}(trials) / \text{Number}(trials)$ ;
   $min \leftarrow n(mean - p)^2 / p$ 
   $position \leftarrow 0$ 
  if  $1 < mean$  then
    for  $i = 1$  to  $\text{LogInt}(fieldsize, p) \cdot (grouporder - 1)$  do
       $index \leftarrow (n * (mean - p^i)^2) / (p^i * (p^i - 1))$ 
      if  $index < min$  then
         $min \leftarrow index$ 
         $position \leftarrow i$ 
      end if
    end for
  end if
  return  $fieldsize^{(grouporder-1)} / p^{position}$ 
end function

```

Using package RAMEGA we can estimate the order of the $*$ -unitary subgroups for the groups of order 2^5 within a reasonable time. For the sake of convenience G_i

represents the group that is returned by the GAP function $SmallGroup(2^5, i)$ using the library of small groups of GAP [18].

Conjecture 1. *Let G be a non-abelian group of order 2^5 and F is a finite field of characteristic two. Then $|V_*(FG)| = \Theta \cdot |F|^{\frac{1}{2}(|G|+|G\{2\}|)-1}$, where*

- (i) $\Theta = 1$ if $G \in \{G_{18}, G_{27}, G_{28}, G_{34}, G_{39}, G_{42}, G_{43}, G_{46}, G_{48}, G_{49}, G_{50}\};$
- (ii) $\Theta = 2$ if

$$G \in \{G_5, G_6, G_7, G_9, G_{11}, G_{17}, G_{19}, G_{22}, G_{25}, G_{30}, G_{31}, G_{37}, G_{38}, G_{40}, G_{44}\};$$

- (iii) $\Theta = 4$ if

$$G \in \{G_2, G_4, G_8, G_{10}, G_{12}, G_{13}, G_{14}, G_{15}, G_{20}, G_{23}, G_{24}, G_{29}, G_{33}, G_{41}, G_{47}\};$$

- (iii) $\Theta = 8$ if $G \in \{G_{26}, G_{32}, G_{35}\}.$

Proposition 3 (Lemma 2.6, [22]). *Let G be a finite group and A an elementary abelian 2-group. If $|V_*(FG)| = \Theta \cdot |F|^{\frac{1}{2}(|G|+|G\{2\}|)-1}$, then*

$$|V_*(F(G \times A))| = \Theta \cdot |F|^{\frac{1}{2}(|G \times A|+|(G \times A)\{2\}|)-1}.$$

Let H be a normal subgroup of G . Let us define S_H to be the set

$$\{xx^* \mid \Psi(x) \in V_*(F\overline{G})\},$$

where $\overline{G} = G/H$ and Ψ is the natural homomorphism from FG to $F\overline{G}$. We will use $\widehat{H}$ to denote the sum of the elements of H in FG .

Theorem 1. *Conjecture 1 is true for the following groups $G_2, G_5, G_{17}, G_{18}, G_{20}, G_{22}, G_{23}, G_{37}, G_{39}, G_{40}, G_{41}, G_{46}, G_{47}, G_{48}$.*

Proof. According to Proposition 3 and Theorem 1.4 in [2]

- (i) $\Theta = 1$ if $G \in \{G_{48} \cong D_8 \times C_4 \times C_2, G_{39} \cong D_{16} \times C_2, G_{46} \cong D_8 \times C_2 \times C_2\};$
- (ii) $\Theta = 2$ if $G \in \{G_{37} \cong M_{16} \times C_2, G_{40} \cong D_{16}^- \times C_2, G_{22} \cong H_{16} \times C_2\};$
- (iii) $\Theta = 4$ if $G \in \{G_{41} \cong Q_{16} \times C_2, G_{23} \cong C_4 \times C_4 \times C_2, G_{47} \cong Q_8 \times C_2 \times C_2\}.$

Let $G = G_2$. Then $G' \cong C_2$, $\overline{G} = G/G' \cong C_4 \times C_4$ and $S_{G'} = \langle 1 + \alpha(g + g^{-1})\widehat{G'} \mid \alpha \in F, g \in G \setminus G\{2\} \rangle$. According to Lemma 1 in [6]

$$|V_*(FG)| = |F|^{|G|} \cdot \frac{|V_*(F\overline{G})|}{|S_{G'}|} = |F|^{\frac{1}{2}|G|} \cdot \frac{|V_*(F\overline{G})|}{|F|^{\frac{1}{4}(|G|-|G\{2\}|)}} = |F|^{\frac{1}{4}(|G|+|G\{2\}|)} \cdot |V_*(F\overline{G})|.$$

By Theorem 2 in [11] and the fact that $|G\{2\}| = 2|\overline{G}\{2\}|$ we have

$$|V_*(F\overline{G})| = |\overline{G}^2\{2\}| \cdot |F|^{\frac{1}{2}(|\overline{G}|+|\overline{G}\{2\}|)-1} = 4 \cdot |F|^{\frac{1}{2}(|\overline{G}|+|\overline{G}\{2\}|)-1} = 4 \cdot |F|^{\frac{1}{4}(|G|+|G\{2\}|)-1}.$$

Therefore

$$|V_*(FG)| = |F|^{\frac{1}{4}(|G|+|G\{2\}|)} \cdot |V_*(F\overline{G})| = 4 \cdot |F|^{\frac{1}{2}(|G|+|G\{2\}|)-1},$$

which proves that $\Theta = 4$.

Let $G = G_5$. Then $G' \cong C_2$, $\overline{G} = G/G' \cong C_8 \times C_2$ and $S_{G'} = \langle 1 + \alpha(g + g^{-1})\widehat{G'} \mid \alpha \in F, g \in G \setminus G\{2\} \rangle$. According to Lemma 1 in [6]

$$|V_*(FG)| = |F|^{\frac{1}{4}(|G|+|G\{2\}|)} \cdot |V_*(F\overline{G})|.$$

By Theorem 2 in [11] and the fact that $|G\{2\}| = 2|\overline{G}\{2\}|$ we have

$$|V_*(F\overline{G})| = |\overline{G}^2\{2\}| \cdot |F|^{\frac{1}{2}(|\overline{G}|+|\overline{G}\{2\}|)-1} = 2 \cdot |F|^{\frac{1}{2}(|\overline{G}|+|\overline{G}\{2\}|)-1} = 2 \cdot |F|^{\frac{1}{4}|G|+\frac{|G\{2\}|}{4}-1}.$$

Therefore

$$|V_*(FG)| = |F|^{\frac{1}{4}(|G|+|G\{2\}|)} \cdot |V_*(F\overline{G})| = 2 \cdot |F|^{\frac{1}{2}|G|+\frac{|G\{2\}|}{4}+\frac{|G\{2\}|}{4}-1},$$

which proves that $\Theta = 2$.

Let $G = G_{17}$. According to Theorem 1.1 in [5] $|V_*(FG)| = 2 \cdot |F|^{\frac{1}{2}|G|+1}$. Since $|G\{2\}| = 4$ we have $|V_*(FG)| = 2 \cdot |F|^{\frac{1}{2}(|G|+|G\{2\}|)-1}$ so $\Theta = 2$.

Let $G = G_{18} \cong D_{32}$. Then $|G\{2\}| = \frac{|G|}{2} + 2$ and by Corollary 2 in [15]

$$|V_*(FG)| = |F|^{3\frac{|G|}{4}} = |F|^{\frac{|G|}{4}+\frac{|G|}{2}}.$$

Therefore

$$|V_*(FG)| = |F|^{\frac{|G|}{4}-\frac{|G\{2\}|}{2}+1} \cdot |F|^{\frac{1}{2}(|G|+|G\{2\}|)-1}.$$

By Proposition 2 $\Theta = |F|^{\frac{|G|}{4}-\frac{|G\{2\}|}{2}+1} = |F|^{\frac{|G|}{4}-\frac{|G|}{4}-1+1} = 1$.

Let $G = G_{20} \cong Q_{32}$. By Corollary 2 in [15] and the fact that $|G\{2\}| = 2$

$$|V_*(FG)| = 4 \cdot |F|^{\frac{|G|}{2}} = 4 \cdot |F|^{\frac{1}{2}(|G|+|G\{2\}|)-1}.$$

Therefore $\Theta = 4$ by Proposition 2. □

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HYERS-ULAM-RASSIAS STABILITY OF SOME SEQUENTIAL NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS WITH CAPUTO-HADAMARD FRACTIONAL DERIVATIVE

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Abstract. In this article, we employ a fixed point theory to investigate the stability in the sense of Hyers-Ulam-Rassias of some sequential neutral functional differential equations with Caputo-Hadamard fractional derivative. We present two examples to illustrate our main results. In this way, we generalize several earlier outcomes.

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1. INTRODUCTION

Functional equations grow in an exponential way in many applications like networks (see e.g. [11, 14]). Particular cases of functional equations are functional differential equations that are also of central importance in many disciplines such as control theory, neural networks, epidemiology, etc. [1]. Fractional derivatives are capable of describing hereditary and memory effects in many processes and materials like e.g., bioengineering, biology, aerodynamics, and chemistry. So the study of neutral functional differential equations in presence of fractional derivatives constitutes an important area of research. For more details, see the text [31].

Recently, there has been a considerable progress in fractional calculus, and many other problems of fractional differential equations, see e.g. [21], and Podlubny [23]. It should be noted that, much of the work on the topic involves Caputo and Riemann-Liouville type fractional derivatives (see e.g. [4, 9, 26, 29, 30]). Besides the mentioned fractional derivatives, there is another fractional derivative introduced by Hadamard in 1892 (see [16]). The new derivative introduced by Hadamard is known as Hadamard derivative and differs from above mentioned derivatives in the presence of logarithmic function of arbitrary exponent in its kernel. It should be remarked that Jarad

et.al. in [19] modified the fractional derivative of the mentioned Hadamard type into a more suitable one with physically interpretable initial conditions comparable to the Caputo setting and named it Caputo-Hadamard fractional derivative. For somehow detailed explanation of Hadamard fractional derivative and integral, the reader is advised to see the articles [12].

Stability theory arose as a consequence of the famous talk delivered by Ulam at a conference held in Wisconsin University in 1940 (see e.g. [17]). Ulam's type of stability is useful, because it guarantees that there is a close exact solution. Nowadays, it becomes a research trend in many applications (see e.g. [18] for more references and details). The famous stability question presented by Ulam can be rewritten as follows:

If $\mathfrak{G}$ is a group and $(\mathfrak{G}^*, \rho)$ is a metric group. Is it true that for $\varepsilon > 0$, there exist a $\delta > 0$ such that if $F : \mathfrak{G} \rightarrow \mathfrak{G}^*$ satisfies

$$\rho(F(ab), F(a)F(b)) < \delta$$

for all $a, b \in \mathfrak{G}$, then a homomorphism $g : \mathfrak{G} \rightarrow \mathfrak{G}^*$ exists such that

$$\rho(F(a), g(a)) < \varepsilon$$

for all $a \in \mathfrak{G}$?

The open problem mentioned above has been handled by many mathematicians as follows. In 1941, D. H. Hyers introduced a positive answer to it in case of Banach spaces. Since then, the stability problem is called Ulam-Hyers or Hyers-Ulam stability problem. The most important result after Hyers seems to be that introduced in 1978 by Rassias (see [25]). The idea of Rassias is simply a generalization of the result of Hyers by considering the stability in the case of unbounded Cauchy differences. The result obtained by Rassias can be rewritten as follows (see [25]):

Theorem 1. *Consider two Banach spaces $\mathfrak{B}, \mathfrak{B}^*$, and a continuous mapping f from $\mathbb{R}$ into $\mathfrak{B}^*$. Suppose that there exists $\omega \geq 0$ and $\vartheta \in [0, 1)$ such that*

$$\|f(b_1 + b_2) - f(b_1) - f(b_2)\| \leq \omega(\|b_1\|^\vartheta + \|b_2\|^\vartheta), \quad b_1, b_2 \in \mathfrak{B}^* \setminus \{0\}.$$

Then there exists a unique solution $f^ : \mathfrak{B} \rightarrow \mathfrak{B}^*$ of the Cauchy equation with*

$$\|f(b_1) - f^*(b_1)\| \leq \frac{2\omega\|b_1\|^\vartheta}{|2 - 2^\vartheta|}, \quad b_1 \in \mathfrak{B}^* \setminus \{0\}.$$

Since then, the stability problem is known as the Ulam-Hyers-Rassias or the Hyers-Ulam-Rassias stability (see also [10]).

During the last three decades, stability of differential equations has been a focus of scientific investigations by many researchers see e.g. [6, 22]. As a consequence of the interesting results presented in this direction, many articles devoted to this subject have been written (see e.g. [8, 20, 24]. In 2019, Shikhare and Kucche (see [28]), employed weakly Picard operator to investigate the stability of some kind of equations in Banach Spaces in the sense of Hyers-Ulam. Furthermore, they obtained stability in

the sense of Ulam-Hyers-Rassias for such kind of equations via Pachpatte's integral inequalities. Also Shah and Zada in 2019 (see [27]) used a fixed point approach to investigate the stability of impulsive Volterra integral equation. In 2020, the authors in [5] investigated the stability of some general differential equation using fixed point approach. See also [15], where the authors studied the stability of some Caputo fractional differential equations using fixed point approach. Note that in the case of the Riemann-Liouville fractional derivative only the case without any delays is studied (see e.g. [7]). The authors in [2] investigated the Hyers-Ulam stability of the non-linear fractional stochastic neutral differential equations system. In [3], the authors studied the Ulam stability of Caputo type fractional stochastic neutral differential equations.

The article is divided into three sections. In the next section we recall some preliminaries, section 3 shows the main results: stability results in Hyers-Ulam-Rassias sense, in section 4, we used two examples to illustrate our results, and in section 5 we conclude our work.

2. PRELIMINARIES

We use this section to introduce notations, some definitions, and recall some well-known results. We start to recall the notions of the Hadamard fractional integral and derivative respectively as follows. Throughout the paper, we use $\mathbb{R}$ to denote the set of real numbers and $C([a, b], \mathbb{R})$ to denote the set of continuous functions from an interval $[a, b]$ into the set $\mathbb{R}$.

Definition 1. [21] The Hadamard fractional integral of order λ for a function h is defined as

$$I^\lambda h(t) = \frac{1}{\Gamma(\lambda)} \int_1^t \left(\log \frac{t}{v} \right)^{\lambda-1} \frac{h(v)}{v} dv, \quad \lambda > 0,$$

provided the integral exists.

Definition 2. ([21]) The Hadamard derivative of fractional order $\lambda \in (0, 1)$ for a function $h : [1, \infty) \rightarrow \mathbb{R}$ is defined as

$$D^\lambda h(t) = \frac{1}{\Gamma(1-\lambda)} \left(t \frac{d}{dt} \right) \int_1^t \left(\log \frac{t}{v} \right)^{-\lambda} \frac{h(v)}{v} dv,$$

with $\log(\cdot) = \log_e(\cdot)$.

Now, we recall the notion of the Caputo-Hadamard fractional derivative as follows.

Definition 3. [21] The Caputo-Hadamard derivative of fractional order $\lambda \in (0, 1)$ for a function $h : [1, \infty) \rightarrow \mathbb{R}$ is defined as

$${}^C D^\lambda h(t) = D^\lambda [h(t) - h(1)].$$

Now, the concept of contractive operator (see e.g. [13])

Definition 4. ([13]) Let $(X, \varkappa)$ be a generalized complete metric space. The mapping $\Pi : X \rightarrow X$ is called a contraction if there exists a constant q_1 with $0 < q_1 < 1$ such that whenever $\varkappa(x_1, x_2) < \infty$ one has

$$\varkappa(\Pi x_1, \Pi x_2) \leq q_1 \varkappa(x_1, x_2).$$

The following is an important results that will play an important role in our analysis. We recall it from [13].

Theorem 2. Suppose (F, d) is a complete metric space and $L : F \rightarrow F$ is a contraction (for some $\delta \in [0, 1]$), $d(L(x), L(y)) \leq \delta d(x, y)$ for all $x, y \in F$. Suppose that there exists $v \in F$, $\chi > 0$ and $d(v, L(v)) \leq \chi$. Then, there exists a unique $k \in F$ such that $k = L(k)$. Moreover,

$$d(v, k) \leq \frac{\chi}{1 - \delta}.$$

The purpose of the article is to study the stability of the solution of the following Caputo-Hadamard sequential fractional order neutral functional differential equations

$$D^{\nu_1} [D^{\nu_2} y(\sigma) - f_2(\sigma, y_\sigma)] = f_1(\sigma, y_\sigma), \quad \sigma \in I := [1, b], \quad (2.1)$$

with initial conditions $y(\sigma) = \phi(\sigma)$, $\sigma \in [1 - \kappa, 1]$, $D^{\nu_2} y(1) = \eta \in \mathbb{R}$, where $\phi \in C([1 - \kappa, 1], \mathbb{R})$, $0 < \nu_1, \nu_2 < 1$, $f_1, f_2 : I \times C([- \kappa, 0], \mathbb{R}) \rightarrow \mathbb{R}$ are given functions.

For any function y defined on $[1 - \kappa, b]$ and any $t \in I$, we denote by y_σ the element of $C_\kappa := C([- \kappa, 0], \mathbb{R})$ defined by $y_\sigma(s) = y(\sigma + s)$, $s \in [- \kappa, 0]$, with norm $\|y_\sigma\| = \sup\{y(\sigma + s); -\kappa \leq s \leq 0\}$.

3. MAIN RESULTS

Here, we apply the Banach fixed point theory to study stability of (2.1) in the sense of Hyers-Ulam-Rassias. The following theorem is our main theorem, where we show that under some conditions functions that satisfy (2.1) approximately (in some sense) are close (in some way) to the solutions of (2.1). In other words, we investigate the Hyers-Ulam-Rassias stability of (2.1).

Theorem 3. Let $f_1 : I \times \mathbb{R} \rightarrow \mathbb{R}$, $f_2 : I \times \mathbb{R} \rightarrow \mathbb{R}$ are some continuous functions satisfying the following conditions

$$|f_i(\sigma, \phi_1) - f_i(\sigma, \phi_2)| \leq l_i \|\phi_1 - \phi_2\|,$$

for all $\sigma \in I$, $\phi_1, \phi_2 \in C_\kappa$ and for some $l_i > 0$ $i = 1, 2$.

If $z \in C^2([1 - \kappa, b], \mathbb{R})$ satisfies

$$|D^{\nu_1} [D^{\nu_2} z(\sigma) - f_2(\sigma, z_\sigma)] - f_1(\sigma, z_\sigma)| \leq \varepsilon \gamma(\sigma),$$

for all $\sigma \in [1, b]$, where $\varepsilon > 0$ and $\gamma(\sigma)$ is a positive, nondecreasing, continuous function, then there exists a unique solution z^* of (2.1) with $z^*(\sigma) = z(\sigma)$, $\sigma \in$

$[1 - \kappa, 1]$, such that

$$|z(\sigma) - z^*(\sigma)| \leq \left(\frac{1}{1 - \left(\frac{l_1}{\mu^{v_1+v_2}} + \frac{l_2}{\mu^{v_1}} \right)} \right) \frac{Mb^\mu}{\Gamma(v_1 + v_2 + 1)} \varepsilon \gamma(\sigma), \quad \forall \sigma \in [1, b],$$

where $M = \sup_{s \in [1, b]} \left(\frac{(\log s)^{v_1+v_2}}{s^\mu} \right)$ and μ is a positive constant such that

$$\left(\frac{l_1}{\mu^{v_1+v_2}} + \frac{l_2}{\mu^{v_1}} \right) < 1.$$

Proof. Consider the metric d on $E = C([1 - \kappa, b], \mathbb{R})$ by

$$d(y_1, y_2) = \inf \left\{ k \in [0, \infty) : \frac{|y_1(\sigma) - y_2(\sigma)|}{\psi(\sigma)} \leq k \tilde{\gamma}(\sigma), \forall \sigma \in [1 - \kappa, b] \right\},$$

with $\psi(\sigma) = \sigma^\mu$ for $\sigma \in [1, b]$ and $\psi(\sigma) = 1$ for $\sigma \in [1 - \kappa, 1]$, and $\tilde{\gamma}(\sigma) = \gamma(\sigma)$ for $\sigma \in [1, b]$ and $\tilde{\gamma}(\sigma) = \gamma(1)$ for $\sigma \in [1 - \kappa, 1]$.

We consider the operator $\mathcal{B} : E \rightarrow E$ such that $(\mathcal{B}y)(\sigma) = z(\sigma)$, for $\sigma \in [1 - \kappa, 1]$, and

$$\begin{aligned} (\mathcal{B}y)(\sigma) &= z(1) + (D^{v_2}z(1) - f_2(1, z(1))) \frac{(\log \sigma)^{v_2}}{\Gamma(v_2 + 1)} \\ &\quad + \frac{1}{\Gamma(v_1)} \int_1^\sigma \left(\log \frac{\sigma}{s} \right)^{v_1-1} \frac{f_2(s, y_s)}{s} ds \\ &\quad + \frac{1}{\Gamma(v_1 + v_2)} \int_1^\sigma \left(\log \frac{\sigma}{s} \right)^{v_1+v_2-1} \frac{f_2(s, y_s)}{s} ds. \end{aligned}$$

Let $y_1, y_2 \in E$, we have $(\mathcal{B}y_1)(\sigma) - (\mathcal{B}y_2)(\sigma) = 0$, for all $\sigma \in [1 - \kappa, 1]$.

For $\sigma \in [1, b]$, we get

$$\begin{aligned} & \left| (\mathcal{B}y_1)(\sigma) - (\mathcal{B}y_2)(\sigma) \right| \\ & \leq \frac{1}{\Gamma(v_1)} \int_1^\sigma \left(\log \frac{\sigma}{s} \right)^{v_1-1} \left| \frac{f_2(s, y_{1s}) - f_2(s, y_{2s})}{s} \right| ds \\ & \quad + \frac{1}{\Gamma(v_1 + v_2)} \int_1^\sigma \left(\log \frac{\sigma}{s} \right)^{v_1+v_2-1} \left| \frac{f_1(s, y_{1s}) - f_1(s, y_{2s})}{s} \right| ds \\ & \leq \frac{l_2}{\Gamma(v_1)} \int_1^\sigma \left(\log \frac{\sigma}{s} \right)^{v_1-1} \frac{\|y_{1s} - y_{2s}\|}{s} ds \\ & \quad + \frac{l_1}{\Gamma(v_1 + v_2)} \int_1^\sigma \left(\log \frac{\sigma}{s} \right)^{v_1+v_2-1} \frac{\|y_{1s} - y_{2s}\|}{s} ds. \end{aligned}$$

For $s \in [1, \sigma]$, there is $q \in [-\kappa, 0]$ such that $\|y_{1s} - y_{2s}\| = |y_1(s+q) - y_2(s+q)|$. Therefore,

$$\begin{aligned} \|y_{1s} - y_{2s}\| &= \frac{|y_1(s+q) - y_2(s+q)|}{\Psi(s+q)\tilde{\gamma}(s+q)} \Psi(s+q)\tilde{\gamma}(s+q) \\ &\leq d(y_1, y_2) \Psi(s+q)\tilde{\gamma}(s+q) \leq d(y_1, y_2) \Psi(s)\gamma(s). \end{aligned}$$

Therefore,

$$\begin{aligned} \left| (\mathcal{B}y_1)(\sigma) - (\mathcal{B}y_2)(\sigma) \right| &\leq \frac{l_2}{\Gamma(v_1)} d(y_1, y_2) \gamma(\sigma) \int_1^\sigma \left(\log \frac{\sigma}{s} \right)^{v_1-1} \frac{s^\mu}{s} ds \\ &\quad + \frac{l_1}{\Gamma(v_1 + v_2)} d(y_1, y_2) \gamma(\sigma) \int_1^\sigma \left(\log \frac{\sigma}{s} \right)^{v_1+v_2-1} \frac{s^\mu}{s} ds. \end{aligned}$$

For $\lambda > 0$, by using the change of variable $u = \mu \log \sigma - \mu \log s$, we get

$$\int_1^\sigma \left(\log \frac{\sigma}{s} \right)^{\lambda-1} \frac{s^\mu}{s} ds = \int_0^{\mu \log \sigma} \left(\frac{u}{\mu} \right)^{\lambda-1} \sigma^\mu \frac{e^{-u}}{\mu} du \leq \frac{\sigma^\mu}{\mu^\lambda} \Gamma(\lambda).$$

Then,

$$\begin{aligned} \left| (\mathcal{B}y_1)(\sigma) - (\mathcal{B}y_2)(\sigma) \right| &\leq l_2 d(y_1, y_2) \gamma(\sigma) \frac{\sigma^\mu}{\mu^{v_1}} + l_1 d(y_1, y_2) \gamma(\sigma) \frac{\sigma^\mu}{\mu^{v_1+v_2}} \\ &\leq \left(\frac{l_1}{\mu^{v_1+v_2}} + \frac{l_2}{\mu^{v_1}} \right) d(y_1, y_2) \gamma(\sigma) \sigma^\mu \end{aligned}$$

Then, $\mathcal{B}$ is contractive (see Definition 4). For $\sigma \in [1 - \kappa, 1]$, we have

$$(\mathcal{B}z)(\sigma) - z(\sigma) = 0.$$

We have

$$|D^{v_1}[D^{v_2}z(\sigma) - f_2(\sigma, z_\sigma)] - f_1(\sigma, z_\sigma)| \leq \varepsilon \gamma(\sigma), \forall \sigma \in [1, b].$$

By using Lemma 2.3 in [1], we get

$$\begin{aligned} |z(\sigma) - \mathcal{B}z(\sigma)| &\leq \frac{\varepsilon}{\Gamma(v_1 + v_2)} \int_1^\sigma \left(\log \frac{\sigma}{s} \right)^{v_1+v_2-1} \frac{\gamma(s)}{s} ds \\ &\leq \frac{\varepsilon \gamma(\sigma)}{\Gamma(v_1 + v_2)} \int_1^\sigma \left(\log \frac{\sigma}{s} \right)^{v_1+v_2-1} \frac{1}{s} ds \\ &\leq \frac{\varepsilon \gamma(\sigma)}{\Gamma(v_1 + v_2 + 1)} (\log \sigma)^{v_1+v_2}, \forall \sigma \in [1, b]. \end{aligned}$$

Hence

$$\begin{aligned} \frac{|z(\sigma) - \mathcal{B}z(\sigma)|}{\sigma^\mu} &\leq \frac{\varepsilon}{\Gamma(v_1 + v_2 + 1)} \gamma(\sigma) \frac{(\log \sigma)^{v_1+v_2}}{\sigma^\mu} \\ &\leq \frac{\varepsilon M}{\Gamma(v_1 + v_2 + 1)} \gamma(\sigma), \quad \forall \sigma \in [1, b], \end{aligned}$$

then

$$d(z, \mathcal{B}z) \leq \varepsilon \frac{M}{\Gamma(\mathfrak{v}_1 + \mathfrak{v}_2 + 1)}.$$

Using Theorem 2, there exists a solution z^* of (2.1) such that

$$d(z, z^*) \leq \varepsilon \left(\frac{1}{1 - \left(\frac{l_1}{\mu^{\mathfrak{v}_1 + \mathfrak{v}_2}} + \frac{l_2}{\mu^{\mathfrak{v}_1}} \right)} \right) \frac{M}{\Gamma(\mathfrak{v}_1 + \mathfrak{v}_2 + 1)},$$

so that

$$|z(\sigma) - z^*(\sigma)| \leq \left(\frac{1}{1 - \left(\frac{l_1}{\mu^{\mathfrak{v}_1 + \mathfrak{v}_2}} + \frac{l_2}{\mu^{\mathfrak{v}_1}} \right)} \right) \frac{Mb^\mu}{\Gamma(\mathfrak{v}_1 + \mathfrak{v}_2 + 1)} \varepsilon \gamma(\sigma),$$

for all $\sigma \in [1, b]$. $\square$

Remark 1. It should be remarked that in our analysis, we proved the existence as a part of the stability results in Theorem 3. Moreover, we get rid off the essential condition in Theorem 3.2 in [1].

The following theorem represents the stability of (2.1) in the sense of Ulam-Hyers.

Theorem 4. Let $f_1 : I \times \mathbb{R} \rightarrow \mathbb{R}$, $f_2 : I \times \mathbb{R} \rightarrow \mathbb{R}$ are some continuous functions satisfying the following conditions

$$|f_i(\sigma, \phi_1) - f_i(\sigma, \phi_2)| \leq l_i \|\phi_1 - \phi_2\|,$$

for all $\sigma \in I$, $\phi_1, \phi_2 \in C_K$ and for some $l_i > 0$ $i = 1, 2$.

If $z \in C^2([1 - \kappa, b], \mathbb{R})$ satisfies

$$|D^{\mathfrak{v}_1}[D^{\mathfrak{v}_2}z(\sigma) - f_2(\sigma, z_\sigma)] - f_1(\sigma, z_\sigma)| \leq \varepsilon,$$

for all $\sigma \in [1, b]$, where $\varepsilon > 0$, then there exists a unique solution z^* of (2.1) with $z^*(\sigma) = z(\sigma)$, $\sigma \in [1 - \kappa, 1]$, such that

$$|z(\sigma) - z^*(\sigma)| \leq \left(\frac{1}{1 - \left(\frac{l_1}{\mu^{\mathfrak{v}_1 + \mathfrak{v}_2}} + \frac{l_2}{\mu^{\mathfrak{v}_1}} \right)} \right) \frac{Mb^\mu}{\Gamma(\mathfrak{v}_1 + \mathfrak{v}_2 + 1)} \varepsilon, \forall \sigma \in [1, b],$$

where $M = \sup_{s \in [1, b]} \left(\frac{(\log s)^{\mathfrak{v}_1 + \mathfrak{v}_2}}{s^\mu} \right)$ and μ is a positive constant such that

$$\left(\frac{l_1}{\mu^{\mathfrak{v}_1 + \mathfrak{v}_2}} + \frac{l_2}{\mu^{\mathfrak{v}_1}} \right) < 1.$$

Proof. The proof is similar to Theorem 3. $\square$

4. EXAMPLES

Two illustrative examples are given to show the validity of our results.

Example 1. Consider the equation (2.1) for $\kappa = 0.5$, $v_1 = 0.5$, $v_2 = 0.3$, $b = 2$, $f_1(\sigma, \phi) = \sigma^3 \cos(\phi(-\kappa))$ and $f_2(\sigma, \phi) = \sigma^2 \sin(\phi(-\kappa))$.

We have

$$|f_1(\sigma, \phi_1) - f_1(\sigma, \phi_2)| \leq 8\|\phi_1 - \phi_2\|, \quad \forall \sigma \in [1, 2], \phi_1, \phi_2 \in C_{0.5},$$

and

$$|f_2(\sigma, \phi_1) - f_2(\sigma, \phi_2)| \leq 4\|\phi_1 - \phi_2\|, \quad \forall \sigma \in [1, 2], \phi_1, \phi_2 \in C_{0.5},$$

Then $l_1 = 8$ and $l_2 = 4$.

Suppose that $z \in C^2([0.5, 2], \mathbb{R})$ satisfies

$$|D^{0.5}[D^{0.3}z(\sigma) - f_2(\sigma, z_\sigma)] - f_1(\sigma, z_\sigma)| \leq \sigma,$$

for all $\sigma \in [1, 2]$.

Here, $\gamma(\sigma) = \sigma$ and $\varepsilon = 1$. Using Theorem 3 there is a solution z^* of the fractional differential equations and $K > 0$ such that

$$|z(\sigma) - z^*(\sigma)| \leq K\sigma, \quad \forall \sigma \in [1, 2].$$

Example 2. Consider the equation (2.1) for $\kappa = 0.5$, $v_1 = 0.6$, $v_2 = 0.3$, $b = 5$, $f_1(\sigma, \phi) = \sin^2(\phi(-\kappa))$ and $f_2(\sigma, \phi) = \sigma \cos(\phi(-\kappa))$.

We have

$$|f_1(\sigma, \phi_1) - f_1(\sigma, \phi_2)| \leq 2\|\phi_1 - \phi_2\|, \quad \forall \sigma \in [1, 5], \phi_1, \phi_2 \in C_{0.5},$$

and

$$|f_2(\sigma, \phi_1) - f_2(\sigma, \phi_2)| \leq 5\|\phi_1 - \phi_2\|, \quad \forall \sigma \in [1, 2], \phi_1, \phi_2 \in C_{0.5},$$

Then $l_1 = 2$ and $l_2 = 5$.

Suppose that $z \in C^2([0.5, 5], \mathbb{R})$ satisfies

$$|D^{0.6}[D^{0.3}z(\sigma) - f_2(\sigma, z_\sigma)] - f_1(\sigma, z_\sigma)| \leq 0.01,$$

for all $\sigma \in [1, 5]$.

Here, $\varepsilon = 0.01$. Using Theorem 4 there is a solution z^* of the fractional differential equations and $K > 0$ such that

$$|z(\sigma) - z^*(\sigma)| \leq 0.01K, \quad \forall \sigma \in [1, 5].$$

5. CONCLUSION

We managed to use a version of Banach fixed point theorem to prove that under certain conditions, functions that satisfy some sequential neutral functional differential equations with Caputo-Hadamard fractional derivative approximately, are close in some sense to the exact solutions of such problems. In other words, we present stability results for some fractional differential equations in Ulam-Hyers-Rassias sense. In order to show the validity of our results, we presented two examples. Potential future work could be to invent a new method to obtain such stability results or to investigate the stability of a much more complicated fractional differential equations.

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DYNAMICS OF A DISCRETE-TIME PREDATOR-PREY SYSTEM WITH RATIO-DEPENDENT FUNCTIONAL RESPONSE

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Abstract. The aim of this article is to analyze the qualitative behavior of a discrete-time predator-prey system with ratio-dependent functional response. We express algebraically the conditions for the existence of a positive fixed point, and determining its stability.

We also prove that the system has a Neimark-Sacker bifurcation when some parametric conditions are satisfied. The Chaos control is nicely studied by applying the hybrid control method. Finally, we present some numerical simulations in order to verify the theoretical results we have shown.

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1. INTRODUCTION

In 1838, Verhulst introduced an equation to describe the growth of a population in an environment with limited resources. The curve that describes the evolution of this population is known as the logistic curve [29]. In 1887, S. A. Forbes concluded that the interactions between different species that make up an ecosystem are so strong that one cannot be altered without causing a change in the others [14]. However, in 1925, Lotka introduced a pair of differential equations to describe the dynamics of the interaction between two populations, namely, a prey and a predator. Furthermore, Volterra independently developed and generalized the Lotka model [24, 30] in 1927. The equations proposed by Volterra [30] present the existence of two species, one of which (the prey, whose population density is denoted by N) has a positive growth rate. The second species, the predator P , would tend to disappear if the population density of the prey is zero, due to the lack of food. However, if both species coexist and the second feeds on the first, both species can survive, and the system of differential

equations that models their evolution can be written as

$$\begin{aligned}\frac{dN(\tau)}{d\tau} &= N(\tau)(\alpha - \omega P(\tau)), \\ \frac{dP(\tau)}{d\tau} &= P(\tau)(\omega\delta N(\tau) - \beta).\end{aligned}$$

In this system, $N(\tau)$ and $P(\tau)$ denote the population densities of prey and predator, respectively, at time τ . The natural growth rate of the prey is denoted by $\alpha > 0$, and $\beta > 0$ is the natural death rate of the predator in the absence of prey. Moreover, $\omega > 0$ represents the effect of predation on the prey, and $\delta > 0$ is the conversion rate of prey into predator. This simple model was modified in order to introduce intra-specific competition of prey population as follows:

$$\begin{aligned}\frac{dN(\tau)}{d\tau} &= N(\tau)(\alpha - \nu N(\tau) - \omega P(\tau)), \\ \frac{dP(\tau)}{d\tau} &= P(\tau)(\omega\delta N(\tau) - \beta),\end{aligned}$$

where $\nu > 0$. In order to have a better accuracy in the interactions between prey and predators, the so-called “trophic function” was introduced, which describes the amount of prey that a predator consumes per unit of time when the densities of populations of prey and predators are given by N and P , respectively. Most of recent works, such as [8, 10, 13, 22], have focused on analyzing how this function varies with the prey density. Arditi and Ginzburg [3] called “prey-dependence” to this kind of dependence. However, the effect of the predator density in the functional response has been recognized in the last decades [3, 5, 17]. In fact, Arditi and Ginzburg [3] suggested that the trophic function depends on the ratio of prey to predator abundances which is called ratio-dependence. The trophic function depends on the ratio N/P . The introduction of this new kind of dependency has managed to model the interactions between predators and prey in real situations in a much better way [2, 4]. The general model introduced by Arditi and Ginzburg is given by the equations

$$\begin{aligned}\frac{dN}{d\tau} &= rN\phi(N) - g\left(\frac{N}{P}\right)P, \\ \frac{dP}{d\tau} &= ePg\left(\frac{N}{P}\right) - qP,\end{aligned}\tag{1.1}$$

where the trophic function is $g(x) = g(N/P)$ and

$$\phi(N) = 1 - \frac{N}{K}.$$

Arditi and Ginzburg assumed that

$$g(x) = \alpha \frac{x}{1 + \alpha h x},$$

so that

$$g\left(\frac{N}{P}\right) = \alpha \frac{N}{P + \alpha h N}.$$

With the use of this trophic function, Equation (1.1) can be rewritten as

$$\begin{aligned} \frac{dN}{d\tau} &= rN \left(1 - \frac{N}{K}\right) - \alpha \frac{NP}{P + \alpha h N}, \\ \frac{dP}{d\tau} &= e \frac{NP}{P + \alpha h N} - qP, \end{aligned} \quad (1.2)$$

where the parameters r, K, α, h, e, q are positive. The change of variables $(N, P, \tau) \rightarrow (x, y, t)$ defined by

$$N = Kx, \quad P = K\alpha hy, \quad \tau = \frac{t}{r},$$

together with the corresponding definitions of parameters given by

$$a = \frac{\alpha}{r}, \quad b = \frac{q}{r}, \quad c = \frac{e}{rh},$$

converts Equations (1.2) into

$$\begin{aligned} \dot{x}(t) &= x(1-x) - \frac{axy}{x+y}, \\ \dot{y}(t) &= -by + \frac{cxy}{x+y}, \end{aligned} \quad (1.3)$$

where $a, b, c > 0$. In Equations (1.3), b denotes the death rate of the (new defined) predators y and the trophic function is described by parameters a and c , where a stands for the maximum asymptotic prey death rate due to predation, for a finite density of predators and c denotes the maximum asymptotic predator growth rate for an infinite density of prey.

The Lotka-Volterra model described by differential equations are appropriate when the different generations can coexist, while the difference equations work better when there is no overlap between different generations. Discrete models are a very powerful tool from a computational point of view in order to perform numerical simulations. Moreover, discrete systems present dynamics more complex and rich than the corresponding continuous system. In recent years, an increasing number of researchers have studied the existence of Neimark-Sacker bifurcations in different discrete Lotka-Volterra systems [1, 6, 8, 10, 13, 16–19, 21–23]. In this paper, we discretize Equations (1.3) using the piecewise constant arguments method [22, 26] to obtain the discrete system

$$\begin{aligned} x_{n+1} &= x_n \exp \left(1 - x_n - \frac{ay_n}{x_n + y_n} \right), \\ y_{n+1} &= y_n \exp \left(-b + \frac{cx_n}{x_n + y_n} \right), \end{aligned} \quad (1.4)$$

where $a, b, c > 0$. We aim to discuss the qualitative study of this system.

The paper is organized as follows. In the Section 2, we calculate the fixed points of Equations (1.4) and establish their stability. In Section 3, we give the conditions under which a Neimark-Sacker bifurcation occurs. Chaos control is discussed in Section 4. Finally, in Section 5, we illustrate all the theoretical results we have obtained with a numerical simulation of the system.

2. STABILITY ANALYSIS

The non trivial equilibrium points of Equations (1.4) are

$$E = (1, 0), \quad P = \left(\frac{ab + c - ac}{c}, \frac{(c - b)(ab + c - ac)}{bc} \right).$$

The positive coexistence equilibrium P exists if $c > b$ and $ab + c > ac$.

Lemma 1. *The extinction equilibrium $E = (1, 0)$ of Equations (1.4) is locally asymptotically stable (sink) if $b > c$, unstable (saddle) if $b < c$, and a non-hyperbolic point if $b = c$.*

Proof. The Jacobian matrix of the Equations (1.4) at the point E is given by

$$J(E) = \begin{pmatrix} 0 & -a \\ 0 & e^{c-b} \end{pmatrix},$$

and its eigenvalues are $\mu_1 = 0 < 1$ and $\mu_2 = e^{c-b}$. Thus, $E = (1, 0)$ is locally asymptotically stable if $b > c$, unstable if $b < c$, and non-hyperbolic if $b = c$. $\square$

In order to analyze the stability of the positive coexistence equilibrium point P , we apply the following Lemma [7, 8, 23].

Lemma 2. *Consider the polynomial $\rho(\lambda) = \lambda^2 - T\lambda + D$, where $\rho(1) > 0$, and λ_1 and λ_2 are the two roots of $\rho(\lambda) = 0$. Then,*

- (1) $|\lambda_1| < 1$ and $|\lambda_2| < 1$ if and only if $\rho(-1) > 0$ and $\rho(0) < 1$.
- (2) $|\lambda_1| < 1$ and $|\lambda_2| > 1$ (or $|\lambda_1| > 1$ and $|\lambda_2| < 1$) if and only if $\rho(-1) < 0$.
- (3) $|\lambda_1| > 1$ and $|\lambda_2| > 1$ if and only if $\rho(-1) > 0$ and $\rho(0) > 1$.
- (4) λ_1 and λ_2 are complex numbers and $|\lambda_1| = |\lambda_2| = 1$ if and only if $T^2 - 4D < 0$ and $\rho(0) = 1$.

The Jacobian matrix of Equations (1.4) evaluated at the equilibrium P reads

$$J(P) = \begin{pmatrix} \frac{a(c^2 - b^2)}{c^2} & -\frac{ab^2}{c^2} \\ \frac{(c - b)^2}{c} & 1 - \frac{b(c - b)}{c} \end{pmatrix},$$

and its characteristic equation is given by

$$\rho(\mu) = \mu^2 - \left(a + 1 - b - \frac{ab^2}{c^2} + \frac{b^2}{c} \right) \mu + \frac{a(c - b)(c + b - bc + b^2)}{c^2}. \quad (2.1)$$

From Equation (2.1), we can obtain

$$\rho(0) = \frac{a(c-b)(c+b-bc+b^2)}{c^2}, \quad \rho(1) = \frac{b(c^2-bc-a(c-b)^2)}{c^2}$$

and

$$\rho(-1) = 2 + \frac{2a(c^2-b^2) + b(bc-c^2-a(c-b)^2)}{c^2}.$$

Then, taking into account that $c^2 > bc + a(c-b)^2$, it follows that $\rho(1) > 0$ and, thus, we can apply Lemma 2 to state the following result:

Lemma 3. Assume that $c > b$ and $a < \frac{c}{(c-b)}$. Then,

$$P = \left(\frac{ab+c-ac}{c}, \frac{(c-b)(ab+c-ac)}{bc} \right),$$

is a positive coexistence equilibrium point of Equations (1.4) and $\rho(1) > 0$. Moreover,

(1) P is locally asymptotically stable (sink stable) if and only if

$$a < \frac{c}{c-b} \quad \text{for } c \geq b+1,$$

and

$$a < \frac{c^2}{(c-b)(c+b(1-c)+b^2)} \quad \text{for } a < c \leq b+1.$$

or

$$b+c-bc+b^2 \leq 0,$$

and

$$\rho(-1) = \frac{c^2(a+1)(2-b) + cb^2(2a+1) - ab(b^2+2b)}{c^2} > 0.$$

(2) P is saddle point if and only if

$$b+c-bc+b^2 \leq 0,$$

and

$$\rho(-1) = \frac{c^2(a+1)(2-b) + cb^2(2a+1) - ab(b^2+2b)}{c^2} < 0.$$

(3) P is unstable (source) if and only if

$$b < c < b+1,$$

and

$$a > \frac{c^2}{(c-b)(c+b-bc+b^2)}.$$

- (4) *The roots of Equation (2.1) are complex conjugates with unit modulus if and only if*

$$b < c < b + 1, \quad (2.2)$$

and

$$a = \frac{c^2}{(c-b)(c+b-bc+b^2)}. \quad (2.3)$$

3. NEIMARK-SACKER BIFURCATION

In this section, we study the existence of a Neimark-Sacker bifurcation for Equations (1.4). A Neimark-Sacker bifurcation occurs when a closed invariant curve emerges from an equilibrium point in a discrete dynamical system and, then, the stability of the equilibrium changes via a pair of complex eigenvalues with unit modulus [27, 28]. We now consider this system around the positive coexistence equilibrium P . According to Lemma 2, the characteristic equation of the linearization of Equations (1.4) at P has two conjugate complex roots with modulus one if the conditions (2.2) and (2.3) are satisfied. Thus, P presents a Neimark-Sacker bifurcation if the parameters (a, b, c) vary in a neighborhood of the set

$$\mathcal{B} = \left\{ (a, b, c) \in \mathbb{R}_+^3 : a = \frac{c^2}{(c-b)(c+b-bc+b^2)}, b < c < b + 1 \right\}.$$

Let $(a, b, c) \in \mathcal{B}$ and assume that the change of variables is given by

$$u_n = x_n - x_0, \quad v_n = y_n - y_0,$$

with

$$x_0 = \frac{ab+c-ac}{c}, \quad y_0 = \frac{(c-b)(ab+c-ac)}{bc},$$

transforms the fixed point $P = (x_0, y_0)$ into the origin $O = (0, 0)$. Equations (1.4) is also changed into

$$\begin{aligned} u_{n+1} &= (u_n + x_0) \exp \left(1 - u_n - x_0 - \frac{a(v_n + y_0)}{(u_n + v_n + x_0 + y_0)} \right) - x_0, \\ v_{n+1} &= (v_n + y_0) \exp \left(-b + \frac{c(u_n + x_0)}{(u_n + v_n + x_0 + y_0)} \right) - y_0, \end{aligned} \quad (3.1)$$

where $a, b, c > 0$.

Let $\bar{a}$ denote a small perturbation in a , with $|\bar{a}| \ll 1$, i.e., $\bar{a} = a - a_0$. Then, the perturbation of Equations (3.1) can be arranged as follows:

$$\begin{aligned} u_{n+1} &= (u_n + x_0) \exp \left(1 - u_n - x_0 - \frac{(\bar{a} + a_0)(v_n + y_0)}{(u_n + v_n + x_0 + y_0)} \right) - x_0, \\ v_{n+1} &= (v_n + y_0) \exp \left(-b + \frac{c(u_n + x_0)}{(u_n + v_n + x_0 + y_0)} \right) - y_0. \end{aligned} \quad (3.2)$$

The characteristic equation of the linearization of Equations (3.2) at the origin can be written as

$$\mu^2 - T(\bar{a})\mu + D(\bar{a}) = 0, \quad (3.3)$$

where

$$T(\bar{a}) = \left((\bar{a} + a_0) + 1 - b - \frac{(\bar{a} + a_0)b^2}{c^2} + \frac{b^2}{c} \right),$$

and

$$D(\bar{a}) = \frac{(\bar{a} + a_0)(c - b)(c + b - bc + b^2)}{c^2}.$$

It is easy to obtain that $T^2(\bar{a}) - 4D(\bar{a}) < 0$ and $(a, b, c) \in \mathcal{B}$. Thus, the complex conjugate roots with unit modulus of Equation (3.3) are

$$\mu_{1,2}(\bar{a}) = \frac{T(\bar{a})}{2} \pm \frac{i}{2} \sqrt{4D(\bar{a}) - T^2(\bar{a})}.$$

Next, we can calculate

$$(|\mu_{1,2}(\bar{a})|)|_{\bar{a}=0} = \sqrt{D(\bar{a})}|_{\bar{a}=0} = \sqrt{\frac{a_0(c-b)(c+b-bc+b^2)}{c^2}} = 1,$$

which implies that

$$\left(\frac{d|\mu_1|}{d\bar{a}} \right)_{\bar{a}=0} = \left(\frac{d|\mu_2|}{d\bar{a}} \right)_{\bar{a}=0} = \left(\frac{\sqrt{(c-b)(c+b-bc+b^2)}}{2c\sqrt{a_0}} \right) > 0.$$

The existence of a Neimark-Sacker bifurcation needs the following conditions to be satisfied:

$$\left(\frac{d|\mu_{1,2}(\bar{a})|}{d\bar{a}} \right)_{\bar{a}=0} \neq 0,$$

and

$$\mu_{1,2}(0)^n \neq 1, \quad n = 1, 2, 3, 4.$$

One can easily calculate $T(0)$ as follows:

$$\begin{aligned} T(0) &= 1 + \frac{c^2 + bc + b(b-c)(c+b-bc+b^2)}{c(c+b-bc+b^2)}, \\ &= \frac{c^2 + bc + (b^2 - bc + c)(c+b-bc+b^2)}{2c(c+b-bc+b^2)}, \end{aligned}$$

and $D(0) = 1$. Thus,

$$\mu_{1,2} = \frac{T(0)}{2} \pm \frac{i}{2} \sqrt{4 - T^2(0)}.$$

Then, we can state that $\mu_{1,2}(0)^n \neq 1$, for any $n = 1, 2, 3, 4$. According to [31], all the conditions for the existence of a Neimark-Sacker bifurcation hold. In order to

determine the normal form of Equations (3.2) [25], we compute its Taylor expansion to the third order around the origin as follows:

$$\begin{aligned} u_{n+1} &= g_{11}u_n + g_{12}v_n + F_1(u_n, v_n), \\ v_{n+1} &= g_{21}u_n + g_{22}v_n + F_2(u_n, v_n), \end{aligned} \quad (3.4)$$

where

$$\begin{aligned} F_1(u_n, v_n) &= \frac{1}{2}g_{13}u_n^2 + 2g_{14}u_nv_n + \frac{1}{2}g_{15}v_n^2 \\ &\quad + \frac{1}{6}g_{16}u_n^3 + \frac{1}{2}g_{17}u_n^2v_n + \frac{1}{2}g_{18}u_nv_n^2 + \frac{1}{6}g_{19}v_n^3 + \mathcal{R}_{F_1,4}(u_n, v_n), \\ F_2(u_n, v_n) &= \frac{1}{2}g_{23}u_n^2 + 2g_{24}u_nv_n + \frac{1}{2}g_{25}v_n^2 \\ &\quad + \frac{1}{6}g_{26}u_n^3 + \frac{1}{2}g_{27}u_n^2v_n + \frac{1}{2}g_{28}u_nv_n^2 + \frac{1}{6}g_{29}v_n^3 + \mathcal{R}_{F_2,4}(u_n, v_n). \end{aligned}$$

Here, $\mathcal{R}_{F_1,4}(u_n, v_n)$ and $\mathcal{R}_{F_2,4}(u_n, v_n)$ denote the terms of order larger than 3 in the Taylor expansion. Furthermore, we have

$$\begin{aligned} g_{11} &= \frac{a_0(c^2 - b^2)}{c^2}, \\ g_{12} &= -\frac{a_0b^2}{c^2}, \\ g_{13} &= -2 + \frac{2a_0(c-b)b^2}{c^2(a_0b+c-a_0c)} + \frac{a_0b+c-a_0c}{c} + \frac{a_0^2(c-b)^2b^2}{c^3(a_0b+c-a_0c)} - \frac{2(c-b)b}{c^2}, \\ g_{14} &= -\frac{2a_0(c-b)b^2}{c^2(a_0b+c-a_0c)} + \frac{a_0b^2}{c^2} - \frac{a_0^2(c-b)b^3}{c^3(a_0b+c-a_0c)}, \\ g_{15} &= \frac{a_0b^3(2c-a_0b)}{c^3(a_0b+c-a_0c)}, \\ g_{16} &= \frac{a_0(c-b)b^2}{c(a_0b+c-a_0c)^2} \left(2 + a_0 - \frac{6b}{c} - \frac{5a_0b}{c} + \frac{4a_0b^2}{c^2} \right) + \frac{4(c-b)b^2}{c^2(a_0b+c-a_0c)^2} \\ &\quad + \left(-1 + \frac{2a_0(c-b)b^2}{c^2(a_0b+c-a_0c)} - \frac{a_0(c-b)}{c} + \frac{a_0(c-b)^2b^2}{c^2(a_0b+c-a_0c)^2} \right. \\ &\quad \left. - \frac{2(c-b)b}{c^2} \right) \times \left(-1 + \frac{a_0(c-b)b}{c(a_0b+c-a_0c)} \right) + 1 - \frac{2(c-b)b}{c^2}, \\ g_{17} &= \frac{a_0(c-b)b^2}{c(a_0b+c-a_0c)} \left(4 - \frac{2a_0}{c} - \frac{2}{a_0b+c-a_0c} + \frac{6b}{c(a_0b+c-a_0c)} \right) \\ &\quad + \left(-1 + \frac{a_0(c-b)b}{c(a_0b+c-a_0c)} \right) \left(\frac{2a_0}{c^2} - \frac{2a_0(c-b)b^2}{c(a_0b+c-a_0c)} \right) \end{aligned}$$

$$\begin{aligned}
& -\frac{a_0^2(c-b)b^3}{c^3(a_0b+c-a_0c)} \Bigg), \\
g_{18} &= \frac{a_0b^3}{c(a_0b+c-a_0c)^2} \left(4 - \frac{6b}{c} + \frac{3a_0b}{c} - \frac{4a_0b^2}{c^2} + \frac{1}{c} \right. \\
& \quad \left. \left(2 + \frac{a_0b}{c} \right) \left(-1 + \frac{a_0(c-b)b}{c^2(a_0b+c-a_0c)} \right) \right), \\
g_{19} &= \frac{a_0b^2}{(a_0b+c-a_0c)^2} \left(-6 - \frac{6a_0b^3}{c^3} - \frac{a_0^2b^4}{c^3} \right), \\
g_{23} &= \frac{(c-b)^2b}{a_0b+c-a_0c} \left(c-b - \frac{2}{c} \right), \\
g_{24} &= \frac{(c-b)b^2(2-c+b)}{c(a_0b+c-a_0c)}, \\
g_{25} &= \frac{b^3(c-b-2)}{c(a_0b+c-a_0c)}, \\
g_{26} &= \frac{(c-b)^2b^2(-6c+6b+(c-b)^2)}{c(a_0b+c-a_0c)^2}, \\
g_{27} &= \frac{(c-b)b^2}{a_0b+c-a_0c} \left(3b-c+2 - \frac{2b(2b+3)}{c} \right) \\
& \quad + \frac{(c-b)^2b^2}{c^2} \left(c-b + \frac{2b}{a_0b+c-a_0c} \right), \\
g_{28} &= \frac{b^3(10b-8c-3(c-b)^2)}{c(a_0b+c-a_0c)^2}, \\
g_{29} &= \frac{9-3(c-b)b^3-(c-b)b^5}{c(a_0b+c-a_0c)^2}.
\end{aligned}$$

We now assume that $J(P)$ is the Jacobian matrix of Equations (3.4) about the point P . Hence,

$$J(P) = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} \frac{a(c^2-b^2)}{c^2} & -\frac{ab^2}{c^2} \\ \frac{(c-b)^2}{c} & 1 - \frac{b(c-b)}{c} \end{pmatrix}.$$

The eigenvalues of $J(P)$ are given by

$$\mu_{1,2} = \xi \pm i\pi,$$

where $\xi = T(0)/2$ and $\pi = \sqrt{4D(0) - T^2(0)}/2$. Now, we define the transformation

$$\begin{pmatrix} u \\ v \end{pmatrix} = T \begin{pmatrix} w \\ z \end{pmatrix}, \tag{3.5}$$

where

$$T = \begin{pmatrix} -\frac{ab^2}{c^2} & 0 \\ \xi - \frac{a(c^2 - b^2)}{c^2} & -\pi \end{pmatrix}.$$

If we carry out the transformation (3.5) into Equations (3.4), we obtain

$$\begin{aligned} w_{n+1} &= \xi w_n - \pi z_n + \tilde{P}(w, z), \\ z_{n+1} &= \pi w_n + \xi z_n + \tilde{Q}(w, z), \end{aligned}$$

where

$$\begin{aligned} \tilde{P}(w, z) &= \frac{g_{13}}{2g_{12}}u^2 + \frac{2g_{14}}{g_{12}}uv + \frac{g_{15}}{2g_{12}}v^2 + \frac{g_{16}}{6g_{12}}u^3 + \frac{g_{17}}{2g_{12}}u^2v \\ &\quad + \frac{g_{18}}{2g_{12}}uv^2 + \frac{g_{19}}{6g_{12}}v^3 + \mathcal{R}_{\tilde{P},4}(u, v), \end{aligned}$$

and

$$\begin{aligned} \tilde{Q}(w, z) &= \left(\frac{g_{13}(\xi - g_{11})}{2g_{12}} - \frac{g_{23}}{2\pi} \right) u^2 + \left(\frac{2g_{14}(\xi - g_{11})}{g_{12}} - \frac{2g_{24}}{\pi} \right) uv \\ &\quad + \left(\frac{g_{15}(\xi - g_{11})}{2g_{12}} - \frac{g_{25}}{2\pi} \right) v^2 + \left(\frac{g_{16}(\xi - g_{11})}{6g_{12}} - \frac{g_{26}}{6\pi} \right) u^3 \\ &\quad + \left(\frac{g_{17}(\xi - g_{11})}{2g_{12}} - \frac{g_{27}}{2\pi} \right) u^2v + \left(\frac{g_{18}(\xi - g_{11})}{2g_{12}} - \frac{g_{28}}{2\pi} \right) uv^2 \\ &\quad + \left(\frac{g_{19}(\xi - g_{11})}{6g_{12}} - \frac{g_{29}}{6\pi} \right) v^3 + \mathcal{R}_{\tilde{Q},4}(u, v). \end{aligned}$$

Here, $\mathcal{R}_{\tilde{P},4}(u, v)$ and $\mathcal{R}_{\tilde{Q},4}(u, v)$ present the terms of order larger than 3 in the Taylor expansion of $\tilde{P}$ and $\tilde{Q}$, respectively. Moreover, we have

$$u = g_{12}w, \quad v = (\xi - g_{11})w - \pi z.$$

Next, in order to determine the direction of the appearance of the invariant curve in a system exhibiting a Neimark–Sacker bifurcation, we consider the first Lyapunov coefficients at the point $(w, z) = (0, 0)$ which is given by

$$L = \left(\operatorname{Re}(\mu_2 \tau_{21}) - \operatorname{Re} \left(\frac{(1 - 2\mu_1)\mu_2^2}{1 - \mu_1} \tau_{20} \tau_{11} \right) - \frac{1}{2} |\tau_{11}|^2 - |\tau_{02}|^2 \right)_{\tilde{a}=0}, \quad (3.6)$$

where

$$\begin{aligned} \tau_{20} &= \frac{1}{8} [\tilde{P}_{ww} - \tilde{P}_{zz} + 2\tilde{Q}_{wz} + i(\tilde{Q}_{ww} - \tilde{Q}_{zz} - 2\tilde{P}_{wz})], \\ \tau_{11} &= \frac{1}{4} [\tilde{P}_{ww} + \tilde{P}_{zz} + i(\tilde{Q}_{ww} + \tilde{Q}_{zz})], \\ \tau_{02} &= \frac{1}{8} [\tilde{P}_{ww} - \tilde{P}_{zz} - 2\tilde{Q}_{wz} + i(\tilde{Q}_{ww} - \tilde{Q}_{zz} + 2\tilde{P}_{wz})], \end{aligned}$$

$$\tau_{21} = \frac{1}{16} [\tilde{P}_{www} + \tilde{P}_{wzz} + \tilde{Q}_{wwz} + \tilde{Q}_{zzz} + i(\tilde{Q}_{www} + \tilde{Q}_{wzz} - \tilde{P}_{wwz} - \tilde{P}_{zzz})].$$

According to the computations described above, we can summarize the results of the existence of a Neimark-Sacker bifurcation in the following theorem [9, 11, 20, 27, 28, 31].

Theorem 1. *If $c > b$, $a < \frac{c}{c-b}$ and condition (3.6) hold with $L \neq 0$. Then, Equations (1.4) has a Neimark-Sacker bifurcation at the positive coexistence equilibrium point*

$$P = \left(\frac{ab + c - ac}{c}, \frac{(c - b)(ab + c - ac)}{bc} \right),$$

when the parameter a varies in a small neighborhood of

$$a_0 = \frac{c^2}{(c - b)(c + b - bc + b^2)}.$$

If $L < 0$, the equilibrium point bifurcates in an attracting invariant closed curve for $a > a_0$. If $L > 0$, a repelling invariant closed curve bifurcates from the equilibrium point when $a < a_0$.

4. CHAOS CONTROL

In order to control chaotic behavior, small perturbations must be added, and in result, the randomness change into order. Therefore, chaos control techniques improve the predictability and stability of chaotic orbits [12, 15]. The stabilization of a perturbed system needs to employ chaos control methods as a means to reduce fluctuation and unpredictability.

Here, we discuss the Chaos control of Equations (1.4) using the hybrid control method. In order to apply the hybrid control method to Equations (1.4), we write the corresponding control model in the following from:

$$\begin{aligned} x_{n+1} &= \delta x_n \exp \left(1 - x_n - \frac{ay_n}{x_n + y_n} \right) + (1 - \delta)x_n, \\ y_{n+1} &= \delta y_n \exp \left(-b + \frac{cx_n}{x_n + y_n} \right) + (1 - \delta)y_n, \end{aligned} \quad (4.1)$$

where $0 < \delta < 1$ is a control parameter for the hybrid control method. The Jacobian matrix of Equations (4.1) evaluated at the positive coexistence equilibrium point P is given by

$$J(P) = \begin{pmatrix} 1 + \delta \left(\frac{a(c^2 - b^2) - c^2}{c^2} \right) & -\frac{a\delta b^2}{c^2} \\ \frac{\delta(c - b)^2}{c} & 1 - \frac{\delta b(c - b)}{c} \end{pmatrix}, \quad (4.2)$$

Hence, the characteristic equation of the matrix (4.2) is

$$\rho(\mu) = \mu^2 - T\mu + D,$$

where

$$T = 2 + \delta \left(a - 1 - b - \frac{ab^2}{c^2} + \frac{b^2}{c} \right),$$

$$D = \left(1 + \delta \left(\frac{a(c^2 - b^2) - c^2}{c^2} \right) \right) \left(1 - \frac{\delta b(c - b)}{c} \right) + \frac{\delta^2 ab(c - b)}{c^3}.$$

Lemma 4. *Let $c > b$ and $ab + c > ac$. Then, the unique positive coexistence equilibrium point $P = \left(\frac{ab+c-ac}{c}, \frac{(c-b)(ab+c-ac)}{bc} \right)$ of Equations (4.1) is locally asymptotically stable if the following condition satisfies*

$$|T| < 1 + D < 2.$$

5. NUMERICAL SIMULATIONS

This section presents some numerical examples to verify the obtained theoretical results.

Example 1. Consider the particular case of Equations (1.4) defined by the following values of the parameters:

$$b = 0.7, \quad c = 1.5, \quad a \in [1.7, 1.75],$$

where a the bifurcation parameter. When $(a, b, c) = (1.714939, 0.7, 1.5)$, the Jacobian matrix of Equations (1.4) at coexistence equilibrium P is given by

$$J(P) = \begin{pmatrix} 1.341 & -0.373 \\ 0.426 & 0.626 \end{pmatrix},$$

whose characteristic equation is

$$\rho(\mu) = \mu^2 - 1.9676\mu + 0.99869 = 0.$$

The eigenvalues are given by

$$\mu_{1,2} = 0.98509 \pm 0.175808i,$$

with unit modulus. Furthermore, it is easy to see that

$$\rho(1) = 0.031877 > 0, \quad \left(\frac{d|\mu_{1,2}|}{d\bar{a}} \right)_{\bar{a}=0} = 0.291558 > 0, \quad T(0) = 1.9676 \neq 0, 1,$$

which leads to $\mu_{1,2}^n \neq 1$, for any $n = 1, 2, 3, 4$. In this example, F_1 and F_2 are given by

$$F_1(x, y) = 2.19x^2 - 11.577xy + 2.857y^2 - 4.224x^3 + 2.541x^2y + 36.659xy^2 \\ - 21.278y^3 + \mathcal{R}_{F_1,4}(x, y),$$

and

$$F_2(x, y) = -1.400x^2 + 4.898xy - 1.070y^2 + 3.913x^3 + 4.092x^2y - 16.130xy^2 + 8.359y^3 + \mathcal{R}_{F_2,4}(x, y).$$

Furthermore, the first Lyapunov exponent for these parameters is given by $L \leq -0.0001898$, which proves the correctness of Theorem 1. The bifurcation diagrams for x_n and y_n are depicted in Figure 1, and the maximum Lyapunov exponent is plotted in Figure 2. It is easy to observe that the positive coexistence equilibrium P of Equations (1.4) is locally asymptotically stable for $1.7 \leq a < 1.714939$ (see Figures 3, 4, 5). For $a = 1.714939$, the point P loses the property of local asymptotic stability and a closed invariant curve Γ_s appears around P (see Figures 7, 8 and 9). Thus, a Neimark-Sacker bifurcation occurs at $a = 1.714939$. In Figures 7, 8 and 9, the orbit with initial condition $(0.089, 0.097)$, colored in green, leaves the unstable equilibrium point P and tends asymptotically to the stable invariant curve Γ_s . The orbit colored in red, with initial condition $(0.17, 0.22)$, also tends asymptotically to Γ_s . In Figure 9 we have depicted also an orbit, colored in cyan, with initial condition $(0.3, 0.25)$. From all the above, we can conclude that all the orbits with initial conditions within the invariant curve, except the equilibrium point P , tends asymptotically from the inside towards Γ_s , and all the orbits starting outside the invariant curve also tends asymptotic to Γ_s . The invariant curve Γ_s tends to expand as the value of the parameter $a > a_0$ grows (see Figures 2 (right panel)), and it tends to narrow when we decrease the value of a . For values of the parameter smaller or equal than a_0 , the invariant curve has collapsed into the equilibrium point P .

Example 2. In order to assess the performance of the hybrid control method in improving chaotic (unstable) Equations (1.4), we take the same parameter values as given in Example 1 ($b = 0.7$, $c = 1.5$) with $a = 1.7167$. Then, it shows that the coexistence equilibrium point P of Equations (1.4) is unstable (see Figure 10 (right)). However, this equilibrium point is stable for the control Equations (4.1) if $0 < \delta < 0.965$ (see Figure 10 (left)) and unstable if $0.965 < \delta < 1$. This is confirmed by the bifurcation diagrams of x_n and y_n in Figure 11.

6. CONCLUDING REMARKS

This paper focuses on a qualitative analysis of a discrete-time Lotka-Volterra system with ratio-dependence, given by Equations (1.4). This system depends on three parameters, a , b and c , and has one positive equilibrium P . We have analyzed the stability of P in Lemma 3. In this Lemma, we have presented some conditions under which the point P is sink, unstable saddle, source, or non-hyperbolic. It is clear that there is a unique positive coexistence equilibrium P of Equations (1.4) with $c > b$ and $ab + c > ac$. We have also proved the existence of a Neimark-Sacker bifurcation

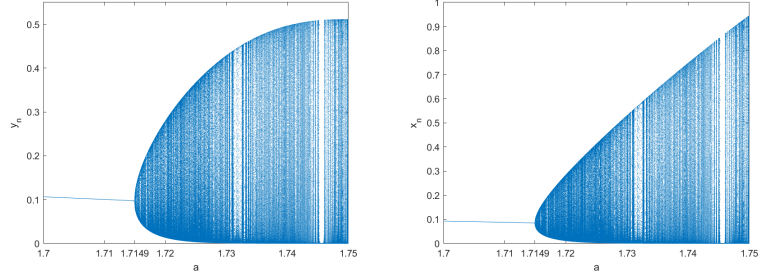


FIGURE 1. Bifurcation diagram for x_n (right) and y_n (left), for the initial condition $(0.089, 0.097)$, and $a \in [1.7, 1.75]$.

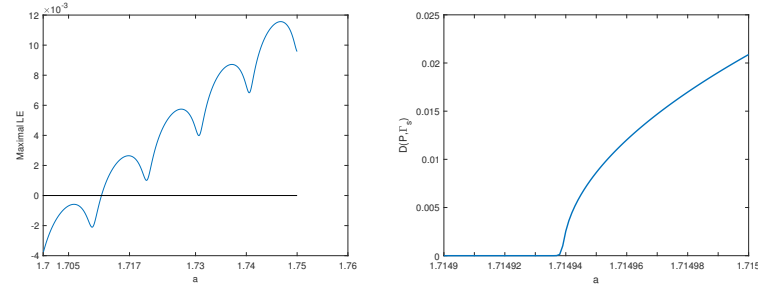


FIGURE 2. In the left panel, maximum Lyapunov exponent for the system given in Equations (1.4), with $a \in [1.7, 1.75]$, $(b, c) = (0.7, 1.5)$ and initial conditions $(0.0853, 0.09756)$. In the right panel, evolution of the distance between the equilibrium point P and the invariant closed curve Γ_s in terms of the bifurcation parameter a .

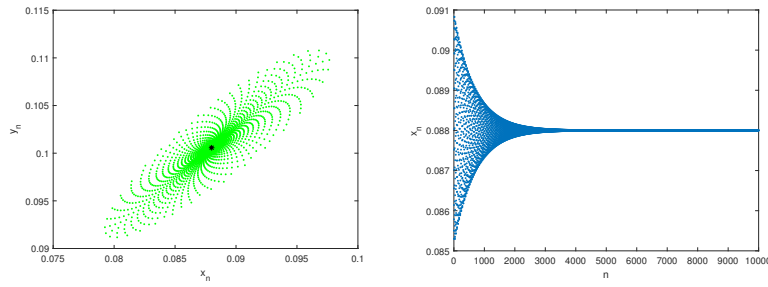


FIGURE 3. Phase portrait for $(a, b, c) = (1.71, 0.7, 1.5)$ (left panel) and evolution of x_n for $a = 1.71$ (right panel).

at the coexistence equilibrium P through the analysis of the normal form of the system, concluding this type of bifurcation occurs when the parameters (a, b, c) vary on

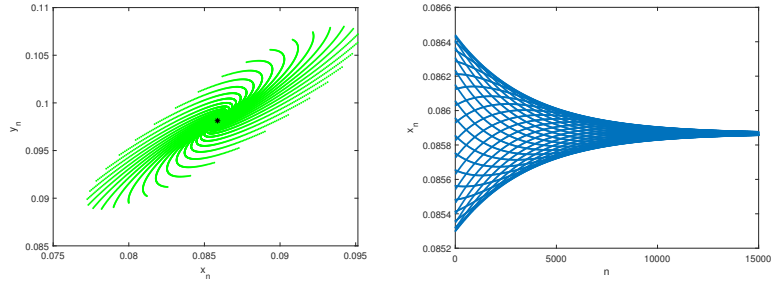


FIGURE 4. Phase portrait for $(a,b,c) = (1.714, 0.7, 1.5)$ (left panel), and evolution of x_n for $a = 1.714$ (right panel).

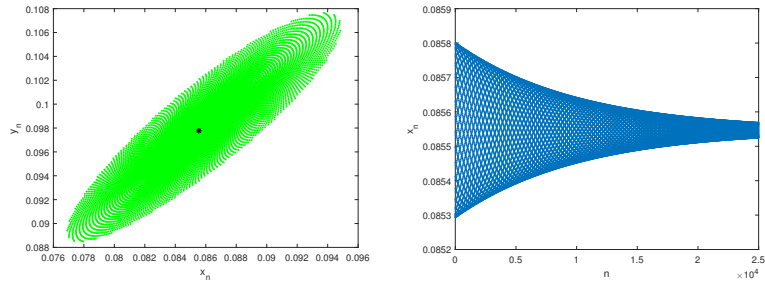


FIGURE 5. Phase portrait for $(a,b,c) = (1.7146, 0.7, 1.5)$ (left panel), and evolution of x_n for $a = 1.7146$ (right panel).

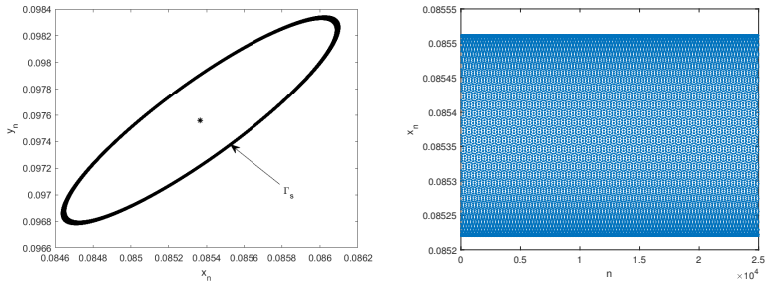


FIGURE 6. Invariant closed curve Γ_s (left panel) and evolution of x_n (right panel) for $(a,b,c) = (1.714939, 0.7, 1.5)$.

the neighborhood $\mathcal{B}$. In the context of biology, Equations (1.4) can be viewed as a predator-prey system interaction. In terms of the latter, the existence of a Neimark-Sacker bifurcation in Equations (1.4) implies that both the prey and predator populations can oscillate around some mean values, and these oscillations will continue

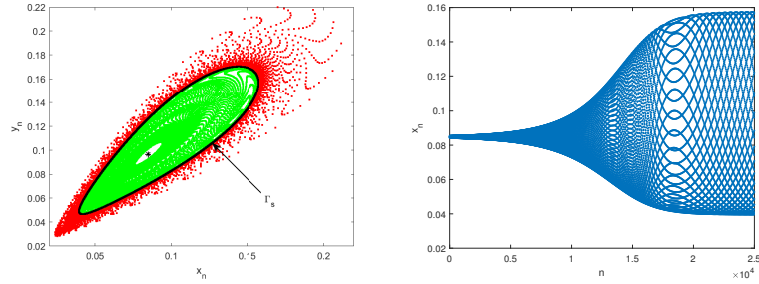


FIGURE 7. Phase portrait (left panel) and evolution of x_n (right panel) for $a = 1.71599$.

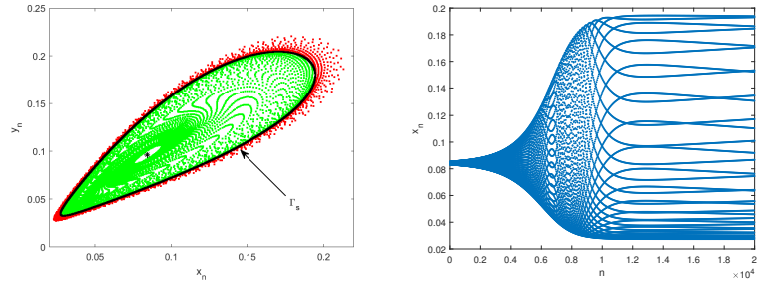


FIGURE 8. Phase portrait (left panel) and evolution of x_n (right panel) for $a = 1.71699$.

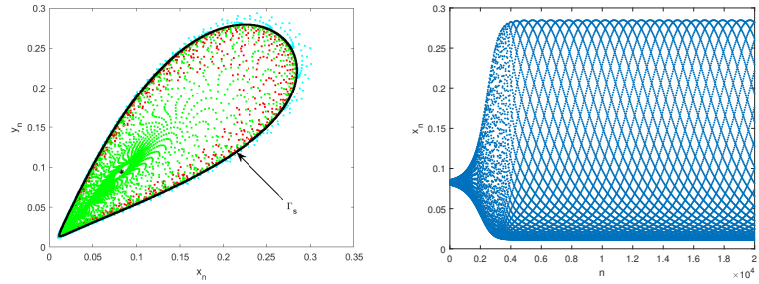


FIGURE 9. Phase portrait (left panel) and evolution of x_n (right panel) for $a = 1.72$.

indefinitely under suitable conditions. These results show far richer dynamics of the discrete model compared to the continuous model. The chaos control of Equations (1.4) has been successfully explored using the hybrid control method. Finally, we have illustrated this theoretical results with the help of a numerical examples.

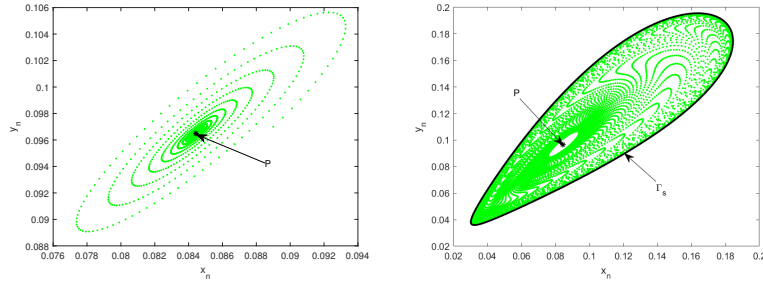


FIGURE 10. Phase portrait of controlled Equations (4.1) for $a = 1.7167$, $\delta = 0.4$ (left) and phase portrait of Equations (1.4) for $a = 1.7167$ (right).

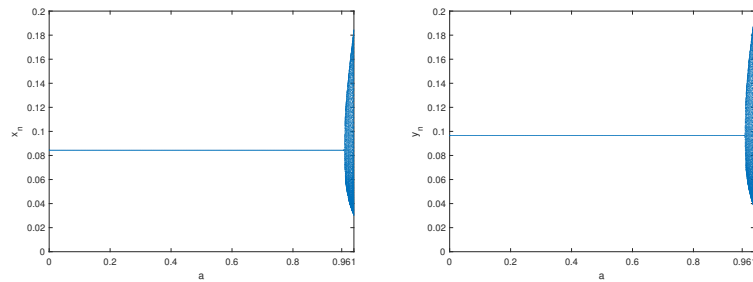


FIGURE 11. Bifurcation diagram of x_n (left) and y_n (right) for controlled Equations (4.1) with $a = 1.7167$, $b = 0.7$, $c = 1.5$ and $\delta \in [0, 1]$.

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ŁUKASIEWICZ FUZZY SUB-HOOPS

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Abstract. With the purpose of applying the Łukasiewicz fuzzy set to sub-hoop in hoops, the concept of Łukasiewicz fuzzy sub-hoop is introduced, and its properties are investigated. The relationship between fuzzy sub-hoop and Łukasiewicz fuzzy sub-hoop are discussed. Conditions for Łukasiewicz fuzzy set to be a Łukasiewicz fuzzy sub-hoop are provided, and characterizations of Łukasiewicz fuzzy sub-hoop are displayed. Conditions under which $\in$ -set, q -set, and O -set can be sub-hoops are investigated.

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1. INTRODUCTION

Hoops, which are introduced by Bosbach and Bosbach (see [6, 7]), are a nice algebraic structure to research the many-valued logical system whose propositional value is given in a lattice. Several properties of hoops are displayed in [1–3, 5, 8, 12, 13]. Substructures in hoops are a very important factor in studying hoops, and studies have been conducted on various types of substructures. The sub-hoop is the most basic substructure of hoops. The fuzzy set acts as a bridge so that algebra theory can be applied to applied sciences. Various kinds of fuzzy sets have been used in the study of substructures such as ideals in BCK/BCI-algebras Borzooei et al. [4] studied fuzzy sub-hoops. Łukasiewicz logic is the logic of the Łukasiewicz t -norm, and it is a non-classical and many-valued logic. It was originally defined in the early 20th century by Jan Łukasiewicz as a three-valued logic. Using the idea of Łukasiewicz t -norm, Y. B. Jun constructed the concept of Łukasiewicz fuzzy sets based on a given fuzzy set and applied it to BCK-algebras and BCI-algebras (see [9, 11]).

For the purpose of applying the Łukasiewicz fuzzy set to the sub-hoop in hoops, we introduce the concept of Łukasiewicz fuzzy sub-hoop and study its properties. We discuss the relationship between Łukasiewicz fuzzy sub-hoop and fuzzy sub-hoop. We provide conditions for Łukasiewicz fuzzy set to be a Łukasiewicz fuzzy

sub-hoop. We discuss characterizations of Łukasiewicz fuzzy sub-hoop, and explore the conditions under which $\in$ -set, q -set, and O -set can be sub-hoops.

2. PRELIMINARIES

2.1. Basic concepts about hoops

By a *hoop* we mean an algebra $(H, \odot, \rightarrow, 1)$ in which $(H, \odot, 1)$ is a commutative monoid and the following assertions are valid.

- (H1) $(\forall a \in H)(a \rightarrow a = 1)$,
- (H2) $(\forall a, b \in H)(a \odot (a \rightarrow b) = b \odot (b \rightarrow a))$,
- (H3) $(\forall a, b, c \in H)(a \rightarrow (b \rightarrow c) = (a \odot b) \rightarrow c)$.

On hoop H , we define $a \leq b$ if and only if $a \rightarrow b = 1$. It is easy to see that $\leq$ is a partial order relation on H .

By a *sub-hoop* of a hoop H we mean a subset F of H which satisfies the condition:

$$(\forall a, b \in H)(a, b \in F \Rightarrow a \odot b \in F, a \rightarrow b \in F).$$

Note that every non-empty sub-hoop contains the element 1.

Every hoop H satisfies the following conditions (see [6, 7]).

$$\begin{aligned} &(\forall a, b \in H)(a \odot b \leq c \Leftrightarrow a \leq b \rightarrow c). \\ &(\forall a, b \in H)(a \odot b \leq a, b). \\ &(\forall a, b \in H)(a \leq b \rightarrow a). \\ &(\forall a \in H)(a \rightarrow 1 = 1). \end{aligned}$$

$$(\forall a \in H)(1 \rightarrow a = a). \quad (2.1)$$

2.2. Basic concepts about (Łukasiewicz) fuzzy set informations

A fuzzy set λ in a set H of the form

$$\lambda(b) := \begin{cases} t \in (0, 1] & \text{if } b = a, \\ 0 & \text{if } b \neq a, \end{cases}$$

is said to be a *fuzzy point* with support a and value t and is denoted by $\langle a/t \rangle$.

For a fuzzy set λ in a set H , we say that a fuzzy point $\langle a/t \rangle$ is

- (i) *contained* in λ , denoted by $\langle a/t \rangle \in \lambda$, (see [10]) if $\lambda(a) \geq t$.
- (ii) *quasi-coincident* with λ , denoted by $\langle a/t \rangle q \lambda$, (see [10]) if $\lambda(a) + t > 1$.

If $\langle a/t \rangle \alpha \lambda$ is not established for $\alpha \in \{\in, q\}$, it is denoted by $\langle a/t \rangle \bar{\alpha} \lambda$.

Definition 1 ([4, Definition 1]). A fuzzy set λ in H is called a *fuzzy sub-hoop* of H (see [4]) if it satisfies:

$$(\forall a, b \in H) \left(\begin{cases} \lambda(a \odot b) \geq \min\{\lambda(a), \lambda(b)\} \\ \lambda(a \rightarrow b) \geq \min\{\lambda(a), \lambda(b)\} \end{cases} \right).$$

Definition 2 ([9, Definition 3.1]). Let λ be a fuzzy set in a set H and let $\delta \in (0, 1)$. A function

$$\mathbb{L}_\lambda^\delta: H \rightarrow [0, 1], x \mapsto \max\{0, \lambda(x) + \delta - 1\}$$

is called the *Łukasiewicz fuzzy set* of λ in H .

Let λ be a fuzzy set in a set H . For the Łukasiewicz fuzzy set $\mathbb{L}_\lambda^\delta$ of λ in H and $t \in (0, 1]$, consider the sets

$$(\mathbb{L}_\lambda^\delta, t)_\in := \{x \in H \mid \langle x/t \rangle \in \mathbb{L}_\lambda^\delta\} \text{ and } (\mathbb{L}_\lambda^\delta, t)_q := \{x \in H \mid \langle x/t \rangle q \mathbb{L}_\lambda^\delta\},$$

which are called the $\in$ -set and q -set, respectively, of $\mathbb{L}_\lambda^\delta$ (with value t). Also, consider a set:

$$O(\mathbb{L}_\lambda^\delta) := \{x \in H \mid \mathbb{L}_\lambda^\delta(x) > 0\},$$

which is called the O -set of $\mathbb{L}_\lambda^\delta$. It is observed that

$$O(\mathbb{L}_\lambda^\delta) = \{x \in H \mid \lambda(x) + \delta - 1 > 0\}.$$

3. ŁUKASIEWICZ FUZZY SUB-HOOPS

In what follows, let H denote a hoop unless otherwise specified.

Definition 3. Let λ be a fuzzy set in H and δ an element of $(0, 1)$. Then its Łukasiewicz fuzzy set $\mathbb{L}_\lambda^\delta$ in H is called a *Łukasiewicz fuzzy sub-hoop* of H if it satisfies:

$$(\forall x, y \in H)(\forall t_a, t_b \in (0, 1]) \left(\begin{array}{l} \langle x/t_a \rangle \in \mathbb{L}_\lambda^\delta, \langle y/t_b \rangle \in \mathbb{L}_\lambda^\delta \\ \Rightarrow \left\{ \begin{array}{l} \langle (x \odot y)/\min\{t_a, t_b\} \rangle \in \mathbb{L}_\lambda^\delta \\ \langle (x \rightarrow y)/\min\{t_a, t_b\} \rangle \in \mathbb{L}_\lambda^\delta \end{array} \right. \end{array} \right). \quad (3.1)$$

Example 1. Let $H = \{0, a, b, c, d, 1\}$ be a set with binary operations “ $\odot$ ” and “ $\rightarrow$ ” in Table 1 and Table 2, respectively.

TABLE 1. Cayley table for the binary operation “ $\odot$ ”

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	d	0	d	a
b	0	d	c	c	0	b
c	0	0	c	c	0	c
d	0	d	0	0	0	d
1	0	a	b	c	d	1

TABLE 2. Cayley table for the binary operation “ $\rightarrow$ ”

$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	c	1	b	c	b	1
b	d	a	1	b	a	1
c	a	a	1	1	a	1
d	b	1	1	b	1	1
1	0	a	b	c	d	1

Then $(H, \odot, \rightarrow, 1)$ is a hoop. Define a fuzzy set λ in H as follows:

$$\lambda: H \rightarrow [0, 1], x \mapsto \begin{cases} 0.58 & \text{if } x = 0, \\ 0.77 & \text{if } x = a, \\ 0.37 & \text{if } x = b, \\ 0.58 & \text{if } x = c, \\ 0.37 & \text{if } x = d, \\ 0.83 & \text{if } x = 1. \end{cases}$$

Given $\delta := 0.58$, the Łukasiewicz fuzzy set $\mathbb{L}_\lambda^\delta$ of λ in H is given as follows:

$$\mathbb{L}_\lambda^\delta: H \rightarrow [0, 1], x \mapsto \begin{cases} 0.16 & \text{if } x = 0, \\ 0.35 & \text{if } x = a, \\ 0.00 & \text{if } x = b, \\ 0.16 & \text{if } x = c, \\ 0.00 & \text{if } x = d, \\ 0.41 & \text{if } x = 1. \end{cases}$$

It is routine to verify that $\mathbb{L}_\lambda^\delta$ is a Łukasiewicz fuzzy sub-hoop of H .

Theorem 1. *If λ is a fuzzy sub-hoop of H , then its Łukasiewicz fuzzy set $\mathbb{L}_\lambda^\delta$ in H is a Łukasiewicz fuzzy sub-hoop of H .*

Proof. Assume that λ is a fuzzy sub-hoop of H . Let $x, y \in H$ and $t_a, t_b \in (0, 1]$ be such that $\langle x/t_a \rangle \in \mathbb{L}_\lambda^\delta$ and $\langle y/t_b \rangle \in \mathbb{L}_\lambda^\delta$. Then $\mathbb{L}_\lambda^\delta(x) \geq t_a$ and $\mathbb{L}_\lambda^\delta(y) \geq t_b$, so

$$\begin{aligned} \mathbb{L}_\lambda^\delta(x \odot y) &= \max\{0, \lambda(x \odot y) + \delta - 1\} \\ &\geq \max\{0, \min\{\lambda(x), \lambda(y)\} + \delta - 1\} \\ &= \max\{0, \min\{\lambda(x) + \delta - 1, \lambda(y) + \delta - 1\}\} \\ &= \min\{\max\{0, \lambda(x) + \delta - 1\}, \max\{0, \lambda(y) + \delta - 1\}\} \end{aligned}$$

$$= \min\{\mathbb{L}_\lambda^\delta(x), \mathbb{L}_\lambda^\delta(y)\} \geq \min\{t_a, t_b\}.$$

Hence $\langle (x \odot y) / \min\{t_a, t_b\} \rangle \in \mathbb{L}_\lambda^\delta$. The similar way induces $\langle (x \rightarrow y) / \min\{t_a, t_b\} \rangle \in \mathbb{L}_\lambda^\delta$. Therefore, $\mathbb{L}_\lambda^\delta$ is a Łukasiewicz fuzzy sub-hoop of H . $\square$

The following example shows that the converse of Theorem 1 may not be true.

Example 2. Consider the hoop $H = \{0, a, b, c, d, 1\}$ in Example 1 and define a fuzzy set λ in H as follows:

$$\lambda: H \rightarrow [0, 1], x \mapsto \begin{cases} 0.22 & \text{if } x = 0, \\ 0.17 & \text{if } x = a, \\ 0.85 & \text{if } x = b, \\ 0.90 & \text{if } x = c, \\ 0.27 & \text{if } x = d, \\ 0.96 & \text{if } x = 1. \end{cases}$$

Given $\delta := 0.73$, the Łukasiewicz fuzzy set $\mathbb{L}_\lambda^\delta$ of λ in H is given as follows:

$$\mathbb{L}_\lambda^\delta: H \rightarrow [0, 1], x \mapsto \begin{cases} 0.69 & \text{if } x = 1, \\ 0.58 & \text{if } x = b, \\ 0.63 & \text{if } x = c, \\ 0.00 & \text{if } x \in \{0, a, d\}. \end{cases}$$

It is routine to verify that $\mathbb{L}_\lambda^\delta$ is a Łukasiewicz fuzzy sub-hoop of H . But λ is not a fuzzy sub-hoop of H since $\lambda(d \odot c) = \lambda(0) = 0.22 \not\geq 0.27 = \min\{\lambda(d), \lambda(c)\}$.

Lemma 1. *If λ is a fuzzy sub-hoop of H , then its Łukasiewicz fuzzy set $\mathbb{L}_\lambda^\delta$ satisfies:*

$$(\forall x \in H)(\forall t \in (0, 1]) \left(\langle x/t \rangle \in \mathbb{L}_\lambda^\delta \Rightarrow \langle 1/t \rangle \in \mathbb{L}_\lambda^\delta \right). \quad (3.2)$$

Proof. If λ is a fuzzy sub-hoop of H , then its Łukasiewicz fuzzy set $\mathbb{L}_\lambda^\delta$ is a Łukasiewicz fuzzy sub-hoop of H by Theorem 1. Hence the combination of (H1) and (3.1) induces (3.2). $\square$

Remark 1. Let λ be a fuzzy sub-hoop of H , and assume that the condition (3.2) is valid. Since $\langle x/\mathbb{L}_\lambda^\delta(x) \rangle \in \mathbb{L}_\lambda^\delta$ for all $x \in H$, we have $\langle 1/\mathbb{L}_\lambda^\delta(x) \rangle \in \mathbb{L}_\lambda^\delta$ by (3.2). Hence $\mathbb{L}_\lambda^\delta(1) \geq \mathbb{L}_\lambda^\delta(x)$ for all $x \in H$. Conversely, suppose $\mathbb{L}_\lambda^\delta$ satisfies $\mathbb{L}_\lambda^\delta(1) \geq \mathbb{L}_\lambda^\delta(x)$ for all $x \in H$. Let $x \in H$ and $t \in (0, 1]$ be such that $\langle x/t \rangle \in \mathbb{L}_\lambda^\delta$. Then $\mathbb{L}_\lambda^\delta(1) \geq \mathbb{L}_\lambda^\delta(x) \geq t$, that is, $\langle 1/t \rangle \in \mathbb{L}_\lambda^\delta$. This shows that if λ is a fuzzy sub-hoop of H , then the condition (3.2) is equivalent to the condition below:

$$(\forall x \in H) \left(\mathbb{L}_\lambda^\delta(1) \geq \mathbb{L}_\lambda^\delta(x) \right).$$

Proposition 1. *For a Łukasiewicz fuzzy set L_λ^δ of λ in H , if λ is a fuzzy sub-hoop of H , then the following assertions are equivalent.*

$$(\forall x \in H) \left(L_\lambda^\delta(x) = L_\lambda^\delta(1) \right). \quad (3.3)$$

$$(\forall x, y \in H)(\forall t \in (0, 1]) \left(\left\{ \langle y/t \rangle \in L_\lambda^\delta \Rightarrow \begin{cases} \langle (y \odot x)/t \rangle \in L_\lambda^\delta \\ \langle (y \rightarrow x)/t \rangle \in L_\lambda^\delta \end{cases} \right\} \right). \quad (3.4)$$

Proof. Let λ be a fuzzy sub-hoop of H . Then its Łukasiewicz fuzzy set L_λ^δ is a Łukasiewicz fuzzy sub-hoop of H (see Theorem 1). Assume that (3.3) is valid. Let $x, y \in H$ and $t \in (0, 1]$ be such that $\langle y/t \rangle \in L_\lambda^\delta$. Then $\langle 1/t \rangle \in L_\lambda^\delta$ by Lemma 1. Since $\langle x/L_\lambda^\delta(x) \rangle \in L_\lambda^\delta$ and $\langle y/L_\lambda^\delta(y) \rangle \in L_\lambda^\delta$ for all $x, y \in H$, we have

$$\langle (y \odot x)/\min\{L_\lambda^\delta(y), L_\lambda^\delta(x)\} \rangle \in L_\lambda^\delta, \quad \langle (y \rightarrow x)/\min\{L_\lambda^\delta(y), L_\lambda^\delta(x)\} \rangle \in L_\lambda^\delta$$

by (3.1). It follows from Remark 1 and (3.3) that

$$L_\lambda^\delta(y \odot x) \geq \min\{L_\lambda^\delta(y), L_\lambda^\delta(x)\} = \min\{L_\lambda^\delta(1), L_\lambda^\delta(x)\} = L_\lambda^\delta(x) \geq t$$

and

$$L_\lambda^\delta(y \rightarrow x) \geq \min\{L_\lambda^\delta(y), L_\lambda^\delta(x)\} = \min\{L_\lambda^\delta(1), L_\lambda^\delta(x)\} = L_\lambda^\delta(x) \geq t.$$

Hence $\langle (y \odot x)/t \rangle \in L_\lambda^\delta$ and $\langle (y \rightarrow x)/t \rangle \in L_\lambda^\delta$ for all $x, y \in H$ and $t \in (0, 1]$.

Conversely, suppose that (3.4) is valid. Since $\langle 1/L_\lambda^\delta(1) \rangle \in L_\lambda^\delta$, it follows from (2.1) and (3.4) that $\langle x/L_\lambda^\delta(1) \rangle = \langle (1 \rightarrow x)/L_\lambda^\delta(1) \rangle \in L_\lambda^\delta$, that is, $L_\lambda^\delta(x) \geq L_\lambda^\delta(1)$ for all $x \in H$. Combining this with Remark 1 leads to $L_\lambda^\delta(x) = L_\lambda^\delta(1)$ for all $x \in H$. $\square$

Theorem 2. *Let λ be a fuzzy set in H . If the Łukasiewicz fuzzy set L_λ^δ of λ in H satisfies:*

$$(\forall x, y, z \in H)(\forall t_b, t_c \in (0, 1]) \left(\begin{cases} z \leq x, \langle y/t_b \rangle \in L_\lambda^\delta, \langle z/t_c \rangle \in L_\lambda^\delta \\ \Rightarrow \begin{cases} \langle (x \odot y)/\min\{t_b, t_c\} \rangle \in L_\lambda^\delta \\ \langle (x \rightarrow y)/\min\{t_b, t_c\} \rangle \in L_\lambda^\delta \end{cases} \end{cases} \right), \quad (3.5)$$

then L_λ^δ is a Łukasiewicz fuzzy sub-hoop of H .

Proof. Let $x, y \in H$ and $t_a, t_b \in (0, 1]$ be such that $\langle x/t_a \rangle \in L_\lambda^\delta$ and $\langle y/t_b \rangle \in L_\lambda^\delta$. Since $x \leq x$ for all $x \in H$, it follows from (3.5) that $\langle (x \odot y)/\min\{t_a, t_b\} \rangle \in L_\lambda^\delta$ and $\langle (x \rightarrow y)/\min\{t_a, t_b\} \rangle \in L_\lambda^\delta$. Therefore, L_λ^δ is a Łukasiewicz fuzzy sub-hoop of H . $\square$

Corollary 1. *Let λ be a fuzzy set in H . If the Łukasiewicz fuzzy set L_λ^δ of λ in H satisfies:*

$$(\forall x, y, z \in H)(\forall t_b, t_c \in (0, 1]) \left(\begin{cases} z \leq x, \langle y/t_b \rangle \in L_\lambda^\delta, \langle z/t_c \rangle \in L_\lambda^\delta \\ \Rightarrow \begin{cases} \langle (x \odot y)/\max\{t_b, t_c\} \rangle \in L_\lambda^\delta \\ \langle (x \rightarrow y)/\max\{t_b, t_c\} \rangle \in L_\lambda^\delta \end{cases} \end{cases} \right),$$

then L_λ^δ is a Łukasiewicz fuzzy sub-hoop of H .

Theorem 3. Let λ be a fuzzy set in H . If the Łukasiewicz fuzzy set L_λ^δ of λ in H satisfies:

$$(\forall x, y, z \in H)(\forall t_b, t_c \in (0, 0.5]) \left(\begin{array}{l} z \leq x, \langle y/t_b \rangle \in L_\lambda^\delta, \langle z/t_c \rangle \in L_\lambda^\delta \\ \Rightarrow \begin{cases} \langle (x \odot y)/\min\{t_b, t_c\} \rangle q L_\lambda^\delta \\ \langle (x \rightarrow y)/\min\{t_b, t_c\} \rangle q L_\lambda^\delta \end{cases} \end{array} \right), \quad (3.6)$$

then L_λ^δ is a Łukasiewicz fuzzy sub-hoop of H .

Proof. Let $x, y \in H$ and $t_a, t_b \in (0, 0.5] \subseteq (0, 1]$ be such that $\langle x/t_a \rangle \in L_\lambda^\delta$ and $\langle y/t_b \rangle \in L_\lambda^\delta$. Since $x \leq x$ for all $x \in H$, we have

$$\langle (x \odot y)/\min\{t_a, t_b\} \rangle q L_\lambda^\delta \quad \langle (x \rightarrow y)/\min\{t_a, t_b\} \rangle q L_\lambda^\delta$$

by (3.6). Since $\min\{t_a, t_b\} \leq 0.5$, it follows that

$$L_\lambda^\delta(x \odot y) > 1 - \min\{t_a, t_b\} \geq \min\{t_a, t_b\}$$

and

$$L_\lambda^\delta(x \rightarrow y) > 1 - \min\{t_a, t_b\} \geq \min\{t_a, t_b\}.$$

This shows that $\langle (x \odot y)/\min\{t_a, t_b\} \rangle \in L_\lambda^\delta$ and $\langle (x \rightarrow y)/\min\{t_a, t_b\} \rangle \in L_\lambda^\delta$. Therefore, L_λ^δ is a Łukasiewicz fuzzy sub-hoop of H . $\square$

Corollary 2. Let λ be a fuzzy set in H . If the Łukasiewicz fuzzy set L_λ^δ of λ in H satisfies:

$$(\forall x, y, z \in H)(\forall t_b, t_c \in (0, 0.5]) \left(\begin{array}{l} z \leq x, \langle y/t_b \rangle \in L_\lambda^\delta, \langle z/t_c \rangle \in L_\lambda^\delta \\ \Rightarrow \begin{cases} \langle (x \odot y)/\max\{t_b, t_c\} \rangle q L_\lambda^\delta \\ \langle (x \rightarrow y)/\max\{t_b, t_c\} \rangle q L_\lambda^\delta \end{cases} \end{array} \right),$$

then L_λ^δ is a Łukasiewicz fuzzy sub-hoop of H .

Theorem 4. Let λ be a fuzzy set in H . If the Łukasiewicz fuzzy set L_λ^δ of λ in H satisfies:

$$(\forall x, y, z \in H)(\forall t_b, t_c \in (0.5, 1]) \left(\begin{array}{l} z \leq x, \langle y/t_b \rangle q L_\lambda^\delta, \langle z/t_c \rangle q L_\lambda^\delta \\ \Rightarrow \begin{cases} \langle (x \odot y)/\min\{t_b, t_c\} \rangle \in L_\lambda^\delta \\ \langle (x \rightarrow y)/\min\{t_b, t_c\} \rangle \in L_\lambda^\delta \end{cases} \end{array} \right), \quad (3.7)$$

then L_λ^δ is a Łukasiewicz fuzzy sub-hoop of H .

Proof. Let $x, y \in H$ and $t_a, t_b \in (0.5, 1] \subseteq (0, 1]$ be such that $\langle x/t_a \rangle \in \mathbf{L}_\lambda^\delta$ and $\langle y/t_b \rangle \in \mathbf{L}_\lambda^\delta$. Then $\mathbf{L}_\lambda^\delta(x) \geq t_a > 1 - t_a$ and $\mathbf{L}_\lambda^\delta(y) \geq t_b > 1 - t_b$, that is, $\langle x/t_a \rangle q \mathbf{L}_\lambda^\delta$ and $\langle y/t_b \rangle q \mathbf{L}_\lambda^\delta$. Since $x \leq x$ for all $x \in H$, it follows from (3.7) that $\langle (x \odot y)/\min\{t_a, t_b\} \rangle \in \mathbf{L}_\lambda^\delta$ and $\langle (x \rightarrow y)/\min\{t_a, t_b\} \rangle \in \mathbf{L}_\lambda^\delta$. Hence $\mathbf{L}_\lambda^\delta$ is a Łukasiewicz fuzzy sub-hoop of H . $\square$

Corollary 3. *Let λ be a fuzzy set in H . If the Łukasiewicz fuzzy set $\mathbf{L}_\lambda^\delta$ of λ in H satisfies:*

$$(\forall x, y, z \in H)(\forall t_b, t_c \in (0.5, 1]) \left(\begin{array}{l} z \leq x, \langle y/t_b \rangle q \mathbf{L}_\lambda^\delta, \langle z/t_c \rangle q \mathbf{L}_\lambda^\delta \\ \Rightarrow \left\{ \begin{array}{l} \langle (x \odot y)/\max\{t_b, t_c\} \rangle \in \mathbf{L}_\lambda^\delta \\ \langle (x \rightarrow y)/\max\{t_b, t_c\} \rangle \in \mathbf{L}_\lambda^\delta \end{array} \right. \end{array} \right),$$

then $\mathbf{L}_\lambda^\delta$ is a Łukasiewicz fuzzy sub-hoop of H .

We explore the conditions under which the $\in$ -set, q -set, and O -set of Łukasiewicz fuzzy set can be sub-hoops.

Theorem 5. *Let $\mathbf{L}_\lambda^\delta$ be a Łukasiewicz fuzzy set of a fuzzy set λ in H . Then the $\in$ -set $(\mathbf{L}_\lambda^\delta, t)_\in$ of $\mathbf{L}_\lambda^\delta$ is a sub-hoop of H for all $t \in (0.5, 1]$ if and only if $\min\{\mathbf{L}_\lambda^\delta(x), \mathbf{L}_\lambda^\delta(y)\}$ is less than or equal to*

$$\max\{\mathbf{L}_\lambda^\delta(x \odot y), 0.5\} \text{ and } \max\{\mathbf{L}_\lambda^\delta(x \rightarrow y), 0.5\}.$$

Proof. Assume that the $\in$ -set $(\mathbf{L}_\lambda^\delta, t)_\in$ of $\mathbf{L}_\lambda^\delta$ is a sub-hoop of H for all $t \in (0.5, 1]$. If $\min\{\mathbf{L}_\lambda^\delta(x), \mathbf{L}_\lambda^\delta(y)\} > \max\{\mathbf{L}_\lambda^\delta(x \odot y), 0.5\}$ for some $x, y \in H$, then

$$t := \min\{\mathbf{L}_\lambda^\delta(x), \mathbf{L}_\lambda^\delta(y)\} \in (0.5, 1], \quad \langle x/t \rangle \in \mathbf{L}_\lambda^\delta, \quad \langle y/t \rangle \in \mathbf{L}_\lambda^\delta.$$

Hence $x, y \in (\mathbf{L}_\lambda^\delta, t)_\in$, and so $x \odot y \in (\mathbf{L}_\lambda^\delta, t)_\in$. But $\langle (x \odot y)/t \rangle \notin \mathbf{L}_\lambda^\delta$ implies $x \odot y \notin (\mathbf{L}_\lambda^\delta, t)_\in$, a contradiction. Hence $\min\{\mathbf{L}_\lambda^\delta(x), \mathbf{L}_\lambda^\delta(y)\} \leq \max\{\mathbf{L}_\lambda^\delta(x \odot y), 0.5\}$ for all $x, y \in H$. In the same way, we can verify $\min\{\mathbf{L}_\lambda^\delta(x), \mathbf{L}_\lambda^\delta(y)\} \leq \max\{\mathbf{L}_\lambda^\delta(x \rightarrow y), 0.5\}$ for all $x, y \in H$.

Conversely suppose that

$$\begin{aligned} \min\{\mathbf{L}_\lambda^\delta(x), \mathbf{L}_\lambda^\delta(y)\} &\leq \max\{\mathbf{L}_\lambda^\delta(x \rightarrow y), 0.5\}, \\ \min\{\mathbf{L}_\lambda^\delta(x), \mathbf{L}_\lambda^\delta(y)\} &\leq \max\{\mathbf{L}_\lambda^\delta(x \odot y), 0.5\} \end{aligned} \tag{3.8}$$

for all $x, y \in H$. Let $t \in (0.5, 1]$ and $x, y \in H$ be such that $x \in (\mathbf{L}_\lambda^\delta, t)_\in$ and $y \in (\mathbf{L}_\lambda^\delta, t)_\in$. Then $\mathbf{L}_\lambda^\delta(x) \geq t$ and $\mathbf{L}_\lambda^\delta(y) \geq t$, which imply from (3.8) that

$$0.5 < t \leq \min\{\mathbf{L}_\lambda^\delta(x), \mathbf{L}_\lambda^\delta(y)\} \leq \max\{\mathbf{L}_\lambda^\delta(x \odot y), 0.5\}$$

and

$$0.5 < t \leq \min\{\mathbf{L}_\lambda^\delta(x), \mathbf{L}_\lambda^\delta(y)\} \leq \max\{\mathbf{L}_\lambda^\delta(x \rightarrow y), 0.5\}.$$

Hence $\langle (x \odot y)/t \rangle \in \mathbb{L}_\lambda^\delta$ and $\langle (x \rightarrow y)/t \rangle \in \mathbb{L}_\lambda^\delta$, i.e., $x \odot y \in (\mathbb{L}_\lambda^\delta, t)_\in$ and $x \rightarrow y \in (\mathbb{L}_\lambda^\delta, t)_\in$. Therefore $(\mathbb{L}_\lambda^\delta, t)_\in$ is a sub-hoop of H for $t \in (0.5, 1]$. $\square$

Theorem 6. Let $\mathbb{L}_\lambda^\delta$ be a Łukasiewicz fuzzy set of a fuzzy set λ in H . If λ is a fuzzy sub-hoop of H , then the q -set $(\mathbb{L}_\lambda^\delta, t)_q$ of $\mathbb{L}_\lambda^\delta$ is a sub-hoop of H for all $t \in (0, 1]$.

Proof. If λ is a fuzzy sub-hoop of H , then $\mathbb{L}_\lambda^\delta$ is a Łukasiewicz fuzzy sub-hoop of H (see Theorem 1). Let $t \in (0, 1]$ and $x, y \in (\mathbb{L}_\lambda^\delta, t)_q$. Then $\langle x/t \rangle q \mathbb{L}_\lambda^\delta$ and $\langle y/t \rangle q \mathbb{L}_\lambda^\delta$, that is, $\mathbb{L}_\lambda^\delta(x) + t > 1$ and $\mathbb{L}_\lambda^\delta(y) + t > 1$. Since $\langle x/\mathbb{L}_\lambda^\delta(x) \rangle \in \mathbb{L}_\lambda^\delta$ and $\langle y/\mathbb{L}_\lambda^\delta(y) \rangle \in \mathbb{L}_\lambda^\delta$ for all $x, y \in H$, we have

$$\langle (x \odot y)/\min\{\mathbb{L}_\lambda^\delta(x), \mathbb{L}_\lambda^\delta(y)\} \rangle \in \mathbb{L}_\lambda^\delta, \quad \langle (x \rightarrow y)/\min\{\mathbb{L}_\lambda^\delta(x), \mathbb{L}_\lambda^\delta(y)\} \rangle \in \mathbb{L}_\lambda^\delta$$

by (3.1). It follows that

$$\mathbb{L}_\lambda^\delta(x \odot y) + t \geq \min\{\mathbb{L}_\lambda^\delta(x), \mathbb{L}_\lambda^\delta(y)\} + t = \min\{\mathbb{L}_\lambda^\delta(x) + t, \mathbb{L}_\lambda^\delta(y) + t\} > 1$$

and

$$\mathbb{L}_\lambda^\delta(x \rightarrow y) + t \geq \min\{\mathbb{L}_\lambda^\delta(x), \mathbb{L}_\lambda^\delta(y)\} + t = \min\{\mathbb{L}_\lambda^\delta(x) + t, \mathbb{L}_\lambda^\delta(y) + t\} > 1.$$

Hence $\langle (x \odot y)/t \rangle q \mathbb{L}_\lambda^\delta$ and $\langle (x \rightarrow y)/t \rangle q \mathbb{L}_\lambda^\delta$, that is, $x \odot y \in (\mathbb{L}_\lambda^\delta, t)_q$ and $x \rightarrow y \in (\mathbb{L}_\lambda^\delta, t)_q$. Thus, $(\mathbb{L}_\lambda^\delta, t)_q$ is a sub-hoop of H . $\square$

Proposition 2. Let $\mathbb{L}_\lambda^\delta$ be a Łukasiewicz fuzzy set of a fuzzy set λ in H .

(i) If the q -set $(\mathbb{L}_\lambda^\delta, t)_q$ is a sub-hoop of H for all $t \in (0, 0.5]$, then $\mathbb{L}_\lambda^\delta$ satisfies:

$$(\forall x, y \in H)(\forall t_a, t_b \in (0, 0.5]) \left(\begin{array}{l} \langle x/t_a \rangle q \mathbb{L}_\lambda^\delta, \langle y/t_b \rangle q \mathbb{L}_\lambda^\delta \\ \Rightarrow \left\{ \begin{array}{l} \langle (x \odot y)/\max\{t_a, t_b\} \rangle \in \mathbb{L}_\lambda^\delta \\ \langle (x \rightarrow y)/\max\{t_a, t_b\} \rangle \in \mathbb{L}_\lambda^\delta \end{array} \right. \end{array} \right).$$

(ii) If the $\in$ -set $(\mathbb{L}_\lambda^\delta, t)_\in$ is a sub-hoop of H for all $t \in (0.5, 1]$, then $\mathbb{L}_\lambda^\delta$ satisfies:

$$(\forall x, y \in H)(\forall t_a, t_b \in (0.5, 1]) \left(\begin{array}{l} \langle x/t_a \rangle \in \mathbb{L}_\lambda^\delta, \langle y/t_b \rangle \in \mathbb{L}_\lambda^\delta \\ \Rightarrow \left\{ \begin{array}{l} \langle (x \odot y)/\min\{t_a, t_b\} \rangle q \mathbb{L}_\lambda^\delta \\ \langle (x \rightarrow y)/\min\{t_a, t_b\} \rangle q \mathbb{L}_\lambda^\delta \end{array} \right. \end{array} \right).$$

Proof.

(i) Assume that $(\mathbb{L}_\lambda^\delta, t)_q$ is a sub-hoop of H for all $t \in (0, 0.5]$. Let $x, y \in H$ and $t_a, t_b \in (0, 0.5]$ be such that $\langle x/t_a \rangle q \mathbb{L}_\lambda^\delta$ and $\langle y/t_b \rangle q \mathbb{L}_\lambda^\delta$. Then $x \in (\mathbb{L}_\lambda^\delta, t_a)_q \subseteq (\mathbb{L}_\lambda^\delta, \max\{t_a, t_b\})_q$ and $y \in (\mathbb{L}_\lambda^\delta, t_b)_q \subseteq (\mathbb{L}_\lambda^\delta, \max\{t_a, t_b\})_q$. Hence $x \odot y \in (\mathbb{L}_\lambda^\delta, \max\{t_a, t_b\})_q$ and $x \rightarrow y \in (\mathbb{L}_\lambda^\delta, \max\{t_a, t_b\})_q$. Since $\max\{t_a, t_b\} \leq 0.5$, it follows that

$$\mathbb{L}_\lambda^\delta(x \odot y) > 1 - \max\{t_a, t_b\} \geq \max\{t_a, t_b\}$$

and

$$\mathbb{L}_\lambda^\delta(x \rightarrow y) > 1 - \max\{t_a, t_b\} \geq \max\{t_a, t_b\}.$$

Therefore, $\langle (x \odot y) / \max\{t_a, t_b\} \rangle \in \mathbb{L}_\lambda^\delta$ and $\langle (x \rightarrow y) / \max\{t_a, t_b\} \rangle \in \mathbb{L}_\lambda^\delta$.

- (ii) Assume that $(\mathbb{L}_\lambda^\delta, t)_\in$ is a sub-hoop of H for all $t \in (0.5, 1]$. Let $x, y \in H$ and $t_a, t_b \in (0.5, 1]$ be such that $\langle x/t_a \rangle \in \mathbb{L}_\lambda^\delta$ and $\langle y/t_b \rangle \in \mathbb{L}_\lambda^\delta$. Then $\mathbb{L}_\lambda^\delta(x) \geq t_a \geq \min\{t_a, t_b\}$ and $\mathbb{L}_\lambda^\delta(y) \geq t_b \geq \min\{t_a, t_b\}$, which imply that

$$x \in (\mathbb{L}_\lambda^\delta, \min\{t_a, t_b\})_\in, \quad y \in (\mathbb{L}_\lambda^\delta, \min\{t_a, t_b\})_\in.$$

Hence

$$x \odot y \in (\mathbb{L}_\lambda^\delta, \min\{t_a, t_b\})_\in, \quad x \rightarrow y \in (\mathbb{L}_\lambda^\delta, \min\{t_a, t_b\})_\in.$$

Since $\min\{t_a, t_b\} > 0.5$, it follows that

$$\mathbb{L}_\lambda^\delta(x \odot y) \geq \min\{t_a, t_b\} > 1 - \min\{t_a, t_b\}$$

and

$$\mathbb{L}_\lambda^\delta(x \rightarrow y) \geq \min\{t_a, t_b\} > 1 - \min\{t_a, t_b\}$$

Therefore, $\langle (x \odot y) / \min\{t_a, t_b\} \rangle q \mathbb{L}_\lambda^\delta$ and $\langle (x \rightarrow y) / \min\{t_a, t_b\} \rangle q \mathbb{L}_\lambda^\delta$.

□

Theorem 7. If $\mathbb{L}_\lambda^\delta$ satisfies:

$$(\forall x, y \in H)(\forall t \in (0, 0.5]) \left(\begin{array}{l} \langle x/t \rangle \in \mathbb{L}_\lambda^\delta, \langle y/t \rangle \in \mathbb{L}_\lambda^\delta \\ \Rightarrow \left\{ \begin{array}{l} \langle (x \odot y)/t \rangle q \mathbb{L}_\lambda^\delta \\ \langle (x \rightarrow y)/t \rangle q \mathbb{L}_\lambda^\delta \end{array} \right. \end{array} \right), \quad (3.9)$$

then the $\in$ -set $(\mathbb{L}_\lambda^\delta, t)_\in$ is a sub-hoop of H for all $t \in (0, 0.5]$.

Proof. Let $t \in (0, 0.5]$ and assume that $\mathbb{L}_\lambda^\delta$ satisfies the condition (3.9). If $x, y \in (\mathbb{L}_\lambda^\delta, t)_\in$, then $\langle x/t \rangle \in \mathbb{L}_\lambda^\delta$ and $\langle y/t \rangle \in \mathbb{L}_\lambda^\delta$, and thus $\langle (x \odot y)/t \rangle q \mathbb{L}_\lambda^\delta$ and $\langle (x \rightarrow y)/t \rangle q \mathbb{L}_\lambda^\delta$ by (3.9). Since $t \leq 0.5$, it follows that $\mathbb{L}_\lambda^\delta(x \odot y) > 1 - t \geq t$ and $\mathbb{L}_\lambda^\delta(x \rightarrow y) > 1 - t \geq t$. Thus $x \odot y \in (\mathbb{L}_\lambda^\delta, t)_\in$ and $x \rightarrow y \in (\mathbb{L}_\lambda^\delta, t)_\in$ which shows that $(\mathbb{L}_\lambda^\delta, t)_\in$ is a sub-hoop of H for all $t \in (0, 0.5]$. □

Theorem 8. If $\mathbb{L}_\lambda^\delta$ satisfies:

$$(\forall x, y \in H)(\forall t \in (0.5, 1]) \left(\begin{array}{l} \langle x/t \rangle q \mathbb{L}_\lambda^\delta, \langle y/t \rangle q \mathbb{L}_\lambda^\delta \\ \Rightarrow \left\{ \begin{array}{l} \langle (x \odot y)/t \rangle \in \mathbb{L}_\lambda^\delta \\ \langle (x \rightarrow y)/t \rangle \in \mathbb{L}_\lambda^\delta \end{array} \right. \end{array} \right), \quad (3.10)$$

then the q -set $(\mathbb{L}_\lambda^\delta, t)_q$ is a sub-hoop of H for all $t \in (0.5, 1]$.

Proof. Suppose that $\mathbf{L}_\lambda^\delta$ satisfies the condition (3.10) and let $t \in (0.5, 1]$ and $x, y \in H$ be such that $x, y \in (\mathbf{L}_\lambda^\delta, t)_q$. Then $\langle x/t \rangle q \mathbf{L}_\lambda^\delta$ and $\langle y/t \rangle q \mathbf{L}_\lambda^\delta$. Hence $\langle (x \odot y)/t \rangle \in \mathbf{L}_\lambda^\delta$ and $\langle (x \rightarrow y)/t \rangle \in \mathbf{L}_\lambda^\delta$ by (3.10). Since $t > 0.5$, it follows that $\mathbf{L}_\lambda^\delta(x \odot y) \geq t > 1 - t$ and $\mathbf{L}_\lambda^\delta(x \rightarrow y) \geq t > 1 - t$, that is, $x \odot y \in (\mathbf{L}_\lambda^\delta, t)_q$ and $x \rightarrow y \in (\mathbf{L}_\lambda^\delta, t)_q$. Therefore, $(\mathbf{L}_\lambda^\delta, t)_q$ is a sub-hoop of H for all $t \in (0.5, 1]$. $\square$

Theorem 9. Let $\mathbf{L}_\lambda^\delta$ be a Łukasiewicz fuzzy set of a fuzzy set λ in H . If λ is a fuzzy sub-hoop of H , then the O -set $O(\mathbf{L}_\lambda^\delta)$ of $\mathbf{L}_\lambda^\delta$ is a sub-hoop of H .

Proof. Let $x, y \in O(\mathbf{L}_\lambda^\delta)$. Then $\lambda(x) + \delta - 1 > 0$ and $\lambda(y) + \delta - 1 > 0$. If λ is a fuzzy sub-hoop of H , then $\mathbf{L}_\lambda^\delta$ is a Łukasiewicz fuzzy sub-hoop of H (see Theorem 1). It follows that

$$\mathbf{L}_\lambda^\delta(x \odot y) \geq \min\{\mathbf{L}_\lambda^\delta(x), \mathbf{L}_\lambda^\delta(y)\} = \min\{\lambda(x) + \delta - 1, \lambda(y) + \delta - 1\} > 0$$

and

$$\mathbf{L}_\lambda^\delta(x \rightarrow y) \geq \min\{\mathbf{L}_\lambda^\delta(x), \mathbf{L}_\lambda^\delta(y)\} = \min\{\lambda(x) + \delta - 1, \lambda(y) + \delta - 1\} > 0.$$

Hence $x \odot y \in O(\mathbf{L}_\lambda^\delta)$ and $x \rightarrow y \in O(\mathbf{L}_\lambda^\delta)$, and therefore $O(\mathbf{L}_\lambda^\delta)$ is a sub-hoop of H . $\square$

Theorem 10. Let λ be a fuzzy set in H . If a Łukasiewicz fuzzy set $\mathbf{L}_\lambda^\delta$ of λ in H satisfies:

$$(\forall x, y \in H)(\forall t_a, t_b \in (0, 1]) \left(\begin{array}{l} \langle x/t_a \rangle \in \mathbf{L}_\lambda^\delta, \langle y/t_b \rangle \in \mathbf{L}_\lambda^\delta \\ \Rightarrow \left\{ \begin{array}{l} \langle (x \odot y)/\max\{t_a, t_b\} \rangle q \mathbf{L}_\lambda^\delta \\ \langle (x \rightarrow y)/\max\{t_a, t_b\} \rangle q \mathbf{L}_\lambda^\delta \end{array} \right. \end{array} \right). \quad (3.11)$$

then the O -set $O(\mathbf{L}_\lambda^\delta)$ of $\mathbf{L}_\lambda^\delta$ is a sub-hoop of H .

Proof. Assume that $\mathbf{L}_\lambda^\delta$ satisfies the condition (3.11). If $x, y \in O(\mathbf{L}_\lambda^\delta)$, then $\lambda(x) + \delta - 1 > 0$ and $\lambda(y) + \delta - 1 > 0$. Since $[x/\mathbf{L}_\lambda^\delta(x)] \in \mathbf{L}_\lambda^\delta$ and $[y/\mathbf{L}_\lambda^\delta(y)] \in \mathbf{L}_\lambda^\delta$, it follows from (3.11) that

$$\langle (x \odot y)/\max\{\mathbf{L}_\lambda^\delta(x), \mathbf{L}_\lambda^\delta(y)\} \rangle q \mathbf{L}_\lambda^\delta \text{ and } \langle (x \rightarrow y)/\max\{\mathbf{L}_\lambda^\delta(x), \mathbf{L}_\lambda^\delta(y)\} \rangle q \mathbf{L}_\lambda^\delta.$$

If $x \odot y \notin O(\mathbf{L}_\lambda^\delta)$ or $x \rightarrow y \notin O(\mathbf{L}_\lambda^\delta)$, then $\mathbf{L}_\lambda^\delta(x \odot y) = 0$ or $\mathbf{L}_\lambda^\delta(x \rightarrow y) = 0$. Hence

$$\begin{aligned} \mathbf{L}_\lambda^\delta(x \odot y) + \max\{\mathbf{L}_\lambda^\delta(x), \mathbf{L}_\lambda^\delta(y)\} &= \max\{\mathbf{L}_\lambda^\delta(x), \mathbf{L}_\lambda^\delta(y)\} \\ &= \max\{\max\{0, \lambda(x) + \delta - 1\}, \max\{0, \lambda(y) + \delta - 1\}\} \\ &= \max\{\lambda(x) + \delta - 1, \lambda(y) + \delta - 1\} \\ &= \max\{\lambda(x), \lambda(y)\} + \delta - 1 \\ &\leq 1 + \delta - 1 = \delta \leq 1, \end{aligned}$$

or

$$\mathbf{L}_\lambda^\delta(x \rightarrow y) + \max\{\mathbf{L}_\lambda^\delta(x), \mathbf{L}_\lambda^\delta(y)\} = \max\{\mathbf{L}_\lambda^\delta(x), \mathbf{L}_\lambda^\delta(y)\}$$

$$\begin{aligned}
&= \max\{\max\{0, \lambda(x) + \delta - 1\}, \max\{0, \lambda(y) + \delta - 1\}\} \\
&= \max\{\lambda(x) + \delta - 1, \lambda(y) + \delta - 1\} \\
&= \max\{\lambda(x), \lambda(y)\} + \delta - 1 \\
&\leq 1 + \delta - 1 = \delta \leq 1,
\end{aligned}$$

that is, $\langle (x \odot y) / \max\{\mathbf{L}_\lambda^\delta(x), \mathbf{L}_\lambda^\delta(y)\} \rangle \bar{q} \mathbf{L}_\lambda^\delta$ or $\langle (x \rightarrow y) / \max\{\mathbf{L}_\lambda^\delta(x), \mathbf{L}_\lambda^\delta(y)\} \rangle \bar{q} \mathbf{L}_\lambda^\delta$. This is a contradiction, and thus $x \odot y, x \rightarrow y \in O(\mathbf{L}_\lambda^\delta)$. Therefore, $O(\mathbf{L}_\lambda^\delta)$ is a sub-hoop of H . $\square$

Theorem 11. *Let λ be a fuzzy set in H . If a Łukasiewicz fuzzy set $\mathbf{L}_\lambda^\delta$ of λ in H satisfies:*

$$(\forall x, y \in H)(\forall t_a, t_b \in (0, 1]) \left(\left\{ \begin{array}{l} \langle x/t_a \rangle q \mathbf{L}_\lambda^\delta, \langle y/t_b \rangle q \mathbf{L}_\lambda^\delta \\ \Rightarrow \left\{ \begin{array}{l} \langle (x \odot y) / \max\{t_a, t_b\} \rangle \in \mathbf{L}_\lambda^\delta \\ \langle (x \rightarrow y) / \max\{t_a, t_b\} \rangle \in \mathbf{L}_\lambda^\delta \end{array} \right. \right\} \right), \quad (3.12)$$

then the O -set $O(\mathbf{L}_\lambda^\delta)$ of $\mathbf{L}_\lambda^\delta$ is a sub-hoop of H .

Proof. Assume that $\mathbf{L}_\lambda^\delta$ satisfies the condition (3.12). Let $x, y \in O(\mathbf{L}_\lambda^\delta)$. Then $\lambda(x) + \delta - 1 > 0$ and $\lambda(y) + \delta - 1 > 0$. Hence

$$\mathbf{L}_\lambda^\delta(x) + 1 = \max\{0, \lambda(x) + \delta - 1\} + 1 = \lambda(x) + \delta - 1 + 1 = \lambda(x) + \delta > 1$$

and

$$\mathbf{L}_\lambda^\delta(y) + 1 = \max\{0, \lambda(y) + \delta - 1\} + 1 = \lambda(y) + \delta - 1 + 1 = \lambda(y) + \delta > 1,$$

that is, $\langle x/1 \rangle q \mathbf{L}_\lambda^\delta$ and $\langle y/1 \rangle q \mathbf{L}_\lambda^\delta$. It follows from (3.12) that

$$\begin{aligned}
\langle (x \odot y)/1 \rangle &= \langle (x \odot y) / \max\{1, 1\} \rangle \in \mathbf{L}_\lambda^\delta, \\
\langle (x \rightarrow y)/1 \rangle &= \langle (x \rightarrow y) / \max\{1, 1\} \rangle \in \mathbf{L}_\lambda^\delta.
\end{aligned} \quad (3.13)$$

If $x \odot y \notin O(\mathbf{L}_\lambda^\delta)$ or $x \rightarrow y \notin O(\mathbf{L}_\lambda^\delta)$, then $\mathbf{L}_\lambda^\delta(x \odot y) = 0 < 1$ or $\mathbf{L}_\lambda^\delta(x \rightarrow y) = 0 < 1$, that is, $\langle (x \odot y)/1 \rangle \bar{\in} \mathbf{L}_\lambda^\delta$ or $\langle (x \rightarrow y)/1 \rangle \bar{\in} \mathbf{L}_\lambda^\delta$. This is a contradiction to (3.13), and thus $x \odot y, x \rightarrow y \in O(\mathbf{L}_\lambda^\delta)$. Therefore, $O(\mathbf{L}_\lambda^\delta)$ is a sub-hoop of H . $\square$

Theorem 12. *If a Łukasiewicz fuzzy set $\mathbf{L}_\lambda^\delta$ in H satisfies:*

$$(\forall x, y \in H) \left(\left\{ \langle x/\delta \rangle q \mathbf{L}_\lambda^\delta, \langle y/\delta \rangle q \mathbf{L}_\lambda^\delta \Rightarrow \left\{ \begin{array}{l} \langle (x \odot y)/\delta \rangle \in \mathbf{L}_\lambda^\delta \\ \langle (x \rightarrow y)/\delta \rangle \in \mathbf{L}_\lambda^\delta \end{array} \right. \right\} \right). \quad (3.14)$$

then the O -set of $\mathbf{L}_\lambda^\delta$ is a sub-hoop of H .

Proof. Let $x, y \in O(\mathbf{L}_\lambda^\delta)$. Then $\lambda(x) + \delta > 1$ and $\lambda(y) + \delta > 1$, i.e., $\langle x/\delta \rangle q \mathbf{L}_\lambda^\delta$ and $\langle y/\delta \rangle q \mathbf{L}_\lambda^\delta$. It follows from (3.14) that $\langle (x \odot y)/\delta \rangle \in \mathbf{L}_\lambda^\delta$ and $\langle (x \rightarrow y)/\delta \rangle \in \mathbf{L}_\lambda^\delta$, which

shows $\mathbb{L}_\lambda^\delta(x \odot y) \geq \delta > 0$ and $\mathbb{L}_\lambda^\delta(x \rightarrow y) \geq \delta > 0$. Hence $x \odot y, x \rightarrow y \in O(\mathbb{L}_\lambda^\delta)$, and therefore $O(\mathbb{L}_\lambda^\delta)$ is a sub-hoop of H . $\square$

Theorem 13. *Let $\mathbb{L}_\lambda^\delta$ be a Łukasiewicz fuzzy set in H that satisfies:*

$$(\forall x, y \in H)(\forall t_a, t_b \in [\delta, 1]) \left(\begin{array}{l} \langle x/t_a \rangle q \lambda, \langle y/t_b \rangle q \lambda \\ \Rightarrow x \odot y, x \rightarrow y \in (\mathbb{L}_\lambda^\delta, \delta)_\in \end{array} \right). \quad (3.15)$$

Then the O -set of $\mathbb{L}_\lambda^\delta$ is a sub-hoop of H .

Proof. Assume that $\mathbb{L}_\lambda^\delta$ satisfies (3.15). If $x, y \in O(\mathbb{L}_\lambda^\delta)$, then $\lambda(x) + t_a \geq \lambda(x) + \delta > 1$ and $\lambda(y) + t_b \geq \lambda(y) + \delta > 1$. Thus $\langle x/t_a \rangle q \lambda$ and $\langle y/t_b \rangle q \lambda$. Using (3.15) leads to $x \odot y, x \rightarrow y \in (\mathbb{L}_\lambda^\delta, \delta)_\in$. Hence $\mathbb{L}_\lambda^\delta(x \odot y) \geq \delta > 0$ and $\mathbb{L}_\lambda^\delta(x \rightarrow y) \geq \delta > 0$, that is, $x \odot y, x \rightarrow y \in O(\mathbb{L}_\lambda^\delta)$. Consequently, $O(\mathbb{L}_\lambda^\delta)$ is a sub-hoop of H . $\square$

Corollary 4. *Let $\mathbb{L}_\lambda^\delta$ be a Łukasiewicz fuzzy set in H that satisfies:*

$$(\forall x, y \in H) \left(\begin{array}{l} \langle x/\delta \rangle q \lambda, \langle y/\delta \rangle q \lambda \\ \Rightarrow x \odot y, x \rightarrow y \in (\mathbb{L}_\lambda^\delta, \delta)_\in \end{array} \right).$$

Then the O -set of $\mathbb{L}_\lambda^\delta$ is a sub-hoop of H .

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SOME BOUNDS OF HERMITE–HADAMARD-TYPE INEQUALITIES BASED ON CONFORMABLE FRACTIONAL INTEGRALS

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Abstract. In this research, we establish the above and below bounds via the left and right sides of Hermite–Hadamard-type inequalities including conformable fractional integrals with the aid of the mappings whose second derivatives are bounded. Instead of using the convexity condition in these obtained inequalities, we used condition $f'(a+b-t) - f'(t) \geq 0$, $t \in \left[a, \frac{a+b}{2}\right]$. We have presented examples of the inequalities acquired. We have given the graph showing the correctness of the presented examples.

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1. INTRODUCTION AND PRELIMINARIES

Convex function theory has many uses in the fields of mathematics, physics, and engineering. Let I be convex set on $\mathbb{R}$. The mapping $f: I \rightarrow \mathbb{R}$ is called convex function on I , if it satisfies the following inequality:

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

for all $(x, y) \in I$ and $t \in [0, 1]$. (see, e.g. [11], and the reference therein). There is a great connection between inequalities and the theory of convex functions. This bond emerged starting from the definition of the convex function and its acceleration is increased by the researches made afterwards. The Hermite–Hadamard-type inequalities, which are obtained by using convex functions and have been the subject of many studies, has been studied in the literature extensively. For more information on the Hermite-Hadamard type inequality obtained for different classes of convex, please refer to the references [5, 15–17]. The right-hand side of the Hermite–Hadamard-type inequality is called trapezoid-type inequality in the literature. Dragomir and Agarwal proved trapezoid-type inequalities based on convex mappings in [8]. The

left side of the Hermite–Hadamard-type inequality is called midpoint-type inequality. Kirmacı acquired midpoint-type inequalities with the aid of the convex mappings in [20]. Researchers began to come up with ideas on how to obtain these inequalities when functions are not convex. With this motivation, many researchers have worked on this subject. Dragomir et al. presented trapezoid-type and midpoint-type extensions by means of the bounds of the twice-differentiable rather than the condition of convexity in [9] and [10], respectively. More precisely, Dragomir et al. obtained new bounds for classical midpoint-type and trapezoid-type inequalities.

Theorem 1 ([10, page 6], [9, page 8]). *Assume that $f: [a, b] \rightarrow \mathbb{R}$ is a twice-differentiable mapping such that there exists real constants m and M so that $m \leq f'' \leq M$. Then, the following inequalities hold:*

$$m \frac{(b-a)^2}{24} \leq \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) \leq M \frac{(b-a)^2}{24} \quad (1.1)$$

and

$$m \frac{(b-a)^2}{24} \leq \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \leq M \frac{(b-a)^2}{24}. \quad (1.2)$$

For different studies obtained with this motivation, the following references can be consulted [2, 6, 25].

Fractional analysis, "Can fractional derivatives and fractional integrals be taken?" emerged from the question. Today, it is the subject of study by many researchers. Many types of fractional integrals are studied in the literature. Especially, there are many results obtained with the help of Riemann-Liouville fractional integrals, generalized fractional integrals, and conformable fractional integrals.

The Euler Gamma function and Euler Beta function are defined

$$\Gamma(x) := \int_0^{\infty} t^{x-1} e^{-t} dt$$

and

$$\mathcal{B}(x, y) := \int_0^1 t^{x-1} (1-t)^{y-1} dt,$$

respectively for $x, y \in \mathbb{R}$. Let us give the definitions of Riemann-Liouville fractional integrals in the literature.

Definition 1 (see [19, page 71], [21]). Let us consider $f \in L_1[a, b]$. The Riemann-Liouville fractional integrals of order $\beta > 0$ are described by

$$J_{a+}^{\beta} f(x) = \frac{1}{\Gamma(\beta)} \int_a^x (x-t)^{\beta-1} f(t) dt, \quad x > a$$

and

$$J_{b-}^{\beta} f(x) = \frac{1}{\Gamma(\beta)} \int_x^b (t-x)^{\beta-1} f(t) dt, \quad x < b.$$

Here, Γ is the Euler Gamma function and $J_{a+}^0 f(x) = J_{b-}^0 f(x) = f(x)$.

Many studies contributing to the literature have been considered, especially using the convexity of the function. Using Riemann-Liouville fractional integrals for functions of one variable, Hermite–Hadamard-type inequalities for convex functions were obtained by Sarikaya et al. in [23]. Sarikaya first [23] presented the Hermite–Hadamard-type inequalities involving Riemann-Liouville integrals as follows:

Theorem 2. *Let $f: [a, b] \rightarrow \mathbb{R}$ be a positive mapping with $f \in L_1[a, b]$. If f is a convex mapping on $[a, b]$, then the following inequalities hold:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\beta+1)}{2(b-a)^{\beta}} \left[J_{a+}^{\beta} f(b) + J_{b-}^{\beta} f(a) \right] \leq \frac{f(a) + f(b)}{2}$$

with $\beta > 0$.

Riemann-Liouville fractional integrals for convex functions of two variables were given by Sarikaya [22] and Hermite–Hadamard-type inequalities for convex functions were proved in the coordinates obtained using these integrals. Then obtained the midpoint-type and trapezoid-type inequality for Riemann-Liouville fractional integrals by using the condition $f'(a+b-x) \geq f'(x)$, $x \in [a, \frac{a+b}{2}]$ instead of the convexity of f in paper [7].

Theorem 3 (see [7, page 2,3]). *Consider $f: [a, b] \rightarrow \mathbb{R}$ is a positive, twice-differentiable function and $f \in L_1[a, b]$. If f'' is bounded $[a, b]$, then we derive*

$$\begin{aligned} m \frac{(b-a)^2 (\beta^2 - \beta + 2)}{8(\beta+1)(\beta+2)} &\leq \frac{\Gamma(\beta+1)}{2(b-a)^{\beta}} \left[J_{a+}^{\beta} f(b) + J_{b-}^{\beta} f(a) \right] - f\left(\frac{a+b}{2}\right) \\ &\leq \frac{(b-a)^2 (\beta^2 - \beta + 2)}{8(\beta+1)(\beta+2)} M \end{aligned}$$

and

$$\begin{aligned} m \frac{(b-a)^2 \beta}{2(\beta+1)(\beta+2)} &\leq \frac{f(a) + f(b)}{2} - \frac{\Gamma(\beta+1)}{2(b-a)^{\beta}} \left[J_{a+}^{\beta} f(b) + J_{b-}^{\beta} f(a) \right] \\ &\leq \frac{(b-a)^2 \beta}{2(\beta+1)(\beta+2)} M \end{aligned}$$

for $\beta > 0$. Here, $m = \inf_{t \in [a, b]} f''(t)$ and $M = \sup_{t \in [a, b]} f''(t)$.

Budak et al. acquired the left and right-hand sides of fractional Hermite–Hadamard-type inequalities with the aid of the bounds of the second derivative in [3].

Theorem 4 (see [3, page 3,8]). *Let $f: [a, b] \rightarrow \mathbb{R}$ denote a positive, twice-differentiable function and $f \in L_1[a, b]$. If f'' is bounded i.e. $m \leq f''(t) \leq M$, $t \in [a, b]$, $m, M \in \mathbb{R}$, then we have the inequalities*

$$\begin{aligned} m \frac{(b-a)^2}{4(\beta+1)(\beta+2)} &\leq \frac{2^{\beta-1}\Gamma(\beta+1)}{(b-a)^\beta} \left[J_{\left(\frac{a+b}{2}\right)^+}^\beta f(b) + J_{\left(\frac{a+b}{2}\right)^-}^\beta f(a) \right] - f\left(\frac{a+b}{2}\right) \\ &\leq \frac{(b-a)^2}{4(\beta+1)(\beta+2)} M, \end{aligned} \quad (1.3)$$

$$\begin{aligned} m \frac{(b-a)^2 \beta(\beta+3)}{8(\beta+1)(\beta+2)} &\leq \frac{f(a)+f(b)}{2} - \frac{2^{\beta-1}\Gamma(\beta+1)}{(b-a)^\beta} \left[J_{\left(\frac{a+b}{2}\right)^+}^\beta f(b) + J_{\left(\frac{a+b}{2}\right)^-}^\beta f(a) \right] \\ &\leq \frac{(b-a)^2 \beta(\beta+3)}{8(\beta+1)(\beta+2)} M, \end{aligned} \quad (1.4)$$

$$\begin{aligned} m \frac{\beta(b-a)^2}{8(\beta+2)} &\leq \frac{2^{\beta-1}\Gamma(\beta+1)}{(b-a)^\beta} \left[J_{a^+}^\beta f\left(\frac{a+b}{2}\right) + J_{b^-}^\beta f\left(\frac{a+b}{2}\right) \right] \\ &\quad - f\left(\frac{a+b}{2}\right) \leq \frac{\beta(b-a)^2}{8(\beta+2)} M \end{aligned}$$

and

$$\begin{aligned} m \frac{(b-a)^2}{4(\beta+2)} &\leq \frac{f(a)+f(b)}{2} - \frac{2^{\beta-1}\Gamma(\beta+1)}{(b-a)^\beta} \left[J_{a^+}^\beta f\left(\frac{a+b}{2}\right) + J_{b^-}^\beta f\left(\frac{a+b}{2}\right) \right] \\ &\leq \frac{(b-a)^2}{4(\beta+2)} M \end{aligned}$$

for $\beta > 0$.

Budak et al. obtained some trapezoid-type and midpoint-type inequalities via generalized fractional integrals with the help of the functions whose second derivatives are bounded [4].

In 2017, Jarad et al. [14] defined the conformable fractional integral operators. For studies on the conformable fractional approach and conformable fractional integrals, see references [1, 13, 18].

Definition 2 (see [14, page 5]). For $f \in L_1[a, b]$, the conformable fractional integral operators of order $\beta \in \mathbb{C}$, $\text{Re}(\beta) > 0$ and $\alpha \in (0, 1]$ are given by

$$\begin{aligned} {}^\beta \Upsilon_{a^+}^\alpha f(x) &= \frac{1}{\Gamma(\beta)} \int_a^x \left(\frac{(x-a)^\alpha - (t-a)^\alpha}{\alpha} \right)^{\beta-1} \frac{f(t)}{(t-a)^{1-\alpha}} dt, \\ {}^\beta \Upsilon_{b^-}^\alpha f(x) &= \frac{1}{\Gamma(\beta)} \int_x^b \left(\frac{(b-x)^\alpha - (b-t)^\alpha}{\alpha} \right)^{\beta-1} \frac{f(t)}{(b-t)^{1-\alpha}} dt. \end{aligned}$$

Set et al. proved Hermite–Hadamard-type inequalities by means of conformable fractional integral operators in [24].

Theorem 5 (see [12, page 35]). *Note that f is a convex function on $[a, b]$. Then, the following double inequality is satisfied*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{2^{\alpha\beta-1}\Gamma(\beta+1)\alpha^\beta}{(b-a)^{\alpha\beta}} \left[{}^\beta_{\left(\frac{a+b}{2}\right)^+} \Upsilon^\alpha f(b) + {}^\beta \Upsilon^\alpha_{\left(\frac{a+b}{2}\right)^-} f(a) \right] \\ &\leq \frac{f(a) + f(b)}{2}. \end{aligned} \quad (1.5)$$

Here, $\beta > 0$, $\alpha \in (0, 1]$ and Γ is Euler Gamma function.

This study consists of three sections including the introduction and preliminaries. In section 2, we will acquire some new versions midpoint-type and trapezoid-type inequalities based on conformable fractional integrals. In some of these acquired inequalities, we will use the bounds of the second derivative of the function rather than the convexity of the function. We also acquired the new version of Hermite–Hadamard-type inequalities via conformable fractional integrals. In this obtained inequality, we used the condition $f'(a+b-t) - f'(t) \geq 0$, $t \in [a, \frac{a+b}{2}]$ rather than the convexity condition.

2. MAIN RESULTS

In this section, we obtain two new midpoint-type inequalities for the case of conformable fractional integrals. We also establish two new versions of trapezoid-type inequalities with the help of conformable fractional integrals. In these obtained four inequalities, the bounds of the second derivative of the function are used instead of the convexity of the function. We also obtained some new Hermite–Hadamard-type inequalities for the case of conformable fractional integrals. In this obtained inequality, we used the condition $f'(a+b-x) \geq f'(x)$ for all $x \in [a, \frac{a+b}{2}]$ instead of the convexity of f .

Theorem 6. *Let us consider that $f: [a, b] \rightarrow \mathbb{R}$ is a twice-differentiable mapping. Let us also consider that there exist real constants m and M so that $m \leq f'' \leq M$ and $\beta > 0$, $\alpha \in (0, 1]$. The following midpoint-type inequalities*

$$\begin{aligned} &m \frac{\beta(b-a)^2}{8} \mathcal{B}\left(\frac{2}{\alpha} + 1, \beta\right) \\ &\leq \frac{2^{\alpha\beta-1}\Gamma(\beta+1)\alpha^\beta}{(b-a)^{\alpha\beta}} \left[{}^\beta_{\left(\frac{a+b}{2}\right)^+} \Upsilon^\alpha f(b) + {}^\beta \Upsilon^\alpha_{\left(\frac{a+b}{2}\right)^-} f(a) \right] - f\left(\frac{a+b}{2}\right) \\ &\leq M \frac{\beta(b-a)^2}{8} \mathcal{B}\left(\frac{2}{\alpha} + 1, \beta\right) \end{aligned} \quad (2.1)$$

are valid. Here, $\mathcal{B}$ is Euler Beta function.

Proof. By using the Definition 2, we establish

$$\begin{aligned} & \frac{2^{\alpha\beta-1}\Gamma(\beta+1)\alpha^\beta}{(b-a)^{\alpha\beta}} \left[{}^\beta_{\left(\frac{a+b}{2}\right)^+} \Upsilon^\alpha f(b) + {}^\beta \Upsilon^\alpha_{\left(\frac{a+b}{2}\right)^-} f(a) \right] \\ &= \frac{2^{\alpha\beta-1}\Gamma(\beta+1)\alpha^\beta}{(b-a)^{\alpha\beta}} \left[\frac{1}{\Gamma(\beta)} \int_{\frac{a+b}{2}}^b \left(\frac{\left(\frac{b-a}{2}\right)^\alpha - \left(t - \frac{a+b}{2}\right)^\alpha}{\alpha} \right)^{\beta-1} \frac{f(t)}{\left(t - \frac{a+b}{2}\right)^{1-\alpha}} dt \right. \\ & \quad \left. + \frac{1}{\Gamma(\beta)} \int_a^{\frac{a+b}{2}} \left(\frac{\left(\frac{b-a}{2}\right)^\alpha - \left(\frac{a+b}{2} - t\right)^\alpha}{\alpha} \right)^{\beta-1} \right. \\ & \quad \left. \times \frac{f(t)}{\left(\frac{a+b}{2} - t\right)^{1-\alpha}} dt \right]. \end{aligned}$$

Taking advantage of the change of variables, we acquire

$$\begin{aligned} & \frac{2^{\alpha\beta-1}\Gamma(\beta+1)\alpha^\beta}{(b-a)^{\alpha\beta}} \left[{}^\beta_{\left(\frac{a+b}{2}\right)^+} \Upsilon^\alpha f(b) + {}^\beta \Upsilon^\alpha_{\left(\frac{a+b}{2}\right)^-} f(a) \right] \\ &= \frac{2^{\alpha\beta-1}\Gamma(\beta+1)\alpha^\beta}{(b-a)^{\alpha\beta}} \left[\frac{1}{\Gamma(\beta)} \int_a^{\frac{a+b}{2}} \left(\frac{\left(\frac{b-a}{2}\right)^\alpha - \left(\frac{a+b}{2} - t\right)^\alpha}{\alpha} \right)^{\beta-1} \frac{f(a+b-t)}{\left(\frac{a+b}{2} - t\right)^{1-\alpha}} dt \right. \\ & \quad \left. + \frac{1}{\Gamma(\beta)} \int_a^{\frac{a+b}{2}} \left(\frac{\left(\frac{b-a}{2}\right)^\alpha - \left(\frac{a+b}{2} - t\right)^\alpha}{\alpha} \right)^{\beta-1} \frac{f(t)}{\left(\frac{a+b}{2} - t\right)^{1-\alpha}} dt \right] \\ &= \frac{2^{\alpha\beta-1}\beta\alpha^\beta}{(b-a)^{\alpha\beta}} \int_a^{\frac{a+b}{2}} \left(\frac{\left(\frac{b-a}{2}\right)^\alpha - \left(\frac{a+b}{2} - t\right)^\alpha}{\alpha} \right)^{\beta-1} \frac{(f(t) + f(a+b-t))}{\left(\frac{a+b}{2} - t\right)^{1-\alpha}} dt. \end{aligned}$$

With the help of the inequality (1.5), we obtain

$$\begin{aligned} & \frac{2^{\alpha\beta-1}\Gamma(\beta+1)\alpha^\beta}{(b-a)^{\alpha\beta}} \left[{}^\beta_{\left(\frac{a+b}{2}\right)^+} \Upsilon^\alpha f(b) + {}^\beta \Upsilon^\alpha_{\left(\frac{a+b}{2}\right)^-} f(a) \right] - f\left(\frac{a+b}{2}\right) \quad (2.2) \\ &= \frac{2^{\alpha\beta-1}\beta\alpha^\beta}{(b-a)^{\alpha\beta}} \int_a^{\frac{a+b}{2}} \left(\frac{\left(\frac{b-a}{2}\right)^\alpha - \left(\frac{a+b}{2} - t\right)^\alpha}{\alpha} \right)^{\beta-1} \\ & \quad \times \frac{(f(t) + f(a+b-t) - 2f\left(\frac{a+b}{2}\right))}{\left(\frac{a+b}{2} - t\right)^{1-\alpha}} dt. \end{aligned}$$

If we take advantage of the facts that

$$f(t) - f\left(\frac{a+b}{2}\right) = \int_{\frac{a+b}{2}}^t f'(s) ds$$

and

$$f(a+b-t) - f\left(\frac{a+b}{2}\right) = \int_{\frac{a+b}{2}}^{a+b-t} f'(s) ds,$$

then we acquire

$$\begin{aligned} f(t) + f(a+b-t) - 2f\left(\frac{a+b}{2}\right) &= \int_{\frac{a+b}{2}}^t f'(s) ds + \int_{\frac{a+b}{2}}^{a+b-t} f'(s) ds \\ &= \int_{\frac{a+b}{2}}^{a+b-t} f'(s) ds - \int_{\frac{a+b}{2}}^{a+b-t} f'(a+b-s) ds \\ &= \int_{\frac{a+b}{2}}^{a+b-t} [f'(s) - f'(a+b-s)] ds. \end{aligned} \quad (2.3)$$

We also acquire

$$f'(s) - f'(a+b-s) = \int_{a+b-s}^s f''(y) dy. \quad (2.4)$$

With the help of the condition $m \leq f''(y) \leq M$ for all $y \in [a, b]$ and from the equality (2.4), we derive

$$\int_{a+b-s}^s m dy \leq \int_{a+b-s}^s f''(y) dy \leq \int_{a+b-s}^s M dy,$$

which gives

$$m(2s-a-b) \leq f'(s) - f'(a+b-s) \leq M(2s-a-b).$$

From the equality (2.3), we obtain

$$m \int_{\frac{a+b}{2}}^{a+b-t} (2s-a-b) \leq \int_{\frac{a+b}{2}}^{a+b-t} [f'(s) - f'(a+b-s)] ds \leq M \int_{\frac{a+b}{2}}^{a+b-t} (2s-a-b).$$

Hence, it yields

$$m \left(\frac{a+b}{2} - t \right)^2 \leq f(t) + f(a+b-t) - 2f\left(\frac{a+b}{2}\right) \leq M \left(\frac{a+b}{2} - t \right)^2. \quad (2.5)$$

Multiplying the inequality (2.5) by $\frac{2^{\alpha\beta-1}\beta\alpha^\beta}{(b-a)^{\alpha\beta}} \left(\frac{(\frac{b-a}{2})^\alpha - (\frac{a+b}{2}-t)^\alpha}{\alpha} \right)^{\beta-1} \left(\frac{a+b}{2} - t \right)^{\alpha-1}$ and then integrating with respect to t on the interval $[a, \frac{a+b}{2}]$, we obtain

$$\begin{aligned} &m \frac{2^{\alpha\beta-1}\beta\alpha^\beta}{(b-a)^{\alpha\beta}} \int_a^{\frac{a+b}{2}} \left(\frac{(\frac{b-a}{2})^\alpha - (\frac{a+b}{2}-t)^\alpha}{\alpha} \right)^{\beta-1} \left(\frac{a+b}{2} - t \right)^{\alpha+1} dt \\ &\leq \frac{2^{\alpha\beta-1}\beta\alpha^\beta}{(b-a)^{\alpha\beta}} \int_a^{\frac{a+b}{2}} \left(\frac{(\frac{b-a}{2})^\alpha - (\frac{a+b}{2}-t)^\alpha}{\alpha} \right)^{\beta-1} \end{aligned}$$

$$\begin{aligned}
& \times \frac{(f(t) + f(a+b-t) - 2f(\frac{a+b}{2}))}{(\frac{a+b}{2} - t)^{1-\alpha}} dt \\
& \leq M \frac{2^{\alpha\beta-1} \beta \alpha^\beta}{(b-a)^{\alpha\beta}} \int_a^{\frac{a+b}{2}} \left(\frac{(\frac{b-a}{2})^\alpha - (\frac{a+b}{2} - t)^\alpha}{\alpha} \right)^{\beta-1} \left(\frac{a+b}{2} - t \right)^{\alpha+1} dt.
\end{aligned}$$

Finally, we obtain

$$\begin{aligned}
& m \frac{\beta(b-a)^2}{8} \mathcal{B}\left(\frac{2}{\alpha} + 1, \beta\right) \\
& \leq \frac{2^{\alpha\beta-1} \Gamma(\beta+1) \alpha^\beta}{(b-a)^{\alpha\beta}} \left[{}^\beta_{(\frac{a+b}{2})^+} \Upsilon^\alpha f(b) + {}^\beta \Upsilon^\alpha_{(\frac{a+b}{2})^-} f(a) \right] - f\left(\frac{a+b}{2}\right) \\
& \leq M \frac{\beta(b-a)^2}{8} \mathcal{B}\left(\frac{2}{\alpha} + 1, \beta\right).
\end{aligned}$$

Finally, the proof of Theorem 6 is completed. $\square$

Remark 1. If we choose $\alpha = 1$ in Theorem 6, then the inequalities (2.1) become to (1.3).

Remark 2. If we assign $\alpha = 1$ and $\beta = 1$ in Theorem 6, then the inequalities (2.1) reduce to the inequalities (1.1).

Example 1. If the mapping $f: [a, b] = [0, 1] \rightarrow \mathbb{R}$ is defined as $f(t) = t^3 + t^2$ such that $2 \leq f''(t) \leq 5$ for $t \in [0, 1]$. Under these conditions, we obtain the mid-term of inequality (2.1) as follows

$$\begin{aligned}
& \frac{2^{\alpha\beta-1} \Gamma(\beta+1) \alpha^\beta}{(b-a)^{\alpha\beta}} \left[{}^\beta_{(\frac{a+b}{2})^+} \Upsilon^\alpha f(b) + {}^\beta \Upsilon^\alpha_{(\frac{a+b}{2})^-} f(a) \right] - f\left(\frac{a+b}{2}\right) \\
& = 2^{\alpha\beta-1} \Gamma(\beta+1) \alpha^\beta \left[{}^\beta_{\frac{1}{2}^+} \Upsilon^\alpha f(1) + {}^\beta \Upsilon^\alpha_{\frac{1}{2}^-} f(0) \right] - f\left(\frac{1}{2}\right) \\
& = 2^{\alpha\beta-1} \Gamma(\beta+1) \alpha^\beta \left[\frac{1}{\Gamma(\beta)} \int_{\frac{1}{2}}^1 \left(\frac{(\frac{1}{2})^\alpha - (t - \frac{1}{2})^\alpha}{\alpha} \right)^{\beta-1} \frac{(t^3 + t^2)}{(t - \frac{1}{2})^{1-\alpha}} dt \right. \\
& \quad \left. + \frac{1}{\Gamma(\beta)} \int_0^{\frac{1}{2}} \left(\frac{(\frac{1}{2})^\alpha - (\frac{1}{2} - t)^\alpha}{\alpha} \right)^{\beta-1} \frac{(t^3 + t^2)}{(\frac{1}{2} - t)^{1-\alpha}} dt \right] \\
& \quad - \left(\frac{1}{2^3} + \frac{1}{2^2} \right).
\end{aligned}$$

By using the change of variables, we have

$$2^{\alpha\beta-1} \Gamma(\beta+1) \alpha^\beta \left[{}^\beta_{\frac{1}{2}^+} \Upsilon^\alpha f(1) + {}^\beta \Upsilon^\alpha_{\frac{1}{2}^-} f(0) \right] - f\left(\frac{1}{2}\right)$$

$$\begin{aligned}
&= 2^{\alpha\beta-1} \beta \alpha^\beta \int_0^{\frac{1}{2}} \left(\frac{\left(\frac{1}{2}\right)^\alpha - \left(\frac{1}{2}-t\right)^\alpha}{\alpha} \right)^{\beta-1} \\
&\quad \times \frac{\left(t^3 + t^2 + (1-t)^3 + (1-t)^2 - 2\left(\frac{1}{2^3} + \frac{1}{2^2}\right)\right)}{\left(\frac{1}{2}-t\right)^{1-\alpha}} dt \\
&= \frac{5\beta}{8} \mathcal{B}\left(\frac{2}{\alpha} + 1, \beta\right).
\end{aligned}$$

As a result, the inequalities (2.1) can be found as follows

$$\frac{\beta}{4} \mathcal{B}\left(\frac{2}{\alpha} + 1, \beta\right) \leq \frac{5\beta}{8} \mathcal{B}\left(\frac{2}{\alpha} + 1, \beta\right) \leq \frac{5\beta}{8} \mathcal{B}\left(\frac{2}{\alpha} + 1, \beta\right).$$

Theorem 7. Note that $f: [a, b] \rightarrow \mathbb{R}$ is a twice-differentiable mapping such that there exists real constants m and M so that $m \leq f'' \leq M$. Then, we derive

$$\begin{aligned}
&m \frac{\beta(b-a)^2}{8} \left[\frac{1}{\beta} - \mathcal{B}\left(\frac{2}{\alpha} + 1, \beta\right) \right] \\
&\leq \frac{f(a) + f(b)}{2} - \frac{2^{\alpha\beta-1} \Gamma(\beta+1) \alpha^\beta}{(b-a)^{\alpha\beta}} \left[{}^\beta_{\left(\frac{a+b}{2}\right)^+} \Upsilon^\alpha f(b) + {}^\beta \Upsilon^\alpha_{\left(\frac{a+b}{2}\right)^-} f(a) \right] \\
&\leq M \frac{\beta(b-a)^2}{8} \left[\frac{1}{\beta} - \mathcal{B}\left(\frac{2}{\alpha} + 1, \beta\right) \right],
\end{aligned} \tag{2.6}$$

where $\mathcal{B}$ is Euler Beta function.

Proof. From the equality (1.5), we acquire

$$\begin{aligned}
&\frac{f(a) + f(b)}{2} - \frac{2^{\alpha\beta-1} \Gamma(\beta+1) \alpha^\beta}{(b-a)^{\alpha\beta}} \left[{}^\beta_{\left(\frac{a+b}{2}\right)^+} \Upsilon^\alpha f(b) + {}^\beta \Upsilon^\alpha_{\left(\frac{a+b}{2}\right)^-} f(a) \right] \\
&= \frac{f(a) + f(b)}{2} - \frac{2^{\alpha\beta-1} \beta \alpha^\beta}{(b-a)^{\alpha\beta}} \int_a^{\frac{a+b}{2}} \left(\frac{\left(\frac{b-a}{2}\right)^\alpha - \left(\frac{a+b}{2}-t\right)^\alpha}{\alpha} \right)^{\beta-1} \\
&\quad \times \frac{(f(t) + f(a+b-t))}{\left(\frac{a+b}{2}-t\right)^{1-\alpha}} dt \\
&= \frac{2^{\alpha\beta-1} \beta \alpha^\beta}{(b-a)^{\alpha\beta}} \int_a^{\frac{a+b}{2}} \left(\frac{\left(\frac{b-a}{2}\right)^\alpha - \left(\frac{a+b}{2}-t\right)^\alpha}{\alpha} \right)^{\beta-1} \\
&\quad \times \frac{(f(a) + f(b) - f(t) - f(a+b-t))}{\left(\frac{a+b}{2}-t\right)^{1-\alpha}} dt.
\end{aligned} \tag{2.7}$$

With the help of the equalities

$$f(a) - f(t) = - \int_a^t f'(s) ds$$

and

$$f(b) - f(a+b-t) = \int_{a+b-t}^b f'(s) ds,$$

we derive

$$\begin{aligned} f(a) + f(b) - (f(t) + f(a+b-t)) &= \int_{a+b-t}^b f'(s) ds - \int_a^t f'(s) ds \\ &= \int_a^t f'(a+b-s) ds - \int_a^t f'(s) ds \\ &= \int_a^t [f'(a+b-s) - f'(s)] ds. \end{aligned} \quad (2.8)$$

We also provide

$$f'(a+b-s) - f'(s) = \int_s^{a+b-s} f''(y) dy. \quad (2.9)$$

From the equality (2.9) and the condition $m \leq f'' \leq M$, we get

$$m(a+b-2s) \leq f'(s) - f'(a+b-s) \leq M(a+b-2s). \quad (2.10)$$

With the help of the equality (2.8) and the inequality (2.10), we have

$$\int_a^t m(a+b-2s) ds \leq \int_a^t [f'(s) - f'(a+b-s)] ds \leq \int_a^t M(a+b-2s) ds$$

such that

$$m(b-t)(t-a) \leq f(a) + f(b) - (f(t) + f(a+b-t)) \leq M(b-t)(t-a). \quad (2.11)$$

Multiplying the inequality (2.11) by $\frac{2^{\alpha\beta-1}\beta\alpha^\beta}{(b-a)^{\alpha\beta}} \left(\frac{(\frac{b-a}{2})^\alpha - (\frac{a+b}{2}-t)^\alpha}{\alpha} \right)^{\beta-1} (\frac{a+b}{2}-t)^{\alpha-1}$ and then integrating with respect to t on the interval $[a, \frac{a+b}{2}]$, we readily obtain

$$\begin{aligned} &m \frac{2^{\alpha\beta-1}\beta\alpha^\beta}{(b-a)^{\alpha\beta}} \int_a^{\frac{a+b}{2}} \left(\frac{(\frac{b-a}{2})^\alpha - (\frac{a+b}{2}-t)^\alpha}{\alpha} \right)^{\beta-1} \left(\frac{a+b}{2}-t \right)^{\alpha-1} (b-t)(t-a) dt \\ &\leq \frac{2^{\alpha\beta-1}\beta\alpha^\beta}{(b-a)^{\alpha\beta}} \int_a^{\frac{a+b}{2}} \left(\frac{(\frac{b-a}{2})^\alpha - (\frac{a+b}{2}-t)^\alpha}{\alpha} \right)^{\beta-1} \\ &\quad \times \frac{(f(a) + f(b) - f(t) - f(a+b-t))}{(\frac{a+b}{2}-t)^{1-\alpha}} dt \end{aligned}$$

$$\leq M \frac{2^{\alpha\beta-1} \beta \alpha^\beta}{(b-a)^{\alpha\beta}} \int_a^{\frac{a+b}{2}} \left(\frac{\left(\frac{b-a}{2}\right)^\alpha - \left(\frac{a+b}{2} - t\right)^\alpha}{\alpha} \right)^{\beta-1} \left(\frac{a+b}{2} - t \right)^{\alpha-1} (b-t)(t-a) dt.$$

By using the above inequalities, we can easily get

$$\begin{aligned} & m \frac{\beta(b-a)^2}{8} \left[\frac{1}{\beta} - \mathcal{B}\left(\frac{2}{\alpha} + 1, \beta\right) \right] \\ & \leq \frac{f(a) + f(b)}{2} - \frac{2^{\alpha\beta-1} \Gamma(\beta+1) \alpha^\beta}{(b-a)^{\alpha\beta}} \left[{}^\beta_{\left(\frac{a+b}{2}\right)^+} \Upsilon^\alpha f(b) + {}^\beta \Upsilon^\alpha_{\left(\frac{a+b}{2}\right)^-} f(a) \right] \\ & \leq M \frac{\beta(b-a)^2}{8} \left[\frac{1}{\beta} - \mathcal{B}\left(\frac{2}{\alpha} + 1, \beta\right) \right]. \end{aligned}$$

Finally, the proof of Theorem 7 is accomplished. $\square$

Remark 3. If we choose $\alpha = 1$ in Theorem 7, then the inequalities (2.6) coincides with the inequalities (1.4).

Remark 4. If we take $\alpha = \beta = 1$ in Theorem 7, then Theorem 7 equals to (1.2).

Example 2. Let us note that the function $f: [a, b] = [0, 1] \rightarrow \mathbb{R}$ is defined as $f(t) = t^2 - t^4$ such that $-10 \leq f''(t) \leq 2$ for $t \in [0, 1]$. By these conditions, the right-side of (2.6) becomes as follows

$$\begin{aligned} & \frac{f(a) + f(b)}{2} - \frac{2^{\alpha\beta-1} \Gamma(\beta+1) \alpha^\beta}{(b-a)^{\alpha\beta}} \left[{}^\beta_{\left(\frac{a+b}{2}\right)^+} \Upsilon^\alpha f(b) + {}^\beta \Upsilon^\alpha_{\left(\frac{a+b}{2}\right)^-} f(a) \right] \\ & = \frac{f(0) + f(1)}{2} - 2^{\alpha\beta-1} \Gamma(\beta+1) \alpha^\beta \left[{}^\beta_{\left(\frac{1}{2}\right)^+} \Upsilon^\alpha f(1) + {}^\beta \Upsilon^\alpha_{\left(\frac{1}{2}\right)^-} f(0) \right] \\ & = -2^{\alpha\beta-1} \beta \alpha^\beta \int_0^{\frac{1}{2}} \left(\frac{\left(\frac{1}{2}\right)^\alpha - \left(\frac{1}{2} - t\right)^\alpha}{\alpha} \right)^{\beta-1} \frac{(f(t) + f(1-t))}{\left(\frac{1}{2} - t\right)^{1-\alpha}} dt \\ & = 2^{\alpha\beta-1} \beta \alpha^\beta \int_a^{\frac{a+b}{2}} \left(\frac{\left(\frac{1}{2}\right)^\alpha - \left(\frac{1}{2} - t\right)^\alpha}{\alpha} \right)^{\beta-1} \frac{(-t^2 + t^4 - (1-t)^2 + (1-t)^4)}{\left(\frac{1}{2} - t\right)^{1-\alpha}} dt. \\ & = \frac{\beta}{16} \left[\left(\mathcal{B}\left(\frac{4}{\alpha} + 1, \beta\right) + 2\mathcal{B}\left(\frac{2}{\alpha} + 1, \beta\right) - \frac{3}{\beta} \right) \right]. \end{aligned}$$

Lastly, the inequality (2.6) becomes to

$$\begin{aligned} \frac{5\beta}{4} \left[\mathcal{B}\left(\frac{2}{\alpha} + 1, \beta\right) - \frac{1}{\beta} \right] & \leq \frac{\beta}{16} \left[\left(\mathcal{B}\left(\frac{4}{\alpha} + 1, \beta\right) + 2\mathcal{B}\left(\frac{2}{\alpha} + 1, \beta\right) - \frac{3}{\beta} \right) \right] \\ & \leq \frac{\beta}{4} \left[\frac{1}{\beta} - \mathcal{B}\left(\frac{2}{\alpha} + 1, \beta\right) \right]. \end{aligned}$$

To illustrate the correctness of Example 2, one can refer to the Figure 1.

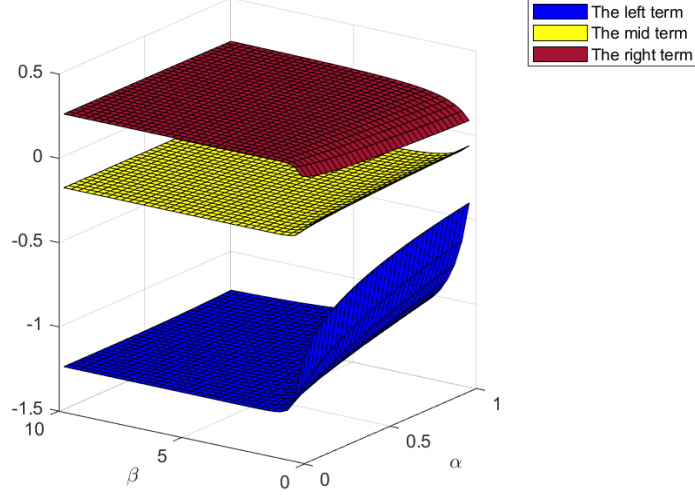


FIGURE 1. Graph for the result of Example 2 computed and plotted in MATLAB.

Theorem 8. Let $f: [a, b] \rightarrow \mathbb{R}$ be a differentiable mapping and $f \in L_1[a, b]$. If $f'(a + b - x) \geq f'(x)$ for all $x \in [a, \frac{a+b}{2}]$. Then, the following Hermite–Hadamard-type inequality for conformable fractional integrals

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{2^{\alpha\beta-1}\Gamma(\beta+1)\alpha^\beta}{(b-a)^{\alpha\beta}} \left[{}^\beta_{\left(\frac{a+b}{2}\right)^+} \Upsilon^\alpha f(b) + {}^\beta \Upsilon^\alpha_{\left(\frac{a+b}{2}\right)^-} f(a) \right] \\ &\leq \frac{f(a) + f(b)}{2} \end{aligned} \quad (2.12)$$

is valid.

Proof. From the equalities (2.2) and (2.3), we have

$$\begin{aligned} &\frac{2^{\alpha\beta-1}\Gamma(\beta+1)\alpha^\beta}{(b-a)^{\alpha\beta}} \left[{}^\beta_{\left(\frac{a+b}{2}\right)^+} \Upsilon^\alpha f(b) + {}^\beta \Upsilon^\alpha_{\left(\frac{a+b}{2}\right)^-} f(a) \right] - f\left(\frac{a+b}{2}\right) \\ &= \frac{2^{\alpha\beta-1}\beta\alpha^\beta}{(b-a)^{\alpha\beta}} \int_a^{\frac{a+b}{2}} \left(\frac{\left(\frac{b-a}{2}\right)^\alpha - \left(\frac{a+b}{2} - t\right)^\alpha}{\alpha} \right)^{\beta-1} \left(\frac{a+b}{2} - t \right)^{\alpha-1} \\ &\quad \times \left(f(t) + f(a+b-t) - 2f\left(\frac{a+b}{2}\right) \right) dt \\ &= \frac{2^{\alpha\beta-1}\beta\alpha^\beta}{(b-a)^{\alpha\beta}} \int_a^{\frac{a+b}{2}} \left(\frac{\left(\frac{b-a}{2}\right)^\alpha - \left(\frac{a+b}{2} - t\right)^\alpha}{\alpha} \right)^{\beta-1} \left(\frac{a+b}{2} - t \right)^{\alpha-1} \end{aligned}$$

$$\begin{aligned}
& \times \left[\int_{\frac{a+b}{2}}^{a+b-t} [f'(s) - f'(a+b-s)] ds \right] dt \\
& = \frac{2^{\alpha\beta-1} \beta \alpha^\beta}{(b-a)^{\alpha\beta}} \int_a^{\frac{a+b}{2}} \left(\frac{\left(\frac{b-a}{2}\right)^\alpha - \left(\frac{a+b}{2}-t\right)^\alpha}{\alpha} \right)^{\beta-1} \left(\frac{a+b}{2} - t \right)^{\alpha-1} \\
& \quad \times \left[\int_t^{\frac{a+b}{2}} [f'(a+b-u) - f'(u)] ds \right] dt \geq 0,
\end{aligned}$$

which presents the first inequality in (2.12).

Similar to foregoing process, by the equalities (2.7) and (2.8), we obtain

$$\begin{aligned}
& \frac{f(a) + f(b)}{2} - \frac{2^{\alpha\beta-1} \Gamma(\beta+1) \alpha^\beta}{(b-a)^{\alpha\beta}} \left[{}^\beta_{\left(\frac{a+b}{2}\right)^+} \Upsilon^\alpha f(b) + {}^\beta \Upsilon_{\left(\frac{a+b}{2}\right)^-}^\alpha f(a) \right] \\
& = \frac{2^{\alpha\beta-1} \beta \alpha^\beta}{(b-a)^{\alpha\beta}} \int_a^{\frac{a+b}{2}} \left(\frac{\left(\frac{b-a}{2}\right)^\alpha - \left(\frac{a+b}{2}-t\right)^\alpha}{\alpha} \right)^{\beta-1} \left(\frac{a+b}{2} - t \right)^{\alpha-1} \\
& \quad \times [f(a) + f(b) - (f(t) + f(a+b-t))] dt \\
& = \frac{2^{\alpha\beta-1} \beta \alpha^\beta}{(b-a)^{\alpha\beta}} \int_a^{\frac{a+b}{2}} \left(\frac{\left(\frac{b-a}{2}\right)^\alpha - \left(\frac{a+b}{2}-t\right)^\alpha}{\alpha} \right)^{\beta-1} \left(\frac{a+b}{2} - t \right)^{\alpha-1} \\
& \quad \times \left[\int_a^t [f'(a+b-s) - f'(s)] ds \right] dt \geq 0.
\end{aligned}$$

Thus, the proof of Theorem 8 is finished. $\square$

Theorem 9. Note that f is a convex mapping on $[a, b]$. Then, the following double inequality holds:

$$\begin{aligned}
f\left(\frac{a+b}{2}\right) & \leq \frac{2^{\alpha\beta-1} \Gamma(\beta+1) \alpha^\beta}{(b-a)^{\alpha\beta}} \left[{}^\beta_{a^+} \Upsilon^\alpha f\left(\frac{a+b}{2}\right) + {}^\beta \Upsilon_{b^-}^\alpha f\left(\frac{a+b}{2}\right) \right] \\
& \leq \frac{f(a) + f(b)}{2}.
\end{aligned} \tag{2.13}$$

Here, $\beta > 0$, $\alpha \in (0, 1]$ and Γ is Euler Gamma function.

Theorem 10. Assume that $f: [a, b] \rightarrow \mathbb{R}$ is a twice-differentiable mapping such that there exists real constants m and M so that $m \leq f'' \leq M$ and $\beta > 0$, $\alpha \in (0, 1]$. The following double inequality

$$\begin{aligned}
& m \frac{\beta(b-a)^2}{8} \left[\frac{1}{\beta} - 2\mathcal{B}\left(\frac{1}{\alpha} + 1, \beta\right) + \mathcal{B}\left(\frac{2}{\alpha} + 1, \beta\right) \right] \\
& \leq \frac{2^{\alpha\beta-1} \Gamma(\beta+1) \alpha^\beta}{(b-a)^{\alpha\beta}} \left[{}^\beta_{a^+} \Upsilon^\alpha f\left(\frac{a+b}{2}\right) + {}^\beta \Upsilon_{b^-}^\alpha f\left(\frac{a+b}{2}\right) \right] - f\left(\frac{a+b}{2}\right)
\end{aligned} \tag{2.14}$$

$$\leq M \frac{\beta(b-a)^2}{8} \left[\frac{1}{\beta} - 2\mathcal{B}\left(\frac{1}{\alpha} + 1, \beta\right) + \mathcal{B}\left(\frac{2}{\alpha} + 1, \beta\right) \right]$$

is valid. Here, $\mathcal{B}$ is Euler Beta function.

Proof. With the help of the Definition 2, we have

$$\begin{aligned} & \frac{2^{\alpha\beta-1}\Gamma(\beta+1)\alpha^\beta}{(b-a)^{\alpha\beta}} \left[{}^\beta_{a^+}\Upsilon^\alpha f\left(\frac{a+b}{2}\right) + {}^\beta_{b^-}\Upsilon^\alpha f\left(\frac{a+b}{2}\right) \right] \\ &= \frac{2^{\alpha\beta-1}\Gamma(\beta+1)\alpha^\beta}{(b-a)^{\alpha\beta}} \left[\frac{1}{\Gamma(\beta)} \int_a^{\frac{a+b}{2}} \left(\frac{(\frac{b-a}{2})^\alpha - (t-a)^\alpha}{\alpha} \right)^{\beta-1} \frac{f(t)}{(t-a)^{1-\alpha}} dt \right. \\ & \quad \left. + \frac{1}{\Gamma(\beta)} \int_{\frac{a+b}{2}}^b \left(\frac{(\frac{b-a}{2})^\alpha - (b-t)^\alpha}{\alpha} \right)^{\beta-1} \frac{f(t)}{(b-t)^{1-\alpha}} dt \right]. \end{aligned}$$

With the aid of the change of variables, we acquire

$$\begin{aligned} &= \frac{2^{\alpha\beta-1}\Gamma(\beta+1)\alpha^\beta}{(b-a)^{\alpha\beta}} \left[\frac{1}{\Gamma(\beta)} \int_a^{\frac{a+b}{2}} \left(\frac{(\frac{b-a}{2})^\alpha - (t-a)^\alpha}{\alpha} \right)^{\beta-1} \frac{f(a+b-t)}{(t-a)^{1-\alpha}} dt \right. \\ & \quad \left. + \frac{1}{\Gamma(\beta)} \int_a^{\frac{a+b}{2}} \left(\frac{(\frac{b-a}{2})^\alpha - (t-a)^\alpha}{\alpha} \right)^{\beta-1} \frac{f(t)}{(t-a)^{1-\alpha}} dt \right] \\ &= \frac{2^{\alpha\beta-1}\beta\alpha^\beta}{(b-a)^{\alpha\beta}} \int_a^{\frac{a+b}{2}} \left(\frac{(\frac{b-a}{2})^\alpha - (t-a)^\alpha}{\alpha} \right)^{\beta-1} \frac{1}{(t-a)^{1-\alpha}} (f(t) + f(a+b-t)) dt. \end{aligned}$$

From the inequality (2.13), we have

$$\begin{aligned} & \frac{2^{\alpha\beta-1}\Gamma(\beta+1)\alpha^\beta}{(b-a)^{\alpha\beta}} \left[{}^\beta_{a^+}\Upsilon^\alpha f\left(\frac{a+b}{2}\right) + {}^\beta_{b^-}\Upsilon^\alpha f\left(\frac{a+b}{2}\right) \right] - f\left(\frac{a+b}{2}\right) \quad (2.15) \\ &= \frac{2^{\alpha\beta-1}\beta\alpha^\beta}{(b-a)^{\alpha\beta}} \int_a^{\frac{a+b}{2}} \left(\frac{(\frac{b-a}{2})^\alpha - (t-a)^\alpha}{\alpha} \right)^{\beta-1} \frac{(f(t) + f(a+b-t) - 2f(\frac{a+b}{2}))}{(t-a)^{1-\alpha}} dt. \end{aligned}$$

If we take advantage of the facts that

$$f(t) - f\left(\frac{a+b}{2}\right) = \int_{\frac{a+b}{2}}^t f'(s) ds$$

and

$$f(a+b-t) - f\left(\frac{a+b}{2}\right) = \int_{\frac{a+b}{2}}^{a+b-t} f'(s) ds,$$

then we have

$$\begin{aligned} f(t) + f(a+b-t) - 2f\left(\frac{a+b}{2}\right) &= \int_{\frac{a+b}{2}}^t f'(s) ds + \int_{\frac{a+b}{2}}^{a+b-t} f'(s) ds \\ &= \int_{\frac{a+b}{2}}^{a+b-t} f'(s) ds - \int_{\frac{a+b}{2}}^{a+b-t} f'(a+b-s) ds \\ &= \int_{\frac{a+b}{2}}^{a+b-t} [f'(s) - f'(a+b-s)] ds. \end{aligned} \quad (2.16)$$

We also have

$$f'(s) - f'(a+b-s) = \int_{a+b-s}^s f''(y) dy. \quad (2.17)$$

By using the condition $m \leq f''(y) \leq M$ for all $y \in [a, b]$ and with the help of the equality (2.17), we have

$$\int_{a+b-s}^s m dy \leq \int_{a+b-s}^s f''(y) dy \leq \int_{a+b-s}^s M dy,$$

which gives

$$m(2s - a - b) \leq f'(s) - f'(a+b-s) \leq M(2s - a - b).$$

Since the equality (2.16), we have

$$m \int_{\frac{a+b}{2}}^{a+b-t} (2s - a - b) \leq \int_{\frac{a+b}{2}}^{a+b-t} [f'(s) - f'(a+b-s)] ds \leq M \int_{\frac{a+b}{2}}^{a+b-t} (2s - a - b).$$

This implies that

$$m \left(\frac{a+b}{2} - t \right)^2 \leq f(t) + f(a+b-t) - 2f\left(\frac{a+b}{2}\right) \leq M \left(\frac{a+b}{2} - t \right)^2. \quad (2.18)$$

Multiplying the inequality (2.18) by $\frac{2^{\alpha\beta-1}\beta\alpha^\beta}{(b-a)^{\alpha\beta}} \left(\frac{(\frac{b-a}{2})^\alpha - (t-a)^\alpha}{\alpha} \right)^{\beta-1} (t-a)^{\alpha-1}$ and then integrating with respect to t on the interval $[a, \frac{a+b}{2}]$, we obtain

$$\begin{aligned} &m \frac{2^{\alpha\beta-1}\beta\alpha^\beta}{(b-a)^{\alpha\beta}} \int_a^{\frac{a+b}{2}} \left(\frac{(\frac{b-a}{2})^\alpha - (t-a)^\alpha}{\alpha} \right)^{\beta-1} (t-a)^{\alpha-1} \left(\frac{a+b}{2} - t \right)^2 dt \\ &\leq \frac{2^{\alpha\beta-1}\beta\alpha^\beta}{(b-a)^{\alpha\beta}} \int_a^{\frac{a+b}{2}} \left(\frac{(\frac{b-a}{2})^\alpha - (t-a)^\alpha}{\alpha} \right)^{\beta-1} \frac{(f(t) + f(a+b-t) - 2f(\frac{a+b}{2}))}{(t-a)^{1-\alpha}} dt \\ &\leq M \frac{2^{\alpha\beta-1}\beta\alpha^\beta}{(b-a)^{\alpha\beta}} \int_a^{\frac{a+b}{2}} \left(\frac{(\frac{b-a}{2})^\alpha - (t-a)^\alpha}{\alpha} \right)^{\beta-1} (t-a)^{\alpha-1} \left(\frac{a+b}{2} - t \right)^2 dt. \end{aligned}$$

Then, we have

$$\begin{aligned}
& m \frac{\beta(b-a)^2}{8} \left[\frac{1}{\beta} - 2\mathcal{B} \left(\frac{1}{\alpha} + 1, \beta \right) + \mathcal{B} \left(\frac{2}{\alpha} + 1, \beta \right) \right] \\
& \leq \frac{2^{\alpha\beta-1} \Gamma(\beta+1) \alpha^\beta}{(b-a)^{\alpha\beta}} \left[{}^\beta_{a^+} \Upsilon^\alpha f \left(\frac{a+b}{2} \right) + {}^\beta_{b^-} \Upsilon^\alpha f \left(\frac{a+b}{2} \right) \right] - f \left(\frac{a+b}{2} \right) \\
& \leq M \frac{\beta(b-a)^2}{8} \left[\frac{1}{\beta} - 2\mathcal{B} \left(\frac{1}{\alpha} + 1, \beta \right) + \mathcal{B} \left(\frac{2}{\alpha} + 1, \beta \right) \right].
\end{aligned}$$

Finally, the proof of Theorem 10 is completed. $\square$

Example 3. If the mapping $f: [a, b] = [-1, 1] \rightarrow \mathbb{R}$ is defined as $f(t) = t^3 + 2t^2 + 1$ such that $-2 \leq f''(t) \leq 10$ for $t \in [-1, 1]$. Under these assumptions, the mid-term of inequality (2.14) reduces to

$$\begin{aligned}
& \frac{2^{\alpha\beta-1} \Gamma(\beta+1) \alpha^\beta}{(b-a)^{\alpha\beta}} \left[{}^\beta_{a^+} \Upsilon^\alpha f \left(\frac{a+b}{2} \right) + {}^\beta_{b^-} \Upsilon^\alpha f \left(\frac{a+b}{2} \right) \right] - f \left(\frac{a+b}{2} \right) \\
& = \frac{2^{\alpha\beta-1} \Gamma(\beta+1) \alpha^\beta}{(2)^{\alpha\beta}} \left[{}^\beta_{-1^+} \Upsilon^\alpha f(0) + {}^\beta_{1^-} \Upsilon^\alpha f(0) \right] - f(0) \\
& = \frac{\Gamma(\beta+1) \alpha^\beta}{2} \left[\frac{1}{\Gamma(\beta)} \int_{-1}^0 \left(\frac{1-(t-a)^\alpha}{\alpha} \right)^{\beta-1} \frac{t^3 + 2t^2 + 1}{(t+1)^{1-\alpha}} dt \right. \\
& \quad \left. + \frac{1}{\Gamma(\beta)} \int_0^1 \left(\frac{1-(1-t)^\alpha}{\alpha} \right)^{\beta-1} \frac{t^3 + 2t^2 + 1}{(1-t)^{1-\alpha}} dt \right] \\
& = 2\beta\alpha^\beta \int_{-1}^0 \left(\frac{1-(t+1)^\alpha}{\alpha} \right)^{\beta-1} \frac{t^2}{(t+1)^{1-\alpha}} dt \\
& = 2\beta \left[\frac{1}{\beta} - 2\mathcal{B} \left(\frac{1}{\alpha} + 1, \beta \right) + \mathcal{B} \left(\frac{2}{\alpha} + 1, \beta \right) \right].
\end{aligned}$$

Consequently, the inequality (2.14) can be written as follows

$$\begin{aligned}
& -\beta \left[\frac{1}{\beta} - 2\mathcal{B} \left(\frac{1}{\alpha} + 1, \beta \right) + \mathcal{B} \left(\frac{2}{\alpha} + 1, \beta \right) \right] \\
& \leq 2\beta \left[\frac{1}{\beta} - 2\mathcal{B} \left(\frac{1}{\alpha} + 1, \beta \right) + \mathcal{B} \left(\frac{2}{\alpha} + 1, \beta \right) \right] \\
& \leq 5\beta \left[\frac{1}{\beta} - 2\mathcal{B} \left(\frac{1}{\alpha} + 1, \beta \right) + \mathcal{B} \left(\frac{2}{\alpha} + 1, \beta \right) \right].
\end{aligned}$$

Theorem 11. Let $f: [a, b] \rightarrow \mathbb{R}$ be a twice-differentiable mapping such that there exists real constants m and M so that $m \leq f'' \leq M$. Then, we establish

$$\begin{aligned} & m \frac{\beta(b-a)^2}{8} \left[2\mathcal{B}\left(\frac{1}{\alpha} + 1, \beta\right) - \mathcal{B}\left(\frac{2}{\alpha} + 1, \beta\right) \right] \\ & \leq \frac{f(a) + f(b)}{2} - \frac{2^{\alpha\beta-1} \Gamma(\beta+1) \alpha^\beta}{(b-a)^{\alpha\beta}} \left[{}^\beta\Upsilon_{a^+}^\alpha f\left(\frac{a+b}{2}\right) + {}^\beta\Upsilon_{b^-}^\alpha f\left(\frac{a+b}{2}\right) \right] \\ & \leq M \frac{\beta(b-a)^2}{8} \left[2\mathcal{B}\left(\frac{1}{\alpha} + 1, \beta\right) - \mathcal{B}\left(\frac{2}{\alpha} + 1, \beta\right) \right], \end{aligned} \quad (2.19)$$

where $\mathcal{B}$ is Euler Beta function.

Proof. With the help of the inequality (2.13), we obtain

$$\begin{aligned} & \frac{f(a) + f(b)}{2} - \frac{2^{\alpha\beta-1} \Gamma(\beta+1) \alpha^\beta}{(b-a)^{\alpha\beta}} \left[{}^\beta\Upsilon_{a^+}^\alpha f\left(\frac{a+b}{2}\right) + {}^\beta\Upsilon_{b^-}^\alpha f\left(\frac{a+b}{2}\right) \right] \\ & = \frac{f(a) + f(b)}{2} - \frac{2^{\alpha\beta-1} \beta \alpha^\beta}{(b-a)^{\alpha\beta}} \int_a^{\frac{a+b}{2}} \left(\frac{\left(\frac{b-a}{2}\right)^\alpha - (t-a)^\alpha}{\alpha} \right)^{\beta-1} \\ & \quad \times \frac{(f(t) + f(a+b-t))}{(t-a)^{1-\alpha}} dt \\ & = \frac{2^{\alpha\beta-1} \beta \alpha^\beta}{(b-a)^{\alpha\beta}} \int_a^{\frac{a+b}{2}} \left(\frac{\left(\frac{b-a}{2}\right)^\alpha - (t-a)^\alpha}{\alpha} \right)^{\beta-1} \\ & \quad \times \frac{(f(a) + f(b) - f(t) - f(a+b-t))}{(t-a)^{1-\alpha}} dt. \end{aligned} \quad (2.20)$$

With the aid of the equalities

$$f(a) - f(t) = - \int_a^t f'(s) ds$$

and

$$f(b) - f(a+b-t) = \int_{a+b-t}^b f'(s) ds,$$

we have

$$\begin{aligned} f(a) + f(b) - (f(t) + f(a+b-t)) &= \int_{a+b-t}^b f'(s) ds - \int_a^t f'(s) ds \\ &= \int_a^t f'(a+b-s) ds - \int_a^t f'(s) ds \\ &= \int_a^t [f'(a+b-s) - f'(s)] ds. \end{aligned} \quad (2.21)$$

We also obtain

$$f'(a+b-s) - f'(s) = \int_s^{a+b-s} f''(y) dy. \quad (2.22)$$

By the equality (2.22) and the condition $m \leq f'' \leq M$, we get

$$m(a+b-2s) \leq f'(s) - f'(a+b-s) \leq M(a+b-2s). \quad (2.23)$$

From the equality (2.21) and the inequality (2.23), we acquire

$$\int_a^t m(a+b-2s) ds \leq \int_a^t [f'(s) - f'(a+b-s)] ds \leq \int_a^t M(a+b-2s) ds$$

so that

$$m(b-t)(t-a) \leq f(a) + f(b) - (f(t) + f(a+b-t)) \leq M(b-t)(t-a). \quad (2.24)$$

Multiplying the inequality (2.24) by $\frac{2^{\alpha\beta-1}\beta\alpha^\beta}{(b-a)^{\alpha\beta}} \left(\frac{(\frac{b-a}{2})^\alpha - (t-a)^\alpha}{\alpha} \right)^{\beta-1} (t-a)^{\alpha-1}$ and then integrating with respect to t on the interval $[a, \frac{a+b}{2}]$, we get

$$\begin{aligned} & m \frac{2^{\alpha\beta-1}\beta\alpha^\beta}{(b-a)^{\alpha\beta}} \int_a^{\frac{a+b}{2}} \left(\frac{(\frac{b-a}{2})^\alpha - (t-a)^\alpha}{\alpha} \right)^{\beta-1} (t-a)^\alpha (b-t) dt \\ & \leq \frac{2^{\alpha\beta-1}\beta\alpha^\beta}{(b-a)^{\alpha\beta}} \int_a^{\frac{a+b}{2}} \left(\frac{(\frac{b-a}{2})^\alpha - (t-a)^\alpha}{\alpha} \right)^{\beta-1} \\ & \quad \times \frac{(f(a) + f(b) - f(t) - f(a+b-t)) dt}{(t-a)^{1-\alpha}} \\ & \leq M \frac{2^{\alpha\beta-1}\beta\alpha^\beta}{(b-a)^{\alpha\beta}} \int_a^{\frac{a+b}{2}} \left(\frac{(\frac{b-a}{2})^\alpha - (t-a)^\alpha}{\alpha} \right)^{\beta-1} (t-a)^\alpha (b-t) dt. \end{aligned}$$

So that

$$\begin{aligned} & m \frac{\beta(b-a)^2}{8} \left[2\mathcal{B}\left(\frac{1}{\alpha} + 1, \beta\right) - \mathcal{B}\left(\frac{2}{\alpha} + 1, \beta\right) \right] \\ & \leq \frac{f(a) + f(b)}{2} - \frac{2^{\alpha\beta-1}\Gamma(\beta+1)\alpha^\beta}{(b-a)^{\alpha\beta}} \left[{}^\beta\Upsilon_{a^+}^\alpha f\left(\frac{a+b}{2}\right) + {}^\beta\Upsilon_{b^-}^\alpha f\left(\frac{a+b}{2}\right) \right] \\ & \leq M \frac{\beta(b-a)^2}{8} \left[2\mathcal{B}\left(\frac{1}{\alpha} + 1, \beta\right) - \mathcal{B}\left(\frac{2}{\alpha} + 1, \beta\right) \right]. \end{aligned}$$

Hence, the proof of Theorem 11 is completed. $\square$

Example 4. Let us consider that the function $f: [a, b] = [-2, 3] \rightarrow \mathbb{R}$ is described as $f(t) = 2t^3 - t^2 + t - 1$ such that $-24 \leq f''(t) \leq 36$ for $t \in [-2, 3]$. By these assumptions, the right-side of (2.19) becomes as follows

$$\begin{aligned} & \frac{f(a) + f(b)}{2} - \frac{2^{\alpha\beta-1}\Gamma(\beta+1)\alpha^\beta}{(b-a)^{\alpha\beta}} \left[{}^\beta\Upsilon_{a^+}^\alpha f\left(\frac{a+b}{2}\right) + {}^\beta\Upsilon_{b^-}^\alpha f\left(\frac{a+b}{2}\right) \right] \\ &= \frac{f(-2) + f(3)}{2} - \frac{2^{\alpha\beta-1}\Gamma(\beta+1)\alpha^\beta}{(5)^{\alpha\beta}} \left[{}^\beta\Upsilon_{-2^+}^\alpha f\left(\frac{1}{2}\right) + {}^\beta\Upsilon_{b^-}^\alpha f\left(\frac{1}{2}\right) \right] \\ &= \frac{25\beta}{2} \left[2\mathcal{B}\left(\frac{1}{\alpha} + 1, \beta\right) - \mathcal{B}\left(\frac{2}{\alpha} + 1, \beta\right) \right]. \end{aligned}$$

Finally, the inequality (2.19) can be written as follows

$$\begin{aligned} & -75\beta \left[2\mathcal{B}\left(\frac{1}{\alpha} + 1, \beta\right) - \mathcal{B}\left(\frac{2}{\alpha} + 1, \beta\right) \right] \\ & \leq \frac{25\beta}{2} \left[2\mathcal{B}\left(\frac{1}{\alpha} + 1, \beta\right) - \mathcal{B}\left(\frac{2}{\alpha} + 1, \beta\right) \right] \\ & \leq \frac{225\beta}{2} \left[2\mathcal{B}\left(\frac{1}{\alpha} + 1, \beta\right) - \mathcal{B}\left(\frac{2}{\alpha} + 1, \beta\right) \right]. \end{aligned}$$

Theorem 12. Let $f: [a, b] \rightarrow \mathbb{R}$ be a differentiable mapping and $f \in L_1[a, b]$. If $f'(a+b-x) \geq f'(x)$ for all $x \in [a, \frac{a+b}{2}]$. Then, the following Hemite–Hadamard-type inequality for conformable fractional integrals holds

$$\begin{aligned} f\left(\frac{a+b}{2}\right) & \leq \frac{2^{\alpha\beta-1}\Gamma(\beta+1)\alpha^\beta}{(b-a)^{\alpha\beta}} \left[{}^\beta\Upsilon_{a^+}^\alpha f\left(\frac{a+b}{2}\right) + {}^\beta\Upsilon_{b^-}^\alpha f\left(\frac{a+b}{2}\right) \right] \\ & \leq \frac{f(a) + f(b)}{2}. \end{aligned}$$

Proof. Since the equalities (2.15) and (2.16), we can write

$$\begin{aligned} & \frac{2^{\alpha\beta-1}\Gamma(\beta+1)\alpha^\beta}{(b-a)^{\alpha\beta}} \left[{}^\beta\Upsilon_{a^+}^\alpha f\left(\frac{a+b}{2}\right) + {}^\beta\Upsilon_{b^-}^\alpha f\left(\frac{a+b}{2}\right) \right] - f\left(\frac{a+b}{2}\right) \\ &= \frac{2^{\alpha\beta-1}\beta\alpha^\beta}{(b-a)^{\alpha\beta}} \int_a^{\frac{a+b}{2}} \left(\frac{\left(\frac{b-a}{2}\right)^\alpha - (t-a)^\alpha}{\alpha} \right)^{\beta-1} (t-a)^{\alpha-1} \\ & \quad \times \left(f(t) + f(a+b-t) - 2f\left(\frac{a+b}{2}\right) \right) dt \\ &= \frac{2^{\alpha\beta-1}\beta\alpha^\beta}{(b-a)^{\alpha\beta}} \int_a^{\frac{a+b}{2}} \left(\frac{\left(\frac{b-a}{2}\right)^\alpha - (t-a)^\alpha}{\alpha} \right)^{\beta-1} (t-a)^{\alpha-1} \end{aligned}$$

$$\begin{aligned}
& \times \left[\int_{\frac{a+b}{2}}^{a+b-t} [f'(s) - f'(a+b-s)] ds \right] dt \\
& = \frac{2^{\alpha\beta-1} \beta \alpha^\beta}{(b-a)^{\alpha\beta}} \int_a^{\frac{a+b}{2}} \left(\frac{\left(\frac{b-a}{2}\right)^\alpha - (t-a)^\alpha}{\alpha} \right)^{\beta-1} (t-a)^{\alpha-1} \\
& \quad \times \left[\int_t^{\frac{a+b}{2}} [f'(a+b-u) - f'(u)] ds \right] dt \\
& \geq 0,
\end{aligned}$$

which presents the first inequality in (2.13).

Likewise, by the equalities (2.20) and (2.21), we establish

$$\begin{aligned}
& \frac{f(a) + f(b)}{2} - \frac{2^{\alpha\beta-1} \Gamma(\beta+1) \alpha^\beta}{(b-a)^{\alpha\beta}} \left[{}^\beta_{\left(\frac{a+b}{2}\right)^+} \Upsilon^\alpha f(b) + {}^\beta \Upsilon^\alpha_{\left(\frac{a+b}{2}\right)^-} f(a) \right] \\
& = \frac{2^{\alpha\beta-1} \beta \alpha^\beta}{(b-a)^{\alpha\beta}} \int_a^{\frac{a+b}{2}} \left(\frac{\left(\frac{b-a}{2}\right)^\alpha - (t-a)^\alpha}{\alpha} \right)^{\beta-1} (t-a)^{\alpha-1} \\
& \quad \times [f(a) + f(b) - (f(t) + f(a+b-t))] dt \\
& = \frac{2^{\alpha\beta-1} \beta \alpha^\beta}{(b-a)^{\alpha\beta}} \int_a^{\frac{a+b}{2}} \left(\frac{\left(\frac{b-a}{2}\right)^\alpha - (t-a)^\alpha}{\alpha} \right)^{\beta-1} (t-a)^{\alpha-1} \\
& \quad \times \left[\int_a^t [f'(a+b-s) - f'(s)] ds \right] dt \geq 0.
\end{aligned}$$

This ends the proof of Theorem 12. □

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MILD SOLUTIONS FOR FRACTIONAL DIFFERENTIAL EQUATIONS IN COLOMBEAU ALGEBRA

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Abstract. The objective of this study is to investigate the unique existence of a generalized solution to an abstract Cauchy problem where the initial data is singular and a generalized real number. Our main theorems were established using the Laplace transform for Caputo fractional derivatives. In addition, we demonstrate the applicability of these results with an illustrative example.

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1. INTRODUCTION

The theory of fractional differential equations has become an interesting area of exploration in recent years, the reader can see, for example [4–7, 19], and the references therein. Currently, in the mathematical literature, several authors study the existence and uniqueness of solutions to nonlinear fractional problems, which are more significant than those involving classical derivatives, [3, 8–11, 14]. The algebra of generalized functions $\mathcal{G}$ was introduced by Colombeau in the early 1980s to address the issue of multiplication distributions, as explained in references [1, 2]. This algebra serves as a differential over the inclusive Schwartz distributions space $\mathcal{D}'$, and it allows for more general non-linear operations than just multiplication. As a result, $\mathcal{G}$ is a valuable tool for discovering and examining solutions to nonlinear differential equations that involve singular data and coefficients. It is instrumental in determining the multiplication of distributions, as described in references [12] and [18]. Furthermore, this algebra is an extension of distribution theory that deals with nonlinearities and singularities in PDE theory [18]. The algebra contains the space of distributions $\mathcal{D}'$ as a subspace, with an embedding achieved through convolution with an appropriate mollifier. The elements of $\mathcal{G}$ are classes of smooth functions,

known as moderate functions, in relation to a set of negligible functions. This regularity is necessary for solving nonlinear problems that involve derivatives of arbitrary real order and singularities. In this paper, we investigate the existence and uniqueness of the solution to the Cauchy problem given by

$$\begin{cases} D^\gamma v(t) + \mathcal{A}v(t) = \Phi(t, v(t)), & 0 < \gamma < 1, \\ v(0) = v_0, \end{cases} \quad (1.1)$$

in the setting of Colombeau algebra of generalized functions, where $v_0 \in (\tilde{\mathbb{R}})^n$ is a generalized real number, $-\mathcal{A}$ be the infinitesimal generator of an analytic generalized semigroup $(Q(t))_{t \geq 0}$ of uniformly bounded linear operators on a class of Colombeau algebras $\mathcal{G}(\mathbb{R}^n)$. In addition, regarding works on the Colombeau semigroup, we refer to [16, 17] and the references therein. Our idea is inspired by the one presented in [21], where the author proved the existence of the Cauchy problem (1.1) under two hypotheses concerning the infinitesimal generator $-\mathcal{A}$ of an analytic semigroup, also we have shown without making assumptions on the generator $-\mathcal{A}$ that the problem (1.1) admits a unique solution in the extension $\mathcal{G}^e(\mathbb{R}^n)$ of Colombeau's algebra.

The organization of the paper is as follows. In Section 2, we recall some fundamental properties of the generalized functions theory. The new notion of generalized semigroup and the proof of the existence and uniqueness of generalized solution in Colombeau algebra to the problem given in (1.1) take place in Section 3. In Section 4 we have introduced an example to illustrate our results.

2. PRELIMINARIES

This section provides a review of the essential characteristics of generalized functions theory in the Colombeau framework. The regularization approaches based on the Colombeau method involve the approximation of nonsmooth things using nets of smooth functions with moderate asymptotic bounds. The objective is to identify regularizing nets whose differences, relative to the moderateness scale, can be neglect. These regularizations are grouped into equivalence classes based on their similarity to negligible nets. Elements of Colombeau generalized functions refer to these equivalence classes, which consist of sequences of smooth functions that meet the asymptotic conditions for the regularization parameter epsilon. Let $n \in \mathbb{N}^*$, as in [6], we define the set

$$\mathcal{E}(\mathbb{R}^n) = (C^\infty(\mathbb{R}^n))^{(0,1)}.$$

The set of moderate functions is given as follows

$$\begin{aligned} \mathcal{E}_M(\mathbb{R}^n) = \Big\{ (v_\epsilon)_{\epsilon > 0} \subset \mathcal{E}(\mathbb{R}^n) : \\ \forall K \subset \subset \mathbb{R}^n, \forall \gamma \in \mathbb{N}_0^n, \exists N \in \mathbb{N}, \sup_{x \in K} |\partial^\gamma v_\epsilon(x)| = O_{\epsilon \rightarrow 0}(\epsilon^{-N}) \Big\}. \end{aligned}$$

The ideal of negligible functions is defined by

$$\mathcal{N}(\mathbb{R}^n) = \left\{ (v_\epsilon)_{\epsilon>0} \subset \mathcal{E}(\mathbb{R}^n) : \right. \\ \left. \forall K \subset\subset \mathbb{R}^n, \forall \gamma \in \mathbb{N}_0^n, \forall p \in \mathbb{N}, \sup_{x \in K} |\partial^\gamma v_\epsilon(x)| = O_{\epsilon \rightarrow 0}(\epsilon^p) \right\},$$

with operations defined component wise, e.g., $(v_\epsilon) + (\mu_\epsilon) = (v_\epsilon + \mu_\epsilon)$. The Colombeau algebra is defined as a factor set

$$\mathcal{G}(\mathbb{R}^n) = \mathcal{E}_M(\mathbb{R}^n) / \mathcal{N}(\mathbb{R}^n).$$

The ring of all generalized real numbers is given by the following set

$$\widetilde{\mathbb{R}} = \mathcal{E}(\mathbb{R}) / I(\mathbb{R}),$$

where

$$\mathcal{E}(\mathbb{R}) = \left\{ (v_\epsilon)_\epsilon \in \mathbb{R}^{(0,1)} : \exists m \in \mathbb{N}, |v_\epsilon| = O_{\epsilon \rightarrow 0}(\epsilon^{-m}) \right\}$$

and

$$I(\mathbb{R}) = \left\{ (v_\epsilon)_\epsilon \in \mathbb{R}^{(0,1)} : \forall m \in \mathbb{N}, |v_\epsilon| = O_{\epsilon \rightarrow 0}(\epsilon^m) \right\}.$$

We note that $\widetilde{\mathbb{R}}$ is a ring obtained by factoring moderate families of real numbers with respect to negligible families. The space $\mathcal{E}(\mathbb{R})$ is an algebra, and $I(\mathbb{R})$ is an ideal of $\mathcal{E}(\mathbb{R})$. The extended Colombeau algebras of generalized functions $\mathcal{G}^e(\Omega)$ on open subset Ω of $\mathbb{R}^n$ is defined in the sense of the extension of the entire derivatives to be a fractional ones, which introduced for the first time by Stojanovic (see [20], for more detail). Let $\mathcal{E}(\Omega)$ be the algebra of all nets $(v_\epsilon)_{\epsilon>0}$ of real valued functions $v_\epsilon \in C^\infty(\Omega)$, the algebra of extended moderate functions is given by

$$\mathcal{E}_M^e(\Omega) = \left\{ (v_\epsilon)_{\epsilon>0} \in (\mathcal{E}(\Omega))^{(0,1)} : \right. \\ \left. \forall K \subset\subset \mathbb{R}, \forall \gamma \in \mathbb{R}_+ \cup \{0\}, \exists N \in \mathbb{N}, \sup_{x \in K} |D^\gamma v_\epsilon(x)| = O_{\epsilon \rightarrow 0}(\epsilon^{-N}) \right\}$$

and the set of negligible functions is defined this time by

$$\mathcal{N}^e(\Omega) = \left\{ (v_\epsilon)_{\epsilon>0} \in (\mathcal{E}(\Omega))^{(0,1)} : \right. \\ \left. \forall K \subset\subset \mathbb{R}, \forall \gamma \in \mathbb{R}_+ \cup \{0\}, \forall q \in \mathbb{N}, \sup_{x \in K} |D^\gamma v_\epsilon(x)| = O_{\epsilon \rightarrow 0}(\epsilon^q) \right\}.$$

$\mathcal{G}^e(\Omega)$ is given by the factor algebras $\mathcal{G}^e(\Omega) = \mathcal{E}_M^e(\Omega) / \mathcal{N}^e(\Omega)$. Here, D^γ is the Caputo fractional derivative, where $m - 1 \leq \gamma < m$, $m \in \mathbb{N}$, for the fractional derivatives and fractional integral we can see [19] and the references therein. Colombeau algebras for tempering generalized functions, $\mathcal{G}\tau(\mathbb{R}^n)$ which was introduced by Colombeau to develop the Fourier transform theory in the algebra of generalized functions. We first define

$$O_M(\mathbb{R}^n) = \left\{ f \in C^\infty(\mathbb{R}^n) : \forall \gamma \in \mathbb{N}_0^n, \exists N \in \mathbb{N}, \sup_{x \in \mathbb{R}^n} \{ \langle x \rangle^{-N} |\partial^\gamma f(x)| \} < \infty \right\},$$

where $\langle x \rangle = (1 + \|x\|)^{2N}$. The Colombeau algebra of tempered generalized functions is given by

$$\mathcal{G}\tau(\mathbb{R}^n) = \mathcal{E}'_\tau(\mathbb{R}^n) / \mathcal{N}'_\tau(\mathbb{R}^n)$$

with

$$\begin{aligned} \mathcal{E}'_\tau(\mathbb{R}^n) = \Big\{ (\mathbf{v}_\varepsilon)_{\varepsilon>0} \in (O_M(\mathbb{R}^n))^{(0,1)} : \\ \forall \gamma \in \mathbb{R}_+ \cup \{0\}, \exists N \geq 0, \sup_{x \in \mathbb{R}^n} |D^\gamma \mathbf{v}_\varepsilon(x)| = O_{\varepsilon \rightarrow 0}(\varepsilon^{-N}) \Big\} \end{aligned}$$

and

$$\begin{aligned} \mathcal{N}'_\tau(\mathbb{R}^n) = \Big\{ (\mathbf{v}_\varepsilon)_{\varepsilon>0} \in (O_M(\mathbb{R}^n))^{(0,1)} : \\ \forall \gamma \in \mathbb{R}_+ \cup \{0\}, \forall q \geq 0, \sup_{x \in \mathbb{R}^n} |D^\gamma \mathbf{v}_\varepsilon(x)| = O_{\varepsilon \rightarrow 0}(\varepsilon^q) \Big\}, \end{aligned}$$

where D^γ is the Caputo fractional derivative, $m-1 \leq \gamma < m$, $m \in \mathbb{N}_0$. Embedding of the Schwartz distributions space $\mathcal{S}'(\mathbb{R}^n)$ into $\mathcal{G}'_\tau(\mathbb{R}^n)$ is given by $\mathbf{v} \rightarrow [(\mathbf{v} * \phi_\varepsilon)_{\varepsilon \in (0,1)}]$, such that $\phi_\varepsilon(x) = \frac{1}{\varepsilon} \phi(\frac{x}{\varepsilon})$, where $\phi \in C_0^\infty(\mathbb{R})$ and

$$\phi_\varepsilon(x) = \frac{1}{\varepsilon} \phi\left(\frac{x}{\varepsilon}\right), \phi(x) \geq 0, \int_{\mathbb{R}} \phi(x) dx = 1, \int_{\mathbb{R}} x^\gamma \phi(x) dx = 0, \forall \gamma \in \mathbb{N}^n, |\gamma| > 0. \quad (2.1)$$

2.1. Caputo derivative

The Caputo fractional integral is defined as follows

$$I_0^\gamma x(t) = \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} x(s) ds, \quad (2.2)$$

with $m-1 < \gamma < m$ and $m \in \mathbb{N}^*$. Fractional derivative in the Caputo sense of order γ of a function x is defined by

$$D_0^\gamma x(t) = \frac{1}{\Gamma(m-\gamma)} \int_0^t (t-s)^{m-\gamma-1} x^{(m)}(s) ds, \quad (2.3)$$

where $m-1 < \gamma < m$ and $m \in \mathbb{N}_0$, provided that this integral convergent which is the case when x belongs to the class of absolutely continuous functions and its of class C^m , see [13].

2.2. Embedding of the Caputo fractional derivative into Colombeau algebra

Inspired by Colombeau's classical theory for entire derivatives, to prove the embedding of the distribution in the extended Colombeau algebra, we must show that all derivatives are moderate, including fractional derivatives, i.e. we have to show that

$\tilde{D}^\gamma v_\varepsilon(t) = D^\gamma v_\varepsilon * \varphi_\varepsilon(t)$ is moderate. For $\gamma \in (0, 1)$, we have

$$\begin{aligned} \tilde{D}^\gamma v_\varepsilon &= D^\gamma v_\varepsilon * \varphi_\varepsilon(t) \\ &= \frac{1}{\Gamma(1-\gamma)} \left(\int_0^t \frac{v'_\varepsilon(s) ds}{(t-s)^\gamma} \right) * (\varphi_\varepsilon(t)) \\ &\leq \frac{1}{\Gamma(1-\gamma)} \sup_{t \in [0, T]} \left| \int_0^t \frac{v'_\varepsilon(s) ds}{(t-s)^\gamma} \right| \|(\varphi_\varepsilon(t))\|_{L^1} \\ &\leq \frac{C}{\Gamma(1-\gamma)} \sup_{t \in [0, T]} |v'_\varepsilon(t)| \frac{T^{1-\gamma}}{1-\gamma} \leq C_{\gamma, T} \varepsilon^{-N}. \end{aligned}$$

Then there exists $N > 0$, such that

$$\tilde{D}^\gamma v_\varepsilon \leq C_{\gamma, T} \varepsilon^{-N}. \quad (2.4)$$

Hence

$$\left| (\tilde{D}^\gamma v_\varepsilon)' \right| = |D^\gamma v_\varepsilon(t) * \varphi'_\varepsilon(t)| \leq \frac{C}{\varepsilon} \sup_{t \in [0, T]} |D^\gamma v_\varepsilon(t)|,$$

where the last expression is given with (2.4). Let $\beta \in (0, 1)$. For higher fractional derivatives we use the semigroup property of fractional differentiation: $D^\beta (D^\gamma v_\varepsilon) = D^{\beta+\gamma} v_\varepsilon$. We have

$$D^\beta (\tilde{D}^\gamma v_\varepsilon) = D^\beta (D^\gamma v_\varepsilon * \varphi_\varepsilon) = D^{\beta+\gamma} v_\varepsilon * \varphi_\varepsilon.$$

Then, there exist $N > 0$ and $t \in [0, T)$, $T > 0$ such that

$$\begin{aligned} (D^{\beta+\gamma} v_\varepsilon) * \varphi_\varepsilon(t) &= \frac{1}{\Gamma(1-\gamma)} \int_0^t \frac{v'_\varepsilon(s) ds}{(t-s)^{\gamma+\beta}} \\ &\leq \frac{1}{\Gamma(1-\gamma)} \sup_{t \in [0, T]} |v'_\varepsilon(t)| \left| \frac{t^{1-(\gamma+\beta)}}{1-(\gamma+\beta)} * \varphi_\varepsilon(t) \right| \\ &\leq \frac{1}{\Gamma(1-\gamma)} \sup_{t \in [0, T]} |v'_\varepsilon(t)| \frac{C}{\varepsilon} \frac{T^{2-(\gamma+\beta)}}{2-(\gamma+\beta)} \leq C_{T, \gamma, \beta} \varepsilon^{-N}. \end{aligned}$$

3. MAIN RESULTS

Let us consider the Cauchy problem in the framework of Colombeau algebras

$$\begin{cases} D^\gamma v(t) + \mathcal{A}v(t) = \Phi(t, v(t)), \\ v(0) = v_0 \in \tilde{\mathbb{R}}, \end{cases} \quad (3.1)$$

where $v \in (\mathcal{G}^e(\mathbb{R}))^n$, $-\mathcal{A}$ is an infinitesimal generator of a generalized Colombeau semigroup $(Q(t))_{t \geq 0} = [((Q_\varepsilon(t))_{t \geq 0})_\varepsilon]$, $\Phi \in (\mathcal{G}^e(\mathbb{R}))^{n+1}$.

Definition 1. Consider the one-sided stable probability density given in [15] by

$$\psi_\gamma(\xi) = \frac{1}{\pi} \sum (-1)^{n-1} \xi^{-m-1} \frac{\Gamma((n+1)\gamma+1)}{\Gamma(n+1)} \sin(n\pi\gamma), \quad \xi \in (0, \infty) \quad (3.2)$$

and the probability density

$$\phi_\gamma(\xi) = \frac{1}{\gamma} \xi^{1-\frac{1}{\gamma}} \psi_\gamma(\xi^{-\frac{1}{\gamma}}).$$

Theorem 1. Assume that $\Phi \in (\mathcal{G}_\tau^e(\mathbb{R}))^{n+1}$, and $|\nabla_x \Phi| \leq C|\ln(\varepsilon)|$, $\varepsilon \in (0, 1)$. Then the Cauchy problem (3.1) has a unique solution in the extended Colombeau algebra $(\mathcal{G}^e(\mathbb{R}))^n$.

Let us consider the following problem

$$\begin{cases} D^\gamma \mathbf{v}_\varepsilon(t) + \mathcal{A}_\varepsilon \mathbf{v}_\varepsilon(t) = \Phi_\varepsilon(t, \mathbf{v}_\varepsilon(t)), \\ \mathbf{v}_\varepsilon(0) = \mathbf{v}_{0\varepsilon} \in \mathcal{E}(\mathbb{R}). \end{cases} \quad (3.3)$$

According to Definitions 2.2 and 2.3, we write the Cauchy problem in the integral equation

$$\begin{cases} \mathbf{v}_\varepsilon(t) = \mathbf{v}_{0\varepsilon} + \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} [-\mathcal{A}_\varepsilon \mathbf{v}_\varepsilon(s) + \Phi_\varepsilon(s, \mathbf{v}_\varepsilon(s))] ds, \\ \mathbf{v}_\varepsilon(0) = \mathbf{v}_{0\varepsilon}. \end{cases} \quad (3.4)$$

The proof of the theorem requires the two lemmas below.

Lemma 1. If (3.4) holds, then there is a probability density function ϕ_γ defined on $(0, +\infty)$ such that

$$\begin{cases} \mathbf{v}_\varepsilon(t) = \int_0^\infty \phi_\gamma Q_\varepsilon(t^\gamma \xi) \mathbf{v}_{0\varepsilon} d\xi \\ \quad + \gamma \int_0^t \int_0^\infty \xi(t-s)^{\gamma-1} \phi_\gamma(\xi) Q_\varepsilon((t-s)^\gamma \xi) \Phi_\varepsilon(s, \mathbf{v}_\varepsilon(s)) d\xi ds, \\ \mathbf{v}_\varepsilon(0) = \mathbf{v}_{0\varepsilon}. \end{cases}$$

Proof. Applying the Laplace transform to the first equation in (3.4), we can get

$$\mathcal{L} \mathbf{v}_\varepsilon(\lambda) = \frac{1}{\lambda} \mathbf{v}_{0\varepsilon} + \frac{1}{\lambda^\gamma} \mathcal{A}_\varepsilon \mathcal{L}(\mathbf{v}_\varepsilon(\lambda)) + \frac{1}{\lambda^\gamma} \mathcal{L}(\Phi_\varepsilon(\lambda, \mathbf{v}_\varepsilon(\lambda))).$$

On the other hand, we have

$$\begin{aligned} \lambda^\gamma R(\lambda^\gamma, -\mathcal{A}_\varepsilon) \mathcal{L} \mathbf{v}_\varepsilon(\lambda) &= \lambda^{\gamma-1} (\lambda^{\gamma-1} I + \mathcal{A}_\varepsilon)^{-1} \mathbf{v}_{0\varepsilon} + (\lambda^{\gamma-1} I + \mathcal{A}_\varepsilon)^{-1} \\ &= \lambda^{\gamma-1} (\lambda^\gamma I + \mathcal{A}_\varepsilon)^{-1} \mathbf{v}_{0\varepsilon} \\ &\quad + (\lambda^\gamma I + \mathcal{A}_\varepsilon)^{-1} \mathcal{L}(e^{-\lambda s} \Phi_\varepsilon(s, \mathbf{v}_\varepsilon(s))) (\lambda) \\ &= \lambda^{\gamma-1} \int_0^\infty e^{-\lambda^\gamma s} Q_\varepsilon(s) \mathbf{v}_{0\varepsilon} ds + \int_0^\infty e^{-\lambda^\gamma s} Q_\varepsilon(s) \omega(\lambda) ds, \end{aligned}$$

where I is the identity operator, and $\omega(\lambda)$ is the Laplace transform of $\Phi_\varepsilon(s, v_\varepsilon(s))$. The probability density given in (3.2) has the Laplace transform given by

$$\mathcal{L}\psi_\gamma(\lambda) = \int_0^\infty e^{-\lambda\xi} \psi_\gamma(\xi) d\xi = e^{-\lambda^\gamma}, \text{ with } \gamma \in (0, 1).$$

Then,

$$\begin{aligned} \lambda^{\gamma-1}(\lambda^{\gamma-1}I + A_\varepsilon)^{-1}v_{0\varepsilon} &= \lambda^{\gamma-1} \int_0^\infty e^{-\lambda^\gamma s} Q_\varepsilon(s) v_{0\varepsilon} ds \\ &= \int_0^\infty \gamma(\lambda t)^{\gamma-1} e^{-(\lambda t)^\gamma} Q_\varepsilon(t^\gamma) v_{0\varepsilon} ds \\ &= \int_0^\infty \frac{-1}{\lambda} \frac{d}{dt} \left[e^{-(\lambda t)^\gamma} \right] Q_\varepsilon(t^\gamma) v_{0\varepsilon} ds \\ &= \int_0^\infty \left[\int_0^\infty \xi \psi_\gamma(\xi) e^{-(\lambda t^\gamma \xi)} Q_\varepsilon(t^\gamma) v_{0\varepsilon} d\xi \right] dt \\ &= \int_0^\infty e^{-\lambda t} \left[\int_0^\infty \psi_\gamma(\xi) Q_\varepsilon\left(\frac{t^\gamma}{\xi^\gamma}\right) v_{0\varepsilon} d\xi \right] dt. \end{aligned}$$

For the second term, we have

$$\begin{aligned} \int_0^\infty e^{-\lambda^\gamma s} Q_\varepsilon(s) \omega(\lambda) ds &= \int_0^\infty \left[\int_0^\infty \gamma t^{\gamma-1} e^{-(\lambda t)^\gamma} Q_\varepsilon(t^\gamma) e^{-\lambda s} \Phi_\varepsilon(s, v_\varepsilon(s)) ds \right] dt \\ &= \int_0^\infty \int_0^\infty \int_0^\infty \gamma \psi_\gamma(\xi) e^{-\lambda t^\gamma \xi} Q_\varepsilon(t^\gamma) e^{-\lambda s} t^{\gamma-1} \Phi_\varepsilon(s, v_\varepsilon(s)) d\xi ds dt \\ &= \int_0^\infty e^{-\lambda t} \left[\gamma \int_0^t \int_0^\infty \psi_\gamma(\xi) Q_\varepsilon\left(\frac{(t-s)^\gamma}{\xi^\gamma}\right) \Phi_\varepsilon(s, v_\varepsilon(s)) \frac{(t-s)^\gamma}{\xi^\gamma} d\xi ds \right] dt. \end{aligned}$$

According to the last equalities, we obtain

$$\begin{aligned} \mathcal{L}v_\varepsilon(\lambda) &= \int_0^\infty \left[e^{-\lambda t} \int_0^\infty \phi_\gamma(\xi) Q_\varepsilon(t^\gamma \xi) v_{0\varepsilon} d\xi \right. \\ &\quad \left. + \gamma \int_0^t \int_0^\infty \xi (t-s)^{\gamma-1} \phi_\gamma(\xi) Q_\varepsilon((t-s)^\gamma \xi) \Phi_\varepsilon(s, v_\varepsilon(s)) d\xi ds \right] dt, \end{aligned}$$

where $\phi_\gamma(\xi) = \frac{1}{\gamma^\gamma} \psi_\gamma(\beta(\xi))$ and β is the reciprocal function of the function $x \mapsto x^\gamma$. And the integral solution of (3.3) becomes

$$\begin{aligned} v_\varepsilon(t) &= \int_0^\infty \phi_\gamma(\xi) Q_\varepsilon(t^\gamma \xi) v_{0\varepsilon} d\xi \\ &\quad + \gamma \int_0^t \int_0^\infty \xi (t-s)^{\gamma-1} \phi_\gamma(\xi) Q_\varepsilon((t-s)^\gamma \xi) \Phi_\varepsilon(s, v_\varepsilon(s)) d\xi ds. \end{aligned}$$

Now, we define a representative semigroup $(Q_\varepsilon^\gamma)_{t \in \mathbb{R}_+}$ by

$$Q_\varepsilon^\gamma(t)v_\varepsilon = \gamma \int_0^\infty \xi \phi_\gamma(\xi) Q_\varepsilon(t^\gamma \xi) v_\varepsilon d\xi. \quad (3.5)$$

Finally the integral solution of the Cauchy (3.1) becomes

$$v_\varepsilon(t) = S_\varepsilon^\gamma(t)v_{0\varepsilon} + \int_0^t (t-s)^{\gamma-1} Q_\varepsilon^\gamma(t-s) \Phi_\varepsilon(s, v_\varepsilon(s)) ds, \quad (3.6)$$

where $(S_\varepsilon^\gamma(t))_{t \in \mathbb{R}_+}$ by

$$S_\varepsilon^\gamma(t)v_\varepsilon = \int_0^\infty \phi_\gamma(\xi) Q_\varepsilon(t^\gamma \xi) v_\varepsilon d\xi. \quad (3.7)$$

□

Lemma 2. For any fixed $t \in [0, T]$, $T > 0$, $(Q_\varepsilon^\gamma(t))_{t \geq 0}$, $(S_\varepsilon^\gamma(t))_{t \geq 0}$ are linear and bounded operators for every $\varepsilon \in (0, 1)$.

Proof. We will give the proof for (Q_ε^γ) because that of (S_ε^γ) is similar. For fixed $t > 0$, $Q_\varepsilon^\gamma(t)$ is linear operator since $Q_\varepsilon(t)$ is a linear operator. Let $\eta \in [0, 1]$ we have the following property (see [15])

$$\int_0^\infty \frac{1}{\xi^\eta} \psi_\gamma(\xi) d\xi = \frac{\Gamma(1+\eta/\gamma)}{\Gamma(1+\eta)},$$

then we have,

$$\int_0^\infty \xi^\eta \phi_\gamma(\xi) d\xi = \int_0^\infty \frac{1}{\xi^\eta} \psi_\gamma(\xi) d\xi = \frac{\Gamma(1+\eta)}{\Gamma(1+\eta\gamma)}.$$

Since $Q_\varepsilon(t)$ has a moderate bounds, then there exists a positive real number a such that

$$\sup_{t \in [0, T]} \|Q_\varepsilon(t)\| = O(\varepsilon^{-a}) \text{ as } \varepsilon \rightarrow 0, \quad (3.8)$$

then there exists $c > 0$ such that for all $t \in [0, T]$, $v_\varepsilon \in \mathcal{E}_M^e(\mathbb{R})$, we have $\|Q_\varepsilon(t)v_\varepsilon\| \leq c\varepsilon^{-a}$, then

$$\begin{aligned} |Q_\varepsilon^\gamma(t)v_\varepsilon| &= \left| \gamma \int_0^\infty \xi \phi_\gamma(\xi) Q_\varepsilon(t^\gamma \xi) v_\varepsilon d\xi \right| \\ &\leq \sup_{t \in [0, T]} |Q_\varepsilon(t^\gamma \xi) v_\varepsilon| \times \gamma \frac{1}{\Gamma(1+\gamma)}, \end{aligned} \quad (3.9)$$

since $(Q_\varepsilon(t))_\varepsilon$ is a representative of the generalized semigroup $((Q(t))_t)$ and by using (3.8), we get

$$|Q_\varepsilon^\gamma(t)v_\varepsilon| \leq \left(C\varepsilon^{-a} \times \gamma \frac{1}{\Gamma(1+\gamma)} \right) |v_\varepsilon| = C_\gamma |v_\varepsilon|,$$

where $C_\gamma = C\varepsilon^{-a} \times \gamma \frac{1}{\Gamma(1+\gamma)}$, which proves that $Q_\varepsilon^\gamma(t)$ is a linear and bounded operator. □

Proof of Theorem 1 For any $\varepsilon \in (0, 1)$ and $\gamma \in (0, 1)$, we have to show that the integral solution $(v_\varepsilon)_\varepsilon$ given in (3.6) of Lemma 1 is an element of $\mathcal{E}_M^e(\mathbb{R})$. First we have the estimation

$$\begin{aligned} |v_\varepsilon(t)| &= \left| S_\varepsilon^\gamma(t) v_{0\varepsilon} + \int_0^t (t-s)^{\gamma-1} Q_\varepsilon^\gamma(t-s) \Phi_\varepsilon(s, v_\varepsilon(s)) ds \right| \\ &\leq |S_\varepsilon^\gamma(t) v_{0\varepsilon}| + \int_0^t |(t-s)^{\gamma-1} Q_\varepsilon^\gamma(t-s) \Phi_\varepsilon(s, v_\varepsilon(s))| ds \\ &\leq |S_\varepsilon^\gamma(t) v_{0\varepsilon}| + \int_0^t (t-s)^{\gamma-1} |Q_\varepsilon^\gamma(t-s) \Phi_\varepsilon(s, v_\varepsilon(s))| ds. \end{aligned}$$

The approximation of the first order to Φ_ε yields

$$\Phi_\varepsilon(t, x(t)) = \Phi_\varepsilon(t, 0) + |\nabla_x \Phi_\varepsilon| v_\varepsilon(t) + N_\varepsilon(t), \quad (3.10)$$

where $N_\varepsilon(t)$ is the negligible part of this approximation. By Lemma 2, and the fact that $(v_\varepsilon) \in \mathcal{E}_M^e(\mathbb{R})$ there are positives constants c, c_1, c_2, N_1 and N_2 such that

$$\begin{aligned} |v_\varepsilon(t)| &\leq c c_2 \varepsilon^{-N_2} + \int_0^t (t-s)^{\gamma-1} \frac{\gamma c_1 \varepsilon^{-N_1}}{\Gamma(1+\gamma)} |\Phi_\varepsilon(s, v_\varepsilon(s))| ds \\ &\leq c c_2 \varepsilon^{-N_2} + \int_0^t (t-s)^{\gamma-1} \frac{\gamma c_1 \varepsilon^{-N_1}}{\Gamma(1+\gamma)} |\Phi_\varepsilon(s, 0) + |\nabla_x \Phi_\varepsilon| v_\varepsilon(s) + N_\varepsilon(s)| ds. \end{aligned}$$

By the Gronwall lemma, we obtain

$$|v_\varepsilon(t)| \leq (c c_2 \varepsilon^{-N_2} + c_T \varepsilon^{-N_1}) \exp(-T \ln \varepsilon).$$

Hence, there are positive constants $\tilde{c}, \tilde{N}$ such that $|v_\varepsilon(t)| \leq \tilde{c} \varepsilon^{-\tilde{N}}$ which proves the moderateness of the solution. Let us prove the uniqueness of the solution in $(\mathcal{G}^e(\mathbb{R}))^n$. Suppose that there are two solutions $v_{1,\varepsilon}, v_{2,\varepsilon}$ to the regularized problem (3.3). Let μ_ε their difference, we have

$$\mu_\varepsilon(t) = \int_0^t (t-s)^{\gamma-1} Q_\varepsilon^\gamma(t-s) [\Phi_\varepsilon(s, v_{1,\varepsilon}) - \Phi_\varepsilon(s, v_{2,\varepsilon})] ds.$$

Now, using the approximation (3.10) of Φ_ε , yields

$$|\mu_\varepsilon(t)| \leq \int_0^t \frac{T^\gamma}{\gamma} |Q_\varepsilon^\gamma(t-s) \nabla_x \Phi_\varepsilon| [(v_{1,\varepsilon}(s) - v_{2,\varepsilon}(s)) + N_\varepsilon(s)] ds,$$

by using the boundedness of the linear operator $Q_\varepsilon^\gamma(t)$, $t \geq 0$ and the Gronwall lemma, and the fact that $v_{1,\varepsilon}(s) - v_{2,\varepsilon}(s)$ is of order $O(\varepsilon^N)$ and the same applies to the negligible part N_ε , it follows that every $N \geq 0$, we have $|\mu_\varepsilon(t)| = O_{\varepsilon \rightarrow 0}(\varepsilon^N)$, which proves the uniqueness of the solution in the algebra $(\mathcal{G}^e(\mathbb{R}))^n$.

4. EXAMPLE

Consider the following fractional evolution equation in the space $Y = L^2([0, \pi])$:

$$\begin{cases} D_{0+}^\gamma v(t, x) = \frac{\partial^2 v}{\partial x^2}(t, x) + v(t, x) D^\gamma H(t), & (t, x) \in [0, 1] \times [0, \pi], \\ v(t, 0) = v(t, \pi) = 0, & t \in [0, 1], \\ v(0, x) = \delta(x), \end{cases} \quad (4.1)$$

where H is the Heaviside distribution and the initial data is the Dirac distribution. Let (v_ε) be a representative of the generalized solution. We define the operator $\mathcal{A}: D(\mathcal{A}) \subset Y \rightarrow Y$ by

$$D(\mathcal{A}) := \left\{ v \in Y : v, v' \text{ are absolutely continuous and } v'' \in Y, v(0) = v(\pi) = 0 \right\}$$

and

$$\mathcal{A}v = \frac{\partial^2 v}{\partial x^2}.$$

It is well known that $\mathcal{A}$ has a discrete spectrum, with eigenvalues $-n^2$, $n \in \mathbb{N}$, and the corresponding normalized eigenvectors $e_n(z) = \sqrt{\frac{2}{\pi}} \sin(nz)$. It is an orthonormal basis for the Hilbert space $L_\pi^2(\mathbb{R})$ of π -periodic square integrable functions on $\mathbb{R}$, that can be naturally identified with the Hilbert space $L^2(\mathbb{T})$ of square integrable functions on the half unit circle $\mathbb{T}$ via the mapping $\Phi f(t) = f(e^{2i\pi t})$. It follows

$$\mathcal{A}v = \sum_{n=1}^{\infty} -n^2 \langle v, e_n \rangle e_n, \quad v \in D(\mathcal{A}).$$

Furthermore, $\mathcal{A}$ generates a uniformly bounded analytic semigroup $\{Q(t)\}_{t \geq 0}$ in Y and it is given by

$$Q(t)v = \sum_{n=1}^{\infty} e^{-n^2 t} \langle v, e_n \rangle e_n, \quad v \in Y,$$

where

$$\langle v, e_n \rangle = \int_0^\pi v(t) e_n(t) dt.$$

Since $\|Q(t)\| \leq e^{-t}$ for all $t \geq 0$, $M = 1$, implying

$$\sup_{t \in [0, \infty)} \|Q(t)\| = 1.$$

The integral solution to the problem (4.1) can be given by

$$\begin{aligned} v_\varepsilon(t, x) &= \int_0^\infty \phi_\gamma(\xi) Q_\varepsilon(t^\gamma \xi) \delta_\varepsilon(\xi) d\xi \\ &\quad + \gamma \int_0^t \int_0^\infty \xi(t-s)^{\gamma-1} \phi_\gamma(\xi) Q_\varepsilon((t-s)^\gamma \xi) v_\varepsilon(s, \xi) \tilde{D}^\gamma h_\varepsilon(s) d\xi ds, \end{aligned}$$

where $(h_\epsilon)_\epsilon$, $(\delta_\epsilon)_\epsilon$ are representatives of the Heaviside function H and the Dirac measure δ respectively, given by $h_\epsilon(t) = \int_{-\infty}^t \varphi_\epsilon(\xi) d\xi$, $\delta_\epsilon(\xi) = \varphi_\epsilon(\xi) \quad \forall \epsilon \in (0, 1)$, where φ_ϵ is given as in (2.1) and $Q_\epsilon = Q$. Using Definitions 3.5 and 3.7, we can write the solution with the following formula

$$v_\epsilon(t, x) = S_\epsilon^\gamma(t) \varphi_\epsilon(x) + \int_0^t (t-s)^{\gamma-1} Q_\epsilon^\gamma(t-s) v_\epsilon(s, x) \tilde{D}^\gamma h_\epsilon(s) ds.$$

We have to establish the moderateness of $v_\epsilon(t, x)$ with the sup-norm, we have the estimation

$$\begin{aligned} |v_\epsilon(t, x)| &\leq |S_\epsilon^\gamma(t) \varphi_\epsilon(x)| + \int_0^t (t-s)^{\gamma-1} |Q_\epsilon^\gamma(t-s) v_\epsilon(s, x) \tilde{D}^\gamma h_\epsilon(s)| ds \\ &\leq |S_\epsilon^\gamma(t) \varphi_\epsilon(x)| + \int_0^t (t-s)^{\gamma-1} \frac{\gamma}{\Gamma(1+\gamma)} |v_\epsilon(s, x)| |\tilde{D}^\gamma h_\epsilon(s)| ds. \end{aligned}$$

Therefore,

$$\begin{aligned} \sup_{t \in [0, T)} |v_\epsilon(t, x)| &\leq \sup_{t \in [0, T)} |S_\epsilon^\gamma(t) \varphi_\epsilon(x)| \\ &\quad + \sup_{t \in [0, T)} \int_0^t (t-s)^{\gamma-1} \frac{\gamma}{\Gamma(1+\gamma)} |v_\epsilon(s, x)| |\tilde{D}^\gamma h_\epsilon(s)| ds \\ &\leq \|S_\epsilon^\gamma(t) \varphi_\epsilon\|_\infty + \int_0^t (t-s)^{\gamma-1} \frac{\gamma}{\Gamma(1+\gamma)} |v_\epsilon(s, x)| \|\tilde{D}^\gamma h_\epsilon(s)\|_\infty ds, \end{aligned}$$

since we use the estimates (3.9) of the semigroups $S^\gamma(t)$ and $Q^\gamma(t)$, and the fact that $Q(t)$ is a contraction semigroup. According to the estimation (2.4) of $\tilde{D}^\gamma$, we can write $\|\tilde{D}^\gamma v_\epsilon\| \leq C_{\gamma, T} \epsilon^{-N} \leq C_\gamma \epsilon^{-N}$ $\epsilon \in (0, 1)$, $N > 0$. By Gronwall inequality it yields

$$\begin{aligned} |v_\epsilon(t, x)| &\leq \|\varphi_\epsilon\|_\infty + \int_0^t (t-s)^{\gamma-1} C_\gamma \epsilon^{-N} ds, \\ &\leq \|\varphi_\epsilon\|_\infty \times \int_0^t \exp((t-s)^{\gamma-1} C_\gamma \epsilon^{-N}) ds \end{aligned}$$

since the integral in the last inequality is convergent, there are $N > 0$, C_γ such that $|v_\epsilon(t, x)| \leq C_\gamma \epsilon^{-N}$, which proves the moderateness of $(v_\epsilon)_\epsilon$.

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ON ALGEBRAS WHOSE CONGRUENCES HAVE A CLASS THAT IS A SUBUNIVERSE

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Abstract. We study algebras and varieties where every non-trivial congruence has some class being a non-trivial subuniverse of the algebra in question. Then we focus on algebras where this non-trivial class is a unique non-singleton class of a congruence. In particular, we investigate Rees algebras, pseudo-Rees algebras and algebras having the one-block property. Many examples are included. At the end, we will also deal with quotient algebras.

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1. INTRODUCTION

Congruences having the property that one of its classes is a subuniverse were studied by B. Csákány [6] and R. F. Tichy [8] who introduced the concept of Rees congruences, see also [2]. This concept was generalized for algebras and varieties satisfying the so-called one-block property in [1]. The aim of the present paper is to modify these concepts and prove some results also for single algebras.

Let $\mathbf{A} = (A, F)$ be an algebra. The algebra $\mathbf{A}$ is called *non-trivial* if $|A| > 1$. Denote by $\mathbf{Con} \mathbf{A} = (\mathbf{Con} \mathbf{A}, \subseteq)$ the congruence lattice of $\mathbf{A}$ and by $\omega_{\mathbf{A}}$ the least congruence on $\mathbf{A}$. A congruence Θ on $\mathbf{A}$ is called *non-trivial* if $\Theta \neq \omega_{\mathbf{A}}$.

B. Csákány [6] characterized those varieties $\mathcal{V}$ having the property that every congruence on an algebra $\mathbf{A}$ of $\mathcal{V}$ has some class being a subuniverse of $\mathbf{A}$. His result is as follows (see also Theorem 4.2.1 in [3]).

Proposition 1. *For a variety $\mathcal{V}$ the following are equivalent:*

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- (i) Each congruence on a member of $\mathcal{V}$ has some class being a subuniverse.
- (ii) There exists some (at most) unary term $v(x)$ such that $\mathcal{V}$ satisfies the identity

$$f(v(x), \dots, v(x)) \approx v(x)$$

for all fundamental operations f .

The best known examples of varieties satisfying the conditions of Proposition 1 are the variety of groups where one can define v as the nullary operation, i.e. the neutral element of the group, or the variety of idempotent algebras (e.g. lattices or semilattices) where one can take $v(x) := x$. Recall that an *algebra* is called *idempotent* if it satisfies the identity $f(x, \dots, x) \approx x$ for every fundamental operation f and that a *variety* is called *idempotent* if every of its members has this property.

However, it may happen that there exists some non-trivial algebra $\mathbf{A}$ and some non-trivial congruence Θ on $\mathbf{A}$ such that the only class of Θ which is a subuniverse of $\mathbf{A}$ is a singleton.

2. NON-TRIVIAL SUBUNIVERSES AS CONGRUENCE CLASSES

This cannot happen if the variety is congruence uniform. Recall that an algebra $\mathbf{A}$ is called *congruence uniform* if for each congruence Θ on $\mathbf{A}$ all classes of Θ have the same cardinality. A *variety* $\mathcal{V}$ is called *congruence uniform* if each of its members has this property. The best known examples of such varieties are the variety of groups, the variety of rings and the variety of Boolean algebras.

An immediate consequence of Proposition 1 is the following result.

Corollary 1. *For a congruence uniform variety $\mathcal{V}$ the following are equivalent:*

- (i) Each congruence on a member of $\mathcal{V}$ has some class being a subuniverse which is non-trivial if the congruence is non-trivial.
- (ii) There exists some (at most) unary term $v(x)$ such that $\mathcal{V}$ satisfies the identity

$$f(v(x), \dots, v(x)) \approx v(x)$$

for all fundamental operations f .

Remark 1. Let $\mathbf{A} = (A, F)$ be an idempotent algebra. Then every congruence class of $\mathbf{A}$ is a subuniverse of $\mathbf{A}$ and hence every non-trivial congruence on $\mathbf{A}$ has some class being a non-trivial subuniverse.

In particular, we have: Every non-trivial congruence on a lattice (semilattice) has some class being a non-trivial sublattice (subsemilattice).

The following lemma is almost evident.

Lemma 1. *Let $\mathbf{A} = (A, F)$ be an algebra, $a, b \in A$ and p_0 a binary term and assume $\mathbf{A}$ to satisfy the identity*

$$(1) \quad f(p_0(x, x), \dots, p_0(x, x)) \approx p_0(x, x)$$

for every fundamental operation f . Then $[p_0(a, b)]\Theta(a, b)$ is a subuniverse of $\mathbf{A}$.

Proof. If f is an m -ary fundamental operation and $a_1, \dots, a_m \in [p_0(a, b)]\Theta(a, b)$ then

$$\begin{aligned} f(a_1, \dots, a_m) &\in [f(p_0(a, b), \dots, p_0(a, b))]\Theta(a, b) \\ &= [f(p_0(a, a), \dots, p_0(a, a))]\Theta(a, b) \\ &= [p_0(a, a)]\Theta(a, b) = [p_0(a, b)]\Theta(a, b). \end{aligned}$$

□

We now return to varieties with congruences having a class being a non-trivial subuniverse.

Theorem 1. *Let $\mathcal{V}$ be a variety where every non-trivial congruence on a member of $\mathcal{V}$ has some class being a non-trivial subuniverse. Then there exists a binary term p_0 such that*

- $\mathcal{V}$ satisfies identity (1) for every fundamental operation f .
- The class $[p_0(x, y)]\Theta(x, y)$ is a non-trivial subuniverse of $F_{\mathcal{V}}(x, y)$.

Proof. Consider the congruence $\Theta(x, y)$ on $F_{\mathcal{V}}(x, y)$. Because $\Theta(x, y)$ is non-trivial, there exists some $p_0 \in F_{\mathcal{V}}(x, y)$ such that $[p_0]\Theta(x, y)$ is a non-trivial subuniverse of $F_{\mathcal{V}}(x, y)$. Since $F_{\mathcal{V}}(x, y)$ is the free algebra of $\mathcal{V}$ with two free generators x and y , we have $p_0 = p_0(x, y)$ for some binary term $p_0(x, y)$. Let g denote the endomorphism of $F_{\mathcal{V}}(x, y)$ satisfying $g(x) = g(y) = x$. Then $g(t(x, y)) = t(x, x)$ for every $t(x, y) \in F_{\mathcal{V}}(x, y)$. Moreover, $(x, y) \in \ker g$ whence $\Theta(x, y) \subseteq \ker g$. Since $[p_0(x, y)]\Theta(x, y)$ is a subuniverse of $F_{\mathcal{V}}(x, y)$ we have $f(p_0(x, y), \dots, p_0(x, y)) \Theta(x, y) p_0(x, y)$ for all fundamental operations f and hence

$$f(p_0(x, x), \dots, p_0(x, x)) = g\left(f(p_0(x, y), \dots, p_0(x, y))\right) = g(p_0(x, y)) = p_0(x, x)$$

for all fundamental operations f . □

In the following theorem we provide sufficient conditions for the fact that every non-trivial congruence on a member of $\mathcal{V}$ has some class being a non-trivial subuniverse.

Theorem 2. *Let $\mathcal{V}$ be a variety such that there exists some positive integer n and binary terms $p_0, \dots, p_n$ satisfying the following two conditions:*

- (i) $p_0(x, y) = \dots = p_n(x, y)$ if and only if $x = y$.
- (ii) $\mathcal{V}$ satisfies identity (1) for every fundamental operation f .

Then for every $\mathbf{A} = (A, F) \in \mathcal{V}$ and every non-trivial congruence Θ on $\mathbf{A}$ there exists some $a \in A$ such that $[p_0(a, a)]\Theta$ is a non-trivial subuniverse of $\mathbf{A}$.

Proof. Let $\mathbf{A} = (A, F) \in \mathcal{V}$ and Θ be a non-trivial congruence on $\mathbf{A}$. Then there exists some $a \in A$ such that $[a]\Theta$ is a non-trivial class of Θ . Put $C := [p_0(a, a)]\Theta$. If $c_1, \dots, c_m \in C$ and f is an m -ary fundamental operation then, according to (ii),

$$f(c_1, \dots, c_m) \in [f(p_0(a, a), \dots, p_0(a, a))]\Theta = [p_0(a, a)]\Theta = C$$

showing C to be a subuniverse of $\mathbf{A}$. Let $b \in [a]\Theta$ with $b \neq a$. Since $p_0(a, a) = \dots = p_n(a, a)$ according to (i) we have $p_i(a, b) \in [p_i(a, a)]\Theta = [p_0(a, a)]\Theta = C$ for $i = 0, \dots, n$. According to Theorem 3.4 in [7] condition (i) implies that there exists some positive integer k , ternary terms $t_1, \dots, t_k$ and $u_1, \dots, u_k, v_1, \dots, v_k \in \{p_0, \dots, p_n\}$ such that

$$\begin{aligned} x &\approx t_1(u_1(x, y), x, y), \\ t_i(v_i(x, y), x, y) &\approx t_{i+1}(u_{i+1}(x, y), x, y) \quad \text{for } i = 1, \dots, k-1, \\ t_k(v_k(x, y), x, y) &\approx y. \end{aligned}$$

Assume now C to be a singleton. Then $p_i(a, b) = p_j(a, b)$ for all $i, j = 0, \dots, n$ and hence $u_i(a, b) = v_j(a, b)$ for all $i, j = 1, \dots, k$ and we obtain

$$\begin{aligned} a &= t_1(u_1(a, b), a, b) = t_1(v_1(a, b), a, b) = t_2(u_2(a, b), a, b) = \dots \\ &= t_k(v_k(a, b), a, b) = b, \end{aligned}$$

a contradiction. Hence C is a non-trivial subuniverse of $\mathbf{A}$. $\square$

If the term $p_0(x, y)$ of (ii) in Theorem 2 is a constant term (as in Corollary 2) then, by Theorem 3.9 in [7], the variety $\mathcal{V}$ is congruence modular and n -permutable (for some $n \geq 2$). This is not true if $p_0(x, y)$ is a non-constant term as in the variety of semilattices where we can put $n := 1$, $p_0(x, y) := x$ and $p_1(x, y) := y$ in order to satisfy the assumptions of Theorem 2. Recall that a class $\mathcal{K}$ of algebras of the same type is called n -permutable if for all $\mathbf{A} \in \mathcal{K}$ and all $\Theta, \Phi \in \text{Con } \mathbf{A}$ we have

$$\Theta \circ \Phi \circ \Theta \circ \Phi \circ \dots = \Phi \circ \Theta \circ \Phi \circ \Theta \circ \dots$$

where on both sides of this equality there are n congruences.

Recall that an algebra $\mathbf{A}$ with an equationally definable constant 1 is called *weakly regular with respect to 1* if for all $\Theta, \Phi \in \text{Con } \mathbf{A}$, $[1]\Theta = [1]\Phi$ implies $\Theta = \Phi$. A variety is called *weakly regular with respect to 1* if any of its members has this property.

We recall the following characterization of varieties being weakly regular with respect to 1 by B. Csákány [5], see also Theorem 6.4.3 in [3].

Proposition 2. *A variety with an equationally definable constant 1 is weakly regular with respect to 1 if and only if there exists some positive integer n and binary terms $t_1, \dots, t_n$ such that*

$$t_1(x, y) = \dots = t_n(x, y) = 1 \text{ if and only if } x = y.$$

Corollary 2. *Let $\mathcal{V}$ be a weakly regular variety with respect to the equationally definable constant 1 satisfying the identity $f(1, \dots, 1) \approx 1$ for every fundamental operation f . Then every non-trivial congruence on a member of $\mathcal{V}$ has some class being a non-trivial subuniverse.*

Proof. According to Proposition 2 there exists some positive integer n and binary terms $t_1, \dots, t_n$ such that

$$t_1(x, y) = \dots = t_n(x, y) = 1 \text{ if and only if } x = y.$$

If we take $p_0 := 1$ and $p_i := t_i$ for $i = 1, \dots, n$ then the assumptions of Theorem 2 are satisfied. $\square$

Example 1. Recall that a *loop* is an algebra $(L, \cdot, /, \backslash, 1)$ of type $(2, 2, 2, 0)$ satisfying the following identities:

$$(x/y)y \approx x, \quad (xy)/y \approx x, \quad x(x \backslash y) \approx y, \quad x \backslash (xy) \approx y, \quad x1 \approx 1x \approx x.$$

Every non-trivial congruence on a loop has some class being a non-trivial subuniverse. This can be seen as follows. Put

$$n := 1, \\ t_1(x, y) := x/y.$$

If $t_1(x, y) = 1$ then $x/y = 1$ and hence $x = (x/y)y = 1y = y$. If, conversely, $x = y$ then $t_1(x, y) = x/x = (1x)/x = 1$. Moreover, $1 \cdot 1 = 1$, $1/1 = (1 \cdot 1)/1 = 1$ and $1 \backslash 1 = 1 \backslash (1 \cdot 1) = 1$. Now apply Corollary 2.

Example 2. An *implication algebra* is a groupoid $(I, \cdot)$ satisfying the identities

$$(xy)x \approx x, (xy)y \approx (yx)x \text{ and } x(yz) \approx y(xz).$$

It is well-known that the identity $xx \approx yy$ holds in every implication algebra. Hence this element is an equationally definable constant denoted by 1. Further, the binary relation $\leq$ on I defined by $x \leq y$ if and only if $xy = 1$ ($x, y \in I$) is a partial order relation on I , see e.g. Section 2.2 in [3]. Every non-trivial congruence on an implication algebra has some class being a non-trivial subuniverse. This can be seen as follows. Put

$$n := 2, \\ t_1(x, y) := xy, \\ t_2(x, y) := yx.$$

Then $t_1(x, y) = t_2(x, y) = 1$ if and only if $x = y$. Moreover, $1 \cdot 1 = 1$. Now apply Corollary 2.

3. REES CONGRUENCES

The concept of a Rees congruence was introduced in [8], but firstly used for semigroups by D. Rees in 1940 under a different name. A *Rees congruence* on an algebra $\mathbf{A} = (A, F)$ is a congruence of the form $B^2 \cup \omega_A$ where B is a subuniverse of $\mathbf{A}$. A *Rees algebra* is an algebra $\mathbf{A} = (A, F)$ such that $B^2 \cup \omega_A \in \text{Con } \mathbf{A}$ for every subuniverse B of $\mathbf{A}$. A *Rees variety* is a variety consisting of Rees algebras only.

It was shown in [3] that every subalgebra and every homomorphic image of a Rees algebra is a Rees algebra again. Moreover, the following is proved (Theorem 12.2.6).

Proposition 3. *For an algebra $\mathbf{A} = (A, F)$ the following conditions are equivalent:*

- (i) *$\mathbf{A}$ is a Rees algebra.*
- (ii) *Every subalgebra of $\mathbf{A}$ generated by two elements is a Rees algebra.*
- (iii) *$(p(a), p(b)) \in \langle \{a, b\} \rangle^2 \cup \omega_A$ for any $a, b \in A$ and every $p \in P_1(\mathbf{A})$.*

Here $\langle \{a, b\} \rangle$ denotes the subalgebra of $\mathbf{A}$ generated by $\{a, b\}$.

Results on Rees algebras are collected in Chapter 12 of [3] which contains also the following characterization of Rees varieties (Theorem 12.2.7).

Proposition 4. *A variety is a Rees variety if and only if it is at most unary.*

Recall that a variety $\mathcal{V}$ is called *at most unary* if every proper term of $\mathcal{V}$ is essentially unary or nullary.

The concept of a Rees algebra is rather strong. Hence we modify it as follows.

Definition 1. An algebra $\mathbf{A} = (A, F)$ is called a *pseudo-Rees algebra* if for every non-trivial congruence Θ on $\mathbf{A}$ there exists some class C of Θ being a non-trivial subuniverse of $\mathbf{A}$ such that $C^2 \cup \omega_A \in \text{Con } \mathbf{A}$.

The following is an immediate consequence of the definition of a Rees algebra and a pseudo-Rees algebra, respectively.

Lemma 2. *Let $\mathbf{A} = (A, F)$ be an idempotent Rees algebra. Then $\mathbf{A}$ is a pseudo-Rees algebra.*

Proof. Let Θ be a non-trivial congruence on $\mathbf{A}$. Then there exists some non-trivial class C of Θ . According to Remark 1, C is a subuniverse of $\mathbf{A}$ and, since $\mathbf{A}$ is a Rees algebra, $C^2 \cup \omega_A \in \text{Con } \mathbf{A}$. $\square$

We are going to find another idempotent class of pseudo-Rees algebras.

Recall from [4] that a *(join-)directoid* is a groupoid $(D, \sqcup)$ satisfying the identities

$$\begin{aligned} x \sqcup x &\approx x, & (x \sqcup y) \sqcup x &\approx x \sqcup y, \\ y \sqcup (x \sqcup y) &\approx x \sqcup y, & x \sqcup ((x \sqcup y) \sqcup z) &\approx (x \sqcup y) \sqcup z. \end{aligned}$$

Hence the class of (join-)directoids forms a variety. In particular, every (join-)semi-lattice is a (join-)directoid. It is well-known that the binary operation $\leq$ on D defined by $x \leq y$ if $x \sqcup y = y$ is a partial order relation on D , called the *order induced by $\mathbf{D}$* , and that $x, y \leq x \sqcup y$.

We can prove the following result.

Theorem 3. *Let $\mathbf{D} = (D, \sqcup)$ be a finite join-directoid. Then $\mathbf{D}$ is a pseudo-Rees algebra.*

Proof. Let Θ be a non-trivial congruence on $\mathbf{D}$. Denote by $\leq$ the order induced by $\mathbf{D}/\Theta$. According to Remark 1 there exists some class of Θ being a non-trivial subuniverse of $\mathbf{D}$. Let C be a maximal (with respect to $\leq$) class with this property. Since D is finite, such a class C exists. Moreover, for each $E \in D/\Theta$ with $C < E$ we have $|E| = 1$. (Clearly, E is a subuniverse of $\mathbf{D}$ for every $E \in D/\Theta$.) Put $\Phi := C^2 \cup \omega_S$. Of course, Φ is an equivalence relation on D and $\Phi \subseteq \Theta$. Suppose $(a, b), (c, d) \in \Phi$.

Case 1. $a = b$ and $c = d$: Then clearly $(a \sqcup c, b \sqcup d) \in \Phi$.

Case 2. $a \neq b$ and $c \neq d$: Then $a, b, c, d \in C$. But C is a subuniverse of $\mathbf{D}$. Thus also $a \sqcup c, b \sqcup d \in C$ and hence $(a \sqcup c, b \sqcup d) \in \Phi$.

Case 3. $a \neq b$ and $c = d$: Then $C = [a]\Theta = [b]\Theta \leq [a \sqcup c]\Theta = [b \sqcup d]\Theta$. By the assumption on C , either $a \sqcup c, b \sqcup d \in C$ or $a \sqcup c = b \sqcup d$. Hence $(a \sqcup c, b \sqcup d) \in \Phi$.

Case 4. $a = b$ and $c \neq d$: This case is symmetric to Case 3.

Altogether, $\Phi \in \text{Con } \mathbf{S}$, i.e. $\mathbf{D}$ is a pseudo-Rees algebra. $\square$

Corollary 3. According to Remark 1 and Theorem 3 the variety $\mathcal{D}$ of (join-)directoids and, in particular, the variety $\mathcal{S}$ of (join-)semilattices has the following property: For each $\mathbf{D} = (D, \sqcup) \in \mathcal{D}$ and each non-trivial congruence Θ on $\mathbf{D}$ there exists some class of Θ being a non-trivial subdirectoid of $\mathbf{D}$ and if D is finite then there exists such a class C of Θ such that $C^2 \cup \omega_D$ is a congruence on $\mathbf{D}$. For each $\mathbf{S} = (S, \sqcup) \in \mathcal{S}$ and each non-trivial congruence Θ on $\mathbf{S}$ there exists some class of Θ being a non-trivial subsemilattice of $\mathbf{S}$ and if S is finite then there exists such a class C such that $C^2 \cup \omega_S$ is a congruence on $\mathbf{S}$.

3.1. One-block property

Congruences having exactly one class that is not a singleton were treated in [1].

Definition 2. An algebra $\mathbf{A}$ has the *one-block property* if every atom of $\text{Con } \mathbf{A}$ has exactly one class which is not a singleton.

Recall that a congruence Θ on $\mathbf{A} = (A, F)$ is called an *atom* of $\text{Con } \mathbf{A}$ if $\omega_A \prec \Theta$.

It was shown in [1] (see also Theorem 12.1.7 in [3]) that a variety satisfying the one-block property is congruence semimodular, but it need not be congruence modular.

The following result can be easily checked.

Proposition 5. Let $\mathbf{A} = (A, F)$ be a pseudo-Rees algebra. Then it has the one-block property.

Proof. Let Θ be an atom of $\text{Con } \mathbf{A}$. Since $\mathbf{A}$ is a pseudo-Rees algebra, there exists some class C of Θ being a non-trivial subuniverse of $\mathbf{A}$ such that $\Phi := C^2 \cup \omega_A \in \text{Con } \mathbf{A}$. Clearly, Φ is a non-trivial congruence on $\mathbf{A}$ with $\Phi \subseteq \Theta$. Since Θ is an atom of $\text{Con } \mathbf{A}$ we have $\Theta = \Phi$. Hence $\mathbf{A}$ has the one-block property. $\square$

We can prove the following characterization of algebras having the one-block property.

Theorem 4. *Let $\mathbf{A} = (A, F)$ be an algebra. Then the following are equivalent:*

- (i) $\mathbf{A}$ has the one-block property.
- (ii) If $a, b \in A$ and $(a, b) \in \Theta(x, y)$ for all $(x, y) \in \Theta(a, b)$ with $x \neq y$ then $x, y \in [a]\Theta(a, b)$ for all $(x, y) \in \Theta(a, b)$ with $x \neq y$.

Proof.

(i) $\Rightarrow$ (ii): Assume $a, b \in A$ and $(a, b) \in \Theta(x, y)$ for all $(x, y) \in \Theta(a, b)$. If $a = b$ then $\Theta(a, b) = \omega_A$ and (ii) is valid. Suppose $a \neq b$. Take $(c, d) \in \Theta(a, b)$ with $c \neq d$. Let Φ be a non-trivial congruence on $\mathbf{A}$ with $\Phi \subseteq \Theta(a, b)$. Then there exists some $(e, f) \in \Phi$ with $e \neq f$. We have $\Theta(e, f) \subseteq \Phi \subseteq \Theta(a, b)$. Because of $(e, f) \in \Theta(a, b)$ and $e \neq f$ we conclude $(a, b) \in \Theta(e, f)$ and hence $\Theta(a, b) \subseteq \Theta(e, f)$. Together we obtain $\Theta(e, f) = \Theta(a, b)$ and therefore $\Phi = \Theta(a, b)$ showing that $\Theta(a, b)$ is an atom of $\mathbf{Con} \mathbf{A}$. According to (i), $[a]\Theta(a, b)$ is the unique class of $\Theta(a, b)$ which is not a singleton and hence $c, d \in [a]\Theta(a, b)$.

(ii) $\Rightarrow$ (i): Assume Θ to be an atom of $\mathbf{Con} \mathbf{A}$. Then there exist $a, b \in A$ with $a \neq b$ and $\Theta(a, b) = \Theta$. Suppose $(c, d) \in \Theta(a, b)$ and $c \neq d$. Then $\theta(c, d)$ is a non-trivial congruence on $\mathbf{A}$ with $\theta(c, d) \subseteq \Theta(a, b)$. Since $\Theta(a, b)$ is an atom of $\mathbf{Con} \mathbf{A}$ we have $\theta(c, d) = \Theta(a, b)$ and hence $(a, b) \in \theta(c, d)$. Because of (ii) we have $x, y \in [a]\Theta(a, b)$ for all $(x, y) \in \Theta(a, b)$ with $x \neq y$. This shows that $[a]\Theta(a, b)$ is the unique class of Θ which is not a singleton. □

3.2. Absorbing element

In the following let $P_1(\mathbf{A})$ denote the set of unary polynomial functions on the algebra $\mathbf{A}$.

In the proof of the next lemma we use the well-known fact that an equivalence relation Θ on the universe of an algebra $\mathbf{A}$ is a congruence on $\mathbf{A}$ if and only if $(x, y) \in \Theta$ implies $(p(x), p(y)) \in \Theta$ for all $p \in P_1(\mathbf{A})$.

The proof of the following lemma is straightforward.

Lemma 3. *Let $\mathbf{A} = (A, F)$ be an algebra and $B \subseteq A$. Then the following are equivalent:*

- (i) $B^2 \cup \omega_A \in \mathbf{Con} \mathbf{A}$.
- (ii) If $(a, b) \in B^2$ and $p \in P_1(\mathbf{A})$ then $(p(a), p(b)) \in B^2 \cup \omega_A$.

Next we describe varieties $\mathcal{V}$ such that for any $\mathbf{A} = (A, F) \in \mathcal{V}$ and any $\Theta \in \mathbf{Con} \mathbf{A}$ there exists some class B of Θ being a subuniverse of $\mathbf{A}$ satisfying $B^2 \cup \omega_A \in \mathbf{Con} \mathbf{A}$.

A variety with an absorbing element 1 is a variety with a unique nullary operation 1 satisfying the identities

$$f(x_1, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_n) \approx 1$$

for all n -ary fundamental operations f and for all $i = 1, \dots, n$. It can be easily shown by induction on term complexity that then also the identities

$$t(x_1, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_n) \approx 1$$

hold for all n -ary terms t and for all $i = 1, \dots, n$. It is easy to see that for an algebra $\mathbf{A}$ the following are equivalent:

- (i) $\mathbf{A}$ has an absorbing element 1.
- (ii) 1 is the unique nullary operation of $\mathbf{A}$, and each $p \in P_1(\mathbf{A})$ is either a constant function or $p(1) = 1$.

Theorem 5. *Let $\mathcal{V}$ be a variety with an absorbing element 1. Then for every $\mathbf{A} = (A, F) \in \mathcal{V}$ and every $\Theta \in \text{Con } \mathbf{A}$, $[1]\Theta$ is the unique class of Θ being a subuniverse of $\mathbf{A}$ and, moreover, $([1]\Theta)^2 \cup \omega_{\mathbf{A}} \in \text{Con } \mathbf{A}$.*

Proof. Let $\mathbf{A} = (A, F) \in \mathcal{V}$ and $\Theta \in \text{Con } \mathbf{A}$ and put $B := [1]\Theta$. Since $\mathbf{A}$ satisfies the identity $f(1, \dots, 1) \approx 1$ for every fundamental operation f , B is a subuniverse of $\mathbf{A}$. Because 1 is a nullary fundamental operation, no other class of Θ is a subuniverse of $\mathbf{A}$. Now let $(a, b) \in B^2$ and $p \in P_1(\mathbf{A})$. Then p either is a constant function or $p(1) = 1$. If p is a constant function then $(p(a), p(b)) \in \omega_{\mathbf{A}} \subseteq B^2 \cup \omega_{\mathbf{A}}$. Otherwise, $(p(a), p(b)) \in ([p(1)]\Theta)^2 = ([1]\Theta)^2 = B^2 \subseteq B^2 \cup \omega_{\mathbf{A}}$. According to Lemma 3, $B^2 \cup \omega_{\mathbf{A}} \in \text{Con } \mathbf{A}$. $\square$

Example 3. Examples of varieties with an absorbing element 1 are join-semilattices with top element 1, join-directoids with a top element 1 and semigroups with an absorbing element, usually denoted by 0.

Corollary 4. *Let $\mathcal{V}$ be a variety with an absorbing element 1. If $\mathbf{A} \in \mathcal{V}$ and $|[1]\Theta| > 1$ for each non-trivial congruence Θ on $\mathbf{A}$ then $\mathbf{A}$ is a pseudo-Rees algebra.*

Example 4. The join-semilattice $\mathbf{A} = (A, \vee, 1)$ with top element 1 depicted in Figure 1 has the absorbing element 1.

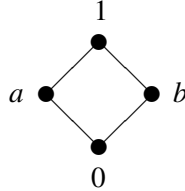


Fig. 1

The congruence lattice of $\mathbf{A}$ is visualized in Figure 2:

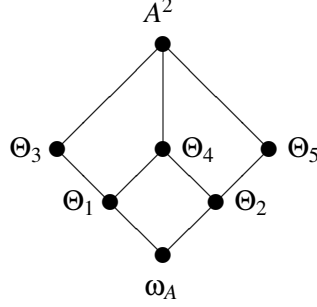


Fig. 2

with

$$\begin{aligned}\Theta_1 &:= \{0\}^2 \cup \{a\}^2 \cup \{b, 1\}^2, \\ \Theta_2 &:= \{0\}^2 \cup \{b\}^2 \cup \{a, 1\}^2, \\ \Theta_3 &:= \{0, a\}^2 \cup \{b, 1\}^2, \\ \Theta_4 &:= \{0\}^2 \cup \{a, b, 1\}^2, \\ \Theta_5 &:= \{0, b\}^2 \cup \{a, 1\}^2.\end{aligned}$$

It is easy to see that $\mathbf{A}$ is a pseudo-Rees algebra having the one-block property. In accordance with the remark after Definition 2, $\mathbf{Con} \mathbf{A}$ is semimodular, but not modular as can easily be checked.

3.3. Quotient algebras

Next we describe Rees congruences on a quotient algebra $\mathbf{A}/\Theta$ of some algebra $\mathbf{A} = (A, F)$ with respect to a congruence Θ on $\mathbf{A}$. Keep in mind that then a subset of A/Θ is in fact a set of congruence classes, i.e. of subsets of A .

Theorem 6. *Let $\mathbf{A}$ be an algebra, $\Theta \in \mathbf{Con} \mathbf{A}$ and $B \subseteq A/\Theta$ and define $C := \bigcup_{X \in B} X$.*

Then the following hold:

- (i) *The relation $B^2 \cup \omega_{\mathbf{A}/\Theta}$ on A/Θ is a congruence on $\mathbf{A}/\Theta$ if and only if $C^2 \cup \Theta \in \mathbf{Con} \mathbf{A}$. In this case we have*

$$B^2 \cup \omega_{\mathbf{A}/\Theta} = (C^2 \cup \Theta)/\Theta.$$

- (ii) *The subset B of A/Θ is a subuniverse of $\mathbf{A}/\Theta$ if and only if C is a subuniverse of $\mathbf{A}$.*

Proof. Assertion (i) follows immediately from the Isomorphism Theorems of Universal Algebra describing the natural bijective correspondence between the set of all congruences on $\mathbf{A}/\Theta$ on the one hand and the set of all congruences on $\mathbf{A}$ including Θ on the other hand. The proof of (ii) is straightforward. $\square$

Using Theorem 6 we can characterize pseudo-Rees quotient algebras as follows.

Corollary 5. *Let $\mathbf{A} = (A, F)$ be an algebra and $\Theta \in \text{Con } \mathbf{A}$. Then the following are equivalent:*

- (i) $\mathbf{A}/\Theta$ is a pseudo-Rees algebra.
- (ii) Every congruence on $\mathbf{A}$ strictly including Θ has some class C being a subuniverse of $\mathbf{A}$, but not a class of Θ , such that $C^2 \cup \Theta \in \text{Con } \mathbf{A}$.

Also the next theorem is a direct consequence of the Isomorphism Theorem mentioned in the proof of Theorem 6.

Theorem 7. *Let $\mathbf{A} = (A, F)$ be some algebra and $\Theta \in \text{Con } \mathbf{A}$. Then the following are equivalent:*

- $\mathbf{A}/\Theta$ has the one-block property.
- For every congruence Φ on $\mathbf{A}$ covering Θ there exists some subset B of A/Θ with $(\bigcup_{X \in B} X)^2 \cup \Theta = \Phi$.

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RELATIVELY HEREDITARY RADICAL CLASSES OF RINGS

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Abstract. Radical classes of rings are studied which, while not hereditary, are closed with respect to ideals of some kind: maximal, prime and finite index ideals among others. In some, but not all cases, the ideal property is characterized by the corresponding class of factor rings; for instance maximal ideals are characterized by the simple rings. Such characterizations sometimes make it possible to prove results for several types of ideal simultaneously. Several results concerning hereditary radicals are generalized to various types of relatively hereditary ones, e.g. if $\mathcal{R}$ is hereditary then for $I \triangleleft A$ we have $\mathcal{R}(I) = I \cap \mathcal{R}(A)$ and hereditary classes define hereditary lower radical classes. In the construction of some examples, use is made of A -radical classes and this leads to some consideration of radical classes of abelian groups.

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1. INTRODUCTION

Although stronger versions of the hereditary property for radical (and other) classes of rings, e.g. *left hereditary* (hereditary for left ideals) and *strongly hereditary* (hereditary for subrings) have been extensively studied, almost no attention has been given to *weak* versions of hereditariness for rings or other structures. Thus nothing seems to be known about radical classes which are hereditary for maximal ideals, or for prime ideals for example.

In fact to our knowledge the only weak hereditary property for radical classes of rings that has been studied previously is that considered by Sands [13]: he defined a radical class $\mathcal{R}$ to be *radically hereditary* if for every $A \in \mathcal{R}$ we have $\mathcal{U}(A) \in \mathcal{R}$ for every radical class $\mathcal{U}$.

For other structures the story is similar. In a paper by the second author written over half a century ago [4] the radical classes of abelian groups which are hereditary

for *pure* subgroups were described and there were some results for other types of subgroups (generalized purity) in [6]. A different characterization of the pure-hereditary radical classes – as kernels of tensor product functors – was later given in [15]. On the other hand, nothing seems to be known in the case of groups. Radical classes of groups which are hereditary for *characteristic* or *fully invariant* subgroups might be worth looking at.

For a radical class $\mathcal{R}$ with semi-simple class $\mathcal{S}$, the hereditariness of $\mathcal{R}$ is equivalent to each of the following.

- (1) $\mathcal{R}(I) = I \cap \mathcal{R}(A)$ for every ideal I of every ring A .
- (2) $\mathcal{S}$ is closed under essential extensions.
- (3) $\mathcal{R}$ is hereditary for essential ideals.

Also the hereditary property is preserved under the lower radical construction in the sense that if $\mathcal{M}$ is a hereditary class, then its lower radical class $L(\mathcal{M})$ is also hereditary.

Among other things we shall seek generalizations of (1)-(3) and the lower radical result to classes with weaker properties. For example the analogue of (1) holds for radical classes which are hereditary for maximal, prime or finite index ideals, while that of (3) holds in the maximal and finite index cases but the prime case remain open. If a class is hereditary for maximal or finite index ideals then its lower radical class has the same property and again the prime case is open. On the other hand, if a class is hereditary for persistent ideals or radically hereditary, its lower radical class need not enjoy the same property. Some of our examples of relatively hereditary radical classes are obtained as intersections of $\mathcal{A}$ -radical classes and hereditary radical classes. Using some results from the literature and such examples we are then able to deduce a couple of “sporadic” results: normal radical classes and lower radical classes defined by zerorings are hereditary for ideals of finite index.

Some types of relative hereditariness can be treated simultaneously by means of the following concept.

For a non-empty class $\mathcal{C}$ of non-zero rings which is hereditary for non-zero ideals an ideal I of a ring A is called a $\mathcal{C}^\leftarrow$ ideal if $A/I \in \mathcal{C}$.

When $\mathcal{C}$ is the class of simple (respectively prime, respectively non-zero finite) rings, the $\mathcal{C}^\leftarrow$ ideals the maximal(respectively prime, respectively finite index) ideals (with the last not including the whole ring).

We prefer not to use the term $\mathcal{C}$ ideal (by analogy with prime ideals and prime rings) because of conflict with the usage “ $\mathcal{R}$ -ideal” meaning an ideal in a radical class $\mathcal{R}$, and to some extent with similar terminology elsewhere. On occasion we impose an extra condition on the class $\mathcal{C}$ – that it be closed under non-zero homomorphic images – in order to prove something, but it remains unknown whether this extra condition is necessary or not.

Recall that an ideal I of a ring A is *essential* if $I \cap J \neq 0$ for every non-zero ideal J of A .

We call an ideal I of a ring A *persistent* if it satisfies the condition

$$A \triangleleft B \Rightarrow I \triangleleft B.$$

Since we shall only consider associative rings, persistent ideals include radicals of rings (by (ADS)), idempotent ideals and semiprime ideals (by the Andrunakievich Lemma). Every ideal of a ring with identity is persistent as a consequence of a ring with identity being a direct summand whenever it's an ideal.

We shall use the following notation:

- $I \triangleleft A$: I is an ideal of A ;
- $I \triangleleft' A$: I is a prime ideal of A ;
- $I \triangleleft_m A$: I is a maximal ideal of A ;
- $I \triangleleft_{fi} A$: I is an ideal of A with finite index;
- $I \triangleleft^\bullet A$: I is an essential ideal of A ;
- $I \triangleleft_p A$: I is a persistent ideal of A .

Expressions such as “ $\mathcal{R}$ is hereditary for maximal ideals” mean, of course, that if A is in $\mathcal{R}$ then so is any maximal ideal of A etc.

A radical class $\mathcal{R}$ is said to be *radically hereditary* (Sands [13]) if for every $A \in \mathcal{R}$ we have $\mathcal{U}(A) \in \mathcal{R}$ for every radical class $\mathcal{U}$.

The term “radical class” is always used in the Kurosh-Amitsur sense of a homomorphically closed class $\mathcal{R}$ of rings such that each ring A has a largest ideal $\mathcal{R}(A) \in \mathcal{R}$ and $\mathcal{R}(A/\mathcal{R}(A)) = 0$ for each A . Equivalently, a radical class is a non-empty class which is homomorphically closed, closed under extensions and closed under unions of ascending chains of ideals,

We also deal with radical classes of abelian groups, which can be defined the same way *mutatis mutandis*, but more conveniently as non-empty classes closed under homomorphic images, extensions and direct sums. For unexplained terms and results pertaining to radical classes of rings, see [9], for abelian groups see [3] and radical classes of abelian groups are discussed in [8].

The following notation will be used at several places in the paper. The (radical) class of torsion abelian groups will be called $\mathcal{T}$ and that of abelian p -groups (p prime) $\mathcal{T}_p$. If a ring A is an algebra over a commutative ring K with identity, $A * K$ will denote the unital extension of A given by the adjunction of the identity of K , i.e. the ring on $A \times K$ with componentwise addition and multiplication given by

$$(a, k)(b, \ell) = (ab + kb + \ell a, k\ell).$$

In several places we need to consider additive groups of rings and ring structures on abelian groups. There we shall use the following notation.

- $(\cdot)^+$: additive group of a ring;
- $(\cdot)^0$: zeroing (all products zero) on the additive group of a ring and more generally, on an abelian group.

All rings are associative and “ring” does not mean “ring with identity”.

2. RESULTS AND EXAMPLES

Proposition 1. *A radical class of rings is hereditary for principal ideals if and only if it is hereditary.*

Proof. If $\mathcal{R}$ is hereditary for principal ideals then for $I \triangleleft A \in \mathcal{R}$, $[i]$ (= principal ideal of A generated by i) is in $\mathcal{R}$ for every $i \in i$ and then $I = \sum_{i \in I} [i] \in \mathcal{R}$. $\square$

The only other case we are aware of where some sort of relative hereditariness implies hereditariness is contained in the next result.

Theorem 1. *The following conditions are equivalent for a radical class $\mathcal{R}$ of rings.*

- (i) $\mathcal{R}$ is hereditary.
- (ii) $\mathcal{R}$ is hereditary for essential ideals.
- (iii) $I \triangleleft A \Rightarrow \mathcal{R}(I) = I \cap \mathcal{R}(A)$.
- (iv) $I \triangleleft^\bullet A \Rightarrow \mathcal{R}(I) = I \cap \mathcal{R}(A)$.

Proof.

(i) $\Rightarrow$ (ii): Clearly.

(ii) $\Rightarrow$ (i): If $\mathcal{R}$ is hereditary for essential ideals and $I \triangleleft A \in \mathcal{R}$, then as $0 \cap I = 0$, Zorn's Lemma ensures that A has an ideal M which is maximal with respect to having zero intersection with I . Then

$$I \cong I/(I \cap M) \cong (I + M)/M \triangleleft^\bullet A/M \in \mathcal{R}.$$

(i) $\Leftrightarrow$ (iii): See [9], p.46, Corollary 3.2.4.

(iii) $\Rightarrow$ (iv): This is clear.

(iv) $\Rightarrow$ (ii): If $I \triangleleft^\bullet A \in \mathcal{R}$, then $\mathcal{R}(I) = I \cap \mathcal{R}(A) = I \cap A = I$, i.e. $I \in \mathcal{R}$. $\square$

If $\mathcal{R}$ is a radical class with semi-simple class $\mathcal{S}$, then $\mathcal{R}$ is hereditary if and only if $\mathcal{S}$ is closed under essential extensions. This result is due to Armendariz [1] and Ryabukhin [12]; see also [9], p.47, Proposition 3.2.6. Hence we have

Corollary 1. *For a radical class $\mathcal{R}$ with semi-simple class $\mathcal{S}$, the following conditions are equivalent.*

- (i) $I \triangleleft^\bullet A$ and $A \in \mathcal{R} \Rightarrow I \in \mathcal{R}$.
- (ii) $I \triangleleft^\bullet A$ and $I \in \mathcal{S} \Rightarrow A \in \mathcal{S}$.

It would be interesting to know if there are any other kinds of ideal that satisfy the analogous "duality". Of course the conditions of Corollary 1 are equivalent to those of Theorem 1. Certainly arbitrary ideals don't satisfy the duality condition. For if $\mathcal{R}$ is a hereditary radical class which is neither $\{0\}$ nor the class of all rings, $\mathcal{S}$ its semi-simple class, let A, B be non-zero rings in $\mathcal{R}, \mathcal{S}$ respectively. Then $B \in \mathcal{S}$ and

$B \triangleleft A \oplus B$ but $A \oplus B \notin \mathcal{S}$ (though $\mathcal{R}$ is hereditary). In the same way, any type of ideal which includes direct summands fails to satisfy the condition.

Before considering some examples, we introduce an idea which makes it possible to treat several relatively hereditary properties simultaneously. For a non-empty class $\mathcal{C}$ of non-zero rings which is hereditary for non-zero ideals, an ideal I of a ring A will be called a $\mathcal{C}^\leftarrow$ ideal if $A/I \in \mathcal{C}$.

For example, if $\mathcal{C}$ is the class of simple (respectively prime, respectively non-zero finite) rings then the $\mathcal{C}^\leftarrow$ ideals are the maximal (respectively prime, respectively finite index) ideals. We exclude the ring 0 from $\mathcal{C}$ so that the ring itself is not a $\mathcal{C}^\leftarrow$ ideal, consistent with the standard usage for maximal and prime ideals.

We now obtain some generalizations of (1) of Section 1.

Theorem 2. *Let $\mathcal{C}$ be a non-empty class of non-zero rings which is hereditary for non-zero ideals. Then a radical class $\mathcal{R}$ is hereditary for $\mathcal{C}^\leftarrow$ ideals if and only if $\mathcal{R}(I) = I \cap \mathcal{R}(A)$ for every $\mathcal{C}^\leftarrow$ ideal I of every ring A .*

Proof. “If” is proved by the usual simple argument. If $A \in \mathcal{R}$ and I is a $\mathcal{C}^\leftarrow$ ideal of A , then $\mathcal{R}(I) = I \cap \mathcal{R}(A) = I \cap A = I$.

“Only if”. Suppose now that $\mathcal{R}$ is hereditary for $\mathcal{C}^\leftarrow$ ideals. If I is a $\mathcal{C}^\leftarrow$ ideal of a ring A , then

$$\mathcal{R}(A)/(I \cap \mathcal{R}(A)) \cong (\mathcal{R}(A) + I)/I \triangleleft A/I \in \mathcal{C}.$$

Hence $\mathcal{R}(A)/(I \cap \mathcal{R}(A)) = 0$ or $\mathcal{R}(A)/(I \cap \mathcal{R}(A)) \in \mathcal{C}$. In the first case, $\mathcal{R}(A) = I \cap \mathcal{R}(A) \triangleleft I$ so $\mathcal{R}(A) \subseteq \mathcal{R}(I)$ and hence

$$\mathcal{R}(A) = \mathcal{R}(I) = I \cap \mathcal{R}(I) = I \cap \mathcal{R}(A).$$

In the second case, $I \cap \mathcal{R}(A)$ is a $\mathcal{C}^\leftarrow$ ideal of $\mathcal{R}(A) \in \mathcal{R}$, so by assumption, $I \cap \mathcal{R}(A)$ is in $\mathcal{R}$. Also $I \cap \mathcal{R}(A) \triangleleft I$ so

$$I \cap \mathcal{R}(A) \subseteq \mathcal{R}(I) \subseteq I \cap \mathcal{R}(A).$$

Hence $\mathcal{R}(I) = I \cap \mathcal{R}(A)$. □

From our theorem above we can now deduce

Corollary 2. *If a radical class $\mathcal{R}$ is hereditary for maximal (respectively prime, respectively finite index) ideals, then $\mathcal{R}(I) = I \cap \mathcal{R}(A)$ for every maximal (respectively prime, respectively finite index) ideal I of every ring A .*

We have a generalization of (2) of Section 1 using classes $\mathcal{C}$, but we may need an extra condition on $\mathcal{C}$.

Theorem 3. *If $\mathcal{C}$ is a class of non-zero rings which is hereditary for non-zero ideals and closed under non-zero homomorphic images, then a radical class $\mathcal{R}$ is hereditary for $\mathcal{C}^\leftarrow$ ideals if and only if it is hereditary for essential $\mathcal{C}^\leftarrow$ ideals.*

Proof. If $\mathcal{R}$ is hereditary for essential $C^\leftarrow$ ideals and I is a $C^\leftarrow$ ideal of $A \in \mathcal{R}$, then as $0 \cap I = 0$, Zorn's Lemma ensures that A has an ideal M which is maximal with respect to having zero intersection with I . Then

$$I \cong I/(I \cap M) \cong (I + M)/M \triangleleft^\bullet A/M \in \mathcal{R}. \quad (*)$$

and $(A/M)/((I + M)/M) \in C$ or $A/M = (I + M)/M$, and hence $A = I + M = I \oplus M$. In the former case $I \in \mathcal{R}$ by $(*)$ and in the latter, I as a direct summand of A , is again in $\mathcal{R}$. $\square$

Problem 1. *Do we need the extra condition in Theorem 3?*

Note that special classes of prime rings satisfy the original conditions imposed on C in Theorem 2 and non-empty classes of simple rings satisfy the conditions of C in Theorem 3. On the other hand, the approach based on classes C has its limitations, as some types of ideals can't be defined in terms of classes of rings.

Proposition 2. *There is no class C of rings defining either*

- (i) *the essential ideals or*
 - (ii) *the persistent ideals*
- as the $C^\leftarrow$ ideals.*

Proof.

- (i): Suppose the essential ideals are the $C^\leftarrow$ ideals for some C . Then for any prime p we have $p\mathbb{Z} \triangleleft^\bullet \mathbb{Z}$ so $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z} \in C$. But for any ring R , we have $R \oplus \mathbb{Z}_p/R \cong \mathbb{Z}_p$, while R is a non-essential ideal.
- (ii): Suppose there is a class C such the persistent ideals are the $C^\leftarrow$ ideals. Let $A = \mathbb{Z} \oplus (\mathbb{Q}/\mathbb{Z})^0$. Then $\mathbb{Z}$ is an idempotent ideal of A and therefore a persistent ideal, so $(\mathbb{Q}/\mathbb{Z})^0 \in C$.

Now in $\mathbb{Q}[X]$ let

$$I = \{a_1X + a_2X^2 + \dots : a_1 \in \mathbb{Z}, a_2, \dots \in \mathbb{Q}\}.$$

Then I is an ideal of (X) , the ideal of $\mathbb{Q}[X]$ generated by X . Define

$$f : (X) \rightarrow \mathbb{Q}^+/\mathbb{Z}^+; \quad b_1X + b_2X^2 + \dots \mapsto b_1 + \mathbb{Z}.$$

This a surjective abelian group homomorphism with kernel I . If

$$\gamma = c_1X + c_2X^2 + \dots, \quad \delta = d_1X + d_2X^2 + \dots \in (X),$$

then

$$\gamma\delta = c_1d_1X^2 + (c_1d_2 + c_2d_1)X^3 + \dots,$$

so $f(\gamma\delta) = 0$. If we now view $\mathbb{Q}/\mathbb{Z}$ as $(\mathbb{Q}/\mathbb{Z})^0$ we have $f(\gamma\delta) = 0 = f(\gamma)f(\delta)$ for all $\gamma, \delta \in (X)$ so f becomes a ring homomorphism with kernel I and $(X)/I \cong (\mathbb{Q}/\mathbb{Z})^0$, so I is a $C^\leftarrow$ ideal of (X) . But $(X) \triangleleft \mathbb{Q}[X]$ and, e.g. $3X \in I$ but $\frac{1}{2}3X = \frac{3}{2}X \notin I$ (though $\frac{1}{2} \in \mathbb{Q}[X]$). Hence I is not an ideal of $\mathbb{Q}[X]$ and consequently not a persistent ideal of (X) .

□

By Theorem 1 radical classes which are hereditary for essential ideals satisfy the conclusion of Theorem 3, so by Proposition 2 the existence of a class $\mathcal{C}$ as in Theorem 3 is not *necessary* for the conclusion of that result.

Problem 2. *Do the persistent ideals satisfy the conclusion of Theorem 3, i.e. if a radical class is hereditary for essential persistent ideals, must it be hereditary for all persistent ideals?*

Proposition 3. *If a radical class $\mathcal{R}$ contains $\mathbb{Z}$ and is hereditary for persistent ideals, then it is hereditary.*

Proof. If $A \in \mathcal{R}$ then $A * \mathbb{Z} \in \mathcal{R}$ and if $I \triangleleft A$ then I is a persistent ideal of $A * \mathbb{Z}$, so I is in $\mathcal{R}$. □

Proposition 4. *Let $\mathcal{R}$ be a radical class consisting of idempotent rings such that each ring in $\mathcal{R}$ is an ideal of a ring with identity in $\mathcal{R}$. If $\mathcal{R}$ is hereditary for persistent ideals then it is hereditary.*

Proof. If $A \in \mathcal{R}$, let $A \triangleleft B$ where B has an identity and is in $\mathcal{R}$. For an ideal I of A , let I^* be the ideal of B generated by I . Since B has an identity, I^* is a persistent ideal and hence is in $\mathcal{R}$. But then I^* is idempotent, while by the Andrunakievich Lemma $(I^*)^3 \subseteq I$. Hence $I = I^* \in \mathcal{R}$. □

This is well and good, but we have not yet given any examples of relatively hereditary radical classes which are not hereditary. To produce some examples we begin with *A-radical classes*.

For every radical class $\mathcal{U}$ of abelian groups we get a radical class $\mathcal{U}^*$ of rings by defining

$$\mathcal{U}^* = \{A : A^+ \in \mathcal{U}\}.$$

A radical class $\mathcal{R}$ of rings is called an *A-radical class* it satisfies the condition

$$A \in \mathcal{R} \text{ and } A^+ \cong B^+ \Rightarrow B \in \mathcal{R}.$$

The *A-radical classes* are precisely the classes $\mathcal{U}^*$ defined above. For all this see [9], pp.165-166.

Let S be a set of primes, S^* the multiplicative semigroup generated by S . An abelian group G is *S-divisible* if $nG = G$ for all $n \in S^*$, i.e. if it satisfies the condition

$$((\forall g \in G)(\forall n \in S^*))(\exists g_n \in G)(ng_n = g).$$

Clearly G is *S-divisible* if and only if $pG = G$ for every $p \in S$.

A subgroup H of an abelian group G is *S-pure* if $nH = H \cap nG$ for all $n \in S^*$. This is *not* equivalent to the condition $pH = H \cap pG$ for all $p \in S$; cf. *neat* subgroups [3]. We call an abelian group *S-torsion* if each of its elements has order in S^* and *S-torsion-free* if none of its non-zero elements has order in S^* . When S is the set of

all primes we get the standard notions *divisible group*, *pure subgroup*, *torsion* and *torsion-free group*.

Example 1. The class $\mathcal{D}_S$ of S -divisible abelian groups is a radical class of abelian groups. We shall consider the $\mathcal{A}$ -radical class $\mathcal{D}_S^*$.

If $I \triangleleft_m R \in \mathcal{D}_S^*$, then R/I is a simple ring in $\mathcal{D}_S^*$. Only simple rings of characteristic 0 or a prime not in S are additively S -divisible, and these are also additively S -torsion-free. But then I^+ is an S -pure subgroup of R^+ and therefore S -divisible (since R^+ is). Hence $I \in \mathcal{D}_S^*$, so that $\mathcal{D}_S^*$ is hereditary for maximal ideals.

But unless $S = \emptyset$, $\mathcal{D}_S^*$ is not hereditary; let $\mathbb{Z}^0, \mathbb{Q}^0$ be the zerorings on $\mathbb{Z}^+, \mathbb{Q}^+$ respectively. Then $\mathbb{Z}^0 \triangleleft \mathbb{Q}^0 \in \mathcal{D}_S^*$, but $\mathbb{Z}^0 \notin \mathcal{D}_S^*$.

Clearly the intersection of two radical classes which are hereditary for maximal ideals is itself hereditary for maximal ideals. This gives us a family of examples.

If $\mathcal{R}$ is a hereditary radical class of rings then $\mathcal{R} \cap \mathcal{D}_S^$ is hereditary for maximal ideals.*

We need to be a little cautious with this example. If $\mathcal{R}$ contains all nilpotent rings, or equivalently, if $\mathbb{Z}^0 \in \mathcal{R}$, then we can use the $\mathbb{Z}^0, \mathbb{Q}^0$ example to see that $\mathcal{R} \cap \mathcal{D}_S^*$ is not hereditary. On the other hand, for the radical class $\mathcal{V}$ of *regular rings*, $\mathcal{V} \cap \mathcal{D}_S^*$ is also the class of S -torsion-free regular rings and this is hereditary. (See, e.g. [3] for the additive structure of regular rings.)

Example 2. The radical class $\mathcal{D}_S^*$ is hereditary for prime ideals. For if $I \triangleleft A \in \mathcal{D}_S^*$, then A/I is an S -divisible divisible prime ring. But its S -torsion ideal is then contained in the annihilator and hence is zero. Thus A/I is additively S -torsion-free and we can argue as in the previous example.

Example 3. The class $\mathcal{D}_S^*$ is hereditary for ideals of finite index. For if $I \triangleleft A \in \mathcal{D}_S^*$ and A/I is finite, then as $A/I \in \mathcal{D}_S^*$, it must be $\{p : p \notin S\}$ -torsion and hence S -torsion-free.

As with Example 2 we see that for every hereditary radical class $\mathcal{R}$, the class $\mathcal{R} \cap \mathcal{D}_S^$ is hereditary for prime ideals and for ideals of finite index.*

Problem 3. *Is there a radical class which is hereditary for prime ideals but not maximal ideals or the other way around?*

Using a method like that used for the divisible radical and its relatives, we can get a family of radical classes which are hereditary for ideals of finite index. We shall make use of the following two assertions concerning abelian groups.

- (α) *If B is a subgroup of a torsion-free abelian group C and C/B is a torsion group, then every radical class $\mathcal{U}$ of abelian groups which contains B also contains C . (Corollary 1.2 of [5])*
- (β) *If a radical class of abelian groups contains a group L , it also contains its torsion subgroup. (Theorem 5.2 of [2])*

The proof of the following result is based on arguments in [5] used to prove that radical classes of abelian groups are closed under quasi-isomorphisms.

Theorem 4. *Every radical class $\mathcal{U}$ of abelian groups is hereditary for subgroups of finite index.*

Proof. Let $\mathcal{U}$ be a radical class of abelian groups, G a group in $\mathcal{U}$ and H a subgroup with $|G/H| = n$ (finite). Then $\mathcal{T}(G) \in \mathcal{U}$ by (β) and $\mathcal{T}(H) = H \cap \mathcal{T}(G) \subseteq \mathcal{T}(G)$. Also

$$\mathcal{T}(G)/\mathcal{T}(H) = \mathcal{T}(G)/\mathcal{T}(G) \cap H \cong (\mathcal{T}(G) + H)/H \subseteq G/H,$$

so $|\mathcal{T}(G)/\mathcal{T}(H)| \leq |G/H| = n$, and thus $n\mathcal{T}(G) \subseteq \mathcal{T}(H)$. From this it is clear that $n\mathcal{T}_p(G) \subseteq \mathcal{T}_p(H)$ for every prime p , where $\mathcal{T}_p(G)$ is the largest p -subgroup of G and so on. If $\mathcal{U}$ contains no non-zero p -groups, then by (β) , $\mathcal{T}_p(G) = 0$ and thus $\mathcal{T}_p(H) = 0 \in \mathcal{U}$. If $\mathcal{U}$ contains only the divisible p -groups, then

$$\mathcal{T}_p(G) = n\mathcal{T}_p(G) \subseteq \mathcal{T}_p(H) \subseteq \mathcal{T}_p(G),$$

so $\mathcal{T}_p(H) = \mathcal{T}_p(G) \in \mathcal{U}$ (by (β) again). The remaining possibility is that $\mathcal{U}$ contains all p -groups ([2], Theorem 2.6) and then $\mathcal{T}_p(H) \in \mathcal{U}$ anyway. Hence $\mathcal{T}(H) \in \mathcal{U}$.

Now $H/\mathcal{T}(H) = H/(H \cap \mathcal{T}(G)) \cong (H + \mathcal{T}(G))/\mathcal{T}(G)$ so we have an exact sequence

$$0 \rightarrow H/\mathcal{T}(H) \rightarrow G/\mathcal{T}(G) \rightarrow G/(H + \mathcal{T}(G)) \rightarrow 0.$$

Since G is in $\mathcal{U}$, so also is $G/(H + \mathcal{T}(G))$. Also $|G/(H + \mathcal{T}(G))| \leq |G/H| = n$.

Let f denote the monomorphism from our exact sequence. Then $|(G/\mathcal{T}(G))/f(H/\mathcal{T}(H))| \leq n$. Hence

$$G/\mathcal{T}(G) \cong n(G/\mathcal{T}(G)) \subseteq f(H/\mathcal{T}(H)),$$

the isomorphism being given by “multiplication by n ”. (This is an isomorphism since $G/\mathcal{T}(G)$ is torsion-free.) Clearly $nf(H/\mathcal{T}(H)) \subseteq n(G/\mathcal{T}(G))$, so in the exact sequence

$$0 \rightarrow n(G/\mathcal{T}(G)) \rightarrow f(H/\mathcal{T}(H)) \rightarrow f(H/\mathcal{T}(H))/n(G/\mathcal{T}(G)) \rightarrow 0$$

the first two terms are torsion-free and the third is a torsion group. Moreover $n(G/\mathcal{T}(G)) \cong G/\mathcal{T}(G) \in \mathcal{U}$, so by (α) we have $H/\mathcal{T}(H) \cong f(H/\mathcal{T}(H)) \in \mathcal{U}$. Since, as shown above, $\mathcal{T}(H) \in \mathcal{U}$, we have $H \in \mathcal{U}$ and the theorem is proved. $\square$

Corollary 3. *Every A-radical class $\mathcal{U}^*$ is hereditary for ideals of finite index.*

As with S -divisible rings and maximal ideals, we can now give further examples of radical classes which are hereditary for ideals of finite index.

Example 4. For every hereditary radical class $\mathcal{R}$ of rings and every A-radical class $\mathcal{U}^*$, $\mathcal{R} \cap \mathcal{U}^*$ is a radical class which is hereditary for ideals of finite index.

A radical class $\mathcal{R}$ is *normal* if for every Morita context (A, V, W, B) we have $V\mathcal{R}(B)W \subseteq \mathcal{R}(A)$. An *N-radical* class is a radical class which contains all nilpotent rings and is left and right hereditary and left and right strong. For a discussion of these concepts see pp. 149-158 of [9]. Note that *N-radical* classes are hereditary.

Jaegermann and Sands have shown that every normal radical class is an intersection of an *N-radical class* and an *A-radical class*. ([11], Theorem 10). Thus as a consequence we have

Theorem 5. *Every normal radical class is hereditary for ideals of finite index.*

It was shown in [7] that the lower radical class defined by any class of zerorings is the intersection of an *A-radical class* with the prime (= Baer) radical class. This gives us another result.

Theorem 6. *Every class of zerorings defines a lower radical class that is hereditary for ideals of finite index.*

3. THE LOWER RADICAL CONSTRUCTION

It was first shown by Hoffman and Leavitt [10] that if a class $\mathcal{M}$ of rings is hereditary, then so is its lower radical class $L(\mathcal{M})$. Along with (1) – (3) of Section 1, the possibility of generalizing *this* result for relatively hereditary classes is a significant question.

Here we show that a class which is hereditary for $C^{\leftarrow}$ ideals determines a lower radical class with the same property, but at the expense of an extra assumption about C . In the final section of the paper we shall see examples in which relative hereditary properties are not preserved under the lower radical construction.

Recall the Tangeman-Kreiling lower radical construction ([14]; see also [9], pp.28-29).

Let $\mathcal{M}$ be a class of rings. We inductively define, for each ordinal γ , a class $\mathcal{M}_\gamma$ as follows.

$\mathcal{M}_1$ is the homomorphic closure of $\mathcal{M}$.

$\mathcal{M}_\mu$ being defined for all $\mu < \lambda$,

- (i) if λ is a successor $\nu + 1$, let $\mathcal{M}_\lambda$ be the class of rings B with an ideal $I \in \mathcal{M}_\nu$ such that also $B/I \in \mathcal{M}_\nu$, and
- (ii) if λ is a limit, let $\mathcal{M}_\lambda$ consist of all rings which are unions of ascending chains of ideals from various $\mathcal{M}_\mu$ with $\mu < \lambda$.

Then the lower radical class $L(\mathcal{M})$ is the union of all the $\mathcal{M}_\gamma$.

Theorem 7. *Let C be a non-empty class of non-zero rings which is hereditary for non-zero ideals and **closed under non-zero homomorphic images**. If a class $\mathcal{M}$ of rings is hereditary for $C^{\leftarrow}$ ideals, then so is its lower radical class $L(\mathcal{M})$.*

Proof. If $A/I \in \mathcal{M}_1$ ($A \in \mathcal{M}$) and J/I is a $\mathcal{C}^{\leftarrow}$ ideal of A/I , then $A/J \cong (A/I)/(J/I) \in \mathcal{C}$ so J is a $\mathcal{C}^{\leftarrow}$ ideal of A , whence $J \in \mathcal{M}$, $J/I \in \mathcal{M}_1$ and finally $\mathcal{M}_1$ is hereditary for $\mathcal{C}^{\leftarrow}$ ideals.

If $\mathcal{M}_\gamma$ is hereditary for $\mathcal{C}^{\leftarrow}$ ideals and $B \in \mathcal{M}_{\gamma+1}$, let A be an ideal of B such that both A and B/A are in $\mathcal{M}_\gamma$. If I is a $\mathcal{C}^{\leftarrow}$ ideal of B , then

$$A/A \cap I \cong (A+I)/I \triangleleft B/I \in \mathcal{C},$$

so $A/A \cap I \in \mathcal{C}$ or $A/A \cap I = 0$. In the former case $A \cap I$ is a $\mathcal{C}^{\leftarrow}$ ideal of $A \in \mathcal{M}_\gamma$, so $A \cap I \in \mathcal{M}_\gamma$. In the latter, $A \cap I = A \in \mathcal{M}_\gamma$ so in any case we have $A \cap I \in \mathcal{M}_\gamma$.

Now

$$B/I \in \mathcal{C}, \text{ so } B/(I+A) \in \mathcal{C} \text{ or } B/(I+A) = 0. \quad (\bullet)$$

If it is in $\mathcal{C}$, then $(I+A)/A$ is a $\mathcal{C}^{\leftarrow}$ ideal of $B/A \in \mathcal{M}_\gamma$ whence

$$I/A \cap I \cong (I+A)/A \in \mathcal{M}_\gamma.$$

But then since $A \cap I \in \mathcal{M}_\gamma$, it follows that $I \in \mathcal{M}_{\gamma+1}$. If, on the other hand, $B/(I+A) = 0$, then

$$I/A \cap I \cong (I+A)/A = B/A \in \mathcal{M}_\gamma,$$

so again I is in $\mathcal{M}_{\gamma+1}$. This class is therefore hereditary for $\mathcal{C}^{\leftarrow}$ ideals.

Now let λ be a limit ordinal such that $\mathcal{M}_\mu$ is hereditary for $\mathcal{C}^{\leftarrow}$ ideals for all $\mu < \lambda$. Let B be in $\mathcal{M}_\lambda$. Then $B = \bigcup I_x$ where the I_x form a chain of ideals of B , and for each x , $I_x \in \mathcal{M}_{\mu_x}$ for some $\mu_x < \lambda$. If J is a $\mathcal{C}^{\leftarrow}$ ideal of B , then $J = \bigcup J \cap I_x$ where each $J \cap I_x \triangleleft J$. Now

$$I_x/I_x \cap J \cong (I_x+J)/J \triangleleft B/J \in \mathcal{C},$$

so $I_x/J \cap I_x \in \mathcal{C}$ or $I_x/J \cap I_x = 0$. If the former, then $J \cap I_x$ is a $\mathcal{C}^{\leftarrow}$ ideal of $I_x \in \mathcal{M}_{\mu_x}$ and so $J \cap I_x \in \mathcal{M}_{\mu_x}$. On the other hand, if $I_x/J \cap I_x = 0$, then

$$J \cap I_x = I_x \in \mathcal{M}_{\mu_x}.$$

Since $J \cap I_x \in \mathcal{M}_{\mu_x}$ for all x , J is in $\mathcal{M}_\lambda$.

We have now shown that each $\mathcal{M}_\gamma$ is hereditary for $\mathcal{C}^{\leftarrow}$ ideals, and hence the same is true of the lower radical class $L(\mathcal{M})$. □

Again it's unclear whether we really need the extra hypothesis of non-zero homomorphic closure; $(\bullet)$ indicates the only point of the proof at which we used it. Meanwhile, as with some other questions, the case of prime ideals remains open. In the final section we shall show that some relative properties are not preserved by the lower radical construction.

4. CONNECTIONS BETWEEN RELATIVE HEREDITARY PROPERTIES

We saw earlier that a radical class is hereditary for essential ideals (respectively principal ideals) if and only if it is hereditary. If a radical class is hereditary for maximal ideals, must it be hereditary for prime ideals? This, and similar questions will be our concern in this section. Our first result connects maximal and finite index ideals, and its proof relies on several applications of a lemma which we shall soon prove. First, though, we need some terminology.

For a property $(*)$ of ideals, let $C(*)$ be the class of rings B such that if $\mathcal{R}$ is a radical class which is hereditary for $(*)$ -ideals, $I \triangleleft A \in \mathcal{R}$ and $A/I \cong B$, then $I \in \mathcal{R}$. Then by definition, any radical class which is hereditary for $(*)$ -ideals is hereditary for $C(*)$ -ideals.

This notation will be retained in the following result and its proof.

Lemma 1. *Let B be a ring with a finite series*

$$0 = J_0 \subseteq J_1 \subseteq \cdots \subseteq J_n \subseteq J_{n+1} = B$$

of ideals such that each $J_{i+1}/J_i \in C()$. Then $B \in C(*)$.*

Proof. Supposing the assertion is true for $n = k$, consider a chain

$$0 = L_0 \subseteq L_1 \subseteq \cdots \subseteq L_k \subseteq L_{k+1} \subseteq L_{k+2} = A$$

of ideals of a ring A such that each $L_{i+1}/L_i \in C(*)$. Then

$$0 = L_0 \subseteq L_1 \subseteq \cdots \subseteq L_k \subseteq L_{k+1}$$

is a chain of ideals of L_{k+1} and all the relevant factors are in $C(*)$, so $L_{k+1} \in C(*)$. Also $A/L_{k+1} = L_{k+2}/L_{k+1} \in C(*)$.

Let $\mathcal{R}$ be a radical class which is hereditary for $(*)$ -ideals. If $I \triangleleft R \in \mathcal{R}$ and $R/I \cong A$, then R has an ideal S such that $S/I \cong L_{k+1}$ and $(R/I)/(S/I) \cong A/L_{k+1}$. Hence $R/S \cong A/L_{k+1} \in C(*)$, so $S \in \mathcal{R}$. But then as $S/I \cong L_{k+1} \in C(*)$ it follows that $I \in \mathcal{R}$. This means that $A \in C(*)$. Since the case $n = 0$ is clear, we have the result by induction. $\square$

Corollary 4. *If $B_1, B_2, \dots, B_n \in C(*)$ (n finite), then $B_1 \oplus B_2 \oplus \cdots \oplus B_n \in C(*)$.*

Theorem 8. *Every radical class which is hereditary for maximal ideals is also hereditary for ideals of finite index.*

Proof. In the notation of the previous two results, let $(*)$ be the property of maximality. Then we have to show that all finite rings are in $C(*)$. We do this by combining a structure theorem for finite rings with multiple uses of the lemma. Recall that a finite ring A has a largest nilpotent ideal N and A/N is isomorphic to a finite direct sum of matrix rings over finite fields.

(i): *For each prime p , the zeroing on the cyclic group of order p^n is in $C(*)$ for $n = 1, 2, 3, \dots$*

If a is a group element of order p^n then we have a series

$$0 = \langle p^n a \rangle \subseteq \langle p^{n-1} a \rangle \subseteq \cdots \subseteq \langle pa \rangle \subseteq \langle a \rangle$$

of cyclic groups with each $\langle p^{i-1} a \rangle / \langle p^i a \rangle$ cyclic of order p , so the corresponding series

$$0 = \langle p^n a \rangle^0 \subseteq \langle p^{n-1} a \rangle^0 \subseteq \cdots \subseteq \langle pa \rangle^0 \subseteq \langle a \rangle^0$$

of ideals of the zeroring $\langle a \rangle^0$ has simple factors which are in $C(*)$ whence also $\langle a \rangle^0 \in C(*)$.

(ii): Every finite p -zeroring is in $C(*)$, as each such ring is a direct sum of finitely many zerorings on cyclic p -groups.

If R is a nilpotent p -ring with $R^n = 0$, we have a series

$$0 = R^n \subseteq R^{n-1} \subseteq R^2 \subseteq R$$

of ideals of R for which all factors are zerorings, so

(iii): all finite nilpotent p -rings are in $C(*)$ for every prime p .

Every finite nilpotent ring is a direct sum of nilpotent p -rings for finitely many primes p . Hence

(iv): every finite nilpotent ring is in $C(*)$.

As matrix rings over fields are simple,

(v): all finite direct sums of matrix rings over finite fields are in $C(*)$.

The result now follows from (iv), (v) and the lemma. $\square$

We do not know whether radical classes which are hereditary for maximal ideals must be hereditary for prime ideals or *vice versa*. Adding ideals of finite index to the mix gives us the following result.

Theorem 9. *If a radical class $\mathcal{R}$ is hereditary for both prime ideals and ideals of finite index, then $\mathcal{R}$ is hereditary for maximal ideals.*

Proof. If $I \triangleleft_m A \in \mathcal{R}$ then A/I simple so is prime or a finite zeroring and either implies that $I \in \mathcal{R}$. $\square$

Theorem 10. *A radical class which is hereditary for ideals of finite index need not be hereditary for prime ideals.*

Proof. Let $\mathcal{R}$ be the upper radical class defined by $\{\mathbb{Z}^0\}$. Since $\mathbb{Z}^0$ is isomorphic to all its non-zero ideals, $\mathcal{R}$ is the class of rings which don't have $\mathbb{Z}^0$ as a homomorphic image or, alternatively, the class of rings with no non-zero homomorphisms to $\mathbb{Z}^0$.

Let $I \triangleleft A \in \mathcal{R}$ with $n(A/I) = 0$ for some integer $n \neq 0$ and suppose $I \notin \mathcal{R}$. Then (using (ADS)!) we have $I/T^*(I) \triangleleft A/T^*(I) \in \mathcal{R}$ and every non-zero homomorphism from I to $\mathbb{Z}^0$ must factor through $I/T^*(I)$ which accordingly is not in $\mathcal{R}$. Consequently we lose no generality by assuming that I is additively torsion-free.

Now $I \cap \mathcal{T}^*(A) = \mathcal{T}^*(I) = 0$. It follows that A has an ideal M which contains $\mathcal{T}^*(A)$ and is maximal with respect to having zero intersection with I . Then

$$I \cong I/I \cap M \cong (I+M)/M \triangleleft^\bullet A/M \in \mathcal{R}$$

and

$$|(A/M)/((I+M)/M)| = |A/(I+M)| \leq |A/I| = n.$$

Moreover, as I is in the semi-simple class corresponding to the hereditary radical class $\mathcal{T}^*$, its essential extension A/M is also additively torsion-free. Thus we may assume that A is torsion-free also.

Now $nA \subseteq I$, so $nA \triangleleft I$. Hence there is a surjective homomorphism $f : nA \rightarrow \mathbb{Z}^0$ (since only the zero map from I to $\mathbb{Z}^0$ can take nI to 0, $\mathbb{Z}^0$ being torsion-free, and $nI = nA$ as I^+ is a pure subgroup of A^+). Define $g : A \rightarrow \mathbb{Z}^0$ by setting $g(a) = f(na)$ for all $a \in A$. For all $a, b \in A$ we have $g(a+b) = f(n(a+b)) = f(na+nb) = f(na) + f(nb) = g(a) + g(b)$. As well, $ng(ab) = nf(nab) = f(n^2ab) = f(na \cdot nb) = f(na)f(nb) = 0$ ($\mathbb{Z}^0$ is a zeroring!). But $\mathbb{Z}^0$ is torsion-free, so $g(ab) = 0 = g(a)g(b)$. Thus g is a homomorphism (surjective in fact) from $A \in \mathcal{R}$ to $\mathbb{Z}^0$ and with this contradiction we see that $\mathcal{R}$ is indeed hereditary for ideals of finite index.

Consider now the ring $\mathbb{Z}^0 * \mathbb{Z}$ obtained by the adjunction of the identity of $\mathbb{Z}$ to $\mathbb{Z}^0$. As a non-zero ring with identity, this is in $\mathcal{R}$, but it has $\mathbb{Z}^0$ as a prime ideal. Hence $\mathcal{R}$ is not hereditary for prime ideals. $\square$

The classes $\mathcal{D}_S^*$ with $S \neq \emptyset$ have been useful to us and enjoy many relatively hereditary properties. We end this section by showing that $\mathcal{D}_S^*$ is not hereditary for persistent ideals, though it is radically hereditary.

- Proposition 5.** (i) If $A \in \mathcal{D}_S^*$ and I is an idempotent ideal of A , then $I \in \mathcal{D}_S^*$.
(ii) $\mathcal{D}_S^*$ is radically hereditary.
(iii) $\mathcal{D}_S^*$ is hereditary for semiprime ideals.
(iv) $\mathcal{D}_S^*$ is not hereditary for persistent ideals.

Proof.

(i): If

$$I^2 = I \triangleleft A \in \mathcal{D}_S^*$$

and $x \in I$, Then $x = i_1 j_1 + i_2 j_2 + \cdots + i_n j_n$ for some $i_1, j_1, \dots, i_n, j_n \in I$. For $m \in S^*$, by the S -divisibility of A there exist $a_1, a_2, \dots, a_n \in A$ such that $j_1 = ma_1, j_2 = ma_2, \dots, j_n = ma_n$ and then

$$x = m(i_1 a_1 + i_2 a_2 + \cdots + i_n a_n),$$

where the bracketed term is in I . Thus I is S -divisible.

(ii): This follows from Lemma 16 of [11].

(iii): Like the proof in Example 2.

(iv): The zeroring $\mathbb{Z}(p^\infty)^0$ is in $\mathcal{D}_S^*$, but for $p \in S$, $\mathbb{Z}(p)^0 \cong \{x \in \mathbb{Z}(p^\infty)^0 : px = 0\}$ is not, though it is a persistent ideal.

□

5. A USEFUL GROUP

Let V be a two dimensional $\mathbb{Q}$ -vector space with basis $\{x, y\}$. We define our group G as a subgroup of V . Let

$$G = \langle p^{-\infty}x, q^{-\infty}y, t^{-\infty}(x+y) \rangle,$$

where $\langle . \rangle$ means “group generated by”, p, q and t are distinct primes and $p^{-\infty}$ is shorthand for a list of all negative powers of p and so on.

For a set S of primes, let $\mathbb{Q}(S) = \{\frac{m}{n} : m \in \mathbb{Z}, n \in S^*\}$. When S has a single element p we write $\mathbb{Q}(p)$ rather than $\mathbb{Q}(\{p\})$ and when S is the set of all primes, we write $\mathbb{Q}$ as usual. We shall first examine some radical classes of abelian groups and then pass to radical classes of rings *via* A-radicals.

Clearly $\langle p^{-\infty}x \rangle$ is p -divisible. On the other hand, routine calculations show that y and $x+y$ have zero p -height. Thus G is not p -divisible, so as it has rank 2 we see that

$$\mathbb{Q}(p) \cong \langle p^{-\infty}x \rangle = \mathcal{D}_p(G).$$

Moreover, $\mathcal{D}_p(G)$ is a pure subgroup (by direct calculation or reference to Proposition 1.1 of [5]), so $G / \langle p^{-\infty}x \rangle = \langle q^{-\infty}\bar{y}, t^{-\infty}\bar{y} \rangle$ where $\bar{y}$ is the coset of y . This is torsion-free of rank 1 and some more routine calculation shows that it is isomorphic to $\mathbb{Q}(\{q, t\})$. Summarizing, then, we have an exact sequence

$$0 \rightarrow \mathbb{Q}(p) \rightarrow G \rightarrow \mathbb{Q}(\{q, t\}) \rightarrow 0.$$

Let $\mathcal{U}$ denote the lower radical class defined by $\{\mathbb{Q}(p), \mathbb{Q}(\{q, t\})\}$. Then $G \in \mathcal{U}$, but by arguments like those above for p , we have $\mathbb{Q}(q) \cong \mathcal{D}_q(G)$. Comparing the types of the rational groups concerned, we see that $\mathcal{D}_q(G) \notin \mathcal{U}$.

By considering A-radicals, we can use the foregoing to get an informative example.

Example 5. For $\mathcal{U}$ etc. as above we have

$$G^0 \in \mathcal{U} \text{ but } \mathcal{D}_q^*(G^0) \cong \mathbb{Q}(q) \notin \mathcal{U}^*,$$

so $\mathcal{U}^*$ is not radically hereditary. Furthermore, for primes $s \neq p$ (including q and t) $\mathcal{U}$ contains non-zero cyclic s -groups and therefore all s -groups. Similarly it contains all s -groups for $s \neq q, t$, including p . It follows that $\mathcal{T} \subseteq \mathcal{U}$, where $\mathcal{T}$ is the class of torsion groups. Thus for abelian group radicals, $\mathcal{U}$ is the lower radical class defined by the homomorphically closed class $\{\mathbb{Q}(p), \mathbb{Q}(\{q, t\})\} \cup \mathcal{T}$.

Proposition 6. *For a non-empty set S of primes, every non-zero persistent ideal of $\mathbb{Q}(S)^0$ is isomorphic to $\mathbb{Q}(S)^0$.*

Proof. Let $\mathbb{Q}(S)^*$ denote the subring of the rationals whose additive group is $\mathbb{Q}(S)$. If $H^0 \neq 0$ is a persistent ideal of $\mathbb{Q}(S)^0$ then $H^0 \triangleleft \mathbb{Q}(S)^0 * (\mathbb{Q}(S))^*$. But then, for $h \in H$,

$p \in S$ and $n \in \mathbb{Z}^+$ we have (writing elements of $\mathbb{Q}(S)^0 * \mathbb{Q}(S)^*$ as ordered pairs for clarity),

$$\frac{h}{p^n} = (h, 0)(0, p^{-n}) \in H^0,$$

so H is S -divisible and therefore has type at least as great of that of $\mathbb{Q}(S)$. (For types, see [3].) The types must therefore be equal so that $H \cong \mathbb{Q}(S)$. $\square$

As $\mathcal{T}$ is hereditary, the class $\{\mathbb{Q}(p), \mathbb{Q}(\{q, t\})\} \cup \mathcal{T}$ is hereditary for persistent ideals, but its lower radical class $\mathcal{U}$ is not radically hereditary (and hence not hereditary for persistent ideals). Turning to ring radicals we see that the lower radical class $L(\{H^0 : H \in \mathcal{U}\})$ is not radically hereditary though it is determined by a subclass which is hereditary for persistent ideals. This gives us a two-part theorem.

Theorem 11. *A class of rings which is radically hereditary (respectively hereditary for persistent ideals) may define a lower radical class which is not.*

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NEW ASYMPTOTICALLY ISOMETRIC PROPERTIES THAT IMPLY THE FAILURE OF THE FIXED POINT PROPERTY IN COPIES OF ℓ^1

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Abstract. In this study, we introduce three new notions which may occur for some Banach spaces. We call these new properties AAI1, AAI2 and AAI3 where AAI stands for “alternative asymptotically isometric”. We prove that if a Banach space has any of them, then it fails to have the fixed point property for nonexpansive mappings. We provide alternative ways of detecting if a Banach space has any of these properties. We show that AAI1 is an equivalent property for a Banach space to have an asymptotically isometric copy of ℓ^1 . That is, a Banach space contains an asymptotically isometric copy of ℓ^1 if and only if it has the property AAI1. In fact, we obtain generalized version of property AAI1, which we call property AAI3, obtained as a conclusion of property AAI2. We prove that all properties AAI1, AAI2 and AAI3 are equivalent. We also support our ideas with some examples and remarks for our equivalent properties to contain asymptotically isometric copy of ℓ^1 .

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1. PRELIMINARIES AND INTRODUCTION

A Banach space is said to have the fixed point property for nonexpansive mappings [fpp(ne)] if every self-map of any closed, bounded and convex domain in that space satisfying the condition of nonexpansiveness has a fixed point (see for example page 37 in [8]). Here, by nonexpansive map we mean a map $T : C \rightarrow C$ such that $\|Tx - Ty\| \leq \|x - y\|$ for every $x, y \in C$.

It is a fact that either c_0 or ℓ^1 is almost isometrically embedded in any nonreflexive Banach spaces with an unconditional basis (see e.g. [9]). Thus, all of the classical nonreflexive spaces fail fpp(ne); that is, there exists a closed, bounded and convex subset and a nonexpansive self-map T defined on that set such that T is fixed point free. This result depends on well-known facts (Theorems 1.c.12 in [12] and 1.c.5 in

[13]) stated by the following: a Banach lattice or a Banach space with an unconditional basis is reflexive if and only if it contains no isomorphic copies of c_0 or ℓ^1 . Hence, if it can be shown that neither c_0 nor ℓ^1 can be renormed to have fpp(ne), it would follow that fpp(ne) in either a Banach lattice or in a Banach space with an unconditional basis would imply reflexivity.

In this matter, for many years, researchers have asked the question whether or not either ℓ^1 or c_0 can be equivalently renormed so that the renormed space has fpp(ne). For ℓ^1 , there is a fact suggesting the contrary by Lin [11].

In [9], James showed that if a Banach space contains an isomorphic copy of ℓ^1 (respectively, c_0), then it contains almost isometric copies of ℓ^1 (respectively, c_0) and then he provided a tool that helped researches investigate the question of whether ℓ^1 or c_0 can be renormed to have fpp(ne). Using this tool and strengthening that, Dowling, Lennard and Turett, in several articles, have inquired relations between spaces containing nice copies of c_0 or ℓ^1 and the failure of fpp(ne). After strengthening James Distortion Theorems, they investigated these structures and then worked on the notion of a Banach space containing an asymptotically isometric (ai) copy of ℓ^1 or c_0 which are used as important tools in identifying failure of fpp(ne) for Banach spaces. It can be said that at first, their aim of introducing the notion of Banach spaces containing ai copies of ℓ^1 was to prove the fact that nonreflexive subspaces of $L^1[0, 1]$ fail fpp(ne). This result can be seen in [3] by Dowling and Lennard.

Later, Dowling, Lennard and Turett [6] showed that if a Banach space contains an ai c_0 copy, then it fails fpp(ne). They also showed in [5] that if a Banach space contains an ai copy of c_0 then its isometric dual contains an ai copy of ℓ^1 . Furthermore, in [6], they prove that there exists a Banach space failing fpp(ne) whereas it does not contain any ai copy of c_0 . In fact, in 2010 [7], they show that any infinite dimensional subspace of their example of asymptotically isometric c_0 free space fails fpp(ne). Hence, one can inquire of existence of a property which may strengthen the notion of ai copy of c_0 so that it can help see failure of fpp(ne) in the case of lack of ai c_0 copy.

For example, in order to obtain alternative methods for detecting failure of fpp(ne), in 2011, Lennard and Nezir [10] proved that if a Banach space contains an ai c_0 -summing basic sequence, it fails fpp(ne). In fact, they showed that closed convex hull of any ai c_0 -summing basic sequence fails fpp(ne). So there was another tool of detecting spaces failing fpp(ne). Moreover, recently, a nice property was developed by Álvaro, Cembranos and Mendoza [1] to detect failure of fpp(ne). They introduced a new property, N1, that implies an isomorphic copy of c_0 fails fpp(ne) such that their property is more general than the property of Banach spaces to contain ai copies of c_0 .

Similarly to their works for Banach spaces containing ai copies of c_0 , starting in 1996, Dowling, Lennard and Turett explored Banach spaces containing ai copies of ℓ^1 and they obtained important results leading researchers to test failure of fpp(ne) in

Banach spaces they work on. For example, it can be said that Lin was inspired by the norm developed by Dowling, Johnson, Lennard and Turett [2] when they showed ℓ^1 with that norm does not contain any ai copy of ℓ^1 whereas Lin later proved in [11] that ℓ^1 considered with a special case of the norm has fpp(ne). As there are interests to search alternative properties for detecting asymptotically isometric copy of c_0 , it must be significant to find better or even equivalent property for a Banach space to contain an ai copy of ℓ^1 .

Hence, aiming to find different tools to detect failure of fpp(ne) in Banach spaces, in this study, we introduce new notions which may occur for some Banach spaces. We call these new properties AAI1, AAI2 and AAI3 where AAI stands for “alternative asymptotically isometric”. We prove that if a Banach space has any of them, then it fails to have the fixed point property for nonexpansive mappings [fpp(ne)]. We provide alternative ways of detecting if a Banach space has any of these properties. We show that AAI1 is an equivalent property for a Banach space to have an ai copy of ℓ^1 . That is, a Banach space contains an ai copy of ℓ^1 if and only if it has the property AAI1. In fact, we obtain generalized version of property AAI1, which we call property AAI3, obtained as a conclusion of property AAI2. We prove that all properties AAI1, AAI2 and AAI3 are equivalent. Moreover, we support our ideas with some examples and remarks for our equivalent properties to contain ai copy of ℓ^1 . For example, we demonstrate an example of a Banach space satisfying property AAI1 where we show that the sequence allowing us to have the property is also an ai ℓ^1 sequence. Furthermore, as our final example, which is an obvious remark due to our result, we give an example of a Banach space failing AAI1 property.

We need to note that in [14], Nezir introduced another alternative asymptotically isometric property inside copies of ℓ^1 but our properties AAI1, AAI2 and AAI3 are still more general than that.

Now, we give some preliminaries required for our work.

Throughout the paper our scalar field is $\mathbb{R}$, c_0 represents the Banach space of scalar sequences converging to 0, ℓ^1 is for the Banach space of absolutely summable sequences, and c_{00} is for the space of sequences of finitely nonzero terms. Furthermore, in the paper, we will be using the unconditional basis $(e_n)_{n \in \mathbb{N}}$ for both $(c_0, \|\cdot\|_\infty)$ and $(\ell^1, \|\cdot\|_1)$ such that it has 1 in its n^{th} coordinate, and 0 in its other coordinates for each $n \in \mathbb{N}$.

Firstly, we will remind the definition of an ai ℓ^1 sequence and we will give the theorem providing alternative way of detecting if there is any as in [4] by Dowling, Lennard and Turett.

Definition 1. [4] We call a Banach space $(X, \|\cdot\|)$ contains an ai copy of ℓ^1 if there exist a sequence $(x_n)_n$ in X and a null sequence $(\varepsilon_n)_n$ in $(0, 1)$ so that

$$\sum_{n=1}^{\infty} (1 - \varepsilon_n) |a_n| \leq \left\| \sum_{n=1}^{\infty} a_n x_n \right\| \leq \sum_{n=1}^{\infty} |a_n|, \text{ for all } (a_n)_n \in \ell^1.$$

Theorem 1. [4] A Banach space $(X, \|\cdot\|)$ contains an ai copy of ℓ^1 if and only if there is a sequence $(x_n)_n$ in X and there are constants $0 < m < M < \infty$ such that for all $(t_n)_n \in \ell^1$,

$$m \sum_{n=1}^{\infty} |t_n| \leq \left\| \sum_{n=1}^{\infty} t_n x_n \right\| \leq M \sum_{n=1}^{\infty} |t_n|$$

and $\lim_{n \rightarrow \infty} \|x_n\| = m$.

Then, the following theorem is given as a result in [3].

Theorem 2. If a Banach space $(X, \|\cdot\|)$ contains an ai copy of ℓ^1 , then X fails $fpp(ne)$.

2. ALTERNATIVE ASYMPTOTICALLY ISOMETRIC PROPERTIES

In this section, we will introduce our AAI properties that will provide alternatives for well-known notions of ai copy of ℓ^1 .

Definition 2. We will say that a Banach space $(X, \|\cdot\|)$ has property AAI1 if there exist a sequence $(x_n)_n$ in X and a null sequence $(\varepsilon_n)_n$ in $(0, 1)$ so that

$$\begin{aligned} \sqrt{\left(\sum_{n=1}^{\infty} (1 - \varepsilon_n) |t_n| \right)^2 + \left(\sum_{n=1}^{\infty} (1 - \varepsilon_n) \frac{|t_n|}{2^n} \right)^2} &\leq \left\| \sum_{n=1}^{\infty} t_n x_n \right\| \\ &\leq \sqrt{\left(\sum_{n=1}^{\infty} |t_n| \right)^2 + \left(\sum_{n=1}^{\infty} \frac{|t_n|}{2^n} \right)^2}, \end{aligned}$$

for all $(t_n)_n \in \ell^1$.

Definition 3. We will say that a Banach space $(X, \|\cdot\|)$ has property AAI2 if there exist a sequence $(x_n)_n$ in X , a null sequence $(\varepsilon_n)_n$ in $(0, 1)$, a sequence $(\rho_n)_n \in \ell^1 \setminus c_{00}$ and an equivalent lattice norm ρ to $\|\cdot\|$ given by $\|x\| = \sum_{k=1}^{\infty} |\rho_k \xi_k|$ for any $x = (\xi_k)_k \in c_0$ so that

$$\sum_{n=1}^{\infty} (1 - \varepsilon_n) |t_n| + \rho((1 - \varepsilon_n) t_n)_n \leq \left\| \sum_{n=1}^{\infty} t_n x_n \right\| \leq \sum_{n=1}^{\infty} |t_n| + \rho((t_n)_n),$$

for all $(t_n)_n \in \ell^1$.

Definition 4. We will say that a Banach space $(X, \|\cdot\|)$ has property AAI3 if there exist a sequence $(x_n)_n$ in X , a null sequence $(\varepsilon_n)_n$ in $(0, 1)$, a sequence $(\rho_n)_n \in \ell^1 \setminus c_{00}$ and an equivalent lattice norm ρ to $\|\cdot\|$ given by $\|x\| = \sum_{k=1}^{\infty} |\rho_k \xi_k|$ for any $x = (\xi_k)_k \in c_0$ so that

c_0 so that

$$\begin{aligned} \sqrt{\left(\sum_{n=1}^{\infty} (1 - \varepsilon_n) |t_n|\right)^2 + [\rho(((1 - \varepsilon_n)t_n)_n)]^2} &\leq \left\| \sum_{n=1}^{\infty} t_n x_n \right\| \\ &\leq \sqrt{\left(\sum_{n=1}^{\infty} |t_n|\right)^2 + [\rho((t_n)_n)]^2}, \end{aligned}$$

for all $(t_n)_n \in \ell^1$.

In this section, we explore alternative properties to the notion of ai ℓ^1 copy. We will see that our properties will imply the failure of fpp(ne) in copies of ℓ^1 . We prove that each of AAI1, AAI2 and AAI3 is equivalent property to have ai copy of ℓ^1 . Then, in the final subsection, we give an example of a Banach space containing an ai ℓ^1 sequence such that the same sequence satisfies the condition of property AAI1. Furthermore, we show an example of a Banach space without AAI1 property or any ai ℓ^1 sequence by a quick remark since we know that a Banach space's failure of having property AAI1 is equivalent to failure of containing an ai copy of ℓ^1 .

2.1. Property AAI1

In this section, we explore property AAI1 and show that property AAI1 implies having ai copy of ℓ^1 and the converse is also true: that is, both properties imply each other.

First, we give an alternative way of detecting our property AAI1 which will help us prove that a Banach space contains an ai copy of ℓ^1 if and only if it has property AAI1.

Theorem 3. *A Banach space $(X, \|\cdot\|)$ has property AAI1 if and only if there is a sequence $(x_n)_n$ in X such that there exists $R \in [1, \infty)$ so that for any $(t_n)_n \in \ell^1$,*

$$\begin{aligned} \sqrt{\left(\sum_{n=1}^{\infty} |t_n|\right)^2 + \left(\sum_{n=1}^{\infty} \frac{|t_n|}{2^n}\right)^2} &\leq \left\| \sum_{n=1}^{\infty} t_n x_n \right\| \\ &\leq R \sqrt{\left(\sum_{n=1}^{\infty} |t_n|\right)^2 + \left(\sum_{n=1}^{\infty} \frac{|t_n|}{2^n}\right)^2} \end{aligned} \quad (2.1)$$

and

$$\lim_{n \rightarrow \infty} \|x_n\| = 1. \quad (2.2)$$

Proof. Suppose that $(X, \|\cdot\|)$ has property AAI1. Then, there exist a null sequence $(\varepsilon_n)_n$ in $(0, 1)$ and a sequence $(x_n)_n$ in X so that

$$\begin{aligned} \sqrt{\left(\sum_{n=1}^{\infty} (1-\varepsilon_n)|t_n|\right)^2 + \left(\sum_{n=1}^{\infty} (1-\varepsilon_n)\frac{|t_n|}{2^n}\right)^2} &\leq \left\|\sum_{n=1}^{\infty} t_n x_n\right\| \\ &\leq \sqrt{\left(\sum_{n=1}^{\infty} |t_n|\right)^2 + \left(\sum_{n=1}^{\infty} \frac{|t_n|}{2^n}\right)^2}, \end{aligned} \quad (2.3)$$

for all $(t_n)_n \in \ell^1$.

We may assume $(\varepsilon_n)_{n \in \mathbb{N}}$ to be a decreasing sequence since we may replace that with $\zeta_n := \max_{j \geq n} \varepsilon_j$, for all $n \in \mathbb{N}$. Let $y_n = (1 - \varepsilon_n)^{-1} x_n$ for each $n \in \mathbb{N}$. Then, for all $(t_n)_n \in \ell^1$,

$$\begin{aligned} \sqrt{\left(\sum_{n=1}^{\infty} |t_n|\right)^2 + \left(\sum_{n=1}^{\infty} \frac{|t_n|}{2^n}\right)^2} &\leq \left\|\sum_{n=1}^{\infty} t_n y_n\right\| \\ &\leq \sqrt{\left(\sum_{n=1}^{\infty} \frac{|t_n|}{1-\varepsilon_n}\right)^2 + \left(\sum_{n=1}^{\infty} \frac{|t_n|}{(1-\varepsilon_n)2^n}\right)^2}. \end{aligned}$$

Let $R = \frac{1}{1-\varepsilon_1}$. Then, condition (2.1) is satisfied for the sequence $(y_n)_n$ in X . Also, it is easy to check (2.2) for the sequence $(y_n)_n$ too since in inequality (2.3), taking $(t_n)_n$ as the unit basis $(e_n)_n$ of c_0 we obtain that $\lim_{n \rightarrow \infty} \|x_n\| = 1$ and so $\lim_{n \rightarrow \infty} \|y_n\| = 1$.

Conversely, assume that there exist a sequence $(x_n)_n$ in X and $R \in [1, \infty)$ so that for all $(t_n)_n \in \ell^1$,

$$\begin{aligned} \sqrt{\left(\sum_{n=1}^{\infty} |t_n|\right)^2 + \left(\sum_{n=1}^{\infty} \frac{|t_n|}{2^n}\right)^2} &\leq \left\|\sum_{n=1}^{\infty} t_n x_n\right\| \\ &\leq R \sqrt{\left(\sum_{n=1}^{\infty} |t_n|\right)^2 + \left(\sum_{n=1}^{\infty} \frac{|t_n|}{2^n}\right)^2} \end{aligned} \quad (2.4)$$

and $\lim_{n \rightarrow \infty} \|x_n\| = 1$.

Let $(\varepsilon_n)_n$ be a null sequence in $(0, 1)$. Since $\lim_{n \rightarrow \infty} \|x_n\| = 1$, and $\|x_n\| \geq 1$ for all $n \in \mathbb{N}$, by passing to subsequences, if necessary, we can assume that $1 \leq \|x_n\| \leq 1 + \varepsilon_n$ for all $n \in \mathbb{N}$. Define $y_n = \frac{x_n}{1+\varepsilon_n}$ for all $n \in \mathbb{N}$. Then, since $\|y_n\| \leq 1$, we have

$$\left\|\sum_{n=1}^{\infty} t_n y_n\right\| \leq \sum_{n=1}^{\infty} |t_n| \leq \sqrt{\left(\sum_{n=1}^{\infty} |t_n|\right)^2 + \left(\sum_{n=1}^{\infty} \frac{|t_n|}{2^n}\right)^2}, \text{ for all } (t_n)_n \in \ell^1.$$

Also, from the left hand side inequality of (2.4), we have

$$\begin{aligned} \left\| \sum_{n=1}^{\infty} t_n y_n \right\| &= \left\| \sum_{n=1}^{\infty} t_n \frac{x_n}{(1 + \varepsilon_n)} \right\| \geq \sqrt{\left(\sum_{n=1}^{\infty} \frac{|t_n|}{1 + \varepsilon_n} \right)^2 + \left(\sum_{n=1}^{\infty} \frac{|t_n|}{(1 + \varepsilon_n) 2^n} \right)^2} \\ &\geq \sqrt{\left(\sum_{n=1}^{\infty} (1 - \varepsilon_n) |t_n| \right)^2 + \left(\sum_{n=1}^{\infty} (1 - \varepsilon_n) \frac{|t_n|}{2^n} \right)^2}. \end{aligned}$$

□

Now, we show property AAI1 is equivalent for a Banach space to contain an ai copy of ℓ^1 .

Theorem 4. *Let $(X, \|\cdot\|)$ be a Banach space. Then, X contains an ai copy of ℓ^1 if and only if X has property AAI1.*

Proof. Assume that a Banach space X contains an ai copy of ℓ^1 . Then, by Theorem 1, there exist a sequence $(x_n)_n$ in X with $\lim_{n \rightarrow \infty} \|x_n\| = 1$ and $M \in [1, \infty)$ such that for all $(t_n)_n \in \ell^1$,

$$\sum_{n=1}^{\infty} |t_n| \leq \left\| \sum_{n=1}^{\infty} t_n x_n \right\| \leq M \sum_{n=1}^{\infty} |t_n|.$$

Then, letting $y_n = (1 + \frac{1}{2^n}) x_n$ for each $n \in \mathbb{N}$ and $R = \frac{3M}{2}$, we have

$$\begin{aligned} \sqrt{\left(\sum_{n=1}^{\infty} |t_n| \right)^2 + \left(\sum_{n=1}^{\infty} \frac{|t_n|}{2^n} \right)^2} &\leq \sum_{n=1}^{\infty} |t_n| + \sum_{n=1}^{\infty} \frac{|t_n|}{2^n} \\ &\leq \left\| \sum_{n=1}^{\infty} t_n y_n \right\| \leq M \left(\sum_{n=1}^{\infty} |t_n| + \sum_{n=1}^{\infty} \frac{|t_n|}{2^n} \right). \end{aligned}$$

Hence,

$$\begin{aligned} \sqrt{\left(\sum_{n=1}^{\infty} |t_n| \right)^2 + \left(\sum_{n=1}^{\infty} \frac{|t_n|}{2^n} \right)^2} &\leq \left\| \sum_{n=1}^{\infty} t_n y_n \right\| \leq R \sum_{n=1}^{\infty} |t_n| \\ &\leq R \sqrt{\left(\sum_{n=1}^{\infty} |t_n| \right)^2 + \left(\sum_{n=1}^{\infty} \frac{|t_n|}{2^n} \right)^2} \end{aligned}$$

and $\lim_n \|y_n\| = 1$.

Thus, by the previous theorem, X has property AAI1.

Conversely assume that X has property AAI1. Then, there exist a sequence $(x_n)_n$ in X satisfying $\lim_{n \rightarrow \infty} \|x_n\| = 1$ and a constant $R \in [1, \infty)$ so that for any $(t_n)_n \in \ell^1$,

$$\sqrt{\left(\sum_{n=1}^{\infty} |t_n|\right)^2 + \left(\sum_{n=1}^{\infty} \frac{|t_n|}{2^n}\right)^2} \leq \left\| \sum_{n=1}^{\infty} t_n x_n \right\| \leq R \sqrt{\left(\sum_{n=1}^{\infty} |t_n|\right)^2 + \left(\sum_{n=1}^{\infty} \frac{|t_n|}{2^n}\right)^2}$$

and so

$$\sum_{n=1}^{\infty} |t_n| \leq \left\| \sum_{n=1}^{\infty} t_n x_n \right\| \leq \frac{3R}{2} \sum_{n=1}^{\infty} |t_n|.$$

Hence, we are done by Theorem 1. $\square$

2.2. A more generalized property, property AAI2

Firstly, we give an alternative way of detecting our property AAI2 which will help us prove that a Banach space contains an ai copy of ℓ^1 if and only if it has property AAI2.

Theorem 5. *A Banach space $(X, \|\cdot\|)$ has property AAI2 if and only if there exist a sequence $(x_n)_n$ in X , a sequence $(\rho_n)_n \in \ell^1 \setminus c_{00}$ and an equivalent lattice norm ρ to $\|\cdot\|$ given by $\|x\| = \sum_{k=1}^{\infty} |\rho_k \xi_k|$ for any $x = (\xi_k)_k \in c_0$ such that there exists $R \in [1, \infty)$ so that for any $(t_n)_n \in \ell^1$,*

$$\sum_{n=1}^{\infty} |t_n| + \rho((t_n)_n) \leq \left\| \sum_{n=1}^{\infty} t_n x_n \right\| \leq R \left[\sum_{n=1}^{\infty} |t_n| + \rho((t_n)_n) \right] \quad (2.5)$$

and

$$\lim_{n \rightarrow \infty} \|x_n\| = 1. \quad (2.6)$$

Proof. Suppose that $(X, \|\cdot\|)$ has property AAI2. Then, there exist a sequence $(x_n)_n$ in X , a decreasing null sequence $(\varepsilon_n)_n$ in $(0, 1)$, a sequence $(\rho_n)_n \in \ell^1 \setminus c_{00}$ and an equivalent lattice norm ρ to $\|\cdot\|$ given by $\|x\| = \sum_{k=1}^{\infty} |\rho_k \xi_k|$ for any $x = (\xi_k)_k \in c_0$ so that

$$\sum_{n=1}^{\infty} (1 - \varepsilon_n) |t_n| + \rho(((1 - \varepsilon_n)t_n)_n) \leq \left\| \sum_{n=1}^{\infty} t_n x_n \right\| \leq \sum_{n=1}^{\infty} |t_n| + \rho((t_n)_n), \quad (2.7)$$

for all $(t_n)_n \in \ell^1$.

Let $y_n = (1 - \varepsilon_n)^{-1} x_n$ for each $n \in \mathbb{N}$. Then, for all $(t_n)_n \in \ell^1$, taking ρ being a lattice norm into consideration,

$$\sum_{n=1}^{\infty} |t_n| + \rho((t_n)_n) \leq \left\| \sum_{n=1}^{\infty} t_n y_n \right\| \leq \sum_{n=1}^{\infty} \frac{|t_n|}{1 - \varepsilon_n} + \rho\left(\left(\frac{|t_n|}{1 - \varepsilon_n}\right)_n\right).$$

Let $R = \frac{1}{1-\varepsilon_1}$. Then, condition (2.5) is satisfied for the sequence $(y_n)_n$ in X . Also, it is easy to check (2.6) for the sequence $(y_n)_n$ too since in inequality (2.7), taking $(t_n)_n$ as the unit basis $(e_n)_n$ of c_0 and recalling ρ is equivalent to $\|\cdot\|$, we obtain that $\lim_{n \rightarrow \infty} \|x_n\| = 1$ and so $\lim_{n \rightarrow \infty} \|y_n\| = 1$.

Conversely, assume that there exist $R \in [1, \infty)$, a sequence $(x_n)_n$ in X , a sequence $(\rho_n)_n \in \ell^1 \setminus c_{00}$ and an equivalent lattice norm ρ to $\|\cdot\|$ given by $\|x\| = \sum_{k=1}^{\infty} |\rho_k \xi_k|$ for any $x = (\xi_k)_k \in c_0$ so that for all $(t_n)_n \in \ell^1$,

$$\sum_{n=1}^{\infty} |t_n| + \rho((t_n)_n) \leq \left\| \sum_{n=1}^{\infty} t_n x_n \right\| \leq R \left[\sum_{n=1}^{\infty} |t_n| + \rho((t_n)_n) \right] \quad (2.8)$$

and $\lim_{n \rightarrow \infty} \|x_n\| = 1$.

Let $(\varepsilon_n)_n$ be a null sequence in $(0, 1)$. Since $\lim_{n \rightarrow \infty} \|x_n\| = 1$, and $\|x_n\| \geq 1$ for all $n \in \mathbb{N}$, by passing to subsequences, if necessary, we can assume that $1 \leq \|x_n\| \leq 1 + \varepsilon_n$ for all $n \in \mathbb{N}$. Define $y_n = \frac{x_n}{1 + \varepsilon_n}$ for all $n \in \mathbb{N}$. Then, since $\|y_n\| \leq 1$, we have

$$\left\| \sum_{n=1}^{\infty} t_n y_n \right\| \leq \sum_{n=1}^{\infty} |t_n| \leq \sum_{n=1}^{\infty} |t_n| + \rho((t_n)_n) \quad \text{for all } (t_n)_n \in \ell^1.$$

Also from the left hand side inequality of (2.8) and recalling ρ is a lattice norm, we have

$$\begin{aligned} \left\| \sum_{n=1}^{\infty} t_n y_n \right\| &= \left\| \sum_{n=1}^{\infty} t_n \frac{x_n}{1 + \varepsilon_n} \right\| \geq \sum_{n=1}^{\infty} \frac{|t_n|}{1 + \varepsilon_n} + \rho\left(\left(\frac{|t_n|}{1 + \varepsilon_n}\right)_n\right) \\ &\geq \sum_{n=1}^{\infty} (1 - \varepsilon_n) |t_n| + \rho(((1 - \varepsilon_n)t_n)_n). \end{aligned}$$

□

Theorem 6. *Let $(X, \|\cdot\|)$ be a Banach space. Then, X has property AAI2 if and only if X contains an ai copy of ℓ^1*

Proof. Let $(X, \|\cdot\|)$ be a Banach space and assume that X has property AAI2. So there exist a sequence $(x_n)_n$ in X satisfying $\lim_{n \rightarrow \infty} \|x_n\| = 1$, a constant $R \in [1, \infty)$, a sequence $(\rho_n)_n \in \ell^1 \setminus c_{00}$ and an equivalent lattice norm ρ to $\|\cdot\|$ given by $\|x\| = \sum_{k=1}^{\infty} |\rho_k \xi_k|$ for any $x = (\xi_k)_k \in c_0$ so that for any $(t_n)_n \in \ell^1$,

$$\sum_{n=1}^{\infty} |t_n| + \rho((t_n)_n) \leq \left\| \sum_{n=1}^{\infty} t_n x_n \right\| \leq R \left[\sum_{n=1}^{\infty} |t_n| + \rho((t_n)_n) \right].$$

Then, there exist constants $0 < A \leq B < \infty$ such that for any $x = (\xi_n)_n \in c_0$, $A\rho(x) \leq \sum_{k=1}^{\infty} |\rho_k \xi_k| \leq B\rho(x)$. Let $M = R \left[1 + \sup_n \frac{|\rho_n|}{A} \right]$. Then,

$$\sum_{n=1}^{\infty} |t_n| \leq \left\| \sum_{n=1}^{\infty} t_n x_n \right\| \leq M \sum_{n=1}^{\infty} |t_n|.$$

So $(x_n)_n$ is an ai ℓ^1 sequence in X by Theorem 1.

Conversely assume that X contains an ai copy of ℓ^1 . Then, by Theorem 1, there exist a sequence $(x_n)_n$ in X with $\lim_n \|x_n\| = 1$ and $M \in [1, \infty)$ such that for all $(t_n)_n \in \ell^1$,

$$\sum_{n=1}^{\infty} |t_n| \leq \left\| \sum_{n=1}^{\infty} t_n x_n \right\| \leq M \sum_{n=1}^{\infty} |t_n|.$$

Now, fix a sequence $(\rho_n)_n \in \ell^1 \setminus c_{00}$ and fix an equivalent lattice norm ρ to $\|\cdot\|$ given by $\|x\| = \sum_{k=1}^{\infty} |\rho_k \xi_k|$ for any $x = (\xi_k)_k \in c_0$. For example, we could consider any equivalent norm to $\|\cdot\|$ given by $\|x\| = \sum_{k=1}^{\infty} \frac{|\xi_k|}{2^k}$ for any $x = (\xi_k)_k \in c_0$ but we may work on an arbitrary one. Then, letting $y_n = \left(1 + \frac{|\rho_n|}{A}\right) x_n$ for each $n \in \mathbb{N}$ and $R = \frac{BM}{A}$, we have

$$\sum_{n=1}^{\infty} |t_n| + \rho((t_n)_n) \leq \left\| \sum_{n=1}^{\infty} t_n y_n \right\| \leq R \left[\sum_{n=1}^{\infty} |t_n| + \rho((t_n)_n) \right].$$

Hence, by the previous theorem, X has property AAI2. $\square$

2.3. Property AAI3; generalizing AAI1

Using the ideas of the previous section, we may get a generalized version of property AAI1, which we call property AAI3. Indeed, one can easily obtain the following conclusions.

Corollary 1. *A Banach space $(X, \|\cdot\|)$ has property AAI3 if and only if there exist a sequence $(x_n)_n$ in X , a sequence $(\rho_n)_n \in \ell^1 \setminus c_{00}$ and an equivalent lattice norm ρ to $\|\cdot\|$ given by $\|x\| = \sum_{k=1}^{\infty} |\rho_k \xi_k|$ for any $x = (\xi_k)_k \in c_0$ such that there exists $R \in [1, \infty)$ so that for any $(t_n)_n \in \ell^1$,*

$$\sqrt{\left(\sum_{n=1}^{\infty} |t_n|\right)^2 + [\rho((t_n)_n)]^2} \leq \left\| \sum_{n=1}^{\infty} t_n x_n \right\| \leq R \sqrt{\left(\sum_{n=1}^{\infty} |t_n|\right)^2 + [\rho((t_n)_n)]^2}$$

and $\lim_{n \rightarrow \infty} \|x_n\| = 1$.

Corollary 2. *Let $(X, \|\cdot\|)$ be a Banach space. Then, X has property AAI3 if and only if X contains an ai copy of ℓ^1*

2.4. Some Remarks and Examples for Alternatives to ai ℓ^1 Copy

Example 1. We consider a renorming of ℓ^1 as follows. Consider $(\ell^1, \|\cdot\|)$ with the norm defined by the following way: for every $x = (\xi_n)_n \in \ell^1$,

$$\|x\| = \sup_n \frac{2^n}{1+2^n} \sum_{k=n}^{\infty} |\xi_k| + \sum_{n=1}^{\infty} \frac{|\xi_n|}{2^n} + \frac{1}{3} \sum_{n=1}^{\infty} |\xi_n|.$$

Then, we can see that $(\ell^1, \|\cdot\|)$ contains a sequence that satisfies both conditions of property AAI1 and an ai ℓ^1 sequence.

Proof. Fix an increasing sequence $(m_i)_i$ in $\mathbb{N}$ and define for each $k \in \mathbb{N}$, $M_k := \sum_{i=1}^k m_i$ with $M_0 := 0$ and $y_k = \frac{1}{1+m_k} \sum_{n=M_{k-1}+1}^{M_k} e_n$. Then, we get for each $k \in \mathbb{N}$,

$$\begin{aligned} \|y_k\| &\leq \frac{1}{m_k+1} \left(1 + \frac{5}{6}(M_k - M_{k-1})\right) \leq \frac{1}{m_k+1} \left(1 + \frac{5}{6}m_k\right) \\ &\leq \frac{1}{m_k+1} (1 + m_k) = 1 \end{aligned}$$

and thus, for all $(t_n)_n \in \ell^1$,

$$\left\| \sum_{n=1}^{\infty} t_n y_n \right\| \leq \sum_{n=1}^{\infty} |t_n| \leq \sqrt{\left(\sum_{n=1}^{\infty} |t_n| \right)^2 + \left(\sum_{n=1}^{\infty} \frac{|t_n|}{2^n} \right)^2}.$$

On the other hand, we have

$$\begin{aligned} \left\| \sum_{n=1}^{\infty} t_n y_n \right\| &= \left\| t_1 \frac{1}{m_1+1} \sum_{j=1}^{M_1} e_j + t_2 \frac{1}{m_2+1} \sum_{j=M_1+1}^{M_2} e_j \right. \\ &\quad \left. + t_3 \frac{1}{m_3+1} \sum_{j=M_2+1}^{M_3} e_j + \cdots + t_n \frac{1}{m_n+1} \sum_{j=M_{n-1}+1}^{M_n} e_j + \cdots \right\| \\ &\geq \frac{2}{3} \sum_{n=1}^{\infty} \frac{|t_n| m_n}{m_n+1} + \sum_{n=1}^{\infty} \frac{m_n |t_n|}{2^n (m_n+1)} + \frac{1}{3} \sum_{n=1}^{\infty} \frac{m_n |t_n|}{m_n+1} \\ &= \sum_{n=1}^{\infty} |t_n| \frac{m_n}{m_n+1} \left(1 + \frac{1}{2^n}\right) \\ &\geq \sqrt{\left(\sum_{n=1}^{\infty} |t_n| \frac{m_n}{m_n+1} \right)^2 + \left(\sum_{n=1}^{\infty} \frac{|t_n|}{2^n} \frac{m_n}{m_n+1} \right)^2} \end{aligned}$$

Then, letting $m_n \leq 2^n$, $\varepsilon_n = \frac{1}{m_n+1}$ and $\delta_n = \frac{2^n - m_n}{2^n(m_n+1)}$ for each $n \in \mathbb{N}$, we obtain that for all $(t_n)_n \in \ell^1$,

$$\sum_{n=1}^{\infty} |t_n|(1 - \delta_n) \leq \left\| \sum_{n=1}^{\infty} t_n y_n \right\| \leq \sum_{n=1}^{\infty} |t_n|$$

and

$$\begin{aligned} \sqrt{\left(\sum_{n=1}^{\infty} (1 - \varepsilon_n) |t_n| \right)^2 + \left(\sum_{n=1}^{\infty} (1 - \varepsilon_n) \frac{|t_n|}{2^n} \right)^2} &\leq \left\| \sum_{n=1}^{\infty} t_n y_n \right\| \\ &\leq \sqrt{\left(\sum_{n=1}^{\infty} |t_n| \right)^2 + \left(\sum_{n=1}^{\infty} \frac{|t_n|}{2^n} \right)^2}. \end{aligned}$$

Thus, $(\ell^1, \|\cdot\|)$ has property AAI1 and it contains ai copy of ℓ^1 such that both properties share a common sequence to exist. $\square$

Remark 1. Dowling, Johnson, Lennard and Turett [2] constructed the following equivalent norm $\|\cdot\|^\sim$ on ℓ^1 and they showed that $(\ell^1, \|\cdot\|^\sim)$ does not contain an ai copy of ℓ^1 and later Lin [11] showed that ℓ^1 can be renormed to have fpp(ne) with a special version of the norm $\|\cdot\|^\sim$.

Equivalent norm $\|\cdot\|^\sim$ on ℓ^1 is given as follows: for $x = (\xi_k)_k \in \ell^1$, write $\|x\|^\sim := \sup_{k \in \mathbb{N}} \gamma_k \sum_{j=k}^{\infty} |\xi_j|$ where $\gamma_k \uparrow_k 1$, γ_k is strictly increasing.

Due to our result in previous section, having property AAI1 is equivalent for a Banach space to contain an ai copy of ℓ^1 , we have a straight result that $(\ell^1, \|\cdot\|^\sim)$ does not satisfy AAI1 property.

Remark 2. Note that [5, Theorem 4.1 on page 87] says that if a Banach space contains an ai copy of c_0 then its isometric dual contains an ai copy of ℓ^1 . Also note that the converse does not generally hold; that is, a dual Banach space may contain an ai copy of ℓ^1 but this does not generally imply its isometric predual contains an ai copy of c_0 ; e.g. as a predual, we can consider ℓ^1 with its canonical norm which does not contain an ai copy of c_0 but ℓ^∞ contains an isometric copy of ℓ^1 so it contains an ai copy of ℓ^1 . Using the same theorem of Dowling, Lennard and Turett, we can say that if a Banach space fails AAI1 property, then its predual does not contain any ai copy of c_0 .

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PREDICTOR/ESTIMATOR COMPUTATIONS UNDER A CONSTRAINED MULTIVARIATE LINEAR MODEL AND SOME RELATED REDUCED MODELS

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Abstract. This study deals with basic prediction/estimation issues involving a constrained multivariate linear model (CMLM) and some related reduced models. By reparameterizing these models, the authors create unconstrained multivariate linear models (UMLMs). After that, the authors use some quadratic matrix optimization methods to derive analytical formulas for computing the best linear unbiased predictors/estimators (BLUPs/BLUEs) of all unknown parameter matrices. This provides a broad perspective on BLUPs.

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1. INTRODUCTION

As follows, we introduce the notations that will be used throughout this study. We will write $\Gamma \in \mathbb{R}_{k,t}$ if Γ is a $k \times t$ real matrix, $\Gamma \in \mathbb{R}_k^s$ if $\Gamma \in \mathbb{R}_{k,k}$ and is symmetric, and $\Gamma \in \mathbb{R}_k^{\geq}$ if $\Gamma \in \mathbb{R}_k^s$ and is positive semi-definite. We will use Γ' , $\mathcal{C}(\Gamma)$, and $r(\Gamma)$ as symbols to present the transpose, the column space, and the rank of $\Gamma \in \mathbb{R}_{k,t}$, respectively. $\Gamma^\perp = \mathbf{I}_k - \Gamma\Gamma^+$ and $\mathbf{F}_\Gamma = \mathbf{I}_t - \Gamma^+\Gamma$ stand for orthogonal projectors, where $\mathbf{I}_k \in \mathbb{R}_k^s$ and $\mathbf{I}_t \in \mathbb{R}_t^s$ are the identity matrices and Γ^+ is the Moore–Penrose generalized inverse of $\Gamma \in \mathbb{R}_{k,t}$. The symbol $i_+(\Gamma)$ represents positive inertia of $\Gamma \in \mathbb{R}_k^s$. The vectorization operation (vec operation) of a matrix $\Gamma = [\gamma_1, \dots, \gamma_t]$ is defined to be $\vec{\Gamma} = [\gamma_1', \dots, \gamma_t']'$, where $\gamma_1, \dots, \gamma_t$ are the columns of $\Gamma \in \mathbb{R}_{k,t}$. A well-known property on the vec operation of a triple matrix product is $\vec{\Gamma_1 \Lambda \Gamma_2} = (\Gamma_2' \otimes \Gamma_1') \vec{\Lambda}$ for matrices Λ , Γ_1 , and Γ_2 .

One of the most frequently used essential statistical tools, both in theory and in practice, is linear models (LMs). The multivariate LM (MLM) is a generalization of the univariate LM, which normally means that a response variable on a given set of regressors is regressed to several response variables on regressors. There may be

certain restrictions on unknown parameters that are often added to model assumptions in statistical theory and its applications. Such restrictions arise in some cases, such as linear hypothesis testing on parameters. In such cases, MLM becomes a constrained MLM (CMLM).

In the present study, we consider a CMLM, formulated by

$$\mathcal{M} : \mathbf{Y} = \mathbf{X}\Theta + \Psi = \mathbf{X}_1\Theta_1 + \mathbf{X}_2\Theta_2 + \Psi, \mathbf{C}\Theta = \mathbf{C}_1\Theta_1 + \mathbf{C}_2\Theta_2 = \mathbf{D}, \quad (1.1)$$

where $\mathbf{Y} \in \mathbb{R}_{n,m}$ is a matrix of observable dependent variables, $\mathbf{X} = [\mathbf{X}_1, \mathbf{X}_2] \in \mathbb{R}_{n,p}$ with $\mathbf{X}_i \in \mathbb{R}_{n,p_i}$, $\mathbf{C} = [\mathbf{C}_1, \mathbf{C}_2] \in \mathbb{R}_{s,p}$ with $\mathbf{C}_i \in \mathbb{R}_{s,p_i}$, and $\mathbf{D} \in \mathbb{R}_{s,m}$ are known matrices of arbitrary ranks, $\Theta = [\Theta'_1, \Theta'_2] \in \mathbb{R}_{p,m}$ with $\Theta_i \in \mathbb{R}_{p_i,m}$ is a matrix of fixed but unknown parameters, $i = 1, 2$, $p_1 + p_2 = p$, $\Psi \in \mathbb{R}_{n,m}$ is a matrix of randomly distributed error terms with mean matrix $E(\Psi) = \mathbf{0}$ and dispersion matrix $D(\vec{\Psi}) = \sigma^2(\Sigma_2 \otimes \Sigma_1)$, where $\Sigma_1 = (\sigma_{1ij}) \in \mathbb{R}_n^{\times}$, $\Sigma_2 = (\sigma_{2ij}) \in \mathbb{R}_m^{\times}$, and σ^2 is an unknown positive number. Further, $\Sigma_2 \otimes \Sigma_1$ means that $\vec{\Psi}$ has a Kronecker product structured covariance matrix.

A CMLM in partitioned form is usually considered for statistical inferences of partial parameters in regression analysis. Model transformations are a common occurrence, and we utilize them to extrapolate and infer the model's parameter space. One of the simplest transformations is a linear transformation, which is achieved by premultiplying both sides of (1.1) by two matrices $\mathbf{X}_1^\perp$ and $\mathbf{X}_2^\perp$ to yield the following pair of transformed CMLMs:

$$\begin{aligned} \mathcal{M}_1 : \mathbf{X}_2^\perp \mathbf{Y} &= \mathbf{X}_2^\perp \mathbf{X}_1 \Theta_1 + \mathbf{X}_2^\perp \Psi, \mathbf{C}_1 \Theta_1 = \mathbf{D}, \\ E(\mathbf{X}_2^\perp \Psi) &= \mathbf{0}, D(\mathbf{X}_2^\perp \Psi) = \sigma^2(\Sigma_2 \otimes \mathbf{X}_2^\perp \Sigma_1 \mathbf{X}_2^\perp) \end{aligned} \quad (1.2)$$

and

$$\begin{aligned} \mathcal{M}_2 : \mathbf{X}_1^\perp \mathbf{Y} &= \mathbf{X}_1^\perp \mathbf{X}_2 \Theta_2 + \mathbf{X}_1^\perp \Psi, \mathbf{C}_2 \Theta_2 = \mathbf{D}, \\ E(\mathbf{X}_1^\perp \Psi) &= \mathbf{0}, D(\mathbf{X}_1^\perp \Psi) = \sigma^2(\Sigma_2 \otimes \mathbf{X}_1^\perp \Sigma_1 \mathbf{X}_1^\perp), \end{aligned} \quad (1.3)$$

which are frequently referred to as correctly-reduced versions of $\mathcal{M}$ in (1.1); for the related expositions, see, [14]. Each of the model equations in $\mathcal{M}_1$ and $\mathcal{M}_2$ is not equal to the whole model equation in $\mathcal{M}$ due to the two linear transformations $\mathbf{X}_1^\perp \mathbf{Y}$ and $\mathbf{X}_2^\perp \mathbf{Y}$ are unique. It is advantageous to formulate $\mathcal{M}_1$ and $\mathcal{M}_2$ since they do not contain the partial parameter matrices Θ_1 and Θ_2 found in $\mathcal{M}$. As a result, using $\mathcal{M}_1$ and $\mathcal{M}_2$, we may estimate Θ_1 and Θ_2 in $\mathcal{M}$ separately.

In many cases, we are forced to use these derived models in the hopes that the inference outcomes obtained from $\mathcal{M}_1$ and $\mathcal{M}_2$ are equal to those that would correspond to $\mathcal{M}$. The linear sufficiency problem, initially stated by [2] and taken into consideration in the statistical literature, is the search for relationships between estimators under original models and their reduced models.

We can take the following matrix into consideration in order to develop some results on predictors of all unknown matrices under $\mathcal{M}$.

$$\begin{aligned}\Phi &= \mathbf{H}\Theta + \mathbf{J}\Psi = \mathbf{H}_1\Theta_1 + \mathbf{H}_2\Theta_2 + \mathbf{J}\Psi, \text{ or,} \\ \vec{\Phi} &= (\mathbf{I}_m \otimes \mathbf{H}) \vec{\Theta} + (\mathbf{I}_m \otimes \mathbf{J}) \vec{\Psi} \\ &= (\mathbf{I}_m \otimes \mathbf{H}_1) \vec{\Theta}_1 + (\mathbf{I}_m \otimes \mathbf{H}_2) \vec{\Theta}_2 + (\mathbf{I}_m \otimes \mathbf{J}) \vec{\Psi},\end{aligned}\tag{1.4}$$

in accordance with the partition considered in (1.1) for given matrices $\mathbf{H} = [\mathbf{H}_1, \mathbf{H}_2] \in \mathbb{R}_{k,p}$ with $\mathbf{H}_i \in \mathbb{R}_{k,p_i}$, $\Theta = [\Theta_1, \Theta_2] \in \mathbb{R}_{p,m}$ with $\Theta_i \in \mathbb{R}_{p_i,m}$, $i = 1, 2$, $p_1 + p_2 = p$, and $\mathbf{J} \in \mathbb{R}_{k,n}$.

For statistical inference under the CMLM and its two reduced models, in addition to (1.4), we are also able to create the next three matrices linked to the constrained reduced models $\mathcal{M}_i$, $i = 1, 2$, respectively.

$$\begin{aligned}\Phi_1 &= \mathbf{H}_1\Theta_1 + \mathbf{J}\Psi = [\mathbf{H}_1, \mathbf{0}] \Theta + \mathbf{J}\Psi, \text{ or,} \\ \vec{\Phi}_1 &= (\mathbf{I}_m \otimes \mathbf{H}_1) \vec{\Theta}_1 + (\mathbf{I}_m \otimes \mathbf{J}) \vec{\Psi},\end{aligned}\tag{1.5}$$

and

$$\begin{aligned}\Phi_2 &= \mathbf{H}_2\Theta_2 + \mathbf{J}\Psi = [\mathbf{0}, \mathbf{H}_2] \Theta + \mathbf{J}\Psi, \text{ or,} \\ \vec{\Phi}_2 &= (\mathbf{I}_m \otimes \mathbf{H}_2) \vec{\Theta}_2 + (\mathbf{I}_m \otimes \mathbf{J}) \vec{\Psi}.\end{aligned}\tag{1.6}$$

Then, we obtain

$$\begin{aligned}E(\Phi) &= \mathbf{H}\Theta, D(\vec{\Phi}) = \sigma^2(\mathbf{I}_m \otimes \mathbf{J})(\Sigma_2 \otimes \Sigma_1)(\mathbf{I}_m \otimes \mathbf{J})', \\ \text{cov}(\vec{\Phi}, \vec{\mathbf{Y}}) &= \text{cov}(\vec{\Phi}_1, \vec{\mathbf{Y}}) = \text{cov}(\vec{\Phi}_2, \vec{\mathbf{Y}}) = \sigma^2(\mathbf{I}_m \otimes \mathbf{J})(\Sigma_2 \otimes \Sigma_1),\end{aligned}\tag{1.7}$$

$$\begin{aligned}E(\Phi_1) &= \mathbf{H}_1\Theta_1, D(\vec{\Phi}_1) = \sigma^2(\mathbf{I}_m \otimes \mathbf{J})(\Sigma_2 \otimes \Sigma_1)(\mathbf{I}_m \otimes \mathbf{J})', \\ \text{cov}(\vec{\Phi}_1, \vec{\mathbf{X}}_1^\perp \vec{\mathbf{Y}}) &= \sigma^2(\mathbf{I}_m \otimes \mathbf{J})(\Sigma_2 \otimes \mathbf{X}_1^\perp \Sigma_1 \mathbf{X}_1^\perp), \\ \text{cov}(\vec{\Phi}_1, \vec{\mathbf{X}}_2^\perp \vec{\mathbf{Y}}) &= \sigma^2(\mathbf{I}_m \otimes \mathbf{J})(\Sigma_2 \otimes \mathbf{X}_2^\perp \Sigma_1 \mathbf{X}_2^\perp),\end{aligned}\tag{1.8}$$

and

$$\begin{aligned}E(\Phi_2) &= \mathbf{H}_2\Theta_2, D(\vec{\Phi}_2) = \sigma^2(\mathbf{I}_m \otimes \mathbf{J})(\Sigma_2 \otimes \Sigma_1)(\mathbf{I}_m \otimes \mathbf{J})', \\ \text{cov}(\vec{\Phi}_2, \vec{\mathbf{X}}_1^\perp \vec{\mathbf{Y}}) &= \sigma^2(\mathbf{I}_m \otimes \mathbf{J})(\Sigma_2 \otimes \mathbf{X}_1^\perp \Sigma_1 \mathbf{X}_1^\perp), \\ \text{cov}(\vec{\Phi}_2, \vec{\mathbf{X}}_2^\perp \vec{\mathbf{Y}}) &= \sigma^2(\mathbf{I}_m \otimes \mathbf{J})(\Sigma_2 \otimes \mathbf{X}_2^\perp \Sigma_1 \mathbf{X}_2^\perp).\end{aligned}\tag{1.9}$$

Note that the linear restriction equations $\mathbf{C}\Theta = \mathbf{D}$ and $\mathbf{C}_i\Theta_i = \mathbf{D}$ in $\mathcal{M}$ and $\mathcal{M}_i$, $i = 1, 2$ are consistent, respectively. The general solutions of these matrix equations can be written as $\Theta = \mathbf{C}^+\mathbf{D} + \mathbf{F}_C\Omega$ and $\Theta_i = \mathbf{C}_i^+\mathbf{D} + \mathbf{F}_{C_i}\Omega_i$, respectively, where $\Omega \in \mathbb{R}_{p,m}$ and $\Omega_i \in \mathbb{R}_{p_i,m}$ are reparameterized but arbitrary matrices. Substituting these solutions

into the model equations in $\mathcal{M}$ and $\mathcal{M}_i$, $i = 1, 2$, yields the following reparameterized MLMs:

$$\mathcal{R} : \Upsilon = \mathbf{X}\mathbf{F}_C\Omega + \Psi, \quad (1.10)$$

$$\mathcal{R}_1 : \mathbf{X}_2^\perp \Upsilon_1 = \mathbf{X}_2^\perp \mathbf{X}_1 \mathbf{F}_{C_1} \Omega_1 + \mathbf{X}_2^\perp \Psi, \quad (1.11)$$

$$\mathcal{R}_2 : \mathbf{X}_1^\perp \Upsilon_2 = \mathbf{X}_1^\perp \mathbf{X}_2 \mathbf{F}_{C_2} \Omega_2 + \mathbf{X}_1^\perp \Psi, \quad (1.12)$$

where $\Upsilon = \mathbf{Y} - \mathbf{X}\mathbf{C}^+\mathbf{D}$ and $\Upsilon_i = \mathbf{Y} - \mathbf{X}_i\mathbf{C}_i^+\mathbf{D}$, $i = 1, 2$, thus, predictions under $\mathcal{M}$ and $\mathcal{M}_i$ can be derived from $\mathcal{R}$ and $\mathcal{R}_i$, $i = 1, 2$, respectively. Correspondingly, Φ in (1.4) and Φ_i in (1.5)-(1.6) become the following reparameterized matrices:

$$\begin{aligned} \Delta &= \mathbf{H}\mathbf{F}_C\Omega + \mathbf{J}\Psi = \mathbf{H}_1\mathbf{F}_{C_1}\Omega_1 + \mathbf{H}_2\mathbf{F}_{C_2}\Omega_2 + \mathbf{J}\Psi, \text{ or,} \\ \vec{\Delta} &= (\mathbf{I}_m \otimes \mathbf{H}\mathbf{F}_C) \vec{\Omega} + (\mathbf{I}_m \otimes \mathbf{J}) \vec{\Psi}, \end{aligned} \quad (1.13)$$

$$\begin{aligned} &= (\mathbf{I}_m \otimes \mathbf{H}_1\mathbf{F}_{C_1}) \vec{\Omega}_1 + (\mathbf{I}_m \otimes \mathbf{H}_2\mathbf{F}_{C_2}) \vec{\Omega}_2 + (\mathbf{I}_m \otimes \mathbf{J}) \vec{\Psi}, \\ \Delta_1 &= \mathbf{H}_1\mathbf{F}_{C_1}\Omega_1 + \mathbf{J}\Psi = [\mathbf{H}_1\mathbf{F}_{C_1}, \quad \mathbf{0}] \Omega + \mathbf{J}\Psi, \text{ or,} \\ \vec{\Delta}_1 &= (\mathbf{I}_m \otimes \mathbf{H}_1\mathbf{F}_{C_1}) \vec{\Omega}_1 + (\mathbf{I}_m \otimes \mathbf{J}) \vec{\Psi}, \end{aligned} \quad (1.14)$$

and

$$\begin{aligned} \Delta_2 &= \mathbf{H}_2\mathbf{F}_{C_2}\Omega_2 + \mathbf{J}\Psi = [\mathbf{0}, \quad \mathbf{H}_2\mathbf{F}_{C_2}] \Omega + \mathbf{J}\Psi, \text{ or,} \\ \vec{\Delta}_2 &= (\mathbf{I}_m \otimes \mathbf{H}_2\mathbf{F}_{C_2}) \vec{\Omega}_2 + (\mathbf{I}_m \otimes \mathbf{J}) \vec{\Psi}, \end{aligned} \quad (1.15)$$

where $\Delta = \Phi - \mathbf{H}\mathbf{C}^+\mathbf{D}$ and $\Delta_i = \Phi_i - \mathbf{H}_i\mathbf{C}_i^+\mathbf{D}$, $i = 1, 2$. Then, we obtain

$$\begin{aligned} E(\Delta) &= \mathbf{H}\mathbf{F}_C\Omega, \quad D(\vec{\Delta}) = \sigma^2(\mathbf{I}_m \otimes \mathbf{J})(\Sigma_2 \otimes \Sigma_1)(\mathbf{I}_m \otimes \mathbf{J})', \\ \text{cov}(\vec{\Delta}, \vec{\Upsilon}) &= \text{cov}(\vec{\Delta}_1, \vec{\Upsilon}) = \text{cov}(\vec{\Delta}_2, \vec{\Upsilon}) = \sigma^2(\mathbf{I}_m \otimes \mathbf{J})(\Sigma_2 \otimes \Sigma_1). \end{aligned} \quad (1.16)$$

$$\begin{aligned} E(\Delta_1) &= \mathbf{H}_1\mathbf{F}_{C_1}\Omega_1, \quad D(\vec{\Delta}_1) = \sigma^2(\mathbf{I}_m \otimes \mathbf{J})(\Sigma_2 \otimes \Sigma_1)(\mathbf{I}_m \otimes \mathbf{J})', \\ \text{cov}(\vec{\Delta}_1, \vec{\mathbf{X}}_2^\perp \Upsilon_1) &= \sigma^2(\mathbf{I}_m \otimes \mathbf{J})(\Sigma_2 \otimes \mathbf{X}_2^\perp \Sigma_1 \mathbf{X}_2^\perp), \\ \text{cov}(\vec{\Delta}_1, \vec{\mathbf{X}}_1^\perp \Upsilon_2) &= \sigma^2(\mathbf{I}_m \otimes \mathbf{J})(\Sigma_2 \otimes \mathbf{X}_1^\perp \Sigma_1 \mathbf{X}_1^\perp). \end{aligned} \quad (1.17)$$

$$\begin{aligned} E(\Delta_2) &= \mathbf{H}_2\mathbf{F}_{C_2}\Omega_2, \quad D(\vec{\Delta}_2) = \sigma^2(\mathbf{I}_m \otimes \mathbf{J})(\Sigma_2 \otimes \Sigma_1)(\mathbf{I}_m \otimes \mathbf{J})', \\ \text{cov}(\vec{\Delta}_2, \vec{\mathbf{X}}_2^\perp \Upsilon_1) &= \sigma^2(\mathbf{I}_m \otimes \mathbf{J})(\Sigma_2 \otimes \mathbf{X}_2^\perp \Sigma_1 \mathbf{X}_2^\perp), \\ \text{cov}(\vec{\Delta}_2, \vec{\mathbf{X}}_1^\perp \Upsilon_2) &= \sigma^2(\mathbf{I}_m \otimes \mathbf{J})(\Sigma_2 \otimes \mathbf{X}_1^\perp \Sigma_1 \mathbf{X}_1^\perp). \end{aligned} \quad (1.18)$$

In the present study, we consider the CMLM in (1.1) and some related reduced models in (1.2) and (1.3) with general assumptions. After reparameterizing these models given as $\mathcal{R}$ and $\mathcal{R}_i$, $i = 1, 2$ in (1.10)-(1.12), we derive analytic formulas for computing the best linear unbiased predictor/estimator (BLUP/BLUE).

In the statistical literature, LMs with exact restrictions on unknown parameters have been studied under various model assumptions, see e.g., [3, 5, 8–10, 12, 13, 18, 20, 23, 25]. We may refer to the studies [6, 7, 11, 19, 24], among others, in which both an CMLM and unconstrained MLM (UMLM) on unknown parameters has been considered from different perspectives.

2. PRELIMINARY

For the models that are taken into consideration in the work, consistency is given the first definition; see [17]. In the second definition, predictability is given; see [1]. The last definition is related to the BLUP and the BLUE.

Definition 1. The model $\mathcal{R}$ is consistent $\iff Y \in \mathcal{C}[\mathbf{X}\mathbf{F}_C, \Sigma_2 \otimes \Sigma_1]$ holds with probability (wp) 1.

For reduced models $\mathcal{R}_i$, $i = 1, 2$, it is possible to make the following consistency assumptions in accordance with Definition 1.

- (1) $\mathcal{R}_1$ is consistent $\iff \mathbf{X}_2^\perp Y_1 \in \mathcal{C}[\mathbf{X}_2^\perp \mathbf{X}_1 \mathbf{F}_{C_1}, \Sigma_2 \otimes \mathbf{X}_2^\perp \Sigma_1 \mathbf{X}_2^\perp]$ holds wp 1.
- (2) $\mathcal{R}_2$ is consistent $\iff \mathbf{X}_1^\perp Y_2 \in \mathcal{C}[\mathbf{X}_1^\perp \mathbf{X}_2 \mathbf{F}_{C_2}, \Sigma_2 \otimes \mathbf{X}_1^\perp \Sigma_1 \mathbf{X}_1^\perp]$ holds wp 1.

We note that $\mathcal{R}_i$ are consistent under the assumption of consistency of $\mathcal{R}$, $i = 1, 2$.

Definition 2. The predictability requirement of matrix Δ under $\mathcal{R}$ is described as holding the inclusion $\mathcal{C}((\mathbf{H}\mathbf{F}_C)') \subseteq \mathcal{C}((\mathbf{X}\mathbf{F}_C)')$. This requirement also corresponds to the estimability of $\mathbf{H}\mathbf{F}_C\Omega$ under $\mathcal{R}$. Further, the matrix Ψ is always predictable under $\mathcal{R}$.

For models $\mathcal{R}$ and $\mathcal{R}_i$, $i = 1, 2$, the following predictability/estimability requirements of Δ_i and its special cases are given in accordance with Definition 2. Let $\hat{\mathbf{H}}_1 = [\mathbf{H}_1 \mathbf{F}_{C_1}, \mathbf{0}]$ and $\hat{\mathbf{H}}_2 = [\mathbf{0}, \mathbf{H}_2 \mathbf{F}_{C_2}]$. Then

- (1) Δ_1 is predictable by Y in $\mathcal{R} \iff \mathcal{C}(\hat{\mathbf{H}}_1') \subseteq \mathcal{C}((\mathbf{X}\mathbf{F}_C)')$.
- (2) Δ_2 is predictable by Y in $\mathcal{R} \iff \mathcal{C}(\hat{\mathbf{H}}_2') \subseteq \mathcal{C}((\mathbf{X}\mathbf{F}_C)')$.
- (3) Δ_1 is predictable by $\mathbf{X}_2^\perp Y_1$ in $\mathcal{R}_1 \iff \mathcal{C}((\mathbf{H}_1 \mathbf{F}_{C_1})') \subseteq \mathcal{C}((\mathbf{X}_2^\perp \mathbf{X}_1 \mathbf{F}_{C_1})')$.
- (4) Δ_2 is predictable by $\mathbf{X}_1^\perp Y_2$ in $\mathcal{R}_2 \iff \mathcal{C}((\mathbf{H}_2 \mathbf{F}_{C_2})') \subseteq \mathcal{C}((\mathbf{X}_1^\perp \mathbf{X}_2 \mathbf{F}_{C_2})')$.

We note that if Δ_i is predictable under $\mathcal{R}_i$, $i = 1, 2$, then it is predictable under $\mathcal{R}$.

Definition 3. If a matrix $\mathbf{K} \in \mathbb{R}_{k,n}$ and $\mathbf{K}_1 \in \mathbb{R}_{k,m}$, or $\hat{\mathbf{K}} \in \mathbb{R}_{km,nm}$ and $\hat{\mathbf{K}}_1 \in \mathbb{R}_{km,1}$ exists with

$$\begin{aligned} D(\overline{\mathbf{K}\mathbf{Y} + \mathbf{K}_1 - \vec{\Phi}}) &= \min \text{ s.t. } E(\mathbf{K}\mathbf{Y} + \mathbf{K}_1 - \vec{\Phi}) = \mathbf{0}, \text{ or,} \\ D(\hat{\mathbf{K}}\vec{\mathbf{Y}} + \hat{\mathbf{K}}_1 - \vec{\Phi}) &= \min \text{ s.t. } E(\hat{\mathbf{K}}\vec{\mathbf{Y}} + \hat{\mathbf{K}}_1 - \vec{\Phi}) = \mathbf{0} \end{aligned} \quad (2.1)$$

in the Löwner partial ordering (LPO), the linear statistic $\mathbf{K}\mathbf{Y} + \mathbf{K}_1$ is said to be the BLUP of Φ and is represented by $\mathbf{K}\mathbf{Y} + \mathbf{K}_1 = \text{BLUP}_{\mathcal{M}}(\mathbf{H}\Theta + \mathbf{J}\Psi)$. If $\mathbf{J} = \mathbf{0}$ in Φ ,

$\mathbf{KY} + \mathbf{K}_1$ is the well-known the BLUE of $\mathbf{H}\Theta$, represented by $\text{BLUE}_{\mathcal{M}}(\mathbf{H}\Theta)$, under $\mathcal{M}$.

Suppose that Δ or $\vec{\Delta}$ in (1.13) is predictable under $\mathcal{R}$. $\mathbf{KY}$ is defined to be the BLUP of Δ if there exists $\mathbf{KY}$ such that

$$D(\overrightarrow{\mathbf{KY} - \Delta}) = \min \text{ s.t. } E(\mathbf{KY} - \Delta) = \mathbf{0} \quad (2.2)$$

holds in the LPO, and is represented by

$$\mathbf{KY} = \text{BLUP}_{\mathcal{R}}(\Delta) = \text{BLUP}_{\mathcal{R}}(\mathbf{H}\mathbf{F}_C\Omega + \mathbf{J}\Psi).$$

If $\mathbf{J} = \mathbf{0}$ in Δ , $\mathbf{KY}$ corresponds the BLUE of $\mathbf{H}\mathbf{F}_C\Omega$, represented by $\text{BLUE}_{\mathcal{R}}(\mathbf{H}\mathbf{F}_C\Omega)$. Further, if $\mathbf{H} = \mathbf{0}$ and $\mathbf{J} = \mathbf{I}_n$ in Δ , $\mathbf{KY}$ corresponds the BLUP of Ψ , i.e., $\text{BLUP}_{\mathcal{R}}(\Psi)$, under $\mathcal{R}$.

The following results are given in accordance with Definition 3. Let Δ_1 or $\vec{\Delta}_1$ be predictable under $\mathcal{R}$ and $\mathcal{R}_i$, $i = 1, 2$. Then

- (1) $\text{BLUP}_{\mathcal{R}}(\Delta_1) = \mathbf{K}_1\Upsilon \iff D(\overrightarrow{\mathbf{K}_1\Upsilon - \Delta_1}) = \min \text{ s.t. } E(\mathbf{K}_1\Upsilon - \Delta_1) = \mathbf{0}.$
- (2) $\text{BLUP}_{\mathcal{R}}(\Delta_2) = \mathbf{K}_2\Upsilon \iff D(\overrightarrow{\mathbf{K}_2\Upsilon - \Delta_2}) = \min \text{ s.t. } E(\mathbf{K}_2\Upsilon - \Delta_2) = \mathbf{0}.$
- (3) $\text{BLUP}_{\mathcal{R}_1}(\Delta_1) = \mathbf{G}_1\mathbf{X}_2^\perp\Upsilon_1$
 $\iff D(\overrightarrow{\mathbf{G}_1\mathbf{X}_2^\perp\Upsilon_1 - \Delta_1}) = \min \text{ s.t. } E(\mathbf{G}_1\mathbf{X}_2^\perp\Upsilon_1 - \Delta_1) = \mathbf{0}.$
- (4) $\text{BLUP}_{\mathcal{R}_2}(\Delta_2) = \mathbf{G}_2\mathbf{X}_1^\perp\Upsilon_2$
 $\iff D(\overrightarrow{\mathbf{G}_2\mathbf{X}_1^\perp\Upsilon_2 - \Delta_2}) = \min \text{ s.t. } E(\mathbf{G}_2\mathbf{X}_1^\perp\Upsilon_2 - \Delta_2) = \mathbf{0}.$

Finally, we give the following two lemmas, which are required in the sections that follow. For the first lemma; see, [15], and for the last one; see, [20].

Lemma 1. *The linear matrix equation $\mathbf{H}\mathbf{X} = \mathbf{D}$ is consistent $\Leftrightarrow \mathbf{r}[\mathbf{H}, \mathbf{D}] = \mathbf{r}(\mathbf{H})$, or equivalently, $\mathbf{H}\mathbf{H}^+\mathbf{D} = \mathbf{D}$. Then, the general solution of this equation can be written as $\mathbf{X} = \mathbf{H}^+\mathbf{D} + (\mathbf{I} - \mathbf{H}^+\mathbf{H})\mathbf{V}$, where $\mathbf{V}$ is an arbitrary matrix.*

Lemma 2. *Let $\mathbf{B} \in \mathbb{R}_{m,p}$, $\mathbf{A} \in \mathbb{R}_{n,p}$ be given matrices, and let $\mathbf{Q} \in \mathbb{R}_n^{\succ}$. Suppose that there exists $\mathbf{X}_0 \in \mathbb{R}_{m,n}$ such that $\mathbf{X}_0\mathbf{B} = \mathbf{A}$. Then the maximal positive inertia of $\mathbf{X}_0\mathbf{Q}\mathbf{X}_0' - \mathbf{X}\mathbf{Q}\mathbf{X}'$ subject to all solutions of $\mathbf{X}\mathbf{B} = \mathbf{A}$ is*

$$\max_{\mathbf{X}\mathbf{B}=\mathbf{A}} i_+(\mathbf{X}_0\mathbf{Q}\mathbf{X}_0' - \mathbf{X}\mathbf{Q}\mathbf{X}') = \mathbf{r} \begin{bmatrix} \mathbf{X}_0\mathbf{Q} \\ \mathbf{B}' \end{bmatrix} - \mathbf{r}(\mathbf{B}) = \mathbf{r}(\mathbf{X}_0\mathbf{Q}\mathbf{B}^\perp).$$

Hence a solution $\mathbf{X}_0$ of $\mathbf{X}_0\mathbf{B} = \mathbf{A}$ exists such that $\mathbf{X}_0\mathbf{Q}\mathbf{X}_0' \preccurlyeq \mathbf{X}\mathbf{Q}\mathbf{X}'$ holds for all solutions of $\mathbf{X}\mathbf{B} = \mathbf{A} \Leftrightarrow$ both the equations $\mathbf{X}_0\mathbf{B} = \mathbf{A}$ and $\mathbf{X}_0\mathbf{Q}\mathbf{B}^\perp = \mathbf{0}$ are satisfied by $\mathbf{X}$.

3. BLUPS/BLUES UNDER A CMLM WITH SOME RELATED REDUCED MODELS

The following theorems, respectively, are collections of the basic findings on BLUP of Δ and Δ_i in (1.10)-(1.12) and its sub-cases. These theorems use a method described in [4]. For different approaches; see, e.g., [16, 22].

Theorem 1. Suppose that Δ is predictable under $\mathcal{R}$ in (1.11). For Δ under $\mathcal{R}$, let $\mathbf{K}\Upsilon$ and $\mathbf{L}\Upsilon$ be unbiased linear predictors. Then the maximal positive inertia of $D(\overrightarrow{\mathbf{K}\Upsilon - \Delta}) - D(\overrightarrow{\mathbf{L}\Upsilon - \Delta})$ subject to $\mathbf{L}\mathbf{X}\mathbf{F}_C = \mathbf{H}\mathbf{F}_C$ is

$$\max_{E(\mathbf{L}\Upsilon - \Delta) = \mathbf{0}} i_+(D(\overrightarrow{\mathbf{K}\Upsilon - \Delta}) - D(\overrightarrow{\mathbf{L}\Upsilon - \Delta})) = r \left([\mathbf{K}, -\mathbf{I}_k] \begin{bmatrix} \mathbf{I}_n \\ \mathbf{J} \end{bmatrix} \text{cov}(\Upsilon) \begin{bmatrix} \mathbf{I}_n \\ \mathbf{J} \end{bmatrix}' \begin{bmatrix} \mathbf{X}\mathbf{F}_C \\ \mathbf{H}\mathbf{F}_C \end{bmatrix}^\perp \right). \quad (3.1)$$

Hence,

$$\begin{aligned} D(\overrightarrow{\mathbf{K}\Upsilon - \Delta}) &= \min \text{ s.t. } E(\mathbf{K}\Upsilon - \Delta) = \mathbf{0} \Leftrightarrow \mathbf{K}\Upsilon = \text{BLUP}_{\mathcal{R}}(\Delta) \\ &\Leftrightarrow \mathbf{K} [\mathbf{X}\mathbf{F}_C, \Sigma_1(\mathbf{X}\mathbf{F}_C)^\perp] = [\mathbf{H}\mathbf{F}_C, \mathbf{J}\Sigma_1(\mathbf{X}\mathbf{F}_C)^\perp]. \end{aligned} \quad (3.2)$$

(3.2) is consistent and the BLUP of Δ under $\mathcal{R}$ can be written as follows by considering the general solution of this equation:

$$\mathbf{K}\Upsilon = \text{BLUP}_{\mathcal{R}}(\Delta) \Leftrightarrow \mathbf{K} [\mathbf{X}\mathbf{F}_C, \Sigma_1(\mathbf{X}\mathbf{F}_C)^\perp] = [\mathbf{H}\mathbf{F}_C, \mathbf{J}\Sigma_1(\mathbf{X}\mathbf{F}_C)^\perp]$$

and the general solution of this consistent equation can be written as

$$\begin{aligned} \text{BLUP}_{\mathcal{R}}(\Delta) &= \mathbf{K}\Upsilon = \left([\mathbf{H}\mathbf{F}_C, \mathbf{J}\Sigma_1(\mathbf{X}\mathbf{F}_C)^\perp] \mathbf{W}_r^+ + \mathbf{U}_r \mathbf{W}_r^\perp \right) \Upsilon, \\ \overrightarrow{\text{BLUP}_{\mathcal{R}}(\Delta)} &= (\mathbf{I}_m \otimes \mathbf{K}) \vec{\Upsilon} \\ &= \left(\mathbf{I}_m \otimes \left([\mathbf{H}\mathbf{F}_C, \mathbf{J}\Sigma_1(\mathbf{X}\mathbf{F}_C)^\perp] \mathbf{W}_r^+ + \mathbf{U}_r \mathbf{W}_r^\perp \right) \right) \vec{\Upsilon}, \end{aligned} \quad (3.3)$$

where $\mathbf{U}_r \in \mathbb{R}_{k,n}$ is an arbitrary matrix and $\mathbf{W}_r = [\mathbf{X}\mathbf{F}_C, \Sigma_1(\mathbf{X}\mathbf{F}_C)^\perp]$. In particular,

$$\begin{aligned} \text{BLUE}_{\mathcal{R}}(\mathbf{H}\mathbf{F}_C\Omega) &= \mathbf{K}\Upsilon = \left([\mathbf{H}\mathbf{F}_C, \mathbf{0}] \mathbf{W}_r^+ + \mathbf{U}_r \mathbf{W}_r^\perp \right) \Upsilon, \\ \overrightarrow{\text{BLUE}_{\mathcal{R}}(\mathbf{H}\mathbf{F}_C\Omega)} &= (\mathbf{I}_m \otimes \mathbf{K}) \vec{\Upsilon} \\ &= \left(\mathbf{I}_m \otimes \left([\mathbf{H}\mathbf{F}_C, \mathbf{0}] \mathbf{W}_r^+ + \mathbf{U}_r \mathbf{W}_r^\perp \right) \right) \vec{\Upsilon}, \end{aligned} \quad (3.4)$$

and

$$\begin{aligned} \text{BLUP}_{\mathcal{R}}(\Psi) &= \mathbf{K}\Upsilon = \left([\mathbf{0}, \Sigma_1(\mathbf{X}\mathbf{F}_C)^\perp] \mathbf{W}_r^+ + \mathbf{U}_r \mathbf{W}_r^\perp \right) \Upsilon, \\ \overrightarrow{\text{BLUP}_{\mathcal{R}}(\Psi)} &= (\mathbf{I}_m \otimes \mathbf{K}) \vec{\Upsilon} \\ &= \left(\mathbf{I}_m \otimes \left([\mathbf{0}, \Sigma_1(\mathbf{X}\mathbf{F}_C)^\perp] \mathbf{W}_r^+ + \mathbf{U}_r \mathbf{W}_r^\perp \right) \right) \vec{\Upsilon}. \end{aligned} \quad (3.5)$$

Further, the matrix $\mathbf{W}_r$ satisfies

$$r(\mathbf{W}_r) = r[\mathbf{X}\mathbf{F}_C, \Sigma_1] \text{ and } \mathcal{C}(\mathbf{W}_r) = \mathcal{C}[\mathbf{X}\mathbf{F}_C, \Sigma_1]. \quad (3.6)$$

Also, the following equalities hold.

$$\begin{aligned} D[\overrightarrow{\text{BLUP}_{\mathcal{R}}(\Delta)}] &= \sigma^2 \Sigma_2 \otimes [\mathbf{H}\mathbf{F}_C, \mathbf{J}\Sigma_1(\mathbf{X}\mathbf{F}_C)^\perp] \mathbf{W}_r^+ \Sigma_1 \\ &\quad \times ([\mathbf{H}\mathbf{F}_C, \mathbf{J}\Sigma_1(\mathbf{X}\mathbf{F}_C)^\perp] \mathbf{W}_r^+)', \end{aligned}$$

$$\begin{aligned}
\text{cov}\{\overrightarrow{\text{BLUP}_{\mathcal{R}}(\Delta)}, \overrightarrow{\Delta}\} &= \sigma^2 \Sigma_2 \otimes [\mathbf{H}\mathbf{F}_C, \mathbf{J}\Sigma_1(\mathbf{X}\mathbf{F}_C)^\perp] \mathbf{W}_r^+ \Sigma_1 \mathbf{J}', \\
\mathbf{D}[\overrightarrow{\Delta} - \overrightarrow{\text{BLUP}_{\mathcal{R}}(\Delta)}] &= \sigma^2 \Sigma_2 \otimes ([\mathbf{H}\mathbf{F}_C, \mathbf{J}\Sigma_1(\mathbf{X}\mathbf{F}_C)^\perp] \mathbf{W}_r^+ - \mathbf{J}) \\
&\quad \times \Sigma_1 ([\mathbf{H}\mathbf{F}_C, \mathbf{J}\Sigma_1(\mathbf{X}\mathbf{F}_C)^\perp] \mathbf{W}_r^+ - \mathbf{J})'.
\end{aligned} \tag{3.7}$$

Proof of Theorem 1. Suppose that $\mathbf{K}\Upsilon$ and $\mathbf{L}\Upsilon$ are two unbiased linear predictors for Δ in $\mathcal{R}$. Then, the expected value and covariance matrix of $\mathbf{K}\Upsilon - \Delta$ are written as

$$\mathbf{E}(\mathbf{K}\Upsilon - \Delta) = \mathbf{0} \Leftrightarrow \mathbf{K}\mathbf{X}\mathbf{F}_C = \mathbf{H}\mathbf{F}_C \Leftrightarrow [\mathbf{K}, -\mathbf{I}_k] \begin{bmatrix} \mathbf{X}\mathbf{F}_C \\ \mathbf{H}\mathbf{F}_C \end{bmatrix} = \mathbf{0} \tag{3.8}$$

and

$$\begin{aligned}
\mathbf{D}(\overrightarrow{\mathbf{K}\Upsilon - \Delta}) &= (\mathbf{I}_m \otimes (\mathbf{K} - \mathbf{J})) \text{cov}(\Psi) (\mathbf{I}_m \otimes (\mathbf{K} - \mathbf{J}))' \\
&= \sigma^2 (\mathbf{I}_m \otimes (\mathbf{K} - \mathbf{J})) (\Sigma_2 \otimes \Sigma_1) (\mathbf{I}_m \otimes (\mathbf{K} - \mathbf{J}))' \\
&= \sigma^2 \Sigma_2 \otimes (\mathbf{K} - \mathbf{J}) \Sigma_1 (\mathbf{K} - \mathbf{J})' \\
&= \sigma^2 \Sigma_2 \otimes [\mathbf{K}, -\mathbf{I}_k] \begin{bmatrix} \mathbf{I}_n \\ \mathbf{J} \end{bmatrix} \Sigma_1 \begin{bmatrix} \mathbf{I}_n \\ \mathbf{J} \end{bmatrix}' [\mathbf{K}, -\mathbf{I}_k]' := \Sigma_2 \otimes f(\mathbf{K}),
\end{aligned} \tag{3.9}$$

where

$$f(\mathbf{K}) = [\mathbf{K}, -\mathbf{I}_k] \begin{bmatrix} \mathbf{I}_n \\ \mathbf{J} \end{bmatrix} \Sigma_1 \begin{bmatrix} \mathbf{I}_n \\ \mathbf{J} \end{bmatrix}' [\mathbf{K}, -\mathbf{I}_k]'$$

By using $\mathbf{L}$ in place of $\mathbf{K}$, the equivalent formulas as in (3.8) and (3.9) may also be given for the other unbiased linear predictor $\mathbf{L}\Upsilon$ for Δ under $\mathcal{R}$. In order to obtain solution $\mathbf{K}$ of the consistent linear matrix equation $\mathbf{K}\mathbf{X}\mathbf{F}_C = \mathbf{H}\mathbf{F}_C$, the matrix minimization problem described in Definition 3 for finding the BLUP of Δ under $\mathcal{R}$ can be expressed such that

$$\Sigma_2 \otimes f(\mathbf{K}) \preceq \Sigma_2 \otimes f(\mathbf{L}) \text{ s.t. } \mathbf{L}\mathbf{X}\mathbf{F}_C = \mathbf{H}\mathbf{F}_C \tag{3.10}$$

or equivalently,

$$f(\mathbf{K}) \preceq f(\mathbf{L}) \text{ s.t. } \mathbf{L}\mathbf{X}\mathbf{F}_C = \mathbf{H}\mathbf{F}_C$$

because Σ_2 is a non-null matrix. According to Lemma 2, (3.10) is a typical constrained quadratic matrix-valued function optimization problem in the LPO. Lemma 2 gives us the basic formula for the BLUP of Δ in (3.2), and Lemma 1 gives us the expression for the BLUP of Δ under $\mathcal{R}$ in (3.3). Setting $\mathbf{J} = \mathbf{0}$, and $\mathbf{H} = \mathbf{0}$ and $\mathbf{J} = \mathbf{I}_n$ at (3.3) yields (3.4) and (3.5), respectively. The expressions in (3.6) are well-known results; see also [21, Lemma 2.1(a)]. From (1.17) and (3.3),

$$\begin{aligned}
\mathbf{D}[\overrightarrow{\text{BLUP}_{\mathcal{R}}(\Delta)}] &= (\mathbf{I}_m \otimes \mathbf{K}) \text{cov}(\Psi) (\mathbf{I}_m \otimes \mathbf{K})' = \sigma^2 (\mathbf{I}_m \otimes \mathbf{K}) (\Sigma_2 \otimes \Sigma_1) (\mathbf{I}_m \otimes \mathbf{K})' \\
&= \sigma^2 \Sigma_2 \otimes \mathbf{K} \Sigma_1 \mathbf{K}' \\
&= \sigma^2 \Sigma_2 \otimes ([\mathbf{H}\mathbf{F}_C, \mathbf{J}\Sigma_1(\mathbf{X}\mathbf{F}_C)^\perp] \mathbf{W}_r^+ + \mathbf{U}_r \mathbf{W}_r^\perp)
\end{aligned}$$

$$\begin{aligned}
& \times \Sigma_1 \left([\mathbf{H}\mathbf{F}_C, \mathbf{J}\Sigma_1(\mathbf{X}\mathbf{F}_C)^\perp] \mathbf{W}_r^+ + \mathbf{U}_r \mathbf{W}_r^\perp \right)' \\
& = \sigma^2 \Sigma_2 \otimes [\mathbf{H}\mathbf{F}_C, \mathbf{J}\Sigma_1(\mathbf{X}\mathbf{F}_C)^\perp] \mathbf{W}_r^+ \\
& \quad \times \Sigma_1 \left([\mathbf{H}\mathbf{F}_C, \mathbf{J}\Sigma_1(\mathbf{X}\mathbf{F}_C)^\perp] \mathbf{W}_r^+ \right)', \\
\text{cov}\{\overrightarrow{\text{BLUP}_{\mathcal{R}}(\Delta)}, \overrightarrow{\Delta}\} & = (\mathbf{I}_m \otimes \mathbf{K}) \text{cov}(\Psi) (\mathbf{I}_m \otimes \mathbf{J})' \\
& = \sigma^2 (\mathbf{I}_m \otimes \mathbf{K}) (\Sigma_2 \otimes \Sigma_1) (\mathbf{I}_m \otimes \mathbf{J})' = \sigma^2 \Sigma_2 \otimes \mathbf{K} \Sigma_1 \mathbf{J}' \\
& = \sigma^2 \Sigma_2 \otimes [\mathbf{H}\mathbf{F}_C, \mathbf{J}\Sigma_1(\mathbf{X}\mathbf{F}_C)^\perp] \mathbf{W}_r^+ \Sigma_1 \mathbf{J}', \\
D[\overrightarrow{\Delta} - \overrightarrow{\text{BLUP}_{\mathcal{R}}(\Delta)}] & = \sigma^2 \Sigma_2 \otimes \left([\mathbf{H}\mathbf{F}_C, \mathbf{J}\Sigma_1(\mathbf{X}\mathbf{F}_C)^\perp] \mathbf{W}_r^+ + \mathbf{U}_r \mathbf{W}_r^\perp - \mathbf{J} \right) \\
& \quad \times \Sigma_1 \left([\mathbf{H}\mathbf{F}_C, \mathbf{J}\Sigma_1(\mathbf{X}\mathbf{F}_C)^\perp] \mathbf{W}_r^+ + \mathbf{U}_r \mathbf{W}_r^\perp - \mathbf{J} \right)' \\
& = \sigma^2 \Sigma_2 \otimes ([\mathbf{H}\mathbf{F}_C, \mathbf{J}\Sigma_1(\mathbf{X}\mathbf{F}_C)^\perp] \mathbf{W}_r^+ - \mathbf{J}) \\
& \quad \times \Sigma_1 ([\mathbf{H}\mathbf{F}_C, \mathbf{J}\Sigma_1(\mathbf{X}\mathbf{F}_C)^\perp] \mathbf{W}_r^+ - \mathbf{J})'.
\end{aligned}$$

Thus establishing equalities in (3.7). $\square$

Theorem 2. Suppose that Δ_i is predictable under $\mathcal{R}_i$, $i = 1, 2$ in (1.11)-(1.12) (also predictable under $\mathcal{R}$ in (1.10)). Thus, the results that follow are provided.

- (1) $\overrightarrow{\text{BLUP}_{\mathcal{R}}(\Delta_1)} = (\mathbf{I}_m \otimes ([\hat{\mathbf{H}}_1, \mathbf{J}\Sigma_1(\mathbf{X}\mathbf{F}_C)^\perp] \mathbf{W}_r^+ + \mathbf{P}_1 \mathbf{W}_r^\perp)) \overrightarrow{\mathbf{Y}}$, where $\hat{\mathbf{H}}_1 = [\mathbf{H}_1 \mathbf{F}_{C_1}, \mathbf{0}]$, $\mathbf{W}_r = [\mathbf{X}\mathbf{F}_C, \Sigma_1(\mathbf{X}\mathbf{F}_C)^\perp]$ and $\mathbf{P}_1 \in \mathbb{R}_{s,n}$ is an arbitrary matrix. Then,

$$\begin{aligned}
D[\overrightarrow{\text{BLUP}_{\mathcal{R}}(\Delta_1)}] & = \sigma^2 \Sigma_2 \otimes [\hat{\mathbf{H}}_1, \mathbf{J}\Sigma_1(\mathbf{X}\mathbf{F}_C)^\perp] \mathbf{W}_r^+ \Sigma_1 \\
& \quad \times ([\hat{\mathbf{H}}_1, \mathbf{J}\Sigma_1(\mathbf{X}\mathbf{F}_C)^\perp] \mathbf{W}_r^+)' , \tag{3.11}
\end{aligned}$$

$$\begin{aligned}
D[\overrightarrow{\Delta_1} - \overrightarrow{\text{BLUP}_{\mathcal{R}}(\Delta_1)}] & = \sigma^2 \Sigma_2 \otimes ([\hat{\mathbf{H}}_1, \mathbf{J}\Sigma_1(\mathbf{X}\mathbf{F}_C)^\perp] \mathbf{W}_r^+ - \mathbf{J}) \\
& \quad \times \Sigma_1 ([\hat{\mathbf{H}}_1, \mathbf{J}\Sigma_1(\mathbf{X}\mathbf{F}_C)^\perp] \mathbf{W}_r^+ - \mathbf{J})' . \tag{3.12}
\end{aligned}$$

- (2) $\overrightarrow{\text{BLUP}_{\mathcal{R}}(\Delta_2)} = (\mathbf{I}_m \otimes ([\hat{\mathbf{H}}_2, \mathbf{J}\Sigma_1(\mathbf{X}\mathbf{F}_C)^\perp] \mathbf{W}_r^+ + \mathbf{P}_2 \mathbf{W}_r^\perp)) \overrightarrow{\mathbf{Y}}$, where $\hat{\mathbf{H}}_2 = [\mathbf{0}, \mathbf{H}_2 \mathbf{F}_{C_2}]$, $\mathbf{W}_r = [\mathbf{X}\mathbf{F}_C, \Sigma_1(\mathbf{X}\mathbf{F}_C)^\perp]$ and $\mathbf{P}_2 \in \mathbb{R}_{s,n}$ is an arbitrary matrix. Then,

$$\begin{aligned}
D[\overrightarrow{\text{BLUP}_{\mathcal{R}}(\Delta_2)}] & = \sigma^2 \Sigma_2 \otimes [\hat{\mathbf{H}}_2, \mathbf{J}\Sigma_1(\mathbf{X}\mathbf{F}_C)^\perp] \mathbf{W}_r^+ \Sigma_1 \\
& \quad \times ([\hat{\mathbf{H}}_2, \mathbf{J}\Sigma_1(\mathbf{X}\mathbf{F}_C)^\perp] \mathbf{W}_r^+)' , \tag{3.13}
\end{aligned}$$

$$\begin{aligned}
D[\overrightarrow{\Delta_2} - \overrightarrow{\text{BLUP}_{\mathcal{R}}(\Delta_2)}] & = \sigma^2 \Sigma_2 \otimes ([\hat{\mathbf{H}}_2, \mathbf{J}\Sigma_1(\mathbf{X}\mathbf{F}_C)^\perp] \mathbf{W}_r^+ - \mathbf{J}) \\
& \quad \times \Sigma_1 ([\hat{\mathbf{H}}_2, \mathbf{J}\Sigma_1(\mathbf{X}\mathbf{F}_C)^\perp] \mathbf{W}_r^+ - \mathbf{J})' . \tag{3.14}
\end{aligned}$$

- (3) $\overrightarrow{\text{BLUP}_{\mathcal{R}_1}(\Delta_1)} = (\mathbf{I}_m \otimes (\mathbf{M}_1 \mathbf{W}_{r_1}^+ + \mathbf{U}_1 \mathbf{W}_{r_1}^\perp)) \overrightarrow{\mathbf{Y}}_1$,
where $\mathbf{U}_1 \in \mathbb{R}_{s,n}$ is arbitrary matrix, $\mathbf{M}_1 = [\mathbf{H}_1 \mathbf{F}_{C_1}, \mathbf{J}\Sigma_1 \mathbf{X}_2^\perp (\mathbf{X}_2^\perp \mathbf{X}_1 \mathbf{F}_{C_1})^\perp]$

and $\mathbf{W}_{r_1} = [\mathbf{X}_2^\perp \mathbf{X}_1 \mathbf{F}_{C_1}, \quad \mathbf{X}_2^\perp \Sigma_1 \mathbf{X}_2^\perp (\mathbf{X}_2^\perp \mathbf{X}_1 \mathbf{F}_{C_1})^\perp]^\top$. Then,

$$D[\overrightarrow{\text{BLUP}_{\mathcal{R}_1}(\Delta_1)}] = \sigma^2 \Sigma_2 \otimes \mathbf{M}_1 \mathbf{W}_{r_1}^+ \mathbf{X}_2^\perp \Sigma_1 \mathbf{X}_2^\perp (\mathbf{M}_1 \mathbf{W}_{r_1}^+)', \quad (3.15)$$

$$D[\overrightarrow{\Delta_1} - \overrightarrow{\text{BLUP}_{\mathcal{R}_1}(\Delta_1)}] = \sigma^2 \Sigma_2 \otimes (\mathbf{M}_1 \mathbf{W}_{r_1}^+ \mathbf{X}_2^\perp - \mathbf{J}) \Sigma_1 (\mathbf{M}_1 \mathbf{W}_{r_1}^+ \mathbf{X}_2^\perp - \mathbf{J})'. \quad (3.16)$$

$$(4) \overrightarrow{\text{BLUP}_{\mathcal{R}_2}(\Delta_2)} = (\mathbf{I}_m \otimes (\mathbf{M}_2 \mathbf{W}_{r_2}^+ + \mathbf{U}_2 \mathbf{W}_{r_2}^\perp)) \overrightarrow{\Upsilon_2},$$

where $\mathbf{U}_2 \in \mathbb{R}_{s,n}$ is arbitrary matrix, $\mathbf{M}_2 = [\mathbf{H}_2 \mathbf{F}_{C_2}, \quad \mathbf{J} \Sigma_1 \mathbf{X}_1^\perp (\mathbf{X}_1^\perp \mathbf{X}_2 \mathbf{F}_{C_2})^\perp]$ and $\mathbf{W}_{r_2} = [\mathbf{X}_1^\perp \mathbf{X}_2 \mathbf{F}_{C_2}, \quad \mathbf{X}_1^\perp \Sigma_1 \mathbf{X}_1^\perp (\mathbf{X}_1^\perp \mathbf{X}_2 \mathbf{F}_{C_2})^\perp]^\top$. Then,

$$D[\overrightarrow{\text{BLUP}_{\mathcal{R}_2}(\Delta_2)}] = \sigma^2 \Sigma_2 \otimes \mathbf{M}_2 \mathbf{W}_{r_2}^+ \mathbf{X}_1^\perp \Sigma_1 \mathbf{X}_1^\perp (\mathbf{M}_2 \mathbf{W}_{r_2}^+)', \quad (3.17)$$

$$D[\overrightarrow{\Delta_2} - \overrightarrow{\text{BLUP}_{\mathcal{R}_2}(\Delta_2)}] = \sigma^2 \Sigma_2 \otimes (\mathbf{M}_2 \mathbf{W}_{r_2}^+ \mathbf{X}_1^\perp - \mathbf{J}) \Sigma_1 (\mathbf{M}_2 \mathbf{W}_{r_2}^+ \mathbf{X}_1^\perp - \mathbf{J})'. \quad (3.18)$$

Proof of Theorem 2. For BLUP of Δ_i under $\mathcal{R}_i$, the fundamental equations can be proved similarly to Theorem 1. $\square$

The basic conclusions are reached by changing (3.3) in Theorem 1 into (3.9), accordingly.

Corollary 1. $\mathcal{M}$ and $\mathcal{R}$ be as given (1.1) and (1.11), respectively. Let Φ and Δ be as given in (1.4) and (1.13), respectively, and suppose that Δ is predictable under $\mathcal{R}$. Then

$$\begin{aligned} \text{BLUP}_{\mathcal{M}}(\Phi) &= \mathbf{H} \mathbf{C}^+ \mathbf{D} + \text{BLUP}_{\mathcal{R}}(\Delta) \\ &= \mathbf{H} \mathbf{C}^+ \mathbf{D} + \left([\mathbf{H} \mathbf{F}_C, \quad \mathbf{J} \Sigma_1 (\mathbf{X} \mathbf{F}_C)^\perp] \mathbf{W}_r^+ + \mathbf{U}_r \mathbf{W}_r^\perp \right) \\ &\quad \times (\mathbf{Y} - \mathbf{X} \mathbf{C}^+ \mathbf{D}) \end{aligned}$$

where $\mathbf{U}_r \in \mathbb{R}_{k,n}$ is arbitrary matrix, and $\mathbf{W}_r = [\mathbf{X} \mathbf{F}_C, \quad \Sigma_1 (\mathbf{X} \mathbf{F}_C)^\perp]^\top$. In particular,

$$\begin{aligned} \text{BLUP}_{\mathcal{M}}(\mathbf{H}\Theta) &= \mathbf{H} \mathbf{C}^+ \mathbf{D} + \text{BLUP}_{\mathcal{R}}(\mathbf{H} \mathbf{F}_C \Omega) \\ &= \mathbf{H} \mathbf{C}^+ \mathbf{D} + \left([\mathbf{H} \mathbf{F}_C, \quad \mathbf{0}] \mathbf{W}_r^+ + \mathbf{U}_r \mathbf{W}_r^\perp \right) (\mathbf{Y} - \mathbf{X} \mathbf{C}^+ \mathbf{D}). \end{aligned}$$

Corollary 2. $\mathcal{M}_i$ and $\mathcal{R}_i$ be as given (1.2)-(1.3) and (1.11)-(1.12), respectively. Let Φ_i and Δ_i be as given in (1.5)-(1.6) and (1.14)-(1.15), respectively, and suppose that Δ_i , $i = 1, 2$ is predictable under $\mathcal{R}_i$. Then

$$\begin{aligned} \text{BLUP}_{\mathcal{M}_1}(\Phi_1) &= \mathbf{H}_1 \mathbf{C}_1^+ \mathbf{D} + \text{BLUP}_{\mathcal{R}_1}(\Delta_1) \\ &= \mathbf{H}_1 \mathbf{C}_1^+ \mathbf{D} + \left(\mathbf{M}_1 \mathbf{W}_{r_1}^+ + \mathbf{U}_1 \mathbf{W}_{r_1}^\perp \right) (\mathbf{Y} - \mathbf{X}_1 \mathbf{C}_1^+ \mathbf{D}) \end{aligned}$$

where $\mathbf{U}_1 \in \mathbb{R}_{k,n}$ is arbitrary matrix, $\mathbf{M}_1 = [\mathbf{H}_1 \mathbf{F}_{C_1}, \quad \mathbf{J} \Sigma_1 \mathbf{X}_2^\perp (\mathbf{X}_2^\perp \mathbf{X}_1 \mathbf{F}_{C_1})^\perp]$ and $\mathbf{W}_{r_1} = [\mathbf{X}_2^\perp \mathbf{X}_1 \mathbf{F}_{C_1}, \quad \mathbf{X}_2^\perp \Sigma_1 \mathbf{X}_2^\perp (\mathbf{X}_2^\perp \mathbf{X}_1 \mathbf{F}_{C_1})^\perp]^\top$. In particular,

$$\text{BLUP}_{\mathcal{M}_1}(\mathbf{H}_1 \Theta_1) = \mathbf{H}_1 \mathbf{C}_1^+ \mathbf{D} + \text{BLUP}_{\mathcal{R}_1}(\mathbf{H}_1 \mathbf{F}_{C_1} \Omega_1)$$

$$= \mathbf{H}_1 \mathbf{C}_1^+ \mathbf{D} + \left([\mathbf{H}_1 \mathbf{F}_{C_1}, \quad \mathbf{0}] \mathbf{W}_{r_1}^+ + \mathbf{U}_1 \mathbf{W}_{r_1}^\perp \right) (\mathbf{Y} - \mathbf{X}_1 \mathbf{C}_1^+ \mathbf{D})$$

and

$$\begin{aligned} \text{BLUP}_{\mathcal{M}_2}(\Phi_2) &= \mathbf{H}_2 \mathbf{C}_2^+ \mathbf{D} + \text{BLUP}_{\mathcal{R}_2}(\Delta_2) \\ &= \mathbf{H}_2 \mathbf{C}_2^+ \mathbf{D} + \left(\mathbf{M}_2 \mathbf{W}_{r_2}^+ + \mathbf{U}_2 \mathbf{W}_{r_2}^\perp \right) (\mathbf{Y} - \mathbf{X}_2 \mathbf{C}_2^+ \mathbf{D}) \end{aligned}$$

where $\mathbf{U}_2 \in \mathbb{R}_{k,n}$ is arbitrary matrix, $\mathbf{M}_2 = [\mathbf{H}_2 \mathbf{F}_{C_2}, \quad \mathbf{J} \Sigma_1 \mathbf{X}_1^\perp (\mathbf{X}_1^\perp \mathbf{X}_2 \mathbf{F}_{C_2})^\perp]$ and

$$\mathbf{W}_{r_2} = [\mathbf{X}_1^\perp \mathbf{X}_2 \mathbf{F}_{C_2}, \quad \mathbf{X}_1^\perp \Sigma_1 \mathbf{X}_1^\perp (\mathbf{X}_1^\perp \mathbf{X}_2 \mathbf{F}_{C_2})^\perp].$$

In particular,

$$\begin{aligned} \text{BLUP}_{\mathcal{M}_2}(\mathbf{H}_2 \Theta_2) &= \mathbf{H}_2 \mathbf{C}_2^+ \mathbf{D} + \text{BLUP}_{\mathcal{R}_2}(\mathbf{H}_2 \mathbf{F}_{C_2} \Omega_2) \\ &= \mathbf{H}_2 \mathbf{C}_2^+ \mathbf{D} + \left([\mathbf{H}_2 \mathbf{F}_{C_2}, \quad \mathbf{0}] \mathbf{W}_{r_2}^+ + \mathbf{U}_2 \mathbf{W}_{r_2}^\perp \right) (\mathbf{Y} - \mathbf{X}_2 \mathbf{C}_2^+ \mathbf{D}). \end{aligned}$$

4. SUMMARY COMMENTS

This study presents a general approach to CMLMs and some related reduced models. We use the approach of reparameterization of CMLMs subject to exact linear restrictions. Another popular approach to handling a CMLM is to construct a new combined model by merging two given parts of the CMLM, i.e., the model part and the restriction part, into a combined form. According to this approach, the explicitly constrained model $\mathcal{M}$ and some related reduced models $\mathcal{M}_i$, $i = 1, 2$ are converted into the following implicitly CMLM and some related reduced models:

$$\widehat{\mathcal{M}}: \begin{bmatrix} \mathbf{Y} \\ \mathbf{D} \end{bmatrix} = \begin{bmatrix} \mathbf{X} \\ \mathbf{C} \end{bmatrix} \Theta + \begin{bmatrix} \Psi \\ \mathbf{0} \end{bmatrix} \quad \text{with} \quad \mathbf{D} \begin{bmatrix} \mathbf{Y} \\ \mathbf{D} \end{bmatrix} = \sigma^2 \begin{bmatrix} \Sigma_2 \otimes \Sigma_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad (4.1)$$

$$\widehat{\mathcal{M}}_i: \begin{bmatrix} \mathbf{X}_j^\perp \mathbf{Y} \\ \mathbf{D} \end{bmatrix} = \begin{bmatrix} \mathbf{X}_j^\perp \mathbf{X}_i \\ \mathbf{C}_i \end{bmatrix} \Theta_i + \begin{bmatrix} \mathbf{X}_j^\perp \Psi \\ \mathbf{0} \end{bmatrix} \quad \text{with} \quad \mathbf{D} \begin{bmatrix} \mathbf{X}_j^\perp \mathbf{Y} \\ \mathbf{D} \end{bmatrix} = \sigma^2 \begin{bmatrix} \Sigma_2 \otimes \mathbf{X}_j^\perp \Sigma_1 \mathbf{X}_j^\perp & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad (4.2)$$

$i, j = 1, 2$ and $i \neq j$. Then, by taking into account the model $\widehat{\mathcal{M}}$ and $\widehat{\mathcal{M}}_i$, $i = 1, 2$ in (4.1) and (4.2), respectively, equivalent results for the BLUP of Φ may be established.

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CONVERGENCE ANALYSIS OF THE LEGENDRE WAVELETS METHOD FOR A CLASS OF FREDHOLM-VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS OF FRACTIONAL-ORDER

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Abstract. In this paper, we develop a Legendre wavelets method for the numerical solution of Fredholm-Volterra integro-differential equations of fractional order. The Caputo sense is used to explain the fractional derivative operator. The proposed scheme is verified by presenting examples that their exact solutions are available. Numerical results show that the approximation errors decay exponentially in L^2 -norm.

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1. INTRODUCTION

Many natural phenomena in science and engineering may be modeled using mathematical tools in the form of fractional calculus. The integro-differential calculus theory of fractional order allows explaining the natural phenomena more precisely [17, 24]. The integro-differential equations of fractional order has many applications in various scientific fields, such as physics, engineering, economics, biology, etc. Hence, solving these equations is a crucially important task. Moreover, most of these problems cannot be thoroughly solved, so finding a proper approximate solution by implementing numerical methods can be very beneficial. In the last few decades, some numerical methods have been implemented to solve differential and fractional integro-differential equations of fractional order; for example, the spectral method [28], reproducing kernel Hilbert space method [5, 14], wavelet method [18, 19, 27, 30], collocation method [6, 25], fractional differential transform method [2, 4, 10], variational iteration method [22], homotopy perturbation method [1, 22, 26], and Adomian decomposition method [21].

In this paper, a new computational method based on the Legendre wavelets is introduced to solve a class of Fredholm-Volterra integro-differential equations of fractional order in the following form:

$$\begin{cases} c_1 y(x) + c_2 y'(x) = f(x) + \lambda \int_0^1 p(x,t) D_{0+}^{\alpha_1} y(t) dt + \mu \int_0^x q(x,t) D_{0+}^{\alpha_2} y(t) dt, & x \in [0, 1], \\ y(0) = c_0, & 0 < \alpha_i < 1, \quad i = 1, 2, \end{cases} \quad (1.1)$$

where D_{0+}^{α} is the Caputo's derivative operator of fractional order α , y is an unknown function, f is a known function on the interval $[0, 1]$, p, q are known functions over the range $[0, 1] \times [0, 1]$, considered to be smooth enough, and $c_0, c_1, c_2, \lambda, \mu$ are constants.

In [9], the existence and uniqueness of the solution to equation (1.1) are investigated. In [3], the authors use a backward and central-difference formula for approximating solutions at mesh points. Yusefi et al. in [28] apply a Chebyshev-Legendre spectral method to approximate the solution of equation (1.1).

In this paper, inspired by [11], we develop an approach on the basis of Legendre wavelet method to solve equation (1.1). Furthermore, we present a precise analysis on the convergence of this method and examine the convergence of the approximate solutions with L^2 -norm.

The rest of the paper is organized as follows. In Section 2, we review some of the basic definitions and mathematical tools of the fractional calculus theory needed for later use. We devote Section 3 to introduce the basic definitions of wavelets, the Legendre polynomials of order m and functions approximation using Legendre wavelets. Additionally, in this section, we obtain the integral and derivative operational matrices of equation (1.1). We investigate the convergence of the approximate solution of equation (1.1) using the Legendre wavelets method in Section 4. In Section 5, we introduce some numerical examples of Fredholm-Volterra integro-differential equations of fractional order for describing the proposed algorithm, and in the final section, we give a brief summary of the paper.

2. PRELIMINARIES

In this section, we review the definitions and auxiliary results regarding fractional calculus that will be required later in the paper. The properties of the following fractional integral and derivative operators can be found in [12, 13, 15, 16, 20, 24].

2.1. Riemann-Liouville fractional integral operator

The Riemann-Liouville fractional integral operator of order α , $\alpha \geq 0$, on the usual Lebesgue space $L[0, b]$ is given as follows:

$$I_{0+}^{\alpha} u(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} u(t) dt, \quad \alpha > 0, \quad x > 0, \quad (2.1)$$

where Γ is the well-known Gamma function, and $I_{0+}^0 u(x) = u(x)$ for $x > 0$. Some of the important properties of the Riemann-Liouville fractional integral operator are as follows:

- (a1) $I_{0+}^\alpha I_{0+}^\beta u(x) = I_{0+}^{\alpha+\beta} u(x),$
- (a2) $I_{0+}^\alpha I_{0+}^\beta u(x) = I_{0+}^\beta I_{0+}^\alpha u(x),$
- (a3) $I_{0+}^\alpha x^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} x^{\alpha+\beta}.$

2.2. Caputo's fractional derivative operator

Let $m \in \mathbb{N}$, then the smallest integer greater than or equal to α , that is, the Caputo's fractional derivative operator of order $\alpha > 0$, is obtained as follows:

$$D_{0+}^\alpha u(x) = \begin{cases} I_{0+}^{m-\alpha} D^m u(x), & m-1 < \alpha < m, \\ D^m u(x), & \alpha = m \end{cases} \quad (2.2)$$

for $x > 0$. A few important features of the Caputo's fractional derivative operator are as follows:

- (b1) $I_{0+}^\alpha D_{0+}^\alpha u(x) = u(x) - \sum_{k=0}^{m-1} u^{(k)}(0^+) \frac{x^k}{k!},$
- (b2) $D_{0+}^\alpha I_{0+}^\alpha u(x) = u(x),$
- (b3) $D_{0+}^\alpha x^\beta = \begin{cases} 0, & \beta \in \mathbb{N} \cup \{0\} \text{ and } \beta < m, \\ \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} x^{\beta-\alpha}, & \beta \in \mathbb{N} \cup \{0\} \text{ and } \beta \geq m. \end{cases}$

When $\alpha \in \mathbb{N}$, the Caputo's differential operator is the same as the ordinary differential operator. Similar to the ordinary differential operator, Caputo's fractional differentiation operator is also a linear operator, namely,

$$D_{0+}^\alpha (C_1 u_1(x) + C_2 u_2(x)) = C_1 D_{0+}^\alpha u_1(x) + C_2 D_{0+}^\alpha u_2(x),$$

where C_1, C_2 are constants.

3. IMPLEMENTATION

Wavelets comprise a class of functions formed by translation and dilation of a single function called mother wavelet $\psi(x)$ that is given as:

$$\psi_{a,b}(x) = \frac{1}{\sqrt{|a|}} \psi\left(\frac{x-b}{a}\right), \quad a, b \in \mathbb{R}, \quad a \neq 0, \quad (3.1)$$

where a and b are the dilation and translation parameters, respectively.

Limiting a and b to discrete values as $a = a_0^{-j}$ and $b = kb_0 a_0^{-j}$, where $a_0 > 1$, $b_0 > 1$, $j, k \in \mathbb{N}$, we can find a class of discrete wavelets as follows:

$$\psi_{j,k}(x) = |a_0|^{-\frac{j}{2}} \psi(a_0^j x - kb_0). \quad (3.2)$$

This set of wavelets forms an orthogonal base for $L^2(\mathbb{R})$.

In particular, when $a_0 = 2$ and $b_0 = 1$, $\Psi_{j,k}(x)$ constructs an orthonormal basis, that is

$$\langle \Psi_{j,k}, \Psi_{l,m} \rangle = \delta_{jl} \delta_{km}, \quad (3.3)$$

where δ_{jl} denotes the Kronecker delta.

3.1. Legendre wavelets

The Legendre polynomial of order m represented by $p_m(x)$ is defined on the interval $[-1, 1]$ by the following recurrence formula:

$$\begin{aligned} p_0(x) &= 1, \\ p_1(x) &= x, \\ p_{m+2}(x) &= \frac{2m+3}{m+2} x p_{m+1}(x) - \frac{m+1}{m+2} p_m(x) \quad m = 0, 1, 2, \dots \end{aligned}$$

The Legendre wavelets are given over the range $[0, 1]$ as:

$$\Psi_{nm}(x) = \begin{cases} (m + \frac{1}{2})^{\frac{1}{2}} 2^{\frac{l}{2}} p_m(2^l x - 2n + 1), & \frac{n-1}{2^{l-1}} \leq x < \frac{n}{2^{l-1}}, \\ 0, & \text{otherwise,} \end{cases}$$

where $l \in \mathbb{N}$, $n = 1, 2, 3, \dots, 2^{l-1}$ and $m = 0, 1, 2, \dots, M-1$; m denotes the order of the Legendre polynomial, and M is a positive integer.

3.2. Approximating functions using Legendre wavelets

Every function y with square integrability over the range $[0, 1]$ can be expanded in terms of the Legendre wavelet polynomials as follows:

$$y(x) = \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} y_{ij} \Psi_{ij}(x), \quad (3.4)$$

where the Fourier coefficients y_{ij} are given as follows:

$$y_{ij} = \langle y(x), \Psi_{ij}(x) \rangle.$$

If the infinite series in (3.4) is truncated, equation (3.4) can be rewritten as:

$$y(x) \cong y_N(x) = \sum_{i=1}^{2^{l-1}} \sum_{j=0}^{M-1} y_{ij} \Psi_{ij}(x) = \Psi^T(x) \mathbf{Y}, \quad (3.5)$$

where $\mathbf{Y}$ and $\Psi(x)$ are two $N \times 1$, ($N = 2^{l-1} \times M$) matrices given by

$$\begin{aligned} \mathbf{Y} &= [y_{10}, y_{11}, \dots, y_{1M-1}, y_{20}, y_{21}, \dots, y_{2M-1}, \dots, y_{2^{l-1}0}, \dots, y_{2^{l-1}M-1}]^T, \\ \Psi(x) &= [\Psi_{10}, \Psi_{11}, \dots, \Psi_{1M-1}, \Psi_{20}, \dots, \Psi_{2M-1}, \dots, \Psi_{2^{l-1}0}, \dots, \Psi_{2^{l-1}M-1}]^T. \end{aligned} \quad (3.6)$$

Correspondingly, to expand functions including $f \in L^2([0, 1])$ and $k, h \in L^2([0, 1] \times [0, 1])$ in terms of Legendre wavelet functions, we have:

$$\begin{aligned} f(x) &\cong f_N(x) = \sum_{i=1}^N f_i \psi_i(x) = \Psi^T(x) \mathbf{F}, \\ k(x, t) &\cong k_N(x, t) = \sum_{i=1}^N \sum_{j=1}^N k_{ij} \psi_i(x) \psi_j(t) = \Psi^T(x) \mathbf{K} \Psi(t), \\ h(x, t) &\cong h_N(x, t) = \sum_{i=1}^N \sum_{j=1}^N h_{ij} \psi_i(x) \psi_j(t) = \Psi^T(x) \mathbf{H} \Psi(t), \end{aligned} \quad (3.7)$$

where $\mathbf{K}$ and $\mathbf{H}$ are $N \times N$ matrices, and $\mathbf{F}$ is $N \times 1$, given as follows:

$$\begin{aligned} \mathbf{K} &= [k_{ij}]_{N \times N}, \quad k_{ij} = \left\langle \psi_i(x), \langle k(x, t), \psi_j(t) \rangle \right\rangle, \quad i, j = 1, 2, \dots, N, \\ \mathbf{H} &= [h_{ij}]_{N \times N}, \quad h_{ij} = \left\langle \psi_i(x), \langle h(x, t), \psi_j(t) \rangle \right\rangle, \quad i, j = 1, 2, \dots, N, \\ \mathbf{F} &= [f_{10}, \dots, f_{1M-1}, f_{20}, \dots, f_{2M-1}, \dots, f_{2^{l-1}0}, \dots, f_{2^{l-1}M-1}]^T, \\ f_{ij} &= \langle f(x), \psi_{ij}(x) \rangle, \quad i, j = 1, 2, \dots, N. \end{aligned}$$

In the following, we will apply our method to Fredholm-Volterra integro-differential equations of fractional order (1.1). First, using the Caputo's fractional derivative operator (2.2), equation (1.1) is obtained as follows:

$$\begin{aligned} c_1 y(x) + c_2 y'(x) &= f(x) + \frac{\lambda}{\Gamma(1-\alpha_1)} \int_0^1 p(x, t) \int_0^t \frac{y'(s)}{(t-s)^{\alpha_1}} ds dt \\ &\quad + \frac{\mu}{\Gamma(1-\alpha_2)} \int_0^x q(x, t) \int_0^t \frac{y'(s)}{(t-s)^{\alpha_2}} ds dt. \end{aligned} \quad (3.8)$$

Then, by changing the order of integration in equation (3.8), we have:

$$\begin{aligned} c_1 y(x) + c_2 y'(x) &= f(x) + \frac{\lambda}{\Gamma(1-\alpha_1)} \int_0^1 y'(s) \int_s^1 \frac{p(x, t)}{(t-s)^{\alpha_1}} dt ds \\ &\quad + \frac{\mu}{\Gamma(1-\alpha_2)} \int_0^x y'(s) \int_s^x \frac{q(x, t)}{(t-s)^{\alpha_2}} dt ds. \end{aligned} \quad (3.9)$$

However, we rewrite equation (3.9) as follows:

$$c_1 y(x) + c_2 y'(x) = f(x) + \lambda \int_0^1 k(x, s) y'(s) ds + \mu \int_0^x h(x, s) y'(s) ds, \quad (3.10)$$

where $k(x, s) = \frac{1}{\Gamma(1-\alpha_1)} \int_s^1 \frac{p(x, t)}{(t-s)^{\alpha_1}} dt$ and $h(x, s) = \frac{1}{\Gamma(1-\alpha_2)} \int_s^x \frac{q(x, t)}{(t-s)^{\alpha_2}} dt$.

Now, by substituting equations (3.5) and (3.7) into equation (3.10) we obtain

$$\begin{aligned} c_1 \Psi^T(x) \mathbf{Y} + c_2 \Psi^T(x) \mathbf{D} \mathbf{Y} &\cong \Psi^T(x) \mathbf{F} + \lambda \int_0^1 \Psi^T(x) \mathbf{K} \Psi(s) \Psi^T(s) \mathbf{D} \mathbf{Y} \, ds \\ &+ \mu \int_0^x \Psi^T(x) \mathbf{H} \Psi(s) \Psi^T(s) \mathbf{D} \mathbf{Y} \, ds. \end{aligned} \quad (3.11)$$

Since $\Psi(s) \Psi^T(s) (\mathbf{D} \mathbf{Y}) = \overline{\mathbf{D} \mathbf{Y}} \Psi(s)$, where $\overline{\mathbf{D} \mathbf{Y}}$ is an $N \times N$ matrix, we have

$$\begin{aligned} c_1 \Psi^T(x) \mathbf{Y} + c_2 \Psi^T(x) \mathbf{D} \mathbf{Y} &\cong \Psi^T(x) \mathbf{F} + \lambda \Psi^T(x) \mathbf{K} \int_0^1 \overline{\mathbf{D} \mathbf{Y}} \Psi(s) \, ds \\ &+ \mu \Psi^T(x) \mathbf{H} \int_0^x \overline{\mathbf{D} \mathbf{Y}} \Psi(s) \, ds. \end{aligned} \quad (3.12)$$

Moreover, $\int_0^1 \Psi(s) \, ds = \mathbf{w}$ and $\int_0^x \Psi(s) \, ds = \mathbf{P} \Psi(x)$, where $\mathbf{w}$ and $\mathbf{P}$ are $N \times 1$ and $N \times N$ matrices, respectively. Therefore, we have

$$\Psi^T(x) [c_1 \mathbf{Y} + c_2 \mathbf{D} \mathbf{Y}] \cong \Psi^T(x) \mathbf{F} + \lambda \Psi^T(x) (\mathbf{K} \overline{\mathbf{D} \mathbf{Y}} \mathbf{w}) + \mu \Psi^T(x) (\mathbf{H} \overline{\mathbf{D} \mathbf{Y}} \mathbf{P}) \Psi(x). \quad (3.13)$$

If we assume $\mathbf{K} \overline{\mathbf{D} \mathbf{Y}} \mathbf{w} = \mathbf{W} \mathbf{Y}$ and $\Psi^T(x) (\mathbf{H} \overline{\mathbf{D} \mathbf{Y}} \mathbf{P}) \Psi(x) = \Psi^T(x) \mathbf{U} \mathbf{Y}$, where $\mathbf{W}$ and $\mathbf{U}$ are $N \times N$ matrices, we can write

$$\Psi^T(x) [c_1 \mathbf{Y} + c_2 \mathbf{D} \mathbf{Y}] \cong \Psi^T(x) [\mathbf{F} + \lambda \mathbf{W} \mathbf{Y} + \mu \mathbf{U} \mathbf{Y}]. \quad (3.14)$$

Now, by replacing $\cong$ with $=$ in equation (3.14) and using orthonormality of $\Psi(x)$, we have

$$c_1 \mathbf{Y} + c_2 \mathbf{D} \mathbf{Y} = \mathbf{F} + \lambda \mathbf{W} \mathbf{Y} + \mu \mathbf{U} \mathbf{Y}. \quad (3.15)$$

From equation (3.15), we obtain the following system:

$$(c_1 \mathbf{I} + c_2 \mathbf{D} - \lambda \mathbf{W} - \mu \mathbf{U}) \mathbf{Y} = \mathbf{F}. \quad (3.16)$$

Theorem 1 ([11]). *To solve Fredholm-Volterra integral equation, the operational matrix of integration, $\mathbf{P}$, is given as:*

$$\mathbf{P}_{N \times N} = \frac{1}{2^l} \begin{bmatrix} T & S & S & \dots & S \\ O & T & S & \dots & S \\ O & O & T & \dots & S \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O & O & O & \dots & T \end{bmatrix},$$

where O , S and T are square matrices of order M given as follows:

$$O_{M \times M} = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad S_{M \times M} = \begin{bmatrix} 2 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$T_{M \times M} = \begin{bmatrix} 1 & \frac{1}{\sqrt{3}} & 0 & 0 & \dots & 0 & 0 \\ -\frac{\sqrt{3}}{3} & 0 & \frac{\sqrt{3}}{3\sqrt{5}} & 0 & \dots & 0 & 0 \\ 0 & -\frac{\sqrt{5}}{5\sqrt{3}} & 0 & \frac{\sqrt{5}}{5\sqrt{7}} & \dots & 0 & 0 \\ 0 & 0 & -\frac{\sqrt{7}}{7\sqrt{5}} & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & \frac{\sqrt{2M-3}}{(2M-3)\sqrt{2M-1}} \\ 0 & 0 & 0 & 0 & \dots & -\frac{\sqrt{2M-1}}{(2M-1)\sqrt{2M-3}} & 0 \end{bmatrix}.$$

Theorem 2. To solve Fredholm-Volterra integro-differential equations of fractional order, the operational matrix of ordinary derivative, $\mathbf{D}$, is given as follows:

$$\mathbf{D}_{N \times N} = 2^l \begin{bmatrix} L & O & O & \dots & O \\ O & L & O & \dots & O \\ O & O & L & \dots & O \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O & O & O & \dots & L \end{bmatrix},$$

where O and L are square matrices of order M given by

$$O_{M \times M} = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$L = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots & 0 & 0 & \dots \\ \frac{0}{\sqrt{3 \times 1}} & 0 & 0 & 0 & \dots & 0 & 0 & \dots \\ 0 & \frac{\sqrt{5 \times 3}}{\sqrt{7 \times 1}} & 0 & 0 & \dots & 0 & 0 & \dots \\ \frac{0}{\sqrt{7 \times 1}} & 0 & \frac{0}{\sqrt{7 \times 5}} & 0 & \dots & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \frac{\sqrt{(2i-3) \times 3}}{\sqrt{(2i-1) \times 1}} & 0 & \frac{\sqrt{(2i-3) \times 7}}{\sqrt{(2i-1) \times 5}} & \dots & \frac{\sqrt{(2i-3)(2j-3)}}{0} & 0 & \dots \\ \frac{0}{\sqrt{(2i-1) \times 1}} & 0 & \frac{0}{\sqrt{(2i-1) \times 5}} & 0 & \dots & 0 & \frac{0}{\sqrt{(2i-1)(2j-1)}} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \end{bmatrix}.$$

Proof. First, we find the operational matrix of ordinary derivative for $l = 2$ and $M = 3$. Next, we obtain the general matrix $\mathbf{D}$. The basis functions $\psi_{nm}(x)$, $n = 1, 2$, $m = 0, 1, 2$, and as a result, the matrix

$$\Psi_{6 \times 1}(x) = [\psi_{10}(x) \ \psi_{11}(x) \ \psi_{12}(x) \ \psi_{20}(x) \ \psi_{21}(x) \ \psi_{22}(x)]^T$$

for $l = 2$ and $M = 3$, is obtained as follows:

$$\left. \begin{aligned} \psi_{10}(x) &= \sqrt{2}, \\ \psi_{11}(x) &= \sqrt{6}(4x-1), \\ \psi_{12}(x) &= \sqrt{10} \left[\frac{3}{2}(4x-1)^2 - \frac{1}{2} \right], \end{aligned} \right\} \quad 0 \leq x < \frac{1}{2}, \quad (3.17)$$

and

$$\left. \begin{aligned} \psi_{20}(x) &= \sqrt{2}, \\ \psi_{21}(x) &= \sqrt{6}(4x-3), \\ \psi_{22}(x) &= \sqrt{10} \left[\frac{3}{2}(4x-3)^2 - \frac{1}{2} \right], \end{aligned} \right\} \quad \frac{1}{2} \leq x < 1. \quad (3.18)$$

By deriving from equations (3.17) and (3.18), we obtain the following equations:

$$\begin{aligned} \frac{d\psi_{10}(x)}{dx} &= 0 \\ &= 0\psi_{10}(x) + 0\psi_{11}(x) + 0\psi_{12}(x) + 0\psi_{20}(x) + 0\psi_{21}(x) + 0\psi_{22}(x) \\ &= 2^2 \begin{bmatrix} 0, 0, 0, 0, 0, 0 \end{bmatrix} \Psi_{6 \times 1}(x). \end{aligned}$$

$$\begin{aligned} \frac{d\psi_{11}(x)}{dx} &= 4\sqrt{3} \\ &= 4\sqrt{3}\psi_{10}(x) + 0\psi_{11}(x) + 0\psi_{12}(x) + 0\psi_{20}(x) + 0\psi_{21}(x) + 0\psi_{22}(x) \\ &= 2^2 \begin{bmatrix} \sqrt{3}, 0, 0, 0, 0, 0 \end{bmatrix} \Psi_{6 \times 1}(x). \end{aligned}$$

$$\begin{aligned} \frac{d\psi_{12}(x)}{dx} &= 12\sqrt{10}(4x-1) \\ &= 0\psi_{10}(x) + 4\sqrt{15}\psi_{11}(x) + 0\psi_{12}(x) + 0\psi_{20}(x) + 0\psi_{21}(x) + 0\psi_{22}(x) \\ &= 2^2 \begin{bmatrix} 0, \sqrt{5 \times 3}, 0, 0, 0, 0 \end{bmatrix} \Psi_{6 \times 1}(x). \end{aligned}$$

Similarly, we have

$$\begin{aligned} \frac{d\psi_{20}(x)}{dx} &= 0 \\ &= 0\psi_{10}(x) + 0\psi_{11}(x) + 0\psi_{12}(x) + 0\psi_{20}(x) + 0\psi_{21}(x) + 0\psi_{22}(x) \\ &= 2^2 \begin{bmatrix} 0, 0, 0, 0, 0, 0 \end{bmatrix} \Psi_{6 \times 1}(x). \end{aligned}$$

$$\begin{aligned}
\frac{d\psi_{21}(x)}{dx} &= 4\sqrt{3} \\
&= 4\sqrt{3}\psi_{10}(x) + 0\psi_{11}(x) + 0\psi_{12}(x) + 0\psi_{20}(x) + 0\psi_{21}(x) + 0\psi_{22}(x) \\
&= 2^2 \begin{bmatrix} \sqrt{3} \times 1, 0, 0, 0, 0, 0 \end{bmatrix} \Psi_{6 \times 1}(x).
\end{aligned}$$

$$\begin{aligned}
\frac{d\psi_{22}(x)}{dx} &= 12\sqrt{10}(4x-3) \\
&= 0\psi_{10}(x) + 4\sqrt{15}\psi_{11}(x) + 0\psi_{12}(x) + 0\psi_{20}(x) + 0\psi_{21}(x) + 0\psi_{22}(x) \\
&= 2^2 \begin{bmatrix} 0, \sqrt{5} \times 3, 0, 0, 0, 0 \end{bmatrix} \Psi_{6 \times 1}(x).
\end{aligned}$$

Therefore, we have

$$\frac{d\Psi_{6 \times 1}(x)}{dx} = \mathbf{D}_{6 \times 6} \Psi_{6 \times 1}(x),$$

where $\mathbf{D}_{6 \times 6}$ is a square matrix of order 6 that can be structured as a block matrix as follows:

$$\mathbf{D}_{6 \times 6} = 2^2 \begin{bmatrix} L_{3 \times 3} & O_{3 \times 3} \\ O_{3 \times 3} & L_{3 \times 3} \end{bmatrix},$$

where

$$L_{3 \times 3} = \begin{bmatrix} 0 & 0 & 0 \\ \sqrt{3} & 0 & 0 \\ 0 & \sqrt{5} \times 3 & 0 \end{bmatrix} \quad \text{and} \quad O_{3 \times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

By assuming that the theorem is true for $l-1$ and $M-1$, it can be easily shown that for general l and M , we can write

$$\frac{d\Psi_{6 \times 1}(x)}{dx} = \mathbf{D}_{N \times N} \Psi_{N \times 1}(x), \quad (N = 2^{l-1} \times M), \quad (3.19)$$

where $\mathbf{D}$ is a square matrix of order N that can be structured as a block matrix as follows:

$$\mathbf{D}_{N \times N} = 2^l \begin{bmatrix} L & O & O & \dots & O \\ O & L & O & \dots & O \\ O & O & L & \dots & O \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O & O & O & \dots & L \end{bmatrix},$$

where O and L are square matrices of order M as follows:

$$O_{M \times M} = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$L = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots & 0 & 0 & \dots \\ \sqrt{3 \times 1} & 0 & 0 & 0 & \dots & 0 & 0 & \dots \\ 0 & \sqrt{5 \times 3} & 0 & 0 & \dots & 0 & 0 & \dots \\ \sqrt{7 \times 1} & 0 & \sqrt{7 \times 5} & 0 & \dots & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\ 0 & \sqrt{(2i-3) \times 3} & 0 & \sqrt{(2i-3) \times 7} & \dots & \sqrt{(2i-3)(2j-3)} & 0 & \dots \\ \sqrt{(2i-1) \times 1} & 0 & \sqrt{(2i-1) \times 5} & 0 & \dots & 0 & \sqrt{(2i-1)(2j-1)} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \end{bmatrix}.$$

□

4. ERROR ESTIMATION

In this section, we prove the convergence of the proposed numerical scheme for fractional Fredholm-Volterra integro-differential equations in L^2 -norm. We also review some of the properties and elementary lemmas, required for obtaining the main results.

Lemma 1 ([8]). Assume that $u, \omega, \beta \in C([0, 1])$ with $\beta(x) \geq 0$. If u satisfies the inequality

$$u(x) \leq \omega(x) + \int_0^x \beta(t)u(t)dt, \quad x \in [0, 1],$$

then

$$u(x) \leq \omega(x) + \int_0^x \beta(t) \left(\omega(t) \exp \left(\int_t^x \beta(s)ds \right) \right) dt, \quad x \in [0, 1].$$

In other words, if ω is non-decreasing over $[0, 1]$, the inequality mentioned above reduces to

$$u(x) \leq \omega(x) \exp \left(\int_t^x \beta(s)ds \right), \quad x \in [0, 1].$$

Lemma 2 ([8]). Suppose that k is a given kernel function over $[0, 1] \times [0, 1]$. If $u \in L^p(0, 1)$, for $1 \leq p \leq \infty$, the integral

$$K(u(x)) = \int_0^{x \text{ or } 1} k(x, t)u(t)dt$$

is well-defined in $L^p(0, 1)$, and there exists a γ_0 so that

$$\|Ku\|_p \leq \gamma_0 \|u\|_p.$$

Suppose that p_N is the interpolation projection operator from $L^2(0, 1)$ to $\mathbb{P}_N(0, 1)$, namely, $p_N: L^2(0, 1) \rightarrow \mathbb{P}_N(0, 1)$ such that, for each $u \in L^2(0, 1)$, we have $p_N(u(x)) = \sum_{j=0}^N u_j \psi_j(x)$, and

$$\int_0^1 (u(x) - p_N(u(x)))v(x)dx = 0, \quad \forall v \in \mathbb{P}_N(0, 1).$$

Moreover, the following inequality for interpolation in Legendre wavelet polynomials and shifted Gauss-Legendre nodal points $r \geq 1$ (or for any fixed $r \leq N$) can be easily written as

$$\begin{aligned} \|u - p_N(u)\|_{H^J(0,1)} &\leq \gamma^* N^{2J - \frac{1}{2} - r} |u|_{r;N}, \\ \|u - p_N(u)\|_{H^J(0,1)} &\leq \delta^* N^{-r} |u|_{r;N}, \end{aligned} \quad (4.1)$$

where $u \in H^I(0,1)$, γ^* and δ^* are constants independent of N and $0 \leq J \leq I$, see [7].

Theorem 3. *For values of N that are large enough, the Legendre wavelet polynomial approximations y_N converge to exact solution in L^2 -norm, namely,*

$$\|y - y_N\| \rightarrow 0 \quad \text{as } N \rightarrow \infty.$$

Proof. Suppose that $y_N(x)$ is an approximate solution obtained by using the Legendre wavelet method for equation (3.10). Then, we have

$$c_1 y_N(x) + c_2 y'_N(x) = f_N(x) + \lambda \int_0^1 k_N(x,t) y'_N(t) dt + \mu \int_0^x h_N(x,t) y'_N(t) dt. \quad (4.2)$$

Consequently, $f_N(x)$, $k_N(x,t)$ and $h_N(x,t)$ are Legendre interpolation polynomials defined for functions $f(x)$, $k(x,t)$ and $h(x,t)$, respectively. Now, by integrating the sides of equations (3.10) and (4.2), we have

$$\begin{aligned} c_1 \int_0^x y(t) dt + c_2 y(x) - c_2 y(0) &= \int_0^x f(t) dt + \lambda \int_0^x \int_0^1 k(t,s) y'(s) ds dt \\ &\quad + \mu \int_0^x \int_0^t h(t,s) y'(s) ds dt, \end{aligned} \quad (4.3)$$

$$\begin{aligned} c_1 \int_0^x y_N(t) dt + c_2 y_N(x) - c_2 y(0) &= \int_0^x f_N(t) dt + \lambda \int_0^x \int_0^1 k_N(t,s) y'_N(s) ds dt \\ &\quad + \mu \int_0^x \int_0^t h_N(t,s) y'_N(s) ds dt. \end{aligned} \quad (4.4)$$

By subtracting equation (4.4) from equation (4.3), we have

$$\begin{aligned} E_N(y(x)) &= -\frac{c_1}{c_2} \int_0^x E_N(y(t)) dt + \frac{1}{c_2} \int_0^x E_N(f(t)) dt \\ &\quad + \frac{\lambda}{c_2} \int_0^x \int_0^1 E_N(k(t,s)) y'_N(s) ds dt + \frac{\mu}{c_2} \int_0^x \int_0^t E_N(h(t,s)) y'_N(s) ds dt \\ &\quad + \frac{\lambda}{c_2} \int_0^x \int_0^1 k(t,s) DE_N(y(s)) ds dt + \frac{\mu}{c_2} \int_0^x \int_0^t h(t,s) DE_N(y(s)) ds dt, \end{aligned} \quad (4.5)$$

therefore,

$$\begin{aligned} E_N(y(x)) &= y(x) - y_N(x), \\ DE_N(y(s)) &= y'(s) - y'_N(s), \\ E_N(f(t)) &= f(t) - f_N(t), \\ E_N(k(t, s)) &= k(t, s) - k_N(t, s), \\ E_N(r(t, s)) &= h(t, s) - h_N(t, s). \end{aligned}$$

From equation (4.5), we can obtain

$$\begin{aligned} |E_N(y(x))| &\leq \left| \frac{c_1}{c_2} \right| \int_0^x |E_N(y(t))| dt + \left| \frac{1}{c_2} \right| \int_0^x |E_N(f(t))| dt \\ &\quad + \left| \frac{\lambda}{c_2} \right| \int_0^x \left| \int_0^1 E_N(k(t, s)) y'_N(s) ds \right| dt + \left| \frac{\mu}{c_2} \right| \int_0^x \left| \int_0^t E_N(h(t, s)) y'_N(s) ds \right| dt \\ &\quad + \left| \frac{\lambda}{c_2} \right| \int_0^x \left| \int_0^1 k(t, s) DE_N(y(s)) ds \right| dt + \left| \frac{\mu}{c_2} \right| \int_0^x \left| \int_0^t h(t, s) DE_N(y(s)) ds \right| dt \\ &\leq \gamma_1 \int_0^x |E_N(y(t))| dt + \gamma_2 \int_0^x |E_N(f(t))| dt \\ &\quad + \gamma_3 \int_0^x \left| \int_0^1 E_N(k(t, s)) y'_N(s) ds \right| dt + \gamma_4 \int_0^x \left| \int_0^t E_N(h(t, s)) y'_N(s) ds \right| dt \\ &\quad + \gamma_5 \int_0^x \left| \int_0^1 k(t, s) DE_N(y(s)) ds \right| dt + \gamma_6 \int_0^x \left| \int_0^t h(t, s) DE_N(y(s)) ds \right| dt. \end{aligned} \tag{4.6}$$

Applying Lemma 1 leads to

$$\begin{aligned} |E_N(y(x))| &\leq \exp \left(\int_0^x \gamma_1 dt \right) \left\{ \gamma_2 \int_0^x |E_N(f(t))| dt + \gamma_3 \int_0^x \left| \int_0^1 E_N(k(t, s)) y'_N(s) ds \right| dt \right. \\ &\quad + \gamma_4 \int_0^x \left| \int_0^t E_N(h(t, s)) y'_N(s) ds \right| dt + \gamma_5 \int_0^x \left| \int_0^1 k(t, s) DE_N(y(s)) ds \right| dt \\ &\quad + \gamma_6 \int_0^x \left| \int_0^t h(t, s) DE_N(y(s)) ds \right| dt \left. \right\} \\ &\leq \gamma_7 \int_0^x |E_N(f(t))| dt + \gamma_8 \int_0^x \left| \int_0^1 E_N(k(t, s)) y'_N(s) ds \right| dt \\ &\quad + \gamma_9 \int_0^x \left| \int_0^t E_N(h(t, s)) y'_N(s) ds \right| dt + \gamma_{10} \int_0^x \left| \int_0^1 k(t, s) DE_N(y(s)) ds \right| dt \\ &\quad + \gamma_{11} \int_0^x \left| \int_0^t h(t, s) DE_N(y(s)) ds \right| dt. \end{aligned} \tag{4.7}$$

Equivalently, by using L^2 -norm, we obtain

$$\begin{aligned} \|E_N(y)\|_2 &\leq \gamma_7 \left\| \int_0^x |E_N(f(t))| dt \right\|_2 + \gamma_8 \left\| \int_0^x \left| \int_0^1 E_N(k(t,s)) y'_N(s) ds \right| dt \right\|_2 \\ &\quad + \gamma_9 \left\| \int_0^x \left| \int_0^t E_N(h(t,s)) y'_N(s) ds \right| dt \right\|_2 + \gamma_{10} \left\| \int_0^x \left| \int_0^1 k(t,s) DE_N(y(s)) ds \right| dt \right\|_2 \\ &\quad + \gamma_{11} \left\| \int_0^x \left| \int_0^t h(t,s) DE_N(y(s)) ds \right| dt \right\|_2. \end{aligned} \quad (4.8)$$

Using Lemma 2, the above inequality reduces to

$$\begin{aligned} \|E_N(y)\|_2 &\leq \gamma_{12} \|E_N(f)\|_2 + \gamma_{13} \left\| \int_0^1 E_N(k(.,s)) y'_N(s) ds \right\|_2 \\ &\quad + \gamma_{14} \left\| \int_0^t E_N(h(.,s)) y'_N(s) ds \right\|_2 + \gamma_{15} \left\| \int_0^1 k(.,s) DE_N(y(s)) ds \right\|_2 \\ &\quad + \gamma_{16} \left\| \int_0^t h(.,s) DE_N(y(s)) ds \right\|_2. \end{aligned} \quad (4.9)$$

Since the linear derivative operator is continuous and bounded [23], a constant $\delta_0 \geq 0$ exists such that

$$\|y'_N\|_{r;N} \leq \delta_0 \|y_N\|_{r;N}, \quad r \in \mathbb{N} \text{ and } r \leq N. \quad (4.10)$$

According to equations (4.1) and (4.10) and Lemma 2, we can write

$$\begin{aligned} \left\| \int_0^1 E_N(k(.,s)) y'_N(s) ds \right\|_2 &\leq \delta^* N^{-r} \left\| \int_0^1 k(.,s) y'_N(s) ds \right\|_{r;N} \\ &\leq \delta^* \gamma_0 N^{-r} \|y'_N\|_{r;N} \\ &\leq (\delta^* \gamma_0 \delta_0) N^{-r} \|y_N\|_{r;N} \\ &\leq \delta_1 N^{-r} (\|y\|_{r;N} + \|E_N(y)\|_{r;N}) \\ &\leq \delta_1 N^{-r} (\|y\|_{r;N} + \|E_N(y)\|_{1;N}) \\ &\leq \delta_1 N^{-r} (\|y\|_{r;N} + \gamma^* N^{\frac{3}{2}-r} \|y\|_{r;N}). \end{aligned}$$

Therefore,

$$\left\| \int_0^1 E_N(k(.,s)) y'_N(s) ds \right\|_2 \leq \delta_1 N^{-r} \|y\|_{r;N} + \delta_2 N^{\frac{3}{2}-2r} \|y\|_{r;N}, \quad (4.11)$$

where $\delta_2 = \delta_1 \gamma^*$. Similarly,

$$\left\| \int_0^t E_N(h(.,s)) y'_N(s) ds \right\|_2 \leq \delta_3 N^{-r} \|y\|_{r;N} + \delta_4 N^{\frac{3}{2}-2r} \|y\|_{r;N}. \quad (4.12)$$

In the same way, from Lemma 2 and equation (4.10), we obtain

$$\begin{aligned} \left\| \int_0^1 k(.,s) DE_N(y(s)) ds \right\|_2 &\leq \gamma_0 \|DE_N(y)\|_2 \\ &\leq \gamma_0 \delta_0 \|E_N(y)\|_2 \\ &\leq (\gamma_0 \delta_0 \delta^*) N^{-r} |y|_{r;N}. \end{aligned}$$

Therefore,

$$\left\| \int_0^1 k(.,s) DE_N(y(s)) ds \right\|_2 \leq \delta_5 N^{-r} |y|_{r;N}, \quad (4.13)$$

where $\delta_3 = \gamma_0 \delta_0 \delta^*$. Similarly,

$$\left\| \int_0^t h(.,s) DE_N(y(s)) ds \right\|_2 \leq \delta_6 N^{-r} |y|_{r;N}. \quad (4.14)$$

In a manner similar to equation (4.1), we may write

$$\|E_N(f)\|_2 \leq \gamma^* N^{-r} |f|_{r;N}. \quad (4.15)$$

Ultimately, by substituting (4.11)-(4.15) in (4.9), we get the following result:

$$\begin{aligned} \|E_N(y)\|_2 &\leq \gamma_{12} \left(\gamma^* N^{-r} |f|_{r;N} \right) + \gamma_{13} \left(\delta_1 N^{-r} \|y\|_{r;N} + \delta_2 N^{\frac{3}{2}-2r} |y|_{r;N} \right) \\ &\quad + \gamma_{14} \left(\delta_3 N^{-r} \|y\|_{r;N} + \delta_4 N^{\frac{3}{2}-2r} |y|_{r;N} \right) + \gamma_{15} \left(\delta_5 N^{-r} |y|_{r;N} \right) \\ &\quad + \gamma_{16} \left(\delta_6 N^{-r} |y|_{r;N} \right). \end{aligned}$$

Hence,

$$\|E_N(y)\|_2 \leq C_1 N^{-r} |f|_{r;N} + C_2 N^{-r} \|y\|_{r;N} + C_3 N^{\frac{3}{2}-2r} |y|_{r;N} + C_4 N^{-r} |y|_{r;N} \quad (4.16)$$

where $C_1 = \gamma^* \gamma_{12}$, $C_2 = \delta_1 \gamma_{13} + \delta_3 \gamma_{14}$, $C_3 = \delta_2 \gamma_{13} + \delta_4 \gamma_{14}$ and $C_4 = \delta_5 \gamma_{15} + \delta_6 \gamma_{16}$.

Equation (4.16) proves the convergence of the approximate solution in L^2 -norm. The proof is completed here. $\square$

5. NUMERICAL EXAMPLES

In this section, we use the Legendre wavelet method for solving Fredholm-Volterra integro-differential equations of fractional order. In order to show the efficiency and accuracy of the presented method, four test equations are taken into account as examples. Numerical results in Table 1 as well as Figures 1 and 2 show that approximate solutions converge to exact solutions. All computations were conducted using the Maple software package.

Example 1 ([29]). We take into account the following Fredholm integro-differential equation of fractional order

$$y'(x) = f(x) + \int_0^1 xt D_{0+}^{\frac{1}{2}} y(t) dt, \quad 0 < x < 1,$$

TABLE 1. Difference in absolute error of exact and approximate solutions for Examples 1 and 4.

x	Example 1 for $k = 2$ and $M = 3$	Example 4 for $k = 2$ and $M = 3$
0.0	$1.33679267695312 \times 10^{-9}$	$4.95687117107094 \times 10^{-10}$
0.1	$1.11257569166759 \times 10^{-9}$	$4.15374405495941 \times 10^{-10}$
0.2	$9.40240333784253 \times 10^{-10}$	$3.31229605283749 \times 10^{-10}$
0.3	$8.19786603303097 \times 10^{-10}$	$2.43252716470520 \times 10^{-10}$
0.4	$7.51214500224126 \times 10^{-10}$	$1.51443739056252 \times 10^{-10}$
0.5	$7.34524024547339 \times 10^{-11}$	$5.58026730409458 \times 10^{-11}$
0.6	$7.69715176272737 \times 10^{-10}$	$4.36704815753983 \times 10^{-10}$
0.7	$8.56787955400318 \times 10^{-10}$	$1.46975724792781 \times 10^{-10}$
0.8	$9.95742361930083 \times 10^{-10}$	$2.54113056611201 \times 10^{-10}$
0.9	$1.18657839586203 \times 10^{-9}$	$3.65082477030659 \times 10^{-10}$
1.0	$1.42929605719617 \times 10^{-9}$	$4.79883986051156 \times 10^{-10}$

where $y(0) = 0$ and $y(x) = 14x$ are the initial condition and the exact solution, respectively;

$$f(x) = 14 \left(1 - \frac{x}{2.5\Gamma(1.5)} \right).$$

Using the presented method, for $k = 2$ and $M = 3$, the approximate solution is obtained as follows:

$$\begin{aligned} y_{\text{numeric}}(x) &= -5.26142047709630 \times 10^{-10} + 13.9999999999075x \\ &\quad - 8.10650176803679 \times 10^{-11}(4x-1)^2 - 8.10650679514575 \times 10^{-11}(4x-3)^2 \\ &\cong 14x. \end{aligned}$$

This is almost the exact solution to the problem. The absolute error between the numerical and exact solution is tabulated in Table 1 for Example 1. The maximum errors with infinite- and L^2 -norms are given as follows:

$$\begin{aligned} \|y_{\text{numeric}} - y_{\text{exact}}\|_{\infty} &= 1.429296057 \times 10^{-9}, \\ \|y_{\text{numeric}} - y_{\text{exact}}\|_{L^2} &= 9.705274462 \times 10^{-10}. \end{aligned}$$

Example 2 ([29]). We consider the following Fredholm integro-differential equation of fractional order

$$y'(x) = f(x) + \int_0^1 x^2 t^2 D_{0+}^{\frac{1}{4}} y(t) dt, \quad 0 < x < 1,$$

with the initial condition $y(0) = 0$ and the exact solution $y(x) = 2x^4 - x^{\frac{3}{2}}$. The numerical results for this example with $f(x) = 8x^3 - \frac{3}{2}x^{\frac{1}{2}} - \left(\frac{48}{6.75\Gamma(4.75)} - \frac{\Gamma(2.5)}{4.25\Gamma(2.25)} \right)x^2$ plus values of $k = 2$ and $M = 5$ are shown in Figure 1.

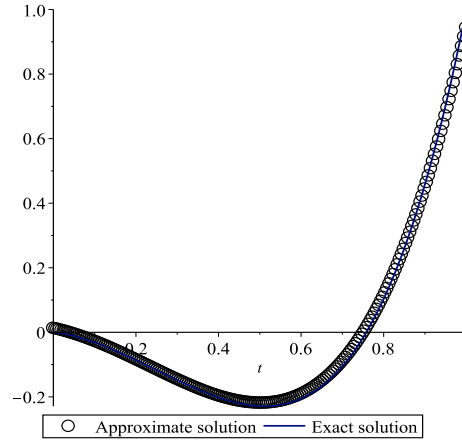


FIGURE 1. Exact and approximate solutions compared for $k = 2$ and $M = 5$, Example 2.

Example 3. We consider the following Fredholm integro-differential equation of fractional order

$$2y'(x) + y(x) = f(x) + \frac{1}{3} \int_0^x x^3 t D_{0+}^{\frac{1}{2}} y(t) dt, \quad 0 < x < 1,$$

with the initial condition $y(0) = 1$, the exact solution $y(x) = 8x + 3x^3$ and $f(x) = 16 + 8x + 18x^2 + (3 - \frac{128}{45\Gamma(0.5)})x^3$. The exact and numerical solutions are shown in Figure 2 for $k = 2$ and $M = 5$.

Example 4. We consider the following Volterra integro-differential equation of fractional order

$$y'(x) + y(x) = 2 + x + \frac{x^{\frac{3}{2}}}{\Gamma(2.5)} - \int_0^x D_{0+}^{\frac{1}{2}} y(t) dt, \quad 0 < x < 1,$$

with the initial condition $y(0) = 1$ and the exact solution $y(x) = x + 1$. For $k = 2$ and $M = 3$, the approximate solution obtained using our method is given as follows:

$$y_{\text{numeric}}(x) = 1.00000000053072 + 0.999999999074120x - 9.09330911332095 \\ \times 10^{-12}(4x - 1)^2 - 2.88196776492321 \times 10^{-12}(4x - 3)^2 \cong x + 1,$$

which is equivalent to the exact solution. For Example 4, the absolute error between the numerical and exact solutions is reported in Table 1. The maximum error with infinite- and L^2 -norms are as follows:

$$\|y_{\text{numeric}} - y_{\text{exact}}\|_{\infty} = 4.956871174 \times 10^{-10}, \\ \|y_{\text{numeric}} - y_{\text{exact}}\|_{L^2} = 2.847848599 \times 10^{-10}.$$

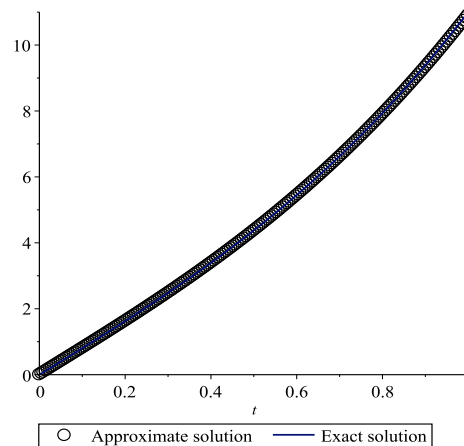


FIGURE 2. Exact and approximate solutions compared for $k = 2$ and $M = 5$, Example 3.

6. CONCLUSIONS

In this paper, we presented a Legendre wavelet approximation of a class of fractional Fredholm-Volterra integro-differential equations. The main purpose of this paper was to show that the approximation errors decay exponentially in L^2 -norm. We proved that the method presented in this paper is effective for solving Fredholm-Volterra integro-differential equations of fractional order, and this method has a high convergence rate. The numerical results of the presented method were compared with exact solutions. The acceptable results are in a very good agreement with the exact solutions only for a small number of Legendre wavelet polynomials.

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***M*–ESTIMATION IN PERIODIC THRESHOLD *GARCH* MODELS: CONSISTENCY AND ASYMPTOTIC NORMALITY**

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Abstract. The present paper derives convergence rates and asymptotic normality of a class of *M*–estimators in the periodic asymmetric *GARCH* model. Simulation studies are conducted for evaluating the performance of the estimator. Finally, an empirical study on the exchange rates of the Algerian Dinar against the U.S-dollar and the single European currency (Euro) illustrates the usefulness of the periodic *TGARCH* model.

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1. INTRODUCTION

Periodic Generalized AutoRegressive Conditional Heteroscedastic (*PGARCH*) models have been vastly used to analyze the volatility in economic and financial time series (c.f., [1], [2]). Later Bibi and Ghezal [3] proposed an asymmetric of the *PGARCH* model called the *PTGARCH* model. A stochastic process $(X_n)_{n \in \mathbb{Z}}$ is said to follow a *PTGARCH*(p, q) model if

$$X_n = \sigma_n^{\frac{1}{2}} \eta_n, \quad (1.1)$$

where $(\eta_n)_{n \in \mathbb{Z}}$ is a sequence of independent identically distributed (*i.i.d.*) random variables with zero mean and unit variance and η_k is independent of X_n for $k > n$, and conditionally on the σ –field $\mathfrak{F}_{n-1} = \sigma(X_{n-i}, i \geq 1)$, σ_n satisfy

$$\sigma_n = a_0(n) + \sum_{i=1}^q (a_i(n)X_{n-i}^{+2} + b_i(n)X_{n-i}^{-2}) + \sum_{j=1}^p c_j(n)\sigma_{n-j}, \quad (1.2)$$

where $X_n^{\pm} = \max(\pm X_n, 0)$, $X_n^{\pm 2} = (X_n^{\pm})^2$ so, $X_n = X_n^+ - X_n^-$ and $|X_n| = X_n^+ + X_n^-$. Now, setting $n = st + v$, $X_{st+v} = X_t(v)$, $\sigma_{st+v} = \sigma_t(v)$ and $\eta_{st+v} = \eta_t(v)$, Model

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(1.1) – (1.2) may be equivalently written as

$$\begin{cases} X_t(v) = \sigma_t^{\frac{1}{2}}(v) \eta_t(v) \\ \sigma_t(v) = a_0(v) + \sum_{i=1}^q (a_i(v) X_t^+(v-i) + b_i(v) X_t^-(v-i)) + \sum_{j=1}^p c_j(v) \sigma_t(v-j) \end{cases} \quad (1.3)$$

In (1.3), $a_0(v)$, $a_i(v)$, $b_i(v)$ and $c_j(v)$ with $i \in \{1, \dots, q\}$ and $j \in \{1, \dots, p\}$ are positive coefficients with $a_0(v) > 0$ for any $v \in \{1, \dots, s\}$, and $X_t(v)$ refers to X_t during the v -th regime $v \in \{1, \dots, s\}$ of cycle t . For the convenience, $X_t(v) = X_{t-1}(v+s)$, $\sigma_t(v) = \sigma_{t-1}(v+s)$ and $\eta_t(v) = \eta_{t-1}(v+s)$ if $v < 0$. The non-periodic notations (X_t) , (σ_t) , (η_t) etc. will be used interchangeably with the periodic notations $(X_t(v))$, $(\sigma_t(v))$, $(\eta_t(v))$ etc. Some results on *PTGARCH* model can be found in literature. Guerbyenne and Kessira [10] gave a necessary and sufficient condition which ensure the existence of a strictly periodically stationary (*SPS*) and periodically ergodic (*PE*) solution to *PTGARCH* process, and established its strong consistency and asymptotic normality of the quasi maximum likelihood estimator (*QMLE*), See also Bibi and Ghezal [3]. As a consequence, *QMLE* is currently a widely used method for estimating the unknown parameters in symmetric or asymmetric *PGARCH* models. But, as everyone knows, the *QMLE* is asymptotic if the innovation has finite four moments, while stringent moment conditions may not hold in many situations, for this reason, we propose an *M*-estimation to reduce the requirement on moments of the innovations. Many symmetric standard or periodic *GARCH* models have been considered in the literature to estimate parameter using *M*-estimation (see, [13], [18], [20]). Now, in this paper we present a class of *M*-estimators of asymmetric *PGARCH* models.

The rest of the paper is organized as follows. In section 2, we define the class of *M*-estimators. Section 3 is devoted to the asymptotic properties of *M*-estimation and gives an estimator of asymptotic variance. Simulation results and an illustrative application on real data are reported in Sections 4 and 5. Section 6 concludes the paper.

2. *M*-ESTIMATION

In this section, we investigate the problem of *M*-estimation of some function of the model parameter vector $\underline{\theta}' := (\underline{a}', \underline{b}', \underline{c}') := (\underline{\theta}'(1), \dots, \underline{\theta}'(s)) \in \Theta \subset]0, +\infty]^s \times [0, +\infty]^{s(2q+p)}$ where $\underline{a}' := (\underline{a}'_0, \underline{a}'_1, \dots, \underline{a}'_q)$, $\underline{b}' := (\underline{b}'_1, \dots, \underline{b}'_q)$, $\underline{c}' := (\underline{c}'_1, \dots, \underline{c}'_p)$ and $\underline{\theta}'(v) := (a_0(v), a_1(v), \dots, a_q(v), b_1(v), \dots, b_q(v), c_1(v), \dots, c_p(v))$, $v = 1, \dots, s$ with $\underline{a}'_i := (a_i(1), \dots, a_i(s))$, $\underline{b}'_k := (b_k(1), \dots, b_k(s))$ and $\underline{c}'_j := (c_j(1), \dots, c_j(s))$ for all $0 \leq i \leq q$, $1 \leq k \leq q$ and $1 \leq j \leq p$ based on the realization $\{X_1, \dots, X_n; n = sN\}$ from the unique, causal and *SPS* solution of (1.3), and let $\sigma_t^2(\underline{\theta})$ be the conditional variance of X_t given $\mathcal{F}_{t-1}$ where $\mathcal{F}_t := \sigma(e_u; u \leq t)$. The true parameter value is unknown and is denoted by $\underline{\theta}'_0 := (\underline{a}'_0, \underline{b}'_0, \underline{c}'_0) \in \Theta \subset]0, +\infty]^s \times [0, +\infty]^{s(2q+p)}$. Now, by a similar argument to Lemma 2.3 and Theorem 2.1 of Berkes et al. [12], then *PTGARCH*(p, q)

can be transformed to an infinite order $PTARCH(\infty)$, i.e., almost surely

$$\sigma_t(v) = \kappa_0(v) + \sum_{i \geq 1} (\kappa_i(v) X_t^{+2}(v-i) + \tau_i(v) X_t^{-2}(v-i)),$$

where the coefficients $\kappa_i(v)$, $\tau_k(v)$, $i \geq 0$, $k \geq 1$ are given by

$$\begin{aligned} \kappa_i(v) &= \frac{a_0(v)}{1 - \sum_{j=1}^p c_j(v)} \mathbb{I}_{\{i=0\}} + \frac{d^i}{dx^i} \left(\frac{\sum_{j=1}^q a_j(v) x^j}{1 - \sum_{j=1}^p c_j(v) x^j} \right)_{x=0} \mathbb{I}_{\{i>0\}}, \quad i \geq 0, \\ \tau_k(v) &= \frac{d^k}{dx^k} \left(\frac{\sum_{j=1}^q b_j(v) x^j}{1 - \sum_{j=1}^p c_j(v) x^j} \right)_{x=0} \mathbb{I}_{\{k>0\}}, \quad k \geq 1, \end{aligned}$$

where $\mathbb{I}_{\{\cdot\}}$ denotes the indicator function. If f denotes the error density of η_t , then the conditional density of X_t given the information available up to time $t-1$ will be $\sigma_{t,\underline{\theta}_0} f(\sigma_{t,\underline{\theta}_0}^{-1} X_t)$, $1 \leq t \leq sN$. Hence, we can get a minimizer of the negative log-likelihood function

$$(sN)^{-1} \sum_{t=0}^{N-1} \sum_{v=1}^s \left(\frac{1}{2} \log \sigma_{t,\underline{\theta}}^2(v) - f(\sigma_{t,\underline{\theta}}^{-1}(v) X_{st+v}) \right), \underline{\theta} \in \Theta,$$

or, as a solution to the equation

$$\sum_{t=0}^{N-1} \sum_{v=1}^s \frac{1}{2} \left(1 + K^* \left(\sigma_{t,\underline{\theta}}^{-1}(v) X_{st+v} \right) \right) \sigma_{t,\underline{\theta}}^{-2}(v) \nabla_{\underline{\theta}} \sigma_{t,\underline{\theta}}^2(v) = 0,$$

where $K^*(x) = x f^{-1}(x) \nabla_x f(x)$, $\nabla_x f(x)$ denotes its derivative or gradient of $f(x)$. More generally, since the density f is unknown, for a score function K which satisfies some constraints, we can then define an estimator $\hat{\underline{\theta}}_{sN}$ as a solution of the following equation

$$\sum_{t=0}^{N-1} \sum_{v=1}^s \frac{1}{2} \left(1 + K \left(\sigma_{t,\underline{\theta}}^{-1}(v) X_{st+v} \right) \right) \sigma_{t,\underline{\theta}}^{-2}(v) \nabla_{\underline{\theta}} \sigma_{t,\underline{\theta}}^2(v) = 0.$$

Remark 1. We now discuss the score function K , in [18], let $K(x) = x\varphi(x)$, $x \in \mathbb{R}$, where $\varphi: \mathbb{R} \rightarrow \mathbb{R}$ is differentiable in all but finite number of points and an odd, i.e., satisfies the skew-symmetric function $\varphi(-x) = -\varphi(x)$, $\forall x \in \mathbb{R}^*$. Let $\Delta \subset \mathbb{R}$ be the set of points on which φ is differentiable and $\bar{\Delta}$ denote its complement. Examples of such score function are reported in Table 1.

Names	Expressions of φ	Expressions of K	Set $\bar{\Delta}$
1. Least absolute deviation (<i>LAD</i>) score function	$sign(x)$	$ x $	$\{0\}$
2. Huber's k -score function, $k > 0$	$x\mathbb{I}_{\{ x \leq k\}} + ksign(x)\mathbb{I}_{\{ x >k\}}$	$x^2\mathbb{I}_{\{ x \leq k\}} + k x \mathbb{I}_{\{ x >k\}}$	$\{-k, k\}$
3. <i>QMLE</i> score function	x	x^2	$\emptyset$
4. Score function for the maximum likelihood estimation (<i>MLE</i>)	$-f^{-1}(x)\nabla_x f(x)$, f is the true density of η_t	$x(-f^{-1}(x)\nabla_x f(x))$	According to f
5. Score function for the exponential pseudo <i>MLE</i>	$a x ^{b-1}sign(x)$ $a > 0, 1 < b \leq 2$	$a x ^b$	$\{0\}$
6. λ -score function, $\lambda > 1$	$\lambda sign(x)(1+ x)^{-1}$	$\lambda x (1+ x)^{-1}$	$\{0\}$
7. Score function for the Cauchy	$2x(1+x^2)^{-1}$	$2x^2(1+x^2)^{-1}$	$\{0\}$

Table 1. Examples of score functions.

Note however that $\hat{\theta}_{sN}$'s are noncomputable because $\sigma_{t,\underline{\theta}}^2$'s are unobservable. We define observable approximations $\{\sigma_{t,\underline{\theta}}^2, t \geq 1\}$ to the variance functions $\{\tilde{\sigma}_{t,\underline{\theta}}^2, t \geq 1\}$ as

$$\tilde{\sigma}_{t,\underline{\theta}}(v) = \kappa_{0,\underline{\theta}}(v) + \mathbb{I}_{\{st+v \geq 2\}} \sum_{i=1}^{st+v-1} (\kappa_{i,\underline{\theta}}(v)X_t^{+2}(v-i) + \tau_{i,\underline{\theta}}(v)X_t^{-2}(v-i)), \quad \underline{\theta} \in \Theta.$$

Then an M -estimator $\tilde{\theta}_{sN}$ based on the score function K is a measurable solution to the following equation,

$$\sum_{t=0}^{N-1} \sum_{v=1}^s \frac{1}{2} \left(1 + K \left(\tilde{\sigma}_{t,\underline{\theta}}^{-1}(v) X_{st+v} \right) \right) \tilde{\sigma}_{t,\underline{\theta}}^{-2}(v) \nabla_{\underline{\theta}} \tilde{\sigma}_{t,\underline{\theta}}^2(v) = 0.$$

Remark 2. For $K(x) = x^2$ of Example 3 in Table 1, $\hat{\theta}_{sN}$ is the renowned *QMLE* as debated by Guerbyenne and Kessira [10] and Bibi and Ghezal [3].

3. ASYMPTOTIC PROPERTIES FOR M -ESTIMATORS OF *PTGARCH* MODELS

To study the strong consistency and the asymptotic normality of $\hat{\theta}_{sN}$, we first define the polynomials $\mathcal{A}_{0,v}(z) = \sum_{i=1}^q a_{0,i}(v)z^i$, $\mathcal{B}_{0,v}(z) = \sum_{i=1}^q b_{0,i}(v)z^i$ and $\mathcal{C}_{0,v}(z) = 1 - \sum_{i=1}^p c_{0,i}(v)z^i$, by convention $\mathcal{A}_{0,v}(z) = 0$, $\mathcal{B}_{0,v}(z) = 0$ if $q = 0$ and $\mathcal{C}_{0,v}(z) = 1$ if $p = 0$, for all $v \in \{1, \dots, s\}$. Now, consider the following regularities assumptions (for more details and discussions, see Iqbal [13]),

H.0 For the score function K , there exists a unique constant $c_K > 0$ satisfying

$$E \left\{ K \left(c_K^{-1/2} \eta_0 \right) \right\} = 1.$$

H.1 $\underline{\theta}_0 \in \Theta$ and Θ is a compact subset of $]0, +\infty]^s \times [0, +\infty[^{s(2q+p)}$ and $\underline{\theta}_0, \underline{\theta}_0^{(K)} \in \overset{\circ}{\Theta}$, with $\overset{\circ}{\Theta}$ denotes the interior of Θ , where

$$\underline{\theta}^{(K)} := (\underline{a}'_{(K)}, \underline{b}'_{(K)}, \underline{c}'_{(K)}) = c_K (\underline{a}', \underline{b}', \underline{c}').$$

H.2 The moment conditions:

$$E \left\{ K^2 \left(c_K^{-1/2} \eta_0 \right) \right\} < \infty \text{ and } 0 < E \left\{ c_K^{-1/2} \eta_0 \nabla_{\underline{\theta}^{(K)}} K \left(c_K^{-1/2} \eta_0 \right) \right\} < \infty.$$

H.3 $(\eta_t)_{t \in \mathbb{Z}}$ is non-degenerate and $P(\eta_t > 0) \in (0, 1)$.

H.4 If $p > 0$, $\mathcal{A}_{0,v}(z)$ and $\mathcal{B}_{0,v}(z)$ have no common roots with $\mathcal{C}_{0,v}(z)$ for all v . Moreover, $\mathcal{A}_{0,v}(1) + \mathcal{B}_{0,v}(1) \neq 0$ and $a_{0,q}(v) + b_{0,q}(v) + c_{0,p}(v) \neq 0$ for all $v \in \{1, \dots, s\}$.

H.5 $\gamma_L^{(s)}(\Lambda_0) < 0$ and $\rho \left(\prod_{v=1}^s \Omega_v \right) < 1$ where $\gamma_L(\Lambda_0)$ is the Lyapunov exponent as-

sociated with the random matrix $\Lambda_0(\underline{\eta}_t)$, with $\Lambda_0(\underline{\eta}_t) := \prod_{v=0}^{s-1} \Gamma_{0,s-v}(\eta_t(s-v))$,

where $\Gamma_{0,v}(\eta_t(v))$ is appropriate $(2q+p) \times (2q+p)$ matrix easily obtained and uniquely determined by

$$\{a_{0,i}(v), b_{0,i}(v), c_{0,j}(v), \eta_t^+(v), \eta_t^-(v), 1 \leq i \leq q, 1 \leq j \leq p\}$$

and Ω_v is appropriate $p \times p$ matrix easily obtained and uniquely determined by $\{c_j(v), 1 \leq j \leq p\}$.

H.6 There exist functions g, h and l satisfying

- i. $|K(uv) - K(v)| \leq g(v) |u^2 - 1|, v \in \mathbb{R}, u > 0,$
- ii. $|\nabla K(uv) - \nabla K(v)| \leq h(v) |u - 1|, v \in \mathbb{R}, u > 0, uv, v \in \Theta,$
- iii. $|h(v + uv) - h(v)| \leq l(v) u, v \in \mathbb{R}, u > 0,$

where $E \left\{ \max \left(\log \left(g \left(c_K^{-1/2} \eta_0 \right) \right), 0 \right) \right\} < \infty, E \left\{ \left| c_K^{-1/2} \eta_0 \right| h \left(c_K^{-1/2} \eta_0 \right) \right\} < \infty$ and $E \left\{ \max \left(\log \left(l \left(c_K^{-1/2} \eta_0 \right) \right), 0 \right) \right\} < \infty.$

We are now in a position to state the following result.

Theorem 1. Under Assumptions **H.0** – **H.6**, then $\widehat{\underline{\theta}}_{Ns} \xrightarrow{P} \underline{\theta}_0^{(K)}$ as $N \rightarrow \infty$ and $\sqrt{Ns} \left(\widehat{\underline{\theta}}_{Ns} - \underline{\theta}_0^{(K)} \right) \rightsquigarrow \mathcal{N}(\underline{0}, \sigma_K^2 J^{-1})$ as $N \rightarrow \infty$ where the matrix J given by

$$J := \sum_{v=1}^s E_{\underline{\theta}_0^{(K)}} \left\{ \sigma_{t, \underline{\theta}_0^{(K)}}^{-2}(v) \nabla_{\underline{\theta}^{(K)}} \sigma_{t, \underline{\theta}_0^{(K)}}(v) \nabla_{\underline{\theta}^{(K)'}} \sigma_{t, \underline{\theta}_0^{(K)}}(v) \right\},$$

and $\sigma_K^2 = 4 \text{Var}_{\underline{\theta}_0^{(K)}} \left(K \left(c_K^{-1/2} \eta_0 \right) \right) \left(E_{\underline{\theta}_0^{(K)}} \left\{ c_K^{-1/2} \eta_0 \nabla_{\underline{\theta}^{(K)'}} K \left(c_K^{-1/2} \eta_0 \right) \right\} \right)^{-2}.$

To prove Theorem 1, we make the following assertions gathered in the following proposition (see, [12], [18], for more details)

Proposition 1. *Under Assumptions H.0 – H.6, we have*

- (1) Let $\{A_t^{(i)}, i = 1, 2, 3, t \geq 0\}$ be a sequence of identically distributed random variables. If $\sum_{i=1}^3 E \left\{ \max \left(\log A_0^{(i)}, 0 \right) \right\} < +\infty$, then for any $|r| < 1$, $\sum_{t \geq 0} \left(A_t^{(1)} + A_t^{(2)} A_t^{(3)} \right) r^t$ converges with probability one.
- (2) For all $v \in \{1, \dots, s\}$, $i \geq 0$, $t \in \mathbb{Z}$, we have $\sigma_{t, \theta_0}^{(K)} = c_K \sigma_{t, \theta_0}$, and the coefficients $\{\kappa_{i, \theta}(v), \tau_{i, \theta}(v)\}$ and the functions $\{\sigma_{t, \theta}\}$ are differentiable in the interior $\mathring{\Theta}$ of Θ .
- (3) There is a number $r \in (0, 1)$ such that $\forall \theta \in \mathring{\Theta}$, $v \in \{1, \dots, s\}$, $\forall i \geq 0$,

$$\begin{aligned} \kappa_1 \varsigma^i &< \kappa_{i, \theta}(v) < \kappa_2 r^i, & |\nabla_{\theta} \kappa_{i, \theta}(v)| &< \kappa_3 r^i, & \left| \nabla_{\theta}^2 \kappa_{i, \theta}(v) \right| &< \kappa_4 r^i, \\ \tau_1 \varsigma^i &< \tau_{i, \theta}(v) < \tau_2 r^i, & |\nabla_{\theta} \tau_{i, \theta}(v)| &< \tau_3 r^i, & \left| \nabla_{\theta}^2 \tau_{i, \theta}(v) \right| &< \tau_4 r^i, \end{aligned}$$
 where $\varsigma = \{a_0(v), a_1(v), \dots, a_q(v), b_1(v), \dots, b_q(v), c_1(v), \dots, c_p(v); v \in \{1, \dots, s\}, \theta \in \Theta\} \in (0, 1)$.
- (4) There exist random variables e_0 and e_1 , both independent of $\{\eta_t, t \geq 1\}$, such that $\forall \theta \in \mathring{\Theta}$, $0 < \sigma_{t, \theta_0} - \tilde{\sigma}_{t, \theta_0} < r^t e_0$ and $|\nabla_{\theta}(\sigma_{t, \theta_0} - \tilde{\sigma}_{t, \theta_0})| < r^t e_1$.
- (5) For any $d \geq 1$ and $v \in \{1, \dots, s\}$,

$$\begin{aligned} E \left\{ \sup \left\{ |\sigma_{t, \theta}(v)|^{-d} |\nabla_{\theta} \sigma_{t, \theta}(v)|^d, \theta \in \mathring{\Theta} \right\} \right\} &< +\infty, \\ E \left\{ \sup \left\{ |\sigma_{t, \theta}(v)|^{-d} |\nabla_{\theta}^2 \sigma_{t, \theta}(v)|^d, \theta \in \mathring{\Theta} \right\} \right\} &< +\infty. \end{aligned}$$
- (6) Let $R_1(x) = x \nabla_x K(x)$. The following two inequalities hold,

$$\begin{aligned} R_1(uv) - R_1(v) &\leq |u - 1| |v| (|\nabla_x K(v)| + u h(v)), \\ R_1(uv) - R_1(wv) &\leq |u - w| |v| (|u - 1| h(v) + |\nabla_x K(v)| + |h(wv)|). \end{aligned}$$

Now, we use the modified result of [16] because we want to establish consistency and asymptotic properties of $\hat{\theta}_{N_s}$. Let function $G_n : \Theta \rightarrow \mathbb{R}$ be twice differentiable with respect to a in a μ -neighborhood $\{\vartheta : \|\vartheta - \vartheta_0\| \leq \mu\} \subset \Theta$, $\vartheta_0 \in \Theta$, we define a sequence of estimators $\hat{\vartheta}_n \in \Theta$, as a solution to the equation

- i. $\nabla_{\vartheta} G_n(\hat{\vartheta}_n) = 0$,
- ii. $G_n(\vartheta) - G_n(\vartheta_0) = (\vartheta - \vartheta_0)' \nabla_{\vartheta} G_n(\vartheta_0) + 2^{-1} (\vartheta - \vartheta_0)' \nabla_{\vartheta}^2 G_n(\vartheta_0) (\vartheta - \vartheta_0) + 2^{-1} (\vartheta - \vartheta_0)' \nabla_{\vartheta}^2 F_n(\vartheta^*) (\vartheta - \vartheta_0)$,

where ϑ^* satisfies $\|\vartheta^* - \vartheta_0\| \leq \mu$ and $F_n(\vartheta^*) = \nabla_{\vartheta}^2 (G_n(\vartheta^*) - G_n(\vartheta_0))$. Moreover, the true parameter value ϑ_0 satisfies the following properties,

- iii. $n^{-1} \nabla_{\vartheta} G_n(\vartheta_0) \xrightarrow{p} 0, n \rightarrow +\infty$,
- iv. $(2n)^{-1} \nabla_{\vartheta}^2 G_n(\vartheta_0) \xrightarrow{p} \Sigma_G, n \rightarrow +\infty$,

for some positive definite matrix Σ_G ,

$$\mathbf{v.} \quad \lim_{n \rightarrow \infty} \limsup_{\mu \rightarrow 0} \left\{ (2n)^{-1} |F_n(\underline{\vartheta}^*)|; \|\underline{\vartheta}^* - \underline{\vartheta}_0\| \leq \mu \right\} < +\infty, \text{ a.s.}$$

From the above mentioned we have the following theorem which is much similar to that of Klimko and Nelson [16]

Theorem 2. Suppose that **ii.** – **v.** hold, then the following results on asymptotics of $\hat{\underline{\vartheta}}_n$ hold.

- (1) For $\hat{\underline{\vartheta}}_n$ satisfies the equation **i.**, then, $\hat{\underline{\vartheta}}_n \xrightarrow{P} \underline{\vartheta}_0$.
- (2) In addition, if for some positive definite matrix Γ_G , $(2n^{1/2})^{-1} \nabla_{\underline{\vartheta}} G_n(\underline{\vartheta}_0) \sim N(\underline{O}, \Gamma_G)$, then $n^{1/2} (\hat{\underline{\vartheta}}_n - \underline{\vartheta}_0) \sim \mathcal{N}(\underline{O}, \Sigma_G^{-1} \Gamma_G \Sigma_G^{-1})$.

The proof of Theorem 1. Define a function ϕ by $\phi(x) = \int_0^{|x|} \varphi(t) dt$, $\forall x \in \mathbb{R}$ and define

$$\begin{aligned} m_t(\underline{\theta}) &= \sum_{v=1}^s \left(\phi \left(\sigma_{t,\underline{\theta}}^{-1/2}(v) X_t(v) \right) + 2^{-1} \log(\sigma_{t,\underline{\theta}}(v)) \right), t = 0, \dots, N-1, \\ \tilde{m}_t(\underline{\theta}) &= \sum_{v=1}^s \left(\phi \left(\tilde{\sigma}_{t,\underline{\theta}}^{-1/2}(v) X_t(v) \right) + 2^{-1} \log(\tilde{\sigma}_{t,\underline{\theta}}(v)) \right), t = 0, \dots, N-1, \\ M_N(\underline{\theta}) &= \sum_{t=0}^{N-1} m_t(\underline{\theta}), \tilde{M}_N(\underline{\theta}) = \sum_{t=0}^{N-1} \tilde{m}_t(\underline{\theta}). \end{aligned}$$

Then, the first partial derivatives of $m_t(\underline{\theta})$ and $\tilde{m}_t(\underline{\theta})$ (resp., $M_N(\underline{\theta})$ and $\tilde{M}_N(\underline{\theta})$) are given by

$$\begin{aligned} \nabla_{\underline{\theta}} m_t(\underline{\theta}) &= 2^{-1} \sum_{v=1}^s \left(-K \left(\sigma_{t,\underline{\theta}}^{-1/2}(v) X_t(v) \right) + 1 \right) \sigma_{t,\underline{\theta}}^{-1}(v) \nabla_{\underline{\theta}} \sigma_{t,\underline{\theta}}(v), \\ \nabla_{\underline{\theta}} \tilde{m}_t(\underline{\theta}) &= 2^{-1} \sum_{v=1}^s \left(-K \left(\tilde{\sigma}_{t,\underline{\theta}}^{-1/2}(v) X_t(v) \right) + 1 \right) \tilde{\sigma}_{t,\underline{\theta}}^{-1}(v) \nabla_{\underline{\theta}} \tilde{\sigma}_{t,\underline{\theta}}(v), \\ \nabla_{\underline{\theta}} M_N(\underline{\theta}) &= 2^{-1} \sum_{t=0}^{N-1} \sum_{v=1}^s \left(-K \left(\sigma_{t,\underline{\theta}}^{-1/2}(v) X_t(v) \right) + 1 \right) \sigma_{t,\underline{\theta}}^{-1}(v) \nabla_{\underline{\theta}} \sigma_{t,\underline{\theta}}(v), \\ \nabla_{\underline{\theta}} \tilde{M}_N(\underline{\theta}) &= 2^{-1} \sum_{t=0}^{N-1} \sum_{v=1}^s \left(-K \left(\tilde{\sigma}_{t,\underline{\theta}}^{-1/2}(v) X_t(v) \right) + 1 \right) \tilde{\sigma}_{t,\underline{\theta}}^{-1}(v) \nabla_{\underline{\theta}} \tilde{\sigma}_{t,\underline{\theta}}(v). \end{aligned}$$

Thus, our proof depends on checking the conditions of the Theorem 1 for the criterion functions $\tilde{M}_N$. In the first step, we verify the above-mentioned conditions of Theorem 2 for the criterion functions M_N . The second partial derivatives of $m_t(\underline{\theta})$ and $\tilde{m}_t(\underline{\theta})$

are given by

$$\begin{aligned}\nabla_{\underline{\theta}}^2 m_t(\underline{\theta}) &= 2^{-2} \sum_{v=1}^s R_1 \left(\sigma_{t,\underline{\theta}}^{-1/2}(v) X_t(v) \right) \sigma_{t,\underline{\theta}}^{-2}(v) \nabla_{\underline{\theta}} \sigma_{t,\underline{\theta}}(v) \nabla_{\underline{\theta}'} \sigma_{t,\underline{\theta}}(v) \\ &\quad + \sum_{v=1}^s R_2 \left(\sigma_{t,\underline{\theta}}^{-1/2}(v) X_t(v) \right) \sigma_{t,\underline{\theta}}^{-2}(v) \left(\sigma_{t,\underline{\theta}}(v) \nabla_{\underline{\theta}}^2 \sigma_{t,\underline{\theta}}(v) \right. \\ &\quad \left. - \nabla_{\underline{\theta}} \sigma_{t,\underline{\theta}}(v) \nabla_{\underline{\theta}'} \sigma_{t,\underline{\theta}}(v) \right), \\ \nabla_{\underline{\theta}}^2 \tilde{m}_t(\underline{\theta}) &= 2^{-2} \sum_{v=1}^s R_1 \left(\tilde{\sigma}_{t,\underline{\theta}}^{-1/2}(v) X_t(v) \right) \tilde{\sigma}_{t,\underline{\theta}}^{-2}(v) \nabla_{\underline{\theta}} \tilde{\sigma}_{t,\underline{\theta}}(v) \nabla_{\underline{\theta}'} \tilde{\sigma}_{t,\underline{\theta}}(v) \\ &\quad + \sum_{v=1}^s R_2 \left(\tilde{\sigma}_{t,\underline{\theta}}^{-1/2}(v) X_t(v) \right) \tilde{\sigma}_{t,\underline{\theta}}^{-2}(v) \left(\tilde{\sigma}_{t,\underline{\theta}}(v) \nabla_{\underline{\theta}}^2 \tilde{\sigma}_{t,\underline{\theta}}(v) \right. \\ &\quad \left. - \nabla_{\underline{\theta}} \tilde{\sigma}_{t,\underline{\theta}}(v) \nabla_{\underline{\theta}'} \tilde{\sigma}_{t,\underline{\theta}}(v) \right),\end{aligned}$$

where $R_2(x) = 2^{-1}(1 - K(x))$. Hence, **ii.** holds. For the rest of the proofs are very similar to those of Fan et al. [20]. $\square$

The following result gives an estimator of the asymptotic variance

Theorem 3. Assume that Assumptions **H.0** – **H.6**, $E \left\{ g \left(c_K^{-1/2} \eta_0 \right) \right\} < +\infty$ and $E \left\{ X_{st+v}^2 \right\} < +\infty$, $1 \leq v \leq s$ hold, then

$$\hat{\sigma}_{K,N}^2 \hat{J}_N^{-1} \xrightarrow{P} \sigma_K^2 J^{-1}, N \longrightarrow +\infty,$$

where

$$\begin{aligned}\hat{\sigma}_{K,N}^2 &= 4 \left((Ns)^{-1} \sum_{t=0}^{N-1} \sum_{v=1}^s K^2 \left(X_t(v) \hat{\sigma}_{t,\hat{\underline{\theta}}_{Ns}}^{-1/2}(v) \right) \right. \\ &\quad \left. - \left((Ns)^{-1} \sum_{t=0}^{N-1} \sum_{v=1}^s K \left(X_t(v) \hat{\sigma}_{t,\hat{\underline{\theta}}_{Ns}}^{-1/2}(v) \right) \right)^2 \right) \\ &\quad \times \left((Ns)^{-1} \sum_{t=0}^{N-1} \sum_{v=1}^s X_t(v) \hat{\sigma}_{t,\hat{\underline{\theta}}_{Ns}}^{-1/2}(v) \nabla K \left(X_t(v) \hat{\sigma}_{t,\hat{\underline{\theta}}_{Ns}}^{-1/2}(v) \right) \right)^{-2}\end{aligned}$$

$$\text{and } \hat{J}_N := N^{-1} \sum_{t=0}^{N-1} \sum_{v=1}^s \hat{\sigma}_{t,\hat{\underline{\theta}}_{Ns}}^{-2}(v) \nabla \hat{\sigma}_{t,\hat{\underline{\theta}}_{Ns}}(v) \left(\nabla \sigma_{t,\hat{\underline{\theta}}_{Ns}}(v) \right)'.$$

Proof. The proof is very similar to those of Mukherjee [18]. It suffices to replace the stationarity and ergodicity arguments by the *SPS* and *PE* ones, respectively. For this we omit the proof. $\square$

Remarks 1. (1) The large class of *M*–estimators for estimating the parameters of the *PTGARCH* model includes the *QMLE* as a special case and *LAD*

- estimator as well as many other useful estimators. It is worth noting that M -estimators also contain less-known estimators (see, Examples 2, 6 and 7 in Table 1), which can be considered an attractive alternative to the $QMLE$.
- (2) Regarding the closed-form expression of the coefficients c_K in Assumption $H.0$ are $c_K^{1/2} = E\{|\eta_0|\}$ for the LAD score and $c_K = E\{\eta_0^2\}$ for the $QMLE$ score, $c_K = 1$ for the MLE score and $c_K = \left(aE\{|\eta_0|^b\}\right)^{2/b}$ for the exponential pseudo MLE score. For the other scores, c_K does not have a closed-form expression but the corresponding numerical values can be computed from $H.0$ and $H.3$ for various error distributions (for further discussion see [11]).
 - (3) It is worth noting that the condition $\gamma_L^{(s)}(\Lambda_0) < 0$ in $H.5$ provide the existence of a SPS solution to the $PTGARCH$ models. Moreover, the solution is unique and PE . Now, the non stationarity condition stems from the fact that the SPS condition is not met, i.e., $\gamma_L^{(s)}(\Lambda_0) > 0$. Thus, it is useful and fruitful the study the asymptotic properties of M -estimators for non-stationary $PTGARCH$ models (cf., [19], [14], [15], when $K(x) = x^2$).
 - (4) Francq and Zakoïan [7] established the asymptotic distribution of the $QMLE$ when some coefficients are equal to zero, thus, it would be beneficial to generalize the asymptotic properties of M -estimators for $PTGARCH$ when $\underline{\theta}_0$ is on the boundary.
 - (5) Various authors have supposed that the period s is known and fixed (see Ghezal et al. [3], [6], [5], [4], [8], [9] and references therein), if otherwise, the period s is obviously an important parameter, hence one can estimate the parameter s using the periodogram, as we can consider it the most important method to determine the presence of periodicities (see Martin and Kedem [17] for more details).

4. SIMULATION STUDY

Now, we illustrate the M -estimators described in previous sections (at least for a moderate periodicity coefficients $s = 2$ say), we provide some numerical results from Monte Carlo experiment. We simulate 500 independent trajectories via some specifications of $PTGARCH$ (1, 1) models with length $n \in \{1000, 3000\}$ with standard $\mathcal{N}(0, 1)$ and Student- $t_{(10)}$ innovation distributions and computed the $QMLE$, LAD , and Huber's estimates with $k = 1.5$ for each replication with vector $\underline{\theta}$ of parameters described in the bottom of each table below which is chosen to satisfy the SPS condition. For each trajectory, the vector $\underline{\theta}$ has been estimated with these three M -estimators noted as $\hat{\underline{\theta}}_n$. In Table below, the columns are correspond to the average of the parameters estimates over the 500 simulations. In order to show the performance of three M -estimators, the roots mean square error ($RMSE$) of the each $\hat{\underline{\theta}}_n(v)$, $v = 1, \dots, s$, (results between bracket), are reported in Table 2 below.

		$\mathcal{N}(0,1)$						$t_{(10)}$					
		$QMLE$		LAD		Huber, $k = 1.5$		$QMLE$		LAD		Huber, $k = 1.5$	
		1	$2/\pi$	1	$2/\pi$	1	$2/\pi$	1	$2/\pi$	1	$2/\pi$	1	$2/\pi$
Parameters	$c_K \setminus n$	1000	3000	1000	3000	1000	3000	1000	3000	1000	3000	1000	3000
$a_0^{(K)}(1)$	c_K	0.9964 (0.0826)	1.0037 (0.0679)	0.6329 (0.0490)	0.6389 (0.0450)	0.8211 (0.0688)	0.8278 (0.0392)	0.9949 (0.0749)	1.0039 (0.0612)	0.5453 (0.0610)	0.5529 (0.0511)	0.7198 (0.0804)	0.7292 (0.0594)
$a_0^{(K)}(2)$	$0.5c_K$	0.5032 (0.0559)	0.4984 (0.0456)	0.3216 (0.0844)	0.3167 (0.0723)	0.4157 (0.0908)	0.4108 (0.0551)	0.5099 (0.0607)	0.4971 (0.0573)	0.2846 (0.0932)	0.2722 (0.0820)	0.3650 (0.0921)	0.3601 (0.0604)
$a_1^{(K)}(1)$	$0.5c_K$	0.4995 (0.0643)	0.4996 (0.0397)	0.3177 (0.0364)	0.3180 (0.0223)	0.4119 (0.0600)	0.4121 (0.0343)	0.5039 (0.0648)	0.4994 (0.0407)	0.2788 (0.0372)	0.2796 (0.0230)	0.3672 (0.0613)	0.3620 (0.0360)
$a_1^{(K)}(2)$	$0.25c_K$	0.2459 (0.0678)	0.2501 (0.0402)	0.1552 (0.0372)	0.1594 (0.0220)	0.2018 (0.0642)	0.2065 (0.0391)	0.2419 (0.0686)	0.2502 (0.0403)	0.1301 (0.0388)	0.1379 (0.0228)	0.1767 (0.0646)	0.1816 (0.0395)
$b_1^{(K)}(1)$	$0.25c_K$	0.2479 (0.0501)	0.2493 (0.0303)	0.1569 (0.0281)	0.1584 (0.0169)	0.2042 (0.0445)	0.2056 (0.0247)	0.2462 (0.0504)	0.2481 (0.0305)	0.1338 (0.0289)	0.1355 (0.0174)	0.1794 (0.0472)	0.1806 (0.0275)
$b_1^{(K)}(2)$	$0.45c_K$	0.4429 (0.0799)	0.4466 (0.0467)	0.2802 (0.0440)	0.2853 (0.0259)	0.3682 (0.0754)	0.3702 (0.0452)	0.4424 (0.0799)	0.4468 (0.0470)	0.2400 (0.0453)	0.2461 (0.0265)	0.3239 (0.0760)	0.3283 (0.0461)
$c_1^{(K)}(1)$	$0.15c_K$	0.1391 (0.0269)	0.1426 (0.0266)	0.0885 (0.0562)	0.0904 (0.0529)	0.1129 (0.0225)	0.1284 (0.0131)	0.1377 (0.0290)	0.1424 (0.0268)	0.0722 (0.0675)	0.0769 (0.0608)	0.0993 (0.0441)	0.1016 (0.0437)
$c_1^{(K)}(2)$	$0.55c_K$	0.5525 (0.0297)	0.5518 (0.0283)	0.3532 (0.0416)	0.3524 (0.0379)	0.4553 (0.0365)	0.4556 (0.0192)	0.5527 (0.0355)	0.5622 (0.0290)	0.3054 (0.0354)	0.3047 (0.0299)	0.4011 (0.0387)	0.3996 (0.0238)

Table 2: Average and $RMSE$ of 500 simulations of three M -estimators for $PTGARCH(1,1)$

Now, a few comments are in order. Table 2 compares the asymptotic parameters estimates and their $RMSE$ over 500 independent simulations of the periodic $TGARCH(1,1)$ for sample sizes $n = 1000$ and $n = 3000$. As expected the estimates of the periodic $TGARCH(1,1)$ coefficients based on $QMLE$, LAD , and Huber's displayed $RMSE$ decrease as the sample size increases. We can also see that the values of the estimates of the $QMLE$, LAD , and Huber's corresponding to $\mathcal{N}(0,1)$ are generally found to be more efficient than t_{10} ones. Furthermore, LAD -estimator performs even better compared to all its competitors in terms of even smaller $RMSE$.

5. AN APPLICATION TO EXCHANGE RATES

In this section, we apply our model for modeling the two daily time series $(X_{1t}, X_{2t})_{t \geq 1}$ of exchange rates: Euro/Algerian dinar (EUR/DZD) and U.S. dollar/ Algerian dinar (USD/DZD) provided by the Bank of Algeria. We first removed all the days where market was closed (i.e., holidays and weekends). The observations cover the period from January 3, 2000 to September 29, 2011. The final sample consists of 3055 observations for each series. In order to do so, we collect the two exchange rates at time t ,

$$X_{1t} = EUR/DZD_t, X_{2t} = USD/DZD_t,$$

and their corresponding log-return series r_{1t} and r_{2t} , where $r_{kt} = 100 \log(X_{kt}/X_{kt-1})$, for $k = 1, 2$. The graphics of prices X_{1t} and X_{2t} , the daily returns series of prices r_{1t}, r_{2t} , squad and absolute returns are plotted in Fig 1

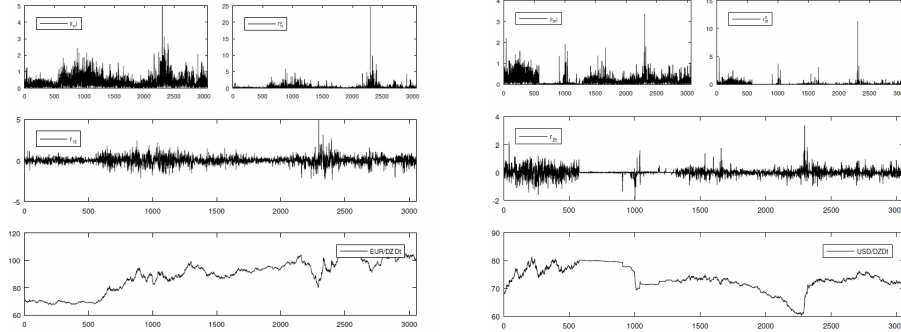


FIGURE 1. The plots of the price series (X_{kt}) , (r_{kt}) , (r_{kt}^2) and $(|r_{kt}|)$, $k = 1, 2$.

Some elementary descriptive statistics are provided for the two log –return series in Table 3.

Series	mean	Std. Dev	Median	Skewness	Kurtosis	Min	Max
r_{1t}	0.1000	5.0000	0.1000	0.4000	9.0000	−23.300	49.700
r_{2t}	0.0000	3.0000	0.0000	1.0000	13.000	−19.000	34.000

Table 3: Summary statistics for returns series $(r_{1t}, r_{2t})_{t \geq 1}$.

Continuation of Table 3

Series	Arch(300)	J. Bera	LBtest
r_{1t}	100%	4.597×10^3	87.33%
r_{2t}	100%	1.301×10^4	100%

The findings indicate that for the EUR/DZD (resp. USD/DZD), the lowest returns (−23.300) (resp. −19.000) and the highest returns (49.700) (resp. 34.000). The skewness for two log –return series is positive. Moreover, one of the features which prominently stands out most from Table 3 is that the kurtosis for two log –returns is much larger than 3, which indicates that the models based on the Gaussian assumption may not well describe the data.

From Fig 2, that $(r_{kt})_{t \geq 1, k=1,2}$ presents a Taylor-effect (characterized by $\hat{\rho}_{|r_{kt}|}(h) > \hat{\rho}_{r_{kt}^2}(h)$ for some lag $h \geq 1$). Now, we will propose the 5–periodic $TGARCH(1,1)$ model that allows for the description of intraweek effect in the daily exchange rate, where the parameters are allowed to vary with the day of the week ($v = 1, 2, 3, 4$ and 5)

$$v = \begin{cases} 1 & \text{if the day corresponding is a Monday} \\ \vdots & \\ 5 & \text{if the day corresponding is a Friday} \end{cases}.$$

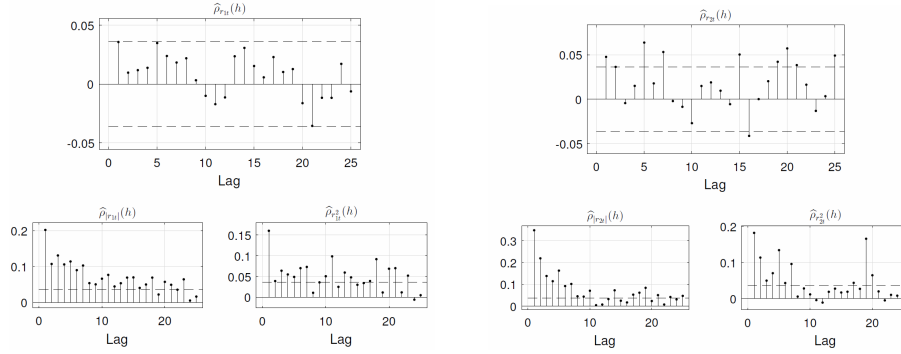


FIGURE 2. Sample autocorrelations of returns associated to Euro and Dollar.

The estimated parameters of the 5—periodic $TGARCH(1, 1)$ model and their $RMSE$ are reported in Tables 4 – 5.

Days	<i>QMLE</i>				<i>LAD</i>			
	$\hat{a}_0^K$	$\hat{a}_1^K$	$\hat{b}_1^K$	$\hat{c}_1^K$	$\hat{a}_0^K$	$\hat{a}_1^K$	$\hat{b}_1^K$	$\hat{c}_1^K$
Monday	0.0308 (0.0403)	0.0239 (0.0165)	0.0259 (0.0177)	0.0986 (0.1130)	0.0013 (0.0008)	0.0089 (0.0018)	0.0067 (0.0016)	0.3464 (0.1043)
Tuesday	0.0434 (0.0370)	0.0277 (0.0135)	0.0363 (0.0145)	0.1154 (0.0893)	0.0051 (0.0004)	0.0031 (0.0007)	0.0028 (0.0008)	0.2154 (0.0560)
Wednesday	0.0489 (0.0311)	0.0255 (0.0145)	0.0232 (0.0127)	0.1348 (0.0713)	0.0048 (0.0006)	0.0095 (0.0009)	0.0091 (0.0008)	3.3459 (0.1415)
Thursday	0.0413 (0.0316)	0.0196 (0.0146)	0.0180 (0.0127)	0.1105 (0.0825)	0.0117 (0.0012)	0.0106 (0.0013)	0.0096 (0.0014)	5.0442 (0.1880)
Friday	0.0291 (0.0331)	0.0183 (0.0148)	0.0234 (0.0179)	0.0884 (0.0964)	0.0048 (0.0009)	0.0031 (0.0014)	0.0086 (0.0020)	0.5705 (0.1103)

Table 4. *QMLE*, *LAD*, Huber's estimates from the Euro data and their $RMSE$.

Days	<i>QMLE</i>				<i>LAD</i>			
	$\hat{a}_0^K$	$\hat{a}_1^K$	$\hat{b}_1^K$	$\hat{c}_1^K$	$\hat{a}_0^K$	$\hat{a}_1^K$	$\hat{b}_1^K$	$\hat{c}_1^K$
Monday	0.0480 (0.0062)	0.0749 (0.0295)	0.0870 (0.0224)	0.2028 (0.0606)	0.0332 (0.0153)	0.0933 (0.0301)	0.0792 (0.0377)	0.0952 (0.0183)
Tuesday	0.0301 (0.0058)	0.0800 (0.0168)	0.0453 (0.0128)	0.1269 (0.0412)	0.0051 (0.0006)	0.0697 (0.0327)	0.2578 (0.0670)	0.1409 (0.0217)
Wednesday	0.0220 (0.0051)	0.1063 (0.0173)	0.0526 (0.0174)	0.0792 (0.0381)	0.0037 (0.0018)	0.1128 (0.0508)	0.0259 (0.0097)	0.0458 (0.0195)
Thursday	0.0219 (0.0045)	0.0755 (0.0193)	0.0529 (0.0127)	0.1016 (0.0440)	0.0100 (0.0086)	0.1159 (0.0991)	0.0903 (0.0341)	0.3014 (0.1665)
Friday	0.0308 (0.0055)	0.0968 (0.0159)	0.0562 (0.0191)	0.1458 (0.0485)	0.0222 (0.0119)	0.0468 (0.0249)	0.0521 (0.0139)	0.0560 (0.0069)

Table 5. *QMLE*, *LAD*, Huber's estimates from the Dollar data and their $RMSE$.

Huber, $k = 1.5$				
Days	$\hat{a}_0^K$	$\hat{a}_1^K$	$\hat{b}_1^K$	$\hat{c}_1^K$
Monday	0.1808 (0.0333)	0.1061 (0.0155)	0.6063 (0.0835)	0.3197 (0.1352)
Tuesday	0.1849 (0.0368)	0.6622 (0.0271)	4.0544 (0.0672)	0.1452 (0.0934)
Wednesday	0.8388 (0.0146)	5.0880 (0.0493)	0.5673 (0.0230)	0.5964 (0.0350)
Thursday	0.4958 (0.0416)	0.2560 (0.0725)	0.1840 (0.0558)	1.9462 (0.1735)
Friday	0.1753 (0.0216)	0.2997 (0.0373)	0.2316 (0.0758)	0.3591 (0.0608)
Continuous of Table 4.				
Huber, $k = 1.5$				
Days	$\hat{a}_0^K$	$\hat{a}_1^K$	$\hat{b}_1^K$	$\hat{c}_1^K$
Monday	0.0390 (0.0270)	0.0999 (0.0369)	0.0670 (0.0437)	0.1465 (0.0323)
Tuesday	0.0198 (0.0137)	0.2140 (0.0739)	0.2896 (0.0804)	0.0510 (0.0304)
Wednesday	0.0038 (0.0004)	0.0920 (0.0215)	0.0114 (0.0016)	0.0039 (0.0018)
Thursday	0.0088 (0.0062)	0.1271 (0.0731)	0.1389 (0.0615)	0.2626 (0.1380)
Friday	0.0439 (0.0140)	0.7237 (0.0301)	0.1481 (0.0916)	0.0658 (0.0363)
Continuous of Table 5.				

6. CONCLUSION

In this paper, we consider estimating the parameters of the periodic asymmetric *GARCH*(p, q) model using M -estimators. Regardless of the usually-used *QMLE*, alternative estimators based on the *LAD* and Huber's score functions are used. Simulation results expose that some of these estimators can perform better than the *QMLE*. Finally, the methodology is illustrated through a simulation study and an empirical application of two daily time series of exchange rates: Euro/Algerian dinar (*EUR/DZD*) and U.S. dollar/ Algerian dinar (*USD/DZD*) provided by the Bank of Algeria.

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SOME FIXED POINT RESULTS FOR GENERALIZED MODIFIED F_α -CONTRACTIONS IN MODULAR METRIC SPACES AND ITS APPLICATIONS

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Abstract. This paper introduces new generalizations of modified F -contractions through α -admissible mappings. It expands the fixed-point results of modified F -contractions within the framework of modular metric spaces. Several theorems are presented regarding the existence of solutions, along with a condition that ensures the uniqueness of a solution. These theorems are applied to specific mappings to verify the results. Additionally, the paper presents significant outcomes, demonstrating the existence of solutions to a general form of an integral equation. Finally, a particular example of an integral equation is discussed.

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1. INTRODUCTION

Wardowski [20] acquainted the prominent idea of F -contraction.

Definition 1. Let Ω be a nonempty set, and d be a usual metric on Ω . Any mapping $H: \Omega \rightarrow \Omega$ is called an F -contraction provided that for a given $\kappa > 0$

$$d(H\tau_0, H\tau_1) > 0 \implies \kappa + F(d(H\tau_0, H\tau_1)) \leq F(d(\tau_0, \tau_1)) \quad (1.1)$$

is satisfied for all $\tau_0, \tau_1 \in \Omega$ and the function $F: R^+ \rightarrow R$ yields the followings:

- (1_F) For all $a, b \in R^+$, $a < b$ implies $F(a) < F(b)$, that is, F is an increasing function.
- (2_F) $\lim_{n \rightarrow \infty} a_n = 0$ if and only if $\lim_{n \rightarrow \infty} F(a_n) = -\infty$ where $\{a_n\}_{n=1}^\infty$ is a sequence in R^+ .
- (3_F) For some $r \in (0, 1)$, $\lim_{n \rightarrow \infty} a^n F(a) = 0$ holds.

The set of all functions satisfying $(1_F) - (3_F)$ is specified by $\mathfrak{F}$. It is easy to figure out that $A(\xi) = \ln(\xi)$, $B(\xi) = \xi + \ln(\xi)$, and $C(\xi) = -\frac{1}{\sqrt{\xi}}$ are satisfying the conditions $(1_F) - (3_F)$, i.e., $A(\xi), B(\xi), C(\xi) \in \mathfrak{F}$.

By defining F -contraction, Wardowski [20] generalized the Banach contraction principle [3] and demonstrated the subsequent theorem.

Theorem 1. Assume that (Ω, d) is a complete metric space. If $H: \Omega \rightarrow \Omega$ is an F -contraction, then H admits a fixed point.

There are many studies related to F -contraction; see [9, 12, 15–19, 21].

Definition 2. Let ψ be defined and continuous on $[0, \infty)$. If ψ yields

- (1) ψ is non-decreasing,
- (2) For all $\xi > 0$, $\sum_{i=1}^{\infty} \psi^i(\xi) < \infty$,

then it is named as c -comparison function.

It is clear that $\psi(\xi) < \xi$ and also $\psi(0) = 0$. Ψ represents the set of functions providing the conditions (1) – (2). For further properties and examples of comparison functions, see [4, 5, 13].

Since the notion of α -admissible mappings were defined, they have been used in many metric fixed point studies and generalized in various directions.

Definition 3 ([14]). Let $\Omega \neq \emptyset$ and $\alpha: \Omega \times \Omega \rightarrow [0, \infty)$ be a mapping. If for all $a, b \in \Omega$,

$$\alpha(a, b) \geq 1 \implies \alpha(Ha, Hb) \geq 1, \quad (1.2)$$

then $H: \Omega \rightarrow \Omega$ is said to be α -admissible.

F -contraction has been reconstructed via α -admissible structure as in the following.

Definition 4 ([2]). Consider the metric space (Ω, d) . If for a given $\kappa > 0$ the following expression holds for all $\tau_0, \tau_1 \in \Omega$:

$$d(H\tau_0, H\tau_1) > 0 \implies \kappa + F(\alpha(\tau_0, \tau_1)d(H\tau_0, H\tau_1)) \leq F(\psi(d(\tau_0, \tau_1))), \quad (1.3)$$

where $\psi \in \Psi$ and $F \in \mathfrak{F}$, then the α -admissible mapping $H: \Omega \rightarrow \Omega$ is called a modified F_α -contraction.

Recall that $F(\xi) = \ln(\xi) \in \mathfrak{F}$. Then, inequality (1.3) turns into

$$\alpha(\tau_0, \tau_1)d(H\tau_0, H\tau_1) \leq e^{-\kappa}\psi(d(\tau_0, \tau_1)) \leq \psi(d(\tau_0, \tau_1)) \quad (1.4)$$

for all $\tau_0, \tau_1 \in \Omega$ and $H\tau_0 \neq H\tau_1$.

In [2] the following result has been verified.

Theorem 2. Consider a complete metric space (Ω, d) and let $H: \Omega \rightarrow \Omega$ be a modified F_α -contraction. If the followings are satisfied:

- (1*) for a given s_0 $\alpha(s_0, Hs_0) \geq 1$ holds,
- (2*) H is α -admissible,
- (3*) H is continuous,

then H has a fixed point.

All the above motivate us to investigate the fixed point results for modified F_α -contraction mappings in the setting modular metric spaces (MMS).

The primary purpose of this paper is to give some fixed point theorems and results in a MMS for modified F_α -contractions. The existence of solutions for an integral equation is investigated for an application. As an example, a solution for a given integral equation is obtained.

This paper is organized as follows: Section 2 offers a comprehensive introduction to MMS, establishing the foundation for subsequent discussions. Section 3 defines modified F -contractions and presents the main results of the paper, along with several important outcomes. Section 4 is dedicated to exploring applications that support the results obtained. Finally, Section 5 provides a summary of the key insights and conclusions of the present work.

2. FUNDAMENTALS OF MODULAR SPACES

We consider an example to understand the theory of modular metric spaces [1]. Take Ω as the set of all points above water on the earth's surface. The average speed needed to travel directly over land from any point τ_0 to another point τ_1 in a time t is denoted by $m_t(\tau_0, \tau_1)$.

Now we shall consider what kind of characteristics the function $m_t(\tau_0, \tau_1)$ should have. If we determine τ_0 and τ_1 , then $m_t(\tau_0, \tau_1)$ becomes a non-increasing function of t , and also non-negative. Clearly, $m_t(\tau_0, \tau_1)$ is symmetric in τ_0 and τ_1 . Till now, we assume that the points are in same ground. What if τ_0 and τ_1 are in separate landmasses? As it is needed to travel from τ_0 to τ_1 by land, it is not possible to travel in any time t independently of the speed. In all situations, the speed function $m_t(\tau_0, \tau_1)$ should be defined so it can be allowed to get extended real values, i.e., $m_t(\tau_0, \tau_1) = \infty$. In short, the speed function is non-increasing in t , symmetric in $\tau_0, \tau_1 \in \Omega$, and also takes values in $[0, \infty]$ for $t > 0$. Further property of the function $m_t(\tau_0, \tau_1)$ is given by

$$m_{t+s}(\tau_0, \tau_1) \leq m_t(\tau_0, \tau_2) + m_s(\tau_2, \tau_1) \quad (2.1)$$

for all $\tau_0, \tau_1, \tau_2 \in \Omega$ and $t, s > 0$.

Consider a function $m: (0, \infty) \times \Omega \times \Omega \rightarrow [0, \infty]$ where $\Omega \neq \emptyset$ and $\eta > 0$. It can be rewritten as $m(\eta, \tau_0, \tau_1) = m_\eta(\tau_0, \tau_1)$ for all $\eta > 0$ and $\tau_0, \tau_1 \in \Omega$ so that $m_\eta: \Omega \times \Omega \rightarrow [0, \infty]$.

Definition 5 ([6, 7]). Assume that $m_\eta: \Omega \times \Omega \rightarrow [0, \infty]$ satisfies the followings:

$$\begin{aligned} m1) \quad & \tau_0 = \tau_1 \iff m_\eta(\tau_0, \tau_1) = 0, \\ m2) \quad & m_\eta(\tau_0, \tau_1) = m_\eta(\tau_1, \tau_0), \\ m3) \quad & m_{\eta+\mu}(\tau_0, \tau_1) \leq m_\eta(\tau_0, \tau_2) + m_\mu(\tau_2, \tau_1) \end{aligned} \quad (2.2)$$

for all $\tau_0, \tau_1, \tau_2 \in \Omega$, and $\eta, \mu > 0$. Then, m_η is called a modular metric (*MM*) on Ω .

m is called pseudomodular if it satisfies $m_\eta(\tau, \tau) = 0, \forall \eta > 0$ instead of $m1$).

If instead of $m3$), m grants the following

$$m_{\eta+\mu}(\tau_0, \tau_1) \leq \frac{\eta}{\eta+\mu} m_\eta(\tau_0, \tau_2) + \frac{\mu}{\eta+\mu} m_\mu(\tau_2, \tau_1), \quad \forall \eta, \mu > 0, \quad (2.3)$$

then it is a convex *MM*. Furthermore, any convex *MM* yields

$$m_\eta(\tau_0, \tau_1) \leq \frac{\mu}{\eta} m_\mu(\tau_0, \tau_1) \leq m_\mu(\tau_0, \tau_1) \quad \forall \eta, \mu > 0 \quad (2.4)$$

for all $\tau_0, \tau_1 \in \Omega$ and $0 < \mu \leq \eta$ [6]. In general,

$$m_{\eta_2}(\tau_0, \tau_1) \leq m_{\eta_1}(\tau_0, \tau_1) \quad (2.5)$$

holds for $0 < \eta_1 \leq \eta_2$ and $\forall \tau_0, \tau_1 \in \Omega$.

Definition 6 ([6, 7]). m is said to be strict on Ω provided that for $\tau_1, \tau_2 \in \Omega$ with $\tau_1 \neq \tau_2$, $m_\eta(\tau_1, \tau_2) > 0, \forall \eta > 0$, or equivalently if $m_\eta(\tau_1, \tau_2) = 0$ for some $\eta > 0$, then $\tau_1 = \tau_2$.

Let (Ω, d) be a metric space consisting of at least two elements. We shall define some modular metrics on Ω .

Example 1. Consider the *MM*

$$m_\eta(\tau_0, \tau_1) = \frac{d(\tau_0, \tau_1)}{\eta}, \quad (2.6)$$

where d is a metric on Ω . Here $m_\eta(\tau_0, \tau_1)$ can be taken as the average speed needed to travel from τ_0 to τ_1 in time η . It is easy to see that the *MM* (2.6) is convex.

Example 2. Define the *MM*

$$m_\eta(\tau_0, \tau_1) = \begin{cases} \infty, & \eta < d(\tau_0, \tau_1), \\ 0, & \eta \geq d(\tau_0, \tau_1), \end{cases} \quad (2.7)$$

where d is a metric on Ω . Here $m_\eta(\tau_0, \tau_1)$ can be considered as the extreme condition of the speed metaphor. If $\eta \geq d(\tau_0, \tau_1)$, then we can travel immediately, yet in the case of $\eta < d(\tau_0, \tau_1)$, it is not possible to travel from τ_0 to τ_1 . The *MM* (2.7) is also convex.

Definition 7 ([6]). Take an m on Ω and $\tau_0 \in \Omega$. The followings are *MMS* around τ_0 :

$$\Omega_m = \Omega_m(\tau_0) = \{\tau \in \Omega: m_\eta(\tau, \tau_0) \rightarrow 0 \text{ as } \eta \rightarrow \infty\}, \quad (2.8)$$

$$\Omega_m^* = \Omega_m^*(\tau_0) = \{\tau \in \Omega: \exists \eta = \eta(\tau) > 0, \text{ s.t. } m_\eta(\tau, \tau_0) < \infty\}. \quad (2.9)$$

Definition 8 ([6, 8]). For Ω_m and Ω_m^* the followings hold.

- The sequence $\{h_n\}$ in Ω_m is m -convergent to a point $h \in \Omega$, named as the modular limit of $\{h_n\}$, if and only if $m_\eta(h_n, h) \rightarrow 0$ as $n \rightarrow \infty$ for some $\eta > 0$.
- $\{h_n\}$ in Ω_m is m -Cauchy if $m_\eta(h_n, h_m) \rightarrow 0$ as $n, m \rightarrow \infty$ for some $\eta > 0$.
- Consider S to be a nonempty subset of Ω_m . Provided that every m -Cauchy sequence belonging to S is m -convergent in S , then S is m -complete.

Now, we give some information about metrics on modular sets. If we take a modular m on Ω and a modular set Ω_m defined by m , then

$$d_m(\tau_0, \tau_1) = \inf\{\eta > 0: m_\eta(\tau_0, \tau_1) \geq \eta\}, \quad \forall \tau_0, \tau_1 \in \Omega_m \quad (2.10)$$

describes a metric on Ω_m [6].

In the case when m is convex, the modular space can be equipped with the following metric:

$$d_m^*(\tau_0, \tau_1) = \inf\{\eta > 0: m_\eta(\tau_0, \tau_1) \leq 1\}, \quad \forall \tau_0, \tau_1 \in \Omega_m. \quad (2.11)$$

The metrics given above on Ω_m are strongly equivalent, i.e., $d_m \leq d_m^* \leq 2d_m$.

Lemma 1 ([8]). Suppose that m is convex on Ω_m . For a sequence h_n belonging to Ω_m and $h \in \Omega_m^*$, the following holds:

$$\lim_{n \rightarrow \infty} d_m^*(h_n, h) = 0 \iff \lim_{n \rightarrow \infty} m_\eta(h_n, h) = 0 \quad (2.12)$$

for all $\eta > 0$.

In [8], it is demonstrated that modular convergence is strictly weaker than metric convergence.

Lemma 2 ([8]). For any pseudomodular metric m on Ω , the (2.8)-(2.9) are closed w.r.t m -convergent. Moreover, if m is strict then the limit is unique whenever it exists.

3. MAIN RESULTS

This part introduces new definitions for generalized modified F_α -contraction mappings. Then, some theorems and related results about fixed points for the mentioned mappings are proved in modular metric spaces.

Definition 9 ([10, 11]). Let m be a modular metric on a nonempty set Ω . Any self-mapping H on Ω_m is named as a generalized modified F_α -contraction of type I if there exists $\kappa > 0$ s.t.

$$\begin{aligned} m_\eta(H\tau_0, H\tau_1) > 0 \quad \text{implies that} \\ \kappa + F(\alpha(\tau_0, \tau_1)m_\eta(H\tau_0, H\tau_1)) \leq \\ F\left(\psi\left(\max\{m_\eta(\tau_0, \tau_1), \frac{m_\eta(\tau_0, H\tau_0)(1+m_\eta(\tau_1, H\tau_1))}{1+m_\eta(\tau_0, \tau_1)}, \frac{m_\eta(\tau_1, H\tau_0)(1+m_\eta(\tau_1, H\tau_0))}{1+m_\eta(\tau_0, \tau_1)}\}\right)\right) \end{aligned}$$

for all $\tau_0, \tau_1 \in \Omega_m$. Here, $F \in \mathfrak{F}$ and $\psi \in \Psi$.

We also introduce the following definition.

Definition 10 ([10, 11]). Let m be a modular metric on a nonempty set Ω . Any self-mapping H on Ω_m is named as a generalized modified F_α -contraction of type II if for all $\tau_0, \tau_1 \in \Omega_m$, there exists $\kappa > 0$ such that the following holds;

$$\begin{aligned} m_\eta(H\tau_0, H\tau_1) > 0 \quad \text{implies that} \\ \kappa + F(\alpha(\tau_0, \tau_1)m_\eta(H\tau_0, H\tau_1)) \leq \\ F\left(\psi\left(\max\{m_\eta(\tau_0, \tau_1), \frac{m_\eta(\tau_0, H\tau_0)(1+m_\eta(\tau_1, H\tau_1))}{1+m_\eta(\tau_0, \tau_1)}, \frac{m_\eta(\tau_1, H\tau_1)m_\eta(\tau_0, H\tau_0)}{1+m_\eta(H\tau_0, H\tau_1)}, \frac{m_\eta(\tau_1, H\tau_1)m_\eta(\tau_1, H\tau_0)}{1+m_\eta(\tau_1, H\tau_0)+m_\eta(\tau_0, H\tau_1)}\}\right)\right), \end{aligned} \quad (3.2)$$

where $F \in \mathfrak{F}$ and $\psi \in \Psi$.

Now, we shall give our main theorems for mappings satisfying the contraction conditions (3.1) and (3.2) in Ω_m (2.8).

Theorem 3. Let m be a strict modular metric in a complete modular metric space Ω_m , and $H: \Omega_m \rightarrow \Omega_m$ which ensures condition (3.1). Assume that

- (C₁) for a given s_0 , $\alpha(s_0, Hs_0) \geq 1$ holds,
- (C₂) H is an α -admissible mapping,
- (C₃) H is a continuous mapping,

then H admits a fixed point.

Proof. Take an initial point $s_0 \in \Omega_m$ satisfying $\alpha(s_0, Hs_0) \geq 1$. Then, construct a sequence $\{s_n\} \in \Omega_m$ as $s_n = H^n s_0$, $\forall n \in \mathbb{N}$.

Presume that $s_n = s_{n+1}$ holds for any $n \in \mathbb{N}$. In this case, s_n is the fixed point of H . Hence, for all $n \in \mathbb{N}$ s_n cannot be equal to s_{n+1} which gives $m_\eta(s_n, s_{n+1}) > 0$. Since $m_\eta(s_n, s_{n+1}) = m_\eta(Hs_{n-1}, Hs_n)$, $m_\eta(Hs_{n-1}, Hs_n) > 0$ holds.

As $m_\eta(s_0, Hs_0) = m_\eta(s_0, s_1) \geq 1$ and H is an α -admissible mapping, we reach $\alpha(s_{n-1}, s_n) \geq 1$ for all n .

Putting $\tau_0 = s_{n-1}$ and $\tau_1 = s_n$ in (3.1) yields

$$\kappa + F(\alpha(s_{n-1}, s_n)m_\eta(Hs_{n-1}, Hs_n)) \leq F\left(\psi\left(\max\{m_\eta(s_{n-1}, s_n), \frac{m_\eta(s_{n-1}, Hs_{n-1})(1+m_\eta(s_n, Hs_n))}{1+m_\eta(s_{n-1}, s_n)}, \frac{m_\eta(s_n, Hs_{n-1})(1+m_\eta(s_n, Hs_{n-1}))}{1+m_\eta(s_{n-1}, s_n)}\}\right)\right). \quad (3.3)$$

Owing to $m_\eta(s_n, Hs_{n-1}) = m_\eta(s_n, s_n) = 0$, we have

$$\begin{aligned} & \kappa + F(\alpha(s_{n-1}, s_n)m_\eta(s_n, s_{n+1})) \\ & \leq F\left(\psi\left(\max\{m_\eta(s_{n-1}, s_n), \frac{m_\eta(s_{n-1}, s_n)(1+m_\eta(s_n, s_{n+1}))}{1+m_\eta(s_{n-1}, s_n)}\}\right)\right). \end{aligned} \quad (3.4)$$

Assume that $m_\eta(s_n, s_{n+1}) > m_\eta(s_{n-1}, s_n)$. Then it is evident that

$$\frac{1+m_\eta(s_n, s_{n+1})}{1+m_\eta(s_{n-1}, s_n)} < \frac{m_\eta(s_n, s_{n+1})}{m_\eta(s_{n-1}, s_n)}. \quad (3.5)$$

Putting (3.5) in the right side of (3.4) we have

$$\kappa + F(\alpha(s_{n-1}, s_n)m_\eta(s_n, s_{n+1})) \leq F\left(\psi\left(\max\{m_\eta(s_{n-1}, s_n), m_\eta(s_n, s_{n+1})\}\right)\right). \quad (3.6)$$

By the assumption $m_\eta(s_n, s_{n+1}) > m_\eta(s_{n-1}, s_n)$, we attain

$$\kappa + F(\alpha(s_{n-1}, s_n)m_\eta(s_n, s_{n+1})) \leq F(\psi(m_\eta(s_n, s_{n+1}))). \quad (3.7)$$

Since $\alpha(s_{n-1}, s_n) \geq 1$ and by (1_F) , F is increasing, we have

$$F(m_\eta(s_n, s_{n+1})) < \kappa + F(m_\eta(s_n, s_{n+1})) \leq \kappa + F(\alpha(s_{n-1}, s_n)m_\eta(s_n, s_{n+1})). \quad (3.8)$$

Moreover, since $\psi(m_\eta(s_n, s_{n+1})) < m_\eta(s_n, s_{n+1})$ and by (1_F) , we gain

$$F(\psi(m_\eta(s_n, s_{n+1}))) < F(m_\eta(s_n, s_{n+1})). \quad (3.9)$$

Combining (3.8) and (3.9) in (3.7) provides

$$F(m_\eta(s_n, s_{n+1})) < F(m_\eta(s_n, s_{n+1})), \quad (3.10)$$

which causes a contradiction. Thus,

$$m_\eta(s_n, s_{n+1}) \leq m_\eta(s_{n-1}, s_n). \quad (3.11)$$

Putting (3.11) into (3.4) grants

$$\kappa + F(m_\eta(s_n, s_{n+1})) \leq F(m_\eta(s_{n-1}, s_n)), \quad \forall n \geq 1. \quad (3.12)$$

Here, we take $a_n = m_\eta(s_n, s_{n+1})$. Then, expression (3.12) can be written as

$$\kappa + F(a_n) \leq F(a_{n-1}), \quad \forall n \geq 1. \quad (3.13)$$

Furthermore, we reach

$$F(a_n) \leq F(a_{n-1}) - \kappa \leq F(a_0) - n\kappa, \quad \forall n \geq 1. \quad (3.14)$$

Taking the limit of both sides in (3.14) gives

$$\lim_{n \rightarrow \infty} F(a_n) = -\infty. \quad (3.15)$$

From (2_F) , we achieve

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} m_\eta(s_n, s_{n+1}) = 0 \quad (3.16)$$

for all $\eta > 0$. For $k \in (0, 1)$, we can also write $\lim_{n \rightarrow \infty} a_n^k F(a_n) = 0$ by (3_F) . Hereby, we get

$$0 \leq a_n^k F(a_n) - a_n^k F(a_0) \leq a_n^k (F(a_0) - n\kappa) - a_n^k F(a_0) = -n\kappa a_n^k \leq 0. \quad (3.17)$$

As $n \rightarrow \infty$, we reach $\lim_{n \rightarrow \infty} n a_n^k = 0$.

By the definition of the limit, one can find $n^* \in N$ s.t.

$$s_n = m_\eta(s_n, s_{n+1}) \leq \frac{1}{n^{\frac{1}{k}}}, \quad (3.18)$$

whenever $n \geq n^*$. Due to $(m3)$, for $m > n \geq n^*$ we write

$$m_\eta(s_n, s_m) \leq m_{\eta_n}(s_n, s_{n+1}) + m_{\eta_{n+1}}(s_{n+1}, s_{n+2}) + \cdots + m_{\eta_{m-1}}(s_{m-1}, s_m), \quad (3.19)$$

where $\eta_n + \eta_{n+1} + \cdots + \eta_{m-1} = \eta$.

Because $0 < \eta_i < \eta$ for $i = n, n+1, \dots, m-1$ and (3.18), we get

$$\begin{aligned} m_\eta(s_n, s_m) &\leq \frac{1}{n^{\frac{1}{k}}} + \frac{1}{(n+1)^{\frac{1}{k}}} + \cdots + \frac{1}{(m-1)^{\frac{1}{k}}} \\ &= \sum_{j=n}^{m-1} \frac{1}{j^{\frac{1}{k}}} \\ &< \sum_{j=1}^{\infty} \frac{1}{j^{\frac{1}{k}}}. \end{aligned} \quad (3.20)$$

Since $\sum_{j=1}^{\infty} \frac{1}{j^{\frac{1}{k}}} < \infty$, we reach $\lim_{m, n \rightarrow \infty} m_\eta(s_n, s_m) = 0$, i.e., $\{s_n\}$ is an m -Cauchy sequence.

Due to the completeness of Ω_m , $\{s_n\}$ is m -convergent to a point s belonging to Ω_m , that is, $\lim_{n \rightarrow \infty} m_\eta(s_n, s) = 0$.

By the continuity of H , we obtain

$$m_\eta(s, Hs) = \lim_{n \rightarrow \infty} m_\eta(s_n, Hs_n) = \lim_{n \rightarrow \infty} m_\eta(s_n, s_{n+1}) = 0. \quad (3.21)$$

As m is strict, we find $s = Hs$. Hence, the proof is completed. $\square$

The following theorem can be verified similarly to Theorem 3.

Theorem 4. *Presume that m is a strict modular on a complete modular metric space Ω_m , and $H: \Omega_m \rightarrow \Omega_m$ satisfies condition (3.2). Assume that*

(C₁) for a given s_0 , $\alpha(s_0, Hs_0) \geq 1$ holds,

(C₂) H is an α -admissible mapping,

(C₃) H is a continuous mapping,

then H has a fixed point.

Example 3. Take the set $\Omega = [0, \infty)$ with $m_\eta(\tau_0, \tau_1) = \frac{|\tau_0 - \tau_1|}{\eta}$ for all $\tau_0, \tau_1 \in [0, \infty)$. Consider a mapping given by

$$H(\tau) = \begin{cases} \frac{\tau+3}{4}, & \tau \in [0, 1], \\ 2, & \tau > 1. \end{cases} \quad (3.22)$$

The mapping $H: [0, \infty) \rightarrow [0, \infty)$ is continuous on Ω_m . Now, we take $\alpha: [0, \infty) \times [0, \infty) \rightarrow [0, \infty)$ as

$$\alpha(\tau_0, \tau_1) = \begin{cases} 1, & \tau_0, \tau_1 \in [0, 1], \\ 0, & \text{otherwise.} \end{cases} \quad (3.23)$$

For $\tau_0, \tau_1 \in \Omega$ s.t. $\alpha(\tau_0, \tau_1) \geq 1$, τ_0, τ_1 must be in $[0, 1]$. We get $H\tau_0, H\tau_1 \in [0, 1]$. By (3.23), we find $\alpha(H\tau_0, H\tau_1) \geq 1$, that is, H is an α -admissible mapping.

For $s_0 = 0$, then $HS_0 = \frac{3}{4}$. Thus, $\alpha(s_0, HS_0) \geq 1$ holds.

Now, we display that (3.1) is satisfied for all $\tau_0, \tau_1 \in [0, \infty)$, where $F(\xi) = \ln(\xi)$ and $\psi(\xi) = \frac{1}{2}\xi \in \Psi$ for $\xi > 0$.

If τ_0 or τ_1 is not in $[0, 1]$, then $\alpha(\tau_0, \tau_1) = 0$ and so (3.1) is satisfied.

For $\tau_0, \tau_1 \in [0, 1]$ with $H\tau_0 \neq H\tau_1$, we can write

$$\begin{aligned} \alpha(\tau_0, \tau_1)m_\eta(H\tau_0, H\tau_1) &= \frac{|H\tau_0 - H\tau_1|}{\eta} = \frac{1}{4\eta}|\tau_0 - \tau_1| \leq \frac{1}{4} \frac{|\tau_0 - \tau_1|}{\eta} \\ &= e^{-0.6} \frac{1}{2} m_\eta(\tau_0, \tau_1) = e^{-0.6} \psi(m_\eta(\tau_0, \tau_1)) \\ &\leq e^{-0.6} \psi(M_1(\tau_0, \tau_1)). \end{aligned} \quad (3.24)$$

Hereby, (3.1) holds for all $\tau_0, \tau_1 \in \Omega$. Thus, all conditions of Theorem 3 are satisfied, which gives the existence of fixed points for H . Here, there exist two fixed points, $H1 = 1$ and $H2 = 2$.

In Theorems 3 and 4, if the condition C_3 is replaced by the following one:

(C_3^*) For the sequence $\{s_n\}$ s.t. $\alpha(s_n, s_{n+1}) \geq 1$ with $\lim_{n \rightarrow \infty} m_\eta(s_n, s) = 0$ where $s \in \Omega_m$, there exists a subsequence $\{s_{n_p}\}$ of $\{s_n\}$ s.t. $\alpha(s_{n_p}, s) \geq 1$ for all p , then the results found therein are still true.

Theorem 5. Presume that Ω is a complete MMS and m is a strict MM. Suppose that $H: \Omega_m \rightarrow \Omega_m$ satisfies the condition (3.1). Assume that (C_1) , (C_2) and (C_3^*) hold. Then H admits a fixed point.

Proof. From Theorem 3, the same sequence $\{s_n\}$ can be constructed in Ω_m so that $\{s_n\}$ is m -Cauchy and m -convergent to $s \in \Omega_m$.

Assume that $s_{n_p} = Hs$ for all $p \in \mathbb{N}$. Then, the uniqueness of the limit gives $s = Hs$ as $n \rightarrow \infty$. Hereby, the proof is done.

Hence, assume that there exists $p_0 \in \mathbb{N}$ s.t. $s_{n_p} \neq Hs$, $\forall p \geq p_0$. Then, for all $p \geq p_0$ we find $Hs_{n_p-1} \neq Hs$, i.e., $m_\eta(Hs_{n_p-1}, Hs) > 0$. By (1.3), we can write

$$\kappa + F(\alpha(s_{n_p-1}, s)m_\eta(Hs_{n_p-1}, Hs)) \leq F(\psi(\max\{m_\eta(s_{n_p-1}, s), \frac{m_\eta(s_{n_p-1}, Hs_{n_p-1})(1+m_\eta(s, Hs))}{1+m_\eta(s_{n_p-1}, s)}, \frac{m_\eta(s, Hs_{n_p-1})(1+m_\eta(s, Hs_{n_p-1}))}{1+m_\eta(s_{n_p-1}, s)}\})). \quad (3.25)$$

Since $\alpha(s_{n_p-1}, s) \geq 1$ and (1_F) , we gain

$$m_\eta(Hs_{n_p-1}, Hs) \leq \psi(\max\{m_\eta(s_{n_p-1}, s), \frac{m_\eta(s_{n_p-1}, Hs_{n_p-1})(1+m_\eta(s, Hs))}{1+m_\eta(s_{n_p-1}, s)}, \frac{m_\eta(s, Hs_{n_p-1})(1+m_\eta(s, Hs_{n_p-1}))}{1+m_\eta(s_{n_p-1}, s)}\}). \quad (3.26)$$

As $n \rightarrow \infty$, we obtain $\lim_{n \rightarrow \infty} m_\eta(s_{n_p}, Hs) = 0$. The uniqueness of the limit provides $s = Hs$. $\square$

The subsequent theorem is verified in a similar manner.

Theorem 6. *Presume that Ω_m is a complete MMS and m is a strict MM. Suppose that $H: \Omega_m \rightarrow \Omega_m$ satisfies condition (3.2), and also the statements (C_1) , (C_2) and (C_3^*) hold. Then H has a fixed point.*

Theorems 3-6 indicate the existence of a fixed point of a mapping, while the following theorem emphasizes the uniqueness of the fixed point of the mapping.

Theorem 7. *Suppose that Ω_m is a complete MMS and m is a strict MM. Assume that $H: \Omega_m \rightarrow \Omega_m$ provides the condition (3.1), and also the statements (C_1) , (C_2) and (C_3) hold. If for all $s, r \in \text{Fix}(H)$, $\alpha(s, r) \geq 1$ holds where $\text{Fix}(H)$ represents the set of fixed points of H . Then the fixed point of H is unique.*

Proof. Assume that $r, s \in \Omega_m$ are two different fixed points of H , i.e., $Hs = s$ and $Hr = r$ with $r \neq s$. Then, $m_\eta(Hs, Hr) > 0$ is granted.

Hence, putting $\tau_0 = s$ and $\tau_1 = r$ in (1.3) gives

$$\kappa + F(\alpha(s, r)m_\eta(Hs, Hr)) \leq F(\psi(\max\{m_\eta(s, r), \frac{m_\eta(s, Hs)(1+m_\eta(r, Hr))}{1+m_\eta(s, r)}, \frac{m_\eta(r, Hs)(1+m_\eta(r, Hs))}{1+m_\eta(s, r)}\})). \quad (3.27)$$

Then, we gain

$$\kappa + F(\alpha(s, r)m_\eta(Hs, Hr)) \leq F(\psi(m_\eta(s, r))) \leq F(m_\eta(s, r)). \quad (3.28)$$

Since $s, r \in \text{Fix}(H)$, $\alpha(s, r) \geq 1$ by (1_F) , we have

$$\kappa + F(\alpha(s, r)m_\eta(s, r)) \leq \kappa + F(m_\eta(s, r)) \leq F(m_\eta(s, r)), \quad (3.29)$$

which gives $\kappa + F(m_\eta(s, r)) \leq F(m_\eta(s, r))$, a contradiction. Hence, the proof is completed. $\square$

Similarly, we have the following theorem.

Theorem 8. Assume that Ω_m is a complete MMS and m is a strict MM. Suppose that $H: \Omega_m \rightarrow \Omega_m$ satisfies condition (3.2), and (C_1) , (C_2) and (C_3) hold. If for all $s, r \in \text{Fix}(H)$, $\alpha(s, r) \geq 1$ holds where $\text{Fix}(H)$ represents the set of fixed points of H . Then the fixed point of H is unique.

An example illustrating the uniqueness of the fixed point is given below.

Example 4. Consider the set $\Omega = [0, \infty)$ with $m_\eta(\tau_0, \tau_1) = \frac{|\tau_0 - \tau_1|}{\eta}$ for all $\tau_0, \tau_1 \in [0, \infty)$. Define the mapping

$$H(\tau) = \begin{cases} \frac{\tau^2 + \tau}{4}, & \tau \in [0, 1], \\ \frac{3}{4}, & \text{otherwise.} \end{cases} \quad (3.30)$$

The mapping $H: [0, \infty) \rightarrow [0, \infty)$ is continuous on Ω_m . Now, $\alpha: [0, \infty) \times [0, \infty) \rightarrow [0, \infty)$ is defined as

$$\alpha(\tau_0, \tau_1) = \begin{cases} 1, & \tau_0, \tau_1 \in [0, 1], \\ 0, & \text{otherwise.} \end{cases} \quad (3.31)$$

For $\tau_0, \tau_1 \in \Omega$ with $\alpha(\tau_0, \tau_1) \geq 1$, we must have $\tau_0, \tau_1 \in [0, 1]$. Then, we get $H\tau_0, H\tau_1 \in [0, 1]$. From (3.31), we find $\alpha(H\tau_0, H\tau_1) \geq 1$, that is, H is an α -admissible mapping.

Here an element $s_0 \in [0, \infty)$ exists s.t. $\alpha(s_0, Hs_0) \geq 1$. If we take $s_0 = \frac{1}{2}$, then $Hs_0 = \frac{3}{16}$. Thus, $\alpha(s_0, Hs_0) = \alpha(\frac{1}{2}, \frac{3}{16}) \geq 1$ holds.

Now, we show that (3.1) is satisfied for all $\tau_0, \tau_1 \in [0, \infty)$, where $F(\xi) = \ln(\xi)$ and $\Psi(\xi) = \frac{3}{2}\xi \in \Psi$ for $\xi > 0$.

If τ_0 or τ_1 is not in $[0, 1]$, then $\alpha(\tau_0, \tau_1) = 0$, and therefore (3.1) is satisfied.

For $\tau_0, \tau_1 \in [0, 1]$ with $H\tau_0 \neq H\tau_1$, we write

$$\begin{aligned} \alpha(\tau_0, \tau_1)m_\eta(H\tau_0, H\tau_1) &= \frac{|H\tau_0 - H\tau_1|}{\eta} \\ &= \frac{1}{4\eta} |\tau_0^2 + \tau_0 - (\tau_1^2 + \tau_1)| \leq \frac{1}{4\eta} (|\tau_0^2 - \tau_1^2| + |\tau_0 - \tau_1|) \\ &\leq \frac{1}{4\eta} (|\tau_0 - \tau_1| |\tau_0 + \tau_1| + |\tau_0 - \tau_1|) \\ &\leq \frac{1}{4\eta} (|\tau_0 - \tau_1| (|\tau_0 + \tau_1| + 1)) \\ &\leq \frac{3}{4} \frac{|\tau_0 - \tau_1|}{\eta} \leq e^{-0.6} \frac{3}{2} m_\eta(\tau_0, \tau_1) = e^{-0.6} \Psi(m_\eta(\tau_0, \tau_1)) \\ &\leq e^{-0.6} \Psi(M_1(\tau_0, \tau_1)). \end{aligned} \quad (3.32)$$

Thus, (3.1) holds for all $\tau_0, \tau_1 \in \Omega$.

For all $\tau_0, \tau_1 \in \text{Fix}(H)$, we need to show that $\alpha(\tau_0, \tau_1) \geq 1$. Let $\tau_0, \tau_1 \in \text{Fix}(H)$. Then, τ_0, τ_1 must be in $[0, 1]$, i.e., $\alpha(\tau_0, \tau_1) \geq 1$. Therefore, all conditions of Theorem 7 are satisfied. Consequently, H has a unique fixed point, i.e., $H0 = 0$.

Now, we introduce some results.

Corollary 1. *Let Ω_m be a complete MMS with a strict MM m . Assume that for a given $\kappa > 0$, $H: \Omega_m \rightarrow \Omega_m$ satisfies the following:*

$$d(H\tau_0, H\tau_1) > 0 \implies \kappa + F(m_\eta(H\tau_0, H\tau_1)) \leq F(\psi(M_1(\tau_0, \tau_1))) \quad (3.33)$$

for all $\tau_0, \tau_1 \in \Omega_m$, where $\psi \in \Psi$, $F \in \mathfrak{F}$, and

$$M_1(\tau_0, \tau_1) = \max\{m_\eta(\tau_0, \tau_1), \frac{m_\eta(\tau_0, H\tau_0)(1 + m_\eta(\tau_1, H\tau_1))}{1 + m_\eta(\tau_0, \tau_1)}, \frac{m_\eta(\tau_1, H\tau_0)(1 + m_\eta(\tau_1, H\tau_1))}{1 + m_\eta(\tau_0, \tau_1)}\} \quad (3.34)$$

Then, H owns a unique fixed point in Ω_m .

Proof. Taking $\alpha(\tau_0, \tau_1) = 1$ in Theorem 7 concludes the proof. $\square$

Corollary 2. *Let Ω_m be a complete MMS with a strict MM m . Assume that for a given $\kappa > 0$, $H: \Omega_m \rightarrow \Omega_m$ satisfies the following*

$$d(H\tau_0, H\tau_1) > 0 \implies \kappa + F(d(H\tau_0, H\tau_1)) \leq F(\psi(M_2(\tau_0, \tau_1))) \quad (3.35)$$

for all $\tau_0, \tau_1 \in \Omega_m$, where $\psi \in \Psi$, $F \in \mathfrak{F}$, and

$$M_2(\tau_0, \tau_1) = \max\{m_\eta(\tau_0, \tau_1), \frac{m_\eta(\tau_0, H\tau_0)(1 + m_\eta(\tau_1, H\tau_1))}{1 + m_\eta(\tau_0, \tau_1)}, \frac{m_\eta(\tau_1, H\tau_1)m_\eta(\tau_0, H\tau_0)}{1 + m_\eta(H\tau_0, F\tau_1)}, \frac{m_\eta(\tau_1, H\tau_1)m_\eta(\tau_1, H\tau_0)}{1 + m_\eta(\tau_1, H\tau_0) + m_\eta(\tau_0, H\tau_1)}\} \quad (3.36)$$

Then, H owns a unique fixed point.

Proof. Putting $\alpha(\tau_0, \tau_1) = 1$ in Theorem 8 completes the proof. $\square$

Now, we consider a partially ordered set (poset) in MMS . Then, we give some fixed point results for posets in MMS .

Definition 11. Let m be a MM on Ω and $(\Omega, \preceq)$ be a poset. A given map H on Ω_m is said to be nondecreasing w.r.t $\preceq$ whenever $\tau_0 \preceq \tau_1 \implies H\tau_0 \preceq H\tau_1$ for all $\tau_0, \tau_1 \in \Omega_m$. Moreover, a sequence $\{h_n\}$ is nondecreasing w.r.t $\preceq$ provided that $h_n \preceq h_{n+1}$ for all n .

Definition 12. Let $(\Omega, \preceq)$ be a poset and m be a MM on Ω . Suppose that $\lim_{n \rightarrow \infty} m_\eta(h_n, h) = 0$ for a given sequence $\{h_n\}$ in Ω_m . If there exists a subsequence $\{h_{n_k}\}$ of $\{h_n\}$ s.t. $\{h_{n_k}\} \preceq h$ for all k , then $(\Omega, \preceq)$ is called regular.

Corollary 3. *Let m be a strict MM in a complete MMS, Ω_m , and $(\Omega_m, \preceq)$ be a poset. Suppose that $H: \Omega_m \rightarrow \Omega_m$ is a nondecreasing mapping w.r.t $\preceq$, and (3.33) is satisfied with $\tau_0 \preceq \tau_1$. If the following statements hold:*

- (C₁^{*}) for a given $s_0 \in \Omega$, $s_0 \preceq Hs_0$ holds,*
- (C₂^{*}) either H is a continuous mapping,*
- (C₃^{*}) or $(\Omega_m, \preceq)$ is regular,*

then H has a fixed point.

Proof. Initially we define $\alpha: \Omega \times \Omega \rightarrow [0, \infty)$ as

$$\alpha(\tau_0, \tau_1) = \begin{cases} 1, & \tau_0 \preceq \tau_1 \text{ or } \tau_1 \preceq \tau_0, \\ 0, & \text{otherwise.} \end{cases} \quad (3.37)$$

Since $\tau_1 \preceq \tau_0$, i.e., $m_\eta(H\tau_0, H\tau_1) \neq 0$, then H is F_α -contraction mapping, that is,

$$\kappa + F(\alpha(\tau_0, \tau_1)d(H\tau_0, H\tau_1)) \leq F(\psi(M_1(\tau_0, \tau_1))), \quad (3.38)$$

where $M_1(\tau_0, \tau_1)$ is given in (3.34).

From (C₁^{*}), we gain $\alpha(s_0, Hs_0) \geq 1$. Furthermore, if $\alpha(\tau_0, \tau_1) \geq 1$ for all $\tau_0, \tau_1 \in \Omega_m$, then either $\tau_0 \preceq \tau_1$ or $\tau_1 \preceq \tau_0$ holds which gives that $H\tau_0 \preceq H\tau_1$ or $H\tau_1 \preceq H\tau_0$. From the definition of $\alpha(\cdot, \cdot)$, we obtain $\alpha(H\tau_0, H\tau_1) \geq 1$. Hence, H is an α -admissible mapping.

Now, if H is continuous, then Theorem 3 assures that H has a fixed point.

Suppose that $(\Omega_m, \preceq)$ is regular. Consider a nondecreasing sequence $\{s_n\}$ in Ω_m , with $\lim_{n \rightarrow \infty} m_\eta(s_n, s) = 0$. Since $s_n \preceq s_{n+1}$ for all n , $\alpha(s_n, s_{n+1}) \geq 1$. In addition, we can find a subsequence $\{s_{n_p}\}$ of $\{s_n\}$ s.t. $s_{n_p} \preceq s$ for all p . Then, we attain $\alpha(s_{n_p}, s) \geq 1$ for all p . Hereby, Theorem 5 provides the existence of a fixed point.

Thus, we conclude the proof. □

We also give the following result.

Corollary 4. *Let m be a strict MM in a complete MMS, Ω_m , and $(\Omega_m, \preceq)$ be a poset. Suppose that $H: \Omega_m \rightarrow \Omega_m$ is a nondecreasing mapping w.r.t $\preceq$, and (3.35) is satisfied with $\tau_0 \preceq \tau_1$. If the followings hold:*

- (C₁^{*}) for a given $s_0 \in \Omega$, $s_0 \preceq Hs_0$ holds,*
- (C₂^{*}) either H is a continuous mapping,*
- (C₃^{*}) or $(\Omega_m, \preceq)$ is regular,*

then H has a fixed point.

4. APPLICATION

This section investigates the existence of a solution for an integral equation given as

$$g(t) = K(t) + \int_a^b M(t, \zeta) Y(\zeta, g(\zeta)) d\zeta, \quad t \in [a, b], \quad (4.1)$$

where the function $Y: [0, T] \times R^+ \rightarrow R$ is continuous and nondecreasing, $M: [a, b] \times [a, b] \rightarrow R^+$ and $K: [a, b] \rightarrow R$ are continuous functions.

We study this equation on the set of a real-valued continuous function on $[a, b]$, i.e., $\Omega = C([a, b])$. We consider the following modular space around g_0 :

$$\Omega_m = \Omega_m(g_0) = \{h \in \Omega = C([a, b]): m_\eta(g, g_0) \rightarrow 0 \text{ as } \eta \rightarrow \infty\}. \quad (4.2)$$

Now, let us take the strict and convex modular metric $m_\eta(g, f) = \frac{d(g, f)}{\eta}$. Here $d(g, f)$ represents the usual metric $d(g, f) = \max_{t \in [a, b]} |g(t) - f(t)|$. Then the space Ω_m (4.2) becomes m -complete. Furthermore, we can equip this space with the partial order defined as $g, f \in C([a, b])$ s.t. $g \leq f \implies g(t) \leq f(t)$ for all $t \in [a, b]$ so that Ω_m becomes partially ordered m -complete w.r.t $' \leq'$.

Now, let us define $H: C([0, T]) \rightarrow C([0, T])$ as

$$(Hg)(t) = K(t) + \int_a^b M(t, \zeta) Y(\zeta, g(\zeta)) d\zeta, \quad t \in [0, T]. \quad (4.3)$$

It is clear that if $g(t)$ is a fixed point of H , then $g \in C([a, b])$ is a solution of the integral equation (4.1). For this aim, we must show that all hypotheses in Corollary 3 hold.

Theorem 9. Suppose that the followings hold:

- (T1) $Y(\zeta, g(\zeta))$ is nondecreasing w.r.t the second variable,
- (T2) for a given $\kappa > 0$, $A \in (0, 1)$ and $\eta \in (0, \infty)$, there exist $z: \Omega_m \times \Omega_m \rightarrow [0, \infty)$ and $\alpha: \Omega_m \times \Omega_m \rightarrow [0, \infty)$ s.t. if $\alpha(g, f) \geq 1$ for all $g, f \in \Omega_m$, then for all $\zeta \in [a, b]$

$$|Y(\zeta, g(\zeta)) - Y(\zeta, f(\zeta))| \leq z(g(\zeta), f(\zeta)) \left| \frac{g(\zeta) - f(\zeta)}{\eta} \right|, \quad (4.4)$$

and

$$\left| \int_a^b M(t, \zeta) z(g, f) d\zeta \right| \leq A e^{-\kappa}, \quad (4.5)$$

- (T3) there exists $g_0 \in \Omega_m$ s.t. $g_0(t) \leq K(t) + \int_a^b M(t, \zeta) Y(\zeta, g_0(\zeta)) d\zeta$, then H has a solution in $C([a, b])$.

Proof. In the first instance define $\alpha: \Omega_m \times \Omega_m \rightarrow [0, \infty)$ as

$$\alpha(g, f) = \begin{cases} 1, & g \leq f, \\ 0, & \text{otherwise.} \end{cases} \quad (4.6)$$

Consider $g, f \in C([a, b])$ s.t. $g \preceq f \iff g(t) \leq f(t)$ for all $t \in [a, b]$. Since $Y(s, g(s))$ is nondecreasing, it easy to see that $Hg \leq Hf$, that is, F is nondecreasing w.r.t ' $\preceq$ '. In addition, $m_\eta(Hg, Gf) > 0$ holds.

If $g \preceq f$, then $\alpha(g, f) \geq 1$. Hence, (T1) provides

$$\begin{aligned} m_\eta(Hg, Hf) &= \frac{d(Hg, Hf)}{\eta} = \frac{1}{\eta} \max_{t \in [a, b]} |Hg(t) - Hf(t)| \\ &\leq \frac{1}{\eta} \max_{t \in [a, b]} \int_a^b |M(t, \zeta)| |Y(\zeta, g(\zeta)) - Y(\zeta, f(\zeta))| d\zeta. \end{aligned} \quad (4.7)$$

By (4.4), we write

$$\begin{aligned} m_\eta(Hg, Hf) &\leq \frac{1}{\eta} \max_{t \in [a, b]} \int_a^b |M(t, \zeta)| z(g, f) \frac{|h(\zeta) - g(\zeta)|}{\eta} d\zeta \\ &\leq \max_{t \in [a, b]} \frac{|h(\zeta) - g(\zeta)|}{\eta} \int_a^b |M(t, \zeta)| z(g, f) d\zeta \\ &\leq m_\eta(g, f) \int_a^b |M(t, \zeta)| z(g, f) d\zeta. \end{aligned} \quad (4.8)$$

From (4.5), we obtain

$$m_\eta(Hg, Hf) \leq m_\eta(g, f) A e^{-\kappa}. \quad (4.9)$$

Hereby, we write for all $g, f \in \Omega_m$ s.t. $g \preceq f$ with $Hg \neq Hf$

$$\alpha(g, f) m_\eta(Hg, Gf) \leq e^{-\kappa} \psi(m_\eta(g, f)) \leq e^{-\kappa} \psi(M_1(g, f)), \quad (4.10)$$

where $\psi(\xi) = A\xi \in \Psi$. Thus, (3.33) is satisfied for $F(\xi) = \ln(\xi)$.

From (4.6), we conclude $\alpha(Hg, Hf) \geq 1$. Thus, H is α -admissible. (T3) guarantees that there exists $g_0 \in \Omega$ s.t. $g_0 \preceq Hg_0$, i.e., $\alpha(g_0, Hg_0) \geq 1$.

We, therefore, have all hypotheses in Corollary 3 fulfilled; that is, H possesses a fixed point. Consequently, $g \in C([a, b])$ is a solution of (4.1). $\square$

Example 5. Let $\Omega = C([0, 1])$ and $m_\eta(g, f) = \max_{t \in [0, 1]} \frac{|g(t) - f(t)|}{\eta}$. Consider the integral equation defined by

$$g(t) = \frac{1}{2} + \frac{1}{2\eta} \int_0^1 \frac{t}{\zeta + 1} \frac{g(\zeta)}{2} d\zeta, \quad t \in [0, 1]. \quad (4.11)$$

Here, $K(t) = \frac{1}{2}$, $M(t, \zeta) = \frac{t}{\zeta + 1}$, and $Y(\zeta, g(\zeta)) = \frac{g(\zeta)}{2\eta}$.

Let us define a mapping $H: C([0, 1]) \rightarrow C([0, 1])$ as

$$(Hg)(t) = \frac{1}{2} + \frac{1}{2\eta} \int_0^1 \frac{t}{\zeta + 1} \frac{g(\zeta)}{2} d\zeta, \quad t \in [0, 1]. \quad (4.12)$$

We take the function α given in (4.6).

For $g \preceq f$, it is clear that $Y(\zeta, g(\zeta)) \leq Y(\zeta, f(\zeta))$, that is, Y is nondecreasing w.r.t the second variable. So, $(T1)$ is done.

If $g \preceq f$, then $\alpha(g, f) \geq 1$. Hence, for all $\zeta \in [0, 1]$

$$|Y(\zeta, g(\zeta)) - Y(\zeta, f(\zeta))| \leq \frac{1}{2} \frac{g(\zeta) - f(\zeta)}{\eta} = z(g, f) \left| \frac{g(\zeta) - f(\zeta)}{\eta} \right|, \quad (4.13)$$

where $z(g, f) = \frac{1}{2}$. Moreover, we obtain

$$\left| \int_0^1 \frac{1}{2} \frac{t}{\zeta + 1} d\zeta \right| \leq \frac{1}{2} \max_{t \in [0, 1]} \{t\} \int_0^1 \frac{1}{\zeta + 1} d\zeta \leq \frac{1}{2} \ln(2) \leq e^{-0.6} \ln(2). \quad (4.14)$$

Hereby, $T2$ is satisfied for $\kappa = -0.6$ and $A = \ln 2 \in (0, 1)$.

We conclude that there exists an element $g_0 = 0$ in $C([0, 1])$ s.t.

$$g_0 = 0 \leq \frac{1}{2} + \frac{1}{2\eta} \int_0^1 \frac{t}{\zeta + 1} \frac{0}{2} d\zeta = \frac{1}{2}.$$

So, $(T3)$ holds.

Since all conditions of Theorem 9 are satisfied, the integral equation (4.11) has a solution in $C([0, 1])$. Indeed, $g(t) = \frac{1}{2} + \frac{t}{2\eta} \in C([0, 1])$ is a solution for (4.11).

5. CONCLUSION

We introduce two generalized modified F_α -contraction of type I and II in MMS . For these contractions, we state the fixed point theorems, which give the existence of the fixed point. Moreover, we put another condition (C_3^*) instead of (C_3) in Theorem 3 and 4. In this case, we keep the existence of the fixed point. Furthermore, we consider an additional condition that provides the uniqueness of the fixed point. To support the outcomes attained here, we give some examples. We also put forward some significant results. Consequently, we consider an integral equation in MMS and prove the existence of this equation. The outcomes obtained herein extend the results for F_α -contraction mappings in metric spaces to MMS . As a final point, in the example $m_\eta(x, y) = \frac{d(x, y)}{\eta}$ if we take $\eta = 1$, all the consequences given are still valid in an ordinary metric structure, which emphasizes that the results obtained here are also a generalizations of the existing literature.

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HERMITE-HADAMARD TYPE INEQUALITIES BY USING NEWTON-COTES QUADRATURE FORMULAS

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Abstract. A convex function $f : [a, b] \rightarrow \mathbb{R}$ satisfies the so-called Hermite-Hadamard inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(t) dt \leq \frac{f(a)+f(b)}{2}.$$

Motivated by the above estimates, in this paper we consider approximately monotone and convex functions, and give upper and lower bounds to the numerical integral mean, i.e., to $\frac{1}{b-a} I_n(f)$, where $I_n(f)$ denotes some of the most popular Newton-Cotes quadrature formulas.

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1. INTRODUCTION

The theory of approximately convex functions goes back to the work of Hyers and Ulam [10] where they first introduced the concept of δ -convexity. A function $f : I \rightarrow \mathbb{R}$ is said to be δ -convex, if for any $x, y \in I$ and for all $t \in [0, 1]$ the following functional inequality holds

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + \delta.$$

They showed that a function satisfying δ -convexity can be decomposed as the algebraic sum of an ordinary convex and a bounded function whose supremum norm is not greater than $\frac{\delta}{2}$. Since then many different versions of approximate convexity were introduced and investigated (see [5], [6], [7] and their references).

The notion of approximate monotonicity often appears during the study of generalized convex functions. A function $f : I \rightarrow \mathbb{R}$ is said to be ε -monotone, if it satisfies

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the following functional inequality

$$f(x) \leq f(y) + \varepsilon \quad \text{for all } x, y \in I \text{ with } x < y.$$

Páles in [12] showed that an ε -monotone function f can also be expressed as $g + h$; where g is nondecreasing and h is a bounded function satisfying $\|h\| \leq \frac{\varepsilon}{2}$.

We are going to see results associated with more generalized versions of approximate monotonicity and convexity. Throughout this paper $I \subseteq \mathbb{R}$ represents the non-empty and non-singleton interval $[a, b]$. The length of the interval I will be denoted by $\ell(I)$. The function $\Phi : [0, \ell(I)] \rightarrow \mathbb{R}_+$ will be termed as error function, where $\mathbb{R}_+$ denotes the set of nonnegative reals. With the help of error function Φ , we can introduce terminologies such as approximately monotone, Hölder, convex and affine functions. These function classes are studied in depth in the papers [5]-[7], [11], where along with structural properties, some characterizations, decompositions, several sandwich type theorems and applications are discussed. For readability purposes, we recall these notions.

A function $f : I \rightarrow \mathbb{R}$ is said to be Φ -monotone if for any $x, y \in I$ with $x < y$, the following inequality holds

$$f(x) \leq f(y) + \Phi(y - x).$$

This version of approximate monotonicity was first introduced in [5]. A more in depth study can be found in [6]. Due to association of the non-negative error term, the class of Φ -monotone functions is bigger than the class of ordinary nondecreasing functions. If both of the functions f and $-f$ satisfy Φ -monotonicity, i.e.,

$$|f(x) - f(y)| \leq \Phi(|y - x|) \quad \text{for all } x, y \in I,$$

then f is termed as Φ -Hölder.

The following definition was introduced in [11]. A function $f : I \rightarrow \mathbb{R}$ is said to be Φ -convex if the functional inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + t\Phi((1-t)|y-x|) + (1-t)\Phi(t|y-x|) \quad (1.1)$$

holds for any $x, y \in I$ and for all $t \in [0, 1]$. It is also evident that any ordinary convex function also possesses Φ -convexity. If both f and $-f$ are Φ -convex, i.e.,

$$|f(tx + (1-t)y) - tf(x) - (1-t)f(y)| \leq t\Phi((1-t)|y-x|) + (1-t)\Phi(t|y-x|)$$

holds for any $x, y \in I$ and for all $t \in [0, 1]$, then we say that f is a Φ -affine function.

In [6], one of the results states that if $f : I \rightarrow \mathbb{R}$ is Φ -monotone, then

$$f(a) - \frac{1}{b-a} \int_0^{b-a} \Phi(s) ds \leq \frac{1}{b-a} \int_a^b f(s) ds \leq f(b) + \frac{1}{b-a} \int_0^{b-a} \Phi(s) ds$$

holds, provided both f and Φ are Lebesgue integrable. Our goal in this paper is to estimate the numerical integral mean of f , i.e., $\frac{1}{b-a} I_n(f)$, where $I_n(f)$ denotes some of the most popular Newton-Cotes quadrature formulas: the Trapezoidal, Simpson's and Simpson's 3/8 rules. Without assuming the Lebesgue integrability of the error

function Φ , we can show that a continuous Φ -monotone function f satisfies the following inequality

$$f(a) - \frac{n}{2}\Phi\left(\frac{b-a}{n}\right) \leq \frac{1}{b-a}I_n(f) \leq f(b) + \frac{n}{2}\Phi\left(\frac{b-a}{n}\right).$$

Let $x_i = a + \frac{(b-a)i}{n}$, $i = 0, 1, \dots, n$ be an equidistant partition of the interval $[a, b]$. For a continuous function $f : I \rightarrow \mathbb{R}$, we recall the Trapezoidal, Simpson's and Simpson's 3/8 rules as follows (see e.g. [3]):

$$\begin{aligned} \mathcal{T}_n(f) &:= \left(\frac{b-a}{2n}\right) \left[f(x_0) + 2 \sum_{i=1}^{n-1} f(x_i) + f(x_n) \right] \quad (n \in \mathbb{N}), \\ \mathcal{S}_n(f) &:= \left(\frac{b-a}{3n}\right) \left[f(x_0) + 2 \sum_{i=1}^{(n/2)-1} f(x_{2i}) + 4 \sum_{i=1}^{n/2} f(x_{2i-1}) + f(x_n) \right] \\ &\quad (n \in \mathbb{N} \text{ is even}), \end{aligned}$$

and

$$\begin{aligned} \mathcal{S}_n^{\frac{3}{8}}(f) &:= \left(\frac{3(b-a)}{8n}\right) \sum_{i=1}^{n/3} [f(x_{3i-3}) + 3f(x_{3i-2}) + 3f(x_{3i-1}) + f(x_{3i})] \\ &\quad (n = 3k, k \in \mathbb{N}). \end{aligned}$$

The classical Hermite-Hadamard inequality was independently introduced by Hermite and Hadamard in [8] and [9]. It states that "In a compact interval, integral mean of a convex function is greater than the functional value at the mid-point, while the same is dominated by the average of the functional values at the extreme points of that interval." Mathematically, for a convex function $f : I \rightarrow \mathbb{R}$, this inequality can be represented as follows

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(t) dt \leq \frac{f(a)+f(b)}{2}. \quad (1.2)$$

Various generalized forms of this inequality were presented for approximate and higher order convex functions. Along with it many associated concepts and applications were also studied (see e.g. [1], [2], [4] and the references therein).

In this paper, we present the numerical version of Hermite-Hadamard inequality for the continuous Φ -convex function $f : I \rightarrow \mathbb{R}$. We show that for any even number $n \in \mathbb{N}$, the following functional inequality holds

$$f\left(\frac{a+b}{2}\right) - \frac{n^2+2}{12}\Phi\left(\frac{b-a}{n}\right) \leq \frac{1}{b-a}\mathcal{T}_n(f) \leq \frac{f(a)+f(b)}{2} + \frac{n^2-1}{6}\Phi\left(\frac{b-a}{n}\right).$$

One can easily observe that it resembles very closely with the original inequality (1.2). We show similar estimates for odd n , see Theorem 4 below.

For the cases of Φ -Hölder and Φ -affine functions we will obtain inequalities with improved lower and upper bounds. Furthermore, in our results, we discuss particular cases where we can omit the dependency on n from the error terms.

The graphs of approximately monotone and convex functions often appear in the stock prices of share market, various financial and business models, population growth of certain regions etc. To have an estimate of mean values for such scenarios, numerical integration techniques are useful. Through this we can neglect the stochastic nature of the associated error term.

2. ON APPROXIMATELY MONOTONE FUNCTIONS

We start with Φ -monotone and Φ -Hölder functions. As discussed before, throughout this section, I will denote the non-empty and non-singleton interval $[a, b]$ with $a < b$.

Theorem 1. *Let $f : I \rightarrow \mathbb{R}$ be a continuous Φ -monotone function. Then for any $n \in \mathbb{N}$, the following inequalities hold*

$$f(a) - \frac{n}{2}\Phi\left(\frac{b-a}{n}\right) \leq \frac{1}{b-a}\mathcal{T}_n(f) \leq f(b) + \frac{n}{2}\Phi\left(\frac{b-a}{n}\right). \quad (2.1)$$

Proof. For the proof, we consider partitioning the interval I into n equal subintervals as follows

$$P = \{a = x_0 < x_1 < \cdots < x_n = b\} \quad \text{with} \quad x_i - x_{i-1} = \frac{b-a}{n}, \quad i \in \{1, \dots, n\}. \quad (2.2)$$

By applying Φ -monotonicity in each of the subintervals $[x_{i-1}, x_i]$ of I , we get the following system of n inequalities

$$\begin{aligned} f(x_0) - f(x_1) &\leq \Phi\left(\frac{b-a}{n}\right) \\ &\vdots \\ f(x_{n-1}) - f(x_n) &\leq \Phi\left(\frac{b-a}{n}\right). \end{aligned} \quad (2.3)$$

First multiplying each inequalities of the above system respectively by $1, 3, \dots, 2n-1$ and then summing those up, we arrive at

$$\begin{aligned} f(x_0) + 2 \sum_{i=1}^{n-1} f(x_i) + f(x_n) &\leq 2nf(x_n) + \left(\sum_{i=1}^n (2i-1)\right) \Phi\left(\frac{b-a}{n}\right) \\ &= 2nf(x_n) + n^2 \Phi\left(\frac{b-a}{n}\right). \end{aligned}$$

Now multiplying both side of the above inequality by $\frac{b-a}{2n}$ and using the definition of the Trapezoidal rule, we obtain

$$\mathcal{T}_n(f) \leq (b-a) \left(f(b) + \frac{n}{2} \Phi \left(\frac{b-a}{n} \right) \right).$$

Dividing both side by $b-a$, we get the second part of inequality (2.1).

Similarly, multiplying each inequalities of (2.3) by $2n-1, 2n-3, \dots, 1$, respectively, and then adding the resultant inequalities, we arrive at

$$(2n)f(x_0) - \left(\sum_{i=1}^n (2(n-i)+1) \right) \Phi \left(\frac{b-a}{n} \right) \leq f(x_0) + 2 \sum_{i=1}^{n-1} f(x_i) + f(x_n).$$

As before, we multiply the both sides of this inequality by $\frac{b-a}{2n}$ and utilizing the definition of the Trapezoidal rule, we get

$$(b-a) \left(f(a) - \frac{n}{2} \Phi \left(\frac{b-a}{n} \right) \right) \leq \mathcal{T}_n(f).$$

This yields the first part of inequality (2.1) and establishes the statement. $\square$

Next we are going to see result associated with trapezoidal integral mean for the Φ -Hölder functions.

Corollary 1. *Let $f : I \rightarrow \mathbb{R}$ be a continuous Φ -Hölder function. Then for any $n \in \mathbb{N}$, the following inequalities hold*

$$\max\{f(a), f(b)\} - \frac{n}{2} \Phi \left(\frac{b-a}{n} \right) \leq \frac{1}{b-a} \mathcal{T}_n(f) \leq \min\{f(a), f(b)\} + \frac{n}{2} \Phi \left(\frac{b-a}{n} \right). \quad (2.4)$$

Proof. For the proof, we consider partitioning the interval I as defined in (2.2). Since f is Φ -Hölder; the function $(-f)$ also possesses Φ -monotonicity. Therefore by Theorem 1, substituting $-f$ instead of f in (2.1), we obtain

$$f(b) - \frac{n}{2} \Phi \left(\frac{b-a}{n} \right) \leq \frac{1}{b-a} \mathcal{T}_n(f) \leq f(a) + \frac{n}{2} \Phi \left(\frac{b-a}{n} \right).$$

This together with (2.1) yields the inequality (2.4) and completes the proof. $\square$

The next corollary shows that under a usual characteristics of error function Φ , the dependency of n can be relaxed in the bounds of the numerical integral. We say that the error function Φ is superadditive if $\Phi(x) + \Phi(y) \leq \Phi(x+y)$ holds for any x, y and $x+y \in [0, \ell(I)]$.

Corollary 2. *Let $f : I \rightarrow \mathbb{R}$ be a continuous Φ -monotone function where Φ possesses superadditivity. Then for any $n \in \mathbb{N}$, the following inequalities hold*

$$f(a) - \frac{1}{2} \Phi(b-a) \leq \frac{1}{b-a} \mathcal{T}_n(f) \leq f(b) + \frac{1}{2} \Phi(b-a). \quad (2.5)$$

Proof. Since $\Phi : [0, \ell(I)] \rightarrow \mathbb{R}_+$ is superadditive, for any $n \in \mathbb{N}$, it satisfies the following functional inequality

$$n\Phi\left(\frac{b-a}{n}\right) = \sum_{i=1}^n \Phi\left(\frac{b-a}{n}\right) \leq \Phi\left(\sum_{i=1}^n \frac{b-a}{n}\right) = \Phi(b-a). \quad (2.6)$$

Using the above inequality at (2.1) of Theorem 1, we obtain the desired result. $\square$

The above corollary can also be formulated for Theorems 2 and 3 under necessary assumptions on n . The next theorems replicate the similar result for Simpson's and Simpson's 3/8 rules.

Theorem 2. *Let $f : I \rightarrow \mathbb{R}$ be a continuous Φ -monotone function. Then for any even number $n \in \mathbb{N}$, the following inequalities hold*

$$f(a) - \frac{n}{2}\Phi\left(\frac{b-a}{n}\right) \leq \frac{1}{b-a}\mathcal{S}_n(f) \leq f(b) + \frac{n}{2}\Phi\left(\frac{b-a}{n}\right). \quad (2.7)$$

Proof. To prove the statement, we consider partitioning the interval I into n equal subintervals as mentioned in (2.2). Utilizing Φ -monotonicity of f in each of the subintervals $[x_{i-1}, x_i]$ of I , we get the system of n inequalities as mentioned in (2.3).

Now we multiply both sides of the odd numbered inequalities of (2.3) by $(6i-5)$ and even numbered inequalities by $(6i-1)$ respectively for $i = 1, \dots, \frac{n}{2}$ and summing up those side by side, we obtain

$$\begin{aligned} & \sum_{i=1}^{n/2} [f(x_{2i-2}) + 4f(x_{2i-1}) + f(x_{2i})] \\ & \leq 3nf(x_n) + \Phi\left(\frac{b-a}{n}\right) \left(\sum_{i=1}^{n/2} (6i-1) + \sum_{i=1}^{n/2} (6i-5) \right). \end{aligned}$$

It can be easily observable that $\sum_{i=1}^{n/2} (6i-1) + \sum_{i=1}^{n/2} (6i-5) = \frac{3}{2}n^2$. Upon multiplying the above inequality by $\frac{b-a}{3n}$ and using the definition of Simpson's rule we get

$$\mathcal{S}_n(f) \leq (b-a) \left(f(b) + \frac{n}{2}\Phi\left(\frac{b-a}{n}\right) \right).$$

Dividing both side of it by $b-a$, we obtain the right most inequality of (2.7).

To show the initial part of inequality (2.7), once again we consider the system of inequalities (2.3). First we multiply both sides of odd numbered inequalities by $6(\frac{n}{2}-i)+5$ and then even numbered inequalities by $6(\frac{n}{2}-i)+1$ for $i = 1, 2, \dots, \frac{n}{2}$ respectively. After adding up the resultant inequalities side by side, we arrive at

$$3nf(x_0) - \Phi\left(\frac{b-a}{n}\right) \left(\sum_{i=1}^{n/2} \left(6\left(\frac{n}{2}-i\right) + 1 \right) + \sum_{i=1}^{n/2} \left(6\left(\frac{n}{2}-i\right) + 5 \right) \right)$$

$$\leq \sum_{i=1}^{n/2} [f(x_{2i-2}) + 4f(x_{2i-1}) + f(x_{2i})].$$

Multiplying the above inequality by $\frac{b-a}{3n}$ and using the definition of Simpson's rule we obtain

$$(b-a) \left(f(a) - \frac{n}{2} \Phi \left(\frac{b-a}{n} \right) \right) \leq S_n(f).$$

This shows the first inequality part of (2.7) and establishes the result. $\square$

The corollary below can be derived by using the above theorem. Hence the proof of it is not included.

Corollary 3. *Let $f : I \rightarrow \mathbb{R}$ be a continuous Φ -Hölder function. Then for any even number $n \in \mathbb{N}$, the following inequalities hold*

$$\max\{f(a), f(b)\} - \frac{n}{2} \Phi \left(\frac{b-a}{n} \right) \leq \frac{1}{b-a} S_n(f) \leq \min\{f(a), f(b)\} + \frac{n}{2} \Phi \left(\frac{b-a}{n} \right).$$

The next corollary shows that in case the error function Φ is superadditive, then we can obtain bounds for the numerical integral mean of f , which are independent of n .

Corollary 4. *Let $f : I \rightarrow \mathbb{R}$ be a continuous Φ -Hölder function where Φ possesses superadditivity. Then*

$$\max\{f(a), f(b)\} - \frac{1}{2} \Phi(b-a) \leq \frac{1}{b-a} S_n(f) \leq \min\{f(a), f(b)\} + \frac{1}{2} \Phi(b-a)$$

hold for any even number $n \in \mathbb{N}$.

Proof. The statement is a direct consequence of Corollary 3 and (2.6). $\square$

We can construct similar results for Φ -Hölder functions for the Trapezoidal and Simpson's 3/8 rule under the appropriate restriction/relaxation on the value of n .

Theorem 3. *Let $f : I \rightarrow \mathbb{R}$ be a continuous Φ -monotone function. Then for any number $n \in \mathbb{N}$ which is a multiple of 3, the inequalities*

$$f(a) - \frac{n}{2} \Phi \left(\frac{b-a}{n} \right) \leq \frac{1}{b-a} S_n^{\frac{3}{8}}(f) \leq f(b) + \frac{n}{2} \Phi \left(\frac{b-a}{n} \right) \quad (2.8)$$

hold.

Proof. To prove the statement, we consider partitioning the interval I into n equal subintervals as mentioned in (2.2). Utilizing Φ -monotonicity of f in each of the subintervals $[x_{i-1}, x_i]$ of I , we get the system of n inequalities as mentioned in (2.3).

Since n is a multiple of 3, we orderly rearrange the system of inequalities in $\frac{n}{3}$ triplets as follows

$$\begin{aligned} f(x_{3k-3}) &\leq f(x_{3k-2}) + \Phi\left(\frac{b-a}{n}\right) \\ f(x_{3k-2}) &\leq f(x_{3k-1}) + \Phi\left(\frac{b-a}{n}\right) \\ f(x_{3k-1}) &\leq f(x_{3k}) + \Phi\left(\frac{b-a}{n}\right). \end{aligned} \quad (k = 1, 2, \dots, n/3) \quad (2.9)$$

Now initially, we multiply the first, second and 3rd inequalities of k^{th} triplet respectively by $8k-7$, $8k-4$ and $8k-1$. After summing up all resultant inequalities side by side, we arrive at the following

$$\begin{aligned} \sum_{k=1}^{n/3} ((8k-7)f(x_{3k-3}) + 3f(x_{3k-2}) + 3f(x_{3k-1}) - (8k-1)f(x_{3k})) \\ \leq \left(\sum_{k=1}^{n/3} (24k-12) \right) \Phi\left(\frac{b-a}{n}\right). \end{aligned}$$

One can easily see that $\sum_{k=1}^{n/3} (24k-12) = \frac{4}{3}n^2$. Now rewriting the above inequality, we get

$$\sum_{i=1}^{n/3} (f(x_{3i-3}) + 3f(x_{3i-2}) + 3f(x_{3i-1}) + f(x_{3i})) \leq \frac{8n}{3}f(b) + \frac{4}{3}n^2\Phi\left(\frac{b-a}{n}\right).$$

Multiplying both sides of the above inequality by $\frac{3(b-a)}{8n}$ and using the definition of Simpson's $3/8$ rule, we obtain the following functional inequality

$$\mathcal{S}_n^{\frac{3}{8}}(f) \leq (b-a) \left(f(b) + \frac{n}{2} \Phi\left(\frac{b-a}{n}\right) \right).$$

By diving both sides of it by $(b-a)$ we get the right most inequality of (2.8).

To obtain the first part of the inequality, we multiplying the first, second and 3rd inequalities of k^{th} triplet in (2.9) by $8(n/3-k)+7$, $8(n/3-k)+4$ and $8(n/3-k)+1$ respectively for $k = 1, \dots, n/3$. After that adding up all the inequalities side by side, we arrive at the following

$$\frac{8n}{3}f(a) - \frac{4}{3}n^2\Phi\left(\frac{b-a}{n}\right) \leq \sum_{i=1}^{n/3} [f(x_{3i-3}) + 3f(x_{3i-2}) + 3f(x_{3i-1}) + f(x_{3i})].$$

Now we multiply both sides of it by $\frac{3(b-a)}{8n}$ and using the definition of Simpson's 3/8 rule, we obtain the functional inequality below

$$(b-a) \left(f(a) - \frac{n}{2} \Phi \left(\frac{b-a}{n} \right) \right) \leq \mathcal{S}_n^{\frac{3}{8}}(f).$$

This yields the first part of inequality (2.8) and completes the proof. $\square$

The establishment of the following result is a direct implication of the above theorem. For that reason, we just state the statement.

Corollary 5. *Let $f : I \rightarrow \mathbb{R}$ be a continuous Φ -Hölder function. Then for any number $n \in \mathbb{N}$ which is a multiple of 3, the function f satisfies the following functional inequalities*

$$\max\{f(a), f(b)\} - \frac{n}{2} \Phi \left(\frac{b-a}{n} \right) \leq \frac{1}{b-a} \mathcal{S}_n^{\frac{3}{8}}(f) \leq \min\{f(a), f(b)\} + \frac{n}{2} \Phi \left(\frac{b-a}{n} \right).$$

In the next section, we discuss results related to Φ -convex and Φ -affine functions.

3. ON APPROXIMATELY CONVEX FUNCTIONS

The main objective of this section is to present a numerical version of Hermite-Hadamard type inequality for Φ -convex functions. Here also, I will stand for non-empty and non-singleton interval $[a, b]$ with $a < b$.

Theorem 4. *Suppose $f : I \rightarrow \mathbb{R}$ is a continuous Φ -convex function. Then for any $n \in \mathbb{N}$, the following inequalities satisfy*

$$f \left(\frac{a+b}{2} \right) - E_n \leq \frac{1}{b-a} \mathcal{T}_n(f) \leq \frac{f(a)+f(b)}{2} + \frac{n^2-1}{6} \Phi \left(\frac{b-a}{n} \right), \quad (3.1)$$

where

$$E_n = \begin{cases} \frac{n^2+2}{12} \Phi \left(\frac{b-a}{n} \right), & \text{if } n \text{ is even,} \\ \frac{n^2-1}{12} \Phi \left(\frac{b-a}{n} \right) + \Phi \left(\frac{b-a}{2n} \right), & \text{if } n \text{ is odd.} \end{cases} \quad (3.2)$$

Proof. Let $x < y$. By substituting $t = \frac{1}{2}$ in (1.1), we arrive at the following

$$2f \left(\frac{x+y}{2} \right) \leq f(x) + f(y) + 2\Phi \left(\frac{y-x}{2} \right). \quad (3.3)$$

We partition the interval I as defined in (2.2). Now instead of x and y ; substituting the pairs (x_{i-1}, x_{i+1}) for all $i = 1, \dots, n-1$ in (3.3), we get the following system of $n-1$ inequalities

$$\begin{aligned} 2f(x_1) &= 2f \left(\frac{x_0+x_2}{2} \right) \leq f(x_0) + f(x_2) + 2\Phi \left(\frac{b-a}{n} \right) \\ &\vdots \end{aligned} \quad (3.4)$$

$$2f(x_{n-1}) = 2f\left(\frac{x_{n-2} + x_n}{2}\right) \leq f(x_{n-2}) + f(x_n) + 2\Phi\left(\frac{b-a}{n}\right)$$

To establish (3.1), we consider two cases. At first we assume that n is odd. Now summing up the all $n-1$ inequalities of (3.4), we obtain the following

$$f(x_1) + f(x_{n-1}) \leq f(x_0) + f(x_n) + 2(n-1)\Phi\left(\frac{b-a}{n}\right).$$

Now excluding the very first and last inequalities from (3.4) and summing up all the remaining $n-3$ inequalities, we arrive at

$$f(x_2) + f(x_{n-2}) \leq f(x_1) + f(x_{n-1}) + 2(n-3)\Phi\left(\frac{b-a}{n}\right).$$

Continuing the same way, we finally have the system of $\frac{n-1}{2}$ inequalities as follows

$$\begin{aligned} f\left(x_{\frac{n-1}{2}}\right) + f\left(x_{\frac{n+1}{2}}\right) &\leq f\left(x_{\frac{n-3}{2}}\right) + f\left(x_{\frac{n+3}{2}}\right) + 4\Phi\left(\frac{b-a}{n}\right) \\ f\left(x_{\frac{n-3}{2}}\right) + f\left(x_{\frac{n+3}{2}}\right) &\leq f\left(x_{\frac{n-5}{2}}\right) + f\left(x_{\frac{n+5}{2}}\right) + 8\Phi\left(\frac{b-a}{n}\right) \\ &\vdots \\ f(x_2) + f(x_{n-2}) &\leq f(x_1) + f(x_{n-1}) + 2(n-3)\Phi\left(\frac{b-a}{n}\right) \\ f(x_1) + f(x_{n-1}) &\leq f(x_0) + f(x_n) + 2(n-1)\Phi\left(\frac{b-a}{n}\right). \end{aligned} \quad (3.5)$$

Now multiplying each of the inequalities of the above system by $2, 4, \dots, n-3, n-1$ respectively and summing those up side by side we have

$$2[f(x_1) + \dots + f(x_{n-1})] \leq (n-1)(f(x_0) + f(x_n)) + 2\left(\sum_{i=1}^{(n-1)/2} (2i)^2\right)\Phi\left(\frac{b-a}{n}\right).$$

Adding $f(x_0) + f(x_n)$ to both sides of the above inequality and multiplying the resulting inequality by $\frac{b-a}{2n}$, we obtain

$$\mathcal{T}_n(f) \leq (b-a)\left[\frac{f(a) + f(b)}{2} + \frac{n^2-1}{6}\Phi\left(\frac{b-a}{n}\right)\right]. \quad (3.6)$$

On the other hand, if n is even; we formulate the following system of $n-1$ inequalities by applying the same method as described above

$$2f\left(x_{\frac{n}{2}}\right) \leq f\left(x_{\frac{n}{2}-1}\right) + f\left(x_{\frac{n}{2}+1}\right) + 2\Phi\left(\frac{b-a}{n}\right)$$

$$\begin{aligned}
f\left(x_{\frac{n}{2}-1}\right) + f\left(x_{\frac{n}{2}+1}\right) &\leq f\left(x_{\frac{n}{2}-2}\right) + f\left(x_{\frac{n}{2}+2}\right) + 6\Phi\left(\frac{b-a}{n}\right) \\
&\vdots \\
f(x_2) + f(x_{n-2}) &\leq f(x_1) + f(x_{n-1}) + 2(n-3)\Phi\left(\frac{b-a}{n}\right) \\
f(x_1) + f(x_{n-1}) &\leq f(x_0) + f(x_n) + 2(n-1)\Phi\left(\frac{b-a}{n}\right).
\end{aligned} \tag{3.7}$$

We multiply each inequalities of the above system respectively by $1, 3, \dots, n-3, n-1$ and then adding the resultant inequalities side by side, we arrive the following

$$2[f(x_1) + \dots + f(x_{n-1})] \leq (n-1)(f(x_0) + f(x_n)) + 2\left(\sum_{i=1}^{n/2} (2i-1)^2\right)\Phi\left(\frac{b-a}{n}\right).$$

By adding $f(x_0) + f(x_n)$, and then multiplying both sides of the resulting inequality by $\frac{b-a}{2n}$ and utilizing the identity $2\sum_{i=1}^{n/2} (2i-1)^2 = \frac{n^3-n}{3}$ we again obtain (3.6). This yields the second inequality of (3.1) for any $n \in \mathbb{N}$ and establishes the first assertion.

To show the first inequality of (3.1), first we assume n is even, and we construct the $n-1$ system of inequalities as described in (3.7). Now first multiplying the inequalities of the system by $n-1, n-3, \dots, 1$ respectively and then summing the resultant inequalities side by side, and using the identity $2\sum_{i=1}^{n/2} (2i-1)(n-(2i-1)) = \frac{n(n^2+2)}{6}$, we obtain

$$2nf\left(x_{\frac{n}{2}}\right) \leq f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n) + \frac{n(n^2+2)}{6}\Phi\left(\frac{b-a}{n}\right).$$

Now we again multiply both sides of the above inequality by $\frac{b-a}{n}$. Applying the trapezoidal rule to the resultant, we get

$$(b-a)\left[f\left(\frac{a+b}{2}\right) - \frac{n^2+2}{12}\Phi\left(\frac{b-a}{n}\right)\right] \leq \mathcal{T}_n(f).$$

Next we suppose n is odd. We multiply the inequalities in (3.5) by $n-2, n-4, \dots, 3, 1$, respectively, and add them up, then we get

$$\begin{aligned}
(n-2)\left[f\left(x_{\frac{n-1}{2}}\right) + f\left(x_{\frac{n+1}{2}}\right)\right] &\leq f(x_0) + 2\left(f(x_1) + \dots + f\left(x_{\frac{n-3}{2}}\right) + f\left(x_{\frac{n+3}{2}}\right) + \dots \right. \\
&\quad \left. + f(x_{n-1})\right) + f(x_n) + 2\sum_{i=1}^{(n-1)/2} 2i(n-2i)\Phi\left(\frac{b-a}{n}\right).
\end{aligned}$$

This yields

$$n \left([f(x_{\frac{n-1}{2}}) + f(x_{\frac{n+1}{2}})] \right) \leq f(x_0) + 2 \sum_{i=1}^{n-1} f(x_i) + f(x_n) + \frac{n^3-n}{6} \Phi \left(\frac{b-a}{n} \right),$$

hence by multiplication with $\frac{1}{2n}$ we obtain

$$\frac{1}{2} \left([f(x_{\frac{n-1}{2}}) + f(x_{\frac{n+1}{2}})] \right) \leq \frac{1}{b-a} \mathcal{T}_n(f) + \frac{n^2-1}{12} \Phi \left(\frac{b-a}{n} \right).$$

Then (3.3) implies

$$\begin{aligned} f\left(\frac{a+b}{2}\right) - \Phi\left(\frac{b-a}{2n}\right) &\leq \frac{1}{2} \left([f(x_{\frac{n-1}{2}}) + f(x_{\frac{n+1}{2}})] \right) \\ &\leq \frac{1}{b-a} \mathcal{T}_n(f) + \frac{n^2-1}{12} \Phi\left(\frac{b-a}{n}\right). \end{aligned}$$

This together with (3.2) establishes the first inequality of (3.1) and completes the proof. $\square$

For the case $\Phi \equiv 0$; the definition of Φ -convex function (in equation (1.1)), turns to be the usual convexity. From the above theorem; it is evident that if $f : I \rightarrow \mathbb{R}$ is convex, then for any $n \in \mathbb{N}$, the function f satisfies the following inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \mathcal{T}_n(f) \leq \frac{f(a)+f(b)}{2}.$$

Next, we are going to study Hermite-Hadamard inequality for Φ -affine functions.

Corollary 6. Suppose $f : I \rightarrow \mathbb{R}$ is a continuous Φ -affine function. Then for any $n \in \mathbb{N}$, the following inequalities are satisfied

$$\begin{aligned} \max \left\{ f\left(\frac{a+b}{2}\right) - E_n, \frac{f(a)+f(b)}{2} - \frac{n^2-1}{6} \Phi\left(\frac{b-a}{n}\right) \right\} \\ \leq \frac{1}{b-a} \mathcal{T}_n(f) \leq \min \left\{ f\left(\frac{a+b}{2}\right) + E_n, \frac{f(a)+f(b)}{2} + \frac{n^2-1}{6} \Phi\left(\frac{b-a}{n}\right) \right\}, \end{aligned} \quad (3.8)$$

where E_n is defined by (3.2).

Proof. Since f is Φ -affine, together with f , the function $-f$ also satisfies Φ -convexity. Thus instead of f , we can replace $-f$ in (3.1) to obtain

$$\frac{f(a)+f(b)}{2} - \frac{n^2-1}{6} \Phi\left(\frac{b-a}{n}\right) \leq \frac{1}{b-a} \mathcal{T}_n(f) \leq f\left(\frac{a+b}{2}\right) + E_n.$$

This inequality together with (3.1) establishes the result. $\square$

We note that by following the method of Theorem 4 we can obtain similar estimates for Simpson's and Simpson's 3/8 rules. Clearly, the proofs of this paper can be easily extended to estimates of the numerical integral mean associated to other Newton-Cotes quadrature formulas.

We close this paper with the following observation. If $f \in C^2[a, b]$, then it is known (see, e.g., [3]) that $\mathcal{I}_n(f) \rightarrow \int_a^b f(x) ds$ as $n \rightarrow \infty$. Therefore, if the error function satisfies

$$\lim_{n \rightarrow \infty} n^2 \Phi \left(\frac{b-a}{n} \right) = 0,$$

then estimates (3.1) imply the classical Hermite-Hadamard inequality (1.2) for Φ -convex functions too.

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ON A MULTIDIMENSIONAL CLOSE-TO-CYCLIC SYSTEM OF DIFFERENCE EQUATIONS

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Abstract. This paper investigates the solvability and dynamic properties of the following multi-dimensional close-to-cyclic system of nonlinear difference equations

$$y_{n+1}^{(i)} = \frac{a_i y_n^{(i+1)} \left(y_{n-k}^{(i+1)} \right)^{p_{i+1}} + b_i}{\left(y_{n-k+1}^{(i)} \right)^{p_i}}; \quad n \in \mathbb{N}_0,$$

where $y_n^{(i+k)} = y_n^{(i)}$, $p_{i+k} = p_i$, $a_{i+k} = a_i$, $b_{i+k} = b_i$; $i = \overline{1, k}$, the initial values $y_{-k}^{(i)}, y_{-k+1}^{(i)}, \dots, y_0^{(i)}$ and a_i and b_i , $i = \overline{1, k}$, are positive real numbers and p_i , $i = \overline{1, k}$, are real numbers. The system, characterized by intricate nonlinear interactions, is analyzed to derive explicit solutions and examine its asymptotic behavior. By leveraging a transformation approach, the multidimensional system is reduced to a simpler form, allowing for a comprehensive analysis of its solutions. The study demonstrates that under specific conditions, the system's equilibrium points are globally attractive, ensuring stability. The theoretical findings are supported by numerical examples that highlight the behavior of solutions under various parameter configurations, illustrating the practical applicability of the results.

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1. INTRODUCTION

Nonlinear difference equations play a fundamental role in modeling various dynamic systems across numerous scientific disciplines. Their solutions offer valuable insights into the behavior and stability of such systems, enabling us to make accurate predictions and devise effective control strategies. In this paper, we delve into the study a close-to-cyclic system of nonlinear difference equations, aiming to represent its solutions and investigate their asymptotic behavior in special cases.

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The close-to-cyclic system under consideration exhibits a rich interplay of nonlinear dynamics, where the variables in the system mutually influence each other. This intricate coupling gives rise to intricate behaviors that often defy straightforward analysis. However, by leveraging advanced mathematical techniques and rigorous reasoning, we aim to unravel the underlying dynamics and shed light on the behavior of the system's solutions.

There has been a great interest in finding solutions to system of nonlinear difference equations. Still, most of the papers published in this aspect were limited to system of two or three dimensions at most, see for example [1–24].

Overall, our aim in this paper is to provide a comprehensive understanding of the solutions and asymptotic behavior of the following close-to-cyclic system of nonlinear difference equations

$$y_{n+1}^{(i)} = \frac{a_i y_n^{(i+1)} \left(y_{n-k}^{(i+1)}\right)^{p_{i+1}} + b_i}{\left(y_{n-k+1}^{(i)}\right)^{p_i}}; \quad n \in \mathbb{N}_0, \quad (1.1)$$

where $y_n^{(i+k)} = y_n^{(i)}$, $p_{i+k} = p_i$, $a_{i+k} = a_i$, $b_{i+k} = b_i$; $i = \overline{1, k}$, the initial values $y_{-k}^{(i)}$, $y_{-k+1}^{(i)}$, $\dots, y_0^{(i)}$ and the parameters a_i and b_i , $i = \overline{1, k}$ are positive real numbers and p_i , $i = \overline{1, k}$, are real numbers.

2. AUXILIARY RESULTS

In this section we will present several results needed to prove the main results in Section 3.

Consider the following k -dimensional linear difference equations system

$$w_{n+1}^{(i)} = a_i w_n^{(i+1)} + b_i, \quad (2.1)$$

where $w_n^{(i+k)} = w_n^{(i)}$ and $w_0^{(i)}$, a_i , b_i , $i = \overline{1, k}$ are positive real numbers.

The following auxiliary result is used several times in the rest of the paper.

Lemma 1. *Let $(w_n^{(i)})_{n \geq 0}$ be the solution to system (2.1). Then for all $n \in \mathbb{N}_0$*

$$w_{kn+j}^{(i)} = \begin{cases} w_j^{(i)} + nT_i, & S = 1, \\ S^n w_j^{(i)} + T_i \left(\frac{S^n - 1}{S - 1} \right), & S \neq 1, \end{cases}$$

where, $i = \overline{1, k}$, $j = \overline{0, k-1}$ and

$$S = \prod_{l=1}^k a_l, \quad T_i = \sum_{r=2}^k \left(\prod_{l=i}^{i+r-2} a_l \right) b_{i+r-1} + b_i. \quad (2.2)$$

Proof. The systems in (2.1) immediately imply, for $i = \overline{1, k}$, the following relations

$$w_{n+k}^{(i)} = a_i w_{n+k-1}^{(i+1)} + b_i$$

$$\begin{aligned}
&= a_i \left[a_{i+1} w_{n+k-2}^{(i+2)} + b_{i+1} \right] + b_i \\
&= a_i a_{i+1} w_{n+k-2}^{(i+2)} + a_i b_{i+1} + b_i \\
&= a_i a_{i+1} \left[a_{i+2} w_{n+k-3}^{(i+3)} + b_{i+2} \right] + a_i b_{i+1} + b_i \\
&= a_i a_{i+1} a_{i+2} w_{n+k-3}^{(i+3)} + a_i a_{i+1} b_{i+2} + a_i b_{i+1} + b_i \\
&= a_i a_{i+1} a_{i+2} a_{i+3} w_{n+k-4}^{(i+4)} + a_i a_{i+1} a_{i+2} b_{i+3} + a_i a_{i+1} b_{i+2} + a_i b_{i+1} + b_i \\
&\vdots \\
&= a_i a_{i+1} \dots a_{i+k-1} w_{n+k-k}^{(i+k)} + a_i a_{i+1} \dots a_{i+k-2} b_{i+k-1} \\
&\quad + a_i a_{i+1} \dots a_{i+k-3} b_{i+k-2} + \dots + a_i b_{i+1} + b_i \\
&= a_i a_{i+1} \dots a_{i+k-1} w_n^{(i)} + a_i a_{i+1} \dots a_{i+k-2} b_{i+k-1} \\
&\quad + a_i a_{i+1} \dots a_{i+k-3} b_{i+k-2} + \dots + a_i b_{i+1} + b_i.
\end{aligned}$$

So, we have

$$w_{n+k}^{(i)} = \left(\prod_{l=1}^k a_l \right) w_n^{(i)} + \left[\sum_{r=2}^k \left(\prod_{l=i}^{i+r-2} a_l \right) b_{i+r-1} \right] + b_i.$$

Putting $S = \prod_{l=1}^k a_l$ and $T_i = \sum_{r=2}^k \left(\prod_{l=i}^{i+r-2} a_l \right) b_{i+r-1} + b_i$. We get

$$w_{n+k}^{(i)} = S w_n^{(i)} + T_i,$$

for $i = \overline{1, k}$, with the initial values $w_j^{(i)}, j = \overline{0, k-1}$.

Consequently, instead of solving system (2.1), we will solve the following equations

$$w_{n+k}^{(i)} = S w_n^{(i)} + T_i, \quad n \in \mathbb{N}_0, \quad (2.3)$$

where $w_j^{(i)}, j = \overline{0, k-1}$, are positive real numbers.

Equation (2.3) yield

$$\begin{aligned}
w_k^{(i)} &= S w_0^{(i)} + T_i, \\
w_{k+1}^{(i)} &= S w_1^{(i)} + T_i, \\
&\vdots \\
w_{2k-1}^{(i)} &= S w_{k-1}^{(i)} + T_i. \\
w_{2k}^{(i)} &= S w_k^{(i)} + T_i = S \left(S w_0^{(i)} + T_i \right) + T_i = S^2 w_0^{(i)} + S T_i + T_i,
\end{aligned}$$

$$\begin{aligned}
w_{2k+1}^{(i)} &= Sw_{k+1}^{(i)} + T_i = S \left(Sw_1^{(i)} + T_i \right) + T_i = S^2 w_1^{(i)} + ST_i + T_i, \\
&\vdots \\
w_{3k-1}^{(i)} &= Sw_{2k-1}^{(i)} + T_i = S \left(Sw_{k-1}^{(i)} + T_i \right) + T_i = S^2 w_{k-1}^{(i)} + ST_i + T_i, \\
w_{3k}^{(i)} &= Sw_{2k}^{(i)} + T_i = S \left(S^2 w_0^{(i)} + ST_i + T_i \right) + T_i = S^3 w_0^{(i)} + S^2 T_i \\
&\quad + ST_i + T_i, \\
w_{3k+1}^{(i)} &= Sw_{2k+1}^{(i)} + T_i = S \left(S^2 w_1^{(i)} + ST_i + T_i \right) + T_i = S^3 w_1^{(i)} + S^2 T_i \\
&\quad + ST_i + T_i, \\
&\vdots \\
w_{4k-1}^{(i)} &= Sw_{3k-1}^{(i)} + T_i = S \left(S^2 w_{k-1}^{(i)} + ST_i + T_i \right) + T_i = S^3 w_{k-1}^{(i)} + S^2 T_i \\
&\quad + ST_i + T_i.
\end{aligned}$$

The inductive argument proves, for $i = \overline{1, k}$, that

$$\begin{aligned}
w_{kn}^{(i)} &= S^n w_0^{(i)} + \sum_{t=0}^{n-1} S^t T_i, \\
w_{kn+1}^{(i)} &= S^n w_1^{(i)} + \sum_{t=0}^{n-1} S^t T_i, \\
w_{kn+2}^{(i)} &= S^n w_2^{(i)} + \sum_{t=0}^{n-1} S^t T_i, \\
&\vdots \\
w_{kn+k-1}^{(i)} &= S^n w_{k-1}^{(i)} + \sum_{t=0}^{n-1} S^t T_i.
\end{aligned}$$

More precisely, for $i = \overline{1, k}$ and $j = 0, 1, \dots, k-1$, we obtain

$$w_{kn+j}^{(i)} = S^n w_j^{(i)} + \sum_{t=0}^{n-1} S^t T_i.$$

Thus, for all $n \in \mathbb{N}_0$ we obtain

$$w_{kn+j}^{(i)} = \begin{cases} w_j^{(i)} + nT_i, & S = 1, \\ S^n w_j^{(i)} + T_i \left(\frac{S^n - 1}{S - 1} \right), & S \neq 1. \end{cases} \quad (2.4)$$

Now, we will prove by induction that the relation (2.4) is true.

A simple verification shows that (2.4) holds for $n = 0$. Suppose that (2.4) holds for n , that is $w_{kn+j}^{(i)} = \begin{cases} w_j^{(i)} + nT_i, & S = 1, \\ S^n w_j^{(i)} + T_i \left(\frac{S^n - 1}{S - 1} \right), & S \neq 1. \end{cases}$

We will prove that (2.4) holds for $n + 1$. We have

- If $S \neq 1$

$$\begin{aligned} w_{k(n+1)+j}^{(i)} &= S w_{kn+j}^{(i)} + T_i = S \left[S^n w_j^{(i)} + T_i \left(\frac{S^n - 1}{S - 1} \right) \right] + T_i \\ &= S^{n+1} w_j^{(i)} + T_i \left[S \left(\frac{S^n - 1}{S - 1} \right) \right] + T_i = S^{n+1} w_j^{(i)} + T_i \left[\frac{S^{n+1} - S + S - 1}{S - 1} \right]. \end{aligned}$$

So

$$w_{k(n+1)+j}^{(i)} = S^{n+1} w_j^{(i)} + T_i \left(\frac{S^{n+1} - 1}{S - 1} \right).$$

- If $S = 1$

$$w_{k(n+1)+j}^{(i)} = w_{kn+j}^{(i)} + T_i = w_j^{(i)} + nT_i + T_i$$

$$w_{k(n+1)+j}^{(i)} = w_j^{(i)} + (n+1)T_i.$$

$$\text{Thus, } w_{k(n+1)+j}^{(i)} = \begin{cases} w_j^{(i)} + (n+1)T_i, & S = 1, \\ S^{n+1} w_j^{(i)} + T_i \left(\frac{S^{n+1} - 1}{S - 1} \right), & S \neq 1. \end{cases}$$

□

3. MAIN RESULTS

In this section we study the solvability of system (1.1) by considering changes of variables that transforms this system to the system of k linear difference equations (2.1).

3.1. Form of solution

Here we show that system of difference equations (1.1) is practically solvable, and following the analysis of each of the systems. By using the changes of variables

$$w_n^{(i)} = y_n^{(i)} \left(y_{n-k}^{(i)} \right)^{p_i}, \quad i = \overline{1, k}, \quad n \in \mathbb{N}_0, \quad (3.1)$$

system (1.1) is transformed to the following one

$$w_{n+1}^{(i)} = a_i w_n^{(i+1)} + b_i,$$

which is the same system studied in the previous section.

Relation (3.1), for $i = \overline{1, k}$, yield

$$y_n^{(i)} = w_n^{(i)} \left(y_{n-k}^{(i)} \right)^{-p_i}, \quad n \in \mathbb{N}_0.$$

So, for $i = \overline{1, k}$, we have

$$\begin{aligned} y_{kn}^{(i)} &= w_{kn}^{(i)} \left(y_{kn-k}^{(i)} \right)^{-p_i} = w_{kn}^{(i)} \left[w_{kn-k}^{(i)} \left(y_{kn-2k}^{(i)} \right)^{-p_i} \right]^{-p_i} \\ &= w_{kn}^{(i)} \left(w_{kn-k}^{(i)} \right)^{-p_i} \left(y_{kn-2k}^{(i)} \right)^{(-p_i)^2} = w_{kn}^{(i)} \left(w_{kn-k}^{(i)} \right)^{-p_i} \left[w_{kn-2k}^{(i)} \left(y_{kn-3k}^{(i)} \right)^{-p_i} \right]^{(-p_i)^2}. \end{aligned}$$

Hence

$$\begin{aligned} y_{kn}^{(i)} &= w_{kn}^{(i)} \left(w_{kn-k}^{(i)} \right)^{-p_i} \left(w_{kn-2k}^{(i)} \right)^{(-p_i)^2} \left(y_{kn-3k}^{(i)} \right)^{(-p_i)^3} \\ &= w_{kn}^{(i)} \left(w_{kn-k}^{(i)} \right)^{-p_i} \left(w_{kn-2k}^{(i)} \right)^{(-p_i)^2} \left[w_{kn-3k}^{(i)} \left(y_{kn-4k}^{(i)} \right)^{-p_i} \right]^{(-p_i)^3} \\ &= w_{kn}^{(i)} \left(w_{kn-k}^{(i)} \right)^{-p_i} \left(w_{kn-2k}^{(i)} \right)^{(-p_i)^2} \left(w_{kn-3k}^{(i)} \right)^{(-p_i)^3} \left(y_{kn-4k}^{(i)} \right)^{(-p_i)^4} \\ &= w_{kn}^{(i)} \left(w_{kn-k.1}^{(i)} \right)^{(-p_i)^1} \left(w_{kn-k.2}^{(i)} \right)^{(-p_i)^2} \cdots \left(w_{kn-k(t-1)}^{(i)} \right)^{(-p_i)^{t-1}} \left(y_{kn-kt}^{(i)} \right)^{(-p_i)^t} \\ &= w_{kn}^{(i)} \left(w_{kn-k}^{(i)} \right)^{-p_i} \left(w_{kn-2k}^{(i)} \right)^{(-p_i)^2} \left(w_{kn-3k}^{(i)} \right)^{(-p_i)^3} \cdots \left(w_{kn-kt}^{(i)} \right)^{(-p_i)^t} \cdots \\ &\quad \times \left(w_k^{(i)} \right)^{(-p_i)^{n-1}} \left(y_0^{(i)} \right)^{(-p_i)^n}. \end{aligned}$$

Hence, we obtain

$$y_{kn}^{(i)} = \left[\prod_{t=0}^{n-1} \left(w_{k(n-t)}^{(i)} \right)^{(-p_i)^t} \right] \left(y_0^{(i)} \right)^{(-p_i)^n}, \quad n \in \mathbb{N}_0. \quad (3.2)$$

By the same argument

$$\begin{aligned} y_{kn+1}^{(i)} &= w_{kn+1}^{(i)} \left(y_{kn+1-k}^{(i)} \right)^{-p_i} = w_{kn+1}^{(i)} \left[w_{kn+1-k}^{(i)} \left(y_{kn+1-2k}^{(i)} \right)^{-p_i} \right]^{-p_i} \\ &= w_{kn+1}^{(i)} \left(w_{kn+1-k}^{(i)} \right)^{-p_i} \left(y_{kn+1-2k}^{(i)} \right)^{(-p_i)^2} \\ &= w_{kn+1}^{(i)} \left(w_{kn+1-k}^{(i)} \right)^{-p_i} \left[w_{kn+1-2k}^{(i)} \left(y_{kn+1-3k}^{(i)} \right)^{-p_i} \right]^{(-p_i)^2} \\ &= w_{kn+1}^{(i)} \left(w_{kn+1-k}^{(i)} \right)^{-p_i} \left(w_{kn+1-2k}^{(i)} \right)^{(-p_i)^2} \left(y_{kn+1-3k}^{(i)} \right)^{(-p_i)^3} \\ &= w_{kn+1}^{(i)} \left(w_{kn+1-k}^{(i)} \right)^{-p_i} \left(w_{kn+1-2k}^{(i)} \right)^{(-p_i)^2} \left[w_{kn+1-3k}^{(i)} \left(y_{kn+1-4k}^{(i)} \right)^{-p_i} \right]^{(-p_i)^3} \\ &= w_{kn+1}^{(i)} \left(w_{kn+1-k}^{(i)} \right)^{-p_i} \left(w_{kn+1-2k}^{(i)} \right)^{(-p_i)^2} \left(w_{kn+1-3k}^{(i)} \right)^{(-p_i)^3} \left(y_{kn+1-4k}^{(i)} \right)^{(-p_i)^4}. \end{aligned}$$

Hence

$$\begin{aligned}
y_{kn+1}^{(i)} &= w_{kn+1}^{(i)} \left(w_{kn+1-k}^{(i)} \right)^{-p_i} \left(w_{kn+1-2k}^{(i)} \right)^{(-p_i)^2} \left(w_{kn+1-3k}^{(i)} \right)^{(-p_i)^3} \cdots \\
&\quad \times \left(w_{kn+1-(t-1)k}^{(i)} \right)^{(-p_i)^{t-1}} \left(y_{kn+1-tk}^{(i)} \right)^{(-p_i)^t} \\
&= w_{kn+1}^{(i)} \left(w_{kn+1-k}^{(i)} \right)^{-p_i} \left(w_{kn+1-2k}^{(i)} \right)^{(-p_i)^2} \left(w_{kn+1-3k}^{(i)} \right)^{(-p_i)^3} \cdots \\
&\quad \times \left(w_{kn+1-tk}^{(i)} \right)^{(-p_i)^t} \cdots \left(w_{k+1}^{(i)} \right)^{(-p_i)^{n-1}} \left(y_1^{(i)} \right)^{(-p_i)^n} \\
&= w_{k(n-0)+1}^{(i)} \left(w_{k(n-1)+1}^{(i)} \right)^{-p_i} \left(w_{k(n-2)+1}^{(i)} \right)^{(-p_i)^2} \left(w_{k(n-3)+1}^{(i)} \right)^{(-p_i)^3} \\
&\quad \times \left(w_{k(n-t)+1}^{(i)} \right)^{(-p_i)^t} \cdots \left(w_{k(n-(n-1))+1}^{(i)} \right)^{(-p_i)^{n-1}} \left(y_1^{(i)} \right)^{(-p_i)^n}.
\end{aligned}$$

So, we get

$$y_{kn+1}^{(i)} = \left[\prod_{t=0}^{n-1} \left(w_{k(n-t)+1}^{(i)} \right)^{(-p_i)^t} \right] \left(y_1^{(i)} \right)^{(-p_i)^n}, \quad n \in \mathbb{N}_0. \quad (3.3)$$

Likewise

$$\begin{aligned}
y_{kn+2}^{(i)} &= w_{kn+2}^{(i)} \left(y_{kn+2-k}^{(i)} \right)^{-p_i} = w_{kn+2}^{(i)} \left[w_{kn+2-k}^{(i)} \left(y_{kn+2-2k}^{(i)} \right)^{-p_i} \right]^{-p_i} \\
&= w_{kn+2}^{(i)} \left(w_{kn+2-k}^{(i)} \right)^{-p_i} \left(y_{kn+2-2k}^{(i)} \right)^{(-p_i)^2} \\
&= w_{kn+2}^{(i)} \left(w_{kn+2-k}^{(i)} \right)^{-p_i} \left[w_{kn+2-2k}^{(i)} \left(y_{kn+2-3k}^{(i)} \right)^{-p_i} \right]^{(-p_i)^2} \\
&= w_{kn+2}^{(i)} \left(w_{kn+2-k}^{(i)} \right)^{-p_i} \left(w_{kn+2-2k}^{(i)} \right)^{(-p_i)^2} \left[w_{kn+2-3k}^{(i)} \left(y_{kn+2-4k}^{(i)} \right)^{-p_i} \right]^{(-p_i)^3} \\
&= w_{kn+2}^{(i)} \left(w_{kn+2-k}^{(i)} \right)^{-p_i} \left(w_{kn+2-2k}^{(i)} \right)^{(-p_i)^2} \left(w_{kn+2-3k}^{(i)} \right)^{(-p_i)^3} \left(y_{kn+2-4k}^{(i)} \right)^{(-p_i)^4} \\
&= w_{kn+2}^{(i)} \left(w_{kn+2-k}^{(i)} \right)^{-p_i} \left(w_{kn+2-2k}^{(i)} \right)^{(-p_i)^2} \left(w_{kn+2-3k}^{(i)} \right)^{(-p_i)^3} \cdots \\
&\quad \times \left(w_{kn+2-(t-1)k}^{(i)} \right)^{(-p_i)^{t-1}} \left(y_{kn+2-tk}^{(i)} \right)^{(-p_i)^t}.
\end{aligned}$$

Hence

$$\begin{aligned}
y_{kn+2}^{(i)} &= w_{kn+2}^{(i)} \left(w_{kn+2-k}^{(i)} \right)^{-p_i} \left(w_{kn+2-2k}^{(i)} \right)^{(-p_i)^2} \left(w_{kn+2-3k}^{(i)} \right)^{(-p_i)^3} \\
&\quad \times \cdots \left(w_{kn+2-tk}^{(i)} \right)^{(-p_i)^t} \cdots \left(w_{k+2}^{(i)} \right)^{(-p_i)^{n-1}} \left(y_2^{(i)} \right)^{(-p_i)^n}
\end{aligned}$$

$$\begin{aligned}
&= w_{k(n-0)+2}^{(i)} \left(w_{k(n-1)+2}^{(i)} \right)^{-p_i} \left(w_{k(n-2)+2}^{(i)} \right)^{(-p_i)^2} \left(w_{k(n-3)+2}^{(i)} \right)^{(-p_i)^3} \cdots \\
&\quad \times \left(w_{k(n-t)+2}^{(i)} \right)^{(-p_i)^t} \cdots \left(w_{k(n-(n-1))+2}^{(i)} \right)^{(-p_i)^{n-1}} \left(y_2^{(i)} \right)^{(-p_i)^n}.
\end{aligned}$$

So, we get

$$y_{kn+2}^{(i)} = \left[\prod_{t=0}^{n-1} \left(w_{k(n-t)+2}^{(i)} \right)^{(-p_i)^t} \right] \left(y_2^{(i)} \right)^{(-p_i)^n}, \quad n \in \mathbb{N}_0. \quad (3.4)$$

By the same argument

$$\begin{aligned}
y_{kn+k-1}^{(i)} &= w_{kn+k-1}^{(i)} \left(y_{kn+k-1-k}^{(i)} \right)^{-p_i} = w_{kn+k-1}^{(i)} \left[w_{kn+k-1-k}^{(i)} \left(y_{kn+k-1-2k}^{(i)} \right)^{-p_i} \right]^{-p_i} \\
&= w_{kn+k-1}^{(i)} \left(w_{kn+k-1-k}^{(i)} \right)^{-p_i} \left(y_{kn+k-1-2k}^{(i)} \right)^{(-p_i)^2} \\
&= w_{kn+k-1}^{(i)} \left(w_{kn+k-1-k}^{(i)} \right)^{-p_i} \left[w_{kn+k-1-2k}^{(i)} \left(y_{kn+k-1-3k}^{(i)} \right)^{-p_i} \right]^{(-p_i)^2} \\
&= w_{kn+k-1}^{(i)} \left(w_{kn+k-1-k}^{(i)} \right)^{-p_i} \left(w_{kn+k-1-2k}^{(i)} \right)^{(-p_i)^2} \\
&\quad \times \left[w_{kn+k-1-3k}^{(i)} \left(y_{kn+k-1-4k}^{(i)} \right)^{-p_i} \right]^{(-p_i)^3} \\
&= w_{kn+k-1}^{(i)} \left(w_{kn+k-1-k}^{(i)} \right)^{-p_i} \left(w_{kn+k-1-2k}^{(i)} \right)^{(-p_i)^2} \left(w_{kn+k-1-3k}^{(i)} \right)^{(-p_i)^3} \\
&\quad \times \left(y_{kn+k-1-4k}^{(i)} \right)^{(-p_i)^4} \\
&= w_{kn+k-1}^{(i)} \left(w_{kn+k-1-k}^{(i)} \right)^{-p_i} \left(w_{kn+k-1-2k}^{(i)} \right)^{(-p_i)^2} \left(w_{kn+k-1-3k}^{(i)} \right)^{(-p_i)^3} \\
&\quad \times \cdots \left(w_{kn+k-1-(t-1)k}^{(i)} \right)^{(-p_i)^{t-1}} \left(y_{kn+k-1-tk}^{(i)} \right)^{(-p_i)^t} \\
&= w_{kn+k-1}^{(i)} \left(w_{kn+k-1-k}^{(i)} \right)^{-p_i} \left(w_{kn+k-1-2k}^{(i)} \right)^{(-p_i)^2} \left(w_{kn+k-1-3k}^{(i)} \right)^{(-p_i)^3} \\
&\quad \times \cdots \left(w_{kn+k-1-tk}^{(i)} \right)^{(-p_i)^t} \cdots \left(w_{2k-1}^{(i)} \right)^{(-p_i)^{n-1}} \left(y_{k-1}^{(i)} \right)^{(-p_i)^n} \\
&= w_{k(n-0)+k-1}^{(i)} \left(w_{k(n-1)+k-1}^{(i)} \right)^{-p_i} \left(w_{k(n-2)+k-1}^{(i)} \right)^{(-p_i)^2} \left(w_{k(n-3)+k-1}^{(i)} \right)^{(-p_i)^3} \\
&\quad \times \cdots \left(w_{k(n-t)+k-1}^{(i)} \right)^{(-p_i)^t} \cdots \left(w_{k(n-(n-1))+k-1}^{(i)} \right)^{(-p_i)^{n-1}} \left(y_{k-1}^{(i)} \right)^{(-p_i)^n}.
\end{aligned}$$

So, we get

$$y_{kn+k-1}^{(i)} = \left[\prod_{t=0}^{n-1} \left(w_{k(n-t)+k-1}^{(i)} \right)^{(-p_i)^t} \right] \left(y_{k-1}^{(i)} \right)^{(-p_i)^n}, \quad n \in \mathbb{N}_0. \quad (3.5)$$

From (3.2), (3.3), (3.4) and (3.5) we can conclude that for $i = \overline{1, k}$ and $j = \overline{0, k-1}$, we obtain

$$y_{kn+j}^{(i)} = \left[\prod_{t=0}^{n-1} \left(w_{k(n-t)+j}^{(i)} \right)^{(-p_i)^t} \right] \left(y_j^{(i)} \right)^{(-p_i)^n}, \quad n \geq 0. \quad (3.6)$$

Now, we will prove by induction that the relation (3.6) is true.

A simple verification shows that (3.6) holds for $n = 0$. Assume that (3.6) holds for n , that is

$$y_{kn+j}^{(i)} = \left[\prod_{t=0}^{n-1} \left(w_{k(n-t)+j}^{(i)} \right)^{(-p_i)^t} \right] \left(y_j^{(i)} \right)^{(-p_i)^n}.$$

We will prove that (3.6) holds for $n+1$. We have

$$\begin{aligned} y_{k(n+1)+j}^{(i)} &= w_{k(n+1)+j}^{(i)} \left(y_{k(n+1)+j-k}^{(i)} \right)^{-p_i} \\ &= w_{k(n+1)+j}^{(i)} \left[w_{k(n+1)+j-k}^{(i)} \left(y_{k(n+1)+j-2k}^{(i)} \right)^{-p_i} \right]^{-p_i} \\ &= w_{k(n+1)+j}^{(i)} \left(w_{k(n+1)+j-k}^{(i)} \right)^{-p_i} \left(y_{k(n+1)+j-2k}^{(i)} \right)^{(-p_i)^2} \\ &= w_{k(n+1)+j}^{(i)} \left(w_{k(n+1)+j-k}^{(i)} \right)^{-p_i} \left[w_{k(n+1)+j-2k}^{(i)} \left(y_{k(n+1)+j-3k}^{(i)} \right)^{-p_i} \right]^{(-p_i)^2} \\ &= w_{k(n+1)+j}^{(i)} \left(w_{k(n+1)+j-k}^{(i)} \right)^{-p_i} \left(w_{k(n+1)+j-2k}^{(i)} \right)^{(-p_i)^2} \\ &\quad \times \left[w_{k(n+1)+j-3k}^{(i)} \left(y_{k(n+1)+j-4k}^{(i)} \right)^{-p_i} \right]^{(-p_i)^3} \\ &= w_{k(n+1)+j}^{(i)} \left(w_{k(n+1)+j-k}^{(i)} \right)^{-p_i} \left(w_{k(n+1)+j-2k}^{(i)} \right)^{(-p_i)^2} \left(w_{k(n+1)+j-3k}^{(i)} \right)^{(-p_i)^3} \\ &\quad \times \left(y_{k(n+1)+j-4k}^{(i)} \right)^{(-p_i)^4}. \end{aligned}$$

Hence

$$\begin{aligned} y_{k(n+1)+j}^{(i)} &= w_{k(n+1)+j}^{(i)} \left(w_{k(n+1)+j-k}^{(i)} \right)^{-p_i} \left(w_{k(n+1)+j-2k}^{(i)} \right)^{(-p_i)^2} \left(w_{k(n+1)+j-3k}^{(i)} \right)^{(-p_i)^3} \\ &\quad \times \dots \left(w_{k(n+1)+j-(t-1)k}^{(i)} \right)^{(-p_i)^{t-1}} \left(y_{k(n+1)+j-tk}^{(i)} \right)^{(-p_i)^t} \\ &= w_{k(n+1)+j}^{(i)} \left(w_{k(n+1)+j-k}^{(i)} \right)^{-p_i} \left(w_{k(n+1)+j-2k}^{(i)} \right)^{(-p_i)^2} \left(w_{k(n+1)+j-3k}^{(i)} \right)^{(-p_i)^3} \end{aligned}$$

$$\begin{aligned}
& \times \dots \left(w_{k(n+1)+j-tk}^{(i)} \right)^{(-p_i)^t} \dots \left(w_{k+j}^{(i)} \right)^{(-p_i)^n} \left(y_j^{(i)} \right)^{(-p_i)^{n+1}} \\
& = w_{k(n+1-0)+j}^{(i)} \left(w_{k(n+1-1)+j}^{(i)} \right)^{-p_i} \left(w_{k(n+1-2)+j}^{(i)} \right)^{(-p_i)^2} \left(w_{k(n+1-3)+j}^{(i)} \right)^{(-p_i)^3} \\
& \quad \times \dots \left(w_{k(n+1-t)+j}^{(i)} \right)^{(-p_i)^t} \dots \left(w_{k(n+1-n)+j}^{(i)} \right)^{(-p_i)^n} \left(y_j^{(i)} \right)^{(-p_i)^{n+1}}.
\end{aligned}$$

So,

$$y_{k(n+1)+j}^{(i)} = \left[\prod_{t=0}^n \left(w_{k(n+1-t)+j}^{(i)} \right)^{(-p_i)^t} \right] \left(y_j^{(i)} \right)^{(-p_i)^{n+1}}.$$

The results below provide an explicit formula for the solution of the system (1.1).

Theorem 1. Let $\{y_n^{(i)}\}_{n \geq -k}$ be a well-defined solution of (1.1). Then, for $i = \overline{1, k}$, $j = \overline{0, k-1}$ and $n \in \mathbb{N}_0$, we have

- If $S \neq 1$

$$y_{kn+j}^{(i)} = \left[\prod_{t=0}^{n-1} \left(S^{n-t} y_j^{(i)} \left(y_{j-k}^{(i)} \right)^{p_i} + T_i \left(\frac{S^{n-t} - 1}{S - 1} \right) \right)^{(-p_i)^t} \right] \left(y_j^{(i)} \right)^{(-p_i)^n}.$$

- If $S = 1$

$$y_{kn+j}^{(i)} = \left[\prod_{t=0}^{n-1} \left(y_j^{(i)} \left(y_{j-k}^{(i)} \right)^{p_i} + (n-t) T_i \right)^{(-p_i)^t} \right] \left(y_j^{(i)} \right)^{(-p_i)^n}.$$

3.2. Asymptotic behavior

In this section, we will study the asymptotic behavior of the equilibrium point of the system (1.1).

The following lemma gives the equilibrium of the system (1.1).

Lemma 2.

If $\left(\overline{y^{(1)}}, \overline{y^{(1)}}, \overline{y^{(1)}}, \overline{y^{(2)}}, \overline{y^{(2)}}, \overline{y^{(2)}}, \dots, \overline{y^{(k)}}, \overline{y^{(k)}}, \overline{y^{(k)}} \right)$ is an equilibrium point of the system (1.1), then it is given by

$$\begin{aligned}
& \left(\left[\frac{T_1}{1-S} \right]^{\frac{1}{p_1+1}}, \left[\frac{T_1}{1-S} \right]^{\frac{1}{p_1+1}}, \left[\frac{T_1}{1-S} \right]^{\frac{1}{p_1+1}}, \left[\frac{T_2}{1-S} \right]^{\frac{1}{p_2+1}}, \left[\frac{T_2}{1-S} \right]^{\frac{1}{p_2+1}}, \right. \\
& \left. \left[\frac{T_2}{1-S} \right]^{\frac{1}{p_2+1}}, \dots, \left[\frac{T_k}{1-S} \right]^{\frac{1}{p_k+1}}, \left[\frac{T_k}{1-S} \right]^{\frac{1}{p_k+1}}, \left[\frac{T_k}{1-S} \right]^{\frac{1}{p_k+1}} \right),
\end{aligned}$$

with $S = \prod_{l=1}^k a_l < 1$.

Proof. Let $(\overline{y^{(1)}}, \overline{y^{(1)}}, \overline{y^{(1)}}, \overline{y^{(2)}}, \overline{y^{(2)}}, \overline{y^{(2)}}, \dots, \overline{y^{(k)}}, \overline{y^{(k)}}, \overline{y^{(k)}})$ is an equilibrium point of the system (1.1). So, for $i = \overline{1, k}$, we have

$$\begin{aligned}
 (\overline{y^{(i)}})^{p_i+1} &= a_i (\overline{y^{(i+1)}})^{p_{i+1}+1} + b_i = a_i \left[a_{i+1} (\overline{y^{(i+2)}})^{p_{i+2}+1} + b_{i+1} \right] + b_i \\
 &= a_i a_{i+1} (\overline{y^{(i+2)}})^{p_{i+2}+1} + a_i b_{i+1} + b_i \\
 &= a_i a_{i+1} \left[a_{i+2} (\overline{y^{(i+3)}})^{p_{i+3}+1} + b_{i+2} \right] + a_i b_{i+1} + b_i \\
 &= a_i a_{i+1} a_{i+2} (\overline{y^{(i+3)}})^{p_{i+3}+1} + a_i a_{i+1} b_{i+2} + a_i b_{i+1} + b_i \\
 &= a_i a_{i+1} a_{i+2} a_{i+3} (\overline{y^{(i+4)}})^{p_{i+4}+1} + a_i a_{i+1} a_{i+2} b_{i+3} \\
 &\quad + a_i a_{i+1} b_{i+2} + a_i b_{i+1} + b_i \\
 &= a_i a_{i+1} a_{i+2} a_{i+3} a_{i+4} (\overline{y^{(i+5)}})^{p_{i+5}+1} + a_i a_{i+1} a_{i+2} a_{i+3} b_{i+4} \\
 &\quad + a_i a_{i+1} a_{i+2} b_{i+3} + a_i a_{i+1} b_{i+2} + a_i b_{i+1} + b_i \\
 &\quad \vdots \\
 &= a_i a_{i+1} \dots a_{i+k-1} (\overline{y^{(i+k)}})^{p_{i+k}+1} + a_i a_{i+1} \dots a_{i+k-2} b_{i+k-1} \\
 &\quad + a_i a_{i+1} \dots a_{i+k-3} b_{i+k-2} + \dots + a_i b_{i+1} + b_i \\
 &= \left(\prod_{l=1}^k a_l \right) (\overline{y^{(i+k)}})^{p_{i+k}+1} + \left[\sum_{r=2}^k \left(\prod_{l=i}^{i+r-2} a_l \right) b_{i+r-1} \right] + b_i \\
 &= \left(\prod_{l=1}^k a_l \right) (\overline{y^{(i)}})^{p_i+1} + \left[\sum_{r=2}^k \left(\prod_{l=i}^{i+r-2} a_l \right) b_{i+r-1} \right] + b_i.
 \end{aligned}$$

So

$$(\overline{y^{(i)}})^{p_i+1} \left(1 - \prod_{l=1}^k a_l \right) = \left[\sum_{r=2}^k \left(\prod_{l=i}^{i+r-2} a_l \right) b_{i+r-1} \right] + b_i,$$

consequently

$$\overline{y^{(i)}} = \left[\frac{\left[\sum_{r=2}^k \left(\prod_{l=i}^{i+r-2} a_l \right) b_{i+r-1} \right] + b_i}{1 - \prod_{l=1}^k a_l} \right]^{\frac{1}{p_i+1}}.$$

Using notation (2.2), we get

$$\overline{y^{(i)}} = \left[\frac{T_i}{1-S} \right]^{\frac{1}{p_i+1}}, \quad i = \overline{1, k}.$$

Note that the condition $S < 1$ implies that $\overline{y^{(i)}}$ is positive whatever the values of p_i , $i = \overline{1, k}$. $\square$

Theorem 2. Consider system (1.1). Assume, for $i = \overline{1, k}$, that $S < 1$ and $|p_i| < 1$. Then the equilibrium point of the system (1.1) is globally attractive.

Proof. Suppose, for $i = \overline{1, k}$, that $S < 1$ and $|p_i| < 1$, so we obtain

$$\begin{aligned} \lim_{n \rightarrow +\infty} y_{kn+j}^{(i)} &= \lim_{n \rightarrow +\infty} \left[\left(\prod_{t=0}^{n-1} \left(S^{n-t} y_j^{(i)} \left(y_{j-k}^{(i)} \right)^{p_i} + T_i \left(\frac{S^{n-t} - 1}{S - 1} \right) \right)^{(-p_i)^t} \right) \left(y_j^{(i)} \right)^{(-p_i)^n} \right] \\ &= \prod_{t \geq 0} \left[T_i \left(\frac{-1}{S-1} \right) \right]^{(-p_i)^t} = \prod_{t \geq 0} \left[\frac{T_i}{1-S} \right]^{(-p_i)^t} = \left[\frac{T_i}{1-S} \right]^{\sum_{t \geq 0} (-p_i)^t}. \end{aligned}$$

Moreover, we have

$$\sum_{t \geq 0} (-p_i)^t = \lim_{s \rightarrow +\infty} \sum_{t=0}^s (-p_i)^t = \lim_{s \rightarrow +\infty} \frac{(-p_i)^{s+1} - 1}{-p_i - 1} = \frac{-1}{-p_i - 1} = \frac{1}{p_i + 1}.$$

So

$$\lim_{n \rightarrow +\infty} y_{kn+j}^{(i)} = \left[\frac{T_i}{1-S} \right]^{\frac{1}{p_i+1}} = \overline{y^{(i)}}.$$

From where the equilibrium is globally attractive. $\square$

4. NUMERICAL EXAMPLES

Here, we present specific examples to illustrate the behavior of solutions for the multidimensional close-to-cyclic system of nonlinear difference equations under varying initial conditions and parameter values. Through these examples, the theoretical findings are validated, and the dynamics of the system, including stability and equilibrium points, are visually demonstrated. The examples are complemented by graphical representations, offering deeper insights into the system's characteristics and confirming the global attractiveness of equilibrium points under specified conditions.

Example 1. Let $k = 2$, $a_1 = 2$, $a_2 = \frac{1}{2}$, $b_1 = 2$, $b_2 = 3$, $p_1 = \frac{1}{2}$ and $p_2 = \frac{1}{3}$, and the initial values $y_{-2}^{(1)} = 4$, $y_{-1}^{(1)} = 4$, $y_0^{(1)} = 3$, $y_{-2}^{(2)} = 8$, $y_{-1}^{(2)} = 8$, $y_0^{(2)} = 6$ in system (1.1),

then we obtain that $S = 1$, and since, for $i = 1, 2$, we have $|p_i| < 1$. So we obtain the following system

$$y_{n+1}^{(1)} = \frac{2y_n^{(2)} \left(y_{n-2}^{(2)}\right)^{\frac{1}{3}} + 2}{\left(y_{n-1}^{(1)}\right)^{\frac{1}{2}}}, \quad y_{n+1}^{(2)} = \frac{\frac{1}{2}y_n^{(1)} \left(y_{n-2}^{(1)}\right)^{\frac{1}{2}} + 3}{\left(y_{n-1}^{(2)}\right)^{\frac{1}{3}}}, \quad n \in \mathbb{N}_0. \quad (4.1)$$

The behavior of the solution of system (4.1) is represented in the figure (1).

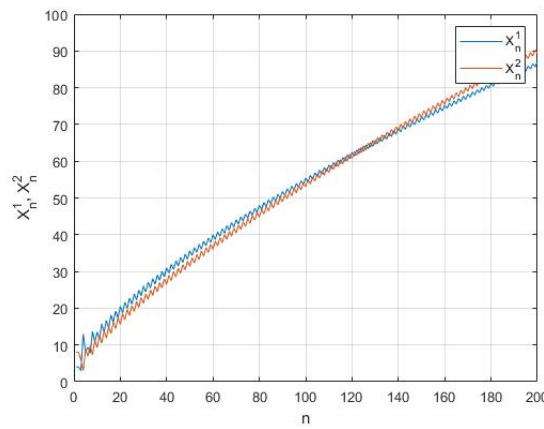


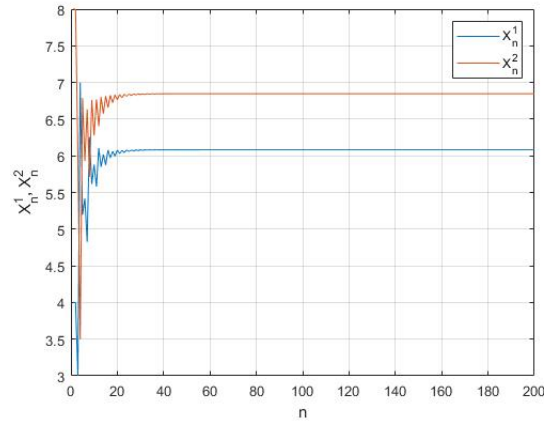
FIGURE 1. The plot of system (4.1) with $S = 1$ and $|p_i| < 1$

Example 2. Let $k = 2$, $a_1 = 1$, $a_2 = \frac{2}{3}$, $b_1 = 2$, $b_2 = 3$, $p_1 = \frac{1}{2}$ and $p_2 = \frac{1}{3}$, and the initial values $y_{-2}^{(1)} = 4$, $y_{-1}^{(1)} = 4$, $y_0^{(1)} = 3$, $y_{-2}^{(2)} = 8$, $y_{-1}^{(2)} = 8$, $y_0^{(2)} = 6$ in system (1.1), then we obtain that $S < 1$, and since, for $i = 1, 2$, we have $|p_i| < 1$. So we obtain the following system

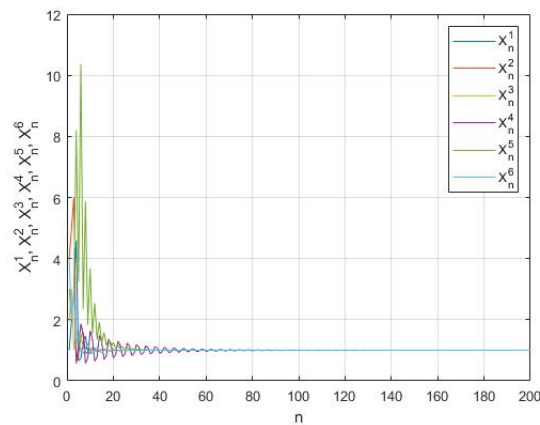
$$y_{n+1}^{(1)} = \frac{y_n^{(2)} \left(y_{n-2}^{(2)}\right)^{\frac{1}{3}} + 2}{\left(y_{n-1}^{(1)}\right)^{\frac{1}{2}}}, \quad y_{n+1}^{(2)} = \frac{\frac{2}{3}y_n^{(1)} \left(y_{n-2}^{(1)}\right)^{\frac{1}{2}} + 3}{\left(y_{n-1}^{(2)}\right)^{\frac{1}{3}}}, \quad n \in \mathbb{N}_0. \quad (4.2)$$

The equilibrium $(\overline{y^{(1)}}, \overline{y^{(1)}}, \overline{y^{(1)}}, \overline{y^{(2)}}, \overline{y^{(2)}}, \overline{y^{(2)}}) = \left(15^{\frac{2}{3}}, 15^{\frac{2}{3}}, 15^{\frac{2}{3}}, 13^{\frac{3}{4}}, 13^{\frac{3}{4}}, 13^{\frac{3}{4}}\right)$ is globally attractive (see Figure (2), Theorem (2)).

Example 3. Let $k = 6$, $a_i = b_i = \frac{1}{2}$, for $i = 1, 2, \dots, 6$ and $p_1 = \frac{1}{2}$, $p_2 = \frac{1}{2}$, $p_3 = \frac{3}{5}$, $p_4 = \frac{9}{10}$, $p_5 = \frac{-7}{10}$, $p_6 = \frac{4}{5}$, and the initial values $y_{-2}^{(1)} = 1$, $y_{-1}^{(1)} = 2$, $y_0^{(1)} = 3$,

FIGURE 2. The plot of system (4.2) with $S < 1$ and $|p_i| < 1$

$y_{-2}^{(2)} = 4, y_{-1}^{(2)} = 5, y_0^{(2)} = 6, y_{-2}^{(3)} = 2, y_{-1}^{(3)} = 3, y_0^{(3)} = 1, y_{-2}^{(4)} = 2, y_{-1}^{(4)} = 2, y_0^{(4)} = 3, y_{-2}^{(5)} = 3, y_{-1}^{(5)} = 2, y_0^{(5)} = 2, y_{-2}^{(6)} = 4, y_{-1}^{(6)} = 2, y_0^{(6)} = 3$ in system (1.1), then we obtain that $S < 1$, and since, for $i = 1, 2, \dots, 6$, we have $|p_i| < 1$. Then the equilibrium $(\overline{y^{(1)}}, \overline{y^{(1)}}, \overline{y^{(1)}}), \dots, (\overline{y^{(6)}}, \overline{y^{(6)}}, \overline{y^{(6)}}) = (1, 1, 1, \dots, 1, 1, 1)$ is globally attractive (see Figure (3), Theorem (2)).

FIGURE 3. The plot of system (1.1) with $S < 1$ and $|p_i| < 1$

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ANALYSIS OF MIXED DIFFERENTIAL EQUATIONS INVOLVING THE CAPUTO AND RIEMANN-LIOUVILLE FRACTIONAL DERIVATIVES

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Abstract. In this paper, we investigate the existence of positive solution as well as the uniqueness results for a fractional order differential equation involving the Caputo fractional derivative and the Riemann-Liouville fractional derivative. First of all, we show the existence and uniqueness of the positive solution by means of the fixed point theory, namely, Banach's contraction Principle. Second of all, we convert the posed problem to a sum of two integral operators, then we apply Kronelskii's fixed point theorem to conclude the existence of nontrivial solutions. As applications, we present examples for the demonstration of our main results.

2010 Mathematics Subject Classification: 34B18; 34B15; 34A08

Keywords: Caputo fractional derivative, Riemann-Liouville fractional derivative, fixed point theorem, existence of positive solution

1. INTRODUCTION

Fractional calculus is an extension of classical calculus and deals with the generalization of integration and differentiation to an arbitrary real order. Boundary value problems for Caputo fractional differential equations, Riemann-Liouville fractional differential equations and mixed fractional differential equations of great importance for the researches due to their applications, such as physics, chemistry, probability, many other branches of engineering. There has been a noticeable development in the study of fractional differential equations in recent years, see the books of Kilbas et al. [6], Podlubny [17], Samko et al. [18] and Miller et al. [15] are mostly cited for the theory and applications of fractional calculus.

The existence of positive solutions for fractional-order nonlinear boundary value problems has been studied by numerous authors by using the fixed-point theorem

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in cones. To identify a few, we refer the reader to [1, 4, 5, 9, 10, 12, 20, 21] and the references therein. Some authors investigated the existence of solutions for a class of mixed fractional differential equations by using different methods. In [2], Agrawal presented the mixed differential equation involving both the Caputo and the Riemann-Liouville fractional derivatives. In [3], Blaszczyk presented the numerical solutions of the mixed boundary value problem. For some interesting results on mixed fractional boundary value problems we refer the literature [8, 11, 12, 14, 16].

In [19], Song and Cui concerned the existence of solutions of nonlinear mixed fractional differential equation with the integral boundary value problem under resonance

$$\begin{cases} {}^c D_{1-}^\alpha D_{0+}^\beta u(t) = f(t, u(t), D_{0+}^{\beta+1} u(t), D_{0+}^\beta u(t)), & t \in (0, 1), \\ u(0) = u'(0) = 0, \quad u(1) = \int_0^1 u(t) dA(t), \end{cases}$$

where $1 < \alpha \leq 2$, $0 < \beta \leq 1$, $f \in C([0, 1] \times \mathbb{R}^3, \mathbb{R})$.

Liu et al. [13] investigated the existence of the unique nontrivial solution for mixed fractional differential equation

$$\begin{cases} {}^c D_{1-}^\alpha D_{0+}^\beta x(t) + f(t, x(t)) = b, & t \in (0, 1), \\ x(0) = 0, \quad x'(1) = D_{0+}^\beta x(1) = 0, \end{cases}$$

where $0 < \alpha \leq 1$, $1 < \beta \leq 2$ and $\alpha + \beta > 2$. Here $f: [0, 1] \times (-\infty, \infty) \rightarrow (-\infty, \infty)$ is continuous, and $b > 0$ is a constant real number.

Guezane et al. [8] investigated the following mixed fractional boundary value problem

$$\begin{cases} -{}^c D_{0+}^\alpha D_{0+}^\beta u(t) + f(t, u(t)) = 0, & t \in (0, 1), \\ u(0) = u'(0) = u(1) = 0, \end{cases}$$

where $0 < \alpha \leq 1$, $1 < \beta \leq 2$. They used Krasnoselskii's fixed point theorem to prove the existence of nontrivial solution.

Motivated by the above papers, we consider the nonlinear mixed fractional differential equation

$$D_{0+}^\alpha {}^c D_{0+}^\beta u(t) + f(t, u(t)) = 0, \quad t \in (0, 1),$$

with the multi-point fractional boundary conditions:

$$\begin{cases} u(0) = u'(0) = {}^c D_{0+}^\beta u(0) = 0, \\ D_{0+}^\gamma u(1) = \sum_{i=1}^{m-2} \eta_i D_{0+}^\gamma u(\tau_i), \end{cases}$$

where $1 < \alpha, \beta \leq 2$, $\gamma < \alpha + \beta - 1$, D_{0+}^α denotes the Riemann-Liouville derivative of order α , ${}^c D_{0+}^\beta$ denotes the Caputo derivative of order β , $0 < \tau_1 < \tau_2 < \dots < \tau_{m-2} < 1$, $0 < \eta_i < 1$ for $i = 1, 2, \dots, m-2$, $f \in C([0, 1] \times [0, \infty), [0, \infty))$.

The organization of this paper is as follows. In Section 2 and Section 3, we provide some definitions and preliminary lemmas which are key tools for our main result. In

Section 4, we give and prove our main results. We give examples to illustrate how the main results can be used in practice.

2. PRELIMINARIES AND LEMMAS

In order to assert our main results, we assemble some necessary definitions and lemmas from the fractional calculus, which can be found in [6, 17, 18].

Definition 1. The fractional integral of order $\alpha > 0$ of a function $y: (0, +\infty) \rightarrow \mathbb{R}$ is given by

$$I_{0+}^{\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds,$$

provided that the right side is pointwise defined on $(0, \infty)$, where

$$\Gamma(\alpha) = \int_0^{+\infty} e^{-x} x^{\alpha-1} dx.$$

Definition 2. For a continuous function $y: (0, +\infty) \rightarrow \mathbb{R}$, the Caputo derivative of fractional order $\alpha > 0$ is defined as

$$D_{0+}^{\alpha} y(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} y(s) ds,$$

where $n = [\alpha] + 1$, provided that the right side is pointwise defined on $(0, +\infty)$.

Definition 3. For a continuous function $y: (0, +\infty) \rightarrow \mathbb{R}$, the Riemann-Liouville derivative of fractional order $\alpha > 0$ is defined as

$$D_{0+}^{\alpha} y(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt} \right)^n \int_0^t (t-s)^{n-\alpha-1} y(s) ds,$$

where $n = [\alpha] + 1$, provided that the right side is pointwise defined on $(0, +\infty)$.

Lemma 1. Let $\alpha > 0$, then

$$I_{0+}^{\alpha} {}^c D_{0+}^{\alpha} u(t) = u(t) + c_0 + c_1 t + c_2 t^2 + \dots + c_{n-1} t^{n-1}$$

for some $c_i \in \mathbb{R}$, $i = 1, \dots, n$, $n = [\alpha] + 1$.

Lemma 2. Let $\alpha > 0$, then

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \dots + c_n t^{\alpha-n}$$

for some $c_i \in \mathbb{R}$, $i = 1, \dots, n$, $n = [\alpha] + 1$.

3. EXISTENCE OF NONTRIVIAL POSITIVE SOLUTIONS

We consider the fractional boundary value problem with mixed derivative given as follows.

$$\begin{cases} D_{0+}^{\alpha} {}^c D_{0+}^{\beta} u(t) + f(t, u(t)) = 0, & t \in (0, 1), \\ u(0) = u'(0) = {}^c D_{0+}^{\beta} u(0) = 0, \\ D_{0+}^{\gamma} u(1) = \sum_{i=1}^{m-2} \eta_i D_{0+}^{\gamma} u(\tau_i), \end{cases} \quad (3.1)$$

where $1 < \alpha, \beta \leq 2$, $\gamma < \alpha + \beta - 1$, D_{0+}^{α} denotes the Riemann-Liouville derivative of order α , ${}^c D_{0+}^{\beta}$ denotes the Caputo derivative of order β , $0 < \tau_1 < \tau_2 < \dots < \tau_{m-2} < 1$, $0 < \eta_i < 1$ for $i = 1, 2, \dots, m-2$, $f \in C([0, 1] \times [0, \infty), [0, \infty))$.

Throughout this paper we assume that the following conditions hold:

$$(H1) \quad 0 < \sum_{i=1}^{m-2} \eta_i \tau_i^{\alpha+\beta-\gamma-1} < 1,$$

$$(H2) \quad f: [0, 1] \times [0, \infty) \rightarrow [0, \infty) \text{ is a continuous function.}$$

Lemma 3. *If $h \in C[0, 1]$, the fractional boundary value problem*

$$\begin{cases} D_{0+}^{\alpha} {}^c D_{0+}^{\beta} u(t) + h(t) = 0, & t \in (0, 1), \\ u(0) = u'(0) = {}^c D_{0+}^{\beta} u(0) = 0, \\ D_{0+}^{\gamma} u(1) = \sum_{i=1}^{m-2} \eta_i D_{0+}^{\gamma} u(\tau_i), \end{cases} \quad (3.2)$$

has an integral expression

$$u(t) = \int_0^1 G(t, s) h(s) ds \quad (3.3)$$

where

$$G(t, s) = G_1(t, s) + G_2(t, s) \quad (3.4)$$

and

$$G_1(t, s) = \frac{1}{\Gamma(\alpha + \beta)} \begin{cases} t^{\alpha+\beta-1} (1-s)^{\alpha+\beta-\gamma-1} - (t-s)^{\alpha+\beta-1}, & 0 \leq s \leq t \leq 1, \\ t^{\alpha+\beta-1} (1-s)^{\alpha+\beta-\gamma-1}, & 0 \leq t \leq s \leq 1, \end{cases} \quad (3.5)$$

$$G_2(t, s) = \frac{\sum_{i=1}^{m-2} \eta_i t^{\alpha+\beta-1}}{\Gamma(\alpha + \beta) \left[1 - \sum_{i=1}^{m-2} \eta_i \tau_i^{\alpha+\beta-\gamma-1} \right]} \times \begin{cases} (\tau_i(1-s))^{\alpha+\beta-\gamma-1} - (\tau_i-s)^{\alpha+\beta-\gamma-1}, & 0 \leq s \leq \tau_i \leq 1, \\ (\tau_i(1-s))^{\alpha+\beta-\gamma-1}, & 0 \leq \tau_i \leq s \leq 1. \end{cases} \quad (3.6)$$

Proof. According to Lemma 2, we can obtain that

$${}^c D_{0+}^{\beta} u(t) = -I_{0+}^{\alpha} h(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2}.$$

We assume that the boundary conditions are satisfied. Firstly, using ${}^c D_{0+}^\beta u(0) = 0$, we have $c_2 = 0$. We may apply Lemma 1, and find that

$$u(t) = -I_{0+}^{\alpha+\beta} h(t) + c_1 I_{0+}^\beta t^{\alpha-1} + d_1 + d_2 t.$$

From $u(0) = u'(0) = 0$ conditions, $d_1 = d_2 = 0$ is found. From condition

$$D_{0+}^\gamma u(1) = \sum_{i=1}^{m-2} \eta_i D_{0+}^\gamma u(\tau_i),$$

we find that

$$c_1 = \frac{\int_0^1 (1-s)^{\alpha+\beta-\gamma-1} h(s) ds - \sum_{i=1}^{m-2} \eta_i \int_0^{\tau_i} (\tau_i - s)^{\alpha+\beta-\gamma-1} h(s) ds}{\Gamma(\alpha) \left[1 - \sum_{i=1}^{m-2} \eta_i \tau_i^{\alpha+\beta-\gamma-1} \right]},$$

substituting this value of c_1 , we have

$$\begin{aligned} u(t) = & \frac{-1}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1} h(s) ds + \frac{t^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \cdot \frac{\int_0^1 (1-s)^{\alpha+\beta-\gamma-1} h(s) ds}{\left[1 - \sum_{i=1}^{m-2} \eta_i \tau_i^{\alpha+\beta-\gamma-1} \right]} \\ & - \frac{\sum_{i=1}^{m-2} \eta_i t^{\alpha+\beta-1} \left[\int_0^{\tau_i} (\tau_i - s)^{\alpha+\beta-\gamma-1} h(s) ds \right]}{\Gamma(\alpha+\beta) \left[1 - \sum_{i=1}^{m-2} \eta_i \tau_i^{\alpha+\beta-\gamma-1} \right]}. \end{aligned}$$

Now let's create the Green function:

$$\begin{aligned} u(t) = & \frac{-1}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1} h(s) ds + \frac{t^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \int_0^1 (1-s)^{\alpha+\beta-\gamma-1} h(s) ds \\ & + \frac{t^{\alpha+\beta-1} \sum_{i=1}^{m-2} \eta_i \tau_i^{\alpha+\beta-\gamma-1}}{\Gamma(\alpha+\beta) \left[1 - \sum_{i=1}^{m-2} \eta_i \tau_i^{\alpha+\beta-\gamma-1} \right]} \int_0^1 (1-s)^{\alpha+\beta-\gamma-1} h(s) ds \\ & - \frac{\sum_{i=1}^{m-2} \eta_i t^{\alpha+\beta-1} \left[\int_0^{\tau_i} (\tau_i - s)^{\alpha+\beta-\gamma-1} h(s) ds \right]}{\Gamma(\alpha+\beta) \left[1 - \sum_{i=1}^{m-2} \eta_i \tau_i^{\alpha+\beta-\gamma-1} \right]} \\ = & \frac{-1}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1} h(s) ds + \frac{t^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \int_0^t (1-s)^{\alpha+\beta-\gamma-1} h(s) ds \\ & + \frac{t^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \int_t^1 (1-s)^{\alpha+\beta-\gamma-1} h(s) ds \\ & + \frac{t^{\alpha+\beta-1} \sum_{i=1}^{m-2} \eta_i \tau_i^{\alpha+\beta-\gamma-1}}{\Gamma(\alpha+\beta) \left[1 - \sum_{i=1}^{m-2} \eta_i \tau_i^{\alpha+\beta-\gamma-1} \right]} \int_0^1 (1-s)^{\alpha+\beta-\gamma-1} h(s) ds \end{aligned}$$

$$\begin{aligned}
& - \frac{\sum_{i=1}^{m-2} \eta_i t^{\alpha+\beta-1} [\int_0^{\tau_i} (\tau_i - s)^{\alpha+\beta-\gamma-1} h(s) ds]}{\Gamma(\alpha+\beta) \left[1 - \sum_{i=1}^{m-2} \eta_i \tau_i^{\alpha+\beta-\gamma-1}\right]} \\
& = \int_0^1 G_1(t, s) h(s) ds + \frac{t^{\alpha+\beta-1} \sum_{i=1}^{m-2} \eta_i \tau_i^{\alpha+\beta-\gamma-1}}{\Gamma(\alpha+\beta) \left[1 - \sum_{i=1}^{m-2} \eta_i \tau_i^{\alpha+\beta-\gamma-1}\right]} \\
& \quad \int_0^1 (1-s)^{\alpha+\beta-\gamma-1} h(s) ds - \frac{\sum_{i=1}^{m-2} \eta_i t^{\alpha+\beta-1} [\int_0^{\tau_i} (\tau_i - s)^{\alpha+\beta-\gamma-1} h(s) ds]}{\Gamma(\alpha+\beta) \left[1 - \sum_{i=1}^{m-2} \eta_i \tau_i^{\alpha+\beta-\gamma-1}\right]} \\
& = \int_0^1 G_1(t, s) h(s) ds + \frac{t^{\alpha+\beta-1} \sum_{i=1}^{m-2} \eta_i}{\Gamma(\alpha+\beta) \left[1 - \sum_{i=1}^{m-2} \eta_i \tau_i^{\alpha+\beta-\gamma-1}\right]} \\
& \quad \times \left[\int_0^1 \tau_i^{\alpha+\beta-\gamma-1} (1-s)^{\alpha+\beta-\gamma-1} h(s) ds - \int_0^{\tau_i} (\tau_i - s)^{\alpha+\beta-\gamma-1} h(s) ds \right] \\
& = \int_0^1 G_1(t, s) h(s) ds + \int_0^1 G_2(t, s) h(s) ds \\
& = \int_0^1 G(t, s) h(s) ds.
\end{aligned}$$

The functions $G(t, s)$, $G_1(t, s)$ and $G_2(t, s)$ are defined in (3.4), (3.5) and (3.6), respectively. $\square$

Lemma 4. The function $G_1(t, s)$ defined by (3.5) satisfies the following properties:

- (i) $G_1(t, s)$ is a continuous function and $G_1(t, s) \geq 0$ for any $(t, s) \in [0, 1] \times [0, 1]$,
- (ii) $G_1(t, s) \leq \frac{1}{\Gamma(\alpha+\beta)}$ for any $(t, s) \in [0, 1] \times [0, 1]$.

Proof. (i) It is obvious.

(ii) Let $s \leq t$.

$$G_1(t, s) = \frac{t^{\alpha+\beta-1} (1-s)^{\alpha+\beta-\gamma-1} - (t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \leq \frac{t^{\alpha+\beta-1} (1-s)}{\Gamma(\alpha+\beta)} \leq \frac{1}{\Gamma(\alpha+\beta)}.$$

Let $t \leq s$.

$$G_1(t, s) = \frac{t^{\alpha+\beta-1} (1-s)^{\alpha+\beta-\gamma-1}}{\Gamma(\alpha+\beta)} \leq \frac{1}{\Gamma(\alpha+\beta)}.$$

In both cases, we obtain $G_1(t, s) \leq \frac{1}{\Gamma(\alpha+\beta)}$. $\square$

Lemma 5. The function $G_2(t, s)$ defined by (3.6) satisfies the following properties:

- (i) $G_2(t, s)$ is a continuous function and $G_2(t, s) \geq 0$ for any $(t, s) \in [0, 1] \times [0, 1]$,
- (ii) $G_2(t, s) \leq \frac{\sum_{i=1}^{m-2} \eta_i \tau_i^{\alpha+\beta-\gamma-1}}{\Gamma(\alpha+\beta) \left[1 - \sum_{i=1}^{m-2} \eta_i \tau_i^{\alpha+\beta-\gamma-1}\right]}$ for any $(t, s) \in [0, 1] \times [0, 1]$.

Proof. (i) It is obvious.

(ii) Let $s \leq \tau_i$.

$$\begin{aligned} G_2(t, s) &= \frac{\sum_{i=1}^{m-2} \eta_i t^{\alpha+\beta-1}}{\Gamma(\alpha+\beta) \left[1 - \sum_{i=1}^{m-2} \eta_i \tau_i^{\alpha+\beta-\gamma-1} \right]} \left((\tau_i(1-s))^{\alpha+\beta-\gamma-1} - (\tau_i - s)^{\alpha+\beta-\gamma-1} \right) \\ &\leq \frac{\sum_{i=1}^{m-2} \eta_i}{\Gamma(\alpha+\beta) \left[1 - \sum_{i=1}^{m-2} \eta_i \tau_i^{\alpha+\beta-\gamma-1} \right]} \tau_i^{\alpha+\beta-\gamma-1}. \end{aligned}$$

Let $\tau_i \leq s$.

$$\begin{aligned} G_2(t, s) &= \frac{\sum_{i=1}^{m-2} \eta_i t^{\alpha+\beta-1}}{\Gamma(\alpha+\beta) \left[1 - \sum_{i=1}^{m-2} \eta_i \tau_i^{\alpha+\beta-\gamma-1} \right]} \left((\tau_i(1-s))^{\alpha+\beta-\gamma-1} \right) \\ &\leq \frac{\sum_{i=1}^{m-2} \eta_i}{\Gamma(\alpha+\beta) \left[1 - \sum_{i=1}^{m-2} \eta_i \tau_i^{\alpha+\beta-\gamma-1} \right]} \tau_i^{\alpha+\beta-\gamma-1}. \end{aligned}$$

The following inequality is obtained from both cases:

$$G_2(t, s) \leq \frac{\sum_{i=1}^{m-2} \eta_i}{\Gamma(\alpha+\beta) \left[1 - \sum_{i=1}^{m-2} \eta_i \tau_i^{\alpha+\beta-\gamma-1} \right]} \tau_i^{\alpha+\beta-\gamma-1}.$$

□

Finally, by means of equation (3.4), we get:

Lemma 6. The $G(t, s)$ function defined by (3.6) provides the following inequality:

$$G(t, s) \leq \frac{1}{\Gamma(\alpha+\beta) \left[1 - \sum_{i=1}^{m-2} \eta_i \tau_i^{\alpha+\beta-\gamma-1} \right]}$$

for any $(t, s) \in [0, 1] \times [0, 1]$.

Proof. This Lemma is obvious from Lemma 4 and Lemma 5. □

4. FIXED POINT THEOREMS

The following fixed point theorems are fundamental and essential to the proofs our main results.

Theorem 1 (Banach contraction mapping principle, [10]). Let (X, d) be a nonempty complete metric space, and let $T: X \rightarrow X$ be a contraction, i.e., there exists a number $0 < \rho < 1$ such that $d(Tx, Ty) \leq \rho d(x, y)$. Then the operator T has a unique fixed point $x^* \in X$.

Theorem 2 (Krasnoselskii's fixed point theorem[7]). *Let M be a closed, convex, bounded and nonempty subset of a Banach space E . Let A, B be the operators such that*

- (i) A is compact and continuous,
- (ii) B is a contraction mapping,
- (iii) $Ax + By \in M$ whenever $x, y \in M$.

Then there exists $z \in M$ such that $z = Az + Bz$.

Theorem 3. *Suppose that $f: [0, 1] \times [0, \infty) \rightarrow [0, \infty)$ is a continuous function, for all $t \in [0, 1]$ and $f(t, 0)$ is not identically null on $[0, 1]$.*

(H3) *For all $t \in [0, 1]$ and $u, v \in [0, \infty)$, we have*

$$|f(t, u) - f(t, v)| \leq \sigma(t) \cdot |u - v|,$$

where $\sigma(t) \in C([0, 1]; [0, \infty))$. Then, there exists a unique positive solution for problem (3.1) under the following condition $r_1 < 1$, where

$$r_1 = \frac{1}{\Gamma(\alpha + \beta) \left[1 - \sum_{i=1}^{m-2} \eta_i \tau_i^{\alpha+\beta-\gamma-1} \right]} \cdot \sigma^* \text{ and } \sigma^* = \int_0^1 \sigma(s) ds.$$

Proof. We consider the Banach space $E = C[0, 1]$ with the maximum norm. Define $T: E \rightarrow E$ by

$$(Tu)(t) = \int_0^1 G(t, s) f(s, u(s)) ds$$

and setting $\sup_{0 \leq t \leq 1} f(t, 0) = F$.

We consider the following set

$$B_r = \{u \in E : \|u\| \leq r\},$$

where

$$r \geq \frac{r_2}{1 - r_1}, \text{ with } r_2 = \frac{1}{\Gamma(\alpha + \beta) \left[1 - \sum_{i=1}^{m-2} \eta_i \tau_i^{\alpha+\beta-\gamma-1} \right]} \cdot F.$$

For each $t \in [0, 1]$ and $u \in B_r$, we have

$$\begin{aligned} |(Tu)(t)| &\leq \left[\frac{1}{\Gamma(\alpha + \beta)} + \frac{\sum_{i=1}^{m-2} \eta_i}{\Gamma(\alpha + \beta) \left[1 - \sum_{i=1}^{m-2} \eta_i \tau_i^{\alpha+\beta-\gamma-1} \right]} \cdot \tau_i^{\alpha+\beta-\gamma-1} \right] \\ &\quad \cdot \left[\int_0^1 |f(s, u(s)) - f(s, 0)| ds \right] + F \\ &\leq \frac{1}{\Gamma(\alpha + \beta) \left[1 - \sum_{i=1}^{m-2} \eta_i \tau_i^{\alpha+\beta-\gamma-1} \right]} \cdot \left[\int_0^1 \sigma(s) ds \cdot \|u\| + F \right] \end{aligned}$$

$$= \frac{1}{\Gamma(\alpha + \beta) \left[1 - \sum_{i=1}^{m-2} \eta_i \tau_i^{\alpha + \beta - \gamma - 1} \right]} \cdot [\sigma^* \cdot \|u\| + F].$$

This means that $\|Tu\| \leq r$. Therefore $TB_r \subseteq B_r$. Next, we prove that T is a contraction mapping for $u, v \in B_r$. We have

$$\begin{aligned} |(Tu)(t) - (Tv)(t)| &\leq \int_0^1 G(t, s) \cdot |f(s, u(s)) - f(s, v(s))| \, ds \\ &\leq \frac{1}{\Gamma(\alpha + \beta) \left[1 - \sum_{i=1}^{m-2} \eta_i \tau_i^{\alpha + \beta - \gamma - 1} \right]} \cdot \int_0^1 \sigma(s) |u(s) - v(s)| \, ds \\ &\leq \frac{1}{\Gamma(\alpha + \beta) \left[1 - \sum_{i=1}^{m-2} \eta_i \tau_i^{\alpha + \beta - \gamma - 1} \right]} \cdot \sigma^* \cdot \|u - v\|. \end{aligned}$$

Since $r_1 < 1$, T is a contraction. Therefore, by Theorem 1, problem (3.1) has a unique positive solution. $\square$

Example 1. Let be $\alpha = \frac{18}{11}$, $\beta = \frac{15}{11}$, $\gamma = \frac{1}{2}$, $m = 3$, $\eta_1 = 1$ and $\tau_1 = \frac{1}{2}$ for the boundary value problem (3.1).

Consider the following boundary value problem:

$$\begin{cases} D_{0+}^{\frac{18}{11}} {}^c D_{0+}^{\frac{15}{11}} u + \frac{t^3}{400} \cdot (1 + u) = 0, & t \in (0, 1), \\ u(0) = u'(0) = {}^c D_{0+}^{\frac{15}{11}} u(0) = 0, \\ D_{0+}^{\frac{1}{2}} u(1) = D_{0+}^{\frac{1}{2}} u\left(\frac{1}{2}\right), \end{cases}$$

where $\sigma(t) = \frac{t^3}{400}$ and $f(t, u) = \frac{t^3}{400} \cdot (1 + u)$.

We obtain $\sigma^* = \frac{1}{1600}$ and $r_1 \cong 0,000483$ by some simple calculations. In this case all the conditions of Theorem 3 are satisfied. Hence, by Theorem 3, we prove that the boundary value problem has one positive solution.

Theorem 4. Assume that (H1)-(H3) hold and $f: [0, 1] \times [0, \infty) \rightarrow [0, \infty)$ is a continuous function. Furthermore, we suppose

(H4) $f(t, u) \leq \theta(t)$, for all $(t, u) \in [0, 1] \times [0, \infty)$ and $\theta \in C([0, 1], \mathbb{R}^+)$.

Then problem (3.1) has at least one positive solution on $[0, 1]$ if $R < 1$, where

$$R = \frac{\sum_{i=1}^{m-2} \eta_i \tau_i^{\alpha + \beta - \gamma - 1}}{\Gamma(\alpha + \beta) \left[1 - \sum_{i=1}^{m-2} \eta_i \tau_i^{\alpha + \beta - \gamma - 1} \right]} \cdot \sigma^*.$$

Proof. We consider the closed ball

$$B_\rho = \{u \in E : \|u\| \leq \rho\}$$

with fixed radius ρ :

$$\rho \geq \frac{\|\theta\|}{\Gamma(\alpha + \beta) \left[1 - \sum_{i=1}^{m-2} \eta_i \tau_i^{\alpha + \beta - \gamma - 1} \right]}.$$

We define the operators T_1 and T_2 on B_ρ as

$$\begin{aligned} (T_1 u)(t) &= \int_0^1 G_1(t, s) f(s, u(s)) ds, \\ (T_2 u)(t) &= \int_0^1 G_2(t, s) f(s, u(s)) ds. \end{aligned}$$

For $u, v \in B_\rho$, we have

$$\begin{aligned} (T_1 u)(t) &\leq \frac{\|\theta\|}{\Gamma(\alpha + \beta)}, \\ (T_2 v)(t) &\leq \frac{\sum_{i=1}^{m-2} \eta_i \tau_i^{\alpha + \beta - \gamma - 1} \cdot \|\theta\|}{\Gamma(\alpha + \beta) \left[1 - \sum_{i=1}^{m-2} \eta_i \tau_i^{\alpha + \beta - \gamma - 1} \right]}. \end{aligned}$$

Consequently,

$$\|T_1 u + T_2 v\| \leq \frac{\|\theta\|}{\Gamma(\alpha + \beta) \left[1 - \sum_{i=1}^{m-2} \eta_i \tau_i^{\alpha + \beta - \gamma - 1} \right]}.$$

Then $T_1 u + T_2 v \in B_\rho$.

In the following, we prove that T_2 is a contraction.

$$\begin{aligned} |(T_2 u)(t) - (T_2 v)(t)| &\leq \int_0^1 G_2(t, s) |f(s, u(s)) - f(s, v(s))| ds \\ &\leq \frac{\sum_{i=1}^{m-2} \eta_i \tau_i^{\alpha + \beta - \gamma - 1}}{\Gamma(\alpha + \beta) \left[1 - \sum_{i=1}^{m-2} \eta_i \tau_i^{\alpha + \beta - \gamma - 1} \right]} \int_0^1 \sigma(s) |u(s) - v(s)| ds \\ &\leq \frac{\sum_{i=1}^{m-2} \eta_i \tau_i^{\alpha + \beta - \gamma - 1}}{\Gamma(\alpha + \beta) \left[1 - \sum_{i=1}^{m-2} \eta_i \tau_i^{\alpha + \beta - \gamma - 1} \right]} \cdot \sigma^* \cdot \|u - v\|. \end{aligned}$$

Since $R < 1$, we conclude that T_2 is a contraction. Now, we show that T_1 is a completely continuous operator. Continuity of f implies that the operator T_1 is continuous. Also, T_1 is uniformly bounded on B_ρ as

$$(T_1 u)(t) \leq \frac{\|\theta\|}{\Gamma(\alpha + \beta)}.$$

Now, we prove the compactness of operator $(T_1 u)$. We have for $u \in B_\rho$, $t_1, t_2 \in [0, 1]$ and $t_1 < t_2$.

$$\begin{aligned} |(T_1 u)(t_1) - (T_1 u)(t_2)| &\leq \int_0^1 |G_1(t_1, s) - G_1(t_2, s)| \cdot |f(s, u(s))| \, ds \\ &\leq \frac{\|\theta\|}{\Gamma(\alpha + \beta)} \left| \int_0^{t_1} (t_1^{\alpha+\beta-1} (1-s)^{\alpha+\beta-\gamma-1} - (t_1-s)^{\alpha+\beta-1} \right. \\ &\quad \left. - (t_2^{\alpha+\beta-1} (1-s)^{\alpha+\beta-\gamma-1} + (t_2-s)^{\alpha+\beta-1}) \, ds \right. \\ &\quad \left. + \int_{t_1}^{t_2} (t_1^{\alpha+\beta-1} (1-s)^{\alpha+\beta-\gamma-1} - t_2^{\alpha+\beta-1} (1-s)^{\alpha+\beta-\gamma-1} \right. \\ &\quad \left. + (t_2-s)^{\alpha+\beta-1}) \, ds + \int_{t_2}^1 (t_1^{\alpha+\beta-1} (1-s)^{\alpha+\beta-\gamma-1} \right. \\ &\quad \left. - t_2^{\alpha+\beta-1} (1-s)^{\alpha+\beta-\gamma-1}) \, ds \right|. \end{aligned}$$

So, $|T_1 u(t_1) - T_1 u(t_2)| \rightarrow 0$ tends to zero when $t_1 \rightarrow t_2$, thus T_1 is relatively compact on B_ρ . Hence, by the Arzela Ascoli theorem, T_1 is compact on B_ρ . Thus, by Krasnoselskii's fixed point theorem, problem (3.1) has one positive solution in E . $\square$

Example 2. Let be $\alpha = \frac{15}{11}$, $\beta = \frac{18}{11}$, $\gamma = \frac{1}{2}$, $m = 3$, $\eta_1 = 1$ and $\tau_1 = \frac{1}{2}$ for the boundary value problem (3.1).

Consider the following boundary value problem:

$$\begin{cases} D_{0+}^{\frac{15}{11}} {}^c D_{0+}^{\frac{18}{11}} u + \frac{t^3}{600} \cdot \frac{u}{1+u} = 0, t \in (0, 1) \\ u(0) = u'(0) = {}^c D_{0+}^{\frac{18}{11}} u(0) = 0, \\ D_{0+}^{\frac{1}{2}} u(1) = D_{0+}^{\frac{1}{2}} u(\frac{1}{2}), \end{cases}$$

where $\sigma(t) = \frac{t^3}{600}$, $f(t, u) = \frac{t^3}{600} \cdot \frac{u}{1+u}$ and $\theta(t) = \frac{t^3}{600}$.

Through calculation, we get $\sigma^* = \frac{1}{2400}$ and $R \cong 0,000113$. We conclude that all the assumptions of Theorem 4 are verified. Thus, problem has at least one positive solution.

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INVERSE TRIGONOMETRICALLY CONVEXITY AND BETTER APPROXIMATIONS

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Abstract. In this paper, we introduce and study the concept of inverse trigonometrically convex functions and their some algebraic properties. We prove some Hermite-Hadamard type integral inequalities for the newly introduced class of functions. We also obtain some refinements of the Hermite-Hadamard inequality for functions whose first derivative in absolute value is inverse trigonometrically convex. Moreover, we proved that Hölder-İşcan and improved power-mean integral inequalities give a better approach than Hölder and power-mean inequalities.

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1. INTRODUCTION

Definition 1. $\varpi: I \rightarrow \mathbb{R}$ is said to be convex function if the inequality

$$\varpi(s\phi + (1-s)\chi) \leq s\varpi(\phi) + (1-s)\varpi(\chi)$$

is valid for all $\phi, \chi \in I$ and $s \in [0, 1]$.

Convexity theory has become a deep research area in pure and applied sciences. In recent years, several extensions and generalizations of classical convexity have been studied by many researchers using novel methods and ideas. See articles [3, 6, 8, 10–13] and the references therein.

Let $\varpi: I \rightarrow \mathbb{R}$ be a convex function. Then

$$\varpi\left(\frac{\phi + \chi}{2}\right) \leq \frac{1}{\chi - \phi} \int_{\phi}^{\chi} \varpi(x) dx \leq \frac{\varpi(\phi) + \varpi(\chi)}{2}$$

holds for all $\phi, \chi \in I$ with $\phi < \chi$. This is well known as the Hermite Hadamard (H-H) integral inequality (see [4]). Some refinements of the Hermite-Hadamard inequality for convex functions have been obtained [2, 15]. This inequality is one of the

most studied result in inequalities via convex functions. It provides a necessary and sufficient condition for a function to be convex.

Definition 2 ([3]). A non-negative $\varpi: I \rightarrow \mathbb{R}$ is said to be a P -function if the inequality

$$\varpi(r\phi + (1-r)\chi) \leq \varpi(\phi) + \varpi(\chi)$$

holds for all $\phi, \chi \in I$ and $r \in [0, 1]$.

Definition 3 ([14, page 304, Definition 4]). Let $h: J \rightarrow \mathbb{R}$ be a non-negative function, $h \neq 0$. We say that $\varpi: I \rightarrow \mathbb{R}$ is an h -convex function, or that ϖ belongs to the class $SX(h, I)$, if ϖ is non-negative and for all $\phi, \chi \in I$, $r \in (0, 1)$ we have

$$\varpi(r\phi + (1-r)\chi) \leq h(r)\varpi(\phi) + h(1-r)\varpi(\chi).$$

In [7], Kadakal gave the concept of trigonometrically convex function and Hermite-Hadamard type inequalities as follows:

Definition 4 ([7, page 20, Definition 4]). A non-negative function $\varpi: I \rightarrow \mathbb{R}$ is called trigonometrically convex if for every $\phi, \chi \in I$ and $r \in [0, 1]$,

$$\varpi(r\phi + (1-r)\chi) \leq \left(\sin \frac{\pi r}{2}\right) \varpi(\phi) + \left(\cos \frac{\pi r}{2}\right) \varpi(\chi).$$

The class of all trigonometrically convex functions is denoted by $TC(I)$ on interval I . We note that, every trigonometrically convex function is a h -convex function for $h(t) = \sin \frac{\pi t}{2}$.

Theorem 1 ([7, page 23, Theorem 4]). Let $\varpi: [\phi, \chi] \rightarrow \mathbb{R}$ be a trigonometrically convex function. If $\phi < \chi$ and $\varpi \in L[\phi, \chi]$, then the following inequality holds:

$$\frac{1}{\chi - \phi} \int_{\phi}^{\chi} \varpi(x) dx \leq \frac{2}{\pi} [\varpi(\phi) + \varpi(\chi)].$$

Theorem 2. Let the function $\varpi: [\phi, \chi] \rightarrow \mathbb{R}$, be a trigonometrically convex function. If $\phi < \chi$ and $\varpi \in L[\phi, \chi]$, then the following inequality holds:

$$\varpi\left(\frac{\phi + \chi}{2}\right) \leq \frac{\sqrt{2}}{\chi - \phi} \int_{\phi}^{\chi} \varpi(x) dx.$$

An refinement of Hölder integral inequality better approach than Hölder integral inequality can be given as follows:

Theorem 3 (Hölder-İşcan Integral Inequality [5, page 2, Theorem 2.1]). Let $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. If f and g are real functions defined on interval $[a, b]$ and if $|f|^p, |g|^q$

are integrable functions on interval $[a, b]$ then

$$\int_a^b |f(x)g(x)| dx \leq \frac{1}{b-a} \left\{ \left(\int_a^b (b-x) |f(x)|^p dx \right)^{\frac{1}{p}} \left(\int_a^b (b-x) |g(x)|^q dx \right)^{\frac{1}{q}} \right. \\ \left. + \left(\int_a^b (x-a) |f(x)|^p dx \right)^{\frac{1}{p}} \left(\int_a^b (x-a) |g(x)|^q dx \right)^{\frac{1}{q}} \right\}.$$

An refinement of power-mean integral inequality better approach than power-mean inequality as a result of the Hölder-İşcan integral inequality can be given as follows:

Theorem 4 (Improved power-mean integral inequality [9, page 444, Theorem 2.1]). *Let $q \geq 1$. If f and g are real functions defined on interval $[a, b]$ and if $|f|$, $|f||g|^q$ are integrable functions on $[a, b]$ then*

$$\int_a^b |f(x)g(x)| dx \leq \frac{1}{b-a} \left\{ \left(\int_a^b (b-x) |f(x)| dx \right)^{1-\frac{1}{q}} \right. \\ \times \left(\int_a^b (b-x) |f(x)||g(x)|^q dx \right)^{\frac{1}{q}} \\ \left. + \left(\int_a^b (x-a) |f(x)| dx \right)^{1-\frac{1}{q}} \left(\int_a^b (x-a) |f(x)||g(x)|^q dx \right)^{\frac{1}{q}} \right\}.$$

2. MAIN RESULTS

In this section, we introduce a new concept, which is called inverse trigonometrically convexity and we give some properties for the inverse trigonometrically convex functions, as follows:

Definition 5. A non-negative function $\varpi: I \rightarrow \mathbb{R}$ is called inverse trigonometrically convex function (or inverse trigonometrically convex) if for every $\phi, \chi \in I$ and $r \in [0, 1]$,

$$\varpi(r\phi + (1-r)\chi) \leq \left(\frac{2}{\pi} \arcsin r \right) \varpi(\phi) + \left(\frac{2}{\pi} \arccos r \right) \varpi(\chi).$$

We will denote by $IT(I)$ the class of all inverse trigonometrically convex functions on interval I .

We discuss some connections between the class of inverse trigonometrically convex functions and other classes of generalized convex functions.

Remark 1. Clearly, if $\varpi(x)$ is a nonnegative function, then every inverse trigonometric convex function is a P -function. Indeed, for every $\phi, \chi \in I$ and $r \in [0, 1]$ we

get

$$\varpi(r\phi + (1-r)\chi) \leq \left(\frac{2}{\pi} \arcsin r\right) \varpi(\phi) + \left(\frac{2}{\pi} \arccos r\right) \varpi(\chi) \leq \varpi(\phi) + \varpi(\chi).$$

Example 1. Every constant function is an inverse trigonometrically convex function.

Theorem 5. Let $\varpi, \omega: [\phi, \chi] \rightarrow \mathbb{R}$. If ϖ and ω are inverse trigonometrically convex functions, then

- (i) $\varpi + \omega$ is inverse trigonometrically convex function,
- (ii) For $c \in \mathbb{R}$ ($c \geq 0$), $c\varpi$ is inverse trigonometrically convex function.

Proof.

- (i) Let ϖ, ω be inverse trigonometrically convex functions, then

$$\begin{aligned} & (\varpi + \omega)(r\phi + (1-r)\chi) \\ & \leq \left(\frac{2}{\pi} \arcsin r\right) \varpi(\phi) + \left(\frac{2}{\pi} \arccos r\right) \varpi(\chi) + \left(\frac{2}{\pi} \arcsin r\right) \omega(\phi) \\ & \quad + \left(\frac{2}{\pi} \arccos r\right) \omega(\chi) \\ & = \left(\frac{2}{\pi} \arcsin r\right) (\varpi + \omega)(\phi) + \left(\frac{2}{\pi} \arccos r\right) (\varpi + \omega)(\chi). \end{aligned}$$

- (ii) Let ϖ be inverse trigonometrically convex function and $c \in \mathbb{R}$ ($c \geq 0$), then

$$\begin{aligned} (c\varpi)(r\phi + (1-r)\chi) & \leq c \left[\left(\frac{2}{\pi} \arcsin r\right) \varpi(\phi) + \left(\frac{2}{\pi} \arccos r\right) \varpi(\chi) \right] \\ & = \left(\frac{2}{\pi} \arcsin r\right) (c\varpi)(\phi) + \left(\frac{2}{\pi} \arccos r\right) (c\varpi)(\chi). \end{aligned}$$

□

Theorem 6. Let $\varpi, \omega: I \rightarrow \mathbb{R}$ are both nonnegative and monotone increasing. If ϖ and ω are inverse trigonometrically convex functions, then $\varpi\omega$ is inverse trigonometrically convex function.

Proof. If $\phi \leq \chi$, then $[\varpi(\phi) - \varpi(\chi)][\omega(\chi) - \omega(\phi)] \leq 0$ which implies

$$\varpi(\phi)\omega(\chi) + \varpi(\chi)\omega(\phi) \leq \varpi(\phi)\omega(\phi) + \varpi(\chi)\omega(\chi). \quad (2.1)$$

On the other hand for $\phi, \chi \in I$ and $r \in [0, 1]$,

$$\begin{aligned} & (\varpi\omega)(r\phi + (1-r)\chi) \\ & \leq \left[\left(\frac{2}{\pi} \arcsin r \right) \varpi(\phi) + \left(\frac{2}{\pi} \arccos r \right) \varpi(\chi) \right] \\ & \quad \times \left[\left(\frac{2}{\pi} \arcsin r \right) \omega(\phi) + \left(\frac{2}{\pi} \arccos r \right) \omega(\chi) \right] \\ & = \frac{4}{\pi^2} (\arcsin^2 r) \varpi(\phi)\omega(\phi) + \frac{4}{\pi^2} (\arcsin r)(\arccos r) [\varpi(\phi)\omega(\chi) + \varpi(\chi)\omega(\phi)] \\ & \quad + \frac{4}{\pi^2} (\arccos^2 r) \varpi(\chi)\omega(\chi). \end{aligned}$$

Using (2.1) in the above inequality, get

$$\begin{aligned} & (\varpi\omega)(r\phi + (1-r)\chi) \\ & \leq \frac{4}{\pi^2} (\arcsin^2 r) \varpi(\phi)\omega(\phi) + \frac{4}{\pi^2} (\arcsin r)(\arccos r) [\varpi(\phi)\omega(\chi) + \varpi(\chi)\omega(\phi)] \\ & \quad + \frac{4}{\pi^2} (\arccos^2 r) \varpi(\chi)\omega(\chi). \end{aligned}$$

Since $\arcsin r + \arccos r = \frac{\pi}{2}$, we get

$$\begin{aligned} (\varpi\omega)(r\phi + (1-r)\chi) & \leq \left(\frac{2}{\pi} \arcsin r \right) \varpi(\phi)\omega(\phi) + \left(\frac{2}{\pi} \arccos r \right) \varpi(\chi)\omega(\chi) \\ & = \left(\frac{2}{\pi} \arcsin r \right) (\varpi\omega)(\phi) + \left(\frac{2}{\pi} \arccos r \right) (\varpi\omega)(\chi). \end{aligned}$$

□

Theorem 7. If $\varpi: I \rightarrow J$ is convex and $\omega: J \rightarrow \mathbb{R}$ is inverse trigonometrically convex function and nondecreasing, then $\omega \circ \varpi: I \rightarrow \mathbb{R}$ is an inverse trigonometrically convex function.

Proof. For $\phi, \chi \in I$ and $r \in [0, 1]$, we get

$$\begin{aligned} (\omega \circ \varpi)(r\phi + (1-r)\chi) & \leq \left(\frac{2}{\pi} \arcsin r \right) \omega(\varpi(\phi)) + \left(\frac{2}{\pi} \arccos r \right) \omega(\varpi(\chi)) \\ & = \left(\frac{2}{\pi} \arcsin r \right) (\omega \circ \varpi)(\phi) + \left(\frac{2}{\pi} \arccos r \right) (\omega \circ \varpi)(\chi). \end{aligned}$$

□

Theorem 8. If $\varpi: I \rightarrow J$ is an inverse trigonometrically convex and $\omega: J \rightarrow \mathbb{R}$ is a nonnegative, nondecreasing and trigonometrically convex function, then $\omega \circ \varpi: I \rightarrow \mathbb{R}$ is an P -function.

Proof. For $\phi, \chi \in I$ and $r \in [0, 1]$, we get

$$\begin{aligned} (\omega \circ \varpi)(r\phi + (1-r)\chi) &\leq \omega\left(\left(\frac{2}{\pi} \arcsin r\right) \varpi(\phi) + \left(\frac{2}{\pi} \arccos r\right) \varpi(\chi)\right) \\ &\leq \sin \frac{\pi}{2} \left(\frac{2}{\pi} \arcsin r\right) \omega(\varpi(\phi)) \\ &\quad + \cos \frac{\pi}{2} \left(\frac{2}{\pi} \arccos r\right) \omega(\varpi(\chi)) \\ &= r[(\omega \circ \varpi)(\phi) + (\omega \circ \varpi)(\chi)] \leq [(\omega \circ \varpi)(\phi) + (\omega \circ \varpi)(\chi)]. \end{aligned}$$

□

Theorem 9. Let $\chi > 0$ and $\varpi_\delta: [\phi, \chi] \rightarrow \mathbb{R}$ be an arbitrary family of inverse trigonometrically convex function and let $\varpi(x) = \sup_\delta \varpi_\delta(x)$. If $J = \{u \in [\phi, \chi] : \varpi(u) < \infty\}$ is nonempty, then J is an interval and ϖ is an inverse trigonometrically convex function on J .

Proof. Let $r \in [0, 1]$ and $\phi, \chi \in J$ be arbitrary. Then

$$\begin{aligned} \varpi(r\phi + (1-r)\chi) &\leq \sup_\delta \left[\left(\frac{2}{\pi} \arcsin r\right) \varpi_\delta(\phi) + \left(\frac{2}{\pi} \arccos r\right) \varpi_\delta(\chi) \right] \\ &\leq \left(\frac{2}{\pi} \arcsin r\right) \varpi(\phi) + \left(\frac{2}{\pi} \arccos r\right) \varpi(\chi) < \infty. \end{aligned}$$

This shows simultaneously that J is an interval, since it contains every point between any two of its points, and that ϖ is an inverse trigonometrically convex function on J . □

3. HERMITE-HADAMARD INEQUALITY FOR INVERSE TRIGONOMETRICALLY CONVEX FUNCTIONS

The goal of this paper is to establish some inequalities of Hermite-Hadamard type for inverse trigonometrically convex functions. In this section, we will denote by $L[\phi, \chi]$ the space of (Lebesgue) integrable functions on $[\phi, \chi]$.

Theorem 10. Let $\varpi: [\phi, \chi] \rightarrow \mathbb{R}$ be an inverse trigonometrically convex function. If $\phi < \chi$ and $\varpi \in L[\phi, \chi]$, then

$$\frac{1}{\chi - \phi} \int_\phi^\chi \varpi(x) dx \leq \left(1 - \frac{2}{\pi}\right) \varpi(\phi) + \frac{2}{\pi} \varpi(\chi).$$

Proof. By using the property of the inverse trigonometrically convex function of the function ϖ , if the variable is changed as $u = r\phi + (1-r)\chi$, then

$$\begin{aligned} \frac{1}{\chi - \phi} \int_{\phi}^{\chi} \varpi(x) dx &= \int_0^1 \varpi(r\phi + (1-r)\chi) dr \\ &\leq \int_0^1 \left\{ \left(\frac{2}{\pi} \arcsin r \right) \varpi(\phi) + \left(\frac{2}{\pi} \arccos r \right) \varpi(\chi) \right\} dr \\ &= \left(1 - \frac{2}{\pi} \right) \varpi(\phi) + \frac{2}{\pi} \varpi(\chi). \end{aligned}$$

□

Theorem 11. Let the function $\varpi: [\phi, \chi] \rightarrow \mathbb{R}$, be an inverse trigonometrically convex function. If $\phi < \chi$ and $\varpi \in L[\phi, \chi]$, then

$$\varpi\left(\frac{\phi + \chi}{2}\right) \leq \frac{1}{\chi - \phi} \int_{\phi}^{\chi} \varpi(x) dx.$$

Proof. From the property of the inverse trigonometrically convex function of ϖ , we get

$$\begin{aligned} \varpi\left(\frac{\phi + \chi}{2}\right) &= \varpi\left(\frac{[r\phi + (1-r)\chi] + [(1-r)\phi + r\chi]}{2}\right) \\ &= \varpi\left(\frac{1}{2}[r\phi + (1-r)\chi] + \frac{1}{2}[(1-r)\phi + r\chi]\right) \\ &\leq \left(\frac{2}{\pi} \arcsin \frac{1}{2}\right) \varpi(r\phi + (1-r)\chi) + \left(\frac{2}{\pi} \arccos \frac{1}{2}\right) \varpi((1-r)\phi + r\chi) \\ &= \frac{\varpi(r\phi + (1-r)\chi)}{3} + \frac{2\varpi((1-r)\phi + r\chi)}{3}. \end{aligned}$$

Now, if we take integral in the last inequality with respect to $r \in [0, 1]$, we deduce that

$$\begin{aligned} \varpi\left(\frac{\phi + \chi}{2}\right) &\leq \left[\frac{1}{3} \int_0^1 \varpi(r\phi + (1-r)\chi) dr + \frac{2}{3} \int_0^1 \varpi((1-r)\phi + r\chi) dr \right] \\ &= \frac{1}{\chi - \phi} \int_{\phi}^{\chi} \varpi(x) dx. \end{aligned}$$

□

4. BETTER APPROXIMATIONS FOR INVERSE TRIGONOMETRICALLY CONVEX FUNCTIONS

The main purpose of this section is to establish new estimates that refine Hermite-Hadamard inequality for functions whose first derivative in absolute value, raised to

a certain power which is greater than one, respectively at least one, is inverse trigonometrically convex function. Moreover, we proved that Hölder-İşcan and improved power-mean integral inequalities give a better approach than Hölder and power-mean inequalities. We will use the following lemma.

Lemma 1 ([1, page 91, Lemma 2.1]). *Let $\varpi: I^\circ \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° , $\phi, \chi \in I^\circ$ with $\phi < \chi$. If $\varpi' \in L[\phi, \chi]$, then*

$$\frac{\varpi(\phi) + \varpi(\chi)}{2} - \frac{1}{\chi - \phi} \int_{\phi}^{\chi} \varpi(x) dx = \frac{\chi - \phi}{2} \int_0^1 (1 - 2r) \varpi'(r\phi + (1 - r)\chi) dr.$$

Theorem 12. *Let the function $\varpi: I \rightarrow \mathbb{R}$ be a differentiable on I° , $\phi, \chi \in I^\circ$ with $\phi < \chi$ and assume that $\varpi' \in L[\phi, \chi]$. If $|\varpi'|$ is an inverse trigonometrically convex function on interval $[\phi, \chi]$, then*

$$\begin{aligned} & \left| \frac{\varpi(\phi) + \varpi(\chi)}{2} - \frac{1}{\chi - \phi} \int_{\phi}^{\chi} \varpi(x) dx \right| \\ & \leq \frac{\chi - \phi}{\pi} \left[\left(\frac{3\sqrt{3}}{4} - 1 \right) |\varpi'(\phi)| + \left(\frac{\pi}{4} - \frac{3\sqrt{3}}{4} + 1 \right) |\varpi'(\chi)| \right] \end{aligned}$$

holds for $r \in [0, 1]$.

Proof. Using Lemma 1 and the inequality

$$|\varpi'(r\phi + (1 - r)\chi)| \leq \left(\frac{2}{\pi} \arcsin r \right) |\varpi'(\phi)| + \left(\frac{2}{\pi} \arccos r \right) |\varpi'(\chi)|,$$

we get

$$\begin{aligned} & \left| \frac{\varpi(\phi) + \varpi(\chi)}{2} - \frac{1}{\chi - \phi} \int_{\phi}^{\chi} \varpi(x) dx \right| \\ & \leq \frac{\chi - \phi}{2} \int_0^1 |1 - 2r| |\varpi'(r\phi + (1 - r)\chi)| dr \\ & \leq \frac{\chi - \phi}{2} \int_0^1 |1 - 2r| \left[\left(\frac{2}{\pi} \arcsin r \right) |\varpi'(\phi)| + \left(\frac{2}{\pi} \arccos r \right) |\varpi'(\chi)| \right] dr \\ & = \frac{\chi - \phi}{\pi} \left[\left(\frac{3\sqrt{3}}{4} - 1 \right) |\varpi'(\phi)| + \left(\frac{\pi}{4} - \frac{3\sqrt{3}}{4} + 1 \right) |\varpi'(\chi)| \right], \end{aligned}$$

where

$$\begin{aligned} \int_0^1 |1 - 2r| \arcsin r dr &= \frac{3\sqrt{3}}{4} - 1, \\ \int_0^1 |1 - 2r| \arccos r dr &= \frac{\pi}{4} - \frac{3\sqrt{3}}{4} + 1. \end{aligned}$$

□

Theorem 13. Let the function $\varpi: I \rightarrow \mathbb{R}$ be a differentiable on I° , $\phi, \chi \in I^\circ$ with $\phi < \chi$, $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and assume that $\varpi' \in L[\phi, \chi]$. If $|\varpi'|^q$ is an inverse trigonometrically convex function on interval $[\phi, \chi]$, then

$$\begin{aligned} & \left| \frac{\varpi(\phi) + \varpi(\chi)}{2} - \frac{1}{\chi - \phi} \int_{\phi}^{\chi} \varpi(x) dx \right| \\ & \leq \frac{\chi - \phi}{2} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left[\frac{(\pi - 2) |\varpi'(\phi)|^q + 2 |\varpi'(\chi)|^q}{\pi} \right]^{\frac{1}{q}} \end{aligned} \quad (4.1)$$

holds for $r \in [0, 1]$.

Proof. By considering Lemma 1, Hölder's integral inequality and the following inequality

$$|\varpi'(r\phi + (1-r)\chi)|^q \leq \left(\frac{2}{\pi} \arcsin r \right) |\varpi'(\phi)|^q + \left(\frac{2}{\pi} \arccos r \right) |\varpi'(\chi)|^q,$$

which is the inverse trigonometrically convex function of $|\varpi'|^q$, we get

$$\begin{aligned} & \left| \frac{\varpi(\phi) + \varpi(\chi)}{2} - \frac{1}{\chi - \phi} \int_{\phi}^{\chi} \varpi(x) dx \right| \\ & \leq \frac{\chi - \phi}{2} \left(\int_0^1 |1 - 2r|^p dr \right)^{\frac{1}{p}} \left(\int_0^1 |\varpi'(r\phi + (1-r)\chi)|^q dr \right)^{\frac{1}{q}} \\ & \leq \frac{\chi - \phi}{2} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left(\int_0^1 \left[\left(\frac{2}{\pi} \arcsin r \right) |\varpi'(\phi)|^q + \left(\frac{2}{\pi} \arccos r \right) |\varpi'(\chi)|^q \right] dr \right)^{\frac{1}{q}} \\ & = \frac{\chi - \phi}{2} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left(\frac{2 |\varpi'(\phi)|^q}{\pi} \int_0^1 \arcsin r dr + \frac{2 |\varpi'(\chi)|^q}{\pi} \int_0^1 \arccos r dr \right)^{\frac{1}{q}} \\ & = \frac{\chi - \phi}{2} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left[\frac{(\pi - 2) |\varpi'(\phi)|^q + 2 |\varpi'(\chi)|^q}{\pi} \right]^{\frac{1}{q}}, \end{aligned}$$

where

$$\begin{aligned} \int_0^1 |1 - 2r|^p dr &= \frac{1}{p+1}, \\ \int_0^1 \arcsin r dr &= \frac{\pi}{2} - 1, \quad \int_0^1 \arccos r dr = 1. \end{aligned}$$

□

Theorem 14. Let the function $\varpi: I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable on I° , $\phi, \chi \in I^\circ$ with $\phi < \chi$, $q \geq 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and assume that $\varpi' \in L[\phi, \chi]$. If $|\varpi'|^q$ is an inverse

trigonometrically convex function on the interval $[\phi, \chi]$, then

$$\begin{aligned} & \left| \frac{\varpi(\phi) + \varpi(\chi)}{2} - \frac{1}{\chi - \phi} \int_{\phi}^{\chi} \varpi(x) dx \right| \\ & \leq \frac{\chi - \phi}{2^{2-\frac{1}{q}}} \left[\frac{2}{\pi} |\varpi'(\phi)|^q \left(\frac{3\sqrt{3}}{4} - 1 \right) + \frac{2}{\pi} |\varpi'(\chi)|^q \left(\frac{\pi}{4} - \frac{3\sqrt{3}}{4} + 1 \right) \right]^{\frac{1}{q}} \end{aligned} \quad (4.2)$$

holds for $r \in [0, 1]$.

Proof. Assume first that $q > 1$. By considering Lemma 1, Hölder integral inequality and the property of the inverse trigonometrically convex function of $|\varpi'|^q$, we obtain

$$\begin{aligned} & \left| \frac{\varpi(\phi) + \varpi(\chi)}{2} - \frac{1}{\chi - \phi} \int_{\phi}^{\chi} \varpi(x) dx \right| \\ & \leq \frac{\chi - \phi}{2} \left(\int_0^1 |1 - 2r| dr \right)^{1-\frac{1}{q}} \left(\int_0^1 |1 - 2r| |\varpi'((r\phi + (1-r)\chi))|^q dr \right)^{\frac{1}{q}} \\ & = \frac{\chi - \phi}{2} \left(\frac{1}{2} \right)^{1-\frac{1}{q}} \left(\int_0^1 |1 - 2r| \left[\left(\frac{2\arcsin r}{\pi} \right) |\varpi'(\phi)|^q + \left(\frac{2\arccos r}{\pi} \right) |\varpi'(\chi)|^q \right] dr \right)^{\frac{1}{q}} \\ & = \frac{\chi - \phi}{2} \left(\frac{1}{2} \right)^{1-\frac{1}{q}} \left[\frac{2|\varpi'(\phi)|^q}{\pi} \left(\frac{3\sqrt{3}}{4} - 1 \right) + \frac{2|\varpi'(\chi)|^q}{\pi} \left(\frac{\pi}{4} - \frac{3\sqrt{3}}{4} + 1 \right) \right]^{\frac{1}{q}} \\ & = \frac{\chi - \phi}{2^{2-\frac{1}{q}}} \left[\frac{2|\varpi'(\phi)|^q}{\pi} \left(\frac{3\sqrt{3}}{4} - 1 \right) + \frac{2|\varpi'(\chi)|^q}{\pi} \left(\frac{\pi}{4} - \frac{3\sqrt{3}}{4} + 1 \right) \right]^{\frac{1}{q}}. \end{aligned}$$

So, the desired result are obtained. For $q = 1$ we use the estimates from the proof of Theorem 12, which also follow step by step the above estimates. $\square$

Corollary 1. Under the assumption of Theorem 14 with $q = 1$, we get the conclusion of Theorem 12.

Theorem 15. Let the function $\varpi: I \rightarrow \mathbb{R}$ be a differentiable on I° , $\phi, \chi \in I^\circ$ with $\phi < \chi$, $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and assume that $\varpi' \in L[\phi, \chi]$. If $|\varpi'|^q$ is an inverse trigonometrically convex function on interval $[\phi, \chi]$, then

$$\begin{aligned} & \left| \frac{\varpi(\phi) + \varpi(\chi)}{2} - \frac{1}{\chi - \phi} \int_{\phi}^{\chi} \varpi(x) dx \right| \\ & \leq \frac{\chi - \phi}{2} \left(\frac{1}{2(p+1)} \right)^{\frac{1}{p}} \left(\frac{2|\varpi'(\phi)|^q}{\pi} \left[\frac{3\pi}{8} - 1 \right] + \frac{2|\varpi'(\chi)|^q}{\pi} \left[1 - \frac{\pi}{8} \right] \right)^{\frac{1}{q}} \\ & \quad + \frac{\chi - \phi}{2} \left(\frac{1}{2(p+1)} \right)^{\frac{1}{p}} \left(\frac{2|\varpi'(\phi)|^q}{\pi} \frac{\pi}{8} + \frac{2|\varpi'(\chi)|^q}{\pi} \frac{\pi}{8} \right)^{\frac{1}{q}}, \end{aligned} \quad (4.3)$$

holds for $r \in [0, 1]$.

Proof. By considering Lemma 1, Hölder-İşcan integral inequality and the property of the inverse trigonometrically convex function of $|\varpi'|^q$, we get

$$\begin{aligned} & \left| \frac{\varpi(\phi) + \varpi(\chi)}{2} - \frac{1}{\chi - \phi} \int_{\phi}^{\chi} \varpi(x) dx \right| \\ & \leq \frac{\chi - \phi}{2} \left(\int_0^1 (1-r) |1-2r|^p dr \right)^{\frac{1}{p}} \left(\int_0^1 (1-r) |\varpi'(r\phi + (1-r)\chi)|^q dr \right)^{\frac{1}{q}} \\ & \quad + \frac{\chi - \phi}{2} \left(\int_0^1 r |1-2r|^p dr \right)^{\frac{1}{p}} \left(\int_0^1 r |\varpi'(r\phi + (1-r)\chi)|^q dr \right)^{\frac{1}{q}} \\ & \leq \frac{\chi - \phi}{2} \left(\int_0^1 (1-r) |1-2r|^p dr \right)^{\frac{1}{p}} \\ & \quad \times \left(\int_0^1 (1-r) \left[\left(\frac{2}{\pi} \arcsin r \right) |\varpi'(\phi)|^q + \left(\frac{2}{\pi} \arccos r \right) |\varpi'(\chi)|^q \right] dr \right)^{\frac{1}{q}} \\ & \quad + \frac{\chi - \phi}{2} \left(\int_0^1 r |1-2r|^p dr \right)^{\frac{1}{p}} \\ & \quad \times \left(\int_0^1 r \left[\left(\frac{2}{\pi} \arcsin r \right) |\varpi'(\phi)|^q + \left(\frac{2}{\pi} \arccos r \right) |\varpi'(\chi)|^q \right] dr \right)^{\frac{1}{q}} \\ & = \frac{\chi - \phi}{2} \left(\frac{1}{2(p+1)} \right)^{\frac{1}{p}} \\ & \quad \times \left(\frac{2|\varpi'(\phi)|^q}{\pi} \int_0^1 (1-r) \arcsin r dr + \frac{2|\varpi'(\chi)|^q}{\pi} \int_0^1 (1-r) \arccos r dr \right)^{\frac{1}{q}} \\ & \quad + \frac{\chi - \phi}{2} \left(\frac{1}{2(p+1)} \right)^{\frac{1}{p}} \\ & \quad \times \left(\frac{2|\varpi'(\phi)|^q}{\pi} \int_0^1 r \arcsin r dr + \frac{2|\varpi'(\chi)|^q}{\pi} \int_0^1 r \arccos r dr \right)^{\frac{1}{q}} \\ & \quad + \frac{\chi - \phi}{2} \left(\frac{1}{2(p+1)} \right)^{\frac{1}{p}} \left(\frac{2|\varpi'(\phi)|^q}{\pi} \frac{\pi}{8} + \frac{2|\varpi'(\chi)|^q}{\pi} \frac{\pi}{8} \right)^{\frac{1}{q}}. \end{aligned}$$

□

Remark 2. The inequality (4.3) is better than the inequality (4.1).

Proof. By using concavity of the function

$$h: [0, \infty) \rightarrow \mathbb{R}, \quad h(x) = x^s, \quad 0 < s \leq 1,$$

we can write the right hand-side of the inequality (4.3) as follow:

$$\begin{aligned}
 & \frac{\chi - \phi}{2} \left(\frac{1}{2(p+1)} \right)^{\frac{1}{p}} \left(\frac{2|\varpi'(\phi)|^q}{\pi} \left[\frac{3\pi}{8} - 1 \right] + \frac{2|\varpi'(\chi)|^q}{\pi} \left[1 - \frac{\pi}{8} \right] \right)^{\frac{1}{q}} \\
 & + \frac{\chi - \phi}{2} \left(\frac{1}{2(p+1)} \right)^{\frac{1}{p}} \left(\frac{2|\varpi'(\phi)|^q}{\pi} \frac{\pi}{8} + \frac{2|\varpi'(\chi)|^q}{\pi} \frac{\pi}{8} \right)^{\frac{1}{q}} \\
 & \leq 2 \frac{\chi - \phi}{2} \left(\frac{1}{2(p+1)} \right)^{\frac{1}{p}} \left[\frac{1}{2} \left(\frac{(\pi-2)|\varpi'(\phi)|^q}{\pi} + \frac{2|\varpi'(\chi)|^q}{\pi} \right) \right] \\
 & = \frac{\chi - \phi}{2} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left[\frac{(\pi-2)|\varpi'(\phi)|^q}{\pi} + \frac{2|\varpi'(\chi)|^q}{\pi} \right].
 \end{aligned}$$

The last statement is the right hand side of (4.3). This completes the proof of Remark. $\square$

Theorem 16. Let the function $\varpi: I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable on I° , $\phi, \chi \in I^\circ$ with $\phi < \chi$, $q \geq 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and assume that $\varpi' \in L[\phi, \chi]$. If $|\varpi'|^q$ is an inverse trigonometrically convex function on the interval $[\phi, \chi]$, then

$$\begin{aligned}
 & \left| \frac{\varpi(\phi) + \varpi(\chi)}{2} - \frac{1}{\chi - \phi} \int_{\phi}^{\chi} \varpi(x) dx \right| \leq \frac{\chi - \phi}{2} \left(\frac{1}{4} \right)^{1 - \frac{1}{q}} \\
 & \times \left\{ \left(\frac{2|\varpi'(\phi)|^q}{\pi} \left[-\frac{10\pi - 81\sqrt{3} + 104}{72} \right] + \frac{2|\varpi'(\chi)|^q}{\pi} \left[\frac{19\pi - 81\sqrt{3} + 104}{72} \right] \right)^{\frac{1}{q}} \right. \\
 & \left. + \left(\frac{2|\varpi'(\phi)|^q}{\pi} \left[\frac{10\pi - 27\sqrt{3} + 32}{72} \right] + \frac{2|\varpi'(\chi)|^q}{\pi} \left[-\frac{\pi - 27\sqrt{3} + 32}{72} \right] \right)^{\frac{1}{q}} \right\},
 \end{aligned} \tag{4.4}$$

holds for $r \in [0, 1]$.

Proof. Assume first that $q > 1$. By considering Lemma 1, improved power-mean integral inequality and the property of the inverse trigonometrically convex function

of $|\varpi'|^q$, we obtain

$$\begin{aligned}
& \left| \frac{\varpi(\phi) + \varpi(\chi)}{2} - \frac{1}{\chi - \phi} \int_{\phi}^{\chi} \varpi(x) dx \right| \\
& \leq \frac{\chi - \phi}{2} \left(\int_0^1 (1-r) |1-2r| dr \right)^{1-\frac{1}{q}} \\
& \quad \times \left(\int_0^1 (1-r) |1-2r| |\varpi'((r\phi + (1-r)\chi))|^q dr \right)^{\frac{1}{q}} \\
& \quad + \frac{\chi - \phi}{2} \left(\int_0^1 r |1-2r| dr \right)^{1-\frac{1}{q}} \left(\int_0^1 r |1-2r| |\varpi'((r\phi + (1-r)\chi))|^q dr \right)^{\frac{1}{q}} \\
& \leq \frac{\chi - \phi}{2} \left(\int_0^1 (1-r) |1-2r| dr \right)^{1-\frac{1}{q}} \\
& \quad \times \left(\frac{2|\varpi'(\phi)|^q}{\pi} \int_0^1 (1-r) |1-2r| \arcsin r dr \right. \\
& \quad \left. + \frac{2|\varpi'(\chi)|^q}{\pi} \int_0^1 (1-r) |1-2r| \arccos r dr \right)^{\frac{1}{q}} + \frac{\chi - \phi}{2} \left(\int_0^1 r |1-2r| dr \right)^{1-\frac{1}{q}} \\
& \quad \times \left(\frac{2|\varpi'(\phi)|^q}{\pi} \int_0^1 r |1-2r| \arcsin r dr + \frac{2|\varpi'(\chi)|^q}{\pi} \int_0^1 r |1-2r| \arccos r dr \right)^{\frac{1}{q}} \\
& = \frac{\chi - \phi}{2} \left(\frac{1}{4} \right)^{1-\frac{1}{q}} \left\{ \left(\frac{2|\varpi'(\phi)|^q}{\pi} \left[-\frac{10\pi - 81\sqrt{3} + 104}{72} \right] + \frac{2|\varpi'(\chi)|^q}{\pi} \left[\frac{19\pi - 81\sqrt{3} + 104}{72} \right] \right)^{\frac{1}{q}} \right. \\
& \quad \left. + \left(\frac{2|\varpi'(\phi)|^q}{\pi} \left[\frac{10\pi - 27\sqrt{3} + 32}{72} \right] + \frac{2|\varpi'(\chi)|^q}{\pi} \left[-\frac{\pi - 27\sqrt{3} + 32}{72} \right] \right)^{\frac{1}{q}} \right\},
\end{aligned}$$

where

$$\begin{aligned}
\int_0^1 (1-r) |1-2r| \arcsin r dr &= -\frac{10\pi - 81\sqrt{3} + 104}{72}, \\
\int_0^1 (1-r) |1-2r| \arccos r dr &= \frac{19\pi - 81\sqrt{3} + 104}{72}, \\
\int_0^1 r |1-2r| \arcsin r dr &= \frac{10\pi - 27\sqrt{3} + 32}{72}, \\
\int_0^1 r |1-2r| \arccos r dr &= -\frac{\pi - 27\sqrt{3} + 32}{72}.
\end{aligned}$$

□

Corollary 2. Under the assumption of Theorem 16 with $q = 1$, we get the following inequality:

$$\begin{aligned} & \left| \frac{\varpi(\phi) + \varpi(\chi)}{2} - \frac{1}{\chi - \phi} \int_{\phi}^{\chi} \varpi(x) dx \right| \\ & \leq \frac{\chi - \phi}{2} \left[|\varpi'(\phi)| \frac{54\sqrt{3} - 72}{36\pi} + |\varpi'(\chi)| \frac{18\pi - 54\sqrt{3} + 72}{36\pi} \right]. \end{aligned}$$

Remark 3. The inequality (4.4) gives better approximation than the inequality (4.2). Really, we can write the right hand side of the inequality (4.4) as

$$\begin{aligned} & \frac{\chi - \phi}{2} \left(\frac{1}{4} \right)^{1 - \frac{1}{q}} \left\{ \left(\frac{2|\varpi'(\phi)|^q}{\pi} \left[-\frac{10\pi - 81\sqrt{3} + 104}{72} \right] + \frac{2|\varpi'(\chi)|^q}{\pi} \left[\frac{19\pi - 81\sqrt{3} + 104}{72} \right] \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left(\frac{2|\varpi'(\phi)|^q}{\pi} \left[\frac{10\pi - 27\sqrt{3} + 32}{72} \right] + \frac{2|\varpi'(\chi)|^q}{\pi} \left[-\frac{\pi - 27\sqrt{3} + 32}{72} \right] \right)^{\frac{1}{q}} \right\} \\ & \leq \frac{\chi - \phi}{2^{2 - \frac{1}{q}}} \left[\frac{|\varpi'(\phi)|^q}{\pi} \left(\frac{3\sqrt{3}}{4} - 1 \right) + \frac{|\varpi'(\chi)|^q}{\pi} \left(\frac{\pi}{4} - \frac{3\sqrt{3}}{4} + 1 \right) \right]^{\frac{1}{q}}. \end{aligned}$$

The last inequality is the right hand side of the inequality (4.2). Therefore, this shows that (4.4) is better than (4.2). This completes the proof of Remark.

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B –MAXIMAL AND B –SINGULAR INTEGRAL OPERATORS IN B –LOCAL MORREY-LORENTZ SPACES

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Abstract. In this article, the boundedness of B –maximal M_γ , B –singular integral T_γ and B –maximal singular integral $\mathcal{T}_\gamma$ operators on B –local Morrey-Lorentz spaces are obtained with the use of rearrangement inequalities and Hardy inequality.

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1. INTRODUCTION

Lorentz spaces have first been introduced by G. G. Lorentz and Lorentz spaces are very useful in the theory of interpolation. These spaces are Banach spaces and generalizations of Lebesgue spaces. The Lorentz space $L_{p,q}(\mathbb{R}^n)$ is known as the set all of measurable functions f such that

$$\|f\|_{L_{p,q}(\mathbb{R}^n)} := \|t^{\frac{1}{p}-\frac{1}{q}} f^*(t)\|_{L_q(0,\infty)} < \infty, \quad 0 < p, q \leq \infty.$$

Here f^* denotes non-increasing rearrangement of f and defined by

$$f^*(t) = \inf \{ \lambda > 0 : |\{y \in \mathbb{R}^n : |f(y)| > \lambda\}| \leq t \}, \quad t \in (0, \infty).$$

Functional $\|\cdot\|_{L_{p,q}}$ is a norm if and only if $1 \leq q \leq p$ or $p = q = \infty$. If $p = q = \infty$, then $L_{\infty,\infty}(\mathbb{R}^n) \equiv L_\infty(\mathbb{R}^n)$. One can easily see that $L_{p,p}(\mathbb{R}^n) \equiv L_p(\mathbb{R}^n)$ and $L_{p,\infty}(\mathbb{R}^n) \equiv WL_p(\mathbb{R}^n)$. It is clear that $L_{p,q} \subset L_p \subset L_{p,r} \subset WL_p$ for $0 < q \leq p \leq q \leq r \leq \infty$. For more details, we refer to [8, 22, 23].

On Lorentz spaces, boundedness problem of classical singular integral operators and singular integral operators related to Laplace-Bessel differential operator

$$\Delta_B := \sum_{i=1}^k \frac{\partial^2}{\partial x_i^2} + \frac{\gamma_i}{x_i} \frac{\partial}{\partial x_i} + \sum_{i=k+1}^n \frac{\partial^2}{\partial x_i^2} \quad 1 \leq k \leq n,$$

have been studied by many researchers [1, 3, 5, 15–19, 22, 23, 26, 29, 30]. Also, boundedness of singular integral operators related to Laplace-Bessel differential operator defined on variable exponent Lorentz spaces have been obtained by Aykol and Kaya [7]. Convolution type operators are generated by generalized shift operator

$$T^\gamma f(x) := C_{\gamma,k} \int_0^\pi \dots \int_0^\pi f[(x_1, y_1)_{\alpha_1}, \dots, (x_k, y_k)_{\alpha_k}, x'' - y''] d\gamma(\alpha),$$

where $C_{\gamma,k} = \pi^{-\frac{k}{2}} \Gamma(\frac{\gamma+1}{2}) [\Gamma(\frac{\gamma}{2})]^{-1}$, $(x_i, y_i)_{\alpha_i} = (x_i^2 - 2x_i y_i \cos \alpha_i + y_i^2)^{\frac{1}{2}}$, $1 \leq i \leq k$, $1 \leq k \leq n$, and $d\gamma(\alpha) = \prod_{i=1}^k \sin^{\gamma_i-1} \alpha_i d\alpha_i$ ([20, 21]).

The B -convolution operator is defined as:

$$(f \otimes g)(x) = \int_{\mathbb{R}_{k,+}^n} f(y) T^\gamma g(x)(y')^\gamma dy.$$

Here, $\mathbb{R}_{k,+}^n = \{x \in \mathbb{R}^n : x_1 > 0, \dots, x_k > 0, 1 \leq k \leq n\}$ denotes the part of the Euclidean space $\mathbb{R}^n$ and $\gamma = (\gamma_1, \dots, \gamma_k)$, $\gamma_1 > 0, \dots, \gamma_k > 0$, $|\gamma| = \gamma_1 + \dots + \gamma_k$. Denote $x = (x', x'')$, $x' = (x_1, \dots, x_k) \in \mathbb{R}^k$, and $x'' = (x_{k+1}, \dots, x_n) \in \mathbb{R}^{n-k}$.

$M_{p,q,\lambda}^{\text{loc}}(\mathbb{R}^n)$ local Morrey-Lorentz spaces have first been introduced by Aykol et al. ([4]) with a finite quasi-norm

$$\|f\|_{M_{p,q,\lambda}^{\text{loc}}(\mathbb{R}^n)} = \sup_{r>0} r^{-\frac{\lambda}{q}} \left\| t^{\frac{1}{p}-\frac{1}{q}} f^*(t) \right\|_{L_q(0,r)}.$$

These spaces are generalizations of Lorentz spaces and $M_{p,q,0}^{\text{loc}}(\mathbb{R}^n) = L_{p,q}(\mathbb{R}^n)$. The boundedness of maximal operator in $M_{p,q,\lambda}^{\text{loc}}(\mathbb{R}^n)$ spaces have been obtained in [4]. Then, in [14], authors have proved that Hardy-Littlewood maximal, Calderón-Zygmund and maximal Calderón-Zygmund operators are bounded in these spaces. Also, in [6], Aykol et al. have shown that Hilbert transform is bounded in local and weak local Morrey-Lorentz spaces. On the other hand, generalized versions of these operators with the Laplace-Bessel differential operator have been studied in various function spaces by many mathematicians [2, 3, 5, 11, 15–18, 20, 21, 24, 25]. The above results inspire us to investigate the boundedness of B -maximal, B -singular integral and B -maximal singular integral operator defined on B -local Morrey-Lorentz spaces. Maximal operators have a crucial role in PDEs, singular integrals and the differentiability properties of functions. In the present study, we consider B -maximal operator defined by (see [13])

$$M_\gamma f(x) = \sup_{r>0} |B_+(0, r)|_\gamma^{-1} \int_{B_+(0, r)} T^\gamma |f(x)| (y')^\gamma dy,$$

where $B_+(x, r) = \{y \in \mathbb{R}_{k,+}^n : |x - y| < r\}$. For a measurable set $B_+(0, r) \subset \mathbb{R}_{k,+}^n$, we have

$$|B_+(0, r)|_\gamma = \int_{B_+(0, r)} (x')^\gamma dx = \omega(n, k, \gamma) r^\mathcal{Q},$$

where $\omega(n, k, \gamma) = \frac{\pi^{\frac{n-k}{2}}}{2^k} \prod_{i=1}^k \frac{\Gamma\left(\frac{\gamma_i+1}{2}\right)}{\left(\frac{\gamma_i}{2}\right)}$, $Q = n + |\gamma|$.

B-singular integral operator is defined as

$$\begin{aligned} T_\gamma f(x) &= p.v. \int_{\mathbb{R}_{k,+}^n} \frac{\Omega(\theta)}{|y|^Q} [T^y f(x)](y')^\gamma dy \\ &= \lim_{\varepsilon \rightarrow 0} \int_{\{y \in \mathbb{R}_{k,+}^n : |y| > \varepsilon\}} \frac{\Omega(\theta)}{|y|^Q} [T^y f(x)](y')^\gamma dy = \lim_{\varepsilon \rightarrow 0} T_{\gamma,\varepsilon} f(x), \end{aligned} \quad (1.1)$$

where $\theta = y/|y|$, and $\Omega(\theta)$ belong to some function spaces on the semi-sphere $S_{k,+} = \{x \in \mathbb{R}_{k,+}^n : |x| = 1\}$ and also satisfy the "cancellation" condition with $d\sigma(\theta)$ is the area element of the sphere $|\theta| = 1$,

$$\int_{S_{k,+}} \Omega(\theta)(\theta')^\gamma d\sigma(\theta) = 0.$$

B-singular integral operator is a convolution type operator, where the kernel of this operator is $K(y) = \frac{\Omega(\theta)}{|y|^Q}$ and thus it can be written as $T_\gamma f(x) = (K \otimes f)(x)$. Kernel of B-singular integral operator $K(x)$ satisfies the followings:

- $\sup_{\theta \in S_{k,+}} |K(\theta)| < \infty$;
- $\int_{S_{k,+}} |K_\varepsilon(x)|(x')^\gamma d\sigma \leq C$ for all ε and $x \in \mathbb{R}_{k,+}^n$, where C is a constant independent of ε and $d\sigma$;
- $\int_{|x| \geq \delta} |K_\varepsilon(x)|(x')^\gamma dx \rightarrow 0$ as $\varepsilon \rightarrow 0$ for any fixed $\delta > 0$;
- The kernel of the B-singular integral operator satisfies Hörmander's condition:

$$\int_{B_+(0, 2|x|)} |K(y) - T^y K(x)|(y')^\gamma dy \leq C, \quad y \neq 0,$$

where C is an absolute constant;

- There exist $\delta > 0$ and $C > 0$ for all distinct $x, y \in \mathbb{R}_{k,+}^n$ with $2|x| < |y|$ such that $|K(x)| \leq C|x|^{-Q}$ and

$$|T^y K(x) - K(y)| \leq C|x|^\delta |y|^{-Q-\delta}.$$

Lemma 1 ([1]). Let $\sup_{x \in \mathbb{R}_{k,+}^n} |\Omega(\theta)| = M < +\infty$. If $|x| \geq |y|/2$, then

$$|T^y K(x) - K(y)| \leq CM|y|^{-Q},$$

where $C > 0$ is a constant.

B –maximal singular integral operator $\mathcal{T}_\gamma$ is one of the important tools of harmonic analysis and its applications that is defined by

$$\mathcal{T}_\gamma f(x) = \sup_{\varepsilon > 0} |T_{\gamma, \varepsilon} f(x)|,$$

and behavior of this operator in Lebesgue spaces have been investigated in [11].

The purpose of this study is to obtain that B –maximal M_γ , B –singular integral T_γ and B –maximal singular integral $\mathcal{T}_\gamma$ operators are bounded on B –local Morrey-Lorentz spaces with the use of rearrangement inequalities and Hardy inequality.

The draft of this study is as follows: Section 1 is devoted to introduction. In Section 2, we recall some basic notions and some known results which we need throughout the paper. In Section 3, we have obtained that B –maximal operators in B –local Morrey-Lorentz spaces are bounded. Section 4 is devoted to the boundedness of B –singular integral and B –maximal singular integral operators.

Throughout the paper, C denotes a positive constant independent of appropriate parameters and not necessary the same at each occurrence.

2. PRELIMINARIES

Given any measurable set E with $|E|_\gamma = \int_E (x')^\gamma dx$ and a measurable function $f: \mathbb{R}_{k,+}^n \rightarrow \mathbb{R}$, the γ –rearrangement of f in decreasing order is defined by

$$f_\gamma^*(t) = \inf \{s > 0 : f_{*,\gamma}(s) \leq t\}, \quad \forall t \in (0, \infty),$$

where $f_{*,\gamma}(s)$ denotes the γ –distribution function of f given by

$$f_{*,\gamma}(s) = |\{x \in \mathbb{R}_{k,+}^n : |f(x)| > s\}|_\gamma.$$

The average function of f_γ^{**} is defined as

$$f_\gamma^{**}(t) = \frac{1}{t} \int_0^t f_\gamma^*(s) ds, \quad t > 0,$$

and

$$(f + g)_\gamma^{**}(t) \leq f_\gamma^{**}(t) + g_\gamma^{**}(t)$$

is valid [10].

Now, we give some characteristics of γ –rearrangement of functions:

- if $0 < p < \infty$, then

$$\int_{\mathbb{R}_{k,+}^n} |f(x)|^p (x')^\gamma dx = \int_0^\infty (f_\gamma^*(t))^p dt;$$

- for any $t > 0$,

$$\sup_{|E|_\gamma=t} \int_E |f(x)| (x')^\gamma dx = \int_0^t f_\gamma^*(s) ds; \quad (2.1)$$

•

$$\int_{\mathbb{R}_{k,+}^n} |f(x)g(x)|(x')^\gamma dx \leq \int_0^\infty f_\gamma^*(t)g_\gamma^*(t)dt;$$

• it is well known that

$$(f+g)_\gamma^*(t) \leq f_\gamma^*(t/2) + g_\gamma^*(t/2) \quad (2.2)$$

holds [8, 10, 28].

Definition 1 ([22]). $L_{p,q,\gamma}(\mathbb{R}_{k,+}^n)$ Lorentz space is the set of all measurable functions $f \in \mathbb{R}_{k,+}^n$ such that

$$\|f\|_{L_{p,q,\gamma}(\mathbb{R}_{k,+}^n)} = \left\| t^{\frac{1}{p}-\frac{1}{q}} f_\gamma^*(t) \right\|_{L_q(0,\infty)} < \infty.$$

If $0 < p \leq \infty$, $q = \infty$, then $L_{p,\infty,\gamma}(\mathbb{R}_{k,+}^n) = WL_{p,\gamma}(\mathbb{R}_{k,+}^n)$, where $WL_{p,\gamma}(\mathbb{R}_{k,+}^n)$ is weak Lebesgue space of all measurable functions f such that

$$\|f\|_{WL_{p,\gamma}(\mathbb{R}_{k,+}^n)} = \sup_{t>0} t^{1/p} f_\gamma^*(t) < \infty, \quad 1 \leq p < \infty.$$

If $p = q = \infty$ or $1 \leq q \leq p$, then functional $\|f\|_{p,q,\gamma}$ is a norm [8, 15, 28]. However if $p = q = \infty$, then $L_{\infty,\infty,\gamma}(\mathbb{R}_{k,+}^n) = L_{\infty,\gamma}(\mathbb{R}_{k,+}^n)$.

In case $0 < p, q \leq \infty$, the functional $\|\cdot\|_{L_{p,q,\gamma}}^*$ is given by

$$\|f\|_{L_{p,q,\gamma}}^* = \|f\|_{L_{p,q,\gamma}(0,\infty)}^* = \|t^{\frac{1}{p}-\frac{1}{q}} f_\gamma^{**}(t)\|_{L_q(0,\infty)},$$

which is a norm on $L_{p,q,\gamma}(\mathbb{R}_{k,+}^n)$ for $1 < p < \infty$, $1 \leq q \leq \infty$ or $p = q = \infty$.

If $1 < p \leq \infty$, $1 \leq q \leq \infty$, then

$$\|f\|_{p,q,\gamma} \leq \|f\|_{p,q,\gamma}^* \leq \frac{p}{p-1} \|f\|_{p,q,\gamma},$$

that is, $\|f\|_{p,q,\gamma}$ and $\|f\|_{p,q,\gamma}^*$ are equivalent.

Definition 2 ([12]). Let $1 \leq p < \infty$, and $0 \leq \lambda \leq Q$. $L_{p,\lambda,\gamma}(\mathbb{R}_{k,+}^n)$ B -Morrey space is the set of all measurable functions with $f \in L_{p,\gamma}^{\text{loc}}(\mathbb{R}_{k,+}^n)$ such that

$$\|f\|_{L_{p,\lambda,\gamma}(\mathbb{R}_{k,+}^n)} = \sup_{x \in \mathbb{R}_{k,+}^n, r>0} r^{-\frac{\lambda}{p}} \|f\|_{L_{p,\gamma}(B_+(x,r))} < \infty.$$

If $\lambda = 0$, then $L_{p,0,\gamma}(\mathbb{R}_{k,+}^n) = L_{p,\gamma}(\mathbb{R}_{k,+}^n)$; if $\lambda < 0$ or $\lambda > Q$, then $L_{p,\lambda,\gamma}(\mathbb{R}_{k,+}^n) = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}_{k,+}^n$. Also, the weak B -Morrey space $WL_{p,\lambda,\gamma}(\mathbb{R}_{k,+}^n)$ is the set of all functions $f \in WL_{p,\gamma}^{\text{loc}}(\mathbb{R}_{k,+}^n)$ with the following norm

$$\|f\|_{WL_{p,\lambda,\gamma}(\mathbb{R}_{k,+}^n)} = \sup_{x \in \mathbb{R}_{k,+}^n, r>0} r^{-\frac{\lambda}{p}} \|f\|_{WL_{p,\gamma}(B_+(x,r))} < \infty.$$

Definition 3 ([9]). Let $0 \leq p < \infty$ and $0 \leq \lambda \leq 1$. $LM_{p,\lambda} \equiv LM_{p,\lambda}(0, \infty)$ local Morrey space is the set of all functions $f \in L_p^{\text{loc}}(0, \infty)$ such that

$$\|f\|_{LM_{p,\lambda}(0,\infty)} = \sup_{r>0} r^{-\frac{\lambda}{p}} \|f\|_{L_p(0,r)} < \infty.$$

Moreover, $WLM_{p,\lambda} \equiv WLM_{p,\lambda}(0, \infty)$ denotes the weak local Morrey space of all functions $f \in WL_p^{\text{loc}}(0, \infty)$ such that

$$\|f\|_{WLM_{p,\lambda}(0,\infty)} = \sup_{r>0} r^{-\frac{\lambda}{p}} \|f\|_{WL_p(0,r)} < \infty.$$

Definition 4. Let $0 < p, q \leq \infty$ and $0 \leq \lambda \leq 1$. $M_{p,q,\lambda,\gamma}^{\text{loc}}(\mathbb{R}_{k,+}^n)$ B -local Morrey-Lorentz space is the set of all measurable functions with quasi-norm

$$\|f\|_{M_{p,q,\lambda,\gamma}^{\text{loc}}(\mathbb{R}_{k,+}^n)} = \sup_{r>0} r^{-\frac{\lambda}{q}} \|t^{\frac{1}{p}-\frac{1}{q}} f_{\gamma}^*(t)\|_{L_q(0,r)} < \infty.$$

If $\lambda < 0$ or $\lambda > 1$, then $M_{p,q,\lambda,\gamma}^{\text{loc}}(\mathbb{R}_{k,+}^n) = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}_{k,+}^n$. Also,

$$M_{p,q,0,\gamma}^{\text{loc}}(\mathbb{R}_{k,+}^n) = L_{p,q,\gamma}(\mathbb{R}_{k,+}^n) \quad \text{and} \quad M_{p,p,\lambda,\gamma}^{\text{loc}}(\mathbb{R}_{k,+}^n) \equiv M_{p,0,\gamma}^{\text{loc}}(\mathbb{R}_{k,+}^n).$$

$WM_{p,q,\lambda,\gamma}^{\text{loc}}(\mathbb{R}_{k,+}^n)$ weak B -local Morrey-Lorentz space is the set of all measurable functions with the quasinorm

$$\|f\|_{WM_{p,q,\lambda,\gamma}^{\text{loc}}(\mathbb{R}_{k,+}^n)} = \sup_{r>0} r^{-\frac{\lambda}{q}} \|t^{\frac{1}{p}-\frac{1}{q}} f_{\gamma}^*(t)\|_{WL_q(0,r)} < \infty.$$

Definition 5 ([27]). Let $\beta \in \mathbb{R}$ and φ be a measurable function on $(0, \infty)$. The weighted Hardy operators H^{β} and $\mathcal{H}^{\beta}$, on which the power weights acts, are defined as

$$H^{\beta}\varphi(t) = t^{\beta-1} \int_0^t \frac{\varphi(y)}{y^{\beta}} dy, \quad \mathcal{H}^{\beta}\varphi(t) = t^{\beta} \int_t^{\infty} \frac{\varphi(y)}{y^{\beta+1}} dy.$$

For the proof of our main theorems, we need the following Hardy inequalities:

Theorem 1 ([6, 27]). Let $\beta \in \mathbb{R}$, $0 \leq \lambda < 1$.

i. If $1 \leq q < \infty$ and $\beta < \frac{1}{q'} + \frac{\lambda}{q}$, then

$$\|H^{\beta}\varphi\|_{LM_{q,\lambda}(0,\infty)} \leq C \|\varphi\|_{LM_{q,\lambda}(0,\infty)}.$$

ii. If $1 < q < \infty$ and $\beta = \frac{1}{q'} + \frac{\lambda}{q}$, then

$$\|H^{\beta}\varphi\|_{WLM_{q,\lambda}(0,\infty)} \leq C \|\varphi\|_{LM_{q,\lambda}(0,\infty)}.$$

iii. If $1 \leq q < \infty$ and $\beta > \frac{\lambda-1}{q}$, then

$$\left\| \mathcal{H}^\beta \Phi \right\|_{LM_{q,\lambda}(0,\infty)} \leq C \|\Phi\|_{LM_{q,\lambda}(0,\infty)}.$$

iv. If $1 < q < \infty$ and $\beta = \frac{\lambda-1}{q}$, then

$$\left\| \mathcal{H}^\beta \Phi \right\|_{WLM_{q,\lambda}(0,\infty)} \leq C \|\Phi\|_{LM_{q,\lambda}(0,\infty)}.$$

3. B-MAXIMAL OPERATORS IN B-LOCAL MORREY-LORENTZ SPACES

We first give sharp rearrangement inequality for B-maximal operator. By using this inequality, we get B-maximal operator is bounded in $M_{p,q,\lambda,\gamma}(\mathbb{R}_{k,+}^n)$.

Lemma 2 ([3]). *Sharp rearrangement inequality for B-maximal operator is given by*

$$(M_\gamma f)_\gamma^*(t) \leq C f_\gamma^{**}(t), \quad t > 0, \quad (3.1)$$

where $C = C(n, \gamma) > 0$ is a constant.

Theorem 2. *Let $1 \leq q \leq \infty$, $0 \leq \lambda < 1$ and $\frac{q}{q+\lambda} \leq p \leq \infty$.*

- (i) *If $\frac{q}{q+\lambda} < p < \infty$, then B-maximal operator M_γ is bounded in B-local Morrey-Lorentz space $M_{p,q,\lambda,\gamma}^{\text{loc}}(\mathbb{R}_{k,+}^n)$.*
- (ii) *If $p = \frac{q}{q+\lambda}$, then B-maximal operator M_γ is bounded from B-local Morrey-Lorentz space $M_{p,q,\lambda,\gamma}^{\text{loc}}(\mathbb{R}_{k,+}^n)$ to weak B-local Morrey-Lorentz space $WM_{p,q,\lambda,\gamma}^{\text{loc}}(\mathbb{R}_{k,+}^n)$.*
- (iii) *If $p = q = \infty$, then B-maximal operator M_γ is bounded in $L_{\infty,\gamma}(\mathbb{R}_{k,+}^n)$.*

Proof. (i) Let $\frac{q}{q+\lambda} < p < \infty$ and $f \in M_{p,q,\lambda,\gamma}^{\text{loc}}(\mathbb{R}_{k,+}^n)$. By Definition 4 and the inequality (3.1), we obtain

$$\begin{aligned} \|M_\gamma f\|_{M_{p,q,\lambda,\gamma}^{\text{loc}}} &= \sup_{r>0} r^{-\frac{\lambda}{q}} \left\| t^{\frac{1}{p}-\frac{1}{q}} (M_\gamma f)_\gamma^*(t) \right\|_{L_q(0,r)} \\ &\leq C \sup_{r>0} r^{-\frac{\lambda}{q}} \left\| t^{\frac{1}{p}-\frac{1}{q}} f_\gamma^{**}(t) \right\|_{L_q(0,r)} \\ &= C \sup_{r>0} r^{-\frac{\lambda}{q}} \left\| t^{\frac{1}{p}-\frac{1}{q}-1} \int_0^t f_\gamma^*(s) ds \right\|_{L_q(0,r)}. \end{aligned}$$

Let $\beta = \frac{1}{p} - \frac{1}{q}$ and $\varphi = t^{\frac{1}{p}-\frac{1}{q}} f_\gamma^*(t)$. Then from Theorem 1, we get

$$\|M_\gamma f\|_{M_{p,q,\lambda,\gamma}^{\text{loc}}} \leq C \sup_{r>0} r^{-\frac{\lambda}{q}} \left\| t^\beta \int_0^t \frac{\varphi(s)}{s^\beta} ds \right\|_{L_q(0,r)}$$

$$\begin{aligned}
&= C \|H^\beta \phi\|_{LM_{q,\lambda}(0,\infty)} \\
&\leq C \|\phi\|_{LM_{q,\lambda}(0,\infty)} \\
&= C \sup_{r>0} r^{-\frac{\lambda}{q}} \|\phi\|_{L_q(0,r)} \\
&= C \sup_{r>0} r^{-\frac{\lambda}{q}} \left\| t^{\frac{1}{p}-\frac{1}{q}} f_\gamma^*(t) \right\|_{L_q(0,r)} \\
&= \|f\|_{M_{p,q,\lambda,\gamma}^{\text{loc}}}.
\end{aligned}$$

Hence, the desired result is obtained.

(ii) Let $p = \frac{q}{q+\lambda}$ and $f \in M_{p,q,\lambda,\gamma}^{\text{loc}}(\mathbb{R}_{k,+}^n)$. From Definition 4 and inequality (3.1), we have

$$\|M_\gamma f\|_{WM_{\frac{q}{q+\lambda},q,\lambda,\gamma}^{\text{loc}}} = C \sup_{r>0} r^{-\frac{\lambda}{q}} \left\| t^{1+\frac{\lambda-1}{q}} \int_0^t f_\gamma^*(s) ds \right\|_{WL_q(0,r)} = C \|H^\beta \phi\|_{WLM_{q,\lambda}(0,\infty)},$$

where $\beta = 1 + \frac{\lambda-1}{q}$ and $\phi = t^{1+\frac{\lambda-1}{q}} f_\gamma^*(t)$. Therefore, we obtain by Theorem 1

$$\|H^\beta \phi\|_{WLM_{q,\lambda}(0,\infty)} \leq C \|\phi\|_{LM_{q,\lambda}(0,\infty)} = C \|f\|_{M_{\frac{q}{q+\lambda},q,\lambda,\gamma}^{\text{loc}}}.$$

Then we get the operator M_γ is bounded from B -local Morrey-Lorentz space $M_{\frac{q}{q+\lambda},q,\lambda,\gamma}^{\text{loc}}$ to weak B -local Morrey-Lorentz space $WLM_{\frac{q}{q+\lambda},q,\lambda,\gamma}^{\text{loc}}$.

(iii) Let $p = \infty$ and $f \in M_{\infty,q,\lambda,\gamma}^{\text{loc}}(\mathbb{R}_{k,+}^n)$. From [14, Remark 2.1], it is well known that $M_{\infty,q,\lambda,\gamma}^{\text{loc}} = \Theta$ for any $0 < q < \infty$. Hence, we now consider the case of $q = \infty$. Since $M_{\infty,q,\lambda,\gamma}^{\text{loc}} = WL_{p,\gamma}$ for $q = \infty$ and M_γ is bounded in $L_{\infty,\gamma}$, we obtain the desired result. $\square$

Corollary 1. (i) If $1 \leq p < \infty$, $0 < \lambda \leq 1$, then B -maximal operator M_γ is bounded in $M_{p,\lambda,\gamma}^{\text{loc}}(\mathbb{R}_{k,+}^n)$.

(ii) If $1 < p < \infty$, $1 \leq q \leq \infty$, then B -maximal operator M_γ is bounded in $L_{p,q,\gamma}(\mathbb{R}_{k,+}^n)$.

(iii) If $1 < q < \infty$, then B -maximal operator M_γ is bounded in $WL_{q,\gamma}(\mathbb{R}_{k,+}^n)$.

4. B -SINGULAR INTEGRAL OPERATORS IN B -LOCAL MORREY-LORENTZ SPACES

We first give sharp rearrangement inequality for B -singular integral operator. By using this inequality, we obtain that B -singular integral is bounded in $M_{p,q,\lambda,\gamma}^{\text{loc}}(\mathbb{R}_{k,+}^n)$.

Theorem 3 ([18]). Let $f, g \in \mathbb{R}_{k,+}^n$. Then

$$(f \otimes g)_\gamma^{**}(t) \leq C_{\gamma,k} \left(f_\gamma^{**}(t) \int_0^t g_\gamma^{**}(u) du + \int_t^\infty f_\gamma^*(u) g_\gamma^{**}(u) du \right), \quad t > 0.$$

Theorem 4 ([18]). *If $K_\alpha \in WL_{Q/(Q-\alpha),\gamma}(\mathbb{R}_{k,+}^n)$, $0 < \alpha < Q$, then*

$$(K_\alpha \otimes g)_\gamma^*(t) \leq (K_\alpha \otimes g)_\gamma^{**}(t) \leq C \left(t^{\alpha/Q-1} \int_0^t f_\gamma^*(s) ds + \int_t^\infty s^{\alpha/Q-1} f_\gamma^*(s) ds \right),$$

where $C = C_{\gamma,k}(Q/\alpha)^2 \|K_\alpha\|_{WL_{Q/(Q-\alpha),\gamma}(\mathbb{R}_{k,+}^n)}$.

Since $T_\gamma f$ is convolution type operator and from the above theorem, we can write

$$(T_\gamma f)_\gamma^*(t) \leq (T_\gamma f)_\gamma^{**}(t) \leq C \left(t^{-1} \int_0^t f_\gamma^*(s) ds + \int_t^\infty s^{-1} f_\gamma^*(s) ds \right). \quad (4.1)$$

Theorem 5. *Suppose that $f \in M_{p,q,\lambda,\gamma}^{\text{loc}}(\mathbb{R}_{k,+}^n)$, $1 \leq q \leq \infty$, $0 \leq \lambda < 1$, $\frac{q}{q+\lambda} \leq p \leq \frac{q}{\lambda}$ and B-singular integral operator $T_\gamma f$ exists a.e. $x \in \mathbb{R}_{k,+}^n$. Moreover,*

- (i) *If $1 \leq q < \infty$, $\frac{q}{q+\lambda} < p < \frac{q}{\lambda}$, then B-singular integral operator is bounded in B-local Morrey-Lorentz space $M_{p,q,\lambda,\gamma}^{\text{loc}}(\mathbb{R}_{k,+}^n)$.*
- (ii) *If $1 < q < \infty$, $p = \frac{q}{q+\lambda}$, then B-singular integral operator is bounded from $M_{p,q,\lambda,\gamma}^{\text{loc}}(\mathbb{R}_{k,+}^n)$ to weak B-local Morrey-Lorentz space $WM_{p,q,\lambda,\gamma}^{\text{loc}}(\mathbb{R}_{k,+}^n)$.*

Proof. Let $1 \leq q \leq \infty$, $0 \leq \lambda < 1$, and $\frac{q}{q+\lambda} \leq p \leq \frac{q}{\lambda}$. Since f satisfies (4.1), B-singular integral operator $T_\gamma f$ exists a.e. $x \in \mathbb{R}_{k,+}^n$.

(i) Let $1 \leq q < \infty$, $0 \leq \lambda < 1$, and $\frac{q}{q+\lambda} \leq p < \frac{q}{\lambda}$ and $f \in M_{p,q,\lambda,\gamma}^{\text{loc}}$. From Definition 4 and by using inequality (4.1) and Minkowski's inequality, we have

$$\begin{aligned} \|T_\gamma f\|_{M_{p,q,\lambda,\gamma}^{\text{loc}}(\mathbb{R}_{k,+}^n)} &= C \sup_{r>0} r^{-\lambda/q} \left\| t^{\frac{1}{p}-\frac{1}{q}} (T_\gamma f)_\gamma^*(t) \right\|_{L_q(0,r)} \\ &\leq C \sup_{r>0} r^{-\lambda/q} \left\| t^{\frac{1}{p}-\frac{1}{q}} (T_\gamma f)_\gamma^{**}(t) \right\|_{L_q(0,r)} \\ &\leq C \sup_{r>0} r^{-\lambda/q} \left\| t^{\frac{1}{p}-\frac{1}{q}} \left[t^{-1} \int_0^t f_\gamma^*(s) ds + \int_t^\infty \frac{f_\gamma^*(s)}{s} ds \right] \right\|_{L_q(0,r)} \\ &\leq C \sup_{r>0} r^{-\lambda/q} \left\| t^{\frac{1}{p}-\frac{1}{q}-1} \int_0^t f_\gamma^*(s) ds \right\|_{L_q(0,r)} \\ &\quad + \sup_{r>0} r^{-\lambda/q} \left\| t^{\frac{1}{p}-\frac{1}{q}} \int_t^\infty \frac{f_\gamma^*(s)}{s} ds \right\|_{L_q(0,r)} \\ &= I_1 + I_2. \end{aligned}$$

I_1 can be estimated using the same method as in the proof of Theorem 2. Let us estimate I_2 :

$$\begin{aligned}
I_2 &= \sup_{r>0} r^{-\lambda/q} \left\| t^{\frac{1}{p}-\frac{1}{q}} \int_t^\infty \frac{f_\gamma^*(s)}{s} ds \right\|_{L_q(0,r)} \\
&= \sup_{r>0} r^{-\lambda/q} \left\| t^\beta \int_t^\infty \frac{\varphi(s)}{s^{\beta+1}} ds \right\|_{L_q(0,r)} \\
&= \sup_{r>0} r^{-\lambda/q} \left\| \mathcal{H}^\beta \varphi \right\|_{L_q(0,r)} \\
&= \left\| \mathcal{H}^\beta \varphi \right\|_{LM_{q,\lambda}(0,\infty)} \\
&= \left\| \varphi \right\|_{LM_{q,\lambda}(0,\infty)} \\
&= C \sup_{r>0} r^{-\lambda/q} \left\| \varphi \right\|_{LM_q(0,r)} \\
&= C \sup_{r>0} r^{-\lambda/q} \left\| t^{\frac{1}{p}-\frac{1}{q}} f_\gamma^*(t) \right\|_{L_q(0,r)} \\
&= \|f\|_{M_{p,q,\lambda,\gamma}^{\text{loc}}}, \tag{4.2}
\end{aligned}$$

where $\varphi(t) = t^{\frac{1}{p}-\frac{1}{q}} f_\gamma^*(t)$. Since $\frac{1}{p} - \frac{\lambda}{q} > 0$, the inequality $\beta > \frac{\lambda}{q} - \frac{1}{q}$ is valid for $\beta = \frac{1}{p} - \frac{1}{q}$. Therefore, we obtain $I_2 \leq C \|f\|_{M_{p,q,\lambda,\gamma}^{\text{loc}}}$. As a consequence, we get the desired result.

(ii) Let $p = \frac{q}{q+\lambda}$, $1 < q < \infty$ and $f \in M_{p,q,\lambda,\gamma}^{\text{loc}}(\mathbb{R}_{k,+}^n)$. From Definition 4 and with the use of inequality (4.1) and Minkowski's inequality, we have

$$\begin{aligned}
\|T_\gamma f\|_{WM_{q/(q+\lambda),q,\lambda,\gamma}^{\text{loc}}(\mathbb{R}_{k,+}^n)} &= C \sup_{r>0} r^{-\lambda/q} \left\| t^{1+\frac{\lambda-1}{q}} (T_\gamma f)_\gamma^*(t) \right\|_{WL_q(0,\infty)} \\
&\leq C \sup_{r>0} r^{-\lambda/q} \left\| t^{1+\frac{\lambda-1}{q}} (T_\gamma f)_\gamma^{**}(t) \right\|_{WL_q(0,\infty)} \\
&\leq C \sup_{r>0} r^{-\lambda/q} \left\| t^{\frac{\lambda-1}{q}} \int_0^t f_\gamma^*(s) ds \right\|_{WL_q(0,\infty)} \\
&\quad + C \sup_{r>0} r^{-\lambda/q} \left\| t^{1+\frac{\lambda-1}{q}} \int_t^\infty \frac{f_\gamma^*(s)}{s} ds \right\|_{WL_q(0,\infty)} \\
&= J_1 + J_2.
\end{aligned}$$

J_1 can be estimated using the same method as in the proof of Theorem 2. Let us estimate J_2 . By Theorem 1, we have

$$J_2 = C \sup_{r>0} r^{-\lambda/q} \left\| t^{1+\frac{\lambda-1}{q}} \int_t^\infty \frac{f_\gamma^*(s)}{s} ds \right\|_{WL_q(0,\infty)}$$

$$\begin{aligned}
 &= C \sup_{r>0} r^{-\lambda/q} \left\| t^\beta \int_t^\infty \frac{\varphi(s)}{s^{\beta+1}} ds \right\|_{WL_q(0,\infty)} \\
 &= \sup_{r>0} r^{-\lambda/q} \left\| \mathcal{H}^\beta \varphi \right\|_{WL_q(0,\infty)} \\
 &= C \left\| \mathcal{H}^\beta \varphi \right\|_{WLM_{q,\lambda}(0,\infty)} \\
 &= \left\| \varphi \right\|_{LM_{q,\lambda}(0,\infty)} \\
 &= C \sup_{r>0} r^{-\lambda/q} \left\| \varphi \right\|_{L_q(0,\infty)} \\
 &= C \sup_{r>0} r^{-\lambda/q} \left\| t^{1+\frac{\lambda-1}{q}} f_\gamma^*(t) \right\|_{L_q(0,\infty)} \\
 &= \left\| f \right\|_{M_{\frac{q}{q+\lambda},q,\lambda,\gamma}^{\text{loc}}},
 \end{aligned}$$

where $\beta = 1 + \frac{(\lambda-1)}{q}$ and $\varphi(t) = t^{1+\frac{\lambda-1}{q}} f_\gamma^*(t)$. Therefore, we obtain that T_γ is bounded from $M_{q/(q+\lambda),q,\lambda,\gamma}^{\text{loc}}$ to $WM_{q/(q+\lambda),q,\lambda,\gamma}^{\text{loc}}$. Thus, the proof is completed. $\square$

Theorem 6. Suppose that $f \in M_{p,q,\lambda,\gamma}^{\text{loc}}(\mathbb{R}_{k,+}^n)$, $1 \leq q \leq \infty$, $0 \leq \lambda < 1$, $\frac{q}{q+\lambda} \leq p \leq \frac{q}{\lambda}$ and (4.1) is valid, then B -maximal singular integral operator $\mathcal{T}_\gamma f$ is finite a.e. $x \in \mathbb{R}_{k,+}^n$. Furthermore,

- (i) If $1 \leq q < \infty$, $\frac{q}{q+\lambda} < p < \frac{q}{\lambda}$, then $\mathcal{T}_\gamma$ is bounded on B -local Morrey-Lorentz space $M_{p,q,\lambda,\gamma}^{\text{loc}}(\mathbb{R}_{k,+}^n)$.
- (ii) If $1 < q < \infty$, $p = \frac{q}{q+\lambda}$, then $\mathcal{T}_\gamma$ is bounded from $M_{p,q,\lambda,\gamma}^{\text{loc}}(\mathbb{R}_{k,+}^n)$ to weak B -local Morrey-Lorentz space $WM_{p,q,\lambda,\gamma}^{\text{loc}}(\mathbb{R}_{k,+}^n)$.

Proof. Let $1 \leq q \leq \infty$, $0 \leq \lambda < 1$, and $\frac{q}{q+\lambda} \leq p \leq \frac{q}{\lambda}$. Since f satisfies (4.1), the B -maximal singular integral operator $\mathcal{T}_\gamma f(x)$ is finite almost everywhere for $x \in \mathbb{R}_{k,+}^n$. By using inequality (4.1), the proof of theorem can be easily obtained in a similar manner of method of Theorem 5. $\square$

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COMPLEMENTS OF NABLA AND DELTA HARDY-COPSON TYPE INEQUALITIES AND THEIR APPLICATIONS

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Abstract. In this paper the classical nabla and delta Hardy-Copson type inequalities, which are derived for $\zeta > 1$, are complemented to the new case $\zeta < 0$. These complements have exactly the same forms as the aforementioned classical inequalities except that the exponent ζ is not greater than one but it is less than zero. The obtained inequalities are not only novel but also unify the continuous and discrete cases for which the case $\zeta < 0$ has not been considered so far either. Moreover one of the applications of Hardy-Copson type inequalities, which is to find nonoscillation criteria for the half linear differential/dynamic/difference equations, are presented by using complementary delta Hardy-Copson type inequalities.

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1. INTRODUCTION

The theory of inequalities containing series or integrals has been given a great importance due to their effective usage in differential equations and their applications after the celebrated discrete and continuous inequalities that Hardy have been obtained. In 1920, when Hardy [24] tried to find a simple and elementary proof of Hilbert's inequality [32]

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m c_n}{m+n} \leq \pi \left(\sum_{m=1}^{\infty} a_m^2 \right)^{1/2} \left(\sum_{n=1}^{\infty} c_n^2 \right)^{1/2},$$

where $a_m, c_n \geq 0$ and $\sum_{m=1}^{\infty} a_m^2$ and $\sum_{n=1}^{\infty} c_n^2$ are convergent, he showed the following pioneering discrete inequality

$$\sum_{j=1}^{\infty} \left(\frac{1}{j} \sum_{i=1}^m c(i) \right)^{\zeta} \leq \left(\frac{\zeta}{\zeta-1} \right)^{\zeta} \sum_{j=1}^{\infty} c^{\zeta}(j), \quad c(j) \geq 0, \zeta > 1 \quad (1.1)$$

and pioneering continuous inequality for a nonnegative function Γ and for a real constant $\zeta > 1$, as

$$\int_0^\infty \left(\frac{1}{t} \int_0^t \Gamma(s) ds \right)^\zeta dt \leq \left(\frac{\zeta}{\zeta-1} \right)^\zeta \int_0^\infty \Gamma^\zeta(t) dt, \quad (1.2)$$

where $\int_0^\infty \Gamma^\zeta(t) dt < \infty$. In fact, Hardy only stated inequality (1.2) in [24] but did not prove it. After that in 1925, the proof of inequality (1.2), which depends on the calculus of variations, was shown by Hardy in [25].

The constant $\left(\frac{\zeta}{\zeta-1} \right)^\zeta$ that appears in the above inequalities also has been found as the best possible one, since if it is replaced by a smaller constant then inequalities (1.1) and (1.2) is not fulfilled anymore for the involved sequences and functions, respectively.

Then Hardy et al. [26, Theorem 330] developed inequality (1.2) and derived the following integral inequality for a nonnegative function Γ as

$$\int_0^\infty \frac{\Psi^\zeta(t)}{t^\theta} dt \leq \left| \frac{\zeta}{\theta-1} \right|^\zeta \int_0^\infty \frac{\Gamma^\zeta(t)}{t^{\theta-\zeta}} dt, \quad \zeta > 1, \quad (1.3)$$

$$\text{where } \Psi(t) = \begin{cases} \int_0^t \Gamma(s) ds, & \text{if } \theta > 1, \\ \int_t^\infty \Gamma(s) ds, & \text{if } \theta < 1. \end{cases}$$

The exhibition of the results which contain the improvements, generalizations and applications of the discrete and continuous Hardy inequalities can be found in the books [8, 26, 32, 33, 37] and references therein.

Since various generalizations and numerous variants of the discrete Hardy inequality (1.1) exist in the literature, all of which can not be covered here, we only focus on the extensions which have been established by Copson [16, Theorem 1.1, Theorem 2.1]. This is why we name our inequalities as Hardy-Copson type inequalities. The discrete Hardy inequality (1.1) or Copson's discrete inequalities were generalized in [10, 15, 20, 34–36] and references therein.

Similar to the discrete Hardy inequality (1.1), the continuous versions (1.2) or (1.3) have attracted many mathematicians' interests and expansions of these continuous inequalities have appeared in the literature. The first continuous refinements were obtained by Copson [17, Theorem 1, Theorem 3] and after these results many papers devoted to continuous analogues and continuous improvements of the discrete Hardy-Copson inequalities, see [9, 27, 38, 40, 42].

Following the development of the time scale concept [7, 11, 12, 21, 22], the analysis of dynamic inequalities have become a popular research area and most classical inequalities have been extended to an arbitrary time scale. The surveys [1, 52] and the monograph [4] can be used to see these extended dynamic inequalities for

delta approach. Although the nabla dynamic inequalities are less attractive compared to the delta ones, some of the nabla dynamic inequalities can be found in [6, 13, 23, 30, 31, 39, 41].

The growing interest to Hardy-Copson type inequalities take place in the time scale calculus as well and delta unifications of these inequalities are established in the books [5] and in the articles [3, 19, 43, 44, 46, 47, 50, 51, 55] for $\zeta > 1$ whereas their reverse versions can be found in [18, 45, 46, 48] for $0 < \zeta < 1$. The nabla unifications of Hardy-Copson type inequalities for $\zeta > 1$ can be seen in [28, 29].

All of the abovementioned articles contain Hardy-Copson type inequalities obtained for $\zeta > 1$. For our further purposes, we will show the delta and nabla Hardy-Copson type inequalities [28, 51] established for $\zeta > 1$ and used in the sequel.

The delta time scale generalizations of the foregoing inequalities in an arbitrary time scale are given in the next four theorems for nonnegative functions z and h and for

$$G(t) = \int_t^\infty z(s)\Delta s, \quad H(t) = \int_a^t z(s)h(s)\Delta s, \quad \overline{G}(t) = \int_a^t z(s)\Delta s, \quad \overline{H}(t) = \int_t^\infty z(s)h(s)\Delta s. \quad (1.4)$$

A delta unification of the discrete inequality obtained by Bennett [10, Corollary 5] or Leindler [35, Proposition 3] and continuous inequality obtained by Saker et al. [51, Corollary 2.2] or Kayar and Kaymakçalan [28, Remark 3.22] is stated as follows.

Theorem 1 ([51]). *Let z, h, G and H be defined as in (1.4). If $\zeta > 1$, $\eta \geq 0$, $\eta + \theta < 1$, then we have*

$$\int_a^\infty \frac{z(t)[H^\sigma(t)]^{\eta+\zeta}}{[G(t)]^{\eta+\theta}}\Delta t \leq \left[\frac{\eta+\zeta}{1-\eta-\theta} \right]^\zeta \int_a^\infty \frac{z(t)h^\zeta(t)[H^\sigma(t)]^\eta}{[G(t)]^{\eta+\theta-\zeta}}\Delta t. \quad (1.5)$$

The following theorem establishes a delta unification of the discrete inequality obtained by Copson [16, Theorem 2.1] or Bennett [10, Corollary 4] or Leindler [35, Proposition 2] and its continuous counterpart obtained by Copson [17, Theorem 3] as well as its continuous generalization obtained in [40, Theorem 1] or [42, Theorem 3]. Moreover the continuous inequality (1.3) obtained by Hardy et al. [26, Theorem 330] is unified.

Theorem 2 ([51]). *Let $z, h, \overline{G}$ and $\overline{H}$ be defined as in (1.4). If $\zeta > 1$, $\eta \geq 0$, $\eta + \theta < 1$, then we have*

$$\int_a^\infty \frac{z(t)[\overline{H}(t)]^{\eta+\zeta}}{[\overline{G}^\sigma(t)]^{\eta+\theta}}\Delta t \leq \left[\frac{\eta+\zeta}{1-\eta-\theta} \right]^\zeta \int_a^\infty \frac{z(t)h^\zeta(t)[\overline{H}(t)]^\eta}{[\overline{G}^\sigma(t)]^{\eta+\theta-\zeta}}\Delta t. \quad (1.6)$$

A delta unification of the discrete inequality obtained by Bennett [10, Corollary 6] or Leindler [35, Proposition 4] and continuous inequality obtained by Saker et al. [51, Corollary 2.3] or Kayar and Kaymakçalan [28, Remark 3.11] is stated as follows.

Theorem 3 ([51]). Let z, h, G and $\bar{H}$ be defined as in (1.4). For $\frac{G^\sigma(t)}{G(t)} \geq \frac{1}{K} > 0$, if $\zeta > 1$, $\eta \geq 0$, $\eta + \theta > 1$, then we have

$$\int_a^\infty \frac{z(t)[\bar{H}(t)]^{\eta+\zeta}}{[G(t)]^{\eta+\theta}} \Delta t \leq \left[K^{\eta+\theta-1} \frac{\eta+\zeta}{\eta+\theta-1} \right]^\zeta \int_a^\infty \frac{z(t)h^\zeta(t)[\bar{H}(t)]^\eta}{[G(t)]^{\eta+\theta-\zeta}} \Delta t. \quad (1.7)$$

The following theorem establishes a delta unification of the discrete inequality obtained by Hardy [24, Theroem B] as inequality (2) and the discrete inequality obtained by Copson [16, Theorem 1.1] or Bennett [10, Corollary 3] or Leindler [35, Proposition 1] and their continuous counterparts obtained by Hardy [24, Theroem B] as inequality (4) and Hardy et al. [26, Theroem 330] and Copson [17, Theorem 1] as well as their continuous generalizations obtained in [40, Theorem 1] or [42, Theorem 1].

Theorem 4 ([28, 51]). Let $z, h, \bar{G}$ and H be defined as in (1.4). For $\frac{\bar{G}(t)}{\bar{G}^\sigma(t)} \geq \frac{1}{J} > 0$, if $\zeta > 1$, $\eta \geq 0$, $\eta + \theta > 1$, then we have

$$\int_a^\infty \frac{z(t)[H^\sigma(t)]^{\eta+\zeta}}{[\bar{G}^\sigma(t)]^{\eta+\theta}} \Delta t \leq \left[J^{\eta+\theta-1} \frac{\eta+\zeta}{\eta+\theta-1} \right]^\zeta \int_a^\infty \frac{z(t)h^\zeta(t)[H^\sigma(t)]^\eta}{[\bar{G}^\sigma(t)]^{\eta+\theta-\zeta}} \Delta t. \quad (1.8)$$

There are more results about delta Hardy-Copson type inequalities different than the previous ones. For example, in [46] it seems that delta Hardy-Copson type inequalities were obtained for $\zeta > 1$ as well as for $0 < \zeta < 1$ and $\zeta < 0$. However there are many errors in this article even for the case $\zeta > 1$. Therefore the obtained inequalities are not correct in [46]. Another result was given in [3] by using a different method in the proof of the theorems. In this article, it seems that delta Hardy-Copson type inequalities were obtained for $\zeta > 1$ as well as $0 < \zeta < 1$. However for $0 < \zeta < 1$, since the obtained inequalities are in reverse directions of those derived for the case $\zeta > 1$, they can not be considered as new delta Hardy-Copson type inequalities. In fact, they already have been established in [48] as delta Bennett-Leindler type inequalities. Therefore for $\zeta < 0$, finding delta Hardy-Copson type inequalities by preserving the directions of the inequalities established for $\zeta > 1$ is still an open problem.

The construction of nabla time scale calculus, which has been introduced simultaneously with delta time scale calculus, can be found in [7, 11, 12, 21, 22].

Contrary to delta case, nabla Hardy-Copson type inequalities have not been considered until 2021. The first results of this case obtained by Kayar and Kaymakçalan in [28].

The nabla time scale generalizations of the foregoing inequalities in an arbitrary time scale are given in the next four theorems for nonnegative functions z and h and

for

$$\begin{aligned} G(t) &= \int_t^\infty z(s) \nabla s, & H(t) &= \int_a^t z(s) h(s) \nabla s, \\ \overline{G}(t) &= \int_a^t z(s) \nabla s, & \overline{H}(t) &= \int_t^\infty z(s) h(s) \nabla s. \end{aligned} \quad (1.9)$$

A nabla unification of the discrete inequality obtained by Bennett [10, Corollary 5] or Leindler [35, Proposition 3] and continuous inequality obtained by Saker et al. [51, Corollary 2.2] or Kayar and Kaymakçalan [28, Remark 3.22] is stated as follows.

Theorem 5 ([28]). *Let z, h, G and H be defined as in (1.9). If $\zeta > 1$, $\eta \geq 0$ and $\eta + \theta < 1$ are real constants, then we have*

$$\int_a^\infty \frac{z(t)[H(t)]^{\eta+\zeta}}{[G^p(t)]^{\eta+\theta}} \nabla t \leq \left[\frac{\eta + \zeta}{1 - \eta - \theta} \right]^\zeta \int_a^\infty \frac{z(t)h^\zeta(t)[H(t)]^\eta}{[G^p(t)]^{\eta+\theta-\zeta}} \nabla t. \quad (1.10)$$

The following theorem establishes a nabla unification of the discrete inequality obtained by Copson [16, Theorem 2.1] or Bennett [10, Corollary 4] or Leindler [35, Proposition 2] and its continuous counterpart obtained by Copson [17, Theorem 3] as well as its continuous generalization obtained in [40, Theorem 1] or [42, Theorem 3]. Moreover the continuous inequality (1.3) obtained by Hardy et al. [26, Theroem 330] is unified.

Theorem 6 ([28]). *Let $z, h, \overline{G}$ and $\overline{H}$ be defined as in (1.9). If $\zeta > 1$, $\eta \geq 0$ and $\eta + \theta < 1$ are real constants, then we have*

$$\int_a^\infty \frac{z(t)[\overline{H}^p(t)]^{\eta+\zeta}}{[\overline{G}(t)]^{\eta+\theta}} \nabla t \leq \left[\frac{\eta + \zeta}{1 - \eta - \theta} \right]^\zeta \int_a^\infty \frac{z(t)h^\zeta(t)[\overline{H}^p(t)]^\eta}{[\overline{G}(t)]^{\eta+\theta-\zeta}} \nabla t. \quad (1.11)$$

A nabla unification of the discrete inequality obtained by Bennett [10, Corollary 6] or Leindler [35, Proposition 4] and continuous inequality obtained by Saker et al. [51, Corollary 2.3] or Kayar and Kaymakçalan [28, Remark 3.11] is stated as follows.

Theorem 7 ([28]). *Let z, h, G and $\overline{H}$ be defined as in (1.9). Suppose that there exists $M > 0$ such that $\frac{G^p(t)}{G(t)} \leq M$ for $t \in (a, \infty)_{\mathbb{T}}$. If $\zeta > 1$, $\eta \geq 0$ and $\eta + \theta > 1$ are real constants, then we have*

$$\int_a^\infty \frac{z(t)[\overline{H}^p(t)]^{\eta+\zeta}}{[G^p(t)]^{\eta+\theta}} \nabla t \leq \left[M^{\eta+\theta-1} \frac{\eta + \zeta}{\eta + \theta - 1} \right]^\zeta \int_a^\infty \frac{z(t)h^\zeta(t)[\overline{H}^p(t)]^\eta}{[G^p(t)]^{\eta+\theta-\zeta}} \nabla t. \quad (1.12)$$

The following theorem establishes a nabla unification of the discrete inequality obtained by Hardy [24, Theroem B] as inequality (2) and the discrete inequality obtained by Copson [16, Theorem 1.1] or Bennett [10, Corollary 3] or Leindler [35, Proposition 1] and their continuous counterparts obtained by Hardy [24, Theroem B] as inequality (4) and Hardy et al. [26, Theroem 330] and Copson [17, Theorem 1] as well their continuous generalizations obtained in [40, Theorem 1] or [42, Theorem 1].

Theorem 8 ([28]). *Let $z, h, \overline{G}$ and H be defined as in (1.9). Suppose that there exists $L > 0$ such that $\frac{\overline{G}(t)}{\overline{G}^p(t)} \leq L$ for $t \in (a, \infty)_{\mathbb{T}}$. If $\zeta > 1$, $\eta \geq 0$ and $\eta + \theta > 1$ are real constants, then we have*

$$\int_a^\infty \frac{z(t)[H(t)]^{\eta+\zeta}}{[\overline{G}(t)]^{\eta+\theta}} \nabla t \leq \left[L^{\eta+\theta-1} \frac{\eta+\zeta}{\eta+\theta-1} \right]^\zeta \int_a^\infty \frac{z(t)h^\zeta(t)[H(t)]^\eta}{[\overline{G}(t)]^{\eta+\theta-\zeta}} \nabla t. \quad (1.13)$$

The reason Hardy-Copson type inequalities have generalized in many directions is that they have wide applications in operator theory and geometry [8, 33], in partial differential equations, such as Fokker-Planck equations and equations modelling fluid dynamics [37], in theory of qualitative behaviour of differential equations, such as finding necessary and sufficient conditions for the existence of positive solutions of half linear equations [9, 43, 49], oscillation theory [14, 55] and references therein.

Although delta and nabla Hardy-Copson type inequalities for the case $\zeta > 1$ have been deeply analyzed, the case $\zeta < 0$ has been investigated neither via nabla and delta approaches nor for continuous and discrete cases. Hence the main contribution of this article is to extend aforementioned Hardy-Copson type inequalities obtained for $\zeta > 1$ to the case $\zeta < 0$ by using nabla and delta time scale calculi without changing the directions of the inequalities derived for $\zeta > 1$. By means of application, we use complementary delta Hardy-Copson type inequalities to obtain necessary and sufficient conditions for the nonoscillation of the related half linear dynamic equations.

Our results are inspired from the papers [28] and [51] which contain nabla and delta Hardy-Copson type inequalities for the case $\zeta > 1$. We notice that when $\zeta < 0$ and $\eta \geq 0$, six choices appear contrary to the three cases in [28] and [51] for $\zeta > 1$ and $\eta \geq 0$. The cases $0 \leq \eta + \theta < 1$, $\eta + \zeta > 1$ and $\eta + \theta > 1$, $\eta + \zeta > 1$ were considered in [28] and [51] whereas the case $\eta + \theta \leq 0$, $\eta + \zeta > 1$ was not investigated. $\zeta < 0$ provides new three cases, namely $\eta + \theta \leq 0$, $0 < \eta + \zeta < 1$ and $0 \leq \eta + \theta < 1$, $0 < \eta + \zeta < 1$ and $\eta + \theta > 1$, $0 < \eta + \zeta < 1$. Therefore in the light of these six options, delta and nabla Hardy-Copson type inequalities, which are presented in Theorem 1-Theorem 8 and obtained for $\zeta > 1$, are extended to the case $\zeta < 0$. These inequalities are established in the same directions of those stated in the above mentioned theorems. Furthermore discrete and continuous Hardy-Copson type inequalities proven for $\zeta > 1$ are expanded to $\zeta < 0$. The application of the delta Hardy-Copson type inequalities provides that the nonoscillation criteria obtained for half linear dynamic equation yield results concerning half linear differential and half linear difference equations.

The organization of this paper can be seen as follows. The nabla time scale calculus and its main properties are introduced in Section 2. The delta version can be obtained similarly. The contribution of Section 3, which includes the main result, is

to extend the recently developed results, which are established for $\zeta > 1$ and presented in [28, 51], to the case $\zeta < 0$ by using the properties of nabla and delta derivatives and integrals. Then the special cases of nabla and delta Hardy-Copson type inequalities, which are continuous and discrete inequalities, are stated. The final section serves as applications of the obtained inequalities in oscillation theory of the half linear dynamic equations.

2. PRELIMINARIES

This section is devoted to present the main definitions and theorems of nabla time scale calculus. We refer the reader to [7, 11] for the concept of time scale calculus in delta and nabla senses.

If $\mathbb{T} \neq \emptyset$ is a closed subset of $\mathbb{R}$, then $\mathbb{T}$ is called a time scale. The backward jump operator ρ is defined as $\rho(t) := \sup(-\infty, t)_{\mathbb{T}}$, for $t \in \mathbb{T}$, provided $\sup \emptyset = \inf \mathbb{T}$. The backward graininess function $v: \mathbb{T} \rightarrow \mathbb{R}_0^+$ is defined by $v(t) := t - \rho(t)$, for $t \in \mathbb{T}$.

The ∇ -derivative of $\Gamma: \mathbb{T} \rightarrow \mathbb{R}$ at the point $t \in \mathbb{T}_{\kappa} = \mathbb{T} \setminus [\inf \mathbb{T}, \sigma(\inf \mathbb{T}))$ denoted by $\Gamma^{\nabla}(t)$ is the number enjoys the property that for all $\varepsilon > 0$, there exists a neighborhood $V \subset \mathbb{T}$ of $t \in \mathbb{T}_{\kappa}$ such that

$$|\Gamma(s) - \Gamma(\rho(t)) - \Gamma^{\nabla}(t)(s - \rho(t))| \leq \varepsilon |s - \rho(t)|$$

for all $s \in V$.

Some important features of the nabla derivative can be listed below.

Lemma 1 ([7, 11]). *Suppose that $\Lambda: \mathbb{T} \rightarrow \mathbb{R}$ and $t \in \mathbb{T}_{\kappa}$.*

- (1) *If Λ is nabla differentiable at t , then Λ is continuous at t .*
- (2) *If Λ is continuous at a left scattered point t , then Λ is nabla differentiable at t with $\Lambda^{\nabla}(t) = \frac{\Lambda(t) - \Lambda(\rho(t))}{v(t)}$.*
- (3) *Λ is nabla differentiable at a left dense point t if and only if the limit $\Lambda^{\nabla}(t) = \lim_{s \rightarrow t} \frac{\Lambda(t) - \Lambda(s)}{t - s}$ exists as a finite number.*
- (4) *If Λ is nabla differentiable at t , then $\Lambda^{\rho}(t) = \Lambda(t) - v(t)\Lambda^{\nabla}(t)$.*

A function $\Gamma: \mathbb{T} \rightarrow \mathbb{R}$ is ld-continuous if it is continuous at each left-dense points in $\mathbb{T}$ and $\lim_{s \rightarrow t^+} \Gamma(s)$ exists as a finite number for all right-dense points in $\mathbb{T}$. The set $C_{ld}(\mathbb{T}, \mathbb{R})$ denotes the class of real, ld-continuous functions defined on a time scale $\mathbb{T}$. If $\Gamma \in C_{ld}(\mathbb{T}, \mathbb{R})$, then there exists a function $\bar{\Gamma}(t)$ such that $\bar{\Gamma}^{\nabla}(t) = \Gamma(t)$ and the nabla integral of Γ is defined by $\int_a^b \Gamma(s) \nabla s = \bar{\Gamma}(b) - \bar{\Gamma}(a)$.

Some of the properties of the nabla integral are gathered next.

Lemma 2 ([7, 11]). *Let $t_1, t_2, t_3 \in \mathbb{T}$ with $t_1 < t_3 < t_2$ and $a, b \in \mathbb{R}$. If $\Lambda, \Gamma: \mathbb{T} \rightarrow \mathbb{R}$ are ld-continuous, then*

- 1) $\int_{t_1}^{t_2} [a\Lambda(s) + b\Gamma(s)] \nabla s = a \int_{t_1}^{t_2} \Lambda(s) \nabla s + b \int_{t_1}^{t_2} \Gamma(s) \nabla s$ and $\int_{t_1}^{t_1} \Lambda(s) \nabla s = 0$,
- 2) $\int_{t_1}^{t_3} \Lambda(s) \nabla s + \int_{t_3}^{t_2} \Lambda(s) \nabla s = \int_{t_1}^{t_2} \Lambda(s) \nabla s = - \int_{t_2}^{t_1} \Lambda(s) \nabla s$,
- 3) integration by parts formula holds: $\int_{t_1}^{t_2} \Lambda(s) \Gamma^\nabla(s) \nabla s = \Lambda(t_2) \Gamma(t_2) - \Lambda(t_1) \Gamma(t_1) - \int_{t_1}^{t_2} \Lambda^\nabla(s) \Gamma(\rho(s)) \nabla s$.

Lemma 3 (Hölder's inequality, [39]). *Let $t_1, t_2 \in \mathbb{T}$. For $\Lambda, \Gamma \in C_{ld}([t_1, t_2]_{\mathbb{T}}, \mathbb{R})$ and for the conjugate numbers $\kappa, \varpi > 1$ satisfying $\frac{1}{\kappa} + \frac{1}{\varpi} = 1$, Hölder's inequality*

$$\int_{t_1}^{t_2} |\Lambda(s) \Gamma(s)| \nabla s \leq \left[\int_{t_1}^{t_2} |\Lambda(s)|^\kappa \nabla s \right]^{1/\kappa} \left[\int_{t_1}^{t_2} |\Gamma(s)|^\varpi \nabla s \right]^{1/\varpi} \text{ holds true.}$$

If $0 < \kappa < 1$ or $\kappa < 0$ with $\frac{1}{\kappa} + \frac{1}{\varpi} = 1$, then the reverse Hölder's inequality

$$\int_{t_1}^{t_2} |\Lambda(s) \Gamma(s)| \nabla s \geq \left[\int_{t_1}^{t_2} |\Lambda(s)|^\kappa \nabla s \right]^{1/\kappa} \left[\int_{t_1}^{t_2} |\Gamma(s)|^\varpi \nabla s \right]^{1/\varpi} \quad (2.1)$$

is satisfied.

Lemma 4 (Chain rule for the nabla derivative, [23]). *If $\Lambda: \mathbb{R} \rightarrow \mathbb{R}$ is continuously differentiable and $\Gamma: \mathbb{T} \rightarrow \mathbb{R}$ is nabla differentiable, then $\Lambda \circ \Gamma$ is nabla differentiable and*

$$(\Lambda \circ \Gamma)^\nabla(s) = \Gamma^\nabla(s) \left[\int_0^1 \Lambda'(\Gamma(\rho(s)) + h\nu(s) \Gamma^\nabla(s)) dh \right].$$

3. HARDY-COPSON TYPE INEQUALITIES

In the sequel, we will obtain several Hardy-Copson type inequalities for non-negative, ld-continuous, ∇ -differentiable and locally nabla integrable functions z and h and for the functions $G, H, \overline{G}$ and $\overline{H}$ defined in (1.9).

The next theorem, which is proven for $\zeta < 0$, $\eta \geq 0$ and $\eta + \theta \leq 0$, provides complements of some of the previous Hardy-Copson type inequalities given for $\zeta > 1$, $\eta \geq 0$ and $0 \leq \eta + \theta < 1$. These previous Hardy-Copson type inequalities are listed as follows:

- (a) The discrete inequality obtained by Bennett [10, Corollary 5] or Leindler [35, Proposition 3].
- (b) The continuous inequality obtained by Saker et al. [51, Corollary 2.2] or Kayar and Kaymakçalan [28, Remark 3.22].
- (c) The delta inequality (1.5) in Theorem 1 obtained by Saker et al. [51, Theorem 2.2].

- (d) The nabla inequality (1.10) in Theorem 5 obtained by Kayar and Kaymakçalan [28, Theorem 3.19].

Theorem 9. For the functions z, h, G and H defined as in (1.9), suppose that there exists $L_1 > 0$ such that $\frac{G^\rho(t)}{G(t)} \leq L_1$ for $t \in (a, \infty)_{\mathbb{T}}$. Let $\zeta < 0$, $\eta \geq 0$ and $\eta + \theta \leq 0$ be real constants.

- (1) If $\eta + \zeta > 1$, then we have

$$\int_a^\infty \frac{z(t)[H^\rho(t)]^{\eta+\zeta}}{[G(t)]^{\eta+\theta}} \nabla t \leq \left[\frac{\eta + \zeta}{1 - \eta - \theta} \right]^\zeta \int_a^\infty \frac{z(t)h^\zeta(t)[H^\rho(t)]^\eta}{[G(t)]^{\eta+\theta-\zeta}} \nabla t \quad (3.1)$$

and

$$\int_a^\infty \frac{z(t)[H^\rho(t)]^{\eta+\zeta}}{[G^\rho(t)]^{\eta+\theta}} \nabla t \leq \left[\frac{L_1^{\eta+\theta-1}(\eta + \zeta)}{1 - \eta - \theta} \right]^\zeta \int_a^\infty \frac{z(t)h^\zeta(t)[H^\rho(t)]^\eta}{[G^\rho(t)]^{\eta+\theta-\zeta}} \nabla t. \quad (3.2)$$

- (2) If $0 < \eta + \zeta < 1$, then we have

$$\int_a^\infty \frac{z(t)[H(t)]^{\eta+\zeta}}{[G(t)]^{\eta+\theta}} \nabla t \leq \left[\frac{\eta + \zeta}{1 - \eta - \theta} \right]^\zeta \int_a^\infty \frac{z(t)h^\zeta(t)[H(t)]^\eta}{[G(t)]^{\eta+\theta-\zeta}} \nabla t \quad (3.3)$$

and

$$\int_a^\infty \frac{z(t)[H(t)]^{\eta+\zeta}}{[G^\rho(t)]^{\eta+\theta}} \nabla t \leq \left[\frac{\eta + \zeta}{1 - \eta - \theta} \right]^\zeta \int_a^\infty \frac{z(t)h^\zeta(t)[H(t)]^\eta}{[G^\rho(t)]^{\eta+\theta-\zeta}} \nabla t. \quad (3.4)$$

Proof. Although we try to obtain nabla Hardy-Copson type inequalities, the same methodology used in the proof of [31, Theorem 3.1] and [31, Theorem 3.4], which present nabla Bennett-Leindler type inequalities, works for the proofs of this theorem. To be clear, we sketch the proof of inequality (3.1). If the same steps are followed in the proof of [31, Theorem 3.1] for the function H and [31, Theorem 3.4] for the function G , we can show that

$$\int_a^\infty \frac{z(t)[H^\rho(t)]^{\eta+\zeta}}{[G(t)]^{\eta+\theta}} \nabla t \geq \frac{\eta + \zeta}{1 - \eta - \theta} \int_a^\infty \frac{z(t)h(t)[H^\rho(t)]^{\eta+\zeta-1}}{[G(t)]^{\eta+\theta-1}} \nabla t. \quad (3.5)$$

The difference of the proofs appear after this step. By applying reverse Hölder inequality (2.1) to the right hand side of inequality (3.5) with the constants $\zeta < 0$ and $0 < \frac{\zeta}{\zeta-1} < 1$, we get

$$\left[\int_a^\infty \frac{z(t)[H^\rho(t)]^{\eta+\zeta}}{[G(t)]^{\eta+\theta}} \nabla t \right]^{1/\zeta} \geq \frac{\eta + \zeta}{1 - \eta - \theta} \left[\int_a^\infty \frac{z(t)h(t)[H^\rho(t)]^\eta}{[G(t)]^{\eta+\theta-\zeta}} \nabla t \right]^{1/\zeta}. \quad (3.6)$$

Then raising both sides of inequality (3.6) to $\zeta < 0$ changes the direction of this inequality, which implies the desired result (3.1). The proofs of inequalities (3.2)-(3.4) can be obtained by following the same method as above. $\square$

Remark 1. The nabla Hardy-Copson type inequalities (3.1)-(3.2) obtained for $\zeta < 0$, $\eta \geq 0$ and $\eta + \theta \leq 0$ are complements of the nabla Hardy-Copson type inequalities given in [28, Theorem 3.19] for $\zeta > 1$, $\eta \geq 0$ and $0 \leq \eta + \theta < 1$.

Note that the condition $\eta + \zeta > 1$ is automatically satisfied in [28, Theorem 3.19] while the other one, $0 < \eta + \zeta < 1$, has not appeared in the literature before. Therefore the nabla inequalities (3.3)-(3.4), which are obtained for $\zeta < 0$, $\eta \geq 0$, $\eta + \theta \leq 0$ and $0 < \eta + \zeta < 1$ for the first time, offer novelties in the current literature.

Corollary 1. *From the inequalities (3.1)-(3.4) obtained by the nabla calculus, we can get the dual inequalities in the delta setting by replacing G^ρ, G, H^ρ, H presented in (1.9) by G, G^σ, H, H^σ defined in (1.4), respectively.*

Let the functions z, h, G and H be defined as in (1.4). Suppose that there exists $M_1 > 0$ such that $\frac{G(t)}{G^\sigma(t)} \leq M_1$ for $t \in (a, \infty)_{\mathbb{T}}$. In this case for $\zeta < 0$, $\eta \geq 0$ and $\eta + \theta \leq 0$, nabla Hardy-Copson type inequalities (3.1)-(3.4) become novel delta Hardy-Copson type inequalities, two of which obtained from (3.2) and (3.4) can be written as follows, respectively. For $\eta + \zeta > 1$, we have

$$\int_a^\infty \frac{z(t)[H(t)]^{\eta+\zeta}}{[G(t)]^{\eta+\theta}} \Delta t \leq \left[\frac{M_1^{\eta+\theta-1}(\eta+\zeta)}{1-\eta-\theta} \right]^\zeta \int_a^\infty \frac{z(t)h^\zeta(t)[H(t)]^\eta}{[G(t)]^{\eta+\theta-\zeta}} \Delta t$$

and for $0 < \eta + \zeta < 1$, we have

$$\int_a^\infty \frac{z(t)[H^\sigma(t)]^{\eta+\zeta}}{[G(t)]^{\eta+\theta}} \Delta t \leq \left[\frac{\eta+\zeta}{1-\eta-\theta} \right]^\zeta \int_a^\infty \frac{z(t)h^\zeta(t)[H^\sigma(t)]^\eta}{[G(t)]^{\eta+\theta-\zeta}} \Delta t.$$

The delta versions of the nabla Hardy-Copson type inequalities (3.1)-(3.2) obtained for $\zeta < 0$, $\eta \geq 0$ and $\eta + \theta \leq 0$ are complements of the delta Hardy-Copson type inequalities given in [51, Theorem 2.2] for $\zeta > 1$, $\eta \geq 0$ and $0 \leq \eta + \theta < 1$.

If $\eta = 0$, then the delta Hardy-Copson type inequalities in [47, Theorem 2.9] obtained for $\zeta > 1$ and $0 \leq \theta < 1$ are extended to the cases $\zeta < 0$ and $\theta \leq 0$ by the delta variant of nabla Hardy-Copson type inequalities (3.1)-(3.2).

Note that the condition $\eta + \zeta > 1$ is automatically satisfied in [51, Theorem 2.2] and [47, Theorem 2.9] while the other one, $0 < \eta + \zeta < 1$, has not appeared in the literature before. Therefore the delta analogues of nabla Hardy-Copson type inequalities (3.3)-(3.4), which are obtained for $\zeta < 0$, $\eta \geq 0$, $\eta + \theta \leq 0$ and $0 < \eta + \zeta < 1$ for the first time, offer novelties in the current literature by this corollary.

Remark 2. If the time scale is the set of real numbers, then for all $t \in \mathbb{R}$, the backward jump operator results in $\rho(t) = t$ and $L_1 = 1$ in (3.1)-(3.4). Hence for $0 < \eta + \zeta$, inequalities (3.1)-(3.4) coincide and their delta versions become exactly the same inequalities as them. Therefore together with its coincident inequalities,

inequality (3.1) reduces to the following inequality as

$$\int_a^\infty \frac{z(t)[H(t)]^{\eta+\zeta}}{[G(t)]^{\eta+\theta}} dt \leq \left[\frac{\eta+\zeta}{1-\eta-\theta} \right]^\zeta \int_a^\infty \frac{z(t)h^\zeta(t)[H(t)]^\eta}{[G(t)]^{\eta+\theta-\zeta}} dt, \quad (3.7)$$

where $\zeta < 0$, $\eta \geq 0$ and $\eta + \theta \leq 0$ and the functions G and H is defined as

$$G(t) = \int_t^\infty z(s) ds \quad \text{and} \quad H(t) = \int_a^t z(s)h(s) ds. \quad (3.8)$$

For the continuous case, when $\zeta > 1$, $\eta \geq 0$ and $0 \leq \eta + \theta < 1$, the first Hardy-Copson type inequality was established in [51, Corollary 2.2] and [28, Remark 3.22] for the given aforementioned functions G and H . By this remark, these results are extended to the cases $\zeta < 0$, $\eta \geq 0$ and $\eta + \theta \leq 0$ by the novel continuous Hardy-Copson type inequality (3.7).

Remark 3. If the time scale is the set of natural numbers, then for all $t \in \mathbb{N}$, the backward jump operator results in $\rho(t) = t - 1$ in (3.1)-(3.4).

Using $\int_t^\infty z(s) \nabla s = \sum_{k=t+1}^\infty z(k)$, we have $G^\rho(t) = G(t-1) = \sum_{k=t}^\infty z(k)$, where $G(t) = \sum_{k=t+1}^\infty z(k)$. Moreover $H(t) = \sum_{k=a+1}^t z(k)h(k)$. Let us assume that there exist $L_1 > 0$

such that $\frac{G(t-1)}{G(t)} \leq L_1$. For $a = 0$, $\zeta < 0$, $\eta \geq 0$ and $\eta + \theta \leq 0$, in the set of natural numbers, inequalities (3.1)-(3.4) become novel discrete Hardy-Copson type inequalities, two of which obtained from (3.2) and (3.4) can be written as follows, respectively. For $\eta + \zeta > 1$, we have

$$\sum_{t=1}^\infty \frac{z(t)[H(t-1)]^{\eta+\zeta}}{[G(t-1)]^{\eta+\theta}} \leq \left[\frac{L_1^{\eta+\theta-1}(\eta+\zeta)}{1-\eta-\theta} \right]^\zeta \sum_{t=1}^\infty \frac{z(t)h^\zeta(t)[H(t-1)]^\eta}{[G(t-1)]^{\eta+\theta-\zeta}}$$

and for $0 < \eta + \zeta < 1$, we have

$$\sum_{t=1}^\infty \frac{z(t)[H(t)]^{\eta+\zeta}}{[G(t-1)]^{\eta+\theta}} \leq \left[\frac{\eta+\zeta}{1-\eta-\theta} \right]^\zeta \sum_{t=1}^\infty \frac{z(t)h^\zeta(t)[H(t)]^\eta}{[G(t-1)]^{\eta+\theta-\zeta}}.$$

The discrete Hardy-Copson type inequality obtained by Bennett [10, Corollary 5] or Leindler [35, Proposition 3] for $\zeta > 1$, $\eta = 0$, $0 \leq \theta < 1$ and the ones obtained in [28, Remark 3.23] for $\zeta > 1$, $\eta \geq 0$, $0 \leq \eta + \theta < 1$, are extended to the cases $\zeta < 0$, $\eta \geq 0$, $\eta + \theta \leq 0$ by Theorem 9 and particularly by this remark.

Note that the condition $\eta + \zeta > 1$ is automatically satisfied in [10, Corollary 5], [35, Proposition 3] and [28, Remark 3.23] while the other one, $0 < \eta + \zeta < 1$, has not appeared in the literature before. Therefore the discrete Hardy-Copson type inequalities, which are obtained for $\zeta < 0$, $\eta \geq 0$, $\eta + \theta \leq 0$ and $0 < \eta + \zeta < 1$ for the first time, offer novelties in the current literature by this remark.

The next theorem, which is proven for $\zeta < 0$, $\eta \geq 0$ and $0 \leq \eta + \theta < 1$, provides complements of some of the previous Hardy-Copson type inequalities given for $\zeta > 1$, $\eta \geq 0$ and $0 \leq \eta + \theta < 1$. These previous Hardy-Copson type inequalities are listed as follows:

- (a) The discrete inequality obtained by Bennett [10, Corollary 5] or Leindler [35, Proposition 3].
- (b) The continuous inequality obtained by Saker et al. [51, Corollary 2.2] or Kayar and Kaymakçalan [28, Remark 3.22].
- (c) The delta inequality (1.5) in Theorem 1 obtained by Saker et al. [51, Theorem 2.2].
- (d) The nabla inequality (1.10) in Theorem 5 obtained by Kayar and Kaymakçalan [28, Theorem 3.19].

Theorem 10. For the functions z, h, G and H defined as in (1.9), suppose that there exists $L_2 > 0$ such that $1 \geq \frac{G(t)}{G^p(t)} \geq L_2$ for $t \in (a, \infty)_{\mathbb{T}}$. Let $\zeta < 0$, $\eta \geq 0$ and $0 \leq \eta + \theta < 1$ be real constants.

- (1) If $\eta + \zeta > 1$, then we have

$$\int_a^\infty \frac{z(t)[H^p(t)]^{\eta+\zeta}}{[G^p(t)]^{\eta+\theta}} \nabla t \leq \left[\frac{L_2(\eta+\zeta)}{1-\eta-\theta} \right]^\zeta \int_a^\infty \frac{z(t)h^\zeta(t)[H^p(t)]^\eta}{[G^p(t)]^{\eta+\theta-\zeta}} \nabla t \quad (3.9)$$

and

$$\int_a^\infty \frac{z(t)[H^p(t)]^{\eta+\zeta}}{[G(t)]^{\eta+\theta}} \nabla t \leq \left[\frac{\eta+\zeta}{1-\eta-\theta} \right]^\zeta \int_a^\infty \frac{z(t)h^\zeta(t)[H^p(t)]^\eta}{[G(t)]^{\eta+\theta-\zeta}} \nabla t. \quad (3.10)$$

- (2) If $0 < \eta + \zeta < 1$, then we have

$$\int_a^\infty \frac{z(t)[H(t)]^{\eta+\zeta}}{[G^p(t)]^{\eta+\theta}} \nabla t \leq \left[\frac{\eta+\zeta}{1-\eta-\theta} \right]^\zeta \int_a^\infty \frac{z(t)h^\zeta(t)[H(t)]^\eta}{[G^p(t)]^{\eta+\theta-\zeta}} \nabla t \quad (3.11)$$

and

$$\int_a^\infty \frac{z(t)[H(t)]^{\eta+\zeta}}{[G(t)]^{\eta+\theta}} \nabla t \leq \left[\frac{\eta+\zeta}{1-\eta-\theta} \right]^\zeta \int_a^\infty \frac{z(t)h^\zeta(t)[H(t)]^\eta}{[G(t)]^{\eta+\theta-\zeta}} \nabla t. \quad (3.12)$$

Proof. The combination of the techniques used in the proof of [31, Theorem 3.1] and in the proof of Theorem 9 work for the proof of this theorem. $\square$

Remark 4. The nabla Hardy-Copson type inequalities (3.9)-(3.10) obtained for $\zeta < 0$, $\eta \geq 0$ and $0 \leq \eta + \theta < 1$ are complements of the nabla Hardy-Copson type inequalities given in [28, Theorem 3.19] for $\zeta > 1$, $\eta \geq 0$ and $0 \leq \eta + \theta < 1$.

Note that the condition $\eta + \zeta > 1$ is automatically satisfied in [28, Theorem 3.19] while the other one, $0 < \eta + \zeta < 1$, has not appeared in the literature before. Therefore the nabla inequalities (3.11)-(3.12), which are obtained for $\zeta < 0$, $\eta \geq 0$, $0 \leq \eta + \theta < 1$ and $0 < \eta + \zeta < 1$ for the first time, offer novelties in the current literature.

Corollary 2. *From the inequalities (3.9)-(3.12) obtained by the nabla calculus, we can get the dual inequalities in the delta setting by replacing G^ρ, G, H^ρ, H presented in (1.9) by G, G^σ, H, H^σ defined in (1.4), respectively.*

Let the functions z, h, G and H be defined as in (1.4). Suppose that there exists $M_2 > 0$ such that $1 \leq \frac{G(t)}{G^\sigma(t)} \leq \frac{1}{M_2}$ for $t \in (a, \infty)_\mathbb{T}$. In this case for $\zeta < 0$, $\eta \geq 0$ and $0 \leq \eta + \theta < 1$, nabla Hardy-Copson type inequalities (3.9)-(3.12) become novel delta Hardy-Copson type inequalities, two of which obtained from (3.9) and (3.11) can be written as follows, respectively. For $\eta + \zeta > 1$, we have

$$\int_a^\infty \frac{z(t)[H(t)]^{\eta+\zeta}}{[G(t)]^{\eta+\theta}} \Delta t \leq \left[\frac{M_2(\eta+\zeta)}{1-\eta-\theta} \right]^\zeta \int_a^\infty \frac{z(t)h^\zeta(t)[H(t)]^\eta}{[G(t)]^{\eta+\theta-\zeta}} \Delta t$$

and for $0 < \eta + \zeta < 1$, we have

$$\int_a^\infty \frac{z(t)[H^\sigma(t)]^{\eta+\zeta}}{[G(t)]^{\eta+\theta}} \Delta t \leq \left[\frac{\eta+\zeta}{1-\eta-\theta} \right]^\zeta \int_a^\infty \frac{z(t)h^\zeta(t)[H^\sigma(t)]^\eta}{[G(t)]^{\eta+\theta-\zeta}} \Delta t.$$

The delta versions of the nabla Hardy-Copson type inequalities (3.9)-(3.10) obtained for $\zeta < 0$, $\eta \geq 0$ and $0 \leq \eta + \theta < 1$ are complements of the delta Hardy-Copson type inequalities given in [51, Theorem 2.2] for $\zeta > 1$, $\eta \geq 0$ and $0 \leq \eta + \theta < 1$.

If $\eta = 0$, then the delta Hardy-Copson type inequalities in [47, Theorem 2.9] obtained for $\zeta > 1$ and $0 \leq \theta < 1$ are extended to the cases $\zeta < 0$ and $\theta < 1$ by the delta variant of Hardy-Copson type inequalities (3.9)-(3.10).

Note that the condition $\eta + \zeta > 1$ is automatically satisfied in [51, Theorem 2.2] and [47, Theorem 2.9] while the other one, $0 < \eta + \zeta < 1$, has not appeared in the literature before. Therefore the delta analogues of nabla Hardy-Copson type inequalities (3.11)-(3.12), which are obtained for $\zeta < 0$, $\eta \geq 0$, $0 \leq \eta + \theta < 1$ and $0 < \eta + \zeta < 1$ for the first time, offer novelties in the current literature by this corollary.

Remark 5. If the time scale is the set of real numbers, then for all $t \in \mathbb{R}$, the backward jump operator results in $\rho(t) = t$ and $L_2 = 1$ in (3.9)-(3.12). Hence for $0 < \eta + \zeta$, inequalities (3.9)-(3.12) coincide and their delta versions become exactly the same inequalities as them. Therefore together with its coincident inequalities, inequality (3.9) reduces to the following inequality as

$$\int_a^\infty \frac{z(t)[H(t)]^{\eta+\zeta}}{[G(t)]^{\eta+\theta}} dt \leq \left[\frac{\eta+\zeta}{1-\eta-\theta} \right]^\zeta \int_a^\infty \frac{z(t)h^\zeta(t)[H(t)]^\eta}{[G(t)]^{\eta+\theta-\zeta}} dt, \quad (3.13)$$

where $\eta \geq 0$, $\zeta < 0$ and $0 \leq \eta + \theta < 1$ and the functions G and H is defined as in (3.8).

For the continuous case, when $\zeta > 1$, $\eta \geq 0$ and $0 \leq \eta + \theta < 1$, the first Hardy-Copson type inequality was established in [51, Corollary 2.2] and [28, Remark 3.22] for the given aforementioned functions G and H . By this remark, these results are

extended to the cases $\zeta < 0$, $\eta \geq 0$ and $\eta + \theta \leq 0$ by the novel continuous Hardy-Copson type inequality (3.13).

Remark 6. If the time scale is the set of natural numbers, then for all $t \in \mathbb{N}$, the backward jump operator results in $\rho(t) = t - 1$ in (3.9)-(3.12). Let us assume that there exist $L_2 > 0$ such that $1 \leq \frac{G(t-1)}{G(t)} \leq \frac{1}{L_2}$. For $a = 0$, $\zeta < 0$, $\eta \geq 0$ and $0 \leq \eta + \theta < 1$, in the set of natural numbers, inequalities (3.9)-(3.12) become novel discrete Hardy-Copson type inequalities, two of which obtained from (3.9) and (3.11) can be written as follows, respectively. For $\eta + \zeta > 1$, we have

$$\sum_{t=1}^{\infty} \frac{z(t)[H(t-1)]^{\eta+\zeta}}{[G(t-1)]^{\eta+\theta}} \leq \left[\frac{L_2(\eta+\zeta)}{1-\eta-\theta} \right]^{\zeta} \sum_{t=1}^{\infty} \frac{z(t)h^{\zeta}(t)[H(t-1)]^{\eta}}{[G(t-1)]^{\eta+\theta-\zeta}}$$

and for $0 < \eta + \zeta < 1$, we have

$$\sum_{t=1}^{\infty} \frac{z(t)[H(t)]^{\eta+\zeta}}{[G(t-1)]^{\eta+\theta}} \leq \left[\frac{\eta+\zeta}{1-\eta-\theta} \right]^{\zeta} \sum_{t=1}^{\infty} \frac{z(t)h^{\zeta}(t)[H(t)]^{\eta}}{[G(t-1)]^{\eta+\theta-\zeta}},$$

where the series G and H are defined as in Remark 3.

The discrete Hardy-Copson type inequality obtained by Bennett [10, Corollary 5] or Leindler [35, Proposition 3] for $\zeta > 1$, $\eta = 0$, $0 \leq \theta < 1$ and the ones obtained in [28, Remark 3.23] for $\zeta > 1$, $\eta \geq 0$, $0 \leq \eta + \theta < 1$, are extended to the cases $\zeta < 0$, $\eta \geq 0$, $0 \leq \eta + \theta < 1$ by Theorem 10 and particularly by this remark.

Note that the condition $\eta + \zeta > 1$ is automatically satisfied in [10, Corollary 5], [35, Proposition 3] and [28, Remark 3.23] while the other one, $0 < \eta + \zeta < 1$, has not appeared in the literature before. Therefore the discrete Hardy-Copson type inequalities, which are obtained for $\zeta < 0$, $\eta \geq 0$, $0 \leq \eta + \theta < 1$ and $0 < \eta + \zeta < 1$ for the first time, offer novelties in the current literature by this remark.

The next theorem, which is proven for $\zeta < 0$, $\eta \geq 0$ and $\eta + \theta \leq 0$, provides complements of some of the previous Hardy-Copson type inequalities given for $\zeta > 1$, $\eta \geq 0$ and $0 \leq \eta + \theta < 1$. These previous Hardy-Copson type inequalities are listed as follows:

- (a) The discrete inequality obtained by Copson [16, Theorem 2.1] and the discrete inequality obtained by Bennett [10, Corollary 4] or Leindler [35, Proposition 2].
- (b) The continuous inequality (1.3) obtained by Hardy et al. [26, Theorem 330], the continuous inequality obtained by Copson [17, Theorem 3] and the continuous inequality obtained by Pachpatte as inequality (6) in [40, Theorem 1] and Pečarić and Hanjš as inequality (9) in [42, Theorem 3].
- (c) The delta inequality (1.6) in Theorem 2 obtained by Saker et al. [51, Theorem 2.1].

- (d) The nabla inequality (1.11) in Theorem 6 obtained by Kayar and Kaymakçalan [28, Theorem 3.13].

Theorem 11. For the functions $z, h, \bar{G}$ and $\bar{H}$ defined as in (1.9), suppose that there exists $L_3 > 0$ such that $1 \leq \frac{\bar{G}(t)}{\bar{G}^p(t)} \leq L_3$ for $t \in (a, \infty)_{\mathbb{T}}$. Let $\zeta < 0$, $\eta \geq 0$ and $\eta + \theta \leq 0$ be real constants.

- (1) If $\eta + \zeta > 1$, then we have

$$\int_a^\infty \frac{z(t)[\bar{H}(t)]^{\eta+\zeta}}{[\bar{G}^p(t)]^{\eta+\theta}} \nabla t \leq \left[\frac{\eta + \zeta}{1 - \eta - \theta} \right]^\zeta \int_a^\infty \frac{z(t)h^\zeta(t)[\bar{H}(t)]^\eta}{[\bar{G}^p(t)]^{\eta+\theta-\zeta}} \nabla t \quad (3.14)$$

and

$$\int_a^\infty \frac{z(t)[\bar{H}(t)]^{\eta+\zeta}}{[\bar{G}(t)]^{\eta+\theta}} \nabla t \leq \left[\frac{L_3^{\eta+\theta-1}(\eta + \zeta)}{1 - \eta - \theta} \right]^\zeta \int_a^\infty \frac{z(t)h^\zeta(t)[\bar{H}(t)]^\eta}{[\bar{G}(t)]^{\eta+\theta-\zeta}} \nabla t. \quad (3.15)$$

- (2) If $0 < \eta + \zeta < 1$, then we have

$$\int_a^\infty \frac{z(t)[\bar{H}^p(t)]^{\eta+\zeta}}{[\bar{G}^p(t)]^{\eta+\theta}} \nabla t \leq \left[\frac{\eta + \zeta}{1 - \eta - \theta} \right]^\zeta \int_a^\infty \frac{z(t)h^\zeta(t)[\bar{H}^p(t)]^\eta}{[\bar{G}^p(t)]^{\eta+\theta-\zeta}} \nabla t \quad (3.16)$$

and

$$\int_a^\infty \frac{z(t)[\bar{H}^p(t)]^{\eta+\zeta}}{[\bar{G}(t)]^{\eta+\theta}} \nabla t \leq \left[\frac{\eta + \zeta}{1 - \eta - \theta} \right]^\zeta \int_a^\infty \frac{z(t)h^\zeta(t)[\bar{H}^p(t)]^\eta}{[\bar{G}(t)]^{\eta+\theta-\zeta}} \nabla t. \quad (3.17)$$

Proof. The combination of the techniques used in the proof of [31, Theorem 3.9] and in the proof of Theorem 9 work for the proof of this theorem. $\square$

Remark 7. The nabla Hardy-Copson type inequalities (3.14)-(3.15) obtained for $\zeta < 0$, $\eta \geq 0$ and $\eta + \theta \leq 0$ are complements of the nabla Hardy-Copson type inequalities given in [28, Theorem 3.13] for $\zeta > 1$, $\eta \geq 0$ and $0 \leq \eta + \theta < 1$.

Note that the condition $\eta + \zeta > 1$ is automatically satisfied in [28, Theorem 3.13] while the other one, $0 < \eta + \zeta < 1$, has not appeared in the literature before. Therefore the nabla inequalities (3.16)-(3.17), which are obtained for $\zeta < 0$, $\eta \geq 0$, $\eta + \theta \leq 0$ and $0 < \eta + \zeta < 1$ for the first time, offer novelties in the current literature.

Corollary 3. From the inequalities (3.14)-(3.17) obtained by the nabla calculus, we can get the dual inequalities in the delta setting by replacing $\bar{G}^p, \bar{G}, \bar{H}^p, \bar{H}$ presented in (1.9) by $\bar{G}, \bar{G}^\sigma, \bar{H}, \bar{H}^\sigma$ defined in (1.4), respectively.

Let the functions $z, h, \bar{G}$ and $\bar{H}$ be defined as in (1.4). Suppose that there exists $M_3 > 0$ such that $\frac{\bar{G}^\sigma(t)}{\bar{G}(t)} \leq M_3$ for $t \in (a, \infty)_{\mathbb{T}}$. In this case for $\zeta < 0$, $\eta \geq 0$ and $\eta + \theta \leq 0$, nabla Hardy-Copson type inequalities (3.14)-(3.17) become novel delta

Hardy-Copson type inequalities, two of which obtained from (3.15) and (3.17) can be written as follows, respectively. For $\eta + \zeta > 1$, we have

$$\int_a^\infty \frac{z(t)[\overline{H}^\sigma(t)]^{\eta+\zeta}}{[\overline{G}^\sigma(t)]^{\eta+\theta}} \Delta t \leq \left[\frac{M_3^{\eta+\theta-1}(\eta+\zeta)}{1-\eta-\theta} \right]^\zeta \int_a^\infty \frac{z(t)h^\zeta(t)[\overline{H}^\sigma(t)]^\eta}{[\overline{G}^\sigma(t)]^{\eta+\theta-\zeta}} \Delta t$$

and for $0 < \eta + \zeta < 1$, we have

$$\int_a^\infty \frac{z(t)[\overline{H}(t)]^{\eta+\zeta}}{[\overline{G}^\sigma(t)]^{\eta+\theta}} \Delta t \leq \left[\frac{\eta+\zeta}{1-\eta-\theta} \right]^\zeta \int_a^\infty \frac{z(t)h^\zeta(t)[\overline{H}(t)]^\eta}{[\overline{G}^\sigma(t)]^{\eta+\theta-\zeta}} \Delta t.$$

The delta versions of the nabla Hardy-Copson type inequalities (3.14)-(3.15) obtained for $\zeta < 0$, $\eta \geq 0$ and $\eta + \theta \leq 0$ are complements of the delta Hardy-Copson type inequalities given in [51, Theorem 2.1] for $\zeta > 1$, $\eta \geq 0$ and $0 \leq \eta + \theta < 1$.

If $\eta = 0$, then the delta Hardy-Copson type inequalities in [47, Theorem 2.5] obtained for $\zeta > 1$ and $0 \leq \theta < 1$ are extended to the cases $\zeta < 0$ and $\theta \leq 0$ by the delta variant of nabla Hardy-Copson type inequalities (3.14)-(3.15).

Note that the condition $\eta + \zeta > 1$ is automatically satisfied in [51, Theorem 2.1] and [47, Theorem 2.5] while the other one, $0 < \eta + \zeta < 1$, has not appeared in the literature before. Therefore the delta analogues of nabla Hardy-Copson type inequalities (3.16)-(3.17), which are obtained for $\zeta < 0$, $\eta \geq 0$, $\eta + \theta \leq 0$ and $0 < \eta + \zeta < 1$ for the first time, offer novelties in the current literature by this corollary.

Remark 8. If the time scale is the set of real numbers, then for all $t \in \mathbb{R}$, the backward jump operator results in $\rho(t) = t$ and $L_3 = 1$ in (3.14)-(3.17). Hence for $0 < \eta + \zeta$, inequalities (3.14)-(3.17) coincide and their delta versions become exactly the same inequalities as them. Therefore together with its coincident inequalities, inequality (3.14) reduces to the following inequality as

$$\int_a^\infty \frac{z(t)[\overline{H}(t)]^{\eta+\zeta}}{[\overline{G}(t)]^{\eta+\theta}} dt \leq \left[\frac{\eta+\zeta}{1-\eta-\theta} \right]^\zeta \int_a^\infty \frac{z(t)h^\zeta(t)[\overline{H}(t)]^\eta}{[\overline{G}(t)]^{\eta+\theta-\zeta}} dt, \quad (3.18)$$

where $\zeta < 0$, $\eta \geq 0$ and $\eta + \theta \leq 0$ and the functions $\overline{G}$ and $\overline{H}$ is defined as

$$\overline{G}(t) = \int_a^t z(s) ds \quad \text{and} \quad \overline{H}(t) = \int_t^\infty z(s)h(s) ds. \quad (3.19)$$

Inequality (3.18) is novel and generalizes the continuous inequality (1.3) obtained by Hardy et al. [26, Theorem 330] and the continuous inequality obtained by Copson [17, Theorem 3] for $\zeta > 1$, $\eta = 0$ and $\theta < 1$ to the cases $\zeta < 0$, $\eta \geq 0$ and $\eta + \theta \leq 0$. Moreover inequality (3.18) extends the continuous inequality obtained by Pachpatte as inequality (6) in [40, Theorem 1] and Pečarić and Hanjš as inequality (9) in [42, Theorem 3] for $\zeta > 1$, $\eta \geq 0$, $\eta + \theta < 1$ to the cases $\zeta < 0$, $\eta \geq 0$ and $\eta + \theta \leq 0$.

Remark 9. If the time scale is the set of natural numbers, then for all $t \in \mathbb{N}$, the backward jump operator results in $\rho(t) = t - 1$ in (3.14)-(3.17).

Using $\overline{G}(t) = \int_a^t z(s) \nabla s = \sum_{k=a+1}^t z(k)$, we have $\overline{G}^p(t) = \overline{G}(t-1) = \sum_{k=a+1}^{t-1} z(k)$.

Moreover $\overline{H}(t) = \sum_{k=t+1}^{\infty} z(k)f(k)$. Let us assume that there exist $L_3 > 0$ such that

$1 \leq \frac{\overline{G}(t)}{\overline{G}(t-1)} \leq L_3$. For $a = 0$, $\zeta < 0$, $\eta \geq 0$ and $\eta + \theta \leq 0$, in the set of natural numbers, inequalities (3.14)-(3.17) become novel discrete Hardy-Copson type inequalities, two of which obtained from (3.15) and (3.17) can be written as follows, respectively. For $\eta + \zeta > 1$, we have

$$\sum_{t=1}^{\infty} \frac{z(t)[\overline{H}(t)]^{\eta+\zeta}}{[\overline{G}(t)]^{\eta+\theta}} \leq \left[\frac{L_3^{\eta+\theta-1}(\eta+\zeta)}{1-\eta-\theta} \right]^{\zeta} \sum_{t=1}^{\infty} \frac{z(t)h^{\zeta}(t)[\overline{H}(t)]^{\eta}}{[\overline{G}(t)]^{\eta+\theta-\zeta}}$$

and for $\eta + \zeta > 1$, we have

$$\sum_{t=1}^{\infty} \frac{z(t)[\overline{H}(t-1)]^{\eta+\zeta}}{[\overline{G}(t)]^{\eta+\theta}} \leq \left[\frac{\eta+\zeta}{1-\eta-\theta} \right]^{\zeta} \sum_{t=1}^{\infty} \frac{z(t)h^{\zeta}(t)[\overline{H}(t-1)]^{\eta}}{[\overline{G}(t)]^{\eta+\theta-\zeta}}.$$

The discrete Hardy-Copson type inequality obtained by Copson [16, Theorem 2.1] and the discrete Hardy-Copson type inequality obtained by Bennett [10, Corollary 4] or Leindler [35, Proposition 2] for $\zeta > 1$, $\eta = 0$, $0 \leq \theta < 1$ and the ones obtained in [28, Remark 3.18] for $\zeta > 1$, $\eta \geq 0$, $0 \leq \eta + \theta < 1$, are extended to the cases $\zeta < 0$, $\eta \geq 0$, $\eta + \theta \leq 0$ by Theorem 11 and particularly by this remark.

Note that the condition $\eta + \zeta > 1$ is automatically satisfied in [16, Theorem 2.1], [10, Corollary 4], [35, Proposition 2] and [28, Remark 3.18] while the other one, $0 < \eta + \zeta < 1$, has not appeared in the literature before. Therefore the discrete Hardy-Copson type inequalities, which are obtained for $\zeta < 0$, $\eta \geq 0$, $\eta + \theta \leq 0$ and $0 < \eta + \zeta < 1$ for the first time, offer novelties in the current literature by this remark.

The next theorem, which is proven for $\zeta < 0$, $\eta \geq 0$ and $0 \leq \eta + \theta < 1$, provides complements of some of the previous Hardy-Copson type inequalities given for $\zeta > 1$, $\eta \geq 0$ and $0 \leq \eta + \theta < 1$. These previous Hardy-Copson type inequalities are listed as follows:

- (a) The discrete inequality obtained by Copson [16, Theorem 2.1] and the discrete inequality obtained by Bennett [10, Corollary 4] or Leindler [35, Proposition 2].
- (b) The continuous inequality (1.3) obtained by Hardy et al. [26, Theorem 330], the continuous inequality obtained by Copson [17, Theorem 3] and the continuous inequality obtained by Pachpatte as inequality (6) in [40, Theorem 1] and Pečarić and Hanjš as inequality (9) in [42, Theorem 3].

- (c) The delta inequality (1.6) in Theorem 2 obtained by Saker et al. [51, Theorem 2.1].
- (d) The nabla inequality (1.11) in Theorem 6 obtained by Kayar and Kaymakçalan [28, Theorem 3.13].

Theorem 12. For the functions $z, h, \bar{G}$ and $\bar{H}$ defined as in (1.9), suppose that there exists $L_4 > 0$ such that $1 \geq \frac{\bar{G}^p(t)}{\bar{G}(t)} \geq L_4$ for $t \in (a, \infty)_{\mathbb{T}}$. Let $\zeta < 0$, $\eta \geq 0$ and $0 \leq \eta + \theta < 1$ be real constants.

- (1) If $\eta + \zeta > 1$, then we have

$$\int_a^\infty \frac{z(t)[\bar{H}(t)]^{\eta+\zeta}}{[\bar{G}(t)]^{\eta+\theta}} \nabla t \leq \left[\frac{L_4(\eta+\zeta)}{1-\eta-\theta} \right]^\zeta \int_a^\infty \frac{z(t)h^\zeta(t)[\bar{H}(t)]^\eta}{[\bar{G}(t)]^{\eta+\theta-\zeta}} \nabla t \quad (3.20)$$

and

$$\int_a^\infty \frac{z(t)[\bar{H}(t)]^{\eta+\zeta}}{[\bar{G}^p(t)]^{\eta+\theta}} \nabla t \leq \left[\frac{\eta+\zeta}{1-\eta-\theta} \right]^\zeta \int_a^\infty \frac{z(t)h^\zeta(t)[\bar{H}(t)]^\eta}{[\bar{G}^p(t)]^{\eta+\theta-\zeta}} \nabla t. \quad (3.21)$$

- (2) If $0 < \eta + \zeta < 1$, then we have

$$\int_a^\infty \frac{z(t)[\bar{H}^p(t)]^{\eta+\zeta}}{[\bar{G}(t)]^{\eta+\theta}} \nabla t \leq \left[\frac{\eta+\zeta}{1-\eta-\theta} \right]^\zeta \int_a^\infty \frac{z(t)h^\zeta(t)[\bar{H}^p(t)]^\eta}{[\bar{G}(t)]^{\eta+\theta-\zeta}} \nabla t \quad (3.22)$$

and

$$\int_a^\infty \frac{z(t)[\bar{H}^p(t)]^{\eta+\zeta}}{[\bar{G}^p(t)]^{\eta+\theta}} \nabla t \leq \left[\frac{\eta+\zeta}{1-\eta-\theta} \right]^\zeta \int_a^\infty \frac{z(t)h^\zeta(t)[\bar{H}^p(t)]^\eta}{[\bar{G}^p(t)]^{\eta+\theta-\zeta}} \nabla t. \quad (3.23)$$

Proof. The combination of the techniques used in the proof of [31, Theorem 3.9] and in the proof of Theorem 9 work for the proof of this theorem. $\square$

Remark 10. The nabla Hardy-Copson type inequalities (3.20)-(3.21) obtained for $\zeta < 0$, $\eta \geq 0$ and $0 \leq \eta + \theta < 1$ are complements of the nabla Hardy-Copson type inequalities given in [28, Theorem 3.13] for $\zeta > 1$, $\eta \geq 0$ and $0 \leq \eta + \theta < 1$.

Note that the condition $\eta + \zeta > 1$ is automatically satisfied in [28, Theorem 3.13] while the other one, $0 < \eta + \zeta < 1$, has not appeared in the literature before. Therefore the nabla inequalities (3.22)-(3.23), which are obtained for $\zeta < 0$, $\eta \geq 0$, $0 \leq \eta + \theta < 1$ and $0 < \eta + \zeta < 1$ for the first time, offer novelties in the current literature.

Corollary 4. From the inequalities (3.20)-(3.23) obtained by the nabla calculus, we can get the dual inequalities in the delta setting by replacing $\bar{G}^p, \bar{G}, \bar{H}^p, \bar{H}$ presented in (1.9) by $\bar{G}, \bar{G}^\sigma, \bar{H}, \bar{H}^\sigma$ defined in (1.4), respectively.

Let the functions $z, h, \bar{G}$ and $\bar{H}$ be defined as in (1.4). Suppose that there exists $M_4 > 0$ such that $1 \geq \frac{\bar{G}(t)}{\bar{G}^\sigma(t)} \geq M_4$ for $t \in (a, \infty)_{\mathbb{T}}$. In this case for $\zeta < 0$, $\eta \geq 0$ and $0 \leq \eta + \theta < 1$, nabla Hardy-Copson type inequalities (3.20)-(3.23) become novel

delta Hardy-Copson type inequalities, two of which obtained from (3.20) and (3.22) can be written as follows, respectively. For $\eta + \zeta > 1$, we have

$$\int_a^\infty \frac{z(t)[\overline{H}^\sigma(t)]^{\eta+\zeta}}{[\overline{G}^\sigma(t)]^{\eta+\theta}} \Delta t \leq \left[\frac{M_4(\eta+\zeta)}{1-\eta-\theta} \right]^\zeta \int_a^\infty \frac{z(t)h^\zeta(t)[\overline{H}^\sigma(t)]^\eta}{[\overline{G}^\sigma(t)]^{\eta+\theta-\zeta}} \Delta t$$

and for $0 < \eta + \zeta < 1$, we have

$$\int_a^\infty \frac{z(t)[\overline{H}(t)]^{\eta+\zeta}}{[\overline{G}^\sigma(t)]^{\eta+\theta}} \Delta t \leq \left[\frac{\eta+\zeta}{1-\eta-\theta} \right]^\zeta \int_a^\infty \frac{z(t)h^\zeta(t)[\overline{H}(t)]^\eta}{[\overline{G}^\sigma(t)]^{\eta+\theta-\zeta}} \Delta t.$$

The delta versions of the nabla Hardy-Copson type inequalities (3.20)-(3.21) obtained for $\zeta < 0$, $\eta \geq 0$ and $0 \leq \eta + \theta < 1$ are complements of the delta Hardy-Copson type inequalities given in [51, Theorem 2.1] for $\zeta > 1$, $\eta \geq 0$ and $0 \leq \eta + \theta < 1$.

If $\eta = 0$, then the delta Hardy-Copson type inequalities in [47, Theorem 2.5] obtained for $\zeta > 1$ and $0 \leq \theta < 1$ are extended to the cases $\zeta < 0$ and $0 \leq \theta < 1$ by the delta variant of nabla Hardy-Copson type inequalities (3.20)-(3.21).

Note that the condition $\eta + \zeta > 1$ is automatically satisfied in [51, Theorem 2.1] and [47, Theorem 2.5] while the other one, $0 < \eta + \zeta < 1$, has not appeared in the literature before. Therefore the delta analogues of nabla Hardy-Copson type inequalities (3.22)-(3.23), which are obtained for $\zeta < 0$, $\eta \geq 0$, $0 \leq \eta + \theta < 1$ and $0 < \eta + \zeta < 1$ for the first time, offer novelties in the current literature by this corollary.

Remark 11. If the time scale is the set of real numbers, then for all $t \in \mathbb{R}$, the backward jump operator results in $\rho(t) = t$ and $L_4 = 1$ in (3.20)-(3.23). Hence for $0 < \eta + \zeta$, inequalities (3.20)-(3.23) coincide and their delta versions become exactly the same inequalities as them. Therefore together with its coincident inequalities, inequality (3.20) reduces to the following inequality as

$$\int_a^\infty \frac{z(t)[\overline{H}(t)]^{\eta+\zeta}}{[\overline{G}(t)]^{\eta+\theta}} dt \leq \left[\frac{\eta+\zeta}{1-\eta-\theta} \right]^\zeta \int_a^\infty \frac{z(t)h^\zeta(t)[\overline{H}(t)]^\eta}{[\overline{G}(t)]^{\eta+\theta-\zeta}} dt, \quad (3.24)$$

where $\zeta < 0$, $\eta \geq 0$ and $0 \leq \eta + \theta < 1$ and the functions $\overline{G}$ and $\overline{H}$ is defined as in (3.19).

Inequality (3.24) is novel and generalizes the continuous inequality (1.3) obtained by Hardy et al. [26, Theorem 330] and the continuous inequality obtained by Copson [17, Theorem 3] for $\zeta > 1$, $\eta = 0$ and $\theta < 1$ to the cases $\zeta < 0$, $\eta \geq 0$ and $0 \leq \eta + \theta < 1$. Moreover inequality (3.24) extends the continuous inequality obtained by Pachpatte as inequality (6) in [40, Theorem 1] and Pečarić and Hanjš as inequality (9) in [42, Theorem 3] for $\zeta > 1$, $\eta \geq 0$, $\eta + \theta < 1$ to the cases $\zeta < 0$, $\eta \geq 0$ and $0 \leq \eta + \theta < 1$.

Remark 12. If the time scale is the set of natural numbers, then for all $t \in \mathbb{N}$, the backward jump operator results in $\rho(t) = t - 1$ in (3.20)-(3.23). Let us assume that

there exist $L_4 > 0$ such that $\frac{\overline{G}(t)}{\overline{G}(t-1)} \leq \frac{1}{L_4}$. For $a = 0$, $\zeta < 0$, $\eta \geq 0$ and $0 \leq \eta + \theta < 1$, in the set of natural numbers, inequalities (3.20)-(3.23) becomes novel discrete Hardy-Copson type inequalities, two of which obtained from (3.20) and (3.22) can be written as follows, respectively. For $\eta + \zeta > 1$, we have

$$\sum_{t=1}^{\infty} \frac{z(t)[\overline{H}(t)]^{\eta+\zeta}}{[\overline{G}(t)]^{\eta+\theta}} \leq \left[\frac{L_4(\eta+\zeta)}{1-\eta-\theta} \right]^{\zeta} \sum_{t=1}^{\infty} \frac{z(t)h^{\zeta}(t)[\overline{H}(t)]^{\eta}}{[\overline{G}(t)]^{\eta+\theta-\zeta}}$$

and for $0 < \eta + \zeta < 1$, we have

$$\sum_{t=1}^{\infty} \frac{z(t)[\overline{H}(t-1)]^{\eta+\zeta}}{[\overline{G}(t)]^{\eta+\theta}} \leq \left[\frac{\eta+\zeta}{1-\eta-\theta} \right]^{\zeta} \sum_{t=1}^{\infty} \frac{z(t)h^{\zeta}(t)[\overline{H}(t-1)]^{\eta}}{[\overline{G}(t)]^{\eta+\theta-\zeta}},$$

where the series $\overline{G}$ and $\overline{H}$ are defined as in Remark 9.

The discrete Hardy-Copson type inequality obtained by Copson [16, Theorem 2.1] and the discrete inequality obtained by Bennett [10, Corollary 4] or Leindler [35, Proposition 2] for $\zeta > 1$, $\eta = 0$, $0 \leq \theta < 1$ and the ones obtained in [28, Remark 3.18] for $\zeta > 1$, $\eta \geq 0$, $0 \leq \eta + \theta < 1$, are extended to the cases $\zeta < 0$, $\eta \geq 0$, $0 \leq \eta + \theta < 1$ by Theorem 12 and particularly by this remark.

Note that the condition $\eta + \zeta > 1$ is automatically satisfied in [16, Theorem 2.1], [10, Corollary 4], [35, Proposition 2] and [28, Remark 3.18] while the other one, $0 < \eta + \zeta < 1$, has not appeared in the literature before. Therefore the discrete Hardy-Copson type inequalities, which are obtained for $\zeta < 0$, $\eta \geq 0$, $0 \leq \eta + \theta < 1$ and $0 < \eta + \zeta < 1$ for the first time, offer novelties in the current literature by this remark.

The next theorem, which is proven for $\zeta < 0$, $\eta \geq 0$ and $\eta + \theta > 1$, provides complements of some of the previous Hardy-Copson type inequalities given for $\zeta > 1$, $\eta \geq 0$ and $\eta + \theta > 1$. These previous Hardy-Copson type inequalities are listed as follows:

- (a) The discrete inequality (1.1) obtained by Hardy as inequality (2) in [24, Theorem B], the discrete inequality obtained by Copson [16, Theorem 1.1] and the discrete inequality obtained by Bennett [10, Corollary 3] or Leindler [35, Proposition 1].
- (b) The continuous inequality (1.2) obtained by Hardy as inequality (4) in [24, Theorem B], the continuous inequality (1.3) obtained by Hardy et al. [26, Theorem 330], the continuous inequality obtained by Copson [17, Theorem 1] and the continuous inequality obtained by Pachpatte as inequality (6) in [40, Theorem 1] and Pečarić and Hanjš as inequality (3) in [42, Theorem 1].
- (c) The delta inequality (1.8) in Theorem 4 obtained by Kayar and Kaymakçalan [28, Remark 3.2] and Saker et al. [51].
- (d) The nabla inequality (1.13) in Theorem 8 obtained by Kayar and Kaymakçalan [28, Theorem 3.1].

Theorem 13. Suppose that the functions $z, h, \bar{G}$ and H are defined as in (1.9) and the constant L_4 is defined as in Theorem 12. Let $\zeta < 0$, $\eta \geq 0$ and $\eta + \theta > 1$ be real numbers.

(1) If $\eta + \zeta > 1$, then we have

$$\int_a^\infty \frac{z(t)[H^\rho(t)]^{\eta+\zeta}}{[\bar{G}(t)]^{\eta+\theta}} \nabla t \leq \left[\frac{\eta + \zeta}{\eta + \theta - 1} \right]^\zeta \int_a^\infty \frac{z(t)h^\zeta(t)[H^\rho(t)]^\eta}{[\bar{G}(t)]^{\eta+\theta-\zeta}} \nabla t \quad (3.25)$$

and

$$\int_a^\infty \frac{z(t)[H^\rho(t)]^{\eta+\zeta}}{[\bar{G}^\rho(t)]^{\eta+\theta}} \nabla t \leq \left[\frac{L_4^{\eta+\theta-1}(\eta + \zeta)}{\eta + \theta - 1} \right]^\zeta \int_a^\infty \frac{z(t)h^\zeta(t)[H^\rho(t)]^\eta}{[\bar{G}^\rho(t)]^{\eta+\theta-\zeta}} \nabla t. \quad (3.26)$$

(2) If $0 < \eta + \zeta < 1$, then we have

$$\int_a^\infty \frac{z(t)[H(t)]^{\eta+\zeta}}{[\bar{G}(t)]^{\eta+\theta}} \nabla t \leq \left[\frac{\eta + \zeta}{\eta + \theta - 1} \right]^\zeta \int_a^\infty \frac{z(t)h^\zeta(t)[H(t)]^\eta}{[\bar{G}(t)]^{\eta+\theta-\zeta}} \nabla t \quad (3.27)$$

and

$$\int_a^\infty \frac{z(t)[H(t)]^{\eta+\zeta}}{[\bar{G}^\rho(t)]^{\eta+\theta}} \nabla t \leq \left[\frac{\eta + \zeta}{\eta + \theta - 1} \right]^\zeta \int_a^\infty \frac{z(t)h^\zeta(t)[H(t)]^\eta}{[\bar{G}^\rho(t)]^{\eta+\theta-\zeta}} \nabla t. \quad (3.28)$$

Proof. The combination of the techniques used in the proof of [31, Theorem 3.12] and in the proof of Theorem 9 work for the proof of this theorem. $\square$

Remark 13. The nabla Hardy-Copson type inequalities (3.25)-(3.26) obtained for $\zeta < 0$, $\eta \geq 0$ and $\eta + \theta > 1$ are complements of the nabla Hardy-Copson type inequalities given in [28, Theorem 3.1] for $\zeta > 1$, $\eta \geq 0$ and $\eta + \theta > 1$.

Note that the condition $\eta + \zeta > 1$ is automatically satisfied in [28, Theorem 3.1] while the other one, $0 < \eta + \zeta < 1$, has not appeared in the literature before. Therefore the nabla inequalities (3.27)-(3.28), which are obtained for $\zeta < 0$, $\eta \geq 0$, $\eta + \theta > 1$ and $0 < \eta + \zeta < 1$ for the first time, offer novelties in the current literature.

Corollary 5. From the inequalities (3.25)-(3.28) obtained by the nabla calculus, we can get the dual inequalities in the delta setting by replacing $\bar{G}^\rho, \bar{G}, H^\rho, H$ presented in (1.9) by $\bar{G}, \bar{G}^\sigma, H, H^\sigma$ defined in (1.4), respectively.

Let the functions $z, h, \bar{G}$ and H be defined as in (1.4) and the constant M_4 be defined as in Corollary 4. In this case for $\zeta < 0$, $\eta \geq 0$ and $\eta + \theta > 1$, nabla Hardy-Copson type inequalities (3.25)-(3.28) become novel delta Hardy-Copson type inequalities, two of which obtained from (3.26) and (3.28) can be written as follows, respectively. For $\eta + \zeta > 1$, we have

$$\int_a^\infty \frac{z(t)[H(t)]^{\eta+\zeta}}{[\bar{G}(t)]^{\eta+\theta}} \Delta t \leq \left[\frac{M_4^{\eta+\theta-1}(\eta + \zeta)}{\eta + \theta - 1} \right]^\zeta \int_a^\infty \frac{z(t)h^\zeta(t)[H(t)]^\eta}{[\bar{G}(t)]^{\eta+\theta-\zeta}} \Delta t$$

and for $0 < \eta + \zeta < 1$, we have

$$\int_a^\infty \frac{z(t)[H^\sigma(t)]^{\eta+\zeta}}{[\overline{G}(t)]^{\eta+\theta}} \Delta t \leq \left[\frac{\eta + \zeta}{\eta + \theta - 1} \right]^\zeta \int_a^\infty \frac{z(t)h^\zeta(t)[H^\sigma(t)]^\eta}{[\overline{G}(t)]^{\eta+\theta-\zeta}} \Delta t.$$

The delta versions of the nabla Hardy-Copson type inequalities (3.25)-(3.26) obtained for $\zeta < 0$, $\eta \geq 0$ and $\eta + \theta > 1$ are complements of the delta Hardy-Copson type inequalities given in [28, Remark 3.2] and [51] for $\zeta > 1$, $\eta \geq 0$ and $\eta + \theta > 1$.

If $\eta = 0$, then the delta Hardy-Copson type inequalities in [47, Theorem 2.1] obtained for $\zeta > 1$ and $\theta > 1$ are extended to the cases $\zeta < 0$ and $\theta > 1$ by the delta variant of nabla Hardy-Copson type inequalities (3.25)-(3.26).

Note that the condition $\eta + \zeta > 1$ is automatically satisfied in [28, Remark 3.2] and [51] while the other one, $0 < \eta + \zeta < 1$, has not appeared in the literature before. Therefore the delta analogues of nabla Hardy-Copson type inequalities (3.27)-(3.28), which are obtained for $\zeta < 0$, $\eta \geq 0$, $\eta + \theta > 1$ and $0 < \eta + \zeta < 1$ for the first time, offer novelties in the current literature by this corollary.

Remark 14. If the time scale is the set of real numbers, then for all $t \in \mathbb{R}$, the backward jump operator results in $\rho(t) = t$ and $L_4 = 1$ in (3.25)-(3.28). Hence for $0 < \eta + \zeta$, inequalities (3.25)-(3.28) coincide and their delta versions become exactly the same inequalities as them. Therefore together with its coincident inequalities, inequality (3.25) reduces to the following inequality as

$$\int_a^\infty \frac{z(t)[H(t)]^{\eta+\zeta}}{[\overline{G}(t)]^{\eta+\theta}} dt \leq \left[\frac{\eta + \zeta}{\eta + \theta - 1} \right]^\zeta \int_a^\infty \frac{z(t)h^\zeta(t)[H(t)]^\eta}{[\overline{G}(t)]^{\eta+\theta-\zeta}} dt, \quad (3.29)$$

where $\zeta < 0$, $\eta \geq 0$ and $\eta + \theta > 1$ and the functions $\overline{G}$ and H is defined as in (3.19) and (3.8), respectively.

Inequality (3.29) is novel and generalizes the continuous inequality (1.2) obtained by Hardy as inequality (4) in [24, Theorem B], the continuous inequality (1.3) obtained by Hardy et al. [26, Theorem 330], the continuous inequality obtained by Copson [17, Theorem 1] for $\zeta > 1$, $\eta = 0$, $\theta > 1$ to the cases $\zeta < 0$, $\eta \geq 0$, $\eta + \theta > 1$. Moreover inequality (3.29) extends the continuous inequality obtained by Pachpatte as inequality (6) in [40, Theorem 1] and Pečarić and Hanjš as inequality (3) in [42, Theorem 1] for $\zeta > 1$, $\eta \geq 0$, $\eta + \theta > 1$ to the case $\zeta < 0$, $\eta \geq 0$, $\eta + \theta > 1$.

Remark 15. If the time scale is the set of natural numbers, then for all $t \in \mathbb{N}$, the backward jump operator results in $\rho(t) = t - 1$ in (3.25)-(3.28). Let the constant L_4 be defined as in Remark 12. For $a = 0$, $\zeta < 0$, $\eta \geq 0$ and $\eta + \theta > 1$, in the set of natural numbers, inequalities (3.25)-(3.28) becomes novel discrete Hardy-Copson type inequalities, two of which obtained from (3.26) and (3.28) can be written as

follows, respectively. For $\eta + \zeta > 1$, we have

$$\sum_{t=1}^{\infty} \frac{z(t)[H(t-1)]^{\eta+\zeta}}{[\overline{G}(t-1)]^{\eta+\theta}} \leq \left[\frac{L_4^{\eta+\theta-1}(\eta+\zeta)}{\eta+\theta-1} \right]^{\zeta} \sum_{t=1}^{\infty} \frac{z(t)h^{\zeta}(t)[H(t-1)]^{\eta}}{[\overline{G}(t-1)]^{\eta+\theta-\zeta}}$$

and for $0 < \eta + \zeta < 1$,

$$\sum_{t=1}^{\infty} \frac{z(t)[H(t)]^{\eta+\zeta}}{[\overline{G}(t-1)]^{\eta+\theta}} \leq \left[\frac{\eta+\zeta}{\eta+\theta-1} \right]^{\zeta} \sum_{t=1}^{\infty} \frac{z(t)h^{\zeta}(t)[H(t)]^{\eta}}{[\overline{G}(t-1)]^{\eta+\theta-\zeta}},$$

where the series $\overline{G}$ and H are defined as in Remark 9 and Remark 3, respectively.

The discrete Hardy-Copson type inequality (1.1) obtained by Hardy as inequality (2) in [24, Theorem B], the discrete inequality obtained by Copson [16, Theorem 1.1] and the discrete inequality obtained by Bennett [10, Corollary 3] or Leindler [35, Proposition 1] for $\zeta > 1$, $\eta = 0$, $\theta > 1$ and the ones obtained in [28, Remark 3.7] for $\zeta > 1$, $\eta \geq 0$, $\eta + \theta > 1$, are extended to the cases $\zeta < 0$, $\eta \geq 0$, $\eta + \theta > 1$ by Theorem 13 and particularly by this remark.

Note that the condition $\eta + \zeta > 1$ is automatically satisfied in [24, Theorem B], [16, Theorem 1.1], [10, Corollary 3], [35, Proposition 1] and [28, Remark 3.7] while the other one, $0 < \eta + \zeta < 1$, has not appeared in the literature before. Therefore the discrete Hardy-Copson type inequalities, which are obtained for $\zeta < 0$, $\eta \geq 0$, $\eta + \theta > 1$ and $0 < \eta + \zeta < 1$ for the first time, offer novelties in the current literature by this remark.

The next theorem, which is proven for $\zeta < 0$, $\eta \geq 0$ and $\eta + \theta > 1$, provides complements of some of the previous Hardy-Copson type inequalities given for $\zeta > 1$, $\eta \geq 0$ and $\eta + \theta > 1$. These previous Hardy-Copson type inequalities are listed as follows:

- (a) The discrete inequality obtained by Bennett [10, Corollary 6] or Leindler [35, Proposition 4].
- (b) The continuous inequality obtained by Saker et al. [51, Corollary 2.3] and by Kayar and Kaymakçalan [28, Remark 3.11].
- (c) The delta inequality (1.7) in Theorem 3 obtained by Saker et al. [51, Theorem 2.3].
- (d) The nabla inequality (1.12) in Theorem 7 obtained by Kayar and Kaymakçalan [28, Theorem 3.8].

Theorem 14. Suppose that the functions z, h, G and $\overline{H}$ are defined as in (1.9) and the constant L_2 is defined as in Theorem 10. Let $\zeta < 0$, $\eta \geq 0$ and $\eta + \theta > 1$ be real numbers.

- (1) If $\eta + \zeta > 1$, then we have

$$\int_a^{\infty} \frac{z(t)[\overline{H}(t)]^{\eta+\zeta}}{[G^p(t)]^{\eta+\theta}} \nabla t \leq \left[\frac{\eta+\zeta}{\eta+\theta-1} \right]^{\zeta} \int_a^{\infty} \frac{z(t)h^{\zeta}(t)[\overline{H}(t)]^{\eta}}{[G^p(t)]^{\eta+\theta-\zeta}} \nabla t \quad (3.30)$$

and

$$\int_a^\infty \frac{z(t)[\overline{H}(t)]^{\eta+\zeta}}{[G(t)]^{\eta+\theta}} \nabla t \leq \left[\frac{L_2^{\eta+\theta-1}(\eta+\zeta)}{\eta+\theta-1} \right]^\zeta \int_a^\infty \frac{z(t)h^\zeta(t)[\overline{H}(t)]^\eta}{[G(t)]^{\eta+\theta-\zeta}} \nabla t. \quad (3.31)$$

(2) If $0 < \eta + \zeta < 1$, then we have

$$\int_a^\infty \frac{z(t)[\overline{H}^p(t)]^{\eta+\zeta}}{[G^p(t)]^{\eta+\theta}} \nabla t \leq \left[\frac{\eta+\zeta}{\eta+\theta-1} \right]^\zeta \int_a^\infty \frac{z(t)h^\zeta(t)[\overline{H}^p(t)]^\eta}{[G^p(t)]^{\eta+\theta-\zeta}} \nabla t \quad (3.32)$$

and

$$\int_a^\infty \frac{z(t)[\overline{H}^p(t)]^{\eta+\zeta}}{[G(t)]^{\eta+\theta}} \nabla t \leq \left[\frac{\eta+\zeta}{\eta+\theta-1} \right]^\zeta \int_a^\infty \frac{z(t)h^\zeta(t)[\overline{H}^p(t)]^\eta}{[G(t)]^{\eta+\theta-\zeta}} \nabla t. \quad (3.33)$$

Proof. The combination of the techniques used in the proof of [31, Theorem 3.4] and in the proof of Theorem 9 work for the proof of this theorem. $\square$

Remark 16. The nabla Hardy-Copson type inequalities (3.30)-(3.31) obtained for $\zeta < 0$, $\eta \geq 0$ and $\eta + \theta > 1$ are complements of the nabla Hardy-Copson type inequalities given in [28, Theorem 3.8] for $\zeta > 1$, $\eta \geq 0$ and $\eta + \theta > 1$.

Note that the condition $\eta + \zeta > 1$ is automatically satisfied in [28, Theorem 3.8] while the other one, $0 < \eta + \zeta < 1$, has not appeared in the literature before. Therefore the nabla inequalities (3.32)-(3.33), which are obtained for $\zeta < 0$, $\eta \geq 0$, $\eta + \theta > 1$ and $0 < \eta + \zeta < 1$ for the first time, offer novelties in the current literature.

Corollary 6. From the inequalities (3.30)-(3.33) obtained by the nabla calculus, we can get the dual inequalities in the delta setting by replacing $G^p, G, \overline{H}^p, \overline{H}$ presented in (1.9) by $G, G^\sigma, \overline{H}, \overline{H}^\sigma$ defined in (1.4), respectively.

Let the functions z, h, G and $\overline{H}$ be defined as in (1.4) and the constant M_2 be defined as in Corollary 2. In this case for $\zeta < 0$, $\eta \geq 0$ and $\eta + \theta > 1$, nabla Hardy-Copson type inequalities (3.30)-(3.33) become novel delta Hardy-Copson type inequalities, two of which obtained from (3.31) and (3.33) can be written as follows, respectively. For $\eta + \zeta > 1$, we have

$$\int_a^\infty \frac{z(t)[\overline{H}^\sigma(t)]^{\eta+\zeta}}{[G^\sigma(t)]^{\eta+\theta}} \Delta t \leq \left[\frac{M_2^{\eta+\theta-1}(\eta+\zeta)}{\eta+\theta-1} \right]^\zeta \int_a^\infty \frac{z(t)h^\zeta(t)[\overline{H}^\sigma(t)]^\eta}{[G^\sigma(t)]^{\eta+\theta-\zeta}} \Delta t$$

and for $0 < \eta + \zeta < 1$, we have

$$\int_a^\infty \frac{z(t)[\overline{H}(t)]^{\eta+\zeta}}{[G^\sigma(t)]^{\eta+\theta}} \Delta t \leq \left[\frac{\eta+\zeta}{\eta+\theta-1} \right]^\zeta \int_a^\infty \frac{z(t)h^\zeta(t)[\overline{H}(t)]^\eta}{[G^\sigma(t)]^{\eta+\theta-\zeta}} \Delta t.$$

The delta versions of the nabla Hardy-Copson type inequalities (3.30)-(3.31) obtained for $\zeta < 0$, $\eta \geq 0$ and $\eta + \theta > 1$ are complements of the delta Hardy-Copson type inequalities given in [51, Theorem 2.3] for $\zeta > 1$, $\eta \geq 0$ and $\eta + \theta > 1$.

If $\eta = 0$, then the delta Hardy-Copson type inequalities in [47, Theorem 2.10] obtained for $\zeta > 1$ and $\theta > 1$ are extended to the cases $\zeta < 0$ and $\theta > 1$ by the delta variant of nabla Hardy-Copson type inequalities (3.30)-(3.31).

Note that the condition $\eta + \zeta > 1$ is automatically satisfied in [51, Theorem 2.3] and [47, Theorem 2.10] while the other one, $0 < \eta + \zeta < 1$, has not appeared in the literature before. Therefore the delta analogues of nabla Hardy-Copson type inequalities (3.32)-(3.33), which are obtained for $\zeta < 0$, $\eta \geq 0$, $\eta + \theta > 1$ and $0 < \eta + \zeta < 1$ for the first time, offer novelties in the current literature by this corollary.

Remark 17. If the time scale is set of real numbers, then for all $t \in \mathbb{R}$, the backward jump operator results in $\rho(t) = t$ and $L_2 = 1$ in (3.30)-(3.33). Hence for $0 < \eta + \zeta$, inequalities (3.30)-(3.33) coincide and their delta versions become exactly the same inequalities as them. Therefore together with its coincident inequalities, inequality (3.30) reduces to the following inequality as

$$\int_a^\infty \frac{z(t)[\overline{H}(t)]^{\eta+\zeta}}{[G(t)]^{\eta+\theta}} dt \leq \left[\frac{\eta + \zeta}{\eta + \theta - 1} \right]^\zeta \int_a^\infty \frac{z(t)h^\zeta(t)[\overline{H}(t)]^\eta}{[G(t)]^{\eta+\theta-\zeta}} dt, \quad (3.34)$$

where $\zeta < 0$, $\eta \geq 0$ and $\eta + \theta > 1$ and the functions G and $\overline{H}$ is defined as in (3.8) and (3.19), respectively.

For the continuous case, when $\zeta > 1$, $\eta \geq 0$ and $\eta + \theta > 1$, the first Hardy-Copson type inequality was established in [51, Corollary 2.3] and [28, Remark 3.11] for the given aforementioned functions G and $\overline{H}$. By this remark, these inequalities are extended to the cases $\zeta < 0$, $\eta \geq 0$ and $\eta + \theta > 1$ by the novel continuous Hardy-Copson type inequality (3.34).

Remark 18. If the time scale is set of natural numbers, then for all $t \in \mathbb{N}$, the backward jump operator results in $\rho(t) = t - 1$ in (3.30)-(3.33). Let the constant L_2 be defined as in Remark 6. For $a = 0$, $\zeta < 0$, $\eta \geq 0$ and $\eta + \theta > 1$, in the set of natural numbers, inequalities (3.30)-(3.33) become novel discrete Hardy-Copson type inequalities, two of which obtained from (3.31) and (3.33) can be written as follows, respectively. For $\eta + \zeta > 1$, we have

$$\sum_{t=1}^\infty \frac{z(t)[\overline{H}(t)]^{\eta+\zeta}}{[G(t)]^{\eta+\theta}} \leq \left[\frac{L_2^{\eta+\theta-1}(\eta + \zeta)}{\eta + \theta - 1} \right]^\zeta \sum_{t=1}^\infty \frac{z(t)h^\zeta(t)[\overline{H}(t)]^\eta}{[G(t)]^{\eta+\theta-\zeta}}$$

and for $0 < \eta + \zeta < 1$, we have

$$\sum_{t=1}^\infty \frac{z(t)[\overline{H}(t-1)]^{\eta+\zeta}}{[G(t)]^{\eta+\theta}} \leq \left[\frac{\eta + \zeta}{\eta + \theta - 1} \right]^\zeta \sum_{t=1}^\infty \frac{z(t)h^\zeta(t)[\overline{H}(t-1)]^\eta}{[G(t)]^{\eta+\theta-\zeta}},$$

where the series $\overline{H}$ and G are defined as in Remark 9 and Remark 3, respectively.

The discrete Hardy-Copson type inequality obtained by Bennett [10, Corollary 6] or Leindler [35, Proposition 4] for $\zeta > 1$, $\eta = 0$, $\theta > 1$ and the ones obtained in

[28, Remark 3.12] for $\zeta > 1$, $\eta \geq 0$, $\eta + \theta > 1$, are extended to the cases $\zeta < 0$, $\eta \geq 0$, $\eta + \theta > 1$ by Theorem 14 and particularly by this remark.

Note that the condition $\eta + \zeta > 1$ is automatically satisfied in [10, Corollary 6], [35, Proposition 4] and [28, Remark 3.12] while the other one, $0 < \eta + \zeta < 1$, has not appeared in the literature before. Therefore the discrete Hardy-Copson type inequalities, which are obtained for $\zeta < 0$, $\eta \geq 0$, $\eta + \theta > 1$ and $0 < \eta + \zeta < 1$ for the first time, offer novelties in the current literature by this remark.

4. APPLICATIONS

This section is devoted to one of the applications of Hardy-Copson inequalities obtained in Section 3. We only employ the delta versions of the nabla inequalities (3.3) and (3.11) presented in Theorem 9 and Theorem 10, respectively, to find nonoscillation criteria for the related half linear dynamic equations. For the other delta inequalities including the term $H^\sigma(t)$, the similar process can be followed and similar theorems can be obtained.

Before proving nonoscillation criteria, let us show Hardy-Copson inequalities that we will use. The nabla inequalities (3.3) and (3.11) presented in Theorem 9 and Theorem 10, respectively, become the following delta Hardy-Copson inequalities.

Corollary 7. *Let the functions z, h, G and H be defined as in (1.4). Let $\zeta < 0$, $\eta \geq 0$ and $0 < \eta + \zeta < 1$.*

(i) *If $\eta + \theta \leq 0$, then we have delta Hardy-Copson inequality*

$$\int_a^\infty \frac{z(t)[H^\sigma(t)]^{\eta+\zeta}}{[G^\sigma(t)]^{\eta+\theta}} \Delta t \leq \left[\frac{\eta + \zeta}{1 - \eta - \theta} \right]^\zeta \int_a^\infty \frac{z(t)h^\zeta(t)[H^\sigma(t)]^\eta}{[G^\sigma(t)]^{\eta+\theta-\zeta}} \Delta t. \quad (4.1)$$

(ii) *If $0 \leq \eta + \theta < 1$, then we have delta Hardy-Copson inequality*

$$\int_a^\infty \frac{z(t)[H^\sigma(t)]^{\eta+\zeta}}{[G(t)]^{\eta+\theta}} \Delta t \leq \left[\frac{\eta + \zeta}{1 - \eta - \theta} \right]^\zeta \int_a^\infty \frac{z(t)h^\zeta(t)[H^\sigma(t)]^\eta}{[G(t)]^{\eta+\theta-\zeta}} \Delta t. \quad (4.2)$$

We focus on the following second order half linear dynamic equations

$$\Delta(r(t)\Phi(x))^\Delta + c(t)\Phi(x^\sigma) = 0 \quad (4.3)$$

and

$$\Delta(R(t)\Phi(y))^\Delta + C(t)\Phi(y^\sigma) = 0 \quad (4.4)$$

where $r, c, R, C \in C_{rd}(I, \mathbb{R})$, $\Phi(x) = |x|^{p-1} \operatorname{sgn} x$ for $p > 1$ and $R(t), r(t) \neq 0$. Moreover, $I := [a, b]_{\mathbb{T}}$ with $a < \rho(b)$ and $I_a := [a, \infty)$ in case $\sup \mathbb{T} = \infty$. For the detailed information about half linear dynamic equations and their oscillation properties, we refer [2, 53, 54]. The oscillation theory of second order half linear dynamic equations has been extensively studied and many techniques have been developed to find conditions for oscillation and nonoscillation of equation (4.3). In order to focus on the two important methods, which are Reid Roundabout Theorem and Sturm Comparison Theorem, we need to give the fundamental definitions of the oscillation theory.

Definition 1 ([53]). A solution x of equation (4.3) has a generalized zero at the point t if $x(t) = 0$. A solution x of equation (4.3) has a generalized zero in $(t, \sigma(t))$ if $r(t)y(t)y(\sigma(t)) < 0$. Equation (4.3) is disconjugate on the interval I , if there is no nontrivial solution of equation (4.3) with two (or more) generalized zeros in I .

Definition 2 ([53]). If there exists $c \in [a, \infty)$ such that equation (4.3) is disconjugate on $[c, d]$ for every $d > c$, then it is called nonoscillatory (on $[a, \infty)$).

Proposition 1 (Reid Roundabout Theorem, [53], Section 5, Theorem 2). *Let*

$$U = U(a; b) = \{\chi: \chi \in C_{rd}^1(I, \mathbb{R}) \text{ with } \chi(a) = \chi(b) = 0\}$$

be the class of admissible functions. The following are equivalent.

(a) *The p -degree functional F on U , which is defined as*

$$F(\chi; a, b) = \int_a^b [r(t)|\chi^\Delta(t)|^p - c(t)|\chi^\sigma(t)|^p \Delta t],$$

is positive definite on U provided $F(\chi) \geq 0$ for all $\chi \in U$ and $F(\chi) = 0$ if and only if $\chi = 0$.

(b) *Equation (4.3) is disconjugate on I and so, it is nonoscillatory.*

Proposition 2 (Sturm type Comparison Theorem, [53], Section 6, Theorem 3). *Suppose that $R(t) \geq r(t)$ and $c(t) \geq C(t)$ for $t \in I^\kappa \subseteq \mathbb{T}^\kappa$. If equation (4.3) is disconjugate (nonoscillatory) on I (on $[a, \infty)$), then so is equation (4.4).*

Now we are ready to present necessary and sufficient condition for the nonoscillation of the half linear dynamic equation (4.3).

Theorem 15. *Suppose that the functions z, h, G and H are defined as in (1.9). Let $\zeta < 0$, $\eta \geq 0$, $0 < \eta + \zeta < 1$ and $p > 1$. We define $E = \left[\frac{1 - \eta - \theta}{\eta + \zeta} \right]^\zeta$.*

(1) *For $\eta + \theta \leq 0$ and for $H(t) \in U$, if $r(t) = \frac{z^{1-p}(t) h^{\zeta-p}(t) [H^\sigma(t)]^\eta}{[G^\sigma(t)]^{\eta+\theta-\zeta}}$ and*

$$c(t) = E \frac{z(t) [H^\sigma(t)]^{\eta+\zeta-p}}{[G^\sigma(t)]^{\eta+\theta}}, \text{ then equation (4.3) is nonoscillatory.}$$

(2) *For $0 \leq \eta + \theta < 1$ and for $H(t) \in U$, if $r(t) = \frac{z^{1-p}(t) h^{\zeta-p}(t) [H^\sigma(t)]^\eta}{[G(t)]^{\eta+\theta-\zeta}}$ and*

$$c(t) = E \frac{z(t) [H^\sigma(t)]^{\eta+\zeta-p}}{[G(t)]^{\eta+\theta}}, \text{ then equation (4.3) is nonoscillatory.}$$

Proof. We follow the technique used in [55, Claim 3.7]. Let us choose $a > 0$ for $a \in \mathbb{T}$. For the nonnegative functions h and z , let us define an admissible function $H \in U$ introduced in Proposition 1 as $H(t) = \int_a^t z(s)h(s)\Delta s$. It is obvious that $\Delta H(t) = z(t)h(t)$.

(1) We have

$$\begin{aligned} \int_a^b [r(t)|H^\Delta(t)|^p - c(t)|H^\sigma(t)|^p] \Delta t &= \int_a^b [r(t)[z(t)h(t)]^p - c(t)[H^\sigma(t)]^p] \Delta t \\ &= \int_a^b \left[\frac{z^{1-p}(t) h^{\zeta-p}(t) [H^\sigma(t)]^\eta}{[G^\sigma(t)]^{\eta+\theta-\zeta}} [z(t)h(t)]^p - E \frac{z(t) [H^\sigma(t)]^{\eta+\zeta-p} (H^\sigma(t))^p}{[G^\sigma(t)]^{\eta+\theta}} \right] \Delta t \\ &= \int_a^b \left[\frac{z(t)h^\zeta(t)[H^\sigma(t)]^\eta}{[G^\sigma(t)]^{\eta+\theta-\zeta}} - E \frac{z(t)[H^\sigma(t)]^{\eta+\zeta}}{[G^\sigma(t)]^{\eta+\theta}} \right] \Delta t \geq 0 \end{aligned}$$

because of the delta Hady-Copson inequality (4.1). Hence, equation (4.3) is nonoscillatory by Proposition 1.

(2) The proof can be obtained from the proof of (1) by replacing $G^\sigma(t)$ by $G(t)$ and by using delta Hady-Copson inequality (4.2).

□

Theorem 16. *Let the assumptions of Theorem 15 hold and $R(t) \geq r(t)$ and $C(t) \leq c(t)$ on I^κ .*

- (i) *If $r(t)$ and $c(t)$ are defined as in (1) of Theorem 15, then equation (4.4) is nonoscillatory.*
- (ii) *If $r(t)$ and $c(t)$ are defined as in (2) of Theorem 15, then equation (4.4) is nonoscillatory.*

Proof. After using the fact that if the sequences $r(t)$ and $c(t)$ are defined as in (1) and (2) of Theorem 15, then equation (4.3) is nonoscillatory, if we employ the Sturm type comparison theorem (Proposition 2), then we can prove (i) and (ii), respectively.

□

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CONVERGENCE ANALYSIS OF SEMI-EXPONENTIAL POST-WIDDER OPERATORS

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Abstract. In the present article, we provide a recurrence relation for the semi-exponential Post-Widder operators (1.1) and estimate the moments for these operators. The next section discusses some convergence results in the Lipschitz-type space and estimates the rate of convergence with the help of the Ditzian-Totik modulus of smoothness and weighted modulus of continuity. At last, we estimate the rate of convergence for the functions whose derivatives are of bounded variation.

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1. INTRODUCTION

In the literature on approximation theory, several generalizations of exponential-type operators have been studied by many authors. The authors have focused on examining the rate of convergence of these generalizations. Recently, some researchers have introduced the concept of semi-exponential operators from the exponential-type operators.

The Post-Widder operators for $n \in \mathbb{N}$ and $x \in [0, \infty)$ considered by D.V. Widder [12] is defined as:

$$P_n(\varpi; x) = \frac{1}{n!} \left(\frac{n}{x}\right)^{n+1} \int_0^\infty \lambda^n e^{-\frac{n\lambda}{x}} \varpi(\lambda) d\lambda.$$

Following [12], For $x \in [0, \infty)$ and a parameter $p > 0$ Rathore and Singh [10] proposed an integral representation of Post-Widder operators as:

$$P_{n,p}(\varpi; x) = \frac{1}{(n+p)!} \left(\frac{n}{x}\right)^{n+p+1} \int_0^\infty \lambda^{n+p} e^{-\frac{n\lambda}{x}} \varpi(\lambda) d\lambda.$$

Recently, for $x \in [0, \infty)$ and $\rho > 0$ using laplace transformation M. Herzog [5] defined the semi exponential Post-Widder operators as:

$$W_n^\rho(\varpi; x) = \frac{n}{\lambda^n e^{\rho\lambda}} \int_0^\infty \frac{\left(\frac{n\lambda}{\rho}\right)^{\frac{n-1}{2}} I_{n-1}(2\sqrt{n\rho\lambda})}{e^{\frac{n\lambda}{\rho}}} \varpi(\lambda) d\lambda.$$

Following [5], an alternative approach of semi exponential Post-Widder operators is given by U. Abel et al. [2] which is defined as:

$$\mathcal{P}_{n,m}^\rho(\varpi; x) = \frac{n^n}{e^{\rho x} x^n} \sum_{m=0}^\infty \frac{(n\rho)^m}{m!} \frac{1}{\Gamma(n+m)} \int_0^\infty \lambda^{n+m-1} e^{-\frac{n\lambda}{x}} \varpi(\lambda) d\lambda \quad (1.1)$$

and an alternative form of operators (1.1) is defined as follows:

$$\mathcal{P}_{n,m}^\rho(\varpi; x) = \frac{n^n}{e^{\rho x} x^n} \int_0^\infty \left(\frac{n\lambda}{\rho}\right)^{\frac{n-1}{2}} e^{-\frac{n\lambda}{x}} I_{n-1}(2\sqrt{n\rho\lambda}) \varpi(\lambda) d\lambda. \quad (1.2)$$

Where I_n represents the modified Bessel function of first kind. For more literature related to this article we may refer [1, 4, 6, 9, 11].

In this article, we study some fundamental properties of the operators (1.1), including the rate of convergence using modulus of continuity, Lipschitz-type space, and weighted space.

2. BASIC PROPERTIES

In this section, we discuss some useful lemmas and results.

Remark 1. For $\rho > 0$, if we denote $H_{n,r}^\rho = \mathcal{P}_{n,m}^\rho(e_r; x)$, $e_r(\lambda) = \lambda^r$, $r = 1, 2, 3, \dots, x > 0$ then

$$nH_{n,r+1}^\rho(x) = x(\rho x + n)H_{n,r}^\rho(x) + x^2 H_{n,r}^{\rho'}(x).$$

Lemma 1. Using the Remark 1, the moments of the operators $\mathcal{P}_{n,k}^\rho$, could be written as:

$$\mathcal{P}_{n,m}^\rho(e_0; x) = 1,$$

$$\mathcal{P}_{n,m}^\rho(e_1; x) = \frac{x}{n}(n + \rho x),$$

$$\mathcal{P}_{n,m}^\rho(e_2; x) = \frac{x^2}{n^2} [n + n^2 + 2\rho x + 2\rho n x + \rho^2 x^2],$$

$$\mathcal{P}_{n,m}^\rho(e_3; x) = \frac{x^3}{n^3} [2n + 3n^2 + n^3 + 6\rho x + 9\rho n x + 3\rho n^2 x + 6\rho^2 x^2 + 3\rho^2 n x^2 + \rho^3 x^3],$$

$$\mathcal{P}_{n,m}^\rho(e_4; x) = \frac{x^4}{n^4} [6n + 11n^2 + 6n^3 + n^4 + 24\rho x + 44\rho n x + 24\rho n^2 x + 4\rho n^3 x + 36\rho^2 x^2 + 30\rho^2 n x^2 + 6\rho^2 n^2 x^2 + 12\rho^3 x^3 + 4\rho^3 n x^3 + \rho^4 x^4].$$

Lemma 2. *Central moments of the operators $\mathcal{P}_{n,m}^{\rho}$, with the help of Lemma 1, are as follows*

$$\begin{aligned}\mathcal{P}_{n,m}^{\rho}((\lambda-x);x) &= \frac{\rho x^2}{n}, \\ \mathcal{P}_{n,m}^{\rho}((\lambda-x)^2;x) &= \frac{x^2}{n^2} [n + 2\rho x + \rho^2 x^2], \\ \mathcal{P}_{n,m}^{\rho}((\lambda-x)^3;x) &= \frac{x^3}{n^3} [2n - 4n^3 + 6\rho x + 3\rho n x - 6\rho n^2 x + 6\rho^2 x^2 + \rho^3 x^3], \\ \mathcal{P}_{n,m}^{\rho}((\lambda-x)^4;x) &= \frac{x^4}{n^4} [6n + 3n^2 + 24\rho x + 20\rho n x \\ &\quad + 36\rho^2 x^2 + 6\rho^2 n x^2 + 12\rho^3 x^3 + \rho^4 x^4].\end{aligned}$$

Lemma 3. *Using Lemma 2, we have*

$$\mathcal{P}_{n,m}^{\rho}((\lambda-x)^2;x) = \frac{x^2}{n^2} (n + 2\rho x + \rho^2 x^2) \leq \frac{x^2}{n} (1 + 2\rho x + \rho^2 x^2) \leq \frac{\chi^2(x)}{n},$$

and

$$\mathcal{P}_{n,m}^{\rho}((\lambda-x)^4;x) \leq C \frac{\chi^4(x)}{n^2},$$

where, C is a large positive constant and $\chi(x) = x(1 + \rho x)$. Moreover

$$\lim_{n \rightarrow \infty} n \mathcal{P}_{n,m}^{\rho}((\lambda-x);x) = \rho x^2, \quad \text{and} \quad \lim_{n \rightarrow \infty} n \mathcal{P}_{n,m}^{\rho}((\lambda-x)^2;x) = x^2.$$

Lemma 4. *For the operators $\mathcal{P}_{n,m}^{\rho}$ and $\varpi \in [0, \infty)$, we have*

$$|\mathcal{P}_{n,m}^{\rho}(\varpi;x)| \leq \|\varpi\|,$$

where norm of the function on the positive half real line is given by $\|\varpi\| = \sup_{x \in [0, \infty)} |\varpi(x)|$.

Proof. From (1.2) and Lemma 1, we have

$$|\mathcal{P}_{n,m}^{\rho}(\varpi;x)| \leq \frac{n^n}{e^{-\rho x} x^n} \sum_{m=0}^{\infty} \frac{(n\rho)^m}{m!} \frac{1}{\Gamma(n+m)} \int_0^{\infty} \lambda^{n+m-1} e^{-\frac{n\lambda}{x}} |\varpi(\lambda)| d\lambda \leq \|\varpi\|.$$

□

Theorem 1. *Let $\varpi \in C_B[0, \infty)$, then $\lim_{n \rightarrow \infty} \mathcal{P}_{n,m}^{\rho}(\varpi;x) = \varpi(x)$, uniformly in every closed interval in $[0, \infty)$.*

Proof. From Lemma 1, $\mathcal{P}_{n,m}^{\rho}(e_0;x) = 1$, $\mathcal{P}_{n,m}^{\rho}(\lambda;x) = x$, $\mathcal{P}_{n,m}^{\rho}(\lambda^2;x) = x^2$, as $n \rightarrow \infty$. Therefore by the Bohman-Korovkin theorem, we get $\mathcal{P}_n^{\rho}(\varpi(\lambda);x) = \varpi(x)$ as $n \rightarrow \infty$, uniformly in every closed subinterval of $[0, \infty)$. □

3. MAIN RESULTS

Here, we assess the rate of convergence by using the Ditzian-Totik modulus of smoothness $\omega_{\chi^\gamma}(\varpi, \delta)$ and Peetre's K -functional $K_{\chi^\gamma}(\varpi, \delta)$, $0 \leq \gamma \leq 1$. For $\varpi \in C_B[0, \infty)$ and $\chi(x) = x(1 + \rho x)$, the Ditzian-Totik modulus of smoothness is explained as

$$\omega_{\chi^\gamma}(\varpi, \delta) = \sup_{0 \leq i \leq \delta} \sup_{x \pm \frac{i\chi^\gamma(x)}{2} \in [0, \infty)} \left| \varpi \left(x + \frac{i\chi^\gamma(x)}{2} \right) - \varpi \left(x - \frac{i\chi^\gamma(x)}{2} \right) \right|,$$

and the Peetre's K -functional is defined as

$$\omega_{\chi^\gamma}(\varpi, \delta) = \inf_{\varphi \in W_\gamma} \{ \|\varpi - \varphi\| + \delta \|\chi^\gamma \varphi'\| \},$$

where W_γ is a subspace of all real valued functions defined on $[0, \infty)$, and $\varphi \in W_\gamma$ which is locally absolutely continuous with norm $\|\varpi^\gamma \varphi'\| \leq \infty$. In [3, Theorem 2.1.1], there exists a constant $\mathcal{D} \geq 0$ such that

$$\mathcal{D}^{-1} \omega_{\chi^\gamma}(\varpi, \delta) \leq K_{\chi^\gamma}(\varpi, \delta) \leq \mathcal{D} \omega_{\chi^\gamma}(\varpi, \delta). \quad (3.1)$$

Theorem 2. For $\varpi \in C_B[0, \infty)$ then, we have

$$|\mathcal{P}_{n,m}^\rho(\varpi; x) - \varpi(x)| \leq \mathcal{D} \omega_{\chi^\gamma} \left(\varpi; \frac{\chi^{2-\gamma}(x)}{\sqrt{n}} \right).$$

Proof. For $\varphi \in W_\gamma$, and calling the representation

$$\varphi(\lambda) = \varphi(x) + \int_x^\lambda \varphi'(s) ds.$$

Applying $\mathcal{P}_{n,m}^\rho$ and using Hölder's inequality, we have

$$\begin{aligned} |\mathcal{P}_{n,m}^\rho(\varphi(\lambda); x) - \varphi(x)| &\leq \mathcal{P}_{n,m}^\rho \left(\int_x^\lambda |\varphi'| ds; x \right) \\ &\leq \|\varpi^\gamma \varphi'\| \mathcal{P}_{n,m}^\rho \left(\left| \int_x^\lambda \frac{ds}{\chi^\gamma(s)} \right|; x \right) \\ &\leq \|\varpi^\gamma \varphi'\| \mathcal{P}_{n,m}^\rho \left(|\lambda - x|^{1-\gamma} \left| \int_x^\lambda \frac{ds}{\chi(s)} \right|^\gamma; x \right). \end{aligned} \quad (3.2)$$

Let $I = \left| \int_x^\lambda \frac{ds}{\chi(s)} \right|$, now first we simplify expression I

$$\begin{aligned} I &\leq \left| \int_x^\lambda \frac{ds}{\sqrt{s}} \right| \left| \left(\frac{1}{1 + \rho x} + \frac{1}{1 + \rho \lambda} \right) \right| \leq 2 |\sqrt{\lambda} - \sqrt{x}| \left(\frac{1}{1 + \rho x} + \frac{1}{1 + \rho \lambda} \right) \\ &\leq 2 \frac{|\lambda - x|}{\sqrt{x}} \left(\frac{1}{1 + \rho x} + \frac{1}{1 + \rho \lambda} \right). \end{aligned} \quad (3.3)$$

Now, we use the inequality $|p+q|^\gamma \leq |p|^\gamma + |q|^\gamma$, $0 \leq \gamma \leq 1$ then from (3.3), we get

$$\left| \int_x^\lambda \frac{ds}{\chi(s)} \right|^\gamma \leq 2^\gamma \frac{|\lambda-x|^\gamma}{x^{\frac{\gamma}{2}}} \left(\frac{1}{(1+\rho x)^{\frac{\gamma}{2}}} + \frac{1}{(1+\rho \lambda)^{\frac{\gamma}{2}}} \right). \quad (3.4)$$

From (3.2) and (3.4) and using Cauchy inequality, we get

$$\begin{aligned} |\mathcal{P}_{n,m}^\rho(\varphi(\lambda);x) - \varphi(x)| &\leq \frac{2^\gamma \|\chi^\gamma \varphi'\|}{x^{\frac{\gamma}{2}}} \mathcal{P}_{n,m}^\rho \left(|\lambda-x| \left(\frac{1}{(1+\rho x)^{\frac{\gamma}{2}}} + \frac{1}{(1+\rho \lambda)^{\frac{\gamma}{2}}} \right); x \right) \\ &= \frac{2^\gamma \|\chi^\gamma \varphi'\|}{x^{\frac{\gamma}{2}}} \left(\frac{1}{(1+\rho x)^{\frac{\gamma}{2}}} (\mathcal{P}_{n,m}^\rho((\lambda-x)^2;x))^{\frac{1}{2}} \right. \\ &\quad \left. + (\mathcal{P}_{n,m}^\rho((\lambda-x)^2;x))^{\frac{1}{2}} \cdot (\mathcal{P}_{n,m}^\rho((1+\rho \lambda)^{-\gamma};x))^{\frac{1}{2}} \right). \end{aligned}$$

From Lemma 2, we may write

$$(\mathcal{P}_{n,m}^\rho((\lambda-x)^2;x))^{\frac{1}{2}} \leq \frac{\chi^2(x)}{\sqrt{n}}, \quad (3.5)$$

where $\chi(x) = x(1+\rho x)$.

For $x \in [0, \infty)$, $\mathcal{P}_{n,m}^\rho((1+\rho \lambda)^{-\gamma};x) \rightarrow (1+\rho x)^{-\gamma}$ as $n \rightarrow \infty$. Thus for $\varepsilon > 0$, we find a number $n_0 \in \mathbb{N}$ such that

$$\mathcal{P}_{n,m}^\rho((1+\rho \lambda)^{-\gamma};x) \leq (1+\rho x)^{-\gamma} + \varepsilon, \quad \text{for all } n \geq n_0.$$

By choosing $\varepsilon = (1+\rho x)^{-\gamma}$, we obtain

$$\mathcal{P}_{n,m}^\rho((1+\rho \lambda)^{-\gamma};x) \leq 2(1+\rho x)^{-\gamma}, \quad \text{for all } n \geq n_0. \quad (3.6)$$

From (3.5) and (3.6), we have

$$\begin{aligned} |\mathcal{P}_{n,m}^\rho(\varphi(\lambda);x) - \varphi(x)| &\leq 2^\gamma \|\chi^\gamma \varphi'\| \frac{\chi^2(x)}{\sqrt{n}} \left(\chi^{-\gamma}(x) + \sqrt{2} x^{-\frac{\gamma}{2}} (1+\rho x)^{-\frac{\gamma}{2}} \right) \\ &\leq 2^\gamma (1+\sqrt{2}) \|\chi^\gamma \varphi'\| \frac{1}{\sqrt{n}} \chi^{2-\gamma}(x). \end{aligned} \quad (3.7)$$

We may write

$$\begin{aligned} |\mathcal{P}_{n,m}^\rho(\varpi(\lambda);x) - \varpi(x)| &\leq |\mathcal{P}_{n,m}^\rho(\varpi(\lambda) - \varphi(\lambda);x)| \\ &\quad + |\mathcal{P}_{n,m}^\rho(\varphi(\lambda);x) - \varphi(x)| + |\varphi(x) - \varpi(x)| \\ &\leq 2\|\varpi - \varphi\| + |\mathcal{P}_{n,m}^\rho(\varphi(\lambda);x) - \varphi(x)|. \end{aligned} \quad (3.8)$$

From (3.7) and (3.8) and for sufficiently large n , we obtain

$$\begin{aligned} |\mathcal{P}_{n,m}^\rho(\varpi(\lambda);x) - \varpi(x)| &\leq 2\|\varpi - \varphi\| + 2^\gamma (1+\sqrt{2}) \|\chi^\gamma \varphi'\| \frac{1}{\sqrt{n}} \chi^{2-\gamma}(x) \\ &\leq C_1 \{ \|\varpi - \varphi\| + \chi^{2-\gamma}(x) \frac{1}{\sqrt{n}} \|\chi^\gamma \varphi'\| \} \leq CK_{\chi^\lambda} \left(\varpi; \chi^{2-\gamma}(x) \frac{1}{\sqrt{n}} \right), \end{aligned} \quad (3.9)$$

where $C_1 = \max\{2, 2^\lambda(1 + \sqrt{2})\}$ and $C = 2C_1$. From (3.1) and (3.9) we may conclude the required result. $\square$

Let $C^B[0, \infty)$ be the class of all absolutely continuous and bounded functions equipped with the norm $\|\varpi\| = \sup\{|\varpi(x)| : x \in [0, \infty)\}$. Then, K-functional in terms of modulus of smoothness is given by:

$$K_2(\varpi, \zeta) = \inf_{h \in W^2} \{\|\varpi - \varphi\| + \zeta\|\varphi''\|\},$$

where $\zeta > 0$ and $W^2 = \{\varphi \in C^B[0, \infty) : \varphi', \varphi'' \in C^B[0, \infty)\}$. Now, we find a constant $C_0 > 0$ such that

$$K_2(\varpi, \zeta) \leq C_0 \omega_2(\varpi, \sqrt{\zeta}), \quad (3.10)$$

where

$$\omega_2(\varpi, \sqrt{\zeta}) = \sup_{0 < \varrho \leq \sqrt{\zeta}} \sup_{x \in [0, \infty)} |\varpi(x + 2\varrho) - 2\varpi(x + \varrho) + \varpi(x)|$$

is known as second order modulus of smoothness of $\varpi \in C^B[0, \infty)$.

Theorem 3. For real valued continuous function $\varpi \in C^B[0, \infty)$, we have

$$|\mathcal{P}_{n,m}^\rho(\varpi; x) - \varpi(x)| \leq C_0 \omega_2(\varpi, \zeta) + \omega\left(\varpi; \left|\frac{x}{n}(n + \rho x)\right|\right).$$

Proof. For any real and continuous function $\xi \in C^B[0, \infty)$, by Taylor's expansion, we have

$$\xi(\lambda) = \xi(x) + (\lambda - x)\xi'(x) + \int_x^\lambda (\lambda - s)\xi''(s)ds. \quad (3.11)$$

Consider an auxiliary operators associated with $\mathcal{P}_{n,m}^\rho$

$$\tilde{\mathcal{P}}_{n,m}^\rho(\varpi; x) = \mathcal{P}_{n,m}^\rho(\varpi; x) - \varpi\left(\frac{x}{n}(n + \rho x)\right) + \varpi(x), \quad (3.12)$$

for $\varpi(\lambda) = 1$, we have $\tilde{\mathcal{P}}_{n,m}^\rho(1; x) = 1$, and for $\varpi(\lambda) = \lambda$

$$\tilde{\mathcal{P}}_{n,m}^\rho(\lambda; x) = \mathcal{P}_{n,m}^\rho(\lambda; x) - \frac{x}{n}(n + \rho x) + x = x.$$

Immediately, we may write

$$\tilde{\mathcal{P}}_{n,m}^\rho((\lambda - x); x) = 0.$$

Now, applying the operators $\tilde{\mathcal{P}}_{n,m}^\rho$ on (3.11) and using (3.12), we have

$$\begin{aligned} \tilde{\mathcal{P}}_{n,m}^\rho(\xi; x) - \xi(x) &= \tilde{\mathcal{P}}_{n,m}^\rho((\lambda - x); x)\xi'(x) + \tilde{\mathcal{P}}_{n,m}^\rho\left(\int_x^\infty (\lambda - s)\xi''(s)ds; x\right) \\ &= \tilde{\mathcal{P}}_{n,m}^\rho\left(\int_x^\infty (\lambda - s)\xi''(s)ds; x\right) \end{aligned}$$

$$= \mathcal{P}_{n,m}^{\rho} \left(\int_x^{\infty} (\lambda - s) \xi''(s) ds; x \right) - \int_x^{\frac{x}{n}(n+\rho x)} \left(\frac{x}{n} (n + \rho x) \right) \xi''(s) ds.$$

Now,

$$\begin{aligned} |\tilde{\mathcal{P}}_{n,m}^{\rho}(\xi; x) - \xi(x)| &\leq \mathcal{P}_{n,m}^{\rho} \left(\frac{1}{2} (\lambda - x)^2 \|\xi''\|; x \right) + \frac{1}{2} \left(\frac{x}{n} (n + \rho x) \right)^2 \|\xi''\| \\ &\leq \frac{1}{2} \left| \mathcal{P}_{n,m}^{\rho}((\lambda - x)^2; x) + \left(\frac{x}{n} (n + \rho x) \right)^2 \right| \|\xi''\| \leq \zeta \|\xi''\|, \end{aligned}$$

where

$$\zeta = \frac{1}{2} \left| \mathcal{P}_{n,m}^{\rho}((\lambda - x)^2; x) + \left(\frac{x}{n} (n + \rho x) \right)^2 \right|.$$

Again, from equation (3.11) and (3.12), we have

$$\begin{aligned} \mathcal{P}_{n,m}^{\rho}(\varpi; x) &= \tilde{\mathcal{P}}_{n,m}^{\rho}(\varpi; x) + \varpi \left(\frac{x}{n} (n + \rho x) \right) - \varpi(x) \\ &= \tilde{\mathcal{P}}_{n,m}^{\rho}((\varpi - \xi); x) + \tilde{\mathcal{P}}_{n,m}^{\rho}(\xi; x) + \varpi \left(\frac{x}{n} (n + \rho x) \right) - \varpi(x) \\ &= \tilde{\mathcal{P}}_{n,m}^{\rho}((\varpi - \xi); x) - (\varpi - \xi)(x) + \tilde{\mathcal{P}}_{n,m}^{\rho}(\xi; x) - \xi(x) \\ &\quad + \varpi \left(\frac{x}{n} (n + \rho x) \right) - \varpi(x). \end{aligned}$$

Now, we have

$$\begin{aligned} |\mathcal{P}_{n,m}^{\rho}(\varpi; x) - \varpi(x)| &\leq \|\varpi - \xi\| + \zeta \|\xi''\| + \left| \varpi \left(\frac{x}{n} (n + \rho x) \right) - \varpi(x) \right| \\ &\leq 2\|\varpi - \xi\| + \zeta \|\xi''\| + \omega \left(\varpi; \left| \frac{x}{n} (n + \rho x) \right| \right) \\ &\leq 2K_2(\varpi, \zeta) + \omega \left(\varpi; \left| \frac{x}{n} (n + \rho x) \right| \right). \end{aligned}$$

In the view of (3.10), we get the required result. $\square$

Let $x \in (0, \infty)$ and $\lambda \in [0, \infty]$, as we can see in Özarslan and Duman [7], the Lipschitz type space is explained as:

$$Lip_M^*(\alpha) = \left\{ \varpi \in C[0, \infty] : |\varpi(\lambda) - \varpi(x)| \leq M \frac{|\lambda - x|^\alpha}{(\lambda + x)^{\frac{\alpha}{2}}} \right\}, \quad \text{where } 0 < \alpha \leq 1.$$

In the following theorem, we obtain the rate of convergence of the operators $\mathcal{P}_{n,m}^{\rho}$ for functions in $Lip_M^*(\alpha)$.

Theorem 4. Let $\varpi \in Lip_M^*(\alpha)$ and $\alpha \in (0, 1]$. Then for all $x \in (0, \infty)$, we have

$$|\mathcal{P}_{n,m}^{\rho}(\varpi(\lambda); x) - \varpi(x)| \leq M \left(\frac{x(n + 2\rho x + \rho^2 x^2)}{n^2} \right)^{\frac{\alpha}{2}}.$$

Proof. From the Lemma 4, we have

$$\begin{aligned} |\mathcal{P}_{n,m}^{\rho}(\mathfrak{w}(\lambda);x) - \mathfrak{w}(x)| &\leq \mathcal{P}_{n,m}^{\rho}(|\mathfrak{w}(\lambda) - \mathfrak{w}(x)|;x) \leq M \mathcal{P}_{n,m}^{\rho}\left(\frac{|\lambda - x|^{\alpha}}{(\lambda + x)^{\frac{\alpha}{2}}};x\right) \\ &\leq \frac{M}{x^{\frac{\alpha}{2}}} \mathcal{P}_{n,m}^{\rho}(|\lambda - x|^{\alpha};x). \end{aligned} \quad (3.13)$$

Taking $p = \frac{2}{\alpha}$ and $q = \frac{2}{2-\alpha}$ and applying Hölder's inequality, we obtain

$$\begin{aligned} \mathcal{P}_{n,m}^{\rho}(\mathfrak{w})(|\lambda - x|^{\alpha};x) &\leq \left\{ \mathcal{P}_{n,m}^{\rho}(|\lambda - x|^2;x) \right\}^{\frac{\alpha}{2}} \cdot \left\{ \mathcal{P}_{n,m}^{\rho}\left(1^{\frac{2}{2-\alpha}};x\right) \right\}^{\frac{2-\alpha}{2}} \\ &\leq \left\{ \mathcal{P}_{n,m}^{\rho}(|\lambda - x|^2;x) \right\}^{\frac{\alpha}{2}}. \end{aligned} \quad (3.14)$$

On combining (3.13), (3.14) and using the Lemma 2, we get the required result. $\square$

For $c, d > 0$, Özarslan and Aktuğlu [8] considered the Lipschitz-type space with two parameters, as follows

$$Lip_M^{(c,d)}(\alpha) = \left(\mathfrak{w} \in C[0, \infty) : |\mathfrak{w}(\lambda) - \mathfrak{w}(x)| \leq M \frac{|\lambda - x|^{\alpha}}{(\lambda + cx^2 + dx)^{\frac{\alpha}{2}}} \right),$$

where M is a positive constant and $0 < \alpha \leq 1$.

Theorem 5 (Point-wise Estimate). *For $f \in Lip_M^{(c,d)}(\alpha)$. Then, for all $x > 0$, we have*

$$|\mathcal{P}_{n,m}^{\rho}(\mathfrak{w};x) - \mathfrak{w}(x)| \leq M \frac{\chi(x)}{(cx^2 + dx)^{\frac{\alpha}{2}}}.$$

Proof. First we prove the theorem for $\alpha = 1$. Then for $\mathfrak{w} \in Lip_M^{(c,d)}(1)$ and $x \in [0, \infty)$, we have

$$\begin{aligned} |\mathcal{P}_{n,m}^{\rho}(\mathfrak{w};x) - \mathfrak{w}(x)| &\leq \mathcal{P}_{n,m}^{\rho}(|\mathfrak{w}(\lambda) - \mathfrak{w}(x)|;x) \leq M \left\{ \mathcal{P}_{n,m}^{\rho}\left(\frac{|\lambda - x|}{(\lambda + cx^2 + dx)^{\frac{1}{2}}};x\right) \right\} \\ &\leq \frac{M}{(cx^2 + dx)^{\frac{1}{2}}} \mathcal{P}_{n,m}^{\rho}(|\lambda - x|;x). \end{aligned}$$

By applying Cauchy-Schwartz inequality and using Lemma 3

$$|\mathcal{P}_{n,m}^{\rho}(\mathfrak{w};x) - \mathfrak{w}(x)| \leq \frac{M}{(cx^2 + dx)^{\frac{1}{2}}} \left\{ \mathcal{P}_{n,m}^{\rho}(|\lambda - x|^2;x) \right\}^{\frac{1}{2}} \leq M \left(\frac{\chi^2(x)}{cx^2 + dx} \right)^{\frac{1}{2}}.$$

This shows that result is true for $\alpha = 1$. Next we prove the stated result for $0 < \alpha < 1$, we have

$$|\mathcal{P}_{n,m}^{\rho}(\mathfrak{w};x) - \mathfrak{w}(x)| \leq \frac{M}{(cx^2 + dx)^{\frac{\alpha}{2}}} \mathcal{P}_{n,m}^{\rho}(|\lambda - x|^{\alpha};x).$$

Assume $p = \frac{1}{\alpha}$, $q = \frac{1}{1-\alpha}$, on applying the Hölder's inequality, we have

$$|\mathcal{P}_{n,m}^p(\varpi; x) - \varpi(x)| \leq \frac{M}{(cx^2 + dx)^{\frac{\alpha}{2}}} (\mathcal{P}_{n,m}^p(|\lambda - x|; x))^{\alpha}.$$

Again, by Cauchy-Schwartz inequality and Lemma 3 required result follows. $\square$

Theorem 6. Let $\varpi \in [0, \infty)$ and second order derivative of ϖ exists in $[0, \infty)$, then we have

$$\lim_{n \rightarrow \infty} n [\mathcal{P}_{n,m}^p(\varpi; x) - \varpi(x)] = x^2 (\rho \varpi'(x) + \varpi''(x))$$

Proof. Using Taylor's series expansion, we write

$$\varpi(\lambda) = \varpi(x) + (\lambda - x)\varpi'(x) + \frac{(\lambda - x)^2}{2!}\varpi''(x) + \hbar(\lambda, x)(\lambda - x)^2, \quad (3.15)$$

where $\hbar(\lambda, x) \rightarrow 0$ as $\lambda \rightarrow x$. Applying $\mathcal{P}_{n,m}^p(\cdot; x)$ on both the sides of (3.15), we get

$$\begin{aligned} \mathcal{P}_{n,m}^p(\varpi(\lambda) - \varpi(x); x) &= \varpi'(x)\mathcal{P}_{n,m}^p((\lambda - x); x) + \frac{\varpi''(x)}{2!}\mathcal{P}_{n,m}^p((\lambda - x)^2; x) \\ &\quad + \mathcal{P}_{n,m}^p(\hbar(\lambda, x)(\lambda - x)^2; x). \end{aligned}$$

From the Lemma 2 and applying the $\lim_{n \rightarrow \infty}$, we get

$$\begin{aligned} \lim_{n \rightarrow \infty} n\mathcal{P}_{n,m}^p(\varpi(\lambda) - \varpi(x); x) &= \varpi'(x) \lim_{n \rightarrow \infty} n\mathcal{P}_{n,m}^p((\lambda - x); x) \\ &\quad + \frac{\varpi''(x)}{2!} \lim_{n \rightarrow \infty} n\mathcal{P}_{n,m}^p((\lambda - x)^2; x) \\ &\quad + \lim_{n \rightarrow \infty} n\mathcal{P}_{n,m}^p(\hbar(\lambda, x)(\lambda - x)^2; x). \end{aligned} \quad (3.16)$$

From Theorem 1, Lemma 2 and using cauchy-Schwarz inequality in the last term of (3.16), we have

$$\lim_{n \rightarrow \infty} n\mathcal{P}_{n,m}^p(\hbar(\lambda, x)(\lambda - x)^2; x) = 0. \quad (3.17)$$

Using Lemma 2 and (3.16), (3.17), required result follows. $\square$

4. BOUNDED VARIATION

In the next theorem, we estimate the operators' convergence rate (1.1). Let $\varpi \in \text{DBV}[0, \infty)$ be a continuous function taken from the class of absolutely continuous functions $\text{DBV}(0, \infty)$ and having a derivative ϖ' on the interval $(0, \infty)$ coincides a.e. with a function which is of bounded variation on every finite partition of $[0, \infty)$. It can be observed that for all $\varpi \in \text{DBV}[0, \infty)$, we have

$$\varpi(x) = \varpi(c) + \int_c^x g(s)ds, \quad x \geq c \geq 0,$$

where $g(s)$ is a function of bounded variation on every finite subinterval of $[0, \infty)$. We may write the operators (1.1) in alternate form as

$$\mathcal{P}_{n,m}^p(\mathfrak{w}; x) = \int_0^\infty \mathcal{K}_{n,m}^p(x, \lambda) \mathfrak{w}(\lambda) d\lambda, \quad (4.1)$$

where

$$\mathcal{K}_{n,m}^p(x, \lambda) = \sum_{m=0}^{\infty} \frac{n^n}{e^{\rho x} x^n} \frac{(n\rho)^m}{m!} \frac{1}{\Gamma(n+m)} \lambda^{n+m-1} e^{-\frac{n\lambda}{x}}.$$

Now we define an auxiliary function $\mathfrak{w}_x$ by

$$\mathfrak{w}_x(\lambda) = \begin{cases} \mathfrak{w}(\lambda) - \mathfrak{w}(x^-) & \text{if } 0 \leq \lambda < x, \\ 0 & \text{if } \lambda = x, \\ \mathfrak{w}(\lambda) - \mathfrak{w}(x^+) & \text{if } x < \lambda < \infty. \end{cases}$$

Lemma 5. For $x \in (0, \infty)$ and sufficiently large n , we have

(1) Since $0 \leq y < x$, therefore

$$\eta_n(x, y) = \int_0^y \mathcal{K}_{n,m}^p(x, \lambda) d\lambda \leq \frac{\chi^2(x)}{n(x-y)^2}.$$

(2) If $x < z < \infty$, then

$$1 - \eta_n(x, z) = \int_z^\infty \mathcal{K}_{n,m}^p(x, \lambda) d\lambda \leq \frac{\chi^2(x)}{n(z-x)^2}.$$

Proof. By simple computations and using Lemma 2 and Lemma 3 we get required results. $\square$

Theorem 7. Let $\mathfrak{w} \in \text{DBV}(0, \infty)$ then for all $x \in (0, \infty)$ and sufficiently large n , we have

$$\begin{aligned} |\mathcal{P}_{n,m}^p(\mathfrak{w}; x) - \mathfrak{w}(x)| &\leq \frac{1}{2} (\mathfrak{w}'(x^+) + \mathfrak{w}'(x^-)) \mathcal{P}_{n,m}^p((x-\lambda); x) \\ &\quad + \frac{1}{2} \frac{\chi(x)}{\sqrt{n}} |\mathfrak{w}'(x^+) + \mathfrak{w}'(x^-)| + \frac{\chi^2(x)}{nx} \sum_{m=1}^{[\sqrt{n}]} \left(\bigvee_{x-\frac{x}{m}}^{x+\frac{x}{m}} (\mathfrak{w}'_x) \right) \\ &\quad + \frac{x}{\sqrt{n}} \left(\bigvee_{x-\frac{x}{\sqrt{n}}}^{x+\frac{x}{\sqrt{n}}} (\mathfrak{w}'_x) \right) + \frac{\chi(x)}{\sqrt{n}} \mathfrak{w}'(x^+) \\ &\quad + \frac{\chi^2(x)}{nx^2} |\mathfrak{w}(2x) - \mathfrak{w}(x) - x\mathfrak{w}'(x^+)| + M(\gamma, r, x) \frac{\chi^2(x) |\mathfrak{w}(x)|}{nx^2}. \end{aligned}$$

Where $\bigvee_a^b(x)$ denotes the total variation of $\mathfrak{w}_x$ on $[a, b]$ and

$$M(\gamma, r, x) = M2^\gamma \left(\int_0^\infty (\lambda - x)^{2r} \mathcal{K}_{n,m}^p(x, \lambda) d\lambda \right)^{\frac{\gamma}{2r}}.$$

Proof. Using the operator (4.1) for all $x \in [0, \infty)$, we obtain

$$\begin{aligned} \mathcal{P}_{n,m}^p(\varpi; x) - \varpi(x) &= \int_0^\infty \mathcal{K}_{n,m}^p(x, \lambda) (\varpi(\lambda) - \varpi(x)) d\lambda \\ &= \int_0^\infty \left(\mathcal{K}_{n,m}^p(x, \lambda) \int_x^\lambda \varpi(s) ds \right) d\lambda. \end{aligned} \quad (4.2)$$

For $\varpi \in \text{DBV}[0, \infty)$, we can write

$$\begin{aligned} \varpi'(s) &= \frac{1}{2}(\varpi'(x^+) + \varpi'(x^-)) + \varpi'_x(x) + \frac{1}{2}(\varpi'(x^+) - \varpi'(x^-)) \text{sgn}(x) \\ &\quad + \delta_x(s)(\varpi' - \frac{1}{2}(\varpi'(x^+) + \varpi'(x^-))). \end{aligned} \quad (4.3)$$

Where

$$\delta_x(s) = \begin{cases} 1, & s = x; \\ 0, & s \neq x. \end{cases}$$

It can be easily seen that

$$\int_0^\infty \left(\int_x^\lambda \left(\varpi'(s) - \frac{1}{2}(\varpi'(x^+) + \varpi'(x^-)) \right) \delta_x(s) ds \right) \mathcal{K}_{n,m}^p(x, \lambda) d\lambda = 0.$$

Using the equation (4.1), we obtain

$$\begin{aligned} &\int_0^\infty \left(\int_x^\lambda \frac{1}{2}(\varpi'(x^+) + \varpi'(x^-)) ds \right) \mathcal{K}_{n,m}^p(x, \lambda) d\lambda \\ &= \frac{1}{2}(\varpi'(x^+) + \varpi'(x^-)) \mathcal{P}_{n,m}^p((x - \lambda); x). \end{aligned}$$

Again using (4.1), we have

$$\begin{aligned} &\int_0^\infty \left(\int_x^\lambda \frac{1}{2}(\varpi'(x^+) - \varpi'(x^-)) \text{sgn}(s - x) ds \right) \mathcal{K}_{n,m}^p(x, \lambda) d\lambda \\ &= \int_0^\infty \frac{1}{2}(\varpi'(x^+) - \varpi'(x^-))(\lambda - x) \mathcal{K}_{n,m}^p(x, \lambda) d\lambda \\ &\leq \frac{1}{2} |\varpi'(x^+) - \varpi'(x^-)| \left(\mathcal{P}_{n,m}^p((x - \lambda)^2; x) \right)^{\frac{1}{2}}. \end{aligned} \quad (4.4)$$

From (4.2), (4.4) and Lemma 3, we have

$$\begin{aligned} \mathcal{P}_{n,m}^p(\varpi; x) - \varpi(x) &\leq \frac{1}{2}(\varpi'(x^+) + \varpi'(x^-)) \mathcal{P}_{n,m}^p((x - \lambda); x) \\ &\leq \frac{1}{2}(\varpi'(x^+) + \varpi'(x^-)) \mathcal{P}_{n,m}^p((x - \lambda); x) \\ &\quad + \frac{1}{2} \frac{\chi(x)}{\sqrt{n}} |\varpi'(x^+) + \varpi'(x^-)| + \int_0^\infty \left(\int_x^\lambda \varpi'_x(s) ds \right) \mathcal{K}_{n,m}^p(x, \lambda) d\lambda. \end{aligned}$$

We obtain

$$\begin{aligned} |\mathcal{P}_{n,m}^{\rho}(\mathfrak{w};x) - \mathfrak{w}(x)| &\leq \frac{1}{2} (\mathfrak{w}'(x^+) + \mathfrak{w}'(x^-)) \mathcal{P}_{n,m}^{\rho}((x-\lambda);x) \\ &\quad + \frac{1}{2} \frac{\chi(x)}{\sqrt{n}} |\mathfrak{w}'(x^+) + \mathfrak{w}'(x^-)| + A_{n1}(x) + A_{n2}(x). \end{aligned} \quad (4.5)$$

Where

$$A_{n1}(x) = \left| \int_0^x \left(\int_x^{\lambda} \mathfrak{w}'_x(s) ds \right) \mathcal{K}_{n,m}^{\rho}(x, \lambda) d\lambda \right|,$$

and

$$A_{n2}(x) = \left| \int_x^{\infty} \left(\int_x^{\lambda} \mathfrak{w}'_x(s) ds \right) \mathcal{K}_{n,m}^{\rho}(x, \lambda) d\lambda \right|.$$

Now applying Lemma 5, integrating by parts and taking $y = x - \frac{x}{\sqrt{n}}$, we get

$$\begin{aligned} A_{n1}(x) &= \left| \int_0^x \left(\int_x^{\lambda} \mathfrak{w}'_x(s) ds \right) d_{\lambda} \eta_n(x, \lambda) d\lambda \right| = \left| \int_0^x \eta_n(x, \lambda) \mathfrak{w}'_x(\lambda) d\lambda \right| \\ &\leq \int_0^y |\eta_n(x, \lambda)| |\mathfrak{w}'_x(\lambda)| d\lambda + \int_y^x |\eta_n(x, \lambda)| |\mathfrak{w}'_x(\lambda)| d\lambda \\ &= \int_0^{x - \frac{x}{\sqrt{n}}} \eta_n(x, \lambda) |\mathfrak{w}'_x(\lambda)| d\lambda + \int_{x - \frac{x}{\sqrt{n}}}^x \eta_n(x, \lambda) |\mathfrak{w}'_x(\lambda)| d\lambda. \end{aligned}$$

Since $|\eta_n(x, \lambda)| \leq 1$ and $\mathfrak{w}'_x(x) = 0$, we get

$$\begin{aligned} \int_{x - \frac{x}{\sqrt{n}}}^x \eta_n(x, \lambda) |\mathfrak{w}'_x(\lambda)| d\lambda &= \int_{x - \frac{x}{\sqrt{n}}}^x \eta_n(x, \lambda) |\mathfrak{w}'_x(\lambda) - \mathfrak{w}'_x(x)| d\lambda \\ &\leq \int_{x - \frac{x}{\sqrt{n}}}^x \bigvee_{\lambda} (\mathfrak{w}'_x) d\lambda \leq \frac{x}{\sqrt{n}} \bigvee_{\lambda - \frac{x}{\sqrt{n}}}^x (\mathfrak{w}'_x). \end{aligned}$$

Again using Lemma 5 and substituting $\lambda = x - \frac{x}{s}$,

$$\begin{aligned} \int_0^{x - \frac{x}{\sqrt{n}}} \eta_n(x, \lambda) |\mathfrak{w}'_x(\lambda)| d\lambda &\leq \frac{\chi^2(x)}{n} \int_0^{x - \frac{x}{\sqrt{n}}} \frac{|\mathfrak{w}'_x(\lambda)|}{(x-\lambda)^2} d\lambda \leq \frac{\chi^2(x)}{nx} \int_1^{\sqrt{n}} \bigvee_{x - \frac{x}{s}}^x (\mathfrak{w}'_x) ds \\ &\leq \frac{\chi^2(x)}{nx} \sum_{m=1}^{[\sqrt{n}]} \bigvee_{x - \frac{x}{m}}^x (\mathfrak{w}'_x). \end{aligned}$$

Therefore,

$$A_{n1}(x) \leq \frac{\chi^2(x)}{nx} \sum_{m=1}^{[\sqrt{n}]} \left(\bigvee_{x - \frac{x}{m}}^x (\mathfrak{w}'_x) \right) + \frac{x}{\sqrt{n}} \left(\bigvee_{x - \frac{x}{\sqrt{n}}}^x (\mathfrak{w}'_x) \right). \quad (4.6)$$

Now we observe, integration by parts and applying Lemma 5, we have

$$\begin{aligned} A_{n2}(x) &= \int_x^\infty \mathcal{K}_{n,m}^p(x, \lambda) \left(\int_x^\lambda \varpi'_x(s) ds \right) d\lambda \\ &\leq \left| \int_x^{2x} \mathcal{K}_{n,m}^p(x, \lambda) \left(\int_x^\lambda \varpi'_x(s) ds \right) d\lambda \right| + \left| \int_{2x}^\infty \mathcal{K}_{n,m}^p(x, \lambda) \left(\int_x^\lambda \varpi'_x(s) ds \right) d\lambda \right| \\ &\leq B_{n1}(x) + B_{n2}(x), \end{aligned} \quad (4.7)$$

where,

$$\begin{aligned} B_{n1}(x) &= \left| \int_x^{2x} \mathcal{K}_{n,m}^p(x, \lambda) \left(\int_x^\lambda \varpi'_x(s) ds \right) d\lambda \right|, \\ B_{n2}(x) &= \left| \int_{2x}^\infty \mathcal{K}_{n,m}^p(x, \lambda) \left(\int_x^\lambda \varpi'_x(s) ds \right) d\lambda \right|. \end{aligned}$$

Applying integration by parts, using (4.3) and Lemma 5. Since $1 - \eta_n(x, \lambda) \leq 1$, substituting $\lambda = x + \frac{x}{s}$, we have

$$\begin{aligned} B_{n1}(x) &= \left| \int_x^{2x} \varpi'_x(s) ds (1 - \eta_n(x, 2x)) - \int_x^{2x} (1 - \eta_n(x, \lambda)) \varpi'_x(\lambda) d\lambda \right| \\ &\leq \left| \int_x^{2x} (\varpi'(s) - \varpi'(x^+)) ds \right| |1 - \eta_n(x, 2x)| + \int_x^{2x} |\varpi'_x(\lambda)| |1 - \eta_n(x, \lambda)| \nu \lambda \\ &\leq \frac{\chi^2(x)}{nx^2} |\varpi(2x) - \varpi(x) - x\varpi'(x^+)| + \int_x^{x+\frac{x}{\sqrt{n}}} |\varpi'_x(\lambda)| |1 - \eta_n(x, \lambda)| d\lambda \\ &\quad + \int_{x+\frac{x}{\sqrt{n}}}^{2x} |\varpi'_x(\lambda)| |1 - \eta_n(x, \lambda)| d\lambda \\ &\leq \frac{\chi^2(x)}{nx^2} |\varpi(2x) - \varpi(x) - x\varpi'(x^+)| + \int_x^{x+\frac{x}{\sqrt{n}}} \bigvee_{\lambda}^x (\varpi'_x) d\lambda \\ &\quad + \frac{\chi^2(x)}{n} \int_{x+\frac{x}{\sqrt{n}}}^{2x} \frac{\bigvee_x^\lambda (\varpi'_x)}{(\lambda - x)^2} d\lambda \\ &\leq \frac{\chi^2(x)}{nx^2} |\varpi(2x) - \varpi(x) - x\varpi'(x^+)| + \frac{x}{\sqrt{n}} \bigvee_x^{x+\frac{x}{\sqrt{n}}} (\varpi'_x) \\ &\quad + \frac{\chi^2(x)}{n} \int_{x+\frac{x}{\sqrt{n}}}^{2x} \frac{1}{(\lambda - x)^2} \bigvee_x^\lambda (\varpi'_x) d\lambda \\ &\leq \frac{\chi^2(x)}{nx^2} |\varpi(2x) - \varpi(x) - x\varpi'(x^+)| + \frac{x}{\sqrt{n}} \bigvee_x^{x+\frac{x}{\sqrt{n}}} (\varpi'_x) + \frac{\chi^2(x)}{nx} \sum_{m=1}^{[\sqrt{n}]} \bigvee_x^{x+\frac{x}{k}} (\varpi'_x). \end{aligned} \quad (4.8)$$

And

$$\begin{aligned}
 B_{n2}(x) &= \left| \int_{2x}^{\infty} \left(\int_x^{\lambda} \varpi'(s) - \varpi'(x+) ds \right) \mathcal{K}_{n,m}^p(x, \lambda) d\lambda \right| \\
 &\leq \int_0^{\infty} |\varpi(\lambda) - \varpi(x)| \mathcal{K}_{n,m}^p(x, \lambda) d\lambda + \int_{2x}^{\infty} |\lambda - x| \varpi'(x+) \mathcal{K}_{n,m}^p(x, \lambda) d\lambda \\
 &\leq M \int_{2x}^{\infty} \lambda^{\gamma} \mathcal{K}_{n,m}^p(x, \lambda) d\lambda + |\varpi(x)| \int_{2x}^{\infty} \mathcal{K}_{n,m}^p(x, \lambda) d\lambda + \frac{\chi(x)}{\sqrt{n}} \varpi'(x+).
 \end{aligned}$$

It is obvious that

$$\lambda \leq 2(\lambda - x) \quad \text{and} \quad x \leq \lambda - x, \quad \text{when} \quad \lambda \geq 2x.$$

Applying Holder's inequality, we get

$$\begin{aligned}
 B_{n2}(x) &\leq M 2^{\gamma} \left(\int_0^{\infty} (\lambda - x)^{2r} \mathcal{K}_{n,m}^p(x, \lambda) d\lambda \right)^{\frac{\gamma}{2r}} + \frac{\chi^2(x)}{nx^2} |\varpi(x)| + \sqrt{\frac{1}{n}} \chi(x) \varpi'(x+) \\
 &\leq M(\gamma, r, x) + \frac{\chi^2(x)}{nx^2} |\varpi(x)| + \frac{\chi(x)}{\sqrt{n}} \varpi'(x+). \quad (4.9)
 \end{aligned}$$

From (4.8) and (4.9), we get

$$\begin{aligned}
 A_{n2}(x) &\leq \frac{\chi^2(x)}{nx^2} |\varpi(2x) - \varpi(x) - x\varpi'(x+)| + \frac{x}{\sqrt{n}} \bigvee_x^{x+\frac{x}{n}}(\varpi'_x) \\
 &\quad + \frac{\chi^2(x)}{nx} \sum_{m=1}^{[\sqrt{n}]} \bigvee_x^{x+\frac{x}{m}}(\varpi'_x) + M(\gamma, r, x) \frac{\chi^2(x) |\varpi(x)|}{nx^2} + \sqrt{\frac{1}{n}} \chi(x) \varpi'(x+). \quad (4.10)
 \end{aligned}$$

On combining (4.5)-(4.7) and (4.10), we get required result. $\square$

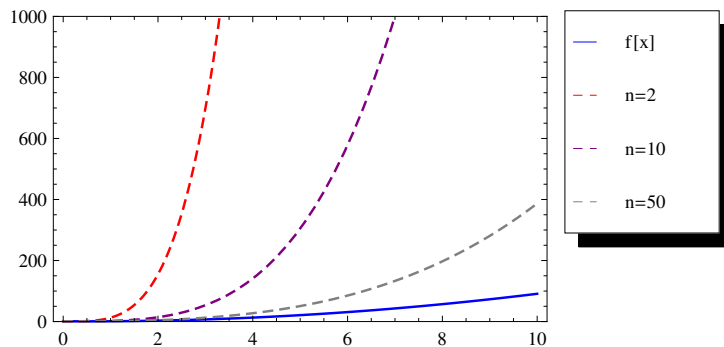


FIGURE 1.

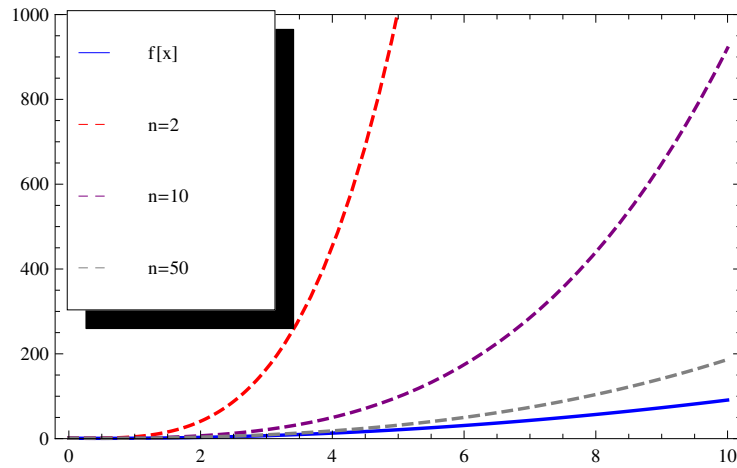


FIGURE 2.

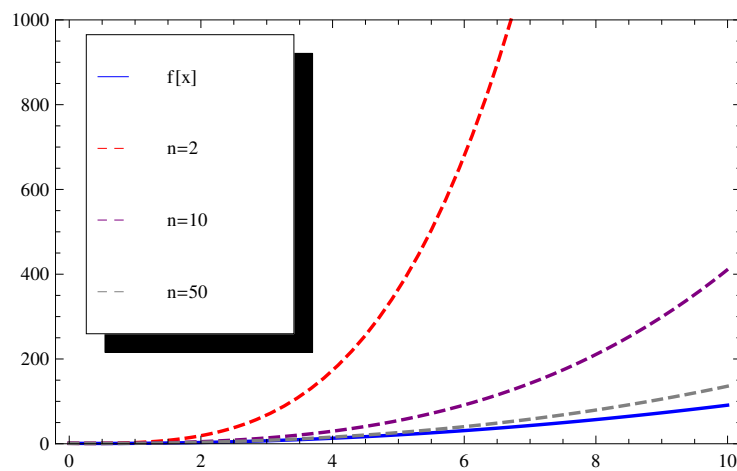


FIGURE 3.

Example 1. The Graphical representation of the convergence of the operators $\mathcal{P}_{n,k}^\beta(f(t);x)$ to the test function $f(x) = x^2 - x + 1$ are given in the Figure-1 for $\beta = 5$, and $n = \{2, 10, 50\}$. Figure-2 shows the convergence of the operators $\mathcal{P}_{n,k}^\beta(f(t);x)$ for $\beta = 2$, and $n = \{2, 10, 50\}$. And Figure-3 represents the convergence of the operators $\mathcal{P}_{n,k}^\beta(f(t);x)$ for $\beta = 1$, and $n = \{2, 10, 50\}$. From the graphical representation we easily seen that the operators converges fast when β decreases.

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WELL-POSEDNESS FOR FRACTIONAL HARDY-HÉNON PARABOLIC EQUATIONS WITH FRACTIONAL BROWNIAN MOTION

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Abstract. In this paper, we consider fractional Hardy-Hénon parabolic equations driven by fractional Brownian motion. The local existence and uniqueness of L^q mild solutions are proved.

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Keywords: fractional Hardy-Hénon parabolic equations, stochastic equations, fractional Brownian motion, L^p solution

1. INTRODUCTION

Stochastic partial differential equations driven by fractional Brownian motion have attracted much attention. Tindel et al. [11] studied linear stochastic evolution equations driven by infinite-dimensional fractional Brownian motion with Hurst parameter in the interval $H \in (0, 1)$. A sufficient and necessary condition for the existence and uniqueness of the solution is established. Maslowski and Nualart [10] studied nonlinear stochastic evolution equations in a Hilbert space driven by a cylindrical fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$ and nuclear covariance operator. Caraballo et al. [3] investigated the existence, uniqueness and exponential asymptotic behavior of mild solutions to a class of stochastic delay evolution equations perturbed by a fractional Brownian motion. Ahmed and Ragusa [1], Chadha and Bora [4], Khan et al. [7] studied some other fractional stochastic equations about the controllability, stability and fractional analysis. Clarke and Olivera [5] studied a semilinear heat equation driven by a Hilbert space-valued fractional Brownian motion, existence and uniqueness of local L^q mild solutions are shown.

In this paper, we study the Cauchy problem

$$\begin{cases} \partial_t u(t) = -(-\Delta)^{\alpha/2} u(t) + |x|^{-\gamma} |u(t)|^{p-1} u(t) + \partial_t B^H(t), & t > 0, \\ u(0) = \varphi, \end{cases} \quad (1.1)$$

where $u \subset \mathbb{R}^d$, $p > 1$, $u_0 \in L^q(\mathbb{R}^d)$, $\gamma \geq 0$, $(-\Delta)^{\alpha/2}$ ($0 < \alpha < 2$) is the fractional Laplacian, $B^H(t)$ is a fractional Brownian motion defined on some complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with Hurst parameter $H \in (\frac{1}{2}, 1)$.

When $\gamma = 0$, existence results of L^p mild solutions for the problem (1.1) without noise was studied in Weissler [12]. In the case $\gamma \in (0, \min(2, d))$, Ben Slimene, Tayachi and Weissler [2] investigated local well-posedness, global existence and large time behavior for the problem (1.1) without noise. When $\alpha = 2$ and $\gamma \in (0, 2)$, Majdoub and Mliki [9] studied the problem (1.1) on $\mathbb{R}^d$ ($d = 2$ or 3), they obtained the local existence and uniqueness of mild L^q solution. The purpose of this paper is to investigate the existence and uniqueness of mild solutions to the equation (1.1). The main results are obtained by using the contraction principle.

Definition 1. A process $u: \Omega \times [0, T]$ is called a mild solution of equation (1.1) if

$$(1) \quad u \in C([0, T]; \mathbb{R}^d),$$

$$(2)$$

$$\begin{aligned} u(t) = & e^{-t(-\Delta)^{\alpha/2}} \varphi + \int_0^t e^{-(t-s)(-\Delta)^{\alpha/2}} (|\cdot|^{-\gamma} |u(s)|^{p-1} u(s)) ds \\ & + \int_0^t e^{-(t-s)(-\Delta)^{\alpha/2}} dB^H(s), \end{aligned} \quad (1.2)$$

with probability one, where $e^{-t(-\Delta)^{\alpha/2}}$ is the linear fractional heat semigroup.

Theorem 1. Let $d \geq 1$ be an integer and let

$$0 < \gamma < \min(\alpha, d), \quad (1.3)$$

$$\max\left(\frac{1}{2}, \frac{1}{q}\right) < H < 1, \quad (1.4)$$

$$\max\left(\frac{dp}{d-\gamma}, \frac{d(p-1)}{\alpha-\gamma}\right) < q < \infty. \quad (1.5)$$

Given $\varphi \in L^q(\mathbb{R}^d)$, then there exist $T > 0$ such that (1.1) has a unique mild solution $u \in C([0, T]; L^q(\mathbb{R}^d))$.

The paper is organized as follows. In Section 2, some basic notations and preliminary facts on stochastic integrals for fractional Brownian motion and smoothing effect for the fractional heat semigroup are given. In Section 3, we give the main results of this paper.

2. PRELIMINARIES

2.1. Fractional Brownian motion

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space.

Definition 2. A centered Gaussian process $\{\beta^H(t), t \geq 0\}$ is called fractional Brownian motion (fBm) of Hurst index $H \in (0, 1)$ if it has the covariance function

$$R_H(t, s) = \mathbb{E}[\beta^H(t)\beta^H(s)] = \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H}).$$

In the following, we assume $\frac{1}{2} < H < 1$. Consider the square integrable kernel

$$K_H(t, s) = c_H s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du,$$

where $c_H = \left[\frac{H(2H-1)}{\beta(2-2H, H-\frac{1}{2})} \right]^{1/2}$ and $t > s$.

Then

$$\frac{\partial K_H}{\partial t}(t, s) = c_H \left(\frac{t}{s} \right)^{H-\frac{1}{2}} (t-s)^{H-\frac{3}{2}}.$$

Consider a fBm $\{\beta^H(t), t \in [0, T]\}$. We denote by ζ the set of step functions on $[0, T]$. Let $\mathcal{H}$ be the Hilbert space defined as the closure of ζ with respect to the scalar product

$$\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} \rangle_{\mathcal{H}} = R_H(t, s).$$

Define the linear operator K_H^* from ζ to $L^2([0, T])$ by

$$(K_H^* \varphi)(s) = \int_s^b \varphi(t) \frac{\partial K_H}{\partial t}(t, s) dt.$$

The operator K_H^* is an isometry between ζ and $L^2([0, T])$ that can be extended to the Hilbert space $\mathcal{H}$.

Consider the process $W = W(t), t \in [0, T]$ defined by

$$W(t) = \beta^H((K_H^*)^{-1} \mathbf{1}_{[0,t]}).$$

Then W is a Wiener process and β^H has the integral representation

$$\beta^H(t) = \int_0^t K_H(t, s) dW(s). \quad (2.1)$$

We have the following relationship between the Wiener integral with respect to fBm and the Wiener integral with respect to the Wiener process W

$$\int_0^t \varphi(s) d\beta^H(s) = \int_0^t (K_H^* \varphi)(s) dW(s), \quad (2.2)$$

for every $t \in [0, T]$ and $\varphi \mathbf{1}_{[0,t]} \in \mathcal{H}$ if and only if $K_H^* \varphi \in L^2([0, T])$. As in [11], we define the standard cylindrical Brownian motion in X as

$$B^H(t) = \sum_{n=0}^{\infty} e_n \beta_n^H(t), \quad (2.3)$$

where $\{e_n\}_{n=1}^{\infty}$ is a complete orthonormal basis in X and β_n^H are real, independent fBm's.

Let f be a deterministic function with values in $\mathcal{L}_2(X, Y)$, the space of Hilbert-Schmidt operators from X to Y . We make the following assumptions on f .

- (i) For each $x \in X$, $f(\cdot)x \in L^p([0, T]; Y)$, for $p > \frac{1}{H}$,
- (ii) $\alpha_H \int_0^T \int_0^T |f(s)|_{\mathcal{L}_2(X, Y)} |f(t)|_{\mathcal{L}_2(X, Y)} |s - t|^{2H-2} ds dt < \infty$,

where $\alpha_H = H(2H - 1)$. The stochastic integral of f with respect to B^H is defined by

$$\int_0^t f(s) dB^H(s) = \sum_{n=1}^{\infty} \int_0^t f(s) e_n d\beta_n^H(s) = \sum_{n=1}^{\infty} (K_H^* f e_n)(s) d\beta_n(s), \quad (2.4)$$

where β_n is the standard Brownian motion used to represent β_n^H as in (2.1) and the series (2.4) is finite if

$$\sum_n \|K_H^*(f e_n)\|_{L^2([0, b]; Y)}^2 = \sum_n \|f e_n\|_{\mathcal{H}}^2 < \infty. \quad (2.5)$$

2.2. Smoothing effect

Let $1 \leq q < \infty$ and denote $L^q(\mathbb{R}^d)$ the space of functions with the norm $\|f\|_q = (\int_{\mathbb{R}^d} |f(x)|^q dx)^{1/q}$. For $t > 0$, $e^{-t(-\Delta)^{\alpha/2}}$ denotes the linear fractional heat semigroup defined by (see [8])

$$(e^{-t(-\Delta)^{\alpha/2}} f)(x) = \int_{\mathbb{R}^d} G_\alpha(t, x - y) f(y) dy, \quad (2.6)$$

where G_α is the fractional heat kernel

$$G_\alpha(t, x) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{ix \cdot \xi - t|\xi|^\alpha} d\xi, \quad t > 0, x \in \mathbb{R}^d, \quad (2.7)$$

and $f \in L^q(\mathbb{R}^d)$, $q \in [1, \infty)$ or $f \in C_0(\mathbb{R}^d)$.

We recall the space-time estimate for the fractional semigroup on Lebesgue spaces,

$$\|e^{-t(-\Delta)^{\alpha/2}} u\|_{s_2} \leq C_1 t^{-\frac{d}{\alpha}(\frac{1}{s_1} - \frac{1}{s_2})} \|u\|_{s_1}, \quad (2.8)$$

for $1 \leq s_1 \leq s_2 \leq \infty$, $t > 0$ and $u \in L^{s_1}(\mathbb{R}^d)$.

Given $\lambda > 0$, we define the dilation operator D_λ by $D_\lambda \varphi(x) = \varphi(\lambda x)$ for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$. This operator is extended by duality to $\mathcal{S}'(\mathbb{R}^d)$.

Lemma 1. D_λ has the following properties:

- (i) $D_\lambda(e^{-\lambda^\alpha t(-\Delta)^{\alpha/2}} \varphi) = e^{-t(-\Delta)^{\alpha/2}}(D_\lambda \varphi)$ for all $\varphi \in \mathcal{S}'(\mathbb{R}^d)$,
- (ii) $D_\lambda(D_{\frac{1}{\lambda}} \varphi) = \varphi$ for all $\varphi \in \mathcal{S}'(\mathbb{R}^d)$,
- (iii) $\|D_\lambda \varphi\|_r = \lambda^{-\frac{d}{r}} \|\varphi\|_r$ for all $\varphi \in L^r(\mathbb{R}^d)$, $r \geq 1$,
- (iv) $D_\lambda(\varphi \psi) = D_\lambda \varphi D_\lambda \psi$ for all $\varphi, \psi, \varphi \psi \in L^1_{loc}(\mathbb{R}^d)$,
- (v) $D_\lambda(|\cdot|^{-\gamma}) = \lambda^{-\gamma} |\cdot|^{-\gamma}$ for all $\gamma > 0$.

Proof. It is obvious that (ii)-(v) hold. We only prove (i). By (2.6) and (2.7),

$$e^{-t(-\Delta)^{\alpha/2}}(D_\lambda \varphi(x)) = \int_{\mathbb{R}^d} G_\alpha(t, x - y) (D_\lambda \varphi(y)) dy$$

$$= \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{i(x-y) \cdot \xi - t|\xi|^\alpha} \varphi(\lambda y) d\xi dy. \quad (2.9)$$

On the other hand,

$$\begin{aligned} e^{-\lambda^{\alpha} t(-\Delta)^{\alpha/2}} \varphi(x) &= \int_{\mathbb{R}^d} G_\alpha(\lambda^{\alpha} t, x-y) \varphi(y) dy \\ &= \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{i(x-y) \cdot \xi - \lambda^{\alpha} t|\xi|^\alpha} \varphi(y) d\xi dy, \end{aligned} \quad (2.10)$$

then

$$\begin{aligned} D_\lambda(e^{-\lambda^{\alpha} t(-\Delta)^{\alpha/2}} \varphi) &= \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{i(\lambda x-y) \cdot \xi - \lambda^{\alpha} t|\xi|^\alpha} \varphi(y) d\xi dy \\ &= \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{i\lambda(x-z) \cdot \xi - t|\lambda \xi|^\alpha} \varphi(\lambda z) \lambda^n d\xi dz \\ &= \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{i(x-z) \cdot \eta - t|\eta|^\alpha} \varphi(\lambda z) d\eta dz. \end{aligned} \quad (2.11)$$

By (2.9) and (2.11), (i) holds. $\square$

Lemma 2. Let $d \geq 1$ be an integer and $0 < \gamma < d$. Let $q_1 \in (1, \infty]$ and $q_2 \in (1, \infty]$ such that

$$0 \leq \frac{1}{q_2} < \frac{\gamma}{d} + \frac{1}{q_1} < 1.$$

Then there exists a constant $C > 0$ depending on $\alpha, d, \gamma, q_1, q_2$ such that

$$\|e^{-t(-\Delta)^{\alpha/2}}(|\cdot|^{-\gamma} u)\|_{q_2} \leq C_2 t^{-\frac{d}{\alpha}(\frac{1}{q_1} - \frac{1}{q_2}) - \frac{\gamma}{\alpha}} \|u\|_{q_1}. \quad (2.12)$$

Proof. Set $m = \frac{d}{\gamma}$ and let $\varepsilon, \delta > 0$ satisfy

$$\varepsilon < m, \quad \frac{1}{q_2} \leq \frac{1}{m+\delta} + \frac{1}{q_1} \leq \frac{1}{m-\varepsilon} + \frac{1}{q_1} \leq 1.$$

In fact, from the following estimate (2.15), we can see that it is reasonable to choose such ε and δ . Consider the following decomposition

$$|\cdot|^{-\gamma} = \psi_1 + \psi_2, \quad \psi_1 \in L^{m-\varepsilon}(\mathbb{R}^d), \quad \psi_2 \in L^{m+\delta}(\mathbb{R}^d).$$

By the Hölder inequality, we have

$$\|\psi_1 u\|_{r_1} \leq \|\psi_1\|_{m-\varepsilon} \|u\|_{q_1}, \quad (2.13)$$

where

$$\frac{1}{r_1} = \frac{1}{m-\varepsilon} + \frac{1}{q_1}$$

and

$$\|\psi_2 u\|_{r_2} \leq \|\psi_2\|_{m+\delta} \|u\|_{q_1}, \quad (2.14)$$

where

$$\frac{1}{r_2} = \frac{1}{m+\delta} + \frac{1}{q_1}.$$

By the Minkowski inequality, (2.8), (2.13) and (2.14),

$$\begin{aligned} \|e^{-(\Delta)^{\alpha/2}}(|\cdot|^{-\gamma}u)\|_{q_2} &\leq \|e^{-(\Delta)^{\alpha/2}}(\Psi_1 u)\|_{q_2} + \|e^{-(\Delta)^{\alpha/2}}(\Psi_2 u)\|_{q_2} \\ &\leq C\|\Psi_1 u\|_{r_1} + C\|\Psi_2 u\|_{r_2} \\ &\leq C(\|\Psi_1\|_{m-\varepsilon} + \|\Psi_2\|_{m+\delta})\|u\|_{q_1}. \end{aligned} \quad (2.15)$$

Thus

$$\|e^{-(\Delta)^{\alpha/2}}(|\cdot|^{-\gamma}u)\|_{q_2} \leq C(\alpha, d, \gamma, q_1, q_2)\|u\|_{q_1}.$$

By (i) and (ii) in Lemma 1, $e^{-\lambda\alpha t(-\Delta)^{\alpha/2}}\varphi = D_{\frac{1}{\lambda}}e^{-t(-\Delta)^{\alpha/2}}D_{\lambda}\varphi$, then

$$e^{-(\Delta)^{\alpha/2}}\varphi = D_{t^{\frac{1}{\alpha}}}e^{-t(-\Delta)^{\alpha/2}}D_{t^{-\frac{1}{\alpha}}}\varphi \quad (2.16)$$

for all $\varphi \in \mathcal{S}'(\mathbb{R}^d)$. From (2.12) and (2.16), it follows that

$$\|D_{t^{\frac{1}{\alpha}}}e^{-t(-\Delta)^{\alpha/2}}D_{t^{-\frac{1}{\alpha}}}(|\cdot|^{-\gamma}u)\|_{q_2} \leq C(\alpha, d, \gamma, q_1, q_2)\|u\|_{q_1}.$$

This together with (iii) in Lemma 1 yield that

$$t^{-\frac{d}{\alpha q_2}}\|e^{-t(-\Delta)^{\alpha/2}}D_{t^{-\frac{1}{\alpha}}}(|\cdot|^{-\gamma}u)\|_{q_2} \leq C(\alpha, d, \gamma, q_1, q_2)\|u\|_{q_1}.$$

By (iv) and (v) in Lemma 1,

$$t^{-\frac{d}{\alpha q_2}}t^{\frac{\gamma}{\alpha}}\|e^{-t(-\Delta)^{\alpha/2}}(|\cdot|^{-\gamma}D_{t^{-\frac{1}{\alpha}}}u)\|_{q_2} \leq C(\alpha, d, \gamma, q_1, q_2)\|u\|_{q_1}.$$

Replacing u by $D_{t^{\frac{1}{\alpha}}}u$,

$$\begin{aligned} t^{-\frac{d}{\alpha q_2}}t^{\frac{\gamma}{\alpha}}\|e^{-t(-\Delta)^{\alpha/2}}(|\cdot|^{-\gamma}u)\|_{q_2} &\leq C(\alpha, d, \gamma, q_1, q_2)\|D_{t^{\frac{1}{\alpha}}}u\|_{q_1} \\ &\leq C(\alpha, d, \gamma, q_1, q_2)t^{-\frac{d}{\alpha q_1}}\|u\|_{q_1}. \end{aligned}$$

Therefore

$$\|e^{-t(-\Delta)^{\alpha/2}}(|\cdot|^{-\gamma}u)\|_{q_2} \leq C(\alpha, d, \gamma, q_1, q_2)t^{-\frac{d}{\alpha}(\frac{1}{q_1}-\frac{1}{q_2})-\frac{\gamma}{\alpha}}\|u\|_{q_1}.$$

□

3. RESULTS

Consider the linear problem

$$\begin{cases} \partial_t z(t) = -(-\Delta)^{\alpha/2} z(t) + \partial_t B^H(t), & t > 0, \\ z(0) = 0, \end{cases} \quad (3.1)$$

the mild solution is

$$z(t) = \int_0^t e^{-(t-s)(-\Delta)^{\alpha/2}} dB^H(s).$$

For $H > \frac{1}{2}$ and $H \geq \frac{1}{q}$, by the result in [6], $z \in C([0, T], L^q(\mathbb{R}^d))$. For $\beta > 0$ and $1 < r < \infty$ to be fixed later, define

$$L(T) = \max\left(\sup_{t \in [0, T]} \|z(t)\|_q, \sup_{t \in (0, T]} t^\beta \|z(t)\|_r\right). \quad (3.2)$$

Proof of Theorem 1. Let q satisfy (1.5) and $1 < q < \infty$. Then there exists $r > q$ such that

$$\frac{1}{qp} - \frac{\gamma}{dp} < \frac{1}{r} < \frac{d-\gamma}{dp}. \quad (3.3)$$

Let

$$\beta = \frac{d}{\alpha q} - \frac{d}{\alpha r}. \quad (3.4)$$

Since $q > \frac{d(p-1)}{\alpha-\gamma}$, then

$$\frac{1}{q} - \frac{\alpha}{dp} < \frac{1}{qp} - \frac{\gamma}{dp}.$$

This together with (3.3) and (3.4) yield that

$$\beta p < 1.$$

Let $\rho > 0$, $M > 0$, $T > 0$ and $\phi \in L^q(\mathbb{R}^d)$ such that

$$\|\phi\|_q \leq \rho, \quad (3.5)$$

$$\max(C_1, 1)\rho + L(T) + \frac{K}{2p} M^p T^{1-\frac{\gamma}{\alpha}-\frac{d(p-1)}{\alpha q}} \leq M, \quad (3.6)$$

$$KM^{p-1} T^{1-\frac{d(p-1)}{\alpha q}-\frac{\gamma}{\alpha}} < 1, \quad (3.7)$$

where K is a positive constant to be fixed later.

Define

$$X = \{u \in C([0, T]; L^q(\mathbb{R}^d)) \cap C((0, T]; L^r(\mathbb{R}^d));$$

$$\max_{t \in [0, T]} \|u(t)\|_q \leq M, \max_{t \in (0, T]} t^\beta \|u(t)\|_r \leq M\}.$$

Endowed with the metric

$$d(u, v) = \max\left(\sup_{t \in [0, T]} \|u(t) - v(t)\|_q, \sup_{t \in (0, T]} t^\beta \|u(t) - v(t)\|_r\right),$$

(X, d) is a complete metric space. Given $u \in X$, set

$$\begin{aligned} F_\Phi(u)(t) &= e^{-t(-\Delta)^{\alpha/2}} \Phi + \int_0^t e^{-(t-s)(-\Delta)^{\alpha/2}} (|\cdot|^{-\gamma} |u(s)|^{p-1} u(s)) ds \\ &\quad + \int_0^t e^{-(t-s)(-\Delta)^{\alpha/2}} dB^H(s), \end{aligned} \quad (3.8)$$

Let $\Phi, \Psi \in L^q(\mathbb{R}^d)$, $u, v \in X$. For $q_1 = \frac{r}{p}$, $q_2 = q$, by Lemma 2 and the Hölder inequality,

$$\begin{aligned} &\|F_\Phi(u)(t) - F_\Psi(v)(t)\|_q \\ &\leq \|e^{-t(-\Delta)^{\alpha/2}} (\Phi - \Psi)\|_q \\ &\quad + \int_0^t \|e^{-(t-s)(-\Delta)^{\alpha/2}} [|\cdot|^{-\gamma} (|u(s)|^{p-1} u(s) - |v(s)|^{p-1} v(s))]\|_q ds \\ &\leq \|\Phi - \Psi\|_q + C_2 \int_0^t (t-s)^{-\frac{d}{\alpha}(\frac{p}{r}-\frac{1}{q})-\frac{\gamma}{\alpha}} \| |u(s)|^{p-1} u(s) - |v(s)|^{p-1} v(s) \|_{\frac{r}{p}} ds \\ &\leq \|\Phi - \Psi\|_q + C_2 p \int_0^t (t-s)^{-\frac{d}{\alpha}(\frac{p}{r}-\frac{1}{q})-\frac{\gamma}{\alpha}} \|u-v\|_r (\|u\|_r^{p-1} + \|v\|_r^{p-1}) ds \\ &\leq \|\Phi - \Psi\|_q + \left(2C_2 p M^{p-1} \int_0^t (t-s)^{-\frac{d}{\alpha}(\frac{p}{r}-\frac{1}{q})-\frac{\gamma}{\alpha}} s^{-\beta p} ds \right) d(u, v). \end{aligned}$$

By (3.4),

$$\begin{aligned} &\|F_\Phi(u)(t) - F_\Psi(v)(t)\|_q \\ &\leq \|\Phi - \Psi\|_q + \left(2C_2 p M^{p-1} t^{1-\frac{\gamma}{\alpha}-\frac{d(p-1)}{\alpha q}} \int_0^1 (1-\tau)^{-\frac{d}{\alpha}(\frac{p}{r}-\frac{1}{q})-\frac{\gamma}{\alpha}} \tau^{-\beta p} d\tau \right) d(u, v). \end{aligned}$$

Since $r > q > \frac{d(p-1)}{\alpha-\gamma}$, then

$$1 - \frac{\gamma}{\alpha} - \frac{d(p-1)}{\alpha q} > 0, \quad \frac{d}{\alpha} \left(\frac{p}{r} - \frac{1}{q} \right) + \frac{\gamma}{\alpha} < \frac{d}{\alpha} \left(\frac{p}{r} - \frac{1}{r} \right) + \frac{\gamma}{\alpha} < 1.$$

We note $\beta p < 1$ and recall the Beta function

$$B(x, y) = \int_0^1 \tau^{x-1} (1-\tau)^{y-1} d\tau, \quad x, y > 0,$$

therefore

$$\|F_\Phi(u)(t) - F_\Psi(v)(t)\|_q \leq \|\Phi - \Psi\|_q + C_3 M^{p-1} T^{1-\frac{\gamma}{\alpha}-\frac{d(p-1)}{\alpha q}} d(u, v), \quad (3.9)$$

where

$$C_3 = 2C_2p \int_0^1 (1-\tau)^{-\frac{d}{\alpha}(\frac{p}{r}-\frac{1}{q})-\frac{\gamma}{\alpha}} \tau^{-\beta p} d\tau \quad (3.10)$$

is a finite positive constant.

For $s_1 = q < s_2 = r$, by (2.8),

$$\|e^{-t(-\Delta)^{\alpha/2}}(\varphi - \psi)\|_r \leq C_1 t^{-\frac{d}{\alpha}(\frac{1}{q}-\frac{1}{r})} \|\varphi - \psi\|_q = C_1 t^{-\beta} \|\varphi - \psi\|_q. \quad (3.11)$$

For $q_1 = \frac{r}{q}$, $q_2 = r$, by (3.11), Lemma 2 and the Hölder inequality,

$$\begin{aligned} & \|F_\varphi(u)(t) - F_\psi(v)(t)\|_r \\ & \leq \|e^{-t(-\Delta)^{\alpha/2}}(\varphi - \psi)\|_r \\ & \quad + \int_0^t \|e^{-(t-s)(-\Delta)^{\alpha/2}}[|\cdot|^{-\gamma}(|u(s)|^{p-1}u(s) - |v(s)|^{p-1}v(s))]\|_r ds \\ & \leq C_1 t^{-\beta} \|\varphi - \psi\|_q + C_2 \int_0^t (t-s)^{-\frac{d}{\alpha}(\frac{p}{r}-\frac{1}{r})-\frac{\gamma}{\alpha}} \| |u(s)|^{p-1}u(s) - |v(s)|^{p-1}v(s) \|_{\frac{r}{p}} ds \\ & \leq C_1 t^{-\beta} \|\varphi - \psi\|_q + C_2 p \int_0^t (t-s)^{-\frac{d}{\alpha}(\frac{p}{r}-\frac{1}{r})-\frac{\gamma}{\alpha}} \|u-v\|_r (\|u\|_r^{p-1} + \|v\|_r^{p-1}) ds \\ & \leq C_1 t^{-\beta} \|\varphi - \psi\|_q + \left(2C_2 p M^{p-1} \int_0^t (t-s)^{-\frac{d}{\alpha}(\frac{p}{r}-\frac{1}{r})-\frac{\gamma}{\alpha}} s^{-\beta p} ds \right) d(u, v). \end{aligned}$$

Then

$$\begin{aligned} & t^\beta \|F_\varphi(u)(t) - F_\psi(v)(t)\|_r \\ & \leq C_1 \|\varphi - \psi\|_q + \left(2C_2 p M^{p-1} t^\beta \int_0^t (t-s)^{-\frac{d}{\alpha}(\frac{p}{r}-\frac{1}{r})-\frac{\gamma}{\alpha}} s^{-\beta p} ds \right) d(u, v) \\ & = C_1 \|\varphi - \psi\|_q + \left(2C_2 p M^{p-1} t^{\beta+1-\frac{d(p-1)}{\alpha r}-\frac{\gamma}{\alpha}-\beta p} \int_0^1 (1-\tau)^{-\frac{d}{\alpha}(\frac{p}{r}-\frac{1}{r})-\frac{\gamma}{\alpha}} \tau^{-\beta p} d\tau \right) d(u, v) \\ & = C_1 \|\varphi - \psi\|_q + \left(2C_2 p M^{p-1} t^{1-\frac{\gamma}{\alpha}-\frac{d(p-1)}{\alpha q}} \int_0^1 (1-\tau)^{-\frac{d}{\alpha}(\frac{p}{r}-\frac{1}{r})-\frac{\gamma}{\alpha}} \tau^{-\beta p} d\tau \right) d(u, v) \\ & = C_1 \|\varphi - \psi\|_q + C_4 M^{p-1} T^{1-\frac{\gamma}{\alpha}-\frac{d(p-1)}{\alpha q}} d(u, v), \end{aligned} \quad (3.12)$$

where

$$C_4 = 2C_2 p \int_0^1 (1-\tau)^{-\frac{d}{\alpha}(\frac{p}{r}-\frac{1}{r})-\frac{\gamma}{\alpha}} \tau^{-\beta p} d\tau \quad (3.13)$$

is a finite positive constant. By (3.9) and (3.17),

$$d(F_\varphi(u), F_\psi(v)) \leq \max(C_1, 1) \|\varphi - \psi\|_q + K M^{p-1} T^{1-\frac{\gamma}{\alpha}-\frac{d(p-1)}{\alpha q}} d(u, v), \quad (3.14)$$

where $K = \max(C_3, C_4)$.

By (3.14), we see that for $u \in X$, $F_\varphi(u) \in C([0, T]; L^q(\mathbb{R}^d)) \cap C((0, T]; L^r(\mathbb{R}^d))$.

Given $u \in X$, by (3.8), (3.2), we have

$$\begin{aligned} \|F_\varphi(u)(t)\|_q &\leq \|e^{-t(-\Delta)^{\alpha/2}} \varphi\|_q + \int_0^t \|e^{-(t-s)(-\Delta)^{\alpha/2}} [|\cdot|^{-\gamma}(|u(s)|^{p-1}u(s))]\|_q ds + \|z(t)\|_q \\ &\leq \|\varphi\|_q + C_2 \int_0^t (t-s)^{-\frac{d}{\alpha}(\frac{p}{r}-\frac{1}{q})-\frac{\gamma}{\alpha}} \| |u(s)|^{p-1}u(s) \|_{\frac{r}{p}} ds + L(T) \\ &\leq \rho + L(T) + C_2 M^p \int_0^t (t-s)^{-\frac{d}{\alpha}(\frac{p}{r}-\frac{1}{q})-\frac{\gamma}{\alpha}} s^{-\beta p} ds \\ &= \rho + L(T) + \frac{C_3}{2p} M^p T^{1-\frac{\gamma}{\alpha}-\frac{d(p-1)}{\alpha q}}. \end{aligned} \quad (3.15)$$

Similarly, by (3.11) and Lemma 2, we have

$$\begin{aligned} \|F_\varphi(u)(t)\|_r &\leq \|e^{-t(-\Delta)^{\alpha/2}} \varphi\|_r + \int_0^t \|e^{-(t-s)(-\Delta)^{\alpha/2}} [|\cdot|^{-\gamma}(|u(s)|^{p-1}u(s))]\|_r ds + \|z(t)\|_r \\ &\leq C_1 \rho t^{-\beta} + C_2 M^p \int_0^t (t-s)^{-\frac{d}{\alpha}(\frac{p}{r}-\frac{1}{r})-\frac{\gamma}{\alpha}} s^{-\beta p} ds + \|z(t)\|_r. \end{aligned} \quad (3.16)$$

Then

$$\begin{aligned} t^\beta \|F_\varphi(u)(t)\|_r &\leq C_1 \rho + \left(C_2 M^p t^\beta \int_0^t (t-s)^{-\frac{d}{\alpha}(\frac{p}{r}-\frac{1}{r})-\frac{\gamma}{\alpha}} s^{-\beta p} ds \right) + t^\beta \|z(t)\|_r \\ &= C_1 \rho + L(T) + \frac{C_4}{2p} M^p T^{1-\frac{\gamma}{\alpha}-\frac{d(p-1)}{\alpha q}}, \end{aligned} \quad (3.17)$$

where C_4 is the same as in (3.13).

By (3.16), (3.17) and (3.6), we get $F_\varphi(u) \in X$. That is F_φ maps X into X .

Letting $\varphi = \psi$ in (3.14), we have

$$d(F_\varphi(u), F_\varphi(v)) \leq K M^{p-1} T^{1-\frac{\gamma}{\alpha}-\frac{d(p-1)}{\alpha q}} d(u, v). \quad (3.18)$$

By (3.7), we conclude that F_φ is a contraction mapping from X into X . Therefore F_φ has a unique fixed point in X , which is the mild solution of (1.1). $\square$

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NUMERICAL SOLUTION OF TIME FRACTIONAL VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. In this work, we develop and analyze a finite difference-based numerical scheme for a system of time-fractional Volterra-type integro-differential equations. The fractional derivative of order α with $\alpha \in (0, 1)$ is considered in the Caputo sense. We provide sufficient conditions to ensure the existence of a unique solution. To construct the difference scheme, classical L1 discretization is employed for the fractional operator, and the integral part is approximated using the composite trapezoidal rule. The convergence analysis and error estimates are discussed. Having a mild singularity in the solution of the system, the proposed scheme achieves only α order accuracy. In the case of sufficiently smooth solutions, it achieves $(2 - \alpha)$ order of accuracy. Finally, several numerical experiments are presented to support our theoretical findings and validate the proposed scheme.

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Keywords: system of integro-differential equation, Caputo derivative, L1 scheme, trapezoidal rule, error analysis

1. INTRODUCTION

Fractional calculus is a generalization of ordinary differentiation and integration to arbitrary orders and has become an essential branch of mathematics. The non-local properties of fractional calculus allow us to model physical phenomena that have a dependence on time instants and previous histories simultaneously. In the last few years, fractional order differential equations (FDEs) and fractional order integro-differential equations (FIDEs) have attracted great attention due to their frequent appearance in various fields such as physics [6], electrochemistry [12], biology [13], viscoelasticity [16], engineering, and other area of applied mathematics. Likewise systems of integro-differential equations play a pivotal role in mathematical modeling of physical phenomena of various field of science and engineering, such as magnetic field, electric-circuit analysis, nano-hydrodynamics, control of systems, glass-forming process, dropwise condensation, statistics of polymer chains, wind ripple in

the desert, modeling the competition between tumor cells and the immune system, modeling of fluid waves in oceanography, the activity of interacting inhibitory and excitatory neurons, hereditary properties of various materials and processes etc. for more details, one may refer [2, 3, 5, 7] and references therein. Consequently, enormous attention has been paid by the researchers to develop approximate and numerical techniques for the solutions of FIDEs of physical interest since most FIDEs has yet to have an exact analytical form of solutions. For instance: Rawashdeh, in [20], applied the spline collocation method to obtain the numerical solution of FIDEs. In [18], Nazari and Shahmorad used the fractional differential transform method to approximate FIDEs with nonlocal boundary conditions. Arqub et al. developed a novel high-order algorithm that reproduces kernel approximation to solve Volterra integro-differential equations involving Atangana–Baleanu (ABC)-fractional derivative in [4]. A class of FIDEs was solved by Panda et al. in [19] using Adomian decomposition and homotopy perturbation methods. The solution of a fractional order differential/integro-differential equation generally has singularity in time [14]. However, the nature of the singularity is dependent on the regularity of the solution. We called the singularity weak/mild if the first derivative of the solution blows up as $t \rightarrow 0^+$. Recently, in [21] the authors proposed a finite difference scheme to approximate the Volterra type FIDEs having weak singularity. In [8], Ghosh and Mohapatra extended the previous work and showed a comparison study using two schemes based on the L1 and L1-2 discretization techniques.

In this article, we consider the following system of fractional order Volterra integro-differential equations (SFVIDEs):

$$\begin{cases} D_t^\alpha \mathbf{U}(t) + \mathbf{A}(t)\mathbf{U}(t) + \lambda \int_0^t \mathbf{K}(t,s)\mathbf{U}(s)ds = \mathbf{F}(t), & t \in (0, T], \\ \mathbf{U}(0) = \mathbf{U}_0, \end{cases} \quad (1.1)$$

where D_t^α denotes the Caputo fractional differential operator of order $\alpha \in (0, 1)$, $T \in \mathbb{R}^+$, $\mathbf{U}(t) = [u^{(1)}(t), u^{(2)}(t), \dots, u^{(n)}(t)]$ is the unknown vector function and

$$\begin{cases} \mathbf{F}(t) = [f^{(1)}(t), f^{(2)}(t), \dots, f^{(n)}(t)], \lambda = [\lambda_1, \lambda_2, \dots, \lambda_n], \mathbf{U}_0 = [U_{01}, U_{02}, \dots, U_{0n}], \\ \mathbf{A}(t) = [a_{ij}(t)], \mathbf{K}(t,s) = [K_{ij}(t,s)], \quad i, j = 1(1)n. \end{cases} \quad (1.2)$$

$a_{ij} : [0, T] \rightarrow \mathbb{R}, K_{ij} : [0, T] \times [0, T] \rightarrow \mathbb{R}$ for $i, j = 1(1)n$, and the vector function $\mathbf{F}(t)$ are known and sufficiently smooth. The parameter $\lambda_i \in \mathbb{R}$. The model problem (1.1) in the present work is an academic example. However, system of integro-differential equations appear in many physical applications [5, 7].

In recent years, a large number of papers have been devoted to approximating a system of linear and nonlinear integro-differential equations numerically; in most cases, finding an exact analytic solution is usually not accessible. To name a few: Momani and Qaralleh [17] implemented the Adomian decomposition method for approximating the solution of system of fractional integro-differential equations. In

[25], Zurigat et al. used the homotopy analysis method to solve a system of linear and nonlinear fractional integro-differential equations. Akbar et al. employed an extension of the optimal homotopy asymptotic method to the fractional order integro-differential equations system in [1]. In [24], Youbi et al. investigated the approximate analytic solutions with the help of reproducing kernel function for a system of Volterra type FIDEs involving Caputo-Fabrizio derivative. In [11], Chebyshev pseudo-spectral collocation method is used to numerically solve a system of linear and nonlinear fractional integro-differential equations of Volterra type. Heydari et al. [10] presented the Chebyshev wavelet computational method by developing the operational matrix with the help of shifted Chebyshev polynomials for solving a class of system of nonlinear singular fractional Volterra integro-differential equations. Hesameddini and Shahbazi obtained the numerical solution of a system of fractional integro-differential equations based on the hybrid Bernstein Block-Pulse functions in [9]. In their study, the Riemann-Liouville fractional integral operator for hybrid Bernstein Block-Pulse functions were proposed and approximated by the Gauss quadrature formula. In [23], Wang et al. established an operational matrix using Bernoulli wavelets for solving coupled system of nonlinear fractional order integro-differential equations.

Observing the literature, we find that the majority of the solution methods are semi-analytical or spectral/collocation based and have limitations. Meanwhile, a little attempt has been made to solve a system of FIDEs using the finite difference technique. The article aims to introduce a numerical scheme by combining the well-known L1 technique and the composite trapezoidal rule to solve the fractional order system (1.1). The advantage of using the L1 technique is it uses the linear interpolation approximation for the integrand function. As a result, twice continuous differentiability is enough to achieve the optimal convergence rate. The simplicity of implementation and the well established convergence and stability analysis are also significant advantages of the L1 technique. The rest of this paper is organized as follows: In Section 2, we introduce few basic definitions of fractional calculus and establish the uniqueness of the solution. The difference scheme is derived in Section 3. Section 4 deals with the convergence analysis and error bounds. In Section 5, numerical experiments have been conducted, and concluding remarks are drawn in Section 6.

Notation 1. On a domain Ω , $C^k(\Omega)$ represents the set of k times continuously differentiable functions. The positive constant C appears in several inequalities. $\Gamma(\cdot)$ denotes Euler Gamma function.

2. PRELIMINARIES

Definition 1. Let $Y = C([0, T], \mathbb{R})$ be the Banach space of all real valued continuous functions, then the supremum norm is defined as:

$$\|y(t)\| := \sup_{t \in [0, T]} |y(t)|.$$

Definition 2. A real function $\phi(x)$ is said to be in the space C_μ , $\mu \in \mathbb{R}$ if there exists a real number $\xi > \mu$ such that $\phi(x) = x^\xi \phi_1(x)$ where $\phi_1(x) \in C(\mathbb{R})$. Clearly, $C_\mu \subset C_\gamma$ if $\gamma \leq \mu$. A function $\phi(x)$ is said to be in the space C_μ^k , $k \in \mathbb{N} \cup \{0\}$, if $\phi^{(k)}(x) \in C_\mu$.

Definition 3. The Riemann-Liouville fractional integral of $\phi(x) \in C[a, b]$ of order $\tilde{\alpha} \in \mathbb{R}^+$ is defined as:

$$J_x^{\tilde{\alpha}} \phi(x) = \frac{1}{\Gamma(\tilde{\alpha})} \int_a^x (x-s)^{\tilde{\alpha}-1} \phi(s) ds.$$

Definition 4. For $m-1 < \tilde{\alpha} \leq m$, the Caputo fractional derivative of $\phi(x) \in C^m[a, b]$ of order $\tilde{\alpha} \in \mathbb{R}^+$ is defined as:

$$D_x^{\tilde{\alpha}} \phi(x) = J_x^{m-\tilde{\alpha}} \phi^{(m)}(x) = \begin{cases} \frac{1}{\Gamma(m-\tilde{\alpha})} \int_a^x \frac{\phi^{(m)}(s)}{(x-s)^{1-m+\tilde{\alpha}}} ds, & \text{if } m-1 < \tilde{\alpha} < m, \\ \phi^{(m)}(x), & \text{if } \tilde{\alpha} = m. \end{cases}$$

Theorem 1. If $\frac{(\alpha+1)\|A\| + \lambda\|K\|}{\Gamma(2+\alpha)} < 1$ then (1.1) has unique solution in $[0, T]$.

Proof. The Banach fixed point theorem is used to prove this theorem. Applying J_x^α from both sides of (1.1), we have $U(t) = \Psi U(t)$ for $t \in [0, T]$, where the operator $\Psi : C([0, T], \mathbb{R}^n) \rightarrow \mathbb{R}^n$ for each $U \in C([0, T], \mathbb{R}^n)$ and $t \in [0, T]$ is defined as:

$$\Psi U(t) \equiv U_0 + J_t^\alpha F(t) - J_t^\alpha \{A(t)U(t)\} - \lambda J_t^\alpha \left\{ \int_0^t K(t,s)U(s) ds \right\}.$$

Our aim is to show that Ψ is a contraction mapping. For any $U_1(t), U_2(t) \in C([0, T], \mathbb{R}^n)$, we have

$$\begin{aligned} & \|\Psi U_1(t) - \Psi U_2(t)\| \\ &= \left\| J_t^\alpha \{A(t)(U_1(t) - U_2(t))\} + \lambda J_t^\alpha \left\{ \int_0^t K(t,s)(U_1(s) - U_2(s)) ds \right\} \right\| \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|A(s)\| \|U_1(s) - U_2(s)\| ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \int_0^s \|K(s,z)\| \|U_1(z) - U_2(z)\| dz ds \\ &\leq \frac{\|A\|}{\Gamma(\alpha+1)} \|U_1(t) - U_2(t)\| + \frac{\lambda\|K\|}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \int_0^s \|U_1(z) - U_2(z)\| dz ds \\ &\leq \frac{(\alpha+1)\|A\| + \lambda\|K\|}{\Gamma(2+\alpha)} \|U_1(t) - U_2(t)\| \leq \|U_1(t) - U_2(t)\|. \end{aligned}$$

Hence, Ψ is a contraction mapping. The Banach fixed point theorem implies the problem (1.1) has a unique solution for all $t \in [0, T]$. $\square$

3. PROPOSED SCHEME

The problem (1.1) is discretized by the combination of the L1 technique and the composite trapezoidal rule. Set $N \in \mathbb{N}$. The uniform mesh $t_j = j\tau$ for $j = 0(1)N$ with equal step length $\tau = T/N$. The computed solution of $u^{(p)}$ at the mesh point t_j is denoted $U_j^{(p)}$.

The Caputo derivative $D_t^\alpha u^{(p)}(t)$ at t_j for $j = 1(1)N$ can be written as:

$$D_t^\alpha u^{(p)}(t_j) = \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{j-1} \int_{s=t_k}^{t_{k+1}} (t_j - s)^{-\alpha} (u^{(p)}(s))' ds,$$

for $p = 1(1)n$, which can be discretized as follows [8, 15, 22]:

$$\begin{aligned} D_N^\alpha u^{(p)}(t_j) &= \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{j-1} \frac{u^{(p)}(t_{k+1}) - u^{(p)}(t_k)}{\tau} \int_{s=t_k}^{t_{k+1}} (t_j - s)^{-\alpha} ds \\ &= \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \sum_{k=0}^{j-1} [u^{(p)}(t_{k+1}) - u^{(p)}(t_k)] b_{j-k} + \epsilon_j^{(p)}, \end{aligned} \quad (3.1)$$

where $\epsilon_j^{(p)} = (D_t^\alpha - D_N^\alpha)u^{(p)}(t_j)$ and $b_q = q^{1-\alpha} - (q-1)^{1-\alpha}$ for $q = 1(1)N$. The integral for $j = 1(1)N$ is

$$I_t u^{(p)}(t_j) = \int_{t_0}^{t_j} \sum_{i=1}^n K_{pi}(t_j, s) u^{(i)}(s) ds = \sum_{i=1}^n \sum_{m=0}^{j-1} \int_{t_m}^{t_{m+1}} K_{pi}(t_j, s) u^{(i)}(s) ds,$$

which can be approximated by applying the composite trapezoidal rule as:

$$I_N u^{(p)}(t_j) = \frac{\tau}{2} \sum_{i=1}^n \sum_{m=0}^{j-1} [K_{pi}(t_j, t_m) u^{(i)}(t_m) + K_{pi}(t_j, t_{m+1}) u^{(i)}(t_{m+1})] + R_j^{(p)}, \quad (3.2)$$

where $R_j^{(p)} = (I_t - I_N)u^{(p)}(t_j)$. Using (3.1) and (3.2), the model (1.1) transformed to

$$\begin{cases} D_N^\alpha u^{(p)}(t_j) + \sum_{i=1}^n a_{pi}(t_j) u^{(i)}(t_j) + \lambda_p I_N u^{(p)}(t_j) = f^{(p)}(t_j) + \mathcal{E}_j^{(p)}, \\ u^{(p)}(t_0) = \eta_p, \end{cases} \quad (3.3)$$

for $j = 1(1)N$, and each $p = 1(1)n$. The remainder term $\mathcal{E}_j^{(p)}$ is given by $\mathcal{E}_j^{(p)} = \epsilon_j^{(p)} + R_j^{(p)}$. Neglecting $\mathcal{E}_j^{(p)}$, the discrete problem (3.3) is reduced to

$$\begin{cases} f_j^{(p)} = \mu \sum_{k=0}^{j-1} (U_{k+1}^{(p)} - U_k^{(p)}) d_{j-k} + \sum_{i=1}^n (a_{pi})_j U_j^{(i)} \\ \quad + \frac{\lambda_p \tau}{2} \sum_{m=0}^{j-1} \sum_{i=1}^n \{ K_{pi}(t_j, t_m) U_m^{(i)} + K_{pi}(t_j, t_{m+1}) U_{m+1}^{(i)} \}, \\ U_0^{(p)} = \eta_p, \end{cases} \quad (3.4)$$

for $j = 1(1)N$, for each $p = 1(1)n$. The difference scheme (3.4) is explicit since one can easily find the solution from the following recursive relation:

$$\left\{ \begin{array}{l} U_j^{(p)} = \Pi^{-1} \left[f_j^{(p)} + \mu \left[d_1 U_{j-1}^{(p)} + \sum_{k=0}^{j-2} (U_k^{(p)} - U_{k+1}^{(p)}) d_{j-k} \right] \right. \\ \quad - \sum_{i=1, i \neq p}^n (a_{pi})_j U_j^{(i)} - \frac{\lambda_p \tau}{2} K_{pp}(t_j, t_{j-1}) U_{j-1}^{(p)} \\ \quad - \frac{\lambda_p \tau}{2} \sum_{m=0}^{j-2} \sum_{i=1}^n \left\{ K_{pi}(t_j, t_m) U_m^{(i)} + K_{pi}(t_j, t_{m+1}) U_{m+1}^{(i)} \right\} \\ \quad \left. - \frac{\lambda_p \tau}{2} \sum_{i=1, i \neq p}^n \left\{ K_{pi}(t_j, t_{j-1}) U_{j-1}^{(i)} + K_{pi}(t_j, t_j) U_j^{(i)} \right\} \right], \\ U_0^{(p)} = \eta_p, \end{array} \right. \quad (3.5)$$

where $\mu = \frac{h^{-\alpha}}{\Gamma(2-\alpha)}$, $\Pi = \mu d_1 + (a_{pp})_j + \frac{\lambda_p \tau}{2} K_{pp}(t_j, t_j)$. Alternatively, it is possible to express (3.4) as a $N \times N$ linear system of equations:

$$G^{(p)} U^{(p)} = F^{(p)} \quad \text{for } p = 1(1)n, \quad (3.6)$$

where $U^{(p)} = [U_1^{(p)}, U_2^{(p)}, \dots, U_N^{(p)}]$, $F^{(p)} = [F_1^{(p)}, F_2^{(p)}, \dots, F_N^{(p)}]$ and for $i = 1(1)N$,

$$\left\{ \begin{array}{l} G_{i,i}^{(p)} = \mu b_1 + (a_{ii})_j + \frac{\lambda_p \tau}{2} K_{ii}(x_j, x_j), \\ G_{i,k}^{(p)} = (a_{ik})_j + \frac{\lambda_p \tau}{2} K_{ik}(x_j, x_j), \quad k = i+1, i+2, \dots, n, \\ G_{l,i}^{(j)} = (a_{il})_j + \frac{\lambda_i h}{2} K_{il}(x_j, x_j), \quad l = 1, 2, \dots, i-1, \\ F_i^{(j)} = f_j^{(i)} + \mu b_1 U_{j-1}^{(i)} + \sum_{k=0}^{j-2} (U_k^{(i)} - U_{k+1}^{(i)}) b_{j-k} \\ \quad - \frac{\lambda_i h}{2} \sum_{k=1}^n \left[\sum_{m=0}^{j-1} K_{ik}(x_j, x_m) U_m^{(i)} - \sum_{m=0}^{j-2} K_{ik}(x_j, x_{m+1}) U_{m+1}^{(i)} \right]. \end{array} \right.$$

4. ERROR ANALYSIS

This section establishes the convergence result and estimates the global error of the proposed scheme (3.4).

Lemma 1. *The truncation error for the discretization of the Caputo fractional derivative for $j = 1(1)N$ satisfies*

$$|\epsilon_j^{(p)}| \leq \begin{cases} C\tau^{2-\alpha}, & \text{if } u^{(p)}(t) \in C^2[0, T], \\ C\tau^\alpha, & \text{if } u^{(p)}(t) \in C_\alpha[0, T]. \end{cases}$$

Proof. Let $u^{(p)}(t) \in \mathcal{C}^2[0, T]$, then we have

$$\begin{aligned} |\varepsilon_j^{(p)}| &= |(D_t^\alpha - D_N^\alpha)u^{(p)}(t_j)| \\ &= \left| \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{j-1} \int_{t_k}^{t_{k+1}} (t_j - s)^{-\alpha} \left[\frac{du^{(p)}(s)}{ds} - \frac{u^{(p)}(t_{k+1}) - u^{(p)}(t_k)}{\tau} \right] ds \right| \\ &\leq C \left| \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{j-1} \int_{t_k}^{t_{k+1}} \frac{t_j + t_{j-1} - 2s}{(t_j - s)^\alpha} ds + O(\tau^2) \right|. \end{aligned}$$

From [15], we get $\left| \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{j-1} \int_{t_k}^{t_{k+1}} \frac{t_j + t_{j-1} - 2s}{(t_j - s)^\alpha} ds \right| \leq 2\tau^{2-\alpha}$. This however means $|\varepsilon_j^{(p)}| \leq C\tau^{2-\alpha}$.

If $u^{(p)}(t) \in C_\alpha[0, T]$, from [22] we have $|\varepsilon_j^{(p)}| = |(D_t^\alpha - D_N^\alpha)u^{(p)}(t_j)| \leq C\tau^\alpha$. Hence, completes the proof. $\square$

Lemma 2. For each $p = 1(1)n$ the remainder $R_j^{(p)}$ satisfies $|R_j^{(p)}| \leq CT\tau^2$ for $j = 1(1)N$.

Proof. Since $K_{ij}(t, s) \in \mathcal{C}^\infty([0, T] \times [0, T])$ for $i, j = 1(1)n$ and $u^{(p)}(t) \in \mathcal{C}^2[0, T] \cap \mathcal{C}^\infty(0, T]$, there exist constants $C_1, C_2 > 0$ such that $|K_{ij}(t, s)| \leq C_1$ and $\left| \frac{d^q u^{(p)}}{dt^q}(t) \right| \leq C_2$ for $q = 0, 1, 2$. Now, we have

$$\begin{aligned} |R_j^{(p)}| &= |(I_t - I_N)u^{(p)}(t_j)| = \sum_{i=1}^n \left| \sum_{m=0}^{j-1} \int_{t_m}^{t_{m+1}} K_{pi}(t_j, s) u^{(i)}(s) ds \right. \\ &\quad \left. - \frac{\tau}{2} \sum_{m=0}^{j-1} \left[K_{pi}(t_j, t_m) U_m^{(i)} + K_{pi}(t_j, t_{m+1}) U_{m+1}^{(i)} \right] \right| \\ &\leq C_1 \sum_{i=1}^n \left| \sum_{m=0}^{j-1} \int_{t_m}^{t_{m+1}} u^{(i)}(s) ds - \frac{\tau}{2} \sum_{m=0}^{j-1} (U_m^{(i)} + U_{m+1}^{(i)}) \right|. \end{aligned}$$

Using the Taylor series, we get

$$\begin{aligned} |R_j^{(p)}| &\leq C_1 \sum_{i=1}^n \left| \sum_{m=0}^{j-1} \left[\left(\tau U_m^{(i)} + \frac{\tau^2}{2!} (U^{(i)})'_m + \frac{\tau^3}{3!} (U^{(i)})''_m + \frac{\tau^4}{4!} (U^{(i)})'''_m + \dots \right) \right. \right. \\ &\quad \left. \left. - \frac{\tau}{2} \left(U_m^{(i)} + U_{m+1}^{(i)} + \tau (U^{(i)})'_m + \frac{\tau^2}{2!} (U^{(i)})''_m + \frac{\tau^3}{3!} (U^{(i)})'''_m + \dots \right) \right] \right| \\ &\leq C_1 \sum_{i=1}^n \left| \sum_{m=0}^{j-1} \left[\frac{\tau^3}{12} (U^{(i)})''_m - \frac{\tau^4}{24} (U^{(i)})'''_m + \dots \right] \right| \end{aligned}$$

$$\leq C_1 C_2 \sum_{i=1}^n j \tau^3 \leq CN \tau^3 \leq CT \tau^2,$$

which is our desired bound. $\square$

Denote $e_j^{(p)} = |u^{(p)}(t_j) - U_j^{(p)}|$ for $j = 1(1)N$. From (3.3) and (3.4), we get

$$\begin{cases} D_N^\alpha e_j^{(p)} + \sum_{i=1}^n a_{pi}(t_j) e_j^{(i)} + \lambda_p I_N e_j^{(p)} = \mathcal{E}_j^{(p)} & \text{for } j = 1(1)N, \\ e_0^{(p)} = 0. \end{cases} \quad (4.1)$$

Lemma 3. For any mesh function $\{U_j^{(p)}\}_{j=0}^N$ with $U_0^{(p)} = 0$, we have

$$|U_j^{(p)}| \leq \max_{k=1(1)j} \left\{ \frac{\tau^\alpha \Gamma(2-\alpha)}{b_k} D_N^\alpha |U_k^{(p)}| \right\} \quad \text{for } j = 1(1)N.$$

Proof. Let $\max_{k=1(1)j} |U_k^{(p)}| = |U_q^{(p)}|$ for some $q \in \{1, 2, \dots, j\}$. Since $U_0^{(p)} = 0$, (3.1) yields

$$D_N^\alpha |U_q^{(p)}| = \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \left\{ |U_q^{(p)}| - \sum_{k=1}^{q-1} (b_k - b_{k+1}) |U_{q-k}^{(p)}| \right\}.$$

Now, $b_1 = 1$ and $b_k > b_{k+1}$ for all $k \geq 1$ implies that

$$D_N^\alpha |U_q^{(p)}| \geq \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \left\{ |U_q^{(p)}| - \sum_{k=1}^{q-1} (b_k - b_{k+1}) |U_q^{(p)}| \right\} \geq \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} b_q |U_q^{(p)}|.$$

Therefore, $|U_q^{(p)}| \leq \frac{\tau^\alpha \Gamma(2-\alpha)}{b_q} D_N^\alpha |U_q^{(p)}|$. Hence, the desired result. $\square$

Theorem 2. If $\{u^{(p)}(t_j)\}_{j=1}^N$ be the true solution and $\{U_j^{(p)}\}_{j=1}^N$ be the approximate solution of the problem (1.1) for each $p = 1(1)n$, then we have

$$|e_j^{(p)}| \leq \begin{cases} CT^\alpha (\tau^{2-\alpha} + T\tau^2), & \text{if } u^{(p)}(t) \in C^2[0, T], \\ CT^\alpha (\tau^\alpha + T\tau^2), & \text{if } u^{(p)}(t) \in C_\alpha[0, T]. \end{cases}$$

Proof. Multiplying (4.1) by $e_j^{(p)}$, then using (3.1), we obtain

$$\begin{aligned} \mu |e_j^{(p)}|^2 + \sum_{i=1}^n |a_{pi}(t_j)| |e_j^{(i)}| |e_j^{(p)}| + \frac{\lambda_p \tau}{2} \sum_{i=1}^n \sum_{m=0}^{j-1} \left[|K_{pi}(t_j, t_m)| |e_m^{(i)}| |e_j^{(p)}| \right. \\ \left. + |K_{pi}(t_j, t_{m+1})| |e_{m+1}^{(i)}| |e_j^{(p)}| \right] \\ \leq |\mathcal{E}_j^{(p)}| |e_j^{(p)}| + \mu \sum_{k=1}^{j-1} (b_k - b_{k+1}) |e_{j-k}^{(p)}| |e_j^{(p)}|. \end{aligned}$$

Therefore, $\mu |e_j^{(p)}|^2 \leq |\mathcal{E}_j^{(p)}| |e_j^{(p)}| + \mu \sum_{k=1}^{j-1} (b_k - b_{k+1}) |e_{j-k}^{(p)}| |e_j^{(p)}|$. Dividing by $|e_j^{(p)}|$, we get $D_N^\alpha |e_j^{(p)}| \leq |\mathcal{E}_j^{(p)}|$. Lemmas 1 and 2 gives

$$|\mathcal{E}_j^{(p)}| \leq |\varepsilon_j^{(p)}| + |R_j^{(p)}| \leq \begin{cases} C(\tau^{2-\alpha} + T\tau^2), & \text{if } u^{(p)}(t) \in \mathcal{C}^2[0, T], \\ C(\tau^\alpha + T\tau^2), & \text{if } u^{(p)}(t) \in \mathcal{C}_\alpha[0, T]. \end{cases}$$

Since $b_1 = 1$ and $b_q > b_{q+1}$ for all $q \geq 1$. The mean value theorem implies that

$$(1 - \alpha)q^{-\alpha} \leq b_q \leq (1 - \alpha)(q - 1)^{-\alpha} \text{ for } q \geq 2.$$

Using Lemma 3, we obtain

$$\begin{aligned} |e_j^{(p)}| &\leq \max_{k=1,2,\dots,j} \left\{ \frac{\tau^\alpha \Gamma(2 - \alpha)}{b_k} |\mathcal{E}_j^{(p)}| \right\} \\ &\leq \frac{\tau^\alpha \Gamma(2 - \alpha)}{(1 - \alpha)N^{-\alpha}} \begin{cases} C(\tau^{2-\alpha} + T\tau^2), & \text{if } u^{(p)}(t) \in \mathcal{C}^2[0, T], \\ C(\tau^\alpha + T\tau^2), & \text{if } u^{(p)}(t) \in \mathcal{C}_\alpha[0, T]. \end{cases} \\ &\leq CT^\alpha \begin{cases} \tau^{2-\alpha} + T\tau^2, & \text{if } u^{(p)}(t) \in \mathcal{C}^2[0, T], \\ \tau^\alpha + T\tau^2, & \text{if } u^{(p)}(t) \in \mathcal{C}_\alpha[0, T]. \end{cases} \end{aligned}$$

This proves the theorem. $\square$

5. NUMERICAL EXPERIMENT

The following two examples are in good agreement with theoretical establishments. The computational results are presented through several tables and graphs.

Example 1. Consider the following system:

$$\begin{cases} D_t^\alpha u^{(1)}(t) + u^{(1)}(t) + u^{(2)}(t) - \int_0^t (t-s)(u^{(1)}(s) + u^{(2)}(s))ds = f^{(1)}(t), \\ D_t^\alpha u^{(2)}(t) + u^{(1)}(t) + u^{(2)}(t) + \int_0^t (s-t)(u^{(1)}(s) - u^{(2)}(s))ds = f^{(2)}(t), \\ \text{for } t \in (0, 1], \quad IC : u^{(1)}(0) = 0 = u^{(2)}(0). \end{cases}$$

We choose $f^{(i)}(t)$, $i = 1, 2$, so that the exact solution is $u^{(1)}(t) = t^{p+\alpha}$, $u^{(2)}(t) = -t^{q+\alpha}$ for $p, q \in \mathbb{R}$. Compute the maximum error and the corresponding rate of convergence defined as: $\Sigma_N = \max_{0 \leq j \leq N} |u^{(i)}(t_j) - u_j^{(i)}|$; $\rho_N = \log_2 \left(\frac{\Sigma_N}{\Sigma_{2N}} \right)$.

Case-I ($p = 2, q = 3$): Figure 1 shows the comparison between the exact and the approximate solution with $\alpha = 0.5$, $N = 32$ for Example 1. The graphs of approximate solutions for different α with $N = 16$ are presented in Figure 2. The log-log plots of the numerical error is displayed in Figure 3. Σ_M and ρ_M are shown in Table 1 for $u^{(1)}(t)$ and $u^{(2)}(t)$.

Case-II ($p = 0, q = 0$): Figure 4 shows the comparison between the exact and approximate solution with $\alpha = 0.5$, $N = 32$ for Example 1. The graphs of approximate

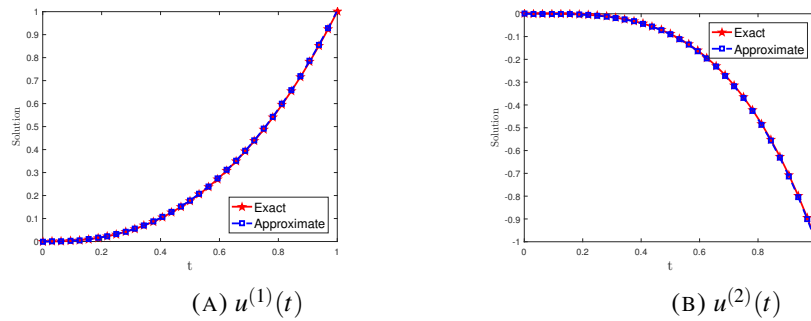


FIGURE 1. The exact and approximate solutions with $N = 32$, $\alpha = 0.5$ for Example 1

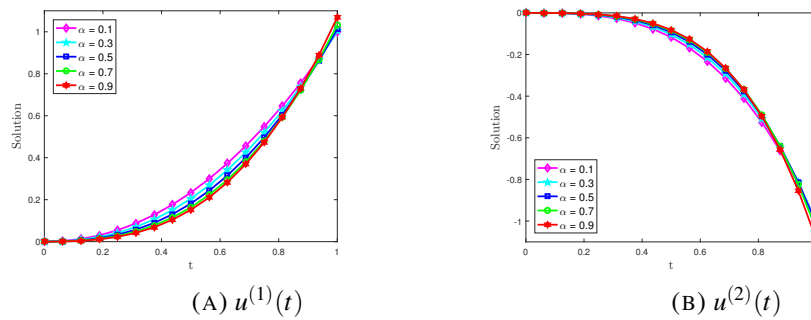


FIGURE 2. The approximate solutions with $N = 16$ for Example 1

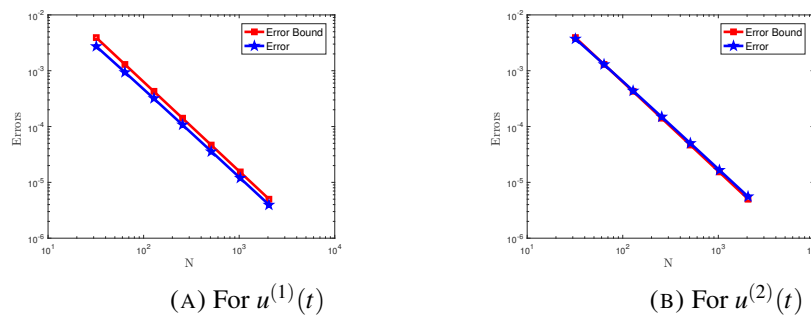


FIGURE 3. Log-log plots of Σ_M with $\alpha = 0.4$ for Example 1

solutions for different α with $N = 16$ are presented in Figure 5. From these graphs, it is clear that the solutions have mild singularity near $t = 0$. The log-log plots of the numerical errors displayed in Figure 6. Σ_N and ρ_N are shown in Table 2 for $u^{(1)}(t)$

TABLE 1. Σ_M and ρ_M with $p = 2$, $q = 3$ for Example 1

α/M		32	64	128	256	512	1024
0.1	$u^{(1)}(t)$	2.054E-04	6.219E-05	1.845E-05	5.387E-06	1.554E-06	4.442E-07
		1.723	1.753	1.776	1.793	1.807	1.818
	$u^{(2)}(t)$	2.563E-04	8.013E-05	2.431E-05	7.223E-06	2.112E-06	6.101E-07
		1.678	1.721	1.751	1.774	1.792	1.806
0.3	$u^{(1)}(t)$	1.440E-03	4.684E-04	1.504E-04	4.784E-05	1.511E-05	4.746E-06
		1.620	1.639	1.653	1.663	1.671	1.677
	$u^{(2)}(t)$	1.970E-03	6.469E-04	2.091E-04	6.683E-05	2.119E-05	6.674E-06
		1.607	1.629	1.646	1.657	1.666	1.673
0.7	$u^{(1)}(t)$	1.302E-02	5.359E-03	2.196E-03	8.969E-04	3.655E-04	1.488E-04
		1.280	1.287	1.292	1.295	1.297	1.298
	$u^{(2)}(t)$	1.752E-02	7.244E-03	2.975E-03	1.217E-03	4.963E-04	2.021E-04
		1.274	1.284	1.290	1.294	1.296	1.298
0.9	$u^{(1)}(t)$	3.196E-02	1.491E-02	6.962E-03	3.250E-03	1.517E-03	7.080E-04
		1.100	1.099	1.099	1.099	1.099	1.100
	$u^{(2)}(t)$	4.221E-02	1.974E-02	9.231E-03	4.313E-03	2.014E-03	9.401E-04
		1.096	1.097	1.098	1.099	1.099	1.100

and $u^{(2)}(t)$. These results show that the approximate solutions converge with the convergence rate α only.

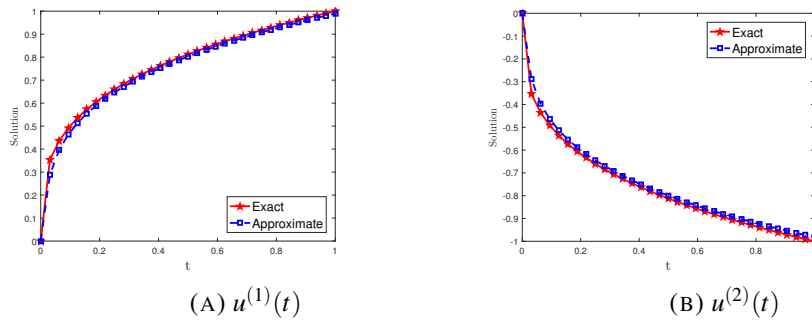
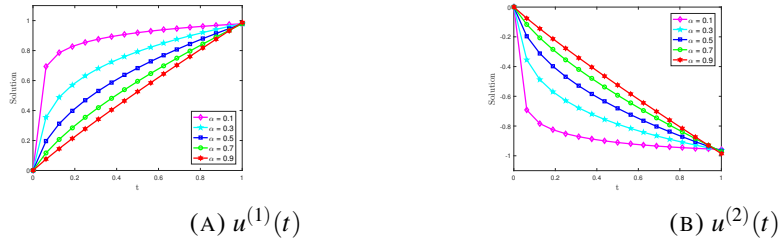


FIGURE 4. The exact and approximate solutions with $N = 32$, $\alpha = 0.3$ for Example 1

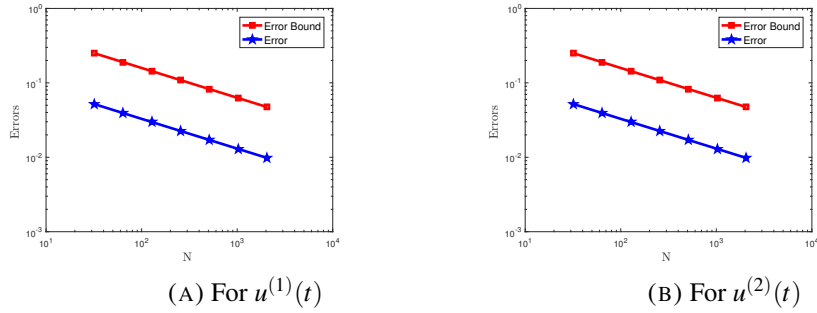
FIGURE 5. The approximate solutions with $N = 16$ for Example 1TABLE 2. Σ_M and ρ_M with $p = 0 = q$ for Example 1

α/M		32	64	128	256	512	1024
0.1	$u^{(1)}(t)$	6.024E-02	5.612E-02	5.234E-02	4.883E-02	4.556E-02	4.251E-02
		0.102	0.100	0.100	0.100	0.100	0.100
	$u^{(2)}(t)$	6.041E-02	5.616E-02	5.235E-02	4.884E-02	4.556E-02	4.251E-02
		0.105	0.101	0.100	0.100	0.100	0.100
0.3	$u^{(1)}(t)$	6.525E-02	5.299E-02	4.304E-02	3.496E-02	2.840E-02	2.307E-02
		0.300	0.300	0.300	0.300	0.300	0.300
	$u^{(2)}(t)$	6.530E-02	5.300E-02	4.304E-02	3.496E-02	2.840E-02	2.307E-02
		0.301	0.300	0.300	0.300	0.300	0.300
0.7	$u^{(1)}(t)$	1.631E-02	1.004E-02	6.180E-03	3.804E-03	2.342E-03	1.442E-03
		0.700	0.700	0.700	0.700	0.700	0.700
	$u^{(2)}(t)$	1.631E-02	1.004E-02	6.180E-03	3.804E-03	2.342E-03	1.442E-03
		0.700	0.700	0.700	0.700	0.700	0.700
0.9	$u^{(1)}(t)$	6.477E-03	3.521E-03	1.895E-03	1.011E-03	5.357E-04	2.819E-04
		0.879	0.893	0.906	0.917	0.926	0.934
	$u^{(2)}(t)$	7.833E-03	4.256E-03	2.297E-03	1.230E-03	6.537E-04	3.451E-04
		0.880	0.890	0.901	0.912	0.922	0.930

Example 2. Consider the following test problem:

$$\begin{cases} D_t^\alpha u^{(1)}(t) + (1-t)u^{(3)}(t) + \int_0^t \{stu^{(1)}(s) + u^{(2)}(t)\}dt = f^{(1)}(t), \\ D_t^\alpha u^{(2)}(t) + tu^{(1)}(t) + \int_0^t \{u^{(2)}(s) + u^{(3)}(t)\}dt = f^{(2)}(t), \\ D_t^\alpha u^{(3)}(t) + t^2u^{(2)}(t) + \int_0^t \{ts^2u^{(1)}(s) + u^{(3)}(t)\}dt = f^{(3)}(t), \\ \text{for } t \in (0, 1], \quad IC : u^{(1)}(0) = u^{(2)}(0) = u^{(3)}(0) = 0. \end{cases}$$

The exact solution is $u^{(1)}(t) = t(t-1)$, $u^{(2)}(t) = e^t$, $u^{(3)}(t) = t^3$ for the suitable choice of $f^{(i)}(t)$. The exact and approximate solutions for Example 2 are shown in Figure 7 and Figure 8 for $\alpha = 0.3$ and $\alpha = 0.7$, respectively. Figure 9 display the

FIGURE 6. Log-log plots of Σ_M with $\alpha = 0.4$ for Example 1

log-log plots of the computational errors for Example 2. Figure 10 illustrates surface plots of the computational error. Table 3 shows Σ_N and ρ_N for $u^{(1)}(t)$, $u^{(2)}(t)$ and $u^{(3)}(t)$. The tabular data shows that the approximate solutions converge with $(2 - \alpha)$ rate of accuracy.

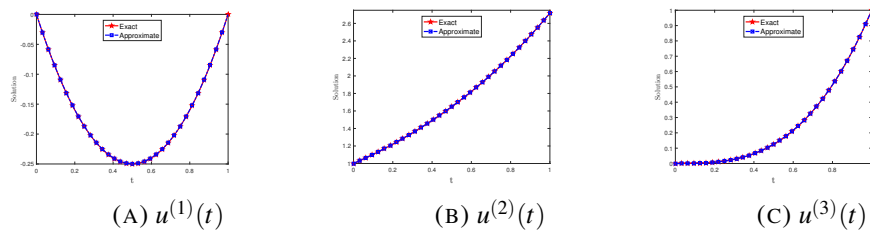
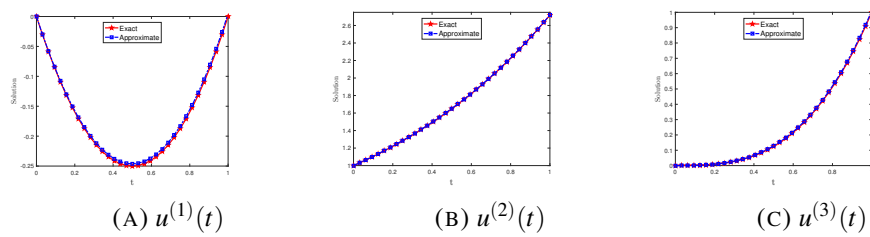
FIGURE 7. Computed solutions with $N = 32$, $\alpha = 0.3$ for Example 2FIGURE 8. Computed solutions with $N = 32$, $\alpha = 0.7$ for Example 2

TABLE 3. Σ_M and ρ_M for Example 2

α/M		32	64	128	256	512	1024
0.1	$u^{(1)}(t)$	6.300E-05	1.909E-05	5.689E-06	1.672E-06	4.854E-07	1.396E-07
		1.722	1.747	1.767	1.784	1.798	1.810
	$u^{(2)}(t)$	2.021E-04	5.228E-05	1.353E-05	3.507E-06	9.096E-07	2.362E-07
		1.951	1.950	1.948	1.947	1.945	1.943
	$u^{(3)}(t)$	1.502E-04	4.822E-05	1.495E-05	4.512E-06	1.335E-06	3.894E-07
		1.640	1.689	1.728	1.757	1.778	1.795
0.5	$u^{(1)}(t)$	1.440E-03	5.201E-04	1.867E-04	6.671E-05	2.376E-05	8.447E-06
		1.469	1.478	1.485	1.489	1.492	1.495
	$u^{(2)}(t)$	6.896E-04	2.564E-04	9.368E-05	3.385E-05	1.215E-05	4.340E-06
		1.427	1.453	1.468	1.478	1.485	1.490
	$u^{(3)}(t)$	3.812E-03	1.394E-03	5.041E-04	1.810E-04	6.470E-05	2.305E-05
		1.451	1.467	1.478	1.484	1.489	1.492
0.9	$u^{(1)}(t)$	1.385E-02	6.511E-03	3.049E-03	1.425E-03	6.654E-04	3.106E-04
		1.089	1.095	1.097	1.099	1.099	1.100
	$u^{(2)}(t)$	6.710E-03	3.203E-03	1.511E-03	7.090E-04	3.317E-04	1.550E-04
		1.067	1.084	1.092	1.096	1.098	1.099
	$u^{(3)}(t)$	2.517E-02	1.182E-02	5.533E-03	2.587E-03	1.208E-03	5.640E-04
		1.091	1.095	1.097	1.098	1.099	1.099

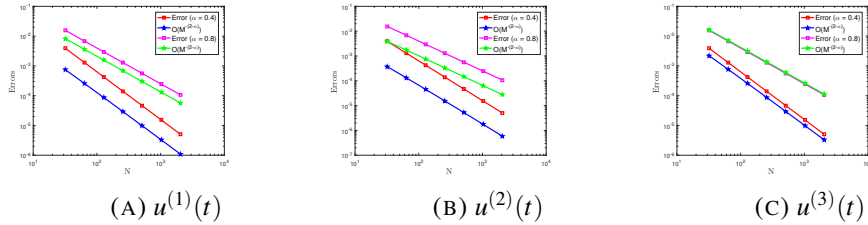


FIGURE 9. Log-log plots for Example 2

6. CONCLUSION

In this work, a novel finite difference scheme is developed for solving a system of fractional order Volterra-type integro-differential equations. The L1 technique is employed to discretize the Caputo fractional derivative, in which the integral component is approximated with the help of the composite trapezoidal rule. The convergence analysis and the error estimation are provided. The theoretical and experimental results confirm the efficiency and the applicability of the proposed scheme.

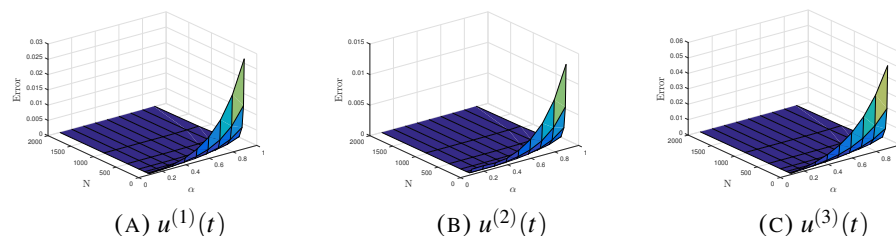


FIGURE 10. Surface plots of pointwise error for Example 2

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AN ESTIMATION FOR THE GENERALIZED TERNARY RING HOMOMORPHISM ON NON-ARCHIMEDEAN TERNARY BANACH ALGEBRAS

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Abstract. In this paper, we introduce the notion of the generalized ternary ring homomorphism on non-Archimedean ternary Banach algebras. Using fixed point methods, we prove the superstability and generalized Hyers-Ulam stability of generalized ternary ring homomorphisms on non-Archimedean ternary Banach algebras associated with a Cauchy-Jensen type functional equation.

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Keywords: generalized ternary ring homomorphism, stability, superstability, non-Archimedean ternary Banach algebra, Cauchy-Jensen functional equation

1. INTRODUCTION

The theory of non-Archimedean spaces has many applications in quantum physics, p -adic strings and superstrings [13]. The methods that are used in non-Archimedean spaces are essentially different from the classical normed space theory [6, 7, 9, 15].

We first review the definition of non-Archimedean spaces [24]. Let $\mathbb{K}$ be a field. A non-Archimedean absolute value on $\mathbb{K}$ is a function $|\cdot|: \mathbb{K} \rightarrow \mathbb{R}$ such that for all $a, b \in \mathbb{K}$ we have

- (1) $|a| \geq 0$ and equality holds if and only if $a = 0$,
- (2) $|ab| = |a||b|$,
- (3) $|a + b| \leq \max\{|a|, |b|\}$.

Condition (3) is the strict triangle inequality. By condition (2) we have $|1| = |-1| = 1$. By induction on n , one can show that $|n| \leq 1$ for each integer n . Note that for $n = 1$ we have $|1| = 1$. Let $|n| \leq 1$ for each integer n . We prove that $|n + 1| \leq 1$. It follows from condition (3) that $|n + 1| \leq \max\{|1|, |n|\} = \max\{1, |n|\} = 1$. A non-Archimedean absolute value $|\cdot|$ is non-trivial, i.e., there is an $a_0 \in \mathbb{K}$ such that $|a_0| \notin \{0, 1\}$.

Let X be a linear space over a scalar field $\mathbb{K}$ with a non-Archimedean non-trivial absolute value $|\cdot|$. A function $\|\cdot\|: X \rightarrow \mathbb{R}$ is a non-Archimedean norm if it satisfies the following conditions:

- (1) $\|x\| = 0$ if and only if $x = 0$,
- (2) $\|rx\| = |r|\|x\|$ for all $r \in \mathbb{K}$ and $x \in X$,
- (3) $\|x+y\| \leq \max\{\|x\|, \|y\|\}$ for all $x, y \in X$.

Then $(X, \|\cdot\|)$ is called a non-Archimedean space. It follows from condition (3) that

$$\|x_m - x_l\| \leq \max\{\|x_{j+1} - x_j\| : l \leq j \leq m-1\}$$

for all $m > l$ and hence a sequence $\{x_m\}$ is Cauchy in X if and only if $\{x_{m+1} - x_m\}$ converges to zero in a non-Archimedean space. A complete non-Archimedean space is a non-Archimedean space that every Cauchy sequence is convergent. A ternary (associative) algebra $(A, [\cdot, \cdot, \cdot])$ is a linear space A over a scalar field $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ equipped with a trilinear mapping, the so-called ternary product, $[\cdot, \cdot, \cdot]: A \times A \times A \rightarrow A$ such that it is associative in the sense that $[[a, b, c], d, e] = [a, [b, c, d], e] = [a, b, [c, d, e]]$ for all $a, b, c, d, e \in A$. This notion is a natural generalization of the binary case. Indeed, if $(A, \circ)$ is a usual (binary) non-Archimedean algebra, then $[a, b, c] := (a \circ b) \circ c$ induced a ternary product making A into a non-Archimedean ternary algebra which will be called trivial. There are other types of non-Archimedean ternary algebras in which one may consider other versions of associativity. Let A be a non-Archimedean vector space. Then $[a, b, c] = a - b + c$ induced a ternary product making A into a non-Archimedean ternary algebra. A non-Archimedean ternary Banach algebra is a complete non-Archimedean ternary algebra A which the norm satisfies $\|[a, b, c]\| \leq \|a\| \cdot \|b\| \cdot \|c\|$ for all $a, b, c \in A$.

The first stability problem concerning group homomorphisms was raised by Ulam [25] and affirmatively solved by Hyers [11]. A generalization of Hyers' problem with unbounded Cauchy differences has been considered by Rassias [20], Bourgin [3], and Găvruta [10]. Moreover, Rassias [19] considered the Cauchy difference controlled by a product of different powers of norm. In 1994, Găvruta [10] promoted the stability result into a simple form and reinstated the upper bound by a general control function. This type of stability result accomplished by Găvruta is known as the generalized Hyers-Ulam stability of functional equation. For the history and various aspects of stability theory we refer to [12, 17, 21–23].

Bourgin [2, 3] is the first mathematician dealing with stability of the (ring) homomorphism $f(xy) = f(x)f(y)$. The stability of the approximate homomorphism, the approximate generalized homomorphism $g(xy) = g(x)f(y)$, where f is a (ring) homomorphism, and the derivation on some suitable Banach spaces was studied by a number of mathematicians, see [1, 5–8, 16, 18] and references therein. Let $(A, [\cdot, \cdot, \cdot])$ and $(B, [\cdot, \cdot, \cdot])$ be two non-Archimedean ternary Banach algebras. A mapping $f: A \rightarrow B$ is a ternary ring homomorphism or a ternary additive homomorphism if f is additive

and satisfies

$$f([x, y, z]) = [f(x), f(y), f(z)]$$

for all $x, y, z \in A$. We introduce the notion of a generalized ternary ring homomorphism between two non-Archimedean ternary Banach algebras A and B as follows. We say that a function $g: A \rightarrow B$ is a generalized ternary ring homomorphism if g is an additive function and there exists a ternary ring homomorphism $f: A \rightarrow B$ satisfying

$$g([x, y, z]) = \alpha[g(x), g(y), f(z)] + \beta[g(x), f(y), g(z)] + \gamma[f(x), g(y), g(z)]$$

for all $x, y, z \in A$ and $\alpha, \beta, \gamma \in \mathbb{R}$ with $\alpha + \beta + \gamma = 1$. It is clear that every ternary ring homomorphism g is a generalized ternary ring homomorphism by taking $f = g$ but the converse is not true in general. So, our results recover the stability of the ternary ring homomorphism, see [6, 8].

We now mention the following fixed point theorem [14]. Let (X, d) be a generalized metric space. An operator $F: X \rightarrow X$ satisfies a Lipschitz condition with Lipschitz constant L if there exists a constant $L \geq 0$ such that $d(Fx, Fy) \leq Ld(x, y)$ for all $x, y \in X$. If the Lipschitz constant L is less than 1, then the operator F is called a strictly contractive operator. Note that the distinction between the generalized metric and the usual metric is that the range of the former is permitted to include the infinity. We recall the following theorem by Margolis and Diaz.

Theorem 1 ([14]). *Let (Ω, d) be a complete generalized metric space and $F: \Omega \rightarrow \Omega$ a strictly contractive mapping with Lipschitz constant L . Then for each $x \in \Omega$, either*

$$d(F^m x, F^{m+1} x) = \infty,$$

for all $m \geq 0$ or there exists a natural number m_0 such that

- (1) $d(F^m x, F^{m+1} x) < \infty$ for all $m \geq m_0$,
- (2) the sequence $\{F^m x\}$ is convergent to a fixed point y^* of F ,
- (3) y^* is the unique fixed point of F in the set $\Lambda = \{y \in \Omega: d(F^{m_0} x, y) < \infty\}$,
- (4) $d(y, y^*) \leq \frac{1}{1-L} d(y, Fy)$ for all $y \in \Lambda$.

Throughout this paper we suppose that $(A, [\cdot, \cdot, \cdot])$ and $(B, [\cdot, \cdot, \cdot])$ are two non-Archimedean ternary Banach algebras. For convenience, we use the following abbreviations for two given functions $f_1: A \rightarrow B$ and $f_2: A \rightarrow B$,

$$\Delta_m f(x, y) = mf\left(\frac{x+y}{m}\right) - f(x) - f(y),$$

$$D_{f_1}(x, y, z) = f_1([x, y, z]) - [f_1(x), f_1(y), f_1(z)],$$

$$D_{f_1, f_2}(x, y, z) = f_2([x, y, z]) - \alpha[f_2(x), f_2(y), f_1(z)] \\ - \beta[f_2(x), f_1(y), f_2(z)] - \gamma[f_1(x), f_2(y), f_2(z)]$$

for all $x, y, z \in A$ and $\alpha, \beta, \gamma \in \mathbb{R}$ with $\alpha + \beta + \gamma = 1$, where m is a natural number.

In this paper, using fixed point methods, we investigate the superstability of generalized ternary ring homomorphisms on non-Archimedean ternary Banach algebras.

Moreover, we prove the generalized Hyers-Ulam stability of generalized ternary ring homomorphisms on non-Archimedean ternary Banach algebras associated with the functional equation

$$\Delta_m f(x, y) = 0.$$

2. SUPERSTABILITY OF GENERALIZED TERNARY RING HOMOMORPHISMS ON NON-ARCHIMEDEAN TERNARY BANACH ALGEBRAS

In this section we establish the superstability of generalized ternary ring homomorphisms on non-Archimedean ternary Banach algebras. We first need to prove the following lemma.

Lemma 1. *Let $f: A \rightarrow B$ be a mapping.*

- (1) *f is additive if and only if $\Delta_m f(x, y) = 0$ for all $x, y \in A$ and all natural numbers $m \neq 2$,*
- (2) *f is additive if and only if $f(0) = 0$ and $\Delta_2 f(x, y) = 0$ for all $x, y \in A$.*

Proof. (1) Since $m \neq 2$, it follows from $\Delta_m f(0, 0) = 0$ that $f(0) = 0$. So, $f(x) = \frac{1}{m}f(mx)$ for all $x \in A$. On the other hand, $\Delta_m f(mx, my) = 0$ for all $x, y \in A$. Therefore, f is additive. The converse is obvious.

(2) It follows from $\Delta_2 f(x, 0) = 0$ that $f(x) = \frac{1}{2}f(2x)$ for all $x \in A$. Since $\Delta_2 f(2x, 2y) = 0$ for all $x, y \in A$, we conclude that f is additive. The converse is obvious. $\square$

In the following two theorems we provide the conditions which give the superstability of generalized ternary ring homomorphisms on non-Archimedean ternary Banach algebras.

Theorem 2. *Let $f_1: A \rightarrow B$ and $f_2: A \rightarrow B$ be two mappings for which there exist some functions $\varphi: A \rightarrow [0, \infty)$ and $\psi_i: A^3 \rightarrow [0, \infty)$ such that*

$$\|\Delta_m f_i(x, y)\| \leq \max\{\varphi(x), \varphi(y)\}, \quad (2.1)$$

$$\|D_{f_1}(x, y, z)\| \leq \psi_1(x, y, z), \quad (2.2)$$

$$\|D_{f_1, f_2}(x, y, z)\| \leq \psi_2(x, y, z) \quad (2.3)$$

for all $x, y, z \in A$, $i \in \{1, 2\}$, and all natural numbers $m > 2$. If there exists a constant $0 < L < 1$ such that

$$\varphi(mx) \leq |m|L\varphi(x), \quad (2.4)$$

$$\psi_i(mx, my, mz) \leq |m|^3 L\psi_i(x, y, z) \quad (2.5)$$

for all $x, y, z \in A$, then f_2 is a generalized ternary ring homomorphism.

Proof. It follows from (2.4) and (2.5) that

$$\lim_{n \rightarrow \infty} \frac{1}{|m|^n} \varphi(m^n x) = 0, \quad (2.6)$$

$$\lim_{n \rightarrow \infty} \frac{1}{|m|^{3n}} \Psi_i(m^n x, m^n y, m^n z) = 0 \quad (2.7)$$

for all $x, y, z \in A$. By (2.6), $\lim_{n \rightarrow \infty} \frac{1}{|m|^n} \varphi(0) = 0$. Hence, $\varphi(0) = 0$. Letting $x = y = 0$ in (2.1), we get $\|(m-2)f_i(0)\| \leq \varphi(0) = 0$. Since $m > 2$, $f_i(0) = 0$.

Let Ω be the set of all mappings $g: A \rightarrow B$ and define a generalized metric on Ω as follows

$$d(g, h) = \inf\{t \in (0, \infty): \|g(x) - h(x)\| \leq t\varphi(x), x \in A\}.$$

It is easy to show that (Ω, d) is a generalized complete metric space [4]. Consider the mapping $F: \Omega \rightarrow \Omega$ defined by $(Fg)(x) = \frac{1}{m}g(mx)$ for all $x \in A$ and all $g \in \Omega$. Let $d(g, h) < t$ for $g, h \in \Omega$ and $t \in (0, \infty)$. Then

$$\|g(x) - h(x)\| \leq t\varphi(x) \quad (2.8)$$

for all $x \in A$. Replace x by mx in (2.8) to find that $\|g(mx) - h(mx)\| \leq t\varphi(mx)$ for all $x \in A$. Using (2.4), we get $\|\frac{1}{m}g(mx) - \frac{1}{m}h(mx)\| \leq Lt\varphi(x)$ for all $x \in A$, which implies $d(Fg, Fh) \leq Lt$. Therefore, $d(Fg, Fh) \leq Ld(g, h)$ for all $g, h \in \Omega$, that is, F is a strictly contractive mapping of Ω with the Lipschitz constant L . Putting $y = 0$ in (2.1), one has

$$\|mf_i(\frac{x}{m}) - f_i(x)\| \leq \max\{\varphi(x), \varphi(0)\} = \varphi(x) \quad (2.9)$$

for all $x \in A$. Replace x by mx in (2.9) and divide both sides by $|m|$ and then use (2.4) to derive

$$\|f_i(x) - \frac{1}{m}f_i(mx)\| \leq \frac{1}{|m|}\varphi(mx) \leq L\varphi(x)$$

for all $x \in A$. So, $d(f_i, Ff_i) \leq L < \infty$. It follows from the fixed point alternative that there exists a fixed point H_i of F in Ω such that

$$H_i(x) = \lim_{n \rightarrow \infty} \frac{1}{m^n} f_i(m^n x) \quad (2.10)$$

for all $x \in A$, since $\lim_{n \rightarrow \infty} d(F^n f_i, H_i) = 0$. On the other hand, it follows from (2.1), (2.6) and (2.10) that

$$\begin{aligned} \|\Delta_m H_i(x, y)\| &= \lim_{n \rightarrow \infty} \frac{1}{|m|^n} \|\Delta_m f_i(m^n x, m^n y)\| \\ &\leq \lim_{n \rightarrow \infty} \frac{1}{|m|^n} \max\{\varphi(m^n x), \varphi(m^n y)\} = 0. \end{aligned}$$

Hence, $\Delta_m H_i(x, y) = 0$ for all $x, y \in A$. This means that H_i is additive. Using (2.2), (2.7) and (2.10) one can deduce

$$\begin{aligned} \|D_{H_1}(x, y, z)\| &= \|H_1([x, y, z]) - [H_1(x), H_1(y), H_1(z)]\| \\ &= \lim_{n \rightarrow \infty} \frac{1}{|m|^{3n}} \left\| f_1([m^n x, m^n y, m^n z]) - [f_1(m^n x), f_1(m^n y), f_1(m^n z)] \right\| \end{aligned}$$

$$\leq \lim_{n \rightarrow \infty} \frac{1}{|m|^{3n}} \Psi_1(m^n x, m^n y, m^n z) = 0.$$

So, $H_1([x, y, z]) = [H_1(x), H_1(y), H_1(z)]$ for all $x, y, z \in A$. Therefore, H_1 is a ternary ring homomorphism. Using (2.3), (2.7) and (2.10) one can also deduce

$$\begin{aligned} \|D_{H_1, H_2}(x, y, z)\| &= \left\| H_2([x, y, z]) - \alpha[H_2(x), H_2(y), H_1(z)] \right. \\ &\quad \left. - \beta[H_2(x), H_1(y), H_2(z)] - \gamma[H_1(x), H_2(y), H_2(z)] \right\| \\ &= \lim_{n \rightarrow \infty} \frac{1}{|m|^{3n}} \left\| f_2([m^n x, m^n y, m^n z]) - \alpha[f_2(m^n x), f_2(m^n y), f_1(m^n z)] \right. \\ &\quad \left. - \beta[f_2(m^n x), f_1(m^n y), f_2(m^n z)] - \gamma[f_1(m^n x), f_2(m^n y), f_1(m^n z)] \right\| \\ &= \lim_{n \rightarrow \infty} \frac{1}{|m|^{3n}} \|D_{f_1, f_2}(m^n x, m^n y, m^n z)\| \\ &\leq \lim_{n \rightarrow \infty} \frac{1}{|m|^{3n}} \Psi_2(m^n x, m^n y, m^n z) = 0. \end{aligned}$$

This indicates that

$$\begin{aligned} H_2([x, y, z]) &= \alpha[H_2(x), H_2(y), H_1(z)] + \beta[H_2(x), H_1(y), H_2(z)] \\ &\quad + \gamma[H_1(x), H_2(y), H_1(z)] \end{aligned}$$

for all $x, y, z \in A$. Therefore, H_2 is a generalized ternary ring homomorphism.

Set $x = 0$ in (2.1) to find that

$$\|mf(\frac{y}{m}) - f(y)\| \leq \varphi(0) = 0.$$

This implies $f(y) = \frac{1}{m}f(my)$ for all $y \in A$. According to the fixed point alternative H_i is the unique fixed point of F in the set

$$\Lambda_i = \{g \in \Omega : d(f_i, g) < \infty\},$$

where $i \in \{1, 2\}$, so $H_i = f_i$. It follows that f_1 is a ternary ring homomorphism and f_2 is a generalized ternary ring homomorphism. $\square$

The proof of the following theorem is similar to that of Theorem 2, if we define the mapping $F : \Omega \rightarrow \Omega$ by $(Fg)(x) = mg(\frac{x}{m})$ for all $x \in A$ and all $g \in \Omega$.

Theorem 3. Let $f_1 : A \rightarrow B$ and $f_2 : A \rightarrow B$ be two mappings for which there exist some functions $\varphi : A \rightarrow [0, \infty)$ and $\psi_i : A^3 \rightarrow [0, \infty)$ such that $\varphi(0) = 0$ and satisfying (2.1), (2.2) and (2.3). If there exists a constant $0 < L < 1$ such that

$$\varphi(\frac{x}{m}) \leq \frac{1}{|m|} L \varphi(x),$$

$$\psi_i\left(\frac{x}{m}, \frac{y}{m}, \frac{z}{m}\right) \leq \frac{1}{|m|^3} L\psi(x, y, z)$$

for all $x, y, z \in A$ and $i \in \{1, 2\}$, then f_2 is a generalized ternary ring homomorphism.

Remark 1. A generalized ternary ring homomorphisms on a non-Archimedean ternary Banach algebra associated with the functional equation $\Delta_2 f(x, y) = 0$ is superstable with an additional assumption $f(0) = 0$.

Corollary 1. Let λ , p , and s be non-negative real numbers and $f_1: A \rightarrow B$ and $f_2: A \rightarrow B$ be two mappings such that $f_i(0) = 0$ and

$$\begin{aligned} \|\Delta_m f_i(x, y)\| &\leq \lambda \max\{\|x\|^p, \|y\|^p\}, \\ \|D_{f_1}(x, y, z)\| &\leq \lambda \|x\|^s \|y\|^s \|z\|^s, \\ \|D_{f_1, f_2}(x, y, z)\| &\leq \lambda \|x\|^s \|y\|^s \|z\|^s \end{aligned}$$

for all $x, y, z \in A$ and $i \in \{1, 2\}$, where $m > 1$ is a natural number. Then each of the following two conditions asserts that f_2 is a generalized ternary ring homomorphism.

- (1) $p > 1$ and $3s - p > 2$,
- (2) $p < 1$ and $3s - p < 2$.

Proof. Define

$$\begin{aligned} \varphi(x) &:= \lambda \|x\|^p, \\ \psi_i(x, y, z) &:= \lambda \|x\|^s \|y\|^s \|z\|^s \end{aligned}$$

for all $x, y, z \in A$ and $i \in \{1, 2\}$. If part (i) holds, then choosing $L = |m|^{p-1}$ and applying Theorem 2, we get the desired result. If part (ii) holds, then the result follows from Theorem 3 by letting $L = |m|^{1-p}$. $\square$

3. STABILITY OF GENERALIZED TERNARY RING HOMOMORPHISMS ON NON-ARCHIMEDEAN TERNARY BANACH ALGEBRAS

In this section we consider the generalized Hyers-Ulam stability of generalized ternary ring homomorphisms on non-Archimedean ternary Banach algebras. In the following two theorems we give the conditions which imply the stability of generalized ternary ring homomorphisms on non-Archimedean ternary Banach algebras.

Theorem 4. Let $f_1: A \rightarrow B$ and $f_2: A \rightarrow B$ be two mappings for which there exist some functions $\varphi: A^2 \rightarrow [0, \infty)$ and $\psi_i: A^3 \rightarrow [0, \infty)$ such that

$$\|\Delta_m f_i(x, y)\| \leq \varphi(x, y), \quad (3.1)$$

$$\|D_{f_1}(x, y, z)\| \leq \psi_1(x, y, z), \quad (3.2)$$

$$\|D_{f_1, f_2}(x, y, z)\| \leq \psi_2(x, y, z) \quad (3.3)$$

for all $x, y, z \in A$, $i \in \{1, 2\}$ and all natural numbers $m > 2$. If there exists a constant $0 < L < 1$ such that

$$\varphi(mx, my) \leq |m|L\varphi(x, y), \quad (3.4)$$

$$\Psi_i(mx, my, mz) \leq |m|^3 L \Psi_i(x, y, z) \quad (3.5)$$

for all $x, y, z \in A$, then there exist a unique ternary ring homomorphism $H_1: A \rightarrow B$ and a unique generalized ternary ring homomorphism $H_2: A \rightarrow B$ such that

$$\|f_1(x) - H_1(x)\| \leq \frac{L}{1-L} \varphi(x, 0), \quad (3.6)$$

$$\|f_2(x) - H_2(x)\| \leq \frac{L}{1-L} \varphi(x, 0) \quad (3.7)$$

for all $x \in A$.

Proof. It follows from (3.4) and (3.5) that

$$\lim_{n \rightarrow \infty} \frac{1}{|m|^n} \varphi(m^n x, m^n y) = 0, \quad (3.8)$$

$$\lim_{n \rightarrow \infty} \frac{1}{|m|^{3n}} \Psi_i(m^n x, m^n y, m^n z) = 0 \quad (3.9)$$

for all $x, y, z \in A$. By a similar argument as in the proof of Theorem 2 we conclude that $f_i(0) = 0$. Define a generalized metric on Ω , the set of all mappings $g: A \rightarrow B$, as follows

$$d(g, h) = \inf\{t \in (0, \infty): \|g(x) - h(x)\| \leq t\varphi(x, 0), x \in A\}.$$

The space (Ω, d) is a generalized complete metric space [4]. Let $F: \Omega \rightarrow \Omega$ be a mapping, given by $(Fg)(x) = \frac{1}{m}g(mx)$ for all $x \in A$ and all $g \in \Omega$. The same method as in the proof of Theorem 2 shows that F is a strictly contractive mapping of Ω with the Lipschitz constant L . Put $y = 0$ in (3.1) and apply (3.4) to obtain

$$\left\| \frac{1}{m} f_i(mx) - f_i(x) \right\| \leq L \varphi(x, 0) \quad (3.10)$$

for all $x \in A$. Thus, $d(f_i, Ff_i) \leq L < \infty$. From the fixed point alternative it follows that there exists a fixed point H_i of F in Ω such that $\lim_{n \rightarrow \infty} d(F^n f_i, H_i) = 0$, so

$$H_i(x) = \lim_{n \rightarrow \infty} \frac{1}{m^n} f_i(m^n x) \quad (3.11)$$

for all $x \in A$. Now, by (3.1), (3.8) and (3.11) we conclude that

$$\begin{aligned} \|\Delta_m H_i(x, y)\| &= \lim_{n \rightarrow \infty} \frac{1}{|m|^n} \|\Delta_m f_i(m^n x, m^n y)\| \\ &\leq \lim_{n \rightarrow \infty} \frac{1}{|m|^n} \varphi(m^n x, m^n y) = 0 \end{aligned}$$

for all $x, y \in A$ and so H_i is additive. The definition of H_1 , (3.2) and (3.9) imply

$$\begin{aligned} \|D_{H_1}(x, y, z)\| &= \|H_1([x, y, z]) - [H_1(x), H_1(y), H_1(z)]\| \\ &= \lim_{n \rightarrow \infty} \frac{1}{|m|^{3n}} \|f_1([m^n x, m^n y, m^n z]) - [f_1(m^n x), f_1(m^n y), f_1(m^n z)]\| \end{aligned}$$

$$\begin{aligned}
&= \lim_{n \rightarrow \infty} \frac{1}{|m|^{3n}} \|D_{f_1}(m^n x, m^n y, m^n z)\| \\
&\leq \lim_{n \rightarrow \infty} \frac{1}{|m|^{3n}} \Psi_1(m^n x, m^n y, m^n z) = 0
\end{aligned}$$

for all $x, y, z \in A$. Therefore, H_1 is a ternary ring homomorphism. The definition of H_2 , (3.3) and (3.9) imply

$$\begin{aligned}
\|D_{H_1, H_2}(x, y, z)\| &= \left\| H_2([x, y, z]) - \alpha[H_2(x), H_2(y), H_1(z)] - \beta[H_2(x), H_1(y), H_2(z)] \right. \\
&\quad \left. - \gamma[H_1(x), H_2(y), H_2(z)] \right\| \\
&= \lim_{n \rightarrow \infty} \frac{1}{|m|^{3n}} \left\| f_2([m^n x, m^n y, m^n z]) - \alpha[f_2(m^n x), f_2(m^n y), f_1(m^n z)] \right. \\
&\quad \left. - \beta[f_2(m^n x), f_1(m^n y), f_2(m^n z)] - \gamma[f_1(m^n x), f_2(m^n y), f_2(m^n z)] \right\| \\
&= \lim_{n \rightarrow \infty} \frac{1}{|m|^{3n}} \|D_{f_1, f_2}(m^n x, m^n y, m^n z)\| \\
&\leq \lim_{n \rightarrow \infty} \frac{1}{|m|^{3n}} \Psi_2(m^n x, m^n y, m^n z) = 0
\end{aligned}$$

for all $x, y, z \in A$. Therefore, H_2 is a generalized ternary ring homomorphism. The fixed point alternative yields H_i is the unique mapping such that

$$\|f_i(x) - H_i(x)\| \leq t\varphi(x, 0)$$

for all $x \in A$ and $t > 0$. Again using the fixed point alternative, we get

$$d(f_i, H_i) \leq \frac{1}{1-L} d(f_i, F f_i) \leq \frac{L}{1-L}$$

and so we deduce that

$$\|f_i(x) - H_i(x)\| \leq \frac{L}{1-L} \varphi(x, 0)$$

for all $x \in A$ and $i \in \{1, 2\}$. This completes the proof. $\square$

Theorem 5. Let $f_1: A \rightarrow B$ and $f_2: A \rightarrow B$ be two mappings with $f_i(0) = 0$ for which there exist some functions $\varphi: A^2 \rightarrow [0, \infty)$ and $\Psi_i: A^3 \rightarrow [0, \infty)$ satisfying (3.1), (3.2), and (3.3). If there exists a constant $0 < L < 1$ such that

$$\begin{aligned}
\varphi\left(\frac{x}{m}, \frac{y}{m}\right) &\leq \frac{1}{|m|} L \varphi(x, y), \\
\Psi_i\left(\frac{x}{m}, \frac{y}{m}, \frac{z}{m}\right) &\leq \frac{1}{|m|^3} L \Psi_i(x, y, z)
\end{aligned}$$

for all $x, y, z \in A$ and $i \in \{1, 2\}$, then there exist a unique ternary ring homomorphism $H_1: A \rightarrow B$ and a unique generalized ternary ring homomorphism $H_2: A \rightarrow B$ such that

$$\|f_i(x) - H_i(x)\| \leq \frac{1}{1-L} \varphi(x, 0)$$

for all $x \in A$.

Proof. Define $(Fg)(x) := mg(\frac{x}{m})$ for all $x \in A$ and $g \in \Omega$ and apply then a similar method as in the proof of Theorem 4 to get the desired result. $\square$

In the following two theorems we show that Theorems 4 and 5 hold for the case where $m = 1, 2$.

Theorem 6. Let $f_1: A \rightarrow B$ and $f_2: A \rightarrow B$ be two mappings for which there exist some functions $\varphi: A^2 \rightarrow [0, \infty)$ and $\psi_i: A^3 \rightarrow [0, \infty)$ satisfying (3.2), (3.3), and

$$\|\Delta_1 f(x, y)\| \leq \varphi(x, y). \quad (3.12)$$

If there exists a constant $0 < L < 1$ such that

$$\varphi(2^r x, 2^r y) \leq |2|^r L \varphi(x, y), \quad (3.13)$$

$$\psi_i(2^r x, 2^r y, 2^r z) \leq |2|^{3r} L \psi_i(x, y, z) \quad (3.14)$$

for all $x, y, z \in A$ and $i \in \{1, 2\}$, then there exist a unique ternary ring homomorphism $H_1: A \rightarrow B$ and a unique generalized ternary ring homomorphism $H_2: A \rightarrow B$ such that

$$\|f_i(x) - H_i(x)\| \leq \frac{L^{\frac{1-r}{2}}}{|2|(1-L)} \varphi(x, x)$$

for all $x \in A$, where $r \in \{-1, 1\}$.

Proof. Let $r \in \{-1, 1\}$. It follows from (3.13) and (3.14) that

$$\lim_{n \rightarrow \infty} \frac{1}{|2|^{nr}} \varphi(2^{nr} x, 2^{nr} y) = \lim_{n \rightarrow \infty} \frac{1}{|2|^{3nr}} \psi_i(2^{nr} x, 2^{nr} y, 2^{nr} z) = 0$$

for all $x, y, z \in A$. Let Ω be the set of all mappings $g: A \rightarrow B$ and define

$$d(g, h) = \inf\{t \in (0, \infty): \|g(x) - h(x)\| \leq t \varphi(x, x), x \in A\}.$$

Clearly, (Ω, d) is a generalized complete metric space. Define $F: \Omega \rightarrow \Omega$ by $(Fg)(x) = \frac{1}{2^r} g(2^r x)$ for all $x \in A$ and all $g \in \Omega$. Letting $x = y$ in (3.12) and applying (3.13), we get

$$\left\| \frac{1}{2^r} f_i(2^r x) - f_i(x) \right\| \leq \frac{L^{\frac{1-r}{2}}}{|2|} \varphi(x, x)$$

for all $x \in A$. Thus, $d(f_i, Ff_i) \leq \frac{L^{\frac{1-r}{2}}}{|2|} < \infty$. The fixed point alternative implies there exists a fixed point H_i of F in Ω such that $\lim_{n \rightarrow \infty} d(F^n f_i, H_i) = 0$, so

$$H_i(x) = \lim_{n \rightarrow \infty} \frac{1}{2^{nr}} f_i(2^{nr} x)$$

for all $x \in A$. By a similar method as in the proof of Theorem 4 one can show that H_1 is a ternary ring homomorphism and H_2 is a generalized ternary ring homomorphism. It follows from the fixed point alternative that H_i is the unique mapping such that

$$d(f_i, H_i) \leq \frac{1}{1-L} d(f_i, Ff_i) \leq \frac{L^{\frac{1-r}{2}}}{1-L}$$

and so we find that

$$\|f_i(x) - H_i(x)\| \leq \frac{L^{\frac{1-r}{2}}}{1-L} \varphi(x, x)$$

for all $x \in A$ and $i \in \{1, 2\}$. This completes the proof. $\square$

Theorem 7. Let $f_1: A \rightarrow B$ and $f_2: A \rightarrow B$ be two mappings with $f_i(0) = 0$ for which there exist some functions $\varphi: A^2 \rightarrow [0, \infty)$ and $\psi_i: A^3 \rightarrow [0, \infty)$ satisfying (3.2), (3.3), and $\|\Delta_2 f(x, y)\| \leq \varphi(x, y)$. If there exists a constant $0 < L < 1$ such that the inequalities (3.13) and (3.14) hold, then there exist a unique ternary ring homomorphism $H_1: A \rightarrow B$ and a unique generalized ternary ring homomorphism $H_2: A \rightarrow B$ such that

$$\|f_i(x) - H_i(x)\| \leq \frac{L^{\frac{r+1}{2}}}{1-L} \varphi(x, 0)$$

for all $x \in A$, where $r \in \{-1, 1\}$ and $i \in \{1, 2\}$.

Proof. The proof is similar to the proof of Theorems 4 and 5 for $m = 2$. $\square$

Corollary 2. Let $\lambda, \eta, \xi, p, q, s$, and t be non-negative real numbers with

$$\max\{2q, s-2, 3t-2\} < p < 1.$$

Let $f_1: A \rightarrow B$ and $f_2: A \rightarrow B$ be two mappings with $f(0) = 0$ and

$$\|\Delta_m f_i(x, y)\| \leq \lambda + \eta(\|x\|^p + \|y\|^p) + \xi\|x\|^q\|y\|^q,$$

$$\|D_{f_1}(x, y, z)\| \leq \lambda + \eta(\|x\|^s + \|y\|^s + \|z\|^s) + \xi\|x\|^t\|y\|^t\|z\|^t,$$

$$\|D_{f_1, f_2}(x, y, z)\| \leq \lambda + \eta(\|x\|^s + \|y\|^s + \|z\|^s) + \xi\|x\|^t\|y\|^t\|z\|^t$$

for all $x, y, z \in A$, $i \in \{1, 2\}$, and all natural numbers m . Then there exist a unique ternary ring homomorphism $H_1: A \rightarrow B$ and a unique generalized ternary ring homomorphism $H_2: A \rightarrow B$ such that

$$\|f_i(x) - H_i(x)\| \leq \begin{cases} \frac{|2|^{1-p}}{|2|(1-|2|^{1-p})}(\lambda + 2\eta\|x\|^p + \xi\|x\|^{2q}), & m = 1, \\ \frac{1}{1-|m|^{1-p}}(\lambda + \eta\|x\|^p), & m \geq 2 \end{cases}$$

for all $x \in A$.

Proof. Define

$$\varphi(x) := \lambda + \eta(\|x\|^p + \|y\|^p) + \xi\|x\|^q\|y\|^q,$$

$$\psi_i(x, y, z) := \lambda + \eta(\|x\|^s + \|y\|^s + \|z\|^s) + \xi\|x\|^t\|y\|^t\|z\|^t$$

for all $x, y, z \in A$ and $i \in \{1, 2\}$. The proof falls naturally into three parts. For $m = 1$ and $m = 2$, apply Theorems 6 and 7, respectively by letting $r = -1$ and $L = |2|^{1-p}$. If $m > 2$, choose $r = -1$ and $L = |m|^{1-p}$ and apply Theorem 5 to get the desired result. $\square$

Corollary 3. Let η, ξ, p, q, s , and t be non-negative real numbers with

$$1 < p < \min\{2q, s - 2, 3t - 2\}.$$

Let $f_1 : A \rightarrow B$ and $f_2 : A \rightarrow B$ be two mappings such that

$$\begin{aligned}\|\Delta_m f_i(x, y)\| &\leq \eta(\|x\|^p + \|y\|^p) + \xi\|x\|^q\|y\|^q, \\ \|D_{f_1}(x, y, z)\| &\leq \eta(\|x\|^s + \|y\|^s + \|z\|^s) + \xi\|x\|^t\|y\|^t\|z\|^t, \\ \|D_{f_1, f_2}(x, y, z)\| &\leq \eta(\|x\|^s + \|y\|^s + \|z\|^s) + \xi\|x\|^t\|y\|^t\|z\|^t\end{aligned}$$

for all $x, y, z \in A$, $i \in \{1, 2\}$, and all natural numbers m . Then there exist a unique ternary ring homomorphism $H_1 : A \rightarrow B$ and a unique generalized ternary ring homomorphism $H_2 : A \rightarrow B$ such that

$$\|f_i(x) - H_i(x)\| \leq \begin{cases} \frac{1}{|2|(1-|2|^{p-1})} (2\eta\|x\|^p + \xi\|x\|^{2q}), & m = 1, \\ \frac{\eta|m|^{p-1}}{1-|m|^{p-1}} \|x\|^p, & m \geq 2. \end{cases}$$

Proof. Define

$$\begin{aligned}\varphi(x) &:= \eta(\|x\|^p + \|y\|^p) + \xi\|x\|^q\|y\|^q, \\ \psi_i(x, y, z) &:= \eta(\|x\|^s + \|y\|^s + \|z\|^s) + \xi\|x\|^t\|y\|^t\|z\|^t\end{aligned}$$

for all $x, y, z \in A$ and $i \in \{1, 2\}$. The proof divides into three parts. For $m = 1$ and $m = 2$, the result follows from Theorems 6 and 7, respectively by choosing $r = 1$ and $L = |2|^{p-1}$. For $m > 2$, it is sufficient to choose $r = 1$ and $L = |m|^{p-1}$ and then apply Theorem 4 to get the desired result. $\square$

We note that when $p = 1$ in Corollaries 2 and 3, the stability result does not hold. Since in this case we have $L = 1$ and the condition $0 < L < 1$ in Theorems 6 and 7 does not fulfill.

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ON QUASI-WEIBULL DISTRIBUTION

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Abstract. Exponential distribution together with a variety of its transformations is permanently used both in probability theory and related fields. The most popular one is the power transformation yielding the Weibull distribution.

In this paper, the power distribution of exponential random variable is supplemented by a logarithmic factor leading to a new distribution called quasi-Weibull. This is a three-parameter distribution, where one parameter is inherited from the underlying exponential distribution, and the others originate from the transformation. The properties of the quasi-Weibull distribution are studied. Specifically, the impact of the parameters on the analyticity of characteristic function, the existence of the moment generating function, the moment-determinacy/indeterminacy and the behaviour of the hazard function are investigated.

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1. INTRODUCTION

Exponential distribution and its various transformations, generalizations and analogues are of great interest not only in probability theory but also in its numerous applications. See, for example, [3, 5, 10] and references therein. The mostly used transformation of an exponential random variable is the power transformation, which possesses the Weibull distribution. Apart from its importance within probability theory and relations to other classical distributions, such as gamma and Rayleigh, the Weibull distribution is a central one used extensively in reliability theory, actuarial science, engineering and finance.

In this work, we study a perturbation of the Weibull distribution, generated by the power-logarithmic transformation of the exponential distribution, as defined below.

Definition 1. Given $X \sim \text{Exp}(\lambda)$ and $\varphi(x) = x^a \ln^b(1+x)$, $x > 0$, where $a \in \mathbb{R}$, $b \geq 0$, the distribution of $Y = Y_{a,b} = \varphi(X)$ is said to be *quasi-Weibull* with parameters a , b and λ .

Notice that when $b = 0$, one recovers the Weibull distribution. For this reason, in the present work, we deal solely with the case $b > 0$. The positive value of the parameter b supplies an additional flexibility for the distribution of Y when describing real data sets as it provides more subtle matching. It has to be emphasized that, power-logarithmic functions of random variables occur in various disciplines. For example, in the nuclear physics, the energy loss of fast particles by ionization is expressed by

$$F = \frac{2\pi N e^4}{E} \ln \left(\frac{4E}{I} \right),$$

where E is a random variable averaging energy of particles, I is a characteristic energy of atom ionization, and N is the density of atoms in the medium. See, e.g. [2].

Let us recall some basic notions of probability theory used in the sequel. Given a random variable X with a distribution function $F_X(x)$, the *reliability function* of X is $R_X(x) := 1 - F_X(x)$. Notice that in the actuarial mathematics, R_X is called a survival function. For $n \in \mathbb{N}_0$, the n -th order moment of X is defined as $M_n := \int_{\mathbb{R}} x^n dF_X(x)$, provided the integral is absolutely convergent. The distribution of X is said to be *moment-determinate* if X has moments of all orders and F_X is the only distribution possessing this moment sequence $\{M_n\}_{n=0}^{\infty}$. Otherwise, the distribution is *moment-indeterminate*. For more information, the reader is referred to, e.g. [9, Ch. 11].

The moment-(in)determinacy of probability distributions is not only one of the classical areas of probability theory owning deep interrelations with other branches of mathematics (see, e.g. [1]), but also it is a significant issue for applications. The respective direction of research, called the moment analysis of distributions, is getting increasingly popular in the fields such as econometrics, quantum physics, actuarial sciences, and mathematical finance. For example, Penson et al. [7] have shown that Stieltjes classes of moment indeterminate distributions emerge when describing coherent states in the quantum physics. Also, Lasserre et al. [4] proposed a novel approach involving the moments of appropriate distributions. For these type of problems, the moment-(in)determinacy is a crucial property. Additional relevant information on the subject can be found in [8] and references therein.

The focus of this work is on the impact of parameters a and b on the properties of the quasi-Weibull distribution. The paper is organized as follows. In section 2, the analyticity of the characteristic function is studied. Section 3 presents the results on the existence of moments for the quasi-Weibull distribution together with the conditions for moment-(in)determinacy in the case when all moments exist. Section 4 provides the conditions on parameters a and b for the quasi-Weibull distribution to be increasing/decreasing failure rate. The comparison with the respective properties of the classical Weibull distribution is performed. The obtained results are illustrated graphically.

2. CHARACTERISTIC FUNCTION

First, let us obtain a density of the quasi-Weibull distribution $Y_{a,b}$. To do this, the following cases have to be considered:

- (i) If $a \leq -b$ or $a \geq 0$, then $\varphi(x)$, $x > 0$, is invertible with $g = \varphi^{-1}$. In this case, the density of Y is given by

$$f_Y(x) = \lambda e^{-\lambda g(x)} |g'(x)|, \quad x > 0, \quad (2.1)$$

and the reliability function by

$$R_Y(x) = \begin{cases} e^{-\lambda g(x)}, & \text{if } a \geq 0, x > 0, \\ 1 - e^{-\lambda g(x)}, & \text{if } a \leq -b, x > 0. \end{cases} \quad (2.2)$$

- (ii) If $-b < a < 0$, then φ is not invertible, and we set $\varphi_1(x) = \varphi(x)$, $x \in (0, x_0)$, $\varphi_2(x) = \varphi(x)$, $x \in (x_0, \infty)$, where x_0 satisfies $\varphi'(x_0) = 0$. Then, one may consider $g_1 = \varphi_1^{-1}$ and $g_2 = \varphi_2^{-1}$. In these notations, the density of Y is given by

$$f_Y(x) = \lambda e^{-\lambda g_1(x)} g_1'(x) - \lambda e^{-\lambda g_2(x)} g_2'(x), \quad 0 < x < \varphi(x_0),$$

Correspondingly, the reliability function equals

$$R_Y(x) = \begin{cases} e^{-\lambda g_1(x)} - e^{-\lambda g_2(x)}, & 0 < x < \varphi(x_0), \\ 0, & x \geq \varphi(x_0), \end{cases}$$

The graphs of the density functions corresponding to $\lambda = 1$ and different values of a and b are demonstrated in Figure 1.

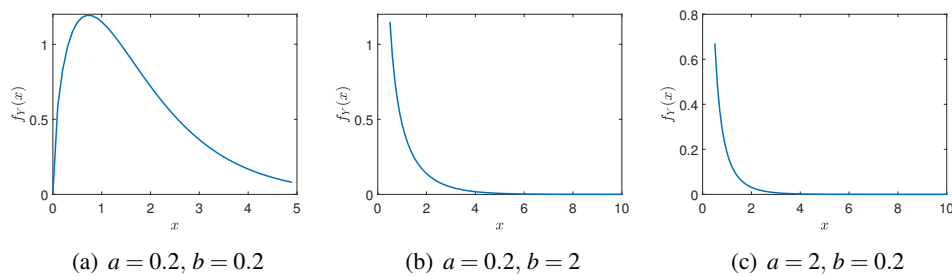


FIGURE 1. Density functions for the quasi-Weibull distribution with $\lambda = 1$.

Our first result is concerned with some analytic properties of the characteristic function $\phi_{a,b}(t)$ of $Y_{a,b}$.

Theorem 1. *If $-b \leq a < 1$, then $\phi_{a,b}(t)$ is entire. Otherwise, it is not analytic in any neighbourhood of 0. Further, if $0 < a < 1$, then $\phi_{a,b}(t)$ is entire of order $\rho = 1/(1-a)$, and type $\sigma = \infty$, while if $-b \leq a < 0$, then $\phi_{a,b}(t)$ is entire of order 1 and type $\sigma = \varphi(x_0)$.*

Proof. By [6, formula (2.2.3)], since $Y_{a,b} \geq 0$, the characteristic function of its probability distribution is analytic in the disc $\mathcal{D}_{\mathcal{R}} = \{z : |z| < \mathcal{R}\}$ if and only if its reliability function satisfies the asymptotic estimate

$$R_Y(x) = O(e^{-rx}), \quad x \rightarrow +\infty \quad \text{for all } 0 < r < \mathcal{R}.$$

When $a \geq 0$, from (2.2), one has, for all $r > 0$,

$$\lim_{x \rightarrow +\infty} e^{rx} R_Y(x) = \lim_{t \rightarrow +\infty} e^{r\varphi(t) - \lambda t} = \begin{cases} 0, & \text{if } a < 1, \\ +\infty, & \text{if } a \geq 1. \end{cases} \quad (2.3)$$

This implies that $\phi_{a,b}(t)$ is entire when $0 \leq a < 1$ and not analytic in any neighbourhood of 0 when $a \geq 1$. To find the order of $\phi_{a,b}(t)$, we use Theorem 2.4.4 of [6]. First, we evaluate

$$\kappa = \lim_{x \rightarrow +\infty} \frac{\ln \ln(1/R_Y(x))}{\ln x} = \lim_{x \rightarrow +\infty} \frac{\ln(\lambda g(x))}{\ln x} = \lim_{t \rightarrow +\infty} \frac{\ln \lambda + \ln t}{a \ln t + b \ln \ln(1+t)} = \frac{1}{a},$$

whence

$$\frac{1}{\rho} = 1 - \frac{1}{\kappa} = 1 - a$$

and the order is $\rho = 1/(1-a)$. As for the type,

$$\tau = \lim_{x \rightarrow +\infty} x^{-\kappa} \ln(1/R_Y(x)) = \lim_{x \rightarrow +\infty} \frac{\lambda g(x)}{x^{1/a}} = \lim_{t \rightarrow +\infty} \frac{\lambda t}{t \ln^{b/a}(1+t)} = 0,$$

yielding that $\phi_{a,b}(t)$ has the maximal type.

Next, let $-b \leq a < 0$. Then, the distribution of $Y_{a,b}$ has bounded support $(0, \varphi(x_0))$, and according to [6], $\phi_{a,b}(t)$ is entire of order 1 and type $\sigma = \varphi(x_0)$. Notice that when $a = -b$, we set $x_0 = 0$.

Finally, let $a < -b$. Using (2.2), one obtains

$$\lim_{x \rightarrow +\infty} e^{rx} R_Y(x) = \lim_{t \rightarrow 0^+} e^{r\varphi(t)} (1 - e^{-\lambda t}) = \lim_{t \rightarrow 0^+} \lambda t e^{r\varphi(t)} = +\infty$$

for all $r > 0$ since $\varphi(t) \sim t^{a+b}$ as $t \rightarrow 0^+$. Combining this with (2.3) completes the proof. $\square$

Corollary 1. *The moment generating function of $Y_{a,b}$ exists if and only if $-b \leq a < 1$. This implies that for other values of the parameters, the quasi-Weibull distribution is heavy-tailed.*

Corollary 2. *For $-b \leq a < 1$, the distribution of $Y_{a,b}$ is moment-determinate by virtue of Cramer's condition.*

3. MOMENT-(IN)DETERMINACY OF QUASI-WEIBULL DISTRIBUTION

Prior to studying the moment-(in)determinacy of the distribution, it is necessary to establish the conditions under which the moments exist.

Theorem 2. *Random variable $Y_{a,b}$ has finite moments of all orders if and only if $a \geq -b$.*

Proof. The moment of order n equals

$$M_n = \lambda \int_0^\infty x^{an} \ln^{bn} (1+x) e^{-\lambda x} dx.$$

When $a \geq 0$, one has $M_n < \infty$ for all $n \in \mathbb{N}_0$. When $a < 0$, since $x^{an} \ln^{bn} (1+x) \sim x^{(a+b)n}$ as $x \rightarrow 0^+$, the integral converges for all $n \in \mathbb{N}_0$ if and only if $a+b \geq 0$. $\square$

In view of the last theorem, the problem of moment-(in)determinacy for $Y_{a,b}$ is well-defined only when $a \geq -b$. In addition, Corollary 2 asserts that for $-b \leq a < 1$, the distribution of $Y_{a,b}$ is moment-determinate. Bearing this in mind, it suffices to examine only the case $a \geq 1$.

Theorem 3. *Let $a \geq 1$. The distribution of $Y_{a,b}$ is moment-determinate if and only if $-b \leq a < 2$ or $a = 2, b \leq 2$.*

Proof. By (2.1), the density is

$$f_{a,b}(x) := f_Y(x) = \lambda e^{-\lambda g(x)} g'(x), \quad x > 0,$$

where $g = \varphi^{-1}$.

First, we apply Krein's condition for moment-indeterminacy of an absolutely continuous distribution with support on $[0, +\infty)$, see, e.g. [9, Ch. 11]:

Let $f(x), x > 0$, be a probability density of a random variable X . If, for some $c > 0$,

$$I[f] = \int_c^\infty \frac{-\ln f(x^2)}{1+x^2} dx < \infty,$$

then the distribution of X is moment-indeterminate. We have, therefore, for f_Y :

$$I[f_{a,b}] = \int_c^\infty \frac{-\ln \lambda + \lambda g(x^2) - \ln g'(x^2)}{1+x^2} dx \sim \frac{1}{2} \int_{g(c^2)}^\infty \frac{[\lambda u + \ln \varphi'(u)] \varphi'(u)}{(1+\varphi(u)) \sqrt{\varphi(u)}} du. \quad (3.1)$$

Since $\varphi'(u) \sim C u^{a-1} \ln^b(1+u)$, $u \rightarrow +\infty$, the numerator of (3.1) satisfies

$$[\lambda u + \ln \varphi'(u)] \varphi'(u) \sim \lambda u^a \ln^b(1+u), \quad u \rightarrow +\infty.$$

Likewise, for the denominator, we obtain

$$(1+\varphi(u)) \sqrt{\varphi(u)} \sim u^{3a/2} \ln^{3b/2}(1+u), \quad u \rightarrow +\infty.$$

As a result, in terms of convergence,

$$I[f_{a,b}] \sim \int_{c_1}^\infty \frac{du}{u^{a/2} \ln^{b/2}(u)}, \quad c_1 > 0.$$

The latter integral converges when $a > 2$ and diverges when $a < 2$ whatever $b > 0$ is. As for $a = 2$, the integral converges only for $b > 2$.

This shows that, for $a > 2$ or $a = 2$, $b > 2$, the distribution of $Y_{a,b}$ is moment-indeterminate. Since Krein's condition is inconclusive in the case when $I[f]$ is divergent, we apply Carleman's condition for the remaining values of parameters.

Recall that Carleman's condition deals with the moments and states that if X is a non-negative random variable with moments M_n such that

$$\sum_{n=1}^{\infty} M_n^{-1/(2n)} = \infty,$$

then the distribution of X is moment-determinate.

To deal with the untouched values of the parameters, we split the rest of the proof into two cases.

Case 1: Let $a < 2$ and $0 < \delta < b$ satisfy $a + \delta < 2$. In this case, $\ln^b(1+x) \leq Cx^\delta$ for $x \geq 0$, and hence,

$$M_n \leq \lambda C^n \int_0^\infty x^{(a+\delta)n} e^{-\lambda x} dx = \frac{C^n}{\lambda^{(a+\delta)n}} \Gamma((a+\delta)n+1) \leq (C_1)^n (2n)!, \quad n \geq n_0.$$

Therefore, by Stirling's formula, $M_n^{1/(2n)} \sim C_2 n$, $n \rightarrow \infty$, and, thus, $\sum_{n=1}^\infty M_n^{-1/(2n)}$ diverges, whence the distribution of $Y_{a,b}$ is moment-determinate.

Case 2: Let $a = 2$ and $b \leq 2$. Then,

$$\begin{aligned} M_n &= \lambda \int_0^\infty x^{2n} \ln^{bn}(1+x) e^{-\lambda x} dx = \lambda^{-2n} \int_0^\infty x^{2n} \ln^{bn} \left(1 + \frac{x}{\lambda}\right) e^{-x} dx \\ &= \lambda^{-2n} \int_0^{10n^2} x^{2n} \ln^{bn} \left(1 + \frac{x}{\lambda}\right) e^{-x} dx + \lambda^{-2n} \int_{10n^2}^\infty x^{2n} \ln^{bn} \left(1 + \frac{x}{\lambda}\right) e^{-x} dx \\ &=: I_1 + I_2. \end{aligned}$$

Clearly, $I_1 \leq \lambda^{-2n} \ln^{bn}(1 + 10n^2/\lambda)(2n)!$. To estimate I_2 , we write

$$I_2 \leq \lambda^{-(2+b)n} \int_{10n^2}^\infty \exp\{-x + (2+b)n \ln x\} dx.$$

It can be readily seen that $v(x) := -x/2 + (2+b)n \ln x$ is strictly decreasing when $x > 2(2+b)n$ and also tends to $-\infty$. Since $2(2+b)n < 8n < 10n^2$ for $n \geq 1$ and $v(10n^2) < 0$ for $n \geq 5$, it follows that

$$(2+b)n \ln x \leq x/2 \quad \text{for all } x \geq 10n^2, \quad n \geq 5.$$

Consequently, for all $n \geq 5$, one has:

$$I_2 \leq \lambda^{-(2+b)n} \int_{10n^2}^\infty e^{-x/2} dx = 2\lambda^{-(2+b)n} e^{-5n^2} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

As a result, for n large enough,

$$M_n = I_1 + I_2 \leq 2I_1 \leq 2\lambda^{-2n} \ln^{bn} (1 + 10n^2/\lambda) (2n)!.$$

The obtained estimate yields, with the help of Stirling's formula, that, when n is large enough,

$$M_n^{1/(2n)} \leq Cn \ln^{b/2} n.$$

By Carleman's condition, the distribution of $Y_{a,b}$ is moment-determinate. □

4. THE HAZARD FUNCTION

In this section, we deal with the hazard function of a quasi-Weibull distribution. As the terminology for this function is rather diverse, we clarify the needed terms here.

Definition 2. For a random variable X possessing an absolutely continuous distribution with a density f_X , the *hazard function* is defined as

$$h_X(x) = \frac{f_X(x)}{R_X(x)},$$

where R_X stands for the reliability function of X .

A hazard function is one of the key characteristics pertinent to lifetime distributions occurring in the reliability theory, queuing, computer science, actuarial and financial mathematics. In different branches, it is also known as hazard rate, force of mortality, intensity rate, instantaneous failure rate, and others. See, e.g. [10, Sec. 3.3]. Notice that the exponential distribution is characterized by a constant hazard function, that is, if $X \sim \text{Exp}(\lambda)$, then $h_X(x) = \lambda$. Its power transformation X^a has the Weibull distribution, whose hazard function is increasing when $a > 1$ and decreasing when $0 < a < 1$.

If the hazard function h_X is increasing, then the distribution of X is said to be *increasing failure rate* (IFR), while if h_X is decreasing, the distribution is said to be *decreasing failure rate* (DFR). Generally speaking, an increasing hazard function corresponds to the situations when conditional survival probability decreases with the ageing of a system whose lifetime is X ; and a decreasing hazard function shows that ageing is beneficial for a system. In practical applications, hazard functions show different monotonicity behaviour on different intervals. The graph of a hazard function can also be called the *mortality curve* or *life characteristic*.

The next results reveal the behaviour of the hazard function of a quasi-Weibull distribution for various values of positive parameters a and b .

Theorem 4. *If $a \geq 1$, then the distribution of $Y_{a,b}$ is DFR for all $x > 0$.*

Proof. Notice that when $a \geq 1$, the functions $\varphi(t)$ and $g(x)$ are increasing on $(0, \infty)$. By virtue of (2.1), the hazard function is

$$h(x) = g'(x) = \frac{1}{\varphi'(t)}, \quad \text{where } x = \varphi(t).$$

Consequently, $h(x)$ decreases if and only if $\varphi'(t)$ increases. As

$$\varphi'(t) = at^{a-1} \ln^b(1+t) + bt^a \frac{\ln^{b-1}(1+t)}{1+t}, \quad (4.1)$$

one has

$$\begin{aligned} \varphi''(t) = & a(a-1)t^{a-2} \ln^b(1+t) + 2abt^{a-1} \frac{\ln^{b-1}(1+t)}{1+t} \\ & + bt^a \frac{\ln^{b-2}(1+t)}{(1+t)^2} [b-1-\ln(1+t)]. \end{aligned} \quad (4.2)$$

Obviously, if $a \geq 1$, then

$$\varphi''(t) \geq bt^{a-1} \frac{\ln^{b-2}(1+t)}{(1+t)^2} \cdot \xi(t),$$

where

$$\begin{aligned} \xi(t) &= 2a(1+t) \ln(1+t) + (b-1)t - t \ln(1+t) \\ &\geq a(2+t) \ln(1+t) + (b-1)t > 0 \quad \text{for all } t > 0. \end{aligned}$$

Thus, $\varphi'(t)$ is increasing for all $t > 0$. □

Theorem 5. If $a + b \leq 1$, then the distribution of $Y_{a,b}$ is IFR for all $x > 0$.

Proof. Using (4.2), we write $\varphi''(t) = \xi_1(t) + \xi_2(t)$, where

$$\xi_1(t) = \frac{at^{a-2}}{1+t} \ln^{b-1}(1+t) [(a-1)(1+t) \ln(1+t) + bt]$$

and

$$\xi_2(t) = \frac{bt^{a-1} \ln^{b-2}(1+t)}{(1+t)^2} [a(1+t) \ln(1+t) + (b-1)t - t \ln(1+t)].$$

Observe that, in both cases, the functions in brackets are decreasing in t and vanish at 0. Thence, both $\xi_1(t) < 0$ and $\xi_2(t) < 0$ for all $t > 0$. The statement is proved. □

Theorem 6. If $a < 1$ and $a + b > 1$, then the distribution of $Y_{a,b}$ is eventually IFR without being IFR for all $x > 0$.

Proof. It follows from (4.1) that

$$\lim_{t \rightarrow 0^+} \varphi'(t) = \lim_{t \rightarrow +\infty} \varphi'(t) = 0,$$

whence $\varphi'(t)$ is not monotone for all $t \in (0, +\infty)$. Meanwhile,

$$\varphi''(t) = t^{a-2} \ln^b(1+t) \xi(t),$$

where

$$\xi(t) = a(a-1) + \frac{2abt}{(1+t)\ln(1+t)} + \frac{bt^2[t-1-\ln(1+t)]}{(1+t)^2 \ln^2(1+t)}.$$

As

$$\lim_{t \rightarrow +\infty} \xi(t) = a(a-1) < 0,$$

it follows that $\varphi''(t) < 0$ for t large enough. $\square$

The graphs of the hazard functions for the quasi-Weibull distribution possessing densities displayed in Figure 1 can be viewed in Figure 2.

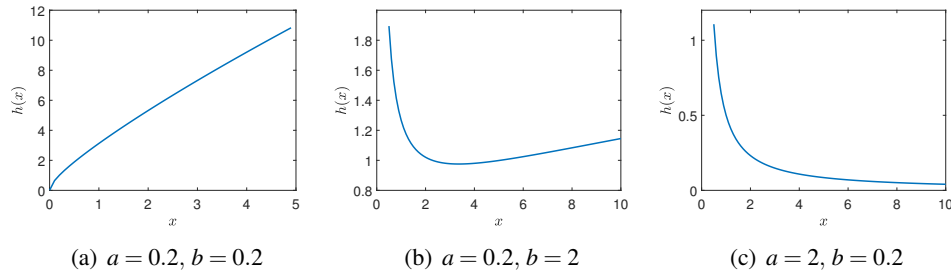


FIGURE 2. Hazard functions for the quasi-Weibull distribution with $\lambda = 1$.

On the whole, although there exist certain similarities between Weibull and quasi-Weibull distributions, in some aspects, their features are quite different. For the latter distribution, first of all, the density function of $Y_{a,b}$ does not appear in a closed form. Next, the moment generating function, for $a = 1$, does not exist in distinction to the Weibull distribution. Both distributions are moment-determinate when $a < 2$ and moment-indeterminate when $a > 2$. When it comes to the case $a = 2$, the Weibull distribution is moment-determinate, whereas the moment-(in)determinacy of $Y_{2,b}$ depends on b with the distribution being moment-determinate only when $b \leq 2$. The list of similarities and distinctions extends to the behaviour of the hazard function. The properties of $Y_{a,b}$ with respect to the behaviour of its hazard function are illustrated in Figure 3.

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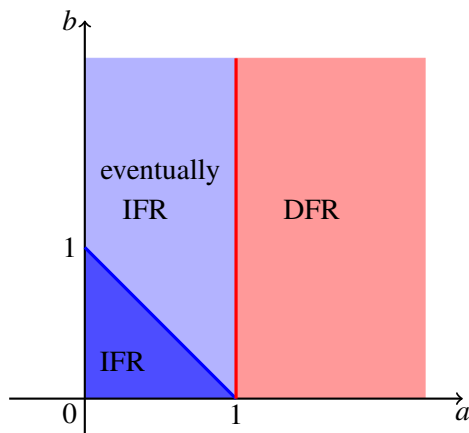


FIGURE 3. Regions of DFR/IFR for the quasi-Weibull distribution

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STUDY OF BI-F-HARMONIC CURVE ALONG IMMERSIONS

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Abstract. In this paper, we characterize the bi-f-harmonic curve on surfaces and then we study the submanifold of a Riemannian manifold using the bi-f-harmonic curve. The conditions for curvature and torsion of bi-f-harmonic curve on surface, ruled surface and 3-dimensional space are derived. In addition, the geometry of the submanifold is studied by taking a bi-f-harmonic curve with immersion from the submanifold to the ambient space. Moreover, the conditions are given for isotropic submanifolds, totally geodesic submanifolds and umbilical submanifolds, so that the immersed curve is a bi-f-harmonic curve in ambient space. Finally, we investigate some important results for a particular case $f = 1$ of the bi-f-harmonic curve.

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Keywords: bi-f-harmonic curves, isotropic submanifold, totally geodesic submanifold, umbilical submanifold

1. INTRODUCTION

Harmonic maps are the generalization of geodesics, minimal surfaces and harmonic functions. These mappings have important applications and relationships in different fields of mathematics and physics with nonlinear partial differential equations and the concept of the stochastic procedure. Let (M_1, g_1) and (M_2, g_2) are Riemannian manifolds, then a harmonic map $\varphi: (M_2, g_2) \rightarrow (M_1, g_1)$ is a critical point of the energy functional,

$$E(\varphi) = \frac{1}{2} \int_{\Gamma_2} |d\varphi|^2 v_{g_2},$$

where Γ_2 is some compact domain of M_2 and $\tau(\varphi) = \text{Trace}_{g_2}(\nabla d\varphi)$ is tension field of φ . The harmonic map equation is an Euler-Lagrange equation of the functional [6] $\tau(\varphi) \equiv \text{Trace}_{g_2}(\nabla d\varphi) = 0$, where $\tau(\varphi) = \text{Trace}_{g_2}(\nabla d\varphi)$ is a tension field of φ . In

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1964, J. Eells and J. H. Sampson [6], generalized the concept of harmonic maps to bi-harmonic maps. A bi-harmonic map $\varphi: (M_2, g_2) \rightarrow (M_1, g_1)$ is a critical point of the bi-energy functional, $E_2(\varphi) = \frac{1}{2} \int_{\Gamma_2} |\tau(\varphi)|^2 v_{g_2}$, where Γ_2 is a compact domain of M_2 . The bi-harmonic map equation is an Euler-Lagrange equation of the functional [7],

$$\tau_2(\varphi) \equiv \text{Trace}_{g_2} \left(\nabla^\varphi \nabla^\varphi - \nabla_{\nabla M_2}^\varphi \right) \tau(\varphi) - \text{Trace}_{g_2} R^{M_1}(\text{d}\varphi, \tau(\varphi)) \text{d}\varphi = 0,$$

where $R^{M_1} = \left[\nabla_X^{M_1}, \nabla_Y^{M_1} \right] Z - \nabla_{[X,Y]}^{M_1} Z$, is a curvature operator of (M_2, g_2) . Bi-harmonic submanifold of the Euclidean space was introduced by B. Y. Chen in 1991 [3]. If the inclusion map is bi-harmonic isometric immersion, then a submanifold is a bi-harmonic submanifold. In [10], the author introduced the f-bi-harmonic maps by combining the bi-harmonic maps and f-harmonic maps. A f-bi-harmonic map is a critical point of the bi-f-energy functional, $\frac{1}{2} \int_{\Gamma_2} |\tau_f(\varphi)|^2 v_{g_2}$. The bi-f-harmonic equation for curves in Euclidean space, hyperbolic space, sphere and hypersurfaces of manifolds were considered in [18]. A f-harmonic map $\varphi: (M_2, g_2) \rightarrow (M_1, g_1)$ is a critical point of the f-energy functional, $E_f(\varphi) = \frac{1}{2} \int_{\Gamma_2} f |\text{d}\varphi|^2 v_{g_2}$, where Γ_2 is a compact domain of M. The f-harmonic map equation is an Euler-Lagrange equation of the functional [5, 16],

$$\tau_f(\varphi) \equiv f\tau(\varphi) + \text{d}\varphi(\text{grad}(f)) = 0,$$

where $\tau(\varphi) = \text{Trace}_{g_2}(\nabla \text{d}\varphi)$, is the tension field of φ . An f-bi-harmonic map from a compact Riemannian manifold to a curved manifold with constant f-bienergy density is a harmonic map [15].

A smooth map $f: (M_1^{m_1}, g_1) \rightarrow (M_2^{m_2}, g_2)$ is said to be an immersion if the differential map $df_{p_1}: T_{p_1} M_1 \rightarrow T_{f(p_1)} M_2$ is one-one for all $p_1 \in M_1$. The theory of isometric immersions is one of the active research areas in differential geometry [1, 2, 4]. In 1974, the authors showed that if a circle is mapped by immersion from a submanifold to the ambient manifold, then the submanifold is totally umbilical with a parallel mean curvature vector field [9]. In [8], the author characterizes the helix by a differential equation and studies the effect on the submanifold by moving a helix from the submanifold to the ambient manifold by immersion. By using isotropic immersion, the circles in complex projective space and complex hyperbolic space were studied in [11, 12, 17]. Recently authors studied the characterization of submanifold by taking the hyperelastic curves along an immersion [19].

We organize our paper as follows: In section 2, we give some basic and useful relations. In section 3, we study bi-f-harmonic curves on surfaces. We prove that if $\gamma(s)$ is a bi-f-harmonic curve on ruled surface $S \subset \mathbb{R}^3$ with $f = \text{constant} \neq 0$, then $\gamma(s)$ must be a straight line. Section 4, is divided into two parts; the first part deals with the characterization of submanifolds by moving the bi-f-harmonic curve from the submanifold to the ambient space by immersion. Then we show that an isometric

immersion is a bi-f-harmonic immersion in the case of a totally geodesic submanifold. The second part of the section, discusses the characterization of a submanifold by taking a bi-harmonic map by immersion from the submanifold to the ambient space.

2. PRELIMINARIES

A bi-f-harmonic map $\varphi: (M_2, g_2) \rightarrow (M_1, g_1)$ from Riemannian manifold M_2 to a Riemannian manifold M_1 is a critical point of the bi-f-energy functional,

$$E_f^2(\varphi) = \frac{1}{2} \int_{\Gamma_2} |\tau_f(\varphi)|^2 v_{g_2},$$

where Γ_2 is a compact domain of M_2 . The bi-f-harmonic map equation is an Euler-Lagrange equation of the functional,

$$\tau_f^2(\varphi) \equiv fJ^\varphi(\tau_f(\varphi)) - \nabla_{\text{grad}(\varphi)}^\varphi \tau_f(\varphi) = 0,$$

where $\tau_f(\varphi)$ is the f-tension field of φ and J^φ is the Jacobi operator of the map defined by $J^\varphi(X) = -[Tr_{g_2} \nabla^\varphi \nabla^\varphi X - \nabla_{\nabla_{M_1} X}^\varphi X - R^{M_2}(d\varphi, X)d\varphi]$ [15, 16].

Let $\phi: I \rightarrow M_1$ be a curve in (M_1, g_1) , then ϕ is a bi-f-harmonic curve on M_1 if and only if the following condition holds [18]

$$(ff''' + f'f'')T_1 + (3ff'' + 2f'^2)\nabla_{T_1}^1 T_1 + 4ff'\nabla_{T_1}^{1,2} T_1 + f^2\nabla_{T_1}^{1,3} T_1 + f^2R^1(\nabla_{T_1}^1 T_1, T_1)T_1 = 0, \quad (2.1)$$

where $f: I \rightarrow (0, \infty)$ is a smooth function, $\nabla_{T_1}^{1,2} T_1 = \nabla_{T_1}^1 \nabla_{T_1}^1 T_1$, $\nabla_{T_1}^{1,3} T_1 = \nabla_{T_1}^1 \nabla_{T_1}^1 \nabla_{T_1}^1 T_1$ and R^1 is a Riemann curvature tensor of M_1 .

Let M be a Riemannian manifold of dimension m and M_1 is its submanifold of dimension m_1 with Riemannian connection ∇ on M and induced Riemannian connection ∇^1 on M_1 . The set of all vector fields on M and M_1 , are represented as $\Gamma(TM)$ and $\Gamma(TM_1)$, respectively. TM_1 , $\Gamma(TM_1)^\perp$ and $TM_1^\perp$ will express the set of tangent vector bundle of M_1 , all normal vector fields of M_1 and normal vector bundle of M_1 , respectively. The Gauss and Weingarten formulae for M and M_1 , are [20]

$$\nabla_{X_1} Y_1 = \nabla_{X_1}^1 Y_1 + h^1(X_1, Y_1), \quad \forall X_1 Y_1 \in \Gamma(TM_1), \quad (2.2)$$

$$\nabla_{X_1} W_1 = -A_{W_1} X_1 + D_{X_1} W_1, \quad \forall W_1 \in \Gamma(TM_1)^\perp, \quad (2.3)$$

where D is a connection in normal bundle, $h^1(X_1, Y_1)$ is second fundamental form of M_1 and A is the shape operator of M_1 . The tangential component and normal component in Gauss formula are $\nabla_{X_1}^1 Y_1$ and $h^1(X_1, Y_1)$, respectively. Whereas in Weingarten formula the tangential component and normal component are $A_{W_1} X_1$ and $D_{X_1} W_1$, respectively. The second fundamental form is bilinear and symmetric in X_1 and Y_1 . The

relation between shape operator and second fundamental form is given by

$$\langle A_{W_1} X_1, Y_1 \rangle = \langle h^1(X_1, Y_1), W_1 \rangle. \quad (2.4)$$

If R and R^1 are the Riemannian curvature tensor fields of M and M^1 , then from (2.2) and (2.3), we have

$$\begin{aligned} R(X_1, Y_1)Z_1 &= R^1(X_1, Y_1)Z_1 - A_{h^1(Y_1, Z_1)}X_1 + A_{h^1(Y_1, Z_1)}Y_1 \\ &\quad + (\nabla_{X_1} h^1)(Y_1, Z_1) - (\nabla_{Y_1} h^1)(X_1, Z_1), \end{aligned}$$

for all $X_1, Y_1, Z_1 \in \Gamma TM_1$. The covariant derivative of second fundamental form and shape operator are, respectively

$$(\nabla_{X_1} h^1)(Y_1, Z_1) = D_{X_1} h^1(Y_1, Z_1) - h^1(\nabla_{X_1}^1 Y_1, Z_1) - h^1(Y_1, \nabla_{X_1}^1 Z_1), \quad (2.5)$$

and

$$(\nabla_{X_1} A)_{W_1} Y_1 = \nabla_{X_1}^1 (A_{W_1} Y_1) - A_{D_{X_1} W_1} Y_1 - A_{W_1} (\nabla_{X_1}^1 Y_1). \quad (2.6)$$

The covariant derivative of h and A , satisfies the relation

$$\langle (\nabla_{X_1} h^1)(Y_1, Z_1), W_1 \rangle = \langle (\nabla_{X_1} A)_{W_1} Y_1, Z_1 \rangle. \quad (2.7)$$

The submanifold M_1 is totally geodesic in M if its second fundamental form is identically zero i.e. $h^1 = 0$ ($A = 0$) and M_1 is totally umbilical if h^1 satisfies

$$h^1(X_1, Y_1) = \langle X_1, Y_1 \rangle H. \quad (2.8)$$

If, $\langle h^1(X_{1p}, X_{1p}), h^1(X_{1p}, X_{1p}) \rangle = \text{constant}$, $\forall X_{1p} \in T_p M_1$, then M_1 is said to be isotropic at p . The necessary and sufficient condition for M_1 to be isotropic is that

$$\langle h^1(X_{1p}, X_{1p}), h^1(X_{1p}, Y_{1p}) \rangle = 0, \quad \forall X_{1p}, Y_{1p} \in T_p M_1.$$

Also, if $h^1(X_{1p}, Y_{1p}) = 0$, $\forall X_{1p}, Y_{1p} \in T_p M_1$, then M_1 is totally umbilical [14].

3. CHARACTERIZATION OF BI-F- HARMONIC CURVES

Let $\gamma(s)$ be a smooth unit speed curve immersed on a surface S , with $T = \gamma'(s)$ is the unit tangent vector and $N = JT$ is the unit normal vector along $\gamma(s)$ [13]. Where J is the anti-clockwise rotation by an angle $\frac{\pi}{2}$ defined in the tangent bundle of S . Let ∇ be the Levi-Civita connection on S , then the Frenet-Serret equation is

$$\nabla_T T = \kappa_g N, \quad (3.1)$$

where κ_g is a geodesic curvature.

Theorem 1. A curve $\gamma(s)$ on a surface S is a bi-f-harmonic curve if and only if its geodesic curvature satisfies the equations

$$f f''' + f' f'' - 4\kappa_g^2 - 3\kappa_g \kappa_g' f^2 = 0, \quad (3.2)$$

$$3\kappa_g f f'' + 2\kappa_g f'^2 + 4\kappa_g' f f' + (\kappa_g'' - \kappa_g^3) f^2 + \kappa_g f^2 K_S = 0, \quad (3.3)$$

where $K_S = \langle R(T, N)N, T \rangle$ is the Gaussian curvature of surface S along γ and $()'$ represents the derivative with respect to parameter s .

Proof. Let $\gamma(s)$ be a curve on a surface S then from (3.1), we have

$$\begin{cases} \nabla_T^2 T = -\kappa_g^2 T + \kappa_g' N, \\ \nabla_T^3 T = -3\kappa_g \kappa_g' T + (\kappa_g'' - \kappa_g^3) N, \end{cases} \quad (3.4)$$

where T and N are tangent and normal along $\gamma(s)$. Substituting equation (3.4) in (2.1), we get

$$\begin{aligned} (ff''' + f'f'' - 4\kappa_g^2 ff' - 3\kappa_g \kappa_g' f^2) T + f^2 \kappa_g R(N, T) T \\ + (3\kappa_g ff'' + 2\kappa_g f'^2 + 4\kappa_g' ff' + f^2 \kappa_g'' - f^2 \kappa_g^3) N = 0. \end{aligned} \quad (3.5)$$

Then the inner product of equation (3.5) with tangent and normal provide the required conditions. $\square$

Let S be a ruled surface in $\mathbb{R}^3$ with parametrization $x(s, t) = \gamma(s) + tN(s)$, where $N(s)$ is a normal of curve in $\mathbb{R}^3$. Then the Gaussian curvature along $\gamma(s)$ satisfies $K_S(\gamma(s)) = -\tau^2(s)$. Also, the geodesic curvature of $\gamma(s)$, as a curve on S , is $\kappa_g = \pm\kappa(s)$, where $\kappa(s)$ is a curvature of $\gamma(s)$ in $\mathbb{R}^3$.

Theorem 2. A curve $\gamma(s)$ is a bi-f-harmonic curve on ruled surface S if and only if curvature $(\kappa(s))$ and torsion $(\tau(s))$ of the curve satisfies the equations

$$\kappa(s) = f^{-\frac{4}{3}} \left(\frac{2}{3} \int f^{\frac{2}{3}} (ff''' + f'f'') ds + C \right)^{\frac{1}{2}}, \quad (3.6)$$

$$\begin{aligned} \tau(s) = \frac{1}{3\kappa^2 f^2} \left[27\kappa^4 f^3 f'' + 18\kappa^4 f^3 f'^2 + 3\kappa^2 (f^2 \{ff''' + f'f'' - 4\kappa^2 ff'\})' \right. \\ \left. - (ff''' + f'f'' - 4\kappa^2 ff')^2 \right]^{\frac{1}{2}}, \end{aligned} \quad (3.7)$$

where C is some constant.

Proof. Let $\gamma(s)$ be a bi-f-harmonic curve on a ruled surface S . Taking $\kappa_g = \kappa$ and $K_S = \langle R(T, N)N, T \rangle = -\tau^2$ in equation (3.2) and (3.3), we get

$$ff''' + f'f'' - 4\kappa^2 ff' - 3\kappa \kappa' f^2 = 0, \quad (3.8)$$

$$3\kappa ff'' + 2\kappa f'^2 + 4\kappa' ff' + (\kappa'' - \kappa^3) f^2 - \kappa f^2 \tau^2 = 0. \quad (3.9)$$

Solution of the equation (3.8), provides the value of the curvature $\kappa(s)$ (using mapple). Also from equation (3.8), we have

$$\kappa' = \frac{1}{3f^2 \kappa} (ff''' + f'f'' - 4\kappa^2 ff'). \quad (3.10)$$

The derivative of κ' from (3.10),

$$\kappa'' = -\frac{\kappa'^2}{\kappa} + \frac{1}{3\kappa} \left(\frac{1}{f^2} (ff''' + f'f'' - 4\kappa^2 ff') \right)'.$$

Substituting the values of κ' and κ'' from above equations in (3.9), we get the required value of torsion $\tau(s)$.

Conversely, assume that $\gamma(s)$ be a curve on a ruled surface S , with curvature and torsion given by (3.6) and (3.7), respectively. Then it is easy to see that κ and τ satisfies the equation (3.8) and (3.9), hence $\gamma(s)$ is a bi- f -harmonic curve on ruled surface S . $\square$

Corollary 1. *If $\gamma(s)$ is a bi- f -harmonic curve with non zero curvature and $f = \text{constant}$ lying on a ruled surface S in $\mathbb{R}^3$, then f must be a zero function (i.e., $f = 0$).*

Proof. Let $\gamma(s)$ is a bi- f -harmonic curve on ruled surface S with $f(s) = \text{constant} \neq 0$, then from equation (3.8), we have $\kappa = C$, where C is constant.

Since $f(s) = \text{constant} \neq 0$ and $\kappa = C$, therefore from equation (3.9), we have $\tau = iC$, which is not possible. $\square$

Hence from the above corollary, we can conclude that

Remark 1. If $\gamma(s)$ is a bi- f -harmonic curve lying on a ruled surface S in $\mathbb{R}^3$ with $f = \text{constant} \neq 0$, then $\gamma(s)$ is a straight line.

Corollary 2. *Let $\gamma(s)$ be a bi- f -harmonic curve lying on a ruled surface S , with $f(s) = \kappa(s) \neq 0$, then $\gamma(s)$ is a Frenet helix.*

Proof. Since $\gamma(s)$ is a bi- f -harmonic curve on a ruled surface S with $f(s) = \kappa(s) \neq 0$, therefore from (3.8), we get

$$\kappa\kappa''' + \kappa'\kappa'' - 4\kappa^3\kappa' - 3\kappa^3\kappa' = 0. \quad (3.11)$$

On integrating (3.11), we obtain

$$\kappa\kappa'' - \frac{7}{4}\kappa^4 = C_1,$$

where C_1 is some constant. Multiplying above equation with $2\frac{\kappa'}{\kappa}$ and then integrating,

$$\kappa'^2 = \frac{7}{8}\kappa^4 + 2C_1 \log \kappa + C_2,$$

where C_2 is another arbitrary constant, for simplicity taking $C_1 = C_2 = 0$, we have

$$\kappa = \sqrt{\frac{8}{7}} \frac{1}{s}.$$

Substituting the value of $f = \kappa = \sqrt{\frac{8}{7}} \frac{1}{s}$ in (3.9), we get

$$\tau = 3\sqrt{\frac{10}{7}} \frac{1}{s}.$$

Since the ratio of κ and τ is constant, hence γ is a Frenet helix. $\square$

Theorem 3. Let γ be a bi-f-harmonic curve on a surface S with Gaussian curvature $K_S = -\frac{1}{9f^2} (70f'^2 + 39C)$ and $ff'' = \text{constant}$, then γ is a geodesic on S .

Proof. Since γ is a bi-f-harmonic curve and $ff'' = \text{constant}(C)$, therefore from (3.2), $\kappa'_g = \frac{4}{3} \left(\frac{\kappa_g f'}{f} \right)$. Further the derivative of κ'_g ,

$$\kappa''_g = \frac{4}{3} \left(\frac{\kappa_g f f'' + \kappa'_g f f' - \kappa_g f'^2}{f^2} \right).$$

Substituting κ'_g and κ''_g in (3.3), we have

$$\kappa_g = \pm \frac{\sqrt{39ff'' + 70f'^2 + 9f^2 K_S}}{3f}, \quad (3.12)$$

where K_S is a Gaussian curvature. Thus, from (3.12) and $K_S = -\frac{1}{9f^2} (70f'^2 + 39C)$, we have $\kappa_g = 0$. Hence γ is a geodesics on surface S . $\square$

Let $\gamma(s)$ be a bi-f-harmonic curve immersed on a Riemannian manifold M^3 and $\{T(s), N(s), B(s)\}$ is a Frenet frame along $\gamma(s)$, where $\gamma'(s) = T(s)$ is a unit tangent vector field. Then the Frenet equations are

$$\begin{cases} \nabla_T T(s) = \kappa N, \\ \nabla_T N(s) = -\kappa T + \tau B, \\ \nabla_T B(s) = -\tau N, \end{cases} \quad (3.13)$$

where $\kappa = \kappa(s)$ and $\tau = \tau(s)$ are the curvature and the torsion of γ , respectively. Now, substituting equation (3.13) in equation (2.1) and taking corresponding tangent, normal and binormal components of the equation (2.1), we obtain

$$\begin{cases} 3\kappa f f'' + 2\kappa f'^2 + 4\kappa' f f' + \kappa'' f^2 + \kappa f^2 \langle R(N, T) T, N \rangle = \kappa^3 f^2 + \kappa \tau^2 f^2, \\ f f''' + f' f'' - 4\kappa^2 f f' - 3\kappa \kappa' f^2 = 0, \\ 4\kappa \tau f f' + 2\kappa' \tau f^2 + \kappa \tau' f^2 + \kappa f^2 \langle R(N, T) T, B \rangle = 0. \end{cases} \quad (3.14)$$

Theorem 4. If a curve $\gamma(s)$ in three dimensional Riemannian manifold M^3 is a bi-f-harmonic curve, then the torsion ($\tau(s)$) and curvature ($\kappa(s)$) of $\gamma(s)$, are

$$\tau(s) = \frac{1}{3\kappa^2 f^2} \left(-9\kappa^6 f^4 + (27f^3 f'' + 36f^2 f'^2 + 9f^4 K_S) \kappa^4 + 3\kappa^2 (f^2 \{f f''' + f' f'' - 4\kappa^2 f f'\})' - (f f''' + f' f'' - 4\kappa^2 f f')^2 \right)^{\frac{1}{2}}, \quad (3.15)$$

$$\kappa(s) = f^{-\frac{4}{3}} \left(\frac{2}{3} \int f^{\frac{2}{3}} (f f''' + f' f'') ds + \frac{3}{2} C \right)^{\frac{1}{2}}, \quad (3.16)$$

where $\kappa = \kappa(s)$, $\tau = \tau(s)$, and c is some constant.

Proof. Let γ be a bi-f-harmonic curve on a Riemannian manifold M^3 , then the solution of equation (3.14), gives us (3.16) (obtain by using mapple). On the other hand, the first part of equation (3.14), provide us

$$\kappa' = \frac{1}{3f^2\kappa} (ff''' + f'f'' - 4\kappa^2 ff'). \quad (3.17)$$

Taking the derivative of (3.17), we get

$$\kappa'' = -\frac{\kappa'^2}{\kappa} + \frac{1}{3\kappa} \left(\frac{1}{f^2} (ff''' + f'f'' - 4\kappa^2 ff') \right)'. \quad (3.18)$$

Substituting the values of κ' and κ'' from equations (3.17) and (3.18) in the second part of (3.14), we get the equation (3.15). $\square$

4. CHARACTERIZATION OF THE SUBMANIFOLD OF A RIEMANNIAN MANIFOLD BY BI-F-HARMONIC CURVES

Let $\gamma_1: I \rightarrow M_1$ be a curve in an m_1 -dimensional Riemannian manifold M_1 . Let $\{N_1^0, N_1^1, N_1^2, \dots, N_1^{m_1-1}\}$ be an orthonormal frame in ΓTM_1 , where $N_1^0 = T_1$, $N_1^1 = N_1$ and $N_1^2 = B_1$ are the unit tangent vector, the unit normal vector and the unit binormal vector of γ_1 , respectively. Then the corresponding Frenet equations are

$$\nabla_{T_1}^1 N_1^i = -\kappa_i N_1^{i-1} + \kappa_{i+1} N_1^{i+1}, \quad 0 \leq i \leq m_1 - 1,$$

where $\kappa_0 = \kappa_n = 0$, $\kappa_1 = \|\nabla_{T_1}^1 N_1^0\|$ is curvature and $\tau_1 = \kappa_2 = -\langle \nabla_{T_1}^1 N_1, N_1^2 \rangle$ is torsion of γ_1 in M_1 , respectively. Next, we introduce the concept of bi-f-harmonic immersion

Definition 1. Let $i: M_1 \rightarrow M$ be an isometric immersion from Riemannian manifold M_1 to a Riemannian manifold M such that γ_1 is a bi-f-harmonic curve on M_1 . If the curve $\gamma = i \circ \gamma_1$ is also a bi-f-harmonic curve on M , then the immersion i is a bi-f-harmonic immersion.

Lemma 1. Let $i: M_1 \rightarrow M$ be an isometric immersion from Riemannian manifold M_1 to a Riemannian manifold M . Then for a curve γ_1 on M_1 and the curve $\gamma(s) = i \circ \gamma_1(s)$ with curvature κ on M , satisfies the following equations

$$\begin{aligned} \nabla_{T_1}^3 T_1 &= h^1 \left(T_1, \nabla_{T_1}^{1,2} T_1 \right) - A_{h^1(T_1, \nabla_{T_1}^1 T_1)} T_1 + D_{T_1} h^1(T_1, \nabla_{T_1}^1 T_1) \\ &\quad - A_{D_{T_1} h^1(T_1, T_1)} T_1 - \nabla_{T_1}^1 (A_{h^1(T_1, T_1)} T_1) \\ &\quad - h^1(T_1, A_{h^1(T_1, T_1)} T_1) + D_{T_1}^2 h^1(T_1, T_1) + \nabla_{T_1}^{1,3} T_1, \end{aligned} \quad (4.1)$$

$$\begin{aligned} R(\nabla_{T_1} T_1, T_1) T_1 &= R^1(\nabla_{T_1}^1 T_1, T_1) T_1 - A_{h^1(T_1, T_1)} \nabla_{T_1}^1 T_1 + A_{h^1(\nabla_{T_1}^1 T_1, T_1)} T_1 \\ &\quad + \left(\nabla_{\nabla_{T_1}^1 T_1} h \right) (T_1, T_1) - (\nabla_{T_1} h) (\nabla_{T_1}^1 T_1, T_1), \end{aligned} \quad (4.2)$$

where $\nabla_{T_1}^{1,2} = \nabla_{T_1}^1 \nabla_{T_1}^1$ and $\nabla_{T_1}^{1,3} = \nabla_{T_1}^1 \nabla_{T_1}^1 \nabla_{T_1}^1$.

Proof. From Gauss and Weingarten formula, we have

$$\begin{aligned}\nabla_{T_1}^2 T_1 &= \nabla_{T_1} (\nabla_{T_1}^1 T_1, h^1(T_1, T_1)) \\ &= \nabla_{T_1}^{1,2} T_1 + h^1(T_1, \nabla_{T_1}^1 T_1) - A_{h^1(T_1, T_1)} T_1 + D_{T_1} h^1(T_1, T_1).\end{aligned}$$

Further taking the covariant derivative of above equation with respect to T and using (2.2) and (2.3), we get the equation (4.1). Whereas equation (4.2) is directly followed from equations (2.2) and (2.6). $\square$

Let γ be a bi-f-harmonic curve on M , then from (2.1), we have

$$\begin{aligned}(ff''' + f'f'') T_1 + (3ff'' + 2f'^2) \nabla_{T_1} T_1 + 4ff' \nabla_{T_1}^2 T_1 + f^2 \nabla_{T_1}^3 T_1 \\ + f^2 R(\nabla_{T_1} T_1, T_1) T_1 = 0.\end{aligned}\quad (4.3)$$

Using Lemma (1) in the above equation, we get

$$\begin{aligned}(ff''' + f'f'') T_1 + (3ff'' + 2f'^2) (\nabla_{T_1}^1 T_1) + (3ff'' + 2f'^2) h^1(T_1, T_1) \\ + 4ff' \nabla_{T_1}^{1,2} T_1 + 4ff' h^1(T_1, \nabla_{T_1}^1 T_1) - 4ff' A_{h^1(T_1, T_1)} T_1 + 4ff' D_{T_1} h^1(T_1, T_1) \\ + f^2 \nabla_{T_1}^{1,3} T_1 + f^2 h^1(T_1, \nabla_{T_1}^{1,2} T_1) - f^2 A_{h^1(T_1, \nabla_{T_1}^1 T_1)} T_1 + f^2 D_{T_1} h^1(T_1, \nabla_{T_1}^1 T_1) \\ - f^2 \nabla_{T_1}^1 (A_{h^1(T_1, T_1)} T_1) - f^2 h^1(T_1, A_{h^1(T_1, T_1)} T_1) - f^2 A_{D_{T_1} h^1(T_1, T_1)} T_1 \\ + f^2 D_{T_1}^2 h^1(T_1, T_1) + f^2 R(\nabla_{T_1}^1 T_1, T_1) T_1 - f^2 A_{h^1(T_1, T_1)} \nabla_{T_1}^1 T_1 + f^2 A_{h^1(\nabla_{T_1}^1 T_1, T_1)} T_1 \\ + f^2 (\nabla_{\nabla_{T_1}^1 T_1} h)(T_1, T_1) - f^2 (\nabla_{T_1} h^1)(\nabla_{T_1}^1 T_1, T_1) = 0.\end{aligned}\quad (4.4)$$

The tangent part of above equation, is

$$\begin{aligned}(ff''' + f'f'') T_1 + (3ff'' + 2f'^2) \nabla_{T_1}^1 T_1 + 4ff' \nabla_{T_1}^{1,2} T_1 - 4ff' A_{h^1(T_1, T_1)} T_1 \\ + f^2 \nabla_{T_1}^{1,3} T_1 - f^2 \nabla_{T_1}^1 (A_{h^1(T_1, T_1)}) - f^2 A_{D_{T_1} h^1(T_1, T_1)} T_1 + f^2 R^1(\nabla_{T_1}^1 T_1, T_1) T_1 \\ - f^2 A_{h^1(T_1, T_1)} \nabla_{T_1}^1 T_1 = 0.\end{aligned}\quad (4.5)$$

Substituting the equations (2.6) and (2.5) in (4.5), we have

$$\begin{aligned}(ff''' + f'f'') T_1 + (3ff'' + 2f'^2) \nabla_{T_1}^1 T_1 + 4ff' \nabla_{T_1}^{1,2} T_1 - 4ff' A_{h^1(T_1, T_1)} T_1 \\ + f^2 \nabla_{T_1}^{1,3} T_1 - f^2 (\nabla_{T_1}^1 A)_{h^1(T_1, T_1)} T_1 - 2f^2 A_{(\nabla_{T_1} h^1)(T_1, T_1)} T_1 - 4f^2 A_{h^1(\nabla_{T_1}^1 T_1, T_1)} T_1 \\ - 2f^2 A_{h^1(T_1, T_1)} \nabla_{T_1}^1 T_1 + f^2 R^1(\nabla_{T_1}^1 T_1, T_1) T_1 = 0.\end{aligned}\quad (4.6)$$

By using Frenet Serret equations of γ_1 , (4.6) reduces to

$$\begin{aligned}(ff''' + f'f'' - 4\kappa_1^2 ff' - 3\kappa_1 \kappa_1' f^2) T_1 + (4\kappa_1 \tau_1 ff' + 2\kappa_1' \tau_1 f^2 + \kappa_1 \tau_1' f^2) B_1 \\ + (3\kappa_1 ff'' + 2\kappa_1 f'^2 + 4\kappa_1' ff' - \kappa_1^3 f^2 + \kappa_1'' f^2 - \kappa_1 \tau_1'^2 f^2) N_1 - 4ff' A_{h^1(T_1, T_1)} T_1\end{aligned}$$

$$\begin{aligned}
& -f^2 (\nabla_{T_1}^1 A)_{h^1(T_1, T_1)} T_1 - 2f^2 A_{(\nabla_{T_1} h^1)(T_1, T_1)} T_1 - 4\kappa_1 f^2 A_{h^1(N_1, T_1)} T_1 \\
& - 2\kappa_1 f^2 A_{h^1(T_1, T_1)} N_1 + \kappa_1 f^2 R^1(N_1, T_1) T_1 = 0.
\end{aligned}$$

Taking inner product of above equation with T_1 and using equation (2.5), we have

$$\begin{aligned}
& f f''' + f' f'' - 3\kappa_1 \kappa_1' f^2 - 4\kappa_1^2 f f' - 4f f' \langle A_{h^1(T_1, T_1)} T_1, T_1 \rangle \\
& - f^2 \langle (\nabla_{T_1}^1 A)_{h^1(T_1, T_1)} T_1, T_1 \rangle - 2f^2 \langle A_{D_{T_1} h^1(T_1, T_1)} T_1, T_1 \rangle + 4f^2 \langle A_{h^1(\nabla_{T_1} T_1, T_1)} T_1, T_1 \rangle \\
& - 4\kappa_1 f^2 \langle A_{h^1(N_1, T_1)} T_1, T_1 \rangle - 2\kappa_1 f^2 \langle A_{h^1(T_1, T_1)} N_1, T_1 \rangle = 0.
\end{aligned} \tag{4.7}$$

From equations (2.4) and (4.7), we obtain

$$\begin{aligned}
& f f''' + f' f'' - 4\kappa_1^2 f f' - 3\kappa_1 \kappa_1' f^2 - 4f f' \langle h^1(T_1, T_1), h^1(T_1, T_1) \rangle \\
& - 2f^2 \langle h^1(T_1, T_1), D_{T_1} h^1(T_1, T_1) \rangle - 2\kappa_1 f^2 \langle h^1(N_1, T_1), h^1(T_1, T_1) \rangle \\
& - f^2 \langle (\nabla_{T_1}^1 A)_{h^1(T_1, T_1)} T_1, T_1 \rangle = 0.
\end{aligned} \tag{4.8}$$

Now, taking $\langle h^1(T_1, T_1), D_{T_1} h^1(T_1, T_1) \rangle = \frac{1}{2} D_{T_1} \langle h^1(T_1, T_1), h^1(T_1, T_1) \rangle$ and equation (2.7) in equation (4.8), we get

$$\begin{aligned}
& f f''' + f' f'' - 3\kappa_1 \kappa_1' f^2 - 4\kappa_1^2 f f' - 4f f' \langle h^1(T_1, T_1), h^1(T_1, T_1) \rangle \\
& - f^2 D_{T_1} \langle h^1(T_1, T_1), h^1(T_1, T_1) \rangle - 2\kappa_1 f^2 \langle h^1(N_1, T_1), h^1(T_1, T_1) \rangle \\
& - f^2 \langle (\nabla_{T_1}^1 h^1)(T_1, T_1), h^1(T_1, T_1) \rangle = 0.
\end{aligned} \tag{4.9}$$

Using the equation (2.5) in equation (4.9), we have

$$\begin{aligned}
& f f''' + f' f'' - 3\kappa_1 \kappa_1' f^2 - 4\kappa_1^2 f f' - 4f f' \|h^1(T_1, T_1)\|^2 \\
& - f^2 D_{T_1} \|h^1(T_1, T_1)\|^2 - 2\kappa_1 f^2 \langle h^1(N_1, T_1), h^1(T_1, T_1) \rangle \\
& - f^2 \langle D_{T_1} h^1(T_1, T_1), h^1(T_1, T_1) \rangle + 2f^2 \langle h^1(\nabla_{T_1}^1 T_1, T_1), h^1(T_1, T_1) \rangle = 0.
\end{aligned} \tag{4.10}$$

Since,

$$-2f^2 \langle h^1(\nabla_{T_1}^1 T_1, T_1), h^1(T_1, T_1) \rangle = -2f^2 \kappa_1 \langle h^1(N_1, T_1), h^1(T_1, T_1) \rangle,$$

and,

$$f^2 \langle D_{T_1} h^1(T_1, T_1), h^1(T_1, T_1) \rangle = \frac{1}{2} f^2 D_{T_1} \|h^1(T_1, T_1)\|^2,$$

then equation (4.10), reduces to

$$\begin{aligned}
& f f''' + f' f'' - 3\kappa_1 \kappa_1' f^2 - 4\kappa_1^2 f f' \\
& = 4f f' \|h^1(T_1, T_1)\|^2 + \frac{3}{2} f^2 D_{T_1} \|h^1(T_1, T_1)\|^2.
\end{aligned} \tag{4.11}$$

Again the normal part of equation (4.4),

$$(3f f'' + 2f'^2) h^1(T_1, T_1) + 4f f' h^1(T_1, \nabla_{T_1}^1 T_1) + 4f f' D_{T_1} h^1(T_1, T_1)$$

$$\begin{aligned}
& + f^2 h^1 \left(T_1, \nabla_{T_1}^{1,2} T_1 \right) + 7f^2 D_{T_1} h^1 \left(T_1, \nabla_{T_1}^1 T_1 \right) - f^2 h^1 \left(T_1, A_{h^1(T_1, T_1)} T_1 \right) \\
& + f^2 D_{T_1}^2 h^1 \left(T_1, T_1 \right) + f^2 \left(\nabla_{\nabla_{T_1}^1 T_1} h^1 \right) \left(T_1, T_1 \right) - f^2 \left(\nabla_{T_1} h^1 \right) \left(\nabla_{T_1}^1 T_1, T_1 \right) = 0. \quad (4.12)
\end{aligned}$$

Thus equation (2.5) can be written as

$$\begin{aligned}
D_{T_1}^2 h^1 \left(T_1, T_1 \right) + 2h^1 \left(\nabla_{T_1}^1 T_1, \nabla_{T_1}^1 T_1 \right) + 2h^1 \left(T_1, \nabla_{T_1}^{1,2} T_1 \right) \\
= \left(\nabla_{T_1}^2 h^1 \right) \left(T_1, T_1 \right) + 4D_{T_1} h^1 \left(T_1, \nabla_{T_1}^1 T_1 \right). \quad (4.13)
\end{aligned}$$

So, from equations (4.13) and (4.12), we have

$$\begin{aligned}
& (3ff'' + 2f'^2) h^1 \left(T_1, T_1 \right) + 4ff' h^1 \left(T_1, \nabla_{T_1}^1 T_1 \right) + 4ff' D_{T_1} h^1 \left(T_1, T_1 \right) \\
& - f^2 h^1 \left(T_1, \nabla_{T_1}^{1,2} T_1 \right) + 5f^2 D_{T_1} h^1 \left(T_1, \nabla_{T_1}^1 T_1 \right) - f^2 h^1 \left(T_1, A_{h^1(T_1, T_1)} T_1 \right) \\
& + f^2 \left(\nabla_{T_1}^2 h^1 \right) \left(T_1, T_1 \right) - 2f^2 h^1 \left(\nabla_{T_1}^1 T_1, \nabla_{T_1}^1 T_1 \right) + f^2 \left(\nabla_{\nabla_{T_1}^1 T_1} h^1 \right) \left(T_1, T_1 \right) \\
& - f^2 \left(\nabla_{T_1} h^1 \right) \left(\nabla_{T_1}^1 T_1, T_1 \right) = 0. \quad (4.14)
\end{aligned}$$

Equation (2.5) and equation (4.14), gives us

$$\begin{aligned}
& (3ff'' + 2f'^2) h^1 \left(T_1, T_1 \right) + 12ff' h^1 \left(T_1, \nabla_{T_1}^1 T_1 \right) + 4ff' \left(\nabla_{T_1} h^1 \right) \left(T_1, T_1 \right) \\
& + 4f^2 h^1 \left(T_1, \nabla_{T_1}^{1,2} T_1 \right) + 3f^2 h^1 \left(\nabla_{T_1}^1 T_1, \nabla_{T_1}^1 T_1 \right) + 5f^2 \left(\nabla_{T_1} h^1 \right) \left(T_1, \nabla_{T_1}^1 T_1 \right) \\
& - f^2 h^1 \left(T_1, A_{h^1(T_1, T_1)} T_1 \right) + f^2 \left(\nabla_{T_1}^{1,2} h^1 \right) \left(T_1, T_1 \right) + f^2 \left(\nabla_{\nabla_{T_1}^1 T_1} h^1 \right) \left(T_1, T_1 \right) \\
& - f^2 \left(\nabla_{T_1} h^1 \right) \left(\nabla_{T_1}^1 T_1, T_1 \right) = 0. \quad (4.15)
\end{aligned}$$

Using Frenet Serret equation in (4.15), we have

$$\begin{aligned}
& (12ff'\kappa_1 + 4f^2\kappa_1') h^1 \left(T_1, N_1 \right) + 4f^2\kappa_1\tau_1 h^1 \left(T_1, B_1 \right) + 4f^2\kappa_1 \left(\nabla_{T_1} h^1 \right) \left(T_1, N_1 \right) \\
& + 3f^2\kappa_1^2 h^1 \left(N_1, N_1 \right) + 4ff' \left(\nabla_{T_1} h^1 \right) \left(T_1, T_1 \right) + \kappa_1 f^2 \left(\nabla_{N_1} h^1 \right) \left(T_1, T_1 \right) \\
& = (4f^2\kappa_1^2 - 3ff'' - 2f'^2) h^1 \left(T_1, T_1 \right) + f^2 \left(h^1 \left(T_1, A_{h^1(T_1, T_1)} T_1 \right) - \left(\nabla_{T_1}^2 h^1 \right) \left(T_1, T_1 \right) \right). \quad (4.16)
\end{aligned}$$

Replacing B_1 with $-B_1$ in above equation and subtracting from equation (4.16), we obtain

$$h^1 \left(T_1, B_1 \right) = 0.$$

Now from all the above discussion, we get the characterization of the submanifold of a Riemannian manifold with the help of bi-f-harmonic curves.

Proposition 1. *Let M_1 be an isotropic submanifold of a Riemannian manifold M and $i: M_1 \rightarrow M$ be an isometric immersion, such that the curve $\gamma_1(s)$ on M_1 and the curve $\gamma(s) = i \circ \gamma_1(s)$ is a bi-f-harmonic curve with curvature κ on M , then*

$$ff''' + f'f'' - 3\kappa_1\kappa_1'f^2 - 4ff'(\kappa_1^2 + C) = 0, \quad (4.17)$$

where $\|h^1(T_1, T_1)\|^2 = C$ is some constant.

Proof. Using the properties of isotropic submanifold in (4.4), we obtain the result. $\square$

Corollary 3. *Let $i: M_1 \rightarrow M$ be an isometric immersion between Riemannian manifold M_1 to a Riemannian manifold M , such that $\gamma(s) = i \circ \gamma_1(s)$ is a bi-f-harmonic curve with curvature κ on M corresponding to curve $\gamma_1(s)$ on M_1 . Suppose that M_1 is an isotropic submanifold and f is constant then curvature κ_1 on M_1 is constant.*

Proof. From equation (4.17), we have

$$\kappa_1 \kappa'_1 = 0. \quad (4.18)$$

Integrating equation (4.18) with respect to parameter s , we obtain

$$\kappa_1 = C_1,$$

where C_1 is a constant. $\square$

Theorem 5. *Let M_1 be a totally geodesic submanifold of Riemannian manifold M and $i: M_1 \rightarrow M$ be an isometric immersion such that $\gamma(s) = i \circ \gamma_1(s)$ is a bi-f-harmonic curve with curvature κ on M corresponding to $\gamma_1(s)$ on M_1 , then*

$$f f''' + f' f'' - 3\kappa_1 \kappa'_1 f^2 - 4\kappa_1^2 f f' = 0.$$

Proof. Since M_1 is a totally geodesic submanifold of M , so taking $h^1 = 0$ in the tangent part of equation (4.11), we get the required result. $\square$

Theorem 6. *Let $i: M_1 \rightarrow M$ be a bi-f-harmonic immersion between Riemannian manifold M_1 to a Riemannian manifold M , then M_1 is a totally umbilical submanifold of Riemannian manifold M . Then mean curvature (H) , holds the equation*

$$D_{T_1}^2 H = \left(\|H\|^2 + \kappa_1^2 - 2\frac{f'^2}{f^2} - 3\frac{f''}{f} \right) H - 4\frac{f'}{f} D_{T_1} H. \quad (4.19)$$

Conversely, if $i: M_1 \rightarrow M$ be an isometric immersion and M_1 is totally geodesic, then i is a bi-f-harmonic immersion.

Proof. Let i be a bi-f-harmonic immersion from Riemannian manifold M_1 to a Riemannian manifold M . Since M_1 is a totally umbilical submanifold, so equation (4.15), reduces to

$$\begin{aligned} & 4f^2 \kappa_1 (\nabla_{T_1} h^1)(T_1, N_1) + 3f^2 \kappa_1^2 H + 4ff' D_{T_1} H + \kappa_1 f^2 (\nabla_{N_1} h^1)(T_1, T_1) \\ &= (4f^2 \kappa_1^2 - 3ff'' - 2f'^2) H + f^2 h^1(T_1, A_H T_1) - f^2 (\nabla_{T_1}^2 h^1)(T_1, T_1). \end{aligned} \quad (4.20)$$

Replacing N_1 with $-N_1$ in (4.20) and using (4.20), we get

$$4\kappa_1 f^2 (\nabla_{T_1} h^1)(T_1, N_1) + \kappa_1 f^2 (\nabla_{N_1} h^1)(T_1, T_1) = 0. \quad (4.21)$$

Substituting equation (4.21) from equation (4.20), we obtain

$$(-\kappa_1^2 f^2 + 3ff'' + 2f'^2) H$$

$$= f^2 h^1(T_1, A_H T_1) - f^2 (\nabla_{T_1}^2 h^1)(T_1, T_1) - 4ff' D_{T_1} H. \quad (4.22)$$

Using the equations (2.4) and (4.13) in equation (4.22), we get

$$f^2 D_{T_1}^2 H = (f^2 \|H\|^2 + f^2 \kappa_1^2 - 3ff'' - 2f'^2) H - 4ff' D_{T_1} H.$$

Therefore

$$D_{T_1}^2 H = \left(\|H\|^2 + \kappa_1^2 - 3\frac{f''}{f} - 2\frac{f'^2}{f^2} \right) H - 4\frac{f'}{f} D_{T_1} H.$$

Conversely, we suppose that the M_1 is totally geodesic and $i: M_1 \mapsto M$ is an isometric immersion. Let γ_1 be a bi-f-harmonic curve with curvature κ_1 on M_1 , and $\gamma(s) = i \circ \gamma_1$ be a curve on M . Then from (4.3) and (4.4), we have

$$\begin{aligned} & (ff''' + f'f'') T_1 + (3ff'' + 2f'^2) \nabla_{T_1} T_1 + 4ff' \nabla_{T_1}^2 T_1 + f^2 \nabla_{T_1}^3 T_1 + f^2 \mathbf{R}(\nabla_{T_1} T_1, T_1) T_1 \\ &= (ff''' + f'f'') T_1 + 4ff' \left(\nabla_{T_1}^{1,2} + h^1(T_1, \nabla_{T_1}^1 T_1) - A_{h^1(T_1, T_1)} T_1 + D_{T_1} h^1(T_1, T_1) \right) \\ &+ (3ff'' + 2f'^2) \nabla_{T_1}^1 T_1 + (3ff'' + 2f'^2) h^1(T_1, T_1) + f^2 \left(\nabla_{T_1}^{1,3} T_1 + h^1(T_1, \nabla_{T_1}^{1,2} T_1) \right. \\ &- A_{h^1(T_1, \nabla_{T_1}^1 T_1)} T_1 + D_{T_1} h^1(T_1, \nabla_{T_1}^1 T_1) - (\nabla_{T_1}^1 A)_{h^1(T_1, T_1)} T_1 - h^1(T_1, A_{h^1(T_1, T_1)} T_1) \\ &- A_{D_{T_1} h^1(T_1, T_1)} T_1 + D_{T_1}^2 h^1(T_1, T_1) + \mathbf{R}^1(\nabla_{T_1}^1 T_1, T_1) T_1 - A_{h^1(T_1, T_1)} \nabla_{T_1}^1 T_1 \\ &\left. + A_{h^1(\nabla_{T_1}^1 T_1, T_1)} T_1 + (\nabla_{\nabla_{T_1}^1 T_1} h^1)(T_1, T_1) - (\nabla_{T_1} h^1)(\nabla_{T_1}^1 T_1, T_1) \right). \end{aligned} \quad (4.23)$$

Using Frenet Serret equation and taking second fundamental form $h^1 = A = 0$ in equation (4.23), we obtain

$$\begin{aligned} & (ff''' + f'f'') T_1 + (3ff'' + 2f'^2) \nabla_{T_1} T_1 + 4ff' \nabla_{T_1}^2 T_1 + f^2 \mathbf{R}(\nabla_{T_1} T_1, T_1) T_1 + f^2 \nabla_{T_1}^3 T_1 \\ &= (ff''' + f'f'') T_1 + (3ff'' + 2f'^2) \nabla_{T_1}^1 T_1 + 4ff' \nabla_{T_1}^{1,2} T_1 + f^2 \nabla_{T_1}^{1,3} T_1 \\ &+ f^2 \mathbf{R}^1(\nabla_{T_1}^1 T_1, T_1) T_1 = 0. \end{aligned}$$

Hence γ is a bi-f-harmonic curve on M , and i is a bi-f-harmonic immersion. $\square$

Corollary 4. Let M_1 be a totally umbilical submanifold of Riemannian manifold M and $i: M_1 \rightarrow M$ be an isometric immersion such that the mean curvature H satisfies the equations (4.19) and

$$f^2 D_{\nabla_{T_1}^1 T_1} H = 4ff' \|H\|^2 T_1 + 2f^2 \|H\|^2 \nabla_{T_1}^1 T_1 + 2f^2 h^1 \left(\nabla_{\nabla_{T_1}^1 T_1} T_1, T_1 \right). \quad (4.24)$$

Then i is a bi-f-harmonic immersion.

Proof. Let $\gamma_1(s)$ be a bi-f harmonic curve on M_1 and $i: M_1 \rightarrow M$ is an isometric immersion on M such that M_1 is a totally umbilical submanifold, then for a curve $\gamma(s)$ on M , from equations (4.3) and (4.4), we have

$$(ff''' + f'f'') T_1 + (3ff'' + 2f'^2) \nabla_{T_1} T_1 + 4ff' \nabla_{T_1}^2 T_1 + f^2 \mathbf{R}(\nabla_{T_1} T_1, T_1) T_1 + f^2 \nabla_{T_1}^3 T_1$$

$$\begin{aligned}
&= (ff''' + f'f'') T_1 + (3ff'' + 2f'^2) \nabla_{T_1}^1 T_1 + f^2 \nabla_{T_1}^{1,3} T_1 + (3ff'' + 2f'^2) h^1(T_1, T_1) \\
&\quad + 4ff' \nabla_{T_1}^{1,2} T_1 + 4ff'h^1(T_1, \nabla_{T_1}^1 T_1) - 4ff'A_{h^1(T_1, T_1)} T_1 + 4ff'D_{T_1} h^1(T_1, T_1) \\
&\quad + f^2 h^1(T_1, \nabla_{T_1}^{1,2} T_1) - f^2 A_{h^1(T_1, \nabla_{T_1}^1 T_1)} T_1 + f^2 D_{T_1} h^1(T_1, \nabla_{T_1}^1 T_1) \\
&\quad - f^2 \nabla_{T_1}^1 (A_{h^1(T_1, T_1)} T_1) - f^2 h^1(T_1, A_{h^1(T_1, T_1)} T_1) - f^2 A_{D_{T_1} h^1(T_1, T_1)} T_1 \\
&\quad + f^2 D_{T_1}^2 h^1(T_1, T_1) + f^2 R^1(\nabla_{T_1}^1 T_1, T_1) T_1 - f^2 A_{h^1(T_1, T_1)} \nabla_{T_1}^1 T_1 + f^2 A_{h^1(\nabla_{T_1}^1 T_1, T_1)} T_1 \\
&\quad + f^2 (\nabla_{\nabla_{T_1} T_1} h^1)(T_1, T_1) - 7f^2 (\nabla_{T_1} h^1)(\nabla_{T_1}^1 T_1, T_1). \tag{4.25}
\end{aligned}$$

Using the Frenet- Serret formulae and (2.8) in equation (4.25), we get

$$\begin{aligned}
&(ff''' + f'f'') T_1 + (3ff'' + 2f'^2) \nabla_{T_1} T_1 + 4ff' \nabla_{T_1}^2 T_1 + f^2 R(\nabla_{T_1} T_1, T_1) T_1 + f^2 \nabla_{T_1}^3 T_1 \\
&= (ff''' + f'f'') T_1 + (3ff'' + 2f'^2) \nabla_{T_1}^1 T_1 + 4ff' \nabla_{T_1}^{1,2} T_1 + (3ff'' + 2f'^2) H \\
&\quad - 4ff'A_H T_1 + 4ff'D_{T_1} H + f^2 D_{T_1}^2 H + f^2 \nabla_{T_1}^{1,3} T_1 - \kappa_1^2 f^2 - f^2 \nabla_{T_1}^1 (A_H T_1) \\
&\quad - f^2 h^1(T_1, A_H T_1) - f^2 A_{D_{T_1} H} T_1 + f^2 R^1(\nabla_{T_1}^1 T_1, T_1) T_1 - f^2 \kappa_1 A_H N_1 \\
&\quad + f^2 D_{\nabla_{T_1}^1 T_1} H - 2f^2 h^1\left(\nabla_{\nabla_{T_1}^1 T_1} T_1, T_1\right). \tag{4.26}
\end{aligned}$$

Using equations (4.19) in (4.26), we obtain

$$\begin{aligned}
&(ff''' + f'f'') T_1 + (3ff'' + 2f'^2) \nabla_{T_1} T_1 + 4ff' \nabla_{T_1}^2 T_1 + f^2 R(\nabla_{T_1} T_1, T_1) T_1 + f^2 \nabla_{T_1}^3 T_1 \\
&= (ff''' + f'f'') T_1 + (3ff'' + 2f'^2) \nabla_{T_1}^1 T_1 + 4ff' \nabla_{T_1}^{1,2} T_1 + f^2 \nabla_{T_1}^{1,3} T_1 \\
&\quad + f^2 R^1(\nabla_{T_1}^1 T_1, T_1) T_1 - 4ff'A_H T_1 - f^2 \nabla_{T_1}^1 (A_H T_1) - f^2 A_{D_{T_1} H} T_1 \\
&\quad - \kappa_1 f^2 A_H N_1 + f^2 (\nabla_{\nabla_{T_1}^1 T_1} H^1)(T_1, T_1) - f^2 (\nabla_{T_1} h^1)(\nabla_{T_1}^1 T_1, T_1). \tag{4.27}
\end{aligned}$$

Since $A_H T_1 = \|H\|^2 T_1$, $A_H N_1 = \|H\|^2 N_1$, $\nabla_{T_1}^1 (A_H T_1) = \|H\|^2 \nabla_{T_1}^1 T_1$, $A_{D_{T_1} H} T_1 = 0$, $(\nabla_{\nabla_{T_1}^1 T_1} h^1)(T_1, T_1) = D_{\nabla_{T_1}^1 T_1} H - 2h^1\left(\nabla_{\nabla_{T_1}^1 T_1} T_1, T_1\right)$ and $(\nabla_{T_1} h^1)(\nabla_{T_1}^1 T_1, T_1) = 0$, therefore (4.27), gives us

$$\begin{aligned}
&(ff''' + f'f'') T_1 + (3ff'' + 2f'^2) \nabla_{T_1} T_1 + 4ff' \nabla_{T_1}^2 T_1 + f^2 R(\nabla_{T_1} T_1, T_1) T_1 + f^2 \nabla_{T_1}^3 T_1 \\
&= (ff''' + f'f'') T_1 + (3ff'' + 2f'^2) \nabla_{T_1}^1 T_1 + 4ff' \nabla_{T_1}^{1,2} T_1 + f^2 \nabla_{T_1}^{1,3} T_1 \\
&\quad + f^2 R^1(\nabla_{T_1}^1 T_1, T_1) T_1 - 4ff'\|H\|^2 T_1 - 2f^2\|H\|^2 \nabla_{T_1}^1 T_1 + f^2 D_{\nabla_{T_1}^1 T_1} H \\
&\quad - 2f^2 h^1(\nabla_{\nabla_{T_1}^1 T_1} T_1, T_1). \tag{4.28}
\end{aligned}$$

As γ_1 is a bi-f-harmonic curve on M_1 , so equation (4.28), implies that

$$(ff''' + f'f'') T_1 + (3ff'' + 2f'^2) \nabla_{T_1} T_1$$

$$\begin{aligned}
& + 4ff'\nabla_{T_1}^2 T_1 + f^2\nabla_{T_1}^3 T_1 + f^2\mathbf{R}(\nabla_{T_1} T_1, T_1) T_1 \\
& = -4ff'\|H\|^2 T_1 - 2f^2\|H\|^2 \nabla_{T_1}^1 T_1 + f^2 D_{\nabla_{T_1}^1 T_1} H - 2f^2 h^1 \left(\nabla_{\nabla_{T_1}^1 T_1}^1 T_1, T_1 \right). \quad (4.29)
\end{aligned}$$

The equations (4.24) and (4.29), together complete the proof. $\square$

4.1. Characterization of submanifold of a Riemannian manifold by bi-harmonic curves

In this subsection, we study the characterization of bi-1-harmonic curve (bi-harmonic curve). A bi-harmonic curve (bi-1-harmonic curve) is a special case of bi-f-harmonic curve, where $f = 1$. Taking $f = 1$ in (2.1), we have

$$\nabla_{T_1}^{1,3} T_1 + \mathbf{R}^1(\nabla_{T_1}^1 T_1, T_1) T_1 = 0.$$

Let $i: M_1 \rightarrow M$ is an isometric immersion from the Riemannian manifold M_1 to a Riemannian manifold M , and $\gamma(s)$ is a bi-harmonic curve on M , then

$$\nabla_{T_1}^3 T_1 + \mathbf{R}(\nabla_{T_1}^1 T_1, T_1) T_1 = 0. \quad (4.30)$$

The tangential part of (4.30) and from equation (4.11), we get

$$-3\kappa_1 \kappa_1' f^2 = \frac{3}{2} f^2 D_{T_1} \|h^1(T_1, T_1)\|^2. \quad (4.31)$$

Also from equation (4.16) and normal part of (4.30), we obtain

$$\begin{aligned}
& -4\kappa_1^2 h^1(T_1, T_1) + 4\kappa_1' h^1(T_1, N_1) + 4\kappa_1 \tau_1 h^1(T_1, B_1) + 3\kappa_1^2 h^1(N_1, N_1) \\
& + 4\kappa_1 (\nabla_{T_1} h^1)(T_1, N_1) - h^1(T_1, A_{h^1(T_1, T_1)} T_1) + (\nabla_{T_1}^2 h^1)(T_1, T_1) \\
& + \kappa_1 (\nabla_{N_1} h^1)(T_1, T_1) = 0. \quad (4.32)
\end{aligned}$$

Changing B_1 into $-B_1$ in equation (4.32) and then subtracting from equation (4.32), we get

$$h^1(T_1, B_1) = 0.$$

Theorem 7. *Let $i: M_1 \rightarrow M$ be an isometric immersion between Riemannian manifolds M_1 and M such that γ_1 is a curve with curvature κ_1 in M_1 and $\gamma(s) = i \circ \gamma_1(s)$ is a bi-f-harmonic curve with curvature κ in M . Then M_1 is an isotropic submanifold if and only if its curvature is constant.*

Proof. Let γ_1 is a curve with constant curvature in M_1 , then from (4.31), we have

$$D_{T_1} \|h^1(T_1, T_1)\|^2 = 0.$$

Thus $\|h^1(T_1, T_1)\|$ is constant, hence M_1 is an isotropic submanifold.

Conversely, let M_1 be an isotropic submanifold, then from equation (4.31), we get $\kappa_1 = \text{constant}$. $\square$

Theorem 8. *Let i be a bi-harmonic immersion from Riemannian manifold M_1 to a Riemannian manifold M and let M_1 be a totally umbilical submanifold of M . Then the mean curvature vector field H satisfies*

$$D_{T_1}^2 H = (\kappa_1^2 + \|H\|^2) H, \quad (4.33)$$

Conversely, Let $i: M_1 \mapsto M$ be an isometric immersion from Riemannian manifold M_1 to a Riemannian manifold M , then

- (1) *If M_1 be a totally geodesic, then i is a bi-harmonic immersion.*
- (2) *If M_1 be a totally umbilical, such that H satisfies the equation (4.33) and $D_{\nabla_{T_1}^1 T_1} H = 2\nabla_{T_1}^1 (\|H\|^2 T_1)$, then i is a bi-harmonic immersion.*

Proof. Let i be a bi-harmonic immersion between Riemannian manifold M_1 and M . Since M_1 is a totally umbilical submanifold, therefore equation (4.16), reduces to

$$\begin{aligned} 4\kappa_1 (\nabla_{T_1} h^1) (T_1, N_1) + 3\kappa_1^2 H + \kappa_1 (\nabla_{N_1} h^1) (T_1, T_1) \\ = 4\kappa_1^2 H + h^1 (T_1, A_H T_1) - (\nabla_{T_1}^2 h^1) (T_1, T_1). \end{aligned} \quad (4.34)$$

Replacing N_1 into $-N_1$ in equation (4.34) and then subtracting from equation (4.34), we get

$$4\kappa_1 (\nabla_{T_1} h^1) (T_1, N_1) + \kappa_1 (\nabla_{N_1} h^1) (T_1, T_1) = 0. \quad (4.35)$$

Substituting equation (4.35) in equation (4.34), we have

$$3\kappa_1^2 H = 4\kappa_1^2 H + \langle A_H T_1, T_1 \rangle H - (\nabla_{T_1}^2 h^1) (T_1, T_1). \quad (4.36)$$

Using the equations (2.4) and (4.13) in equation (4.36), we have

$$D_{T_1}^2 H = (\kappa_1^2 + \|H\|^2) H.$$

Conversely, (1) let $i: M_1 \mapsto M$ be an isometric immersion such that M_1 is a totally geodesic submanifold of M and γ_1 be a bi-harmonic curve with curvature κ_1 on M_1 . Taking $f = 1$ in (4.23), we have

$$\begin{aligned} \nabla_{T_1}^3 T_1 + R(\nabla_{T_1} T_1, T_1) T_1 &= \nabla_{T_1}^{1,3} T_1 + h^1(T_1, \nabla_{T_1}^{1,2} T_1) - A_{h^1(T_1, \nabla_{T_1}^1 T_1)} T_1 \\ &+ D_{T_1} h^1(T_1, \nabla_{T_1}^1 T_1) - \nabla_{T_1}^1 (A_{h^1(T_1, T_1)} T_1) - h^1(T_1, A_{h^1(T_1, T_1)} T_1) - A_{D_{T_1} h^1(T_1, T_1)} T_1 \\ &+ D_{T_1}^2 h^1(T_1, T_1) + R^1(\nabla_{T_1}^1 T_1, T_1) T_1 - A_{h^1(T_1, T_1)} \nabla_{T_1}^1 T_1 + A_{h^1(\nabla_{T_1}^1 T_1, T_1)} T_1 \\ &+ (\nabla_{\nabla_{T_1}^1 T_1} h^1) (T_1, T_1) - (\nabla_{T_1} h^1) (\nabla_{T_1}^1 T_1, T_1). \end{aligned} \quad (4.37)$$

Using the fact that M_1 is totally geodesic and the Frenet Serret formulae in equation (4.37), we get

$$\nabla_{T_1}^3 T_1 + R(\nabla_{T_1} T_1, T_1) T_1 = \nabla_{T_1}^{1,3} T_1 + R^1(\nabla_{T_1}^1 T_1, T_1) T_1. \quad (4.38)$$

Since γ_1 is a bi-harmonic curve on M_1 , so from equation (4.38), we can say that $\gamma(s) = i \circ \gamma_1$ is bi-harmonic curve on M .

Now for (2), let M_1 be a totally umbilical submanifold and satisfies the equation (4.33), then

$$\nabla_{T_1}^3 T_1 = \nabla_{T_1}^{1,3} T_1 - \nabla_{T_1}^1 (||H||^2 T_1) - (\kappa_1^2 + ||H||^2) H + D_{\nabla_{T_1}^1 T_1}^2 H, \quad (4.39)$$

and

$$R(\nabla_{T_1} T_1, T_1) T_1 = R^1(\nabla_{T_1}^1 T_1, T_1) T_1 - \nabla_{T_1}^1 (||H||^2 T_1) + D_{\nabla_{T_1}^1 T_1} H. \quad (4.40)$$

Adding the equations (4.39) and (4.40), we obtain

$$\begin{aligned} \nabla_{T_1}^3 T_1 + R(\nabla_{T_1} T_1, T_1) T_1 &= \nabla_{T_1}^{1,3} T_1 - \nabla_{T_1}^1 (2||H||^2 T_1) - (\kappa_1^2 + ||H||^2) H \\ &\quad + D_{\nabla_{T_1}^1 T_1}^2 H + R^1(\nabla_{T_1}^1 T_1, T_1) T_1 + D_{\nabla_{T_1}^1 T_1} H. \end{aligned} \quad (4.41)$$

Using the fact that γ_1 is a bi-harmonic curve on M_1 , equation (4.41) can be written as

$$\nabla_{T_1}^3 T_1 + R(\nabla_{T_1} T_1, T_1) T_1 = -\nabla_{T_1}^1 (2||H||^2 T_1) + D_{\nabla_{T_1}^1 T_1} H.$$

Thus from above equation, we have

$$\nabla_{T_1}^3 T_1 + (\nabla_{T_1} T_1, T_1) T_1 = 0,$$

if and only if $D_{\nabla_{T_1}^1 T_1} H = 2\nabla_{T_1}^1 (||H||^2 T_1)$, hence i is a bi-harmonic immersion. $\square$

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SOME NEW FILTERS AND THEIR PROPERTIES ON BL-ALGEBRAS

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Abstract. In this paper we introduce the notion of a DC-prime filter in BL-algebras and obtain some theorems in order to determine the relationships between this notion and some types of filters in a BL-algebra. By some examples, we show that each of these concepts is different from the other. Also, using the notion of DC-prime filters, a new class of BL-algebras are defined and studied.

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1. INTRODUCTION

BL-algebras are the algebraic structures for Hájek's basic logic. His motivations for introducing BL-algebras were of two kinds. The first one was to provide an algebraic counterpart of a propositional logic, called Basic Logic, which embodies a fragment common to some of the most important many-valued logics, namely Łukasiewicz Logic, Gödel Logic and Product Logic. This Basic Logic (BL for short) is proposed as "the most general" many-valued logic with truth values in $[0, 1]$ and BL-algebras are the corresponding Lindenbaum-Tarski algebras. Hájek's second motivation was to provide an algebraic mean for the study of continuous t -norms (or triangular norms) on $[0, 1]$. BL-algebras arise as Lindenbaum algebras from certain logical axioms in a similar manner that Boolean algebras or MV-algebras do from classical logic or Łukasiewicz logic, respectively [12]. Today this many valued logic has been developed into a fuzzy logic, which meets together with the motivations from the beginning of the twentieth century in one theory that connects quantum mechanics, mathematical logic, probability theory, computer science, algebra, soft computing, and many other important aspects of our modern world. Filters theory plays an important role in studying these logical algebras and ordered semigroups.

From logical point of view, various filters correspond to various sets of provable formulas. Based on these results, some types of filters on BL-algebras have been widely studied and some important results have been obtained. This was the motivation for the researchers of this study to introduce some new filters in BL-algebras.

The structure of the paper is as follows: Section 2 is a recall of some definitions and results about BL-algebras that are used in the paper. In Section 3, a new filter (DC-prime filter) in BL-algebras and a new class of BL-algebras (DCP-BL-algebra) are introduced and some of their basic properties are provided. Section 4 is the main section of the paper. In this section, many new properties of DC-prime filters in BL-algebras, are obtained. Also, the relationships between this new filter and other types of filters in BL-algebras are investigated. In Section 5, many different characterizations and many important properties of DC-prime filters in BL-algebras, are proved. Also, some new properties for prime filters, maximal filters and primary filters in BL-algebras were established.

2. PRELIMINARIES

Definition 1 ([9, Definition 2.3.3]). A BL-algebra is an algebra $(A, \wedge, \vee, *, \rightarrow, 0, 1)$ with four binary operations $\wedge, \vee, *, \rightarrow$ and two constants $0, 1$ such that:

- (BL₁) $(A, \wedge, \vee, 0, 1)$ is a bounded lattice $L(A)$;
- (BL₂) $(A, *, 1)$ is a commutative monoid;
- (BL₃) $*$ and $\rightarrow$ form an adjoint pair i.e. $c \leq a \rightarrow b$ if and only if $a * c \leq b$, for all $a, b, c \in A$;
- (BL₄) $a \wedge b = a * (a \rightarrow b)$;
- (BL₅) $(a \rightarrow b) \vee (b \rightarrow a) = 1$.

For $a \in A$ and a natural number n we define $a^n = a * a * \cdots * a$, (n -times), and $a^- = a \rightarrow 0$.

Theorem 1. *In any BL-algebra A , the following properties hold for all $x, y, z \in A$:*

- (1) $x \leq y$, if and only if $x \rightarrow y = 1$, [9, Lemma 2.2.8-2.2.10];
- (2) $x \rightarrow (y \rightarrow z) = (x * y) \rightarrow z = y \rightarrow (x \rightarrow z)$, [9, Lemma 2.2.8-2.2.10];
- (3) if $x \leq y$, then $y \rightarrow z \leq x \rightarrow z$ and $z \rightarrow x \leq z \rightarrow y$, [9, Lemma 2.2.8-2.2.10];
- (4) $x * y \leq x, y$, hence $x * y \leq x \wedge y$ and $x * 0 = 0$, [9, Lemma 2.2.8-2.2.10];
- (5) $x \leq y$ implies $x * z \leq y * z$, [9, Lemma 2.2.8-2.2.10];
- (6) $1 \rightarrow x = x$, $x \rightarrow x = 1$, $x \leq y \rightarrow x$, $x \rightarrow 1 = 1$, $0 \rightarrow x = 1$ and $x * x^- = 0$, [9, Lemma 2.2.8-2.2.10];
- (7) $x * y = 0$, if and only if $x \leq y^-$ and $x \leq y$ implies $y^- \leq x^-$, [20, Theorem 13,14];
- (8) $x \leq x^{--}$, $1^- = 0$, $0^- = 1$, $x^{---} = x^-$ and $x^{--} \leq x^- \rightarrow x$, [20, Theorem 13,14];
- (9) if $x^{--} \leq x^{---} \rightarrow x$, then $x^{--} = x$, [20, Theorem 13,14];
- (10) $x \vee y = ((x \rightarrow y) \rightarrow y) \wedge ((y \rightarrow x) \rightarrow x)$, [9, Lemma 2.2.8-2.2.10];

- (11) $x * (y \vee z) = (x * y) \vee (x * z)$, [6];
- (12) $(x * y)^- = x \rightarrow y^-$, [6];
- (13) $(x \wedge y) * (y \wedge z) \leq (x \wedge z)$, [6];
- (14) $x \vee (y * z) \geq (x \vee y) * (x \vee z)$ and we have $x^m \vee y^n \geq (x \vee y)^{mn}$, for any $m, n \in \mathbb{N}$, [7];
- (15) $(y \wedge z) \rightarrow x = (y \rightarrow x) \vee (z \rightarrow x)$, [6].

Definition 2. A BL-algebra A is called:

- (1) an MV-algebra, if for all $x \in A$, $x^{--} = x$, [9, Definition 3.2.2];
- (2) a Gödel algebra, if for all $x \in A$, $x^2 = x$, [9, Definition 4.2.12];
- (3) a Boolean algebra, if for all $x \in A$, $x^2 = x$ and $x^{--} = x$, [9, Definition 1.5.9];
- (4) a special BL-algebra, if for all $0 \neq a, b \in A$, $(a \rightarrow b)^- = (b \rightarrow a)^-$, [13, Definition 3.1];
- (5) an integral BL-algebra, if for any $x, y \in A$, $x * y = 0$, implies $x = 0$ or $y = 0$, [5, Definition 3.1];
- (6) a semi-integral BL-algebra, if for any $x, y \in A$, $x * y = 0$, implies $x = 0$ or $y^n = 0$, for some $n \in \mathbb{N}$, [14, Definition 4].

From now, it is assumed that a BL-algebra A is non-trivial i.e. A has at least two members. In order to simplify the notation, a BL-algebra $(A, \wedge, \vee, *, \rightarrow, 0, 1)$ will be referred to its support set, A .

Recall that $D_s(A) = \{x \in A : x^- = 0\}$ is the set of the dense elements of a BL-algebra A . Clearly $D_s(A)$ is a proper filter of A .

Recall that if $x^{--} = 1$, for $x \in A$, then we say that x is an almost top element of A . Also, $N(A) = \{x \in A : x \text{ is an almost top element of } A\}$, [3, Definition 4.4].

Definition 3 ([9, Definition 2.3.13]). Let F be a non-empty subset of a BL-algebra A and $a, b \in A$. Then the following conditions are equivalent:

- (i) F is a proper filter of A .
- (ii) $a, b \in F$ imply $a * b \in F$ and if $a \in F$ and $a \leq b$ then $b \in F$.
- (iii) $1 \in F$, and if $a, a \rightarrow b \in F$ imply $b \in F$.

Definition 4 ([19, Page 82]). Let A be a BL-algebra and X be a non-empty subset of A . Then $\langle X \rangle = \cap \{F \subseteq A : X \subseteq F, F \text{ is a filter of } A\}$. Clearly, $\langle X \rangle$ is the smallest filter of A , containing X .

Theorem 2 ([19, Proposition 3.15, Remark 3.16, Lemma 3.18]). Let $x \in A$, $X \subseteq A$ and F, G be filters of a BL-algebra A . Recall that

- (1) $\langle x \rangle = \{a \in A : a \geq x^n, \text{ for some } n \in \mathbb{N}\}$;
- (2) $\langle X \rangle = \{a \in A : a \geq x_1 * x_2 * \dots * x_n, \text{ for some } x_1, x_2, \dots, x_n \in X\}$;
- (3) $F \vee \langle x \rangle = \langle F \cup \langle x \rangle \rangle = \{a \in A : a \geq f * x^n, \text{ for some } f \in F \text{ and } n \in \mathbb{N}\}$;
- (4) $\langle F \cup G \rangle = \{a \in A : a \geq f * g, \text{ for some } f \in F \text{ and } g \in G\}$;
- (5) $F \vee G := \langle F \cup G \rangle$ and $F \wedge G := F \cap G$;
- (6) $F \rightarrow G = \{a \in A : F \cap \langle a \rangle \subseteq G\}$.

Definition 5. A non-empty subset F of a BL-algebra A is called:

- (1) a prime filter of A , if F is a proper filter and for all $x, y \in A$, $x \vee y \in F$ implies $x \in F$ or $y \in F$, [9, Definition 2.3.13];
- (2) a maximal filter of A , if F is a proper filter and for all $x \in A$, if $x \notin F$, there exists $n \in \mathbb{N}$ such that $(x^n)^- \in F$, [9, Theorem 3.43];
- (3) a Boolean filter of A , if F is a filter and for all $x \in A$, $x \vee x^- \in F$, [9, Definition 2.9];
- (4) a primary filter of A , if F is a proper filter and for all $x, y \in A$, if $(x * y)^- \in F$ implies $(x^n)^- \in F$ or $(y^n)^- \in F$, for some $n \in \mathbb{N}$, [21, Definition 1];
- (5) an implicative filter of A , if $1 \in F$ and for all $x, y, z \in A$, if $x \rightarrow (y \rightarrow z) \in F$, $x \rightarrow y \in F$ imply $x \rightarrow z \in F$, [10, Definition 3.1];
- (6) a positive implicative filter of A , if $1 \in F$ and for all $x, y, z \in A$, if $x \rightarrow ((y \rightarrow z) \rightarrow y) \in F$, $x \in F$ imply $y \in F$, [10, Definition 3.9];
- (7) a fantastic filter of A , if $1 \in F$ and for all $x, y, z \in A$, if $z \rightarrow (y \rightarrow x) \in F$ and $z \in F$ imply $((x \rightarrow y) \rightarrow y) \rightarrow x \in F$, [10, Definition 4.1];
- (8) a normal filter of A , if F is a filter of A and if $1 \in F$ and for all $x, y, z \in A$, if $z \rightarrow ((y \rightarrow x) \rightarrow x) \in F$ and $z \in F$ imply $(x \rightarrow y) \rightarrow y \in F$, [2, Definition 9];
- (9) an obstinate filter of A , if F is a proper filter of A and for all $x, y, z \in A$, if $x, y \notin F$ imply $x \rightarrow y \in F$ and $y \rightarrow x \in F$, [4, Definition 3.1];
- (10) a semi maximal filter of A , if F is a filter of A and $\text{Rad}(F) = F$, [18, Definition 4.1];
- (11) a special filter of A , if F is a proper filter and $(x \rightarrow y)^- = (y \rightarrow x)^-$, for all $x, y \in F$, [13, Definition 4.1];
- (12) an easy filter of A , if F is a filter and for all $x, y, z \in A$, if $x^{--} \rightarrow (y \rightarrow z) \in F$ and $x^{--} \rightarrow y \in F$ imply $x^{--} \rightarrow z \in F$, [8, Definition 13];
- (13) an integral filter of A , if F is a proper filter and for all $x, y \in A$, if $(x * y)^- \in F$ imply $x^- \in F$ or $y^- \in F$, [5, Definition 4.1];
- (14) a semi-integral filter of A , if F is a proper filter and for all $x, y \in A$, if $(x * y)^- \in F$ then there exists $n \in \mathbb{N}$ such that $x^- \in F$ or $(y^n)^- \in F$, [14, Definition 5].

Definition 6 ([18, Definition 3.1]). If F is a proper filter of a BL-algebra A , the intersection of all maximal filters of L containing F is called the radical of F and it is denoted by $\text{Rad}(F)$ and $\text{Rad}(F) = \{a \in A : (a^n)^- \rightarrow a \in F, \text{ for all } n \in \mathbb{N}\}$.

Definition 7 ([3, Definition 3.1]). If F is a filter of a BL-algebra A , the set of double complemented elements of F , is defined by $D(F) = \{x \in A : x^{--} \in F\}$.

Definition 8 ([9, Lemma 2.3.14]). Let F be a proper filter of a BL-algebra A . The relation $\sim_F$ defined on a BL-algebra A , by $(x, y) \in \sim_F$ if and only if $x \rightarrow y \in F$ and $y \rightarrow x \in F$, is a congruence relation on A . The quotient algebra $A / \sim_F$ denoted by A / F becomes a BL-algebra in a natural way, with the operations induced from those of A . So, the order relation on A / F is given by $x / F \leq y / F$ if and only if $x \rightarrow y \in F$. Hence $x / F = 1 / F$ if and only if $x \in F$ and $x / F = 0 / F$ if and only if $x^- \in F$.

3. DC-PRIME FILTERS AND DCP-BL-ALGEBRAS

In this section, a new filter in BL-algebras and a new class of BL-algebras are introduced and also characterized.

Definition 9. A proper filter F of a BL-algebra A is called a "double complemented prime filter" or briefly "DC-prime filter" of A , if for any $x, y \in A$, $x \vee y \in F$ then $x^{--} \in F$ or $y^{--} \in F$.

A proper filter F of a BL-algebra A is a DC-prime filter if and only if for any $x, y \in A$, $x \vee y \in F$ implies $x \in D(F)$ or $y \in D(F)$.

Example 1. Let $A = \{0, a, b, c, d, 1\}$, where $0 < b < c, d < 1$ and $0 < a < c < 1$. Operations $*$ and $\rightarrow$ are defined as follows:

$*$	0	a	b	c	d	1	$\rightarrow$	0	a	b	c	d	1
0	0	0	0	0	0	0	0	1	1	1	1	1	1
a	0	a	0	a	0	a	a	d	1	d	1	d	1
b	0	0	b	b	b	b	b	a	a	1	1	1	1
c	0	a	b	c	b	c	c	0	a	d	1	d	1
d	0	0	b	b	d	d	d	a	a	c	c	1	1
1	0	a	b	c	d	1	1	0	a	b	c	d	1

Then $(A, \wedge, \vee, *, \rightarrow, 0, 1)$ is a BL-algebra ([16, Example 4.4]). Clearly $F = \{b, c, d, 1\}$ is a DC-prime filter of A and $G = \{c, 1\}$ is not a DC-prime filter (since $a \vee b = c \in G$ and $b^{--}, a^{--} \notin G$).

The following theorem is an important property of DC-prime filters in BL-algebras.

Theorem 3. Let F be a proper filter of a BL-algebra A . Then F is a DC-prime filter of A if and only if for all $x, y \in A$, $x^{--} \rightarrow y^{--} \in F$ or $y^{--} \rightarrow x^{--} \in F$.

Proof. Let F be a DC-prime filter of A and $x, y \in A$. So according to $(x \rightarrow y) \vee (y \rightarrow x) = 1 \in F$, then $x^{--} \rightarrow y^{--} \in F$ or $y^{--} \rightarrow x^{--} \in F$. Now assume that for all $x, y \in A$, $x^{--} \rightarrow y^{--} \in F$ or $y^{--} \rightarrow x^{--} \in F$ and for $x, y \in A$, $x \vee y \in F$. Then as $x \vee y \leq (x \rightarrow y) \rightarrow y$ and $x \vee y \leq (y \rightarrow x) \rightarrow x$, so $(x \rightarrow y) \rightarrow y \in F$ and $(y \rightarrow x) \rightarrow x \in F$. Hence $(x^{--} \rightarrow y^{--}) \rightarrow y^{--} \in F$ and $(y^{--} \rightarrow x^{--}) \rightarrow x^{--} \in F$. So according to hypothesis it follows that $y^{--} \in F$ or $x^{--} \in F$. Therefore F is a DC-prime filter of A . $\square$

We can deduce the following theorem according to Theorem 3:

Theorem 4 (Extension property for DC-prime filters). Let F be a DC-prime filter of a BL-algebra A and G be a proper filter of A containing F . Then G is a DC-prime filter of A .

Remark 1. Let F be a filter of a BL-algebra A . It well known that $D(F)$ is a proper filter if and only if F is a proper filter if and only if $\text{Rad}(F)$ is a proper filter.

According to Theorem 4, we have the following corollary.

Corollary 1. *Let F be a filter of a BL-algebra A . Then the following conclusions hold:*

- (i) $\{1\}$ is a DC-prime filter of A if and only if all proper filters of A are DC-prime.
- (ii) F is a DC-prime filter of A if and only if $D(F)$ is a DC-prime filter of A .
- (iii) If F is a DC-prime filter, then $\text{Rad}(F)$ is a DC-prime filter of A .

Definition 10. A BL-algebra A is called a DCP-BL-algebra if for any $x, y \in A$, $x \vee y = 1$ implies $x^{--} = 1$ or $y^{--} = 1$.

Example 2.

- (i) Let $A = \{0, a, b, 1\}$, where $0 < a < b < 1$. Operations $*$ and $\rightarrow$ are defined as follows:

$*$	0	a	b	1	$\rightarrow$	0	a	b	1
0	0	0	0	0	0	1	1	1	1
a	0	0	a	a	a	a	1	1	1
b	0	a	b	b	b	0	a	1	1
1	0	a	b	1	1	0	a	b	1

Then $(A, \wedge, \vee, *, \rightarrow, 0, 1)$ is a BL-algebra ([17, Example 3.4]) which is a DCP-BL-algebra.

- (ii) Let $A = \mathbb{Z} \cup \{-\infty\} \cup \{0, a, b, 1\}$, where $\mathbb{Z}$ is the set of negative integer numbers and $-\infty < \dots < -2 < -1 < 0 < a, b < 1$. Operations $*$ and $\rightarrow$ are defined as follows:

$*$	$-\infty$	$\dots$	-3	-2	-1	0	a	b	1
$-\infty$	$-\infty$	$\dots$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$
$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
-3	$-\infty$	$\dots$	-6	-5	-4	-3	-3	-3	-3
-2	$-\infty$	$\dots$	-5	-4	-3	-2	-2	-2	-2
-1	$-\infty$	$\dots$	-4	-3	-2	-1	-1	-1	-1
0	$-\infty$	$\dots$	-3	-2	-1	0	0	0	0
a	$-\infty$	$\dots$	-3	-2	-1	0	a	0	a
b	$-\infty$	$\dots$	-3	-2	-1	0	0	b	b
1	$-\infty$	$\dots$	-3	-2	-1	0	a	b	1

$\rightarrow$	$-\infty$	$\dots$	-3	-2	-1	0	a	b	1
$-\infty$	1	$\dots$	1	1	1	1	1	1	1
$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
-3	$-\infty$	$\dots$	1	1	1	1	1	1	1
-2	$-\infty$	$\dots$	-1	1	1	1	1	1	1
-1	$-\infty$	$\dots$	-1	1	1	1	1	1	1
0	$-\infty$	$\dots$	-3	-2	-1	1	1	1	1
a	$-\infty$	$\dots$	-3	-2	-1	b	1	b	1
b	$-\infty$	$\dots$	-3	-2	-1	a	a	1	1
1	$-\infty$	$\dots$	-3	-2	-1	0	a	b	1

Then $(A, \wedge, \vee, *, \rightarrow, -\infty, 1)$ is a BL-algebra ([11, Page 185]). All filters of A are $F_1 = \{1\}$, $F_2 = \{a, 1\}$, $F_3 = \{b, 1\}$, $F_4 = \{0, a, b, 1\}$ and $F_5 = \{\dots, -3, -2, -1, 0, a, b, 1\} - \{-\infty\}$. It is clear that F_1, F_2, F_3, F_4 and F_5 are DC-prime filters of A . So A is an infinite DCP-BL-algebra.

- (iii) BL-algebra A in Example 1 is not a DCP-BL-algebra, since $a \vee d = 1$ while $a^{--} \neq 1$ and $d^{--} \neq 1$.

Remark 2. It can be easily proved that a BL-algebra A is a DCP-BL-algebra, if and only if for any $x, y \in A$, $x \vee y = 1$ implies $x \in D(\{1\})$ or $y \in D(\{1\})$.

Theorem 5. The following conditions are equivalent:

- (i) A is a DCP-BL-algebra.
- (ii) $\{1\}$ is a DC-prime filter of a BL-algebra A .
- (iii) Every proper filter of a BL-algebra A is a DC-prime filter.

Proof.

- (i) $\Leftrightarrow$ (ii): According to Definition 10, the proof is clear.
- (ii) $\Leftrightarrow$ (iii): According to Theorem 4, the proof is easy.

□

Remark 3. Let F and G be proper filters of a BL-algebra A such that $F \subseteq G$. Then G is a DC-prime filter of A if and only if G/F is a DC-prime filter of a BL-algebra A/F .

Corollary 2. Let F be a proper filter of a BL-algebra A . Then the following conditions are equivalent:

- (i) A/F is a DCP-BL-algebra.
- (ii) F is a DC-prime filter of A .
- (iii) Any proper filter of A/F is a DC-prime filter.

Proof.

- (i) $\Leftrightarrow$ (iii): According to Theorem 5, the proof is clear.

- (ii) $\Rightarrow$ (iii): Let F be a DC-prime filter of A . We need to show that $\{1/F\}$ is a DC-prime filter of A/F . Assume that $x/F \vee y/F \in \{1/F\}$, for $x/F, y/F \in A/F$. Hence $x \vee y \in F$ so based on hypothesis, $x^{--} \in F$ or $y^{--} \in F$. Thus $x^{--}/F \in \{1/F\}$ or $y^{--}/F \in \{1/F\}$ and so $\{1/F\}$ is a DC-prime filter of A/F . Therefore according to Theorem 5, any proper filter of A/F is a DC-prime filter.
- (iii) $\Rightarrow$ (ii): Let any proper filter of A/F be a DC-prime filter so $\{1/F\}$ is a DC-prime filter. Also let $x \vee y \in F$, for $x, y \in A$. Hence $x/F \vee y/F \in \{1/F\}$. Thus $x^{--}/F \in \{1/F\}$ or $y^{--}/F \in \{1/F\}$ and so $x^{--} \in F$ or $y^{--} \in F$. Therefore F is a DC-prime filter of A . □

For a BL-algebra A , take $A^{--} = \{x^{--} : x \in A\}$.

Proposition 1. *The following conditions are equivalent:*

- (i) A is a DCP-BL-algebra.
- (ii) A^{--} is linearly ordered.
- (iii) $a \leq b^{--}$ or $b \leq a^{--}$, for all $a, b \in A$.

Proof.

- (i) $\Leftrightarrow$ (ii): According to Theorem 5:

A is a DCP-BL-algebra $\Leftrightarrow \{1\}$ is a DC-prime filter of A ,

$$\begin{aligned} &\Leftrightarrow a^{--} \rightarrow b^{--} = 1 \text{ or } b^{--} \rightarrow a^{--} = 1, \text{ for all } a, b \in A, \\ &\Leftrightarrow a^{--} \leq b^{--} \text{ or } b^{--} \leq a^{--}, \text{ for all } a, b \in A, \\ &\Leftrightarrow A^{--} \text{ is linearly ordered.} \end{aligned}$$

- (ii) $\Rightarrow$ (iii): Let A^{--} be linearly ordered. So for all $a, b \in A$, $a^{--} \leq b^{--}$ or $b^{--} \leq a^{--}$. Hence for all $a, b \in A$, $a \leq a^{--} \leq b^{--}$ or $b \leq b^{--} \leq a^{--}$.
- (iii) $\Rightarrow$ (ii): Let $a \leq b^{--}$ or $b \leq a^{--}$, for all $a, b \in A$. Hence $a^{--} \leq (b^{--})^{--} = b^{--}$ or $b^{--} \leq (a^{--})^{--} = a^{--}$, for all $a, b \in A$. □

A node of a poset A is an element of A which is comparable with every element of A . The set of all node elements of A is denoted by $\text{nod}(A)$. A filter F of a BL-algebra A is called a nodal filter if it is a node of $F(A)$, [1, Definition 3.8]. We denote $DF(A) := \{D(F) : F \in F(A)\}$.

Lemma 1. *For any BL-algebra A , $DF(A)$ is a poset.*

Theorem 6. *A is a DCP-BL-algebra if and only if $\text{nod}(DF(A)) = DF(A)$.*

Proof. Let A be a DCP-BL-algebra and $DF(A) \not\subseteq \text{nod}(DF(A))$. Hence there exists $H \in DF(A)$ such that $H \notin \text{nod}(DF(A))$. So there exists $G \in DF(A)$ such that $H \not\leq G$ and $G \not\leq H$. As $G, H \in DF(A)$, there exist $G', H' \in F(A)$ such that $H = D(H')$ and

$G = D(G')$. Also there exists $g \in G - H$ and $h \in H - G$. According to Proposition 1, A^{--} is linearly ordered and so $g^{--} < h^{--}$ or $h^{--} < g^{--}$, (if $g^{--} = h^{--}$, then $h^{--} \in G$ and so $h \in D(G) = D(D(G')) = D(G') = G$, which is a contradiction). So $h \in D(H')$, hence $h^{--} \in H'$. Also $g \in D(G')$, hence $g^{--} \in G'$. On the other hand since $g \notin H = D(H')$ and $h \notin G = D(G')$, thus $h^{--} \notin G'$ and $g^{--} \notin H'$, which is a contradiction with $g^{--} < h^{--}$ or $h^{--} < g^{--}$. Therefore $DF(A) \subseteq \text{nod}(DF(A))$, i.e. $\text{nod}(DF(A)) = DF(A)$. Now assume that $\text{nod}(DF(A)) = DF(A)$ and $a \vee b = 1$, for $a, b \in A$. Then $D(\langle a \rangle) \subseteq D(\langle b \rangle)$ or $D(\langle b \rangle) \subseteq D(\langle a \rangle)$. As $a \in \langle a \rangle \subseteq D(\langle a \rangle)$ and $b \in \langle b \rangle \subseteq D(\langle b \rangle)$, so $a \in D(\langle b \rangle)$ or $b \in D(\langle a \rangle)$. Thus $a^{--} \in \langle b \rangle$ or $b^{--} \in \langle a \rangle$. Hence $a^{--} \geq b^n$ or $b^{--} \geq a^m$, for some $n, m \in \mathbb{N}$. So $a^{--} \vee b^n = a^{--}$ or $b^{--} \vee a^m = b^{--}$, for some $n, m \in \mathbb{N}$. Since $a \vee b = 1$, then $a^n \vee b^n = 1$ and $a^m \vee b^m = 1$ for all $m, n \in \mathbb{N}$. On the other hand

$$\begin{aligned} a^{--} &= a^{--} \vee b^n \geq a \vee b^n \geq a^n \vee b^n = 1 \text{ and} \\ b^{--} &= b^{--} \vee a^m \geq a^m \vee b \geq a^m \vee b^m = 1. \end{aligned}$$

Therefore $a^{--} = 1$ or $b^{--} = 1$ and so $\{1\}$ is a DC-prime filter. Hence based on Theorem 5, A is a DCP-BL-algebra. $\square$

Corollary 3. *Let A be a BL-algebra and $\text{nod}(F(A)) = F(A)$. Then A is a DCP-BL-algebra.*

Proposition 2. *Let A be a BL-algebra and for any $0 \neq a \in A$, $a^- = 0$. Then*

- (i) *A is a DCP-BL-algebra.*
- (ii) *for any proper filter F of A , A/F is a DCP-BL-algebra.*

Proof.

(i): Let for any $0 \neq a \in A$, $a^- = 0$ and $x \vee y = 1$, for $x, y \in A$. Then

- If $x = 0$ then $y = 1$, so $y^{--} = 1$.
- If $x \neq 0$ then $x^- = 0$. So $x^{--} = 1$.

Therefore A is a DCP-BL-algebra.

(ii): Let $0/F \neq a/F \in A/F$. Then $a \neq 0$ and so according to hypothesis $a^- = 0$. Hence $a^-/F = 0/F$. Therefore according to Part (i), A/F is a DCP-BL-algebra. $\square$

Remark 4. BL-algebra A in Example 2(i) is a DCP-BL-algebra while $a^- \neq 0$. Also $F = \{b, 1\}$ is a DC-prime filter hence A/F is a DCP-BL-algebra, while $a^- \neq 0$. Therefore the converse of Parts (i) and (ii) of Proposition 2 are not true in general.

The following corollary is a straightforward consequence of Proposition 2(i).

Corollary 4. *Let A be a BL-algebra. Then the following conclusions hold:*

- (i) *If for any $0 \neq a, b \in A$, $(a \rightarrow b)^- = (b \rightarrow a)^-$, then A is a DCP-BL-algebra.*
- (ii) *If for any $0 \neq a, b \in A$, $a \rightarrow b^{--} = b \rightarrow a^{--}$, then A is a DCP-BL-algebra.*

- (iii) If A is a special BL-algebra, then A is a DCP-BL-algebra.
 (iv) If $A = D_s(A) \cup \{0\}$ then A is a DCP-BL-algebra.

In the following example it is shown that the converse of the parts of Corollary 4 is not true in general.

Example 3. Let $A = \{0, a, b, c, d, 1\}$ where $0 < d < c < a, b < 1$. Operations $*$ and $\rightarrow$ are defined as follows:

$*$	0	a	b	c	d	1	$\rightarrow$	0	a	b	c	d	1
0	0	0	0	0	0	0	0	1	1	1	1	1	1
a	0	a	c	c	d	a	a	0	1	b	b	d	1
b	0	c	b	c	d	b	b	0	a	1	a	d	1
c	0	c	c	c	d	c	c	0	1	1	1	d	1
d	0	d	d	d	0	d	d	d	1	1	1	1	1
1	0	a	b	c	d	1	1	0	a	b	c	d	1

Then $(A, \wedge, \vee, *, \rightarrow, 0, 1)$ is a BL-algebra ([14, Example 1]), and $F_1 = \{1\}, F_2 = \{a, 1\}, F_3 = \{b, 1\}, F_4 = \{a, b, c, 1\}$ are all filters of A . Clearly A is a DCP-BL-algebra, while $(b \rightarrow d)^- \neq (d \rightarrow b)^-, b \rightarrow d^{--} \neq d \rightarrow b^{--}$ and $d^- \neq 0$. Also, A is not a special BL-algebra and $A \neq D_s(A) \cup \{0\}$.

Lemma 2. Let A be a Boolean algebra and DCP-BL-algebra. Then $A = D_s(A) \cup \{0\}$.

Proof. Let A be a DCP-BL-algebra and $0 \neq x \in A$. Then all filters of A are DC-prime. As A is a Boolean algebra, $x \vee x^- = 1$, so $x^{--} = 1$ or $x^- = 1$. As $x \neq 0$ so $x^- \neq 1$, hence $x^{--} = 1$, i.e. $x \in D_s(A)$. $\square$

Lemma 3. Let A be a Boolean algebra. Then A is a DCP-BL-algebra if and only if $A = \{0, 1\}$.

Proof. Let A be a Boolean algebra and DCP-BL-algebra. Then according to hypothesis $x \vee x^- = 1$, for any $x \in A$. Hence $x^{--} = 1$ or $x^- = 1$, since A is a DCP-BL-algebra. So $x = 1$ or $x^- = 1$, i.e. $x = 1$ or $x = 0$. Therefore $A = \{0, 1\}$. The converse is clear. $\square$

4. THE RELATIONSHIPS BETWEEN DC-PRIME FILTERS AND OTHER TYPES OF FILTERS IN BL-ALGEBRAS

What follows is a discussion of the relations between DC-prime filters and some other filters in BL-algebras.

Theorem 7. Every prime filter in BL-algebras, is a DC-prime filter.

Proof. Let F be a prime filter of a BL-algebra A and $a, b \in A$. So $a \rightarrow b \in F$ or $b \rightarrow a \in F$. As $a \rightarrow b \leq (a \rightarrow b)^{--}$ and $b \rightarrow a \leq (b \rightarrow a)^{--}$, then $(a \rightarrow b)^{--} \in F$ or

$(b \rightarrow a)^{--} \in F$, for all $a, b \in A$. Therefore according to Theorem 3 F is a DC-prime filter of A . $\square$

The following example is shown that the converse of Theorem 7 does not correct in general.

Example 4. Consider BL-algebra A in Example 3. Clearly $F = \{1\}$ is a DC-prime filter while is not a prime filter, since $a \rightarrow b \notin F$ and $b \rightarrow a \notin F$.

For a BL-algebra A , $MV(A) = \{x \in A : x^{--} = x\}$.

According to Theorems 7 and 5:

Corollary 5.

- (i) If A is a linearly ordered BL-algebra, then A is a DCP-BL-algebra.
- (ii) If A is a DCP-BL-algebra and $MV(A) = A$ then A is a linearly ordered BL-algebra.

Proof.

- (i): Let A be a linearly ordered BL-algebra. Then all proper filters of A are prime filter. Hence according to Theorem 7, all proper filters of A are DC-prime. Therefore based on Theorem 5, A is a DCP-BL-algebra.
- (ii): Let A be a DCP-BL-algebra and $MV(A) = A$. Then all proper filters of A are DC-prime filter and $x^{--} = x$, for $x \in A$. Therefore based on the definition of DC-prime filters, all proper filters of A are prime filter and so A is a linearly ordered BL-algebra. $\square$

Remark 5. The condition " $MV(A) = A$ " is necessary in Corollary 5(ii) and the converse of Corollary 5(i) is not true in general. Consider Example 3. It is easy to check that A is a DCP-BL-algebra and $a^{--} \neq a$, while A is not a linearly ordered.

Corollary 6.

- (i) Every maximal filter in BL-algebras, is a DC-prime filter.
- (ii) Every obstinate filter in BL-algebras, is a DC-prime filter.

In the following example it is shown that the converse of Corollary 6 is not correct in general.

Example 5. Let $A = \{0, a, b, 1\}$, where $0 < a < b < 1$. Operations $*$ and $\rightarrow$ are defined as follows:

$*$	0	a	b	1	$\rightarrow$	0	a	b	1
0	0	0	0	0	0	1	1	1	1
a	0	a	a	a	a	0	1	1	1
b	0	a	b	b	b	0	a	1	1
1	0	a	b	1	1	0	a	b	1

Then $(A, \wedge, \vee, *, \rightarrow, 0, 1)$ is a BL-algebra ([17, Example 5.2]). Clearly, $F = \{b, 1\}$ is a DC-prime filter while is not a maximal filter nor an obstinate filter of A .

Theorem 8. *Let F be a proper filter of a BL-algebra A . Then the following conditions are equivalent:*

- (i) F is a DC-prime filter and Boolean filter of A .
- (ii) F is a maximal filter of A .
- (iii) F is an obstinate filter of A .

Proof.

- (i) $\Rightarrow$ (ii): Let F be a DC-prime filter and Boolean filter of A and $x \notin F$. As F is a Boolean filter, $x \vee x^- \in F$. Hence $x^{--} \in F$ or $x^- \in F$, since F is a DC-prime filter. If $x^{--} \in F$, since F is a Boolean filter, then $x \in F$, which is a contradiction. Therefore $x^- \in F$, i.e. F is a maximal filter of A .
- (ii) $\Rightarrow$ (i): It is known that every maximal filter, in BL-algebras, is a Boolean filter. Hence according to Corollary 6(i), the proof is completed.
- (i) $\Rightarrow$ (iii): Let F be a DC-prime filter and Boolean filter of A . Then according to ((i) $\Leftrightarrow$ (ii)), F is a maximal filter of A . Hence based on [4, Theorem 4.4], the proof is completed.
- (iii) $\Rightarrow$ (i): Let F be an obstinate filter. Then based on [4, Theorem 4.1], F is a maximal filter. Then according to ((ii) $\Leftrightarrow$ (i)), the proof is completed. □

Remark 6. The condition "Boolean filter" in Theorem 8(i) is necessary. Consider Example 5, clearly $F = \{b, 1\}$ is a DC-prime filter while it is not a Boolean filter and nor a maximal filter.

A BL-algebra A is called degenerate if $0 = 1$ in A , otherwise A is non-degenerate.

The following lemma is a straightforward consequence of [20, Theorems 1,2,3] and Theorem 7.

Lemma 4. *Let A be a BL-algebra. Then the following conclusions hold:*

- (i) A non-degenerate BL-algebra contains a DC-prime filter.
- (ii) Any proper filter F of a non-degenerate BL-algebra A can be extended to a DC-prime filter.
- (iii) In a non-degenerate BL-algebra A , any proper filter can be extended to a maximal and DC-prime filter.

Theorem 9. *Let F be a filter of a BL-algebra A . The the following conditions are equivalent.*

- (i) F is a DC-prime filter of A .
- (ii) $D(F)$ is a prime filter of A .
- (iii) $D(F)$ is a DC-prime filter of A .
- (iv) $A/D(F)$ is a linearly ordered BL-algebra.

(v) $A/D(F)$ is a DCP-BL-algebra.

Proof.

(i) $\Rightarrow$ (ii): Let F be a DC-prime filter of A . Then $x^{--} \rightarrow y^{--} \in F$ or $y^{--} \rightarrow x^{--} \in F$, for any $x, y \in A$. Hence $x \rightarrow y \in D(F)$ or $y \rightarrow x \in D(F)$, for any $x, y \in A$. Therefore $D(F)$ is a prime filter of A .

(ii) $\Rightarrow$ (iii): It is proved in Theorem 7.

(iii) $\Rightarrow$ (i): Let $D(F)$ be a DC-prime filter of A . Then $x^{--} \rightarrow y^{--} \in D(F)$ or $y^{--} \rightarrow x^{--} \in D(F)$, for any $x, y \in A$. Hence $x^{--} \rightarrow y^{--} \in F$ or $y^{--} \rightarrow x^{--} \in F$, for any $x, y \in A$. Therefore F is a DC-prime filter of A .

(ii) $\Leftrightarrow$ (iv), (iii) $\Leftrightarrow$ (v): The proofs are obvious. $\square$

Corollary 7. Let F be a proper filter of a BL-algebra A . F is a DC-prime filter of A if and only if $\text{nod}(F(A/D(F))) = F(A/D(F))$.

Proof. Let F be a DC-prime filter. Then based on Theorem 9, $D(F)$ is a prime filter and so $A/D(F)$ is a linearly ordered BL-algebra. Hence according to [15, Lemma 3.8], $\text{nod}(F(A/D(F))) = F(A/D(F))$. Now let $\text{nod}(F(A/D(F))) = F(A/D(F))$ and $a \vee b \in F$, for $a, b \in A$. So $a \vee b \in D(F)$, hence $a/D(F) \vee b/D(F) = 1/D(F) \in \{1/D(F)\}$. Based on [15, Proposition 3.10], $\{1/D(F)\}$ is a prime filter of $A/D(F)$ and so based on Theorem 7, $\{1/D(F)\}$ is a DC-prime filter of $A/D(F)$. Hence $a^{--}/D(F) \in \{1/D(F)\}$ or $b^{--}/D(F) \in \{1/D(F)\}$. So $a^{--} \in D(F)$ or $b^{--} \in D(F)$. Then $a^{--} \in F$ or $b^{--} \in F$. Therefore F is a DC-prime filter of A . $\square$

Corollary 8. (i) If $D(F)$ is a maximal filter of a BL-algebra A , then F is a DC-prime filter of A .

(ii) If F is a DC-prime filter of a BL-algebra A , then $D(F)$ is a primary filter of A .

(iii) If F is a DC-prime filter of a BL-algebra A , then $\text{Rad}(D(F))$ is a maximal filter of A .

Proof. According to Theorem 9 and [16, Theorem 3.3], the proofs are obvious. $\square$

Theorem 10. Every DC-prime filter in BL-algebras is a primary filter.

Proof. Let F be a DC-prime filter of a BL-algebra A and $(x * y)^- \in F$, for $x, y \in A$. Then according to hypothesis and Theorem 3, $x^{--} \rightarrow y^{--} \in F$ or $y^{--} \rightarrow x^{--} \in F$. Assume that $x^{--} \rightarrow y^{--} \in F$ so

$$\begin{aligned} (x^{--} \rightarrow y^{--}) * (x * y)^- &= (x^{--} \rightarrow y^{--}) * (y^{--} \rightarrow x^-) \\ &\leq x^{--} \rightarrow x^- = (x * x)^- \\ &= (x^2)^- \in F. \end{aligned}$$

Similarly, if $y^{--} \rightarrow x^{--} \in F$ and $(x * y)^- \in F$ then $(y^2)^- \in F$. Therefore F is a primary filter of A . $\square$

Open Problem 1. *Is every primary filter in BL-algebras, a DC-prime filter?*

Corollary 9. *Let F be a DC-prime filter of a BL-algebra A . Then the following hold:*

- (i) $\text{Rad}(F)$ is a primary filter of A .
- (ii) $\text{Rad}(F)$ is a maximal filter of A .
- (iii) $\text{Rad}(F)$ is a prime filter of A .

Proof.

- (i): Let F be a DC-prime filter of A . Then based on Theorem 10, F is a primary filter of A . As $F \subseteq \text{Rad}(F)$ so $\text{Rad}(F)$ is a primary filter of A .
- (ii): Let F be a DC-prime filter of A . According to Theorem 10, F is a primary filter of A . Hence based on [16, Theorem 3.3], $\text{Rad}(F)$ is a maximal filter.
- (iii): Based on Part (ii), the proof is obvious. □

Corollary 10. *Let F be a DC-prime filter of a BL-algebra A such that it satisfies in at least one of the following conditions. Then F is a prime filter of A .*

- (i) F is a normal filter of A .
- (ii) F is a fantastic filter of A .
- (iii) F is a Boolean filter of A .
- (iv) F is a positive implicative filter of A .
- (v) $D(F) \subseteq B(A)$.
- (vi) $\text{Rad}(F) \subseteq B(A)$.
- (vii) Any almost top element of A/F is trivial.

Proof.

- (i), (ii), (iii), (iv), (v): Let F be a DC-prime filter of A such that satisfies in at least one of conditions (i), (ii), (iii), (iv) and (v). Hence according to [3, Theorems 3.25, 3.33] and [3, Corollary 3.28], $D(F) = F$. Based on Theorem 9, $D(F)$ is a prime filter, therefore F is a prime filter of A .
- (vi): Let F be a DC-prime filter of A such that it satisfies in condition (vi). Hence according to [18, Theorem 4.6] F is a semi-maximal filter of A , i.e. $\text{Rad}(F) = F$. Based on Corollary 9, $\text{Rad}(F)$ is a prime filter. Therefore F is a prime filter of A .
- (vii): Let F be a DC-prime filter of A . Then according to Theorem 9, $D(F)$ is a prime filter. Hence to close the proof, we need to show that $F = D(F)$. Assume that $x \in D(F)$. Then $x^{--} \in F$ and so $x^{--}/F = 1/F$, i.e. x/F is an almost top element of A/F . Hence based on hypothesis $x/F = 1/F$, so $x \in F$. Then $F = D(F)$. Therefore F is a prime filter of A . □

The following examples show that the conditions in Corollary 10 are necessary.

Example 6.

- (i) Let $A = \{0, a, b, c, 1\}$, where $0 < c < a, b < 1$. Operations $*$ and $\rightarrow$ are defined as follows:

$*$	0	a	b	c	1	$\rightarrow$	0	a	b	c	1
0	0	0	0	0	0	0	1	1	1	1	1
a	0	a	c	c	a	a	0	1	b	b	1
b	0	c	b	c	b	b	0	a	1	a	1
c	0	c	c	c	c	c	0	1	1	1	1
1	0	a	b	c	1	1	0	a	b	c	1

Then $(A, \wedge, \vee, *, \rightarrow, 0, 1)$ is a BL-algebra ([16, Example 3.2]). $F = \{1\}$ is a DC-prime filter while is not a fantastic filter, nor a normal filter and nor a prime filter.

- (ii) Consider Example 3. It is easy to check that $F = \{1\}$ is a DC-prime filter while is not a prime filter (since $a \rightarrow b \notin F$ and $b \rightarrow a \notin F$) and is not a Boolean filter (since $a \vee a^- = a$) so is not a positive implicative filter. Also $D(F) = \text{Rad}(F) = \{a, b, c, 1\}$ and $B(A) = \{0, 1\}$ so $\text{Rad}(F) \not\subseteq B(A)$ and $D(F) \not\subseteq B(A)$. Also clearly $N(A/F) \neq \{1/F\}$.

Proposition 3. Let F be a proper filter of a BL-algebra A and for any $x \in A$, $x^{--} \in F$ or $x^- \in F$. Then F is a DC-prime filter of A .

Proof. Let $a \vee b \in F$, for $a, b \in A$. According to hypothesis, $a^{--} \in F$ or $a^- \in F$ and $b^{--} \in F$ or $b^- \in F$. Hence there are several possibilities:

- (i): $a^{--} \in F$ and $b^{--} \in F$.
- (ii): $a^{--} \in F$ and $b^- \in F$.
- (iii): $a^- \in F$ and $b^{--} \in F$.
- (iv): $a^- \in F$ and $b^- \in F$. Then $a^- \wedge b^- = (a \vee b)^- \in F$. So $a \vee b \notin F$, since F is a proper filter. That is a contradiction.

Therefore F is a DC-prime filter of A . □

Remark 7. The converse of Proposition 3 is not true in general. Consider $F = \{b, 1\}$ in Example 2. Clearly F is a DC-prime filter of A while $a^{--} \notin F$ and $a^- \notin F$.

Lemma 5. Let F be a DC-prime and easy filter of a BL-algebra A . Then for all $x \in A$, $x^- \in F$ or $x^{--} \in F$.

Proof. Let F be a DC-prime and easy filter and $x \in A$. It is known that $(x^{--} \rightarrow x^-) \vee (x^- \rightarrow x^{--}) = 1 \in F$, hence $(x^{--} \rightarrow x^-)^{-} \in F$ or $(x^- \rightarrow x^{--})^{-} \in F$. So $x^{--} \rightarrow x^- \in F$ or $x^- \rightarrow x^{--} \in F$. If $x^{--} \rightarrow x^- \in F$, then $x^{--} \rightarrow x^- = x^{--} \rightarrow x^{--} = x^{--} \rightarrow (x^{--} \rightarrow 0) = (x^{--})^2 \rightarrow 0 \in F$. Thus $((x^{--})^2)^- \in F$, so according to hypothesis and [8, Proposition 51], $x^- \in F$. If $x^- \rightarrow x^{--} \in F$, then $x^{--} \rightarrow (x^- \rightarrow 0) \in F$. So $(x^-)^{-} \rightarrow (x^- \rightarrow 0) \in F$. As F is an easy filter and $(x^-)^{-} \rightarrow x^- =$

$x^- \rightarrow x^- = 1 \in F$, it is concluded that $(x^-)^{-} \rightarrow 0 = x^{-} \in F$. Therefore the proof is completed. $\square$

Theorem 11. *Let F be a filter of a BL-algebra A . Then the following statements are equivalent:*

- (i) F is a primary filter of A .
- (ii) $\text{Rad}(F)$ is a DC-prime of A .
- (iii) $\text{Rad}(F)$ is a primary filter of A .

Proof.

- (i) $\Rightarrow$ (ii): Let F be a primary filter of A . Then based on [16, Theorem 3.3], $\text{Rad}(F)$ is a maximal filter of A and according to Corollary 6(i), $\text{Rad}(F)$ is a DC-prime filter of A .
- (ii) $\Rightarrow$ (i): Let $\text{Rad}(F)$ be a DC-prime filter of A . Then based on Corollary 9, $\text{Rad}(\text{Rad}(F)) = \text{Rad}(F)$ is a maximal filter of A . Then based on [16, Theorem 3.3], F is a primary filter of A .
- (ii) $\Rightarrow$ (iii): Let $\text{Rad}(F)$ be a DC-prime filter of A . Based on ((ii) $\Leftrightarrow$ (i)), F is a primary filter of A . As $F \subseteq \text{Rad}(F)$, $\text{Rad}(F)$ is a primary filter of A .
- (iii) $\Rightarrow$ (ii): Let $\text{Rad}(F)$ be a primary filter of A . Then based on [16, Theorem 3.3], $\text{Rad}(\text{Rad}(F)) = \text{Rad}(F)$ is a maximal filter of A and according to Corollary 6(i), $\text{Rad}(F)$ is a DC-prime filter.

$\square$

If F is a filter of a BL-algebra A , then $F \subseteq (F : x) = \{r \in A : r \vee x \in F\}$. It is known that $(F : x)$ is a proper filter of A , where $x \notin F$, [16, Proposition 4.2].

Lemma 6. *Let F be a DC-prime filter of a BL-algebra A . Then for any $x \in A - F$, the following hold:*

- (i) $\text{Rad}(F) = \text{Rad}(F : x)$.
- (ii) $(F : x) \subseteq \text{Rad}(F)$.

Proof.

- (i): Let F be a DC-prime filter of A . Then based on Theorem 10, F is a primary filter and so according to [16, Theorem 4.5], $\text{Rad}(F) = \text{Rad}((F : x))$, where $x \in A - F$.
- (ii): As $(F : x) \subseteq \text{Rad}(F : x)$, the proof gets by (i).

$\square$

Remark 8. The inclusion of Lemma 6(ii) is not equal in general. Consider $F = \{a, 1\}$ in Example 6(i). Clearly $(F : b) = \{a, 1\}$ and $\text{Rad}(F) = \{a, b, c, 1\}$.

Theorem 12. *Let F be a proper filter of a BL-algebra A . Then the following conditions are equivalent:*

- (i) F is a DC-prime filter of A .

- (ii) $D(F) = (D(F) : x)$, where $x \in A - D(F)$.
- (iii) $D(F) = D((F : x))$, where $x \in A - D(F)$.
- (iv) $(F : x) \subseteq D(F)$, where $x \in A - D(F)$.

Proof.

- (i) $\Rightarrow$ (ii): Let F be a DC-prime filter of A and $y \in (D(F) : x)$, where $x \in A - D(F)$. Then $x \vee y \in D(F)$, hence $x^{--} \vee y^{--} \in F$. Since $x \notin D(F)$, so $x^{--} \notin F$. Thus $y^{--} \in F$, i.e. $y \in D(F)$. Hence $(D(F) : x) \subseteq D(F)$. Therefore $D(F) = (D(F) : x)$.
- (ii) $\Rightarrow$ (i): Let for $x \in A - D(F)$, $D(F) = (D(F) : x)$ and also assume that $a \vee b \in F$ and $a^{--} \notin F$. So $a \notin D(F)$. Using Part (ii), $D(F) = (D(F) : a)$. As $a \vee b \in F \subseteq D(F)$ so $b \in (D(F) : a) = D(F)$. Therefore $b \in D(F)$, hence $b^{--} \in F$, i.e. F is a DC-prime filter of A .
- (i) $\Rightarrow$ (iii): Let F be a DC-prime filter of A . Since $F \subseteq (F : x)$, then $D(F) \subseteq D((F : x))$. On the other hand if $y \in D((F : x))$ then $x^{--} \vee y^{--} \in F$. So $x \vee y \in D(F)$, i.e. $y \in (D(F) : x)$. Thus $D((F : x)) \subseteq (D(F) : x) = D(F)$, using $((i) \Leftrightarrow (ii))$.
- (iii) $\Rightarrow$ (i): Let $D(F) = D((F : x))$, for any $x \in A - D(F)$ and $a \vee b \in F$, for $a, b \in A$. If $a^{--} \notin F$ ($a \notin D(F)$), then $D(F) = D((F : a))$. As $a \vee b \in F$, so $b \in (F : a) \subseteq D((F : a)) = D(F)$. Therefore $b \in D(F)$, i.e. F is a DC-prime filter of A .
- (i) $\Rightarrow$ (iv): Let F be a DC-prime filter of A . It is known that $(F : x) \subseteq (D(F) : x)$. So $(F : x) \subseteq D(F)$, using $((i) \Leftrightarrow (ii))$.
- (iv) $\Rightarrow$ (i): Let $(F : x) \subseteq D(F)$, where $x \in A - D(F)$ and $a \vee b \in F$ such that $a^{--} \notin F$ ($a \notin D(F)$). So $(F : a) \subseteq D(F)$. Then $b \in (F : a) \subseteq D(F)$, hence $b \in D(F)$. Therefore $b^{--} \in F$, i.e. F is a DC-prime filter of A .

□

According to Corollary 9(i) and [14, Proposition 8]:

Lemma 7. *If F is a DC-prime filter then there exists an unique maximal filter M of a BL-algebra A containing F .*

Remark 9. It can be easily proved that if BL-algebra A is an MV-algebra then F is a DC-prime filter of A if and only if F is a prime filter of A . So according to [18, Proposition 3.22]:

Corollary 11.

- (i) *Let F and G be proper filters of a BL-algebra A such that $D_s(A) \subseteq F \subseteq G$. Then G/F is a prime filter of A/F if and only if G/F is a DC-prime filter of A/F .*
- (ii) *Let F be a proper filter of a BL-algebra A and $D_s(A) = \{1\}$. Then F is a prime filter of A if and only if F is a DC-prime filter of A .*
- (iii) *All DC-prime filters of a BL-algebra $A/D_s(A)$ are prime filters.*

Lemma 8. *Let F be a proper filter of a BL-algebra A . If F is a semi-integral and positive implicative filter then F is a DC-prime filter.*

Proof. Based on Corollary 6 and [14, Theorem 4], the proof is clear. $\square$

Example 7. Let $A = \{0, a, b, c, d, e, f, g, 1\}$, where $0 < a < b, d, e, g < 1$, $0 < b < e < 1$, $0 < c < d, e, g, f < 1$, $0 < d < e, g < 1$, $0 < f < g < 1$. Operations $*$ and $\rightarrow$ are defined as follows:

$*$	0	a	b	c	d	e	f	g	1
0	0	0	0	0	0	0	0	0	0
a	0	a	a	0	a	a	0	a	a
b	0	a	b	0	a	b	0	a	b
c	0	0	0	c	c	c	c	c	c
d	0	a	a	c	d	d	c	d	d
e	0	a	b	c	d	e	c	d	e
f	0	0	0	c	c	c	f	f	f
g	0	a	a	c	d	d	f	g	g
1	0	a	b	c	d	e	f	g	1

$\rightarrow$	0	a	b	c	d	e	f	g	1
0	1	1	1	1	1	1	1	1	1
a	f	1	1	f	1	1	f	1	1
b	f	g	1	f	g	1	f	g	1
c	b	b	b	1	1	1	1	1	1
d	0	b	b	f	1	1	f	1	1
e	0	a	b	f	g	1	f	g	1
f	b	b	b	e	e	e	1	1	1
g	0	b	b	c	e	e	f	1	1
1	0	a	b	c	d	e	f	g	1

Then $(A, \wedge, \vee, *, \rightarrow, 0, 1)$ is a BL-algebra ([16, Example 4.3]). Clearly $F_1 = \{1\}$, $F_2 = \{e, 1\}$, $F_3 = \{g, 1\}$, $F_4 = \{b, e, 1\}$, $F_5 = \{f, g, 1\}$, $F_6 = \{d, e, g, 1\}$, $F_7 = \{d, e, g, a, b, 1\}$ and $F_8 = \{d, e, g, f, c, 1\}$ are all filters of A . It is clear that F_4 , F_5 , F_7 and F_8 are DC-prime filters of A . Clearly, F_6 is a positive implicative filter while is not a DC-prime filter (since $a^{--} \rightarrow c^{--} \notin F_6$ and $c^{--} \rightarrow a^{--} \notin F_6$) also F_6 is not a semi-integral filter (since $(a * c)^- \in F_6$ and $(a^n)^-, (b^n)^- \notin F_6$, for all $n \in \mathbb{N}$).

Corollary 12. *Let F be a proper filter of a BL-algebra A . Then the following conclusions hold:*

- (i) *If F is an integral and positive implicative filter of A then F is a DC-prime filter of A .*
- (ii) *If A is a Gödel algebra, F is a primary and positive implicative filter of A then F is a DC-prime filter of A .*

- (iii) If $D(F) = F$, F is an implicative and semi-integral filter then F is a DC-prime filter of A .
- (iv) If A is a finite BL-algebra, F is a positive implicative and perfect filter then F is a DC-prime filter of A .
- (v) If $F = A - \{0\}$, then F is a DC-prime filter of A .

Proof.

- (i): This part is a straightforward consequence of Lemma 8 and [14, Corollary 1].
- (ii): According to [14, Proposition 7] and Lemma 8, the proof is clear.
- (iii): Based on [3, Corollary 3.28] and Lemma 8, the proof is obvious.
- (iv): This part is a straightforward consequence of Lemma 8 and [14, Theorem 5].
- (v): Based on the definition of a DC-prime filter, the proof is obvious.

□

In the following, we show the necessity of the conditions in the Corollary 12.

Example 8.

- (i) Let $A = \{0, a, b, 1\}$, where $0 < a, b < 1$. Operations $*$ and $\rightarrow$ are defined as follows:

$*$	0	a	b	1	$\rightarrow$	0	a	b	1
0	0	0	0	0	0	1	1	1	1
a	0	a	0	a	a	b	1	b	1
b	0	0	b	b	b	a	a	1	1
1	0	a	b	1	1	0	a	b	1

Then $(A, \wedge, \vee, *, \rightarrow, 0, 1)$ is a BL-algebra ([11, Page163]). It is clear that $F = \{1\}$ is a positive implicative filter while is not an integral filter (since $(a * b)^- \in F$ and $a^-, b^- \notin F$), nor a DC-prime filter (since $a^{--} \rightarrow b^{--} \notin F$ and $b^{--} \rightarrow a^{--} \notin F$).

- (ii) Consider Gödel algebra A in Example 1. Clearly $F = \{c, 1\}$ is a positive implicative filter while is not a DC-prime filter and nor a primary filter.
- (iii) Consider Example 7. F_6 is an implicative filter, $D(F_6) = F_6$ while is not a semi-integral and nor a DC-prime filter (since $a^{--} \rightarrow c^{--} \notin F_6$ and $c^{--} \rightarrow a^{--} \notin F_6$).
- (iv) Consider Example 7. F_6 is a positive implicative filter while is not a perfect filter (since it is not a semi-integral filter) and nor a DC-prime filter (since $a^{--} \rightarrow c^{--} \notin F_6$ and $c^{--} \rightarrow a^{--} \notin F_6$).
- (v) Consider Example 2(i). $F = \{b, 1\} \neq A - \{0\}$ is a DC-prime filter of A .

Proposition 4. Let F be a proper filter of a BL-algebra A . If $A/D(F)$ is a special BL-algebra then F is a DC-prime filter of A .

Proof. Let $A/D(F)$ be a special BL-algebra. Then based on Corollary 4(iii), $A/D(F)$ is a DCP-BL-algebra so $\{1/D(F)\}$ is a DC-prime filter of $A/D(F)$. Assume that $x \vee y \in F$, for $x, y \in A$. So $x \vee y \in D(F)$, hence $x \vee y/D(F) = 1/D(F)$. Thus $(x \vee y)^{-}/D(F) = 1/D(F)$. Hence $x^{-}/D(F) = 1/D(F)$ or $y^{-}/D(F) = 1/D(F)$. So $x^{-} \in D(F)$ or $y^{-} \in D(F)$ since $\{1/D(F)\}$ is a DC-prime filter of $A/D(F)$. Therefore $x^{-} \in F$ or $y^{-} \in F$, i.e. F is a DC-prime filter of A . $\square$

Recall that if F is a filter of a BL-algebra A , $F^{-} = \{a \in A : a^{-} \in F\}$.

Corollary 13. *Let F be a DC-prime and Boolean filter of a BL-algebra A . Then*

- (i) $F \cup F^{-} = A$.
- (ii) A/F has no zero divisor element.

Proof.

- (i): Let F be a DC-prime and Boolean filter of A and $x \in A$. Based on Lemma 5, $x^{-} \in F$ or $x^{-} \in F$. If $x^{-} \in F$, hence $x \in F^{-}$, and so $x \in F \cup F^{-}$. If $x^{-} \in F$, since F is Boolean filter, then $x \in F$. Hence $x \in F \cup F^{-}$. Therefore $F \cup F^{-} = A$.
- (ii): Let $[x], [y] \in A/F$ and $[x] * [y] = [0]$. Then $(x * y)^{-} \in F$. As $(x * y)^{-} = x \rightarrow y^{-} \leq (x \rightarrow y^{-})^{-} = x^{-} \rightarrow y^{-}$ hence $x^{-} \rightarrow y^{-} \in F$. Based on Lemma 5, $x^{-} \in F$ or $x^{-} \in F$. If $x^{-} \in F$, then $y^{-} \in F$, so $[y] = [0]$. If $x^{-} \in F$, then $[x] = [0]$. Therefore A/F has no zero divisor element. $\square$

Example 9. Consider Example 6. $F = \{a, 1\}$ is a DC-prime filter while is not a Boolean filter (since $b \vee b^{-} \notin F$) and $F \cup F^{-} \neq A$ (since $F^{-} = \{0\}$).

The following examples show that there are no relationships between DC-prime filters and some other types of filters in BL-algebras.

Example 10.

- (i) Consider Example 7. Clearly, F_6 is a special filter, positive implicative filter (so is a Boolean filter, implicative filter, easy filter, fantastic filter and normal filter) while is not a DC-prime filter (since $a^{-} \rightarrow c^{-} \notin F_6$ and $c^{-} \rightarrow a^{-} \notin F_6$), nor a semi-integral filter (since $(a * c)^{-} \in F_6$ and $(a^n)^{-}, (b^n)^{-} \notin F_6$, for all $n \in \mathbb{N}$. $G = \{b, e, 1\}$ is a DC-prime filter while is not a special filter, nor a Boolean filter (since $D_s(G) = \{e, 1\} \neq G$ and $a \vee a^{-} \notin G$).
- (ii) Consider Example 6. $F = \{1\}$ is a DC-prime filter while F is not a fantastic filter, nor a normal filter and nor a positive implicative filter.
- (iii) Consider Example 1. Clearly $F = \{c, 1\}$ is a positive implicative filter while is not a DC-prime filter.
- (iv) Consider Example 5. $F = \{b, 1\}$ is a DC-prime filter while is not a semi-maximal filter, since $\text{Rad}(F) = \{a, b, 1\}$.

(v) Let $A = \{0, a, b, c, d, 1\}$, where $0 < a, b < c < 1$ and $0 < b < d < 1$. Operations $*$ and $\rightarrow$ are defined as follows:

$*$	0	a	b	c	d	1	$\rightarrow$	0	a	b	c	d	1
0	0	0	0	0	0	0	0	1	1	1	1	1	1
a	0	a	0	a	0	a	a	d	1	d	1	d	1
b	0	0	0	0	b	b	b	c	c	1	1	1	1
c	0	a	0	a	b	c	c	b	c	d	1	d	1
d	0	0	b	b	d	d	d	a	a	c	c	1	1
1	0	a	b	c	d	1	1	0	a	b	c	d	1

Then $(A, \wedge, \vee, *, \rightarrow, 0, 1)$ is a BL-algebra ([16, Example 4.3]). $F_1 = \{1\}$, $F_2 = \{d, 1\}$ and $F_3 = \{a, c, 1\}$ are all filters of A . Clearly F_1 is a nodal filter while is not a DC-prime filter, since $a^{--} \rightarrow b^{--} \notin F_1$ and $b^{--} \rightarrow a^{--} \notin F_1$.

In the following, we see that the intersection of two DC-prime filters is not a DC-prime filter in general.

Example 11. Let $A = \{0, a, b, c, d, 1\}$, where $0 < b < a < 1$ and $0 < d < a, c < 1$. Define $*$ and $\rightarrow$ by the following

$*$	1	a	b	c	d	0
1	1	a	b	c	d	0
a	a	b	b	d	0	0
b	b	b	b	0	0	0
c	c	d	0	c	d	0
d	d	0	0	d	0	0
0	0	0	0	0	0	0

$\rightarrow$	1	a	b	c	d	0
1	1	a	b	c	d	0
a	1	1	a	c	c	d
b	1	1	1	c	c	c
c	1	a	b	1	a	b
d	1	1	a	1	1	a
0	1	1	1	1	1	1

Then $(A, \wedge, \vee, *, \rightarrow, 0, 1)$ is a BL-algebra, [11, Example 4.3]. Clearly, $F_1 = \{a, b, 1\}$, $F_2 = \{c, 1\}$ are DC-prime filters while $F_1 \cap F_2 = \{1\}$ is not DC-prime. Since $a \vee c \in \{1\}$ but $a^{--}, c^{--} \notin \{1\}$.

Lemma 9. Let F_1 and F_2 be DC-prime filters and F_2 be a maximal filter of an integral BL-algebra A . Then $F_1 \wedge F_2$ is a DC-prime filter of A .

Proof. Let $a \vee b \in F_1 \wedge F_2$. As $F_1 \wedge F_2 \leq F_1, F_2$ so $a \vee b \in F_1$ and $a \vee b \in F_2$. Hence $a^{--} \in F_1$ or $b^{--} \in F_1$ and $a^{--} \in F_2$ or $b^{--} \in F_2$. The following cases are considered:

Case (1): If $a^{--} \in F_1$ and $a^{--} \in F_2$, then $a^{--} \in F_1 \wedge F_2$.

Case (2): If $a^{--} \in F_1$ and $a^{--} \notin F_2$. As F_2 is a maximal filter, $F_2 \vee \langle a^{--} \rangle = A$. So $0 \in F_2 \vee \langle a^{--} \rangle$ hence there exists $f_2 \in F_2$ and $n \in \mathbb{N}$ such that $0 = f_2 * (a^{--})^n$. As A is an integral BL-algebra, $(a^{--})^n = 0$ or $f_2 = 0$. If $f_2 = 0$, then $0 \in F_2$, which is a contradiction. If $(a^{--})^n = 0$ then $0 \in F_1$, since $(a^{--})^n \in F_1$ which is a contradiction. So $a^{--} \in F_1$ and $a^{--} \in F_2$. Therefore $a^{--} \in F_1 \wedge F_2$.

Case (3): If $a^{--} \in F_2$ and $a^{--} \notin F_1$. It only needs to be proven that $b^{--} \in F_1 \wedge F_2$. Let $b^{--} \notin F_1 \wedge F_2$. As $b^{--} \in F_1$ so $b^{--} \notin F_2$. Hence $F_2 \vee \langle b^{--} \rangle = A$. So there exists $n \in \mathbb{N}$ and $f_3 \in F_2$ such that $0 = (b^{--})^n * f_3$. As A is an integral BL-algebra, $f_3 = 0$ or $(b^{--})^n = 0$. If $f_3 = 0$, then $0 \in F_2$. That is a contradiction. If $(b^{--})^n = 0$ then $0 \in F_1$, since $b^{--} \in F_1$. That is a contradiction. Therefore $b^{--} \in F_1 \wedge F_2$.

Case (4): If $b^{--} \in F_1$, $b^{--} \in F_2$, $a^{--} \notin F_1$ and $a^{--} \notin F_2$, then $b^{--} \in F_1 \wedge F_2$.

Therefore $F_1 \wedge F_2$ is a DC-prime filter of A . $\square$

Lemma 10. *Let F and G be proper filters of a BL-algebra A such that $D(F) = D(G)$. Then F and G are DC-prime filters if and only if $F \wedge G$ is a DC-prime filter.*

Proof. Let $x \vee y \in F \wedge G$, for $x, y \in A$ and $x^{--} \notin F \wedge G$. Suppose that $x^{--} \notin F$. As F is a DC-prime so $y^{--} \in F$. Then $y \in D(F) = D(G) = D(F) \wedge D(G) = D(F \wedge G)$. Hence $y \in D(F \wedge G)$. Therefore $y^{--} \in F \wedge G$, i.e. $F \wedge G$ is a DC-prime filter of A . Now let $F \wedge G$ be a DC-prime filter and $D(F) = D(G)$. Then $D(F \wedge G) = D(F) \wedge D(G) = D(F) = D(G)$ is a DC-prime filter. Hence based on Theorem 9, F and G are DC-prime filters of A . $\square$

Lemma 11. *Let $\{F_i : i \in I\}$ be a non-empty totally ordered subset of DC-prime filters in a BL-algebra A . Then $\bigcap_{i \in I} F_i$ and $\bigcup_{i \in I} F_i$ are DC-prime filters of A .*

Proof. Suppose that $x \vee y \in \bigcap_{i \in I} F_i$, for $x, y \in A$ and $x^{--} \notin \bigcap_{i \in I} F_i$. So there exists $j \in I$ such that $x^{--} \notin F_j$. As F_j is a DC-prime, $y^{--} \in F_j$. Now take $t \in I$. If $F_t \subseteq F_j$, then $x^{--} \notin F_t$ and so $y^{--} \in F_t$. If $F_j \subseteq F_t$, then $y^{--} \in F_t$. So for all $t \in I$, $y^{--} \in F_t$ and therefore $y^{--} \in \bigcap_{i \in I} F_i$ i.e. $\bigcap_{i \in I} F_i$ is a DC-prime filter. Clearly $\bigcup_{i \in I} F_i$ is a DC-prime filter. $\square$

5. CHARACTERIZATIONS OF DC-PRIME FILTERS IN BL-ALGEBRAS

In this section, the properties of a DC-prime filter in BL-algebras are studied, exactly. Also, some equivalent conditions are presented for it.

Theorem 13. *Let E_1 and E_2 be filters of a BL-algebra A and F be a proper filter of A . Then the following statements are equivalent:*

- (i) *If $F = E_1 \cap E_2$, then $D(F) = D(E_1)$ or $D(F) = D(E_2)$.*
- (ii) *F is a DC-prime filter.*

Proof.

- (i) $\Rightarrow$ (ii): Let $a, b \in A$ and $a \vee b \in F$. Hence based on [19, Proposition 1.40], $(F \vee \langle a \rangle) \wedge (F \vee \langle b \rangle) = F$. So according to Part (i), $D(F \vee \langle a \rangle) = D(F)$ or

$D(F \vee \langle b \rangle) = D(F)$. As $a \in F \vee \langle a \rangle \subseteq D(F \vee \langle a \rangle)$ and $b \in F \vee \langle b \rangle \subseteq D(F \vee \langle b \rangle)$, so $a \in D(F)$ or $b \in D(F)$. Therefore $a^{--} \in F$ or $b^{--} \in F$, i.e. F is a DC-prime filter of A .

(ii) $\Rightarrow$ (i): Let F be a DC-prime filter and $E_1 \cap E_2 = F$. Assume that $D(F) \neq D(E_1)$ and $D(F) \neq D(E_2)$. Then there exists $a \in D(E_1)$ such that $a \notin D(F)$ and there exists $b \in D(E_2)$ such that $b \notin D(F)$. So $a^{--} \vee b^{--} \in E_1 \cap E_2 = F$. Then $a^{--} \in F$ or $b^{--} \in F$, since F is a DC-prime filter. Hence $a \in D(F)$ or $b \in D(F)$, which is a contradiction. Therefore the proof is completed. $\square$

According to Theorem 13:

Corollary 14. *Let E_i be filters of a BL-algebra A such that $i \in I$ and I be a finite set, and F be a proper filter of A . Then the following statements are equivalent:*

- (i) *If $F = \bigcap_{i \in I} E_i$, then there exists $i \in I$ such that $D(F) = D(E_i)$.*
- (ii) *F is a DC-prime filter.*

Theorem 14. *Let E_1 and E_2 be filters of a BL-algebra A and F be a DC-prime filter of A . If $E_1 \cap E_2 \subseteq F$, then $D(E_1) \subseteq D(F)$ or $D(E_2) \subseteq D(F)$.*

Proof. Let F be a DC-prime filter and $E_1 \cap E_2 \subseteq F$. Also assume that $D(E_1) \not\subseteq D(F)$ and $D(E_2) \not\subseteq D(F)$. Hence there exist $a \in D(E_1) - D(F)$ and $b \in D(E_2) - D(F)$. So $a^{--} \in E_1$ and $b^{--} \in E_2$ then $a^{--} \vee b^{--} \in E_1 \cap E_2 \subseteq F$. As F is a DC-prime filter, $a^{--} \in F$ or $b^{--} \in F$. Hence $a \in D(F)$ or $b \in D(F)$. Which is a contradiction. So the proof is completed. $\square$

Example 12. Consider Example 6. Clearly, $F = \{a, 1\}$ is a DC-prime filter and $E_1 = \{b, 1\}$ and $E_2 = \{a, b, c, 1\}$ are filters of A . So the converse of above theorem, is not true in general.

Corollary 15. *Let F_1 and F_2 be filters of a BL-algebra A and F be a DC-prime filter of A . Then $D(F_1) \cap D(F_2) \subseteq F$ implies $D(F_1) \subseteq D(F)$ or $D(F_2) \subseteq D(F)$.*

Lemma 12. *Let A be a DCP-BL-algebra. Then for any proper filters F and G of A , $D(F) \subseteq D(G)$ or $D(G) \subseteq D(F)$.*

Proof. Let A be a DCP-BL-algebra. Then $D(F \cap G)$ is a DC-prime filter of A . As $D(F \cap G) = D(F) \cap D(G)$, according to Theorem 13, $D(D(F \cap G)) = D(D(F))$ or $D(D(F \cap G)) = D(D(G))$. Hence $D(F \cap G) = D(F)$ or $D(F \cap G) = D(G)$. Therefore as $D(F \cap G) = D(F) \cap D(G)$, $D(F) \subseteq D(G)$ or $D(G) \subseteq D(F)$, for any proper filters F and G of A . $\square$

Example 13. Consider Example 1. $F_1 = \{1\}$, $F_2 = \{c, 1\}$ and $F_3 = \{b, c, d, 1\}$ are all filters of A . Also $D(F_1) = D(F_2) = F_2$ and $D(F_3) = F_3$. So $D(F_1) = D(F_2) \subseteq D(F_3)$ while F_1 is not a DC-prime filter (since $a^{--} \rightarrow b^{--} \notin F_1$ and $b^{--} \rightarrow a^{--} \notin F_1$). So A is not a DCP-BL-algebra.

Proposition 5. *Let F be a DC-prime filter of a BL-algebra A . Then the set $\{D(E) : F \subseteq E, E \text{ is a proper filter of } A\}$ is linearly ordered with respect to set theoretical inclusion.*

Proof. Let $D(E), D(G) \in T = \{D(E) : F \subseteq E, E \text{ is a proper filter of } A\}$. Assume $D(E) \not\subseteq D(G)$ and $D(G) \not\subseteq D(E)$. Hence there exist $a, b \in A$ such that $a \in D(E) - D(G)$ and $b \in D(G) - D(E)$. As F is a DC-prime filter, $a^{--} \rightarrow b^{--} \in F$ or $b^{--} \rightarrow a^{--} \in F$. If $a^{--} \rightarrow b^{--} \in F$ then $a^{--} \rightarrow b^{--} \in E \subseteq D(E)$. Then $b^{--} \in D(E)$, since $a \in D(E)$. Hence $b \in D(D(E)) = D(E)$ which is a contradiction since $b \notin D(E)$. If $b^{--} \rightarrow a^{--} \in F$ then $b^{--} \rightarrow a^{--} \in G \subseteq D(G)$. So $a^{--} \in D(G)$, since $b \in D(G)$. Hence $a \in D(D(G)) = D(G)$ which is a contradiction since $a \notin D(G)$. Therefore the proof is complete. $\square$

Proposition 6.

- (i) *Let A be a DCP-BL-algebra and F be a proper fantastic filter of A . Then for all $x, y \in A - F$, there exists $z \in A - F$ such that $x^{--} \leq z$ and $y^{--} \leq z$.*
- (ii) *Let F be a proper filter of a BL-algebra A . If for all $x, y \in A - F$ there is $z \in A - F$ such that $x^{--} \leq z$ and $y^{--} \leq z$, then F is a DC-prime filter.*

Proof.

- (i): Let F be a fantastic filter of a DCP-BL-algebra A . According to Theorem 5, F is a DC-prime filter of A . Also, assume that there is $x, y \in A - F$ such that for all $z \in A - F$, $z < x^{--}$ or $z < y^{--}$. Since $x^{--} \leq x^{--} \vee y^{--}$ and $y^{--} \leq x^{--} \vee y^{--}$, so $x^{--} \vee y^{--} \in F$. As F is a DC-prime filter then $x^{--} \in F$ or $y^{--} \in F$. Hence $x \in F$ or $y \in F$, since F is a fantastic filter. That is a contradiction. Therefore the proof is completed.
- (ii): Let for all $x, y \in A - F$ there is $z \in A - F$ such that $x^{--} \leq z$ and $y^{--} \leq z$. Since $x \leq x^{--} \leq z$ and $y \leq y^{--} \leq z$, then $x \leq z$ and $y \leq z$. So according to [19, Proposition 1.45], F is a prime filter of A , hence based on Theorem 7, F is a DC-prime filter.

$\square$

Proposition 7. *Let A be a DCP-BL-algebra and F be a proper filter of A . Then for any $x, y \in A - D(F)$, there exists $z \in A - D(F)$ such that $x \leq z$ and $y \leq z$.*

Proof. Let F be a proper filter of a DCP-BL-algebra A . According to Theorem 5, F is a DC-prime filter of A . Also, assume that there are $x, y \in A - D(F)$ such that for all $z \in A - D(F)$, $z < x$ or $z < y$. Since $x \leq x \vee y$ and $y \leq x \vee y$, so $x \vee y \in D(F)$. Then $x^{--} \vee y^{--} \in F$, so $x^{--} \in F$ or $y^{--} \in F$, since F is a DC-prime filter. Hence $x \in D(F)$ or $y \in D(F)$, which is a contradiction. $\square$

Proposition 8. *Let F be a filter of a BL-algebra A and for any $x, y \in A - D(F)$, there exists $z \in A - D(F)$ such that $x \leq z$ and $y \leq z$. Then F is a DC-prime filter of A .*

Proof. According to Theorem 13, it only needs to be proven that if $F = E_1 \cap E_2$, then $D(F) = D(E_1)$ or $D(F) = D(E_2)$, for filters E_1 and E_2 of A . Now let $F = E_1 \cap E_2$ and $D(F) \neq D(E_1)$ and $D(F) \neq D(E_2)$. So there exists $x \in D(E_1)$ such that $x \notin D(F)$ and there exists $y \in D(E_2)$ such that $y \notin D(F)$. So according to hypothesis there exists $z \in A - D(F)$ such that $x \leq z$ and $y \leq z$. Since $x \in D(E_1)$ and $y \in D(E_2)$ so $z \in D(E_1) \cap D(E_2) = D(E_1 \cap E_2) = D(F)$. Therefore $z \in D(F)$ which is a contradiction. Thus the proof is completed. $\square$

Corollary 16. *Let A be a DCP-BL-algebra and F be a proper filter of A . Then for any $x, y \in A/D(F)$ such that $x, y \neq 1/D(F)$, there exists $w \in A/D(F)$ such that $w \neq 1/D(F)$, $x \leq w$ and $y \leq w$.*

Proof. Let F be a proper filter of a DCP-BL-algebra A . According to Theorem 5, F is a DC-prime filter of A . Also, assume that $x, y \in A/D(F)$ such that $x, y \neq 1/D(F)$. Hence $x = a/D(F)$ and $y = b/D(F)$ for some $a, b \in A$, such that $a/D(F) \neq 1/D(F)$ and $b/D(F) \neq 1/D(F)$, and so $a, b \notin D(F)$. So according to Proposition 7, there exists $z \in A - D(F)$ such that $a \leq z$ and $b \leq z$. Hence $a \rightarrow z = 1$ and $b \rightarrow z = 1$. So $a/D(F) \leq z/D(F)$ and $b/D(F) \leq z/D(F)$. Therefore $x \leq w$ and $y \leq w$. $\square$

Corollary 17. *Let F be a filter of a BL-algebra A and for any $x, y \in A/D(F)$ such that $x, y \neq 1/D(F)$, there exists $w \in A/D(F)$ such that $w \neq 1/D(F)$, $x \leq w$ and $y \leq w$. Then F is a DC-prime filter of A .*

Proof. According to Theorem 13, it only needs to be proven that if $F = E_1 \cap E_2$, then $D(F) = D(E_1)$ or $D(F) = D(E_2)$, for filters E_1 and E_2 of A . Now let $F = E_1 \cap E_2$ and $D(F) \neq D(E_1)$ and $D(F) \neq D(E_2)$. So there exists $a \in D(E_1)$ such that $a \notin D(F)$ and there exists $b \in D(E_2)$ such that $b \notin D(F)$. Thus $x = a/D(F) \neq 1/D(F)$ and $y = b/D(F) \neq 1/D(F)$. So according to hypothesis there exists $w = c/D(F) \in A/D(F)$ such that $w \neq 1/D(F)$, $x \leq w$ and $y \leq w$. Then $a/D(F) \leq c/D(F)$ and $b/D(F) \leq c/D(F)$. Hence $c/D(F) \in (D(E_1) \cap D(E_2))/D(F)$. As $F = E_1 \cap E_2$, $D(F) = D(E_1) \cap D(E_2)$. Therefore $w = c/D(F) = 1/D(F)$, which is a contradiction. Thus the proof is completed. $\square$

Proposition 9. *Let A be a DCP-BL-algebra and F be a proper filter of A . Then $\langle x^{--} \rangle \cap \langle y^{--} \rangle \subseteq F$ implies $x^{--} \in F$ or $y^{--} \in F$, for all $x, y \in A$.*

Proof. According to Theorem 5, F is a DC-prime filter of A . Also, assume that $\langle x^{--} \rangle \cap \langle y^{--} \rangle \subseteq F$, for $x, y \in A$. Suppose that $x^{--} \notin F$ and $y^{--} \notin F$ so $x \notin D(F)$ and $y \notin D(F)$. Then based on Proposition 7, there is $z \in A - D(F)$ such that $x \leq z$ and $y \leq z$. Then $x^{--} \leq z^{--}$ and $y^{--} \leq z^{--}$. Hence $\langle z^{--} \rangle \subseteq \langle x^{--} \rangle$ and $\langle z^{--} \rangle \subseteq \langle y^{--} \rangle$. So $z^{--} \in \langle x^{--} \rangle \cap \langle y^{--} \rangle \subseteq F$ thus $z \in D(F)$, which is a contradiction. $\square$

Proposition 10. *Let F be a filter of a BL-algebra A and for all $x, y \in A$, $\langle x^{--} \rangle \cap \langle y^{--} \rangle \subseteq F$ implies $x^{--} \in F$ or $y^{--} \in F$. Then F is a DC-prime filter of A .*

Proof. Assume that for any $x, y \in A$, $\langle x^{--} \rangle \cap \langle y^{--} \rangle \subseteq F$ implies $x^{--} \in F$ or $y^{--} \in F$. Also assume that $x \vee y \in F$. Hence $\langle x \vee y \rangle \subseteq F$ so $\langle x \rangle \cap \langle y \rangle \subseteq F$. As $\langle x^{--} \rangle \subseteq \langle x \rangle$ and $\langle y^{--} \rangle \subseteq \langle y \rangle$ thus $\langle x^{--} \rangle \cap \langle y^{--} \rangle \subseteq \langle x \rangle \cap \langle y \rangle \subseteq F$. According to hypothesis $x^{--} \in F$ or $y^{--} \in F$. Therefore F is a DC-prime filter of A . $\square$

Proposition 11. *Let A be a DCP-BL-algebra and F be a proper filter of A . Then $\langle x \rangle \cap \langle y \rangle \subseteq F$ implies $x \in D(F)$ or $y \in D(F)$, for all $x, y \in A$.*

Proof. According to Theorem 5, F is a DC-prime filter of A . Also, assume that for any $x, y \in A$, $\langle x \rangle \cap \langle y \rangle \subseteq F$. So $\langle x^{--} \rangle \cap \langle y^{--} \rangle \subseteq F$. According to Proposition 9, $x \in D(F)$ or $y \in D(F)$. $\square$

Proposition 12. *Let F be a filter of a BL-algebra A and for all $x, y \in A$, $\langle x \rangle \cap \langle y \rangle \subseteq F$ implies $x \in D(F)$ or $y \in D(F)$. Then F is a DC-prime filter of A .*

Proof. Let for any $x, y \in A$, if $\langle x \rangle \cap \langle y \rangle \subseteq F$ then $x \in D(F)$ or $y \in D(F)$ and assume that $x \vee y \in F$. So $\langle x \vee y \rangle \subseteq F$. Then $\langle x \rangle \cap \langle y \rangle \subseteq F$, since $\langle x \rangle \cap \langle y \rangle = \langle x \vee y \rangle$. Hence according to hypothesis $x \in D(F)$ or $y \in D(F)$. Therefore $x^{--} \in F$ or $y^{--} \in F$, i.e. F is a DC-prime filter of A . $\square$

Recall that the set of all filters in a BL-algebra A is denoted by $F(A)$.

Corollary 18. *If $F(A)$ is a chain, then all proper filters of a BL-algebra A are DC-prime filters, i.e. A is a DCP-BL-algebra.*

Proof. Let $F(A)$ be a chain and F be a proper filter of A . Also assume that for any $x, y \in A$, $\langle x \rangle \cap \langle y \rangle \subseteq F$. Based on the condition of this corollary, there are two possibilities:

- (i): If $\langle x \rangle \subseteq \langle y \rangle$. Hence $x \in \langle x \rangle = \langle x \rangle \cap \langle y \rangle \subseteq F \subseteq D(F)$.
- (ii): If $\langle y \rangle \subseteq \langle x \rangle$. Hence $y \in \langle y \rangle = \langle x \rangle \cap \langle y \rangle \subseteq F \subseteq D(F)$.

Therefore according to Proposition 12, F is a DC-prime filter of A . $\square$

Remark 10. In Example 2(ii), the converse of Corollary 18 is not true.

Recall that ${}^\perp X = \{a \in A : a \vee x = 1, \text{ for all } x \in X\}$ is a filter of a BL-algebra A , where X is a non-empty subset of A , [20, Definition 11]. Also recall that a filter F of A is called a linearly ordered filter if for any $x, y \in F$, either $x \leq y$ or $y \leq x$.

Lemma 13. *Let $F \neq \{1\}$ be a linearly ordered filter of A . Then ${}^\perp F$ is a DC-prime filter of A .*

Proof. According to the hypothesis $F \neq \{1\}$, hence ${}^\perp F \neq A$, i.e. ${}^\perp F$ is a proper filter. Now let $a \vee b \in {}^\perp F$, for $a, b \in A$, such that $a^{--} \notin {}^\perp F$ and $b^{--} \notin {}^\perp F$. So $a \notin {}^\perp F$, $b \notin {}^\perp F$. Then there exist $x, y \in F$ such that $a \vee x \neq 1$ and $b \vee y \neq 1$. Denote $z = x \wedge y$. Hence $z \in F$, $a \vee z \in F$ and $b \vee z \in F$, moreover $a \vee z \neq 1$ and $b \vee z \neq 1$ (if $a \vee z = 1$ then $a \vee x = 1$ and if $b \vee z = 1$ then $b \vee y = 1$, which are contradictions). As F is a linearly ordered filter, assume that $b \vee z \leq a \vee z$. Therefore as $z \in F$ and

$a \vee b \in {}^\perp F$, $1 = (a \vee b) \vee z = a \vee (b \vee z) \leq a \vee (a \vee z) = a \vee z$, hence $a \vee z = 1$, which is contradiction. Thus the proof is completed. $\square$

A BL-algebra A is called a RS-BL-algebra, if for all non-empty subset X of A , $X_r = \{a \in A : x \rightarrow a = a, \text{ for all } x \in X\}$ is a filter of A [17, Definition 3.1].

According to Lemma 13:

Corollary 19. *Let A be a RS-BL-algebra, $X \neq \{1\}$ be a non-empty linearly ordered upper subset of A . Then X_r is a DC-prime filter of A .*

Lemma 14. *Let x be an idempotent element of a BL-algebra A and $a, b \in A$. If $a \vee b = x$ implies $a = x$ or $b = x$ ($\vee$ -irreducible), then $F_x = \{y \in A : x \leq y\}$ is a DC-prime filter of A .*

Proof. Clearly F_x is a proper filter of A . Let $y_1 \vee y_2 \in F_x$, for $y_1, y_2 \in A$. So $x \leq y_1 \vee y_2$. Then

$$\begin{aligned} x &= x * x \leq x * (y_1 \vee y_2) \leq x * (y_1 \vee y_2)^{--} \\ &= x * (y_1^{--} \vee y_2^{--}) \\ &= (x * y_1^{--}) \vee (x * y_2^{--}) \\ &\leq x \vee x = x. \end{aligned}$$

Then $x = (x * y_1^{--}) \vee (x * y_2^{--})$. Since x is an $\vee$ -irreducible so $x * y_1^{--} = x$ or $x * y_2^{--} = x$. It is known that $x * y_1^{--} \leq y_1^{--}$ and $x * y_2^{--} \leq y_2^{--}$, thus $x \leq y_1^{--}$ or $x \leq y_2^{--}$. Therefore $y_1^{--} \in F_x$ or $y_2^{--} \in F_x$ and so the proof is completed. $\square$

Lemma 15. *Let F be a DC-prime filter of a BL-algebra A and G be a filter of A such that $G \subseteq F$. The set of all DC-prime filters F' of A such that $G \subseteq F' \subseteq F$ contains a minimal element.*

Proof. Take the set $\Sigma := \{F' : F' \text{ is DC-prime filter such that } G \subseteq F' \subseteq F\}$. Clearly $F \in \Sigma$. The relation $\leq$, on Σ , is defined $F' \leq G'$ if and only if $G' \subseteq F'$, for all $F', G' \in \Sigma$. Clearly the relation $\leq$ is a partially ordered on Σ . Now let T be a chain of Σ . Take $d = \bigcap_{F' \in T} F'$. It is clear that d is a filter and for all $F' \in T$, $d \subseteq F'$. Then for all $F' \in T$, $F' \leq d$. We need to prove that d is a DC-prime filter. Now let for $x, y \in A$, $x \vee y \in d$ and $x^{--} \notin d$ and $y^{--} \notin d$. So there exist $F', G' \in T$ such that $x^{--} \notin F'$ and $y^{--} \notin G'$. Since T is a chain, so $G' \subseteq F'$ or $F' \subseteq G'$. If $F' \subseteq G'$, then $y^{--} \notin F'$. Hence based on $x \vee y \in d \subseteq F'$ and since F' is a DC-prime filter, $x^{--} \in F'$. That is a contradiction. Therefore d is a DC-prime filter. Also if $G' \subseteq F'$, the process is similarly. So d is an upper bound for T . Then by Zorn's lemma, Σ contains a maximal element, hence it contains a minimal element. $\square$

Proposition 13. *Let F be a proper filter of a BL-algebra A . Then for any $a \in A - D(F)$, there exists a DC-prime filter P such that $F \subseteq D(P)$ and $a \notin D(P)$.*

Proof. Consider the set $T := \{G : G \text{ is a filter of } A, F \subseteq D(G), a \notin D(G)\}$. Clearly $F \in T$. Now we define the relation $\leq$ on T as follows: for all $G, G' \in T$, $G \leq G'$ if and only if $G \subseteq G'$. It is easy to check that the relation $\leq$ is a partially ordered on T . Let L be a chain of T so $H = \bigcup_{G \in L} G$ is a proper filter and for all $G \in L$, $G \subseteq H$ i.e. $G \leq H$ and $F \subseteq D(G)$. So $F \subseteq \bigcup_{G \in L} D(G) \subseteq D(\bigcup_{G \in L} G) = D(H)$. Also for $G \in L$, $a \notin D(G)$. Hence for all $G \in L$, $a^{--} \notin G$, then $a^{--} \notin H = \bigcup_{G \in L} G$ thus $a \notin D(H)$, so $H \in T$. Therefore H is an upper bound for L . Then by Zorn's lemma the set T contains a maximal element like P , (clearly P is a proper filter of A). Now suppose that P is not a DC-prime filter. Thus there exist $x, y \in A$ such that $x \vee y \in P$ and $x^{--} \notin P$ and $y^{--} \notin P$. As $P \subsetneq P \vee \langle x^{--} \rangle$ and $P \subsetneq P \vee \langle y^{--} \rangle$ so $P \vee \langle x^{--} \rangle \notin T$ and $P \vee \langle y^{--} \rangle \notin T$, since P is maximal element of T . Hence as $F \subseteq D(P) \subseteq D(P \vee \langle x^{--} \rangle)$ and $F \subseteq D(P) \subseteq D(P \vee \langle y^{--} \rangle)$, then $a \in D(P \vee \langle x^{--} \rangle)$ and $a \in D(P \vee \langle y^{--} \rangle)$. Thus $a^{--} \in P \vee \langle x^{--} \rangle$ and $a^{--} \in P \vee \langle y^{--} \rangle$. So $a^{--} \geq \alpha * (x^{--})^n$ and $a^{--} \geq \beta * (y^{--})^m$, for some $\alpha, \beta \in P$ and for some $n, m \in \mathbb{N}$. According to rules on BL-algebras,

$$\begin{aligned} a^{--} &\geq (\alpha * (x^{--})^n) \vee (\beta * (y^{--})^m) \\ &\geq (\alpha \vee (\beta * (y^{--})^m)) * ((x^{--})^n \vee (\beta * (y^{--})^m)) \\ &\geq (\alpha \vee \beta) * (\alpha \vee (y^{--})^m) * ((x^{--})^n \vee \beta) * ((x^{--})^n \vee (y^{--})^m) \\ &\geq (\alpha \vee \beta) * (\alpha \vee (y^{--})^m) * ((x^{--})^n \vee \beta) * (x^{--} \vee y^{--})^{mn} \\ &\geq (\alpha \vee \beta) * \alpha * \beta * (x \vee y)^{mn}. \end{aligned}$$

It is obtained that $a^{--} \in P$. So $a \in D(P)$, which is a contradiction. Therefore P is a DC-prime filter of A . Hence the proof is completed. $\square$

Corollary 20. *Let F be a proper filter of a BL-algebra A . Then*

$$D(F) = \bigcap_{F \subseteq D(G)} D(G),$$

where G is a DC-prime filter of A .

Proof. Let G be a DC-prime filter of A such that $F \subseteq D(G)$. So $D(F) \subseteq D(D(G)) = D(G)$. Hence $D(F) \subseteq \bigcap_{F \subseteq D(G)} D(G)$, where G is a DC-prime filter of A . Now assume that $x \in \bigcap_{F \subseteq D(G)} D(G)$ and $x \notin D(F)$, where G is a DC-prime filter of A . According to Proposition 13, there exists a DC-prime filter P of A such that $F \subseteq D(P)$ and $x \notin D(P)$. That is a contradiction, therefore the proof is completed. $\square$

Lemma 16. *Let F be a DC-prime filter of a BL-algebra A . Then for all $G \in F(A)$, $D(G \rightarrow D(F)) = D(F)$ or $D(G) \subseteq D(F)$.*

Proof. Based on [19, Corollary 1.37], $F(A)$ is a Heyting algebra. Hence based on the definition of Heyting algebra, $(G \rightarrow D(F)) \cap ((G \rightarrow D(F)) \rightarrow D(F)) = (G \rightarrow$

$D(F)) \cap D(F)$. As $D(F) \cap G \subseteq D(F)$, so according to [19, Lemma 1.35], $D(F) \subseteq G \rightarrow D(F)$. Then $D(F) \cap (G \rightarrow D(F)) = D(F)$, and so

$$(G \rightarrow D(F)) \cap ((G \rightarrow D(F)) \rightarrow D(F)) = D(F).$$

As F is a DC-prime filter, so $D(F)$ is a DC-prime filter, thus based on Theorem 13, $D(G \rightarrow D(F)) = D(D(F)) = D(F)$ or $D((G \rightarrow D(F)) \rightarrow D(F)) = D(D(F)) = D(F)$. On other hand by [19, Lemma 1.35], $D(F) = D((G \rightarrow D(F)) \rightarrow D(F)) = D(\sup\{H : (G \rightarrow D(F)) \cap H \subseteq D(F)\})$. First prove that $G \in \{H : (G \rightarrow D(F)) \cap H \subseteq D(F)\}$. Assume that $a \in (G \rightarrow D(F)) \cap G$. As $a \in G$, so $\langle a \rangle \subseteq G$ and since $a \in (G \rightarrow D(F))$, so $G \cap \langle a \rangle \subseteq D(F)$. Thus $\langle a \rangle \subseteq D(F)$ hence $a \in D(F)$. Therefore $G \in \{H : (G \rightarrow D(F)) \cap H \subseteq D(F)\}$. Then $G \subseteq \sup\{H : (G \rightarrow D(F)) \cap H \subseteq D(F)\} = D(F)$. Therefore $G \subseteq D(F)$ and so $D(G) \subseteq D(F)$. Thus the proof is completed. $\square$

Lemma 17. Let F be a DC-prime filter of a BL-algebra A . Then for all $G \in F(A)$, $D(G \rightarrow F) = D(F)$ or $D(G) \subseteq D(F)$.

Proof. Based on [19, Corollary 1.37], $F(A)$ is a Heyting algebra. Hence $(G \rightarrow F) \cap ((G \rightarrow F) \rightarrow F) = F \cap (G \rightarrow F)$. Since $F \cap G \subseteq F$ so according to [19, Lemma 1.35], $F \subseteq G \rightarrow F$, then $F \cap (G \rightarrow F) = F$. So $(G \rightarrow F) \cap ((G \rightarrow F) \rightarrow F) = F$. Based on Theorem 13, $D(G \rightarrow F) = D(F)$ or $D((G \rightarrow F) \rightarrow F) = D(F)$. On other hand by [19, Lemma 1.35], $(G \rightarrow F) \rightarrow F = \sup\{H : (G \rightarrow F) \cap H \subseteq F\}$. So $D((G \rightarrow F) \rightarrow F) = D(\sup\{H : (G \rightarrow F) \cap H \subseteq F\}) = D(F)$. First prove that $G \in \{H : (G \rightarrow F) \cap H \subseteq F\}$. Assume that $a \in (G \rightarrow F) \cap G$. As $a \in G$, so $\langle a \rangle \subseteq G$ and since $a \in (G \rightarrow F)$. Then $G \cap \langle a \rangle \subseteq F$, so $\langle a \rangle \subseteq F$. Therefore $a \in F$, hence $G \in \{H : (G \rightarrow F) \cap H \subseteq F\}$. Then $G \subseteq \sup\{H : (G \rightarrow F) \cap H \subseteq F\}$. So $D(G) \subseteq D(\sup\{H : (G \rightarrow F) \cap H \subseteq F\}) = D(F)$. So $D(G) \subseteq D(F)$. Thus the proof is completed. $\square$

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SOME NEW RESULTS ASSOCIATED WITH RAMANUJAN'S NUMBERS $\tau(n)$

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Abstract. In this article, the authors derive more precise limits in the recursion formula of D. H. Lehmer for Ramanujan's numbers $\tau(n)$, which are generated by

$$\sum_{n=1}^{\infty} \tau(n) z^n = z \prod_{n=1}^{\infty} (1 - z^n)^{24}, \quad (|z| < 1).$$

A presumably new recursion formula for Ramanujan's numbers $\tau(n)$ is also presented.

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1. INTRODUCTION AND MOTIVATION

Ramanujan's numbers $\tau(n)$ are defined by the following generating functions:

$$\sum_{n=1}^{\infty} \tau(n) z^n = z \prod_{n=1}^{\infty} (1 - z^n)^{24} = z \left(\sum_{n=0}^{\infty} (-1)^n (2n+1) z^{\frac{n(n+1)}{2}} \right)^8, \quad (1.1)$$

each of which is valid for $z \in \mathbb{C}$ such that $|z| < 1$. The following recursion formula for the numbers $\tau(n)$ is due to Ramanujan (see [1, p. 14]):

$$(n-1)\tau(n) = \sum_{1 \leq m \leq b_n} (-1)^{m+1} (2m+1) \left(n-1 - \frac{9}{2}m(m+1) \right) \cdot \tau \left(n - \frac{m}{2}(m+1) \right), \quad (1.2)$$

where

$$b_n := \frac{1}{2} (\sqrt{8n+1} - 1).$$

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Subsequently, Lehmer showed that (see [9] and [10]; see also [11, p. 14])

$$(n-1)\tau(n) = \sum_{1 \leq |m| \leq a_n} \left(n-1 - \frac{25}{2}m(3m+1) \right) \cdot \tau \left(n - \frac{m}{2}(3m+1) \right), \quad (1.3)$$

where

$$a_n = \frac{1}{6}(\sqrt{24n+1} + 1).$$

The formula (1.3) contains a typographical error. In Section 3, we propose to state the exact formula and a more precise limit for a_n . The proposed new limits will exclude the cases when

$$\tau(-n) = 0, \quad (n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}),$$

$\mathbb{N}$ being the set of positive integers. We also give new recursion formula for Ramanujan's numbers $\tau(n)$. In Section 2, we give an elementary proof of the formula (1.2).

The present investigation is motivated essentially by the various past and recent developments involving Ramanujan's function $\tau(n)$ (see, for details, [2–8, 14] and [15]; see also [13]).

2. PRELIMINARY RESULTS AND THE NOTATIONS USED

We begin this section by presenting our proof of the generating functions in the equation (1.2). Indeed, since

$$\sum_{n=1}^{\infty} \tau(n)z^n = z \left(\sum_{n=0}^{\infty} (-1)^n (2n+1) z^{\frac{n(n+1)}{2}} \right)^8 = z \left(\sum_{n=0}^{\infty} \omega(n)z^n \right)^8,$$

where the sequence $\omega(k)$ is given for $z \neq 0$ by (see [12])

$$\omega(k) = \begin{cases} (-1)^{\frac{1}{2}(\sqrt{8k+1}-1)} \sqrt{8k+1}, & \left(k = \frac{m(m+1)}{2}; m \in \mathbb{N}_0 \right), \\ 0, & \left(k \neq \frac{m(m+1)}{2}; m \in \mathbb{N}_0 \right), \end{cases}$$

we have

$$\frac{d}{dz} \left\{ \ln \left(\sum_{n=1}^{\infty} \tau(n)z^n \right) \right\} = \frac{d}{dz} \left\{ \ln z + \ln \left(\sum_{n=0}^{\infty} \omega(n)z^n \right)^8 \right\}.$$

We thus find that

$$\sum_{n=1}^{\infty} n\tau(n)z^n \sum_{n=0}^{\infty} \omega(n)z^n = \sum_{n=1}^{\infty} \tau(n)z^n \sum_{n=0}^{\infty} \omega(n)z^n + 8 \sum_{n=1}^{\infty} \tau(n)z^n \sum_{n=1}^{\infty} n\omega(n)z^n.$$

Furthermore, since $\omega(0) = 1$ for $|z| < 1$, we obtain

$$\sum_{n=1}^{\infty} (n-1)\tau(n)z^n = \sum_{n=1}^{\infty} \left(\sum_{k=1}^n \omega(k)(9k-n+1)\tau(n-k) \right) z^n.$$

Consequently, we get

$$(n-1)\tau(n) = \sum_{k=1}^n \omega(k)(9k-n+1)\tau(n-k),$$

which, for $k = \frac{1}{2}m(m+1)$, yields the formula (1.2).

Next, in order to introduce the notations used in our investigation, we define the integer sequence $\{\alpha_n\}_{n=0}^\infty$ (that is, the sequence A001318 in [1]), the integer sequence $\{\beta_n\}_{n=0}^\infty$ (that is, sequence A087960 in [1]), $\omega_1(k)$, $\omega_2(k)$, $\gamma(k)$ and $\delta(m)$ (that is, the sequence A002107 in [1]) as follows:

$$\alpha_n \in \{0, 1, 2, 5, 7, 12, 15, 22, \dots\}, \quad \alpha_{2n} = \frac{n(3n+1)}{2} \text{ and } \alpha_{2n-1} = \frac{n(3n-1)}{2}, \quad (2.1)$$

$$\beta_n \in \{1, -1, -1, 1, 1, -1, -1, \dots\} \quad \text{and} \quad \beta_n = (-1)^{\frac{n(n+1)}{2}}. \quad (2.2)$$

$$\omega_1(k) = \begin{cases} \frac{1}{6}(\sqrt{24k+1}-1), & \left(k = \frac{m(3m+1)}{2}; m \in \mathbb{N}_0\right), \\ 0, & \left(k \neq \frac{m(3m+1)}{2}; m \in \mathbb{N}_0\right), \end{cases} \quad (2.3)$$

$$\omega_2(k) = \begin{cases} \frac{1}{6}(\sqrt{24k+1}+1), & \left(k = \frac{m(3m-1)}{2}; m \in \mathbb{N}_0\right), \\ 0, & \left(k \neq \frac{m(3m-1)}{2}; m \in \mathbb{N}_0\right), \end{cases} \quad (2.4)$$

$$\gamma(k) = (-1)^{\omega_1(k)} \frac{6\omega_1(k)}{\sqrt{24k+1}-1} + (-1)^{\omega_2(k)} \frac{6\omega_2(k)}{\sqrt{24k+1}+1}, \quad (2.5)$$

$(k \in \mathbb{N}; \gamma(k) \in \{-1, 0, 1\})$

and

$$\delta(1) = -2, \quad \delta(2) = -1, \dots, \delta(m) = 2\gamma(m) + \sum_{k=1}^{m-1} \gamma(k)\gamma(m-k), \quad (m \in \mathbb{N} \setminus \{1, 2\}). \quad (2.6)$$

3. THE MAIN RESULTS

Our first main result is asserted by Theorem 1 below.

Theorem 1. *Let $n \in \mathbb{N} \setminus \{1\}$. Then*

$$(n-1)\tau(n) = \sum_{1 \leq m \leq c_n} (-1)^{m+1} \left(n-1 - \frac{25m(3m+1)}{2}\right) \cdot \tau\left(n - \frac{m(3m+1)}{2}\right) \\ + \sum_{1 \leq m \leq d_n} (-1)^{m+1} \left(n-1 - \frac{25m(3m-1)}{2}\right) \cdot \tau\left(n - \frac{m(3m-1)}{2}\right),$$

where

$$c_n = \frac{\sqrt{24n-23}-1}{6} \quad \text{and} \quad d_n = \frac{\sqrt{24n-23}+1}{6}.$$

Proof. Since

$$\sum_{n=1}^{\infty} \tau(n)z^n = z \prod_{n=1}^{\infty} (1-z^n)^{24} = z(1-z-z^2+z^5+z^7-z^{12}-z^{15}+\dots)^{24}, \quad (|z| < 1),$$

by applying (2.1) and (2.2), we have

$$\sum_{n=1}^{\infty} \tau(n)z^n = z \left(1 + \sum_{n=1}^{\infty} \beta_{2n} z^{\alpha_{2n}} + \sum_{n=1}^{\infty} \beta_{2n-1} z^{\alpha_{2n-1}} \right)^{24},$$

which, in view of (2.5), yields

$$\sum_{n=1}^{\infty} \tau(n)z^n = z \left(1 + \sum_{n=1}^{\infty} \gamma(n)z^n \right)^{24}. \quad (3.1)$$

Thus, if we repeat the procedure as in the proof of (1.2), we find that

$$(n-1)\tau(n) = \sum_{k=1}^{n-1} \gamma(k)(25k-n+1)\tau(n-k).$$

Moreover, since

$$\gamma\left(\frac{m(3m+1)}{2}\right) = \gamma\left(\frac{m(3m-1)}{2}\right) = (-1)^m,$$

we readily arrive at the result asserted by Theorem 1. $\square$

In our second main result (Theorem 2 below), we give a presumably new recurrence relation for the Ramanujan numbers $\tau(n)$.

Theorem 2. Suppose that $n \in \mathbb{N} \setminus \{1\}$. Then

$$(n-1)\tau(n) = \sum_{m=1}^{n-1} (13m+1-n)\delta(m)\tau(n-m), \quad (3.2)$$

where $\delta(m)$ is defined in (2.6).

Proof. Equation (3.1) implies that

$$\sum_{n=1}^{\infty} \tau(n)z^n = z \left(1 + \sum_{n=1}^{\infty} \delta(n)z^n \right)^{12}. \quad (3.3)$$

Thus, if we repeat the procedure as in the proof of (1.2), we get the result asserted by Theorem 2. $\square$

4. THE CORRECTED LEHMER'S FORMULA (1.3) AND AN ILLUSTRATIVE EXAMPLE

In this section, we first state the following corrected version of Lehmer's formula (1.3):

$$(n-1)\tau(n) = \sum_{1 \leq |m| \leq a_n} (-1)^{|m|+1} \left(n-1 - \frac{25}{2}m(3m+1) \right) \cdot \tau \left(n - \frac{m}{2}(3m+1) \right). \quad (4.1)$$

It is obvious that $c_n \leq a_n$ and that $d_n \leq a_n$. Furthermore, for

$$n \in \{5, 6, 12, 22, \dots\}$$

we have

$$c_n + 1 = a_n = d_n + 1.$$

Then, in the case when $n = 5$, we get

$$a_5 = 2, \quad b_5 = 2 \quad \text{and} \quad c_5 = d_5 = 1.$$

Let us now calculate $\tau(5)$ by using Ramanujan's formula (1.2) and Lehmer's formula (1.3) in its corrected form (4.1). By appealing appropriately to Theorem 1 and Theorem 2, we find that

$$4\tau(5) = -15\tau(4) + 115\tau(2),$$

and

$$4\tau(5) = 121\tau(0) - 21\tau(4) - 46\tau(3) + 171\tau(-2),$$

and

$$4\tau(5) = -21\tau(4) - 46\tau(3),$$

and

$$4\tau(5) = -18\tau(4) - 22\tau(3) + 70\tau(2) + 48\tau(1),$$

respectively.

5. CONCLUDING REMARKS AND OBSERVATIONS

Motivated essentially by the importance and usefulness of Ramanujan's function $\tau(n)$ in the past and current literature on the subject, in our present investigation, we have derived more precise limits in the recursion formula of D. H. Lehmer for Ramanujan's numbers $\tau(n)$, which are generated by

$$\sum_{n=1}^{\infty} \tau(n)z^n = z \prod_{n=1}^{\infty} (1 - z^n)^{24}, \quad (|z| < 1).$$

We have also presented a presumably new recursion formula for Ramanujan's numbers $\tau(n)$. With a view to encouraging further researches on the subject, which is dealt with herein, we have chosen to cite a number of recently-published developments dealing extensively with the Ramanujan function $\tau(n)$.

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SOME INEQUALITIES INVOLVING GENERALIZED HYPERGEOMETRIC FUNCTIONS VIA DISCRETE JENSEN'S TYPE INEQUALITIES

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Abstract. Recently, an enormous amount of effort has been devoted to extending the gamma and beta functions because of their nice properties and interesting applications. The contribution of this paper falls within this framework. We devote our attention here to investigate some approximations for a class of Gauss hypergeometric functions by the use of some discrete Jensen's type inequalities.

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1. INTRODUCTION

In various areas of applied mathematics, several types of special functions become essential tools for scientists and engineers. For instance, we cite the gamma and beta functions that are very useful in modeling many phenomena arising in mathematical physics, probability theory, information theory and so on. Thus, many researchers have been interested by generalizing and extending these special functions, as we can find in a wide bibliography as for instance in [2, 3, 6–10] and in the related references cited therein.

In the current work, we are interested in giving further approximations for some classes of these generalizations. So, we will provide an overview on some generalized Gauss hypergeometric functions that will be used. We begin by recalling the Gauss hypergeometric function (GHF) that was defined in [9, p.1]

$${}_2F_1(a, b; c; z) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \quad \Re(a) > 0, \Re(b) > 0, \Re(c) > 0, \quad (1.1)$$

provided that this series is convergent. Here, the notation $(\lambda)_n$, for $\Re(\lambda) > 0$, refers to the Pochhammer symbol defined by

$$(\lambda)_n := \lambda(\lambda+1)\dots(\lambda+n-1) = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} \text{ and } (\lambda)_0 = 1,$$

where $\Gamma(x) := \int_0^\infty t^{x-1} e^{-t} dt$, $\Re(x) > 0$, is the standard gamma function.

The (GHF) can be written in terms of the classical beta function as follows [3]

$${}_2F_1(a, b; c; z) := \frac{1}{B(b, c-b)} \sum_{n=0}^{\infty} (a)_n B(b+n, c-b) \frac{z^n}{n!}, \quad (1.2)$$

provided that $0 < \Re(b) < \Re(c)$, where $B(x, y) := \int_0^1 t^{x-1} (1-t)^{y-1} dt$, $\Re(x) > 0$, $\Re(y) > 0$, is the standard beta function. The formula (1.2) was deduced from the following integral representation, [9, p.20]

$${}_2F_1(a, b; c; z) = \frac{1}{B(b, c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-zt)^{-a} dt, \quad |z| \leq 1. \quad (1.3)$$

The confluent hypergeometric function (CHF), also called Kummer's function, was defined in [4] by the following formula

$${}_1F_1(b; c; z) := \sum_{n=0}^{\infty} \frac{(b)_n}{(c)_n} \frac{z^n}{n!}, \quad \Re(b) > 0, \Re(c) > 0, \quad (1.4)$$

provided that this series is convergent. It can be written in terms of the classical beta function as follows [1, p.504]

$${}_1F_1(b; c; z) = \frac{1}{B(b, c-b)} \sum_{n=0}^{\infty} B(b+n, c-b) \frac{z^n}{n!}, \quad 0 < \Re(b) < \Re(c). \quad (1.5)$$

In integral form we have, [6]

$${}_1F_1(b; c; z) = \frac{1}{B(b, c-b)} \int_0^1 u^{b-1} (1-u)^{c-b-1} e^{zu} du. \quad (1.6)$$

The extension of Euler's beta function given by Chaudhy et al. in [2] has motivated Özergin et al. to introduce in [6] the following generalized beta function

$$B_p^{(\alpha, \beta)}(x, y) := \int_0^1 t^{x-1} (1-t)^{y-1} {}_1F_1\left(\alpha; \beta; \frac{-p}{t(1-t)}\right) dt, \quad (1.7)$$

where $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(p) > 0$.

Based on (1.7), Özergin et al. defined in [6] the generalized Gauss hypergeometric function (GGHF) and the generalized confluent hypergeometric function (GCHF), respectively, as follows

$${}_2F_1^{(\alpha, \beta; p)}(a, b; c; z) := \frac{1}{B(b, c-b)} \sum_{n=0}^{\infty} (a)_n B_p^{(\alpha, \beta)}(b+n, c-b) \frac{z^n}{n!}, \quad (1.8)$$

$${}_1F_1^{(\alpha, \beta; p)}(b; c; z) := \frac{1}{B(b, c-b)} \sum_{n=0}^{\infty} B_p^{(\alpha, \beta)}(b+n, c-b) \frac{z^n}{n!}. \quad (1.9)$$

It is clear that

$${}_2F_1^{(\alpha, \beta; 0)}(a, b; c; z) = {}_2F_1(a, b; c; z) \text{ and } {}_1F_1^{(\alpha, \beta; 0)}(b; c; z) = {}_1F_1(b; c; z). \quad (1.10)$$

In [6], the authors gave integral representations of (GGHF) and (GCHF) as follows

$${}_2F_1^{(\alpha, \beta; p)}(a, b; c; z) \quad (1.11)$$

$$= \frac{1}{B(b, c-b)} \times \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-zt)^{-a} {}_1F_1\left(\alpha; \beta; \frac{-p}{t(1-t)}\right) dt, \quad |z| \leq 1,$$

$${}_1F_1^{(\alpha, \beta; p)}(b; c; z) \quad (1.12)$$

$$= \frac{1}{B(b, c-b)} \times \int_0^1 t^{b-1} (1-t)^{c-b-1} e^{zt} {}_1F_1\left(\alpha; \beta; \frac{-p}{t(1-t)}\right) dt.$$

The fundamental goal of this paper is to state some inequalities concerning the special functions evoked above by the use of discrete Jensen's type inequalities.

2. STATEMENT OF MAIN RESULTS

To establish our main results, we will need the following lemma which concerns the so-called discrete Jensen's inequality [5, p.6].

Lemma 1. *Let I be a nonempty interval of $\mathbb{R}$ and let $f: I \rightarrow \mathbb{R}$ be a convex function. For any $x_0, x_1, x_2, \dots, x_n \in I$ and $m_0, m_1, m_2, \dots, m_n \geq 0$, with $\sum_{k=0}^n m_k > 0$, we have*

$$f\left(\frac{\sum_{k=0}^n m_k x_k}{\sum_{k=0}^n m_k}\right) \leq \frac{\sum_{k=0}^n m_k f(x_k)}{\sum_{k=0}^n m_k}. \quad (2.1)$$

If f is concave then inequality (2.1) is reversed.

Our first main result reads as follows.

Theorem 1. *Let $\alpha > 0, \beta > 0, 0 < b < c, z_1 > 0$ and $z_2 \in \mathbb{R}$. Then the following inequality*

$$\left({}_1F_1^{(\alpha, \beta; p)}(b; c; z_1 z_2)\right)^r \leq {}_1F_1^{(\alpha, \beta; p)}(b; c; z_1^r z_2) \times \left({}_1F_1^{(\alpha, \beta; p)}(b; c; z_2)\right)^{r-1} \quad (2.2)$$

holds for any $r \in (-\infty, 0] \cup [1, \infty)$. In particular, we have

$$\left({}_1F_1(b; c; z_1 z_2)\right)^r \leq {}_1F_1(b; c; z_1^r z_2) \times \left({}_1F_1(b; c; z_2)\right)^{r-1}. \quad (2.3)$$

If $r \in [0, 1]$ then (2.2) and (2.3) are reversed.

Proof. We consider the following positive sequences

$$x_k = z_1^k \text{ and } m_k = B_p^{(\alpha, \beta)}(b+k, c-b) \frac{z_2^k}{k!}, \quad k \geq 0,$$

and the function f defined on $(0, +\infty)$ by $f(t) = t^r$ which is convex if $r \in (-\infty, 0] \cup [1, \infty)$, and concave for $r \in [0, 1]$.

Assume that $r \in (-\infty, 0] \cup [1, \infty)$. Applying (2.1), we get

$$\left(\sum_{k=0}^n m_k x_k \right)^r \leq \left(\sum_{k=0}^n m_k x_k^r \right) \times \left(\sum_{k=0}^n m_k \right)^{r-1}. \quad (2.4)$$

In another part, we have

$$\sum_{k=0}^{\infty} m_k = \sum_{k=0}^{\infty} B_p^{(\alpha, \beta)}(b+k, c-b) \frac{z_2^k}{k!} = B(b, c-b) {}_1F_1^{(\alpha, \beta; p)}(b; c; z_2), \quad (2.5)$$

$$\sum_{k=0}^{\infty} m_k x_k = \sum_{k=0}^{\infty} B_p^{(\alpha, \beta)}(b+k, c-b) \frac{(z_1 z_2)^k}{k!} = B(b, c-b) {}_1F_1^{(\alpha, \beta; p)}(b; c; z_1 z_2) \quad (2.6)$$

and

$$\sum_{k=0}^{\infty} m_k x_k^r = \sum_{k=0}^{\infty} B_p^{(\alpha, \beta)}(b+k, c-b) \frac{(z_1^r z_2)^k}{k!} = B(b, c-b) {}_1F_1^{(\alpha, \beta; p)}(b; c; z_1^r z_2).$$

Letting $n \rightarrow \infty$ in (2.4) and then substituting these three latter expressions in (2.4) we get the desired inequality. Taking $p = 0$ in (2.2), we obtain (2.3).

If $r \in [0, 1]$, f is concave and Lemma 1 asserts that (2.2) and (2.3) should be reversed. $\square$

Note that, if $r \in [1, \infty)$ is integer then (2.2) is still valid for any $z_1 \in \mathbb{R}$. In another part, a simple mathematical induction shows that (2.2) can be generalized as recited in the following result.

Corollary 1. Let $\alpha > 0, \beta > 0, 0 < b < c, z_1 > 0$ and $z_2, z_3, \dots, z_n \in \mathbb{R}$. Then we have

$$\left({}_1F_1^{(\alpha, \beta; p)}\left(b; c; \prod_{i=1}^n z_i\right) \right)^r \leq {}_1F_1^{(\alpha, \beta; p)}\left(b; c; z_1^r \prod_{i=2}^n z_i\right) \times \left({}_1F_1^{(\alpha, \beta; p)}\left(b; c; \prod_{i=2}^n z_i\right) \right)^{r-1} \quad (2.7)$$

for any $r \in (-\infty, 0] \cup [1, \infty)$. In particular, we have

$$\left({}_1F_1\left(b; c; \prod_{i=1}^n z_i\right) \right)^r \leq {}_1F_1\left(b; c; z_1^r \prod_{i=2}^n z_i\right) \times \left({}_1F_1\left(b; c; \prod_{i=2}^n z_i\right) \right)^{r-1}. \quad (2.8)$$

If $r \in [0, 1]$ then (2.7) and (2.8) are reversed.

It is worth mentioning that (2.7) and (2.8) can be reiterated many times as explained in the following example.

Example 1. Let $r = 3$ and $z_1, z_2, z_3 \in \mathbb{R}$. If we apply (2.7) two times, we get

$$\begin{aligned} \left({}_1F_1^{(\alpha, \beta; p)}(b; c; z_1 z_2 z_3) \right)^3 &\leq {}_1F_1^{(\alpha, \beta; p)}(b; c; z_1^3 z_2 z_3) \times \left({}_1F_1^{(\alpha, \beta; p)}(b; c; z_2 z_3) \right)^2 \\ &\leq {}_1F_1^{(\alpha, \beta; p)}(b; c; z_1^3 z_2 z_3) \times {}_1F_1^{(\alpha, \beta; p)}(b; c; z_2^2 z_3) \\ &\quad \times {}_1F_1^{(\alpha, \beta; p)}(b; c; z_3). \end{aligned}$$

If moreover $0 \leq z_1 \leq 1$, $0 \leq z_2 \leq 1$ and $z_3 \geq 0$ then there holds

$$\begin{aligned} \left({}_1F_1^{(\alpha, \beta; p)}(b; c; z_1 z_2 z_3) \right)^3 &\leq {}_1F_1^{(\alpha, \beta; p)}(b; c; z_1 z_2 z_3) \times {}_1F_1^{(\alpha, \beta; p)}(b; c; z_2 z_3) \\ &\quad \times {}_1F_1^{(\alpha, \beta; p)}(b; c; z_3). \end{aligned}$$

Now, we state our second main result.

Theorem 2. Let $\alpha > 0, \beta > 0, a > 0, 0 < b < c, 0 < z_1 < 1$ and $|z_2| < 1$. Then the following inequality

$$\left({}_2F_1^{(\alpha, \beta; p)}(a, b; c; z_1 z_2) \right)^r \leq {}_2F_1^{(\alpha, \beta; p)}(a, b; c; z_1^r z_2) \times \left({}_2F_1^{(\alpha, \beta; p)}(a, b; c; z_2) \right)^{r-1} \quad (2.9)$$

holds for any $r \in (-\infty, 0] \cup [1, \infty)$. In particular, we have

$$\left({}_2F_1(a, b; c; z_1 z_2) \right)^r \leq {}_2F_1(a, b; c; z_1^r z_2) \times \left({}_2F_1(a, b; c; z_2) \right)^{r-1}. \quad (2.10)$$

If $r \in [0, 1]$ then (2.9) and (2.10) are reversed.

Proof. Here, we consider the following positive sequences,

$$x_k = z_1^k \text{ and } m_k = (a)_k B_p^{(\alpha, \beta)}(b+k, c-b) \frac{z_2^k}{k!}, \quad k \geq 0,$$

with the same function $f(t) = t^r$, $t \in (0, \infty)$. The details are similar to those of the proof of Theorem 2 and therefore omitted here for the reader. $\square$

We also left to the reader the routine task for formulating analogous statements as those of Corollary 1 and Example 1 when ${}_1F_1(b; c; z)$ is replaced by ${}_2F_1(a, b; c; z)$. Otherwise, another main result is stated in the following.

Theorem 3. Let $\alpha > 0, \beta > 0, 0 < b < c$ and $z_1, z_2 > 0$. Then we have

$$\frac{{}_1F_1^{(\alpha, \beta; p)}(b; c; z_1 z_2)}{{}_1F_1^{(\alpha, \beta; p)}(b; c; z_2)} \geq \exp \left(\frac{b}{c} z_2 (\log z_1) \frac{{}_1F_1^{(\alpha, \beta; p)}(b+1; c+1; z_2)}{{}_1F_1^{(\alpha, \beta; p)}(b; c; z_2)} \right). \quad (2.11)$$

Proof. We also consider the following sequences

$$x_k = z_1^k \text{ and } m_k = B_p^{(\alpha, \beta)}(b+k, c-b) \frac{z_2^k}{k!}, \quad k \geq 0$$

and the concave function defined on $(0, +\infty)$ by $f(t) = \log t$. Applying the reverse of (2.1), we get (after a simple manipulation)

$$\log \left(\sum_{k=0}^n m_k x_k \right) \geq \frac{\sum_{k=0}^n m_k \log x_k}{\sum_{k=0}^n m_k} + \log \left(\sum_{k=0}^n m_k \right). \quad (2.12)$$

In another part, we have

$$\begin{aligned} \sum_{k=0}^{\infty} m_k \log x_k &= (\log z_1) \sum_{k=1}^{\infty} B_p^{(\alpha, \beta)}(b+k, c-b) \frac{z_2^k}{(k-1)!} \\ &= z_2 (\log z_1) \sum_{k=1}^{\infty} B_p^{(\alpha, \beta)}(b+k, c-b) \frac{z_2^{k-1}}{(k-1)!} = z_2 (\log z_1) \sum_{k=0}^{\infty} B_p^{(\alpha, \beta)}(b+k+1, c-b) \frac{z_2^k}{k!}. \end{aligned}$$

This, with (1.9) and the fact that $B(b+1, c-b) = \frac{b}{c} B(b, c-b)$, respectively yields

$$\begin{aligned} \sum_{k=0}^{\infty} m_k \log x_k &= z_2 (\log z_1) B(b+1, c-b) {}_1F_1^{(\alpha, \beta; p)}(b+1; c+1; z_2) \\ &= \frac{b}{c} z_2 (\log z_1) B(b, c-b) {}_1F_1^{(\alpha, \beta; p)}(b+1; c+1; z_2). \end{aligned} \quad (2.13)$$

Letting $n \rightarrow \infty$ in (2.12) and then utilizing (2.5), (2.6) and (2.13), we get

$$\begin{aligned} &\log \left(B(b, c-b) {}_1F_1^{(\alpha, \beta; p)}(b; c; z_1 z_2) \right) \\ &\geq \frac{b}{c} \frac{z_2 (\log z_1) {}_1F_1^{(\alpha, \beta; p)}(b+1; c+1; z_2)}{{}_1F_1^{(\alpha, \beta; p)}(b; c; z_2)} + \log \left(B(b, c-b) {}_1F_1^{(\alpha, \beta; p)}(b; c; z_2) \right), \end{aligned}$$

which, after simple algebraic operations, is equivalent to (2.11). $\square$

From the previous theorem we can deduce the following result.

Corollary 2. Let $\alpha > 0, \beta > 0, 0 < b < c$ and $z_1, z_2, \dots, z_n > 0$. Then we have

$$\frac{{}_1F_1^{(\alpha, \beta; p)}(b; c; \prod_{i=1}^n z_i)}{{}_1F_1^{(\alpha, \beta; p)}(b; c; z_n)} \geq \prod_{i=1}^n \exp \left(\frac{b}{c} z_n (\log z_i) \frac{{}_1F_1^{(\alpha, \beta; p)}(b+1; c+1; z_n)}{{}_1F_1^{(\alpha, \beta; p)}(b; c; z_n)} \right). \quad (2.14)$$

Proof. According to (2.11) we have

$$\frac{{}_1F_1^{(\alpha, \beta; p)}(b; c; \prod_{i=1}^n z_i)}{{}_1F_1^{(\alpha, \beta; p)}(b; c; z_n)} \geq \exp \left(\frac{b}{c} z_n \left(\log \prod_{i=1}^{n-1} z_i \right) \frac{{}_1F_1^{(\alpha, \beta; p)}(b+1; c+1; z_n)}{{}_1F_1^{(\alpha, \beta; p)}(b; c; z_n)} \right),$$

from which the desired inequality follows by simple elementary properties of logarithm and exponential. $\square$

Remark 1. It is worth noting that inequalities similar to (2.11) can be established by considering logarithmic functions of different bases.

Finally, the following main result is also stated.

Theorem 4. Let $\alpha > 0, \beta > 0, a > 0, 0 < b < c$ and $0 < z_1, z_2 < 1$. Then we have

$$\frac{{}_2F_1^{(\alpha, \beta; p)}(a, b; c; z_1 z_2)}{{}_2F_1^{(\alpha, \beta; p)}(a, b; c; z_2)} \geq \exp \left(\frac{ab}{c} z_2 (\log z_1) \frac{{}_2F_1^{(\alpha, \beta; p)}(a+1, b+1; c+1; z_2)}{c B(b, c-b) {}_2F_1^{(\alpha, \beta; p)}(a, b; c; z_2)} \right). \quad (2.15)$$

Proof. We consider here the following positive sequences

$$x_k = z_1^k \text{ and } m_k = (a)_k B_p^{(\alpha, \beta)}(b+k, c-b) \frac{z_2^k}{k!}, \quad k \geq 0$$

and also the concave function $f(t) = \log t, t \in (0, \infty)$. The remainder of the proof is similar to that of the previous theorem and we therefore omitted here. $\square$

Remark 2. By similar way, we can deduce from (2.15) a generalized inequality analogous to (2.14). We left it to the reader.

3. CONCLUSION

In this paper, we focused on establishing some approximations for the generalized Gauss hypergeometric and the generalized confluent hypergeometric functions. Employing the discrete Jensen's inequality, several analytic inequalities involving these functions have been provided. Related to these inequalities, we consider that it will be interesting to study in a future work the following two subjects. The tightness of the bounds in the inequalities pointed out throughout this paper and their applications in studying the behavior of the evoked hypergeometric functions.

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GENERALIZED STONEAN BE-ALGEBRAS

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Abstract. In this paper, the notion of a generalized Stonean BE -algebra is introduced. A set of equivalent conditions is given for every quasi-complemented BE -algebra to become a generalized Stonean BE -algebra. A necessary and sufficient condition is stated for a self-distributive and commutative BE -algebra to become a generalized Stonean BE -algebra. The concept of Stonean filters is introduced and then generalized Stonean BE -algebras are characterized by Stonean filters. The notion of hyper Stonean BE -algebras is introduced and then a characterization theorem in terms of prime filters is given.

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1. INTRODUCTION

The notion of BE -algebras was introduced and extensively studied by H.S. Kim and Y.H. Kim in [3]. These classes of BE -algebras were introduced as a generalization of the class of BCK -algebras of K. Iseki and S. Tanaka [2]. Some properties of filters of BE -algebras were studied by S.S. Ahn and Y.H. Kim in [1] and by B.L. Meng in [8]. In [12], A. Walendziak discussed some significant properties of commutative BE -algebras. He also investigated the relationship between BE -algebras, implicative algebras and J -algebras. In [8], Meng introduced the notion of prime filters in BCK -algebras, and then gave a description of the filter generated by a set, and obtained some of fundamental properties of prime filters. Some properties of prime ideals are investigated in BCK -algebras [2]. The first author studied some properties of prime filters in BE -algebras. Also, the author extensively studied the algebraic as well as the topological properties of prime filters of commutative BE -algebras [11]. The notion of dual annihilators of commutative BE -algebra is introduced and studied extensively the properties of these dual annihilators [4]. In 2020, the notions of regular filters [6] and O -filters [5] in commutative BE -algebras are introduced and studied the interconnection between those two special classes of filters. A. Soleimani

Nasab and A. Borumand Saeid introduced the notion of Stonean implicative filters in Hilbert algebras and characterized the Stonean Hilbert algebras with the help of Stonean implicative filters [9].

In this paper, certain properties of dual annihilator filters, prime filters, and minimal prime filters of commutative *BE*-algebras are investigated. The notion of generalized Stonean *BE*-algebras is introduced. It is proved that a self-distributive and commutative *BE*-algebra will become a generalized Stonean *BE*-algebra whenever every prime filter of the *BE*-algebra is minimal. It is observed that every generalized Stonean *BE*-algebra is a quasi-complemented *BE*-algebra, the other direction is not always true. However, a set of equivalent conditions is given for every quasi-complemented *BE*-algebra to become a generalized Stonean *BE*-algebra. Some necessary conditions of generalized Stonean *BE*-algebras are derived. Generalized Stonean *BE*-algebras are also characterized with the help of σ -filters, dual annihilator filters and regular filters.

Filters are important substructures in a *BE*-algebra and play an important role. It is well understood that filters are the kernels of congruences. Filter theory is crucial in the study of any class of logical algebras. From a logical standpoint, different filters correspond to different sets of valid formulas in an appropriate logic. Designing various types of filters in some logical algebra, on the other hand, is also algebraically interesting. With this motivation, we introduce the concept of Stonean filters is introduced in commutative *BE*-algebras. Some sufficient conditions are derived for every filter of a self-distributive and commutative *BE*-algebra to become a Stonean filter. The notion of hyper Stonean *BE*-algebras is introduced and observed that every hyper Stonean *BE*-algebra is generalized Stonean *BE*-algebra. Though every generalized Stonean *BE*-algebra need not to be a hyper *BE*-algebra, however, a sufficient condition is derived for every generalized Stonean *BE*-algebra to become a hyper *BE*-algebra. The class of hyper Stonean *BE*-algebras is characterized with the help of prime filters of self-distributive and commutative *BE*-algebras. It is observed that every maximal filter of a self-distributive and commutative *BE*-algebra is Stonean filter. Some equivalent conditions are given for every filter of a commutative *BE*-algebra to become a Stonean filter which leads to a characterization of generalized Stonean *BE*-algebras. Finally, an extension property of Stonean filters of commutative *BE*-algebras is derived.

2. PRELIMINARIES

In this section, we present certain definitions and results which are taken mostly from the papers [1], [3], [11], [7], [4], [6], and [5] for the ready reference.

Definition 1. [3] An algebra $(X, *, 1)$ of type $(2, 0)$ is called a *BE*-algebra if it satisfies the following properties:

- (1) $x * x = 1$,
- (2) $x * 1 = 1$,

- (3) $1 * x = x$,
- (4) $x * (y * z) = y * (x * z)$ for all $x, y, z \in X$.

A *BE*-algebra X is called *self-distributive* if $x * (y * z) = (x * y) * (x * z)$ for all $x, y, z \in X$. A *BE*-algebra X is called *transitive* if $y * z \leq (x * y) * (x * z)$ for all $x, y, z \in X$. A *BE*-algebra X is called *commutative* if $(x * y) * y = (y * x) * x$ for all $x, y \in X$. Every commutative *BE*-algebra is transitive. For any $x, y \in X$, define $x \vee y = (y * x) * x$. If X is commutative, then $(X, \vee)$ is a semilattice [12]. We introduce a relation $\leq$ on a *BE*-algebra X by $x \leq y$ if and only if $x * y = 1$ for all $x, y \in X$. Clearly $\leq$ is reflexive. If X is commutative, then $\leq$ is transitive, anti-symmetric and hence a partial order on X .

Theorem 1. [3] *Let X be a transitive *BE*-algebra and $x, y, z \in X$. Then*

- (1) $1 \leq x$ implies $x = 1$,
- (2) $y \leq z$ implies $x * y \leq x * z$ and $z * x \leq y * x$.

Definition 2. [1] A non-empty subset F of a *BE*-algebra X is called a filter of X if, for all $x, y \in X$, it satisfies the following properties:

- (1) $1 \in F$,
- (2) $x \in F$ and $x * y \in F$ imply that $y \in F$.

For any non-empty subset A of a transitive *BE*-algebra X , the set $\langle A \rangle = \{x \in X \mid a_1 * (a_2 * (\dots * (a_n * x) \dots)) = 1 \text{ for some } a_1, a_2, \dots, a_n \in A\}$ is the smallest filter containing A . For any $a \in X$, $\langle a \rangle = \{x \in X \mid a^n * x = 1 \text{ for some } n \in \mathbb{N}\}$, where $a^n * x = a * (a * (\dots * (a * x) \dots))$ with the repetition of a is n times, is called *the principal filter generated a* . If X is self-distributive, then $\langle a \rangle = \{x \in X \mid a * x = 1\}$. If X is commutative and self-distributive, then $\langle a \rangle \cap \langle b \rangle = \langle a \vee b \rangle$ for any $a, b \in X$. Let F be a filter of a transitive *BE*-algebra and $a \in X$, then $\langle F \cup \{a\} \rangle = \{x \in X \mid a^n * x \in F \text{ for some } n \in \mathbb{N}\}$. A proper filter P of a *BE*-algebra X is called *prime* [11] if $F \cap G \subseteq P$ implies $F \subseteq P$ or $G \subseteq P$ for any two proper filters F, G of X . A proper filter P of a *BE*-algebra is prime if and only if $\langle x \rangle \cap \langle y \rangle \subseteq P$ implies $x \in P$ or $y \in P$ for any $x, y \in X$. A proper filter M of a transitive *BE*-algebra is called *maximal* if there exists no proper filter Q such that $M \subset Q$. Every maximal filter of a commutative *BE*-algebra is prime.

Theorem 2. [11] *Let F and G be two filters of a transitive *BE*-algebra X . Then*

$$F \vee G = \{x \in X \mid a * (b * x) = 1 \text{ for some } a \in F, b \in G\}$$

is the supremum of F and G . Hence the set $\mathcal{F}(X)$ of all filters of X is a lattice with respect to the operation $\vee$.

Lemma 1. [4] *Let X be a commutative *BE*-algebra. Then for any $x, y, a \in X$*

- (1) $y * z \leq (z * x) * (y * x)$,
- (2) $(x * y) \vee a \leq (x \vee a) * (y \vee a)$.

For any non-empty subset A of a BE -algebra X , the *dual annihilator* [4] of A is defined as $A^+ = \{x \in X \mid x \vee a = 1 \text{ for all } a \in A\}$. In a commutative BE -algebra X , the set A^+ forms a filter of X such that $A \cap A^+ = \{1\}$. In case of $A = \{a\}$, we have $(a)^+ = \{x \in X \mid a \vee x = 1\}$. Clearly $X^+ = \{1\}$ and $\{1\}^+ = X$. An element $a \in X$ is called *dual dense* if $(a)^+ = \{1\}$.

Proposition 1. [4] Let X be a commutative BE -algebra and $\emptyset \neq A, B \subseteq X$. Then

- (1) if $A \subseteq B$, then $B^+ \subseteq A^+$,
- (2) $A \subseteq A^{++}$,
- (3) $A^+ = A^{+++}$.

Proposition 2. [4] Let F, G be two filters of a commutative BE -algebra X . Then

- (1) $F \cap G = \{1\}$ if and only if $F \subseteq G^+$,
- (2) $(F \vee G)^+ = F^+ \cap G^+$,
- (3) $(F \cap G)^{++} = F^{++} \cap G^{++}$.

Proposition 3. [4] Let X be a commutative BE -algebra and $a, b \in X$. Then

- (1) $\langle a \rangle \subseteq (a)^{++}$,
- (2) $a \leq b$ implies $(a)^+ \subseteq (b)^+$,
- (3) $a \in (b)^{++}$ implies $(b)^+ \subseteq (a)^+$.

A prime filter P of a commutative BE -algebra X is called *minimal* [7] if it is minimal in the class of all prime filters of X .

Theorem 3. [7] Let X be a self-distributive and commutative BE -algebra. A prime filter P of X is minimal if and only if to each $x \in P$, there exists $y \notin P$ such that $x \vee y = 1$.

Proposition 4. [6] Let X be a self-distributive and commutative BE -algebra. Then for any $x \in X$, we have

$$(x)^+ = \bigcap \{P \mid P \text{ is a minimal prime filter such that } x \notin P\}$$

A filter F of a commutative BE -algebra X is called a *dual annihilator filter* [4] if $F = F^{++}$. A filter F of a commutative BE -algebra X is called a *regular filter* [6] if $(x)^{++} \subseteq F$ whenever $x \in F$. Every dual annihilator filter is a regular filter.

Proposition 5. [6] Every minimal prime filter of a self-distributive and commutative BE -algebra is a regular filter.

A filter F of a commutative BE -algebra X is called an *O-filter* [5] if $F = O(S)$ for some $\vee$ -closed subset S of X . Every O-filter of a commutative BE -algebra is a regular filter. A commutative BE -algebra X is called *quasi-complemented* [5] if to each $x \in X$, there exists $y \in X$ such that $x \vee y = 1$ and $(x)^+ \cap (y)^+ = \{1\}$.

Theorem 4. [5] A commutative BE -algebra X is quasi-complemented if and only if to each $x \in X$, there exists $y \in X$ such that $(x)^{++} = (y)^+$

A filter F of a commutative BE -algebra X is called a σ -filter [10] of X if $\sigma(F) = F$ where $\sigma(F) = \{x \in X \mid (x)^+ \vee F = X\}$. Every σ -filter of a commutative BE -algebra is a regular filter and every σ -filter of a commutative BE -algebra is an O -filter.

3. PROPERTIES OF GENERALIZED STONEAN BE-ALGEBRAS

In this section, the notion of generalized Stonean BE -algebras is introduced. A set of equivalent conditions is derived for a quasi-complemented BE -algebra to become a generalized Stonean BE -algebra. It is proved that every quasi-complemented BE -algebra is a generalized Stonean if and only if $\mathcal{D}^+(X) = \{(x)^+ \mid x \in X\}$ is a Boolean algebra.

Lemma 2. *Let X be a commutative BE -algebra and $a, b \in X$. Then the following properties hold:*

- (1) $(a)^+ = (\langle a \rangle)^+$;
- (2) $a \in (b)^+$ if and only if $\langle a \rangle \subseteq (b)^+$;
- (3) $\langle a \rangle \cap \langle b \rangle = \{1\}$ if and only if $\langle a \rangle \subseteq (b)^+$;
- (4) $(a)^+ \cap (a * b)^+ \subseteq (b)^+$;
- (5) $(a)^{++} \cap (b)^{++} = (a \vee b)^{++}$.

Proof. (1) Since $\{a\} \subseteq \langle a \rangle$, we get $(\langle a \rangle)^+ \subseteq (a)^+$. Conversely, let $x \in (a)^+$. Then $a \vee x = 1$. For any $c \in \langle a \rangle$, we get $a^n * c = 1$ for some positive integer n . Now

$$1 = a^n * c \leq (a^n * c) \vee x \leq (a \vee x)^n * (c \vee x) = 1 \vee (c \vee x) = c \vee x$$

Hence $c \vee x = 1$ for any $c \in \langle a \rangle$. Then $x \in (\langle a \rangle)^+$. Therefore $(a)^+ \subseteq (\langle a \rangle)^+$.

- (2) Assume that $a \in (b)^+$. Then $a \vee b = 1$. Let $x \in \langle a \rangle$. Then there exists a positive integer n such that $a^n * x = 1$. Now, we get

$$\begin{aligned} 1 &= a^n * x \leq (a^n * x) \vee b \leq (a \vee b)^n * (x \vee b) \quad \text{by Lemma 1(2)} \\ &= 1 * (x \vee b) = x \vee b \end{aligned}$$

which yields that $x \in (b)^+$. Therefore $\langle a \rangle \subseteq (b)^+$. Converse is clear since $a \in \langle a \rangle$.

- (3) Let $a, b \in L$. Assume that $\langle a \rangle \cap \langle b \rangle = \{1\}$. Let $x \in \langle a \rangle$. For any $y \in \langle b \rangle$, we get $x \vee y \in \langle a \rangle \cap \langle b \rangle = \{1\}$. Hence $x \vee y = 1$ for any $y \in \langle b \rangle$. Thus $x \in (\langle b \rangle)^+ = (b)^+$. Therefore $\langle a \rangle \subseteq (b)^+$. Conversely, assume that $\langle a \rangle \subseteq (b)^+$. Let $x \in \langle a \rangle \cap \langle b \rangle$. Then $x \in \langle a \rangle \subseteq (b)^+$ and $x \in \langle b \rangle$. Hence $x \in \langle b \rangle \cap (b)^+ = \{1\}$. Thus $x = 1$. Therefore $\langle a \rangle \cap \langle b \rangle = \{1\}$.

- (4) Let $x \in (a)^+ \cap (a * b)^+$. Then $a \vee x = 1$ and $(a * b) \vee x = 1$. Hence

$$\begin{aligned} 1 &= (a * b) \vee x \leq (a \vee x) * (b \vee x) \quad \text{by Lemma 1(2)} \\ &= 1 * (b \vee x) = b \vee x \end{aligned}$$

which means $b \vee x = 1$. Hence $x \in (b)^+$. Therefore $(a)^+ \cap (a * b)^+ \subseteq (b)^+$.

- (5) Let $a, b \in X$. Since $a, b \leq a \vee b$, we get $(a)^+, (b)^+ \subseteq (a \vee b)^+$. Hence $(a \vee b)^{++} \subseteq (a)^{++}, (b)^{++}$. Thus $(a \vee b)^{++} \subseteq (a)^{++} \cap (b)^{++}$. Conversely, let $x \in (a)^{++} \cap (b)^{++}$. Suppose $y \in (a \vee b)^+$ be an arbitrary element. Since $y \in (a \vee b)^+$, we get

$$\begin{aligned} y \vee (a \vee b) = 1 &\Rightarrow y \vee a \in (b)^+ \Rightarrow x \vee y \vee a = 1 && \text{since } x \in (b)^{++} \\ &\Rightarrow x \vee y \in (a)^+ \Rightarrow x \vee (x \vee y) = 1 && \text{since } x \in (a)^{++} \\ &\Rightarrow x \vee y = 1 && \text{for all } y \in (a \vee b)^+ \end{aligned}$$

which means that $x \in (a \vee b)^{++}$. Therefore $(a)^{++} \cap (b)^{++} \subseteq (a \vee b)^{++}$. $\square$

Proposition 6. *Let X be a commutative BE-algebra and $a, b \in X$. Then the following assertions are equivalent:*

- (1) $a \vee b = 1$;
- (2) $(a)^{++} \cap \langle b \rangle = \{1\}$;
- (3) $(a)^{++} \cap (b)^{++} = \{1\}$.

Proof. (1) $\Rightarrow$ (2): Let $a, b \in X$. Assume that $a \vee b = 1$. Then $b \in (a)^+$. By Lemma 2(1), we get $\langle b \rangle \subseteq (a)^+$. Hence $(a)^{++} \cap \langle b \rangle \subseteq (a)^{++} \cap (a)^+ = \{1\}$.
 (2) $\Rightarrow$ (3): Assume that $(a)^{++} \cap \langle b \rangle = \{1\}$ for any $a, b \in X$. By Lemma 2(3), we get $(a)^{++} \subseteq (b)^+$. Therefore $(a)^{++} \cap (b)^{++} \subseteq (b)^+ \cap (b)^{++} = \{1\}$.
 (3) $\Rightarrow$ (1): Assume that $(a)^{++} \cap (b)^{++} = \{1\}$ for any $a, b \in X$. By Lemma 2(3), we get $(a)^{++} \subseteq (b)^{++} = (b)^+$. Hence $a \in (a)^{++} \subseteq (b)^+$, which means $a \vee b = 1$. $\square$

Lemma 3. *Every prime filter of a BE-algebra contains a minimal prime filter.*

Proof. Let P be a prime filter of a BE-algebra X . Consider

$$\mathfrak{S} = \{Q \mid Q \text{ is a prime filter such that } Q \subseteq P\}.$$

Clearly $P \in \mathfrak{S}$ and hence $\mathfrak{S} \neq \emptyset$. By Zorn's Lemma, $\mathfrak{S}$ has a minimal element, say P_0 . Clearly P_0 is the required minimal prime filter in X . $\square$

Proposition 7. *The intersection of all minimal prime filters of a BE-algebra is $\{1\}$.*

Proof. Clearly $\{1\} \subseteq \bigcap \{P \mid P \text{ is a minimal prime filter}\}$. Let $x \neq 1$ or $x \notin \langle 1 \rangle$. Then there exists a prime filter P such that $x \notin P$. By Lemma 3, there exists a minimal prime filter P_0 of X such that $P_0 \subseteq P$. Since $x \notin P$, we must have $x \notin P_0$. Hence $x \notin \bigcap \{P \mid P \text{ is a minimal prime filter}\}$. Thus $\bigcap \{P \mid P \text{ is a minimal prime filter}\} \subseteq \{1\}$. Therefore $\{1\} = \bigcap \{P \mid P \text{ is a minimal prime filter}\}$. $\square$

Theorem 5. *A prime filter P of a self-distributive and commutative BE-algebra X is minimal if and only if it satisfies the following condition:*

$$x \notin P \text{ if and only if } (x)^+ \subseteq P$$

Proof. Assume that P is minimal. Let $x \notin P$. Then clearly $(x)^+ \subseteq P$. Conversely, let $(x)^+ \subseteq P$. Suppose $x \in P$. Since P is minimal, there exists $y \notin P$ such that $x \vee y = 1$. Hence $y \in (x)^+$ and $y \notin P$. Thus $(x)^+ \not\subseteq P$, which is a contradiction. Therefore $x \notin P$.

Conversely, assume that X satisfies in the above condition. Let $x \in P$. By the assumed condition, we get $(x)^+ \not\subseteq P$. Hence, there exists $y \in (x)^+$ such that $y \notin P$. Thus $y \vee x = 1$ where $y \notin P$. By Theorem 5, P is a minimal prime filter of X . $\square$

Definition 3. A commutative BE -algebra X is called a generalized Stonean BE -algebra if $(x)^+ \vee (x)^{++} = X$ for all $x \in X$.

Example 1. Let $X = \{1, a, b, c\}$ be a set. Define a binary operation $*$ on X and then deduce the operation $\vee$ from $*$ as given in the following tables:

$*$	1	a	b	c
1	1	a	b	c
a	1	1	b	c
b	1	a	1	c
c	1	a	b	1

$\vee$	1	a	b	c
1	1	1	1	1
a	1	a	1	1
b	1	1	b	1
c	1	1	1	c

Then clearly $(X, *, \vee, 1)$ is a commutative BE -algebra. Clearly $(a)^+ = \{1, b, c\}$; $(b)^+ = \{1, a, c\}$ and $(c)^+ = \{1, a, b\}$. Hence, it can be seen that $(a)^{++} = \{1, a\}$; $(b)^{++} = \{1, b\}$; $(c)^{++} = \{1, c\}$. It can be easily verified that $(a)^+ \vee (a)^{++} = X$; $(b)^+ \vee (b)^{++} = X$ and $(c)^+ \vee (c)^{++} = X$. Therefore X is a generalized Stonean BE -algebra.

Example 2. Let $X = \{1, a, b, c, d\}$ and define a binary operation $*$ on X and then deduce the operation $\vee$ from $*$ as given in the following tables:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	b
b	1	a	1	b	a
c	1	a	1	1	a
d	1	1	1	b	1

$\vee$	1	a	b	c	d
1	1	1	1	1	1
a	1	a	1	1	a
b	1	1	b	d	b
c	1	1	d	c	b
d	1	a	b	b	d

Clearly $(X, *, \vee, 1)$ is a commutative BE -algebra. It is easy to check that $(b)^+ = \{1, a\}$ and $(b)^{++} = \{1, b, c\}$. Hence $(b)^+ \vee (b)^{++} = \{1, a\} \vee \{1, b, c\} = X$. Similarly, we can see that $(a)^+ \vee (a)^{++} = X$; $(c)^+ \vee (c)^{++} = X$ and $(d)^+ \vee (d)^{++} = X$. Therefore X is a generalized Stonean BE -algebra.

Proposition 8. If every prime filter of a self-distributive and commutative BE -algebra X is minimal, then X is a generalized Stonean BE -algebra.

Proof. Assume that every prime filter of a self-distributive and commutative BE -algebra X is minimal. Let $x \in X$. Suppose $(x)^+ \vee (x)^{++} \neq X$. Then there exists a maximal filter P of X such that $(x)^+ \vee (x)^{++} \subseteq P$. Since every maximal filter is prime, P is a prime filter of X . Hence $(x)^+ \subseteq P$ and $(x)^{++} \subseteq P$. Since $(x)^+ \subseteq P$, by Theorem

5, we get $x \notin P$. Clearly $x \in (x)^{++} \subseteq P$. Hence $x \in P$, which is a contradiction. Thus $(x)^+ \vee (x)^{++} = X$. Therefore X is a generalized Stonean BE -algebra. $\square$

The converse of Proposition 8 is not true. That is, every generalized Stonean BE -algebra need not to have all of its prime filters to be minimal. Indeed, consider the following example:

Example 3. Let $X = \{1, a, b, c\}$ be a set. Define a binary operation $*$ on X and then deduce the operation $\vee$ from $*$ as given in the following tables:

$*$	1	a	b	c
1	1	a	b	c
a	1	1	1	1
b	1	b	1	1
c	1	c	c	1

$\vee$	1	a	b	c
1	1	1	1	1
a	1	a	b	c
b	1	b	b	c
c	1	c	c	c

Then clearly $(X, *, \vee, 1)$ is a self-distributive and commutative BE -algebra. Observe that $(x)^+ = \{1\}$ for all $x \in X$. Hence $(x)^+ \vee (x)^{++} = X$ for all $x \in X$. Therefore X is a generalized Stonean BE -algebra. Clearly $\{1\}$ is a prime filter of X . It can be easily verified that $P = \{1, c\}$ is a prime filter of X which is not minimal.

Proposition 9. *Every generalized Stonean BE -algebra with a dual dense element is a quasi-complemented BE -algebra.*

Proof. Suppose X is generalized Stonean. Let $d \in X$ be such that $(d)^+ = \{1\}$. Let $x \in X$. Then $(x)^+ \vee (x)^{++} = X$. Hence $d \in (x)^+ \vee (x)^{++}$, which implies $a * (b * d) = 1$ for some $a \in (x)^+$ and $b \in (x)^{++}$. Since $b \in (x)^{++}$, we get $(b)^{++} \subseteq (x)^{++}$. Thus

$$\begin{aligned}
 a * (b * d) = 1 &\Rightarrow a \leq b * d \Rightarrow (a)^+ \subseteq (b * d)^+ \\
 &\Rightarrow (a)^+ \cap (b)^+ \subseteq (b)^+ \cap (b * d)^+ \\
 &\Rightarrow (a)^+ \cap (b)^+ \subseteq (d)^+ = \{1\} && \text{by Lemma 2(4)} \\
 &\Rightarrow (a)^+ \cap (b)^+ = \{1\} \Rightarrow (a)^+ \subseteq (b)^{++} && \text{by Lemma 2(3)} \\
 &\Rightarrow (a)^+ \subseteq (x)^{++} && \text{since } (b)^{++} \subseteq (x)^{++} \\
 &\Rightarrow (a)^+ \cap (x)^+ = \{1\} && \text{by Lemma 2(3)}
 \end{aligned}$$

Since $a \in (x)^+$, we get $a \vee x = 1$. Therefore L is quasi-complemented. $\square$

The importance of the sufficient condition of having a dual-dense element can be seen in Example 1. Clearly X is a generalized Stonean BE -algebra. Observe that X has no dual-dense element. For $a \in X$, there exists no $x \in X$ such that $a \vee x = 1$ and $(a)^+ \cap (x)^+ = \{1\}$. Therefore X is not a quasi-complemented BE -algebra.

A filter F of a BE -algebra X is called a factor of X if there exists a proper filter G such that $F \cap G = \{1\}$ and $F \vee G = X$. Denote $\mathcal{D}^{++}(X) = \{(x)^{++} \mid x \in X\}$ and $\mathcal{D}^+(X) = \{(x)^+ \mid x \in X\}$. The converse of Proposition 9 need not be true. However,

in the following, a set of equivalent conditions is given to show that every quasi-complemented *BE*-algebra to become a generalized Stonean *BE*-algebra.

Theorem 6. *Let X be a quasi-complemented *BE*-algebra. Then the following assertions are equivalent:*

- (1) X is a generalized Stonean *BE*-algebra;
- (2) each $(x)^+$ is a factor of X ;
- (3) for each $x \in X$, there exists $x' \in X$ such that $(x)^+ \vee (x')^+ = X$;
- (4) for any $x, y \in X$, $(x)^+ \vee (y)^+ = (x \vee y)^+$;
- (5) $\mathcal{D}^{++}(X)$ is a sublattice of the lattice $\mathcal{F}(X)$ of all filters of X .

Proof. (1) $\Rightarrow$ (2): Assume that X is generalized Stonean. Let $x \in X$. Clearly $(x)^+ \cap (x)^{++} = \{1\}$. By (1), we get $(x)^+ \vee (x)^{++} = X$. Therefore $(x)^+$ is a factor of X .
 (2) $\Rightarrow$ (3): Assume condition (2). Let $x \in X$. Since X is quasi-complemented, there exists $x' \in X$ such that $(x)^{++} = (x')^+$. Since $(x)^+$ is a factor of X , there exists a filter G such that $(x)^+ \cap G = \{1\}$ and $(x)^+ \vee G = X$. Since $(x)^+ \cap G = \{1\}$, we get $G \subseteq (x)^{++} = (x')^+$. Therefore $X = (x)^+ \vee G \subseteq (x)^+ \vee (x')^+$. Hence $(x)^+ \vee (x')^+ = X$.
 (3) $\Rightarrow$ (4): Assume condition (3). Let $x, y \in X$. By (3), there exists $x' \in X$ such that $(x)^+ \vee (x')^+ = X$. Clearly $(x)^+ \vee (y)^+ \subseteq (x \vee y)^+$. Conversely, let $a \in (x \vee y)^+$. Then $a \vee x \vee y = 1$, which gives $a \vee y \in (x)^+$. By Proposition 1.7(2) and Lemma 3.1, we get

$$\begin{aligned}
 a \vee y \in (x)^+ &\Rightarrow (x)^{++} \subseteq (a \vee y)^+ \Rightarrow (x)^{++} \cap (a \vee y)^{++} = \{1\} && \text{by Lemma 2(3)} \\
 &\Rightarrow (x)^{++} \cap (a)^{++} \cap (y)^{++} = \{1\} && \text{by Lemma 2(5)} \\
 &\Rightarrow (x)^{++} \cap (a)^{++} \subseteq (y)^+ && \text{by Lemma 2(3)} \\
 &\Rightarrow (x')^+ \cap (a)^{++} \subseteq (y)^+ \Rightarrow (x')^+ \cap \langle a \rangle \subseteq (y)^+ && \text{by Proposition 3(1)}
 \end{aligned}$$

Clearly $(x)^+ \cap \langle a \rangle \subseteq (x)^+$. Hence $a \in \langle a \rangle = X \cap \langle a \rangle = \{(x)^+ \vee (x')^+\} \cap \langle a \rangle = \{(x)^+ \cap \langle a \rangle\} \vee \{(x')^+ \cap \langle a \rangle\} \subseteq (x)^+ \vee (y)^+$. Therefore $(x \vee y)^+ \subseteq (x)^+ \vee (y)^+$.

(4) $\Rightarrow$ (5): For any $x, y \in X$, it is clear that $(x)^{++} \cap (y)^{++} = (x \vee y)^{++}$. Since X is quasi-complemented, there exist $x', y' \in X$ such that $(x)^{++} = (x')^+$ and $(y)^{++} = (y')^+$. Hence $(x)^{++} \vee (y)^{++} = (x')^+ \vee (y')^+ = (x' \vee y')^+ = (c)^{++}$ for some $c \in X$, as X is quasi-complemented. Therefore $\mathcal{D}^{++}(X)$ is a sublattice of $\mathcal{F}(X)$.

(5) $\Rightarrow$ (1): Assume that the condition (5) holds. Let $x \in X$. Since X is quasi-complemented, there exists $y \in X$ such that $(x)^{++} = (y)^+$. Since $\mathcal{D}^{++}(X)$ is a sublattice of $\mathcal{F}(X)$, we get $(x)^{++} \vee (y)^{++} = (t)^{++}$ for some $t \in X$. Thus $\langle x \rangle \vee \langle y \rangle \subseteq (x)^{++} \vee (y)^{++} = (t)^{++}$. Therefore, by Proposition 2(2), we get

$$(t)^+ = (t)^{+++} \subseteq (\langle x \rangle \vee \langle y \rangle)^+ = (x)^+ \cap (y)^+ = (x)^+ \cap (x)^{++} = \{1\}$$

which implies that $(t)^{++} = \{1\}^+ = X$. Hence $(x)^+ \vee (x)^{++} = (y)^{++} \vee (x)^{++} = (t)^{++} = X$. Therefore X is a generalized Stonean *BE*-algebra. $\square$

The following corollaries state the properties of generalized Stonean BE -algebra in terms of minimal prime filters. Two filters F and G of a BE -algebra X are called *comaximal* if $F \vee G = L$.

Corollary 1. *If X is a self-distributive and generalized Stonean BE -algebra, then any two distinct minimal prime filters of X are comaximal.*

Proof. Assume that X is a generalized Stonean BE -algebra. By condition (2) of the main theorem, each $(x)^+$ is a factor of X . Let P and Q be two distinct minimal prime filters of X . Choose $a \in P - Q$. Hence $(a)^+ \subseteq Q$. Since P is minimal, by Proposition 5, we get that $(a)^{++} \subseteq P$. Since $(a)^+$ is a factor of X , there exists a filter G such that $(a)^+ \cap G = \{1\}$ and $(a)^+ \vee G = X$. Hence $G \subseteq (a)^{++} \subseteq P$. Thus $X = (a)^+ \vee G \subseteq Q \vee P$. Therefore P and Q are comaximal. $\square$

Corollary 2. *If X is a self-distributive and generalized Stonean BE -algebra, then every prime filter contains a unique minimal prime filter.*

Proof. Let P be a prime filter of X . Suppose P contains two distinct minimal prime filters, say Q_1 and Q_2 . Then $Q_1 \vee Q_2 \subseteq P$. Since X is generalized Stonean, by Corollary 1, we get $Q_1 \vee Q_2 = X$. Hence $X = Q_1 \vee Q_2 \subseteq P$, which is a contradiction. Therefore P contains a unique minimal prime filter. $\square$

From Theorem 6, it can be easily observed that $\mathcal{D}^+(X)$ is a semilattice with respect to operation $\vee$ of filters. We now define that $(\mathcal{D}^+(X), \vee)$ is complemented if to each $(a)^+ \in \mathcal{D}^+(X)$, there exists $(b)^+ \in \mathcal{D}^+(X)$ such that $(a)^+ \cap (b)^+ = \{1\}$ and $(a)^+ \vee (b)^+ = X$.

Theorem 7. *Let X be a quasi-complemented BE -algebra. Then X is a generalized Stonean BE -algebra if and only if $\mathcal{D}^+(X)$ is a complemented semilattice.*

Proof. Assume that X is a generalized Stonean BE -algebra. Let $(x)^+, (y)^+ \in \mathcal{D}^+(X)$. Since X is a generalized Stonean BE -algebra, by Theorem 6, we get $(x)^+ \vee (y)^+ = (x \vee y)^+$. Hence $(\mathcal{D}^+(X), \vee)$ is a semilattice. Let $(x)^+ \in \mathcal{D}^+(X)$ where $x \in X$. Since X is quasi-complemented, there exists $x' \in X$ such that $(x)^{++} = (x')^+$. Clearly $(x)^+ \cap (x')^+ = (x)^+ \cap (x)^{++} = \{1\}$. Since X is generalized Stonean, we get $(x)^+ \vee (x')^+ = X$. Hence $(x)^+ \vee (x')^+ = X$. Thus $(x')^+$ is the complement of $(x)^+$ in $\mathcal{D}^+(X)$. Therefore $\mathcal{D}^+(X)$ is a complemented semilattice.

Conversely, assume that $\mathcal{D}^+(X)$ is a complemented semilattice. Let $x \in X$. Then $(x)^+ \in \mathcal{D}^+(X)$. Since $(\mathcal{D}^+(X), \vee)$ is complemented, there exists $(x')^+ \in \mathcal{D}^+(X)$ such that $(x)^+ \cap (x')^+ = \{1\}$ and $(x)^+ \vee (x')^+ = X$. Since $(x)^+ \cap (x')^+ = \{1\}$, we get $(x')^+ \subseteq (x)^{++}$. Hence $X = (x)^+ \vee (x')^+ \subseteq (x)^+ \vee (x)^{++}$, which gives $(x)^+ \vee (x)^{++} = X$. Therefore X is a generalized Stonean BE -algebra. $\square$

In [10], the notion of σ -filters is introduced in commutative BE -algebras and the properties of σ -filters are studied. In the following theorem, generalized Stonean BE -algebras are characterized with the help of σ -filters and dual annihilator filters.

Theorem 8. *The following assertions are equivalent in a commutative BE-algebra:*

- (1) *X is generalized Stonean;*
- (2) *every regular filter is a σ -filter;*
- (3) *every dual annihilator filter is a σ -filter.*

Proposition 10. *Every prime σ -filter of a self-distributive and commutative BE-algebra X is a minimal prime filter.*

Proof. Let P be a prime σ -filter of X . Since P is proper, there exists $a_0 \in X - P$. Let $x \in P$. Then $x \in \sigma(P)$ and hence $(x)^+ \vee P = X$. Since $a_0 \in X$, there exists $a \in (x)^+$ and $b \in P$ such that $a * (b * a_0) = 1 \in P$. Since $a \in (x)^+$, we get $a \vee x = 1$. Suppose $a \in P$. Since $b \in P$, we must have $a_0 \in P$ because of $a * (b * a_0) \in P$. But $a_0 \in P$ contradicts the fact that $a_0 \in X - P$. Hence $a \notin P$ such that $a \vee x = 1$. By Proposition 5, P is a minimal prime filter of X . $\square$

The converse of Proposition 10 is true in particular condition.

Theorem 9. *If X is a self-distributive and generalized Stonean BE-algebra, then every minimal prime filter of X is a σ -filter.*

Proof. Suppose that X is a generalized Stonean BE-algebra. Let P be a minimal prime filter of X . By Proposition 5, P is a regular filter of X . Since X is generalized Stonean, by Theorem 8, P is a σ -filter of X . $\square$

4. STONEAN FILTERS OF BE-ALGEBRAS

In this section, the notion of Stonean filters is introduced in commutative BE-algebras. Some properties of Stonean filters are derived in commutative BE-algebras. A set of equivalent conditions is given for every filter of a commutative BE-algebra to become a Stonean filter.

Definition 4. A proper filter F of a commutative BE-algebra X is called a *Stonean filter* of X if $F^+ \subseteq (x)^+ \vee (x)^{++}$ for all $x \in X$.

Example 4. Let $X = \{1, a, b, c\}$ be a set. Define a binary operation $*$ on X and then deduce the operation $\vee$ from $*$ as given in the following tables:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	1	b
b	1	a	1	1	a
c	1	a	b	1	d
d	1	1	1	1	1

$\vee$	1	a	b	c	d
1	1	1	1	1	1
a	1	a	b	c	1
b	1	b	b	1	1
c	1	c	1	c	1
d	1	1	1	1	d

Then clearly $(X, *, \vee, 1)$ is a commutative BE-algebra. It can be easily seen that $(a)^+ = \{1, d\}$, $(b)^+ = \{1, c, d\}$; $(c)^+ = \{1, b, d\}$ and $(d)^+ = \{1, a, b, c\}$. Now, we see that $(a)^{++} = \{1, a, b, c\}$; $(b)^{++} = \{1, b\}$; $(c)^{++} = \{1, c\}$ and $(d)^{++} = \{1, d\}$.

Consider the filter $F = \{1, c\}$. Then $F^+ = \{1, b, d\}$. It can be easily verified that $F^+ \subseteq (x)^+ \vee (x)^{++}$ for all $x \in X$. Therefore F is a Stonean filter of X .

Example 5. Let $X = \{1, a, b, c\}$ be a set. Define a binary operation $*$ on X and then deduce the operation $\vee$ from $*$ as given in the following tables:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	1	1	d
b	1	c	1	c	d
c	1	b	b	1	d
d	1	a	b	c	1

$\vee$	1	a	b	c	d
1	1	1	1	1	1
a	1	a	b	c	1
b	1	b	b	1	1
c	1	c	1	c	1
d	1	1	1	1	d

Then clearly $(X, *, \vee, 1)$ is a commutative *BE*-algebra. It can be easily seen that $(a)^+ = \{1, d\}$, $(b)^+ = \{1, c, d\}$; $(c)^+ = \{1, b, d\}$ and $(d)^+ = \{1, a, b, c\}$. Now, we see that $(a)^{++} = \{1, a, b, c\}$; $(b)^{++} = \{1, b\}$; $(c)^{++} = \{1, c\}$ and $(d)^{++} = \{1, d\}$. Consider the filter $F = \{1, c\}$. Then $F^+ = \{1, b, d\}$. It can be easily verified that $F^+ \subseteq (x)^+ \vee (x)^{++}$ for all $x \in X$. Therefore F is a Stonean filter of X .

Proposition 11. *Every prime filter of a self-distributive and commutative BE-algebra is a Stonean filter.*

Proof. Let P be a prime filter of a self-distributive and commutative *BE*-algebra X . Suppose $x \in P$. Then $\langle x \rangle \subseteq P$. Hence, by Proposition 1(1), we get $P^+ \subseteq (x)^+$. Thus $P^+ \subseteq (x)^+ \subseteq (x)^+ \vee (x)^{++}$. Suppose $x \notin P$. Since P is prime, we get $(x)^+ \subseteq P$. Hence $P^+ \subseteq (x)^{++}$. Thus $P^+ \subseteq (x)^{++} \subseteq (x)^+ \vee (x)^{++}$. Therefore P is Stonean. $\square$

The converse of Proposition 11 is not true, which means that every Stonean filter of a *BE*-algebra need not be prime. For, consider the self-distributive and commutative *BE*-algebra given in Example 5. It can be easily noticed that the filter $F = \{1, c\}$ is Stonean but not prime because of $a \vee d = 1 \in F$ but neither $a \in F$ nor $d \in F$.

Since every maximal filter of a self-distributive and commutative *BE*-algebra is a prime filter, the following corollary is a direct consequence of Proposition 11.

Corollary 3. *Every maximal filter of a self-distributive and commutative BE-algebra is a Stonean filter.*

Definition 5. A commutative *BE*-algebra X is called a *hyper Stonean BE-algebra* if $\langle x \rangle \vee (x)^+ = X$ for all $x \in X$.

Since $\langle x \rangle \subseteq (x)^{++}$ for all $x \in X$, it can be easily verified that every hyper Stonean *BE*-algebra is a generalized Stonean *BE*-algebra. However, every generalized Stonean *BE*-algebra need not be a hyper Stonean *BE*-algebra. For consider the following example:

Example 6. Let $X = \{1, a, b, c\}$ be a set. Define a binary operation $*$ on X and then deduce the operation $\vee$ from $*$ as given in the following tables:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	a	c	d
b	1	1	1	c	d
c	1	a	b	1	d
d	1	1	1	c	1

$\vee$	1	a	b	c	d
1	1	1	1	1	1
a	1	a	a	1	a
b	1	a	b	1	b
c	1	1	1	c	1
d	1	a	b	1	d

Clearly $(X, *, \vee, 1)$ is a commutative *BE*-algebra. It is easily seen that $(a)^+ = (b)^+ = \{1, c\}$; $(c)^+ = \{1, a, b, d\}$ and $(d)^+ = \{1, c\}$. Hence X has no dual-dense element. Now, we see that $(a)^{++} = (b)^{++} = \{1, a, b, d\}$; $(c)^{++} = \{1, c\}$ and $(d)^{++} = X$. It is routinely verified that X is a generalized Stonean *BE*-algebra. It is clear that $\langle a \rangle = \{1, a\}$. Hence

$$\langle a \rangle \vee (a)^+ = \{1, a\} \vee \{1, b\} = \{1, a, b\} \neq X.$$

Therefore X is not a hyper Stonean *BE*-algebra.

In the following result, a sufficient condition is derived for every generalized Stonean *BE*-algebra to become a hyper Stonean *BE*-algebra.

Proposition 12. *Let X be a self-distributive generalized Stonean *BE*-algebra. If every filter of X is a dual annihilator filter, then X is a hyper Stonean *BE*-algebra.*

Proof. Let $x \in X$. By hypothesis, we get $(x)^{++} = \langle x \rangle$. Since X is generalized Stonean, we get $(x)^+ \vee \langle x \rangle = (x)^+ \vee (x)^{++} = X$. Therefore X is a hyper Stonean *BE*-algebra. $\square$

In the following theorem, a set of equivalent conditions is given for every self-distributive and commutative *BE*-algebra to become a hyper Stonean *BE*-algebra.

Theorem 10. *Let X be a self-distributive and commutative *BE*-algebra. Then the following assertions are equivalent:*

- (1) X is a hyper Stonean *BE*-algebra;
- (2) every prime filter is maximal;
- (3) every prime filter is minimal.

Proof. (1) $\Rightarrow$ (2): Assume that X is a hyper Stonean *BE*-algebra. Let P be a prime filter of X . Suppose P is not maximal. Then there exists a proper filter Q of X such that $P \subset Q$. Choose $x \in Q - P$. Since X is hyper Stonean, we get $\langle x \rangle \vee (x)^+ = X$. Since $x \in Q$, we get $\langle x \rangle \subseteq Q$. Since $x \notin P$ and P is prime, we get $(x)^+ \subseteq P$. Hence $X = \langle x \rangle \vee (x)^+ \subseteq Q \vee P = Q$, which is a contradiction. Therefore P is maximal.

(2) $\Rightarrow$ (3): Since every maximal filter is prime, it is clear.

(3) $\Rightarrow$ (1): Assume that every prime filter is minimal. Let $x \in X$. Suppose $\langle x \rangle \vee (x)^+ \neq X$. Then there exists a prime filter P such that $\langle x \rangle \vee (x)^+ \subseteq P$. Hence $x \in \langle x \rangle \subseteq P$ and

$(x)^+ \subseteq P$. Since P is minimal, by Theorem 5, we get $x \notin P$ which is a contradiction. Hence $\langle x \rangle \vee (x)^+ = X$. Therefore X is a hyper Stonean BE -algebra $\square$

The converse of Corollary 3 is not true. For, consider the self-distributive and commutative BE -algebra given in Example 4. Note that $\{1\}$ is a Stonean filter but not a prime filter. However, in the following theorem, a set of equivalent conditions is given for every Stonean filter of a BE -algebra to become a maximal filter.

Theorem 11. *Let X be a self-distributive hyper Stonean BE -algebra and F be a proper filter of X . Then the following assertions are equivalent:*

- (1) F is a maximal filter;
- (2) for each $x \in X$, $x \notin F$ implies $(x)^+ \subseteq F$;
- (3) F is a prime Stonean filter.

Proof. (1) $\Rightarrow$ (2): Assume that F is a maximal filter of X . Let $x \in X$ be such that $x \notin F$. Since F is a prime filter of X , we get that $(x)^+ \subseteq F$.

(2) $\Rightarrow$ (3): Assume condition (2). Let $x \in X$. Suppose $x \in F$. Then $\langle x \rangle \subseteq F$. Hence, by Proposition 1(1), we get $F^+ \subseteq (x)^+ \subseteq F$. Thus $F^+ \subseteq (x)^+ \subseteq (x)^+ \vee (x)^{++}$. Suppose $x \notin F$. By condition (2), we get $(x)^+ \subseteq F$. Hence $F^+ \subseteq (x)^{++}$. Thus $F^+ \subseteq (x)^{++} \subseteq (x)^+ \vee (x)^{++}$. Therefore F is Stonean. We now prove that F is prime. Let $x, y \in X$ be such that $x \vee y \in F$. Suppose $x \notin F$. By condition (2), we have $(x)^+ \subseteq F$. Now,

$$\begin{aligned} (x)^+ \vee \langle y \rangle &= X \cap \{(x)^+ \vee \langle y \rangle\} \\ &= \{(x)^+ \vee \langle x \rangle\} \cap \{(x)^+ \vee \langle y \rangle\} && \text{since } X \text{ is hyper Stonean} \\ &= (x)^+ \vee \{\langle x \rangle \cap \langle y \rangle\} = (x)^+ \vee \langle x \vee y \rangle \subseteq F && \text{since } (x)^+ \subseteq F \text{ and } x \vee y \in F \end{aligned}$$

which gives $\langle y \rangle \subseteq (x)^+ \vee \langle y \rangle \subseteq F$. Hence $y \in F$. Therefore F is a prime Stonean filter.

(3) $\Rightarrow$ (1): Assume that F is a prime Stonean filter of X . Suppose F is not maximal. Then there exists a proper filter F' of X such that $F \subset F'$. Choose $x \in F' - F$. Since F is Stonean, we get $F^+ \subseteq (x)^+ \vee (x)^{++}$. Since F is prime and $x \notin F$, we get $(x)^+ \subseteq F \subset F'$. Since $x \in F'$, we get $\langle x \rangle \subseteq F'$. Since X is hyper Stonean, we get $X = \langle x \rangle \vee (x)^+ \subseteq F'$, which is a contradiction. Therefore F is a maximal filter. $\square$

In the following theorem, a set of equivalent conditions is given for every filter of a commutative BE -algebra to become a Stonean filter.

Theorem 12. *Let X be a commutative BE -algebra. Then the following are equivalent:*

- (1) X is a generalized Stonean;
- (2) every filter is a Stonean filter;
- (3) $\{1\}$ is a Stonean filter.

Proof. (1) $\Rightarrow$ (2): Assume that X is Stonean. Let F be a filter of X . Since X is Stonean, we have $F^+ \subseteq X = (x)^+ \vee (x)^{++}$ for all $x \in X$. Hence F is a Stonean filter

of X .

(2) $\Rightarrow$ (3): It is clear.

(3) $\Rightarrow$ (1): Assume that $\{1\}$ is a Stonean filter of X . Hence $X = \{1\}^+ \subseteq (x)^+ \vee (x)^{++}$. Therefore X is a generalized Stonean BE -algebra. $\square$

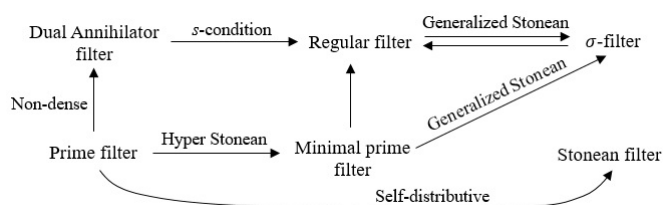
Theorem 13. (*Extension property of Stonean filters*) Let F and G be two filters of a commutative BE -algebra such that $F \subseteq G$. If F is a Stonean filter, then so is G .

Proof. If $F \subseteq G$, by Proposition 1(1), $G^+ \subseteq F^+$. Since F is a Stonean filter, then $G^+ \subseteq (x)^+ \vee (x)^{++}$ for all $x \in X$, hence G is a Stonean filter. $\square$

5. CONCLUSION

In this work, we have considered the Stonean BE -algebras. The notion of Stonean filter has been introduced and considered them in detail. These type of filter play a basic role. Based on these facts, we give a classification for BE -algebras. The notion of hyper Stonean BE -algebras had been introduced and we show that these structures are particular cases of commutative BE -algebras. We think such results are very useful for the further characterization of generalized Stonean BE -algebra in terms of congruences of this structure. In future, we plan to investigate the topological properties of generalized Stonean BE -algebras. Further properties of Stonean filters and their interconnections between various filters existed in BE -algebras can also be investigated.

In the following diagram, we show the relationships between some filters of BE -algebras. The notion " $A \rightarrow B$ " means A should be B .



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ON SOME INEQUALITIES RELATED TO OPEN NEWTON-COTES FORMULAS FOR DIFFERENTIABLE CONVEX FUNCTIONS WITH APPLICATIONS

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Abstract. In this paper, first, we prove a novel integral identity involving a single time-differentiable function. Then, we prove some new inequalities associated with one of the open Newton-Cotes formulas for differentiable convex functions. The newly established inequalities can be helpful in finding the error bounds of one of the open Newton-Cotes formulas. Finally, some applications of the inequalities are also presented in the context of open Newton-Cotes formulas.

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Keywords: open Newton-Cotes formulas, convex functions, integral inequalities

1. INTRODUCTION

The area of mathematics known as mathematical analysis covers the theory of measure, limits, differentiation, integration, and convex functions. Convex functions are fundamental as positive or increasing functions, and they have emerged as a key topic in the field of mathematical analysis research.

Inequalities are at the core of mathematical analysis, and they have developed into a crucial tool in that process up until the early 20th century, when we started to view them as a separate field of modern mathematics. The pioneering work in this field was the book "Inequalities" [12] by Hardy, Littlewood and Pólya. Other books (see, e.g., [19, 20]) are of great value in this field as well.

In recent years, many researchers have developed numerical integration formulas and found their error bounds using different techniques. To determine the error bounds of numerical integration formulas, mathematical inequalities are used, and the authors used various functions such as convex functions, bounded functions, Lipschitzian functions, and so on. For example, some error bounds for trapezoidal

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and midpoint formulas of numerical integration using the convex functions were found in [6, 15]. A number of papers have been published on the error bounds of Simpson's formula using the convex functions in different calculi and some of these bounds can be found in [1, 3, 5, 7–9, 17, 21]. Some error bounds for Newton's formula in numerical integration have also been established by using the convex functions in different calculi and these bounds can be found in [10, 11, 13, 14, 16, 22, 23]. In open Newton-Cotes formulas, Milne's formula is very important and its error bounds for four times twice differentiable functions were found in [2]. In [18], the authors used a general form of the convexity and established some new Maclaurin's formula type inequalities, and discussed their applications.

Inspired by the ongoing studies, we use differentiable convex functions and prove some integral inequalities related to one of the open Newton-Cotes formulas for $n = 1$ (see, [4, p. 200]). These inequalities can help us to find the error bounds for the discussed formula. Finally, we give some applications to quadrature formulas.

2. MAIN RESULTS

In this section, we establish and prove some important integral inequalities for differentiable convex functions. For this, we will prove the following lemma first.

Lemma 1. *Let $F : [\pi_1, \pi_2] \rightarrow \mathbb{R}$ be a differentiable function over (π_1, π_2) . If $F' \in L_1[\pi_1, \pi_2]$, then the following equality holds:*

$$\begin{aligned} & \frac{1}{2} \left[F \left(\frac{2\pi_1 + \pi_2}{3} \right) + F \left(\frac{\pi_1 + 2\pi_2}{3} \right) \right] - \frac{1}{\pi_2 - \pi_1} \int_{\pi_1}^{\pi_2} F(\tau) d\tau \\ &= (\pi_2 - \pi_1) \left[\int_0^{\frac{1}{3}} \varkappa F'(\varkappa\pi_2 + (1 - \varkappa)\pi_1) d\varkappa \right. \\ & \quad \left. + \int_{\frac{1}{3}}^{\frac{2}{3}} \left(\varkappa - \frac{1}{2} \right) F'(\varkappa\pi_2 + (1 - \varkappa)\pi_1) d\varkappa + \int_{\frac{2}{3}}^1 (\varkappa - 1) F'(\varkappa\pi_2 + (1 - \varkappa)\pi_1) d\varkappa \right] \end{aligned} \quad (2.1)$$

Proof. With simple computations, we have

$$\begin{aligned} I_1 &= (\pi_2 - \pi_1) \int_0^{\frac{1}{3}} \varkappa F'(\varkappa\pi_2 + (1 - \varkappa)\pi_1) d\varkappa \\ &= \frac{1}{3} F \left(\frac{2\pi_1 + \pi_2}{3} \right) - \int_0^{\frac{1}{3}} F(\varkappa\pi_2 + (1 - \varkappa)\pi_1) d\varkappa, \end{aligned} \quad (2.2)$$

$$\begin{aligned} I_2 &= (\pi_2 - \pi_1) \int_{\frac{1}{3}}^{\frac{2}{3}} \left(\varkappa - \frac{1}{2} \right) F'(\varkappa\pi_2 + (1 - \varkappa)\pi_1) d\varkappa \\ &= \frac{1}{6} F \left(\frac{\pi_1 + 2\pi_2}{3} \right) + \frac{1}{6} F \left(\frac{2\pi_1 + \pi_2}{3} \right) - \int_0^1 F(\varkappa\pi_2 + (1 - \varkappa)\pi_1) d\varkappa \end{aligned} \quad (2.3)$$

and

$$\begin{aligned} I_3 &= (\pi_2 - \pi_1) \int_{\frac{2}{3}}^1 (\varkappa - 1) F'(\varkappa \pi_2 + (1 - \varkappa) \pi_1) d\varkappa \\ &= \frac{1}{3} F\left(\frac{\pi_1 + 2\pi_2}{3}\right) - \int_{\frac{2}{3}}^1 F(\varkappa \pi_2 + (1 - \varkappa) \pi_1) d\varkappa. \end{aligned} \quad (2.4)$$

Thus, we have the following required equality by adding (2.2)-(2.4):

$$I_1 + I_2 + I_3 = \frac{1}{2} \left[F\left(\frac{2\pi_1 + \pi_2}{3}\right) + F\left(\frac{\pi_1 + 2\pi_2}{3}\right) \right] - \frac{1}{\pi_2 - \pi_1} \int_{\pi_1}^{\pi_2} F(\tau) d\tau.$$

The proof is completed. $\square$

Theorem 1. *If all conditions of Lemma 1 hold and $|F'|$ is a convex function, then the following inequality holds:*

$$\begin{aligned} & \left| \frac{1}{2} \left[F\left(\frac{2\pi_1 + \pi_2}{3}\right) + F\left(\frac{\pi_1 + 2\pi_2}{3}\right) \right] - \frac{1}{\pi_2 - \pi_1} \int_{\pi_1}^{\pi_2} F(\tau) d\tau \right| \\ & \leq \frac{5(\pi_2 - \pi_1)}{72} [|F'(\pi_1)| + |F'(\pi_2)|]. \end{aligned}$$

Proof. By taking modulus in (2.1) and using convexity of $|F'|$, we have

$$\begin{aligned} & \left| \frac{1}{2} \left[F\left(\frac{2\pi_1 + \pi_2}{3}\right) + F\left(\frac{\pi_1 + 2\pi_2}{3}\right) \right] - \frac{1}{\pi_2 - \pi_1} \int_{\pi_1}^{\pi_2} F(\tau) d\tau \right| \\ & \leq (\pi_2 - \pi_1) \left[\int_0^{\frac{1}{3}} \varkappa |F'(\varkappa \pi_2 + (1 - \varkappa) \pi_1)| d\varkappa \right. \\ & \quad + \int_{\frac{1}{3}}^{\frac{2}{3}} \left| \varkappa - \frac{1}{2} \right| |F'(\varkappa \pi_2 + (1 - \varkappa) \pi_1)| d\varkappa \\ & \quad \left. + \int_{\frac{2}{3}}^1 (1 - \varkappa) |F'(\varkappa \pi_2 + (1 - \varkappa) \pi_1)| d\varkappa \right] \\ & \leq (\pi_2 - \pi_1) \left[\int_0^{\frac{1}{3}} \varkappa [\varkappa |F'(\pi_2)| + (1 - \varkappa) |F'(\pi_1)|] d\varkappa \right. \\ & \quad + \int_{\frac{1}{3}}^{\frac{2}{3}} \left| \varkappa - \frac{1}{2} \right| [\varkappa |F'(\pi_2)| + (1 - \varkappa) |F'(\pi_1)|] d\varkappa \\ & \quad \left. + \int_{\frac{2}{3}}^1 (1 - \varkappa) [\varkappa |F'(\pi_2)| + (1 - \varkappa) |F'(\pi_1)|] d\varkappa \right] \\ & = \frac{5(\pi_2 - \pi_1)}{72} [|F'(\pi_1)| + |F'(\pi_2)|]. \end{aligned}$$

Thus, the proof is completed. $\square$

Theorem 2. *If all conditions of Lemma 1 hold and $|F'|^q$, $q \geq 1$ is a convex function, then the following inequality holds:*

$$\begin{aligned} & \left| \frac{1}{2} \left[F \left(\frac{2\pi_1 + \pi_2}{3} \right) + F \left(\frac{\pi_1 + 2\pi_2}{3} \right) \right] - \frac{1}{\pi_2 - \pi_1} \int_{\pi_1}^{\pi_2} F(\tau) d\tau \right| \\ & \leq \frac{(\pi_2 - \pi_1)}{18} \left[\left(\frac{2|F'(\pi_2)|^q + 7|F'(\pi_1)|^q}{9} \right)^{\frac{1}{q}} + \frac{1}{2} \left(\frac{|F'(\pi_2)|^q + |F'(\pi_1)|^q}{2} \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left(\frac{7|F'(\pi_2)|^q + 2|F'(\pi_1)|^q}{9} \right)^{\frac{1}{q}} \right]. \end{aligned}$$

Proof. Taking modulus of equality (2.1) and using power mean inequality, we have

$$\begin{aligned} & \left| \frac{1}{2} \left[F \left(\frac{2\pi_1 + \pi_2}{3} \right) + F \left(\frac{\pi_1 + 2\pi_2}{3} \right) \right] - \frac{1}{\pi_2 - \pi_1} \int_{\pi_1}^{\pi_2} F(\tau) d\tau \right| \\ & \leq (\pi_2 - \pi_1) \left[\left(\int_0^{\frac{1}{3}} \mathcal{K} d\mathcal{K} \right)^{1-\frac{1}{q}} \left(\int_0^{\frac{1}{3}} \mathcal{K} |F'(\mathcal{K}\pi_2 + (1-\mathcal{K})\pi_1)|^q d\mathcal{K} \right)^{\frac{1}{q}} \right. \\ & \quad + \left(\int_{\frac{1}{3}}^{\frac{2}{3}} \left| \mathcal{K} - \frac{1}{2} \right| d\mathcal{K} \right)^{1-\frac{1}{q}} \left(\int_{\frac{1}{3}}^{\frac{2}{3}} \left| \mathcal{K} - \frac{1}{2} \right| |F'(\mathcal{K}\pi_2 + (1-\mathcal{K})\pi_1)|^q d\mathcal{K} \right)^{\frac{1}{q}} \\ & \quad \left. + \left(\int_{\frac{2}{3}}^1 (1-\mathcal{K}) d\mathcal{K} \right)^{1-\frac{1}{q}} \left(\int_{\frac{2}{3}}^1 (1-\mathcal{K}) |F'(\mathcal{K}\pi_2 + (1-\mathcal{K})\pi_1)|^q d\mathcal{K} \right)^{\frac{1}{q}} \right]. \end{aligned}$$

We have the following relation by using the convexity of $|F'|^q$

$$\begin{aligned} & \left| \frac{1}{2} \left[F \left(\frac{2\pi_1 + \pi_2}{3} \right) + F \left(\frac{\pi_1 + 2\pi_2}{3} \right) \right] - \frac{1}{\pi_2 - \pi_1} \int_{\pi_1}^{\pi_2} F(\tau) d\tau \right| \\ & \leq (\pi_2 - \pi_1) \left[\left(\int_0^{\frac{1}{3}} \mathcal{K} d\mathcal{K} \right)^{1-\frac{1}{q}} \left(\int_0^{\frac{1}{3}} \mathcal{K} [\mathcal{K}|F'(\pi_2)|^q + (1-\mathcal{K})|F'(\pi_1)|^q] d\mathcal{K} \right)^{\frac{1}{q}} \right. \\ & \quad + \left(\int_{\frac{1}{3}}^{\frac{2}{3}} \left| \mathcal{K} - \frac{1}{2} \right| d\mathcal{K} \right)^{1-\frac{1}{q}} \left(\int_{\frac{1}{3}}^{\frac{2}{3}} \left| \mathcal{K} - \frac{1}{2} \right| [\mathcal{K}|F'(\pi_2)|^q + (1-\mathcal{K})|F'(\pi_1)|^q] d\mathcal{K} \right)^{\frac{1}{q}} \\ & \quad \left. + \left(\int_{\frac{2}{3}}^1 (1-\mathcal{K}) d\mathcal{K} \right)^{1-\frac{1}{q}} \left(\int_{\frac{2}{3}}^1 (1-\mathcal{K}) [\mathcal{K}|F'(\pi_2)|^q + (1-\mathcal{K})|F'(\pi_1)|^q] d\mathcal{K} \right)^{\frac{1}{q}} \right] \\ & = (\pi_2 - \pi_1) \left[\left(\frac{1}{18} \right)^{1-\frac{1}{q}} \left(\frac{|F'(\pi_2)|^q}{81} + \frac{7|F'(\pi_1)|^q}{162} \right)^{\frac{1}{q}} \right. \end{aligned}$$

$$+ \left(\frac{1}{36} \right)^{1-\frac{1}{q}} \left(\frac{|F'(\pi_2)|^q}{72} + \frac{|F'(\pi_1)|^q}{72} \right)^{\frac{1}{q}} \\ + \left(\frac{1}{18} \right)^{1-\frac{1}{q}} \left(\frac{7|F'(\pi_2)|^q}{162} + \frac{|F'(\pi_1)|^q}{81} \right)^{\frac{1}{q}} \Bigg].$$

Thus, the proof is completed. $\square$

Theorem 3. If all conditions of Lemma 1 hold and $|F'|^q$, $q > 1$ is a convex function, then the following inequality holds:

$$\left| \frac{1}{2} \left[F \left(\frac{2\pi_1 + \pi_2}{3} \right) + F \left(\frac{\pi_1 + 2\pi_2}{3} \right) \right] - \frac{1}{\pi_2 - \pi_1} \int_{\pi_1}^{\pi_2} F(\tau) d\tau \right| \\ \leq (\pi_2 - \pi_1) \left[\left(\frac{1}{3^{p+1}(p+1)} \right)^{\frac{1}{p}} \left(\frac{|F'(\pi_2)|^q + 5|F'(\pi_1)|^q}{18} \right)^{\frac{1}{q}} \right. \\ + \left(\frac{2}{6^{p+1}(p+1)} \right) \left(\frac{|F'(\pi_2)|^q + |F'(\pi_1)|^q}{6} \right)^{\frac{1}{q}} \\ \left. + \left(\frac{1}{3^{p+1}(p+1)} \right)^{\frac{1}{p}} \left(\frac{5|F'(\pi_2)|^q + |F'(\pi_1)|^q}{18} \right)^{\frac{1}{q}} \right],$$

where $p + q = pq$.

Proof. Taking modulus of equality (2.1) and using Hölder's inequality, we have

$$\left| \frac{1}{2} \left[F \left(\frac{2\pi_1 + \pi_2}{3} \right) + F \left(\frac{\pi_1 + 2\pi_2}{3} \right) \right] - \frac{1}{\pi_2 - \pi_1} \int_{\pi_1}^{\pi_2} F(\tau) d\tau \right| \\ \leq (\pi_2 - \pi_1) \left[\left(\int_0^{\frac{1}{3}} \mathcal{K}^p d\mathcal{K} \right)^{\frac{1}{p}} \left(\int_0^{\frac{1}{3}} |F'(\mathcal{K}\pi_2 + (1-\mathcal{K})\pi_1)|^q d\mathcal{K} \right)^{\frac{1}{q}} \right. \\ + \left(\int_{\frac{1}{3}}^{\frac{2}{3}} \left| \mathcal{K} - \frac{1}{2} \right|^p d\mathcal{K} \right)^{\frac{1}{p}} \left(\int_{\frac{1}{3}}^{\frac{2}{3}} |F'(\mathcal{K}\pi_2 + (1-\mathcal{K})\pi_1)|^q d\mathcal{K} \right)^{\frac{1}{q}} \\ \left. + \left(\int_{\frac{2}{3}}^1 (1-\mathcal{K})^p d\mathcal{K} \right)^{\frac{1}{p}} \left(\int_{\frac{2}{3}}^1 |F'(\mathcal{K}\pi_2 + (1-\mathcal{K})\pi_1)|^q d\mathcal{K} \right)^{\frac{1}{q}} \right].$$

We have the following relation by using the convexity of $|F'|^q$

$$\left| \frac{1}{2} \left[F \left(\frac{2\pi_1 + \pi_2}{3} \right) + F \left(\frac{\pi_1 + 2\pi_2}{3} \right) \right] - \frac{1}{\pi_2 - \pi_1} \int_{\pi_1}^{\pi_2} F(\tau) d\tau \right| \\ \leq (\pi_2 - \pi_1) \left[\left(\int_0^{\frac{1}{3}} \mathcal{K}^p d\mathcal{K} \right)^{\frac{1}{p}} \left(\int_0^{\frac{1}{3}} [\mathcal{K}|F'(\pi_2)|^q + (1-\mathcal{K})|F'(\pi_1)|^q] d\mathcal{K} \right)^{\frac{1}{q}} \right.$$

$$\begin{aligned}
& + \left(\int_{\frac{1}{3}}^{\frac{2}{3}} \left| \varkappa - \frac{1}{2} \right|^p d\varkappa \right)^{\frac{1}{p}} \left(\int_{\frac{1}{3}}^{\frac{2}{3}} [\varkappa |F'(\pi_2)|^q + (1-\varkappa) |F'(\pi_1)|^q] d\varkappa \right)^{\frac{1}{q}} \\
& + \left(\int_{\frac{2}{3}}^1 (1-\varkappa)^p d\varkappa \right)^{\frac{1}{p}} \left(\int_{\frac{2}{3}}^1 [\varkappa |F'(\pi_2)|^q + (1-\varkappa) |F'(\pi_1)|^q] d\varkappa \right)^{\frac{1}{q}} \Big] \\
& = (\pi_2 - \pi_1) \left[\left(\frac{1}{3^{p+1}(p+1)} \right)^{\frac{1}{p}} \left(\frac{|F'(\pi_2)|^q + 5|F'(\pi_1)|^q}{18} \right)^{\frac{1}{q}} \right. \\
& \quad + \left(\frac{2}{6^{p+1}(p+1)} \right) \left(\frac{|F'(\pi_2)|^q + |F'(\pi_1)|^q}{6} \right)^{\frac{1}{q}} \\
& \quad \left. + \left(\frac{1}{3^{p+1}(p+1)} \right)^{\frac{1}{p}} \left(\frac{5|F'(\pi_2)|^q + |F'(\pi_1)|^q}{18} \right)^{\frac{1}{q}} \right].
\end{aligned}$$

Thus, the proof is completed. $\square$

3. APPLICATIONS

Let P be a partition of the interval $[\pi_1, \pi_2]$ as:

$$\pi_1 = \tau_0 < \tau_1 < \tau_2 < \dots < \tau_n = \pi_2,$$

and we consider the quadrature formula

$$\int_{\pi_1}^{\pi_2} F(\tau) d\tau = \Lambda(F, P) + R(F, P), \quad (3.1)$$

where

$$\Lambda(F, P) = \sum_{i=0}^{n-1} \frac{\tau_{i+1} - \tau_i}{2} \left[F\left(\frac{2\tau_i + \tau_{i+1}}{3}\right) + F\left(\frac{\tau_i + 2\tau_{i+1}}{3}\right) \right] \quad (3.2)$$

and $R(F, P)$ is the associated approximation error.

Proposition 1. Let $F : [\pi_1, \pi_2] \rightarrow \mathbb{R}$ be a differentiable function over (π_1, π_2) with $0 \leq \pi_1 < \pi_2$ and $F' \in L_1[\pi_1, \pi_2]$. If $|F'|$ is a convex function, then we have

$$|R(F, P)| \leq \sum_{i=0}^{n-1} \frac{5(\tau_{i+1} - \tau_i)^2}{144} [|F'(\tau_i)| + |F'(\tau_{i+1})|]. \quad (3.3)$$

Proof. From Theorem 1 over the subintervals $[\tau_i, \tau_{i+1}]$ of partition P , we get

$$\begin{aligned}
& \left| \frac{1}{2} \left[F\left(\frac{2\tau_i + \tau_{i+1}}{3}\right) + F\left(\frac{\tau_i + 2\tau_{i+1}}{3}\right) \right] - \frac{1}{\tau_{i+1} - \tau_i} \int_{\tau_i}^{\tau_{i+1}} F(\varkappa) d\varkappa \right| \\
& \leq \frac{5(\tau_{i+1} - \tau_i)}{144} [|F'(\tau_i)| + |F'(\tau_{i+1})|].
\end{aligned} \quad (3.4)$$

Thus, we obtain the required inequality by multiplying (3.4) with $(\tau_{i+1} - \tau_i)$ and then summing the resultant inequality for all $i = 0, 1, 2, \dots, n-1$. $\square$

Proposition 2. Let $F : [\pi_1, \pi_2] \rightarrow \mathbb{R}$ be a differentiable function over (π_1, π_2) with $0 \leq \pi_1 < \pi_2$ and $F' \in L_1[\pi_1, \pi_2]$. If $|F'|^q$, $q \geq 1$ is a convex function, then we have

$$|R(F, P)| \leq \sum_{i=0}^{n-1} \frac{(\tau_{i+1} - \tau_i)^2}{18} \left[\left(\frac{2|F'(\tau_{i+1})|^q + 7|F'(\tau_i)|^q}{9} \right)^{\frac{1}{q}} + \frac{1}{2} \left(\frac{|F'(\tau_{i+1})|^q + |F'(\tau_i)|^q}{2} \right)^{\frac{1}{q}} + \left(\frac{7|F'(\tau_{i+1})|^q + 2|F'(\tau_i)|^q}{9} \right)^{\frac{1}{q}} \right]. \quad (3.5)$$

Proof. From Theorem 2 and method used in the proof of (3.3), we get the desired inequality (3.5). $\square$

Proposition 3. Let $F : [\pi_1, \pi_2] \rightarrow \mathbb{R}$ be a differentiable function over (π_1, π_2) with $0 \leq \pi_1 < \pi_2$ and $F' \in L_1[\pi_1, \pi_2]$. If $|F'|^q$, $q > 1$ is a convex function, then we have

$$|R(F, P)| \leq (\tau_{i+1} - \tau_i) \left[\left(\frac{1}{3^{p+1}(p+1)} \right)^{\frac{1}{p}} \left(\frac{|F'(\tau_{i+1})|^q + 5|F'(\tau_i)|^q}{18} \right)^{\frac{1}{q}} + \left(\frac{2}{6^{p+1}(p+1)} \right) \left(\frac{|F'(\tau_{i+1})|^q + |F'(\tau_i)|^q}{6} \right)^{\frac{1}{q}} + \left(\frac{1}{3^{p+1}(p+1)} \right)^{\frac{1}{p}} \left(\frac{5|F'(\tau_{i+1})|^q + |F'(\tau_i)|^q}{18} \right)^{\frac{1}{q}} \right], \quad (3.6)$$

where $p + q = pq$.

Proof. From Theorem 3 and method used in the proof of (3.3), we get the desired inequality (3.6). $\square$

4. CONCLUSION

In this work, we have proved some error bounds for one of the open Newton-Cotes formulas for differentiable convex functions. We also gave some applications of newly established results in the context of open Newton-Cotes formulas. It is an interesting and new problem that the upcoming researchers can obtain similar inequalities for fractional integrals and coordinated convex functions.

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CONJOINED LORENZ-LIKE ATTRACTORS COINED

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Abstract. In contrast to most other periodically forced chaotic systems with infinitely many isolated coexisting strange attractors, little seems to be known about the ones that possess conjoined Lorenz-like attractors with potential existence of infinitely many pairs of wings/scrolls. To achieve this target, this note proposes a new periodically forced extended Lorenz-like system, which generates infinitely many singularly degenerate heteroclinic cycles or heteroclinic orbits to any two equilibria of a family of non-hyperbolic lines, the collapses of which create not only the desirable conjoined Lorenz-like attractors, but also infinitely many isolated coexisting ones. What is more, the state variable x of that conjoined Lorenz-like attractor presents stochastic behaviors, confirming the links between long period and chaos. This also generalizes the classical concept of boundedness of chaos, i.e., the system orbits beginning from one sub-two-scroll of that conjoined Lorenz-like attractor might tend to the ones at infinity. Apart from those, the existence of an invariant surface and a family of infinitely many pairs of symmetrical heteroclinic orbits are proved by utilizing the Lyapunov function, the definitions of both α -limit set and ω -limit set.

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1. INTRODUCTION

Since the introduction of Lorenz system, scholars made many attempts to shed some light on the forming mechanism of various strange attractors, including self-excited and hidden ones, etc. Combining contraction map and boundary problem [8],

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Shilnikov et al. formulated an effective method to determine the existence of homoclinic and heteroclinic orbits and further classified chaos occurring in 3D quadratic autonomous differential systems into the following four cases:

- (1) chaos of the Shilnikov homoclinic-orbit type;
- (2) chaos of the Shilnikov heteroclinic-orbit type;
- (3) chaos of the hybrid type with both Shilnikov homoclinic and heteroclinic orbits;
- (4) chaos of other types.

After the appearance of powerful computational tools, the computer-assisted proof for the Lorenz attractor became a reality [9, 12, 13]. Kokubu and Roussarie firstly found that the broken of singularly degenerate heteroclinic cycles (i.e., each of which consists of invariant sets formed by a non-hyperbolic (i.e., at least one of eigenvalues is null) line of equilibria together with heteroclinic orbits connecting two of the equilibria) created strange attractors [3]. Liao put forward two sufficient conditions to guarantee a continuous system to exhibit chaotic motions, i.e., the existence of at least one positive Lyapunov exponent and one ultimate bound set [7]. Specifically, the first one repels the trajectories of studied systems from inside to outside, while the second one attracts the trajectories from outside to inside. Kuznetsov et al. searched a new hidden Lorenz-like attractor by applying homotopy and numerical continuation to synthesise a new transition scenario among the well-known Glukhovskiy-Dolghansky, the Lorenz system and the Rabinovich system, and analytical/numerical methods based on the continuation and perpetual points [5]. Further, Leonov and Kuznetsov generalized the second part of the celebrated Hilbert's 16th problem [2]: the number and mutual disposition of attractors and repellers in the chaotic multidimensional dynamical systems, and, in particular, their dependence on the degree of polynomials in the model [4]. Zhang and Chen then also formulated a chaotic analog when searching for infinitely many chaotic attractors in lower-dimensional autonomous systems, and with an extension to the fractional-order setting [19]. Meanwhile, they also found some new phenomena that generalized the classical concept of sensitive dependence on the initial conditions, i.e., the system orbits beginning from two very close initial points might either converge and then stay on one attractor or diverge to two different attractors. Along the same line, by introducing sine functions, Yang and Yang proposed a new 3D autonomous system with infinitely many coexisting chaotic attractors and infinitely many coexisting periodic attractors in the following three cases:

- (i) no equilibria,
- (ii) only infinitely many nonhyperbolic double-zero equilibria,
- (iii) both infinitely many hyperbolic saddles and nonhyperbolic pure-imaginary equilibria [17].

Based on the guess that the decrease of powers of some variable states may widen the ranges of some parameters for which hidden attractors exist, Wang et al. numerically

found multitudinous potential hidden attractors in a sub-quadratic Lorenz-like system [15].

Remarkably, all of aforementioned attractors are isolated ones. To this end, by carrying out numerical simulation on a kind of complex Lorenz-type systems, Zhang proposed an open question: There might exist an infinitely many-scroll attractor for that system, where the attractor oscillates as well as it goes to infinity along the x_1 -axis. Since the well-known chaotic Lorenz systems have sensitive dependence on initial conditions, we cannot simulate the systems for a long time, and the numerical experiments might not reflect the true orbits. This is a very strange and interesting phenomenon for future work [18, Question 4.1, p.2150101-21].

As the question itself stated, it may be hard to answer at present. Therefore, the existence of conjoined Lorenz-like attractors may prove to be the next best thing. Now, how to generate a conjoined Lorenz-like attractor is not only theoretically significant but also practically important, motivating the work to be presented in this paper.

The new introduced system should satisfy the following three principles:

- (1) The system should be a Lorenz-like one with infinitely many lines of non-hyperbolic equilibria.
- (2) The system should generate singularly degenerate heteroclinic cycles or heteroclinic orbits with nearby chaotic attractors.
- (3) The system should be sensitive dependence on the initial conditions as the one in [19], which guarantees potential existence of strong connection among different isolated chaotic attractors.

On the basis of these three simple tips, one tries to construct a new 3D periodically forced extended Lorenz-like system and find the parameters of conjoined Lorenz-like attractors by a trial-and-error process.

As far as we know, very few research on the conjoined Lorenz-like attractor is available in the published literatures. Accordingly, it is a demanding work to reveal the forming mechanism of it. The main contributions are as follows:

- (1) Proposing a new 3D periodically forced extended Lorenz-like system family which generates singularly degenerate heteroclinic cycles or heteroclinic orbits with nearby conjoined Lorenz-like attractors.
- (2) Illustrating another new kind of extreme sensitivity to initial conditions, i.e., the system's orbits starting from two very close initial points of any one line of non-hyperbolic equilibria might converge to another equilibria of either the same line of non-hyperbolic equilibria and then create singularly degenerate heteroclinic cycles, or the neighbouring ones and thus forms heteroclinic orbits.
- (3) Proving the existence of an invariant surface and infinitely many pairs of heteroclinic orbits.

The rest of the paper is structured as follows. In Section 2, some basic concepts are introduced. Section 3 formulates a novel periodically forced extended Lorenz-like system and presents conjoined Lorenz-like attractors. The basic dynamics, such as stability, Hopf bifurcation and invariant surface, are analyzed in Section 4. Section 5 illustrates singularly degenerate heteroclinic cycles and heteroclinic orbits with nearby infinitely many isolated coexisting Lorenz-like attractors. In Section 6, one gives a rigorous proof for the existence of infinitely many pairs of symmetric heteroclinic orbits.

2. PRELIMINARY

Consider the differential system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \xi)$, where $\mathbf{x} \in \mathbb{R}^n$ and $\xi \in \mathbb{R}^m$ are vectors representing phase variables and control parameters respectively. Assume that $\mathbf{f}$ is of class C^∞ in $\mathbb{R}^n \times \mathbb{R}^m$. Suppose that system has an equilibrium point $\mathbf{x} = \mathbf{x}_0$ at $\xi = \xi_0$. If at least one eigenvalue of the Jacobian matrix associated with linearized vector field about x_0 is zero or has a zero real part, then x_0 is said to be non-hyperbolic.

Let $\mathbb{R}[x, y, z]$ be the ring of real polynomials in the variables x, y and z . We say that $Q(x, y, z) \in \mathbb{R}[x, y, z]$ is a Darboux polynomial of a three dimensional ODEs system if the time derivative of it satisfies

$$\frac{dQ(x, y, z)}{dt} = \frac{\partial Q(x, y, z)}{\partial x} \dot{x} + \frac{\partial Q(x, y, z)}{\partial y} \dot{y} + \frac{\partial Q(x, y, z)}{\partial z} \dot{z} = k(x, y, z)Q(x, y, z),$$

where $k(x, y, z)$ is a real polynomial called the cofactor of $Q(x, y, z)$. If $Q(x, y, z)$ is a Darboux polynomial, then the surface $Q(x, y, z) = 0$ is known as invariant algebraic surface. If $Q(x, y, z)$ contains trigonometric functions, then we call it invariant surface.

Let the set of points: S (either connected or disconnected) be equilibria of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \xi)$ and $D \subset \mathbb{R}^n$ to be a domain containing S . Let $V: D \rightarrow \mathbb{R}$ be a continuously differentiable function such that $V(S) = 0$ and $V(x) > 0$ in $D \setminus S$, $\dot{V}(x) \leq 0$ in D . The derivative of $V(x)$ along the trajectories of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \xi)$, denoted by $\dot{V}(x)$, is given by $\dot{V}(x) = \sum_{i=1}^n \frac{\partial V}{\partial x_i} \dot{x}_i = \sum_{i=1}^n \frac{\partial V}{\partial x_i} f_i(x)$. Then, S is stable. If $\dot{V}(x) < 0$ in $D \setminus S$, then S is asymptotically stable. Moreover, if $D = \mathbb{R}^n$, then S is globally asymptotically stable.

3. CONJOINED LORENZ-LIKE ATTRACTORS

In light of Lorenz-like systems that easily generate singularly degenerate heteroclinic cycles with nearby bifurcated strange attractors, we firstly replace the state variable x of the following 3D extended Lorenz system [6]:

$$\begin{cases} \dot{x} = y, \\ \dot{y} = mx - ny - mxz - px^3, \\ \dot{z} = -az + bx^2, \end{cases}$$

with $\sin(2x)$ and $\sin(x)$, and formulates the following new periodically forced analogue:

$$\begin{cases} \dot{x} = y, \\ \dot{y} = \sin(2x) [a + c_1 z + c_2 \sin^2(x)] - by, \\ \dot{z} = -d_1 z + d_2 \sin^2(x), \end{cases} \quad (3.1)$$

where $a, b, c_1, c_2, d_1, d_2 \in \mathbb{R}$.

Secondly, based on the dynamics of E_z in Proposition 3 in Section 4, we choose the parameters $(a, b, c_1, c_2, d_1, d_2) = (12, 1.0, -4, -1, 0, 1)$ and initial values $(x_0^{1,2}, y_0^{1,2}, z_0^{1,2}) = (\pm 1.3, \pm 13, \pm 16) \times 10^{-8} + (0, 0, 2.3710)$ and $(x_0^{1,2}, y_0^{1,2}, z_0^{1,2}) = (\pm 1.3, \pm 13, \pm 16) \times 10^{-8} + (0, 0, 2.3711)$. Fig. 1 shows heteroclinic orbits and singularly degenerate heteroclinic cycles and nearby conjoined Lorenz-like attractors of system (3.1), which extends the classical concept of sensitive dependence on the initial conditions, i.e., the trajectories from two very close initial points $(0, 0, 2.3710)$ and $(0, 0, 2.3711)$ might converge to $(\pm\pi, 0, 6.6552)$ and then create heteroclinic orbits, or $(0, 0, 6.2848)$ and thus forms singularly degenerate heteroclinic cycles.

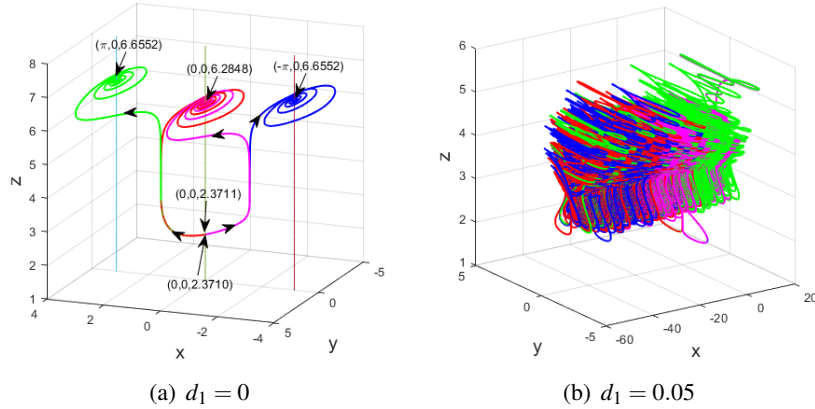


FIG. 1. (a) Heteroclinic orbits to $(0, 0, 2.3710)$ and $(\pi, 0, 6.6552)$, $(0, 0, 2.3710)$ and $(-\pi, 0, 6.6552)$; singularly degenerate heteroclinic cycles consisting of $(0, 0, 2.3711)$ and $(0, 0, 6.2848)$ of system (3.1), and (b) nearby conjoined Lorenz-like attractors when $(a, b, c_1, c_2, d_2) = (12, 1.0, -4, -1, 1)$, generalizing the classical concept of sensitive dependence on the initial conditions.

Choose other initial conditions $(\pm\pi, 0, 1.5)$, $(0, 0, 1.5)$, $(\pm\pi, 0, 2)$, $(0, 0, 2)$, $(\pm 2\pi, 0, 2.5)$, $(0, 0, 2.5)$ and $(\pm\pi, 0, 2.5)$. Figs. 2-4 also depict ten heteroclinic orbits and five pairs of singularly degenerate heteroclinic cycles with nearby bifurcated conjoined Lorenz-like attractors. Fig. 5 illustrates time series of the variable x of conjoined Lorenz-like attractors with initial conditions $(0, 0, 2)$ and $(0, 0, 1000\pi)$, verifying the complex structures of conjoined Lorenz-like attractors having the characteristics of random walk. Moreover, conjoined Lorenz-like attractors may extend the

classical concept of boundedness of chaos, i.e., the system orbits beginning from one sub-two-scroll of that conjoined Lorenz-like attractor might tend to infinity.

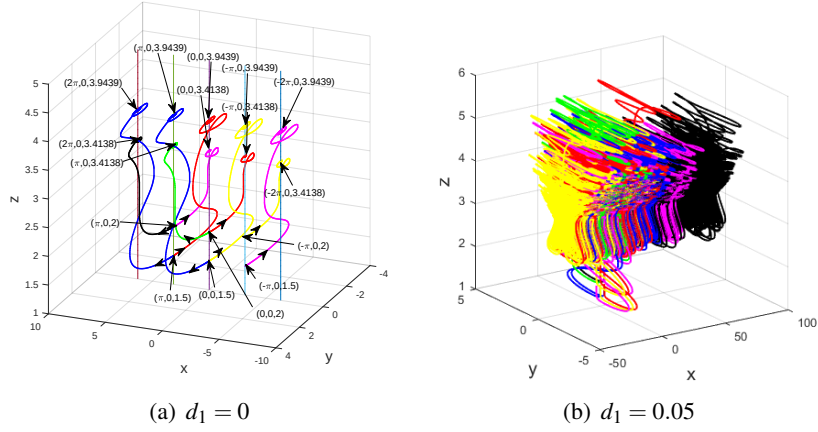


FIG. 2. (a) Heteroclinic orbits to $(\pi, 0, 1.5)$ and $(2\pi, 0, 3.9439)$, $(\pi, 0, 2)$ and $(2\pi, 0, 3.3148)$, $(\pi, 0, 1.5)$ and $(0, 0, 3.9439)$, $(\pi, 0, 2)$ and $(2\pi, 0, 3.3148)$, $(0, 0, 1.5)$ and $(\pi, 0, 3.9439)$, $(0, 0, 2)$ and $(\pi, 0, 3.3148)$, $(0, 0, 1.5)$ and $(-\pi, 0, 3.9439)$, $(0, 0, 2)$ and $(2\pi, 0, 3.3148)$, $(-\pi, 0, 1.5)$ and $(-2\pi, 0, 3.9439)$, $(-\pi, 0, 2)$ and $(-2\pi, 0, 3.3148)$ of system (3.1), and (b) nearby conjoined Lorenz-like attractors when $(a, b, c_1, c_2, d_2) = (12, 1.0, -4, -1, 1)$

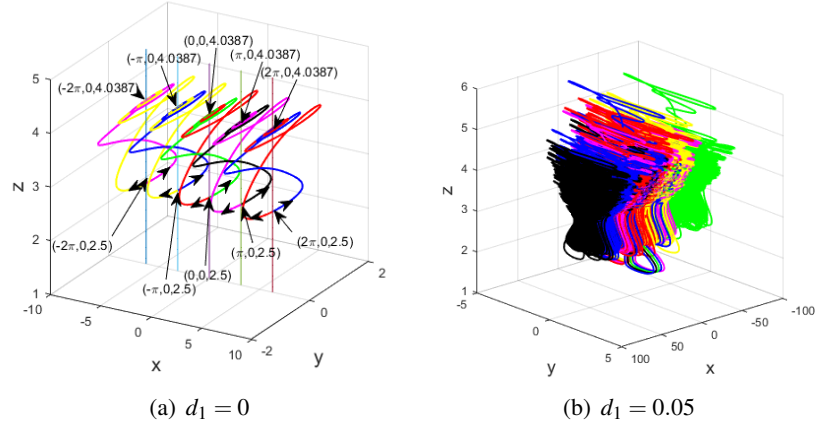


FIG. 3. (a) Singularly degenerate heteroclinic cycles consisting of $(-2\pi, 0, 2.5)$ and $(-2\pi, 0, 4.0387)$, $(-\pi, 0, 2.5)$ and $(-\pi, 0, 4.0387)$, $(0, 0, 2.5)$ and $(0, 0, 4.0387)$, $(\pi, 0, 2.5)$ and $(\pi, 0, 4.0387)$, $(2\pi, 0, 2.5)$ and $(2\pi, 0, 4.0387)$ of system (3.1), and (b) nearby conjoined Lorenz-like attractors when $(a, b, c_1, c_2, d_2) = (12, 1.0, -4, -1, 1)$.

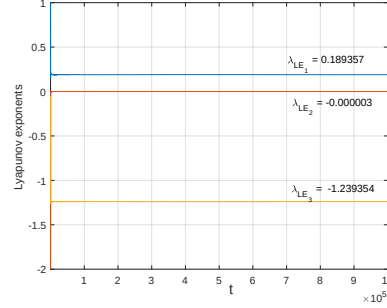


FIG. 4. Lyapunov exponents of system (3.1) with $(a, b, c_1, c_2, d_1, d_2) = (12, 1.0, -4, -1, 0.05, 1)$ and $(x_0^1, y_0^1, z_0^1) = (1.3, 1.3, 16) \times 10^{-8}$.

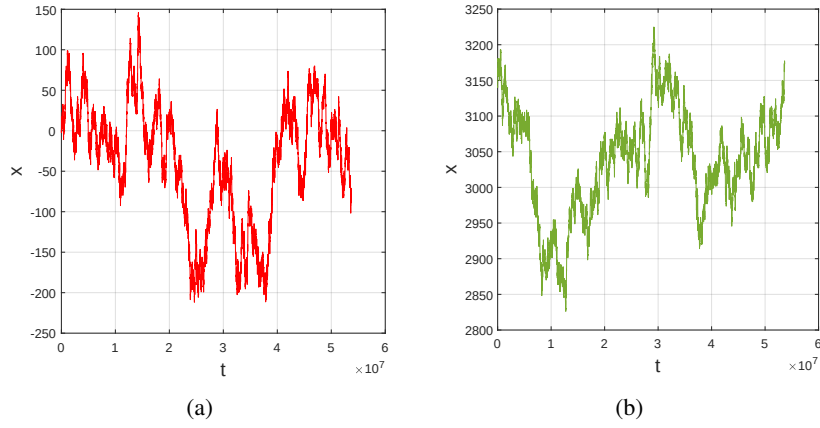


FIG. 5. Time series of the variable x of system (3.1) when $(a, b, c_1, c_2, d_1, d_2) = (12, 1.0, -4, -1, 0.05, 1)$, time interval $T = [0, 2 \times 10^7]$, $(y_0^2, z_0^2) = (1.3 \times 10^{-8}, 2 + 16 \times 10^{-8})$ and (a) $x_0^2 = 2 + 16 \times 10^{-8}$, (b) $x_0^3 = 1000\pi + 1.3 \times 10^{-8}$, which suggest stochastic behaviors of conjoined Lorenz-like attractors.

4. BASIC BEHAVIORS

In this section, we mainly give some results on some basic dynamics of system (3.1), i.e., the distribution of equilibrium points, stability, Hopf bifurcation, etc.

First of all, the distribution of equilibrium points easily follows from the algebraic structure of system (3.1) and is listed in the following proposition.

Proposition 1.

- (1) When $d_1 = 0$, system (3.1) has a family of parallel lines of non-hyperbolic equilibria $E_z = \{(k\pi, 0, z) | k \in \mathbb{N}, z \in \mathbb{R}\}$.

- (2) When $a = 0$, $c_1 = d_1 \neq 0$ and $c_2 = -d_2 \neq 0$, or $c_1 = -d_1 \neq 0$ and $c_2 = d_2 \neq 0$, $E_p = \{(x, 0, \frac{d_2 \sin^2 x}{d_1}) | x \in \mathbb{R}\}$ is a family of curves of non-hyperbolic equilibria.
- (3) When $d_1 \neq 0$, $E_0 = \{(k\pi, 0, 0) | k \in \mathbb{N}\}$ is an infinite set of isolated equilibria in system (3.1) for $\frac{-ad_1}{c_1 d_2 + c_2 d_1} > 1$ or $\frac{-ad_1}{c_1 d_2 + c_2 d_1} < 0$; while $0 < \frac{-ad_1}{c_1 d_2 + c_2 d_1} \leq 1$, $E_{\pm} = \{(k\pi \pm \frac{1}{2} \arcsin \sqrt{\frac{-ad_1}{c_1 d_2 + c_2 d_1}}, 0, -\frac{ad_2}{c_1 d_2 + c_2 d_1}) | k \in \mathbb{N}\}$ are infinitely many pairs of symmetrical isolated equilibria.

Remark 1. When $d_1 \neq 0$, a passes through the null value and $0 < \frac{-ad_1}{c_1 d_2 + c_2 d_1} \leq 1$, system (3.1) undergoes a generic pitchfork bifurcation at points of E_0 . While $a \neq 0$, d_1 crosses the null value and $0 < \frac{-ad_1}{c_1 d_2 + c_2 d_1} \leq 1$, system (3.1) undergoes a degenerate pitchfork bifurcation at points of E_z .

Next, in order to determine the stability and bifurcation of equilibria, we have to calculate Jacobian matrix associated vector field of system (3.1):

$$J = \begin{pmatrix} 0 & 1 & 0 \\ 2 \cos 2x(a + c_1 z + c_2 \sin^2 x) + c_2 \sin^2 2x & -b & c_1 \sin 2x \\ d_2 \sin 2x & 0 & -d_1 \end{pmatrix}.$$

One can easily calculate the characteristic equations of points of E_z , E_0 , E_p and $E_{\pm}$:

- (1) The one of each of E_z is

$$\lambda [\lambda^2 + b\lambda - 2(a + c_1 z)] = 0,$$

$$\text{with } \lambda_1 = 0, \lambda_{2,3} = \frac{-b \pm \sqrt{b^2 + 8(a + c_1 z)}}{2}.$$

- (2) The one of each of E_0 is

$$(\lambda + d_1) [\lambda^2 + b\lambda - 2a] = 0,$$

$$\text{with } \lambda_1 = -d_1, \lambda_{2,3} = \frac{-b \pm \sqrt{b^2 + 8a}}{2}.$$

- (3) For $c_1 = d_1$ and $c_2 = -d_2$ (resp. $c_1 = -d_1$ and $c_2 = d_2$), the one of each of E_p is

$$\lambda [\lambda^2 + (b + d_1)\lambda + bd_1 + d_2 \sin^2 2x] = 0,$$

$$\text{with } \lambda_1 = 0, \lambda_{2,3} = \frac{-(b + d_1) \pm \sqrt{(b + d_1)^2 - 4(bd_1 + d_2 \sin^2 2x)}}{2} \text{ (resp.}$$

$$\lambda [\lambda^2 + (b + d_1)\lambda + bd_1 - d_2 \sin^2 2x] = 0,$$

$$\text{with } \lambda_1 = 0, \lambda_{2,3} = \frac{-(b + d_1) \pm \sqrt{(b + d_1)^2 - 4(bd_1 - d_2 \sin^2 2x)}}{2}).$$

- (4) The one of points of $E_{\pm}$ is:

$$\begin{aligned} & \lambda^3 + (b + d_1)\lambda^2 + d_1 \left[b + \frac{4ac_2}{c_1 d_2 + c_2 d_1} \left(1 + \frac{ad_1}{c_1 d_2 + c_2 d_1} \right) \right] \lambda \\ & + 4ad_1 \left(1 + \frac{ad_1}{c_1 d_2 + c_2 d_1} \right) = 0. \end{aligned} \quad (4.1)$$

TABLE 1. The dynamical behaviors of points of E_z .

b	$a + c_1 z$	Property of E_z
> 0	< 0	A 2D W_{loc}^s and a 1D W_{loc}^c
	$= 0$	A 1D W_{loc}^s and a 2D W_{loc}^c
	> 0	A 1D W_{loc}^s , a 1D W_{loc}^c and a 1D W_{loc}^u
$= 0$	< 0	A 3D W_{loc}^c
	$= 0$	A 3D W_{loc}^c
	> 0	A 1D W_{loc}^s , a 1D W_{loc}^c and a 1D W_{loc}^u
< 0	< 0	A 1D W_{loc}^c and a 2D W_{loc}^u
	$= 0$	A 2D W_{loc}^c and a 1D W_{loc}^u
	> 0	A 1D W_{loc}^s , a 1D W_{loc}^c and a 1D W_{loc}^u

TABLE 2. The dynamical behaviors of points of E_p .

$b + d_1$	$-(bd_1 \pm d_2 \sin^2 2x)$	Property of E_p
> 0	< 0	A 2D W_{loc}^s and a 1D W_{loc}^c
	$= 0$	A 1D W_{loc}^s and a 2D W_{loc}^c
	> 0	A 1D W_{loc}^s , a 1D W_{loc}^c and a 1D W_{loc}^u
$= 0$	< 0	A 3D W_{loc}^c
	$= 0$	A 3D W_{loc}^c
	> 0	A 1D W_{loc}^s , a 1D W_{loc}^c and a 1D W_{loc}^u
< 0	< 0	A 1D W_{loc}^c and a 2D W_{loc}^u
	$= 0$	A 2D W_{loc}^c and a 1D W_{loc}^u
	> 0	A 1D W_{loc}^s , a 1D W_{loc}^c and a 1D W_{loc}^u

TABLE 3. The dynamical behaviors of points of E_0 .

b	a	Property of E_0
> 0	< 0	A 2D W_{loc}^s and a 1D W_{loc}^c
	$= 0$	A 1D W_{loc}^s and a 2D W_{loc}^c
	> 0	A 1D W_{loc}^s , a 1D W_{loc}^c and a 1D W_{loc}^u
$= 0$	< 0	A 3D W_{loc}^c
	$= 0$	A 3D W_{loc}^c
	> 0	A 1D W_{loc}^s , a 1D W_{loc}^c and a 1D W_{loc}^u
< 0	< 0	A 1D W_{loc}^c and a 2D W_{loc}^u
	$= 0$	A 2D W_{loc}^c and a 1D W_{loc}^u
	> 0	A 1D W_{loc}^s , a 1D W_{loc}^c and a 1D W_{loc}^u

Proposition 2.

- (1) When $a, b, c_1, c_2, d_1, d_2 \in \mathbb{R}$, the local dynamical behaviors of points of E_z are totally summarized in Table 1. While $a = 0$, $c_1 = d_1 \neq 0$, $c_2 = -d_2 \neq 0$ (resp.

$c_1 = -d_1$ and $c_2 = d_2$), $-(bd_1 + d_2 \sin^2 2x) \in \mathbb{R}$ (resp. $-(bd_1 - d_2 \sin^2 2x) \in \mathbb{R}$), Table 2 lists the local dynamics of each of E_p .

- (2) Moreover, for $a = 0$, $b > 0$, $d_1 = -c_1 > 0$ and $d_2 = c_2 < 0$, each of E_p is globally asymptotically stable. In particular, when

$$b = d_1, \quad Q = \frac{1}{2} [-d_2 y^2 + (-d_1 z + d_2 \sin^2 x)^2]$$

is an invariant surface with cofactor $-2b$.

Proof.

- (1) Firstly, the local stability of points of E_z and E_p easily follow from the linear analysis and are omitted here.
- (2) Secondly, we discuss the global stability of points of E_p , i.e., each point of E_p is globally stable. For $a = 0$, $b > 0$, $d_1 = -c_1 > 0$ and $d_2 = c_2 < 0$, set the following Lyapunov function:

$$Q = \frac{1}{2} [-d_2 y^2 + (-d_1 z + d_2 \sin^2 x)^2],$$

with

$$\frac{dQ}{dt} \Big|_{(3.1)} = bd_2 y^2 - d_1 (-d_1 z + d_2 \sin^2 x)^2,$$

which yields

$$\dot{Q} = 0 \quad \Leftrightarrow \quad y = -d_1 z + d_2 \sin^2 x = 0,$$

which implies the stability of points of E_p . According to LaSalle theorem [1], each of E_p is also globally asymptotically stable.

In particular, for $b = d_1$ and $Q_0 = \frac{1}{2} [-d_2 y_0^2 + (-d_1 z_0 + d_2 \sin^2 x_0)^2]$, we arrive at

$$\frac{dQ}{dt} \Big|_{(3.1)} = -b [-d_2 y^2 + (-d_1 z + d_2 \sin^2 x)^2] = -2bQ,$$

which leads to

$$0 \leq Q = Q_0 e^{-2b(t-t_0)} \rightarrow 0, \quad t \rightarrow +\infty.$$

Namely, Q is an invariant surface with cofactor $-2b$. The proof is finished. $\square$

Proposition 3.

- (1) If $d_1 < 0$, then each of E_0 is unstable.
- (2) If $d_1 = 0$, then the dynamics of points of E_0 are the same to the ones of points of E_z and listed in Table 1.
- (3) If $d_1 > 0$, then the dynamics of points of E_0 are presented in Table 3.

Proof. The local stability of points of E_0 easily follows from the linear analysis and are omitted here. $\square$

Proposition 4. Set $W = \{(a, b, c_1, c_2, d_1, d_2) \in \mathbb{R}^6 | 0 < \frac{-ad_1}{c_1d_2 + c_2d_1} \leq 1\}$, $W_2 = W \setminus W_1$
 $W_1 = \{(a, b, c_1, c_2, d_1, d_2) \in W | b + d_1 > 0,$

$$d_1 \left[b + \frac{4ac_2}{c_1d_2 + c_2d_1} \left(1 + \frac{ad_1}{c_1d_2 + c_2d_1} \right) \right] > 0,$$

$$4ad_1 \left(1 + \frac{ad_1}{c_1d_2 + c_2d_1} \right) > 0\}$$

and

$$W_1^1 = \{(a, b, c_1, c_2, d_1, d_2) \in W_1 : \Sigma < 0\},$$

$$W_1^2 = \{(a, b, c_1, c_2, d_1, d_2) \in W_1 : \Sigma = 0\},$$

$$W_1^3 = \{(a, b, c_1, c_2, d_1, d_2) \in W_1 : \Sigma > 0\},$$

where $\Sigma = d_1 \left[(b + d_1) \left(b + \frac{4ac_2}{c_1d_2 + c_2d_1} \left(1 + \frac{ad_1}{c_1d_2 + c_2d_1} \right) \right) - 4a \left(1 + \frac{ad_1}{c_1d_2 + c_2d_1} \right) \right]$.

The following two assertions hold.

- (1) Points of $E_{\pm}$ are unstable when $(a, b, c_1, c_2, d_1, d_2) \in W_1^1 \cup W_2$ whereas points of $E_{\pm}$ are asymptotically stable when $(a, b, c_1, c_2, d_1, d_2) \in W_1^3$.
- (2) While $(a, b, c_1, c_2, d_1, d_2) \in W_1^2$, system (3.1) simultaneously undergoes Hopf bifurcation at points of $E_{\pm}$.

Proof.

- (1) The proof of stability of points of $E_{\pm}$ easily follows from Routh-Hurwitz criterion and is omitted here.
- (2) Assume $\lambda_1 = -(b_* + d_1) < 0$, $\lambda_{2,3} = \pm \omega i$ with

$$\omega = \sqrt{d_1 \left[b_* + \frac{4ac_2}{c_1d_2 + c_2d_1} \left(1 + \frac{ad_1}{c_1d_2 + c_2d_1} \right) \right]},$$

where b_* satisfies

$$d_1 \left[(b_* + d_1) \left(b_* + \frac{4ac_2}{c_1d_2 + c_2d_1} \left(1 + \frac{ad_1}{c_1d_2 + c_2d_1} \right) \right) - 4a \left(1 + \frac{ad_1}{c_1d_2 + c_2d_1} \right) \right] = 0.$$

Calculating the derivatives on both sides of Eq. (4.1) with respect to b and substituting λ with ωi lead to

$$\frac{dRe(\lambda_2)}{db} \Big|_{b=b_*} = -\frac{\omega^2 + d_1(b_* + d_1)}{2[\omega^2 + (b_* + d_1)^2]} \neq 0,$$

which validates the condition of the transversality.

Consequently, Hopf bifurcation simultaneously happens at points of $E_{\pm}$, as shown in Fig. 6. The proof is completed.

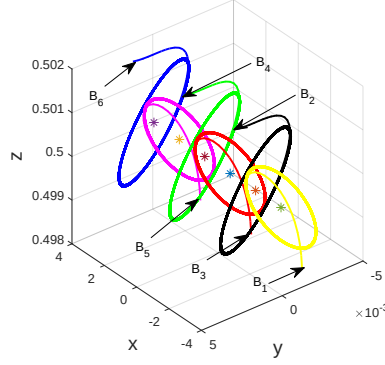


FIG. 6. Hopf bifurcation of system (3.1) when $(a, b, c_1, c_2, d_1, d_2) = (1, -1 + \sqrt{2}, -1, -1, 1, 1)$ and $(x_0^{4,5}, y_0^{4,5}, z_0^{4,5}) = (\pm \frac{\pi}{4}, 0, \frac{1}{2}) + (\pm 1.3, \pm 1.3, \pm 1.6) \times 10^{-3}$, $(x_0^{6,7}, y_0^{6,7}, z_0^{6,7}) = (\pm \frac{3\pi}{4}, 0, \frac{1}{2}) + (\pm 1.3, \pm 1.3, \pm 1.6) \times 10^{-3}$, $(x_0^{8,9}, y_0^{8,9}, z_0^{8,9}) = (\pm \frac{5\pi}{4}, 0, \frac{1}{2}) + (\pm 1.3, \pm 1.3, \pm 1.6) \times 10^{-3}$.

5. SINGULARLY DEGENERATE HETEROCLINIC CYCLES WITH NEARBY ISOLATED LORENZ-LIKE ATTRACTORS

Except for conjoined Lorenz-like attractors shown in Figs. 1-3, with suitable choices of parameters and initial conditions, this section detects singularly degenerate heteroclinic cycles and heteroclinic orbits, whose broken generates infinitely many isolated coexisting Lorenz-like attractors as the ones in [17, 19]. Unfortunately, we are not able to give a rigorous proof of the existence of singularly degenerate heteroclinic cycles of system (3.1) by now. To achieve this target, we only resort to the numerical simulation. Aiming at coining some new phenomena, the following numerical results are illustrated.

Case (1): $d_1 = 0, 0.06$, $(a, b, c_1, c_2, d_2) = (2, 0.5, -3, -1, 1)$, $E_z^1 = (0, 0, -3)$, $E_z^2 = (0, 0, 0.1)$, $E_z^3 = (0, 0, 0.4)$, $E_z^4 = (0, 0, -2)$, $E_z^5 = (0, 0, 0)$.

Numerical Result. 4.1:

- (1) For $d_1 = 0$, the 1D $W^u(E_z^{1,2,3})$ of each normally hyperbolic saddles $E_z^{1,2,3}$ tend toward one of the normally hyperbolic stable foci $(0, 0, 1.4133)$, $(0, 0, 2.8513)$ and $(0, 0, 1.2887)$ in the line $(0, 0, z)$ as $t \rightarrow \infty$, which creates singularly degenerate heteroclinic cycles, while 1D $W^u(E_z^{4,5})$ of each normally hyperbolic saddles $E_z^{4,5}$ tend toward the normally hyperbolic stable foci $(\pm \pi, 0, 1.4344)$ and $(\pm \pi, 0, 2.4165)$ in the line $(\pm \pi, 0, z)$ as $t \rightarrow \infty$, which forms heteroclinic orbits, as shown in Fig. 7(a).

- (2) For $d_1 = 0.06$, the broken of singularly degenerate heteroclinic cycles consisting of E_z^3 and $(0, 0, 1.2887)$, and heteroclinic orbits to E_z^4 and $(\pm\pi, 0, 1.4344)$ creates chaotic attractors that circle around the line $(0, 0, z)$; the broken of singularly degenerate heteroclinic cycles consisting of $E_z^{1,2}$, and heteroclinic orbits to E_z^5 and $(\pm\pi, 0, 2.4165)$ creates chaotic attractors that circle around $(\pm\pi, 0, z)$, as depicted in Fig. 7(b).
- Case (2):** $d_1 = 0, 0.06$, $b = 0.5, 0.738, 0.759$, $(a, c_1, c_2, d_2) = (2, -3, -1, 1)$, $E_z^6 = (0, 0, -1)$, $E_z^{7,8} = (\pm\pi, 0, -1)$, $E_z^{9,10} = (\pm\pi, 0, 0)$, $E_z^{11,12} = (\pm\pi, 0, -2)$, $E_z^{13} = (0, 0, -2)$.

Numerical Result. 4.2:

- (3) For $d_1 = 0$ and $b = 0.5$, the $1D W^u(E_z^{6,7,8})$ of each normally hyperbolic saddles $E_z^{6,7,8}$ tend toward one of the normally hyperbolic stable foci $(0, 0, 2.2345)$ and $(\pm\pi, 0, 2.2345)$ as $t \rightarrow \infty$, which forms singularly degenerate heteroclinic cycles; while $1D W^u(E_z^{9,10,11,12,13,5})$ of each normally hyperbolic saddles $E_z^{9,10,11,12,13,5}$ tend toward the normally hyperbolic stable foci $(\pm 2\pi, 0, 2.4165)$, $(0, 0, 2.4165)$, $(\pm 2\pi, 0, 1.4344)$, $(0, 0, 1.4344)$, $(\pm\pi, 0, 1.4344)$ and $(\pm\pi, 0, 2.4165)$, as $t \rightarrow \infty$, which creates heteroclinic orbits, as illustrated in Fig. 8(a).
- (4) For $d_1 = 0.06$, the broken of singularly degenerate heteroclinic cycles consisting of $E_z^{6,7,8}$, and heteroclinic orbits to $E_z^{9,10}$ (resp. E_z^5) and $(\pm 2\pi, 0, 2.4165)$ and $(0, 0, 2.4165)$ (resp. $(\pm\pi, 0, 2.4165)$) creates chaotic attractors that circle around $(\pm 2\pi, 0, z)$ and $(0, 0, z)$ (resp. $(\pm\pi, 0, z)$); the broken of heteroclinic orbits to $E_z^{11,12,13}$ generates chaotic attractors that circle around $(\pm\pi, 0, z)$ and $(0, 0, z)$, as depicted in Fig. 8(b).

Numerical Result. 4.3:

- (5) For $d_1 = 0$ and $b = 0.738$, the $1D W^u(E_z^{5,9,10})$ of each normally hyperbolic saddles $E_z^{5,9,10}$ tend toward normally hyperbolic stable foci $(0, 0, 4.0264)$ and $(\pm\pi, 0, 4.0264)$ as $t \rightarrow \infty$, which forms singularly degenerate heteroclinic cycles; while $1D W^u(E_z^{6,7,8,11,12,13})$ of each normally hyperbolic saddles $E_z^{6,7,8,11,12,13}$ tends toward the normally hyperbolic stable foci $(\pm\pi, 0, 2.0143)$, $(\pm 2\pi, 0, 2.0143)$, $(0, 0, 2.0143)$, $(\pm 2\pi, 0, 2.0641)$, $(\pm\pi, 0, 2.0641)$ and $(0, 0, 2.0641)$ as $t \rightarrow \infty$, which generates heteroclinic orbits, as displayed in Fig. 9(a).
- (6) For $d_1 = 0.06$, the broken of singularly degenerate heteroclinic cycles consisting of $E_z^{5,9,10}$ generates chaotic attractors that circle around $(\pm\pi, 0, z)$, $(\pm 2\pi, 0, z)$ and $(0, 0, z)$; the broken of heteroclinic orbits to $E_z^{6,7,8}$ (resp. $E_z^{11,12,13}$) creates chaotic attractors that circle around $(\pm\pi, 0, z)$, $(\pm 2\pi, 0, z)$ and $(0, 0, z)$ (resp. $(\pm\pi, 0, z)$ and $(0, 0, z)$), as depicted in Fig. 9(b).

Numerical Result. 4.4:

- (7) For $d_1 = 0$ and $b = 0.759$, the 1D $W^u(E_z^{5,9,10})$ of each normally hyperbolic saddles $E_z^{5,9,10}$ tend toward normally hyperbolic stable foci $(0, 0, 2.4165)$ and $(\pm\pi, 0, 2.4165)$ as $t \rightarrow \infty$, which forms singularly degenerate heteroclinic cycles, while 1D $W^u(E_z^{6,7,8,11,12,13})$ of each normally hyperbolic saddles $E_z^{6,7,8,11,12,13}$ tends toward the normally hyperbolic stable foci $(\pm\pi, 0, 2.2345)$, $(\pm 2\pi, 0, 2.2345)$, $(0, 0, 2.2345)$, $(\pm 2\pi, 0, 1.4344)$, $(\pm\pi, 0, 1.4344)$ and $(0, 0, 1.4344)$ as $t \rightarrow \infty$, which generates heteroclinic orbits, as displayed in Fig. 10(a).
- (8) For $d_1 = 0.06$, the broken of singularly degenerate heteroclinic cycles consisting of $E_z^{5,9,10}$ generates chaotic attractors that circle around $(\pm\pi, 0, z)$ and $(0, 0, z)$; the broken of heteroclinic orbits to $E_z^{6,7,8}$ (respectively $E_z^{11,12,13}$) creates chaotic attractors that circle around $(\pm\pi, 0, z)$, $(\pm 2\pi, 0, z)$ and $(0, 0, z)$ (resp. $(\pm\pi, 0, z)$ and $(0, 0, z)$), as depicted in Fig. 10(b).

Remark 2. Numerical Result. 4.1-4.4 demonstrate that the creation of singularly degenerate heteroclinic cycles and heteroclinic orbits with nearby isolated Lorenz-like attractors depends not only the initial conditions but also parameters.

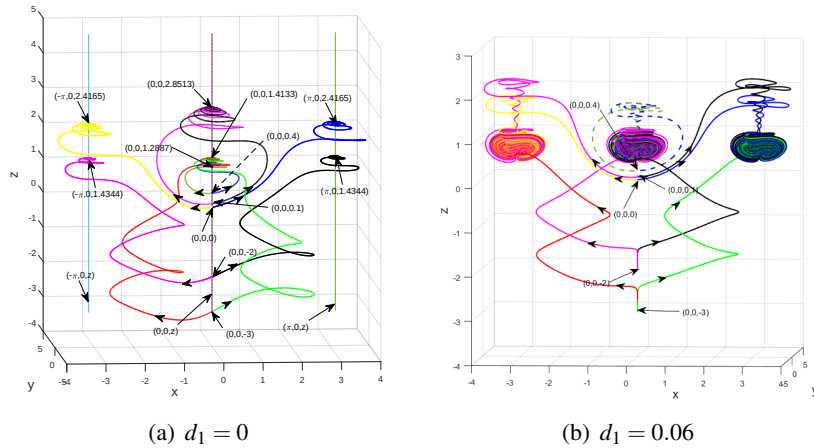


FIG. 7. Coexistence of singularly degenerate heteroclinic cycles and heteroclinic orbits with nearby isolated strange attractors of system (3.1) when $(a, b, c_1, c_2, d_2) = (2, 0.5, -3, -1, 1)$ and initial conditions $E_z^1 = (0, 0, -3)$, $E_z^2 = (0, 0, 0.1)$, $E_z^3 = (0, 0, 0.4)$, $E_z^4 = (0, 0, -2)$, $E_z^5 = (0, 0, 0)$.

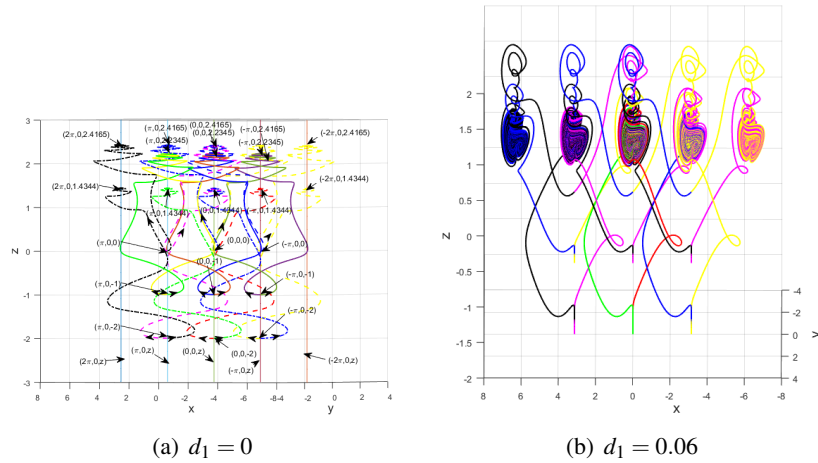


FIG. 8. Coexistence of singularly degenerate heteroclinic cycles and heteroclinic orbits with nearby isolated strange attractors of system (3.1) when $(a, b, c_1, c_2, d_2) = (2, 0.5, -3, -1, 1)$, $E_z^6 = (0, 0, -1)$, $E_z^{7,8} = (\pm\pi, 0, -1)$, $E_z^{9,10} = (\pm\pi, 0, 0)$, $E_z^{11,12} = (\pm\pi, 0, -2)$, $E_z^{13} = (0, 0, -2)$ and $E_z^5 = (0, 0, 0)$.

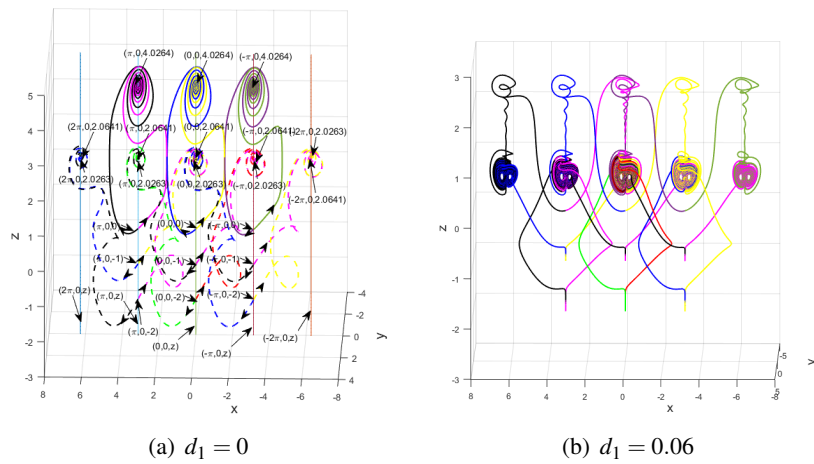


FIG. 9. Coexistence of singularly degenerate heteroclinic cycles with and heteroclinic orbits nearby isolated strange attractors of system (3.1) when $(a, b, c_1, c_2, d_2) = (2, 0.738, -3, -1, 1)$, $E_z^6 = (0, 0, -1)$, $E_z^{7,8} = (\pm\pi, 0, -1)$, $E_z^{9,10} = (\pm\pi, 0, 0)$, $E_z^{11,12} = (\pm\pi, 0, -2)$, $E_z^{13} = (0, 0, -2)$ and $E_z^5 = (0, 0, 0)$.

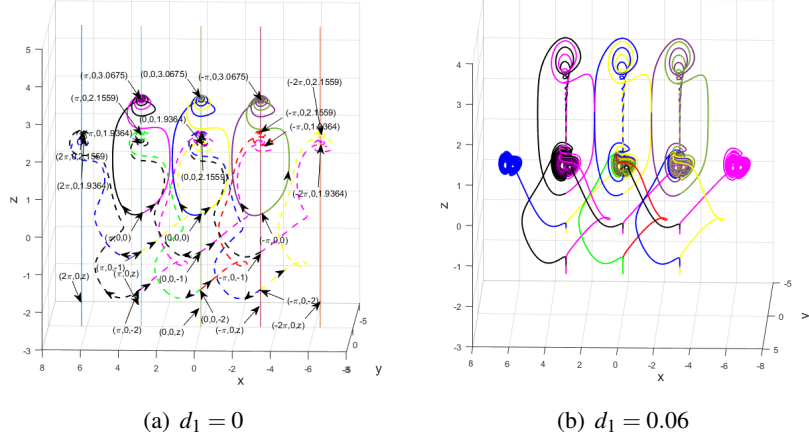


FIG. 10. Coexistence of singularly degenerate heteroclinic cycles and heteroclinic orbits with nearby isolated strange attractors of system (3.1) when $(a, b, c_1, c_2, d_2) = (2, 0.759, -3, -1, 1)$, $E_z^6 = (0, 0, -1)$, $E_z^{7,8} = (\pm\pi, 0, -1)$, $E_z^{9,10} = (\pm\pi, 0, 0)$, $E_z^{11,12} = (\pm\pi, 0, -2)$, $E_z^{13} = (0, 0, -2)$ and $E_z^5 = (0, 0, 0)$.

6. EXISTENCE OF HETEROCLINIC ORBITS

In this section, we prove that there exists an infinite set of heteroclinic orbits in system (3.1) when $a > 0$, $b > 0$, $d_1 > 0$, $c_1 d_2 > 0$, $c_1 d_2 + c_2 d_1 < 0$ and $0 < \frac{-ad_1}{c_1 d_2 + c_2 d_1} \leq 1$, as [10, 11, 14–16].

To facilitate derivation, denote by $\phi_t(q_0) = (x(t; x_0), y(t; y_0), z(t; z_0))$ any one solution of system (3.1) with the initial point $q_0 = (x_0, y_0, z_0) \in \mathbb{R}^3$. Since in this case W_{loc}^u is one-dimensional according to Proposition 3, we can denote $\gamma = \{\phi_t(q_0) | t \in \mathbb{R}\}$ be the unstable manifold of system (3.1) at points of E_0 .

Firstly, set the first Lyapunov function

$$V(\phi_t(q_0)) = \frac{1}{2} \left[-\frac{d_1}{c_1 d_2 + c_2 d_1} y^2 - \frac{c_1}{d_2(c_1 d_2 + c_2 d_1)} (-d_1 z + d_2 \sin^2(x))^2 + \left(\frac{ad_1}{c_1 d_2 + c_2 d_1} + \sin^2(x) \right)^2 \right].$$

Then, the derivative of V along the trajectories of system (3.1) is calculated as follows:

$$\begin{aligned} \frac{dV(\phi_t(q_0))}{dt} \Big|_{(3.1)} &= \frac{c_1 d_1}{d_2(c_1 d_2 + c_2 d_1)} (-d_1 z + d_2 \sin^2(x))^2 \\ &\quad + \frac{bd_1}{c_1 d_2 + c_2 d_1} y^2. \end{aligned} \tag{6.1}$$

First of all, one introduces the following result.

Proposition 5. *When $a > 0$, $b > 0$, $d_1 > 0$, $c_1 d_2 > 0$, $c_1 d_2 + c_2 d_1 < 0$ and $0 < \frac{-ad_1}{c_1 d_2 + c_2 d_1} \leq 1$, we derive the following two assertions.*

- (a) *If $\exists t_{1,2}$, such that $t_1 < t_2$ and $V(\phi_{t_1}(q_0)) = V(\phi_{t_2}(q_0))$, then q_0 is one of the equilibrium points of system (3.1).*
- (b) *If $\lim_{t \rightarrow -\infty} \phi_t(q_0) \in E_0$ and $q_0 \neq E_0$, then $V(E_0) > V(\phi_t(q_0))$ for all $t \in \mathbb{R}$.*

Proof.

- (a) Since $\frac{dV(\phi_t(q_0))}{dt} \Big|_{(3.1)} \leq 0$ when $a > 0$, $b > 0$, $d_1 > 0$, $c_1 d_2 > 0$, $c_1 d_2 + c_2 d_1 < 0$ and $0 < \frac{-ad_1}{c_1 d_2 + c_2 d_1} \leq 1$, it follows from Eq. (6.1) that $\frac{dV(\phi_t(q_0))}{dt} \Big|_{(3.1)} = 0$, $\forall t \in (t_1, t_2)$, which thus suggests that q_0 is one of equilibria, i.e.,

$$\dot{x}(t; x_0) \equiv \dot{y}(t; y_0) \equiv \dot{z}(t; z_0) \equiv 0. \quad (6.2)$$

Actually, $\dot{x}(t; x_0) = y = 0$ implies $x(t) = x_0$ and $\dot{y}(t; y_0) = 0$, $y(t) = y_0 = 0$, $\forall t \in \mathbb{R}$.

In a word, $\phi_t(q_0) \in \{y = 0\} \cap \{-d_1 z + d_2 \sin^2 x = 0\}$ leads to (6.2).

- (b) We first prove that $V(E_0) > V(\phi_t(q_0))$ for all $t \in \mathbb{R}$. Suppose that $V(E_0) \leq V(p(t; q_0))$ for some $t \in \mathbb{R}$. Then the above result (a) reads that q_0 is one of equilibria of system (3.1) and $q_0 \notin E_0$. This contradicts the fact that $\lim_{t \rightarrow -\infty} \phi_t(q_0) \in E_0$. Hence, it follows that $V(E_0) > V(\phi_t(q_0))$ for all $t \in \mathbb{R}$.

□

Using Proposition 5, the existence of heteroclinic orbits to points of E_0 and $E_{\pm}$ is derived in the following statement.

Proposition 6. *Consider $a > 0$, $b > 0$, $d_1 > 0$, $c_1 d_2 > 0$, $c_1 d_2 + c_2 d_1 < 0$ and $0 < \frac{-ad_1}{c_1 d_2 + c_2 d_1} \leq 1$. Then the following statements are true.*

- (i) *Neither homoclinic orbits nor heteroclinic orbits to points of E_+ or E_- exist in system (3.1).*
- (ii) *System (3.1) has infinitely many heteroclinic orbits to points of E_0 and $E_{\pm}$.*

Proof.

- (i) Let us first prove that there is no heteroclinic/homoclinic orbits to points of E_+ or E_- in system (3.1) when $a > 0$, $b > 0$, $d_1 > 0$, $c_1 d_2 > 0$, $c_1 d_2 + c_2 d_1 < 0$ and $0 < \frac{-ad_1}{c_1 d_2 + c_2 d_1} \leq 1$.

Suppose that $p(t) = (x, y, z)$ (it belongs to the set of $\phi_t(q_0)$) is a homoclinic orbit of system (3.1) or a heteroclinic orbit to E'_+ or E'_- , where $\forall E'_+ \in E_+$ and $\forall E'_- \in E_-$. Namely, $p(t)$ is a solution of system (3.1) such that $\lim_{t \rightarrow -\infty} p(t) = e_-$, $\lim_{t \rightarrow +\infty} p(t) = e_+$, where points e_- and e_+ satisfy either $e_- = e_+ \in \{E'_-, E'_0, E'_+\}$ or $\{e_-, e_+\} = \{E'_-, E'_+\}$ with $\forall E'_0 \in E_0$. From (6.1), one has

$$V(e_-) \geq V(p(t)) \geq V(e_+).$$

In either case, one only has the relation $V(e_-) = V(e_+)$, which suggests $V(p(t)) \equiv V(e_-)$. According to the assertion (a) of Proposition 5, $p(t)$ is just one of the equilibrium points of system (3.1). Therefore, neither homoclinic orbits nor heteroclinic orbits to points of E_+ and E_- exist in system (3.1).

- (ii) Next, let us show that γ is any one heteroclinic orbit to E'_0 and E'_+ , i.e., $\lim_{t \rightarrow +\infty} p(t) = E'_+$. From the definition of γ and the conclusion (ii) of Proposition 5 one can get $V(E'_0) > V(p(t; q_0))$. This demonstrates that $p(t)$ does not approach E'_0 as $t \rightarrow +\infty$. Therefore, $\lim_{t \rightarrow +\infty} p(t) = E'_+$.

Lastly, we prove that, if system (3.1) has a heteroclinic orbit to E'_0 and E'_+ , then this orbit belongs to γ .

Denote by $p^*(t) = (x^*(t), y^*(t), z^*(t))$ be a solution of system (3.1) such that

$$\lim_{t \rightarrow -\infty} p^*(t) = e_1^-, \quad \lim_{t \rightarrow +\infty} p^*(t) = e_1^+,$$

where e_1^- and e_1^+ satisfy $\{e_1^-, e_1^+\} = \{E'_0, E'_+\}$. Like for (6.2), one obtains from (6.1) that for all $t \in \mathbb{R}$, $V(e_1^-) \geq V(p^*(t)) \geq V(e_1^+)$. Since $V(E'_0) > V(E'_+)$, we have $e_1^- = E'_0$ and $e_1^+ = E'_+$, i.e.,

$$\lim_{t \rightarrow -\infty} p^*(t) = E'_0, \quad \lim_{t \rightarrow +\infty} p^*(t) = E'_+,$$

which yields $p^*(t) \in \gamma$ based on the assertion (b) of Proposition 5. As the orbits are symmetrical w.r.t. the z -axis, there still exists a unique heteroclinic orbit $q^*(t) \in \gamma$ symmetrical to $p^*(t)$, which satisfies

$$\lim_{t \rightarrow -\infty} q^*(t) = E'_0, \quad \lim_{t \rightarrow +\infty} q^*(t) = E'_-.$$

As E_0 and $E_{\pm}$ contain infinitely many isolated stationary points E'_0 and $E'_{\pm}$, there exist infinitely many heteroclinic orbits to points of E_0 and $E_{\pm}$. This theoretical result is also verified via numerical simulation, as shown in Fig. 11. The proof is over. $\square$

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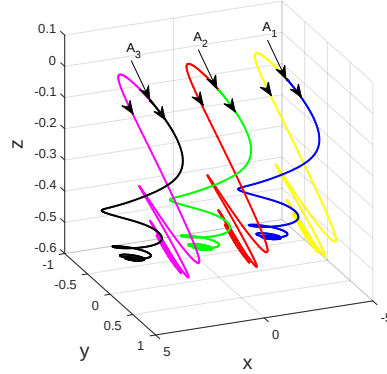


FIG. 11. Infinitely many pairs of heteroclinic orbits to points of E_0 and $E_{\pm}$ of system (3.1) $(a, b, c_1, c_2, d_1, d_2) = (3, 0.1, -3, -9, 1, -1)$ and $(A_1) (x_0^{1,2}, y_0^{1,2}, z_0^{1,2}) = (-\pi, 0, 0) + (\pm 1.3, \pm 13, \pm 16) \times 10^{-8}$, $(A_2) (x_0^{3,4}, y_0^{3,4}, z_0^{3,4}) = (\pm 1.3, \pm 13, \pm 16) \times 10^{-8}$, $(A_3) (x_0^{5,6}, y_0^{5,6}, z_0^{5,6}) = (\pi, 0, 0) + (\pm 1.3, \pm 13, \pm 16) \times 10^{-8}$.

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APPROXIMATE SOLUTION OF THE CONFORMABLE INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. In this paper, fractional linear and nonlinear integro-differential equations are solved by using an iteration method. Fractional derivative and fractional integral are considered in the conformable sense. The conformable integro-differential equation is converted to a conformable integral equation. Then, the conformable integral equation leads to an iteration sequence, the limit of which is a solution of the conformable integro-differential equation. In addition, stability and convergence analysis of the presented method are investigated. The applicability of the presented method is also shown by using numerical examples.

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1. INTRODUCTION

In recent years, the theory of fractional calculus has attracted the attention of many researchers in various disciplines due to the fact that fractional calculus is more useful for formulation of the problems in physics, chemistry and engineering such as the field of electromagnetic waves, viscoelasticity, dielectric polarization and diffusion equations [7, 10, 12, 13, 16, 18, 19, 21, 24].

Conformable fractional derivative has been defined by Khalil et al. ([14]) and has also some applications in the real world (see, for example, [3, 15, 20, 23]). In the last decades, solutions of the conformable fractional integro-differential equations have been investigated by researchers. In [4], existence of the solutions of periodic boundary value problems for impulsive fractional integro-differential equations has been investigated. In [5], fractional linear Volterra-Fredholm integro-differential equations have been solved by using the Sinc-Collocation Method. In [17], fractional linear integral and integro-differential equations have been solved by using the conformable Laplace transform. Laguerre polynomial solutions of the linear fractional integro differential equations have been obtained in [9]. Conformable linear and nonlinear

fractional integro-differential systems of second order have been solved by using the reproducing kernel Hilbert space method in [6]. Variational iteration method has been applied to the conformable linear and nonlinear fractional partial integro-differential equation in [8]. A computational approach based on the shifted Legendre polynomials has been presented for solving the conformable linear and nonlinear fractional partial integro-differential equation in [2].

In the present paper, we consider conformable fractional integro-differential equations in the following form

$$T_{\alpha}^a y(t) = f(t) + \lambda (I_{\alpha}^a F(t, x, y(x))) (t), \quad 0 < \alpha < 1, \quad (1.1)$$

with initial condition

$$y(a) = c, \quad (1.2)$$

where $f(t)$ is continuous for $\forall t \in [a, T]$ and $F(x, t, y(t))$ is continuous for $\forall (x, t) \in [a, T] \times [a, T]$, $\forall y \in \Re$, $T > 0$, T_{α}^a is conformable fractional derivative of order α and I_{α}^a is conformable fractional integral of order α .

The rest of the paper is organized as follows. In Section 2, we present the basic properties of the conformable fractional derivative and integral. In Section 3, we construct an iteration method. In addition, we present stability and convergence analysis of the iteration method. In Section 4, we show accuracy of the method by using numerical examples. We give conclusion in the last section.

2. DESCRIPTION OF THE CONFORMABLE FRACTIONAL DERIVATIVE AND INTEGRALS

For a function $f: (a, \infty) \rightarrow \mathbb{R}$, the conformable fractional derivative of f of order $0 < \alpha < 1$ in variable t is defined as follows: (see, for example, [14])

$$T_{\alpha}^a f(t) = \lim_{\varepsilon \rightarrow 0} \frac{f(t + \varepsilon(t-a)^{1-\alpha}) - f(t)}{\varepsilon}.$$

Lemma 1 ([14, Theorem 2.2]). *Let $\alpha \in (0, 1]$ and f, g be α differentiable at a point $t > 0$. Then*

$$\begin{aligned} T_{\alpha}^a (c_1 f + c_2 g)(t) &= c_1 T_{\alpha}^a f(t) + c_2 T_{\alpha}^a g(t), \quad \forall c_1, c_2 \in \mathbb{R}, \\ T_{\alpha}^a (fg)(t) &= f(t) T_{\alpha}^a g(t) + g(t) T_{\alpha}^a f(t), \\ T_{\alpha}^a (f')(t) &= (t-a)^{1-\alpha} f'(t). \end{aligned}$$

Conformable fractional integral of f of order $0 < \alpha < 1$ is defined by (see, for example, [1])

$$(I_{\alpha}^a f)(t) = \int_a^t f(x) d_{\alpha}(x, a) = \int_a^t f(x) (x-a)^{\alpha-1} dx.$$

Note that for $a = 0$, the conformable fractional derivative of f is denoted by $T_\alpha f(t)$ and the conformable fractional integral of f is denoted by $(I_\alpha f)(t) = \int_0^t f(x) d_\alpha(x)$.

Lemma 2 ([1, Lemma 2.8]). *Let $f : (a, b) \rightarrow \mathbb{R}$ be differentiable and $0 < \alpha \leq 1$. Then, for all $t > a$ we have*

$$(I_\alpha^a T_\alpha^a f)(t) = f(t) - f(a).$$

3. ITERATIVE METHOD

Theorem 1. *Consider the following conformable integro-differential equation with initial condition*

$$\begin{aligned} T_\alpha^a y(t) &= f(t) + \lambda (I_\alpha^a F(t, x, y(x))) (t), \quad 0 < \alpha < 1, \\ y(a) &= c. \end{aligned} \quad (3.1)$$

Let $y(t)$ be differentiable for $\forall t \in (a, T)$. Then the general solution of the above equation is equivalent to the general solution of the following equation

$$y(t) = y(a) + (I_\alpha^a f)(t) + \lambda \left(I_\alpha^a \int_a^u F(u, x, y(x)) d_\alpha(x, a) \right) (t), \quad 0 < \alpha < 1. \quad (3.2)$$

Proof. Applying the operator I_α^a to Eq. (3.1) and using Lemma 2, we have

$$y(t) - y(a) = (I_\alpha^a f)(t) + \lambda \left(I_\alpha^a \int_a^u F(u, x, y(x)) d_\alpha(x, a) \right) (t).$$

□

Using Eq. (3.2), the iteration formula for the solution of the problem (1.1)-(1.2) is constructed as follows

$$\begin{aligned} y_k(t) &= y_0(t) + \lambda \left(I_\alpha^a \int_a^u F(u, x, y_{k-1}(x)) d_\alpha(x, a) \right) (t), \quad k = 1, 2, \dots, \\ y_0(t) &= y(a) + (I_\alpha^a f)(t). \end{aligned} \quad (3.3)$$

Theorem 2. *The sequence $y_k(t)$ defined by (3.3) is convergent for $\forall t \in [a, T]$ when $F(x, t, y(t))$ is continuous; $\|F(x, t, y(t))\|_\infty < M$; $\|F(x, t, y_1(t)) - F(x, t, y_2(t))\|_\infty < L \cdot \|y_1(t) - y_2(t)\|_\infty$, $\forall y_1(t), y_2(t) \in \mathfrak{R}$, $\forall (x, t) \in [a, T] \times [a, T]$ and some $L, M > 0$.*

Proof. Taking $k = 1$ in the iteration formula (3.3), we have

$$|y_1(t) - y_0(t)| \leq |\lambda| \left(I_\alpha^a \int_a^u |F(u, x, y_0(x))| d_\alpha(x, a) \right) (t) \leq |\lambda| M \frac{|t - a|^{2\alpha}}{2\alpha^2}$$

and by induction

$$|y_k(t) - y_{k-1}(t)| \leq \frac{M}{L} |\lambda|^k L^k \cdot \frac{|t-a|^{2k\alpha}}{(2k)! \alpha^{2k}}$$

is obtained. Since $y_k(t) = y_0(t) + \sum_{i=1}^k (y_i(t) - y_{i-1}(t))$ and

$$\sum_{k=0}^{\infty} |\lambda|^k L^k \cdot \frac{|t-a|^{2k\alpha}}{(2k)! \alpha^{2k}} = E_{2,1} \left(L |\lambda| \cdot \frac{|t-a|^{2\alpha}}{\alpha^2} \right)$$

in the whole line, the sequence $y_k(t)$ is convergent for $t \in [a, T]$. Here $E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}$, $\alpha, \beta > 0$, is the Mittag-Leffler function (see, for example, [22]). $\square$

Theorem 3. Let $\varphi(t) = \lim_{k \rightarrow \infty} y_k(t)$ for $\forall t \in [a, T]$. Assume that $F(x, t, y(t))$ is continuous; $\|F(x, t, y(t))\|_{\infty} < M$; $\|F(x, t, y_1(t)) - F(x, t, y_2(t))\|_{\infty} < L \cdot \|y_1(t) - y_2(t)\|_{\infty}$, $\forall y_1(t), y_2(t) \in \mathfrak{R}$, $\forall (x, t) \in [a, T] \times [a, T]$ and some $L, M > 0$. Then for $\forall t \in [a, T]$

$$\lim_{k \rightarrow \infty} \left(I_{\alpha}^a \int_a^u F(u, x, y_k(x)) d_{\alpha}(x, a) \right) (t) = \left(I_{\alpha}^a \int_a^u F(u, x, \varphi(x)) d_{\alpha}(x, a) \right) (t). \quad (3.4)$$

Proof. Since

$$y_k(t) = y_0(t) + \sum_{i=1}^k (y_i(t) - y_{i-1}(t))$$

and

$$\varphi(t) = y_0(t) + \sum_{i=1}^{\infty} (y_i(t) - y_{i-1}(t))$$

we can write

$$|y_k(t) - \varphi(t)| \leq \sum_{i=k+1}^{\infty} |y_i(t) - y_{i-1}(t)| \leq \sum_{i=k+1}^{\infty} \frac{M}{L} |\lambda|^i L^i \cdot \frac{|t-a|^{2i\alpha}}{(2i)! \alpha^{2i}}. \quad (3.5)$$

Let $z = |\lambda| L \frac{|t-a|^{2\alpha}}{\alpha^2}$. Since $E_{2,1}(z) = \frac{\exp(\sqrt{z}) + \exp(-\sqrt{z})}{2} = \sum_{i=0}^{\infty} \frac{z^i}{(2i)!}$ (see, for example, [11]) and $E_{2,1}(z) = 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^k}{(2k)!} + \exp(\theta) \frac{z^{k+1}}{(2k+2)!}$, $|\theta| < \sqrt{z}$, we have

$$|E_{2,1}(z) - (1 + z + \frac{z^2}{2!} + \cdots + \frac{z^k}{(2k)!})| \leq \exp(\sqrt{z}) \frac{|z|^{k+1}}{(2k+2)!}. \quad (3.6)$$

From (3.6), (3.5) can be written as

$$|y_k(t) - \varphi(t)| \leq \frac{M}{L} \exp \left(\sqrt{|\lambda| L} \frac{|T-a|^{\alpha}}{|\alpha|} \right) \frac{L^{k+1} |\lambda|^{k+1} |T-a|^{2\alpha(k+1)}}{\alpha^{2k+2} (2k+2)!}. \quad (3.7)$$

From (3.7), we have

$$\left| \left(I_{\alpha}^a \int_a^u F(u, x, y_k(x)) d_{\alpha}(x, a) \right) (t) - \left(I_{\alpha}^a \int_a^u F(u, x, \varphi(x)) d_{\alpha}(x, a) \right) (t) \right| \\ \leq M \cdot \exp\left(\sqrt{|\lambda|L} \frac{|T-a|^{\alpha}}{|\alpha|}\right) \frac{L^{k+1} |\lambda|^{k+1} |T-a|^{2\alpha(k+2)}}{2\alpha^{2k+4} (2k+2)!}.$$

Since $\exp\left(\sqrt{|\lambda|L} \frac{|T-a|^{\alpha}}{|\alpha|}\right) \frac{L^{k+1} |\lambda|^{k+1} |T-a|^{2\alpha(k+2)}}{2\alpha^{2k+4} (2k+2)!} \rightarrow 0$ as $k \rightarrow \infty$, for $k \rightarrow \infty$

$$\left(I_{\alpha}^a \int_a^u F(u, x, y_k(x)) d_{\alpha}(x, a) \right) (t) \rightarrow \left(I_{\alpha}^a \int_a^u F(u, x, \varphi(x)) d_{\alpha}(x, a) \right) (t).$$

□

Theorem 4. Assume that $F(x, t, y(t))$ is continuous; $\|F(x, t, y(t))\|_{\infty} < M$; $\|F(x, t, y_1(t)) - F(x, t, y_2(t))\|_{\infty} < L$. $\|y_1(t) - y_2(t)\|_{\infty}$, $\forall y_1(t), y_2(t) \in \mathfrak{R}$, $\forall (x, t) \in [a, T] \times [a, T]$ and some $L, M > 0$. If the sequence $y_k(t)$ defined by (3.3) converges to $\varphi(t)$, then $\varphi(t)$ is the exact solution of problem (1.1)-(1.2) for $\forall t \in [a, T]$.

Proof. Taking limit as k approaches to infinity in the iteration formula (3.3) and using (3.4) we have

$$\varphi(t) = y_0(t) + \lambda \left(I_{\alpha}^a \int_a^u F(u, x, \varphi(x)) d_{\alpha}(x, a) \right) (t).$$

□

Theorem 5. Assume that the exact solution of the problem (1.1)-(1.2) is $\varphi(t)$ for $\forall t \in [a, T]$ and the approximate solution is $y_k(t)$ for $\forall t \in [a, T]$, where $y_k(t)$ is the k_{th} order approximation, $F(x, t, y(t))$ is continuous; $\|F(x, t, y(t))\|_{\infty} < M$; $\|F(x, t, y_1(t)) - F(x, t, y_2(t))\|_{\infty} < L$. $\|y_1(t) - y_2(t)\|_{\infty}$, $\forall y_1(t), y_2(t) \in \mathfrak{R}$, $\forall (x, t) \in [a, T] \times [a, T]$ and some $L, M > 0$. Then for $\forall t \in [a, T]$ the maximum error is estimated as

$$|y_k(t) - \varphi(t)| \leq \frac{M}{L} \exp\left(\sqrt{|\lambda|L} \frac{|T-a|^{\alpha}}{|\alpha|}\right) \cdot E$$

where

$$E = \frac{L^{k+1} |\lambda|^{k+1} |T-a|^{2\alpha(k+1)}}{\alpha^{2k+2} (2k+2)!}.$$

Proof. The proof can be shown from (3.7). □

Theorem 6. Let $y(t)$, $\bar{y}(t)$ be differentiable for $\forall t \in (a, T)$ and be the solutions of the following problems

$$T_{\alpha}^a y(t) = f(t) + \lambda (I_{\alpha}^a F(t, x, y(x))) (t), \quad 0 < \alpha < 1,$$

$$y(a) = c. \quad (3.8)$$

$$\begin{aligned} T_\alpha^a \bar{y}(t) &= g(t) + \lambda(I_\alpha^a F(t, x, \bar{y}(x)))(t), \quad 0 < \alpha < 1, \\ \bar{y}(a) &= d. \end{aligned} \quad (3.9)$$

Assume that there exists constants H and L such that $\|f(t) - g(t)\|_\infty \leq H$, $\forall t \in [a, T]$; $\|F(x, t, y_1(t)) - F(x, t, y_2(t))\|_\infty < L$. $\|y_1(t) - y_2(t)\|_\infty$ for $\forall y_1(t), y_2(t) \in \mathfrak{R}$ and $\forall (x, t) \in [a, T] \times [a, T]$, then the following inequality is satisfied

$$|y(t) - \bar{y}(t)| \leq C.E_{2,1} \left(L|\lambda| \frac{(t-a)^{2\alpha}}{\alpha^2} \right), \quad \forall t \in [a, T],$$

where $C = |c - d| + H \frac{|T-a|^\alpha}{\alpha}$.

Proof. From Thm.(1), the iteration formulas for Eq. (3.8) and Eq. (3.9) can be written in the following form

$$y_k(t) = y_0(t) + \lambda \left(I_\alpha^a \int_a^u F(u, x, y_{k-1}(x)) d_\alpha(x, a) \right) (t), \quad k = 1, 2, \dots, \quad (3.10)$$

$$y_0(t) = c + (I_\alpha^a f)(t) \quad (3.11)$$

and

$$\bar{y}_k(t) = \bar{y}_0(t) + \lambda \left(I_\alpha^a \int_a^u F(u, x, \bar{y}_{k-1}(x)) d_\alpha(x, a) \right) (t), \quad k = 1, 2, \dots, \quad (3.12)$$

$$\bar{y}_0(t) = d + (I_\alpha^a g)(t). \quad (3.13)$$

From Eq. (3.11) and Eq. (3.13), we have

$$|y_0(t) - \bar{y}_0(t)| \leq |c - d| + H \frac{|t-a|^\alpha}{\alpha} \leq |c - d| + H \frac{|T-a|^\alpha}{\alpha} = C.$$

From Eq. (3.10) and Eq. (3.12) for $k = 1$ we have

$$\begin{aligned} |y_1(t) - \bar{y}_1(t)| &\leq |y_0(t) - \bar{y}_0(t)| \\ &\quad + |\lambda| \left(I_\alpha^a \int_a^u |F(u, x, y_0(x)) - F(u, x, \bar{y}_0(x))| d_\alpha(x, a) \right) (t) \\ &\leq C + |\lambda|.L.C \frac{|t-a|^{2\alpha}}{2\alpha^2}. \end{aligned}$$

From Eq. (3.10) and Eq. (3.12) for $k = 2$ we have

$$\begin{aligned} |y_2(t) - \bar{y}_2(t)| &\leq |y_0(t) - \bar{y}_0(t)| \\ &\quad + |\lambda| \left(I_\alpha^a \int_a^u |F(u, x, y_1(x)) - F(u, x, \bar{y}_1(x))| d_\alpha(x, a) \right) (t) \end{aligned}$$

$$\leq C + |\lambda|.L.C \frac{|t-a|^{2\alpha}}{2\alpha^2} + |\lambda|^2.L^2.C \frac{|t-a|^{4\alpha}}{4!\alpha^4}.$$

Similarly, k^{th} term is written as follows

$$|y_k(t) - \bar{y}_k(t)| \leq C \sum_{i=0}^k |\lambda|^i . L^i . \frac{|t-a|^{2i\alpha}}{(2i)!\alpha^{2i}}. \quad (3.14)$$

Since $y(t) = \lim_{k \rightarrow \infty} y_k(t)$ and $\bar{y}(t) = \lim_{k \rightarrow \infty} \bar{y}_k(t)$, taking the limit of (3.14) as $k \rightarrow \infty$ we have

$$|y(t) - \bar{y}(t)| \leq C \sum_{i=0}^{\infty} |\lambda|^i . L^i . \frac{|t-a|^{2i\alpha}}{(2i)!\alpha^{2i}} = C.E_{2,1} \left(L|\lambda| \frac{(t-a)^{2\alpha}}{\alpha^2} \right).$$

□

It concludes from this theorem that small changes in initial condition and non-homogenous term cause only small changes of the obtained solution.

4. APPLICATIONS

Example 1. Let us consider the following conformable integro-differential equation

$$T_{\frac{1}{2}} y(t) = 1 + \int_0^t y(x) d_{\frac{1}{2}}(x) \quad (4.1)$$

with initial condition

$$y(0) = 1. \quad (4.2)$$

Applying the operator $I_{\frac{1}{2}}$ to Eq. (4.1) we have the following equation

$$y(t) = 1 + (I_{\frac{1}{2}} 1)(t) + \left(I_{\frac{1}{2}} \int_0^u y(x) d_{\frac{1}{2}}(x) \right) (t)$$

and its iteration formula is obtained by

$$y_k(t) = y_0(t) + \left(I_{\frac{1}{2}} \int_0^u y_{k-1}(x) d_{\frac{1}{2}}(x) \right) (t), \quad k = 1, 2, \dots,$$

$$y_0(t) = 1 + 2\sqrt{t}.$$

Therefore, we have the following approximations

$$y_1(t) = 1 + 2\sqrt{t} + 2t + \frac{4}{3}t^{3/2},$$

$$y_2(t) = 1 + 2\sqrt{t} + 2t + \frac{4}{3}t^{3/2} + \frac{4}{15}t^{5/2} + \frac{2}{3}t^2,$$

$$\begin{aligned}
y_3(t) &= 1 + 2\sqrt{t} + 2t + \frac{4}{3}t^{3/2} + \frac{4}{15}t^{5/2} + \frac{2}{3}t^2 + \frac{4}{45}t^3 + \frac{8}{315}t^{7/2}, \\
y_4(t) &= 1 + 2\sqrt{t} + 2t + \frac{4}{3}t^{3/2} + \frac{4}{15}t^{5/2} + \frac{2}{3}t^2 + \frac{4}{45}t^3 + \frac{8}{315}t^{7/2} \\
&\quad + \frac{2}{315}t^4 + \frac{4}{2835}t^{9/2}.
\end{aligned}$$

The k^{th} approximation is obtained as follows

$$y_k(t) = \sum_{i=0}^{2k+1} \frac{2^i t^{i/2}}{i!}.$$

Then $y(t) = \lim_{k \rightarrow \infty} y_k(t) = \sum_{i=0}^{\infty} \frac{2^i t^{i/2}}{i!} = \exp(2\sqrt{t})$ (see, for example, [1]). Note that the $y(t) = \exp(2\sqrt{t})$ is the exact solution of the problem (4.1)-(4.2).

Example 2. Let us consider the following conformable integro-differential equation

$$T_{\frac{1}{3}} y(t) = \frac{1}{3} + \frac{2}{3}t^{1/3} + \frac{9}{10}t^{5/3} + \frac{1}{2}t^2 + \int_0^t (x-t)y(x) d_{\frac{1}{3}}(x) \quad (4.3)$$

with initial condition

$$y(0) = 0. \quad (4.4)$$

Applying the operator $I_{\frac{1}{3}}$ to Eq. (4.3) we have the following equation

$$y(t) = \left(I_{\frac{1}{3}} \left(\frac{1}{3} + \frac{2}{3}t^{1/3} + \frac{9}{10}t^{5/3} + \frac{1}{2}t^2 \right) \right) (t) + \left(I_{\frac{1}{3}} \int_0^t (x-u)y(x) d_{\frac{1}{3}}(x) \right) (t)$$

and its iteration formula is obtained by

$$\begin{aligned}
y_k(t) &= y_0(t) + \left(I_{\frac{1}{3}} \int_0^t (x-u)y_{k-1}(x) d_{\frac{1}{3}}(x) \right) (t), \quad k = 1, 2, \dots, \\
y_0(t) &= t^{1/3} + t^{2/3} + \frac{9}{20}t^2 + \frac{3}{14}t^{7/3}.
\end{aligned}$$

Therefore, we have the following approximations

$$\begin{aligned}
y_1(t) &= t^{1/3} + t^{2/3} - \frac{243}{15400}t^{11/3} - \frac{27}{4928}t^4, \\
y_2(t) &= t^{1/3} + t^{2/3} + \frac{729}{4928000}t^{16/3} + \frac{729}{17425408}t^{17/3}, \\
y_3(t) &= t^{1/3} + t^{2/3} - \frac{6561}{11728640000}t^7 - \frac{729}{5367025664}t^{22/3},
\end{aligned}$$

$$\begin{aligned}
y_4(t) &= t^{1/3} + t^{2/3} + \frac{177147}{167719552000000} t^{26/3} + \frac{729}{3209481347072} t^9, \\
y_5(t) &= t^{1/3} + t^{2/3} - \frac{59049}{51993061120000000} t^{31/3} - \frac{19683}{89146553896271872} t^{32/3}, \\
y_6(t) &= t^{1/3} + t^{2/3} + \frac{177147}{232928913817600000000} t^{12} \\
&\quad + \frac{19683}{145130589743130607616} t^{37/3}.
\end{aligned}$$

The k^{th} approximation can be written in the following form

$$y_k(t) = t^{1/3} + t^{2/3} + 10^{-2k} \cdot c_1 t^{c_2}, \quad c_1, c_2 \in \mathbb{R}.$$

Then $y(t) = \lim_{k \rightarrow \infty} y_k(t) = t^{1/3} + t^{2/3}$. Note that the $y(t) = t^{1/3} + t^{2/3}$ is the exact solution of the problem (4.3)-(4.4).

Example 3. Let us consider the following conformable integro-differential equation

$$\begin{aligned}
T_{\frac{1}{2}} y(t) &= 2t\sqrt{t} \cos(2\sqrt{t}) - \sin(2\sqrt{t}) - t \sin(2\sqrt{t}) + 2t^2 \sin(2\sqrt{t}) \\
&\quad - 2 \int_0^t xy(x) d_{\frac{1}{2}}(x)
\end{aligned} \tag{4.5}$$

with initial condition

$$y(0) = 1. \tag{4.6}$$

Applying the operator $I_{\frac{1}{2}}$ to Eq. (4.5) we have the following equation

$$\begin{aligned}
y(t) &= 1 + \left(I_{\frac{1}{2}} (2u\sqrt{u} \cos(2\sqrt{u}) - \sin(2\sqrt{u}) - u \sin(2\sqrt{u}) + 2u^2 \sin(2\sqrt{u})) \right) (t) \\
&\quad - 2 \left(I_{\frac{1}{2}} \int_0^u xy(x) d_{\frac{1}{2}}(x) \right) (t)
\end{aligned}$$

and its iteration formula is obtained by

$$\begin{aligned}
y_k(t) &= y_0(t) + \left(I_{\frac{1}{2}} \int_0^u xy_{k-1}(x) d_{\frac{1}{2}}(x) \right) (t), \quad k = 1, 2, \dots, \\
y_0(t) &= 1 - 2t + \frac{2t^2}{3} + \frac{16t^3}{45} - \frac{124t^4}{315} + \frac{1076t^5}{14175} - \frac{3076t^6}{467775} + \dots
\end{aligned}$$

Therefore, we have the following approximations

$$y_1(t) = 1 - 2t + \frac{2t^2}{3} - \frac{4t^3}{45} + \frac{2t^4}{315} - \frac{4t^5}{14175} - \frac{5132t^6}{155925} + \dots$$

$$y_2(t) = 1 - 2t + \frac{2t^2}{3} - \frac{4t^3}{45} + \frac{2t^4}{315} - \frac{4t^5}{14175} + \frac{4t^6}{467775} - \frac{8t^7}{42567525} \\ + \frac{2t^8}{638512875} + \frac{95295196t^9}{97692469875} + \dots$$

Note that $\cos(2\sqrt{t}) = \sum_{i=0}^{\infty} (-1)^i \frac{t^{i22i}}{(2i)!}$ (see, for example, [1]). Then $y(t) \simeq \cos(2\sqrt{t})$ is the exact solution of the problem (4.5)-(4.6).

Example 4. Let us consider the following conformable integro-differential equation

$$T_{\frac{1}{2}}y(t) = 2t\sqrt{t} + \frac{2}{45}\sqrt{t}(45 + 36t^2 + 20t^4) \\ - \int_0^t (x^2 + y(x))^2 d_{\frac{1}{2}}(x) \quad (4.7)$$

with initial condition

$$y(0) = 1. \quad (4.8)$$

Applying the operator $I_{\frac{1}{2}}$ to Eq. (4.7) we have the following equation

$$y(t) = 1 + \left(I_{\frac{1}{2}}(2u\sqrt{u} + \frac{2}{45}\sqrt{u}(45 + 36u^2 + 20u^4)) \right)(t) \\ - \left(I_{\frac{1}{2}} \int_0^u (x^2 + y(x))^2 d_{\frac{1}{2}}(x) \right)(t)$$

and its iteration formula is obtained by

$$y_k(t) = y_0(t) + \left(I_{\frac{1}{2}} \int_0^u (x^2 + y_{k-1}(x))^2 d_{\frac{1}{2}}(x) \right)(t), \quad k = 1, 2, \dots, \\ y_0(t) = 1 + 2t + t^2 + \frac{8}{15}t^3 + \frac{8}{45}t^5.$$

Therefore, we have the following approximations

$$y_1(t) = 1 - \frac{t^2}{2} - \frac{8}{15}t^3 - \frac{68}{105}t^4 - \frac{64}{675}t^5 + \dots - \frac{128}{467775}t^{11}, \\ y_2(t) = 1 + t^2 + \frac{16}{45}t^3 + \frac{8}{105}t^4 + \frac{3056}{14175}t^5 + \dots - \frac{32768}{226471921396875}t^{23}, \\ y_3(t) = 1 + t^2 - \frac{16}{315}t^4 - \frac{32}{4725}t^5 - \frac{26272}{467775}t^6 + \dots + 10^{-23}t^{47}, \\ y_4(t) = 1 + t^2 + \frac{64}{14175}t^5 + \frac{64}{155925}t^6 + \frac{295168}{42567525}t^7 + \dots + 10^{-50}t^{95},$$

$$y_5(t) = 1 + t^2 - \frac{128}{467775}t^6 - \frac{256}{14189175}t^7 + \dots - 3 \cdot 10^{-105}t^{191},$$

$$y_6(t) = 1 + t^2 + 10^{-5}t^7 + 10^{-7}t^8 + \dots,$$

$$y_7(t) = 1 + t^2 + 4 \cdot 10^{-7}t^8 + 10^{-8}t^9 + \dots,$$

$$y_8(t) = 1 + t^2 + 10^{-8}t^9 + 10^{-10}t^{10} + \dots.$$

$y(t) = \lim_{k \rightarrow \infty} y_k(t) = 1 + t^2$. Note that the $y(t) = 1 + t^2$ is the exact solution of the problem (4.7)-(4.8).

Example 5. Let us consider the following conformable integro-differential equation

$$\begin{aligned} T_{\frac{1}{2}}y(t) &= \sqrt{t} + \frac{4}{315}t^{\frac{5}{2}}(35t^3 - 135t^2 + 189t - 105) \\ &\quad - 2 \int_0^t xy^3(x) d_{\frac{1}{2}}(x) \end{aligned} \quad (4.9)$$

with initial condition

$$y(0) = -1. \quad (4.10)$$

Applying the operator $I_{\frac{1}{2}}$ to Eq. (4.9) we have the following equation

$$\begin{aligned} y(t) &= -1 + \left(I_{\frac{1}{2}}(\sqrt{u} + \frac{4}{315}u^{\frac{5}{2}}(35u^3 - 135u^2 + 189u - 105)) \right)(t) \\ &\quad - 2 \left(I_{\frac{1}{2}} \int_0^u xy^3(x) d_{\frac{1}{2}}(x) \right)(t) \end{aligned}$$

and its iteration formula is obtained by

$$\begin{aligned} y_k(t) &= y_0(t) + \left(I_{\frac{1}{2}} \int_0^u xt(y_{k-1}(x))^3(x) d_{\frac{1}{2}}(x) \right)(t), \quad k = 1, 2, \dots, \\ y_0(t) &= -1 + t - \frac{4}{9}t^3 + \frac{3}{5}t^4 - \frac{12}{35}t^5 + \frac{2}{27}t^6. \end{aligned}$$

Therefore, we have the following approximations

$$y_1(t) = -1 + t + 0.099t^6 - 0.23t^7 + 0.23t^8 + \dots - 2 \cdot 10^{-6}t^{21},$$

$$y_2(t) = -1 + t - 0.0088t^9 + 0.03t^{10} - 0.045t^{11} + \dots + 4 \cdot 10^{-21}t^{66},$$

$$y_3(t) = -1 + t - 0.00042t^{12} - 0.0019t^{13} + 0.0039t^{14} + \dots - 10^{-66}t^{201},$$

$$y_4(t) = -1 + t - 0.000012t^{15} + 0.000071t^{16} + \dots,$$

$$y_5(t) = -1 + t + 2.510^{-7}t^{18} - 1.710^{-6}t^{19} + \dots.$$

$y(t) = \lim_{k \rightarrow \infty} y_k(t) = t - 1$. Note that the $y(t) = t - 1$ is the exact solution of the problem (4.9)-(4.10).

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